On Gaussian Invariant Measure to First Order PDEs

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Abstract

It is vital to consider invariant measures of dynamical systems induced by partial differential equations with irregular coefficients because of their application and importance in applied and pure mathematics. This thesis investigates the existence of invariant measures for the Lasota equation in threefold.

Firstly, we consider a special choice of the coefficients using the interpolation theory. We show that the law of the Liouville Fractional Brownian Motion with the Hurst parameter H is an invariant measure of the Lasota equation with the drift coefficient $a(x) = x$ and the multiplication parameter $\lambda = H - \frac{1}{2}$. Secondly, we study the existence and the uniqueness of mild solutions and prove the existence of invariant measures to the linear Lasota equation assuming only some basic properties of the coefficients a and c. Lastly, we consider the nonlinear Lasota equation and study the existence and the uniqueness of a global mild solution with a new set of assumptions for the coefficient c and prove the existence of invariant measures.

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Author's Declaration

I declare that this thesis is my own work and carried out under the supervision of Professor Zdzisław Brzeźniak. This work has not previously been presented for an award at this, or any other, University. All sources are acknowledged as References.

In the fond memory of my father

Chapter 1

Introduction

1.1 Overview

The concept of an invariant measure is an essential topic in mathematics and mathematical physics, particularly in Partial Differential Equations (PDEs) and Dynamical Systems. It is mostly simulated or modelled using differential equations. In modern mathematics, both applied and pure, it is important to consider PDEs with irregular coefficients. In the physical system, the importance of invariant measures appears in the Liouville Theorem. The existence of an invariant measure is one of the problems in many PDEs. One of the most important well-known evolution equations is called Lasota equation. The Lasota equation is important due to its properties and applications. For instance, its nonlinear version describes the process of differentiation and reproduction of the population of red blood cells depending on the concentration of hormones at a specific stage. It is important to study the behavior of the solution and the properties of such equation. Many authors have studied the solution and the properties of the equation under different assumptions, for instance, in such work of [7], [6], [27], [42], and [43].

A starting point of our deliberations is a result from the monograph [27] that the law of the Brownian Motion is an invariant measure for the Lasota equation when the multiplication parameter $\lambda = \frac{1}{2}$. The first objective of this thesis is to investigate what happens when $\lambda \neq \frac{1}{2}$. We find that the law of the Liouville Fractional Brownian Motion with the Hurst parameter H is an invariant measure for such an equation when $\lambda = H - \frac{1}{2}$. This is achieved by using fractional integral and derivative operators. In particular, we show that the two semigroups $\{\tilde{S}_t\}_{t\geq 0}$ corresponding to $\lambda = \frac{1}{2} + \alpha$ on the fractional Sobolev space ${}_0H^{\alpha,p}[0,1]$ and $\{S_t\}_{t\geq 0}$ corresponding to $\lambda = \frac{1}{2}$ on space $L^p(0,1)$, commute via the fractional derivative D^α and fractional integral I^α maps. Our work is motivated by the following principle. Suppose we know an invariant measure to one semigroup on one Banach space and we have another Banach space with another semigroup for which those two semigroups are "commuting" in an appropriate sense, then the new semigroup has also an invariant measure. A such generalisation can massively help to understand more PDEs and their properties in more complicated spaces such as interpolation spaces.

One of the fundamental questions to be asked about nonlinear evolution equations is the existence and uniqueness of the solutions. Thus, in the second objective of this thesis, we address this question and assume sufficient conditions for the coefficients of the Lasota equation to find the existence and the uniqueness of mild solutions. Moreover, we study the large-time behavior of the solutions. Studying such solutions along with proving the existence of nontrivial invariant measures and their properties make our work in this thesis more rigorous than Rudnicki in [43]. We study a nonlinear Lasota equation under certain assumptions of the coefficients a and c. Our main equation is the following problem

$$\begin{gathered}\frac{\partial u(t,x)}{\partial t} + a(x)\frac{\partial u(t,x)}{\partial x} = c(u(t,x)), \quad t \geq 0,\ x \in [0,1] \qquad (1.1.1)\\ u(0,x) = u_0(x), \quad x \in [0,1]\end{gathered}$$

where, $u_0 \in {}_0C\big([0,1],\mathbb{R}\big)$. We consider first in Chapter 4 the linear part of the above problem. We assume natural assumptions where the drift coefficient, i.e., the function a is continuous and satisfies the so-called Osgood condition and the function $c = \lambda \cdot u$, for some $\lambda \geq 0$, is constant. We prove the existence and the uniqueness of mild solutions by using the characteristic method. In particular, we prove that a natural family of linear operators associated with equation (1.1.1) with $c = 0$, is a C_0-semigroup on an appropriately chosen Banach space $E = {}_0C([0,1])$. Moreover, we characterise the domain of the infinitesimal generator of this semigroup. Furthermore, we prove the existence of an invariant measure under these natural assumptions and study some properties related to this measure.

In the following Chapter 5 we generalise the linear case to be a nonlinear case and assume our function c to be Lipschitz on balls and of dissipative type. We prove the existence and the uniqueness of mild solutions and analyse the property of these solutions. We also provide an explicit solution by using the characteristic method. This solution allows us later to study the operator's properties, which guides us to prove the existence of an invariant measure to the nonlinear Lasota equation with irregular coefficients. We prove the existence of the invariant measure by following a similar path presented by Rudnicki in [41]. We first define two Banach spaces E and Y with two semigroups. Then we find a stationary process (in our case the Ornstein-Uhlenbeck process) in the space Y. This process induces an invariant measure for the shift semigroup on the space Y. This measure on Y induces a measure on the space E which turns out to be invariant for the semigroup on E.

1.2 Thesis Structure

This thesis comprises six chapters, including the introduction, we describe them as the following.

In Chapter 2, we give basic definitions and notations that are necessary to make this thesis well-contained. These preliminaries contain some definitions and results (presented without proof) about linear operators, semigroup theory with some applications, random variables and stochastic processes with values in Banach spaces, and probability measures in Banach spaces.

In Chapter 3, our main work is motivated by a result presented in the monograph [28] by Lasota and Mackey in 1981. We generalise the existence of the invariant measures for the dynamical systems generated by a first-order PDE. Then, we construct this generalisation by studying the Lasota equation with different parameters. In other words, we extend the parameter to any parameter between $\frac{1}{2}$ and $\frac{3}{2}$ using the interpolation theory. Firstly, we prove the existence of invariant measures for a special case if $\alpha = 1$, i.e., we consider the spaces $L^p(0,1)$ and ${}_0H^{1,p}(0,T)$. Next, after we tested our method on the special case we extend the result for α to be any value between $(0,1)$, i.e, ${}_0H^{\alpha,p}(0,T)$. In the chapter the drift coefficient $a(x)$ in the equation (1.1.1) is equal to x and the function $c(u)$ is linear of the form λu.

Chapter 4 is devoted to studying the question of the existence and the uniqueness of a solution to a first-order PDE. In particular, we study a more general case of the equation presented in Chapter 3 when the coefficient $a(x)$ is no longer equal to x but is allowed to be irregular. We assume that a is only continuous and satisfies the Osgood condition. We consider a linear case of Lasota equation, i.e., when $c(u) = \lambda \cdot u$, for some $\lambda \geq 0$. We prove that our PDE has a unique mild solution defined via characteristic methods corresponding to the Ordinary differential equation (ODE). In particular, we show that the ODE has a unique globally defined solution until $-\infty$, i.e., the solution will be defined on $(-\infty, 0]$. After having established the well-posedness of the equation, we employ methods developed by Rudnicki [43], when the coefficient a was assumed to be a function of C^1-class, to prove the existence of invariant measures using a new set of assumptions and we prove this measure satisfies some property.

Chapter 5 extends what we started in Chapter 4 by considering the nonlinear case under new assumptions on the coefficient c. We assume that the function c is Lipschitz on balls and of dissipative type and we prove the existence and the uniqueness of mild solutions. We also analyse the properties of an appropriate solution. Moreover, we use rigorously the notion of a classical solution to prove the representation Theorem 5.35. Next, we prove the existence of an invariant measure under our assumptions for the coefficients a

and c. At the end of the chapter, we discuss our main contribution concerning the paper by Rudnicki [43].

In Chapter 6, we present related future research questions that are revealed to some of our main results in the thesis.

Chapter 2

Preliminaries

This chapter introduces the most important preliminaries in different topics in mathematics that are used in the rest of the thesis in general.

2.1 Functional Analysis

Functional Analysis is an abstract branch of mathematics that arose from Classical Analysis. Some described it as infinite-dimensional Linear Algebra along with analysis [44]. It heavily relies on vector spaces among other concepts such as metric space and topology. Since this thesis depends on functional spaces, we provide fundamental definitions and some basic theorems that are intensively going to be used.

2.1.1 Banach spaces

A Banach space is a normed vector space that is complete, see [44]. It is named after Polish mathematician Stefan Banach who introduced it for the first time and studied it systematically in the 1920s. We state in this section the most important definitions and properties related to Banach spaces.

Definition 2.1. Assume that X is a real vector space. A real-valued function $\|\cdot\|$ on X is called a norm if and only if for any $x, y \in X$ and for all $\alpha \in \mathbb{R}$, the following properties hold

1. $\|x\| \geqslant 0 \qquad$ (positivity),
2. $\|x\| = 0 \Leftrightarrow x = 0 \qquad$ (definiteness),
3. $\|\alpha x\| = |\alpha|\|x\| \qquad$ (homogeneity),

4. $\|x+y\| \leqslant \|x\| + \|y\|$ (triangle inequality).

Definition 2.2. A Banach space is a complete normed vector space.

Remark 2.3. A Banach space is a metric space and hence a topological space.

Definition 2.4. A topological space (for more detail see [46, Chapter X]) is called separable if it has a countable dense subset.

In the following example, we provide the most important separable Banach space, which we use throughout this thesis in fact our space is ${}_0C([0,1])$. Proofs related to this example are made available in Appendix B.1.

Example 2.5. *Assume that $a, b \in \mathbb{R}$ such that $a < b$. The space $X = C([a,b])$ which defined by the following formula*

$$X = C[a,b] = \{f : [a,b] \to C : f \text{ is continuous } \}. \tag{2.1.1}$$

is a separable Banach space with a norm defined by

$$\|f\| = \sup_{x \in [a,b]} |f(x)|. \tag{2.1.2}$$

Similarly, the space $E = {}_0C([a,b]) := \{f : [a,b] \to C : f \text{ is continuous } : f(a) = 0\}$ is also a separable Banach space with the norm defined on (2.1.2)*. Moreover, the space E is a closed subspace of the space X, see Lemma B.2.*

Definition 2.6. Assume that X is a normed vector space. A function $f : X \to X$ is called continuous at x_0 if

$$\forall \varepsilon > 0 \quad \exists \delta > 0 : x \in X, \|x - x_0\| < \delta \Rightarrow \|f(x) - f(x_0)\| < \varepsilon.$$

If f is continuous at all $x \in X$, then f is said to be continuous in X.

The following definition is taken from monograph [40], see Definition 7.17.

Definition 2.7. A function $f : [a,b] \to \mathbb{R}$ is called absolutely continuous if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for any finite family of intervals $I_j = (a_j, b_j),\ 1 \leq j \leq n$ which are pointwise disjoint, $\bigcup_{j=1^n} I_j \subset [a,b]$ and $\sum_{j=1}^n |I_j| \leq \delta$,

$$\sum_{j=1}^{n} |f(b_j) - f(a_j)| \leq \varepsilon.$$

We also recall [40, Theorem 7.20].

Theorem 2.8. *If a function $f : [a,b] \to \mathbb{R}$ is called absolutely continuous, then f is differentiable at almost all points with respect to the Lebesgue measure of the interval $[a,b]$, the derivative function f' belongs to the space $L^1([a,b])$, and the following Fundamental Theorem of Calculus holds*

$$f(y) - f(x) = \int_x^y f'(t)\,dt, \quad a \le x \le y \le b.$$

Definition 2.9. Assume that (X,d) and (Y,ρ) are two metric spaces. A function $f : X \to Y$ is called continuous at $x \in X$ if for any $\varepsilon > 0$ there exists $\delta > 0$ such that if

$$d(x,y) < \delta \text{ for } \quad \text{then} \quad \rho\big(f(x), f(y)\big) < \varepsilon.$$

Definition 2.10. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence of elements of a metric space (X,d). We say that the sequence $\{x_n\}_{n=1}^{\infty}$ converges to $x \in X$ if

$$\forall\, \varepsilon > 0\; \exists\, N = N_\varepsilon : \forall n \ge N \text{ then } d(x_n, x) \le \varepsilon.$$

We write $\lim_{n\to\infty} x_n = x$.

Definition 2.11. A sequence $\{x_n\}_{n=1}^{\infty}$ of elements of a metric space (X,d) is called a Cauchy sequence if and only if for any $\varepsilon > 0$ there exists $N_\varepsilon \in \mathbb{N}$ such that for all $m, n > N_\varepsilon$ then $d(x_m, x_n) < \varepsilon$.

Definition 2.12. We say a metric space (X,d) is complete if every Cauchy sequence in X is convergent.

Theorem 2.13. *Assume that (X,d) and (Y,d) two metric spaces. Assume for every $n \in \mathbb{N}$ the function $f_n : X \to Y$ is a continuous function and $\{f_n\}_{n\ge 0}$ converges uniformly to a function f on the same metric space, then f is also continuous.*

Definition 2.14. Let $f_n : X \to Y, n \in \mathbb{N}$, where X is a set and (Y,d) is a metric space. We say $f_n \to f$ uniformly if and only if for any $\varepsilon > 0$ there exists N such that for all $n \in \mathbb{N} \in \mathbb{N}$ such that

$$d(f_n(x), f(x)) \le \varepsilon, \quad \text{for every} \quad x \in X.$$

The following Theorem is called Sandwich Theorem for functions [46], which is related to the limits of functions and it is very useful in proving some properties.

Theorem 2.15. *Assume that $I \subset \mathbb{R}$ and a is an accumulation point of I. Let $f, g, h : I \to \mathbb{R}$ be functions such that $g(x) \le f(x) \le h(x)$ for all $x \in I$. If*

$$g(x) \underset{as \quad x\to a}{\to} L \textit{ and } h(x) \underset{as \quad x\to a}{\to} L, \quad \textit{then} \quad f(x) \underset{as \quad x\to a}{\to} L.$$

Definition 2.16. A vector space endowed with an inner product is called Hilbert space.

Definition 2.17. Let (X, d) be a metric space and M is a subset of X. We say that M is compact if every sequence in M has a convergent subsequence with the limit in M.

Definition 2.18. Suppose that (X, d) is a metric space and the set $A \subset X$. Then, we say that A is dense set in X if and only if for every $x \in X$ and every $r > 0$ there exists $y \in A$ such that $d(y, x) < r$.

An equivalent definition to Definition 2.18 is the following:

Definition 2.19. Let (X, d) be a metric space. Then, a set $D \subset X$ is dense set if and only if for every $x \in X$ there exists a sequence $\{x_n\} \subset D$ such that $x_n \to x$.

Theorem 2.20. *Let (E, d) is a metric space, a set $A \subset E$ be dense in E, and that $A \subset B \subset E$. Then $B \subset E$ is dense in E.*

Proof of Theorem 2.20. Let $x \in E$, suppose $r > 0$ is given. Since A is dense in E then there exists $y \in A$ such that $d(x, y) < r$. Since $A \subset B$, then y is also in B. Hence B is dense in E. $\square$

Definition 2.21. Let $(X, \|\cdot\|)$ be a Banach space and let $D \subset X$ be a dense linear subspace of X. A linear map $A : D \to X$ is called closed in $X \times X$ if and only if the graph(A) is closed, i.e., if $(x_n, Ax_n) \to (x, y) \in X \times X$, where $x_n \in D$, then $x \in D$ and $y = Ax$.

2.1.2 Linear bounded operators

Knowing the general form of bounded linear functional in various spaces is very important. Part of our work is mainly based on operators, and therefore, in this section, we introduce some definitions and general properties of linear operators between two normed vector spaces.

Definition 2.22. Let X, Y be vector spaces over $\mathbb{R}$. A function $T : X \to Y$ is called a linear transformation (or mapping) if for any $\alpha, \beta \in \mathbb{R}$ and $x, y \in X$

$$T(\alpha x + \beta y) = \alpha T(x) + \beta T(y).$$

Definition 2.23. Let T be a function from a set X to a set Y and $B \subset Y$. The inverse image of B is defined by

$$T^{-1}(B) := \{x \in X : T(x) \in B\}.$$

Definition 2.24. Assume that X, Y are two normed vector spaces. If $T : X \to Y$ is a linear function then T is called bounded if and only if there exists a real number $C > 0$ such that

$$\|T(x)\| \leq C\|x\|, \quad x \in X.$$

The following theorem plays an important role in proving some properties of linear operators on Banach spaces.

Theorem 2.25. *[40, Theorem 5.4] If $(X, \|\cdot\|)$ and $(Y, \|\cdot\|)$ are two a normed vector spaces and $f : X \to Y$ is a linear operators, then the following are equivalent*

1. *f is bounded,*
2. *f is continuous, i.e., continuous at every $a \in X$,*
3. *f is continuous at one point of X.*

The following two theorems are important results in the proofs related to Sobolev spaces which are discussed in Section 2.3.2.

Theorem 2.26. *[10, Theorem 6.2] If X and Y are two normed vector spaces and $f : X \to Y$ is a linear and bounded (i.e., continuous) map, then the* $\ker f$ *is a closed subspace of X, where*

$$\ker f = \{x \in X : f(x) = 0\}.$$

Theorem 2.27. *[37] If $(X, \|\cdot\|_X)$ is a Banach space and $Y \subset X$ is a closed subspace of X endowed with norm $\|y\|_Y := \|y\|_X$, $y \in Y$, then $(Y, \|\cdot\|_Y)$ is also a Banach space.*

Lemma 2.28. *[37] Let X with norm $\|\cdot\|_X$ be a separable Banach space and Z be a linear subspace of X. Let us endow Z with the norm inherited from X, i.e.,*

$$\|x\|_Z := \|x\|_X, \ \ x \in Z.$$

Then Z with norm $\|\cdot\|_Z$ is a separable normed vector space.

Definition 2.29. Let X and Y be two Banach spaces. A linear map $T : X \to Y$ is called an isomorphism if it is bijective and bounded and its inverse $T^{-1} : Y \to X$ is bounded.

Definition 2.30. A map $T : X \to Y$ is called embedding if it is an injective continuous map.

Corollary 2.31. *Let X, Y, Z and E are Banach spaces and $f : X \to Y$, $g : Y \to Z$ and $h : Z \to E$ are isomorphisms, then $h \circ g \circ f$ is also an isomorphism and*

$$(h \circ g \circ f)^{-1} = f^{-1} \circ g^{-1} \circ h^{-1}.$$

2.2 Measure Theory

Measure theory is a very important concept in mathematics, especially in analysis. Some functions do not behave well under integration and their limits may not exist. Such problems can be addressed by the measure theory which provides more general frameworks to cover these problems. In this section, we introduce the basic notation of measure theory including a brief introduction to the Lebesgue and probability measure. We will give more information regarding the invariant measure in Chapter 3.

2.2.1 Probability measure

Definition 2.32. Let Ω be any non-empty set. A collection $\mathcal{F}$ of subsets of Ω is called a σ-field (or σ-algebra) on Ω if the following conditions are satisfied

1. $\emptyset \in \mathcal{F}$,
2. $\mathcal{F}$ is closed under complements, i.e., if $A \in \mathcal{F}$ then $A^{\mathsf{c}} \in \mathcal{F}$,
3. $\mathcal{F}$ is closed under countable unions, i.e., if $A_1, A_2, \ldots \in \mathcal{F}$, then $\cup_i A_i \in \mathcal{F}$,
4. if $A_1, A_2, \ldots \in \mathcal{F}$, then $\cap_i A_i \in \mathcal{F}$.

Definition 2.33. The Borel σ-field over a metric space (X, d) is the smallest σ-field containing all open sets of X and we write $\mathcal{B}(X)$. In particular, $\mathcal{B}(\mathbb{R})$ is the smallest σ-field on $\mathbb{R}$ that contains all open (or equivalently closed) subsets of $\mathbb{R}$.

Proposition 2.34. *Suppose X, Y are topological spaces with topologies denoted by* $\operatorname{top}(X)$ *and* $\operatorname{top}(Y)$. *Let $\mathcal{B}(X), \mathcal{B}(Y)$ denote the Borel σ-field on X, Y respectively, i.e.,*

$$\mathcal{B}(X) = \sigma\big(\operatorname{top}(X)\big) \text{ and } \mathcal{B}(Y) = \sigma\big(\operatorname{top}(Y)\big).$$

If a map $f : X \to Y$ is continuous then f is $\mathcal{B}(X)/\mathcal{B}(Y)$ (Borel) measurable, i.e.,

$$f^{-1}(B) \in \mathcal{B}(X) \quad \text{for every} \quad B \in \mathcal{B}(Y).$$

Definition 2.35. Let $\mathcal{F}$ be a σ-field on Ω. A probability measure on $\mathcal{F}$ is a function $\mathbb{P} : \mathcal{F} \to [0, 1]$ such that

1. $\mathbb{P}(\Omega) = 1$,
2. if $\{A_i\}_{i=1}^{\infty} \in \Omega$ are pairwise disjoint sets (that is $A_i \cap A_j = \emptyset$ for $i \neq j$), then

$$\mathbb{P}\Big[\bigcup_{i=1}^{\infty} A_i\Big] = \sum_{i=1}^{\infty} \mathbb{P}(A_i).$$

The triple $(\Omega, \mathcal{F}, \mathbb{P})$ is called a probability space, where Ω is the non-empty set, $\mathcal{F}$ is σ-field on Ω, and $\mathbb{P}$ is the probability measure. The sets that belong to $\mathcal{F}$ are called events.

Definition 2.36. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. A function $\xi : \Omega \to \mathbb{R}$ is called a random variable or measurable if and only if $\xi(A)^{-1} \in \mathcal{F}$ for every $A \in \mathcal{B}(\mathbb{R})$.

Definition 2.37. A stochastic process is a family $\xi = \{\xi(t), t \in T\}$ of random variables $\xi(t)$ parametrized by $t \in T$, where $T \subset \mathbb{R}$.

The following definition is classical and we follow here Definition 1.1 from the book [31].

Definition 2.38. Suppose $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space. A stochastic process $w = \big(w(t)\big)_{t\geq 0}$ is called a Brownian Motion (BM for short) if and only if the following conditions hold

1. $w(0) = 0$ almost surely, i.e., there exists a set $\Omega_1 \in \mathcal{F}$ such that $\mathbb{P}(\Omega_1) = 1$ and for all $\omega \in \Omega_1$, $w(0, \omega) = 0$.
2. w has independent increments, i.e., if $0 \leq t_0 < t_1 < \cdots < t_n < \infty$, the random variables $w(t_0), w(t_1) - w(t_0), w(t_2) - w(t_1), \cdots, w(t_n) - w(t_{n-1})$ are independent.
3. if $0 \leq s < t$ then $w(t) - w(s)$ is $N(0, t-s)$, i.e., $w(t) - w(s)$ has a normal distribution with parameters $\mu = 0$ and $\sigma^2 = t - s$. In other words, $w(t) - w(s)$ is an absolutely continuous random variable with density
$$p_{t-s}(x) = \frac{1}{\sqrt{(2\pi)(t-s)}} e^{\frac{-x^2}{2(t-s)}}, \quad x \in \mathbb{R}.$$
4. The trajectories $[0, \infty) \ni t \mapsto w(t)$ are, almost surely, continuous functions.

Theorem 2.39. *[13, Theorem 6.2] Suppose that $\big(w(t)\big)_{t\geq 0}$ is a Brownian Motion on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let $A_1, A_2, \dots, A_n \in \mathcal{B}(\mathbb{R})$, for example $A_1 = (a_1, b_1), A_2 = (a_2, b_2), \dots$. Let $0 < t_1 < t_2 < \cdots < t_n$. Then*

$$\begin{aligned}&\mathbb{P}\big(\{\omega \in \Omega : w(t_1)(\omega) \in A_1, w(t_2)(\omega) \in A_2, \dots, w(t_n)(\omega) \in A_n\}\big)\\ &= \int_{A_1} \cdots \int_{A_n} p_{t_1}(x_1) p_{t_2 - t_1}(x_2 - x_1) p_{t_3} - t_2(x_3 - x_2) \dots p_{t_n - t_n - 1}(x_n - x_{n-1})\, dx_n \dots dx_1.\end{aligned}$$

In Chapters 4 and 5, we use the following sophisticated property mentioned in the book by [31, Theorem 5.1] of the BM called the law of iterated logarithm. We write down the presentation here as the following theorem.

Theorem 2.40. *Let $(w(t))_{t\geq 0}$ be a Brownian motion. Then, almost surely,*

$$\limsup_{t\to\infty} \frac{|w(t)|}{\sqrt{2t \log\log(t)}} = 1.$$

2.2.2 Lebesgue measure

Definition 2.41. Let $\mathcal{F}$ be a σ-field on a set Ω. A measure μ is a function

$$\mu : \mathcal{F} \to [0, \infty]$$

that satisfies the following conditions.

1. $\mu(\emptyset) = 0$,
2. if $\{A_i\}_{i=1}^{\infty} \in \Omega$ are pairwise disjoint sets (that is $A_i \cap A_j = \emptyset$ for $i \neq j$), then

$$\mu[\bigcup_{i=1}^{\infty} A_i] = \sum_{i=1}^{\infty} \mu(A_i).$$

The measure μ could be finite and we write $\mu(\Omega) < \infty$ or infinite we write $\mu(\Omega) = \infty$. Moreover, the triple $(\Omega, \mathcal{F}, \mu)$ is called measure space, see e.g. [2], p. 161 and/or Definition 1.3.5 in [4].

Definition 2.42. A set $X \subset \mathbb{R}$ is called Lebesgue measurable or simply measurable if for any elementary set E, we have that

$$\mu(E \cap X) + \mu(E \setminus X) = \mu(E).$$

The set of all Lebesgue measurable subsets of $\mathbb{R}$ will be denoted by $\mathcal{L}(\mathbb{R})$.

Definition 2.43. If M is a subset of $\mathbb{R}$ such that $M = \cup_{i=1}^{k} P_i$ for some pairwise disjoint intervals $P_1, \cdots P_k$, then M is called an elementary set.

Definition 2.44. The Lebesgue measure on $\mathcal{B}(\mathbb{R})$ is a unique $[0, \infty]$-valued measure $m : \mathcal{B}(\mathbb{R}) \to [0, \infty]$ such that $m\big([a, b]\big) = |b - a|, \quad \text{for all } a, b \in \mathbb{R}, a < b$. Note that: $m\big([0, \infty]\big) = \infty$.

Definition 2.45. The Lebesgue measure on $\mathcal{L}(\mathbb{R})$ is the unique $[0, \infty]$-valued measure $m_L : \mathcal{L}(\mathbb{R}) \to [0, \infty]$ such that $m_L\big([a, b]\big) = |b - a|, \quad \text{for all } a, b \in \mathbb{R}, a < b$.

It can be proved that the measure m from Definition 2.44 is the restriction of the measure m from Definition 2.45, i.e., $m(A) = m_L(A) \;$ for every $\; \mathcal{B}(\mathbb{R})$.

Definition 2.46. A measurable space is a pair $(\Omega, \mathcal{F})$, where Ω is a non-empty set and $\mathcal{F}$ is a σ-field of subset of Ω, [2], p. 161 and/or Definition 1.2.3 in [4].

For any element $S \in \mathcal{F}$ we say that S is a measurable set.

Usually, authors introduce the notion of a strongly measurable X-value function but we do not do this because of the following corollary

Corollary 2.47. *If X is a separable Banach space then the classes of strongly measurable functions and Borel measurable functions are equal.*

2.2.3 Gaussian measures in Banach spaces

Definition 2.48. A Gaussian measure μ on $\mathbb{R}$ is either concentrated at one point $\mu = \delta_m$ or has a density

$$f(x) = \frac{1}{\sqrt{2\pi q}} e^{-\frac{(x-m)^2}{2q}}, \quad x \in \mathbb{R}$$

for some $q > 0$ and $m \in \mathbb{R}$. Such a measure is denoted by $\mathscr{N}(m, q)$.

Definition 2.49. A Borel measure μ has a density $f(x)$ if and only if function f is a Borel measurable and for every $A \in \mathcal{B}(\mathbb{R})$

$$\mu(A) = \int_A f(x)\, dx.$$

Let $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space. A density, if it exists, of random variable $\xi : \Omega \to \mathbb{R}$ is a Borel measurable function $f : \mathbb{R} \to [0, \infty)$ such that

$$\mathbb{P}(\xi \in A) = \int_A f(x)\, dx, \quad \text{for every} \quad A \in \mathcal{B}(\mathbb{R}).$$

Definition 2.50. Suppose that $\big(w(t)\big)_{t \geq 0}$ is BM. We define a cylinder set in the space $E = {}_0C([0,1])$ by $\Gamma = \{x \in E : x(t_1) \in A_1, \cdots x(t_n) \in A_n\}$, where

$$0 < t_1 < t_2 < \cdots < t_n \leq 1 \quad \text{and} \quad A_1 < \cdots < A_n \in \mathcal{B}(\mathbb{R}).$$

The family of all cylinder sets is denoted by $cyl(E)$. The law of BM is the unique Borel probability measure $\mu : \mathcal{B}(E) \to [0,1]$ such that for every $\Gamma \in cyl(E)$

$$\mu(\Gamma) = \int_{A_1} \cdots \int_{A_n} p_{t_1}(x_1) p_{t_2 - t_1}(x_2 - x_1) \cdots p_{t_n - t_n - 1}(x_n - x_n - 1)\, dx_n, \cdots dx_1.$$

Definition 2.51. Gaussian measure on a Banach space. Let E be a separable Banach space with $\mathcal{B}(E)$ the Borel σ-field. A probability measure $\mu : \mathcal{B}(E) \to [0,1]$ is called a Gaussian measure on the space E if and only if the law of an arbitrary linear functional $\varphi \in E^*$ is a Gaussian measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$.

Proposition 2.52. *The law of Brownian motion is a Gaussian measure on the space* $E = {}_0C([0,T])$

2.3 Lebesgue and Sobolev Spaces

Function spaces, in particular, L^p and Sobolev spaces, are considered fundamental tools in modern analysis such as partial differential equations (PDEs). The importance of these spaces appears from the completeness property of the space which helps us to do analysis and find the existence of solutions. In this section, we are not considering the whole theory of such spaces instead, only properties that are useful and directly related to this thesis are given.

2.3.1 Lebesgue spaces

In this subsection, we assume that μ to be the Leagues measure on $\mathcal{L}(\mathbb{R}_+)$. All definitions in this section are valid if $\mathbb{R}_+$ is replaced by any Borel subset of $\mathbb{R}$, for instance, $\mathbb{R}_+ = [0, 1]$

Definition 2.53. We say that a function $f \in \mathcal{L}^p(\mu) = \mathcal{L}^p(\mathbb{R}_+)$ if and only if

1. The function $f : \mathbb{R}_+ \longrightarrow \mathbb{R}$ is Lebesgue measurable,
2. $\int_{\mathbb{R}_+} |f(x)|^p d\mu(x) < \infty$.

Definition 2.54. Let $f \in \mathcal{L}^p(\mathbb{R}_+, \mu)$, we define the equivalence classes of f as the follow $[f]_\sim = [f] = \{g \in \mathcal{L}^p : g \sim f\}$.

Any function $g \in [f]_\sim$ is called a representative of $[f]_\sim$.

Definition 2.55. If $1 \leq p < \infty$, we define L^p space to be the space of all equivalence classes of measurable functions $f : \mathbb{R}_+ \to \mathbb{R}$ such that

$$\|f\|_{L^p(\mathbb{R}_+)} = \Big(\int_0^1 |f(x)|^p \mu\, d(x)\Big)^{\frac{1}{p}} < \infty.$$

Definition 2.56. If $f \in \mathcal{L}^p(\mathbb{R}_+, \mu)$ we can define the norm of the equivalent class of function f as follows

$$\|[f]_\sim\|_p = \|f\|_p = \left(\int_{\mathbb{R}_+} |f(x)|^p \, d\mu(x)\right)^{\frac{1}{p}}.$$

Definition 2.57. Let X be a separable Banach space. We say that $f \in L^1_{loc}(\mathbb{R}_+; X)$ if and only if

1. $f : \mathbb{R}_+ \to X$ is measurable, and
2. $\int_{[0,T]} |f(s)|_X \, ds < \infty$, for every $T > 0$.

Now we state a theorem without proof that contains two basic inequalities called Hölder and Minkowski inequalities. The proof of this theorem can be found in [40, Theorem 3.5].

Theorem 2.58. *Let $p, q \in [1, \infty)$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Assume that $f : \mathbb{R}_+ \to [0, \infty)$ and $g : \mathbb{R} \to [0, \infty)$ are Lebesgue measurable functions. The first one is called the Hölder's and it is as follows*

$$\int_{\mathbb{R}} f(x)g(x)\, d\mu \leq \left(\int_{\mathbb{R}} f^p(x)\, d\mu \right)^{\frac{1}{p}} \left(\int_{\mathbb{R}} g^q(x)\, d\mu \right)^{\frac{1}{q}}. \tag{2.3.1}$$

The second one is called Minkowski and it is as follows

$$\left(\int_{\mathbb{R}} (f(x) + g(x))^p\, d\mu \right)^{\frac{1}{p}} \leq \left(\int_{\mathbb{R}} f^p(x)\, d\mu \right)^{\frac{1}{p}} + \left(\int_{\mathbb{R}} g^p(x)\, d\mu \right)^{\frac{1}{p}}.$$

Theorem 2.59. *The space $L^p(\mathbb{R}_+, \mu)$ is a Banach space.*

The proof of this theorem can be found in [40, Theorem 3.11]. $L^p(\mathbb{R}_+, \mu)$ spaces are important examples of Banach spaces. We use these spaces later in Chapter 3. Note that the elements in the space L^p are equivalent classes, and we suppose that all the elements in L^p are not equivalent classes but functions because if it is true for functions it follows that it will be true for equivalent classes.

2.3.2 Sobolev spaces

As we mentioned before Sobolev spaces are a very useful concept in partial differential equations. Specifically, the existence of a solution with more nice properties. The solution of differential equations if it exists normally belongs to Sobolev spaces. Before we go through the most important definitions and properties of such spaces we need to shed light on notation called weak derivatives. This notation is a generalization of the derivative of functions. Therefore, in this subsection, we start by setting up the definition of a weak derivative. Most of the results in this section have been taken from [5]. We start by setting up some notations. Let $I \subset \mathbb{R}$ be an open interval (or open set). We define the space $C_c^\infty(I)$ as follows:

$$\begin{aligned} C_c^\infty(I) = \ &\{\phi : I \to \mathbb{R} \text{ such that } \phi \text{ is infinitely differentiable and there} \\ &\text{exists a compact interval } K \subset I : \phi(x) = 0 \text{ if } x \notin K\}. \end{aligned}$$

Definition 2.60. Assume that $I \subset \mathbb{R}$ is a Borel subset. A function $u \in L^1(I)$ if and only if $u : I \to \mathbb{R}$ is Lebesgue measurable and

$$\int_I |u(x)| d\mu(x) = \int_{\mathbb{R}} |u(x)| \mathbb{1}_I(x) d\mu(x) < \infty,$$

where

$$\mathbb{1}_I(x) = \begin{cases} x, & \text{if } x \in I \\ 0, & \text{if } x \notin I \end{cases}$$

Definition 2.61. A function $u \in L^1_{loc}(I)$ if and only if $u : I \to \mathbb{R}$ is measurable and for every $K \subset I$ compact interval,

$$\int_K |u(x)| d\mu(x) < \infty.$$

Definition 2.62. Let $u, v \in L^1_{loc}(I)$. We say that $v = Du$ (i.e., v is the weak derivative of u) if and only if for any function $\phi \in C_c^\infty(I)$,

$$\int_I u(x)\phi'(x) d\mu(x) = (-1) \int_I v(x)\phi(x) d\mu(x).$$

We now point out the basic definitions and properties of Sobolev spaces. From now on we consider the interval I to be $(0, 1)$.

Definition 2.63. Let $p \in [1, \infty)$. We say that $u \in H^{1,p}(0, 1)$ if and only if the function u is continuous, $\int_{(0,1)} |u|^p d\mu < \infty$ and there exists a function $v : (0, 1) \to \mathbb{R}$, which is Lebesgue measurable such that $\int_{(0,1)} |v|^p d\mu < \infty$, and $Du = v$ (in a weak sense). The space $H^{1,p}(0, 1)$ is called Sobolev space.

For a function $u \in H^{1,p}(0, 1)$ we define the norm in the space $H^{1,p}(0, 1)$ as follows

$$\|u\|_{H^{1,p}(0,1)} = \Big[\int_{(0,1)} |u|^p \, d\mu + \int_{(0,1)} |Du|^p \, d\mu \Big]^{\frac{1}{p}}. \tag{2.3.2}$$

Note: By definition above, the norm $\|u\|_{H^{1,p}(0,1)}$ is finite.

Proposition 2.64. *[5, Proposition 9.1] The space $H^{1,p}(0, 1)$ with norm defined by (2.3.2) is a separable Banach space.*

In the following, we introduce an equivalent definition to the Definition 2.63 which is an important tool.

Definition 2.65. The space $\tilde{H}^{1,p}(0, 1)$ is the space of all functions $u \in C([0, 1])$ such that the weak derivative Du of function u exists and $Du \in L^p(0, 1)$.

The norm of the space $\tilde{H}^{1,p}(0, 1)$ for any function u is given by

$$\|u\|_{\tilde{H}^{1,p}(0,1)} = \left(|u|^p_{C([0,1])} + |Du|^p_{L^p(0,1)} \right)^{\frac{1}{p}}. \tag{2.3.3}$$

Theorem 2.66. *The space $\tilde{H}^{1,p}(0, 1)$ with norm defined in (2.3.3) is a Banach space.*

It is important to say that those two spaces are equal and their norms are equivalent as we formulate in the next result.

Theorem 2.67. $H^{1,p}(0,1) = \tilde{H}^{1,p}(0,1)$ *and the norms are equivalent, i.e., there exists* $C > 0$*, such that for every* $u \in H^{1,p}(0,1)$ *we have*

$$\frac{1}{c}\|u\|_{H^{1,p}(0,1)} \leq \|u\|_{\tilde{H}^{1,p}(0,1)} \leq C\|u\|_{H^{1,p}(0,1)}.$$

Definition 2.68. The space ${}_0H^{1,p}(0,1)$ is defined as follows

$${}_0H^{1,p}(0,1) := \{u \in H^{1,p}(0,1) : u(0) = 0\}. \tag{2.3.4}$$

Remark 2.69. Since $H^{1,p}(0,1) = \tilde{H}^{1,p}(0,1)$, the equality (2.3.4) makes sense.

Theorem 2.70. *The space* ${}_0H^{1,p}(0,1)$ *is a closed subspace of the space* $H^{1,p}(0,1)$*. Equivalently, the space* ${}_0H^{1,p}(0,1)$ *is a closed subspace of the space* $\tilde{H}^{1,p}(0,1)$*.*

Corollary 2.71. *The space* ${}_0H^{1,p}(0,1)$ *is a Banach space with norm defined in quality* (3.1.3)*, i.e.,*

$$\|u\|_{{}_0H^{1,p}(0,1)} := \|u\|_{H^{1,p}(0,1)}.$$

However, on this space ${}_0H^{1,p}(0,1)$ we can introduce a different norm which is

$$|||u|||_{{}_0H^{1,p}(0,1)} := \left(\int_0^1 |Du|^p\,dx\right)^{\frac{1}{p}} = \|Du\|_{L^p(0,1)} \text{ for all } u \in {}_0H^{1,p}(0,1). \tag{2.3.5}$$

This norm allows us to state the following theorem.

Theorem 2.72. *The space* ${}_0H^{1,p}(0,1)$ *endowed with norm* $|||u|||_{{}_0H^{1,p}(0,1)}$ *is also a Banach space. Moreover, the norms* $|||u|||_{{}_0H^{1,p}(0,1)}$ *and* $\|u\|_{{}_0H^{1,p}(0,1)}$ *are equivalent. That is, there exists* $C > 0$*, such that for every* $u \in {}_0H^{1,p}(0,1)$

$$\frac{1}{C}\|u\|_{{}_0H^{1,p}(0,1)} \leq |||u|||_{{}_0H^{1,p}(0,1)} \leq C\|u\|_{{}_0H^{1,p}(0,1)}.$$

From Definitions 2.63, 2.65 and Theorem 2.67, we deduce the following property

$$H^{1,p}(0,1) \subset C([0,1]). \tag{2.3.6}$$

2.3.2.1 Bochner Integral

We do not define the Bochner integral here see for more details [49, Chapter V], Here we only need the following useful result.

Corollary 2.73. *[49, Corollary 1] Assume that X is a separable Banach space. If a function $f : (a,b) \to X$ is Borel measurable and the function $(a,b) \ni r \mapsto \|f(s)\|_X \in \mathbb{R}$ is integrable, then f is Bochner integrable and*

$$\| \int_{(a,b)} f(s)\, ds\|_X \leq \int_{(a,b)} \|f(s)\|_X\, ds.$$

2.4 C_0-Semigroups

The most important set of preliminaries to mention in this Chapter is related to the theory of semigroups in Banach Spaces. Semigroups can be used to solve some problems in PDEs. In this section, we first give a number of definitions and identify related properties. Next, we give the well-known result called the Hille-Yosida Theorem which helps to find the infinitesimal generator of the strongly continuous semigroup. Finally, we provide some examples of C_0-semigroups and some applications. Most of our materials in this section are inspired by Pazy [33].

2.4.1 Definitions and properties

Definition 2.74. Let X be a Banach space with a norm $\|\cdot\|_X$ and let $\{S_t\}_{t\geq 0}$ be a family of bounded linear operators from X to X. Then $\{S_t\}_{t\geq 0}$ is called a strongly continuous semigroup of bounded linear operators on the space X (or shortly C_0-semigroup on X) if and only if the following conditions are satisfied:

1. $S_0 = I \quad$ (where I is the identity element on X),
2. $S_{t+s} = S_t S_s \quad$ for every $t, s \in [0,\infty)$, and
3. $\|S_t x - x\|_X \to 0 \quad$ as $\quad t \to 0$, for every $x \in X$.

Sometimes we write $S(t), t \geq 0$ to denote the semigroups instead of $\{S_t\}_{t\geq 0}$.

Definition 2.75. A C_0-semigroup $\{S_t\}_{t\geq 0}$ on a Banach space X is called a contraction semigroup if and only if $\|S_t\| \leq 1$, for every $t \in [0,\infty)$.
It is called a uniformly bounded semigroup if and only if there exists $M \geq 1$ such that

$$\|S_t\| \leq M \text{ for } \quad 0 \leq t < \infty.$$

Corollary 2.76. *[33, Corollary 2.3] If $\{S_t\}_{t\geq 0}$ is a C_0-semigroup on a Banach space X, then for every $x \in X$ the trajectory of the semigroup $\{S_t\}_{t\geq 0}$ starting at x, i.e., the function*

$$u : \mathbb{R}_+ = [0,\infty) \ni t \longmapsto S_t x \in X$$

is continuous.

Definition 2.77. If $\{S_t\}_{t\geq 0}$ is a C_0-semigroup on a Banach space X, then we define

$$D(A) = \{x \in X : \lim_{t\to 0^+} \frac{S_t x - x}{t} \text{ exists } \},$$

$$\text{and we put } Ax = \lim_{t\to 0^+} \frac{S_t x - x}{t}, \quad \text{if } x \in D(A).$$

A is called the infinitesimal generator of the semigroup $\{S_t\}_{t\geq 0}$, and $D(A)$ is the domain of A.

Remark 2.78.

The following theorem is one of the important results in the theory of semigroups and it has been used in many places in mathematics. The proof of the theorem can be found in [33, Ch. 1, Theorem 2.4].

Theorem 2.79. *[33] Assume that $\{S(t)\}_{t\geq 0}$ is a C_0-semigroup on a Banach space X and let A be the infinitesimal generator of the semigroup. Then we have the following properties hold:*

1. *If $t \geq 0, x \in X$*
$$\lim_{h\to 0} \frac{1}{h} \int_t^{t+h} T(s)x\, ds = T(t)x.$$
2. *For $t > 0, x \in X$*
$$\int_0^t T(s)x\, ds \in D(A) \quad \textit{and} \quad A\Big(\int_0^t T(s)x\, ds\Big) = T(t)x - x.$$
3. *If $x \in D(A)$ and $t \geq 0$, then $T(t)x \in D(A)$ and $A\big(T(t)x\big) = T(t)(Ax)$.*
4. *If $x \in D(A)$,*
$$T(t)x - T(s)x = \int_s^t T(\tau)Ax\, d\tau = \int_s^t AT(\tau)x\, d\tau.$$
5. *$D(A)$ is a dense subspace of X.*

2.4.2 Useful examples of C_0-Semigroups

In this section, we present examples and applications of different C_0-semigroups in different Banach spaces that are related to our work. We prove the first example in this section with details, and all other models can be proved similarly.

Example 2.80. *Let $X = L^p(0,1)$, where $1 \leq p < \infty$. Define*

$$(T_t x)(s) = x(se^{-t}), \quad x \in X, \quad s \in [0,1), \ t > 0. \tag{2.4.1}$$

The family $\{T_t\}_{t\geq 0}$ is a C_0-semigroup on the space $L^p(0,1)$, Moreover, $\{T_t\}_{t\geq 0}$ is a C_0 contraction type semigroup on $L^p(0,1)$, i.e., precisely the following inequality holds,

$$\|T_t\|_{\mathcal{L}(L^p(0,1))} \leq e^{\frac{t}{p}}, \quad \text{for every } \ t \geq 0. \tag{2.4.2}$$

Proof of Example 2.80. We need to check first if T_t is 1) linear, and 2) bounded operator before we dive in to prove the conditions of C_0-semigroup listed in the Definition 2.74. First, regarding the linearity of T_t, let $x, y \in L^p(0,1)$ and $s \in \mathbb{R}, \alpha, \beta \in \mathbb{R}$, then we have

$$T_t[\alpha x + \beta y](s) = [\alpha x + \beta y](se^{-t}) = \alpha x(se^{-t}) + \beta y(se^{-t}) = \alpha T_t x(s) + \beta T_t y(s).$$

Thus, the operator T_t is linear for every $t \in (0,1)$. For the boundedness, let us choose and fix $t > 0$. For any $x \in L^p(0,1)$. By applying the norm in the space $L^p(0,1)$ and changes of variables we have

$$\left\|T_t x\right\|^p_{L^p(0,1)} = \int_0^1 |x(se^{-t})|^p ds = e^t \int_0^{e^{-t}} |x(r)|^p \, dr \leq e^t \int_0^1 |x(r)|^p \, dr = e^t |x|^p_{L^p(0,1)}.$$

Hence, T_t is bounded and

$$\|T_t\|_{\mathcal{L}(L^p)} \leq e^{\frac{t}{p}}, \quad t \geq 0 \tag{2.4.3}$$

Now we verify the three conditions of the Definition 2.74 of C_0-semigroup. Regarding the first condition, for any $x \in L^p(0,1)$ and any $s \in (0,1)$ we have $T_0 x(s) = x(se^0) = x(s)$, so $T_0 = I$. Regarding the second condition, for every $r, t \geq 0$, and for any $x \in \mathbb{R}$ we have

$$(T_t T_r x)(s) = [T_t(T_r x)](s) = (T_r x)(se^{-1}) = x(se^{-t}e^{-r}) = x(se^{-(t+r)}) = (T_{t+r})(s).$$

Hence, $T_t T_r = T_{t+r}$. Regarding the third condition, we need to show that $\|T_t x - x\|_{L^p(0,1)} \to 0$ as $t \to 0$, for all $x \in L^p(0,1)$. We consider two steps. For the first step, we assume that the function x is more regular, which means that x is a Lipschitz function. Then. for every $t \in (0,1)$ we have

$$\left\|T_t x - x\right\|^p_{L^p(0,1)} = \int_0^1 |x(se^{-t}) - x(s)|^p \, ds.$$

By considering our assumption above that function x is Lipschitz, with change of variables we get

$$\|T_t x - x\|^p = \int_0^1 |x(r) - x(s)|^p ds \leq \int_0^1 L^p |se^{-t} - s|^p ds = L^p \int_0^1 |se^{-t} - s|^p ds.$$

To check if the expression above converges to zero, we consider two cases. The first case, if $p = 1$ then

$$L\int_0^1 |se^{-1} - s|\, ds = L\int_0^1 |s|(1-e^{-t})\, ds = L(1-e^{-t})\int_0^1 |s|\, ds = \frac{1}{2}L(1-e^{-t}).$$

So we proved that

$$0 \leq |T_t x - x| \leq L\big[\frac{1}{2}(1-e^{-t})\big].$$

Therefore, by the Sandwich Theorem 2.15 we get

$$\lim_{t\to 0} \|T_t x - x\| = 0.$$

In the second case, if $p > 1$, we have

$$0 \leq \|T_t x - x\|^p = L^p \int_0^1 |(1-e^{-t})|s|^p\, ds = (1-e^{-t})L^p\big[\frac{1}{p+1}\big] \xrightarrow[t\to 0]{} 0.$$

For the second step of proving the continuity of C_0-semigroup condition, we consider the set of Lipschitz functions is dense in $L^p(0,1)$. Let us choose and fix $x \in L^p(0,1)$ and $\varepsilon > 0$. Then, there exists a Lipschitz function $y : [0,1] \to \mathbb{R}$ such that $\|x - y\|_{L^p} < \frac{\varepsilon}{4}$. Since the function y is Lipschitz, by step 1 we can find $\delta_1 > 0$ such that

$$|T_t y - y| < \frac{\varepsilon}{4} \;\text{ if }\; t \in [0, \delta_1).$$

Moreover, we can find $\delta_2 > 0$ such that $e^{\frac{\delta_2}{p}} \leq 2$. Put $\delta = \min\{\delta_1, \delta_2\}$. Then, if $t \in (0, \delta)$, then by inequality (2.4.3), we get

$$\begin{aligned}
\|T_t x - x\|_{L^p(0,1)} &\leq |T_t(x-y)|_{L^p(0,1)} + |T_t y - y|_{L^p(0,1)} + |y - x|_{L^p(0,1)} \\
&= e^{\frac{\delta_2}{p}}|x - y|_{L^p(0,1)} + |T_t y - y|_{L^p(0,1)} + |y - x|_{L^p(0,1)} \\
&\leq 2\frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \varepsilon.
\end{aligned}$$

By this, the third condition of C_0-semigroup follows.

Hence, we proved that the family $\{T_t\}_{t\geq 0}$ of linear bounded operators defined on the equation (2.4.1) is a C_0-semigroup on the space $L^p(0,1)$. □

Example 2.81. *Let $X = L^p\big([0,\infty)\big) = L^p$. We define for $t \geq 0$,*

$$[T_t f](x) = f(t + x), \;\; f \in L^p,\, x \in [0, \infty).$$

Then the family $\{T_t\}_{t\geq 0}$ on the space L^p is a C_0-semigroup of contraction.
It is straightforward to check that all the conditions in the Definition 2.74 are satisfied.

Example 2.82. *Let $X = {}_0C([0,1])$. Define*

$$[S_t x](s) = e^{\lambda t} x(se^{-t}), \quad x \in X, \quad s \in (0,1), t \geq 0. \tag{2.4.4}$$

Then the family $\{S_t\}_{t\geq 0}$ is a C_0-semigroup on the space X, in fact, This semigroup is contraction type.

Proof of Example 2.82. We need to prove directly that the family $\{S_t\}_{t\geq 0}$ is a C_0-semigroup on space X. Before proving that $\{S_t\}_{t\geq 0}$ is a C_0-semigroup we need to attest if S_t is well-defined and for every $t \geq 0$, $S_t : X \to X$ is a linear bounded operator. Respectively, the following checks these properties. For the first property, we need to show that if $x \in X$ then $S_t x \in X$. Denote $y = S_t x$. By the formula in the equation (2.4.4), we denote the family of semigroups by $y(s) = e^{\lambda t} z(s)$, where $z(s) = x(se^{-t})$, for every $s \in [0,1]$. To prove that $y \in X$, it is sufficient to prove that $z \in X$, because $e^{\lambda t}$ is just a constant. First we notice that $z(0) = x(0e^{-t}) = x(0) = 0$. Secondly, we need to check if the function z is continuous. One easy way to consider z as a composition of two functions as follows $z = x \circ \varphi$, where

$$\varphi : [0,1] \ni s \mapsto se^{-t} \in [0,1].$$

Since the function φ is linear then it is continuous. Hence, we infer that z is also continuous as a composition of two continuous functions (x from our assumption belongs to the space X), see Theorem 4.7 in [39].

For the second property (that is, the linearity), let x, y belong to the space ${}_0C([0,1])$, and $\alpha, \beta \in \mathbb{R}$, then we have

$$S_t[\alpha x + \beta y](s) = e^{\lambda t}[\alpha x + \beta y](se^{-t}) = \alpha S_t x(s) + \beta S_t y(s).$$

From that, we deduce that S_t is a linear operator. For the boundedness of S_t, let us fix $t \geq 0$ then we have

$$\|S_t x\|_{{}_0C([0,1])} = \sup_{s\in[0,1]} |(S_t x)(s)| = \sup_{s\in[0,1]} |e^{\lambda t} x(e^{-t}s)| = e^{\lambda t} \sup_{s\in[0,1]} |x(e^{-t}s)|.$$

After Using a change of variables we get

$$\|S_t x\|_{{}_0C([0,1])} = e^{\lambda t} \sup_{\sigma\in[0,e^{-t}]} |x(\sigma)| \leq e^{\lambda t} \sup_{\sigma\in[0,1]} |x(\sigma)| = e^{\lambda t}\|x\|_{{}_0C([0,1])}.$$

By that, we have found that S_t is a linear bounded operator. After checking the properties of the operator S_t we now attest if the family $\{S_t\}_{t\geq 0}$ is a C_0-semigroup on the space ${}_0C([0,1])$. The first two conditions are apparent. For the third condition of C_0-semigroup, we need to verify the C_0-continuity property. In other words, for every $x \in {}_0C([0,1])$, we have $\|S_t x - x\|_{{}_0C([0,1])} \to 0 \quad \text{as} \quad t \to 0$. Before commencing the proof of this condition, we prepare the reader for the following remarks. We notice that since $x \in {}_0C([0,1])$, then x is continuous. Also, the function x is bounded, that means, for every $s \in [0,1]$ there exists a constant $M > 0$ such that $|x(s)| \leq M$. Furthermore, because the interval $[0,1]$ is compact, then x is uniformly continuous, i.e.,

$$\forall\, \varepsilon' > 0 \quad \exists\, \delta' > 0 : \quad \text{if} \quad |r - s| \leq \delta' \quad \text{then} \quad |x(r) - x(s)| \leq \varepsilon'. \tag{2.4.5}$$

To verify the third condition, our aim is for any $\varepsilon > 0$ we need to find $\delta > 0$ such that

$$0 < t \leq \delta \Rightarrow \|S_t x - x\|_{{}_0C([0,1])} \leq \varepsilon.$$

Let $t > 0$, $s \in [0,1]$, then we have

$$\begin{aligned} \|S_t x(s) - x(s)\|_{{}_0C([0,1])} &= |e^{\lambda t} x(se^{-t}) - x(s)| \\ &= |e^{\lambda t} x(se^{-t}) - e^{\lambda t} x(s) + e^{\lambda t} x(s) - x(s)| \\ &\leq e^{\lambda t} |x(se^{-t}) - x(s)| + |e^{\lambda t} - 1||x(s)| \leq \cdots \end{aligned} \tag{2.4.6}$$

First, we consider the first part of the equation (2.4.6). Since the function $x(s)$ is uniformly continuous, we need to check that $|s - se^{-t}| \leq \delta'$. In other words,

$$|s - se^{-t}| = |s||1 - e^{-t}| \leq |1 - e^{-t}|.$$

We know from [39, Theorem 8.6] that the function $t \mapsto e^{-t}$ is continuous, in particular, continuous at 0. So, let $\delta'' > 0$ such that

$$0 < t \leq \delta'' \Rightarrow |e^{-t} - e^{-0}| \leq \delta'.$$

Therefore, we infer that for every $s \in [0,1]$ we have

$$0 < t \leq \delta'' \Rightarrow |s - se^{-t}| \leq \delta'. \tag{2.4.7}$$

By using the equation(2.4.5) with $\varepsilon' = \frac{\varepsilon}{4}$, we can find $\delta' > 0$ such that

$$|x(r) - x(s)| \leq \frac{\varepsilon}{4} \quad \text{if} \quad |r - s| \leq \delta'. \tag{2.4.8}$$

Hence, we deduce from the equation (2.4.7) and the equation (2.4.8) that for every $s \in [0,1]$ the following

$$0 < t \leq \delta', s \in [0,1] \text{ then } |x(se^{-t}) - x(s)| \leq \frac{\varepsilon}{4}.$$

Moreover, since $t \mapsto e^{\lambda t}$ is continuous and $0 \mapsto e^{0\lambda} = 1$, then we have

$$\exists\, \delta''' > 0 : \quad e^{\lambda t} \leq 2 \text{ if } 0 < t \leq \delta'''.$$

To conclude the first part of equation (2.4.6), if $0 < t \leq \min\{\delta', \delta'''\}$ then for every $s \in [0,1]$ we have

$$e^{\lambda t}|x(se^{-t}) - x(s)| \leq 2 \cdot \frac{\varepsilon}{4} = \frac{\varepsilon}{2}. \tag{2.4.9}$$

Regarding the second part of equation (2.4.6), notice that the function $t \mapsto (e^{\lambda t} - 1)$ is also continuous and $0 \mapsto (e^{\lambda 0} - 1) = 0$. So, there exists $\delta'''' > 0$ such that

$$0 < t \leq \delta'''' \Rightarrow |e^{\lambda t} - 1| \leq \frac{\varepsilon}{2M}. \tag{2.4.10}$$

By substituting equations (2.4.9) and (2.4.10) in the RHS of equation (2.4.6) and because x is bounded we have. Then, if $0 < t \leq \min\{\delta', \delta''', \delta''''\}$, then for every $s \in [0,1]$

$$|S_t x(s) - x(s)| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2M} \cdot M \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

To sum up, we proved the following

$$\|S_t x - x\|_{{}_0C([0,1])} = \sup_{s \in [0,1]} \|S_t x(s) - x(s)\| \leq \varepsilon.$$

Hence we proved that the family $\{S_t\}_{t \geq 0}$ that was defined on equation (2.4.4) is a C_0-semigroup on the space ${}_0C([0,1])$. $\square$

Theorem 2.83. *A family $\{T_t\}_{t \geq 0}$ is defined by*

$$(T_t x)(s) = x(se^{-t}), \text{ for any } s \in (0,1), t > 0 \tag{2.4.11}$$

is a C_0-semigroup on the space ${}_0C^1([0,1])$.

This theorem is a special case of Example 2.82 and the proof can be done in a similar way to that example.

Remark 2.84. The C_0-semigroup on the space ${}_0C^1([0,1])$ from the previous Theorem 2.83 was denoted by symbol $\{T_t\}_{t \geq 0}$. The C_0-semigroup on the space ${}_0H^{1,p}(0,1)$ from the Theorem 2.85 below is denoted by the same symbol $\{T_t\}_{t \geq 0}$. But these are different objects because they are defined in different Banach spaces. Nevertheless, these two semigroups agree on the smaller of these spaces. In the following Example 2.92 the former semigroup will be denoted by $\{\tilde{T}_t\}_{t \geq 0}$. With this notation, the following condition holds:

$$\tilde{T}_t(x) = T_t(x), \quad \text{for every } x \in {}_0C^1([0,1]), \text{ for all } t \in [0,\infty).$$

Theorem 2.85. *A family $\{T_t\}_{t\geq 0}$ defined by the equation* (2.4.11) *is a C_0-semigroup on the space ${}_0H^{1,p}(0,1)$, which has been defined in equation* (2.3.4) *Moreover, $\{T_t\}_{t\geq 0}$ is a C_0-semigroup of contractions on the space ${}_0H^{1,p}(0,1)$.*

Proof of Theorem 2.85. In order to prove that the family $\{T_t\}_{t\geq 0}$ is a C_0-semigroup on the space ${}_0H^{1,p}(0,1)$, firstly we need to verify that for each $t \geq 0$, the operator T_t is a well-defined linear bounded operator.
Regarding the well-defined property, we need to show that:

$$\text{if} \quad x \in {}_0H^{1,p}(0,1) \implies T_t x \in {}_0H^{1,p}(0,1).$$

First of all, suppose that $x \in {}_0H^{1,p}(0,1)$, and denote $y = T_t x$, i.e.,

$$y(s) = x(se^{-t}) \quad \text{for every } s \in [0,1].$$

We need to show that the function y belongs to the space ${}_0H^{1,p}(0,1)$; and to do that, the function must satisfy all conditions in the definition of the space ${}_0H^{1,p}(0,1)$. That is, we need to show the following

1) $y \in C([0,1])$ and $y(0) = 0$

2) the weak derivative Dy exists and Dy belongs to space $L^p(0,1)$.

The first two conditions are straightforward. To prove the second condition, let Dx be a weak derivative of the function x and let us put

$$z(s) := e^{-t}(Dx)(se^{-t}), \;\; s \in [0,1].$$

Since $Dx \in L^p(0,1)$, we easily can check that $z \in L^p(0,1)$ as a multiplication of two measurable functions [39]. The aim is to prove that $Dy = z$. However, it is sufficient to verify only the following equation

$$\int_{(0,1)} z(s)\phi(s)\,ds = -\int_{(0,1)} y(s)\phi'(s)\,ds, \;\; \phi \in C_c^\infty(0,1). \tag{2.4.12}$$

Let us choose and fix $\phi \in C_c^\infty(0,1)$. Then, by the above definition of function z, we get

$$\begin{aligned}\int_{(0,1)} z(s)\phi(s)\,ds &= \int_{(0,1)} e^{-t}(Dx)(se^{-t})\phi(s)\,ds \\ &= e^{-t}\int_{(0,1)} (Dx)(se^{-t})\phi(s)\,ds = \cdots\end{aligned}$$

To calculate the last integral, we use changes of variables as follows $\sigma = se^{-t} \in [0, e^{-t}]$. Note $d\sigma = e^{-t} ds$, then we obtain

$$\cdots = \int_{(0,e^{-t})} (Dx)(\sigma)\phi(e^t\sigma)\, d\sigma$$

Because x is weakly differentiable ($Dx = x$) and function $\phi \in C_c^\infty(0,1)$, then we can move the derivative to the other direction as follows

$$\begin{aligned} &= -\int_{(0,e^{-t})} x(\sigma)\frac{d}{d\sigma}\phi(e^t\sigma)\, d\sigma = -\int_{(0,e^{-t})} x(\sigma)e^t\phi'(e^t\sigma)\, d\sigma \\ &= -\int_{(0,1)} x(se^{-t})\phi'(s)\, ds = -\int_{(0,1)} y(s)\phi'(s)\, ds. \end{aligned}$$

By that, we proved the equation (2.4.12). Thus, the verification of the second condition is complete, and as a consequence $y \in {}_0H^{1,p}(0,1)$, and therefore, the operator T_t is well-defined. Regarding the second main property (boundedness and linearity of the operator T_t), it can be proved in a similar way as in Example 2.80. Moreover, the family $\{T_t\}_{t\geq 0}$ is a contraction C_0-semigroup. After showing that the operator T_t is well-defined, linear and bounded, we are now able to verify whether the family $\{T_t\}_{t\geq 0}$, which was defined by in equation (2.4.11), is a C_0-semigroup on space ${}_0H^{1,p}(0,1)$. The first two conditions are trivial. For the C_0-continuity condition, we take $x \in {}_0H^{1,p}([0,1])$, and then check the following

$$\|T_t x - x\|_{{}_0H^{1,p}([0,1])} \to 0 \text{ as } t \to 0. \tag{2.4.13}$$

To prove the equation (2.4.13) is satisfied, we use the definition of the norm in the space ${}_0H^{1,p}([0,1])$ defined by equation (2.3.5) and we consider two steps. The first step is to attest the continuity condition when $x \in {}_0C^1([0,1])$, which is a subspace in the space ${}_0H^{1,p}([0,1])$, see Example B.1. Then, in the second step we use the density of space ${}_0C^1([0,1])$ in ${}_0H^{1,p}([0,1])$.

Regarding the former, we assume that $x \in {}_0C^1([0,1])$. Let $t \geq 0$, then we have

$$\begin{aligned} \|T_t x - x\|_{{}_0H^{1,p}(0,1)} &= \Big[\int_0^1 \big|\frac{d}{ds}\big(x(se^{-t}) - x(s)\big)\big|^p\, ds\Big]^{1/p} \\ &= \Big[\int_0^1 \big|e^{-t}x'(se^{-t}) - x'(s)\big|^p\, ds\Big]^{1/p} \end{aligned} \tag{2.4.14}$$

Since $x \in {}_0C^1([0,1])$, there exists $\delta > 0$ such that

$$0 < t \leq \delta \implies |e^{-t}x'(se^{-t}) - x'(s)| \leq \frac{2\varepsilon}{3}.$$

By substituting the last equation with the equation (2.4.14), we obtain

$$\|T_t x - x\|_{{}_0H^{1,p}(0,1)} \leq \Big[\int_0^1 \big|\frac{2\varepsilon}{3}\big|^p\, ds\Big]^{1/p} \leq \big|\frac{2\varepsilon}{3}\big|^p.$$

Hence we proved that

$$\|T_t x - x\|_{{}_0H^{1,p}(0,1)} \to 0 \quad \text{as} \quad t \to 0 \tag{2.4.15}$$

Regarding the latter step, the following result is known. One can prove this by slightly modifying the proof of Lemma 6.1 from the book [23].

Lemma 2.86. *The space ${}_0C^1([0,1])$ is a dense subset of the space ${}_0H^{1,p}(0,1)$. Moreover, the natural embedding*[1] ${}_0C^1([0,1]) \hookrightarrow {}_0H^{1,p}(0,1)$ *is continuous.*

Recall that we aim to prove equation (2.4.13). For this aim, we need to choose and fix $x \in {}_0H^{1,p}(0,1)$ and $\varepsilon > 0$. So, we want to find $\delta > 0$ such that

$$\text{if} \quad 0 < t \leq \delta \quad \text{then} \quad \|T_t x - x\|_{{}_0H^{1,p}(0,1)} \leq \varepsilon.$$

By Lemma 2.86 we can find $y \in {}_0C^1([0,1])$ such that

$$\|x - y\|_{{}_0H^{1,p}(0,1)} < \frac{\varepsilon}{3}. \tag{2.4.16}$$

By the equation (2.4.15) applied to the function y we can find $\delta > 0$ such that

$$\text{if} \quad 0 < t \leq \delta \quad \text{then} \quad \|T_t y - y\|_{{}_0H^{1,p}(0,1)} \leq \frac{\varepsilon}{3}. \tag{2.4.17}$$

From the boundedness property, equations (2.4.16) and (2.4.17), we infer that if $0 < t \leq \delta$, then

$$\|T_t x - x\|_{{}_0H^{1,p}(0,1)} \leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$

So we verified that

$$\text{if} \quad 0 < t \leq \delta \quad \text{then} \quad \|T_t x - x\|_{{}_0H^{1,p}(0,1)} \leq \varepsilon.$$

Hence, the family $\{T_t\}_{t\geq 0}$ of linear bounded operators defined in the equation (2.4.11) is a C_0-semigroup in the space ${}_0H^{1,p}(0,1)$. □

Theorem 2.87. *If $\{T_t\}_{t\geq 0}$ is a C_0-semigroup on a Banach space X then a family $\{S_t\}_{t\geq 0}$ defined by $S_t = e^{\lambda t}T_t$ for every $t \geq 0$, i.e.,*

$$S_t x = e^{\lambda t}T_t x, \; \textit{for } t \geq 0, \quad \lambda \in \mathbb{R} \textit{ and } x \in X, \tag{2.4.18}$$

is also a C_0-semigroup on the Banach space X.

[1] By this we mean the map ${}_0C^1([0,1]) \ni x \mapsto x \in {}_0H^{1,p}(0,1)$.

Before we embark on the proof of this result let us formulate the above result in terms of the infinitesimal generators, see [33, Proposition 1.4].

Theorem 2.88. *If $\{T_t\}_{t\geq 0}$ is a C_0-semigroup on a Banach space X with the infinitesimal generator A, then the infinitesimal generator B of the family $\{S_t\}_{t\geq 0}$ which defined by equation* (2.4.18)*, satisfies the following equality*

$$B = A + \lambda I, \quad i.e.,$$

$$D(B) = D(A), \quad and \quad Bx = Ax + \lambda x, \;\; x \in D(B)$$

Proof of Theorem 2.87. We assume that X is a Banach space, $\lambda \in \mathbb{R}$ and that $\{T_t\}_{t\geq 0}$ is a C_0-semigroup on X. It is obvious that for each $t \geq 0$, $S_t : X \to X$ is well-defined, linear and bounded. Moreover, for the C_0-semigroup conditions, the first two conditions are easy to check. For the third condition, we need to prove the following

$$\exists\, \delta > 0 : |S_t x - x|_X \to 0 \quad \text{as} \quad t \to 0.$$

Let $x \in X$ and by using the definition (2.4.18) of the C_0-semigroup $\{S_t\}_{t\geq 0}$ we have

$$|S_t x - x|_X \leq e^{t\lambda}|T_t x - x|_X + |e^{t\lambda} - 1||x|_X. \tag{2.4.19}$$

Starting with the first term of the RHS of equation (2.4.19). Since the function $t \to e^{\lambda t}$ is continuous and $0 \to e^{0\lambda} = 1$, then there exists $\delta' > 0$ such that $e^{\lambda t} \leq \frac{\varepsilon}{4}$. By the assumptions that $\{T_t\}_{t\geq 0}$ is a C_0-semigroup, then there exists $\delta'' > 0$ such that $|T_t x - x| \leq \frac{\varepsilon}{4}$. Hence if $0 < t < min\{\delta', \delta''\}$, then we have

$$e^{\lambda t}|T_t x - x| \leq \frac{\varepsilon}{2}. \tag{2.4.20}$$

For the second term of equation (2.4.19), the function x is continuous, and therefore, bounded. As a consequence, we have for every s there exists a constant $M > 0$ such that

$$|x(s)| \leq M. \tag{2.4.21}$$

We notice that $t \to (e^{\lambda t})$ is also continuous and $0 \to (e^0\lambda) = 0$. Thus, there exists $\delta''' > 0$ such that

$$|e^{\lambda t} - 1| \leq \frac{\varepsilon}{2M}. \tag{2.4.22}$$

Now after all this verification we substitute the equations (2.4.20), (2.4.22) and (2.4.21) in the RHS of the equation (2.4.19) to get the following

$$|S_t x - x|_X \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2M} \cdot M = \varepsilon.$$

Hence, if $0 < t \le \min\{\delta', \delta'', \delta'''\}$ then $|S_t x - x| \le \varepsilon$. To conclude, we proved that

$$\lim_{t \to 0} |S_t x - x|_X = 0 \quad \text{for every} \quad x \in X.$$

As a result, $\{S_t\}_{t \ge 0}$ is a C_0-semigroup in the Banach space X. □

The next example is a generalization of Example 2.80.

Example 2.89. *Let $X = L^p(0,1)$ and $\{S_t\}_{t\ge 0}$ is a family of bounded linear operators defined on X as follows*

$$S_t x(s) = e^{\lambda t} x(se^{-t}). \tag{2.4.23}$$

Then $\{S_t\}_{t\ge 0}$ is a C_0-semigroup on the space X.

Proof of Example 2.89. According to Theorem 2.87 it is sufficient to consider the case $\lambda = 0$. However, when $\lambda = 0$, $S_t = T_t$, for $t \ge 0$, where T_t was defined in Example 2.80. Moreover, we already proved in Example 2.80 that $\{T_t\}_{t\ge 0}$ is a C_0-semigroup in the space $L^p(0,1)$.

□

Corollary 2.90. *The family $\{S_t\}_{t\ge 0}$ that defined by the equation* (2.4.23) *is a C_0-semigroup on spaces ${}_0C^1([0,1])$ and ${}_0H^{1,p}(0,1)$.*

Proof of Corollary 2.90. According to Theorem 2.87 it is sufficient to consider the case $\lambda = 0$. However, when $\lambda = 0$, from Theorem 2.83 and Example 2.82 we infer that $\{S_t\}_{t\ge 0}$ is a C_0-semigroup on the space ${}_0C^1([0,1])$ and in Theorem 2.85 that $\{S_t\}_{t\ge 0}$ is a C_0-semigroup on the space ${}_0H^{1,p}(0,1)$. This completes the proof. □

2.4.3 C_0-Semigroups and applications

This section contains some abstract theorems related to the C_0-semigroup on Banach spaces. Also, we provide named spaces to apply those abstract theorems. The first abstract result that we state in this section is a result of generalising the Theorem 2.83.

Theorem 2.91. *Suppose that X and Z are Banach spaces satisfying the following assumptions.*

(i) $Z \subset X$ is a dense subspace,

(ii) the embedding $Z \hookrightarrow X$ is continuous, that is, there exists a number $M_1 > 0$ such that for every $z \in Z$ the following inequality holds $|z|_X \le M_1 |z|_Z$.

Assume that $\{T_t\}_{t\geq 0}$ is a family of linear operators on X such that the following assumption is satisfied.

(I) for every $t \in [0,\infty)$, T_t is linear bounded operator on X and there exists $M_0 > 0$ such that

$$\sup_{t\in[0,1]} \|T_t\|_{\mathcal{L}(X)} \leq M_0.$$

Assume that $\{\tilde{T}_t\}_{t\geq 0}$ is a family of linear operators on Z such that the following assumptions are satisfied.

(a) if $z \in Z$ and $t \in [0,\infty)$, then $T_t z = \tilde{T}_t z$. In other words, $\tilde{T}_t$ is the restriction of T_t to the space Z.

(b) The map $\tilde{T}_t$ is a bounded linear map on Z.

(c) The family $\{\tilde{T}_t\}_{t\geq 0}$ is a C_0-semigroup on Z.

Then, the family $\{T_t\}_{t\geq 0}$ is a C_0-semigroup on the space X.

Proof of Theorem 2.91. The aim in this proof is to verify that $\{T_t\}_{t\geq 0}$ is a C_0-semigroup on space X. That means, three conditions need to be checked according to the Definition 2.74. Regarding the first condition, we need to show that $T_0 = I_X$, where I_X is the identity map on the space X. We notice that since $\{\tilde{T}_t\}_{t\geq 0}$ is a C_0-semigroup on the space Z we infer that $\tilde{T}_0 = I_Z$, where I_Z is the identity map on the space Z. From assumption (a) we infer that

$$T_0(x) = \tilde{T}_0(x) = x, \text{ for every } x \in Z.$$

Since by assumption (i) above Z is a dense subspace of X and $T_0 : X \to X$ is a linear bounded operator, we infer, see e.g. [36, Theorem I.7], that $T_0 = I_X$. Regarding the second condition, i.e. $T_{t+r} = T_t T_r$, for every $s, r \geq 0$, we know that this condition is satisfied in space Z, i.e. $\tilde{T}_{t+r} = \tilde{T}_t \tilde{T}_r$, $s, r \geq 0$. Since Z is a dense subspace of X, by using assumption (a) and [36, Theorem I.7], we can deduce that this condition is also satisfied. Regarding the C_0 continuity, which is the third condition, let us choose $x \in X$. We want to prove that

$$\lim_{t\mapsto 0} |T_t x - x|_X = 0. \tag{2.4.24}$$

Let us take $\varepsilon > 0$. By assumption (i) there exists $z \in Z$ such that

$$|x - z|_X < \frac{\varepsilon}{2(1+M_0)}. \tag{2.4.25}$$

By assumption (c) there exists $\delta > 0$ such that

$$0 \leq t \leq \delta \implies |\tilde{T}_t z - z|_Z < \frac{\varepsilon}{2M_1}.$$

After the above few verifications, now we apply the triangle inequality and assumption (a) to the equation (2.4.24) to get the following

$$\begin{aligned}\|T_t x - x\|_X &= |T_t x - T_t z + T_t z - z + z - x|_X \\ &\leq |T_t x - T_t z|_X + |T_t z - z|_X + |z - x|_X \\ &\leq |T_t(x - z)|_X + |z - x|_X + |T_t z - z|_X. \end{aligned} \tag{2.4.26}$$

By taking the first term and last them of the equation (2.4.26) and applying assumptions (I) and (ii), we get the following two equations:

$$\|T_t(x - z)\|_X \leq M_0 \|x - z\|_X, \quad t \leq 1, \tag{2.4.27}$$

and by assumptions (ii) and (a) we have

$$\|T_t z - z\|_X \leq M_1 |T_t z - z|_Z = M_1 |\tilde{T}_t z - z|_Z, \quad z \in Z, \quad t \geq 0.$$

Finally, by substituting the equations (2.4.25) and (2.4.27) in the equation (2.4.24) provided $0 \leq t \leq \delta \wedge 1 := \min\{\delta, 1\}$ and we obtain the following

$$\begin{aligned}\|T_t x - x\|_X &= (1 + M_0)|x - z|_X + M_1 |\tilde{T}_t z - z|_Z \\ &\leq (1 + M_0)\frac{\varepsilon}{2(1 + M_0)} + M_1 \frac{\varepsilon}{2M_1} \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.\end{aligned}$$

Hence we proved that

$$0 \leq t \leq \min\{\delta, 1\} \implies |T_t x - x|_X \leq \varepsilon.$$

Hence, we proved the C_0-continuity of the semigroup $\{T_t\}_{t\geq 0}$. As consequence, the family $\{T_t\}_{t\geq 0}$ is a C_0-semigroup on the space X. □

In the following example, we show how one can apply Theorem 2.91 to produce a new proof of the C_0-continuity from Theorem 2.85.

Example 2.92. *This example shows how one can use the abstract Theorem 2.91. Let us consider two specific spaces defined as follows*

$$X = {}_0H^{1,p}(0, 1) \ \textit{and} \ Z = {}_0C^1([0, 1]).$$

Firstly, let us note that by the Sobolev Embedding Theorem, see Lemma 2.86, assumptions (i) and (ii) are satisfied. Let $\{T_t\}_{t\geq 0}$ *be the family defined by formula* (2.4.11) *in Theorem 2.85 on the space* X*. Let* $\{\tilde{T}_t\}_{t\geq 0}$ *be the family defined by formula* (2.4.11) *in Theorem 2.83 on the space* Z*. Note that in that Theorem this family was denoted by*

$\{T_t\}_{t\geq 0}$, see Remark 2.84. By using Theorem 2.85 it follows that the assumptions regarding the family $\{T_t\}_{t\geq 0}$ in the abstract Theorem 2.91 are satisfied. Using Theorem 2.83 it follows that the assumptions regarding the family $\{\tilde{T}_t\}_{t\geq 0}$ in Theorem 2.91 are satisfied. By Remark 2.84 the assumption (a) is also satisfied. Hence we conclude that the family $\{T_t\}_{t\geq 0}$ is a C_0-semigroup on the space X, as claimed.

The real power of the abstract Theorem 2.91 will be seen later when we prove a similar result in the fractional Sobolev spaces.
Before ending this section, we provide important results. This result is a special case of a known theorem called Calderon-Lions interpolation Theorem [30]. The purpose is to show that one can apply Theorem 2.91 to produce a new result by using Calderon-Lions interpolation Theorem. In the following theorem X_α, resp. Y_α, for $\alpha \in [0,1]$, are the interpolation space between X_0 and X_1, resp. Y_0 and Y_1, which are introduced in Appendix IX.4 in the book [30].

Corollary 2.93 (Calderon-Lions Interpolation Theorem)**.** *[30, Theorem IX.20] Assume that X_0, X_1, Y_0 and Y_1 are four complex vector spaces with norms denoted by $|\cdot|_{X_0}$, $|\cdot|_{X_1}$, $|\cdot|_{Y_0}$ and $|\cdot|_{Y_1}$ respectively. Suppose also that $X_1 \subset X_0$ and $Y_1 \subset Y_0$ densely and continuously. Suppose that $T_0 \in \mathcal{L}(X_0, Y_0)$ and $T_1 \in \mathcal{L}(X_1, Y_1)$ with the following properties:*

$$T_0 x = T_1 x \text{ for every } x \in X_1. \tag{2.4.28}$$

Denote

$$M_0 = \|T_0\|_{\mathcal{L}(X_0,Y_0)} \quad \text{and} \quad M_1 = \|T_1\|_{\mathcal{L}(X_1,Y_1)}.$$

Then the following holds.

(i) For every $\alpha \in (0,1)$ we have $T_0 x \in Y_\alpha$ if $x \in X_\alpha$.

(ii) Denote by T_α the restriction of the operator T_0 to the space X_α with range Y_α. By assertion (i), T_α is a linear map from X_α to Y_α.

(iii) The map $T_\alpha : X_\alpha \to Y_\alpha$ is bounded and $\|T_\alpha\|_{\mathcal{L}(X_\alpha,Y_\alpha)} \leq M_0^{1-\alpha} M_1^{\alpha}$.

Remark 2.94. [30] Let us write down the previous result, i.e., Corollary 2.93, in a slightly less rigorous way.
Assume that X_0, X_1, Y_0 and Y_1 are four complex vector spaces with norms denoted by $|\cdot|_{X_0}$, $|\cdot|_{X_1}$, $|\cdot|_{Y_0}$ and $|\cdot|_{Y_1}$ respectively. Suppose also that $X_1 \subset X_0$ and $Y_1 \subset Y_0$ densely and continuously. Suppose that $T \in \mathcal{L}(X_0, Y_0)$, i.e. T is a bounded linear map from X_0 to Y_0 such that T maps the space X_1 to the space Y_1 and, the restriction of T to X_1 is a bounded linear map from X_1 to Y_1.
Denote

$$M_0 = \|T\|_{\mathcal{L}(X_0,Y_0)} \quad \text{and} \quad M_1 = \|T\|_{\mathcal{L}(X_1,Y_1)}.$$

Then the following holds.

(i) For every $t \in (0,1)$ we have
$$Tx \in Y_t \text{ if } x \in X_t.$$

(ii) The restriction of the operator T to the space X_t with range Y_t is a bounded linear map from X_t to Y_t and
$$\|T\|_{\mathcal{L}(X_t,Y_t)} \leq M_0^{1-t} M_1^t.$$

Theorem 2.95. *Assume that $0 < \alpha < 1$ and $p \geq 1$. Then the family $\{T_t\}_{t\geq 0}$ defined in (2.4.11) is a C_0-semigroup on the space ${}_0H^{\alpha,p}(0,1)$, i.e.,*

(i) for every $t \geq 0$, if $x \in {}_0H^{\alpha,p}(0,1)$ then $T_t x \in {}_0H^{\alpha,p}(0,1)$, the map
$${}_0H^{\alpha,p}(0,1) \ni x \mapsto T_t x \in {}_0H^{\alpha,p}(0,1)$$
is linear and bounded,

(ii) $T_0 x = x$ for every $x \in {}_0H^{\alpha,p}(0,1)$,

(iii) $T_{t+s}x = T_t(T_s x)$ for every $x \in {}_0H^{\alpha,p}(0,1)$ and all $t, s \in [0,\infty)$,

(iv) $\lim_{t\to 0+} |T_t x - x|_{{}_0H^{\alpha,p}(0,1)} = 0$.

Proof of Theorem 2.95. To prove that the family $\{T_t\}_{t\geq 0}$ is a C_0-semigroup on the space ${}_0H^{\alpha,p}(0,1)$, we use Theorem 2.91. We choose the following notation
$$X = {}_0H^{\alpha,p}(0,1), \quad Z = {}_0H^{1,p}(0,1).$$

Assume that the C_0-semigroup $\{\tilde{T}_t\}_{t\geq 0}$ on the space X is the semigroup from Theorem 2.85. We need to show that family $\{T_t\}_{t\geq 0}$ on the space X satisfied relevant assumptions of the abstract Theorem 2.91. For this purpose, to prove condition (i) we begin with the following important auxiliary result.

Lemma 2.96. *The condition (i) of Theorem 2.95 is satisfied.*

Proof of Lemma 2.96. Let us choose and fix $t \geq 0$. We apply Calderon-Lions Interpolation Theorem, see Corollary 2.93. For this purpose, we choose the following spaces
$$\begin{aligned} &X_0 = Y_0 = L^p(0,1) \\ &X_1 = Y_1 = {}_0H^{1,p}(0,1), \\ &X_\alpha = Y_\alpha = {}_0H^{\alpha,p}(0,1) := [L^p(0,1), {}_0H^{1,p}(0,1)]_\alpha. \end{aligned}$$

Let T_0 be the linear and bounded map T_t from the space $L^p(0,1)$ to $L^p(0,1)$, see Example 2.80. Let T_1 be the linear and bounded map T_t from the space ${}_0H^{1,p}(0,1)$ to ${}_0H^{1,p}(0,1)$, see Theorem 2.85. Since these maps are defined by the same formula (2.4.11) we infer that assumption (2.4.28) from Corollary 2.93 is satisfied. We can also easily verify all other assumptions of that Corollary 2.93. Therefore, we conclude that the map T_α, defined as the restriction of the map T_0 to the space X_α with range in X_α is a bounded linear map and moreover,

$$\|T_\alpha\|_{\mathcal{L}(X_\alpha)} \leq M_0^{1-\alpha} M_1^\alpha,$$

where

$$M_0 = \|T_0\|_{\mathcal{L}(X_0)} \text{ and } M_1 = \|T_1\|_{\mathcal{L}(X_1)}.$$

On the other hand, by Example 2.80 and by Theorem 2.85, we have

$$M_0 = e^{\frac{t}{p}} \quad \text{and} \quad M_1 = e^{-t(1-\frac{1)}{p}}.$$

Hence, we deduce that

$$\|T_\alpha\|_{\mathcal{L}(X_\alpha)} \leq e^{t(\frac{1}{p}-\alpha)}$$

Thus, the proof of condition (i) is satisfied. $\square$

Regarding the conditions (ii) and (iii) the proof of these conditions is a consequence of the abstract Theorem 2.91. For condition (iv), i.e., the C_0-continuity we apply assumptions of Theorem 2.91 and make sure that they are satisfied. According to equation (3.1.5) in Corollary 3.10, the assumptions (i) and (ii) in Theorem 2.91 are satisfied. Moreover, in Theorem 2.85 we have already checked that assumptions (b) and (c) are hold. Regarding the assumption (I) we proved in the above Lemma 2.96 that T_α is a bounded linear map (T_α is the restriction of the map T_0 to the space X_α with range in X_α is a bounded linear map). According to assumption (i) in Theorem 2.91 and [36, Theorem I.7] we infer that assumption (a) in the abstract Theorem 2.91 is hold. Since all the assumptions of the abstract theorem 2.91 is satisfied we deduce that $\{T_t\}_{t\geq 0}$ is a C_0-semigroup on the space $X = {}_0H^{\alpha,p}(0,1)$. $\square$

2.4.4 Hille Yosida Theorem and applications

Definition 2.97. Let X be a Banach space over $\mathbb{R}$ and $A : D(A) \to X$ is a linear operator with the domain $D(A)$, which is a subspace of X. Then, we define the *resolvent set* of A as the set of all real numbers λ which satisfy the following two conditions

1. $\lambda I - A : D(A) \to X$ is a bijective,
2. $(\lambda I - A)^{-1} : X \to X$ is bounded.

We denote the resolvent set by $\rho(A)$.

Remark 2.98. Condition 1 of Definition 2.97 implies that for every $\lambda \in \rho(A)$, the inverse $(\lambda I - A)^{-1}$ exists and is a bijection from X onto the set $D(A)$. In particular,

$$D(A) = Range\big((\lambda I - A)^{-1}\big), \quad \text{for} \quad \lambda \in \rho(A). \tag{2.4.29}$$

The following Theorem is called the Hille-Yosida Theorem for a C_0-semigroup of contractions.

Theorem 2.99. *[33]*

(I) If $\{S(t)\}_{t\geq 0}$ is a C_0-semigroup of contractions on a Banach space X and if A denotes the infinitesimal generator of this semigroup, then

(i) A is closed and $D(A)$ is dense in X.

(ii) $(0, \infty) \subset \rho(A)$ and $\|(\lambda I - A)^{-1}\| \leq \frac{1}{\lambda}$, for every $\lambda > 0$. Moreover

$$(\lambda I - A)^{-1} f = \int_0^\infty e^{-\lambda t} T_t f \, dt, \quad f \in X. \tag{2.4.30}$$

(II) If A is a linear operator on a Banach space X with domain $D(A)$ such that conditions (i) and (ii) above are satisfied, then there exists a C_0-semigroup of contractions $\{S(t)\}_{t\geq 0}$ such that A is the infinitesimal generator of this semigroup.

The above result can be easily modified to include contraction type semigroups.

Corollary 2.100. *Assume that $\{S(t)\}_{t\geq 0}$ is a C_0-semigroup of contraction type on a Banach space X, in particular there exists $\gamma_0 \in \mathbb{R}$ such that*

$$\|S(t)\| \leq e^{\gamma_0 t}, \quad t \geq 0.$$

Let A be the infinitesimal generator of this semigroup. Then the following hold.

(i) A is closed and $D(A)$ is dense in X;

(ii) $(\gamma_0, \infty) \subset \rho(A)$ and for every $\lambda > \gamma_0$,

$$(\lambda I - A)^{-1} f = \int_0^\infty e^{-\lambda t} T_t f \, dt, \quad f \in X, \quad \|(\lambda I - A)^{-1}\| \leq \frac{1}{\lambda}.$$

The identity (2.4.30) is a byproduct of the proof of the above Theorem given by Pazy see [33, Ch. 1, Theorem 3.1].
The following Lemma is an important result that can be used to prove the infinitesimal generator of the semigroups.

Lemma 2.101. *Assume that A is a densely defined linear operator in a Banach space X. Assume that λ belongs to the resolvent set $\rho(A)$. Then*

(i) $(I - \frac{1}{\lambda}A)^{-1}$ *exists and is bounded and*

$$J_\lambda := (I - \frac{1}{\lambda}A)^{-1} = \lambda R(\lambda, A). \tag{2.4.31}$$

(ii) $\lambda(J_\lambda - I) = AJ_\lambda$, *and*

$$J_\lambda - AR(\lambda, A) = I. \tag{2.4.32}$$

Proof of Lemma 2.101. The first part is the consequence from the assumptions. Regarding the second part (i), let $\lambda > 0$, also $\frac{1}{\lambda} > 0$. From the first part of (i) we infer that the inverse $(I - \frac{1}{\lambda}A)^{-1}$ exists and bounded. Defined $J_\lambda = (I - \frac{1}{\lambda}A)^{-1}$, then we have

$$J_\lambda := (I - \frac{1}{\lambda}A)^{-1} = (\frac{1}{\lambda}\lambda I - \frac{1}{\lambda}A)^{-1} = \big[\frac{1}{\lambda}(\lambda I - A)\big]^{-1} = \lambda R(\lambda, A).$$

Hence we proved equality (2.4.31).

To prove part (ii), we multiply both sides of the equality (2.4.32) by λ then we get the following train of identities

$$\lambda J_\lambda - \lambda AR(\lambda, A) = \lambda I \iff \lambda J_\lambda - AJ_\lambda = \lambda I \iff \lambda(J_\lambda - I) = AJ_\lambda.$$

Regarding the second equality of part (ii), since $\lambda \in \rho(A)$, $(\lambda I - A)(\lambda I - A)^{-1} = I$, therefore, we infer that

$$\lambda IR(\lambda, A) - AR(\lambda, A) = I.$$

So we deduce that equality (2.4.32) is satisfied. By this the proof of Lemma 2.101 is complete. $\square$

Proposition 2.102. *Assume that $p \in [1, \infty)$. Let $\{T_t\}_{t\geq 0}$ be a C_0-semigroup on the Banach space $X = L^p\big([0, \infty)\big)$, defined*

$$[T_t f](x) = f(t + x), \quad x \in [0, \infty), \quad t \geq 0.$$

Let us denote by A the infinitesimal generator of this C_0-semigroup. Then the following holds.

1. $D(A) = \{f \in L^p\big([0, \infty)\big) : Df \in L^p\big([0, \infty)\big)\}$, *where D is the weak derivative.*
2. $Af = Df$, *for* $f \in D(A)$.

To prove Proposition 2.102, we first need to prove the following three auxiliary results: Claims 2.103, 2.104, and 2.105.

Claim 2.103. Let us assume the assumptions and notation from Proposition 2.102. If the function $f \in L^p\big([0,\infty)\big)$ and the number $\lambda > 0$, define a function $g_\lambda \in L^p\big([0,\infty)\big)$ by the following formula

$$g_\lambda := \lambda(\lambda I - A)^{-1} f. \tag{2.4.33}$$

Then

$$g_\lambda(x) = \lambda \int_0^\infty 1_{(0,\infty)}(t-x) e^{-\lambda(t-x)} f(t)\, dt,\ x \in [0,\infty). \tag{2.4.34}$$

Proof of Claim 2.103. Let us choose and fix number $\lambda > 0$. We begin by observing that since $\lambda > 0$, by the Hille-Yosida Theorem, see Theorem 2.99 part (I), the map $(\lambda I - A)^{-1} =: R(\lambda, A)$ exists and by identity (2.4.30) is satisfied. Take function $f \in L^p([0,\infty))$. Since $R(\lambda, A) = (\lambda I - A)^{-1}$ is bounded linear operator, we define $J_\lambda : L^p \to L^p$ is also a bounded linear operator. Assume that $g_\lambda = J_\lambda f \in L^p[0,\infty)$. Moreover, since $R(\lambda, A)$ is linear, then we have $g_\lambda = \lambda R(\lambda, A) f = R(\lambda, A)(\lambda f)$. Hence, the function $g_\lambda \in RangeR(\lambda, A)$. Since $(\lambda I - A)R(\lambda, A) = I$, we infer that the $RangeR(\lambda, A) \subset D(A)$. Thus, $g_\lambda \in D(A)$. From our assumption of function g_λ, we have

$$g_\lambda = J_\lambda f = \lambda R(\lambda, A) f = \lambda \int_0^\infty e^{-\lambda t} T_t f\, dt.$$

Hence, for every $x \in [0,\infty)$ we have

$$g_\lambda(x) = \lambda \int_0^\infty e^{-\lambda t} (T_t f)(x)\, dt = \lambda \int_0^\infty e^{-\lambda t} f(t+x)\, dt. \tag{2.4.35}$$

By invoking the changing of variables of the above equation (2.4.35) as follows $t' = t + x \Rightarrow t = t' - x$, $t' \in [x,\infty)$ and $dt = dt'$, we obtain

$$(g_\lambda)(x) = (J_\lambda f)(x) = \lambda \int_x^\infty e^{-\lambda(t'-x)} f(t')\, dt' = \lambda \int_0^\infty 1_{(0,\infty)}(t-x) e^{-\lambda(t-x)} f(t)\, dt.$$

Therefore, we proved that

$$g_\lambda(x) = \lambda \int_0^\infty 1_{(0,\infty)}(t-x) e^{-\lambda(t-x)} f(t)\, dt, \quad x \in [0,\infty).$$

So the proof of the Claim 2.103 is complete. □

Claim 2.104. Let us assume the assumptions and notation from Proposition 2.102. Assume that $f \in L^p([0,\infty))$ and $\lambda > 0$ and the function $g_\lambda \in L^p([0,\infty))$ be defined by equation (2.4.33). Then g_λ is weakly differentiable on $[0,\infty)$ and the weak derivative Dg_λ of g_λ satisfies the following equality

$$Dg_\lambda = -\lambda f + \lambda g_\lambda. \tag{2.4.36}$$

Proof of Claim 2.104. Let us choose and fix $f \in L^p([0,\infty))$ and $\lambda > 0$. Let $g_\lambda \in L^p([0,\infty))$ be defined by equation (2.4.33). By the previous Claim 2.103, g_λ satisfies identity (2.4.34). We aim to show that g_λ is weakly differentiable on $[0,\infty)$. Let us recall that if the weak derivative exists it is unique [22, Ch-V Lemma 1]. We guess that the following choice of function h_λ is good $h_\lambda = \lambda g_\lambda - \lambda f$. Since by assumptions $f \in L^p[0,\infty)$ and by identity (2.4.33) function g_λ is also belongs to $L^p[0,\infty)$, we infer that the function h_λ defined above, also belongs to $L^p[0,\infty)$. So, it is sufficient to show that for every $\phi \in C_0^\infty$ we have

$$\int_0^\infty g_\lambda(x)\phi'(x)\,d\mu(x) = \lambda \int_0^\infty f(x)\phi(x)\,d\mu(x) - \lambda \int_0^\infty g_\lambda(x)\phi(x)\,d\mu(x). \tag{2.4.37}$$

To prove the above equation (2.4.37), let us take and fix $\phi \in C_0^\infty$. Note that by using equation (2.4.34) we have

$$\int_0^\infty g_\lambda(x)\phi'(x)\,dx = \int_0^\infty \big[\lambda \int_0^\infty 1_{(0,\infty)}(t-x)e^{-\lambda(t-x)}f(t)\,dt\big]\phi'(x)\,dx.$$

It follows, by applying the Fubini Theorem [28, Theorem 2.2.3] to the double integral on the RHS above we infer that

$$\begin{aligned}\int_0^\infty g_\lambda(x)\phi'(x)\,dx &= \lambda \int_0^\infty f(t)\big[\int_0^\infty 1_{(0,\infty)}(t-x)e^{-\lambda(t-x)}\phi'(x)\,dx\big]\,dt \\ &= \lambda \int_0^\infty f(t)e^{-\lambda t}\big[\int_0^t e^{\lambda x}\phi'(x)\,dx\big]\,dt.\end{aligned} \tag{2.4.38}$$

Note that, by using integration by parts for the second integral of equation (2.4.38) we get

$$\int_0^t e^{\lambda x}\phi'(x)\,dx = e^{\lambda t}\phi(t) - e^{\lambda 0}\phi(0) - \lambda \int_0^t e^{\lambda x}\phi(x)\,dx.$$

Since the function ϕ has a compact support in the interval $(0,\infty)$, we infer $\phi(0) = 0$. Hence

$$\int_0^t e^{\lambda x}\phi'(x)\,dx = e^{\lambda t}\phi(t) - \int_0^t \lambda e^{\lambda x}\phi(x)\,dx. \tag{2.4.39}$$

Next we substitute equation (2.4.39) in the second integral of equation (2.4.38) we get

$$\begin{aligned}\int_0^\infty g_\lambda(x)\phi'(x)\,dx &= \lambda \int_0^\infty f(t)e^{-\lambda t}\big[e^{\lambda t}\phi(t) - \int_0^t \lambda e^{\lambda x}\phi(x)\,dx\big]\,dt \\ &= \lambda \int_0^\infty f(t)\phi(t)\,dt - \lambda \int_0^\infty f(t)\big[\int_0^t \lambda e^{-\lambda(t-x)}\phi(x)\,dx\big]\,dt.\end{aligned}$$

By introducing an appropriate indicator function and then using the Fubini Theorem again for the right-hand side we obtain

$$\begin{aligned}\int_0^\infty f(t)\big[\int_0^t \lambda e^{-\lambda(t-x)}\phi(x)\,dx\big]\,dt &= \int_0^\infty f(t)\big[\lambda\int_0^\infty 1_{(0,\infty)}(t-x)e^{-\lambda(t-x)}\phi(x)\,dx\big]\,dt\\ &= \int_0^\infty \phi(x)\big[\lambda\int_0^\infty 1_{(0,\infty)}(t-x)e^{-\lambda(t-x)}f(t)\,dt\big]\,dx.\end{aligned}$$

Note that according to identity (2.4.34) in Proposition 2.104

$$\lambda\int_0^\infty 1_{(0,\infty)}(t-x)e^{-\lambda(t-x)}f(t)\,dt = g_\lambda(x),\ x\in(0,\infty).$$

Therefore,

$$\int_0^\infty g_\lambda(x)\phi'(x)\,dx = \lambda\int_0^\infty f(x)\phi(x)\,dx - \lambda\int_0^\infty g_\lambda(x)\phi(x)\,dx.$$

Thus we proved equality (2.4.37) and this completes the proof of the Claim 2.104. □

Claim 2.105. Let us assume the assumptions and notation from Proposition 2.102. Assume that $f\in L^p([0,\infty))$ and $\lambda>0$ and the function g_λ be defined by equation (2.4.33). Then the function g_λ belongs to $D(A)$ and satisfies

$$Ag_\lambda = Dg_\lambda.$$

Proof of Claim 2.105. Let us choose and fix $\lambda>0$ and $f\in L^p([0,\infty))$. Let us consider the function g_λ be defined by equation (2.4.33). If follows from the Claim 2.104 that $Dg_\lambda\in L^p([0,\infty))$. Also, since by identity (2.4.29) in Remark 2.98, $\mathit{Range}\big((\lambda I-A)^{-1}\big)\subset D(A)$ and since $g_\lambda=\lambda(\lambda I-A)^{-1}f\in \mathit{Range}\big((\lambda I-A)^{-1}\big)$, we deduce that $g_\lambda\in D(A)$. So we have proved the first part of our Claim. Moreover, by equation (2.4.33) we have

$$\begin{aligned}Ag_\lambda &= A\lambda(\lambda I-A)^{-1}f = AJ_\lambda f\\ &= \lambda(J_\lambda-I)f = \lambda J_\lambda f-\lambda f = \lambda g_\lambda-\lambda f.\end{aligned}$$

On the other hand, by identity (2.4.36) in Claim 2.104 we infer that $Dg_\lambda=\lambda g_\lambda-\lambda f$. Therefore, we infer that $Ag_\lambda=Dg_\lambda$. Hence, the proof of Claim 2.105 is complete. □

After we proved necessary Claims 2.103, 2.104 and 2.105, it is now possible to embark with the proof of Proposition 2.102.

Proof of Proposition 2.102. Let us recall that A is the infinitesimal generator of the C_0-semigroup $\{T_t\}_{t\geq 0}$ on the space $L^p([0,\infty))$. Our aim is to prove that properties 1 and 2 are satisfied. Let us choose and fix an auxiliary number $\lambda > 0$. Let us denote

$$V := \{f \in L^p([0,\infty)) : Df \in L^p([0,\infty))\}.$$

First we prove that $D(A) \subset V$ and

$$Ag = Dg, \quad \text{for every } g \in D(A). \tag{2.4.40}$$

Let us choose and fix an arbitrary $g \in D(A)$. Then, by the Hille-Yosida Theorem and by identity (2.4.29) in Remark 2.98, $D(A) \subset \mathit{Range}\big((\lambda I - A)^{-1}\big)$, as well as by Lemma 2.101 part (i), we infer that there exists $f \in L^p([0,\infty))$ such that

$$g = J_\lambda f = \lambda(\lambda I - A)^{-1} f.$$

In other words, because of the definition (2.4.33) of the function g_λ, $g = g_\lambda$. Hence We proved that if $g \in D(A)$ and $\lambda > 0$ then there exists $f \in L^p([0,\infty))$ such that $g = J_\lambda f = \lambda(\lambda I - A)^{-1} f$, that means, $g = g_\lambda$. From Claim 2.104 we infer that the function g is weakly differentiable and $Dg \in L^p([0,\infty))$. That proves that $g \in V$. Also, by Claim 2.105, we infer that $Ag = Ag_\lambda = Dg_\lambda = Dg$ what proves the equation (2.4.40). Secondly, we need to show that $V \subset D(A)$. Let us take and fix an arbitrary function $f \in V$. Define an auxiliary function u as the follows

$$Df - \lambda f = -\lambda u. \tag{2.4.41}$$

Thus, because $L^p([0,\infty))$ is a vector space, $u \in L^p([0,\infty))$. Let us consider a function g_λ defined by identity (2.4.33) with the function f replaced by the function u, i.e. $g_\lambda := \lambda(\lambda I - A)^{-1} u$. By Claim 2.104 we infer that g_λ is weakly differentiable and $Dg_\lambda = -\lambda u + \lambda g_\lambda$, i.e.

$$Dg_\lambda - \lambda g_\lambda = -\lambda u. \tag{2.4.42}$$

We see now that f and g_λ satisfy the same equation and both belong to the space $L^p([0,\infty))$. Our aim is to show that $f = g_\lambda$. For this purpose let us define a function w as follows

$$w := f - g_\lambda.$$

It is sufficient to prove that $w = 0$. By linearity of weak derivative, we deduce that w is weakly differentiable and, next by identities (2.4.41) and (2.4.42) we infer that

$$Dw = Df - Dg_\lambda = \lambda f - \lambda u - (-\lambda u + \lambda g_\lambda) = \lambda(f - g_\lambda) = \lambda w$$

We will show that $Dw = \lambda w$. For this aim, let us define an auxiliary function $Z(x) = e^{-\lambda x} w(x)$. Function $Z(x)$ is measurable and

$$\int_{[a,b]} |Z(x)|^p \, dx < \infty,$$

for every bounded interval $[a,b] \subset [0,\infty)$, which means that $Z \in L^p_{loc}([0,\infty))$. Moreover, one can show that function Z is weakly differentiable. Since we have $Dw = \lambda w$, we get

$$DZ = D\big(e^{-\lambda x} w(x)\big) = D(e^{-\lambda x}) w(x) + e^{-\lambda x} D(w) = 0.$$

That means

$$\int_0^\infty Z(x)\phi'(x) \, dx = 0, \ \phi \in C_0^\infty([0,\infty)).$$

Note that the function $e^{-\lambda x}$ is of C^1-class classically and hence it is differentiable and the weak it is equal to the classical derivative. Therefore, by the Lemma [24, Lemma 2], we infer that there exists a number $C \in \mathbb{R}$ such that

$$Z(x) = C, \text{ for almost all } x \in [0,\infty),$$

where "for almost all" means with respect to the Lebesgue measure. In view of the definition of the function Z we deduce that

$$w(x) = Ce^{\lambda x}, \text{ for almost all } x \in [0,\infty). \tag{2.4.43}$$

Now, we claim that $C = 0$. Suppose by contradiction that $C \neq 0$. Then since w belongs to the space $L^p([0,\infty))$ we have

$$\infty > \int_0^\infty |w(x)|^p \, dx = \int_0^\infty |Ce^{\lambda x}|^p \, dx = |C|^p \int_0^\infty e^{p\lambda x} \, dx = \infty.$$

The last equality holds because $|C|^p > 0$ and, as $p\lambda > 0$, that $\int_0^\infty e^{p\lambda x} \, dx = \infty$. Hence we deduce that

$$\infty > \infty.$$

Thus, it is a contradiction. That means, $C = 0$, and therefore, by identity (2.4.43), we infer that $w = 0$. Hence, $f = g_\lambda$. From the Claim 2.105 that states $g_\lambda \in D(A)$, we infer that $f \in D(A)$, which proves the second property 2.

So we have finished verifying Properties 1 and Property 2. Hence, the proof of the Proposition 2.102 is concluded. $\square$

Proposition 2.106. *Assume that $p \in [1,\infty)$. Let $\{T_t\}_{t\geq 0}$ be a C_0-semigroup on the Banach space $L^p(0,1)$, defined by equality* (2.4.1)*, see Example 2.80. Let us denote by A*

the infinitesimal generator of this C_0-semigroup. Then the following holds.

$$D(A) = \{f \in L^p(0,1) : f \text{ is weakly differentiable and}$$
$$\{(0,1) \ni x \mapsto Df(x)\} \in L^p(0,1), \text{ i.e.}, \int_0^1 |xDf(x)|^p \, dx < \infty\},$$
$$(Af)(x) = -xDf(x), \quad x \in (0,1) \quad \text{for } f \in D(A).$$

In the above, the symbol D represents the weak derivative.

Proposition 2.107. *Assume that $\{T_t\}_{t\geq 0}$ be a C_0-semigroup on the Banach space ${}_0C([0,1])$, defined by equality* (2.4.1)*, see Example 2.80. Let us denote by A the infinitesimal generator of this C_0-semigroup. Then the following holds true.*

$$D(A) = \{f \in {}_0C([0,1]) : f \text{ is continuously differentiable on } (0,1]$$
$$\text{and } \lim_{x\to 0} xDf(x) = 0\};$$

$$\begin{cases} (Af)(x) & = -xDf(x), \quad x \in (0,1) \quad \text{for } f \in D(A). \\ (Af)(0) & = 0 \text{ if } \quad x = 0. \end{cases}$$

The proof of this result can be done similarly to the proof of Proposition 2.106 and hence will be skipped. This result is a special case of Theorem 4.20 from chapter 4 with function $a(x) = x$.

Note that for a function $f \in {}_0C([0,1])$ which is continuously differentiable on $(0,1]$ the condition

$$\lim_{x\to 0} xDf(x) = 0$$

is equivalent to $g \in {}_0C([0,1])$, where

$$g(x) = \begin{cases} xDf(x), \quad x \in (0,1) \quad \text{for } f \in D(A). \\ 0 \text{ if } \quad x = 0. \end{cases}$$

To prove Proposition 2.106, we first need to prove the following three auxiliary results Claim 2.108, Claim 2.109 and Claim 2.110.

Claim 2.108. Let us assume the assumptions and notation from Proposition 2.106. Assume also the function $f \in L^p(0,1)$ and the number $\lambda > \frac{1}{p}$. Define a function $g_\lambda \in L^p(0,1)$ by the following formula

$$g_\lambda := \lambda(\lambda I - A)^{-1} f. \tag{2.4.44}$$

Then

$$g_\lambda(x) = \frac{\lambda}{x^\lambda} \int_0^x y^{\lambda-1} f(y) \, dy, \quad x \in (0,1). \tag{2.4.45}$$

Proof of Claim 2.108. Let us choose and fix a number $\lambda > \frac{1}{p}$. We notice that our current semigroup is not a contraction semigroup. It is only a contraction type semigroup, with constant $\frac{1}{p}$, in other words, it satisfies estimate (2.4.2). Hence, by Corollary 2.100, we infer that since $\lambda > \frac{1}{p}$, $R(\lambda, A) := (\lambda I - A)^{-1}$ exists and by identity (2.4.30),

$$(\lambda I - A)^{-1} f = \int_0^\infty e^{-\lambda t} T_t f \, dt.$$

One can see directly that in order for the above integral is convergent, we need to assume that $\lambda > \frac{1}{p}$. Indeed, then $\lambda - \frac{1}{p} > 0$ and thus $\int_0^\infty e^{-\lambda - \frac{1}{p})t} \, dt = \frac{1}{\lambda - \frac{1}{p}} < \infty$. Therefore,

$$\| \int_0^\infty e^{-\lambda t} T_t f \, dt \| \le \int_0^\infty \| e^{-\lambda t} T_t f \| \, dt \le \int_0^\infty e^{-\lambda t} e^{\frac{t}{p}} \|f\| \, dt = \int_0^\infty e^{-(\lambda - \frac{1}{p})t} \, dt \, \|f\| < \infty.$$

We take $f \in L^p(0,1)$. Since $R(\lambda, A) = (\lambda I - A)^{-1}$ is bounded linear operator, we define $J_\lambda = \lambda R(\lambda, A) : L^p(0,1) \to L^p(0,1)$ is also a bounded linear operator. Assume that $g_\lambda = J_\lambda f \in L^p(0,1)$. Moreover, since $R(\lambda, A)$ is linear, then we have

$$g_\lambda = \lambda R(\lambda, A) f = R(\lambda, A)(\lambda f).$$

Hence, $g_\lambda \in Range R(\lambda, A)$. Since $(\lambda I - A) R(\lambda, A) = I$, we infer that the $Range R(\lambda, A) \subset D(A)$. Thus, $g_\lambda \in D(A)$. From our assumption of function g_λ, we have

$$g_\lambda = J_\lambda f = \lambda R(\lambda, A) f = \lambda \int_0^\infty e^{-\lambda t} T_t f \, dt.$$

Hence, for every $x \in (0,1)$ we have

$$(g_\lambda)(x) = \lambda \int_0^\infty e^{-\lambda t} (T_t f)(x) \, dt = \lambda \int_0^\infty e^{-\lambda t} f(x e^{-t}) \, dt. \tag{2.4.46}$$

By invoking the changing of variables of the above equation (2.4.46) we obtain

$$(g_\lambda)(x) = \lambda \int_x^0 \big(\frac{y}{x}\big)^\lambda f(y) \big(-\frac{1}{y} \, dy \big) = \frac{\lambda}{x^\lambda} \int_0^x y^{\lambda - 1} f(y) \, dy. \tag{2.4.47}$$

Since we assume that the function $f \in L^p(0, \infty)$ we infer that the integral (2.4.47) exists because by Hölder inequality with $(\frac{1}{p} + \frac{1}{p^*} = 1)$,

$$\begin{aligned}
\int_0^x y^{\lambda-1} f(y) \, dy &\le \big(\int_0^x |y^{\lambda-1}|^{p^*} \, dy \big)^{\frac{1}{p^*}} \big(\int_0^x |f(y)|^p \, dy \big)^{\frac{1}{p}} \\
&\le \big(\int_0^1 y^{p^*(\lambda-1)} \, dy \big)^{\frac{1}{p^*}} \big(\int_0^1 |f(y)|^p \, dy \big)^{\frac{1}{p}} \\
&\le \|f\|_{L^p(0,1)} \big(\int_0^1 y^{p^*(\lambda-1)} \, dy \big) < \infty.
\end{aligned}$$

Hence, we proved for $\lambda > \frac{1}{p}$

$$g_\lambda(x) = \frac{\lambda}{x^\lambda} \int_0^x y^{\lambda-1} f(y)\, dy, \quad x \in (0,1).$$

So the proof of the Claim 2.108 is complete. $\square$

The following heuristic argument explains the content of Claim 2.109 Let us differentiate equation (2.4.45), we get, for $x \in (0,1)$,

$$\begin{aligned} \big(Dg_\lambda\big)(x) &= \lambda(-\lambda)\frac{1}{x^{\lambda+1}} \int_0^x y^{\lambda-1} f(y)\, dy + \frac{\lambda}{x^\lambda} x^{\lambda-1} f(x) \\ &= -\frac{\lambda}{x}\frac{\lambda}{x^\lambda} \int_0^x y^{\lambda-1} f(y)\, dy + \frac{\lambda}{x} f(x) \\ &= -\frac{\lambda}{x} g_\lambda(x) + \frac{\lambda}{x} f(x). \end{aligned} \tag{2.4.48}$$

Claim 2.109. Let us assume the assumptions and notation from Proposition 2.106. Assume that $f \in L^p(0,1)$ and $\lambda > \frac{1}{p}$ and the function $g_\lambda \in L^p(0,1)$ be defined by equation (2.4.44). Then g_λ is weakly differentiable on $(0,1)$ and the weak derivative Dg_λ of the function g_λ satisfies the following equality

$$-xDg_\lambda(x) = \lambda g_\lambda(x) - \lambda f(x), \text{ for } x \in (0,1). \tag{2.4.49}$$

Proof of Claim 2.109. Let us choose and fix $f \in L^p(0,1)$ and $\lambda > \frac{1}{p}$. Let $g_\lambda \in L^p(0,1)$ defined by equation (2.4.44). By the previous Claim 2.108, the function g_λ satisfies identity (2.4.45). We aim to show that g_λ is weakly differentiable on $(0,1)$. We guess that the following choice of function h_λ obtained from earlier calculated formula (2.4.48) for the classical derivative of the function g_λ is good:

$$h_\lambda(x) = -\frac{\lambda g_\lambda(x)}{x} + \frac{\lambda f(x)}{x}, \quad \text{for } x \in (0,1).$$

From the above we deduce the following useful version of it:

$$-x\, h_\lambda(x) = \lambda g_\lambda(x) - \lambda f(x), \quad \text{for } x \in (0,1).$$

Since by assumptions $f \in L^p(0,1)$ and by identity (2.4.44) function g_λ is also belongs to the space $L^p(0,1)$, we infer that the function h_λ defined above also belongs to $L^p(0,1)$. So, it is sufficient to show that for every $\phi \in C_0^\infty(0,1)$ we have

$$\int_0^1 g_\lambda(x)\phi'(x)\, dx = -\int_0^1 \frac{\lambda}{x} f(x)\phi(x)\, dx + \int_0^1 g_\lambda(x)\frac{\lambda}{x}\phi(x)\, dx. \tag{2.4.50}$$

To prove the above equation (2.4.50), let us take and fix $\phi \in C_0^\infty(0,1)$. Note that by using equation (2.4.45) we have

$$\int_0^1 g_\lambda(x)\phi'(x)\,dx = \int_0^1 \frac{\lambda}{x^\lambda}\,\big[\int_0^x y^{\lambda-1} f(y)\,dy\big]\,\phi'(x)\,dx.$$

It follows, by applying the Fubini Theorem [28, Theorem 2.2.3] to the double integral on the RHS above we infer that

$$\int_0^1 g_\lambda(x)\phi'(x)\,dx = \int_0^1 y^{\lambda-1} f(y)\big[\int_y^1 \frac{\lambda}{x^\lambda}\phi'(x)\,dx\big]\,dy. \tag{2.4.51}$$

Note that since the function ϕ has a compact support, we infer that $\phi(1) = 0$. If $y \in (0,1)$, using integration by parts for the second integral of equation (2.4.51) we have

$$\int_y^1 \frac{\lambda}{x^\lambda}\phi'(x)\,dx = -\phi(y)\frac{\lambda}{y^\lambda} + \int_y^1 \phi(x)\frac{\lambda^2}{x^{\lambda+1}}\,dx.$$

Next we substitute the last equation in the second integral of equation (2.4.51) and

$$\begin{aligned}
\int_0^1 g_\lambda(x)\phi'(x)\,dx &= \int_0^1 y^{\lambda-1} f(y)\big[-\phi(y)\frac{\lambda}{y^\lambda} + \int_y^1 \phi(x)\frac{\lambda^2}{x^{\lambda+1}}\,dx\big]\,dy\\
= -\int_0^1 y^{\lambda-1} f(y)\phi(y)\frac{\lambda}{y^\lambda}\,dy &+ \int_0^1 \big[\int_0^x y^{\lambda-1} f(y)\,dy\big]\phi(x)\frac{\lambda^2}{x^{\lambda+1}}\,dx\\
= -\int_0^1 f(y)\phi(y)\frac{\lambda}{y}\,dy &+ \int_0^1 g_\lambda(x)\frac{\lambda}{x}\,dx
\end{aligned}$$

Thus we proved equality (2.4.50) as we wanted and this completes the proof of the Claim 2.109. ☐

Claim 2.110. Let us assume the assumptions and notation from Proposition 2.106. Assume that $f \in L^p(0,1)$ and $\lambda > \frac{1}{p}$ and the function g_λ be defined by equation (2.4.44). Then the function g_λ belongs to $D(A)$ and satisfies the following equality

$$(Ag_\lambda)(x) = -x(Dg_\lambda)(x), \quad x \in (0,1].$$

Proof of Claim 2.110. Let us choose and fix $\lambda > \frac{1}{p}$ and $f \in L^p(0,1)$. Let us consider the function g_λ be defined by equation (2.4.44). If follows from the Claim 2.109 that $Dg_\lambda \in L^p(0,1)$. Also, since by identity (2.4.29) in Remark 2.98, $Range\big((\lambda I - A)^{-1}\big) \subset D(A)$ and since $g_\lambda = \lambda(\lambda I - A)^{-1} f \in Range\big((\lambda I - A)^{-1}\big)$, we deduce that $g_\lambda \in D(A)$. So we have proved the first part of our Claim.

Moreover, by Lemma 2.101 we infer that for every $x \in (0,1]$ we have

$$\begin{aligned}
[Ag_\lambda](x) &= [A\lambda(\lambda I - A)^{-1} f](x) = [AJ_\lambda f](x)\\
&= [\lambda(J_\lambda - I)f](x) = [\lambda J_\lambda f](x) - \lambda f(x) = \lambda g_\lambda(x) - \lambda f(x).
\end{aligned}$$

On the other hand, by identity (2.4.49) we have $-xDg_\lambda(x) = \lambda g_\lambda(x) - \lambda f(x),\;\; x \in (0,1]$. Therefore, we infer that $Ag_\lambda(x) = -xDg_\lambda(x)$ for $x \in (0,1]$. Hence, the proof of the Claim 2.110 is complete. □

After we proved necessary Claims 2.108, 2.109 and 2.110, it is now possible to embark with the proof of Proposition 2.106.

Proof of Proposition 2.106. Let us recall that A is the infinitesimal generator of the contraction type C_0-semigroup $\{T_t\}_{t\geq 0}$ on the space $L^p(0,1)$. Our aim is to prove properties 2.106 and 2.106 are satisfied. Let us choose and fix an auxiliary number $\lambda > \frac{1}{p}$. Let us denote

$$V := \{f \in L^p(0,1) : f \text{ is weakly differentiable and } xDf(x) \in L^p(0,1)\}.$$

First we will prove that $D(A) \subset V$ and

$$Ag(x) = -xDg(x), \;\; \text{for every } g \in D(A) \;\text{ and } x \in (0,1). \tag{2.4.52}$$

For this purpose, let us choose and fix an arbitrary $g \in D(A)$. Then, by the Hille-Yosida Theorem and by identity (2.4.29) in Remark 2.98, $D(A) \subset \mathit{Range}\big((\lambda I - A)^{-1}\big)$, as well as Lemma 2.101, we infer that there exists $f \in L^p(0,1)$ such that

$$g(x) = J_\lambda f(x) = \lambda(\lambda I - A)^{-1} f(x), \quad \text{for } x \in (0,1).$$

In other words, because of the definition (2.4.44) of the function g_λ, we see that $g = g_\lambda$. From Claim 2.109 we infer that function g is weakly differentiable and $xDg \in L^p(0,1)$. That proves that $g \in V$. Next, by Claim 2.110, we infer that

$$Ag(x) = Ag_\lambda(x) = -xDg_\lambda(x) = -xDg(x), \text{ for every } x \in (0,1]\ .$$

Hence, we proved equation (2.4.52). Secondly, we need to show that $V \subset D(A)$. To do this, let us take and fix an arbitrary function $f \in V$. Define a function u as follows

$$-xDf - \lambda f = -\lambda u \tag{2.4.53}$$

Thus, because $L^p(0,1)$ is a vector space, $u \in L^p(0,1)$. Let us consider a function g_λ defined by identity (2.4.44) with the function f replaced by the function u, i.e., $g_\lambda := \lambda(\lambda I - A)^{-1}u$. By Claim 2.109 we infer that g_λ is weakly differentiable and

$$-xDg_\lambda(x) - \lambda g_\lambda(x) = -\lambda u(x), \;\; x \in (0,1). \tag{2.4.54}$$

Hence the functions f and g_λ satisfy the same equation and both belong to the space $L^p(0,1)$. Our aim is to show that $f = g_\lambda$. For this purpose let us define a function w as follows

$$w := f - g_\lambda.$$

It is sufficient to prove that $w = 0$. By linearity of weak derivative, we deduce that w is weakly differentiable and by identities (2.4.53) and (2.4.54) we infer that for every $x \in (0,1)$

$$-xDw(x) = -xDf(x) - [-xDg_\lambda(x)] = \lambda(f(x) - g_\lambda(x)) = \lambda w(x).$$

Then, we conclude that

$$-xDw(x) = \lambda w(x), \;\; x \in (0,1). \tag{2.4.55}$$

We aim to prove that $w = 0$. We notice that equation (2.4.55) can be written as

$$-x\frac{dw}{dx} = \lambda w.$$

We solve this equation by the method of separation of variables and we infer that there exists a constant $C \in \mathbb{R}$ such that

$$w(x) = \frac{C}{x^\lambda}, \;\; x \in (0,1).$$

But since $\lambda > \frac{1}{p}$ we have

$$\int_0^1 |\frac{1}{x^\lambda}|^p \, dx = \int_0^1 \frac{1}{x^{p\lambda}} \, dx = \infty.$$

So we proved that if $C \neq 0$ then

$$\|w\|_{L^p(0,1)} = |C| \Big[\int_0^1 |\frac{1}{x^\lambda}|^p \, dx\Big]^{1/p} = \infty.$$

But because $w \in L^p(0,1)$ hence we deduce that $C = 0$. Therefore, $w = 0$. This is what we wanted to prove. Hence, $f = g_\lambda$. From the Claim 2.110 that states $g_\lambda \in D(A)$ we infer that $f \in D(A)$, which proves the second property 2.106. By this, we have finished the proof of Proposition 2.106. $\square$

Chapter 3

Invariant Measures on the Fractional Sobolev Spaces

The existence of an invariant measure is one of the most important problems in the theory of PDEs. In this chapter, we focus on proving the existence of the invariant measures for PDEs and this helps us to understand the properties of PDEs. An important work on the invariant measures was established by Lasota and Mackey [28, Example 11.1.1]. They considered a Wiener measure (Gaussian measure) in the space of all continuous functions $x : [0,1] \to \mathbb{R}$ such that $x(0) = 0$. Also, they considered a C_0-semigroup $\{S_t\}_{t\geq 0}$ on the space $X = {}_0C\big([0,1]\big)$ corresponding to the following partial differential equation:

$$\begin{aligned} \frac{\partial u(t,s)}{\partial t} + s\frac{\partial u(t,s)}{\partial s} &= \frac{1}{2}u(t,s),\ t > 0,\ \ s \in [0,1], \\ u(0,s) &= x(s),\ \ s \in [0,1], \end{aligned} \tag{3.0.1}$$

where $x \in X$.

The solution to this equation can be written explicitly as follows:

$$u(t,s) = e^{\frac{t}{2}}x(se^{-t}), \quad s \in [0,1],\ \ t \geq 0,$$

and the C_0-semigroup $\{S_t\}_{t\geq 0}$ is defined by the analogous equation:

$$S_t x(s) = e^{\frac{t}{2}}x(se^{-t}), \quad s \in [0,1],\, t \geq 0,\ x \in X. \tag{3.0.2}$$

Remark 3.1. It is important to point out that the ”space” variable in this section is denoted by a letter $s \in [0,1]$. While the “space” variable in the following sections are denoted by a letter $x \in [0,1]$. To make matters even more complicated, the letter x is used in the present section to denote the elements of the space of initial data. However, this should not be a problem for an attentive reader and should not lead to any confusion or misunderstanding.

Lasota and Mackey [28] proved that the C_0-semigroup $\{S_t\}_{t\geq 0}$ preserves the Wiener measure. In other words, the classical Wiener measure on the space X is an invariant measure for $\{S_t\}_{t\geq 0}$. We formulate their contribution into the following Lemma. The proof of the Lemma itself is made available in Appendix A.1.1, only pointing out that we add more detail to the proof.

Lemma 3.2. *Let C be any cylinder set on the space ${}_0C\big([0,1]\big)$. If $\{S_t\}_{t\geq 0}$ is a C_0-semigroup in the same space, then the following equality satisfies*

$$\mu\big(S_t^{-1}(C)\big) = \mu(C). \tag{3.0.3}$$

Here, μ is a Gaussian measure, which is the law of Brownian Motion, on the space ${}_0C\big([0,1]\big)$, see Proposition 2.52.

By using the Lemma 3.2, we prove that it is possible to construct a generalisation to the equation (3.0.1) with different parameters. The main novelty is to extend the parameter to any parameter between $\frac{1}{2}$ and $\frac{3}{2}$. A such generalisation can massively help to understand more PDEs and their properties in more complicated spaces such as interpolation spaces.

To find this generalisation, we start by extending Lasota and Mackey's work and define a new Banach space and a new C_0-semigroup related to equation (3.0.1) but with different parameters. We define an isomorphism operator between the two spaces and such an operator needs to satisfy a suitable commutation property. If the operator exists, then the invariant measure that was found in the Lasota and Mackey [28, Example 11.1.1] can be used to define a new invariant measure for the new objects.

Accordingly, this chapter is organised as follows. The first Section 3.1 gives related preliminaries about invariant measures and interpolation spaces. Next, Section 3.2 states our abstract theorem as initially described above, which becomes a foundation for building many of the results in the rest of this chapter, along with applications (using concrete spaces) to the abstract theorem. Lastly, Section 3.3 gives our main result (the generalisation of Lasota and Mackey [28, Example 11.1.1].

3.1 Preliminaries

3.1.1 Introduction to invariant measure

Let us begin with defining a fundamental notion of this thesis, the definition of invariant measures.

Definition 3.3. Let $(X, \mathcal{A}, \mu)$ be a probability space. If $\{T_t\}_{t\geq 0}$ is a measurable semiflow on $(X, \mathcal{A})$. Then μ is called an invariant probability measure for the semiflow $\{T_t\}_{t\geq 0}$ if and only if for every $t \geq 0$ and for every $C \in \mathcal{A}$ the following equality holds

$$\mu(T_t^{-1}(C)) = \mu(C).$$

In what follows we need also the following definition.

Definition 3.4. Let $(X, \mathcal{A})$ be a measurable space. Let Y be the space of all functions defined on the interval $[0, \infty)$ with values in X. For an increasing sequence $s_1, \cdots, s_n \in [0, \infty)$ and a measurable set $A = \prod_{i=1}^{n} A_i \in \mathcal{A}^n$ we define a cylindrical set by the following form

$$C(s_1, \cdots, s_n; A_1, \cdots, A_n) = \{x \in Y : x(s_1) \in A_1, \cdots, x(s_n) \in A_n\}.$$

A special case of the above definition is when $(X, \mathcal{A})$ is equal to $\big(\mathbb{R}, \mathcal{B}(\mathbb{R})\big)$.

3.1.2 Interpolation spaces

An interpolation space is a space who intermediate between two spaces [29]. One of the most important applications of interpolation space is Sobolev spaces. In this section, we define our main space $H^{\alpha,p}(0,1)$ by

$$H^{\alpha,p}(0,1) := [L^p(0,1), H^{1,p}(0,1)]_\alpha.$$

The following theorem generalise equation (2.3.6) in the following manner

Theorem 3.5. *[45, Theorem 29] If $\alpha \in (\frac{1}{p}, 1)$, then*

$$H^{\alpha,p}(0,1) \subset C([0,1]). \tag{3.1.1}$$

Moreover, there exists a constant $C = C(\alpha, p) > 0$ such that

$$\|u\|_{C([0,1])} \leq C\|u\|_{H^{\alpha,p}(0,1)}.$$

Also, similarly to the Definition 2.68 of the space ${}_0H^{1,p}(0,1)$ we can also define the space ${}_0H^{\alpha,p}(0,1)$ as the following

Definition 3.6. Let $\alpha \in (\frac{1}{p}, 1)$, we define the space ${}_0H^{\alpha,p}(0,1)$ as

$${}_0H^{\alpha,p}(0,1) = \{u \in H^{\alpha,p}(0,1) : u(0) = 0\}. \tag{3.1.2}$$

We can apply Theorem 2.27 to the following spaces. Let $X = H^{1,p}(0,1)$ and $Y = {}_0H^{1,p}(0,1)$. We know from Theorem 2.70 that space Y is a closed subspace of space

X. So, we deduce that Y is a Banach space and the norm of an element $u \in Y$ is given by

$$\|u\|_Y = \|u\|_X = \left(|u|^p_{L^p(0,1)} + |Du|^p_{L^p(0,1)}\right)^{\frac{1}{p}}. \tag{3.1.3}$$

Since the space $H^{\alpha,p}(0,1)$ is subspace of the space $C([0,1])$ we infer that the natural embedding, call it i,

$$i : H^{\alpha,p}(0,1) \ni u \mapsto u \in C([0,1]), \tag{3.1.4}$$

is well-defined. It is easy to observe that i is a linear map. Thus, condition (3.1.1) implies that the natural embedding i is well defined linear map. Moreover, if $u \in H^{\alpha,p}(0,1)$, then

$$\|i(u)\|_{C([0,1])} = \|u\|_{C([0,1])} \leq C\|u\|_{H^{\alpha,p}(0,1)},$$

which means that i is a bounded map. Hence we can summarize that, the natural embedding i defined in (3.1.4) is not only well-defined and a linear map but also bounded. As a consequence of this, we can formulate the following Corollary.

Corollary 3.7. *If $\alpha \in (\frac{1}{p}, 1)$ then the natural embedding map i defined in formula* (3.1.4) *is well defined, linear and bounded.*

As the earlier proofs, we can prove the following theorem.

Theorem 3.8. *Let $\alpha \in (\frac{1}{p}, 1)$, then the space ${}_0H^{\alpha,p}(0,1)$ is a closed subspace of $H^{\alpha,p}(0,1)$. Moreover, ${}_0H^{\alpha,p}(0,1)$ endowed with the norm inherited from the space $H^{\alpha,p}(0,1)$ is a Banach space.*

Proof of Theorem 3.8. The proof of this theorem comes with three steps as follows

1. Prove that the space ${}_0H^{\alpha,p}(0,1)$ is well-defined.

2. The space ${}_0H^{\alpha,p}(0,1)$ is a closed subspace of the space $H^{\alpha,p}(0,1)$.

3. ${}_0H^{\alpha,p}(0,1)$ is a Banach space.

Step 1: We would like to show that the space ${}_0H^{\alpha,p}(0,1)$ is well-defined. Let us fix $\alpha > \frac{1}{p}$. The reason we assume that $\alpha > \frac{1}{p}$ is that in this case Theorem 3.5 holds, i.e. ${}_0H^{\alpha,p}(0,1) \subset C([0,1])$. Hence, if $u \in H^{\alpha,p}(0,1)$, then $u \in C([0,1])$ so that $u(t)$ makes sense for all $t \in [0,1]$. In particular, $u(0)$ also makes sense. Hence we show that the space ${}_0H^{\alpha,p}(0,1)$ is well-defined.
Step 2: Let us fix $\alpha > \frac{1}{p}$. According to Corollary 3.7 above the natural embedding $i : H^{\alpha,p}(0,1) \hookrightarrow C([0,1])$ is bounded. Next, we consider the evaluation map j_0 as follows

$$j_0 : C([0,1]) \ni u \mapsto u(0) \in \mathbb{R}.$$

Obviously, j_0 is well defined and linear. Moreover, j_0 is bounded because

$$\|j_0(u)\| = |u(0)| \leq \sup_{t\in[0,1]} = \|u\|_{C([0,1])}.$$

Hence,

$$\|j_0(u)\|_{\mathbb{R}} \leq \|u\|_{C([0,1])}.$$

Then j_0 is linear and bounded. Therefore the composition $j_0 \circ i$ is also linear and bounded. Denote $i_0 = j_0 \circ i$, i.e.,

$$i_0 : H^{\alpha,p}(0,1) \ni u \mapsto u(0) \in \mathbb{R}.$$

So we proved that $i_0(u) = u(0)$. Hence i_0 is bounded and linear.
Notice: we already know the definition of the space ${}_0H^{\alpha,p}(0,1)$ is

$$\begin{aligned} {}_0H^{\alpha,p}(0,1) &= \{u \in H^{\alpha,p}(0,1) : u(0) = 0\} \\ &= \{u \in H^{\alpha,p}(0,1) : i_0(u) = 0\} = \ker i. \end{aligned}$$

Using Theorem 2.26 we deduce that ${}_0H^{\alpha,p}(0,1)$ is a closed subspace of space $H^{\alpha,p}(0,1)$.
Step 3: In order to prove that the space ${}_0H^{\alpha,p}(0,1)$ is a Banach space we apply Theorem 2.27 again for the following spaces. We choose $X = H^{\alpha,p}(0,1)$ and $Y = {}_0H^{\alpha,p}(0,1)$. Since the space $H^{\alpha,p}(0,1)$ is a Banach space and ${}_0H^{\alpha,p}(0,1)$ is a closed subspace of $H^{\alpha,p}(0,1)$ we deduce that ${}_0H^{\alpha,p}(0,1)$ is also a Banach space. □

Theorem 3.9. *Suppose that X and F are Banach spaces and $V \subset X$ is a dense subspace of X. Suppose that $\mathbf{A}, \mathbf{B} : X \to F$ are bounded linear operators such that*

$$\mathbf{A}(x) = \mathbf{B}(x) \textit{ for every } x \in V,$$

then

$$\mathbf{A}(x) = \mathbf{B}(x), \textit{ for every } x \in X.$$

Proof of theorem 3.9. The proof of this theorem is very simple. We start by assuming that $x \in X$. By the density of the set V in X there exists sequence $\{x_n\}_{n\in\mathbb{N}} \subset V$ such that $x_n \to x$ in X. By assumption, both $\mathbf{A}$ and $\mathbf{B}$ are bounded and hence continuous. Thus, $\mathbf{A}x_n \to \mathbf{A}x$ and $\mathbf{B}x_n \to \mathbf{B}x$. On the other hand, $\mathbf{A}x_n = \mathbf{B}x_n$ for every n. Hence, by the uniqueness of the limit in F, we infer that $\mathbf{A}x = \mathbf{B}x$, for every $x \in X$. □

We conclude this section by stating an important corollary that helps us to prove some properties.

Corollary 3.10. *[45, corollary 23] If $1 > \alpha_2 > \alpha > 0$, then*

$$H^{1,p}(0,1) \subset H^{\alpha_2,p}(0,1) \subset H^{\alpha,p}(0,1) \subset L^p(0,1).$$

Moreover, each of those embedding is continuous, i.e., there exists $C > 0$ *such that*

$$\begin{aligned} \|u\|_{H^{\alpha_2,p}} &\le C\|u\|_{H^{1,p}} \qquad \forall u \in H^{1,p} \\ \|u\|_{H^{\alpha,p}} &\le C\|u\|_{H^{\alpha_2,p}} \qquad \forall u \in H^{\alpha_2,p} \\ \|u\|_{L^p} &\le C\|u\|_{H^{\alpha,p}} \qquad \forall u \in H^{\alpha,p}. \end{aligned}$$

And,

$$\begin{aligned} H^{1,p} &\quad \textit{is dense in} \quad H^{\alpha_2,p} \\ H^{\alpha_2,p} &\quad \textit{is dense in} \quad H^{\alpha,p} \\ H^{\alpha,p} &\quad \textit{is dense in} \quad L^p. \end{aligned} \tag{3.1.5}$$

Let us begin this section by recalling a definition of a positive operator in a Banach space.

Definition 3.11. [47, p. 1.14.1] Let Y be a complex Banach space and let Λ be a linear closed operator in Y with a dense domain $D(\Lambda) \subset Y$, i.e., $\Lambda : D(\Lambda) \to Y$. The operator Λ is said to be positive, if and only if the following two conditions are satisfies

(i) the interval $(-\infty, 0]$ is a subset of the resolvent set $\rho(\Lambda)$ of Λ, and

(ii) there exists a constant $C \ge 0$, such that

$$\|(\Lambda - \lambda I)^{-1}\| \le \frac{C}{1 + |\lambda|}, \quad \lambda \in (-\infty, 0].$$

The importance of positive operators stems from the fact that one can define fractional powers of such operators, see [47, section 1.15.1] for the definition and section 1.15.2 therein for basic properties. It follows that the definition of fractional powers of positive operators is closed. The theorem below will play a fundamental rôle in this thesis.

Theorem 3.12. *[47, Theorem 1.15.3] Let* Λ *be a positive operator. It is supposed that there exist two positive numbers* ε *and* C *such that* Λ^{it} *is bounded operator for* $t \in [-\varepsilon, \varepsilon]$ *and*

$$\|\Lambda^{it}\| \le C \quad \textit{for all } t \in [-\varepsilon, \varepsilon]. \tag{3.1.6}$$

If α *and* β *are two complex numbers,* $0 \le \operatorname{Re}\alpha < \operatorname{Re}\beta < \infty$ *and* $0 < \theta < 1$ *then*

$$[D(\Lambda^{\alpha}), D(\Lambda^{\beta})]_{\theta} = D\big(\Lambda^{\alpha(1-\theta)+\beta\theta}\big). \tag{3.1.7}$$

Let $X = L^p(0,1;\mathbb{C})$. We know that the space X is a Banach space over field $\mathbb{C}$. Define

$$B : D(B) \ni u \mapsto u' = Du \in L^p(0,1;\mathbb{C}).$$

$$D(B) = \{u \in H^{1,p}(0,1) : u(0) = 0\}.$$

$$Bu = u' = Du.$$

The map B is well defined linear operator map. Note that the operator B is associated with a certain C_0-semigroup. Consider the following family of bounded linear operators $\{T_t\}_{t\geq 0}$ such that if $t \in [0,1]$ then

$$\big(T_t f\big)(s) = \begin{cases} f(t+s), & \text{if } s \in [0, 1-t], \\ 0, & \text{if } s \in [1-t, 1], \end{cases}$$

and $T_t = 0$ if $t \geq 1$.

Proposition 3.13. *The family $\{T_t\}_{t\geq 0}$ is a C_0-semigroup on the space $X = L^p(0,1;\mathbb{C})$.*

Note that we proved in Example 2.81 that the $[T_t f](x) = f(t+s)$ is a C_0-semigroup on the space $X = L^p\big([0,\infty)\big)$.

Theorem 3.14. *[21, Theorem 3.1] Let Y be a complex ζ-convex Banach space, let $T \in (0,+\infty), p \in (1,+\infty)$ and $X = L^p(0,1;Y)$. Set $D(B) = \{u \in H^{1,p}(0,1;Y); u(0) = 0\}$ and the map $B : D(B) \to X$, $B(u) = u'$. Consider B is a closed operator in X. Then*

i) $\mathbb{R}^- \cup \{0\} \subseteq \rho(B)$ *and*

$$\|(\lambda - B)^{-1}\| \leq \frac{C_0}{1+|\lambda|}, \quad \forall \lambda \in (-\infty, 0]$$

ii) $\forall \zeta \in \mathbb{R}$, *the operator $B^{i\zeta}$ is bounded. The family $\big(B^{i\zeta}\big)_{\zeta\in\mathbb{R}}$ is a strongly continuous group in $\mathcal{L}(X)$ and there exists a constant $C_1 > 0$ such that*

$$\|B^{i\zeta}\| \leq C_1(1+\zeta^2)e^{\frac{\pi}{2}|\zeta|}, \quad \zeta \in \mathbb{R}.$$

If $Y = \mathbb{C}$ then we get the following simple consequence of Theorem 3.14.

Theorem 3.15. *Let $p \in (1,+\infty)$ and $X = L^p(0,1;\mathbb{C})$. Set*

$$D(B) = \{u \in H^{1,p}(0,1;\mathbb{C}) : u(0) = 0\},$$

where the operator B is defined as $\quad B : D(B) \to X, \quad B(u) = u'.$

Consider B is a closed operator in X. Then

i) $\mathbb{R}^- \cup \{0\} \subseteq \rho(B)$ *and*

$$\|(\lambda - B)^{-1}\| \leq \frac{C_0}{1+|\lambda|}, \quad \forall \lambda \in (-\infty, 0].$$

ii) $\forall \zeta \in \mathbb{R}$, *the operator* $B^{i\zeta}$ *is bounded. The family* $\big(B^{i\zeta}\big)_{\zeta\in\mathbb{R}}$ *is a strongly continuous group in* $\mathcal{L}(X)$ *and there exists a constant* $C_1 > 0$ *such that*

$$\|B^{i\zeta}\| \le C_1(1+\zeta^2)e^{\frac{\pi}{2}|\zeta|}, \quad \zeta \in \mathbb{R}. \tag{3.1.8}$$

Corollary 3.16. *In the framework and under assumptions of Theorem 3.15, we have*

$$D(B^\theta) = {}_0H^{\theta,p}(0,1), \quad \theta \in (0,1). \tag{3.1.9}$$

In particular, the operator B^θ *is an isomorphism between the space* ${}_0H^{\theta,p}(0,1)$ *and the space* $L^p(0,1)$.

Proof of Corollary 3.16. Let us choose and fix $\theta \in (0,1)$. The proof of this corollary is divided into four steps:

Step 1. It follows from Theorem 3.15 that the operator $\Lambda = B$ satisfies assumptions of Theorem 3.12. Indeed, if condition (3.1.8) is satisfied, then condition (3.1.6) is satisfied with $\epsilon = 1$ and $C = 2C_1e^{\frac{\pi}{2}}$. Hence it follows that B satisfies equality (3.1.7) (we take a special case here) with $\alpha = 0$ and $\beta = 1$, i.e.,

$$[D(B^0), D(B^1)]_\theta = D\big(B^\theta\big). \tag{3.1.10}$$

Step 2. $D(B^0) = X = L^p(0,1;\mathbb{C})$ and

$$D(B^1) = D(B) = \{u \in H^{1,p}(0,1;\mathbb{C}) : u(0) = 0\}.$$

Therefore, by identity (3.1.10), we infer that

$$D(B^\theta) = [L^p(0,1;\mathbb{C}), \{u \in H^{1,p}(0,1;\mathbb{C}) : u(0) = 0\}]_\theta.$$

Step 3. By [47, Theorem 4.3.3] and the definition (3.1.2) of the space ${}_0H^{\theta,p}(0,1;\mathbb{C})$ we get the following equality

$$\big[L^p(0,1;\mathbb{C}), \{u \in H^{1,p}(0,1;\mathbb{C}) : u(0) = 0\}\big]_\theta = {}_0H^{\theta,p}(0,1;\mathbb{C}).$$

Hence, we infer that equality (3.1.9) holds.

Step 4. By [47, Theorem 1.15.1 part (b)], see also Step 6 in the proof of that result, the operator B^θ is an isomorphism between the space $D(B^\theta)$ and the space $L^p(0,1;\mathbb{C})$. Applying the earlier proved identity (3.1.9), we infer that the operator B^θ is an isomorphism between the space ${}_0H^{\theta,p}(0,1;\mathbb{C})$ and the $L^p(0,1;\mathbb{C})$. The proof is complete. □

3.2 An Abstract Theorem about invariant measures

Theorem 3.17. *Let $(E, \mathcal{E})$ be a measure space and $E \neq \emptyset$ where $\mathcal{E}$ is a σ-field on E. Let the following assumptions be true on the space E:*

1. *Suppose that for all $t \geq 0$, $S_t : E \to E$ is $\mathcal{E}/\mathcal{E}$ measurable.*
2. *Suppose that μ is a probability measure on $\mathcal{E}$ such that*
$$\mu\big(S_t^{-1}(C)\big) = \mu(C), \;\; C \in \mathcal{E}, \; \textit{for all } t \geq 0;$$
3. *Suppose also that $(F, \mathcal{F})$ is another measure space, where $\mathcal{F}$ is a σ-field on F. Assume that a bijection map $\Lambda : F \to E$ satisfies*
$$\Lambda(B) \in \mathcal{E}, \quad \textit{for all } B \in \mathcal{F},$$
so that Λ^{-1} exists and $\Lambda^{-1} : E \to F$ is $\mathcal{E}/\mathcal{F}$ measurable. We assume also that $\Lambda : F \to E$ is $\mathcal{F}/\mathcal{E}$ measurable
4. *Define a measure ν on the space $(F, \mathcal{F})$ as the following formula:*
$$\nu(C) := \mu\big(\Lambda(C)\big), \quad C \in \mathcal{F}. \tag{3.2.1}$$
5. *Define a family $T_t : F \to F$, $t \geq 0$, in the form*
$$T_t(x) = \Lambda^{-1}\Big(S_t\big(\Lambda(x)\big)\Big), \; \textit{for every } t \geq 0$$
i.e.
$$T_t = \Lambda^{-1} \circ S_t \circ \Lambda$$
In other words, the diagram (3.1) is commuting

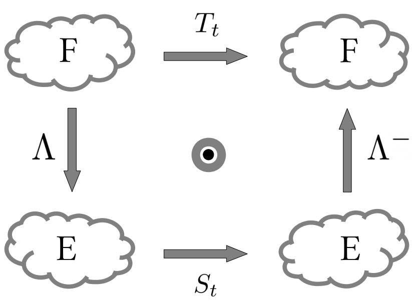


FIGURE 3.1: A graph showing the commuting of the semigroups $\{T_t\}_{t\geq 0}$ on the space F and $\{S_t\}_{t\geq 0}$ on the space E.

Based on this, it is clear that T_t is measurable, for every $t \geq 0$.

We assert the following statements for the measure ν:

(i) ν is a probability measure on the space $(F, \mathcal{F})$,

(ii) For any $C \in \mathcal{F}$ and $t \geq 0$ we have

$$\nu\big(T_t^{-1}(C)\big) = \nu(C).$$

Proof of Theorem 3.17. We need to prove the theorem in regard of two assertions (i) and (ii).

Regarding the first assertion (i), which is related to whether ν is a probability measure, we need to satisfy two conditions that: (1) measure ν of the empty set is zero, and (2) ν is σ-additive. To check the first condition, from the definition of the measure ν in equality (3.2.1), we have

$$\nu(\emptyset) = \mu\big(\Lambda(\emptyset)\big) = \mu(\emptyset) = 0.$$

For the second condition, let $C_n \in \mathcal{F}$ for $n = 1, 2, \dots$ (pairwise-disjoint). We have to show that

$$\nu\big(\cup_{n=1}^{\infty} (C_n)\big) = \sum_{n=1}^{\infty} \nu(C_n).$$

Let us choose any sets C_i, $C_j \in \mathcal{F}$. Since Λ is bijection, we have

$$\Lambda(C_i) \cap \Lambda(C_j) = \Lambda\big(C_i \cap C_j\big) = \Lambda(\emptyset) = \emptyset.$$

Therefore, $\Lambda(C_n)$ are pairwise-disjoint sets. Since μ is σ-additive and Λ is bijection, we infer that

$$\mu\Big(\Lambda\big(\cup_{n=1}^{\infty} (C_n)\big)\Big) = \mu\big(\cup_{n=1}^{\infty} \Lambda(C_n)\big) = \sum_{n=1}^{\infty} \mu\big(\Lambda(C_n)\big).$$

It follows from the definition of the measure ν and the last equation that

$$\nu\Big(\cup_{n=1}^{\infty} (C_n)\Big) = \mu\Big(\Lambda\big(\cup_{n=1}^{\infty} (C_n)\big)\Big) = \sum_{n=1}^{\infty} \mu\big(\Lambda(C_n)\big) = \sum_{n=1}^{\infty} \nu(C_n),$$

which implies that ν is σ-additive. Therefore, ν is a probability measure on the measure space $(F, \mathcal{F})$. This completes the proof of the first assertion (i).

Let us now prove the second assertion. Let us fix $C \in \mathcal{F}$. By assumption (5) in Theorem 3.17, we have

$$T_t^{-1}(C) = \big(\Lambda^{-1} \circ S_t \circ \Lambda\big)^{-1}(C) = \Lambda^{-1}\big(S_t^{-1}(\Lambda(C))\big).$$

Here we take the inverse of T_t and then apply the set C for both sides. If we apply the measure ν to both sides of the above equation, we get

$$\nu\big(T_t^{-1}(C)\big) = \nu\big[\Lambda^{-1}\big(S_t^{-1}(\Lambda(C))\big)\big]. \tag{3.2.2}$$

If we denote $\Lambda(C)$ by C', where C' is an auxiliary set, we obtain

$$C' = \Lambda(C) \Leftrightarrow C = \Lambda^{-1}(C').$$

Therefore, we can re-write the equality (3.2.1) as follows

$$\nu\big(\Lambda^{-1}(C')\big) = \mu(C'). \tag{3.2.3}$$

By taking the above equations (3.2.3), (3.0.3) and the definition of measure ν in equation (3.2.1) and substituting them in equation (3.2.2), we get

$$\begin{aligned}\nu\big(T_t^{-1}(C)\big) &= \nu\big[\Lambda^{-1}\big(S_t^{-1}(\Lambda(C))\big)\big] = \mu\big(S_t^{-1}(\Lambda(C))\big)\\ &= \mu\big(\Lambda(C)\big) = \nu(C),\end{aligned}$$

which concludes the second assertion. Therefore, the proof of Theorem 3.17 is completed. □

3.2.1 Invariant measures on the space ${}_0C^1\big([0,1]\big)$

After we stated the abstract Theorem 3.17 in the previous section, we need to attest and analyse our method in an example. In this section, we intend to define spaces E and F and apply them to the abstract Theorem 3.17 to generate a concrete model (which will later facilitate the application of different spaces other than E and F). However, we need before that to mention some notations and auxiliary properties that are going to lead us towards the application of the defined spaces. The main objective of this section is to prove the existence of an invariant measure on the following defined spaces E and F.
Let $E = {}_0C([0,1])$ be a space defined as the following

$$E = {}_0C\big([0,1]\big) := \{u \in C\big([0,1]\big) : u(0) = 0\}. \tag{3.2.4}$$

The norm of this space E is given by

$$\|u\|_{{}_0C([0,1])} = \sup_{t\in[0,1]} |u(t)|. \tag{3.2.5}$$

Let $F = {}_0C^1\big([0,1]\big)$ be a space defined as the following

$${}_0C^1([0,1]) = \{x : [0,1] \to \mathbb{R} : x \text{ is of } C^1\text{-class, and } x(0) = 0,\ x'(0) = 0\}. \tag{3.2.6}$$

The space F is endowed with the following norm:

$$\|x\|_{{}_0C^1\left([0,1]\right)} = \sup_{s\in[0,1]} |x(s)| + \sup_{s\in[0,1]} |x'(s)|. \tag{3.2.7}$$

Define a new C_0-semigroup called $\{\tilde{S}_t\}_{t\geq 0}$ on the space F by

$$\tilde{S}_t x(s) = e^{\frac{3t}{2}}\, x(se^{-t}), \quad s \in [0,1], \ t \geq 0. \tag{3.2.8}$$

This C_0-semigroup is generated by the equation

$$\frac{\partial u}{\partial t} + s\,\frac{\partial u}{\partial s} = \frac{3}{2}u. \tag{3.2.9}$$

Recall that the C_0-semigroup $\{S_t\}_{t\geq 0}$ on the space E was given in equation (3.0.2).

Theorem 3.18. *Let $E = {}_0C\big([0,1]\big)$ be the separable Banach space defined on* (3.2.4) *and $F = {}_0C^1\big([0,1]\big)$ be the separable Banach space defined on* (3.2.6). *Then the C_0-semigroup $\{\tilde{S}_t\}_{t\geq 0}$ on the Banach space F, generated by the equation* (3.2.9), *has an invariant measure. Moreover, the measure ν defined by*

$$\nu(C) := \mu\big(\Lambda(C)\big), \quad C \in \mathcal{B}(F) \tag{3.2.10}$$

is an invariant measure for $\{\tilde{S}_t\}_{t\geq 0}$, where

$$\Lambda : F \ni x \mapsto x' \in E,$$

is the derivative map. $\mathcal{B}(F)$ is the Borel σ-field on the space F, and the measure μ, which is the law of Brownian motion on E, is the invariant measure for the C_0-semigroup $\{S_t\}_{t\geq 0}$ generated by

$$\frac{\partial v}{\partial t} + s\frac{\partial v}{\partial s} = \frac{1}{2}v.$$

Based on the above Theorem, we can highlight the following properties:

1. E and F are separable Banach spaces,
2. $\{S_t\}_{t\geq 0}$ is a C_0-semigroup on the space E,
3. $\{\tilde{S}_t\}_{t\geq 0}$ is a C_0-semigroup on the space F,
4. The map $\Lambda : F \to E$ is bijective,
5. For every $t \geq 0$, the map Λ is commuting. That is,

$$S_t \circ \Lambda = \Lambda \circ \tilde{S}_t.$$

To reach the objective of this section, we need to verify all the properties listed in Theorem 3.18. For the property 1, the space E is a separable Banach space, according to Lemma B.2 and Theorem 2.27, we deduce that the space E is a separable Banach space. Similarly, the proof that the space F is a separable Banach space can be found in Appendix B.3.

For the property 2, which is extracted from Theorem 3.18, the Example 2.82 contains prove that $\{S_t\}_{t\geq 0}$ defined on equation (3.0.2) is a C_0-semigroup on the Banach space E. Moreover, for the property 3, the Corollary 2.90 contains prove that $\{\tilde{S}_t\}_{t\geq 0}$ defined on (3.2.8) is a C_0-semigroup on the Banach space F.

Now we need to verify the property 4, which is extracted from Theorem 3.18, we need to prove that the map $\Lambda : F \to E$ is bijective.

Proof of property 4. Since Λ is linear, for the injectivity of Λ it is sufficient to prove that if $x \in F$ and $\Lambda(x) = 0$, then $x = 0$. For this aim, we choose and fix $x \in F$ and $\Lambda(x) = 0$. Since Λ is the derivative map and the function x is of C^1-class, we infer that the derivative of x is zero and we deduce that there exists a constant $C \in \mathbb{R}$ such that $x = C$, see [39, Theorem 5.11 part(c)]. In particular, $x(0) = C$. But since $x \in F$, $x(0) = 0$ and thus $C = 0$. Therefore, $x = 0$ as required.

For the surjectivity of the map Λ, let us choose and fix $y \in E$. We put

$$x(s) = \int_0^s y(\sigma)\, d\sigma, \quad s \in [0, 1].$$

and we will show that $x \in F$ $\Lambda(x) = y$. In order to prove that $x \in F$ we need to show that x is continuous, differentiable, $x(0) = 0$ and $x'(0) = 0$. These properties follow from properties of the Riemann integral, in particular, [39, Theorem 6.20 part (c)]. To conclude, we proved that the map Λ is injective and surjective and we infer that it is bijective. Therefore, the property 4 is satisfied. $\square$

For property 5 that states for every $t \geq 0$, the following diagram is commuting. That is,

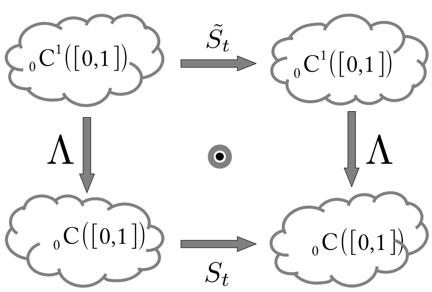


FIGURE 3.2: A graph showing the commuting of the semigroups $\{\tilde{S}_t\}_{t\geq 0}$ on the space ${}_0C^1([0, 1])$ and $\{S_t\}_{t\geq 0}$ on the space ${}_0C([0, 1])$.

$$S_t \circ \Lambda = \Lambda \circ \tilde{S}_t. \tag{3.2.11}$$

Proof of property 5. To verify the property we need to show equality (3.2.11). For this aim, let us choose and fix $t \geq 0$ and $x \in F$. We need to show that the following equation is satisfied.

$$\big[S_t\big(\Lambda(x)\big)\big](s) = \big[\Lambda\big(\tilde{S}_t(x)\big)\big](s), \;\; s \in [0,1]. \tag{3.2.12}$$

Let us start first with the LHS of the equation (3.2.12) as follows:

$$\big[S_t\big(\Lambda(x)\big)\big](s) = e^{\frac{t}{2}}\Lambda(x)\big(se^{-t}\big) = e^{\frac{t}{2}}\, x'\big(se^{-t}\big), \;\; s \in [0,1].$$

On the other hand, for the RHS of equation (3.2.12) by using the definition (3.2.8) of the C_0-semigroup $\{\tilde{S}_t\}_{t\geq 0}$ on the space F, we have, for $s \in [0,1]$, the following train of equalities satisfy

$$\big[\Lambda\big(\tilde{S}_t(x)\big)\big](s) = \frac{d}{ds}\big(\tilde{S}_t(x)(s)\big) = \frac{d}{ds}\big(e^{\frac{3t}{2}}x(se^{-t})\big) = e^{\frac{3t}{2}}\, x'\big(se^{-t}\big)\cdot e^{-t} = e^{\frac{t}{2}}\, x'\big(se^{-t}\big).$$

Thus, we conclude that the RHS and LHS of the equality (3.2.11) are equal. Consequently, we find that the equation (3.2.12) is satisfied, and this completes the property 5. □

To sum up, all properties extracted from Theorem 3.18 are satisfied, and therefore, the defined spaces E and F are applicable to the Abstract Theorem 3.17 stated in Section 3.2. Now we are going to prove our main Theorem in this section.

Proof of Theorem 3.18. We need to use the abstract Theorem 3.17 and verify all the assumptions stated in it to prove Theorem 3.18 but on our concrete spaces E and F. Regarding assumption 1, we need to prove that for all $t \geq 0$ the map $S_t : E \to E$ is $\mathcal{A}/\mathcal{A}$ measurable.

Proof of assumption 1. We proved in Example 2.82 that S_t is bounded and linear for every $t \geq 0$. Thus by [40, Theorem 5.4] we deduce that S_t is continuous and therefore, it is measurable by the Preposition 2.34. Consequently, the assumption 1 is valid. □

Regarding assumption 2, which studies whether μ is a probability measure on $\mathcal{A}$ such that:

$$\mu\big(S_t^{-1}(C)\big) = \mu(C), \quad C \in \mathcal{A}, \text{ for all } t \geq 0;$$

The assumption has already been verified in Lemma 3.2.
For the assumption 3, we need to show that $\Lambda^{-1} : E \to F$ is measurable.

Proof of assumption 3. We do this by showing that it is bounded. Recall the space $E = {}_0C\big([0,1]\big)$, with σ-filed $\mathcal{A} = \mathcal{B}(E)$ and the space $F = {}_0C^1\big([0,1]\big)$, with σ-filed $\mathcal{B} = \mathcal{B}(F)$. Also, recall that $\Lambda : F \ni x \longmapsto x' \in E$ is the derivative map. We already proved in 3.2.1

that the map Λ is bijective, and therefore, the inverse of the map Λ exists.
In the following we need to prove that the map Λ^{-1} satisfies the following equation:

$$\big[\Lambda^{-1}(y)\big](s) = \int_0^s y(r)\,dr, \quad s \in [0,1],\, y \in E. \tag{3.2.13}$$

For this aim let us denote by I the linear operator defined by the RHS of equality (3.2.13). Next we will show that the map I is indeed the inverse of Λ. For this, we need to verify that (1) I is well-defined, and (2) $\Lambda \circ I = id_E$ as well as $I \circ \Lambda = id_F$.
Regarding the first condition, let us suppose that $y \in E$, which means that y is continuous. Since y is continuous then $I(y)$ is differentiable [39, Theorem 6.20], and $[I(y)]'(s) = y(s)$ for every $s \in [0,1]$. Therefore, $I(y)$ is of C^1-class because its derivative is continuous. Moreover, since $I(y) \in E$, we infer that

$$\big[I(y)\big](0) = 0 \quad \text{and} \quad \big[I(y)\big]'(0) = y(0) = 0.$$

Hence, we proved that $I(y) \in F$, and therefore, we deduce that the map I is well-defined. Regarding the second condition, we already know from above that $\Lambda\big[I(y)\big](s) = y(s)$ for all $s \in [0,1]$ because the map Λ is just the derivative. That is, on one hand $\Lambda[I(y)] = y$. On the other hand, we need to prove that if $x \in F \Rightarrow I\big[\Lambda(x)\big] = x$. Let us take $x \in F$. Then, we have

$$\big[\Lambda(x)\big](s) = x'(s), \qquad \text{for all } s \in [0,1].$$

Applying the map I to the derivative of x, and using the Fundamental Theorem of Calculus [40], we have

$$\big[I(x')\big](s) = \int_0^s x'(r)\,dr = x(s) - x(0) = x(s).$$

Hence we proved that $I(x') = x$. In other words, $I\big[\Lambda(x)\big] = x$.

Thus, we proved that the map $I : E \to F$ is well-defined and $\Lambda \circ I = id_E$ and $I \circ \Lambda = id_F$. In other words, $\Lambda^{-1} = I$ or $I^{-1} = \Lambda$. After we have proven that the map Λ^{-1} exists, we need to study whether the map is measurable. According to the Proposition 2.34, it is sufficient to verify if the map Λ^{-1} is bounded. We start by taking $y \in E$, and any $s \in [0,1]$. Then, we have the following

$$\big[I(y)\big](s) = \int_0^s y(r)\,dr.$$

We know that $I(y) \in F$ and by using the norm of the space F that we defined before in equation (3.2.7) we obtain the following

$$\begin{aligned}\|I(y)\|_F &= \sup_{s\in[0,1]} \big|\big[I(y)\big]'(s)\big| + \sup_{s\in[0,1]} \big|\big[I(y)\big](s)\big| \\ &= \sup_{s\in[0,1]} |y(s)| + \sup_{s\in[0,1]} |\int_0^s y(r)\,dr|.\end{aligned}$$

Since

$$\sup_{s\in[0,1]} |\int_0^s y(r)\,dr| = \sup_{s\in[0,1]} \int_0^s |y(r)|\,dr \le \int_0^1 |y(r)|\,dr,$$

we infer that

$$\|I(y)\|_F = \|y\|_E + \int_0^1 |y(r)| dr \le \|y\|_E + \sup_{r\in[0,1]} |y(r)| \int_0^1 |1| dr = \|y\|_E + \|y\|_E = 2\|y\|_E.$$

□

This prove that $\|I(y)\|_F \le 2\|y\|_E$. Therefore, $I : E \to F$ is bounded and the linearity of map I is obvious. Also, since the map Λ^{-1} is bounded linear (i.e., continuous) and using the Proposition 2.34, we deduce that Λ^{-1} is measurable.
Regarding assumption 5, that studies whether the operator $\tilde{S}_t$ is measurable.

Proof of assumption 5. Similar to the above situation, if we can show that $\tilde{S}_t$ is bounded (hence continuous) then the operator will be measurable. Therefore, we just want to verify that $\tilde{S}_t$ is bounded. Let $x \in F$. By using the norm of the space F that we defined before in equation (3.2.7) and the definition of the $\{\tilde{S}_t\}_{t\ge 0}$ we have for every $s \in [0,1]$, the following equations

$$\begin{aligned}\|\tilde{S}_t x\|_F &= \sup_{s\in[0,1]} |\tilde{S}_t x(s)| + \sup_{s\in[0,1]} |[\tilde{S}_t x]'(s)| \\ &= e^{\frac{3}{2}t} \sup_{s\in[0,1]} |x(e^{-t}s)| + \sup_{s\in[0,1]} |e^{\frac{3}{2}t} e^{-t} x'(e^{-t}s)]| \\ &= e^{\frac{3}{2}t} \sup_{\sigma\in[0,e^{-t}]} |x(\sigma)| + e^{\frac{t}{2}} \sup_{\sigma\in[0,e^{-t}]} |x'(\sigma)| \\ &\le e^{\frac{3}{2}t} \sup_{\sigma\in[0,1]} |x(\sigma)| + e^{\frac{t}{2}} \sup_{\sigma\in[0,1]} |x'(\sigma)| \\ &= e^{\frac{3}{2}t} \big[\sup_{\sigma\in[0,1]} |x(\sigma)| + \sup_{\sigma\in[0,1]} |x'(\sigma)| \big] = e^{\frac{3}{2}t} \|x\|_F.\end{aligned}$$

Hence, we proved that $\tilde{S}_t$ is bounded, and therefore, it is continuous. By applying Proposition 2.34, we infer that $\tilde{S}_t$ is measurable. □

Theorem 3.19. *Let $\{S_t^0\}_{t\ge 0}$ be a C_0-semigroup on the space $E_0 = {}_0C([0,1])$ given by*

$$S_t^0 x(s) = e^{\frac{t}{2}} x(se^{-t}), \quad s \in [0,1], \;\; x \in E_0, \tag{3.2.14}$$

and, let $\{\tilde{S}^0_t\}_{t\geq 0}$ *is a* C_0*-semigroup on the space* $F_0 = {}_0C^1([0,1])$ *given by*

$$\tilde{S}^0_t x(s) = e^{\frac{3t}{2}} x(se^{-t}), \quad s \in [0,1], \;\; x \in F_0. \tag{3.2.15}$$

Also, let $\Lambda_0^{-1} : E_0 \to F_0$ *be a map defined by*

$$\big(\Lambda_0^{-1}(x)\big)(s) = \int_0^s x(r)\, dr, \quad r \in [0,1], \, x \in E_0. \tag{3.2.16}$$

Then,

$$\Lambda_0^{-1} \circ S^0_t = \tilde{S}^0_t \circ \Lambda_0^{-1} \;\; \text{on } E_0. \tag{3.2.17}$$

That is, for all $x \in E_0$*, we have for every* $s \in [0,1]$ *the following*

$$\big[\Lambda_0^{-1}\big(S^0_t x\big)\big](s) = \big[\tilde{S}^0_t\big(\Lambda_0^{-1} x\big)\big](s). \tag{3.2.18}$$

Proof of Theorem 3.19. Our aim is to prove the equality (3.2.17). Let us fix $t > 0$, and choose $x \in E_0$. We start with the LHS of the equality. By denoting $y = S^0_t x \in E_0$, and using the Definition (3.2.14) of the C_0-semigroup $S^0_t x$, we have for every $s \in [0,1]$

$$\big(\Lambda_0^{-1} y\big)(s) = \int_0^s y(r)\, dr = \int_0^s e^{\frac{t}{2}} x(e^{-t} r)\, dr = e^{\frac{t}{2}} \int_0^s x(e^{-t} r)\, dr.$$

By applying the change of variables to the last integral, we get

$$\big(\Lambda_0^{-1} y\big)(s) = e^{\frac{t}{2}} \int_0^{e^{-t}s} x(\rho) e^t\, d\rho = e^{\frac{3t}{2}} \int_0^{e^{-t}s} x(\rho)\, d\rho, \;\; s \in [0,1].$$

Thus, we proved

$$\big(\Lambda_0^{-1} S_t x\big)(s) = e^{\frac{3t}{2}} \int_0^{e^{-1}s} x(\rho)\, d\rho, \;\; s \in [0,1].$$

For the RHS of the equality (3.2.18), we know since $\Lambda_0^{-1} x \in F_0$, we can apply the the definition of the C_0-semigroup $\{\tilde{S}^0_t\}_{t\geq 0}$. Thus, by the Definition of $\tilde{S}^0_t y$ in equation (3.2.15), we have

$$\big(\tilde{S}^0_t \Lambda_0^{-1} x\big)(s) = e^{\frac{3t}{2}} \Lambda_0^{-1} x(e^{-t} s) = e^{\frac{3t}{2}} \int_0^{e^{-t}s} x(r)\, dr, \;\; s \in [0,1].$$

Hence, we proved that for every $s \in [0,1]$,

$$\big(\Lambda_0^{-1} S^0_t x\big)(s) = \big(\tilde{S}^0_t \Lambda_0^{-1} x\big)(s).$$

Therefore, the proof of Theorem 3.19 is complete. □

After verifying all the assumptions of the abstract Theorem 3.17, we now in the right position to prove our concrete Theorem 3.18. In other words, we need to prove that the C_0-semigroup $\{\tilde{S}_t\}_{t\geq 0}$ has an invariant measure. Since $\tilde{S}_t = \Lambda^{-1} \circ S_t \circ \Lambda$, then for any set

C and by taking the inverse for both sides we have

$$\begin{aligned} \tilde{S_t}^{-1}(C) &= \big(\Lambda^{-1} \circ S_t \circ \Lambda\big)^{-1}(C) = \Lambda^{-1}\big(S_t^{-1}(\Lambda(C))\big) \\ \nu\big(\tilde{S_t}^{-1}(C)\big) &= \nu\big[\Lambda^{-1}\big(S_t^{-1}(\Lambda(C))\big)\big]. \end{aligned} \tag{3.2.19}$$

From the equation (3.2.10), if we denote $\Lambda(C)$ by C' , then we get

$$C' = \Lambda(C) \Leftrightarrow C = \Lambda^{-1}(C')$$

Therefore, we can re-write the equation (3.2.10) as follows

$$\nu\big(\Lambda^{-1}(C')\big) = \mu(C'). \tag{3.2.20}$$

Hence, applying the above consequence equation (3.2.20) and (3.0.3) along with the definition of the measure ν in equation (3.2.10) to the equation (3.2.19) we obtain

$$\nu\big(\tilde{S_t}^{-1}(C)\big) = \mu\big(S_t^{-1}(\Lambda(C))\big) = \mu\big(\Lambda(C)\big) = \nu(C).$$

To conclude, we have proved that $\nu\big(\tilde{S_t}^{-1}(C)\big) = \nu(C)$, which completes the proof of Theorem 3.18. $\square$

3.3 Existence of Invariant measures on the fractional Sobolev spaces $H^{\alpha,p}(0,1)$

Toward finding the invariant measure, we apply the output of the last section (Sect. 3.2.1) against yet a new spaces $X = L^p(0,1)$ and Sobolev space $F = {}_0H^{\alpha,p}(0,1)$, which are broader spaces than what we have defined previously. But, before that, we are going to attest our new spaces by taking a special case. In other words, we first take $\alpha = 1$ in the space F as in the following section.

3.3.1 Invariant measures for a special case of Sobolev spaces

Let $X = L^p(0,1)$ defined in Definition 2.55 and $F = {}_0H^{1,p}(0,T)$ defined in Definition 2.68. We want to study whether the following operators Λ and I and their properties are valid on the new choice of spaces X and F. The operators Λ and I are defined as follows

$$\Lambda : F \ni u \mapsto Du \in X. \tag{3.3.1}$$

$$I : X \ni u \mapsto \int_0^t u(s)\, ds \in F \ \text{ for every } \ t \in [0,1]. \tag{3.3.2}$$

Note that: the definition of maps I and Λ here use the same formula in the previous Section 3.2.1 but we mean by I here the Lebesgue integral and by Λ the weak derivative Du, whereas before we meant the Riemann integral and the classical derivative respectively.

Proposition 3.20. *The operator* Λ *defined in equation* (3.3.1) *is linear and bounded from the space* F *to the space* X.

Proof of Proposition 3.20. The linearity property is straightforward. For the boundedness property, let $u \in F$. We know from equation (2.3.5) in Corollary 2.71 the following

$$\|\Lambda u\|_E = |\Lambda u|_{L^p(0,1)} = \||u|\|_{{}_0H^{1,p}(0,1)} \leq \||u|\|_{{}_0H^{1,p}(0,1)}.$$

Hence the operator Λ is bounded. Moreover, $\|\Lambda\| \leq 1$. $\square$

Lemma 3.21. *If* $u \in X$*, then for every* $t \in [0,1]$ *the expression* $\big[I(u)\big](t)$ *defined by equation* (3.3.2) *is well-defined.*

Proof of Lemma 3.21. : Let us fix $t \in [0,1]$. We need to show that the integral defined in equation (3.3.2) exists. In fact, we notice that u is not a continuous function because $u \in X$, where $X = L^p(0,1)$ consists of all equivalent classes of functions. Thus, we cannot use Riemann integral. Also, u is not even a function at all because $u = [f]_\sim$. Therefore, we can use $\int_0^t f(s)ds$, where $u = [f]_\sim$, $f : [0,1] \to \mathbb{R}$, which for some function is Lebesgue measurable and

$$\int_{[0,1]} |f(x)|^p \, dx < \infty. \tag{3.3.3}$$

Because the integral is Lebesgue integral, then we can rewrite the original integral as follows.

$$\int_0^t f(s)\, ds = \int_{[0,1]} \mathbb{1}_{[0,t]}(s) f(s)\, ds.$$

To study whether the above integral exists in the Lebesgue sense, we study it within two cases: $p = 1$ and $p > 1$. If $p = 1$ in equation (3.3.3), then have

$$\int_{[0,1]} |\mathbb{1}_{[0,t]}(s) f(s)|\, ds = \int_{[0,1]} \mathbb{1}_{[0,t]}(s) |f(s)|\, ds \leq \int_{[0,1]} |f(s)|\, ds < \infty.$$

Hence, the integral exists. If $p > 1$, by using Hölder Inequality [40], we have

$$\int_{[0,1]} |\mathbb{1}_{[0,t]}(s) f(s)|\, ds \leq \left(\int_{[0,1]} |\mathbb{1}_{[0,t]}(s)|^q\, ds \right)^{\frac{1}{q}} \left(\int_{[0,1]} |f(s)|^p\, ds \right)^{\frac{1}{p}}.$$

By the assumption $\left(\int_{[0,1]} |f(s)|^p\, ds \right)^{\frac{1}{p}} < \infty$, and $\left(\int_{[0,1]} |\mathbb{1}_{[0,t]}|^q\, ds \right)^{\frac{1}{q}} = t^{\frac{1}{q}} < \infty$. Hence, we deduce that the whole integral is finite, and therefore, it exists. To conclude, we proved that

$$\int_0^t f(s)ds \quad \text{exists if} \quad u = [f]_\sim .$$

Suppose we have another representative function for u called g, such that $u = [g]_\sim$, we can claim that

$$\int_0^t f(s)ds = \int_0^t g(s)\, ds.$$

To verify this claim, let us first fix $t \in [0,1]$. Then we have

$$\left| \int_0^t f(s)\, ds - \int_0^t g(s)\, ds \right| = \left| \int_0^t \big[f(s) - g(s)\big]\, ds \right| \le \int_0^t \big|f(s) - g(s)\big|\, ds.$$

We know that $u = [f]_\sim = [g]_\sim \Rightarrow f = g \quad a.e$ with respect to the Lebesgue measure. This means, that there exists $A \subset [0,1]$ Lebesgue measurable such that $\mu\big([0,1] \setminus A\big) = 0$, $f(s) = g(s)$ for $s \in A$. Hence,

$$\left| \int_0^t f(s)\, ds - \int_0^t g(s)\, ds \right| = \int_{[0,1]\setminus A} 0\, ds = 0.$$

So, we proved that if $u \in L^p(0,1)$ then

$$\int_0^t f(s)\, ds = \int_0^t g(s)\, ds, \qquad \text{for all } f, g : u = [f]_\sim = [g]_\sim .$$

As a consequence, we can put

$$\int_0^t u(s)\, ds := \int_0^t f(s)\, ds.$$

To sum up, we proved if $u \in X$ then $\big[I(u)\big](t)$ is well-defined for all $t \in [0,1]$. □

Lemma 3.22. *If $u \in X = L^p(0,1)$, then $I(u)$, which is defined by equation* (3.3.2)*, is a continuous function.*

Proof of Lemma 3.22. Let us take and fix $u \in X$. Our aim is to show that $v := I(u) \in C\big([0,1]\big)$. For this aim, we consider two cases: $p > 1$ and $p = 1$.

Case 1. Assume that $p = 1$. We want to prove that v is right-continuous at every $t_0 \in [0,1)$ and that v is left-continuous at every $t_0 \in (0,1]$. Firstly we choose and fix $t_0 \in [0,1)$. In order to prove that v is right-continuous at t_0, by the Heine (equivalent) definition of continuity, see [39, Theorem 2.41], it is sufficient to prove that if a sequence $(t_n)_{n=1}^\infty$ is decreasing and converges to t_0, then $v(t_n) \to v(t_0)$. So, let us choose and fix such a sequence $(t_n)_{n=1}^\infty$. We have, since $t_n > t_0$, by properties of the Lebesgue integral,

$$\begin{aligned}
|v(t_n) - v(t_0)| &= |\int_0^{t_n} u(s)\,ds - \int_0^{t_0} u(s)\,ds| \\
&= |\int_{t_0}^{t_n} u(s)\,ds| \le \int_{t_0}^{t_n} |u(s)|\,ds \\
&= \int_0^1 1_{[t_0,t_n]}|u(s)|\,ds = \int_0^1 f_n(s)\,ds,
\end{aligned}$$

where

$$f_n(s) := 1_{[t_0,t_n]}|u(s)|, \quad s \in [0,1].$$

Put also

$$f(s) := \begin{cases} 0 & \text{if } s \in [0,1] \setminus \{t_0\}, \\ |u(t_0)|, & \text{if } s = t_0. \end{cases}$$

We observe that if $s = t_0$ then $f_n(s) = |u(s)|$ but if $s \neq t_0$, then $f_n(s) \to f(s)$. Hence

$$f_n(s) \to f(s) \text{ for all } s \in [0,1].$$

Hence assumption (82) of [39, Theorem 11.32] is satisfied. Moreover, for every $n \in \mathbb{N}$

$$|f_n(s)| = f_n(s) \le g(s) := |u(s)|, \quad s \in [0,1],$$

and $\int_0^1 |u(s)|\,ds < \infty$. Hence assumption (83) [39, Theorem 11.32] is satisfied. Hence we can apply the Lebesgue Dominated Convergence Theorem [39, Theorem 11.32] and deduce that

$$\int_0^1 f_n(s)\,ds \to \int_0^1 f(s)\,ds = 0.$$

In summary, we deduce that

$$\lim_{n\to\infty} |v(t_n) - v(t_0)| = 0.$$

Therefore, v is right continuous at t_0.
Secondly we choose and fix $t_0 \in (0,1]$. In order to prove that v is left-continuous at t_0, again by the Heine (equivalent) definition of continuity, it is sufficient to prove that if a sequence $(t_n)_{n=1}^\infty$ is increasing and converges to t_0, then $v(t_n) \to v(t_0)$. So, let us choose and fix such a sequence $(t_n)_{n=1}^\infty$. We have, since $t_n < t_0$, by the properties of the Lebesgue

integral,

$$\begin{aligned}
|v(t_n) - v(t_0)| &= |\int_0^{t_n} u(s)\,ds - \int_0^{t_0} u(s)\,ds| \\
&= |\int_{t_n}^{t_0} u(s)\,ds| \le \int_{t_n}^{t_0} |u(s)|\,ds \\
&= \int_0^1 1_{[t_n,t_0]}(s)|u(s)|\,ds = \int_0^1 f_n(s)\,ds,
\end{aligned}$$

where

$$f_n(s) = 1_{[t_n,t_0]}(s)|u(s)|, \quad s \in [0,1].$$

We put

$$f(s) := \begin{cases} 0 & \text{if } s \in [0,1] \setminus \{t_0\}, \\ |u(t_0)|, & \text{if } s = t_0. \end{cases}$$

We observe that if $s = t_0$ then $f_n(t_0) = |u(t_0)|$ and therefore $f_n(t_0) \to |u(t_0)|$. Moreover, if $s \neq t_n$, then $f_n(s) \to f(s)$. Hence

$$f_n(s) \to f(s) \text{ for all } s \in [0,1].$$

Hence assumption (82) of [39, Theorem 11.32] is satisfied.
Finally, for every $n \in \mathbb{N}$,

$$|f_n(s)| = f_n(s) \le g(s) := |u(s)|, \quad s \in [0,1],$$

and $\int_0^1 |u(s)|\,ds < \infty$. Hence assumption (83) of [39, Theorem 11.32] is satisfied.
Again, by applying the Lebesgue Dominated Convergence Theorem [39, Theorem 11.32] and deduce that

$$\int_0^1 f_n(s)\,ds \to \int_0^1 f(s)\,ds = 0.$$

Hence, we deduce that

$$\lim_{n\to\infty} |v(t_n) - v(t_0)| = 0.$$

Therefore, v is left continuous at t_0. □

Case 2. Assume that $p > 1$. Let us consider $t_1, t_2 \in [0,1]$ such that $0 \le t_1 < t_2 \le 1$. Then we have

$$v(t_1) = \int_0^{t_1} u(s)\,ds, \quad v(t_2) = \int_0^{t_2} u(s)\,ds.$$

By using the Hölder Inequality (2.3.1) for the above v_1 and v_2, we get

$$\begin{aligned}\big|v(t_1)-v(t_2)\big| &= \left|\int_{t_1}^{t_2} u(s)\,ds\right| \le \int_{t_1}^{t_2} |u(s)|\,ds \\ &\le \left(\int_{t_1}^{t_2} 1^q\,ds\right)^{\frac{1}{q}}\left(\int_{t_1}^{t_2} |u(s)|^p\,ds\right)^{\frac{1}{p}} \\ &\le |t_2-t_1|^{\frac{1}{q}}\left(\int_{t_1}^{t_2} |u(s)|^p\,ds\right)^{\frac{1}{p}} \\ &\le |t_2-t_1|^{\frac{1}{q}}\left(\int_{0}^{1} |u(s)|^p\,ds\right)^{\frac{1}{p}} \\ &\le |t_2-t_1|^{\frac{1}{q}}\|u\|_{L^p(0,1)}.\end{aligned}$$

Hence, we proved that $|v(t_2)-v(t_1)| \le |t_2-t_1|^{\alpha}\|u\|_{L^p(0,1)}$, where $\alpha = \frac{1}{q} = 1-\frac{1}{p} > 0$. Therefore, v is Hölder continuous with exponent $\alpha > 0$, so it is continuous. □

Hence, since we proved that v is left continuous on $(0,1]$ and right continuous on $[0,1)$, we infer that v is continuous on $[0,1]$, as claimed. □

Lemma 3.23. *If $u \in X = L^p(0,1)$, then $I(u) \in F$. In other words, the operator I is a mapping from the space X to the space F.*

Proof of Lemma 3.23. Let $u \in X$. To show that $I(u) \in F = {}_0H^{1,p}(0,1)$, we need to show, in view of Definition 2.65, Theorem 2.67 and Definition 2.68, that the following are true:

- **Statement 1**: $I(u) \in C([0,1])$. This is satisfied as it follows from Lemma 3.22.
- **Statement 2**: $\big[I(u)\big](0) = 0$. This is satisfied because $\big[I(u)\big](0) = \int_0^0 u(s)\,ds = 0$.
- **Statement 3**: The weak derivative $D\big(I(u)\big)$ exists and belongs to the space $L^p(0,1)$.

Regarding the last statement, let us denote $v = I(u)$ and let us recall that v has a derivative Dv where $Dv \in L^p(0,1)$, if and only if we can find a function $z \in L^p(0,1)$ such that

$$\int_0^1 v(t)\phi'(t)\,dt = -\int_0^1 z(t)\phi(t)\,dt, \quad \forall \phi \in C_c^\infty(0,1).$$

To find this function, we guess that $z = u$. Since $u \in L^p(0,1)$, it is sufficient to show that

$$\int_0^1 v(t)\phi'(t)\,dt = -\int_0^1 u(t)\phi(t)\,dt, \quad \forall \phi \in C_c^\infty(0,1). \tag{3.3.4}$$

Recall that function $\phi \in C_c^\infty(0,1)$ if and only if $\phi : [0,1] \to \mathbb{R}$ of C^∞-class and there exists $a > 0$ such that $\phi \subset [a, 1-a]$. Also, from the assumption, $u \in L^p(0,1)$.

Now, we assume that $u \in C\big([0,1]\big)$. Then we have

$$v(t) = \big[I(u)\big](t) = \int_0^t u(s)\,ds, \quad t \in [0,1].$$

According to Rudin [39], we infer that in this case $v \in C^1\big([0,1]\big)$ and $v(0) = 0$. Moreover, $v'(t) = u(t)$. Thus, to prove the equation (3.3.4), we use the Fundamental Theorem of Calculus [39, Theorem 11.33]. since the function $t \mapsto v(t)\phi(t)$ is of C^1-class, then we have the following

$$v(1)\phi(1) - v(0)\phi(0) = \int_0^1 \frac{d}{ds} v(s)\phi(s)\,ds.$$

On one hand, the left-hand-side is $v(1)\phi(1) = v(1) \times 0 = 0$ and $v(0)\phi(0) = v(0) \times 0 = 0$ because function ϕ has a compact support. Therefore,

$$\int_0^1 \frac{d}{ds}\big(v(s)\phi(s)\big)\,ds = 0. \tag{3.3.5}$$

On the other hand, the derivative inside the integral on equation (3.3.5) of the right-hand-side can be rewritten as follows

$$\frac{d}{ds}\big(v(s)\phi(s)\big) = v'(s)\phi(s) + v(s)\phi'(s).$$

Hence,

$$\int_0^1 \big[v'(s)\phi(s) + v(s)\phi'(s)\big]\,ds = 0.$$

By distributing the above integral and rearranging it, and Since $v'(s) = u$, we infer that s

$$-\int_0^1 u(s)\phi(s)\,ds = \int_0^1 v(s)\phi'(s)\,ds \tag{3.3.6}$$

To conclude, we proved that the equation (3.3.4) for $u \in C[0,1]$ is satisfied. But, we want that equation (3.3.4) holds for arbitrary $u \in L^p(0,1)$, and to achieve that, we apply the spaces that exist in Theorem 3.9, namely X, F and V, to our special spaces. We choose $X = L^p(0,1)$, $F = F$ and $V = C\big([0,1]\big)$. We know already that $V \subset X$ is dense of X. Fix $\phi \in C_c^\infty(0,1)$ and define two maps $\mathbf{A}$ and $\mathbf{B}$ by

$$\mathbf{A} : L^p(0,1) \ni u \mapsto -\int_0^1 u(s)\phi(s)\,ds \in F;$$

and

$$\mathbf{B} : L^p(0,1) \ni u \mapsto \int_0^1 v(s)\phi'(s)\,ds \in F, \quad \text{where } v = I(u).$$

The maps $\mathbf{A}$ and $\mathbf{B}$ are clearly bounded and linear maps from X to F and the equation (3.3.6) shows that $\mathbf{B}(u) = \mathbf{A}(u)$ for all $u \in C[0,1] = V$. Therefore, by using the

Theorem 3.9, we infer that $\mathbf{B}(u) = \mathbf{A}(u)$, $u \in L^p(0,1) = X$. That is,

$$-\int_0^1 u(s)\phi(s)\, ds = \int_0^1 v(s)\phi'(s)\, ds, \qquad u \in L^p(0,1).$$

Therefore, we proved that the weak derivative of v exists and $Dv = u$. In particular, $Dv \in L^p(0,1)$ and that concludes the proof of Lemma 3.23. $\square$

Lemma 3.24. *The map I, which was defined in equation* (3.3.2)*, is a bounded linear map from $L^p(0,1)$ to ${}_0H^{1,p}(0,1)$. That is, for every $u \in L^p(0,1)$, there exists $C > 0$ such that*

$$\|I(u)\|_{{}_0H^{1,p}(0,1)} \le C\|u\|_{L^p(0,1)}.$$

Proof of Lemma 3.24. The proof of this lemma is similar to the proof of the map Λ^{-1} in the previous section. $\square$

Theorem 3.25. *Assume that $p \in [1,\infty)$. Let $\{S_t^p\}_{t\ge 0}$ be a C_0-semigroup on the space $E = L^p[0,1]$ given by $\big(S_t^p x\big)(s) = e^{\frac{t}{2}}\, x(se^{-t}), s \in [0,1],\ x \in E$. Also, let $\{\tilde{S_t^p}\}_{t\ge 0}$ be a C_0-semigroup on the space $F = {}_0H^{1,p}(0,1)$ given by $\big(\tilde{S_t^p} x\big)(s) = e^{\frac{3t}{2}}\, x(se^{-t}), s \in [0,1],\ x \in F$. In addition, let $\Lambda_p^{-1} : E \to F$ be defined as*

$$\big(\Lambda_p^{-1}(x)\big)(s) = \int_0^s x(r)\, dr, \quad s \in [0,1] \tag{3.3.7}$$

Then, for every $t > 0$, we have

$$\Lambda_p^{-1} \circ S_t^p = \tilde{S_t^p} \circ \Lambda_p^{-1} \ \ \text{on } E. \tag{3.3.8}$$

In other words, the following diagram (3.3) *is commuting*

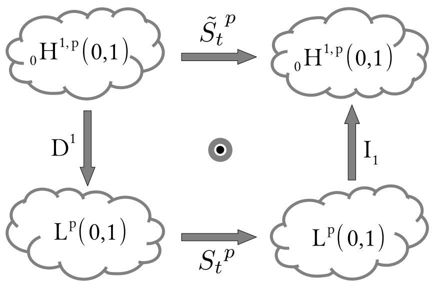


FIGURE 3.3: A graph showing the commuting of the semigroups $\{S_t^p\}_{t\ge 0}$ on the space $L^p[0,1]$ and $\{\tilde{S_t^p}\}_{t\ge 0}$ on the space ${}_0H^{1,p}(0,1)$.

Proof of Theorem 3.25. To prove this Theorem, we are required to ultimately prove the equation (3.3.8). We do this by following the abstract result Theorem 3.9 and naming the abstract spaces X, F, and V. Using the notation from Theorem 3.25 we already have defined the space F and we put $X = E$. This leaves us to define the space V based on the properties of the spaces X and F. Therefore, we choose

$$V = {}_0C[0,1].$$

To commence the proof, we need first to verify if the assumptions of the Theorem 3.9 are satisfied. We choose and fix $t > 0$ and we put

$$\mathbf{A} = \Lambda_p^{-1} \circ S_t^p \quad \text{and} \quad \mathbf{B} = \tilde{S_t}^p \circ \Lambda_p^{-1}. \tag{3.3.9}$$

From the above equation (3.3.9), we observe the following points.

1. The operator $\Lambda_p^{-1} : X \to F$ is a linear bounded operator according to Lemma 3.24.
2. $\tilde{S_t}^p : F \to F$ is a linear bounded operator because $(\tilde{S_t}^p)_{t\geq 0}$ is a C_0-semigroup on the space F, see Corollary 2.90. Hence, $\mathbf{B} = \tilde{S_t}^p \circ \Lambda_p^{-1} : X \to F$ is also a bounded linear operator.
3. $S_t^p : X \to X$ is a linear bounded operator because $(S_t{}^p)_{t\geq 0}$ is a C_0-semigroup on the space X, see Example 2.89.
4. As before the operator $\Lambda_p^{-1} : X \to F$ is linear bounded. Therefore, $\mathbf{A} := \Lambda_p^{-1} \circ S_t^p : X \to F$ is also a bounded linear operator. Hence, the operators on the LHS and RHS in equality (3.3.8) are linear bounded operators.
5. The space V is a dense subspace of the space X.

Let us observe that it follows from Example 2.89 and respectively Corollary 2.90 the families $\{\tilde{S_t}^p\}_{t\geq 0}$ and respectively $\{S_t{}^p\}_{t\geq 0}$ are C_0-semigroups.

By Rudin [40] and Theorem 2.20, the set $C_c(0,1)$ is a dense subspace of $L^p(0,1)$. Thus, we deduce that $C_c(0,1)$ is subset of ${}_0C([0,1])$. Hence, according to Theorem 2.20 we deduce that ${}_0C([0,1])$ is also a dense subspace of $L^p(0,1)$. In other words, the space ${}_0C([0,1])$ is a dense subspace of the space $L^p(0,1)$.
Lastly, after verifying all the assumptions of the Theorem 3.9, we need now to verify the following equation

$$\Lambda_p^{-1} \circ S_t^p = \tilde{S_t}^p \circ \Lambda_p^{-1} \text{ on } \quad V. \tag{3.3.10}$$

By observing the above equation, we had proved something similar in Theorem 3.19, i.e., we proved the identity equation (3.2.17), which we recall as follows

$$\Lambda_0^{-1} \circ S_t^0 = \tilde{S}_t^0 \circ \Lambda_0^{-1} \quad \text{on} \quad V.$$

To prove that the equation (3.3.10) holds on the space V, we need to compare the LHS of the equation (3.3.10) with the corresponding LHS in the equation (3.2.17). Also, we need to do the same comparison for the RHS expressions. For the left-hand sides, we have

$$\Lambda_p^{-1} \circ S_t^p = \Lambda_0^{-1} \circ S_t^0 \text{ on } V. \tag{3.3.11}$$

Similarly, for the right-hand sides, we have

$$\tilde{S_t}^p \circ \Lambda_p^{-1} = \tilde{S_t^0} \circ \Lambda_0^{-1} \text{ on } V. \tag{3.3.12}$$

Toward the equivalence of the two equalities (3.3.11) and (3.3.12), we observe that

$$V \subset X \text{ and } F_0 \subset F.$$

From which, and taking into account the two equalities (3.3.11) and (3.3.12), we obtain three statements as follows

$$\Lambda_0^{-1}(x) = \Lambda_p^{-1}(x) \text{ in } F, \text{ for every } x \in V,$$

$$S_t^0(x) = S_t^p(x) \text{ in } E, \text{ for every } x \in V,$$
$$\tilde{S_t^0}(y) = \tilde{S_t^p}(y) \text{ in } F, \text{ for every } y \in F_0.$$

We know that the first statement is a consequence of the definition (3.3.7) of the operator Λ_p^{-1} and the definition (3.2.16) of the operator Λ_0^{-1} which gives the equality. To justify the second statement, we have two different C_0-semigroups:

(i) $\{S_t^p\}_{t\geq 0}$ on $X = L^p[0,1]$;

(ii) $\{S_t^0\}_{t\geq 0}$ on V.

Since $V \subset X$ and if we propose an element $x \in V$, then $x \in X$. Therefore,

$$S_t^0 x \in V \text{ and } S_t^p x \in X.$$

However, since $V \subset X$, we also have

$$S_t^0 x \in X \text{ and } S_t^p x \in X.$$

We also remember that these two C_0-semigroups are defined by the same formula. Thus, we deduce that not only $S_t^0 x$ and $S_t^p x$ belong to the same space E but also that

$$S_t^0 x = S_t^p x \in X.$$

That satisfies the second statement. Likewise, the third statement can be justified in the same way as the above comparison. By this the assumptions of Theorem 3.9 are satisfied. Hence, we deduce that these three statements imply (3.3.12) and (3.3.11). After all, we see that the assumptions of Theorem 3.9 hold, therefore, the proof of Theorem 3.25 is complete. □

Theorem 3.26. *Let $\{\tilde{S}_t^p\}_{t\geq 0}$ be a C_0-semigroup on the Banach space F and generated by the following PDE*

$$\frac{\partial u}{\partial t} + s\frac{\partial u}{\partial s} = \frac{3}{2}u.$$

Let $\{S_t^p\}_{t\geq 0}$ be a C_0-semigroup on the Banach space E. Then the Borel measure ν defined by the following formula

$$\nu(C) = \mu\big(\Lambda_p^{-1}(C)\big), \;\; C \in \mathcal{B}(F),$$

is an invariant measure of the C_0-semigroup $\{\tilde{S}_t^p\}_{t\geq 0}$ on the space F, where μ is the law of Brownian Motion on the space E, $\Lambda_p^{-1} : E \to F$ is defined as in equation (3.3.7) *and $\mathcal{B}(F)$ is the Borel σ-filed on the space F.*

Proof of Theorem 3.26. The proof of this theorem can be done in a similar way as in Theorem 3.18.

□

3.3.2 Invariant measures for a general case of fractional Sobolev spaces

In the previous section, we applied Theorem 3.18 on a special space $E = L^p(0,1)$ and Sobolev space $F = {}_0H^{1,p}(0,1)$. In this section, we intend to generalise the application to a broader space of the space F. In particular, we keep the same space $E = L^p(0,1)$ and $F = {}_0H^{\alpha,p}(0,1)$, where $\alpha \in (0,1)$. The goal of this section is to prove the existence of an invariant measure for a C_0-semigroup in Sobolev space that can be used to understand the properties of many PDEs. But, before we commence this ultimate generalisation, we are required to generalise the Diagram (3.3) in Theorem 3.25 by using a new operator I^α, defined as the inverse of the operator B^α introduced earlier in Corollary 3.16, i.e.,

$$I^\alpha := (B^\alpha)^{-1}.$$

It can be shown that I^α is the fractional integral operator in the space ${}_0H^{\alpha,p}(0,1)$ defined as

$$\big[I^\alpha(x)\big](s) = \frac{1}{\Gamma(\alpha)}\int_0^s (s-r)^{\alpha-1}x(r)\,dr, \;\; \text{for all } s \in [0,1]. \tag{3.3.13}$$

Moreover, from Corollary 3.16 we get the following result about the operator I^α.

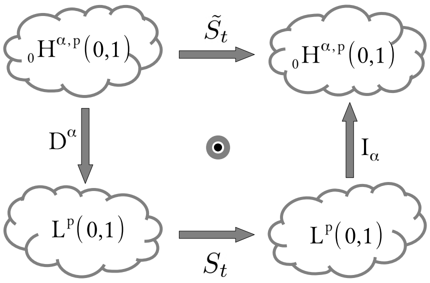


FIGURE 3.4: A graph showing the commuting of the semigroups $\{\tilde{S}_t\}$ on space ${}_0H^{\alpha,p}(0,1)$ and $\{S_t\}$ on spaces and $L^p(0,1)$ via the the fractional derivative D^α and fractional integral I^α maps.

Proposition 3.27. *If $\alpha \in (0,1)$, then*

$$R(B^\alpha) = {}_0H^{\alpha,p}(0,1).$$

In particular, the operator I^α is an isomorphism between the Lebesgue space $L^p(0,1)$ and the Sobolev space ${}_0H^{\alpha,p}(0,1)$.

Before we denoted the operator I by

$$[I(x)](s) = \int_0^s x(r)\,dr, \quad s \in [0,1], \; x \in {}_0C\big([0,1]\big),$$

where the operator I is the inverse of the operator Λ, and Λ is the derivative. However, the operator I requires to be replaced by I^α for $\alpha \in (0,1)$. We apply the same calculation we had before for the property 5 in Section 3.2.1 but with replacing Λ with this fractional integral operator I^α. That is, we need to show the following equality

$$\tilde{S}_t \circ I^\alpha = I^\alpha \circ S_t \text{ on } L^p(0,1). \tag{3.3.14}$$

where

$$\big(S_t x\big)(s) = e^{\frac{t}{2}} x(se^{-t}), \qquad x \in L^p(0,1), \;\; s \in [0,1], \tag{3.3.15}$$

and

$$\big(\tilde{S}_t x\big)(s) = e^{(\frac{1}{2}+\alpha)t} x(se^{-t}), \quad x \in L^p(0,1), \;\; s \in [0,1]. \tag{3.3.16}$$

Note that

$$\tilde{S}_t x = e^{\alpha t} S_t x, \quad x \in L^p(0,1), \;\; t \geq 0.$$

In the following, we need to formulate the equation (3.3.14) as a theorem so we can use it, later on, to prove our main result (Theorem 3.30) in this section.

Lemma 3.28. *We assume that $\alpha \in (0,1)$ and $p \in (1,\infty)$. The family $\{\tilde{S}_t\}_{t\geq 0}$ defined by formula* (3.3.16) *is a C_0-semigroup on the space $F = {}_0H^{\alpha,p}(0,1)$.*

Proof of Lemma 3.28. Follows from Theorems 2.95 and 2.87. □

Theorem 3.29. *We assume that $\alpha \in (0,1)$ and $p \in (1,\infty)$. Let $\{S_t\}_{t\geq 0}$ be the C_0-semigroup on the space $E = L^p(0,1)$ defined in equation (3.3.15), see Example 2.89. Let $\{\tilde{S}_t\}_{t\geq 0}$ be the C_0-semigroup on the space $F = {}_0H^{\alpha,p}(0,1)$ from Lemma 3.28. Let also*

$$I^\alpha : L^p(0,1) \to {}_0H^{\alpha,p}(0,1)$$

be the fractional integral operator of order α, defined by equation (3.3.13), that is an isomorphism, see proposition 3.27. Then, for every $t \geq 0$ the following equality holds

$$\tilde{S}_t \circ I^\alpha = I^\alpha \circ S_t \text{ on } L^p(0,1). \tag{3.3.17}$$

Proof of Theorem 3.29. Let $x \in L^p(0,1)$. We start with an observation that in view of Example 2.89 that is $\{S_t x\} \in L^p(0,1)$. Then, for every $s \in [0,1]$ the right-hand-side of the target equality (3.3.17) is equal to

$$\begin{aligned}
[I^\alpha(S_t x)](s) &= \frac{1}{\Gamma(\alpha)} \int_0^s (s-r)^{\alpha-1} S_t x(r)\, dr \\
&= \frac{1}{\Gamma(\alpha)} \int_0^s (s-r)^{\alpha-1} e^{\frac{t}{2}} x(e^{-t} r)\, dr \\
&= e^{\frac{t}{2}} \frac{1}{\Gamma(\alpha)} \int_0^s (s-r)^{\alpha-1} x(e^{-t} r)\, dr.
\end{aligned} \tag{3.3.18}$$

Using change of variables as follows, we put $\rho = e^{-t} r \Rightarrow r = e^t \rho$ and $0 \leq \rho \leq s e^{-t}, dr = e^t d\rho$ in the above equation (3.3.18), then we obtain the following

$$\begin{aligned}
[I^\alpha(S_t x)](s) &= e^{\frac{t}{2}} \frac{1}{\Gamma(\alpha)} \int_0^{se^{-t}} (s - e^t \rho)^{\alpha-1} x(\rho) e^t\, d\rho \\
&= \frac{e^{\frac{3t}{2}}}{\Gamma(\alpha)} \int_0^{se^{-t}} (e^t s e^{-t} - e^t \rho)^{\alpha-1} x(\rho)\, d\rho \\
&= \frac{e^{\frac{3t}{2}}}{\Gamma(\alpha)} e^{t(\alpha-1)} \int_0^{se^{-t}} (s e^{-t} - \rho)^{\alpha-1} x(\rho)\, d\rho \\
&= \frac{e^{(\frac{1}{2}+\alpha)t}}{\Gamma(\alpha)} \int_0^{se^{-t}} (s e^{-t} - \rho)^{\alpha-1} x(\rho)\, d\rho.
\end{aligned} \tag{3.3.19}$$

Since $x \in L^p(0,1)$ as we assumed before, we deduce from Dore and Venni [21], see Corollary 3.16, that

$$I^\alpha(x) \in {}_0H^{\alpha,p}(0,1).$$

By Lemma 3.28 we can say that $\tilde{S}_t(I^\alpha(x)) \in {}_0H^{\alpha,p}(0,1)$. Moreover, by formula (3.3.16) we have the following equation.

$$\tilde{S}_t\big(I^\alpha(x)\big) = e^{(\frac{1}{2}+\alpha)t}\big(I^\alpha(x)\big)(se^{-t}), \quad s \in [0,1].$$

For the left-hand-side of the equality (3.3.17), we fix $s \in [0,1]$ and we verify that the equality is satisfied. Let $t \geq 0$,

$$\begin{aligned}[\tilde{S}_t I^\alpha(x)](s) &= e^{(\frac{1}{2}+\alpha)t}\frac{1}{\Gamma(\alpha)}\int_0^{se^{-t}}(se^{-t}-r)^{\alpha-1}x(r)\,dr \qquad \text{put } r=\rho \\ &= e^{(\frac{1}{2}+\alpha)t}\frac{1}{\Gamma(\alpha)}\int_0^{se^{-t}}(se^{-t}-\rho)^{\alpha-1}x(\rho)\,d\rho, \quad s\in[0,1]. \qquad (3.3.20)\end{aligned}$$

Hence, from equations (3.3.19) and (3.3.20) we deduced and re-write the equality (3.3.17) as follows

$$[\tilde{S}_t I^\alpha(x)](s) = [I^\alpha(S_t x)](s), \text{ for all } s\in[0,1].$$

Therefore, the proof of Theorem 3.29 is complete. □

Before we formulate the main result in this section, let us denote by μ_0 the Borel probability measure on the space ${}_0C[0,1]$ which is invariant for the semigroup $\{S_t\}_{t\geq 0}$. This measure was earlier denoted by μ.

Theorem 3.30. *Let $\alpha \in (0,1)$ and $\{\tilde{S}_t\}_{t\geq 0}$ be a C_0-semigroup on the Banach space $F = {}_0H^{\alpha,p}(0,1)$, which is defined in equation (3.3.16). This C_0-semigroup is generated by the following PDE*

$$\frac{\partial u}{\partial t} + s\frac{\partial u}{\partial s} = (\frac{1}{2}+\alpha)u \qquad (3.3.21)$$

In addition, the Borel measure ν_α that is defined by the following formula

$$\nu_\alpha(C) = \mu\big(D^\alpha(C)\big), \;\; C\in\mathcal{B}\big({}_0H^{\alpha,p}(0,1)\big), \qquad (3.3.22)$$

is an invariant measure for the C_0-semigroup $\{\tilde{S}_t\}_{t\geq 0}$, where

(i) $D^\alpha := (I^\alpha)^{-1}$ *is the fractional derivative operator of order* α,

(ii) $\mathcal{B}({}_0H^{\alpha,p}(0,1))$ *is the Borel σ-field on the space* ${}_0H^{\alpha,p}(0,1)$, *and*

(iii) μ *is the law of Brownian Motion on space* $L^p(0,1)$.

Remark 3.31. It can be proved that the measure ν_α is the law of fractional Brownian Motion with Hurst parameter $H = \frac{1}{2}+\alpha$ on space $L^p(0,1)$. Details will be presented in a forthcoming paper. Note that when $\alpha = 0$, then $\nu_\alpha = \mu$ is the law of Brownian Motion as proved in the book [28].

Proof of Theorem 3.30. This proof consists of two parts. In the first part we prove that the C_0-semigroup $\{\tilde{S}_t\}_{t\geq 0}$ is generated by equation (3.3.21) and in the second part we prove that ν_α is an invariant measure for the C_0-semigroup $\{\tilde{S}_t\}$.

Part one. The method presented in this proof has been generalized and used later in this thesis in Step I of the proof of Theorem 5.35 in Section 5.3. Let us observe that formula (3.3.23) below is a special of the formula (5.3.20).

Let us choose and fix a function $x \in {}_0H^{\alpha,p}(0,1)$. Define a function u by formula

$$u(t,s) = \big(\tilde{S}_t x\big)(s) = e^{(\frac{1}{2}+\alpha)t} x(se^{-t}), \;\; t \geq 0, \;\; s \in [0,1]. \tag{3.3.23}$$

We need to show that this function u solves the equation (3.3.21) under an additional assumption that the initial data function x is of C^1-class. Indeed, by the chain rule, we have the following

$$\begin{aligned}\frac{\partial u(t,s)}{\partial t} &= (\frac{1}{2}+\alpha)e^{(\frac{1}{2}+\alpha)t}x(se^{-t}) - se^{(\frac{1}{2}+\alpha)t}x'(se^{-t}) \\ &= (\frac{1}{2}+\alpha)u(t,s) - e^{(\frac{1}{2}+\alpha)t}x'(se^{-t})se^{-t}.\end{aligned} \tag{3.3.24}$$

And,

$$\frac{\partial u(t,s)}{\partial s} = e^{-t}e^{(\frac{1}{2}+\alpha)t}x'(se^{-t}). \tag{3.3.25}$$

Applying equations (3.3.24) and (3.3.25) to the equation (3.3.23) we get

$$\begin{aligned}\frac{\partial u(t,s)}{\partial t} + s\frac{\partial u(t,s)}{\partial s} &= (\frac{1}{2}+\alpha)u(t,s) - e^{(\frac{1}{2}+\alpha)t}x'(se^{-t})se^{-t} + e^{-t}e^{(\frac{1}{2}+\alpha)t}x'(se^{-t}) \\ &= (\frac{1}{2}+\alpha)u(t,s).\end{aligned}$$

Hence we proved that the semigroup $\{\tilde{S}_t\}_{t\geq 0}$ defined by equation (3.3.16) is corresponds to equation (3.3.21), i.e., if $u(t,s) := \big(\tilde{S}_t x\big)(s)$ then u solves equation (3.3.21).

Part two. From Theorem 3.29 we have the following equality

$$\tilde{S}_t \circ I^\alpha = I^\alpha \circ S_t.$$

If we apply $(I^\alpha)^{-1}$ for both sides on the right side of the above equality, then we deduce the following

$$\tilde{S}_t = I^\alpha \circ S_t \circ D^\alpha \text{ on } {}_0H^{\alpha,p}(0,1). \tag{3.3.26}$$

For any $C \in \mathcal{B}({}_0H^{\alpha,p}(0,1))$, if we take that the inverse to the equation (3.3.26) and apply Corollary 2.31 then, we infer that

$$\begin{aligned}\tilde{S}_t^{-1}(C) &= \big(I^\alpha \circ S_t \circ D^\alpha\big)^{-1}(C) \\ &= (D^\alpha)^{-1}\Big((S_t)^{-1}\big((I^\alpha)^{-1}(C)\big)\Big) \\ &= I^\alpha\Big((S_t)^{-1}\big(D^\alpha(C)\big)\Big).\end{aligned}$$

The last equality is a consequence of the fact that $(D^\alpha)^{-1} = I^\alpha$. By taking the measure ν_α for both sides, we obtain

$$\nu_\alpha\big(\tilde{S}_t^{-1}(C)\big) = \nu_\alpha\big[I^\alpha\big(S_t^{-1}(D^\alpha(C)\big)\big]. \tag{3.3.27}$$

To understand the last equality (3.3.27), let us choose and fix any arbitrary set C that belongs to the space $\mathcal{B}\big({}_0H^{\alpha,p}(0,1)\big)$ and then consider an auxiliary set C' such that

$$C' = D^\alpha(C) \in \mathcal{B}\big(L^p(0,1)\big).$$

So, we can re-write the equation (3.3.22) as follows

$$\nu_\alpha\big(I^\alpha(C')\big) = \mu(C') \tag{3.3.28}$$

By substituting equations (3.3.28), (3.3.22) and (3.0.3) in the equation (3.3.27), we obtain

$$\begin{aligned}\nu_\alpha\big(\tilde{S}_t^{-1}(C)\big) &= \nu_\alpha\big[I^\alpha\big(S_t^{-1}(D^\alpha(C))\big)\big]\\ &= \mu[S_t^{-1}\big(D^\alpha(C)\big)] = \mu(D^\alpha(C)) = \nu_\alpha(C).\end{aligned}$$

Therefore, we proved that for any $C \in \mathcal{B}\big({}_0H^{\alpha,p}(0,1)\big)$ we have

$$\nu_\alpha\big(\tilde{S}_t^{-1}(C)\big) = \nu_\alpha(C).$$

Hence, the proof of Theorem 3.30 is complete. □

Remark 3.32. The measure μ which is the law of Brownian motion on the space $L^p(0,1)$ is constructed from the measure μ_0 on the space ${}_0C[0,1]$. That is because the space ${}_0C[0,1]$ is dense and subspace of the space $L^p(0,1)$ and the natural embedding

$$i : {}_0C[0,1] \hookrightarrow L^p(0,1)$$

is linear and bounded and hence, it is measurable. These facts allow us to define a Borel probability measure μ on the space $L^p(0,1)$ by the following

$$\mu(A) := \mu_0\big(i^{-1}(A)\big) = \mu_0\big(A \cap {}_0C[0,1]\big). \tag{3.3.29}$$

In other words, μ is the image, or push-forward, of the measure μ_0 via the map i. These measures are different because they are defined in different spaces, but they are closely related. For instance, if $A \in \mathcal{B}(L^p(0,1))$ then $A \cap {}_0C[0,1] \in \mathcal{B}({}_0C[0,1])$. The measure μ_0 makes sense not only in the space ${}_0C[0,1]$ but also in a slightly bigger space and remains invariant.

Note that the Kuratowski Theorem and equality (3.3.29) implies that for every $A \in$

$\mathcal{B}({}_0C[0,1])$,

$$\mu_0(A) = \mu(A).$$

because, if $A \in \mathcal{B}({}_0C[0,1])$ then $A \in \mathcal{B}(L^p(0,1))$ and they are different because the domain of the measure is different $\mu : \mathcal{B}(L^p(0,1)) \to [0,1]$ and $\mu_0 : \mathcal{B}({}_0C[0,1]) \to [0,1]$.

Remark 3.33. Assume that $\alpha \in (0,1)$, $\beta \in [0,\alpha)$ and consider the family $\{\tilde{S}_t\}$ defined by (3.3.16). We proved in Lemma 3.28 that this family is a C_0-semigroup on the Banach space $E = {}_0H^{\alpha,p}(0,1)$. We believe that it can be proven this family is also a C_0-semigroup on the Banach space $F = {}_0H^{\beta,p}(0,1)$. Moreover, we believe that it can be proven that there exists a unique "extension" of the measure ν_α to the space F which is also an invariant measure for this extension semigroup on the space F. The details will need to be worked out.

Chapter 4

Solutions for First Order PDEs under General Assumptions

In this chapter, we study the existence and the uniqueness of a solution to a specific first order Partial Differential Equation (PDE) called the Lasota equation. Due to its properties and applications, the Lasota equation is considered one of the most essential first-order PDEs. For instance, the nonlinear part describes the process of differentiation and reproduction of the population of red blood cells depending on the concentration of hormones at a specific stage. Using semigroup theory, we define a solution to the Lasota equation under some sufficient conditions in a specified Banach space. The main two results of this part of this section are Theorems 4.19 and 4.20 which we prove respectively that a natural family of linear operators associated with equation (4.0.4) with $c = 0$, is a C_0-semigroup on an appropriately chosen Banach space E, and we characterise the domain of the infinitesimal generator of that semigroup. We also study some properties of the solutions and proved the existence of invariant measures of such equations under natural assumptions on the coefficients. The properties of such equation were studied in many papers, see for instance [6], [7], [41], [27] and [43].

Lasota in [27] studied the following problem

$$\frac{\partial v(t,x)}{\partial t} + a(t,x)\,\frac{\partial v(t,x)}{\partial x} = c(t,x,v) \quad \text{for } (t,x) \in [0,\infty) \times [0,1], \tag{4.0.1}$$
$$v(0,x) = v_0(x) \quad \text{for } x \in [0,1].$$

He proved the existence of a unique classical solution by using the characteristics method under certain regularity assumptions on the coefficients a and c. For the convenience of the reader, we state the assumptions here

Assumption 4.1. The functions $a : [0,\infty) \times [0,1] \to \mathbb{R}$ and $c : [0,\infty) \times [0,1] \times \mathbb{R} \to \mathbb{R}$ are of C^1-class and satisfy the following

i) $a(t,x) \geq 0, \quad \text{for } (t,x) \in [0,\infty) \times [0,1];$

ii) $c(t,x,0) \geq 0, \quad \text{for } (t,x) \in [0,\infty) \times [0,1];$

iii) $c(t,x,v) \leq k_1(t)\, v + k_2(t), \quad \text{for } (t,x) \in [0,\infty) \times [0,1],\ v \geq 0,\ k_1, k_2 \in C([0,1]);$

iv) $a(t,0) = 0, \quad \text{for } t \geq 0.$

He also considered the following autonomous version of the problem (4.0.1)

$$\frac{\partial v(t,x)}{\partial t} + a(x)\,\frac{\partial v(t,x)}{\partial x} = c(x,v) \quad \text{for } (t,x) \in [0,\infty) \times [0,1], \tag{4.0.2}$$

$$v(0,x) = v_0(x) \quad \text{for } x \in [0,1]. \tag{4.0.3}$$

He proved that the problem (4.0.2)-(4.0.3) generates a semiflow $\{S_t\}_{t\geq 0}$ on the space $C^1_+\big([0,1]\big)$ (that is, the space of C^1-class nonnegative functions). This semiflow was defined by the following formula

$$S_t v_0 = u(t,x) \ \text{ for } x \in [0,1].$$

where u is the unique classical solution of the problem (4.0.2)-(4.0.3).
The set of Assumptions 4.1, was used in many research papers including Rudnicki [43] and Brunovsky and Komornik [6] to find the properties of the Lasota equation. Rudnicki in [43] studied the problem (4.0.2)-(4.0.3) and he confirmed that the existence of an exact invariant measure for the semiflow $\{S_t\}_{t\geq 0}$ has more properties.

The main objective of this Chapter is to generalise the results of Lasota, Rudnicki, and others by studying linear first-order differential equations without assuming that the drift coefficient $a(x)$ is a smooth function. We only assume that a is a continuous function satisfying so-called the Osgood condition. Then we prove that the informal family of operators derived via informal application of the characteristics method is a C_0-contraction semigroup in the space ${}_0C([0,1])$. Using this fact and applying the Rudnicki approach we prove that our equation has an invariant measure for the coefficient $\lambda > 0$.

Before we start, we need to state our equation and our assumptions. We consider the following problem:

$$\frac{\partial u(t,x)}{\partial t} + a(x)\frac{\partial u(t,x)}{\partial x} = c(u(t,x)), \ \ t \geq 0, \ x \in [0,1] \tag{4.0.4}$$

$$u(0,x) = u_0(x), \quad x \in [0,1] \tag{4.0.5}$$

where $c : \mathbb{R} \to \mathbb{R}$ is linear map and $u_0 \in {}_0C([0,1])$. From now on we assume that the coefficient a satisfies the following assumptions.

Assumption 4.2. Let $a : [0,1] \to \mathbb{R}$.

A1 - The function a is continuous;

A2 - $a(0) = 0$ and $a(x) > 0$ for $x \in (0, 1]$;

A3 - The function a satisfies the Osgood condition, i.e.,

$$|a(x_2) - a(x_1)| \leq \phi(|x_2 - x_1|), \quad \text{for all } x_1, x_2 \in [0, 1], \tag{4.0.6}$$

for an increasing continuous function $\phi : [0, 1] \to [0, \infty)$ such that $\phi(r) > 0$ if $r > 0$, $\phi(0) = 0$ and

$$\int_0^\delta \frac{1}{\phi(r)}\, dr = \infty, \quad \text{for every } \delta > 0,$$

i.e.,

$$\lim_{\varepsilon \to 0^+} \int_\varepsilon^\delta \frac{1}{\phi(r)}\, dr = \infty.$$

Assumptions about the nonlinearity of function c will be listed later in appropriate sections. In this chapter, we consider first the homogeneous linear problem of equation (4.0.4)-(4.0.5), i.e., we assume that $c = 0$ and at the end we assume $c(x) = \lambda u$.
This chapter is organised as follows. In Section 4.1 we state the required preliminaries which we use throughout the chapter. Section 4.2 is devoted to the existence and uniqueness of solution for the homogeneous case of the problem (4.0.4)- (4.0.5). In section 4.3 we use Rudnicki's [43, Theorem 1] as a case study to prove the existence of an invariant measure under Assumption 4.2.

4.1 Preliminaries

In this section, we state the required definitions and properties of the solution to the first order PDE. The definition below follows from [33, Definition IV.2.3] with the difference that the phrase "classical solution" is replaced by a strong solution. A generalisation of Definition 4.3 to in-homogeneous problems will be presented later on in Section 5.1 in Chapter 5.

Definition 4.3. Let X be a Banach space and A be a linear densely defined operator in X, i.e. $D(A)$ is a dense subspace of X and $A : D(A) \to X$ is a linear map. By a strong solution to the following abstract Cauchy problem

$$\frac{du(t)}{dt} = A\,u(t), \quad t > 0, \tag{4.1.1}$$

$$u(0) = x, \tag{4.1.2}$$

where $x \in X$, we mean an X valued function u defined on the interval $[0, \infty)$ such that

(i) $u : [0, \infty) \to X$ is continuous and

(ii) $u(0) = x$,

(iii) $u : (0, \infty) \to X$ is continuously differentiable,

(iv) $u(t) \in D(A)$ for every $t > 0$, and

(v) $u'(t) = Au(t)$, for every $t > 0$.

The following Theorem is based on [33, Theorem IV.1.3].

Theorem 4.4. *Let X be a Banach space and A be the infinitesimal generator of a C_0-semigroup $\{S_t\}_{t\geq 0}$. Then for all $x \in D(A)$ the abstract Cauchy problem* (4.1.1)-(4.1.2) *has a unique strong solution given by the following formula*

$$u(t) = S(t)x, \quad t \geq 0.$$

Definition 4.5. Let E be a Banach space and $U \subset \mathbb{R} \times E$ be an open set. Assume that $f : U \to E$ is a continuous function. Such a function is often called a time-dependent vector field. If $U \subset E$ is open then the function $f : U \to E$ is often called a vector field on U.

Before we state some results about the existence and uniqueness of solutions to our equation, we need to define a solution. We will define a local solution, a local maximal solution and global solutions (for positive and negative times).

Definition 4.6. Let us assume the framework of Definition 4.5. Assume $(t_0, x_0) \in U$. A local solution of the following differential equation

$$\frac{dx}{dt} = f(t, x), \tag{4.1.3}$$

with the following initial condition

$$x(t_0) = x_0 \tag{4.1.4}$$

is a function $\phi : I \to E$, where $I \subset \mathbb{R}$ is an open interval such that $t_0 \in I$, if and only if the following conditions are satisfied

(i) if $t \in I$ then $(t, x) \in U$ (so that $f(t, x)$ makes sense);

(ii) function ϕ is of C^1-class,

(iii) $\phi(t_0) = x_0$, i.e., the equation (4.1.4) is satisfied;

(iv) if $t \in I$ then $\phi'(t) = f(t, \phi(t))$, i.e., the equation (4.1.3) is satisfied for every $t \in I$.

A local solution $\psi : J \to U$ of the problem (4.1.3)-(4.1.4) is said to extend a local solution $\phi : I \to U$ of the problem (4.1.3)-(4.1.4) if and only if the following two conditions are satisfied

(v) $I \subset J$;

(v) if $t \in I$ then $\psi(t) = \phi(t)$.

A local solution $\phi : I \to U$ of the problem (4.1.3)-(4.1.4) is said to be a maximal local solution if and only if it can not be extended to a strictly larger interval than I.
A maximal local solution $\phi : I \to U$ of the problem (4.1.3)-(4.1.4) is said to be global in positive times if and only if the right end of the interval I is equal to ∞.
A maximal local solution $\phi : I \to U$ of the problem (4.1.3)-(4.1.4) is said to be global in negative times if and only if the left end of the interval I is equal to $-\infty$.
A maximal local solution $\phi : I \to U$ of the problem (4.1.3)-(4.1.4) is said to be global if and only if it is global in both positive and negative times, i.e., the left end of the interval I is equal to $-\infty$ and the right end of the interval I is equal to ∞, in other words, the interval I is equal to $\mathbb{R}$.

The following Theorem is based on Osgood's Uniqueness Theorem see [34, Ch.3 Theorem 1], which plays an important role in our proof of the uniqueness of a solution.

Theorem 4.7. *Assume that $a : [0,1] \to \mathbb{R}$ is a continuous function satisfying the Osgood condition. If $x_1, x_2 : [S,T] \to \mathbb{R}$, for some $S < 0 < T$, are two solutions of the following ODE problem*

$$\begin{aligned} x'(t) &= a\big(x(t)\big), \quad t \in [S,T], \\ x(0) &= x_0, \end{aligned}$$

for some $x_0 \in \mathbb{R}$, then

$$x_1(t) = x_2(t), \quad t \in [S,T].$$

We do not provide proof of this theorem since we only need the formulation above.

4.2 Existence and Uniqueness of Solutions of Linear First order PDEs

In this section, under Assumptions 4.2, we prove the existence and the uniqueness of a global solution of the problem (4.0.4)-(4.0.5) in a special case, that is, a linear homogeneous case. In other words, we consider first the following problem:

$$\frac{\partial u(t,x)}{\partial t} + a(x)\frac{\partial u(t,x)}{\partial x} = 0, \;\; t \geq 0, \;\; x \in [0,1] \tag{4.2.1}$$

$$u(0,x) = u_0(x), \;\; x \in [0,1], \tag{4.2.2}$$

where $u_0 \in {}_0C([0,1])$.

In what follows, note that if the function $a : [0,1] \to \mathbb{R}$ satisfies Assumption (A1) we can find a nice extension of the function a to the half-line $[0,\infty)$. For instance, we can define such a continuous extension $\tilde{a} : [0,\infty) \to \mathbb{R}$ by the following formula

$$\tilde{a}(x) = \begin{cases} a(x), & \text{if } x \in [0,1], \\ a(1), & \text{if } x > 1. \end{cases} \tag{4.2.3}$$

But other choices are possible.

In this way, we could have assumed that a is defined on the whole interval $[0,\infty)$. But for our purposes, we only need to know properties of a on the interval $[0,1]$. Hence, although we define below a function G on the whole interval $[0,\infty)$, we only need its properties on the interval $[0,1]$.

To define a solution to equation (4.2.1) we follow the notation of Lasota [27] and use the method of characteristics. Therefore, we consider first the following Ordinary Differential Equation (ODE):

$$\frac{d\,x(t)}{dt} = a(x(t)), \;\; t \in \mathbb{R}, \tag{4.2.4}$$

$$x(0) = x_0, \;\; x_0 \geq 0 \tag{4.2.5}$$

A solution to the above ODE is found in two distinct cases. Firstly, if $x_0 = 0$, then by Assumption (A2) we have $a(0) = 0$, so a function $x(t) = 0$ for $t \in \mathbb{R}_+$ is a solution to the equation (4.2.4).

Secondly, if $x_0 > 0$, then by employing the classical characteristics method we find the following candidate for the solution to problem (4.2.4)-(4.2.5)

$$x(t) = G^{-1}\big(t + G(x_0)\big), \;\; t \in I, \tag{4.2.6}$$

where

$$G(x) = -\int_x^1 \frac{1}{\tilde{a}(r)}\,dr, \quad \text{so that} \quad G'(x) = \frac{1}{\tilde{a}(x)}, \quad x \in (0,\infty) \tag{4.2.7}$$

Since we have not specified the interval of I, let us define a function $x(t), t \in I$ by the above formula (4.2.6) for the largest possible interval I. In other words, we want to determine the domain of the function on the RHS of the formula (4.2.6). It will be shown below that the largest interval I is of the form $I = \big(-\infty, \tau(x_0)\big)$, where $\tau(x_0) > 0$. In some cases $\tau(x_0) = \infty$ and some other cases $\tau(x_0) < \infty$. To make the observation rigorous we need the following result which formulates properties of the function G.
In the following proposition, we list some basic properties that function $G(x)$, which defined by formula (4.2.7), satisfies.

Proposition 4.8. *Assume that $a : [0,1] \to \mathbb{R}$ is a continuous function satisfying Assumption 4.2 and let $\tilde{a}$ be defined by formula* (4.2.3)*. Let $G : (0,\infty) \to \mathbb{R}$ be a function defined by formula* (4.2.7)*. Then the following conditions hold.*

(i) $G(1) = 0$,

(ii) If $x < 1$, then $G(x) < 0$ and if $x > 1$ then $G(x) > 0$,

(iii) $G(x)$ is (strictly) increasing, i.e., if $0 < x_1 < x_2$, then $G(x_1) < G(x_2)$,

(iv) $\lim_{x\to 0^+} G(x) = -\infty$ *and* $\lim_{x\to\infty} G(x) =: G_\infty \in (0,\infty]$ *exists,*

(v) $\lim_{y\to-\infty} G^{-1}(y) = 0$,

(vi) the function G is of C^1-class, and

$$G'(x) = \frac{1}{a(x)} \text{ for every } x \in (0,\infty).$$

(vii)(a) The function G is a bijection from $(0,\infty)$ onto $(-\infty, G_\infty)$,

(b) it maps bijectively the interval $(0,1]$ onto interval $(-\infty,0]$

(c) and the interval $[1,\infty)$ onto interval $[0,G_\infty)$.

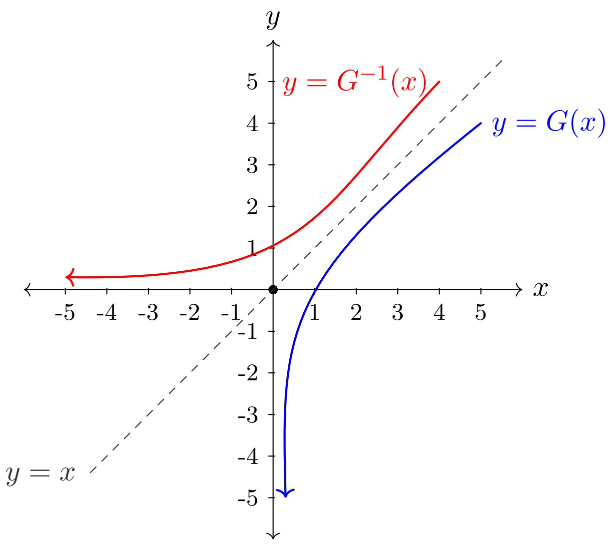
Proof of Proposition 4.8. The proof of the first three conditions (i), (ii) and (iii) is trivial. Proof of condition (iv): Let us recall that from Osgood condition (4.0.6) we know that

$$\int_0^\delta \frac{1}{\phi(r)}\,dr = \infty, \quad \text{for all } \delta > 0, \text{ i.e., } \lim_{\varepsilon\to 0^+}\int_\varepsilon^\delta \frac{1}{\phi(r)}\,dr = \infty.$$

Hence we proved that

$$\lim_{x\to 0^+}[G(x)] = -\infty.$$

FIGURE 4.1: Graph of functions G and G^{-1} when $G_\infty = \infty$.



By part (iii) the function $G : (0, \infty) \to \mathbb{R}$ is increasing, the limit

$$G_\infty := \lim_{x \to \infty} G(x) \in (0, \infty) \cup \{\infty\}$$

exists. Let us stress that this limit is either a positive real number or ∞.

Proof of condition (v): First of all let us observe that since by condition (iii) the function G is (strictly) increasing, the inverse G^{-1} exists and is also (strictly) increasing. By condition (i) the function $G^{-1}(0) = 1$. Moreover, by conditions (ii) and (iv) the image by G of the interval $(1, \infty)$ is equal to the interval $(0, G_\infty)$ and the image by G of the interval $(0, 1)$ is equal to the interval $(-\infty, 0)$. Finally, because by condition (iv) $\lim_{x \to 0^+} G(x) = -\infty$ and G is (strictly) increasing, we infer that $\lim_{y \to -\infty} G^{-1}(y) = 0$.

Proof of condition (vi): Since the function a is continuous on $[0, \infty)$ and $a(x) > 0$ for $x \in (0, \infty)$, by [39, Theorem 4.17] the function $\frac{1}{a}$ is continuous on $(0, \infty)$. From the definition of the function G in formula (4.2.7) and the Fundamental Theorem of Calculus, see [39, Theorem 11.33] we deduce that the function G is differential at every $x \in (0, \infty)$ and

$$G'(x) = \frac{1}{a(x)}, \quad x \in (0, \infty).$$

Obviously by equation (4.2.3)

$$G'(x) = \frac{1}{\tilde{a}(x)}, \quad x \in [0, 1].$$

To finish the proof we observe that since the function $\frac{1}{a}$ is continuous on $(0, \infty)$, also the function $(0, \infty) \ni x \mapsto G'(x)$ is continuous. Hence we proved that the function G is of C^1-class on the interval $(0, \infty)$.

Proof of condition (vii): This has already been proved inside the proof of condition (v). Hence, the proof of proposition 4.8 is completed. □

Theorem 4.9. *Let us assume that the function a satisfies Assumption 4.2. Then for every $x_0 \in [0,1]$ there exists a unique maximal solution of problem* (4.2.4)-(4.2.5)

$$x : (-\infty, \tau(x_0)) \to \mathbb{R},$$

where

$$\tau(x_0) = \begin{cases} \infty, & \text{if } G_\infty = \infty, \\ G_\infty - G(x_0), & \text{if } G_\infty < \infty. \end{cases} \tag{4.2.8}$$

The uniqueness has to be understood in the following way.
If $x_1 : (-\infty, \tau_1) \to \mathbb{R}$ and $x_2 : (-\infty, \tau_2) \to \mathbb{R}$ are two maximal solution of the ODE (4.2.4) *then $\tau_1 = \tau_2$ and $x_1(t) = x_2(t)$ for all $t \in (-\infty, \tau_1)$.*

Proof of Theorem 4.9. Let us choose and fix $x_0 \in (0,1]$ and define $\tau(x_0)$ by formula (4.2.8). Next, let us define a function $x : (-\infty, \tau(x_0)) \to \mathbb{R}$ by formula (4.2.6). Then, by Proposition 4.8, we infer that the function x is of C^1-class and, by the chain rule, see [39, Theorem 5.5], and the equation (4.2.7), its derivative is given by the following formula

$$x'(t) = \frac{1}{G'[G^{-1}(t + G(x_0))]} = \frac{1}{\frac{1}{a[G^{-1}(t+G(x_0))]}} = a\big(x(t)\big), \quad t \in \big(-\infty, \tau(x_0)\big).$$

Hence the function x solves our equation (4.2.4) together with the initial condition (4.2.5). The uniqueness of solution follows from Theorem 4.7 because the function a satisfies the Osgood condition. To be precise, we argue as follows. Suppose by contradiction that $\tau_1 \neq \tau_2$. Without loss of generality we can assume that $\tau_1 < \tau_2$. Since the function a in equation (4.2.4) satisfies the Assumptions 4.2, by Theorem 4.7, we deduce that the solution to the equation (4.2.4) is unique, i.e.,

$$x_1(t) = x_2(t) \quad \text{for all } t \in (-\infty, \tau_1).$$

Therefore, the solution $x_1 : (-\infty, \tau_1) \to \mathbb{R}$ is not a maximal one because it can be extended (by x_2) to a strictly larger interval $(-\infty, \tau_2)$. This contradiction completes the proof of Theorem 4.9. □

Example 4.10. *Here we present an example related to Proposition 4.8 and equation* (4.2.4). *In the equation* (4.2.4), *we choose $\alpha > 0$ and define function a by*

$$a(x) = \alpha x, \quad \text{for } x \in [0,1],$$

The extension $\tilde{a}$ can be defined as before, but it is more natural to consider

$$\tilde{a}(x) = \alpha x, \ \textit{for } x \in [0, \infty).$$

One can check that these functions a and $\tilde{a}$ satisfy all assumptions of Proposition 4.8. Using equation (4.2.7) we can find an explicit formula for the corresponding function $G(x)$.

$$G(x) = -\int_x^1 \frac{1}{\tilde{a}(r)}\, dr = -\int_x^1 \frac{1}{\alpha r}\, dr = \frac{\ln x}{\alpha}, x \in (0, \infty).$$

One can check with bare hands that in this case, the function G satisfies all properties listed in Proposition 4.8. This is of course not surprising.
If we put $G(x) = y$, that implies $\frac{\ln x}{\alpha} = y \Longleftrightarrow \ln x = \alpha y \Longleftrightarrow x = e^{\alpha y}$. Since the inverse function G^{-1} is characterized by

$$G^{-1}(y) = x \Longleftrightarrow G^{-1}(y) = e^{\alpha y},$$

we infer, that the solution to the equation (4.2.4) is given by the following formula,

$$x(t) = e^{\alpha\big(t+G(x_0)\big)} = e^{\alpha t}.e^{\alpha\frac{\ln x_0}{\alpha}} = e^{\alpha t}\, x_0, \ \ t \in (-\infty, \infty).$$

Hence, in this case we have

$$\tau(x_0) = \infty$$

and the solution is global.

The formula (4.2.6) allows us to define a solution to the problem (4.2.4)-(4.2.5) as follows:

$$\varphi(t, x_0) := \begin{cases} G^{-1}\big(t + G(x_0)\big), \ \ -\infty < t < \tau(x_0); & \text{if } x_0 \in (0, 1]; \\ 0, & \text{if } x_0 = 0. \end{cases} \tag{4.2.9}$$

In particular, we deduce that $\varphi(t, x_0)$ is well defined for all $t \in (-\infty, 0]$ and $x_0 \in [0, 1]$. We showed in Theorem 4.9 that for every $x_0 \in [0, 1]$, the function

$$\big(-\infty, \tau(x_0)\big) \ni t \mapsto \varphi(t, x_0) \in \mathbb{R}$$

is a solution to problem (4.2.4). In particular, $\varphi(0, x_0) = 0$. Indeed,

$$\varphi(0, x_0) = G^{-1}\big(0, G(x_0)\big) = G^{-1}\big(G(x_0)\big) = x_0.$$

Using the above formulation we can define a solution to the problem (4.2.1)-(4.2.2). We put

$$u(t, x_0) := u_0\big(\varphi(-t, x_0)\big), \ \ x_0 \in [0, 1], \ t \in [0, \infty). \tag{4.2.10}$$

Inserting formula (4.2.9) into the last formula (4.2.10) we obtain the following expression for a solution to equation (4.2.1) as follows

$$u(t,x) := \begin{cases} u_0\big(G^{-1}(-t+G(x))\big), & x \in (0,1],\ t \in [0,\infty). \\ u_0(0), & x = 0,\ \ t \in [0,\infty). \end{cases} \tag{4.2.11}$$

The above formula is based on the characteristics method. The solution φ of equation (4.2.4) is called the characteristic of the first order PDE (4.0.4) and the equation (4.2.4) is called the characteristic equation.

Remark 4.11. If the function a satisfies the Assumption 4.2 and the function u_0 is of C^1-class, then by direct calculations one can prove that the function $u(t,x)$ for $(t,x) \in [0,\infty) \times [0,1]$, defined above in (4.2.11), satisfies equation (4.2.1) and initial condition (4.2.2) point-wise, i.e.,

(i) at every $(t,x) \in [0,\infty) \times [0,1]$ the partial derivatives $\frac{\partial u}{\partial t}(t,x)$ and $\frac{\partial u}{\partial x}(t,x)$ exist and the equality (4.2.1) is satisfied;

(ii) at every $x \in [0,1]$, the identity (4.2.2) is satisfied.

We say such that function $u(t,x)$ is a classical solution to the problem (4.2.1)-(4.2.2) in the classical sense.

Let us observe that the definition of the function u that is given in equation (4.2.10) is well posed due to the following Corollary.

Corollary 4.12. *If the function φ is defined by formula* (4.2.9) *then the following condition is satisfied.*

$$\varphi(s,x_0) \in [0,1],\ \textit{for all}\ x_0 \in [0,1],\ \ s \leq 0.$$

Proof of Corollary 4.12. Let us choose and fix $x_0 \in [0,1]$ and $s \leq 0$. Then by properties (i) and (ii) of Proposition 4.8, we infer that $G(x_0) \leq 0$. Since $s \leq 0$, then also $s + G(x_0) \leq 0$. Hence, by condition (vii) of the same Proposition, we infer that $G^{-1}(s + G(x_0)) \in [0,1]$. The result follows by using the formula (4.2.9). □

The above corollary is also an important result since we will use it later to prove the boundedness and the C_0-continuity in our main result of this section. The following two results are important to prove the C_0-continuity condition of our main result (Theorem 4.19). Before we state the proposition let us recall that

$$\tau(x_0) > 0, \quad \text{for every } x_0 \in [0,1].$$

Proposition 4.13. *Assume that the function a satisfy Assumption 4.2 and u_0 is a continuous function such that $u_0(0) = 0$. Then the restriction of the function φ to the set $(-\infty, 0] \times [0, 1]$, i.e., the function*

$$\varphi : (-\infty, 0] \times [0, 1] \to \mathbb{R}$$

defined by formula (4.2.9), *is continuous with respect to variable t uniformly with respect to x_0 variable, i.e., for every $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$\textit{if } \ t_1, t_2 \in (-\infty, 0], \ |t_1 - t_2| \le \delta, \ x_0 \in [0, 1] \ \ \textit{then } |\varphi(t_1, x_0) - \varphi(t_2, x_0)| \le \varepsilon. \tag{4.2.12}$$

To prove Proposition 4.13 we need first to formulate two claims. One is standard and the second one is about some properties of the function φ. The following claim is a special case of [39, Theorem 4.19].

Claim 1: If a set K is a compact subset of $\mathbb{R}^2$ and a function $f : K \to \mathbb{R}$ is continuous, then f is uniformly continuous, i.e., for every $\varepsilon > 0$ there exists $\delta > 0$ such that for all $x = (x_1, x_2) \in K$ and $y = (y_1, y_2) \in K$

$$\text{if } d(x, y) \le \delta \ \text{ then } |f(x) - f(y)| \le \varepsilon.$$

Here, d is the classical Euclidean metric on $\mathbb{R}^2$

$$d(x, y) = \sqrt{(y_1 - x_1)^2 + (y_2 - x_2)^2}.$$

We can not directly apply Claim 1 to the proof of Proposition 4.13 because the domain of the function φ is the set $(-\infty, 0] \times [0, 1]$ which is not a compact set. Therefore, to complete the proof we need the following additional claim.

Claim 2: For every $\varepsilon > 0$ there exists $T_\varepsilon < 0$ such that

$$\text{if } t \in (-\infty, T_\varepsilon] \text{ and } x_0 \in [0, 1], \text{ then } |\varphi(t, x_0)| \le \varepsilon. \tag{4.2.13}$$

By formula (4.2.9) we can consider only $x_0 \in (0, 1]$. Moreover, since $\tau(x_0) > 0$ by the same formula we have

$$\varphi(t, x_0) = G^{-1}\big(t + G(x_0)\big), \ \ t \in (-\infty, 0], \ \ x_0 \in (0, 1].$$

Let us now choose and fix $\varepsilon > 0$. For the time being, we consider $T_\varepsilon < 0$. The precise value of $T_\varepsilon < 0$ will be decided below. Since by Proposition 4.8, the function G^{-1} takes only positive values, by the last formula function φ also takes positive values. Moreover, functions G and G^{-1} are increasing. Hence, for $x_0 \in (0,1]$ and $t \leq T_\varepsilon$

$$|\varphi(t,x_0)| = \varphi(t,x_0) = G^{-1}\big(t + G(x_0)\big) \leq G^{-1}\big(T_\varepsilon + G(1)\big) = G^{-1}(T_\varepsilon).$$

To find the value of T_ε we observe that by part (v) of previously used Proposition 4.8 we can find $T_\varepsilon < 0$ such that $G^{-1}(T_\varepsilon) < \varepsilon$. Having done so, we observe that if $x_0 \in (0,1]$ and $t \leq T_\varepsilon$ then

$$|\varphi(t,x_0)| \leq \varepsilon.$$

Thus the proof of Claim 2 is complete.

Proof of Proposition 4.13. Let us now choose and fix $\varepsilon > 0$. By Claim 2 there exists $T_\varepsilon < 0$ such that the formula (4.2.13) holds with $\frac{\varepsilon}{2}$, i.e.,

$$\text{if } t \in (-\infty, T_\varepsilon] \text{ and } x_0 \in [0,1], \text{ then } |\varphi(t,x_0)| \leq \frac{\varepsilon}{2}. \tag{4.2.14}$$

Let $K = [T_\varepsilon - 1, 0] \times [0,1]$. Then K is a compact subset of $\mathbb{R}^2$. Therefore, by applying Claim 1 for function $f = \varphi$, we can find $\delta > 0$ such that

$$\text{if } t_1, t_1 \in [T_\varepsilon - 1, 0],\ x_0 \in [0,1], |t_2 - t_1| \leq \delta \text{ then } |\varphi(t_2,x_0) - \varphi(t_1,x_0)| \leq \varepsilon. \tag{4.2.15}$$

Without loss of generality, we can assume that $\delta < 1$. the assertion of Proposition 4.13 follows from (4.2.14) and (4.2.15).

Thus the proof of Proposition 4.13 is complete. □

The following result follows from Proposition 4.13.

Corollary 4.14. *For every $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$\textit{if } s \in [-\delta, 0] \textit{ and } x_0 \in [0,1], \textit{ then } |\varphi(s,x_0) - x_0| \leq \varepsilon.$$

Proof of Corollary 4.14. Let us choose and fix $\varepsilon > 0$. By the Proposition 4.13, we can find $\delta > 0$ such that (4.2.12) holds. Putting $t_2 = 0$ and $t_1 = s$, then we get the following

$$\text{if } \ s \in (-\infty, 0],\ |s - 0| \leq \delta,\ x_0 \in [0,1] \ \text{ then } |\varphi(s,x_0) - \varphi(0,x_0)| \leq \varepsilon.$$

Because $\varphi(0,x_0) = x_0$ and $s \in [-\delta, 0]$ then $|s - 0| \leq \delta$, and therefore, we deduce the result. □

We present now one of the most important results from the current subsection. We continue to assume Assumption 4.2 hold.

Theorem 4.15. *If a function $u_0 : [0,1] \to \mathbb{R}$ is continuous and $u_0(0) = 0$, then for every $t \geq 0$, the function $u(t,\cdot)$ has the same properties. Moreover, the function u satisfies the initial condition* (4.0.5)*.*

Proof of Theorem 4.15. Firstly, we prove that the function u satisfies equation (4.0.5). From the definition of the function u that given in equation (4.2.10) if we take $t = 0$ and $x_0 \in [0,1]$ then we get $u(0,x_0) = u_0(\varphi(0,x_0)) = u_0(x_0)$ because by equation (4.2.9) $\varphi(0,x_0) = x_0$. So our function u satisfies the initial condition.
Secondly, we need to prove that the function $x \mapsto u(t,x)$ is continuous. Let us choose and fix $t > 0$. We prove that $u(t,\cdot)$ is continuous at $x_0 = 0$. According to [39, Theorem 4.1] to prove this continuity it is sufficient to prove that

$$\lim_{x_0 \to 0^+} u(t,x_0) = u(t,0).$$

But we have already proved that $u(t,0) = 0$, so we need to show that $u(t,x_0) \to 0$ when $x_0 \to 0^+$. Let us notice that in the view of formulae (4.2.9) and (4.2.10) we have

$$u(t,x_0) = u_0\big(G^{-1}(-t + G(x_0))\big), \quad x_0 \in [0,1].$$

It follows from the condition (iv) of Proposition 4.8 that

$$G(x_0) \to -\infty, \quad \text{as } x_0 \to 0^+.$$

Hence, we infer

$$G(x_0) - t \to -\infty, \quad \text{as } x_0 \to 0^+.$$

By applying G^{-1} and using again condition (v) of Proposition 4.8 to the last equation, we infer that

$$G^{-1}(G(x_0) - t) \to 0, \quad \text{as } x_0 \to 0^+.$$

By the assumptions of Theorem 4.15, the function u_0 is continuous at 0 and by [39, Theorem 4.7], we infer that

$$u_0\big(G^{-1}(G(x_0) - t)\big) \to u_0(0) = 0, \quad \text{as } x_0 \to 0^+.$$

Hence we proved that

$$u(t,x_0) \to 0, \quad \text{as } x_0 \to 0^+.$$

To prove that $u(t,\cdot)$ is continuous at $x_0 \in (0,1]$, we use again the following formula

$$u(t,x_0) = u_0\big(G^{-1}(-t + G(x_0))\big), \quad x_0 \in (0,1].$$

Let us observe that $u(t,\cdot)$ is a composition of three functions, namely;

(i) $(0,1] \ni x_0 \mapsto -t + G(x_0) \in (-\infty, 0]$,

(ii) $(-\infty, 0] \ni y \mapsto G^{-1}(y) \in (0,1]$,

(iii) $[0,1] \ni z \mapsto u_0(z) \in \mathbb{R}$.

Since each of these three functions is continuous, by [39, Theorem 4.7] we infer that $u(t,\cdot)$ is also continuous on the interval $(0,1]$. This completes the proof of Theorem 4.15. □

In the following result, we denote the unique maximal local solution to the ODE (4.2.4) by $\big(-\infty, \tau(x_0)\big) \ni t \mapsto \varphi(t, x_0) \in \mathbb{R}$. The proof of this result is based on the condition of the uniqueness of the solutions in Theorem 4.9 and not on the explicit formula for the solution so that it is applicable to equations that do not have explicit solutions.

Corollary 4.16. *If $s < \tau(x_0)$ and $t < \varphi(s, x_0)$ then*

$$\varphi\big(t, \varphi(s, x_0)\big) = \varphi(t+s, x_0). \tag{4.2.16}$$

Proof of Corollary 4.16. Let us fix $s_0 < \tau(x_0)$ and $x_0 \in [0,1]$. Define two functions as follows

$$x_1(t) = \varphi\big(t, \varphi(s, x_0)\big) \quad \text{and} \quad x_2(t) = \varphi(t+s, x_0).$$

By the definition of the function φ in equality (4.2.16), function x_1 is a solution of equation (4.2.4) with initial condition $x_1(0) = \varphi(s, x_0)$, i.e.,

$$\frac{dx_1(t)}{dt} = a\big(x_1(t)\big).$$

Now we claim that x_2 is also a solution of equation (4.2.4) because by the chain rule [39, Theorem 5.5] we have

$$\frac{dx_2(t)}{dt} = \frac{d}{dt}\big[\varphi(t+s, x_0)\big] = \varphi'(t+s, x_0) = a\big(\varphi(t+s, x_0)\big) = a\big(x_2(t)\big).$$

Moreover, $x_2(0) = \varphi(0+s, x_0) = \varphi(s, x_0)$. Hence we have proved that x_1 and x_2 are solutions to the same equation with the same initial condition. By the uniqueness part of Theorem 4.9 we deduce that

$$x_1(t) = x_2(t), \quad \text{for all } t.$$

By the definition of those functions we deduce that $\varphi\big(t, \varphi(s, x_0)\big) = \varphi(t+s, x_0)$. □

We can apply an appropriate result obtained from Corollary 4.16 to deduce a solution to the function u which exists in equation (4.2.10). However, before we formulate this solution, let us recall that $u(t,x_0) := u_0\big(\varphi(-t,x_0)\big)$ for every $x_0 \in [0,1]$ and $t \in [0,\infty)$. Also, let us introduce the following notation

$$[\pi_t(u_0)](x_0) := u(t,x_0),\ x_0 \in [0,1],\ \ t \geq 0, \tag{4.2.17}$$

whenever $u_0 : [0,1] \to \mathbb{R}$ is a continuous function such that $u_0(0) = 0$. Which means that, according to Theorem 4.15, the function $u_0 \in {}_0C([0,1]))$. In view of definition (4.2.17) of the function π_t and by Theorem 4.15, we deduce that $\pi_t(u_0)$ is also belongs to the space ${}_0C([0,1])$. In other words, we proved the following result.

Proposition 4.17. *If the function $u_0 \in {}_0C([0,1])$, then $\pi_t(u_0) \in {}_0C([0,1])$ for every $t \geq 0$.*

In the following theorem, we state some properties that function π satisfies.

Theorem 4.18. *Assume that the function $u_0 : [0,1] \to \mathbb{R}$ such that $u_0(0) = 0$. If $t,s \geq 0$, then the following holds*

$$\pi_{t+s}(u_0) = \pi_t\big(\pi_s(u_0)\big). \tag{4.2.18}$$

Proof of Theorem 4.18. Let us choose and fix $u_0 \in {}_0C([0,1])$ and $t,s \geq 0$. Before we embark on the proof, let us explain why equality (4.2.18) makes sense. For this aim, we observe that in view of Proposition 4.17 the LHS of equation (4.2.18), i.e., $\pi_{t+s}(u_0)$ belongs to ${}_0C([0,1])$. Similarly, $\pi_s(u_0)$ also belongs to ${}_0C([0,1])$ and hence, $\pi_t\big(\pi_s(u_0)\big)$ makes sense and belongs to ${}_0C([0,1])$. So, both sides of the equality (4.2.18) are elements of the space ${}_0C([0,1])$.
Next, let us also choose and fix $x_0 \in [0,1]$ and we need to show that the values of the LHS and the RHS of equality (4.2.18) at x_0 are equal. We start with the RHS of the equality (4.2.18) as follows

$$\begin{aligned}
\big[\pi_t(\pi_s)u_0\big](x_0) &= \pi_s u_0\big(\varphi(-t,x_0)\big) = u_0\big(\varphi(-s,\varphi(-t,x_0))\big) \\
&= u_0\big(\varphi(-s-t,x_0)\big) = u_0\big(\varphi(-(s+t),x_0)\big) \\
&= u(s+t,x_0) = \pi_{s+t}u_0(x_0).
\end{aligned}$$

Hence, by the arbitrariness of x_0 we infer that equality (4.2.18) is true and therefore, the proof of Theorem 4.18 is complete. □

We are now ready to formulate the main result of this current subsection.

Theorem 4.19. *The family $\{\pi_t\}_{t\geq 0}$, which defined by equation (4.2.17), is a C_0-semigroup of linear and bounded operators on the Banach space ${}_0C([0,1])$. In fact, it is a C_0-semigroup of contractions on ${}_0C([0,1])$.*

Proof of Theorem 4.19. Assume that $t \in [0, \infty)$. First of all, we would like to show that the map $\pi_t : {}_0C([0,1]) \to {}_0C([0,1])$ is a linear bounded map. For the linearity, let us choose two elements $u_0, v_0 \in {}_0C([0,1])$. Let u be the solution to equation (4.2.1) given by equation (4.2.10). Let v be also the solution to equation (4.2.1) but with the initial data u_0 replaced by v_0, given by an appropriate modification of the formula (4.2.10), i.e.

$$v(t, x_0) := v_0\big(\varphi(-t, x_0)\big), \;\; x_0 \in [0,1], \; t \in [0, \infty).$$

Note that $u_0 + v_0 \in {}_0C([0,1])$, because the space ${}_0C([0,1])$ is a vector space. Let z be a solution to equation (4.2.1) but with the initial data $u_0 + v_0$ given by an appropriate modification of the formula (4.2.10), i.e.,

$$z(t, x_0) := [u_0 + v_0]\big(\varphi(-t, x_0)\big) = u_0\big(\varphi(-t, x_0)\big) + v_0\big(\varphi(-t, x_0)\big), \;\; x_0 \in [0,1], \; t \in [0, \infty).$$

Therefore, by formula (4.2.17), for every $x_0 \in [0,1]$ we have

$$\begin{aligned}
[\pi_t(u_0 + v_0)](x_0) = z(t, x_0) &= u_0\big(\varphi(-t, x_0)\big) + v_0\big(\varphi(-t, x_0)\big) \\
&= u(t, x_0) + v(t, x_0) = [\pi_t(u_0)](x_0) + [\pi_t(v_0)](x_0) \\
&= [\pi_t(u_0) + \pi_t(v_0)](x_0).
\end{aligned}$$

Since x_0 is an arbitrary element of $[0,1]$, we infer that

$$[\pi_t(u_0 + v_0)] = [\pi_t(u_0) + \pi_t(v_0)].$$

Similarly, for every $x_0 \in [0,1]$ and $\alpha \in \mathbb{R}$ we have

$$[\pi_t(\alpha u_0)](x_0) = \alpha u(t, x_0) = \alpha[\pi_t(u_0)](x_0).$$

Again, because x_0 is an arbitrary element of $[0,1]$ we infer that $[\pi_t(\alpha u_0)] = \alpha[\pi_t(u_0)]$. Hence we proved that π_t is a linear map. For the boundedness, let us choose and fix $t \in [0, \infty)$. By using the definition (3.2.5) of the norm in the space ${}_0C([0,1])$ and the definition (4.2.17) of π_t along with (4.2.10) we have

$$\begin{aligned}
\|\pi_t u_0\|_{{}_0C([0,1])} &= \sup_{x_0 \in [0,1]} |\pi_t u_0(x_0)| \\
&\leq \sup_{x_0 \in [0,1]} \big|u_0\big(\varphi(-t, x_0)\big)\big| \leq \sup_{s \in [0,1]} |u_0(s)| = \|u_0\|.
\end{aligned} \tag{4.2.19}$$

The last inequality is a consequence of Corollary 4.12 and the following observation: if two bounded sets A and B are such that $A \subset B \subset \mathbb{R}$ then $\sup A \leq \sup B$. Therefore, it is bounded. Moreover, from equation (4.2.19) we infer that π_t is contraction, which means that

$$\|\pi_t\|_{\mathcal{L}\left({}_0C([0,1])\right)} \leq 1.$$

Now we need to verify the C_0-semigroup properties. We need to show that the following three properties are satisfied.

(i) $\pi_0 = I$, where I is the identity element in the space ${}_0C([0,1])$,

(ii) $\pi_t \pi_s = \pi_{t+s}$, if $t, s \geq 0$,

(iii) $\|\pi_t u_0 - u_0\|_{{}_0C([0,1])} \to 0$ as $t \to 0$, for every $u_0 \in {}_0C([0,1])$.

Let us begin with the observation that the second property has already been proven in Theorem 4.18 above. Regarding the first property, let $u_0 \in {}_0C([0,1])$ and $x_0 \in [0,1]$, then we have, $[\pi_0 u_0](x_0) = u(0, x_0) = u_0(x_0)$. Hence, $\pi_0 = I$. So the first property is satisfied. Finally, we prove the last property, we aim to prove the following. Let us choose $u_0 \in {}_0C([0,1])$ and $\eta > 0$. We want to find $\delta > 0$ such that

$$0 \leq t \leq \delta \implies \|\pi_t u_0 - u_0\|_{{}_0C([0,1])} \leq \eta.$$

By using the norm on the space ${}_0C([0,1])$ for every function u_0 we have

$$\|\pi_t u_0 - u_0\| = \sup_{x_0 \in [0,1]} |u(t, x_0) - u_0(x_0)| = \sup_{x_0 \in [0,1]} |u_0\big(\varphi(-t, x_0)\big) - u_0(x_0)|. \tag{4.2.20}$$

Note that the function $u_0 : [0,1] \to \mathbb{R}$ is continuous. Therefore, since the interval $[0,1]$ is compact, by [39, Theorem 4.19] we deduce that the function u_0 is uniformly continuous. Hence we can find $\varepsilon > 0$ such that if $x_1, x_2 \in [0,1]$ then

$$|x_1 - x_2| \leq \varepsilon \implies |u_0(x_1) - u_0(x_2)| \leq \eta.$$

Using Corollary 4.14 with this ε, we deduce that there exists $\delta > 0$ such that if $s \in [-\delta, 0]$ and $x_0 \in [0,1]$ then

$$|\varphi(s, x_0) - x_0| \leq \varepsilon.$$

Therefore, we take $t \in [0, \delta]$ and $x_0 \in [0,1]$ then we obtain $s = -t \in [-\delta, 0]$ and $\varphi(s, x_0) \in [0,1]$, by Corollary 4.12. Hence we have

$$|u_0\big(\varphi(-t, x_0)\big) - u_0(x_0)| = |u_0\big(\varphi(s, x_0)\big) - u_0(x_0)| \leq \eta.$$

Provided that $|\varphi(s, x_0) - x_0| \leq \varepsilon$. In other words, we proved the following:
If $\eta > 0$ then there exists $\delta > 0$ such that for $t \in [0, \delta]$ and $x_0 \in [0,1]$ we have

$$|u_0\big(\varphi(-t, x_0)\big) - u_0(x_0)| \leq \eta.$$

By taking the supremum for the above equation we obtain the following

$$\text{if } t \in [0,\delta] \implies \sup_{x_0 \in [0,1]} |u_0\big(\varphi(-t,x_0)\big) - u_0(x_0)| \leq \eta. \tag{4.2.21}$$

Hence by equations (4.2.20) and (4.2.21), we infer that

$$\text{if } t \in [0,\delta] \ \text{ then } \ \|\pi_t u_0 - u_0\|_{{}_0C([0,1])} \leq \eta.$$

Hence the proof of condition (iii) is complete. So we proved that the family $\{\pi_t\}_{t\geq 0}$ is a C_0-semigroup of linear and bounded operators on the space ${}_0C([0,1])$, which completes the proof of Theorem 4.19 □

Theorem 4.20. *The infinitesimal generator A of the C_0-semigroup $\{\pi_t\}_{t\geq 0}$ of linear and bounded operators on the space ${}_0C([0,1])$, see Theorem 4.19, is characterised by the following two equalities:*

$$\begin{aligned} D(A) = \Big\{ & u : [0,1] \to \mathbb{R} : u \textit{ is continuous}, u(0) = 0 \\ & u : (0,1] \to \mathbb{R} : u \textit{ is of } C^1\textit{-class, and } \lim_{x\to 0} a(x)u'(x) = 0 \Big\}, \end{aligned}$$

$$[Au](x) = \begin{cases} -a(x)Du(x), & \textit{if} \quad x \in (0,1],\, u \in D(A), \\ 0 & \textit{if} \quad x = 0, \end{cases}$$

The proof of this theorem can be done similarly to the proof of Proposition 2.106 and will be skipped. Theorem 4.20 is a generalisation of Proposition 2.107. Using Theorem 4.4 we can formulate the following result.

Theorem 4.21. *For every C^1-class function $u_0 : [0,1] \to \mathbb{R}$ satisfying $u_0(0) = 0$ and $\lim_{x\to 0} a(x)u'(x) = 0$, i.e., for every $u_0 \in D(A)$, where A is the infinitesimal generator of the C_0-semigroup $\{\pi_t\}_{t\geq 0}$ on the space $E = {}_0C([0,1])$, see Theorems 4.19 and 4.20, the function $u(t)$ defined by*

$$u(t) = \pi_t(u_0), \ \ t \geq 0$$

is the strong solution, in the sense of Definition 4.3, to the following problem

$$\frac{du}{dt} = Au(t), \quad u(0) = u_0. \tag{4.2.22}$$

Proof of Theorem 4.21. Follows from Theorems 4.4 and 4.20. □

Using a notion of a mild solution introduced in the next chapter we can formulate the following extension of the previous result.

Theorem 4.22. *For every continuous function* $u_0 : [0,1] \to \mathbb{R}$ *satisfying* $u_0(0) = 0$*, i.e., for every* $u_0 \in E = {}_0C([0,1])$*, the function* $u(t)$ *defined by*

$$u(t) = \pi_t(u_0), \quad t \geq 0$$

is the mild solution, in the sense of Definition 5.7, of the problem (4.2.22)*, where, as in Theorem 4.21, A is the infinitesimal generator of the* C_0*-semigroup* $\{\pi_t\}_{t\geq 0}$ *on the space E, see Theorems 4.19 and 4.20.*

Next we will present a notion of a classical solution to equation (4.2.1).

Definition 4.23. A classical solution to equation (4.2.1)-(4.2.2) where $u_0 : [0,1] \to \mathbb{R}$ is a continuous function such that $u_0(0) = 0$, is a continuous function

$$u : [0,\infty) \times [0,1] \to \mathbb{R}$$

which is of C^1-class in $(0,\infty) \times [0,1]$, such that $u(t,0) = 0$ for all $t \in [0,1)$, and initial condition (4.2.2) is satisfied and equation (4.2.1) is satisfied for every $(t,x) \in (0,\infty) \times [0,1]$. A similar definition can be given for an in-homogeneous problem, assuming only that the external force c is a continuous function

$$c : [0,\infty) \times [0,1] \to \mathbb{R}.$$

Recall that, the function $\pi_t(u_0)$ has been defined by equation (4.2.17). Since also $u(t,x_0) := u_0\big(\varphi(-t,x_0)\big)$ for every $x_0 \in [0,1]$ and $t \in [0,\infty)$, we can rewrite the definition of $\pi_t(u_0)$ as follows:

$$[\pi_t(u_0)](x_0) = u_0\big(\varphi(-t,x_0)\big), \ x_0 \in [0,1], \quad t \geq 0.$$

Using the formula (4.2.9) we deuce that

$$[\pi_t(u_0)](x_0) = u_0\big(G^{-1}\big(-t + G(x_0)\big)\big), \ x_0 \in [0,1], \quad t \geq 0.$$

Now we can define a function $u(t,x)$, for $(t,x) \in [0,\infty) \times [0,1]$ given by

$$u(t,x) := [\pi_t(u_0)](x) = u_0\big(G^{-1}\big(-t + G(x)\big)\big), \ x \in [0,1], \quad t \geq 0. \tag{4.2.23}$$

We can prove that if the function u_0 satisfies assumptions of the previous Theorem 4.21, then the function u defined by formula (4.2.23) is a classical solution of the PDE (4.2.1) in the classical sense, i.e.,

(i) u is of C^1-class,

(ii) $u(0,x) = u_0(x)$ for every $x \in [0,1]$;

(iii) equation (4.2.1) is satisfied for all $(t, x) \in [0, \infty) \times [0, 1]$.

4.2.1 Maps for finding properties for the solution

After we prove the existence and the uniqueness of the solution of the equation (4.2.1) defined in Section 4.2, we now find the properties of that solution. Therefore, in this section, we define a set of functions that lead us to those properties. Since we do not have an infinite set of notations, we reused a few notations repeatedly to define the functions between different spaces.

4.2.1.1 Maps Φ and $\hat{\Phi}$ if $c = 0$

To find the properties of the unique solution that we found in the previous section, see equation (4.2.11), we give and recall some important notations that we are going to use throughout the whole section.

- ${}_0C([0,1])$ is the space of all real-valued continuous functions such that $u(0) = 0$.
- $C_{b,uf}\big([0,\infty)\big)$ is the space of all real-valued bounded and uniformly continuous functions.
- $C\big([0,\infty)\big)$ the space of all real-valued continuous functions on the interval $[0,\infty)$.
- $C_0\big([0,\infty)\big)$ is the space of all real-valued continuous functions on $[0,\infty)$ vanishing at infinity, i.e.,
$$C_0[0,\infty) = \{u \in C\big([0,\infty)\big) : \lim_{t\to\infty} u(t) = 0\}. \tag{4.2.24}$$

Let $\{\pi_t\}_{t\geq 0}$ be a C_0-semigroup on the Banach space ${}_0C([0,1])$. We define the shift C_0-semigroup $\{T_t\}_{t\geq 0}$ on the space $C_{b,uf}\big([0,\infty)\big)$ or $C_0\big([0,\infty)\big)$ or $C\big([0,\infty)\big)$ by the following

$$\big(T_t g\big)(s) = g(t+s), \;\; s \in \mathbb{R}_+, t \geq 0. \tag{4.2.25}$$

Define the map $\Phi : {}_0C([0,1]) \to C\big([0,\infty)\big)$ by the following

$$[\Phi(v)](t) := (\pi_t v)(1), \;\; t \in [0,\infty), \;\; v \in {}_0C([0,1]). \tag{4.2.26}$$

First of all, we need to show that this map Φ is well defined and for this purpose, we need to observe that the following lemmata are true.

Lemma 4.24. *Let us assume that $\{\pi_t\}_{t\geq 0}$ is an arbitrary C_0-semigroup on the Banach space ${}_0C([0,1])$. If $v \in {}_0C([0,1])$ then $\Phi(v)$, defined by formula (4.2.26), belongs to the space $C\big([0,\infty)\big)$. In particular, the map $\Phi : {}_0C([0,1]) \to C\big([0,\infty)\big)$ is well defined.*

Proof of Lemma 4.24. Since $v \in {}_0C([0,1])$ and by Theorem 4.19 the family $\{\pi_t\}_{t\geq 0}$ is a C_0-semigroup on the space ${}_0C([0,1])$, then by [33, Corollary 2.3], see also Corollary 2.76, the function

$$\Pi v := \{[0,\infty) \ni t \mapsto \pi_t v \in {}_0C([0,1])\}$$

is also continuous. Let us recall that Πv is the trajectory of the semigroup $(\pi_t)_{t\geq 0}$ starting at v. Define an evaluation map at 1 by

$$e_1 : {}_0C\big([0,1],\mathbb{R}\big) \ni v \mapsto v(1) \in \mathbb{R}.$$

We need to show that the map e_1 is linear and bounded. The proof of linearity is straightforward. For the boundedness, starting with the left-hand side we have $\|e_1(v)\|_{\mathbb{R}} = |v(1)| = \|v\|$. On the other hand, for the RHS we know the norm on space ${}_0C([0,1])$ is given by

$$\|v\|_{{}_0C([0,1])} = \sup_{x\in[0,1]} |v(x)| \geq |v(1)|.$$

Hence

$$\|e_1(v)\| = |v(1)| \leq \|v\|. \tag{4.2.27}$$

So the map e_1 is linear and bounded and hence it is continuous. Moreover, we proved that e_1 is a contraction. Hence, the proof of the Lemma is concluded by observing that

$$\Phi(v) = e_1 \circ \Pi v,$$

so that $\Phi(v)$, as a composition of two continuous functions is a continuous function from $[0,\infty$ to the set $\mathbb{R}$. In other words, $\phi(v) \in C([0,\infty))$ as claimed. $\square$

Note that the above Lemma is an abstract result because it is true for any C_0-semigroup. The next result strengthens the previous one. We formulate these abstract results because they might be used for different equations, e.g, Burger equation (Burgers' equation is a fundamental PDE occurring in various areas of applied mathematics, such as fluid mechanics).

Lemma 4.25. *Let us assume that $\{\pi_t\}_{t\geq 0}$ is an arbitrary uniformly bounded C_0-semigroup on the Banach space ${}_0C([0,1])$. If $v \in {}_0C([0,1])$ then $\Phi(v)$ defined by formula* (4.2.26) *belongs to the space $C\big([0,\infty)\big)$. If $v \in {}_0C([0,1])$ then $\Phi(v)$ belongs to the space $C_{b,uf}\big([0,\infty)\big)$.*

Proof of Lemma 4.25. Let us choose and fix an element $v \in {}_0C([0,1])$. In the previous Lemma, we proved that $\Phi(v)$ belongs to the space $C\big([0,\infty)\big)$. So, now we need to prove that $\Phi(v)$ is bounded and uniformly continuous. Let us begin with the proof of boundedness. By inequalities (4.2.27) and (4.2.19) we get

$$|\Phi(v)(t)| = |e_1\big(\pi_t(v)\big)| \le \|\big(\pi_t(v)\big)\|_{{}_0C([0,1])} \le \|v\|_{{}_0C([0,1])}. \tag{4.2.28}$$

So, we proved that the function $\Phi(v)$ is bounded. Next, we need to prove that it is uniformly continuous. For this aim, let us choose $0 \le t_1 < t_2 < \infty$ and consider the following

$$\begin{aligned}|\Phi(v)(t_2) - \Phi(v)(t_1)| &= |e_1\big(\pi_{t_2}(v)\big) - e_1\big(\pi_{t_1}(v)\big)| = |e_1\big(\pi_{t_2}(v) - \pi_{t_1}(v)\big)| \\ &\le |\pi_{t_2}(v) - \pi_{t_1}(v)|_{{}_0C([0,1])} = |\pi_{t_1}\big(\pi_{t_2-t_1}v - v\big)|_{{}_0C([0,1])} \\ &\le |\pi_{t_2-t_1}v - v|_{{}_0C([0,1])}.\end{aligned}$$

Let us take an arbitrary $\varepsilon > 0$. Then by the C_0-continuity of the semigroup $\{\pi_t\}_{t\ge 0}$ we can find $\delta > 0$ such that

$$\text{if } s \in [0,\delta] \text{ then } |\pi_s v - v|_{{}_0C([0,1])} \le \varepsilon.$$

Therefore, if $0 \le t_2 - t_1 \le \delta$ then we infer that

$$|\Phi(v_2)(t) - \Phi(v)(t_1)| \le |\pi_{t_2-t_1}v - v|_{{}_0C([0,1])} \le \varepsilon.$$

This completes the proof of Lemma 4.25. □

Note that we have used contractivity of the semigroup $\{\pi_t\}_{t\ge 0}$ on the space ${}_0C([0,1])$ many times. We also can strengthen the previous results by proving the following Proposition.

Proposition 4.26. *If $v \in {}_0C([0,1])$ then $\Phi(v)$ belongs to the space $C_0\big([0,\infty)\big)$.*

Proof of Proposition 4.26. Let us choose and fix an arbitrary function $v \in {}_0C([0,1])$. Then, because $G(1) = 0$, we write the definition of map Φ as follows

$$[\Phi(v)](t) = [\pi_t(v)](1) = v\big(G^{-1}(-t)\big), \quad t \in [0,\infty). \tag{4.2.29}$$

We have already proved in Lemma 4.24 that the function $\Phi(v)$ is continuous and thus, to prove that it belongs to $C_0\big([0,\infty)\big)$ we need to prove that

$$\lim_{t\to\infty}[\Phi(v)](t) = 0.$$

By using formula (4.2.29) it follows that to prove the last equality it is sufficient to show that

$$\lim_{t\to\infty} v\big(G^{-1}(-t)\big) = 0. \tag{4.2.30}$$

But we have

$$\lim_{t\to\infty} -t = -\infty.$$

On the other hand, by the properties of the function G we have

$$\lim_{x\to-\infty} G^{-1}(x) = 0.$$

Finally, because $v \in {}_0C([0,1])$,

$$\lim_{s\to 0^+} v(s) = 0.$$

According to [3, Theorem 8.17], by combining the last three equalities we deduce that the identity (4.2.30) as required. $\square$

Proposition 4.27. *The space $C_0[0,\infty)$ is a closed subspace of the Banach space $C_{b,uf}\big([0,\infty)\big)$ and hence it is also a Banach space with the norm induced by the norm from the latter space.*

Claim 4.28. Let $\{T_t\}_{t\geq 0}$ be the shift C_0-semigroup on the space $C_{b,uf}\big([0,\infty)\big)$ or the space $C\big([0,\infty)\big)$ and $\{\pi_t\}_{t\geq 0}$ be an arbitrary uniformly bounded C_0-semigroup on the space ${}_0C([0,1])$, then for every $t \geq 0$ the following equality satisfies

$$T_t \circ \Phi = \Phi \circ \pi_t. \tag{4.2.31}$$

Proof of Claim 4.28. We need first to prove that both sides of the above equality (4.2.31) make sense. It is sufficient to consider only the space $C_{b,uf}\big([0,\infty)\big)$. For this purpose, we recall that

$$\Phi : {}_0C([0,1]) \to C_{b,uf}\big([0,\infty)\big) \text{ and } T_t : C_{b,uf}\big([0,\infty)\big) \to C_{b,uf}\big([0,\infty)\big)$$

Therefore, the composition

$$T_t \circ \Phi : {}_0C([0,1]) \to C_{b,uf}\big([0,\infty)\big).$$

is well-defined. On the other hand,

$$\pi_t : {}_0C([0,1]) \to {}_0C([0,1]) \text{ and } \Phi : {}_0C([0,1]) \to C_{b,uf}\big([0,\infty)\big).$$

Therefore, the composition

$$\Phi \circ \pi_t : {}_0C([0,1]) \to C_{b,uf}\big([0,\infty)\big).$$

Hence, the equality (4.2.31) makes sense. It remains to prove it is true. For this purpose, let us choose and fix $v \in {}_0C([0,1])$ and $t \geq 0$. Then, using properties and definitions stated earlier, we have for every $s \in \mathbb{R}$,

$$\begin{aligned}[\mathrm{LHS}(v)](s) &= \big[T_t\big(\Phi(v)\big)\big](s) \\ &= \big[\Phi(v)\big](t+s) \\ &= \big(\pi_{t+s}v\big)(1) = [\pi_s(\pi_t v)](1) \\ &= \big[\Phi(\pi_t v)\big](s) = \big[\mathrm{RHS}(v)\big](s).\end{aligned}$$

This completes the proof of claim 4.28. □

Let us emphasise that in Claim 4.28 $\{T_t\}_{t\geq 0}$ is the shift semigroup while $\{\pi_t\}_{t\geq 0}$ is an arbitrary C_0-semigroup on the space ${}_0C([0,1])$. It follows from the previous Proposition 4.26 that the function Φ is not only well-defined as a function from the space ${}_0C([0,1])$ to the space $C_{b,uf}\big([0,\infty)\big)$ but also is well defined as a function from the space ${}_0C([0,1])$ to the space $C_0\big([0,\infty)\big)$. This new object we will denote by $\hat{\Phi}$. Let us observe that in view of equality (4.2.29) the new map $\hat{\Phi}$ satisfies the following

$$\hat{\Phi} : {}_0C([0,1]) \to C_0\big([0,\infty)\big) \tag{4.2.32}$$

$$[\hat{\Phi}(v)](t) = \big(\pi_t v\big)(1) = v\big(G^{-1}(-t)\big), \;\; t \in [0,\infty). \tag{4.2.33}$$

The next result list some fundamental properties of the latter function $\hat{\Phi}$.

Lemma 4.29. *The map $\hat{\Phi}$ introduced above in identity* (4.2.32) *is well-defined and also it is linear and bounded. In particular, it is continuous.*

Proof of Lemma 4.29. The proof of the linearity of the map $\hat{\Phi}$ is standard. For boundedness, let us recall that the space $C_0\big([0,\infty)\big)$ is endowed with the supremum norm. Therefore, by inequality (4.2.28), we have for $v \in {}_0C([0,1])$,

$$\|\hat{\Phi}(v)\|_{C_0\left([0,\infty)\right)} = \sup_{t\geq 0} \|\hat{\Phi}(v)(t)\|_{{}_0C([0,1])} \leq \|v\|_{{}_0C([0,1])}.$$

This proves that $\hat{\Phi}$ is bounded, in particular, it is a contraction map. □

Remark 4.30. Let us observe that Lemma 4.29 remains true for the map

$$\Phi : {}_0C([0,1]) \to C_{b,uf}\big([0,\infty)\big)$$

with practically the same proof.

Lemma 4.31. *The family $\{T_t\}_{t\geq 0}$ is a C_0-semigroup of contractions on the space $C_{b,uf}\big([0,\infty)\big)$. Moreover, the infinitesimal generator B of this semigroup satisfies the following*

$$\begin{aligned} D(B) &= \big\{g \in C_{b,uf}\big([0,\infty)\big) : g \text{ is of } C^1\text{-class and } g' \in C_{b,uf}\big([0,\infty)\big)\big\} \\ B(g) &= g', \quad g \in D(B). \end{aligned}$$

See Example 1 in section IX.2 and Example 1 in section IX.5 in the monograph [49].

Remark 4.32. Let us observe that the Claim 4.28 remains true for the map

$$\hat{\Phi} : {}_0C([0,1]) \to C_0\big([0,\infty)\big)$$

with practically the same proof.

4.2.1.2 Inverses of maps Φ and $\hat{\Phi}$

In this subsection we denote the new maps by $Q = \Phi^{-1}$ ($\hat{Q} = \hat{\Phi}^{-1}$ respectively), where Φ^{-1} and $\hat{\Phi}^{-1}$ are the inverse of the maps Φ and $\hat{\Phi}$ respectively defined before. Define a map Q as the following

$$\begin{gathered} Q : C_{b,uf}\big([0,\infty)\big) \to {}_0C([0,1]) \\ [Q(\psi)](x_0) := \psi\big(-G(x_0)\big), \quad \text{where} \quad \psi \in C_{b,uf}([0,\infty)),\ x_0 \in (0,1]. \end{gathered}$$

Note that $Q(\psi)$ is only defined for $x_0 \in (0,1]$ because $0 \notin Dom(G)$.
Also, define the map $\hat{Q}$ by the following

$$\hat{Q} : C_0\big([0,\infty)\big) \to {}_0C([0,1]), \tag{4.2.34}$$

$$[\hat{Q}(\psi)](x_0) := \psi\big(-G(x_0)\big), \quad \text{if } x_0 \in (0,1],\ \psi \in C_0([0,\infty)). \tag{4.2.35}$$

Note that, so far $\hat{Q}(\psi)$ is only defined for $x_0 \in (0,1]$ because $0 \notin Dom(G)$. However, in contrast to the case of map Q, we can additionally define

$$[\hat{Q}(\psi)](0) = 0, \quad \text{if} \quad \psi \in C_0([0,\infty)). \tag{4.2.36}$$

This works, because $\psi \in C_0([0,\infty),\mathbb{R})$ so that $\lim_{t\to\infty}\psi(t) = 0$, contrary to the previous case. This is made more precise in the proof of Proposition 4.33 below. In the following, we need to study some properties of maps Q and $\hat{Q}$.

Proposition 4.33. *Suppose $\psi \in C_0([0,\infty))$ then the map $\hat{Q}\psi$ defined by* (4.2.35) *and* (4.2.36) *is continuous, i.e., $\hat{Q}\psi \in {}_0C([0,1])$. In particular, map $\hat{Q}$ defined by* (4.2.34) *is well defined.*

Proof of Proposition 4.33. Let us choose and fix $\psi \in C_0([0,\infty))$. Our aim is to prove that $\hat{Q}\psi \in {}_0C([0,1])$. To verify this aim, we need to prove two statements. First, we need to prove that $\hat{Q}\psi$ is continuous at x_0 if $x_0 \in (0,1]$. Second, we need to prove that $\hat{Q}\psi$ is continuous at 0. Regarding the first statement, the function $-G : (0,1] \to [0,\infty)$ is continuous and the function $\psi : [0,\infty) \to \mathbb{R}$ is continuous. Hence, the composition $\psi \circ (-G) : (0,1] \to \mathbb{R}$ is also continuous. For the second statement, we need to show that

$$\lim_{x_0 \to 0^+} [\hat{Q}\psi](x_0) = [\hat{Q}\psi](0). \tag{4.2.37}$$

Using the definition of function $\hat{Q}\psi$ we have

$$\lim_{x_0 \to 0^+} [\hat{Q}\psi](x_0) = \lim_{x_0 \to 0^+} \psi(-G(x_0)) \tag{4.2.38}$$

We know from the properties of function $-G(x_0)$ that $\lim_{x \to 0^+}[-G(x_0)] = \infty$. Now in equation (4.2.38) we put $-G(x_0) = t$ and we need to find $\lim_{t\to\infty} \psi(t)$. We observe that if ψ only belongs to the space $C_{b,uf}\big([0,\infty)\big)$ then such a limit may not exists. This means that the space we used $C_{b,uf}\big([0,\infty)\big)$ has to be replaced by the space $C_0([0,\infty))$ which defined before by equality (4.2.24). Thus, since our function $\psi \in C_0([0,\infty))$, then the $\lim_{t\to\infty} \psi(t)$ not only exists but also is equal to 0. Therefore, by [3, Theorem 8.17] about the limit of the composition of functions, we infer that

$$\lim_{x_0 \to 0^+} \psi(-G(x_0)) = \lim_{t \to \lim_{x_0 \to 0^+}(-G(x_0))} \psi(t) = \lim_{t\to\infty} \psi(t) = 0.$$

So we proved that

$$\lim_{x_0 \to 0^+} [\hat{Q}\psi](x_0) = 0.$$

On the other hand, from the equation (4.2.36) we have $[\hat{Q}\psi](0) = 0$ which implies that equality (4.2.37). This together with $0 = [\hat{Q}\psi](0)$ implies that $\hat{Q}\psi \in {}_0C([0,1])$. $\square$

Proposition 4.34. *The map $\hat{Q}$ defined by* (4.2.34) *is not only well defined but also linear and bounded (from $C_0([0,\infty))$ to ${}_0C([0,1])$).*

Proof of Proposition 4.34. Recall that for $v \in C_0([0,\infty))$,

$$\big(\hat{Q}v\big)(x) = \begin{cases} v(-G(x)), & \text{if } x \in (0,1], \\ 0, & \text{if } x = 0. \end{cases}$$

The proof of the linearity is straightforward. To prove the boundedness of the map $\hat{Q}$, let $v \in C_0([0,\infty))$. Since $\hat{Q}v(0) = 0$, by using the definitions of the norms in the spaces

$C_0([0,\infty))$ and ${}_0C([0,1])$, we get

$$\begin{aligned}\|\hat{Q}v\|_{{}_0C([0,1])} &= \sup_{x\in[0,1]} |[\hat{Q}v](x)| = \sup_{x\in(0,1]} |[\hat{Q}v](x)| \\ &= \sup_{x\in(0.1]} |v\big(-G(x)\big)| \le \sup_{t\in[0,\infty)} |v(t)| = \|v\|_{C_0([0,\infty))}.\end{aligned}$$

where the last inequality is a consequence of proved earlier Proposition 4.8, that the function $-G$ maps bijectively $(0,1]$ into interval $[0,\infty)$. Therefore, the proof Proposition 4.34 is complete. □

Proposition 4.35. *Let $\hat{\Phi}$ be the map as in formula* (4.2.32) *and $\hat{Q}$ be the map defined on equation* (4.2.35)*. Then we have $\hat{Q}\circ\hat{\Phi} = \mathrm{id}$ on ${}_0C([0,1])$ and $\hat{\Phi}\circ\hat{Q} = \mathrm{id}$ on $C_0([0,\infty))$. In particular,*

$$\hat{\Phi}^{-1} = \hat{Q} \text{ and } \hat{Q}^{-1} = \hat{\Phi}.$$

Proof of Proposition 4.35. To prove this Proposition, we identify that there are two parts. The first part is related to $\hat{Q}\circ\hat{\Phi} = \mathrm{id}$ on ${}_0C([0,1])$, and the second part is related to $\hat{\Phi}\circ\hat{Q} = \mathrm{id}$ on $C_0([0,\infty))$. However, before we proceed with the proof, we state that the compositions $\hat{Q}\circ\hat{\Phi} : {}_0C([0,1]) \to {}_0C([0,1])$ and $\hat{\Phi}\circ\hat{Q} : C_0([0,\infty)) \to C_0([0,\infty))$ are well defined by Lemma 4.29 and Proposition 4.33.
Now we start with the first part, we need to prove that $\hat{Q}\circ\hat{\Phi} = \mathrm{id}$ on ${}_0C([0,1])$, i.e.,

$$[\hat{Q}\circ\hat{\Phi}](v) = v, \quad \text{for every } v \in {}_0C([0,1]).$$

For this purpose, let us choose and fix an arbitrary element $v \in {}_0C([0,1])$. We need to prove the following equality hold

$$[\hat{Q}\circ\hat{\Phi}](v) = v.$$

Let us notice that both sides of the above equality are functions belonging to the space ${}_0C([0,1])$. In particular, $[\hat{Q}\circ\hat{\Phi}](v)(0) = 0$ and $v(0) = 0$. Therefore, it is enough to prove the following

$$[\hat{Q}\circ\hat{\Phi}](v)(x_0) = v(x_0) \text{ for every } x_0 \in (0,1].$$

For this purpose, we choose and fix $x_0 \in (0,1]$. Then by the definition (4.2.35) of the map $\hat{Q}$ followed by the identity (4.2.33) satisfied by $\hat{\Phi}$, we have

$$\begin{aligned}\big[(\hat{Q}\circ\hat{\Phi})(v)\big](x_0) &= \hat{Q}[\hat{\Phi}(v)](x_0) = [\hat{\Phi}(v)](-G(x_0)) \\ &= v\big(G^{-1}(-(-G(x_0)))\big) = v\big(G^{-1}(G(x_0))\big) = v(x_0).\end{aligned}$$

This concludes the first part of the proof. For the second part, similarly to the previous part, we need to show that $\hat{\Phi} \circ \hat{Q} = \mathrm{id}$ on the space $C_0([0,\infty),\mathbb{R})$. That means,

$$[\hat{\Phi} \circ \hat{Q}](v) = v, \quad \text{for every } v \in C_0\big([0,\infty)\big). \tag{4.2.39}$$

We notice that both sides of the above equality (4.2.39) are functions belonging to the space $C_0([0,\infty))$. So it is enough to prove that

$$\big[\big(\hat{\Phi} \circ \hat{Q}\big)v(t)\big](x_0) = v(x_0) \;\text{ for every } x_0 \in (0,1].$$

For this purpose, let us choose and fix an arbitrary element $v \in C_0([0,\infty))$. Then for every $x_0 \in (0,1]$ we have

$$\begin{aligned}\big[\big(\hat{\Phi} \circ \hat{Q}\big)v(t)\big](x_0) &= \hat{\Phi}\big[\hat{Q}v\big](x_0) = \hat{Q}v(G^{-1}(-x_0)) \\ &= v\big(-G(G^{-1}(-x_0))\big) = v(x_0).\end{aligned}$$

This concludes the second part of the proof and therefore, the proof of Proposition 4.35 is complete. □

The following result is an obvious consequence of Proposition 4.35.

Corollary 4.36. *The map $\hat{\Phi}$ is bijection between ${}_0C([0,1])$ and $C_0([0,\infty))$.*

Let us observe that Claim 4.28 can also be generalised to the map $\hat{\Phi}$ as follows.

Claim 4.37. In the above framework, for every $t \geq 0$,

$$\hat{T}_t \circ \hat{\Phi} = \hat{\Phi} \circ \pi_t, \tag{4.2.40}$$

in the space E, where $\{\hat{T}_t\}_{t\geq 0}$ is the shift semigroup on the space $C_0([0,\infty))$ defined by the following version of identity (4.2.25).

$$\big(\hat{T}_t g\big)(s) = g(t+s), \;\; g \in C_0\big([0,\infty)\big), \;\; s \in \mathbb{R}_+, t \geq 0. \tag{4.2.41}$$

Proof of Claim 4.37. We divide the proof of this claim into two parts. In the first part, we start with the observation that by Proposition 4.27 the space $C_0([0,\infty))$ is a closed subspace of $C_{b,uf}([0,\infty))$. Moreover, for every $t \in [0,\infty)$, $\hat{T}_t$ maps the space $C_0([0,\infty))$ into itself and the family $\{\hat{T}_t\}_{t\geq 0}$ is a restriction of the C_0-semigroup $\{T_t\}_{t\geq 0}$ defined by formula (4.2.25) from $C_{b,uf}([0,\infty))$ to $C_0([0,\infty))$. Since by Lemma 4.31, the family $\{T_t\}_{t\geq 0}$ is a C_0-semigroup of contractions on the space $C_{b,uf}([0,\infty))$, we deduce that the family $\{\hat{T}_t\}_{t\geq 0}$ is also a C_0-semigroup of contractions on the space $C_0([0,\infty))$.

In the second part of the proof, we first need to show that both sides of the equality (4.2.40) make sense. For the left-hand-side, we recall that

$$\hat{\Phi} : {}_0C([0,1]) \to C_0([0,\infty)) \text{ and } \hat{T}_t : C_0([0,\infty)) \to C_0([0,\infty)),$$

so that the composition of those maps satisfies

$$\hat{T}_t \circ \hat{\Phi} : {}_0C([0,1]) \to C_0([0,\infty)).$$

For the right-hand-side, we also recall that

$$\pi_t : {}_0C([0,1]) \to {}_0C([0,1]) \text{ and } \hat{\Phi} : {}_0C([0,1]) \to C_0([0,\infty))$$

so similarly, the composition of both maps $\hat{\Phi}$ and π_t satisfies

$$\hat{\Phi} \circ \pi_t : {}_0C([0,1]) \to C_0([0,\infty)).$$

So, we proved that both sides of the equality (4.2.40) make sense. Next, we want to prove that equality (4.2.40) holds. For this aim let us choose and fix $v \in {}_0C([0,1])$ and $t \geq 0$. Then, by using properties and definitions of the maps $\hat{\Phi}$, $\hat{T}_t$ and π_t we have, for every $s \in \mathbb{R}$,

$$\begin{aligned}
[(\hat{T}_t \circ \hat{\Phi})(v)](s) &= \big[\hat{T}_t(\hat{\Phi}(v))\big](s) = \big[\hat{\Phi}(v)\big](t+s) \\
&= \big(\pi_{t+s}v\big)(1) = [\pi_s(\pi_t v)](1) \\
&= \big[\hat{\Phi}(\pi_t v)\big](s) = \big[\hat{\Phi} \circ \pi_t(v)\big](s).
\end{aligned}$$

Hence, the proof of equality (4.2.40) holds, and thus the proof of the Claim 4.37 is complete. □

4.2.1.3 Maps Φ and Q if $c \neq 0$

In this section we study equation (4.0.4) which is a special case of equation studied by Rudnicki [43]. We know that for $\lambda = 0$ equation (4.0.4) generates a C_0-semigroup $\{\pi_t\}_{t\geq 0}$ on the Banach space ${}_0C([0,1])$. The infinitesimal generator of this semigroup $\{\pi_t\}_{t\geq 0}$ denoted by A, see Theorem 4.20. By Theorem 2.88, the operator $A + \lambda I$ is an infinitesimal generator of a C_0-semigroup on the space ${}_0C([0,1])$. This semigroup is related to the linear part of equation (4.0.4) if $c = \lambda u$, for $\lambda > 0$, denoted by $\{S_t\}_{t\geq 0}$ and defined by the following formula

$$\big[S_t u\big](x) = e^{\lambda t}\big(\pi_t u\big)(x), \quad t \geq 0, x \in [0,1].$$

Given a parameter $\gamma \geq 0$ we introduce a Banach space $Y = Y_\gamma$ defined by the following

$$Y = Y_\gamma := \big\{ v \in C\big([0,\infty)\big) : \sup_{s\in[0,\infty)} |v(s)|e^{-\gamma s} < \infty \big\} \tag{4.2.42}$$

with a norm defined by

$$\|v\|_Y := \sup_{s\in[0,\infty)} |v(s)|e^{-\gamma s}, \; v \in Y.$$

Note also that Y_0 is equal to the space of all continuous and bounded functions $v : [0,\infty) \to \mathbb{R}$.

Let $\{T_t\}_{t\geq 0}$ be a family of linear operators on the space Y defined by

$$\big(T_t g\big)(s) := g(t+s), \;\; t, s \geq 0, v \in Y. \tag{4.2.43}$$

Proposition 4.38. *The space defined by formula* (4.2.42) *is a separable Banach space.*

A standard proof of Proposition 4.38 is omitted. The proof of the next proposition is also standard and hence also omitted.

Proposition 4.39. *Let Y be the separable Banach space defined in formula* (4.2.42)*. Let $\{T_t\}_{t\geq 0}$ be the shift semigroup defined by formula* (4.2.43) *on the space Y. Then the family of linear operators $\{T_t\}_{t\geq 0}$ is a measurable semigroup on the measurable space* $(Y, \mathcal{B}(Y))$*.*

The next result is a generalisation of our previous Claim 4.37. This is an abstract result in the sense that the semigroup $\{S_t\}_{t\geq 0}$ is an arbitrary semigroup. The following corollary will be valid for our concrete semigroup that is defined by the formula (4.2.47).

Proposition 4.40. *Let $\{S_t\}_{t\geq 0}$ be a C_0-semigroup of bounded linear operators on the Banach space $E = {}_0C([0,1])$ such that*

$$\|S_t\|_{\mathcal{L}(E)} \leq Me^{\lambda t}, \quad \textit{for all } t \geq 0, \tag{4.2.44}$$

where $M \geq 1$ and $\lambda \geq 0$ are some constants[1]*. Assume that $\gamma \geq \lambda$ and let $Y = Y_\gamma$ be the Banach space defined earlier in formula* (4.2.42)*. Then the map $\Phi : E \to Y$ defined by*

$$\big(\Phi v\big)(t) = \big(S_t v\big)(1), \;\; t \in [0,\infty), \tag{4.2.45}$$

is a well-defined linear and bounded map from the space E to the space Y. Moreover, the following equality is satisfies

$$\Phi \circ S_t = T_t \circ \Phi \;\; \textit{on} \;\; E \;\; \textit{for every} \;\; t \geq 0. \tag{4.2.46}$$

[1]According to [33, Theorem I.2.2] such constants always exist.

Proof of Proposition 4.40. We divide the proof into four steps.

1. The map Φ is well defined, i.e., if $v \in E$ then $\Phi v \in Y$.
2. The map Φ is linear and bounded from the space E to the space Y.
3. Verify that the equality (4.2.46) makes sense.
4. Verify that the equality (4.2.46) holds true.

Regarding the first step, let us denote by e_1 the evaluation map at position $x = 1$, i.e.,

$$e_1 : E \ni v \mapsto v(1) \in \mathbb{R}.$$

Note that e_1 is a linear contraction and hence a continuous function. Since

$$(\Phi v)(t) = e_1(S_t v), \quad \text{for every } v \in E \text{ and } t \in [0, \infty)$$

and, by [33, Corollary 2.3], the map $[0, \infty) \ni t \mapsto S_t v \in E$ is continuous for every $v \in E$. Therefore, we deduce that for every $v \in E$,

$$\Phi v \in C\big([0, \infty)\big).$$

In order to complete the proof that $\Phi v \in Y$ we only need to check the growth condition. From the definition of the map Φ and the condition (4.2.44), we infer that for any $C > 0$

$$|\Phi v| \leq C e^{\lambda s}.$$

Since by assumptions $\gamma \geq \lambda$, we infer that Φv indeed belongs to the space Y.
Regarding the second step, we need to show that Φ is a linear and bounded map. For the linearity, let $u, v \in E$ and $\alpha, \beta \in \mathbb{R}$. Then we have

$$\Phi[\alpha u + \beta v](t) = S_t(\alpha u + \beta v)(1) = \alpha S_t u(1) + \beta S_t v(1) = \alpha \Phi u + \beta \Phi v.$$

Thus Φ is linear. To prove the boundedness of Φ, let us fix $v \in E$. Then we have

$$\begin{aligned} |\Phi v|_Y &= \sup_{t \geq 0} |\Phi v(t)| e^{-\gamma t} = \sup_{t \geq 0} |[S_t v](1)| e^{-\gamma t} \leq \sup_{t \geq 0} \sup_{x \in [0,1]} |[S_t v](x)| e^{-\gamma t} \\ &= \sup_{t \geq 0} |S_t v|_E e^{-\gamma t} \leq M \sup_{t \geq 0} |v|_E e^{-\gamma t} e^{\lambda t} = M |v|_E \sup_{t \geq 0} e^{(\lambda - \gamma)t} \leq M |v|_E. \end{aligned}$$

Hence, we found that Φ is a linear bounded map.
Regarding the third step, we take $v \in {}_0C([0,1])$. Then $S_t v \in {}_0C([0,1])$ because S_t maps the space ${}_0C([0,1])$ into itself. Thus, by step 1, we infer that $\Phi(S_t v) \in Y$.
Regarding the fourth step (4), let us choose and fix $v \in {}_0C([0,1])$ and $t \geq 0$. Then, by

using properties and definition (4.2.45) of the map Φ and the family $\{T_t\}_{t\geq 0}$, and the semigroup property of the family $\{S_t\}_{t\geq 0}$, we have for every $s \in [0,\infty)$ the following equalities hold.

$$\begin{aligned}[(T_t \circ \Phi)(v)](s) &= \big[T_t(\Phi(v))\big](s) = \big[\Phi(v)\big](t+s)\\ &= \big(S_{t+s}v\big)(1) = [S_s(S_t v)](1)\\ &= \big[\Phi(S_t v)\big](s) = \big[\Phi \circ S_t(v)\big](s).\end{aligned}$$

This concludes the proof of equality (4.2.46). Thus the proof of Proposition 4.40 is complete. $\square$

Corollary 4.41. *Assume that $\lambda > 0$. Let $\{S_t\}_{t\geq 0}$ be the C_0-semigroup of bounded linear operators on the Banach space $E = {}_0C([0,1])$ defined by*

$$\big(S_t v\big)(x) = e^{\lambda t}\pi_t v(x) = e^{\lambda t} v\big(G^{-1}(-t+G(x))\big), \;\; x \in (0,1]. \tag{4.2.47}$$

Assume also that $\gamma \geq \lambda$ and $Y = Y_\gamma$ is the Banach space defined earlier in formula (4.2.42) *with this parameter γ. Let also Φ be a map defined by*

$$\begin{aligned}&\Phi : E \to Y,\\ &\big(\Phi v\big)(t) = \big(S_t v\big)(1) = e^{\lambda t}\pi_t v(1) = e^{\lambda t} v\big(G^{-1}(-t)\big).\end{aligned} \tag{4.2.48}$$

Then all the assertions of Proposition 4.40 hold, i.e., the map Φ is a well-defined linear and bounded (hence continuous) map from the space E to the space Y and the equality (4.2.46) *is satisfied. Moreover, Φ is injective.*[2]

Proof of Corollary 4.41. The proof of almost all parts of this Corollary follows from Proposition 4.40. However, the injectivity of the map Φ is true only in our special semigroup that is defined by the formula (4.2.47). To prove Φ is an injective map, we need to show that

$$\text{if } \; v_1, v_2 \in E \text{ and } \; \Phi(v_1) = \Phi(v_2) \text{ then } v_1 = v_2.$$

Since $\Phi(v_1) = \Phi(v_2)$ then for every $t \geq 0$ we have $\Phi(v_1)(t) = \Phi(v_2)(t)$ which means that for every $t \geq 0$

$$S_t(v_1)(1) = S_t(v_2)(1).$$

Since S_t is linear then we get $\big[S_t(v_1 - v_2)\big](1) = 0$. If we denote $v = v_1 - v_2$, then $\big[S_t(v)\big](1) = 0$. By using the definition of S_t in formula (4.2.48) we obtain the following

$$\big[S_t v\big](1) = e^{\lambda t}\pi_t v(1) = e^{\lambda t} v\big(G^{-1}(-t)\big) = 0, \; t \geq 0.$$

[2]Note that injectivity of Φ is not always true and we didn't assert it in the previous Proposition 4.40.

Hence we have, for every $t \geq 0$,

$$v\big(G^{-1}(-t)\big) = 0. \tag{4.2.49}$$

From the properties of function G in Proposition 4.8 the function G^{-1} maps bijectively the interval $(-\infty, 0]$ onto the interval $(0, 1]$. Hence, we deduce that for every $x \in (0, 1]$, there exists $s \in (-\infty, 0]$ such that $G^{-1}(s) = x$. If we denote s by $-t$, we have for every $x \in (0, 1]$ there exists $t \in (-\infty, 0]$ such that

$$G^{-1}(-t) = x.$$

By substituting the above equation into equation (4.2.49) we get the following

$$v\big(G^{-1}(-t)\big) = v(x) = 0, \text{ for every } x \in (0, 1]$$

Hence, we proved that $v = 0$ which implies that $v_1 = v_2$. Therefore, the map Φ is injective. $\square$

Remark 4.42. We should deduce from the proof above that it is enough to assume that the family $\{S_t\}_{t \geq 0}$ is a continuous semiflow on the Banach space ${}_0C([0, 1])$, i.e., the result is true without assuming that the maps S_t are linear. This generalization of Proposition 4.40 will be done later in Chapter 5.

In the next definition, we defined a map Q which is the inverse of the map Φ that is defined in the equation (4.2.48).

Definition 4.43. Suppose $\psi \in C\big([0, \infty)\big)$ and let us consider the following PDE in the classical sense

$$\begin{aligned} \frac{\partial u(t,x)}{\partial t} + a(x)\,\frac{\partial u(t,x)}{\partial x} &= \lambda u(t,x), t > 0, x \in (0, 1], \\ u(t, 1) = \psi(t),\ t \geq 0. & \end{aligned} \tag{4.2.50}$$

We put $Q\psi(x) = u(0, x)$ for every $x \in [0, 1]$. We know that the solution to equation (4.2.50) with initial data u_0 is given by

$$u(t, x) = e^{\lambda t} u_0\big(G^{-1}(-t + G(x)\big).$$

If we put $x = 1$, then we get

$$u(t, 1) = e^{\lambda t} u_0\big(G^{-1}(-t + G(1)\big) = e^{\lambda t} u_0\big(G^{-1}(-t\big). \tag{4.2.51}$$

But from the second equation in (4.2.50) we have $u(t, 1)) = \psi(t)$, $t \geq 0$. So by rearranging equality (4.2.51) we obtain the following

$$u_0\big(G^{-1}(-t)\big) = e^{-\lambda t}\psi(t). \tag{4.2.52}$$

If we choose $x = G^{-1}(-t) \Leftrightarrow t = -G(x)$ and replace those new variables in equation (4.2.52) we get,

$$u_0\big(x\big) = e^{\lambda G(x)}\psi\big(-G(x)\big).$$

Hence we get that the map $Q : Y \to E$ defined by

$$\big(Q\psi\big)(x) = \begin{cases} e^{\lambda G(x)}\psi(-G(x)), & \text{if } x \in (0,1], \\ 0, & \text{if } x = 0. \end{cases} \tag{4.2.53}$$

This formula is similar to the formulae (4.2.35) and (4.2.36), in which we defined the map $\hat{Q}$.

Next, we need to prove a result, which is similar to Proposition 4.33 in the sense that the space $C_0([0,\infty))$ is replaced by the space $C([0,\infty))$. However, such a result seems to be not true and therefore, we need to use a smaller space than the space $C([0,\infty))$. The space Y which was defined by formula (4.2.42) is a candidate for this purpose we need to prove that it is a good candidate. Let us introduce the following auxiliary space.

$$Y_0 := \big\{y \in C([0,\infty)) : \exists\, C > 0 : |y(t)| \le C \ln(2+2t),\ \ t \in [0,\infty)\big\}. \tag{4.2.54}$$

Proposition 4.44. *If $y \in Y_0$, then the function $Q(y)$ defined by* (4.2.53) *is continuous, i.e., $Q(y) \in {}_0C([0,1])$. In particular, the map $Q : Y_0 \to {}_0C([0,1])$ is well-defined.*

Proof of Proposition 4.44. Let us observe that the last assertion about the map Q and the space Y_0 is an obvious consequence of the first assertion and the definition (4.2.54) of the space Y_0. Let us choose and fix $y \in Y_0$. Then, we infer that y belongs to $C([0,\infty))$ which means that function y is a continuous function. Since by Proposition 4.8 the function G is also a continuous function, so because the composition of continuous functions is continuous, see [39, Theorem 4.7], we infer that functions $Q(y)$ is continuous on the left open interval $(0,1]$. Hence we only need to prove that $Q(y)$ is continuous at 0. Since by the definition of the function Q in formula (4.2.53), $[Q(y)](0) = 0$, it is sufficient to prove that

$$\lim_{x\to 0^+} [Q(y)](x) = 0. \tag{4.2.55}$$

We take $x \in (0,1]$. Then, since the function $y \in Y_0$ we have

$$\lim_{x\to 0^+} |[Q(y)](x)| = \lim_{x\to 0^+} e^{\lambda G(x)}|y(-G(x))| = \lim_{t\to\infty} e^{-\lambda t}|y(t)|,$$

because if $t = -G(x)$ then $x \to 0^+$ if and only if $t \to \infty$. Moreover, because $y \in Y_0$ and by the L'Hospital rule, see [39, Theorem 5.13], we infer that

$$\lim_{t\to\infty} e^{-\lambda t} \ln(2+2t) = 0.$$

By applying the Sandwich Principle, see Theorem 2.15, we deduce that

$$\lim_{x\to 0^+} |[Q(y)](x)| = 0.$$

Hence the equality (4.2.55) follows.

We proved that $Q(y) \in {}_0C([0,1])$ which conclude the proof of Proposition 4.44. $\square$

Proposition 4.45. *Let* Φ *be the map defined on* (4.2.48) *and* Q *be the map defined by formula* (4.2.53)*, then we have*

$$Q \circ \Phi = \text{id} \ \textit{on} \ {}_0C([0,1]),$$

and

$$\Phi \circ Q = \text{id} \ \textit{on} \ Y_0. \tag{4.2.56}$$

Proof of Proposition 4.45. To prove this Proposition, we identify there are two parts. The first part is related to $Q \circ \Phi = \text{id}$ on ${}_0C([0,1])$, and the second part is related to $\Phi \circ Q = \text{id}$ on Y_0. Before we proceed with the proof, we need to ensure that the compositions $Q \circ \Phi : {}_0C([0,1]) \to {}_0C([0,1])$ and $\Phi \circ Q : Y_0 \to Y$ are well defined. This is true because of Propositions 4.40 and 4.44. Now we start with the first part, we need to show that

$$[Q \circ \Phi](v) = v, \quad \text{for every } v \in {}_0C([0,1]). \tag{4.2.57}$$

Since both sides of the above equality (4.2.57) are functions in the space ${}_0C([0,1])$ it is sufficient to show that for every $x_0 \in (0,1]$ we have

$$[Q \circ \Phi(v)](x_0) = v(x_0).$$

For this purpose, let us choose and fix an arbitrary element $v \in {}_0C([0,1])$ and for every $x_0 \in (0,1]$, by applying the definitions of Q and Φ, we have

$$\begin{aligned}\big[Q \circ \Phi(v)\big](x_0) &= Q[\Phi v](x_0) = e^{\lambda G(x_0)}[\Phi v]\big(-G(x_0)\big) \\ &= e^{\lambda G(x_0)}\, e^{\lambda(-G(x_0))} v\big(G^{-1}(G(x_0))\big) = v(x_0),\end{aligned}$$

which completes the first part of the proof. For the second part, we need to show that $\Phi \circ Q = \text{id}$ on Y_0. For this aim, we choose and fix $y \in Y_0$. Let us first observe that in view of the previous Proposition, $Q(y) \in {}_0C([0,1])$ and therefore the composition $[\Phi \circ Q](y)$ is a well-defined element of Y. We need to prove that $\Phi(Q(y)) = y$. Note that by applying the definitions of maps Q and Φ we have, for every $t \in [0, \infty)$

$$\begin{aligned}\big[(\Phi \circ Q)(y)\big](t) &= \Phi[Qy](t) = e^{\lambda t}[Qy\big(G^{-1}(-t)\big) \\ &= e^{\lambda t}\, e^{\lambda G\left(G^{-1}(-t)\right)} y\big(-G\big(G^{-1}(-t)\big)\big) = y(t),\end{aligned}$$

which concludes the second part of the proof, and therefore, the proof of Proposition 4.45 is complete. □

4.3 Applying the Rudnicki Method for Invariant Measures

In Section 4.2.1.3 we defined two maps Φ and Q that are very important to connect our semigroup which was generated from the problem (4.0.4)- (4.0.5) to the shift semigroup. Therefore, we use these maps in this section as well to establish the existence of an invariant measure. Our proof depends on applying the Rudnicki method in [43, Theorem 1]. We follow to prove the existence of the invariant measure but using our assumptions 4.2. Before we state our main theorem, we need to mention some required results that help to achieve our goal.

Proposition 4.46. *The C_0-semigroups $\{\pi_t\}_{t\geq 0}$, which defined by equality* (4.2.17) *on the space ${}_0C([0,1])$ and $\{\hat{T}_t\}_{t\geq 0}$, which defined by equality* (4.2.41) *on the space $C_0([0,\infty))$ are stable, that is, if $v \in {}_0C([0,1])$ and $g \in C_0([0,\infty))$, then*

$$\lim_{t\to\infty} \pi_t v = 0 \ \textit{in}\ {}_0C([0,1]),$$
$$\textit{and}$$
$$\lim_{t\to\infty} \hat{T}_t g = 0 \ \textit{in}\ C_0([0,\infty)).$$

Proof of Proposition 4.46. Starting with the C_0-semigroup $\{\pi_t\}_{t\geq 0}$, we claim that

$$\|\pi_t v - 0\|_{{}_0C([0,1])} \to 0 \ \text{ as } \ t \to \infty.$$

By applying the definition of the norm on the space ${}_0C([0,1])$ we have

$$\sup_{x\in(0,1]} |v\big(G^{-1}(-t+G(x))\big)|_{{}_0C([0,1])} \to 0 \ \text{ as } \ t \to \infty.$$

From the properties of the function G in Proposition 4.8, we have

$$0 < G^{-1}\big(-t + G(x)\big) \leq G^{-1}(-t). \tag{4.3.1}$$

Now, applying the limit of composition of functions Theorem A.6, with choosing the functions $f(t) = G^{-1}\big(-t+G(x)\big)$ and $v(y) = v$. We know that $\lim_{t\to\infty} f(t) = \lim_{s\to\infty} G^{-1}(-s) =$

0. Moreover, $\lim_{y\to 0} v(y) = 0$, because the function v is continuous and $v(0) = 0$. Hence, we get

$$\lim_{t\to\infty} v\big[G^{-1}(-t + G(x))\big] = 0.$$

But we need the uniformly continuous version. Therefore, let us take $\varepsilon > 0$. Since the function v is continuous and $v(0) = 0$, there exists $\delta > 0$ such that $|v(y) - 0| \leq \varepsilon$ if $0 \leq y \leq \delta$. By Proposition 4.8 that the $\lim_{t\to\infty} G^{-1}(-t) = 0$ we infer that there exists $T \geq 0$ such that

$$0 < G^{-1}(-t) \leq \delta \;\text{ if }\; t \geq T.$$

Hence, by inequality (4.3.1) we have, for every $x \in (0, 1]$

$$0 < G^{-1}\big(- t + G(x)\big) \leq \delta \;\text{ if }\; t \geq T.$$

Hence, we have, for every $x \in (0, 1]$ and $t \geq T$,

$$|v\big(G^{-1}(-t + G(x))\big) - 0|_{{}_0C([0,1])} \leq \varepsilon$$

In other words, we proved that for $t \geq T$,

$$\sup_{x\in(0,1]} |v\big(G^{-1}(-t + G(x))\big)|_{{}_0C([0,1])} \to 0 \;\text{ as }\; t \to \infty,$$

which means that

$$\lim_{t\to\infty} \pi_t v = 0 \text{ in } {}_0C([0, 1]).$$

Hence we proved that the C_0-semigroup $\{\pi_t\}_{t\geq 0}$ is stable on the space ${}_0C([0, 1]$.
In a similar way one can show that $\{\hat{T}_t\}_{t\geq 0}$ is stable on the space $C_0([0,\infty))$. □

Remark 4.47. Note that the first part of the Proposition 4.46 generalises Theorem 3.12 in [11], where the authors proved that the semigroup is stable in the case when $a(x) = x$. Moreover, it is relevant to mention that the importance of Proposition 4.46 can appear in the proof of the following corollary.

Corollary 4.48. *The unique invariant measure of the C_0-semigroup $\{\pi_t\}_{t\geq 0}$ on the space ${}_0C([0, 1])$ is the Dirac delta measure at 0, i.e., δ_0. The unique invariant measure of the C_0-semigroup $\{\hat{T}_t\}_{t\geq 0}$ on the space $C_0([0,\infty))$ is the Dirac delta measure at 0, i.e., δ_0.*

Recall that, for any Banach space X with the Borel σ-field denoted by $\mathcal{B}(X)$ the Dirac delta measure at a, where $a \in X$ in the Borel probability measure δ_a is defined by

$$\delta_a(A) = \begin{cases} 1, & \text{if } a \in A \in \mathcal{B}(X), \\ 0, & \text{if } a \notin A \in \mathcal{B}(X). \end{cases}$$

Proof of Corollary 4.48. We prove this corollary in two steps. For the first step we take $A \in \mathcal{B}(E)$ such that there exists $r > 0$ and $B(0,r) \cap A = \emptyset$. We claim that $\mu(A) = 0$. To verify our claim, let us take $t \geq 0$. Then we have

$$\mu(A) = \int_E \mathbb{1}_{\{\pi_t^{-1}(A)\}}\, d\mu(x) = \int_E f_t x\, d\mu(x).$$

where

$$f_t x = \mathbb{1}_{\{\pi_t^{-1}(A)\}} = \begin{cases} 1, & \text{if } t \in \pi_t^{-1}(A), \\ 0, & \text{if } t \notin \pi_t^{-1}(A). \end{cases}$$

In order to show that $f_t x \to 0$ as $t \to \infty$ we choose and fix $x \in E$. By Proposition 4.46 for every $x \in E$, $\pi_t x \to 0$ as $t \to \infty$ and since $|f_t x| \leq 1$ we infer that for every $t \geq t_0$ there exists $t_0 > 0$ such that $\|\pi_t x\|_E < r$. Which means that, $\pi_t x \in B(0,r)$ and therefore, $\pi_t x \notin A$ for every $t \geq t_0$. Hence $f_t x = 0$ as $t \to \infty$, $t \geq t_0$. Next, we use Lebesgue's Dominated Convergence Theorem A.3,

$$\lim_{t \to t_0} \int_E f_t x\, d\mu(x) = \int_E \lim_{t \to t_0} f_t x\, d\mu(x) = \int_E 0\, d\mu(x) = 0.$$

Hence we infer that $\mu(A) = 0$. For the second step, we take $A \in \mathcal{B}(E)$ such that $0 \notin A$. Let $A_k = A \cap \big(E \setminus B(0, \frac{1}{k})\big)$, for $k \in \mathbb{N}$. Then, $A \cap \big(E \setminus B(0, \frac{1}{k})\big) = \emptyset$. By step 1, we have $\mu(A_k) = 0$. Moreover, $A_k \subset A$ and $\bigcup_{k=1}^{\infty} A_k = A$. By [13, Exercise 1.1] we infer that

$$\mu(A) = \lim_{k \to \infty} \mu(A_k) = \lim_{k \to \infty} 0 = 0.$$

In particular, $\mu(E \setminus \{0\} = 0)$. Since μ is a probability measure, we infer that

$$1 = \mu(E) = \mu(E \setminus \{0\} + \{0\}) = 0 + \mu(\{0\}) = 1.$$

Hence we proved that $\mu = \delta_0$. □

Let us recall that the C_0-semigroup $\{\pi_t\}_{t \geq 0}$ on the space ${}_0C([0,1])$ corresponding to the equation (4.2.1). Thus, Corollary 4.48 can be rephrased by saying that the equation (4.2.1) has only a trivial invariant measure on the space ${}_0C([0,1])$ and this invariant measure is the Dirac delta measure at the origin. Hence, we need to modify the equation if we want to find nontrivial invariant measures. So, instead of problem (4.2.1)-(4.2.2) which is a special case of problem (4.0.4)-(4.0.5) with $c(u) = 0$, we will consider another special case of problem (4.0.4)-(4.0.5) with $c(u) = \lambda u,\ u \in \mathbb{R},\ \lambda \geq 0$ is a fixed parameter. In other words, we consider the following problem

$$\begin{aligned} \frac{\partial u(t,x)}{\partial t} + a(x)\frac{\partial u(t,x)}{\partial x} &= \lambda u(t,x), \\ u(0,x) &= u_0(x). \end{aligned} \tag{4.3.2}$$

Thus, if we look for the existence of nontrivial invariant measures we assume that $\lambda > 0$. Before we embark on proving the existence of the invariant measures for the equation (4.3.2) we need to state some useful definitions and properties.

Definition 4.49. [43]. We say that a measurable semiflow $\{T_t\}_{t\geq 0}$ defined on a probability measure space $(X, \mathcal{X}, \mu)$, with μ being an invariant probability measure for $\{T_t\}_{t\geq 0}$, is exact if and only if the σ-field $\bigcap_{t>0} T_t^{-1}(\mathcal{X})$ contains only sets of measure zero or one, i.e.,

$$\text{if } C \in \bigcap_{t>0} T_t^{-1}(\mathcal{X}) \text{ then } \mu(C) \in \{0, 1\}.$$

Definition 4.50. Suppose that $\{T_t\}_{t\geq 0}$ is a semiflow on a set Y. An element $y \in Y$ is called A periodic point of the semiflow $\{T_t\}_{t\geq 0}$ if and only if there exists $t_0 > 0$ such that the following condition is satisfied

$$T_{t_0} y = y. \tag{4.3.3}$$

A number $t_0 > 0$ satisfying condition (4.3.3) is called a period of the periodic point y of the semigroup $\{T_t\}_{t\geq 0}$.

Let us now state the following profound result known as the Kuratowski Theorem which plays an important role in our main proof of finding the invariant measure, see [32, Theorem 3.9 and Corollary 3.3].

Theorem 4.51. *Let X_1 and X_2 are two complete separable metric spaces and $E_1 \subset X_1$. Let $\varphi : E_1 \to X_2$ be injective and Borel measurable. Then, $E_2 := \varphi(E_1) \in \mathcal{B}(X_2)$, i.e., E_2 are Borel subset of X_2.*
Moreover, the map $\varphi : E_1 \to E_2$ is a Borel measurable isomorphism. In particular, the the inverse maps $\varphi^{-1} : E_2 \to E_1$ is Borel measurable.

The following corollary is a simple formulation from Kuratowski Theorem.

Corollary 4.52. *Let X_1 and X_2 are two complete separable metric space and $E_1 \subset X_1$ is Borel set. Let $\varphi : E_1 \to X_2$ is an injective and continuous map. Then the set $E_2 = \varphi(E_1)$ is Borel subset of X_2, i.e., $E_2 := \varphi(E_1) \in \mathcal{B}(X_2)$.*
Moreover, the map $\varphi : E_1 \to E_2$ is a Borel measurable isomorphism. In particular, the the inverse map $\varphi^{-1} : E_2 \to E_1$ is Borel measurable.

Definition 4.53. Assume that $T \subset \mathbb{R}$ and $\xi = \{\xi(t) : t \in T\}$ is a stochastic process. The finite dimensional distribution $\Phi_{t_1,\cdots,t_n}$ at times $t_1, \cdots, t_n \in T$ of the process ξ is a Borel probability measure on $\mathbb{R}^n$ defined by the following formula

$$\Phi_{t_1,\cdots,t_n}(A) := \mathbb{P}\big((\xi_{t_1}, \cdots, \xi_{t_n}) \in A\big), \quad A \in \mathcal{B}(\mathbb{R}^n). \tag{4.3.4}$$

Definition 4.54. Assume that $T \subset \mathbb{R}$. A stochastic processes $\xi = \{\xi(t) : t \in T\}$ defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is called stationary if for any real number h, its finite dimensional distributions are unaffected by a shift through h. In other words, for all $t_1, \cdots, t_n \in T$, $h \in \mathbb{R}$ such that $t_1 + h, \cdots, t_n + h \in T$, the finite dimensional distributions of ξ at times $t_1, \cdots, t_n$ and $t_1 + h, \cdots, t_n + h$ are equal, i.e.,

$$\Phi_{t_1,\cdots,t_n} = \Phi_{t_1+h,\cdots,t_n+h}.$$

It turns out that it is sufficient in Definition 4.54 to consider Borel sets of a special form as is explained in the following result.

Proposition 4.55. *A stochastic processes $\xi = \{\xi(t) : t \in T\}$ defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is stationary if and only if for any real number h and all $t_1, \cdots, t_n \in T$ such that $t_1 + h, \cdots, t_n + h \in T$, the following equality holds*

$$\Phi_{t_1,\cdots,t_n}(A) = \Phi_{t_1+h,\cdots,t_n+h}(A),$$

for every set $A = A_1 \times \cdots \times A_n \in \mathcal{B}(\mathbb{R}^n)$, where $A_i \in \mathcal{B}(\mathbb{R})$, $i = 1, \ldots, n$.

Proof of Proposition 4.55. The proof of this Proposition consists of two parts. The first part is trivial, because if $A_i \in \mathcal{B}(\mathbb{R})$, $i = 1, \ldots, n$ then $A = A_1 \times \cdots \times A_n \in \mathcal{B}(\mathbb{R}^n)$. For the second part, we choose and fix $h > 0$, $t_1, \cdots, t_n \in [0, \infty)$. We use a well-known result [2, Example 10.1], that the Borel σ-field on $\mathbb{R}^n$ is generated by Borel "rectangles", i.e.,

$$\mathcal{B}(\mathbb{R}^n) = \sigma(\mathcal{B}_0(\mathbb{R}^n)), \tag{4.3.5}$$

where

$$\mathcal{B}_0(\mathbb{R}^n) := \Big\{A_1 \times \cdots \times A_n : A_i \in \mathcal{B}(\mathbb{R}), \;\; i = 1, \cdots, n.\Big\}$$

By the "uniqueness of measures" property [2, Theorem 3.3] it follows from equality (4.3.5) that if m_1 and m_2 are two probability measures of $\mathcal{B}(\mathbb{R}^n)$ which coincide on $\mathcal{B}_0(\mathbb{R}^n)$, i.e.,

$$m_1(A) = m_2(A), \quad \text{for every } A \in \mathcal{B}_0(\mathbb{R}^n)$$

then these measures are equal, i.e.,

$$m_1(A) = m_2(A), \quad \text{for every } A \in \mathcal{B}(\mathbb{R}^n).$$

We use this property for two measures

$$m_1 = \Phi_{t_1,\cdots,t_n} \;\text{ and }\; m_2 = \Phi_{t_1+h,\cdots,t_n+h}.$$

Since the measures m_1 and m_2 are equal in $\mathcal{B}_0(\mathbb{R}^n)$ for every set A, we infer that

$$\Phi_{t_1,\cdots,t_n}(A) = \Phi_{t_1+h,\cdots,t_n+h}(A).$$

The proof of Proposition 4.55 is complete. □

4.3.1 An Invariant measure for the shift-semigroup

In this section we construct an invariant probability measure for the shift semigroup $\{T_t\}_{t\geq 0}$ that defined by equation (4.2.43) in the space Y. For the completeness sake, we recall some important results related to the Ornstein-Uhlenbeck process (or for short, OU process).

4.3.1.1 The Ornstein-Uhlenbeck process

Assume that $w = (w(s))_{s\in[0,1]}$ is a Brownian Motion defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. In particular, we assume that the trajectories of w are $\mathbb{P}$-almost surely continuous, which means, there exists a set $\Omega_0 \in \mathcal{F}$ such that $\mathbb{P}(\Omega_0) = 1$ and for every $\omega \in \Omega_0$, the trajectory of w corresponding to ω, i.e.,

$$w(\cdot, \omega) := \big\{[0,1] \ni t \mapsto w(t,\omega) \in \mathbb{R}\big\} \in {}_0C\big([0,1]\big)$$

is continuous. Thus, without loss of generality we assume that

$$\Omega = {}_0C\big([0,1]\big) \tag{4.3.6}$$

and the Brownian Motion w is the canonical process defined by the following formula

$$w(s) : {}_0C\big([0,1]\big) \ni \omega \mapsto \omega(s) \in \mathbb{R}, \quad s \in [0,1].$$

For a set $A \subset [0,\infty)$, we denote by $\mathcal{F}_A$ the σ-field generated by the random variables $w(t)$, for $t \in A$. In other words,

$$\mathcal{F}_A := \sigma\big(\{(w(t_1), \ldots, w(t_n)^{-1}(B) : n \in \mathbb{N},\ t_1, \ldots, t_n \in A,\ B \in \mathcal{B}(\mathbb{R}^n)\}\big).$$

Let $\xi = \{\xi(t) : t \in [0,\infty)\}$ be the OU process defined by the following identity

$$\xi(t) = e^t w(e^{-2t}), \quad t \geq 0. \tag{4.3.7}$$

Proposition 4.56. *Let $\xi = \{\xi(t) : t \in [0,\infty)\}$ be the OU process defined by formula* (4.3.7)*. Then the following conditions are satisfied.*

(i) If $t \in [0,\infty)$, then $\xi(t)$ is $N(0,1)$, i.e., $\xi(t)$ has a normal distribution with parameters $\mu = 0$ and $\sigma^2 = 1$.

(ii) Process ξ is stationary,

(iii) There exists a set $\Omega_0 \in \mathcal{F}$ such that $\mathbb{P}(\Omega_0) = 1$ and for every $\omega \in \Omega_0$ there exists $M = M(\omega) > 0$ such that

$$|\xi(t,\omega)| \leq M \ln(2+2t), \quad t \in [0,\infty). \tag{4.3.8}$$

The proof below uses the following different but equivalent (in some sense) definition of the OU process,

$$\xi(t) = e^{-t} w(e^{2t}), \quad t \geq 0. \tag{4.3.9}$$

Proof of Proposition 4.56. Proof of part (i). Using the alternative definition (4.3.9) we have

$$\mathbb{E}\big(\xi(t)\big) = \mathbb{E}\big(e^{-t} w(e^{2t})\big) = 0,$$

and

$$\mathbb{E}\big(\xi(t)^2\big) = \mathbb{E}\big[\big(e^{-t} w(e^{2t})\big)^2\big] = e^{-2t}\mathbb{E}\big[\big(w(e^{2t})\big)^2\big] = e^{-2t}e^{2t} = 1.$$

Proof of part (ii). This result is standard and well known. One can also calculate, for $t \in \mathbb{R}$ and $h > 0$,

$$\begin{aligned}\mathbb{E}\big(\xi(t)\xi(t+h)\big) &= \mathbb{E}\big[e^{t} w(e^{-2t}) e^{t+h} w(e^{-2(t+h)})\big] \\ &= e^{2t+h}\mathbb{E}\big[w(e^{-2t}) w(e^{-2(t+h)})\big] = e^{2t+h} e^{-2(t+h)} = e^{-h},\end{aligned}$$

Proof of part (iii). In order to prove this condition, we use the law of the iterated logarithm for Brownian motion. According to Definition 2.38 condition (4) and Theorem 5.1 in [31], there exists a set $\Omega_0 \in \mathcal{F}$ such that $\mathbb{P}(\Omega_0) = 1$ and for every $\omega \in \Omega_0$, the trajectory

$$[0,\infty) \ni t \mapsto B(t,\omega) \in \mathbb{R} \text{ is continuous} \tag{4.3.10}$$

and

$$\limsup_{t\to\infty} \frac{|B(t,\omega)|}{\sqrt{2\log\log(2t)}} = 1. \tag{4.3.11}$$

Let us now choose and fix $\omega \in \Omega_0$. Then by equality (4.3.11), for every $\varepsilon > 0$ there exists $T_\varepsilon = T_\varepsilon(\omega) \geq 0$ such that for every $t \geq T_\varepsilon(\omega)$, we have

$$|B(t,\omega)| \leq (1+\varepsilon)\sqrt{2t\log(\log t)}.$$

Let us choose $\varepsilon = 1$ for the remaining proof. Thus, for every $e^{2t} \geq T_1$ we have

$$|B(e^{2t}, \omega)| \leq 2\sqrt{2e^{2t}\log(\log(e^{2t}))} = 2\sqrt{2}e^t\sqrt{\ln(2t)}.$$

Let us observe that $e^{2t} \geq T_\varepsilon$ if and only if $t \geq \frac{1}{2}\log T_1$. Hence we infer that

$$\begin{aligned}|\xi(t,\omega)| = |e^{-t}w(e^{2t},\omega)| &\leq e^{-t}(1+\varepsilon)\sqrt{2}e^t\sqrt{(\log(2+2t))} \\ &\leq 2\sqrt{2}\log(2+2t), \text{ for every } t \in [\frac{1}{2}\log T_1, \infty).\end{aligned} \tag{4.3.12}$$

On the other hand, by condition (4.3.10) the function $W(\cdot,\omega) : [0, T_1] \to \mathbb{R}$ is continuous and therefore, the function $\xi(\cdot,\omega) : [0, \frac{1}{2}\log T_1] \to \mathbb{R}$ is continuous. Because the interval $[0, \frac{1}{2}\log T_1]$ is compact and function $\frac{|\xi(t)|}{\log(2+2t)} : [0, \frac{1}{2}\log T_1] \to \mathbb{R}$ is continuous, we infer that this function is bounded, see [39, Theorem 2.41]. Therefore, there exists $M > 0$ such that

$$\frac{|\xi(t)|}{\log(2+2t)} \leq M, \text{ for every } t \in [0, \frac{1}{2}\log T_1]. \tag{4.3.13}$$

Combining inequalities (4.3.12)-(4.3.13) we infer that there exists $C > 0$ such that

$$|\xi(t)| \leq C\log(2+2t), \quad t \geq 0.$$

□

The importance of the law of the Iterated Logarithm for Brownian motion, see Theorem 2.40, for this thesis lies in the following corollary.

Corollary 4.57. *Let $\xi = \{\xi(t) : t \in [0,\infty)\}$ be the OU process defined by formula* (4.3.7)*. Then $\mathbb{P}$-almost surely the trajectories of the process ξ belong to the set Y_0 which was defined before in formula* (4.2.54)*.*

Let Y be the space defined earlier in formula (4.2.42). Let $\mathcal{B} = \mathcal{B}(Y)$ be the σ-field of Borel subset of Y. Since, the σ-field $\mathcal{B}(Y)$ is equal to the σ-field generated by the family of cylindrical sets one gets the following abstract result. This is a consequence of a general result due to Fernique, see [48, Corollary and Theorem 1.2, p.8]

Theorem 4.58. *Let Y be a separable metric space. Suppose that $\xi = \{\xi(t) : t \in [0,\infty)\}$ is a stochastic process defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ such that $\mathbb{P}$-almost surely, the trajectories of the process ξ belong to Y. Then there exists a unique probability measure m on the measurable space $\big(Y, \mathcal{B}(Y)\big)$ such that for every cylinder set C,* [3] *of the following form*

$$C = C(s_1, \cdots, s_n; A_1, \cdots, A_n) = \{x \in Y : x(s_1) \in A_1, \cdots, x(s_n) \in A_n\}, \tag{4.3.14}$$

[3] Note that every cylinder set C belongs to $\mathcal{B}(Y)$. Let us recall, see Corollary and Theorem 1.2, p.8 in [48], that the Borel σ field $\mathcal{B}(Y)$ on Y, is generated by the family of all cylinder sets.

where $0 \leq s_1 < s_2 < \cdots < s_n < \infty$ *and* $A_i \in \mathcal{B}(\mathbb{R}^n)$, $i = 1, \ldots, n$, *one has*

$$m\big(C\big) = \Phi_{s_1,\cdots,s_n}\big(\prod_{i=1}^{n} A_i\big), \tag{4.3.15}$$

where $\Phi_{s_1,\cdots,s_n}$ *is the finite dimensional distribution of process* ξ *defined in formula* (4.3.4).

Proof of Theorem 4.58. The proof follows from the Hahn's extension theorem, see Theorem 1.1.16 in [15]. □

Definition 4.59. Let Y be the separable Banach space defined in formula (4.2.42). Suppose that $\xi = \{\xi(t) : t \in [0, \infty)\}$ is a stochastic process defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ such that $\mathbb{P}$-almost surely, the trajectories of ξ belong to Y. Then the measure m on the space $\big(Y, \mathcal{B}(Y)\big)$ from Theorem 4.58 is called the law of stochastic process ξ.

Remark 4.60. It is important to note that the meaning of what we say that "m is the law of the OU process" appears implicitly in the course of the next Lemma 4.61. We write

$$m(T_t^{-1}(C)) = \mathbb{P}\big(\big\{\omega \in \Omega : \xi(\cdot, \omega) \in T_t^{-1}(C)\big\}\big), \;\; C \in \mathcal{B}(Y).$$

Denoting the set $T_t^{-1}(C)$ by A we get

$$m(A) = \mathbb{P}\big(\big\{\omega \in \Omega : \xi(\cdot, \omega) \in A\big\}\big), \;\; A \in \mathcal{B}(Y). \tag{4.3.16}$$

The definition of the OU process suggests introducing the following map

$$K : {}_0C\big([0, 1]\big) \ni \omega \mapsto \{[0, \infty) \ni t \mapsto e^t\omega(e^{-2t})\} \in Y \subset C([0, \infty)), \tag{4.3.17}$$

where we use the concrete model (4.3.6) of the sample space Ω.
Using this map we can rewrite the previous identity (4.3.16) in the following way

$$m(A) = \mathbb{P}\big(\big\{\omega \in \Omega : K(\omega) \in A\big\}\big) = \mathbb{P}\big(K^{-1}(A)\big), \;\; A \in \mathcal{B}(Y). \tag{4.3.18}$$

In other words, the measure m is the image of the Wiener measure $\mathbb{P}$ via the transformation K.

Lemma 4.61. *Assume that* $\gamma > 0$ *and* $Y = Y_\gamma$ *is the separable Banach space defined in formula* (4.2.42). *Let* $\{T_t\}_{t\geq 0}$ *be the shift semigroup defined by formula* (4.2.43) *in the space* Y. *Suppose that* $\xi = \{\xi(t) : t \in [0, \infty)\}$ *is a stochastic process defined on a probability space* $(\Omega, \mathcal{F}, \mathbb{P})$ *such that* $\mathbb{P}$*-almost surely, every trajectory of process* ξ *belongs to the space* Y. *Let* m *be the law of the process* ξ. *Then* m *is a Borel probability measure on*

$(Y, \mathcal{B}(Y))$. *Moreover, the measure* m *is an invariant probability measure of the semigroup* $\{T_t\}_{t\geq 0}$ *if and only if the stochastic process* ξ *is stationary.*

Proof of Lemma 4.61. Let us first observe that it is known that m is a Borel probability measure on $(Y, \mathcal{B}(Y))$. We prove the second assertion of this Lemma in two steps.

Step 1. We prove that if ξ is a stationary process then m is an invariant measure of the semigroup $\{T_t\}_{t\geq 0}$. Assume that the process ξ is a stationary process. Let C be a cylindrical set of the following form

$$C = \{C(s_1, \cdots, s_n; A_1, \cdots, A_n)\} = \{x \in Y : x(s_1) \in A_1, \cdots, x(s_n) \in A_n\}$$

for some $0 < s_1 < \cdots < s_n$, and $A_1, \cdots, A_n \in \mathcal{B}(\mathbb{R})$. Let us choose and fix $t > 0$. By the definition of the inverse set and the definition of shift semigroup in equation (4.2.43) we have

$$\begin{aligned} T_t^{-1}(C) &:= \{x \in Y : T_t(x) \in C\} \\ &= \{x \in Y : [T_t(x)](s_i) \in A_i,\ i = 1, \cdots, n\} \\ &= \{x \in Y : x(t + s_i) \in A_i,\ i = 1, \cdots, n\}. \end{aligned}$$

Next, we calculate the measure m for both sides of the above equality to get the following

$$\begin{aligned} m(T_t^{-1}(C)) &= \mathbb{P}\big(\{\omega \in \Omega : \xi(\cdot, \omega) \in T_t^{-1}(C)\}\big) \\ &= \mathbb{P}\big(\{\omega \in \Omega : \xi_{s_{i+t}}(\omega) \in A_i,\ i = 1, \cdots, n\}\big) \\ &= \Phi_{s_1+t, \cdots, s_n+t}(A_1 \times \cdots \times A_n) = \cdots \end{aligned}$$

To move ahead with the proof we use the stationarity of process ξ, so we have

$$\begin{aligned} \cdots &= \Phi_{s_1, \cdots, s_n}(A_1 \times \cdots \times A_n) \\ &= \mathbb{P}\big(s\{\omega \in \Omega : \xi_{s_i}(\omega) \in A_i,\ i = 1, \cdots, n\}\big) \\ &= \mathbb{P}\big(\{\omega \in \Omega : \xi(\cdot, \omega) \in C\}\big) = m(C). \end{aligned}$$

Hence we proved that for any cylindrical set,

$$m(T_t^{-1}(C)) = m(C) \text{ for every } C \in \mathcal{B}(Y).$$

Since the σ-field generated by the cylindrical set is equal to the Borel σ-field $\mathcal{B}(Y)$ we deduce that

$$m(T_t^{-1}(D)) = m(D), \quad \text{for any arbitrary set } D \in \mathcal{B}(Y). \tag{4.3.19}$$

Step 2. We prove that if m is invariant of the semigroup $\{T_t\}_{t\geq 0}$ then ξ is a stationary process. Assume that m is an invariant probability measure for the semiflow $\{T_t\}_{t\geq 0}$.

According to Definition 4.54, in order to prove that ξ is a stationary process we need to show that for all $h > 0$ and $s_1, \cdots, s_n \in [0, \infty)$ the following condition is satisfied

$$\Phi_{s_1,\cdots,s_n}(A) = \Phi_{s_1+h,\cdots,s_n+h}(A), \quad \text{for every } A \in \mathcal{B}(\mathbb{R}^n). \tag{4.3.20}$$

Let us recall that $\Phi_{s_1,\cdots,s_n}(A) = \mathbb{P}\big((\xi_{s_1}, \cdots, \xi_{s_n}) \in A\big)$ and

$$\Phi_{s_1+h,\cdots,s_n+h}(A) = \mathbb{P}\big((\xi_{s_1+h}, \cdots, \xi_{s_n+h}) \in A\big).$$

Let us choose and fix $h > 0$, $t_1, \cdots, t_n \in [0, \infty)$ and a special Borel set $A = A_1 \times \ldots \times A_n \in \mathcal{B}(\mathbb{R}^n)$. Let C be any cylindrical set of the following form

$$C = \{C(s_1, \cdots, s_n; A_1, \cdots, A_n)\} = \{x \in Y : x(s_1) \in A_1, \cdots, x(s_n) \in A_n\}.$$

Note that the set $T_t^{-1}(C)$ is equal to

$$\begin{aligned} T_t^{-1}(C) &= \{x \in Y : T_t x \in C\} = \{x \in Y : x(t + \cdot) \in C\} \\ &= \{x \in Y : x(t + s_1) \in A_1, \cdots, x(t + s_n) \in A_n\}. \end{aligned} \tag{4.3.21}$$

To prove condition (4.3.20) we start with the LHS and we get the following train of equalities.

$$\begin{aligned} \Phi_{s_1,\cdots,s_n}(A_1 \times \cdots \times A_n) &= \mathbb{P}\big(\{\omega \in \Omega : \xi_{s_i}(\omega) \in A_i,\ i = 1, \cdots, n\}\big) \\ &= \mathbb{P}\big(\{\omega \in \Omega : \xi(\cdot, \omega) \in C\}\big) = m(C) \end{aligned}$$

Since m is invariant probability measure of T_t, by using identity (4.3.21), we infer that

$$\begin{aligned} \Phi_{s_1,\cdots,s_n}(A_1 \times \cdots \times A_n) = m(T_t^{-1}(C)) &= \mathbb{P}\big(\{\omega \in \Omega : \xi(\cdot, \omega) \in T_t^{-1}(C)\}\big) \\ &= \mathbb{P}\big(\{\omega \in \Omega : \xi_{s_{i+t}}(\omega) \in A_i,\ i = 1, \cdots, n\}\big) \\ &= \Phi_{s_{1+t},\cdots,s_{n+t}}(A_1 \times \cdots \times A_n). \end{aligned}$$

Hence, by Proposition 4.55, we infer that condition (4.3.20) is satisfied. This implies that ξ is a stationary process. □

The following proposition is related to [43, Proposition 4], where a similar result is proved for a different space.

Proposition 4.62. *Assume that* $\gamma > 0$ *and let* $Y = Y_\gamma$ *be the space defined in formula* (4.2.42)*. Let* $\{T_t\}_{t \geq 0}$ *be the the shift semigroup on the space* Y *defined by equation* (4.2.43)*. Let* m *be the law of the OU process* ξ *so that* m *is a probability measure on* $(Y, \mathcal{B}(Y))$*. Assume that the measure* m *is invariant for the semigroup* $\{T_t\}_{t \geq 0}$*. Then the* *semigroup* $\{T_t\}_{t\geq 0}$ *is exact on* $(Y, \mathcal{B}(Y), m)$*. Moreover, the measure* m *of the set of all periodic points of* $\{T_t\}$ *on the space* Y *is equal to zero.*

Before we embark on the proof of the above result let us state a couple of standard but important results which are a consequence of Corollary and Theorem 1.2, p.8 in [48], that the Borel σ field $\mathcal{B}(Y)$ on Y, is generated by the family of all cylinder sets.

$$\begin{aligned} \mathcal{B}(Y) &:= \sigma\big(\xi(t) : t \in [0, \infty)\big), \\ T_s^{-1}(\mathcal{B}(Y)) &:= \sigma\big(\xi(t) : t \in [s, \infty)\big). \end{aligned} \tag{4.3.22}$$

Proof of Proposition 4.62. To prove that $\{T_t\}_{t\geq 0}$ is exact, we need to verify the following condition

$$\text{if } \ A \in \bigcap_{s>0} T_s^{-1}(\mathcal{B}(Y)) \text{ then } m(A) \in \{0, 1\}.$$

We assume that $T_s^{-1}(\mathcal{B}(Y))$ is σ-field generated by $\xi(t)$ for every $t \geq s$. Let us choose and fix $A \in \bigcap_{s>0} T_s^{-1}(\mathcal{B}(Y))$. Hence, $A \in T_s^{-1}(\mathcal{B}(Y))$ for every $s > 0$. Let us now choose and fix $s > 0$. Hence, by equality (4.3.22), we infer that

$$A \in \sigma\big(\xi(t) : t \in [s, \infty)\big).$$

Digression 4.63. Let us recall that $\sigma\big(\xi(t) : t \in [s, \infty)\big)$ is the smallest σ-field of subsets of Ω such that every $\xi(t)$ is measurable with respect it. Obviously, this σ-field is generated by a family of sets of the form

$$\{\omega \in \Omega : \xi(t_1, \omega) \in A_1, \dots, \xi(t_n, \omega) \in A_n\},$$

where $s \leq t_1 < t_2 < \cdots < t_n$ and $A_1, \dots, A_n \in \mathcal{B}(\mathbb{R})$. Thus this σ-field is generated by a family of all cylindrical sets.

By the definition (4.3.7) of the Ornstein-Uhlenbeck process we infer that

$$A \in \sigma\big(w(t) : t \in (0, e^{-2s}]\big).$$

Since $\sigma\big(w(t) : t \in (0, e^{-2s}]\big) = \mathcal{F}_{e^{-2s}}$ we infer that

$$A \in \bigcap_{s>0} \mathcal{F}_{e^{-2s}} = \bigcap_{0<t<1} \mathcal{F}_t.$$

Applying the Blumenthal's law [31, Theorem 2.7], we infer that $m(A)$ is zero or one. So we proved that T_t is exact.

Now we want to show that the set of periodic points of $\{T_t\}$ has zero measure. In order to show that it is sufficient to show the set of bounded (continuous) functions has zero measure. Let ξ be the OU process defined by (4.3.7). Since ξ is a stationary process, by Lemma 4.61 we infer that the measure m defined by (4.3.19) is an invariant probability measure for the semigroup $\{T_t\}_{t\geq 0}$. From the law of iterated logarithm [31, Theorem 5.1] it follows that

$$\limsup_{t\to\infty} \frac{|\xi(t)|}{\sqrt{2\log(2t)}} = 1 \;\; \mathbb{P}- \text{ almost surely.}$$

This means that there exists a set $\Omega_1 \in \mathcal{F}$ such that $\mathbb{P}(\Omega_1) = 1$ and for every $\omega \in \Omega_1$,

$$\limsup_{t\to\infty} \frac{|\xi(t,\omega)|}{\sqrt{2\log(2t)}} = 1.$$

Therefore we infer that the sets of bounded function has measure zero. Hence we proved that the set of periodic points of $\{T_t\}$ has measure zero. $\square$

Lemma 4.64. $\mathbb{P}$*-almost surely every trajectory of the OU process* ξ *defined by* (4.3.7) *belongs to the following space*

$$\big\{v \in C([0,\infty)) : \sup_{s\in[0,\infty)} \frac{|v(s)|}{1+s} < \infty\big\}. \tag{4.3.23}$$

In particular, if $\gamma > 0$ *and* $Y = Y_\gamma$ *is the space defined in formula* (4.2.42)*, then* $\mathbb{P}$*-almost surely every trajectory of the OU process* ξ *defined by* (4.3.7) *belongs to* Y_γ.

Remark 4.65. If we define the space Y_0 by the formula (4.2.42) with $\gamma = 0$, then the last assertion of Lemma 4.64 is not true for the space Y_0. Indeed, by the law of iterated logarithm, see e.g. equality (4.3.24) below, the trajectories of the O-U process ξ are unbounded. Indeed, since $\lim_{t\to\infty} \sqrt{2\log\log(2t)} = \infty$, we infer that $\limsup_{t\to\infty} |\xi(t,\omega)| = \infty$.

Proof of Lemma 4.64. Because the space defined by formula (4.3.23) is contained in the space Y_γ it is sufficient to prove the first part of the Lemma. For this purpose, let $\xi = \{\xi(t),\ t \geq 0\}$ be the OU process defined by (4.3.7) in term of the Brownian motion $w = (w(t))_{t\geq 0}$. We know from the definition of Brownian motion that $\mathbb{P}$-almost surely the trajectories $t \mapsto w(t)$ are continuous. Thus, as we have written earlier, there exists a set $\Omega_0 \in \mathcal{F}$ such that $\mathbb{P}(\Omega_0) = 1$ and for every $\omega \in \Omega_0$ the trajectory of w corresponding to ω, i.e.,

$$w(\cdot,\omega) := \big\{[0,\infty) \ni t \mapsto w(t,\omega) \in \mathbb{R}\big\}$$

is a continuous function.

Since the composition and the product of continuous functions is a continuous function and the function $t \mapsto e^{-2t}$ is continuous, we infer that for every $\omega \in \Omega_0$ the trajectory of ξ corresponding to ω, is also a continuous function.

Moreover, from the Law of the Iterated Logarithm for Brownian motion [31, Theorem 5.1] it follows that $\mathbb{P}$-almost surely

$$\limsup_{t\to\infty} \frac{|\xi(t)|}{\sqrt{2\log\log(2t)}} = 1.$$

Thus, there exists a set $\Omega_1 \in \mathcal{F}$ such that $\mathbb{P}(\Omega_1) = 1$ and for every $\omega \in \Omega_1$,

$$\limsup_{t\to\infty} \frac{|\xi(t,\omega)|}{\sqrt{2\log\log(2t)}} = 1. \tag{4.3.24}$$

Since by the l'Hospital rule, see e.g. [39, Theorem 5.13],

$$\lim_{t\to\infty} \frac{\sqrt{2\log\log(2t)}}{1+t} = 0,$$

we infer that for every $\omega \in \Omega_1$, there exists $C > 0$ such that

$$|\xi(t,\omega)| \le C(1+t), \quad t \ge 0.$$

Since $\Omega_2 := \Omega_0 \cap \Omega_1 \in \mathcal{F}$ and $\mathbb{P}(\Omega_2) = 1$ we infer that $\mathbb{P}$-almost surely every trajectory of the process ξ belongs to space. $\square$

The second part of Lemma 4.64 and the abstract Theorem 4.58 imply the following result.

Corollary 4.66. *Assume that $\gamma > 0$ and let $Y = Y_\gamma$ be the separable Banach space defined in equation* (4.2.42). *Let ξ be the OU process defined by formula* (4.3.7). *Then there exists a unique probability measure m on the measurable space $\big(Y, \mathcal{B}(Y)\big)$ such that for every cylinder set C of the form* (4.3.14)*, the equality* (4.3.15) *holds.*
Moreover, this measure m is concentrated on the set Y_0, defined in formula (4.2.54)*, which means,*

$$Y_0 \in \mathcal{B}(Y) \quad \textit{and} \quad m(Y_0) = 1.$$

Proof of Corollary 4.66. The proof of the first part is based on applying Theorem 4.58 and the proof of the second part is based on applying Corollary 4.57. $\square$

4.3.2 The main result

In this section, we present the main Theorem. It is a generalisation of [43, Theorem 1] which was proved under stronger assumptions on the coefficients a and c. We begin by recalling our Assumption 4.2 that we listed at the beginning of this chapter.

A1 - The function $a : [0,1] \to \mathbb{R}$ is continuous;

A2 - $a(0) = 0$ and $a(x) > 0$ for $x \in (0,1]$;

A3 - The function a satisfies the Osgood condition, i.e.,

$$|a(x_2) - a(x_1)| \leq \phi(|x_2 - x_1|), \quad \text{for all } x_1, x_2 \in [0,1],$$

for an increasing function $\phi : [0,1] \to [0,\infty)$ such that $\phi(r) > 0$ if $r > 0$ and

$$\int_0^\delta \frac{1}{\phi(r)}\, dr = \infty, \quad \text{for every } \delta > 0. \text{ i.e., } \lim_{\varepsilon \to 0^+} \int_\varepsilon^\delta \frac{1}{\phi(r)}\, dr = \infty.$$

Theorem 4.67. *Assume that the Assumption 4.2 is satisfied. Assume that $\lambda > 0$. Let $\{S_t\}_{t\geq 0}$ be the C_0-semigroup generated by equation* (4.3.2)*, that given in Example 2.82. Then, there exists a Borel probability measure μ on the Banach space $E = {}_0C([0,1])$ satisfying the following conditions:*

(i) μ *is invariant under* $\{S_t\}$,

(ii) $\mu(Per) = 0$, *where* Per *is the set of periodic points of* $\{S_t\}$,

(iii) $\{S_t\}_{t\geq 0}$ *is exact on* $\big(E, \mathcal{B}(E), \mu\big)$,

(iv) each nonempty open subset of the space E has a positive measure,

(v) all moments of μ are finite and even more, i.e. there exists $\beta > 0$ such that

$$\int_E e^{\beta \|v\|_E^2}\, \mu(dv) < \infty.$$

Proof of Theorem 4.67. First of all, we need to find a Borel probability measure μ on the space E which is invariant for the semigroup $\{S_t\}_{t\geq 0}$. After that, We show that μ satisfies the set of conditions stated in the Theorem 4.67. We begin with proposing a candidate for such a measure. For this purpose, let us choose $\gamma \geq \lambda$ (note that this implies that $\gamma > 0$) and let $Y = Y_\gamma$ be the Banach space defined by formula (4.2.42). It follows from Lemma 4.64 and Proposition 4.62 that there exists a Borel probability measure m on Y which is invariant for the shift semigroup $\{T_t\}_{t\geq}$ on Y. Let $\Phi : E \to Y$ be the map defined in formula (4.2.48). Then we define a function μ by

$$\mu : \mathcal{B}(E) \ni A \mapsto m\big(\Phi(A)\big) \in [0,1]. \tag{4.3.25}$$

To start with, we need to verify whether μ defined above in formula (4.3.25) is a Borel probability measure on the space E. In other words, we need to show that μ satisfies the following properties.

(a) μ is a well-defined map from $\mathcal{B}(E)$ to $[0,1]$;

(b) μ is a probability measure on the space E.

Regarding the first property, since $\gamma \geq \lambda$, by Corollary 4.41, the map Φ is injective and continuous. Moreover, E and Y are separable Banach spaces (in particular, separable metric spaces, i.e., Polish spaces). Hence, according to the Kuratowski Theorem, see Corollary 4.52,

$$\Phi(A) \in \mathcal{B}(Y), \quad \text{for every } A \in \mathcal{B}(E).$$

This implies, that for every set $A \in \mathcal{B}(E)$, the RHS of identity (4.3.25) makes sense. Hence the function μ is well-defined. Regarding the property (b), we need to satisfy the following two conditions: (1) μ is σ-additive, and (2) $\mu(E) = 1$.
To check the first condition, we need to show that

$$\mu(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mu(A_i), \ i = 1, \cdots, n.$$

Again use the definition of measure μ in the equation (4.3.25), so we have the following

$$\mu(\bigcup_{i=1}^{\infty} A_i) = m\Big(\Phi(\bigcup_{i=1}^{\infty}(A_i))\Big) = m\big(\bigcup_{i=1}^{\infty} \Phi(A_i)\big) = \sum_{i=1}^{\infty} m\big(\Phi(A_i)\big) = \sum_{i=1}^{\infty} \mu(A_i).$$

Regarding condition (2), we need to show that $\mu(E) = 1$. For this aim, it is sufficient to prove that there exists a set $Y_0 \subset Y$ having the following two properties

(i) $Y_0 \in \mathcal{B}(Y)$,

(ii) $Y_0 \subset \Phi(E)$.

Note that a natural candidate for Y_0 would be the space Y, but unfortunately, this is not a good choice because $\Phi : E \to Y$ is not surjective. Therefore, we consider a set Y_0 defined by an earlier formula (4.2.54), i.e.

$$\begin{aligned} Y_0 &:= \big\{y \in C([0,\infty)) : \exists\, C > 0 : |y(t)| \leq C \ln(2+2t), \ \ t \in [0,\infty)\big\}. \\ &= \big\{y \in C([0,\infty)) : \exists\, k \in \mathbb{N} : |y(t)| \leq k \ln(2+2t), \ \ t \in [0,\infty)\big\} \end{aligned}$$

Obviously, the set Y_0 is a subset of $Y = Y_\gamma$. We start with the proof of property (i). Let us observe that

$$Y_0 = \bigcup_{k=1}^{\infty} Y_{0,k}$$

where, for $k \in \mathbb{N}$ we define

$$Y_{0,k} = \{y \in C([0,\infty)) : |y(t)| \leq k \ln(2+2t), \;\; t \in [0,\infty)\}.$$

Because a closed set is a Borel set and a countable union of Borel sets is a Borel set; in order to prove that Y_0 is a Borel set, it is sufficient to prove that every set $Y_{0,k}$ is a closed subset of the space Y. Equivalently, it is enough to prove that for every $k \in \mathbb{N}$, the set $Y \setminus Y_{0,k}$ is an open subset of Y. For this purpose, let us choose and fix a function $a \in Y \setminus Y_{0,k}$. We need to find $r > 0$ and $n \in \mathbb{N}$ such that

$$\text{if } \; p_n(u-a) < r \text{ then } u \in Y \setminus Y_{0,k}.$$

For this aim, we observe that since $a \in Y \setminus Y_{0,k}$ there exists $t_0 > 0$ such that

$$|a(t_0)| > k \ln(2+2t_0).$$

Put

$$\varepsilon := \frac{|a(t_0)| - k\ln(2+2t_0)}{2} > 0.$$

Then we choose $n \in \mathbb{N}$ such that $n \geq t_0$, i.e. $t_0 \in [0,n]$. Take now an arbitrary $u \in Y$ such that

$$p_n(u-a) < \varepsilon.$$

In view of the definition of p_n,

$$\sup_{t\in[0,n]} |u(t) - a(t)| < \varepsilon.$$

Since $t_0 \in [0,n]$ we infer

$$|u(t_0) - a(t_0)| < \varepsilon.$$

Note that the above inequality together with earlier proven inequality $|a(t_0)| > \varepsilon$ implies that

$$|u(t_0) - a(t_0)| < |a(t_0)|.$$

Now we are going to prove that

$$|u(t_0)| > k\ln(2+2t_0).$$

By applying the fact $||x| - |y|| \leq |x-y|$ we infer that $|u(t_0)|$ satisfies the following

$$\begin{aligned}|u(t_0)| &= |u(t_0) - a(t_0) + a(t_0)| = |a(t_0) - (a(t_0) - u(t_0))| \\ &\geq ||a(t_0)| - |a(t_0) - u(t_0)|| = |a(t_0)| - |a(t_0) - u(t_0)| \\ &> |a(t_0)| - |a(t_0) - \varepsilon > |a(t_0)| - \big(|a(t_0)| - k\ln(2+2t_0)\big) = k\ln(2+2t_0),\end{aligned}$$

which conclude the proof of the above inequality. This inequality implies that $u \in Y \setminus Y_{0,k}$. In other words, we proved that, if $p_n(u-a) < \varepsilon$ then $u \in Y \setminus Y_{0,k}$. This is exactly the proof that the set $Y \setminus Y_{0,k}$ is closed. To verify the property (ii), let us choose and fix $y \in Y_0$. Then by Proposition 4.44, we infer that $Qy \in E$. Hence $x := Qy$ belongs to the space E. Moreover, by equality (4.2.56) in Proposition 4.45, we deduce that

$$\Phi(x) = \Phi(Qy) = y.$$

This implies that $y \in \Phi(E)$. Hence we proved that $Y_0 \subset \Phi(E)$.
From the definition (4.3.25) of μ, the second property $Y_0 \subset \Phi(E)$ and Corollary 4.66 we deduce the following

$$\mu(E) = m\big(\Phi(E)\big) \geq m(Y_0) = 1.$$

From conditions (1) and (2), which are related to the measure μ, we conclude that μ defined by equation (4.3.25) is indeed a Borel probability measure on the space E.
After proving the existence of the measure μ, we now are ready to commence verifying the properties that are stated in the Theorem 4.67. We start with the first Condition that μ is an invariant measure for $\{S_t\}_{t\geq 0}$. That is, we are going to prove that

$$\mu\big(S_t^{-1}(A)\big) = \mu(A), \quad \text{for every } t \geq 0 \text{ and } A \in \mathcal{B}(E). \tag{4.3.26}$$

Proof of equality (4.3.26). Let us recall that $\{S_t\}_{t>0}$ is a C_0-semigroup on the space E and $\{T_t\}_{t>0}$ is the shift semigroup on the space Y. Also, m is the Borel probability measure on the space Y, see Lemma 4.61, which is invariant for $\{T_t\}_{t\geq 0}$. Let us also recall that

$$\Phi \circ S_t = T_t \circ \Phi. \tag{4.3.27}$$

Since $\{S_t\}_{t\geq 0}$ is a C_0-semigroup, we infer that equality (4.3.26) holds for $t = 0$. Let us choose and fix $t > 0$. Let us also put $Z = \Phi(E)$. We claim that $T_t(Z) \subset Z$. For this aim, let us choose $z \in Z$. Then there exists $x \in E$ such that $z = \Phi(x)$. By equality (4.3.27) we obtain the following

$$T_t z = T_t\big(\Phi(x)\big) = \Phi\big(S_t(x)\big).$$

Hence, $T_t z \in \Phi(E)$ because $S_t x \in E$. From this claim, we infer that we can define two maps

$$\tilde{T}_t : Z \ni z \mapsto T_t z \in Z$$

and

$$\tilde{\Phi} : E \ni x \mapsto \Phi(x) \in Z.$$

Since Φ is injective, it is easy to note that the new map $\tilde{\Phi}$ is injective from E to Z and the following equality holds

$$\tilde{\Phi} \circ S_t = \tilde{T}_t \circ \tilde{\Phi}. \tag{4.3.28}$$

Let us recall that by property (b) $\mu(E) = 1$. From equation (4.3.25), $m\big(\Phi(E)\big) = 1$. Since m is an invariant measure we infer that $m(Z) = 1$. Then m is concentrated on the set Z which is a Borel subset of the space Y. Let us define the following family of subsets of the set Z:

$$\mathcal{Z} := \{B \subset Z : B \in \mathcal{B}(Y)\}.$$

Obviously, $\mathcal{Z}$ is a subset of $\mathcal{B}(Y)$. We also define a map $\tilde{m}$ as a restriction of the measure m to the σ-field $\mathcal{Z}$, i.e.,

$$\tilde{m} : \mathcal{Z} \ni B \mapsto m(B) \in [0, 1].$$

The map $\tilde{m}$ is a probability measure and it is an invariant probability measure for the semigroup $\{\tilde{T}_t\}_{t\geq 0}$. In particular,

$$\tilde{m}(\tilde{T}_t^{-1}(B)) = \tilde{m}(B), \;\; B \in \mathcal{Z}. \tag{4.3.29}$$

The measure $\tilde{m}$ is often called the trace of the measure m to the set Z. We also have the following formula related to the formula (4.3.25).

Claim 4.68. We have

$$\mu(C) = \tilde{m}\big(\tilde{\Phi}(C)\big), \;\; C \in \mathcal{B}(E). \tag{4.3.30}$$

The Proof of Claim 4.68 is a direct consequence of the definitions of the map $\tilde{\Phi}$, the measure $\tilde{m}$ and the definition (4.3.25) of the measure μ.
Because the map $\tilde{\Phi}$ is bijection, if we apply $\tilde{\Phi}^{-1}$ to both sides of the equality (4.3.28), we get the following:

$$S_t = \tilde{\Phi}^{-1} \circ \tilde{T}_t \circ \tilde{\Phi}, \;\; t \geq 0.$$

Let us choose and fix an arbitrary set $C \in \mathcal{B}(E)$ and an arbitrary number $t \in [0, \infty)$. Also, let us take the inverse image for both sides in the above equality, then by applying the Corollary 2.31 we get

$$S_t^{-1}(C) = \big(\tilde{\Phi}^{-1} \circ \tilde{T}_t^{-1} \circ \tilde{\Phi}\big)(C) = \tilde{\Phi}^{-1}\big[\tilde{T}_t^{-1}\big(\tilde{\Phi}(C)\big)\big].$$

By taking the measure μ for both sides of the above equality, we get

$$\mu\big(S_t^{-1}(C)\big) = \mu\big[\tilde{\Phi}^{-1}\big(\tilde{T}_t^{-1}(\tilde{\Phi}(C))\big)\big]. \tag{4.3.31}$$

From the equality (4.3.30), if we denote $\tilde{\Phi}(C)$ by C', where C' is an auxiliary set, we obtain

$$C' = \tilde{\Phi}(C) \Leftrightarrow C = \tilde{\Phi}^{-1}(C').$$

As a consequence, we can rewrite the equality (4.3.30) as follows

$$\mu\big(\tilde{\Phi}^{-1}(C')\big) = \tilde{m}(C'). \tag{4.3.32}$$

Using the equality (4.3.32) and the definition of $\tilde{m}$ in the equality (4.3.29) along with the definition of μ in equality (4.3.30) and substitute them in the equality (4.3.31) we get the following final equations:

$$\mu\big(S_t^{-1}(C)\big) = \tilde{m}\big[\tilde{T_t}^{-1}\big(\tilde{\Phi}(C)\big)\big] = \tilde{m}\big(\tilde{\Phi}(C)\big) = \mu(C).$$

$\square$

So far, we have proved the first condition of Theorem 4.67. Next, we need to prove condition (ii) of Theorem 4.67.

Proof of condition (ii). Let us assume the following systems $\big(E, \mathcal{B}(E), \{S_t\}_{t\geq 0}, \mu\big)$ and $\big(Z, \mathcal{Z}, \{\tilde{T}_t\}_{t\geq 0}, \tilde{m}\big)$. Also, we assume that the Proposition 4.62 is true for the system $\big(Z, \mathcal{Z}, \{\tilde{T}_t\}_{t\geq 0}, \tilde{m}\big)$. Recall that the map $\tilde{\Phi} : E \to Z$ is a bijection, measurable and

$$\tilde{\Phi} \circ S_t = \tilde{T}_t \circ \tilde{\Phi}, \quad \text{for any } t \geq 0. \tag{4.3.33}$$

By applying the Kuratowski Theorem 4.51, the inverse $\tilde{\Phi}^{-1} : Z \to E$ is measurable.
Define the following sets

$$\tilde{P} := \{z \in Z : \exists\, t_0 > 0,\ \tilde{T}_{t_0} z = z\}, \tag{4.3.34}$$

$$P := \{e \in E : \exists\, t_0 > 0,\ S_{t_0} e = e\}, \tag{4.3.35}$$

where $\tilde{P}$ is the set of periodic point of the semigroup $\{\tilde{T}_t\}_{t\geq 0}$ and P is the set of periodic point of the semigroup $\{S_t\}_{t\geq 0}$. By using the definition (4.3.30) we have $\mu(P) = \tilde{m}\big(\tilde{\Phi}(P)\big)$. We know that $\tilde{m}(\tilde{P}) = 0$. So, in order to verify that $\mu(P) = 0$, it is sufficient to show that

$$\tilde{\Phi}(P) = \tilde{P}. \tag{4.3.36}$$

To verify the above equality we have two statements. The first statement is $\tilde{\Phi}(P) \subseteq \tilde{P}$ and the second statement is $\tilde{P} \subseteq \tilde{\Phi}(P)$. Regarding the first statement, we assume $z \in \tilde{\Phi}(P)$ and we want to show that $z \in \tilde{P}$. For this purpose, let us take $z \in \tilde{\Phi}(P)$, which means that there exists an element e in P such that $z = \tilde{\Phi}(e)$. Since $e \in P$, by the equality (4.3.35),

e is periodic, i.e.,

$$\exists\, e \in E : \exists\, t_0 > 0,\ S_{t_0} e = e.$$

To show that $z \in \tilde{P}$, we use the definition in formula (4.3.34) of the periodic points in $\tilde{P}$. We need to prove that $\tilde{T}_{t_0} z = z$. Take $z \in Z$ and $t_0 > 0$, then we have

$$\tilde{T}_{t_0} z = \tilde{T}_{t_0}\big(\tilde{\Phi}(e)\big) = \big(\tilde{T}_{t_0} \circ \tilde{\Phi}\big)(e) = \big(\tilde{\Phi} \circ S_{t_0}\big)(e) = \tilde{\Phi}\big(S_{t_0} e\big) = \tilde{\Phi}(e) = z,$$

which implies that $z \in \tilde{P}$. Regarding the second statement that requires $\tilde{P} \subseteq \tilde{\Phi}(P)$, we follow the argument above. Assume $z \in \tilde{P}$ and we want to show $z \in \tilde{\Phi}(P)$. Take $z \in \tilde{P}$. Then by the definition of periodic point, there exists $t_o > 0$ such that $\tilde{T}_{t_0}(z) = z$ and $z \in Z$. Because map $\tilde{\Phi}$ is bijection and $z \in Z$, then there exists $e \in E$ such that $z = \tilde{\Phi}(e)$. To prove that $z \in \tilde{\Phi}(P)$ it is sufficient to prove that $e \in P$. In order to do so, we use again the definition in the equality (4.3.35) and show that $S_{t_0} e = e$.

$$\tilde{\Phi}(e) = z = \tilde{T}_{t_0} z = \tilde{T}_{t_0}\big(\tilde{\Phi}(e)\big) = \big(\tilde{T}_{t_0} \circ \tilde{\Phi}\big)(e) = \big(\tilde{\Phi} \circ S_{t_0}\big)(e) = \tilde{\Phi}\big(S_{t_0}(e)\big).$$

Hence $\tilde{\Phi}(e) = \tilde{\Phi}(S_{t_0} e)$. Thus, we infer that $e = S_{t_0} e$ because the map $\tilde{\Phi}$ is injective. Therefore, $e \in P$. This concludes the proof of the second statement, and consequently, we finish the proof of the equality (4.3.36). Since $\tilde{\Phi}(P) = \tilde{P}$ we deduce that

$$\mu(P) = \tilde{m}(\tilde{P}) = 0.$$

By this, we verified the second condition (ii) of the Theorem 4.67. □

Now we need to prove the condition (iii) of Theorem 4.67, i.e, that the semigroup $\{S_t\}_{t\geq 0}$ is exact on $\big(E, \mathcal{B}(E), \mu\big)$. We will show that

$$\text{if } A \in \bigcap_{t>0} S_t^{-1}(\mathcal{B}(E)) \text{ then } \mu(A) \in \{0,1\}. \tag{4.3.37}$$

Proof of condition (iii) . We start the proof by recalling the following. The map $\tilde{\Phi}$ is bijection and equality (4.3.33) holds. Also, the measure μ is defined by $\mu(A) = \tilde{m}(\tilde{\Phi}(A))$. We assume that the system $\big(Z, \mathcal{Z}, \{\tilde{T}_t\}_{t\geq 0}, \tilde{m}\big)$ is exact. That is,

$$\text{if } C \in \bigcap_{t>0} \tilde{T}_t^{-1}(\mathcal{Z}) \text{ then } \tilde{m}(C) \in \{0,1\}. \tag{4.3.38}$$

In order to prove condition (4.3.37) we take and fix an arbitrary set $A \in \bigcap_{t>0} S_t^{-1}\big(\mathcal{B}(E)\big)$. If we can prove that the set $C := \tilde{\Phi}(A)$ belongs to $\bigcap_{t>0} \tilde{T}_t^{-1}(\mathcal{Z})$, then by the equation (4.3.38) we would deduce that $\tilde{m}(C) \in \{0,1\}$ from which we would infer that

$\mu(A) \in \{0,1\}$. To do this, it is sufficient to prove that for every $t > 0$ we have

$$C := \tilde{\Phi}(A) \in \tilde{T}_t^{-1}(\mathcal{Z}).$$

Since we assume that $A \in \bigcap_{t>0} S_t^{-1}(\mathcal{B}(E))$, then $A \in S_t^{-1}(\mathcal{B}(E))$ for every $t > 0$. Which means that there exists a set $B \in \mathcal{B}(E)$ such that $A = S_t^{-1}(B)$.
So let us choose and fix $t > 0$. Because $\tilde{\Phi} : E \to Z$ is an injective and measurable map, by applying the Kuratowski Theorem 4.51 we infer that $\tilde{\Phi}^{-1}$ is also measurable. Therefore, since $B \in \mathcal{B}(E)$, we infer that

$$\tilde{\Phi}(B) \in \mathcal{Z}. \tag{4.3.39}$$

Now we claim that

$$C = \tilde{T}_t^{-1}(\tilde{\Phi}(B)). \tag{4.3.40}$$

Indeed, by equality (4.3.33), because $\tilde{\Phi}$, $\tilde{T}_t$ and S_t are bijections, their inverses exist. $\tilde{\Phi}$ is surjective because of the choice of Z So, if we take the inverse for both sides of the equality (4.3.33), we get

$$S_t^{-1} \circ \tilde{\Phi}^{-1} = \tilde{\Phi}^{-1} \circ \tilde{T}_t^{-1}.$$

By applying map $\tilde{\Phi}$ for both sides in the above equality from the right we obtain

$$S_t^{-1} = \tilde{\Phi}^{-1} \circ \tilde{T}_t^{-1} \circ \tilde{\Phi}. \tag{4.3.41}$$

Since $C = \tilde{\Phi}(A)$ and $A = S_t^{-1}(B)$, we deduce that $C = \tilde{\Phi}(S_t^{-1}(B))$. By substituting the equality (4.3.41) in the last equality, we get

$$C = \tilde{\Phi}\big[\tilde{\Phi}^{-1} \circ \tilde{T}_t^{-1} \circ \tilde{\Phi}(B)\big] = \tilde{T}_t^{-1} \circ \tilde{\Phi}(B) = \tilde{T}_t^{-1}\big(\tilde{\Phi}(B)\big).$$

Hence we have verified claim (4.3.40). From the equation (4.3.39) we infer that $C \in \tilde{T}_t^{-1}\big(\mathcal{Z}\big)$. Since $t > 0$ was arbitrary, we proved that $C = \tilde{\Phi}(A) \in \bigcap_{t>0} \tilde{T}_t^{-1}(\mathcal{Z})$, and therefore, by the exactness property of the system $\big(Z, \mathcal{Z}, \{\tilde{T}_t\}_{t\geq 0}, \tilde{m}\big)$ in the equation (4.3.38) we infer that

$$\tilde{m}(C) = \tilde{m}\big(\tilde{\Phi}(A)\big) \in \{0,1\}.$$

From the definition of measure μ in equation (4.3.30) we deduce that $\mu(A) \in \{0,1\}$. That means, $\{S_t\}_{t\geq 0}$ is exact on $\big(E, \mathcal{B}(E), \mu\big)$. By this, we verified and completed the third condition (iii) of Theorem 4.67. □

Proof of condition (iv). Let us recall that by formula (4.3.18) of the function K and the definition of the map μ in formula (4.3.25) we have

$$\mu(A) = m\big(\Phi(A)\big), \ A \in \mathcal{B}(E) \text{ and } \ m(B) = \mathbb{P}(K^{-1}(B)), \quad B \in \mathcal{B}(Y).$$

Thus we can rewrite the measure μ as follows

$$\mu(A) = m\big((\Phi^{-1})^{-1}(A)\big), \quad A \in \mathcal{B}(E),$$

and therefore, for every $A \in \mathcal{B}(E)$,

$$\mu(A) = m\big((\Phi^{-1})^{-1}(A)\big) = \mathbb{P}\big(K^{-1}(\Phi^{-1})^{-1}(A)\big) = \mathbb{P}\big(\Phi^{-1} \circ K)^{-1}(A)\big). \quad (4.3.42)$$

Now we need to find the map $\Phi^{-1} \circ K$.
For this purpose let choose and fix $\omega \in \Omega$. We denote $x = (\Phi^{-1} \circ K)(\omega)$ and $y = K(\omega)$. Our aim is to find a formula that relates x to ω. Then $x = \Phi^{-1}(y)$ and so $y = \Phi(x)$. Thus by the definition of the map K in formula (4.3.17), we deduce that for every $t \geq 0$, $y(t) = e^t\omega(e^{-2t})$. By using the changes of variables, we get

$$x(s) = e^{\lambda G(s)} y\big(- G(s)\big), \;\; s \in [0,1].$$

Hence by using the definition of the function y we have

$$x(s) = e^{\lambda G(s)}\big(e^{-G(s)}\omega(e^{-2(-G(s))})\big) = e^{(\lambda-1)G(s)}\omega\big(e^{2G(s)}\big), \; s \in [0,1].$$

We have found a formula for the map $\Phi^{-1} \circ K : \Omega \to E$. To be more precise, if $\omega \in \Omega$, then $x = [\Phi^{-1} \circ K](\omega)$ is given by the formula (4.3.43) below

$$[(\Phi^{-1} \circ K)\omega](s) = e^{(\lambda-1)G(s)}\omega\big(e^{2G(s)}\big), \;\; s \in [0,1], \quad (4.3.43)$$

Also, we can choose the space $\Omega = {}_0C\big([0,1]\big)$. Before proceeding with the proof of condition (iv), we state a remark that will help us with the proof.

Remark 4.69. So far we have chosen $\Omega = {}_0C\big([0,1]\big)$ but we are free to choose different spaces for Ω, for instance, for a set $[0,1] \subset \mathbb{R}$ and parameter $\alpha \in (0,1]$,

$$\begin{aligned}\Omega_\alpha = {}_0C^\alpha\big([0,1]\big) &:= \{\omega \in {}_0C\big([0,1]\big) : \sup_{0\leq s<t\leq 1} \frac{|\omega(t)-\omega(s)|}{|t-s|^\alpha} < \infty\}, \\ \|\omega\|_{{}_0C^\alpha\left([0,1]\right)} &:= \sup_{t\in[0,1]} |\omega(t)| + \sup_{0\leq s<t\leq 1} \frac{|\omega(t)-\omega(s)|}{|t-s|^\alpha}. \quad (4.3.44)\end{aligned}$$

It follows that

$$\begin{aligned}\|\omega\|_{{}_0C^\alpha\left([0,1]\right)} &= \sup_{t\in[0,1]} |\omega(t)| + \sup_{0\leq s<t\leq 1} \frac{|\omega(t)-\omega(s)|}{|t-s|^\alpha} \\ &\geq \sup_{0\leq s<t\leq 1} \frac{|\omega(t)-\omega(s)|}{|t-s|^\alpha} \geq \sup_{0<t\leq 1} \frac{|\omega(t)|}{|t|^\alpha} = \sup_{0<r\leq 1} \frac{|\omega(t)|}{|r|^\alpha}.\end{aligned}$$

We also put

$$ {}_0C^0\big([0,1]\big) := {}_0C\big([0,1]\big). $$

Let us recall that there exists a Borel probability measure on the set Ω which is equal to the law of the Brownian Motion. This probability measure is usually called the standard (classical) Wiener measure and we will denote it by $\mathbb{P}$. Note that, almost surely trajectories of Brownian Motion are continuous. But, a stronger property is known, namely that almost surely the trajectories of Brownian Motion are Hölder continuous with every exponent $\alpha \in (0, \frac{1}{2})$. This is a well-known fact and can be proved by using the Kolmogorov test. It follows that the law of the Brownian Motion induces a Borel probability measure $\mathbb{P}_\alpha$ on the set Ω_α, i.e., the Borel probability measure $\mathbb{P}_\alpha$ is the law of the Brownian Motion on the set Ω_α. Note that by definition $\Omega_\alpha \subset \Omega$. Moreover, both spaces Ω and Ω_α are separable Banach spaces (with naturally defined norms), see e.g. (4.3.44) and that the embedding $i_\alpha : \Omega_\alpha \hookrightarrow \Omega$ is linear and continuous (i.e., bounded). Moreover, it can be easily shown that the measure $\mathbb{P}$ is the image of the measure $\mathbb{P}_\alpha$ via the map i_α, i.e.,

$$ \mathbb{P}(A) = \mathbb{P}_\alpha\big(i_\alpha^{-1}(A)\big), \quad A \in \mathcal{B}(\Omega). $$

Finally, since the map i_α is continuous (as noted above) and obviously injective, by the Kuratowski Theorem, the image by i_α of the set Ω_α is a Borel subset of the set Ω. In other words, since $i_\alpha(\Omega_\alpha) = \Omega_\alpha$, we deduce that Ω_α is a Borel subset of the set Ω and the restriction of the measure $\mathbb{P}$ to the set Ω_α is equal to the measure $\mathbb{P}_\alpha$.
It is known, that $\mathbb{P}_\alpha$ is a Gaussian measure on the separable Banach space Ω_α. Hence, by the celebrated Fernique Theorem, see [19, Theorem 2.7], there exists $\beta_\alpha > 0$ such that

$$ \int_{\Omega_\alpha} e^{\beta_\alpha \|\omega\|^2_{C^\alpha}} \, \mathbb{P}_\alpha(d\omega) < \infty. \tag{4.3.45} $$

It is also important to note that the measure $\mathbb{P}_\alpha$ is non-degenerate, i.e.,

$$ \mathbb{P}_\alpha(B) > 0, \text{ for every non-empty open } B \subset \Omega_\alpha. $$

Now we go back to prove condition (iv) of Theorem 4.67. Our objective is to confirm that the measure μ is positive on open sets. For this aim let us choose and fix $\lambda > 0$ as in the system (4.3.2). Let us choose and fix an auxiliary number $\alpha \in [0, \frac{1}{2})$ such that

$$ \lambda > 2(\frac{1}{2} - \alpha), \tag{4.3.46} $$

i.e.

$$ \frac{\lambda - 1}{2} + \alpha > 0. $$

We consider a map

$$L_\alpha : \Omega_\alpha \to E \tag{4.3.47}$$
$$[L_\alpha(\omega)](s) = e^{(\lambda-1)G(s)}\omega\big(e^{2G(s)}\big), \quad s \in (0,1],$$

and $[(L_\alpha)\omega](0) = 0$.
Note that the RHS of the above equality is the same as RHS of equality (4.3.43). This means that the map L_α is the restriction of the map $\Phi^{-1} \circ K$ to the space Ω_α.

Now it only remains to verify the following four properties of the map L_α.

1. the map L_α is well defined, i.e., if $\omega \in {}_0C^\alpha\big([0,1]\big)$ and $x = L_\alpha\omega$, then $x \in {}_0C\big([0,1]\big)$,
2. the map L_α is linear,
3. the map L_α is continuous, (i.e., bounded);
4. the measure μ satisfies

$$\mu(A) = \mathbb{P}_\alpha\big((L_\alpha)^{-1}(A)\big), \quad \text{for every } A \in \mathcal{B}(E). \tag{4.3.48}$$

Remark 4.70. Using Definition from section 13 of the book [2], see also section 3.6 of [4], formula (4.3.48) above means that the measure μ is equal to the image of the measure $\mathbb{P}_\alpha$ under the mapping L_α. Thus, using notation [2], $\mu = \mathbb{P}_\alpha L_\alpha^{-1}$, and, using Bogachev's notation, $\mu = \mathbb{P}_\alpha \circ L_\alpha^{-1}$. A very important result here is the change of measure Theorem, i.e., Theorem 6.13 in [2]. In our context this theorem says that for every measurable function $f : E \to [0,\infty)$ the two integrals below exist simultaneously and they are equal, i.e.,

$$\int_E f(v)\,\mu(dv) = \int_{\Omega_\alpha} f(L_\alpha(\omega))\,\mathbb{P}_\alpha(d\omega), \tag{4.3.49}$$

or

$$\int_E f\,d\mu = \int_{\Omega_\alpha} f \circ L_\alpha\,d\mathbb{P}_\alpha.$$

To prove the first property (1), we fix $\omega \in {}_0C^\alpha\big([0,1]\big)$ and put $x = L_\alpha\omega$. By the definition of the map L_α we have $x(0) = 0$ and

$$x(s) = \big(e^{2G(s)}\big)^{\frac{(\lambda-1)}{2}}\omega\big(e^{2G(s)}\big), \quad s \in (0,1].$$

Since the composition of continuous functions is continuous and the function $G : (0,1] \to \mathbb{R}$ is continuous, by [39, Theorem 4.9], we infer that the function x is continuous at every $s \in (0,1]$. To prove that x is continuous on the whole closed interval $[0,1]$ it is sufficient to prove that, since $x(0) = 0$,

$$\lim_{s\to 0^+} x(s) = 0.$$

For every $s \in (0,1]$, we put $r = e^{2G(s)}$. Because $G(s) \to -\infty$ when $s \to 0^+$, we infer that $r \to 0$. Moreover,

$$x(s) = z(r) = r^{\frac{(\lambda-1)}{2}}\omega(r).$$

Since $\omega \in {}_0C^\alpha([0,1])$ with α satisfying condition (4.3.46), we infer that

$$\lim_{r \to 0^+} z(r) = 0.$$

Hence we infer that $\lim_{s \to 0^+} x(s) = 0$. Thus we proved the first property.
Now we will prove the second property (2). Recall that $\lambda > 0$ and α satisfies condition (4.3.46). In part (1) we proved that the map

$$L_\alpha : {}_0C^\alpha([0,1]) \ni \omega \mapsto x \in {}_0C([0,1])$$

is well-defined. One can trivially prove that it is linear. Now we will prove the third property (3) that it is bounded (and hence continuous). We begin with recalling the definition of the norm in the space ${}_0C([0,1])$,

$$\|x\|_{{}_0C([0,1])} = \sup_{s \in [0,1]} |x(s)|.$$

Let us choose and fix $\lambda > 0$, and $\alpha \in [0, \frac{1}{2})$ such that $\lambda > 2(\frac{1}{2} - \alpha)$. Hence we have

$$\begin{aligned}\|L_\alpha \omega\|_{{}_0C([0,1])} = \|x\|_{{}_0C([0,1])} &= \sup_{s \in [0,1]} |x(s)| = \sup_{0<s\le 1} |x(s)| = \sup_{0<r\le 1} r^{\frac{\lambda-1}{2}} |\omega(r)| \\ &= \sup_{0<r\le 1} r^{\frac{\lambda-1}{2}+\alpha} \frac{|\omega(r)|}{r^\alpha} \le \sup_{0<r\le 1} \frac{|\omega(r)|}{r^\alpha} \le \|\omega\|_{{}_0C^\alpha([0,1])}.\end{aligned}$$

To verify part (4), we notice that the identity (4.3.48) follows from earlier proven identity (4.3.42). By this, we confirm the property (iv) of Theorem 4.67. That is, we deduce each nonempty open subset of the space ${}_0C([0,1])$ has a positive measure μ. Indeed, since L_α is continuous, $L_\alpha^{-1}(A)$ is an open subset of Ω_α, for every open subset A of E. □

Finally, it remains to prove the last property of our main result of this section, Theorem 4.67.

Proof of condition (v). Let us observe that by the change of measure theorem, the Fernique Theorem, see (4.3.45), the boundedness of the linear map L_α defined by equation (4.3.47) and the Change of Measure Formula (4.3.49) (which follows from the identify

(4.3.48)) we deduce that with function $f(v) = e^{\beta\|v\|_E^2}$,

$$\begin{aligned}\int_E e^{\beta\|v\|_E^2}\,\mu(dv) &= \int_E f(v)\,\mu(dv) = \int_E f(v)\,(\mathbb{P}_\alpha \circ L_\alpha^{-1})(dv)\\ &\le \int_{\Omega_\alpha} e^{\beta\|L_\alpha\|^2_{\mathcal{L}(\Omega_\alpha,E)}\|\omega\|^2_{C^\alpha}}\,\mathbb{P}_\alpha(d\omega)\\ &= \int_{\Omega_\alpha} e^{\beta_\alpha\|\omega\|^2_{C^\alpha}}\,\mathbb{P}_\alpha(d\omega) < \infty\end{aligned}$$

if we choose $\beta > 0$ such that $\beta\|L_\alpha\|^2_{\mathcal{L}(\Omega_\alpha,E)} \le \beta_\alpha$, where β_α is the constant in the Fernique inequality (4.3.45). Hence, the proof of part (v) of Theorem 4.67 is complete. □

Hence, the proof of the whole Theorem is complete. □

Chapter 5

Invariant Measures for Dissipative Nonlinear First Order PDEs

It is known in [43] that the Lasota equation (4.0.2) has an invariant measure under Assumptions 4.1. In this chapter, in order to prove the existence and uniqueness of a mild solution for a nonlinear Lasota equation, we assume a new set of assumptions for the nonlinear case along with what we assumed in Chapter 4. Moreover, we analyse the properties of this solution. In the end, we prove the existence of an invariant measure for such equation. We do not investigate the question of the uniqueness invariant measure. This is an interesting question for future research.

The organization of the present chapter is as follows. We dedicate the first Section 5.1 to state standard facts and definitions of the dissipative and Lipschitz functions. In the second Section 5.2 we study the existence and the uniqueness of mild solutions for the nonlinear evolution equation with Lipschitz nonlinearity. In Section 5.3 we prove the existence and the uniqueness of mild solutions to evolution equations with dissipative nonlinearity and we study the properties of such solutions. In Section 5.4 we present our main theorem regarding the existence of an invariant measure for our nonlinear Lasota equation. Lastly, Section 5.5 provides a discussion related to our work with compare to the paper by Rudnicki [43].

5.1 Preliminaries

In this section, we present all the information needed about the existence and the uniqueness of the solution in order to define a mild solution for our PDE.

Definition 5.1. Suppose E is a Banach space with dual denoted by E^*. A function $f : D(f) \to E$, where $D(f) \subset E$ is a dense subset, is called dissipative if and only if for every $x \in D(f)$ there exists $z^* \in \partial\|x\|$ such that

$$_E\langle f(x), z^* \rangle_{E^*} \leq 0.$$

The above definition is standard, see e.g., [33, Definition I.4.1]
Notation:

- We mean by the dual space E^* the following, $E^* := \{\varphi : E \to \mathbb{R} : \varphi \text{ is linear and bounded}\}$.
- If E^* is dual space of the Banach space E then we use notation

$$\langle x, \varphi \rangle = {}_E\langle x, \varphi \rangle_{E^*} = \varphi(x), \;\; x \in E, \varphi \in E^*.$$

- To define the notation $\partial\|x\|$ which we use in Definition 5.1, there are two cases:

  Case 1: if $x = 0$, then $\partial\|x\| := \{x^* \in E^* : \|x^*\|_{E^*} \leq 1\}$.

  Case 2: if $x \neq 0$, then $\partial\|x\| := \{x^* \in E^* : \langle x, x^* \rangle = \|x\| = \|x^*\| = 1\}$.

Example 5.2. *If E is a Hilbert space, then by the Riesz Lemma [40, Theorem 4.12] and [26, Theorem 3.8-1] the dual space E^* can be identified with the space E itself. Hence in this case Definition 5.1 can be rewritten in the following form, see Pazy [33].*
Suppose E is a real Hilbert space with the inner product $\langle \cdot, \cdot \rangle$. A function $f : D(f) \to E$, where $D(f) \subset E$ is a dense subset, is called dissipative if and only if for every $x \in D(f)$

$$\langle f(x), x \rangle \leq 0.$$

In particular, if $E = \mathbb{R}$, then a function $c : D(c) \to \mathbb{R}$ such that $D(c) \subset \mathbb{R}$ dense subset, is called dissipative if and only if for every $x \in D(c)$

$$\langle c(x), x \rangle_{\mathbb{R}} = c(x)x \leq 0. \tag{5.1.1}$$

An example of such a function is $c(x) = -x^3$, for $x \in \mathbb{R}$. But another important example of function c is a function $c(x) = \lambda x - x^3$, $x \in \mathbb{R}$, where $\lambda \geq 0$ is fixed. This function c satisfies the following generalisation of condition (5.1.1)*:*

$$c(x)x \leq \lambda|x|^2, \;\; x \in D(c). \tag{5.1.2}$$

A function c satisfying condition (5.1.2) will be called of dissipative type. Note that if the function c is of dissipative type with constant $\lambda \geq 0$ then the function $c - \lambda I$ is dissipative.

Definition 5.3. Let X be a normed vector space with the norm denoted by $|\cdot|$.

(1) A function $f : X \to X$ is a globally Hölder with exponent $\alpha \in (0, 1]$ if and only if there exists a constant $C > 0$ such that

$$|f(x_1) - f(x_2)| \leq C|x_1 - x_2|^\alpha, \quad \text{for all } x_1, x_2 \in X.$$

(2) A function $f : X \to X$ which is a globally Hölder with exponent 1 is called globally Lipschitz.

(3) A function $f : X \to X$ is a Lipschitz on balls if and only if for every $R > 0$ there exists a constant $C_R > 0$ such that

$$|f(x_1) - f(x_2)| \leq C_R|x_1 - x_2|, \quad \text{for all } x_1, x_2 \in B_X(0, R),$$

where $B_X(0, R)$ is the closed ball in the space X centred at zero and of radius R.

(4) A function $f : X \to X$ is a Hölder on balls with exponent $\alpha \in (0, 1]$ if and only if for every $R > 0$ there exists a constant $C_R > 0$ such that

$$|f(x_1) - f(x_2)| \leq C_R|x_1 - x_2|^\alpha, \quad \text{for all } x_1, x_2 \in B_X(0, R).$$

The definitions of globally Lipschitz/Hölder functions make sense in every metric space. For instance, we can take $X = [0, 1]$ with metric

$$d(x_1, x_2) = |x_2 - x_1|, \;\; x_1, x_2 \in [0, 1]. \tag{5.1.3}$$

In general metric spaces, instead of using notions of Lipschitz/Hölder functions on balls, one often uses notions of locally Lipschitz/Hölder functions. These two notions are equivalent if X is a finite dimensional normed vector space. But in the case when X is an infinite dimensional normed vector space, these two notions are not equivalent. Let us recall that if (X, d) is a metric space then a function $f : X \to X$ is said to be locally α-Hölder with $\alpha \in (0, 1]$ if and only if for every $x_0 \in X$ there exists $R > 0$ and a constant $C = C(x_0, R)$ such that

$$d\big(f(x_1), f(x_2)\big) \leq Cd(x_1, x_2)^\alpha, \quad \text{for all } x_1, x_2 \in B_R(x_0).$$

A locally 1- Hölder function is called a locally Lipschitz function. Obviously, if X is a normed vector space with norm $|\cdot|$ and the corresponding distance function is defined by formula (5.1.3), then every function which is Hölder on balls with exponent $\alpha \in (0,1]$ is also locally α-Hölder. As we mentioned above, these notions are not equivalent in infinite dimensional spaces.

5.1.1 Initial value problems in Banach space

Assume that X is a Banach space and A is the infinitesimal generator of a C_0-semigroup $\{S_t\}_{t\geq 0}$ of bounded linear operators on X. Consider the following linear in-homogeneous initial value problem

$$\frac{du(t)}{dt} = Au(t) + f(t), \quad t > 0, \tag{5.1.4}$$

$$u(0) = u_0, \tag{5.1.5}$$

where $u_0 \in X$ and $f : [0,\infty) \to X$ is a Bochner integrable function. The following definition is [33, Definition IV.2.1] which gives the classical solution to an abstract in-homogeneous Cauchy problem on the interval $[0,\infty)$. Pazy [33] used a notion of a classical solution. As mentioned in Chapter 4, because we define a classical solution in a more "classical" way, Pazy's "classical solution" will be renamed here as "strong solution".

Definition 5.4. By a strong solution to the abstract in-homogeneous Cauchy problem (5.1.4)-(5.1.5), we mean a continuous function $u : [0,\infty) \to X$ which is continuously differentiable on $(0,\infty)$, $u(t) \in D(A)$ for $t \in (0,\infty)$, equation (5.1.4) is satisfied on $(0,\infty)$, and (5.1.5) holds.

In order to formulate the definition of a mild solution we need the following result (Lemma 5.5) that is stated in Pazy [33] (after Corollary IV.2.2) and in the proof of the main Theorem in the paper by Ball [1]. This result provides motivation to define a mild solution to the initial value problem (5.1.4)-(5.1.5).

Lemma 5.5. *Assume that X is a separable Banach space and $S = \{S(t)\}_{t\geq 0}$ is a C_0-semigroup on the space X. If $f \in L^1(0,T;X)$, then a function z defining by*

$$z : [0,T] \ni t \mapsto \int_0^t S(t-r)f(r)\, dr \in X, \tag{5.1.6}$$

where the integral is meant in the Bochner sense, is well defined and continuous.

Before we start with the proof of this Lemma we need to mention an important result that we use in the proof.

Lemma 5.6. *Assume that $f \in L^1(a,b;X)$ and the function β defined by*

$$\beta : [a,b] \ni r \mapsto \beta(r) \in \mathcal{L}(X,X)$$

is strongly continuous, i.e., for every $x \in X$ the following function

$$\beta(\cdot)x : [a,b] \ni r \mapsto \beta(r)(x) \in X$$

is continuous. Then the function

$$\beta \cdot f : [a,b] \ni r \mapsto \beta(r)f(r) \in X$$

belongs to the space $L^1(a,b;X)$.

Proof of Lemma 5.5. The proof of this lemma is twofold. First, we verify that the function z defined by formula (5.1.6) is well-defined. Second, we prove that the function z is continuous. Regarding the former, we fix an element $t \in [0,T]$. Since $f \in L^1(0,T;X)$ then $f \in L^1(0,t;X)$. Hence we apply Lemma 5.6 with replacing the interval (a,b) by $(0,t)$ and the function $\beta(r)$ replaced by $S(t-r)$ for every $r \in [0,t]$. Note that if $r \in [0,t]$ then also $t-r \in [0,t]$ and $t-r \geq 0$. Therefore, $S(t-r) \in \mathcal{L}(X,X)$. Moreover, by the strong continuity, see [33, Corollary I.2.3], we infer that function β satisfies assumptions of Lemma 5.6. Hence, the function $[0,t] \ni r \mapsto S(t-r)f(r) \in X$ belongs to $L^1(0,T;X)$. In particular, the integral $\int_0^t S(t-r)f(r)\,dr$ exists and belongs to X. Hence we proved that the function z is well-defined.
Regarding the second part, we need to show that the function z defined by formula (5.1.6) is continuous. In order to do that we need to prove the following

(i) $\lim_{t\to T-} x(t) = x(T)$ in X,

(ii) $\lim_{t\to 0+} x(t) = 0$ in X,

(iii) $\lim_{t\to t_0-} x(t) = x(t_0)$ in X and $t_0 \in (0,T)$,

(iv) $\lim_{t\to t_0+} x(t) = x(t_0)$ in X and $t_0 \in (0,T)$.

We only provide proof of part (iv). The proofs of all three remaining parts are similar. Since the limit in the sense of Cauchy is equivalent to the limit in the sense of Heine, see [39, Theorem 4.2], it is sufficient to prove that if $(t_n)_{n\geq 0}$ is such that

$$t_n > t_0 \text{ for every } n, \quad \lim_{t_n\to t_0} t_n = t_0, \quad \text{then} \quad \lim_{n\to\infty} x(t_n) = x(t_0). \tag{5.1.7}$$

Let us choose and fix a sequence $(t_n)_{n\geq 0}$ satisfies the equation (5.1.7). We want to show that

$$\lim_{n\to\infty} x(t_n) = x(t_0). \tag{5.1.8}$$

For this purpose, let us observe that

$$\begin{aligned}
x(t_n) - x(t_0) &= \int_0^{t_n} S(t_n - r) f(r)\, dr - \int_0^{t_0} S(t_0 - r) f(r)\, dr \\
&= \int_0^{t_0} S(t_n - r) f(r)\, dr + \int_{t_0}^{t_n} S(t_n - r) f(r)\, dr - \int_0^{t_0} S(t_0 - r) f(r)\, dr \\
&= \int_0^{t_0} \big[S(t_n - r) f(r) - S(t_0 - r) f(r)\big]\, dr + \int_{t_0}^{t_n} S(t_n - r) f(r)\, dr.
\end{aligned}$$

Thus, by [49, Corollary V. 1], we have

$$\begin{aligned}
|x(t_n) - x(t_0)|_X = &\int_0^{t_0} |S(t_n - r) f(r) - S(t_0 - r) f(r)|_X\, dr \\
&+ \int_{t_0}^{t_n} \|S(t_n - r)\|_{\mathcal{L}(X)} |f(r)|_X\, dr.
\end{aligned} \tag{5.1.9}$$

We first show that the second term of the RHS of the equality (5.1.9) converges to 0 as $n \to \infty$. Since for every $r \in [0, t_n]$ so that $t_n - r \in [0, t_n] \subset [0, T]$, we have for every n

$$\|S(t_n - r)\|_{\mathcal{L}(X)} \leq C_T.$$

Therefore,

$$\int_{t_0}^{t_n} \|S(t_n - r)\|_{\mathcal{L}(X)} |f(r)|_X\, dr \leq C_T \int_{t_0}^{t_n} |f(r)|_X\, dr = \int_0^T \mathbb{1}_{[t_0, t_n]}(r) |f(r)|_X\, dr.$$

By applying the Lebesgue Dominated Convergent Theorem (LDCT) with the following choices $h_n(r) = \mathbb{1}_{[t_0,t_n]}(r)|f(r)|$, $g(r) = |f(r)|$ and $h(r) = 0$ for $r \in [0, T]$. Because $t_n \to t_0$ we easily infer that $h_n(r) \to h(0)$ for every $r \in [0, T]$. Moreover, since the function $f \in L^1(0, T; X)$,

$$\int_0^T g(r)\, dr = \int_0^T |f(r)|_X\, dr < \infty.$$

Hence the assumptions of LDCT are satisfied and therefore,

$$\int_0^T t_n(r)\, dr \to \int_0^T h(r)\, dr = 0.$$

This implies that

$$\int_{t_0}^{t_n} \|S(t_n - r)\|_{\mathcal{L}(X)} |f(r)|_X\, dr \to 0 \;\text{ as } n \to \infty. \tag{5.1.10}$$

By this, we deduce that the second term of the RHS of the equality (5.1.9) converges to 0 as $n \to \infty$. Now we show that the first term of the RHS of the equality (5.1.9) converges to 0 as $n \to \infty$. For this purpose, let us recall that from Lemma 5.6 for each r the function

$$\beta : [0, \infty)] \ni t \mapsto S(t) f(r) \in X$$

is continuous. Since $t_n - r \to t_0 - r$ for every $r \in [0, t_0]$ we infer that $S(t_n - r)f(r)$ is also converges to $S(t_0 - r)f(r)$ in X. That is, for every $r \in [0, t_0]$

$$|S(t_n - r)f(r) - S(t_0 - r)f(r)|_X \to 0.$$

If we put $h_n(r) = |S(t_n - r)f(r) - S(t_0 - r)f(r)|_X$ then $h_n(r) \to 0$ for every $r \in [0, t_0]$. Moreover,

$$\begin{aligned} h_n(r) &\le \|S(t_n - r)\||f(r)|_X + \|S(t_0 - r)\||f(r)|_X \\ &\le C_T|f(r)|_X =: g(r) \;\; r \in [0, T]. \end{aligned}$$

Put $h(r) = 0$, then again the assumptions of LDCT are satisfied and therefore,

$$\int_0^T h_n(r)\, dr \to \int_0^T h(r)\, dr = 0.$$

Hence, we infer that

$$\int_0^{t_0} |S(t_n - r)f(r) - S(t_0 - r)f(r)|_X\, dr \to 0. \tag{5.1.11}$$

Hence, by substituting equations (5.1.10) and (5.1.11) into equation (5.1.9) we infer that equation (5.1.8) holds. Thus, the proof of part (iv) of Lemma 5.5 is complete. □

Now we are ready to define the mild solution to problem (5.1.4)-(5.1.5).

Definition 5.7. Assume that A is a generator of a C_0-semigroup $\{S(t)\}_{t\ge 0}$ of bounded linear operators on a Banach space X. Let a function $f \in L^1(0, T; X)$ for some $T > 0$. A function $u \in C([0, T], X)$ is called a mild solution to the linear in- homogeneous equation (5.1.4)-(5.1.5) if and only if the following equality holds

$$u(t) = S(t)u_0 + \int_0^t S(t - s)\, f(s)\, ds, \;\; \text{for every } t \in [0, T]. \tag{5.1.12}$$

It is important to remark that the function u defined by the formula (5.1.12) belongs to the space $C([0, T], X)$ because of Corollary 2.76 and our Lemma 5.5.
Recall that, if $f = 0$, then equation (5.1.4)-(5.1.5) is called linear homogeneous and in this case, a mild solution is equal to $S(t)u_0$, $t \in [0, T]$.
The same comments apply to the equation on the whole interval $[0, \infty)$. The argument at the bottom of page 105 of [33] implies the following corollary.

Corollary 5.8. *Assume that $T > 0$. If $x \in X$, $f \in L^1(0, T; X)$ and a function $u \in C([0, T], X)$ is a strong solution to the initial value problem* (5.1.4)-(5.1.5) *then u is also a mild solution to the same problem.*

Remark 5.9. A converse to Corollary 5.8 is not true even when $f = 0$. Indeed, one can find a simple counterexample:

We can also define a mild solution for the whole real half-line $[0, \infty)$ as follows.

Definition 5.10. Assume that A is a generator of a C_0-semigroup $\{S(t)\}_{t\geq 0}$ of bounded linear operators on a Banach space X. Assume also that a function f belongs to $L^1_{loc}([0, \infty); X)$. A function $u \in C([0, \infty); X)$ is called a mild solution to the problem (5.1.4)-(5.1.5) if and only if the following equality holds

$$u(t) = S(t)u_0 + \int_0^t S(t-s)\, f(s)\, ds, \quad \text{for every } t \in [0, \infty).$$

Ball in [1] studied mainly a notion of a weak solution and he proved the uniqueness of a weak solution and the equivalence between the notions of weak and mild solutions. The following definition is taken from [1]. In this definition by A^* we denote the dual operator to the operator A, see [33, section 1.10]. The notation we use here is the notation from that book. Whereas, the notion of an absolutely continuous function has been recalled in Definition 2.7 in Chapter 2 of this thesis. An obvious advantage of a notion of a weak solution is that one does not require A to be an infinitesimal generator of a C_0-semigroup. However, under some reasonable assumptions, see Theorem 5.12 below, if for every $x \in X$ there exists a unique weak solution, then A is an infinitesimal generator of the C_0-semigroup.

Definition 5.11. Assume that A is a densely defined closed linear operator on a real or complex Banach space X.
(I) Assume that $T > 0$ and a function $f \in L^1(0, T; X)$ is given. A function $u \in C([0, T]; X)$ is called a weak solution of problem (5.1.4)-(5.1.5) if and only if for every $v \in D(A^*)$ the function $[0, T] \ni t \mapsto \langle u(t), v\rangle \in \mathbb{R}$ is absolutely continuous and

$$\frac{d}{dt}\langle u(t), v\rangle = \langle u(t), A^*v\rangle + \langle f(t), v\rangle$$

for almost all $t \in [0, T]$.[1]
(II) If a function $f \in L^1_{loc}([0, \infty); X)$, i.e., $f \in L^1(0, T; X)$ for every $T > 0$, then a function $u \in C([0, \infty); X)$ is called a weak solution of problem (5.1.4)-(5.1.5) if and only if for every $v \in D(A^*)$ the function $\langle u(t), v\rangle$ is locally absolutely continuous on $[0, \infty)$ and

$$\frac{d}{dt}\langle u(t), v\rangle = \langle u(t), A^*v\rangle + \langle f(t), v\rangle$$

for almost all $t \in [0, \infty)$.

[1] This "almost all" we mean with respect to the Lebesgue measure on $[0, T]$

In the following, we formulate a significant feat presented in [1] about the weak solution and the infinitesimal generator. In the next theorem, Ball [1] gives the existence and the uniqueness of the weak solution of Problem (5.1.4)-(5.1.5).

Theorem 5.12. *[1] Assume that A is densely defined closed linear operator on a real or complex Banach space X, $T > 0$ and function $f \in L^1(0, T; X)$. Then the following two conditions are equivalent.*

(i) For every $u_0 \in X$, there exists a unique weak solution u of problem (5.1.4) -(5.1.5)*;*

(ii) A is the infinitesimal generator of a C_0-semigroup $\{S(t)\}_{t\geq 0}$ on X.

In this case, the solution u is given by the formula (5.1.12) *and hence u is a mild solution to the initial value problem* (5.1.4)-(5.1.5).

One can deduce the following corollary.

Corollary 5.13. *Assume that A is the infinitesimal generator of a C_0-semigroup $\{S(t)\}_{t\geq 0}$ on a Banach space X. Assume that $T > 0$, $u_0 \in X$, $f \in L^1(0, T; X)$ and a function $u \in C([0, T]; X)$. Then the following two conditions are equivalent.*

(i) A function u is a weak solution of problem (5.1.4)-(5.1.5)

(ii) A function u is a mild solution to the problem (5.1.4)-(5.1.5)

Proof of Corollary 5.13. The proof of implication (i) $\implies$ (ii) can be done in a similar way to proof of Proposition 4.2 in [9], see also [16]. The proof of implication (ii) $\implies$ (i) can be found in the paper [1] by Ball. □

In the next result, we will prove that a classical solution to problem (4.2.1)-(4.2.2), is also a weak solution to equation (5.1.4) satisfying the initial condition $u(0) = u_0$.

Proposition 5.14. *Let A be the infinitesimal generator of the C_0-semigroup $\{\pi_t\}_{t\geq 0}$ on the space $E = {}_0C([0, 1])$, see Theorems 4.19 and 4.20. Assume that $u_0 \in E$. Assume that a continuous function $u : [0, \infty) \times [0, 1] \to \mathbb{R}$ is a classical solution to problem* (5.1.5)-(5.1.4) *in the sense of Definition 4.23. Then a function u viewed as a function $u : [0, \infty) \to E$ is a weak solution, in the sense of Definition 5.11, to the problem* (5.1.4)-(5.1.5).

Proof of Proposition 5.14. For simplicity, we assume that $f = 0$. The same proof works in the case when the function $f \neq 0$.
Assume that a continuous function $u : [0, \infty) \times [0, 1] \to \mathbb{R}$ is a classical solution to problem (4.2.1)-(4.2.2). Firstly it is easy to prove that the corresponding function $u : [0, \infty) \to E$ is

well-defined and continuous, Let us take an arbitrary $\eta \in D(A^*)$. According to Definition 5.11, it is sufficient to show that the function $(0,\infty) \ni t \mapsto \langle u(t), \eta\rangle$ is of C^1-class, and satisfies

$$\frac{d}{dt}\langle u(t), A^*\eta\rangle = \langle u(t), A^*\eta\rangle, \quad \text{for every } t \in (0,\infty).$$

Note that

$$\langle u(t), \eta\rangle = \int_0^1 u(t,x)\eta(x)\,dx, \ \ t \geq 0.$$

Therefore, by the theorem about differentiation of a function defined by an integral, see e.g. [39, Theorem 9.42], this function is differentiable and

$$\frac{d}{dt}\langle u(t), \eta\rangle = \frac{d}{dt}\int_0^1 u(t,x)\eta(x)\,dx = \int_0^1 \frac{\partial}{\partial t}u(t,x)\eta(x)\,dx = \cdots$$

Since by the assumption the function u is a classical solution we infer that

$$\begin{aligned}\cdots &= -\int_0^1 a(x)\frac{\partial u(t,x)}{\partial x}\,\eta(x)\,dx = -\int_0^1 \frac{\partial u(t,x)}{\partial x}[a(x)\eta(x)]\,dx \\ &= \int_0^1 u(t,x)\frac{\partial[a(x)\eta(x)]}{\partial x}\,dx = \int_0^1 u(t,x)\frac{d[a(x)\eta(x)]}{dx}\,dx = \langle u(t), A^*\eta\rangle,\end{aligned}$$

where the last equality follows from the Integration by parts formula, see [39, Theorem 6.22]. Hence we proved that the function u is a weak solution. □

5.2 Solutions to Evolution Equations with Lipschitz Nonlinearity

Theorem 5.15. *Assume that X is a Banach space and $\{S(t)\}_{t\geq 0}$ is a C_0-semigroup on X. If $f : X \to X$ is a globally Lipschitz map, then for every $u_0 \in X$, there exists a unique continuous function $u : [0,\infty) \to X$ such that*

$$u(t) = S(t)u_0 + \int_0^t S(t-s)\,f\big(u(s)\big)\,ds, \ \ t \geq 0. \tag{5.2.1}$$

The proof of this Theorem is based on the fact that $\{S(t)\}_{t\geq 0}$ is a C_0-semigroup on the space X and Theorem VI.1.2 in [33] with $f(t,x) = f(x)$.

Remark 5.16. The above Theorem is applicable in the following case: The Banach space $X = {}_0C([0,1])$ and the C_0-semigroup $\{\pi_t\}_{t\geq 0}$ that was defined by equation (4.2.17).

Definition 5.17. Assume that X is a Banach space and A is a generator of a C_0-semigroup $\{S(t)\}_{t\geq 0}$ of bounded linear operators on a Banach space X. Assume also that $f : X \to X$ is a measurable function. A mild solution to the system

$$\frac{du(t)}{dt} = Au(t) + f(u(t)), \quad t > 0, \tag{5.2.2}$$

$$u(0) = u_0, \tag{5.2.3}$$

where $u_0 \in X$, is a function $u \in C\big([0,\infty), X\big)$ satisfying $f \circ u \in L^1_{\text{loc}}([0,\infty), X)$ so that the equality (5.2.1) is satisfied.

Note that if $f : X \to X$ is a continuous function and $u \in C\big([0,\infty), X\big)$ then $f \circ u \in C\big([0,\infty), X\big)$ and hence $f \circ u \in L^1_{\text{loc}}([0,\infty), X)$.

Definition 5.18. A semiflow associated to the system (5.2.2)-(5.2.3) is a family $\{\mathbf{S}(t)\}_{t\geq 0}$ of maps from X to X defined by

$$\mathbf{S}(t)(u_0) = \mathbf{S}(t, u_0) = \mathbf{S}_t(u_0) := u(t), \quad t \in [0,\infty) \text{ and } u_0 \in X,$$

where the function u is a mild solution to the system (5.2.2)-(5.2.3).

Remark 5.19. Theorem 5.15 can be rewritten in the following way:
If $u_0 \in X$ and $f : X \to X$ is a globally Lipschitz map then there exists a unique mild solution to problem (5.2.2)-(5.2.3).

Remark 5.20. Assume that $\{\pi(t)\}_{t\geq 0}$ is a C_0-semigroup on a Banach space X and it's infinitesimal generator A. Assume that $\lambda \in \mathbb{R}$. Let $\{S(t)\}_{t\geq 0}$ be the C_0-semigroup on the space X and it's infinitesimal generator $A + \lambda I$, i.e.,

$$S(t) = e^{\lambda t}\pi(t), \quad t \geq 0,$$

see Theorem 2.88. Assume also that $f : X \to X$ and $f_0 : X \to X$ are measurable functions such that

$$f(x) = \lambda x + f_0(x). \tag{5.2.4}$$

Assume that $u_0 \in X$. Let us consider equation (5.2.2). A mild solution to equation (5.2.2) with the initial condition (5.2.3) by our definition is a function u satisfying

$$u(t) = \pi(t)u_0 + \int_0^t \pi(t-s)\, f\big(u(s)\big)\, ds, \quad t \geq 0.$$

In view of equality (5.2.4), equation (5.2.2) can be written in the following form

$$\frac{du(t)}{dt} = [A + \lambda I]u(t) + f_0(u(t)), \quad t > 0. \tag{5.2.5}$$

Since $A+\lambda I$ is a generator of the semigroup $\{S(t)\}_{t\geq 0}$, we can also define a mild solution of the equation (5.2.5) by

$$u(t) = S(t)u_0 + \int_0^t S(t-s)\, f_0\big(u(s)\big)\, ds, \ \ t \geq 0.$$

One can prove the following **Equivalence Theorem**.
A function $u \in C([0,\infty), X)$ is a mild solution to equation (5.2.2) if and only if it is the mild solution to the equation (5.2.5).

In this section, we considered equations with the nonlinearity f being globally Lipschitz. A function f investigated in the next section is not globally Lipschitz. In the following publications, we will investigate the existence of non-trivial invariant measures for our problem. Let us note that according to the last part of Lemma 5.24, an example of a globally Lipschitz map is provided by the Nemytski map associated with a globally Lipschitz function $c : \mathbb{R} \to \mathbb{R}$.

5.3 Solutions to Evolution Equations with Dissipative Nonlinearity

Our new contribution in this chapter is that we apply the abstract result and definition from Ball [1] to define the mild solution to the special case of the semigroup $\{\pi(t)\}_{t\geq 0}$. This is only possible because we proved in Theorem 4.19 that $\{\pi(t)\}_{t\geq 0}$ is a C_0-semigroup on the Banach space $E = {}_0C([0,1])$. We consider special case of equation (4.0.4)-(4.0.5). In particular, we consider the following nonlinear problem

$$\frac{\partial u(t,x)}{\partial t} + a(x)\frac{\partial u(t,x)}{\partial x} = \lambda u(t,x) - u^3(t,x), \quad t>0,\, x\in[0,1], \tag{5.3.1}$$

$$u(0,x) = u_0(x), \ \ x \in [0,1] \tag{5.3.2}$$

where $u_0 \in E = {}_0C([0,1])$. In this section we consider the C_0-semigroup $\{\pi(t)\}_{t\geq 0}$ from Theorem 4.19 on the space E and it's infinitesimal generator A. Let also $\{S(t)\}_{t\geq 0}$ be the C_0-semigroup on the space E and it's infinitesimal generator $A+\lambda I$, so that $S(t) = e^{\lambda t}\pi(t)$ for $t \geq 0$. The next definition is a definition of a mild solution to Problem (5.3.1) as a special case of the abstract definition .

Definition 5.21. Assume that $u_0 \in E$. A mild solution to the system (5.3.1)-(5.3.2) is a function $u \in C\big([0,\infty), X\big)$ such that equality (5.2.1) is satisfied with $X = E$ and $f(u) = \lambda u - u^3$.

Let us notice that the function $f : E \to E$ used above is of dissipative type. In the following result, we use a notion of the mild solution introduced in Definition 5.17.

Theorem 5.22. *Assume that X is a separable Banach space and A is an infinitesimal generator of a contraction type C_0-semigroup $\{S(t)\}_{t\geq 0}$ on X and assume that $f : X \to X$ is a dissipative map. Then for every $u_0 \in X$, there exists a unique continuous function $u : [0, \infty) \to X$ which is a mild solution to the problem* (5.2.2)*-*(5.2.3)*.*
Moreover, the solution u depends continuously on the initial data u_0. That is, if $(u_0^n)_{n=1}^{\infty}$ is an X-valued sequence and $u_0 \in X$ such that for every $T > 0$ if

$$u_0^n \to u_0 \text{ in } X \quad \text{then } u^n \to u \text{ in } C([0,T], X),$$

where u^n is the unique mild solution to equation (5.2.2) *with the initial data u_0^n, i.e., the unique mild solution to the following equation*

$$\begin{aligned}\frac{du^n(t)}{dt} &= Au^n(t) + f(u^n(t)), \quad t > 0\\ u^n(0) &= u_0^n.\end{aligned}$$

Proof of Theorem 5.22. See the proof of Theorem 5.5.8 in [18]. □

Remark 5.23. In Theorem 5.15 and Theorem 5.22, the equality (5.2.1) hold the same. The only difference is the assumptions in these two Theorems.

In order to apply the above abstract result to our system (5.3.1)-(5.3.2), let us formulate the following auxiliary results.

Lemma 5.24. *Let $c : \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying the following assumption*

$$c(0) = 0. \tag{5.3.3}$$

Let f be a function defined by

$$f : E \ni u \mapsto c \circ u \in E. \tag{5.3.4}$$

Then the function f satisfies the following assertions.

(i) Function f is a well defined map from E to itself.

(ii) Function $f : E \to E$ is continuous.

Moreover, if the function c is globally Lipschitz then f is also globally Lipschitz.

Remark 5.25. Assumption (5.3.3) is needed because our space E consists of continuous functions vanishing at 0. If E were replaced by the space $C([0, 1])$, assumption (5.3.3) would not be required.

Lemma 5.26. *Let $E = {}_0C([0,1])$. Then, the function*

$$f_0 : E \ni u \mapsto -u^3 \in E, \tag{5.3.5}$$

is dissipative, see Definition 5.1 and Lipschitz on balls.

The Proof of Lemma 5.24 is standard, see for instance [8, Proposition 5.1 and Theorem 5.2] for more difficult results. The Proof of Lemma 5.26 is classical and can be found in the book [18, Chapter 6].
Next, we state the following theorem that is a consequence of Theorem 5.22.

Theorem 5.27. *Assume that $\lambda \geq 0$ and the function $f_0 : E \to E$ is defined by formula (5.3.5). Then for every $u_0 \in E$, there exists a unique continuous function $u : [0,\infty) \to E$ such that the following equality is satisfied.*

$$u(t) = S(t)u_0 + \int_0^t S(t-s)\, f_0(u(s))\, ds, \quad t \geq 0, \tag{5.3.6}$$

Moreover, the solution u depends continuously on the initial data u_0. That is, if $(u_0^n)_{n=1}^{\infty}$ is an E-valued sequence and $u_0 \in E$ such that $u_0^n \to u_0$ in E then, for every $T > 0$.

$$u^n \to u \text{ in } C([0,T], E),$$

where u^n is the unique mild solution to equation (5.3.1) with the initial data u_0^n, i.e., the unique mild solution to the following equation

$$u^n(t) = S(t)u_0^n + \int_0^t S(t-s)\, f_0(u^n(s))\, ds. \;\; t \geq 0.$$

We can get rid of the exponential function in the above Theorem by replacing the nonlinear function f_0 by $f = f_0 + \lambda I$, i.e.,

$$f : E \ni u \mapsto \lambda u + f_0(u) = \lambda u - u^3 \in E. \tag{5.3.7}$$

We tacitly assume that f maps E to E. To verify that the map f defined by formula (5.3.7) is well defined, let us assume that $u \in E$. Then we have,

$$[f(u)](x) = \lambda u(x) - \big(u(x)\big)^3, \;\; x \in [0,1].$$

We want to show that $f(u) \in E$. For this aim, let us first observe that if $t = 0$ then $u(0) = 0$ and hence $[f(u)](0) = 0$. Secondly, a composition of continuous functions is a continuous function, see [39, Theorem 4.9], we infer that the function

$$[0,1] \ni x \mapsto [f(u)](x) = \lambda u(x) - \big(u(x)\big)^3 \in \mathbb{R}$$

is continuous. Hence, we showed that the function f is well defined. Using this new object we deduce from Theorem 5.27 the following result.

Corollary 5.28. *For every u_0 belongs to the space E there exists a unique continuous function $u : [0, \infty) \to E$ such that the following equality is satisfied*

$$u(t) = \pi(t)u_0 + \int_0^t \pi(t-s)\, f(u(s))\, ds, \quad t \geq 0. \tag{5.3.8}$$

The Definition and Remark about the mild solution from the previous section 5.2 can be generalised to the current setting as follows.

Definition 5.29. Assume that $u_0 \in E$. A *mild* solution to the system (5.3.1)-(5.3.2) is a function $u \in C\big([0, \infty), X\big)$ such that equality (5.3.6), or equivalently (5.3.8) is satisfied.

Remark 5.30. Theorem 5.27 or Corollary 5.28 can be rewritten in the following way. If $u_0 \in E$ then there exists a unique *mild* solution to problem (5.3.1)-(5.3.2).

Example 5.31. *Assume that $u_0 = 0 \in E$. Then by the definition* (5.3.7) *of the function f, $f(u_0) = 0$. Moreover, since $\{\pi(t)\}_{t \geq 0}$ is a C_0-semigroup on the space E, we infer that $\pi(t)(0) = 0$ for every $t \geq 0$. Hence the constant function u is defined by*

$$u(t) := u_0 = 0, \quad t \in [0, \infty)$$

satisfies equation (5.3.8)*. Hence this constant function u is the unique mild solution to Problem* (5.3.1)-(5.3.2)*.*

We finish these considerations with the following version of Definition 5.18.

Definition 5.32. A semiflow associated to the system (5.3.1)-(5.3.2) is a family $\{\mathbf{S}(t)\}_{t \geq 0}$ of maps from E to E defined by

$$\mathbf{S}(t)(u_0) = \mathbf{S}(t, u_0) = \mathbf{S}_t(u_0) := u(t), \quad t \in [0, \infty) \text{ and } u_0 \in E, \tag{5.3.9}$$

where the function u is the unique mild solution to the system (5.3.1)-(5.3.2), hence with the initial data u_0. The existence and the uniqueness of a mild solution to the system (5.3.1)-(5.3.2) is guaranteed by Remark 5.30.

Remark 5.33. It is important to bear in mind that a mild solution is not necessarily a classical solution in the sense of Remark 4.11. First of all, even in the homogeneous case, i.e., when the external force $f = 0$, the classical solution has been defined only for the initial data function u_0 of C^1-class. Secondly, the classical solution is required to be of C^1-class. On the other hand, the mild solution requires the initial data function u_0 to be only a continuous function, and the mild solution is required itself to be only a continuous function as well. Finally, a classical solution satisfies equation (5.3.1) for every $(t,x) \in [0,\infty) \times [0,1]$ while a mild solution satisfies the integral equation (5.3.8).

Corollary 5.34. *The semiflow associated to the system* (5.3.1)-(5.3.2) *and introduced in Definition 5.32 is continuous with respect to the initial data. That is, if $n \in \mathbb{N}$, $u_0 \in E$ and $\varepsilon > 0$, then there exists $\delta > 0$ such that if $\tilde{u}_0 \in E$ such that $\|\tilde{u}_0 - u_0\|_E \le \delta$ then*

$$\sup_{t\in[0,n]} \|\mathbf{S}_t(\tilde{u}_0) - \mathbf{S}_t(u_0)\|_E \le \varepsilon.$$

Proof of Corollary 5.34. Follows from inequality (6.3.4) in [18, section 6.3]. In fact, in that section a stronger result is proven, i.e., there exists $C \in \mathbb{R}$ such that for all $n \in \mathbb{N}$ and $u_0, \tilde{u}_0 \in E$,

$$\sup_{t\in[0,n]} \|\mathbf{S}_t(\tilde{u}_0) - \mathbf{S}_t(u_0)\|_E \le e^{Cn}\|\tilde{u}_0 - u_0\|_E.$$

□

5.3.1 An explicit formula for a classical solution

In this section, we formulate a result about a special representation for the mild solutions to the system (5.3.1)-(5.3.2). For this aim, we need to diverge a bit and consider the classical characteristics method. We consider the following

$$\frac{\partial u(t,x)}{\partial t} + a(x)\frac{\partial u(t,x)}{\partial x} = c\big(u(t,x)\big) \tag{5.3.10}$$

with the initial condition

$$u(0,x) = u_0(x), \quad x \in [0,1]. \tag{5.3.11}$$

We continue to assume that Assumption 4.2 are satisfied for the function a, whereas the function c depends only on the third variable, i.e., $c : \mathbb{R} \to \mathbb{R}$, and assumed to be Lipschitz on balls and of dissipative type function such that (5.3.3) holds. Let us point out that with a given function c as above we associate the Nemytski map $f : E \to E$ by the formula (5.3.4). In other words,

$$[f(u)](x) = c(u(x)), \quad \text{for every } x \in [0,1].$$

For instance, if

$$c(z) = \lambda z - z^3, \text{ for } z \in \mathbb{R}, \tag{5.3.12}$$

then the above Nemtyski map f is equal to the map f introduced earlier in (5.3.7).
The function f indeed maps the space E to itself, where the space E is always the same

$E = {}_0C([0,1]$ with the sup norm. For more details see Lemma 5.24. Moreover, this function c satisfies condition (5.1.2), i.e. c is of dissipative-type (with constant λ).

The equation (5.3.10) can be solved by the characteristics method. To be precise, we consider a system of two ordinary differential equations in $\mathbb{R}_+ \times \mathbb{R}$ as follows

$$\frac{dx(t)}{dt} = a(x(t)), \tag{5.3.13}$$
$$\frac{dz(t)}{dt} = c(z(t)). \tag{5.3.14}$$

Suppose that u is a function of C^1-class that solves equation (5.3.10) and $x : I \to \mathbb{R}$ is a solution to the equation (5.3.13) satisfying the initial condition $x(0) = x_0$. The solution to the equation (5.3.13) has been discussed in detail before in Section 4.2 and it is given by the following formula

$$x(t) = G^{-1}\big(t + G(x_0)\big), \quad t \geq 0.$$

Suppose that for every $z_0 \in \mathbb{R}$ there exists a unique global solution $z : \mathbb{R}_+ \to \mathbb{R}$ of equation (5.3.14) satisfying the initial condition

$$z(0) = z_0. \tag{5.3.15}$$

where z is the local maximal solution equation (5.3.14) given by

$$z(t) := u\big(t, x(t)\big), \quad t \in I. \tag{5.3.16}$$

This local maximal solution is indeed a global solution when the function $c : \mathbb{R} \to \mathbb{R}$ is Lipschitz on balls and of dissipative type. Indeed, the local Lipschitzianity of the function c implies that Problem (5.3.14)-(5.3.15) has a unique local maximal solution z defined on some time interval $[0, \tau)$. The dissipativity type of function c, i.e. because function c satisfies condition (5.1.2), implies that $\tau = \infty$. Indeed, by assumption (5.1.1) we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\big[e^{-2\lambda t}|z(t)|^2\big] &= \frac{1}{2}\big[-2\lambda e^{-2\lambda t}|z(t)|^2 + e^{-2\lambda t}2z'(t)z(t)\big]\\ &\leq e^{-2\lambda t}\big[-\lambda|z(t)|^2 + \lambda|z(t)|^2\big] = 0, \quad t \in [0, \tau).\end{aligned}$$

Hence the function $[0, \tau) \ni t \mapsto e^{-2\lambda t}|z(t)|^2$ is non-increasing, so that

$$e^{-2\lambda t}|z(t)|^2 \leq e^{-2\lambda 0}|z(0)|^2 = |z_0|^2, \quad t \in [0, \tau).$$

Thus we infer that

$$|z(t)| \leq e^{\lambda t}|z_0|, \quad t \in [0, \tau).$$

So the solution cannot explode on the maximal interval of existence and hence $\tau = \infty$. This solution of equation (5.3.14) satisfying the initial condition (5.3.15) will be denoted by

$$\psi(t,z_0) := z(t), \quad t \in \mathbb{R}_+, \quad z_0 \in \mathbb{R}.$$

Note that because we assume that $c(0) = 0$, the unique solution of problem (5.3.14)-(5.3.15) with $z_0 = 0$ is the constant function $z(t) = 0$ for all $t \in \mathbb{R}_+$. Hence we infer that

$$\psi(t,0) = 0, \quad t \in \mathbb{R}_+. \tag{5.3.17}$$

From equation (5.3.16) we infer that $\psi(t,z_0) = u\big(t,x(t)\big)$. Hence the general solution to the system (5.3.10)-(5.3.11) is given by

$$u(t,x) = u\big(t,x(t)\big) = \psi(t,z_0) = \psi\big(t,u_0(x_0)\big) = \psi\big(t,u_0\big[G^{-1}(G(x)-t)\big]\big).$$

The above formula is a generalisation of equation (4.2.11) in Section 4.2. The above formula has been derived for solutions of the C^1-class. However, we show that it is also valid for the mild solutions. To be precise we have the following representation result. This result is a generalization of the classical characteristics method to the case of the coefficient $a(x)$ in equation (5.3.1) being only continuous and satisfying the Osgood condition and the initial data u_0 being only a continuous function. The following theorem plays a significant role later when we prove the injectivity of the map Ψ, see Proposition 5.42, which is an important tool for finding the invariant measures.

Theorem 5.35. ***Representation Theorem.*** *Assume that $\lambda > 0$ and that $c : \mathbb{R} \to \mathbb{R}$ is a function given by formula* (5.3.12)*. Let, for any $z_0 \in \mathbb{R}$, $\psi(\cdot,z_0) : [0,\infty) \to \mathbb{R}$ be the unique solution $z : [0,\infty) \to \mathbb{R}$ of the problem*

$$\frac{dz}{dt} = c\big(z(t)\big), \quad t \geq 0, \tag{5.3.18}$$

satisfying the initial condition

$$\psi(0,z_0) = z(0) = z_0. \tag{5.3.19}$$

Let us assume that a function f is defined by formula (5.3.7)*. Assume that $u_0 \in E$ and $u : [0,\infty) \to E$ is the unique mild solution to the problem* (5.3.1)-(5.3.2)*, whose existence is guaranteed by Remark 5.30. Then,*

$$u(t,x) = \begin{cases} \psi\big(t,u_0\big[G^{-1}(G(x)-t)\big]\big), & \text{if } (t,x) \in [0,\infty) \times (0,1], \\ 0, & \text{if } (t,x) \in [0,\infty) \times \{0\}. \end{cases} \tag{5.3.20}$$

Note that we need the second line in the formula above (5.3.20) because as we mentioned in Proposition 4.8 the domain of the function G is the interval $(0,1]$. In particular, $0 \notin dom(G)$, so $G(0)$ is not well defined. we have encountered the same issue in the case

of the homogeneous equation, see equation (4.2.11) in Section 4.2.
Before we embark on the proof of Theorem 5.35, we need first to formulate an important result about the existence of the function ψ.

Lemma 5.36. *Let us assume that* $\lambda > 0$ *and a function/vector field* $c : \mathbb{R} \to \mathbb{R}$ *is defined by equality* (5.3.12).

(o) Then for every $z_0 \in \mathbb{R}$*, there exists a unique global solution* $z : [0, \infty) \to \mathbb{R}$*, which is a solution of the ODE* (5.3.18) *and satisfies the initial condition* (5.3.19).

Let $\psi : [0, \infty) \times \mathbb{R} \to \mathbb{R}$ *be the function introduced in Theorem 5.35. Then*

(i) The function ψ *is continuous. In particular, it is uniformly continuous on every rectangle* $[0, T] \times [-M, M]$ *for every* $T > 0$ *and every* $M > 0$*. mb, Moreover, equation* (5.3.17) *is satisfied.*

(ii) For every $t \in [0, \infty)$ *the function* $\psi_t := \psi(t, \cdot) : \mathbb{R} \to \mathbb{R}$*, i.e.*

$$\psi_t(z_0) := \psi(t, z_0), t \in [0, \infty),\ z_0 \in \mathbb{R},$$

is injective,

(iii) The function ψ *is of* C^1*-class and the derivative* $\frac{\partial \psi(t,z_0)}{\partial t}$ *satisfies*

$$\frac{\partial \psi(t, z_0)}{\partial t} = c\big(\psi(t, z_0)\big), \quad t \geq 0,$$

and the derivative $\frac{\partial \psi(t,z_0)}{\partial z_0}$ *solves the following linear equation*

$$\begin{aligned} \frac{\partial \psi(t, z_0)}{\partial z_0} &= c'(\psi(t, z_0)) \frac{\partial \psi(t, z_0)}{\partial z_0}, \quad t \geq 0, \\ \frac{\partial \psi(0, z_0)}{\partial z_0} &= 1. \end{aligned}$$

The function ψ *is often called the flow associated with the ODE* (5.3.18).

(iv) Moreover, if we replace the vector field c *by vector field* $-c$ *and we denote by* ϕ_t *the corresponding flow, the flow associated with the following ODE*

$$\frac{dy}{dt} = -c\big(y(t)\big), \quad t \geq 0,$$

then

$$\phi_t \circ \psi_t = id \text{ and } \psi_t \circ \phi_t = id, \quad i.e.,\ \phi_t = \psi_t^{-1}.$$

Proof of Theorem 5.35. The proof is divided into three steps.

Step I. Assume additionally that the initial data u_0 is of C^1-class. In this step we show that the function u defined by formula (5.3.20), i.e.,

$$u : [0,\infty) \times (0,1] \ni (t,x) \mapsto \psi\big(t, u_0\big[G^{-1}(G(x)-t)\big]\big) \in \mathbb{R}$$

has the following four properties.

(i) If $x \in (0,1]$ then $u(0,x) = \psi\big(0, u_0\big[G^{-1}(G(x))\big]\big) = \psi\big(0, u_0[x]\big) = u_0(x)$, because by Lemma 5.36 and equality (5.3.19), $\psi(0,x) = x$ for all $x \in [0,1]$.

(ii) The function u is of C^1-class.
Indeed, the composition of C^1-class functions is of C^1-class, see [39, Theorem 5.5]. The claim follows because by assumption u_0 is of C^1-class, the functions G and G^{-1} are of of C^1-class, see Proposition 4.8 and function ψ is also of C^1-class by Lemma 5.36.

(iii) The function u from part (ii) satisfies (5.3.1)-(5.3.2) point-wise, i.e., it is a classical solution of that problem.

(iv) By applying Proposition 5.14 and Corollary 5.13 we deduce that the function u viewed as a continuous map from $[0,\infty)$ to the Banach space ${}_0C^1([0,1])$ is a mild solution to problem (5.3.1)-(5.3.2).

Step II. If (u_0^n) is a ${}_0C^1([0,1])$-valued sequence and $u_0 \in {}_0C([0,1])$ such that

$$u_0^n \to u_0 \text{ in } {}_0C([0,1]),$$

then, for every $T > 0$,

$$u^n \to u \text{ in } C([0,T], {}_0C([0,1])),$$

where the function u defined by formula (5.3.20) and u^n is a function defined by formula (5.3.20) with u_0 replaced by u_0^n, i.e.,

$$u^n(t,x) = \begin{cases} \psi\big(t, u_0^n\big[G^{-1}(G(x)-t)\big]\big), & \text{if } (t,x) \in [0,\infty) \times (0,1], \\ 0, & \text{if } (t,x) \in [0,\infty) \times \{0\}, \end{cases} \tag{5.3.21}$$

Proof Step II: Let us observe that we need to prove that for every $T > 0$,

$$u^n \to u \text{ in } C([0,T] \times [0,1]).$$

In other words, we need to prove that for every $T > 0$

$$u^n(t,x) \to u(t,x) \text{ uniformly w.r.t. } (t,x) \in [0,T] \times [0,1].$$

To prove the last assertion let us choose and fix $T > 0$. By applying Theorem A.1, and [35], we choose the following sets

$$X = [0,T] \times [0,1],\ Z = \mathbb{R},\ \psi : [0,\infty) \times \mathbb{R} \to \mathbb{R}.$$

where function ψ has been introduced in the statement of Lemma 5.36.

Note that

$$u^n = \psi \circ v^n \text{ and } u = \psi \circ v$$

where

$$v^n : [0,T]\times[0,1] \ni (t,x) \mapsto \begin{cases} \big(t, u_0^n\big[G^{-1}(G(x)-t)\big]\big), & \text{if } (t,x) \in [0,T] \times (0,1], \\ (t,0), & \text{if } (t,x) \in [0,T] \times \{0\}, \end{cases} \in [0,\infty)\times\mathbb{R}.$$

and

$$v : [0,T]\times[0,1] \ni (t,x) \mapsto \begin{cases} \big(t, u_0\big[G^{-1}(G(x)-t)\big]\big), & \text{if } (t,x) \in [0,T] \times (0,1], \\ (t,0), & \text{if } (t,x) \in [0,T] \times \{0\}, \end{cases} \in [0,\infty)\times\mathbb{R}.$$

Now we need to make sure that all the assumptions are satisfied. In other words, we need to show that

$$\psi \circ v^n \to \psi \circ v \text{ on } [0,T] \times [0,1].$$

According to part (i) of Lemma 5.36, the function ψ is uniformly continuous on every rectangle $[0,T] \times [-M,M]$ for $T > 0$ and $M > 0$. Regarding the functions v^n and v, since their first components are the same, it is sufficient to prove that the second components converge uniformly, i.e., we need to prove that

$$u_0^n(G^{-1}(G(x)-t)) \to u_0(G^{-1}(G(x)-t)) \ \text{ uniformly w.r.t. } (t,x) \in [0,T] \times [0,1].$$

The above formula follows by applying Theorem A.2 with the following choice of notations:

$$\begin{aligned} X &= [0,T] \times [0,1], \quad Y = [0,1] \\ g(t,x) &= G^{-1}(G(x)-t) \in Y, \quad \text{for } \ (t,x) \in X, \\ f_n &= u_0^n, \quad \text{and} \quad f = u_0. \end{aligned}$$

Hence, we deduce that

$$u_0^n(G^{-1}(G(x)-t)) \to u_0(G^{-1}(G(x)-t)) \ \text{ uniformly w.r.t. } (t,x) \in [0,T] \times [0,1].$$

As consequence, we infer that v^n convergent uniformly to v on the set $[0,T] \times [0,1]$. Let us observe that from (ii) in step I, on page 164 since the function u is of C^1-class, we deduce that functions v^n and v are continuous. Moreover, by [39, Theorem 2.41], the set $[0,T]\times[0,1]$ is compact. Hence, see [39, Theorem 4.15] the sets $v([0,T]\times[0,1])$ and $v^n([0,T]\times[0,1])$ are compact. Furthermore, by the uniform convergence $v^n\to v$ on the set $[0,T]\times[0,1]$ and [39, Theorem 2.41] again, we deduce that there exists $M>0$ such for every $n\in\mathbb{N}$

$$v^n([0,T]\times[0,1]), v^n([0,T]\times[0,1])\subset[0,T]\times[-M,M].$$

In order to apply the Theorem A.1 we put

$$X=[0,T]\times[0,1],\ \ Y=[0,T]\times[-M,M],\ \ Z=\mathbb{R},$$

and

$$g_n=v^n,\ \ g=v,\ \ \psi=\psi.$$

Then $u^n=\psi\circ g_n=\psi\circ v^n$ and $u=\psi\circ g=\psi\circ v$. In the proof of Lemma 5.36, the function ψ is uniformly continuous on the set $Y=[0,T]\times[-M,M]$. Therefore, Theorem A.1 is applicable and so we deduce that

$$u^n(t,x)\to u(t,x)\quad\text{uniformly w.r.t.}\quad (t,x)\in[0,T]\times[0,1].$$

By this, the proof of step II is complete.

Step III. Assume that $u_0\in E$ and $u:[0,\infty)\to E$ is the unique mild solution to the problem (5.3.1)-(5.3.2). Our aim here is to prove that u satisfies identity (5.3.20). Recall that $E={}_0C([0,1])$. Let us consider a sequence $(u_0^n)_{n=1}^\infty$ such that u_0^n belongs to the class ${}_0C^1([0,1])$ and

$$u_0^n\to u_0 \text{ in } E={}_0C\big([0,1]\big).$$

Let $u^n:[0,\infty)\to E$ is the unique mild solution to the equation (5.3.1) with initial data u_0 replaced by u_0^n. By the continuity part of Theorem 5.27 we infer that for every $T>0$, the following holds

$$u^n\to u \text{ in } C\big([0,T],{}_0C([0,1])\big).$$

Because u_0^n is regular, by Step I, the function u^n satisfies formula (5.3.21). Let us denote by z a function defined by formula (5.3.20). By Step II, we infer that for every $T>0$,

$$u^n\to z \text{ in } C\big([0,T],{}_0C([0,1])\big).$$

Hence, by the uniqueness of the limit, we infer that for every $T>0$,

$$u=z \text{ in } C([0,T],{}_0C\big([0,1]\big)).$$

Because T is arbitrary, we deduce that $u = z$. From the definition of the function z, we infer that the function u satisfies formula (5.3.20). Therefore, the proof of Theorem 5.35 is complete. □

Proof of Lemma 5.36. The uniform continuity of the function ψ on a rectangle $[0, T] \times [-M, M]$, when $T > 0$ and $M > 0$, is a consequence of continuity of function ψ and compactness of the rectangle $[0, T] \times [-M, M]$, see [39, Theorem 4.19]. But the continuity of the function ψ follows from Theorem 10.8.1 in [20] because the function c is locally Lipschitz which is a consequence of the function c is of C^1-class. This proves part (ii). Part (iii) follows from Theorem 3.4.2 in [14]. Finally, the proof of the last part (iv) follows from [14, Theorem 3.2.1]. □

Below we analyse the properties of the global solution depending on the initial condition z_0.

Proposition 5.37. *Assume that $c : \mathbb{R} \to \mathbb{R}$ is a function given by formula* (5.3.12). *Consider an arbitrary $z_0 \in \mathbb{R}$. Let $z : [0, \infty) \to \mathbb{R}$ be the unique global solution of the problem* (5.3.18)-(5.3.19). *Then*

1) if $z_0 \in \{-\sqrt{\lambda}, 0, \sqrt{\lambda}\}$, then the function c has a unique solution if $c(z_0) = 0$. Hence the constant function $z(t) = z_0$, for every $t \geq 0$, is a solution of the problem (5.3.18)-(5.3.19). *Hence*

$$\psi(t, z_0) = z_0 \text{ for every } t \geq 0.$$

2) if $z_0 > \sqrt{\lambda}$, then since $c(z) < 0$ for all $z > \sqrt{\lambda}$, then function z is decreasing on the maximal interval of the existence and $z(t) > \sqrt{\lambda}$. Hence we get another proof that z is a global solution and

$$z(t) \in (\sqrt{\lambda}, \infty), \quad \text{for every } t \geq 0.$$

Moreover we can show that $z(t) \to \sqrt{\lambda}$ as $t \to \infty$.

3) if $z_0 \in (0, \sqrt{\lambda})$, i.e., $0 < z_0 < \sqrt{\lambda}$, then $c(z) > 0$ for $z \in (0, \sqrt{\lambda})$. Hence, the solution of problem (5.3.18)-(5.3.19) *is increasing and $z(t) < \sqrt{\lambda}$. Thus, we get another proof that z is a global solution and*

$$z(t) \in (0, \sqrt{\lambda}), \quad \text{for every } t \geq 0.$$

Moreover, $z(t) \to \sqrt{\lambda}$ as $t \to \infty$.

4) if $z_0 \in (-\sqrt{\lambda}, 0)$, then the function $c(z) < 0$, so z is decreasing and

$$z(t) \in (-\sqrt{\lambda}, 0), \quad \text{for every } t \geq 0.$$

5) if $z_0 < -\sqrt{\lambda}$*, then the function* $c(z) > 0$*, so* z *is increasing and*

$$z(t) \in (-\infty, -\sqrt{\lambda}), \quad \textit{for every } t \geq 0.$$

In particular, for every $t \in [0, \infty)$*,* ψ_t *maps the set* $(-\sqrt{\lambda}, \sqrt{\lambda})$ *into itself, and the map* ϕ_t *(which is equal to* ψ_t^{-1}*) also maps the set* $(-\sqrt{\lambda}, \sqrt{\lambda})$ *into itself.*

The above Proposition 5.37 is about the properties of the solutions to the ODE (5.3.18). The representation Theorem 5.35 links solutions to the ODE (5.3.18) with solutions to the PDE (5.3.1). From Theorem 5.35 and Proposition 5.37 above we get the following corollary. The Corollary 5.38 is about properties of solutions to the PDE (5.3.1). These properties are the consequence of that relationship. The following corollary about mild solutions to Problem (5.3.1)-(5.3.2) describes solutions in very specific three cases which correspond to parts (1), (3) and (4) of the previous Proposition 5.37.

Corollary 5.38. *Let us assume that* $\lambda > 0$ *and a function* f *is defined by* (5.3.7)*. Assume that* $u_0 \in E$ *and* $u : [0, \infty) \to E$ *be the unique mild solution to problem* (5.3.1)-(5.3.2)*. Then, the following holds.*

1) If $u_0 = 0$ *in* E*, i.e.,*

$$u_0(x) = 0, \quad \textit{for every } x \in [0, 1], \tag{5.3.22}$$

then

$$u(t, x) = 0, \quad \textit{for all } (t, x) \in [0, \infty) \times [0, 1].$$

3) If for every $x \in (0, 1]$*,* $u_0(x) \in (0, \sqrt{\lambda})$*, then* $u(t, x) \in (0, \sqrt{\lambda})$ *for every* $(t, x) \in [0, \infty) \times [0, 1]$*.*

4) If for every $x \in (0, 1]$*,* $u_0(x) \in (-\sqrt{\lambda}, 0)$*, then* $u(t, x) \in (-\sqrt{\lambda}, 0)$ *for every* $(t, x) \in [0, \infty) \times [0, 1]$*.*

Proof of Corollary 5.38. Assume that $u : [0, \infty) \to E$ is the unique mild solution to problem (5.3.1)-(5.3.2).

1) Assume that $u_0 = 0$ in E. Then by Example 5.31, the unique mild solution is the constant function $u(t) = 0$ and hence

$$u(t, x) = 0, \;\; t \geq 0, \; x \in [0, 1].$$

3) Assume that $u_0(x) \in (0, \sqrt{\lambda})$, for all $x \in (0,1]$. By the representation Theorem 5.35 and condition (5.3.22) the solution u satisfies

$$\begin{aligned} u(t,x) &= \psi(t, u_0\big[G^{-1}(G(x)-t)\big]\big) \\ &= \psi(t, z_0), \quad \text{if } (t,x) \in [0,\infty) \times (0,1], \end{aligned}$$

where $\psi(\cdot, z_0) : [0,\infty) \to \mathbb{R}$ is the unique solution $z : [0,\infty) \to \mathbb{R}$ of equation (5.3.18) satisfying the initial condition (5.3.19). By applying part (3) of Proposition 5.37 we infer that

$$\psi(t, u_0\big[G^{-1}(G(x)-t)\big] \in \big(0, \sqrt{\lambda}\big), \;\; t \geq 0, x \in (0,1].$$

4) Assume that $u_0(x) \in (-\sqrt{\lambda}, 0)$, for all $x \in (0,1]$. By the representation Theorem 5.35 and condition condition (5.3.22) the solution u satisfies

$$\begin{aligned} u(t,x) &= \psi(t, u_0\big[G^{-1}(G(x)-t)\big]\big) \\ &= \psi(t, z_0), \quad \text{if } (t,x) \in [0,\infty) \times (0,1], \end{aligned}$$

Note that $\psi(\cdot, z_0) : [0,\infty) \to \mathbb{R}$ is the unique solution $z : [0,\infty) \to \mathbb{R}$ of equation (5.3.18) satisfying the initial condition (5.3.19). Similarly to the above, by applying part (4) of Proposition 5.37 we infer that

$$\psi(t, u_0\big[G^{-1}(G(x)-t)\big] \in \big(-\sqrt{\lambda}, 0\big), \;\; t \geq 0, x \in (0,1].$$

By this, the proof of Corollary 5.38 is complete. □

The above corollary can be used to prove the invariance of the sets W_0, W_0^+ and W_0^- in the following next section.

5.4 Invariant Measure for a Nonlinear PDE

In this section, we want to construct invariant measures for the nonlinear Lasota equation. We consider our special case, the equation (5.3.1)-(5.3.2). We continue to assume the assumptions about the coefficients a and c as we listed before in Section 5.3.1 are hold. Assume that there exists a continuous semiflow $\{\mathbf{S}_t\}_{t\geq 0}$ on the space E generated by the equation (5.3.1)-(5.3.2). Recall that

$$c(u) = \lambda u - u^3, \;\; u \in \mathbb{R}, \; \lambda > 0. \tag{5.4.1}$$

Define $u_+ = \min\{u > 0 : f(u) = 0\}$ and $u_- = \max\{u < 0 : f(u) = 0\}$. In the case of function c that given by equation (5.4.1) we note that $u_+ = \sqrt{\lambda}$ and $u_- = \sqrt{-\lambda}$. We consider a space

$$Y = C\big([0,\infty)\big). \tag{5.4.2}$$

Let us point out that this space Y is different from the space Y defined by formula (4.2.42) in Section 4.2.1.3. We endow the space Y is a topology induced by a sequence $(p_n)_{n=1}^{\infty}$ of seminorms on Y defined by

$$p_n(u) := \sup_{t\in[0,n]} |u(t)|, \quad u \in Y. \tag{5.4.3}$$

The space Y is not a normed vector space. The family $(p_n)_{n=1}^{\infty}$ generates a metric d on Y and (Y,d) is complete, see [38, chapter 1].

Proposition 5.39. *The family $\{T_t\}_{t\geq 0}$ defined by $\big(T_t u\big)(\cdot) = u(t+\cdot)$ for $t \geq 0$ is a C_0-semigroup on the space Y.*

Define the following sets

$$W_0^+ = \{v \in E : 0 < v(x) < u_+, \text{ for every } x \in (0,1]\}.$$

$$W_0^- = \{v \in E : u_- < v(x) < 0, \text{ for every } x \in (0,1]\}.$$

$$W_0 = \{v \in E : u_- < v(x) < u_+, \text{ for every } x \in (0,1]\}. \tag{5.4.4}$$

Note that we consider above $x > 0$ because if $x = 0$ then $v(x) = 0$. Let us also define the following maps

$$\Phi : W_0 \ni v \mapsto \{[0,\infty) \ni t \mapsto \big(\mathbf{S}_t v\big)(1)\} \in C([0,\infty);(u_-,u_+)), \tag{5.4.5}$$

and

$$\Psi : W_0 \ni v \mapsto \{[0,\infty) \ni t \mapsto h^{-1}\big(\mathbf{S}_t v(1)\big)\} \in Y. \tag{5.4.6}$$

where $h : \mathbb{R} \to (u_-,u_+)$ is an increasing function such that $h(0) = 0$ and there exists some $R > 0$ such that

$$h(x) = \begin{cases} u_+ - e^{-x}, & \text{if } x > R, \\ u_- + e^{x}, & \text{if } x < -R. \end{cases} \tag{5.4.7}$$

Let us observe that $h^{-1} : (u_-,u_+) \to \mathbb{R}$ is an increasing function such that $h^{-1}(0) = 0$.

Note that from the definitions of maps Φ and Ψ we infer that

$$\big[\Psi(v)\big](t) = h^{-1}\big(\Phi(v)(t)\big), \quad \text{for every } \ t \geq 0.$$

Proposition 5.40. *The sets W_0^+, W_0^- and W_0 are invariant with respect to the (nonlinear) semiflow $\{\mathbf{S}_t\}_{t\geq 0}$, i.e.,*

$$u_0 \in W_0^+ \implies \mathbf{S}_t u_0 \in W_0^+, \text{ for every } t \geq 0.$$

$$u_0 \in W_0^- \implies \mathbf{S}_t u_0 \in W_0^-, \text{ for every } t \geq 0.$$

$$u_0 \in W_0 \implies \mathbf{S}_t u_0 \in W_0, \text{ for every } t \geq 0.$$

Proof of Proposition 5.40. This result is a direct consequence of Corollary 5.38. □

The following two results are a generalization of Proposition 4.40 and Corollary 4.41 to the nonlinear case. The importance of those propositions is to study the properties of the mapping Φ and Ψ, which help us to prove the existence of the invariant measures.

Proposition 5.41. *Let us assume that $\{\mathbf{S}_t\}_{t\geq 0}$ is the continuous semiflow on the space $E = {}_0C\big([0,1]\big)$, defined by formula* (5.3.9)*. Let $Y = C\big([0,\infty)\big)$ be the topological vector space defined earlier in equation* (5.4.2) *with the corresponding seminorm defined in equation* (5.4.3)*. Let the maps Φ and Ψ defined by* (5.4.5) *and* (5.4.6) *respectively. Then the following statements are satisfied*

(i) if $v \in W_0$ then the function $\{[0,\infty) \ni t \mapsto \big(\mathbf{S}_t v\big)(1) \in (u_-, u_+)\}$ belongs to the space $C\big([0,\infty)(u_-, u_+)\big)$, and hence, the map $\Phi : W_0 \to C\big([0,\infty)(u_-, u_+)\big)$ defined by equation (5.4.5) *is well-defined,*

(ii) the map $\Psi : W_0 \to Y$ is continuous,

(iii) the following equality holds

$$T_t \circ \Psi = \Psi \circ \mathbf{S}_t. \tag{5.4.8}$$

Proof of Proposition 5.41. Proof of item (i): To prove that the map Φ is well-defined, let us take e_1 to be the evaluation map, which means, $e_1 : E \ni v \mapsto v(1) \in \mathbb{R}$. It is known that e_1 is a linear contraction and hence, it is continuous. Since $[\Phi v](t) = e_1(\mathbf{S}_t v)$ for every $v \in E$ and $t \geq 0$, by Corollary 2.76, the map $[0,\infty) \ni t \mapsto \mathbf{S}_t v \in E$ is continuous for every $v \in E$. Therefore, we deduce that for every $v \in E$, $\Phi v \in C([0,\infty))$. Hence the map Φ defined by equation (5.4.5) is well-defined.
Proof of item (ii): Because $\big[\Psi(v)\big](t) = h^{-1}\big(\Phi(v)(t)\big)$, $t \geq 0$, and the function h^{-1} is uniformly continuous on compact subsets of the open interval (u_-, u_+), it is sufficient to prove that the map Φ is continuous. Therefore, it is sufficient to prove that it is continuous with respect to each seminorm p_n. For this purpose, let us fix $n \in \mathbb{N}$, an element $v_0 \in E$ and $\varepsilon > 0$. We want to find $\delta > 0$ with the following property.

$$\text{If } \tilde{u}_0 \in E \text{ and } \|\tilde{u}_0 - u_0\|_E \leq \delta \text{ then } p_n\big(\Phi(\tilde{u}_0) - \Phi(u_0)\big) \leq \varepsilon.$$

Thus, by definitions of map Φ in (5.4.5) and the seminorm in (5.4.3) we have

$$
\begin{aligned}
p_n(\Phi(\tilde{u}_0) - \Phi(u_0)) &= \sup_{t\in[0,n]} |[\Phi(\tilde{u}_0) - \Phi(u_0)](t)| \\
&= \sup_{t\in[0,n]} |[\Phi(\tilde{u}_0)](t) - [\Phi(u_0)](t)| \\
&= \sup_{t\in[0,n]} |\big(\mathbf{S}_t\tilde{u}_0\big)(1) - \big(\mathbf{S}_t u_0\big)(1)| \\
&\le \sup_{t\in[0,n]} \sup_{x\in[0,1]} |\big(\mathbf{S}_t\tilde{u}_0\big)(x) - \big(\mathbf{S}_t u_0\big)(x)|
\end{aligned}
$$

Hence, the result follows by applying Corollary 5.34.

Proof of item (iii): To prove the equality (5.4.8), let us choose and fix $t \ge 0$ and $v \in W_0$. Then by using the definitions of the map Ψ, the semiflow $\{\mathbf{S}_t\}_{t\ge 0}$ and $\{T_t\}_{t\ge 0}$ we have for every $s \ge 0$ and $v \in W_0$,

$$
\begin{aligned}
\Big(\big[T_t \circ \Psi\big](v)\Big)(s) &= \big[T_t\big(\Psi(v)\big)\big](s) \\
&= \big(\Psi(v)\big)(t+s) = h^{-1}\big[(\mathbf{S}_{t+s}(1)\big] \\
&= h^{-1}\big[(\mathbf{S}_t(\mathbf{S}_s v\big)(1)\big] = \big[\Psi(\mathbf{S}_t v)\big](s).
\end{aligned}
$$

Since s and v are arbitrary we deduce that $T_t \circ \Psi = \Psi \circ \mathbf{S}_t$.

Hence, we conclude that all the three statements in Proposition 5.41 is complete. □

Proposition 5.42. *The map* Ψ *defined earlier in equation* (5.4.6) *is injective.*

Proof of Proposition 5.42. Because $\big[\Psi(v)\big](t) = h^{-1}\big(\Phi(v)(t)\big)$, $t \ge 0$, it is sufficient to prove that the map Φ is injective. Using the definition (5.4.5) of equality (5.3.20) from Theorem 5.35, we deduce that if $v \in E$ and $t \in [0,\infty)$ then because $G(1) = 0$ we have

$$
\begin{aligned}
\big(\Phi v\big)(t) &= \big(\mathbf{S}_t v\big)(1) = \psi\big(t, v(G^{-1}(G(1)-t)\big) \\
&= \psi\big(t, v(G^{-1}(-t))\big).
\end{aligned}
\tag{5.4.9}
$$

We use the above established representation (5.4.9) of the map Φ to prove its injectivity. For this aim let us choose and fix $v^1, v^2 \in E$ such that $\Phi(v^1) = \Phi(v^2)$. Thus, by representation (5.4.9) we infer that

$$
\psi\big(t, v^1(G^{-1}(-t))\big) = \psi\big(t, v^2(G^{-1}(-t))\big), \text{ for every } t \ge 0.
$$

Since for every $t \in [0,\infty)$ the map $\psi(t,\cdot) : \mathbb{R} \to \mathbb{R}$ is injective, by part II of Lemma 5.36, we infer that

$$
v^1\big(G^{-1}(-t)\big) = v^2\big(G^{-1}(-t)\big),\ t \in [0,\infty).
$$

Since by Proposition 4.8 part (vii), the function $G^{-1} : (-\infty, 0] \to (0, 1]$ is surjective, we deduce that

$$v^1(x) = v^2(x), \text{ for every } x \in (0,1]. \tag{5.4.10}$$

Because both v^1 and v^2 belong to the space E, we deduce that $v^1(0) = v^2(0)$ and therefore by the just proven identity (5.4.10), we deduce that $v^1 = v^2$. The proof of injectivity is complete. $\square$

Definition 5.43. Let $\mu : \mathcal{B}(W_0) \to [0, 1]$ be a measure μ defined by

$$\mu(A) = m\big(\Psi(A)\big), \quad A \in \mathcal{B}(W_0), \tag{5.4.11}$$

where m is the Borel probability measure which is the law of the stationary process, on the space Y defined by formula (5.4.2), and Ψ the map defined by formula (5.4.6).

Corollary 5.44. *The measure μ defined by equation* (5.4.11) *is $\{\mathbf{S}_t\}_{t\geq 0}$ invariant.*

Proof of corollary 5.44. Let $A \in \mathcal{B}(W_0)$ for $t \geq 0$. By using the definition of the measure μ in equation (5.4.11) we have

$$\begin{aligned} \mu\big(\mathbf{S}_t^{-1}(A)\big) &= m\big(\Psi(\mathbf{S}_t^{-1}(A))\big) \\ &= m\big((\Psi^{-1})^{-1}(\mathbf{S}_t^{-1}(A))\big) = m\big((\mathbf{S}_t \circ \Psi^{-1})^{-1}(A)\big). \end{aligned}$$

Applying the commuting property of the map Ψ and the invariant of the measure m, we obtain

$$\begin{aligned} m\big((\mathbf{S}_t \circ \Psi^{-1})^{-1}(A)\big) &= m\big[(\Psi^{-1} \circ T_t)^{-1}(A)\big] \\ &= m\big[T_t^{-1} \circ \Psi(A)\big] = m\big(\Psi(A)\big) = \mu(A). \end{aligned}$$

Hence we proved $\mu\big(\mathbf{S}_t^{-1}(A)\big) = \mu(A)$, therefore, the proof of Corollary 5.44 is complete. $\square$

Now we are ready to prove our main result in the present section. This result generalises [43, Theorem 1] by allowing coefficient a to satisfy natural weak assumptions and by considering the dissipative type and Lipschitz on balls nonlinearity. One important difference between our work and Rudnicki's proofs is that we do not prove the transformation Φ is a homeomorphism. In fact, we have only been able to prove that it is continuous and injective. Fortunately, because of the Kuratowski Theorem, see Corollary 4.52, this is sufficient to deduce that the inverse map Φ^{-1} is Borel measurable.

Theorem 5.45. *Let $E = {}_0C([0,1])$, the set W_0 be defined in formula* (5.4.4) *and Y be the space defined in formula* (5.4.2)*. Let $\{\mathbf{S}_t\}_{t\geq 0}$ be the semiflow associated to the* *system* (5.3.1)-(5.3.2) *on the space* E *as in Definition 5.32. Then there exists a probability measure* μ *defined on* σ*-field of Borel subsets of* W_0*, such that the following conditions are satisfied:*

(i) μ *is invariant under* $\{\mathbf{S}_t\}_{t\geq 0}$*,*

(ii) $\mu(Per) = 0$*, where* Per *is the set of periodic points of* $\{\mathbf{S}_t\}_{t\geq 0}$*,*

(iii) $\{\mathbf{S}_t\}_{t\geq 0}$ *is exact on* $\big(W_0, \mathcal{B}(W_0), \mu\big)$

(iv) the second moment of μ *is finite, i.e.,*

$$\int_{W_0} \|v\|_E^2 \mu\,(dv) < \infty$$

Before we commence the proof of Theorem 5.45 we need to prove the following auxiliary result.

Claim 5.46. The set W_0 defined in formula (5.4.4) is an open set, and hence, it is a Borel, subset of the space E.

Proof of Claim 5.46. In order to prove that W_0 is an open set in the space E, let us choose and fix an arbitrary element $v_0 \in W_0$. Since v_0 is continuous then by [39, Theorem 4.16] there exists $x_1 \in [0,1]$ and $x_2 \in [0,1]$ such that

$$\max\{v_0(x) : x \in [0,1]\} = v_0(x_1) \quad \text{and} \;\; \min\{v_0(x) : x \in [0,1]\} = v_0(x_2).$$

By the definition of W_0 in formula (5.4.4) we infer that $v_0(x_1) < u_+$ and $v_0(x_2) > u_-$. Let us now take $r = \min\{u_+ - v_0(x_1), v_0(x_2) - u_-\}$. Hence,

$$r \leq u_+ - v_0(x_1) \text{ and } r \leq v_0(x_2) - u_-. \tag{5.4.12}$$

Note that $r > 0$. We need to show that $B(v_0, r) \subset W_0$. In order to prove that, we take $v \in B(v_0, r)$, then

$$\sup_{x\in[0,1]} |v(x) - v_0(x)| < r.$$

Hence for every $x \in [0,1]$ we have $|v(x) - v_0(x)| < r$. That means,

$$v_0(x) - r < v(x) < v_0(x) + r.$$

Obviously $v \in E$ because $B(v_0, r) \in E$. Moreover, from inequality (5.4.12) we infer

$$v(x) < v_0(x) + r \leq v_0(x_1) + u_+ - v_0(x_1) = u_+.$$

Hence, we proved that $v(x) < u_+$ for every $x \in [0,1]$. Similarly, for $v(x) > u_-$ because

$$v(x) > v_0(x_2) - r \geq v_0(x_2) - v_0(x_2) - u_- = -u_-.$$

To conclude, we proved that $v \in W_0$. In other words, W_0 is (an open) Borel subset of E. by this, the proof of Claim 5.46 is complete. □

Proof of Theorem 5.45. From Propositions 5.41 and 5.42 we infer that the map $\Psi : E \to Y$ is continuous and injective. Therefore, we deduce that the restriction map $\Psi : W_0 \to Y$ is also continuous and injective. By applying Corollary 4.52, we put $X_1 = E$ and $X_2 = Y$, both these spaces are separable metric spaces. Also, put $E_1 = W_0$. By claim 5.46, E_1 is a Borel subset of X_1. Hence we infer that the map $\Psi^{-1} : \Psi(E_1) \to E_1$ is measurable. Recall that the set W_0 defined in formula (5.4.4) and the space Y defined in formula (5.4.2). The semiflow $\{\mathbf{S}_t\}_{t\geq 0}$ associated to the system (5.3.1)-(5.3.2) on the space E as in Definition 5.32. Let $\xi = \{\xi_t\}_{t\geq 0}$ be the Ornstein-Uhlenbeck process process and the trajectories of ξ belong to the space Y. Moreover, $\{T_t\}_{t\geq 0}$ is the shift C_0-semigroup on the space Y. The measure $m = law(\xi)$ is the probability measure on the space Y satisfy the following equality

$$m\big(T_t^{-1}(A)\big) = m(A) \text{ for every } A \in \mathcal{B}(Y),$$

see Lemma 4.61. Since by Lemma 4.64, $\mathbb{P}$-almost all trajectories of ξ belong to the space Y, we infer that m is a Borel probability measure on Y. Define a function $\mu : \mathcal{B}(W_0) \to [0,1]$ by the following

$$\mu(A) := m\big(\Psi(A)\big), \quad A \in \mathcal{B}(W_0). \tag{5.4.13}$$

Note that the RHS of the above formula (5.4.13) is meaningless unless the set $\Psi(A)$ belongs to the family $\mathcal{B}(Y)$ of Borel subsets of Y. Also, it can be rewrite equation (5.4.13) in the following equivalent form $\mu = m \circ \Psi$. Note that if $A \in \mathcal{B}(W_0)$ then because Ψ^{-1} is measurable, we infer that

$$\Psi(A) \in \mathcal{B}(Y).$$

Hence we deduce that $m\big(\Psi(A)\big)$ makes sense for every $A \in \mathcal{B}(W_0)$, and thus, we infer that μ which is defined by formula (5.4.13) is well defined. Note that our definition of the measure μ is the same as defined by equation (5.4.11) on Definition 5.43 made earlier. Therefore, we have already proved in the Corollary 5.44 that this measure μ is $\{\mathbf{S}_t\}_{t\geq 0}$ invariant.

To prove that μ is a probability measure, let us define the space Y_0 by

$$Y_0 = \{\psi \in C[0,\infty) : \lim_{t\to\infty} \frac{|\psi(t)|}{t} = 0\}.$$

We need to show that Y_0 satisfies the following three conditions:

(i) Y_0 is a Borel subset of Y,

(ii) $Y_0 \subset \Psi(W_0)$,

(iii) $m(Y_0) = 1$.

Note that the conditions (i) to (iii) rewrittenabove imply the following equality

$$\mu(W_0) = m\big(\Psi(W_0)\big) = 1.$$

The assertion (i) above can be proved in a similar way as we proved that the set W_0 is a Borel subset of the space E, see Claim 5.46. The assertion (iii) above is a consequence of Proposition 4.56 because every continuous function satisfying the growth condition (4.3.8) belongs to the space Y. Indeed,

$$\begin{aligned}\lim_{t\to\infty} \frac{|\psi(t)|}{t} &= \lim_{t\to\infty} \Big[\frac{|\psi(t)|}{\ln(t+M)} \frac{\ln(t+M)}{t}\Big] \\ &\leq \lim_{t\to\infty} \frac{|\psi(t)|}{\ln(t+M)} \lim_{t\to\infty} \frac{\ln(t+M)}{t} = ...\end{aligned}$$

Hence we only need to prove assertion (ii). To do this, we occupy ourselves with the following proposition.

Proposition 5.47. *If $y \in Y_0$, then there exists $v \in W_0$ such that $\Psi(v) = y$.*

Proof of Proposition 5.47. Let $y \in Y_0$. By formula (5.4.9) we have

$$\big(\Psi v\big)(t) = h^{-1}\big[\psi\big(t, v(G^{-1}(-t))\big)\big].$$

So we need to find $v \in W_0$ such that

$$\psi\big(t, v(G^{-1}(-t))\big) = h\big[y(t)\big],\ t \geq 0. \tag{5.4.14}$$

By Lemma 5.36, the function $\psi_t := \psi(t,\cdot)$ is injective for every t and $\phi_t := \psi_t^{-1}$. Moreover, the flow $\{\phi_t\}_{t\geq 0}$ corresponding to the vector field $-c$. By applying ϕ_t to equation (5.4.14) we infer that we need to find $v \in W_0$ such that

$$v\big(G^{-1}(-t)\big) = \phi_t\big(h[y(t)]\big),\ \ t \geq 0.$$

If we denote $x = G^{-1}(-t)$ which implies that $-t = G(x) \iff t = -G(x)$, we get the following equivalent form of equation (5.4.14)

$$v(x) = \phi_{-G(x)}\big(h[y(-G(x))]\big),\ x \in (0,1]. \tag{5.4.15}$$

We want to prove that $v \in W_0$. In view of the last part of Proposition 5.37, it is sufficient to prove that $v \in E$. For this purpose, we need to establish the continuity of the function v on the interval $[0,1]$ and to show that $v(0)=0$. In view of identity (5.4.15) it is sufficient to show that $\lim_{x\to 0^+} v(x)=0$, i.e., it is sufficient to prove that

$$\lim_{x\to 0^+} \phi_{-G(x)}\big(h[y(-G(x))]\big)=0, \quad x\in(0,1]. \tag{5.4.16}$$

Using the transformation $x=G^{-1}(-t)$ and observing that $x\to 0^+$ if and only if $t\to\infty$, we see that equality (5.4.16) is equivalent to

$$\lim_{t\to\infty} \phi_t\big(h[y(t)]\big)=0. \tag{5.4.17}$$

Thus we need to prove the following: If $y\in Y_0$, then condition (5.4.17) is satisfied.

Let us recall that the flow ϕ_t is the one from Lemma 5.36. Our aim is to find condition on the function y such that if v is defined by equality (5.4.15), then equation (5.4.17) is satisfied. Let $-c(z)=-\lambda z+z^3$, and the flow ϕ_t corresponding to vector field $-c$, i.e., for every $z_0\in(0,\sqrt{\lambda})$,

$$\begin{aligned}\frac{d\phi_t(z_0)}{dt} &= -c(\phi_t(z_0))\\ \phi_0(z_0) &= z_0.\end{aligned} \tag{5.4.18}$$

If $z_0\in(0,\sqrt{\lambda})$ then also $\phi_t(z_0)\in(0,\sqrt{\lambda})$ and

$$\phi_t(z_0)\to 0 \text{ as } t\to\infty. \tag{5.4.19}$$

Moreover,

$$\phi_t(z_0)\to\sqrt{\lambda} \text{ as } t\to-\infty.$$

If the function y is bounded, i.e. there exists $C>0$ such that

$$0<y(t)\le C, \;\; ;t\ge 0,$$

then by property (5.4.19) we infer that

$$\phi_t(h[y(t)])\to 0 \text{ as } t\to\infty.$$

Suppose that z_0 is replaced by $h[y(t)]\in(0,\sqrt{\lambda})$. The last observation implies that we can assume that $y(t)\to\infty$ as $t\to\infty$. Thus also $h[y(t)]\to\sqrt{\lambda}$.

We want to investigate the speed of convergence of $h[y(t)]$ towards $\sqrt{\lambda}$ so that condition (5.4.17) holds. For our aim, let us introduce the following auxiliary function V as follows

$$V(x) = \int_{x_0}^{x} \frac{dz}{c(z)}, \quad x \in (0, \sqrt{\lambda}).$$

where x_0 is any fixed number belongs also to the interval $(0, \sqrt{\lambda})$. The function V satisfies the following properties:

(i) $V(x_0) = 0$;

(ii) The function V is increasing,

(iii) $\frac{dV}{dx} = V'(x) = \frac{1}{c(x)} > 0$;

(iv) $V(x) \to +\infty$ when $x \nearrow \sqrt{\lambda}$ and

$$V(x) \geq \frac{1}{2\lambda}\big[-\ln(\sqrt{\lambda} - x) + \ln(\sqrt{\lambda} - x_0)\big], \quad x \in (x_0, \sqrt{\lambda});$$

(v) $-V(x) \to +\infty$ when $x \searrow 0$ and

$$-V(x) \geq \frac{1}{2\lambda}(\ln x_0 - \ln x), \quad x \in (0, x_0);$$

(vi) $V(x) \leq \frac{1}{x_0(\sqrt{\lambda}+x_0)}\big[\ln(\sqrt{\lambda} - x_0) - \ln(\sqrt{\lambda} - x)\big], \quad x \in (x_0, \sqrt{\lambda})$.

Now we observe that the flow ϕ_t, which satisfies (5.4.18), satisfies the following equality

$$V(\phi_t) = -t + C_0, \ \text{ where } C_0 \ \text{ is a constant and } t \in \mathbb{R}.$$

To detect what is the constant C_0 in the above solution we put $t = 0$. Then we obtain $V(\phi_0) = C_0$ and hence,

$$V(\phi_t) = -t + V(\phi_0), \ t \in \mathbb{R}.$$

In other words, we proved that if the function ϕ_t is a solution to the equation (5.4.18) then $V(\phi_t) = -t + V(\phi_0)$ for every $t \in \mathbb{R}$. In particular,

$$V(\phi_t(h[y(t)])) = -t + V(h[y(t)]) \ \text{ for every } \ t \in \mathbb{R}.$$

Using the last assertion and properties (iv) and (v) of function V, we infer that condition (5.4.17) is equivalent to the following one

$$-t + V(h[y(t)]) \to -\infty \text{ as } t \to \infty,$$

or equivalently,

$$t - V(h[y(t)]) \to +\infty \text{ as } t \to \infty, \tag{5.4.20}$$

From property (vi) of the function V above we have

$$t - V(h[y(t)]) \geq t - A + B\ln(\sqrt{\lambda} - h[y(t)]), \tag{5.4.21}$$

where

$$\begin{aligned} A &:= \frac{1}{x_0(\sqrt{\lambda}+x_0)}\ln(\sqrt{\lambda}-x_0) \\ B &:= \frac{1}{x_0(\sqrt{\lambda}+x_0)} \end{aligned}$$

By the definition (5.4.7) of function h, for t large enough, i.e. when $y(t) > R$,

$$\sqrt{\lambda} - h[y(t)] = e^{-y(t)}.$$

Hence, for t large enough,

$$\ln(\sqrt{\lambda} - h[y(t)]) = -y(t).$$

Hence, by inequality (5.4.21) we have

$$t - V(h[y(t)]) \geq t - A - By(t), \text{ for } t \text{ large enough.} \tag{5.4.22}$$

But

$$t - A - By(t) = t(1 - \frac{A}{t} - B\frac{y(t)}{t})$$

Since $\lim_{t\to\infty}\frac{A}{t} = 0$, and, because $y \in Y_0$, $\lim_{t\to\infty}\frac{y(t)}{t} = 0$, we deduce that

$$\lim_{t\to\infty}(1 - \frac{A}{t} - \frac{y(t)}{t}) = 1$$

Hence

$$\lim_{t\to\infty}(t - A - By(t)) = \infty.$$

Hence, by inequality (5.4.22) and Sandwich Principle (or a comparison Lemma), see [39, Theorem 3.19] for a related result, we infer that

$$\lim_{t\to\infty}(t - V(h[y(t)])) = \infty.$$

Hence we proved that condition (5.4.20) is satisfied and therefore we deduce that also condition (5.4.17) holds true.

By this, we conclude the proof of Proposition 5.47. □

Thus, we proved that the measure μ defined by equation (5.4.13) is probability measure which is $\{\mathbf{S}_t\}_{t\geq 0}$ invariant. Therefore, the proof of condition (i) of Theorem 5.45 is complete.

The proof of conditions (ii) and (iii) of Theorem 5.45 can be done in the same fashion as we proved parts (ii) and (iii) of Theorem 4.67 in Chapter 4.

The proof of condition (iv) of Theorem 5.45 follows trivially because W_0 is a bounded subset of E, i.e.,

$$\|v\|_E \leq \max\{\lambda_+, \lambda_-\} = \lambda_0 < \infty, \ \ v \in W_0.$$

Hence

$$\int_{W_0} \|v\|_E^2 \leq \lambda_0^2 \, \mu(W_0) = \lambda_0^2 < \infty.$$

Thus the proof of our main result in this section, i.e., Theorem 5.45, is complete. $\square$

5.5 Discussion

Our main contribution in Chapters 4 and 5 concerning the paper by Rudnicki [43] is that we have been able to formulate and prove the well-posedness of the Lasota equation for irregular drift coefficient $a(x)$. In fact, as in Chapter 4, we assume that a is only continuous and satisfies the Osgood condition. To formulate our results for the nonlinear case of the Lasota equation, we use deeply some of the results from chapter 4, that the corresponding linear equation generates a C_0-semigroup on the Banach space ${}_0C\big([0,1]\big)$ so that a notion of a mild solution to the Lasota equation makes sense. We also use some results about dissipative problems in Banach spaces to prove that the Lasota equation is well-posed.

Rudnicki [43] and others have always assumed that the drift coefficient $a(x)$ is of C^1-class, since their notion of a classical solution was intrinsically dependent on that assumption. One could have suspected that the C^1-class regularity of the function a was a redundant assumption and we proved that this is the case. On the other hand, we also rigorously use the notion of a classical solution to verify the representation Theorem 5.35.

Chapter 6

Open Problems and Future Work

In this Chapter, we outline open problems that can be considered for future research directions. These open problems were encountered in Chapter 3 and in Chapter 4.

6.1 Open Problem of Chapter 3

We identified during our proving of the existence of invariant measures for the first-order PDE, the following two open problems. The first problem is related to Theorem 3.30. The theorem can be generalise to the case $\alpha \in (-\frac{1}{2}, 0)$. We conjecture that our result can be rewritten similarly to Remark 3.31. In this present case, the Hurst parameter $H = \alpha + \frac{1}{2}$ could be an arbitrary element of $(0, \frac{1}{2})$.

The second open problem is that the law of Brownian motion μ is an invariant measure for the semigroup denoted by $\{S^{\frac{1}{2}}(t)\}_{t\geq 0}$ generated by the following equation

$$\frac{\partial u}{\partial t} + s\frac{\partial u}{\partial s} = \frac{1}{2}u \tag{6.1.1}$$

Denote by $\{J^{\lambda}(t)\}_{t\geq 0}$ the semigroup on the E generated by equation (6.1.1) with $\frac{1}{2}$ replaced by λ. Is it true if μ is an invariant measure for $\{J^{\lambda}(t)\}_{t\geq 0}$ then $\lambda = \frac{1}{2}$?

6.2 Open Problem of Chapter 4

We found two main open problems in this chapter. The first one is related to the Theorem 4.19. It is an open problem whether the theorem holds on the Lebesgue spaces $L^p(0,1)$ or Sobolev spaces ${}_0H^{\alpha,p}([0,1])$. In Chapter 3, we proved that this is true when the function $a(x) = x$ for $x \in \mathbb{R}$. This should be easily generalised to the case when the function a is of C^1-class. But the general case of a satisfying only Assumption 4.2 remains open.

The second open problem is related to the Theorem 4.67, which is the main result of Section 4.3. We studied there a problem of the existence of an invariant probability measure for the C_0-semigroup on the space ${}_0C([0,1])$ generated by the equation (4.3.2). We assumed very weak assumptions of the coefficient a but the coefficient c was assumed to be a constant. Our main tool was an earlier Theorem 4.19 that equation (4.3.2) with $c = 0$, generates a C_0-semigroup on the space ${}_0C([0,1])$. Many open questions remain to be investigated, and we are mentioning some of them as follows.

- Q1. Does equation (4.3.2) with $c = 0$, generates a C_0-semigroup on spaces different than ${}_0C([0,1])$? We have tried and failed to prove such a result for the space $E = L^p(0,1)$. On the other hand, as has been established in [11] and [12] that this is true when $a(x) = x$.
- Q2. Under what natural conditions on the function c, equation (4.3.2) generates a C_0-semigroup on space ${}_0C([0,1])$?
  When this is the case, under what additional assumptions, does this semigroup has an invariant measure?
- Q3. Combine Q1 with Q2.
- Q4. Suppose the answer of Q2 is positive, what can be said about linear equation (4.3.2) with λu replaced by $c(x)$? What are the natural assumptions on function c?

Remark 6.1. Lemmata 4.24, 4.25 and Claim 4.28 have been formulated and proved for abstract C_0-semigroups, not only for those used in this thesis. Would these results be of any use in the study of the existence of invariant measures for other equations, e.g., the Burgers Equation. Completely different methods in the case of stochastic Burgers Equations have been used in [17].

We ask if the following problem has a unique mild solution for every $u_0 \in E = {}_0C\big([0,1]\big)$:

$$\begin{aligned}
&\frac{\partial u(t,x)}{\partial t} + a(x)\frac{\partial u(t,x)}{\partial x} = c(u(t,x)), t > 0, \quad x \in [0,1].\\
&u(0,x) = u_0(x).
\end{aligned}$$

To prove the mild solution to the nonlinear above equation, we formulate the following result which is a consequence of the Wintner Theorem, see the book by Hartman [25, Theorem 5.1], see also [34, problem 8, p.36].

Theorem 6.2. *Assume that a function $c : \mathbb{R} \to \mathbb{R}$ is continuous and satisfies*

$$|c(y)| \leq \phi(|y|), \quad y \in \mathbb{R},$$

for some function $\phi : [0, \infty) \to [0, \infty)$, such that $\phi(r) > 0$ if $r > 0$ and

$$\int_N^\infty \frac{1}{\phi(r)} \, dr = \infty, \quad \textit{for every} \quad N > 0.$$

Then every local maximal solution $y(t)$, $t \in [t_0, \tau)$ of the equation

$$y'(t) = c\big(y(t)\big), \quad t \geq t_0,$$

is a global solution, i.e., $\tau = \infty$.

Note that a non-trivial linear function c satisfies assumptions of the above Theorem 6.2. Indeed,

$$\int_N^\infty \frac{dr}{r} = \infty \quad \text{for every } N > 0.$$

Appendix A

Referenced Theorems, Lemmas and Propositions

For the sake of completeness in this thesis, we recall in this appendix important well-known theorems that are used in some parts of our proofs.

A.1 Referenced Theorems, Lemmas and Propositions

Theorem A.1. *[35] Let X be a non-empty set and let Y and Z be two metric spaces. Assume $\{g_n\}_{n\in\mathbb{N}}$ be a sequence of mappings from X to Y such that*

$$g_n \to g \text{ uniformly on } X.$$

Let $\psi : Y \to Z$ be a uniformly continuous function. Then

$$\psi \circ g_n \to \psi \circ g \text{ uniformly on } X.$$

We also need a simpler version of the above theorem.

Theorem A.2. *Let X and Y be two non-empty sets and let (N, d) be a metric space. Assume that $g : X \to Y$. Assume $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of mappings from Y to N such that*

$$f_n \to f \text{ uniformly on } Y.$$

Then

$$f_n \circ g \to f \circ g \text{ uniformly on } X.$$

Theorem A.3 (Dominated Convergence Theorem [2])**.** *Suppose that ϕ is an integrable function and f_k is a sequence of measurable functions such that $|f_k(x)| \le \phi(x)$ almost* *everywhere and suppose that f_k converges to a function f almost everywhere. Then f_k and f are integrable and*

$$\lim_{k\to\infty}\int_{\mathbb{R}^n} f_k\,d\mu = \int_{\mathbb{R}^n} f\,d\mu.$$

Theorem A.4 (Monotone Convergence Theorem [2])**.** *Suppose f_k is a sequence of measurable functions on $\mathbb{R}^n$ such that $0 \le f_k(x) \le f_{k+1}(x)$ for all k and x. Let $f(x) = \lim_{k\to\infty f_k(x)}$, then*

$$\lim_{k\to\infty}\int_{\mathbb{R}^n} f_k\,d\mu = \int_{\mathbb{R}^n} f\,d\mu.$$

Theorem A.5 (Weierstrass Approximation Theorem [39])**.** *Let f be a continuous real-valued function on $[a,b]$ and for any given $\varepsilon > 0$ there exists a polynomial $P \in [a,b]$ such that*

$$|f(x) - P(x)| < \varepsilon, \quad x \in [a,b].$$

Theorem A.6. *[3, Theorem 8.17] Suppose that a continuous functions $f(t) \to y_0$ as $t \to \infty$ and that function $v(y) \to v_0$ as $y \to y_0$. Then the composition $v(f(t)) \to v_0$ as $t \to \infty$.*

A.1.1 Proof of Lemma 3.2

Proof. Fix $t > 0$. Take a cylinder set C of the following form

$$\begin{aligned} C &= C(s_1,\ldots,s_n; A_1,\ldots,A_n) \\ &= \{x \in E : x(s_1) \in A_1,\ldots,x(s_n) \in A_n\} \end{aligned}$$

for some $0 < s_1 < \ldots < s_n \le 1$, and $A_1,\ldots,A_n \in \mathcal{B}(\mathbb{R})$. By the definition of the inverse set we have

$$\begin{aligned} S_t^{-1}(C) &:= \{x \in E : S_t(x) \in C\} \\ &= \{x \in E : \big(S_t x\big)(s_i) \in A_i, \quad i = 1,\ldots,n\} \\ &= \{x \in E : e^{\frac{t}{2}} x(s_i e^{-t}) \in A_i, \quad i = 1,\ldots,n\} \\ &= \{x \in E : x(s_i e^{-t}) \in e^{-\frac{t}{2}} A_i, \quad i = 1,\ldots,n\} \\ &= C(s_1 e^{-t},\ldots,s_n e^{-t}, e^{-\frac{t}{2}} A_1,\ldots,e^{-\frac{t}{2}} A_n) \end{aligned} \tag{A.1.1}$$

Note that sets $e^{\frac{-t}{2}} A_i$ are Borel. So we proved that $S_t^{-1}(C)$ is also cylinder set but with different time and different base sets. Now, we take the measure μ for both sides of equality (A.1.1) and put $\alpha^2 = e^{-t}$, $(\alpha > 0)$ we obtain

$$\begin{aligned}
\mu\big(S_t^{-1}(C)\big) &= \mu\big(C(s_1 e^{-t}, \dots, s_n e^{-t}, e^{-\frac{t}{2}} A_1, \dots, e^{-\frac{t}{2}} A_n)\big) \\
&= \mu\big(C(\alpha^2 s_1, \dots, \alpha^2 s_n, \alpha A_1, \dots, \alpha A_n)\big) \\
&= \frac{1}{\sqrt{(2\pi)^n (\alpha^2 s_1 - 0) \dots (\alpha^2 s_n - \alpha^2 s_{n-1})}} \\
&\int_{\alpha A_1} \dots \int_{\alpha A_n} e^{-\frac{1}{2} \sum_{k=1}^{n} \frac{(x_k - x_{k-1})^2}{(\alpha^2 s_k - \alpha^2 s_{k-1})}} \, dx_1 \dots dx_n
\end{aligned}$$

Using the change of variables formula as follows:

Since $x_1 \in \alpha A_1$ then $x_1 = \alpha y_1$, $dx_1 = \alpha dy_1$ where $y_1 \in A_1$. Also $x_n \in \alpha A_n$ then $x_n = \alpha y_n$, $dx_n = \alpha dy_n$ where $y_n \in A_n$. Hence by substituting those changes into the above equality we obtain

$$\begin{aligned}
\mu\big(S_t^{-1}(C)\big) &= \frac{1}{\sqrt{(2\pi)^n (\alpha^2)^n (s_1 - 0) \dots (s_n - s_{n-1})}} \\
&\int_{A_1} \dots \int_{A_n} \alpha^n \, e^{-\frac{1}{2} \sum_{k=1}^{n} \frac{(y_k - y_{k-1})^2}{(s_k - s_{k-1})}} \, dy_1 \dots dy_n \\
&= \frac{1}{\sqrt{(2\pi)^n (s_1 - 0) \dots (s_n - s_{n-1})}} \\
&\int_{A_1} \dots \int_{A_n} e^{-\frac{1}{2} \sum_{k=1}^{n} \frac{(y_k - y_{k-1})^2}{(s_k - s_{k-1})}} \, dy_1 \dots dy_n \\
&= \mu\big(C(s_1, \dots, s_n; A_1, \dots, A_n)\big) = \mu(C).
\end{aligned}$$

Hence, we proved

$$\mu\big(S_t^{-1}(C)\big) = \mu(C).$$

$\square$

Appendix B

Examples

In this section, we present some of the fundamental examples in functional spaces.

B.1 Examples of Separable Banach spaces

Proof of Example 2.5. To verify that our claim is a Banach space, we need to check whether it is a complete normed space.

Normed Space: The conditions of normed space that were presented in Definition 2.1 must be validated as follows.

N1. $\|f\| \geq 0,$ because $\sup_{y\in[a,b]} |f(x)| \geq 0.$

N2. $\|0\| = \sup_{y\in[a,b]} |0| = 0.$ If $\|f\| = 0$ then $\sup_{y\in[a,b]} |f(x)| = 0$ and so $f = 0.$

N3. Let $f \in C[a,b]$ and for any scalar λ, we have

$$\|\lambda f\| = |\lambda|\|f\|. \tag{B.1.1}$$

In order to verify the third condition, we start with the left-hand-side of the equality (B.1.1).

$$\|\lambda f\| = \sup_{x\in[a,b]} |\lambda f(x)| = |\lambda| \sup_{x\in[a,b]} |f(x)|,$$

where we have here two cases:

Case 1: If $\lambda = 0$. Then

$$\sup_{y\in[a,b]} |0| = 0, \quad \text{and} \quad |0| \sup_{y\in[a,b]} |f(x)| = 0.$$

Case 2: If $\lambda \neq 0$. For every x belongs to $[a,b]$, we have

$$|\lambda f(x)| \leq |\lambda||f(x)| \leq |\lambda| \sup_{y\in[a,b]} |f(y)|.$$

Since the right-hand-side of the above equation is an upper bound of $\{|\lambda f(x)| : x \in [a,b]\}$ then it is larger than the least upper bound, i.e

$$\sup_{y\in[a,b]} |\lambda f(x)| \leq |\lambda| \sup_{y\in[a,b]} |f(y)|. \tag{B.1.2}$$

For the right-hand-side of the equality (B.1.1), we take $x \in [a,b]$. Then

$$\begin{aligned} |\lambda||f(x)| = |\lambda f(x)| &\leq \sup_{y\in[a,b]} |\lambda f(y)| \quad (\text{divide by}|\lambda|) \\ |f(x)| &\leq \frac{1}{|\lambda|} \sup_{y\in[a,b]} |\lambda f(y)| \quad \forall x \in [a,b]. \end{aligned}$$

Hence with the same argument above we deduce the following

$$\begin{aligned} \sup_{y\in[a,b]} |f(x)| &\leq \frac{1}{|\lambda|} \sup_{y\in[a,b]} |\lambda f(y)| \\ |\lambda| \sup_{y\in[a,b]} |f(x)| &\leq \sup_{y\in[a,b]} |\lambda f(y)|. \end{aligned} \tag{B.1.3}$$

So, from equations (B.1.2) and (B.1.3) we conclude that condition (**N3**) is held.

$$\sup_{y\in[a,b]} |\lambda f(x)| = |\lambda| \sup_{y\in[a,b]} |f(x)|.$$

In other words, we proved $\|\lambda f\| = |\lambda|\|f\|$.

N4. We want to show that for every $f, g \in C[a,b]$, we have $\|f+g\| = \|f\| + \|g\|$.

To prove this, let $f, g \in C[a,b]$ and $x \in [a,b]$, then we have

$$\|f+g\| = \sup_{x\in[a,b]} |f(x)+g(x)| \leq \sup_{x\in[a,b]} |f(x)| + \sup_{x\in[a,b]} |g(x)| = \|f\| + \|g\|.$$

Now we would like to verify with more detail the following equation

$$\sup_{x\in[a,b]} \big[|f(x)| + |g(x)|\big] \leq \sup_{x\in[a,b]} |f(x)| + \sup_{x\in[a,b]} |g(x)|. \tag{B.1.4}$$

Let $x \in [a,b]$, then we have

$$|f(x)| \leq \sup_{x\in[a,b]} |f(y)|. \tag{B.1.5}$$

And similarly

$$|g(x)| \leq \sup_{y\in[a,b]} |g(y)|. \tag{B.1.6}$$

So, by adding equations (B.1.5) and (B.1.6) together, we obtain

$$|f(x)| + |g(x)| \leqslant \underset{y\in[a,b]}{sup} |f(y)| + \underset{y\in[a,b]}{sup} |g(y)| \quad ,\forall x \in [a,b]$$

Assume that $\sup_{y\in[a,b]} |f(y)| + \sup_{y\in[a,b]} |g(y)| = M$. That means

$$|f(x)| + |g(x)| \leq M, \quad \forall x \in [a,b]$$

Then, as arguing before in **N3.** M will be an upper bound of

$$\{|f(x)| + |g(x)| \quad ,\forall x \in [a,b]\}.$$

Thus, $\sup_{x\in[a,b]}[|f(x)| + |g(x)|] \leq M$. That is

$$\sup_{x\in[a,b]} \left[|f(x)| + |g(x)|\right] \leq \sup_{x\in[a,b]} |f(x)| + \sup_{x\in[a,b]} |g(x)|$$

Hence the proof of equation (B.1.4) is complete. Finally, Since all conditions of normed space are valid, the function $\|\cdot\|$ is a norm.

Completeness: To verify the completeness of the normed vector space $C[a,b]$, we have to show that every Cauchy sequence in the space $C[a,b]$ is convergent. We start with taking $\{f_n\}$ to be a Cauchy sequence in $C[a,b]$. Our objective is to find a function $f \in C[a,b]$ such that $\|f_n - f\| \to 0$ as $n \to \infty$. That is,

$$\forall \varepsilon > 0 \quad \exists N = N_\varepsilon : \forall n \geqslant N \quad \text{then} \quad \|f_n - f\| \leqslant \varepsilon.$$

In order to meet our objective, we identify four requirements. The first requirement is to find a candidate for f. Take $x_0 \in [a,b]$, then for every $n, m \in \mathbb{N}$ we have

$$|f_n(x_0) - f_m(x_0)| \leq \sup_{x\in[a,b]} |f_n(x_0) - f_m(x_0)| = \|f_n - f_m\|. \tag{B.1.7}$$

Now, take $\varepsilon > 0$. Because $\{f_n\}$ is a Cauchy sequence then there exists $N \in \mathbb{N}$ such that for all $n, m \geqslant N$ then $\|f_n - f_m\| \leqslant \varepsilon$. Hence, by equation (B.1.7), we infer that

$$|f_n(x_0) - f_m(x_0)| \leqslant \varepsilon \qquad , n, m \geqslant N.$$

So, we verified that the sequence $f_n(x_0)$ is a Cauchy sequence in $\mathbb{R}$. Because $\mathbb{R}$ is complete, we deduce that $f_n(x_0)$ is convergent in $\mathbb{R}$. Let us denote the limit by $f(x_0)$. Thus,

$$f_n(x_0) \longrightarrow f(x_0) \quad \text{as } n \to \infty. \tag{B.1.8}$$

After we proved that $f_n(x_0)$ is convergent to $f(x_0)$, the second requirement is to show that f belongs to the space $C[a,b]$. Next of that, the third requirement is to prove that $f_n \to f$ in $C[a,b]$. However, before fulfillment of those requirements, we need to consider and achieve yet another requirement that studies whether f_n converges to f uniformly.

Let $\varepsilon > 0$. Since our sequence is Cauchy in $C[a,b]$, then $\exists N = N_\varepsilon$ such that for all $n, m \geq N$ we have $\|f_n - f_m\| \leq \varepsilon$. So,

$$\sup_{x\in[a,b]} \|f_n(x) - f_m(x)\| \leq \varepsilon. \tag{B.1.9}$$

Fix $x_0 \in [a,b]$. Then, from equations (B.1.7) and (B.1.9) we infer that

$$|f_n(x_0) - f_m(x_0)| \leq \varepsilon, \quad \text{where } n, m \in \mathbb{N}.$$

Now, let us fix $n \geq N_\varepsilon$ and let $m \to \infty$. Then from equation (B.1.8) we deduce that $f_m(x_0) \longrightarrow f(x_0)$. Since $|f_n(x_0) - f_m(x_0)| \leq \varepsilon$, then we can rewrite it as follows

$$-\varepsilon \leq f_n(x_0) - f_m(x_0) \leq \varepsilon.$$

$$f_n(x_0) - \varepsilon \leq f_m(x_0) \leq f_n(x_0) + \varepsilon.$$

By using the Sandwich Theorem 2.15 for functions, we obtain

$$f_n(x_0) - \varepsilon \leq f(x_0) \leq f_n(x_0) + \varepsilon.$$

Thus

$$\|f(x_0) - f_n(x_0)\| \leq \varepsilon.$$

Hence, f_n converges uniformly to f. Regarding requirement 2 studies whether f belongs to the space $C[a,b]$. We know from our assumption that $f_n \in C[a,b]$, and from the results of proving requirement 4 that $f_n \longrightarrow f$ uniformly, we infer by Theorem 2.13 that $f \in C[a,b]$. Regarding requirement 3, let $\varepsilon > 0$. Then, from the result of requirement 4,

$$\exists\, N = N_\varepsilon : \forall x \in [a,b], \forall\, n \geq N \quad \text{then} \quad |f_n(x) - f(x)| \leq \varepsilon$$

Therefore,

$$\sup_{x\in[a,b]} |f_n(x) - f(x)| \leqslant \varepsilon, \quad \forall\, n \geq N_\varepsilon.$$

Hence, the main objective (that studies the completeness of normed space $C[a,b]$) has been proved. That means

$$\forall \varepsilon > 0 \, \exists N = N_\varepsilon : \forall n \geq N \|f_n - f\| \leq \varepsilon.$$

Therefore, the normed space is complete and consequently the space $X = C[a,b]$ is a Banach space. $\square$

Proof of the space X in equation (2.1.1) *is separable.* In order to prove the space $C[a,b]$ is separable, we need to find a subset that is countable dense in the space. For that, let $\mathcal{P}_n$ be polynomials with rational coefficients of degree n. For countability, it is clear that $\mathcal{P}_n$ is a countable set because the rational numbers are countable and $p = \bigcup_{n=1}^{\infty} \mathcal{P}_n$ is a countable union of countable sets. Let $q(x)$ any a polynomial such that

$$q(x) = a_0 + a_1 x + \ldots + a_n x^n \quad \text{where} \;\; a_i \in \mathbb{R}, \; i = 1, \ldots, n.$$

For density, for every $\varepsilon > 0$, we can choose $b_i \in \mathbb{Q}$ such that

$$|a_i - b_i| \, < \, \frac{\varepsilon}{2(n+1)}.$$

For every $\varepsilon > 0$, since rationals are dense, then it follows that $p(x) = b_0 + b_1 x + \ldots + b_n x^n$. Hence for every $x \in [a,b]$, we have

$$\begin{aligned}|P(x) - q(x)| &\leq |b_0 - a_0| + |b_1 - a_1||x| + \ldots + |b_n - a_n||x|^n \\ &\leq |b_0 - a_0| + |b_1 - a_1| + \ldots + |b_n - a_n| \;\text{ as } x \in [0,1] \leq \frac{\varepsilon}{2}.\end{aligned}$$

Now for any $f \in C[a,b]$ and $\varepsilon > 0$, by Weierstrass Approximation Theorem A.5, we can find a polynomial q such that $|f - q| \leq \frac{\varepsilon}{2}$. Then

$$\|P - f\| \leq \|P - q\| + \|q - f\| \leq \varepsilon.$$

Hence, the rational polynomials are dense and countable. Therefore, we can deduce that the space $C[a,b]$ is separable. $\square$

Example B.1. *The space ${}_0C^1([0,1])$ is a subspace of the space ${}_0H^{1,p}(0,1)$.*

Proof of the Example B.1. To verify this example, we need to check two statements:1) The space ${}_0C^1([0,1])$ is subset of space ${}_0H^{1,p}(0,1)$ and 2) The space ${}_0C^1([0,1])$ is vector space itself. For the former, we argue that any element x in the space ${}_0C^1([0,1])$ is also in the space ${}_0H^{1,p}(0,1)$. This means, x has a weak derivative and Both x and x' have a finite L^p norm, which is true since they both continuous function on a closed and bounded interval. For the latter, ${}_0C^1([0,1])$ is a vector subspace because: if $x, y \in {}_0C^1([0,1])$ then

$\alpha x + \beta y \in {}_0C^1([0,1])$ for any α and β in $\mathbb{R}$. It is clear that $\alpha x + \beta y$ is continuously differentiable as x and y are an element in space ${}_0C^1([0,1])$ and addition and scalar multiplication preserve the continuity and differentiability. In more details, $(\alpha x + \beta y)' = \alpha x' + \beta y'$ and $(\alpha x + \beta y)(0) = \big(\alpha x(0) + \beta y(0)\big) = \alpha(0) + \beta(0) = 0$, and similarly for x'

$$(\alpha x + \beta y)'(0) = (\alpha x)'(0) + (\beta y)'(0) = \alpha(0) + \beta(0) = 0.$$

Hence, we proved that

$${}_0C^1([0,1]) \subset {}_0H^{1,p}(0,1).$$

□

Lemma B.2. *The space ${}_0C([0,1])$ is a closed subspace of the space $C([0,1])$.*

Proof of the Lemma B.2. It is obvious to see that ${}_0C([0,1]) \subset C([0,1])$ because any element in the space ${}_0C([0,1])$ is continuous, in particular, $f(0) = 0$. To prove the space ${}_0C([0,1])$ is a vector space itself, let $x, y \in {}_0C([0,1])$ and $\alpha, \beta \in \mathbb{R}$ then we have $\alpha x(s) + \beta y(r)$ is continuous and $\alpha x(0) + \beta y(0) = 0$.

Finally, it remains to show that the space ${}_0C([0,1])$ is closed. Let $\{x_n\}_{n\in\mathbb{N}}$ be a Cauchy sequence in the space ${}_0C([0,1])$ then $x_n \to x \in C([0,1])$ uniformly because the space $C([0,1])$ is complete. In particular, $x_n(0) \to x(0)$. Since $x_n \in {}_0C([0,1])$ this means that $x_n(0) = 0$. Hence $x(0) = 0$. Therefore, $x \in {}_0C([0,1])$. Hence ${}_0C([0,1])$ is closed subspace of the $C([0,1])$. □

Lemma B.3. *The space $F = {}_0C^1\big([0,1]\big)$, defined by formula* (3.2.6) *is a separable Banach space.*

Proof of Lemma B.3. Let us consider the following map:

$$\Gamma : F \ni x \mapsto (x, x') \in E \times E, \quad \text{where } E = {}_0C([0,1]). \tag{B.1.10}$$

Based on the above map (B.1.10), the justification can be expressed in three steps. The first step is to verify that the cartesian product $E \times E$, that is endowed with the norm:

$$\|(x,y)\|_{E\times E} := |x|_E + |y|_E, \tag{B.1.11}$$

is a Banach space. The second step of the justification is to prove that the cartesian product $E \times E$ is separable. The third step is to justify that Γ, which was defined in (B.1.10), is isometric.

Regarding the first step, there are two main conditions for the Banach space that need to be satisfied, which are $E \times E$ is complete. For the former (norm), it is obvious that

$E \times E$ is a vector space with addition and multiplication by scalars defined in a standard way, i.e.,

$$\alpha_1(x_1, y_1) + \alpha_2(x_2, y_2) := (\alpha_1 x_1 + \alpha_2 x_2, \alpha_1 y_1 + \alpha_2 y_2), \quad \alpha_i \in \mathbb{K}, x_i, y_i \in E.$$

Moreover, all the conditions, which are mentioned before in Definition 2.1 are clearly satisfied. Regarding the latter condition, which is related to the completeness of the vector space $E \times E$, we need to study whether the space has a convergent Cauchy sequence. Let (x_n, y_n) be a Cauchy sequence. Let $\varepsilon > 0$ and $N \in \mathbb{N}$ such that for every $n, m \geq N$ we have

$$\|(x_n, y_n) - (x_m, y_m)\| < \varepsilon.$$

From the definition of the norm of the space $E \times E$, which is stated in equality (B.1.11), we have

$$|x_n - x_m|_E + |y_n - y_m|_E < \varepsilon, \quad n, m \geq N.$$

Hence, we see that sequences (x_n) and (y_n) are Cauchy in the space E. Since E is a complete space, we deduce that the sequences (x_n) and (y_n) are convergent in E. That means, there exist $x, y \in E$ such that $x_n \to x$ and $y_n \to y$ in E.
Let us observe that $(x, y) \in E \times E$. Given an $\varepsilon > 0$, we want an N_1 so that $|x_n - x|$ is less than $\frac{\varepsilon}{2}$ and an N_2 so that the same is also true for $|y_n - y|$ and $\frac{\varepsilon}{2}$. If we take $N = \max\{N_1, N_2\}$ then we have

$$|(x_n, y_n) - (x, y)|_{E \times E} < \varepsilon.$$

Thus,

$$\|(x_n, y_n) - (x, y)\|_{E \times E} < \varepsilon, \quad \text{for } n \geq \mathbb{N}.$$

Hence $(x_n, y_n) \to (x, y)$. By this we proved that $E \times E$ is complete.
Regarding the second step that the cartesian product $E \times E$ is separable, we have already proven in Example B.1 that the space $C\big([0,1]\big)$ is a separable Banach space. Since the space ${}_0C\big([0,1]\big)$ is a closed subspace of $C\big([0,1]\big)$ with the same norms, then we deduce from Lemma 2.28 that the space ${}_0C\big([0,1]\big)$ is also separable. Moreover, a cartesian product of two separable normed vector spaces is separable . Hence, we deduce that the space $E \times E$ is separable.
Regarding the third step which is to justify whether Γ, which was defined in equation (B.1.10), is isometric, we start first by showing that the map $\Gamma : F \ni x \mapsto (x, x') \in E \times E$ is injective. This is because:

$$\Gamma(x_1) = \Gamma(x_2) \implies (x_1, x_1') = (x_2, x_2') \implies x_1 = x_2.$$

In addition to that the, map Γ is isometry, that is

$$\|\Gamma(x)\|_{E\times E} = \|x\|_F, \quad \text{for every } x \in F. \tag{B.1.12}$$

To justify that the equality (B.1.12) holds, we start with the LHS. let x belongs to the space F. Then we have

$$\|\Gamma(x)\|_{E\times E} = \|(x, x')\|_{E\times E} = |x|_E + |x'|_E.$$

Similarly, for the RHS of equality (B.1.12), let x belongs to the space F. Then we deduce from the definition of the norm of the space F for an element $x \in F$ the following equality is valid:

$$\|x\|_F = \sup_{s\in[0,1]} |x(s)| + \sup_{s\in[0,1]} |x'(s)| = \|x\|_E + \|x'\|_E$$

As a consequence, the LHS and the RHS of equality (B.1.12) are equal.
However, the map Γ is not surjective. For instance, if we take for example, elements $y(s) = s^2$ and $z(s) = s^3$, then the ordered pair (y, z) belongs to the space $E \times E$ but does not belong to $\Gamma(F)$.
Put

$$Z := \Gamma(F).$$

Let us note we proved earlier that $E \times E$ is a separable Banach space. Let us also note that Z is a closed subspace of $E \times E$. Hence Z is a separable Banach space.
Moreover, we can show that $\Gamma : F \to Z$ is an isomorphism. Note that $\Gamma : F \to E \times E$ is not surjective but we replace the "co-domain" $E \times E$ by the range Z of the function Γ. Hence, with this change Γ becomes surjective. Hence we infer that F is also a separable Banach space. $\square$

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