Integrable initial-boundary value problems and solution methods for the nonlinear Schrödinger equation

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To my parents

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Abstract

In this thesis, we review the inverse scattering transform with zero and non-zero boundary conditions at infinity for the one-dimensional nonlinear Schrödinger equation. The inverse problems are discussed making use of the theory of Riemann-Hilbert problems.

We perform the analysis of the focusing nonlinear Schrödinger equation on the half-line with time-dependent boundary conditions at origin and zero boundary conditions at infinity along the lines of the nonlinear mirror image method with the help of Bäcklund transformations. We find two possible classes of solutions. One class is very similar to the case of Robin boundary conditions whereby solitons are reflected at the boundary, as a result of effective interaction with their images on the other half-line. The new class of solutions supports the existence of one soliton that is not reflected at the boundary but can be either absorbed or emitted by it. We demonstrate that this is a unique feature of time-dependent integrable boundary conditions.

Finally, we present partial results of the analysis for the focusing nonlinear Schrödinger equation on the half-line with Robin boundary conditions at origin and non-zero boundary conditions at infinity using the nonlinear mirror image method in conjunction with Bäcklund transformations.

Abbreviations

NLS	Nonlinear Schrödinger equation
ZBCs	Zero boundary conditions
BCs	Boundary conditions
NZBCs	Non-zero boundary conditions
KdV	Korteweg-de Vries
IST	Inverse scattering transform
PDEs	Partial differential equations
RHP	Riemann-Hilbert Problem
IBVPs	Initial-boundary value problems
IVPs	Intial-value problems
BT	Bäcklund transformation
WKB	Wentzel-Kramers-Brillouin

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Chapter 1

Introduction

1.1 Presentation of the model

The one-dimensional nonlinear Schrödinger (NLS) equation is given by

$$iu_t + u_{xx} - 2\kappa|u|^2u = 0. \tag{1.1.1}$$

The unknown function $u(x,t)$ appearing in this equation is complex-valued; the constant κ is ± 1. The NLS equation (1.1.1) is referred to as *focusing* if $\kappa = -1$ and it is *defocusing* in the case $\kappa = 1$. In this work, the space variable x is considered either in the *full-line* (that is, $x \in \mathbb{R}$) or *half-line* (that is, $x \in [0, +\infty)$), and the time variable t is assumed in $[0, +\infty)$.

The NLS equation (1.1.1) can be used to describe the propagation of light in a nonlinear optical fibre Hasegawa & Tappert (1973). It also has many other interesting physical applications: we refer interested readers to Ablowitz *et al.* (2004), Dauxois & Peyrard (2006) and Sulem & Sulem (2007).

To keep the terminology used in the literature, we refer to solutions of (1.1.1) as *potentials*. We will consider two classes of potentials:

- Potentials $u(x,t)$ such that $u(\cdot,t)$ belong to the Schwartz class $\mathcal{S}(\mathbb{R})$[1] for each $t \geq 0$. We will refer to this as zero boundary conditions (ZBCs) at infinity.

[1] $\mathcal{S}(\mathbb{R})$ is defined as the space of functions whose derivatives (including the functions themselves) decay faster than any power of $|x|$ as $|x| \to \infty$; see Appendix A.2.

- Potentials $u(x,t)$ that tend to a time-dependent function as $x \to \pm\infty$, i.e. for each $t \geq 0$
$$u(x,t) \to u_{\pm} e^{-2i\kappa q_0^2 t}, \quad x \to \pm\infty, \tag{1.1.2}$$
where $u_{\pm}$ are some complex constants such that $|u_+| = |u_-| = q_0 \neq 0$. To tackle the initial-value problem for the NLS equation (1.1.1) with boundary conditions (BCs) (1.1.2), it is ideal to make the BCs time-independent. This can be done by introducing the following transformation $u(x,t) \to u(x,t)e^{2i\kappa q_0^2 t}$. As a consequence of this, (1.1.1) and (1.1.2) become
$$iu_t + u_{xx} - 2\kappa(|u|^2 - q_0^2)u = 0, \tag{1.1.3}$$
$$u(x,t) \to u_{\pm}, \quad x \to \pm\infty. \tag{1.1.4}$$
We will refer to (1.1.4) as non-zero boundary conditions (NZBCs) at infinity.

1.2 Properties of the NLS equation

1.2.1 Group symmetries for the NLS equation

Consider the following change of variables
$$\chi = cx, \quad \tau = c^2 t,$$
for some non-zero constant c. A simple application of the chain rule yields
$$\frac{\partial}{\partial x} = c\frac{\partial}{\partial \chi}, \quad \frac{\partial}{\partial t} = c^2\frac{\partial}{\partial \tau}.$$
A direct calculation shows that (1.1.1) yields
$$iq_\tau + q_{\chi\chi} - 2\kappa|q|^2 q = 0,$$
where $q = cu$. This means that if $u(x,t)$ is a solution of equation (1.1.1) then $cu(cx, c^2t)$ solves the same equation. This is known as the *scaling symmetry group* for the NLS equation (1.1.1).

Another important group symmetry for equation (1.1.1) is the so-called *Galilean transformation group*. It is realised by making the following change of variables
$$x \to x - vt, \quad t \to t,$$
for some non-zero constant v. In this case if $u(x,t)$ solves equation (1.1.1), then $u(x-vt,t)e^{i\frac{v}{2}\left(x-\frac{v}{2}t\right)}$ will also solve the same equation.

1.2.2 Solution of the NLS equation

Since the appearance of the influential and important paper by Zabusky & Kruskal (1965), (multi) solitons have attracted the attention of both physicists and mathematicians. To physicists, they are important to understand phenomena such as the appearance of rogue waves and the propagation of signals in the optical fibre, see for example Dauxois & Peyrard (2006); while to mathematicians, equations that approximately describe these phenomena are good examples of *completely integrable* systems. The mathematical machinery that encodes this integrability property and the construction of complicated soliton solutions is known as the *inverse scattering transform*, which is the central tool in this thesis. To appreciate the importance of the inverse scattering transform, we will discuss the condition under which solitons appear for the NLS equation (1.1.1), and we will provide a direct derivation of the simplest soliton solution.

Consider the NLS equation (1.1.1). We look for solutions of the form

$$u(x,t) = \psi(x)e^{i\phi(t)}, \tag{1.2.1}$$

where $\psi(x)$ and $\phi(t)$ are real-valued functions. Substituting (1.2.1) into (1.1.1) yields

$$-\psi\phi_t + \psi_{xx} - 2\kappa\psi^3 = 0.$$

This equation can be rearranged to obtain

$$\phi_t = \frac{\psi_{xx}}{\psi} - 2\kappa\psi^2.$$

The LHS of this equation is a function of t and the RHS is a function of x. Thus both sides should be equal to a constant, say ϕ_0. We have $\phi_t = \phi_0 \implies \phi(t) = \phi_0 t + \phi_1$, where ϕ_1 is the constant of integration. The second equation can be also rearranged to obtain

$$\psi_{xx} = 2\kappa\psi^3 + \phi_0\psi. \tag{1.2.2}$$

Assume that ψ and its first derivative ψ_x decay to zero as $x \to \pm\infty$. Multiplying (1.2.2) by ψ_x and integrating, yields

$$(\psi_x)^2 = \kappa\psi^4 + \phi_0\psi^2. \tag{1.2.3}$$

We do not have any constant of integration due to the boundary conditions of $\psi(x)$ above. This equation can be seen as

$$\frac{(\psi_x)^2}{2} + \mathcal{U}(\psi) = 0,$$

where $\mathcal{U}(\psi) = -\left(\frac{\kappa\psi^4}{2} + \frac{\phi_0\psi^2}{2}\right)$. Note that since $\psi(x)$ is a real-valued function, we have $(\psi_x)^2 \geq 0$. This implies that $\mathcal{U}(\psi) \leq 0$ for all values of ψ that correspond to a solution. A quick phase-portrait analysis tells us that a bounded motion starting from $\psi = 0$ can only be possible if $\kappa = -1$ and $\phi_0 \geq 0$. This corresponds to a soliton solution for the NLS equation. Now, assume that $\kappa = -1$ and $\phi_0 \geq 0$. Therefore, one can integrate the remaining differential equation by making the following change of variable $\psi = a\,\mathrm{sech}(\theta)$, where $a = \sqrt{\phi_0}$, which yields $\psi(x) = a\,\mathrm{sech}\left(a(x + K_0)\right)$, where $K_0 = \frac{1}{\sqrt{\phi_0}}\,\mathrm{arcsech}\left(\frac{\psi(0)}{\sqrt{\phi_0}}\right)$. Putting everything together, we obtain

$$u(x,t) = a\,\mathrm{sech}\left(ax + K\right) e^{i(a^2 t + \phi_1)},$$

where $K = aK_0$. By applying the Galilean transformation above, one gets

$$u(x,t) = a\,\mathrm{sech}\left(a(x - vt) + K\right) e^{i\left[\frac{v}{2}x + \left(a^2 - \frac{v^2}{4}\right)t + \phi_1\right]}. \tag{1.2.4}$$

Solution (1.2.4) is known as *the four-parameter bright soliton* for the focusing NLS equation (1.1.1).

It is clear that the construction of (multi) soliton solutions is not an easy task. A good insight about the physical system of interest is required to produce an "accurate" ansatz. The situation becomes more complicated when one attempts to explain the interactions between solitons. As we shall see in the next chapter, the inverse scattering transform provides a robust mathematical framework to tackle rigorously these challenges.

Thesis structure In Chapter 2, we discuss in detail the inverse scattering transform for both the focusing and defocusing NLS equation (1.1.1) with zero boundary conditions at infinity. We then review the recent developments of the inverse scattering transform for the focusing NLS equation with non-zero boundary conditions at infinity. In Chapter 3, we study integrable boundary conditions by making use of Sklyanin's formalism. We illustrate Sklyanin's approach with two examples: the well-known Robin boundary conditions (BCs) and, recently discovered, time-dependent BCs. Then, we analyse initial-boundary value problems (IBVPs) for the NLS equation, on the half-line, with Robin BCs at $x = 0$ and ZBCs at infinity through the nonlinear mirror image method; and, we briefly review the unified transform approach to IBVPs by Fokas. We discuss the connection between Sklyanin's formalism, the nonlinear mirror image and the unified transform. In Chapter 4, we apply the nonlinear mirror image method to solve IBVPs for the NLS equation with time-dependent boundary conditions. Finally, we present partial results obtained when implementing the nonlinear mirror image method to solve IBVPs for the NLS equation with Robin BCs at the origin and non-zero boundary conditions at infinity.

Chapter 2

Review of the inverse scattering transform

Gardner, Greene, Kruskal & Miura (1967) introduced a rather strange method that solves initial-value problems (IVPs) for the famous *Korteweg-de Vries equation*

$$u_t - 6uu_x + u_{xxx} = 0. \tag{2.0.1}$$

A year after, Lax (1968) formalised their ideas by developing a general framework known as *Lax pair formalism*. Later on, Shabat & Zakharov (1972) applied similar ideas to the NLS equation (1.1.1) which opened up a door to a huge amount of research in mathematical physics. These ideas were referred to as *inverse scattering transform* (IST) for the first time by Ablowitz, Kaup, Newell & Segur (1974) due to their multiple similitude to the well-known *Fourier transform*. Note that the Fourier transform method is used to solve linear partial differential equations (PDEs). The hallmark of the IST is that it provides a framework to derive *solitons* through a finite number of linear steps. Nonlinear PDEs that can be solved by the use of the inverse scattering transform are said to be *integrable*.

In Section 2.2 we discuss the inverse scattering transform for both focusing and defocusing NLS equation (1.1.1) with zero boundary conditions (ZBCs) at infinity. In Section 2.3 we consider only the IST for the focusing NLS equation (1.1.1) with non-zero boundary conditions (NZBCs) at infinity.

In the sequel, we will sometimes drop out the arguments when everything follows clearly by the context.

2.1 Lax pair formalism

The NLS equation (1.1.1) can be written as the *compatibility condition* of the following auxiliary system of *linear* differential equations

$$\psi_x(x,t,\lambda) = U(x,t,\lambda)\psi(x,t,\lambda), \tag{2.1.1}$$
$$\psi_t(x,t,\lambda) = V(x,t,\lambda)\psi(x,t,\lambda), \tag{2.1.2}$$

where

$$U(x,t,\lambda) = \begin{pmatrix} -i\lambda & u(x,t) \\ \kappa u^*(x,t) & i\lambda \end{pmatrix} \equiv -i\lambda\sigma_3 + Q(x,t), \tag{2.1.3}$$

$$V(x,t,\lambda) = -2i\lambda^2\sigma_3 + 2\lambda Q(x,t) - iQ_x(x,t)\sigma_3 - iQ^2(x,t)\sigma_3, \quad \kappa = \pm 1. \tag{2.1.4}$$

The unknown function $\psi(x,t,\lambda)$ is a 2×1 vector column. The asterisk stands for the complex conjugate, λ is a complex parameter and σ_3 is the third Pauli matrix given by $\sigma_3 = \mathrm{diag}(1,-1)$. In other words, a complex-valued function $u(x,t)$ solves the NLS equation (1.1.1) if and only if $\psi_{xt}(x,t,\lambda) = \psi_{tx}(x,t,\lambda)$ for all λ. This latter condition can be equivalently written as

$$U_t(x,t,\lambda) - V_x(x,t,\lambda) + [U(x,t,\lambda), V(x,t,\lambda)] = 0, \quad \text{for all } \lambda.[1] \tag{2.1.5}$$

Equation (2.1.5) is known in the literature as the *zero curvature condition*. More details about the zero curvature condition and its geometric interpretation can be found, for example, in Faddeev & Takhtajan (2007). The two matrix functions U and V defined above, form the so-called *Lax pair* associated with the NLS equation (1.1.1). Given that the NLS equation is equivalent to the auxiliary system (2.1.1)-(2.1.2), the study of its solution space is fundamental to solving the NLS equation. This is essentially what the inverse scattering transform is about.

[1]The matrix commutator is defined $[A,B] := AB - BA$ for any square matrices A, B.

As mentioned above, functions $\psi(x,t,\lambda)$ that solve the auxiliary system (2.1.1) are 2×1 complex-valued vectors. The solution space for this system would be generated by two linearly independent column-vector solutions. Therefore, it makes sense to build up a 2×2 matrix $\Psi(x,t,\lambda)$ whose columns are vector solutions of (2.1.1)-(2.1.2) and write the system in a matrix form as $\Psi_x = U\Psi$ and $\Psi_t = V\Psi$. A matrix solution $\Psi(x,t,\lambda)$ is said to be *fundamental* if it is invertible, i.e., its determinant is non-zero. In the sequel, unless explicitly stated, whenever we mention solutions of (2.1.1), we refer to matrix solutions. Occasionally, we will refer to equations (2.1.1) and (2.1.2) as the x-part and t-part of the Lax pair, respectively.

Let

$$\sigma_\kappa = \begin{pmatrix} 0 & \kappa \\ 1 & 0 \end{pmatrix} \tag{2.1.6}$$

Owing to the particular form of the matrix $U(x,t,\lambda)$, we have the following general result.

Lemma 2.1 (NLS symmetry)**.** *Let $\Psi(x,t,\lambda)$ be a solution of (2.1.1) and λ a complex parameter. Then we have that $\sigma_\kappa \Psi(x,t,\lambda^*)^* \sigma_\kappa^{-1}$ solves the same equation. In other words,*

$$\sigma_\kappa \Psi(x,t,\lambda^*)^* \sigma_\kappa^{-1} = \Psi(x,t,\lambda) M(t,\lambda), \tag{2.1.7}$$

where $M(t,\lambda)$ is a constant matrix with respect to x.

Proof: The proof follows from $\sigma_\kappa U(x,t,\lambda^*)^* \sigma_\kappa^{-1} = U(x,t,\lambda)$. ∎

The x-part of the Lax pair can be rewritten as an eigenvalue or a spectral problem:

$$\mathcal{L}\Psi = \lambda\Psi, \tag{2.1.8}$$

where

$$\mathcal{L} = i\sigma_3 \left(\partial_x - Q\right). \tag{2.1.9}$$

Owing to this representation, λ is referred to as *spectral parameter*. We will refer to (2.1.1) as the *spectral problem* or *scattering problem*.

The inverse scattering transform can be broken into three steps: direct problem, time evolution and the inverse problem. In the direct problem, we construct the scattering data from a suitable initial condition $u_0(x)$ using the x-part of the

Lax pair. The time evolution step consists of using the t-part of the Lax pair to study how the scattering data evolve with time. Finally, we use the time-dependent scattering data to reconstruct the solution $u(x,t)$ for the NLS equation via a normalised Riemann-Hilbert problem or Gel'fand-Levitan-Marchenko (GLM) equation such that $u(x,0) = u_0(x)$. This is schematized as follows:

$$\begin{tikzcd}
u_0(x) \arrow[rr, "\substack{\text{Direct problem} \\ \text{Via (2.1.1)}}"] \arrow[d, "\text{NLS equation (1.1.1)}"'] & & \text{Scattering data (SD)} \arrow[d, "\substack{\text{Time evolution} \\ \text{Via (2.1.2)}}"] \\
u(x,t) & & \text{Time-dependent of SD} \arrow[ll, "\text{Inverse problem}"', "\text{Riemann-Hilbert problem or GLM}"]
\end{tikzcd}$$

When presenting the inverse scattering transform, it is useful to implement the direct and inverse problems at an initial time, $t = 0$, and then connect them through the time evolution. We will use this approach in this thesis.

Hereafter $\mathbb{C}^+$ and $\mathbb{C}^-$ are, respectively, the upper and lower half of the complex plane, that is, $\mathbb{C}^+ = \{z \in \mathbb{C} \,|\, \mathrm{Im}(z) > 0\}$, $\mathbb{C}^- = \{z \in \mathbb{C} \,|\, \mathrm{Im}(z) < 0\}$. Note also that, we have

$$\overline{\mathbb{C}^\pm} := \mathbb{C}^\pm \cup \mathbb{R} \cup \{\infty\}.$$

Recall that the *point at infinity* ∞ in the complex plane can be taken in any direction. We denote by

$$e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \mathbb{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

2.2 Zero boundary conditions at infinity

In this section, we will describe the inverse scattering transform for the initial-value problem for the NLS equation in the case of ZBCs. We consider our potentials to be in the Schwartz class, i.e. $\mathcal{S}(\mathbb{R})$. We will also assume weaker conditions on the potentials, such as $L^1(\mathbb{R})$ and/or $L^2(\mathbb{R})$, to establish some fundamental results. Other functional spaces can be considered; see for example Beals & Coifman (1984), Faddeev & Takhtajan (2007) and Deift & Park (2011).

2.2.1 Direct problem

Set $u(x, t = 0) = u_0(x)$. Since only the initial condition (at time $t = 0$) is known, then one can only analyse the scattering problem at $t = 0$. Thus, for the rest of this section, every object will be constructed at time $t = 0$. We will drop any dependence on time t. For example, we will write $\Psi_\pm(x, \lambda)$ instead of $\Psi_\pm(x, t = 0, \lambda)$.

The eigenfunctions of the scattering problem (2.1.1) are asymptotic[1] to solutions of

$$\Psi_x(x, \lambda) = -i\lambda\sigma_3\Psi(x, \lambda)$$

because the initial potential $u_0(x)$ decays at infinity. For each λ, a particular solution for this differential equation is $e^{-i\lambda\sigma_3 x}$. Eigenfunctions of (2.1.1) that behave asymptotically as $e^{-i\lambda\sigma_3 x}$ are associated to λ being real. Therefore $\mathbb{R}$ constitutes the *continuous spectrum* for the operator $\mathcal{L}$.

Jost solutions Let $\lambda \in \mathbb{R}$, we denote $\Psi_\pm(x, \lambda)$, solutions of the x-part of the Lax pair such that they behave like $e^{-i\lambda\sigma_3 x}$ as $x \to \pm\infty$. These boundary conditions can be written as

$$\lim_{x\to\pm\infty} \Psi_\pm(x, \lambda)e^{i\lambda x\sigma_3} = \mathbb{I}. \tag{2.2.1}$$

The eigenfunctions $\Psi_\pm(x, \lambda)$ are referred to as *Jost solutions*.

We will now study some properties of the Jost solutions. We start by describing how one can formally construct them. To this end, one introduces the following transformation:

$$\Phi_\pm(x, \lambda) = \Psi_\pm(x, \lambda)e^{i\lambda\sigma_3 x}. \tag{2.2.2}$$

The ordinary differential equations satisfied by $\Phi_\pm(x, \lambda)$ can be directly integrated to obtain Volterra integral equations

$$\Phi_-(x, \lambda) = \mathbb{I} + \int_{-\infty}^{x} e^{i\lambda\sigma_3(y-x)}Q(y)\Phi_-(y, \lambda)e^{-i\lambda\sigma_3(y-x)}\mathrm{d}y, \tag{2.2.3}$$

$$\Phi_+(x, \lambda) = \mathbb{I} - \int_{x}^{\infty} e^{i\lambda\sigma_3(y-x)}Q(y)\Phi_+(y, \lambda)e^{-i\lambda\sigma_3(y-x)}\mathrm{d}y. \tag{2.2.4}$$

[1]The word asymptotic in this context refers of to the behaviour of the functions as x tends to $\pm\infty$.

For convenience, we introduce the following notations: $\Psi_{\pm}^{(1)}$ and $\Psi_{\pm}^{(2)}$ denote the first and second column of $\Psi_{\pm}$, respectively. The same convention holds for $\Phi_{\pm}$.

The next result shows that for $\lambda \in \mathbb{R}$, $\Phi_{\pm}(x, \lambda)$ are given by absolutely and uniformly convergent Neumann series.

Proposition 2.2. *Let $u_0(x)$ be an element of $L^1(\mathbb{R})$. Then the integral equations (2.2.3)-(2.2.4) have unique solutions $\Phi_{\pm}(x, \lambda)$, and these solutions are uniformly bounded on $\mathbb{R}$ for each $\lambda \in \mathbb{R}$. In particular, $\Psi_{\pm}(x, \lambda)$ are unique and uniformly bounded solutions of the scattering problem such that (2.2.1) hold for each $\lambda \in \mathbb{R}$.*

Proof. We prove this result for $\Psi_{-}^{(1)}(x, \lambda)$ in detail. The analysis for the other columns is similar. The first column of Eq. (2.2.3) is

$$\Phi_{-}^{(1)}(x, \lambda) = e_1 + \int_{-\infty}^{x} \operatorname{diag}\left(1, e^{-2i\lambda(y-x)}\right) Q(y) \Phi_{-}^{(1)}(y, \lambda) \mathrm{d}y.$$

To avoid cumbersome notations, we set $\left(f(x, \lambda), e^{2i\lambda x} g(x, \lambda)\right)^T = \Phi_{-}^{(1)}(x, \lambda)$. Thus we have

$$f(x, \lambda) = 1 + \int_{-\infty}^{x}, e^{2i\lambda y} u_0(y) g(y, \lambda) \mathrm{d}y, \;\; g(x, \lambda) = \kappa \int_{-\infty}^{x} e^{-2i\lambda y} u_0^*(y) f(y, \lambda) \mathrm{d}y. \quad (2.2.5)$$

By substituting the second equation into the first, and changing the order of integration

$$\int_{-\infty}^{x} \int_{-\infty}^{y} \kappa e^{-2i\lambda(z-y)} u_0(y) u_0^*(z) f(z, \lambda) \mathrm{d}z \mathrm{d}y = \int_{-\infty}^{x} \int_{z}^{x} \kappa e^{-2i\lambda(z-y)} u_0(y) u_0^*(z) f(z, \lambda) \mathrm{d}y \mathrm{d}z,$$

we obtain the following Volterra integral equation for $f(x, \lambda)$

$$f(x, \lambda) = 1 + \int_{-\infty}^{x} K(x, z, \lambda) f(z, \lambda) \mathrm{d}z, \quad (2.2.6)$$

where the kernel K is given by

$$K(x, z, \lambda) = \kappa u_0^*(z) \int_{z}^{x} e^{-2i\lambda(z-y)} u_0(y) \mathrm{d}y.$$

We now introduce the Neumann series for $f(x, \lambda)$ as

$$f(x, \lambda) = \sum_{n=0}^{\infty} f_n(x, \lambda), \quad (2.2.7)$$

$$f_0(x,\lambda) = 1, \quad f_{n+1}(x,\lambda) = \int_{-\infty}^{x} K(x,z,\lambda) f_n(z,\lambda) \mathrm{d}z, \quad n \geq 0.$$

Assume $u_0(x) \in L^1(\mathbb{R})$ and $\lambda \in \mathbb{R}$. The following estimate is essential for the rest of this proof

$$|K(x,z,\lambda)| \leq C|u_0(z)|, \tag{2.2.8}$$

where $C = \|u\|_{L^1(\mathbb{R})}$. Following the proof of Lemma 2.1 (Ablowitz *et al.*, 2004), we make the following claim: for all integers $n \geq 0$

$$|f_n(x,\lambda)| \leq \frac{C^n N^n(x)}{n!}, \tag{2.2.9}$$

where $N(x) = \int_{-\infty}^{x} |u_0(z)| \mathrm{d}z$. We use the induction principle to prove this claim. It is true for $n = 0$ and $n = 1$. Assume that the claim is true for any integer $n > 1$. Let's prove that it holds for $n+1$. Before we prove this, observe that, for any integer $j \geq 1$, we have

$$\int_{-\infty}^{x} |u_0(z)| N^j(z) \mathrm{d}z = \frac{1}{j+1} \int_{-\infty}^{x} \frac{\mathrm{d}}{\mathrm{d}z} \left[N^{j+1}(z) \right] \mathrm{d}z. \tag{2.2.10}$$

Hence, one gets

$$\begin{aligned} |f_{n+1}(x,\lambda)| &\leq \frac{C^{n+1}}{n!} \int_{-\infty}^{x} |u_0(z)| N^n(z) \mathrm{d}z \\ &= \frac{C^{n+1}}{n!(n+1)} \int_{-\infty}^{x} \frac{\mathrm{d}}{\mathrm{d}z} \left[N^{n+1}(z) \right] \mathrm{d}z = \frac{C^{n+1} N^{n+1}(x)}{(n+1)!}. \end{aligned}$$

Therefore, by the induction principle, the claim holds for any integer n. By comparison with the exponential series, it follows that the Neumann series defining $f(x,\lambda)$ converges absolutely. We have

$$|f(x,\lambda)| \leq \sum_{n=0}^{\infty} |f_n(x,\lambda)| \leq \exp\left(C N(x)\right). \tag{2.2.11}$$

We can replace x with $+\infty$ in (2.2.11) to obtain a uniform convergence:

$$\|f\|_\infty := \sup_{x\in\mathbb{R}} |f(x,\lambda)| \leq \exp\left(C^2\right),$$

for all $\lambda \in \mathbb{R}$. Therefore, we have obtained a uniformly bounded solution f for the integral equation (2.2.6).

This solution is unique. Indeed, consider two solutions $f(x,\lambda)$ and $\tilde{f}(x,\lambda)$ that solve the integral equation (2.2.6) with the same properties. We easily see that $\Delta f(x,\lambda) := \tilde{f}(x,\lambda) - f(x,\lambda)$ solves

$$\Delta f(x,\lambda) = \int_{-\infty}^{x} K(x,z,\lambda)\Delta f(z,\lambda)\mathrm{d}z. \tag{2.2.12}$$

Replace the integral equation (2.2.12) into itself $n-1$ times reads

$$\begin{aligned}\Delta f(x,\lambda) = \int_{-\infty}^{x} K(x,z_n,\lambda)\int_{-\infty}^{z_n} K(z_n,z_{n-1},\lambda)\times\cdots\\ \cdots\times\int_{-\infty}^{z_2} K(z_2,z_1,\lambda)\Delta f(z_1,\lambda)\mathrm{d}z_1\cdots\mathrm{d}z_n.\end{aligned}$$

Note that there exists a positive constant M such that $\|\Delta f\|_\infty = M$. Such a constant exists because $\Delta f(x,\lambda)$ is defined as the difference between two uniformly bounded functions. From this, one gets

$$\begin{aligned}|\Delta f(x,\lambda)| &\leq MC^n \int_{-\infty}^{x} |u_0(y_n)| \int_{-\infty}^{y_n} |u_0(y_n)| \cdots \int_{-\infty}^{y_2} |u_0(y_1)|\mathrm{d}y_1\cdots\mathrm{d}y_n\\ &\leq M\frac{C^{2n}}{n!}.\end{aligned}$$

Note that we used n times the identity (2.2.10) to obtain the last inequality. Recall that $M\sum_{n=0}^{\infty}\frac{C^{2n}}{n!}$ is a convergent series, hence the sequence $\left(\frac{C^{2n}}{n!}\right)$ will converge to zero. This means $\Delta f(x,\lambda) = 0$ for all $x,\lambda \in \mathbb{R}$. This proves the uniqueness of the solution to the Volterra equation (2.2.6). Finally, it follows that $g(x,\lambda)$ is an absolutely continuous and uniformly bounded function on $\mathbb{R}$ for each $\lambda \in \mathbb{R}$.

We have constructed the unique vector $\Phi_-^{(1)}(x,\lambda)$ that solves the first column of the integral equation (2.2.3) for $\lambda \in \mathbb{R}$. Therefore, $\Psi_-^{(1)}(x,\lambda) = \Phi_-^{(1)}(x,\lambda)e^{-i\lambda x}$ is the unique solution of the differential equation (2.1.1) such that $e^{i\lambda x}\Psi_-^{(1)}(x,\lambda) \to e_1$ as $x \to -\infty$. This concludes the proof. ■

Remark 2.3. *In the above lemma, we proved that for each $\lambda \in \mathbb{R}$, the functions $f(x,\lambda)$ and $g(x,\lambda)$ are uniformly bounded on $\mathbb{R}$ for each $\lambda \in \mathbb{R}$. It follows from Theorem A.1(a) that:*

$$\lim_{x\to-\infty} f(x,\lambda) = 1, \quad \lim_{x\to+\infty} f(x,\lambda) = 1 + \int_{-\infty}^{+\infty} K(+\infty,z,\lambda)f(z,\lambda)dz, \tag{2.2.13}$$

and

$$\lim_{x\to-\infty} g(x,\lambda)=0, \qquad \lim_{x\to+\infty} g(x,\lambda)=\kappa\int_{-\infty}^{+\infty} e^{-2i\lambda y}u_0^*(y)f(y,\lambda)\,dy. \tag{2.2.14}$$

Due to the exponential functions appearing in the integral representation of $\Phi_\pm(x,\lambda)$, different columns of a matrix Jost solution will not be well-defined in the same region of the λ-complex plane. For example, $\Psi_-^{(1)}(x,\lambda)$ will be well-defined for $\lambda\in\mathbb{C}^+$ while $\Psi_-^{(2)}(x,\lambda)$ will be well-defined for $\lambda\in\mathbb{C}^-$. In this case, using its integral representation, we can prove that $\Psi_-^{(1)}(x,\lambda)$ (resp. $\Psi_+^{(2)}(x,\lambda)$), seen as a function of the spectral parameter λ, will admit an analytic continuation in $\mathbb{C}^+$ (resp. $\mathbb{C}^-$). These properties will be useful for the inverse problem; see Subsection 2.2.2.

Lemma 2.4 (Analytic continuation)**.** *Let $u_0(x)$ be an element of $L^1(\mathbb{R})$. Let $\Phi_-(x,\lambda)$ and $\Phi_+(x,\lambda)$ be the solutions for the integral equations* (2.2.3)-(2.2.4)*, respectively. Fix $x\in\mathbb{R}$, then $\Phi_\mp^{(1)}(x,\cdot)$ and $\Phi_\pm^{(2)}(x,\cdot)$ are continuous on $\mathbb{R}\cup\mathbb{C}^\pm$ and have an analytic continuation on $\mathbb{C}^\pm$. In particular, if $u_0\in L^1(\mathbb{R})\cap L^2(\mathbb{R})$, we have*

1. *$\Psi_-^{(1)}(x,\cdot)$ and $\Psi_+^{(2)}(x,\cdot)$ are continuous on $\mathbb{R}\cup\mathbb{C}^+$ and analytic on $\mathbb{C}^+$ with property*

$$\Psi_-^{(1)}(x,\lambda)=\mathcal{O}\left(e^{\mathrm{Im}(\lambda)x}\right),\ \textit{as } x\to-\infty,\ \Psi_+^{(2)}(x,\lambda)=\mathcal{O}\left(e^{-\mathrm{Im}(\lambda)x}\right),\ \textit{as } x\to+\infty,\ \lambda\in\mathbb{C}^+,$$

$$e^{i\lambda x}\Psi_-^{(1)}(x,\lambda)=e_1+\mathcal{O}(\lambda^{-1}),\quad e^{-i\lambda x}\Psi_+^{(2)}(x,\lambda)=e_2+\mathcal{O}(\lambda^{-1}),\quad \textit{as } \lambda\to\infty \textit{ and } \lambda\in\mathbb{C}^+.$$

2. *$\Psi_+^{(1)}(x,\cdot)$ and $\Psi_-^{(2)}(x,\cdot)$ are continuous on $\mathbb{R}\cup\mathbb{C}^-$ and analytic on $\mathbb{C}^-$ with property*

$$\Psi_+^{(1)}(x,\lambda)=\mathcal{O}\left(e^{\mathrm{Im}(\lambda)x}\right),\ \textit{as } x\to+\infty,\ \Psi_-^{(2)}(x,\lambda)=\mathcal{O}\left(e^{-\mathrm{Im}(\lambda)x}\right),\ \textit{as } x\to-\infty,\ \lambda\in\mathbb{C}^-,$$

$$e^{i\lambda x}\Psi_+^{(1)}(x,\lambda)=e_1+\mathcal{O}(\lambda^{-1}),\quad e^{-i\lambda x}\Psi_-^{(2)}(x,\lambda)=e_2+\mathcal{O}(\lambda^{-1}),\quad \textit{as } \lambda\to\infty \textit{ and } \lambda\in\mathbb{C}^-.$$

Proof: Again, we will show these results in detail for $\Psi_-^{(1)}(x,\lambda)$. The analysis for the other columns of Jost solutions will be similar. We set $(f(x,\lambda),e^{2i\lambda x}g(x,\lambda))^T=\Phi_-^{(1)}(x,\lambda)$. We know that f and g are solutions of the integral equations (2.2.5) for $\lambda\in\mathbb{R}$. Recall that f is formally defined by the Neumann series

$$f(x,\lambda)=\sum_{n=0}^{\infty} f_n(x,\lambda),$$

$$f_0(x,\lambda)=1, \quad f_{n+1}(x,\lambda)=\int_{-\infty}^{x} K(x,z,\lambda)f_n(z,\lambda)\mathrm{d}z, \quad n\geq 0,$$
$$K(x,z,\lambda)=\kappa u_0^*(z)\int_z^x e^{-2i\lambda(z-y)}u_0(y)\mathrm{d}y.$$

It turns out that this series will still be convergent when the parameter λ belongs to a specific region of the complex plane. Let us assume that $\lambda\in\mathbb{C}$. The key ingredient used to prove that the above Neumann series converges was the estimate for the kernel $K(x,z,\lambda)$. So, let us try to do the same but now λ is a complex parameter:

$$|K(x,y,\lambda)|\leq |u_0(y)|\int_z^x e^{-2\mathrm{Im}(\lambda)(y-z)}|u_0(y)|\mathrm{d}y.$$

We see that we cannot obtain the same estimate for the kernel K (see (2.2.8)) without further assumption on the spectral parameter λ. Note that $y\geq z$, that means $y-z\geq 0$. Hence, by assuming $\mathrm{Im}(\lambda)>0$ (that is $\lambda\in\mathbb{C}^+$), one recovers the same estimate as in (2.2.8) because the exponential is bounded above by 1 and the potential u_0 is considered to be absolutely integrable on $\mathbb{R}$. Therefore, one can repeat the exact construction as above to obtain the existence and uniqueness of the uniformly bounded function $f(x,\lambda)$ that solves Eq. (2.2.6) for each $\lambda\in\mathbb{C}^+$. The limiting values (2.2.13) still hold.

Note that, for $\lambda\in\mathbb{C}^+$, the integral equation defining $g(x,\lambda)$ remains valid. However, the integrand $e^{-2i\lambda x}u_0^*(x)f(x,\lambda)$ is not anymore absolutely integrable on $\mathbb{R}$ because the exponential factor grows without limit as $x\to+\infty$. Hence, $g(\cdot,\lambda)$ is not absolutely continuous, nor uniformly bounded on $\mathbb{R}$ for each $\lambda\in\mathbb{C}^+$. So, in this case, one cannot have the limiting value of $g(x,\lambda)$ at $x=+\infty$ as in (2.2.14). However, the first limit in (2.2.14) holds even for $\lambda\in\mathbb{C}^+$ because the exponential decays as $x\to-\infty$. Moreover, we have

$$\begin{aligned}|g(x,\lambda)| &\leq \int_{-\infty}^{x} e^{2\mathrm{Im}(\lambda)y}|u_0(y)||f(y,\lambda)|\mathrm{d}y\\ &\leq e^{2\mathrm{Im}(\lambda)x}\int_{-\infty}^{x}|u_0(y)||f(y,\lambda)|\mathrm{d}y\\ &\leq e^{2\mathrm{Im}(\lambda)x}C\exp(C^2),\end{aligned}$$

where $C=\|u_0\|_{L^1(\mathbb{R})}$. This means that $g(x,\lambda)$ decays to zero at the same rate as $e^{2\mathrm{Im}(\lambda)x}$ as $x\to-\infty$. It follows that $\Psi_-^{(1)}(x,\lambda)=\mathcal{O}\left(e^{\mathrm{Im}(\lambda)x}\right)$ as $x\to-\infty$.

Let us fix $x \in \mathbb{R}$. Obviously, the function $f_0(x,\lambda)$ is continuous on $\mathbb{R} \cup \mathbb{C}^+$. Assume that for any $n > 1$, $f_{n-1}(x,\lambda)$ is continuous on $\mathbb{R} \cup \mathbb{C}^+$. One has the following estimate

$$|K(x,z,\lambda)f_{n-1}(z,\lambda)| \leq |u(z)|\frac{C^{2n-1}}{(n-1)!}. \tag{2.2.15}$$

Note that the RHS of the above estimate is absolutely integrable on $(-\infty, x)$. Owing to Theorem A.1(a), we conclude that $f_n(x,\lambda)$ is continuous for λ in $\mathbb{R} \cup \mathbb{C}^+$. Therefore, $f(x,\lambda)$ is continuous on $\mathbb{R} \cup \mathbb{C}^+$ since it is given by a uniformly convergent series of continuous functions over the same domain. Now, we will use the same approach to prove analyticity. Again, $f_0(x,\lambda)$ is analytic on $\mathbb{C}^+$. Assume that f_{n-1} is an analytic function of λ in $\mathbb{C}^+$. Let Γ be a piecewise-smooth closed curve contained in $\mathbb{C}^+$. Then, one has

$$\begin{aligned}
\oint_\Gamma f_n(x,\lambda)\mathrm{d}\lambda &= \oint_\Gamma \int_{-\infty}^{x} K(x,z,\lambda)f_{n-1}(z,\lambda)\mathrm{d}z\mathrm{d}\lambda \\
&= \int_{-\infty}^{x} \left[\oint_\Gamma K(x,z,\lambda)f_{n-1}(z,\lambda)\mathrm{d}\lambda\right]\mathrm{d}z, \text{ using (2.2.15)}, \\
&= 0.
\end{aligned} \tag{2.2.16}$$

Note that we used Fubini's Theorem to exchange the order of integration. The last equality is justified by the fact $K(x,z,\lambda)f_{n-1}(z,\lambda)$ is analytic in $\mathbb{C}^+$, by assumption and the fact that the kernel $K(x,y,\lambda)$ is an entire function of λ. We proved above that each $f_n(x,\lambda)$ is continuous on $\mathbb{R} \cup \mathbb{C}^+$ and now we have just shown that for any closed contour Γ in $\mathbb{C}^+$ we have $\oint_\Gamma f_n(x,\lambda)\mathrm{d}\lambda = 0$. It follows by Morera's Theorem that $f_n(x,\lambda)$ is analytic in $\mathbb{C}^+$. By the same argument as above, $f(x,\lambda)$ is analytic on $\mathbb{C}^+$.

Applying similar arguments, we can prove that $g(x,\lambda)$ is a continuous function of $\lambda \in \mathbb{C}^+$. Thus the use of Morera's theorem as above will help us conclude $g(x,\lambda)$ is also analytic on $\mathbb{C}^+$. A direct integration by parts of $f(x,\lambda)$ and $g(x,\lambda)$ yields the following asymptotic expansion in λ

$$f(x,\lambda) = 1 - \frac{\kappa}{2i\lambda}\int_{\infty}^{x} |u_0(y)|^2\mathrm{d}y + \mathcal{O}(\lambda^{-2}), \tag{2.2.17}$$

$$e^{2i\lambda x}g(x,\lambda) = -\frac{\kappa}{2i\lambda}u_0^*(x) + \mathcal{O}(\lambda^{-2}).$$

It follows that $e^{i\lambda x}\Psi_-^{(1)}(x,\lambda) \to e_1$, as $\lambda \to \infty$ and $\lambda \in \mathbb{C}^+$. This concludes the proof. ∎

Remark 2.5. *A direct consequence of the above result is that $\Psi_\pm^{(1)}(x,\lambda)$ and $\Psi_\mp^{(2)}(x,\lambda)$ are Schwartz functions on $\mathbb{R}_\pm$ and $\mathbb{R}_\mp$ for each $\lambda \in \mathbb{C}_\mp$, respectively.*

Lemma 2.6 (Abel's theorem, Coddington & Levinson (1955))**.** *Consider an n-dimensional first-order homogeneous linear ordinary differential equation $y' = A(x)y$, on an interval $I \subset \mathbb{R}$, where $A(x)$ denotes a complex square matrix of order n. Let H be a matrix-valued solution of this equation. If the trace $\operatorname{tr} A(x)$ is a continuous function, then one has*

$$\det H(x) = \det H(x_0) \exp\left[\int_{x_0}^{x} \operatorname{tr} A(\xi)\, d\xi\right], \qquad x, x_0 \in I.$$

Scattering coefficients Recall that the Jost solutions $\Psi_\pm(x,\lambda)$ are 2×2 matrix functions that solve the scattering problem. Since the matrix function $U(x,\lambda)$ is traceless (that is, its trace is identically zero), one can use Abel's theorem and the normalisation of the Jost solutions at $\pm\infty$ to obtain

$$\det\Psi_\pm(x,\lambda) = 1, \qquad \lambda \in \mathbb{R}. \tag{2.2.18}$$

This means that each Jost solution constitutes a fundamental matrix of solutions for the scattering problem. In other terms, they both generate the space of solutions. Note that the space of solutions is a two-dimensional vector space. Therefore, they must be linearly dependent, that is

$$\Psi_+(x,\lambda) = \Psi_-(x,\lambda) S(\lambda), \qquad \lambda \in \mathbb{R}, \tag{2.2.19}$$

where the 2×2 proportionality matrix $S(\lambda) = (s_{ij}(\lambda))_{1\le i,j\le 2}$ is referred to as *scattering matrix* associated with (the potential) $u_0(x)$. The entries of the scattering matrix are called *scattering coefficients* associated with (the potential) $u_0(x)$ and they are complex-valued functions of $\lambda \in \mathbb{R}$. When there is no confusion, we will omit the reference to the potential that gives rise to the scattering data. One deduces from relations (2.2.18) and (2.2.19) that the scattering matrix $S(\lambda)$ has the following property

$$\det S(\lambda) = 1, \qquad \lambda \in \mathbb{R}. \tag{2.2.20}$$

Analogous to columns of the Jost solutions, diagonal entries of the scattering matrix $S(\lambda)$ can be continued analytically to some regions of the λ complex plane whenever the potential $u_0(x)$ is absolutely integrable on $\mathbb{R}$, i.e., $u_0(x) \in L^1(\mathbb{R})$. This is discussed in the following result.

Proposition 2.7. *Let $u_0(x)$ be an element of $L^1(\mathbb{R})$. Then the scattering coefficient $s_{22}(\lambda)$ (respectively, $s_{11}(\lambda)$) is continuous on $\mathbb{R} \cup \mathbb{C}^+$ and has an analytic continuation on $\mathbb{C}^+$ (respectively, is continuous on $\mathbb{R} \cup \mathbb{C}^-$ and has an analytic continuation on $\mathbb{C}^-$). Moreover, the scattering coefficients $s_{12}(\lambda)$ and $s_{21}(\lambda)$ are continuous functions on $\mathbb{R}$ but do not have, in general, analytic continuations in any region of the λ complex plane.*

<u>Proof:</u> The columns of $\Psi_-(x,\lambda) = \Psi_+(x,\lambda)S(\lambda)^{-1}$ are given by

$$\begin{cases} \Psi_-^{(1)}(x,\lambda) = s_{22}(\lambda)\Psi_+^{(1)}(x,\lambda) - s_{21}(\lambda)\Psi_+^{(2)}(x,\lambda), \\ \Psi_-^{(2)}(x,\lambda) = s_{11}(\lambda)\Psi_+^{(2)}(x,\lambda) - s_{12}(\lambda)\Psi_+^{(1)}(x,\lambda). \end{cases} \tag{2.2.21}$$

By calculating the determinant of $\left(\Psi_-^{(1)}(x,\lambda), \Psi_+^{(2)}(x,\lambda)\right)$, one obtains

$$s_{22}(\lambda) = \det\left(\Psi_-^{(1)}(x,\lambda), \Psi_+^{(2)}(x,\lambda)\right). \tag{2.2.22}$$

Similarly, one gets

$$s_{11}(\lambda) = \det\left(\Psi_+^{(1)}(x,\lambda), \Psi_-^{(2)}(x,\lambda)\right). \tag{2.2.23}$$

The results for $s_{11}(\lambda)$ and $s_{22}(\lambda)$ follow from Lemma 2.4. Similar arguments are used to obtain the results for $s_{12}(\lambda)$ and $s_{21}(\lambda)$. ∎

Remark 2.8. *Let $\lambda \in \mathbb{R}$. It follows from the first equation in (2.2.21) and the normalisation of the Jost solutions $\Psi_+(x,\lambda)$ that*

$$\Psi_-^{(1)}(x,\lambda) = s_{22}(\lambda)\begin{pmatrix} e^{-i\lambda x} \\ 0 \end{pmatrix} - s_{21}(\lambda)\begin{pmatrix} 0 \\ e^{i\lambda x} \end{pmatrix} + \mathcal{O}(1), \quad \text{as } x \to +\infty. \tag{2.2.24}$$

As in the proof of Proposition 2.2, we set $e^{i\lambda x}\Psi_-^{(1)}(x,\lambda) = \begin{pmatrix} f(x,\lambda) \\ e^{2i\lambda x} g(x,\lambda) \end{pmatrix}$ where f and g are given by (2.2.6) and the second integral equation in (2.2.5), respectively. It follows from Remark 2.3 that

$$s_{22}(\lambda) = 1 + \int_{-\infty}^{+\infty} K(+\infty, z, \lambda) f(z,\lambda)dz, \quad s_{21}(\lambda) = -\kappa \int_{-\infty}^{+\infty} e^{-2i\lambda y} u^*(y) f(y,\lambda)dy. \tag{2.2.25}$$

Note that, the integral equation for $s_{22}(\lambda)$ still holds even when $\lambda \in \mathbb{C}^+$. However, one cannot say the same for $s_{21}(\lambda)$ when $\lambda \in \mathbb{C}^+$. In addition, we have from (2.2.17)

$$s_{22}(\lambda) \to 1, \quad \text{as } \lambda \to \infty \text{ and } \lambda \in \mathbb{C}^+.$$

In addition, it follows from the properties of $\Psi_-^{(1)}(x,\lambda)$ and the scattering coefficients that

$$e^{i\lambda x}\Psi_-^{(1)}(x,\lambda) = \begin{pmatrix} s_{22}(\lambda) \\ 0 \end{pmatrix} + \mathcal{O}(1), \quad x \to +\infty \text{ and } \lambda \in \mathbb{C}^+. \tag{2.2.26}$$

Hereafter, whenever we mention any column of the Jost matrix solutions $\Psi_\pm(x,\lambda)$ or the scattering coefficients $s_{jj}(\lambda)$ for $j = 1, 2$, we always refer to their analytic continuation in the appropriate region of the λ complex plane.

Corollary 2.9. *Consider $u_0(x) \in L^1(\mathbb{R})$. Then the scattering coefficients satisfy*

$$s_{22}(\lambda) = s_{11}(\lambda^*)^*, \quad \lambda \in \mathbb{R} \cup \mathbb{C}^+, \quad s_{21}(\lambda) = \kappa s_{12}(\lambda)^*, \quad \lambda \in \mathbb{R}. \tag{2.2.27}$$

Proof: Suppose that λ belongs to $\mathbb{R}$. The relation (2.1.7) in terms of the Jost solutions takes the form

$$\sigma_\kappa \Psi_\pm(x,\lambda)^* \sigma_\kappa^{-1} = \Psi_\pm(x,\lambda).$$

Thus the scattering matrix has the property

$$\sigma_\kappa S(\lambda)^* \sigma_\kappa^{-1} = S(\lambda).$$

Elementwise, we have $s_{22}(\lambda) = s_{11}(\lambda)^*$ and $s_{12}(\lambda) = \kappa s_{21}(\lambda)^*$. Now, assume that $\lambda \in \mathbb{C}$. Taking into consideration the analytical continuation of the Jost solutions in the correct regions of the λ complex plane, one obtains

$$\Psi_-^{(1)}(x,\lambda) = \kappa\sigma_\kappa \Psi_-^{(2)}(x,\lambda^*)^*, \quad \Psi_+^{(2)}(x,\lambda) = \sigma_\kappa \Psi_+^{(1)}(x,\lambda^*)^*, \quad \lambda \in \mathbb{C}^+. \tag{2.2.28}$$

Owing to Eqs. (2.2.22) and (2.2.23), one obtains the first in (2.2.27). ∎

We say that $\lambda \in \mathbb{C}\backslash\mathbb{R}$ is an *eigenvalue* for the operator $\mathcal{L}$ if its eigenfunction is an element of $L^2(\mathbb{R}, \mathbb{C}^2)$. We denote by $\mathcal{Z}^\pm$ the set of all $\lambda \in \mathbb{C}^\pm\backslash\mathbb{R}$ with such a property. The *discrete spectrum* for the operator $\mathcal{L}$ is given by

$$\mathcal{Z} = \mathcal{Z}^+ \cup \mathcal{Z}^-.$$

Owing to the relation (2.2.22), one can easily deduce that the set of zeros for the scattering coefficient $s_{22}(\lambda)$ coincides with $\mathcal{Z}^+$. The same correspondence can be drawn between $\mathcal{Z}^-$ and the set of zeros for $s_{11}(\lambda)$ using (2.2.23). We have the following well-known results in the scattering theory; see for example Beals & Coifman (1984).

Lemma 2.10. *Let $u_0(x)$ be an element of $L^1(\mathbb{R})$*

(a) if $\kappa = 1$, the scattering coefficients $s_{11}(\lambda)$ and $s_{22}(\lambda)$ do not vanish on $\mathbb{R} \cup \mathbb{C}^-$ and $\mathbb{R} \cup \mathbb{C}^+$, respectively;

(b) if $\kappa = -1$, there exists an open dense subset $\mathcal{G}$ of $L^1(\mathbb{R})$ such that for all $u_0(x)$ in $\mathcal{G}$, the following hold

- *The scattering coefficients $s_{11}(\lambda)$ and $s_{22}(\lambda)$ do not vanish on $\mathbb{R}$;*
- *$\mathcal{Z}^+$ and $\mathcal{Z}^-$ have a finite number of elements.*

Proof: From (2.2.20) and the symmetries in (2.2.27), one gets

$$|s_{22}(\lambda)|^2 - \kappa |s_{21}(\lambda)|^2 = 1, \quad \lambda \in \mathbb{R}. \tag{2.2.29}$$

If $\kappa = 1$, we have $|s_{22}(\lambda)|^2 \geq 1$. Together with the fact that the operator $\mathcal{L}$ is self-adjoint concludes the proof of part (a). The proof for part (b) can be found in (Beals & Coifman, 1984, Theorem A) ■

In the case $\kappa = -1$, we follow the terminology used in Beals & Coifman (1984): potentials $u_0(x)$ that lead to properties given in Lemma 2.10(b) are called *generic potentials*.

Definition 2.11 (Reflection coefficients)**.** *The* reflection coefficients r *and* $\bar{r}$ *are two functions of the real variable λ defined as*

$$\begin{array}{rccc} r: & \mathbb{R} & \longrightarrow & \mathbb{C} \\ & \lambda & \longmapsto & \frac{s_{21}(\lambda)}{s_{22}(\lambda)} \end{array}, \qquad \begin{array}{rccc} \bar{r}: & \mathbb{R} & \longrightarrow & \mathbb{C} \\ & \lambda & \longmapsto & \frac{s_{12}(\lambda)}{s_{11}(\lambda)} \end{array}$$

A direct consequence of symmetries in (2.2.27) is that the reflection coefficients are linked in the following way

$$\bar{r}(\lambda) = \kappa r(\lambda)^*. \tag{2.2.30}$$

Lemma 2.12 (Beals & Coifman (1984)). *Suppose that $u_0(x)$ is an element of $\mathcal{S}(\mathbb{R})$. Then the reflection coefficient $r(\lambda)$ will also be an element of $\mathcal{S}(\mathbb{R})$ and $\|r\|_\infty < 1$.*

It is important to mention that the condition $\|r\|_\infty < 1$ is essential because it guarantees the existence of a solution for the Riemann-Hilbert problem in the inverse problem.

Norming constants Consider $\kappa = -1$. Let N and $\bar{N}$ be positive integers. Assume that $u_0(x)$ is generic. Set

$$\mathcal{Z}^+ = \{\lambda_1, \lambda_2, \ldots, \lambda_N\}, \quad \mathcal{Z}^- = \left\{\bar{\lambda}_1, \bar{\lambda}_2, \ldots, \bar{\lambda}_{\bar{N}}\right\}. \tag{2.2.31}$$

For each $j = 1, \ldots, N$, it follows from (2.2.22) that $\Psi_-^{(1)}(x, \lambda_j)$ and $\Psi_+^{(2)}(x, \lambda_j)$ are linearly dependent, that is

$$\Psi_-^{(1)}(x, \lambda_j) = \gamma(\lambda_j)\Psi_+^{(2)}(x, \lambda_j), \tag{2.2.32}$$

where $\gamma(\lambda_j)$ is the proportionality constant that depends on λ_j. Each of these proportionality constants $\gamma(\lambda_j)$ introduces a quantity $c(\lambda_j)$ that we refer to as *norming constant* and it is defined as

$$c(\lambda_j) = \frac{\gamma(\lambda_j)}{s_{22}'(\lambda_j)}.$$

It is worth mentioning some authors refer to $\gamma(\lambda_j)$ as norming constants. Similarly, we have

$$\Psi_-^{(2)}\left(x, \bar{\lambda}_j\right) = \gamma\left(\bar{\lambda}_j\right)\Psi_+^{(1)}\left(x, \bar{\lambda}_j\right),$$

where $\gamma\left(\bar{\lambda}_j\right)$ depends on $\bar{\lambda}_j$ and its associated norming constant is

$$c\left(\bar{\lambda}_j\right) = \frac{\gamma\left(\bar{\lambda}_j\right)}{s_{11}'\left(\bar{\lambda}_j\right)}.$$

The symmetries in (2.2.27) affect the zeros and the norming constants associated with them as follows

$$N = \bar{N}, \quad \bar{\lambda}_j = \lambda_j^*, \quad \gamma\left(\bar{\lambda}_j\right) = -\gamma(\lambda_j)^*, \quad c\left(\bar{\lambda}_j\right) = -c(\lambda_j)^*. \tag{2.2.33}$$

Indeed, the first two in (2.2.33) follow directly from $s_{22}(\lambda) = s_{11}(\lambda^*)^*$. Since $\bar{\lambda}_j = \lambda_j^*$ is a simple zero of $s_{11}(\lambda)$, one has $\Psi_-^{(2)}\left(x, \lambda_j^*\right) = \gamma\left(\lambda_j^*\right)\Psi_+^{(1)}\left(x, \lambda_j^*\right)$.

Using both equations in (2.2.28), one obtains $\Psi_-^{(1)}(x,\lambda_j) = -\gamma(\lambda_j^*)^*\Psi_+^{(2)}(x,\lambda_j)$ which gives the third relation in (2.2.33) by comparing it with Eq. (2.2.32). Note that a direct calculation gives $s_{11}'(\lambda_j^*) = s_{22}'(\lambda_j)^*$. From this, we directly deduce the last relation in (2.2.33).

We can express the scattering coefficients $s_{jj}(\lambda)$ in terms of their zeros and the reflection coefficient, for $j = 1, 2$. Let us define

$$\phi_+(\lambda) = s_{22}(\lambda)\prod_{j=1}^{N}\frac{\lambda-\lambda_j^*}{\lambda-\lambda_j}, \quad \phi_-(\lambda) = s_{11}(\lambda)\prod_{j=1}^{N}\frac{\lambda-\lambda_j}{\lambda-\lambda_j^*}. \tag{2.2.34}$$

We can see that $\phi_+(\lambda)$ and $\phi_-(\lambda)$ are analytic in the upper-half and lower-half of the λ complex plane, respectively. By construction, $\phi_\pm(\lambda)$ do not have any singularities in their respective region of analyticity. We have that

$$\phi_+(\lambda)\phi_-(\lambda) = s_{11}(\lambda)s_{22}(\lambda), \quad \lambda \in \mathbb{R}. \tag{2.2.35}$$

The scattering coefficients carry a natural multiplicative Riemann-Hilbert problem on the real axis: A direct calculation from Eq. (2.2.20) gives the following jump condition

$$\phi_+(\lambda)\phi_-(\lambda) = \frac{1}{1-r(\lambda)\bar{r}(\lambda)}, \qquad \lambda \in \mathbb{R}.$$

By taking the logarithm of the above, and then using Plemelj's formulas one obtains

$$\log\phi_+(\lambda) = -\frac{1}{2\pi i}\int_{\mathbb{R}}\frac{\log(1-r(\tau)\bar{r}(\tau))}{\tau-\lambda}\mathrm{d}\tau, \quad \lambda \in \mathbb{C}^+,$$
$$\log\phi_-(\lambda) = \frac{1}{2\pi i}\int_{\mathbb{R}}\frac{\log(1-r(\tau)\bar{r}(\tau))}{\tau-\lambda}\mathrm{d}\tau, \quad \lambda \in \mathbb{C}^-.$$

Therefore, one can replace the function $\phi_+(\lambda)$ by its expression as given in relations (2.2.34) to obtain

$$s_{22}(\lambda) = \prod_{j=1}^{N}\frac{\lambda-\lambda_j}{\lambda-\lambda_j^*}\exp\left[-\frac{1}{2\pi i}\int_{\mathbb{R}}\frac{\log(1-r(\tau)\bar{r}(\tau))}{\tau-\lambda}\mathrm{d}\tau\right], \quad \lambda \in \mathbb{C}^+. \tag{2.2.36}$$

The same can be done to express the scattering coefficient $s_{11}(\lambda)$ in terms of its simple zeros in the lower-half plane and reflection coefficients.

Definition 2.13. *Consider $u_0(x) \in \mathcal{S}(\mathbb{R})$ and generic.*[1] *Let $\mathbb{S}$ be the map that associates to $u_0(x)$ its scattering data, namely*

[1] We need $u_0(x)$ to be generic only when $\kappa = -1$.

1. *If* $\kappa = 1$
$$\mathbb{S} : \{u_0(x)\} \mapsto \{r(\lambda)\}$$
where $r(\lambda)$ *is defined as above.*

2. *If* $\kappa = -1$
$$\mathbb{S} : \{u_0(x)\} \mapsto \{r(\lambda), (\lambda_j, c(\lambda_j)_{1\leq j\leq N})\}$$
where $r(\lambda), \lambda_j$ *and* $c(\lambda_j)$ *are defined as above.*

2.2.2 Inverse scattering problem

In this section, we will address the following problem: Is the map $\mathbb{S}$ *invertible*? That is, given the scattering $\mathbb{S}[u_0(x)]$, can we recover the potential $u_0(x) \in \mathcal{S}(\mathbb{R})$? The answer is Yes!

Originally, Gardner, Greene, Kruskal & Miura (1967), answered this question by using the so-called *Gel'fand-Levitan-Marchenko integral equations*. Later on, it was realised that the eigenfunctions $\Psi_{\pm}(x, \lambda)$ carry a natural structure of a Riemann-Hilbert problem (RHP) with the jump across the real line. In this work, we will take the latter approach to describe the inverse problem. We refer interested readers to Ablowitz, Prinari & Trubatch (2004) and therein references for more details about the Gel'fand-Levitan-Marchenko integral equations viewpoint.

We have the following two theorems, see for examples Beals & Coifman (1984), Zhou (1998), Ablowitz, Prinari & Trubatch (2004) and Faddeev & Takhtajan (2007).

Theorem 2.14 (RHP without poles). *Let* $r(\lambda)$ *be an element of* $\mathcal{S}(\mathbb{R})$ *with the property* $\|r\|_\infty \leq 1$. *Then the following normalised Riemann-Hilbert problem*

- **Analyticity.** $m(x, t, \lambda)$ *is analytic in* $\mathbb{C}\backslash\mathbb{R}$*;*
- **Jump condition.** *It has continuous boundary values*
$$m^{\pm}(x, t, \lambda) = \lim_{\varepsilon\to 0^+} m(x, t, \lambda \pm i\varepsilon), \quad \lambda \in \mathbb{R}, \tag{2.2.37}$$
satisfying the jump condition
$$m^+(x, t, \lambda) = m^-(x, t, \lambda)v(x, t, \lambda), \quad \lambda \in \mathbb{R}, \tag{2.2.38}$$

where the jump matrix *is given by*

$$v(x,t,\lambda) = \begin{pmatrix} 1-|r(\lambda)|^2 & r^*(\lambda)e^{-2i\theta(\lambda)} \\ -r(\lambda)e^{2i\theta(\lambda)} & 1 \end{pmatrix}, \quad \theta(\lambda) = \lambda x + 2\lambda^2 t;$$

- **Normalisation.** $m(x,t,\lambda) = \mathbb{I} + \mathcal{O}(\lambda^{-1})$ *as* $\lambda \to \infty$.

has a unique solution given by

$$m(x,t,\lambda) = \mathbb{I} + \frac{1}{2\pi i}\int_{\mathbb{R}} \frac{m_-(x,t,\xi)(I - v(x,t,\lambda))}{\xi - \lambda} d\xi, \quad \lambda \in \mathbb{C}\backslash\mathbb{R}.$$

Theorem 2.15 (RHP with poles)**.** *Let* $r(\lambda)$ *be an element of* $\mathcal{S}(\mathbb{R})$ *with the property* $\|r\|_\infty \leq 1$ *and* $(\lambda_j, c(\lambda_j)) \in \mathbb{C}^+ \times (\mathbb{C}\backslash\{0\})$, *for each* $j = 1, \dots, N$. *Then the following normalised Riemann-Hilbert problem*

- **Analyticity.** $m(x,t,\lambda)$ *is analytic in* $\mathbb{C}\backslash(\mathbb{R}\cup\mathcal{Z})$*;*
- **Jump condition.** *It has continuous boundary values*

$$m^\pm(x,t,\lambda) = \lim_{\varepsilon\to 0^+} m(x,t,\lambda \pm i\varepsilon), \quad \lambda \in \mathbb{R}, \tag{2.2.39}$$

satisfying the jump condition

$$m^+(x,t,\lambda) = m^-(x,t,\lambda)v(x,t,\lambda), \quad \lambda \in \mathbb{R}, \tag{2.2.40}$$

where the jump matrix *is given by*

$$v(x,t,\lambda) = \begin{pmatrix} 1+|r(\lambda)|^2 & -r^*(\lambda)e^{-2i\theta(\lambda)} \\ -r(\lambda)e^{2i\theta(\lambda)} & 1 \end{pmatrix};$$

- **Residues.** $m(x,t,\lambda)$ *has simple poles at* λ_j, λ_j^* *for* $j = 1, \dots, N$, *and the residues are given by*

$$\operatorname*{Res}_{\lambda=\lambda_j} m(x,t,\lambda) = \lim_{\lambda\to\lambda_j}\left[m(x,t,\lambda)\begin{pmatrix} 0 & 0 \\ c(\lambda_j)e^{2i\theta(\lambda_j)} & 0 \end{pmatrix}\right], \tag{2.2.41}$$

$$\operatorname*{Res}_{\lambda=\lambda_j^*} m(x,t,\lambda) = \lim_{\lambda\to\lambda_j^*}\left[m(x,t,\lambda)\begin{pmatrix} 0 & -c(\lambda_j)^* e^{-2i\theta(\lambda_j^*)} \\ 0 & 0 \end{pmatrix}\right]; \tag{2.2.42}$$

- **Normalisation.** $m(x,t,\lambda) = \mathbb{I} + \mathcal{O}(\lambda^{-1})$ *as* $\lambda \to \infty$.

has a unique solution given by

$$m(x,t,\lambda) = \mathbb{I} + \sum_{n=1}^{N} \left(\frac{\operatorname*{Res}_{\lambda=\lambda_j} m(x,t,\lambda)}{\lambda - \lambda_n} + \frac{\operatorname*{Res}_{\lambda=\lambda_n^*} m(x,t,\lambda)}{\lambda - \lambda_n^*} \right)$$
$$+ \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{m_-(x,t,\xi)(I - v(x,t,\lambda))}{\xi - \lambda} d\xi, \quad \lambda \in \mathbb{C} \backslash \mathbb{R}.$$

Let us consider $u_0(x) \in \mathcal{S}(\mathbb{R})$ and generic (when $\kappa = -1$) with the map $\mathbb{S}$ be given as in Definition 2.13. From the equation (2.2.19), one obtains that

$$\Phi_-(x,\lambda) = \Phi_+(x,\lambda) e^{-i\lambda\sigma_3 x} S(\lambda)^{-1} e^{i\lambda\sigma_3 x}, \quad \lambda \in \mathbb{R}.$$

This relation can be rewritten by grouping together entries that have an analytical continuation in the same λ complex plane region as

$$m^+(x,\lambda) = m^-(x,\lambda) v(x,\lambda), \quad \lambda \in \mathbb{R}, \tag{2.2.43}$$

where

$$m^+(x,\lambda) = \left(\frac{\Phi_-^{(1)}(x,\lambda)}{s_{22}(\lambda)}, \Phi_+^{(2)}(x,\lambda) \right), \quad m^-(x,\lambda) = \left(\Phi_+^{(1)}(x,\lambda), \frac{\Phi_-^{(2)}(x,\lambda)}{s_{11}(\lambda)} \right),$$
$$v(x,\lambda) = \begin{pmatrix} 1 - \kappa|r(\lambda)|^2 & \kappa r^*(\lambda) e^{-2i\lambda x} \\ -r(\lambda) e^{2i\lambda x} & 1 \end{pmatrix}.$$

We used the symmetry (2.2.30). Let us define the following matrix functions:

$$m(x,\lambda) = \begin{cases} m^+(x,\lambda), & \lambda \in \mathbb{C}^+, \\ m^-(x,\lambda), & \lambda \in \mathbb{C}^-. \end{cases} \tag{2.2.44}$$

It is worth mentioning that the superscripts "$\pm$" on $m(x,\lambda)$ in (2.2.44) do not have the same meaning as the ones in (2.2.37) and (2.2.39).

Lemma 2.16. *The 2×2 matrix functions $m(x,\lambda)$ defined in (2.2.44) solves the Riemann-Hilbert problem without poles when $\kappa = 1$ and with poles when $\kappa = -1$ at time $t = 0$. Moreover, the map*

1. *If $\kappa = 1$*
$$\mathbb{P} : \{r(\lambda)\} \mapsto \{u_0(x)\}$$
2. *If $\kappa = -1$*
$$\mathbb{P} : \{r(\lambda), (\lambda_j, c(\lambda_j)_{1 \le j \le N})\} \mapsto \{u_0(x)\}$$

defined by

$$u_0(x) = 2i \lim_{\lambda\to\infty} \lambda \, (m(x,\lambda))_{12} \,, \tag{2.2.45}$$

is the inverse to $\mathbb{S}$.

The reconstruction formula (2.2.45) is obtained by utilising the fact that $m(x,\lambda)$ solves the x-part of the Lax pair for the NLS equation (1.1.1).

2.2.3 Time evolution

Let $u(x,t)$ be the complex-valued solution of the NLS equation such that $u(x,t=0) = u_0(x) \in \mathcal{S}(\mathbb{R})$ and generic (when $\kappa = -1$). For each $\lambda \in \mathbb{R}$, consider $\Psi_\pm(x,t,\lambda)$ the Jost solutions associated to $u(x,t)$, that is $\Psi_\pm(x,t,\lambda)$ solve Eq. (2.1.1) such that

$$\lim_{x\to\pm\infty} \Psi_\pm(x,t,\lambda)e^{i\lambda\sigma_3 x} = \mathbb{I}, \quad \lambda \in \mathbb{R}.$$

Thus, starting with these Jost solutions, we can repeat the same construction as above to obtain the scattering data at any time $t > 0$. In what follows, we will describe the relationship between $\mathbb{S}[u_0(x)]$ and $\mathbb{S}[u(x,t)]$. To achieve this, let us first find out the time derivative for the Jost solutions $\Psi_\pm(x,t,\lambda)$. Since $u(x,t)$ solves the NLS equation, we know that the zero-curvature condition (2.1.5) must be satisfied for all λ. This means there exists a 2×2 matrix solution $Y(x,t,\lambda)$ that solves simultaneously the x-part and t-part of the Lax pair. However, $\Psi_\pm(x,t,\lambda)$ are fundamental matrix solutions for equation (2.1.1), therefore we have

$$Y(x,t,\lambda) = \Psi_\pm(x,t,\lambda)C_\pm(t,\lambda)$$

where $C_\pm(t,\lambda)$ are constant matrices with respect to x. Differentiating this equation with respect to time and using the fact that $Y(x,t,\lambda)$ satisfies equation (2.1.2), one gets

$$\begin{aligned} V(x,t,\lambda)\Psi_\pm(x,t,\lambda)C_\pm(t,\lambda) = \partial_t\big(\Psi_\pm(x,t,\lambda)\big)C_\pm(t,\lambda) \\ + \Psi_\pm(x,t,\lambda)\partial_t\big(C_\pm(t,\lambda)\big). \end{aligned} \tag{2.2.46}$$

Since $u(x,t) \in \mathcal{S}(\mathbb{R})$, for each $t \geq 0$, we can evaluate this equation at $x = +\infty$ to get

$$\left(C_\pm(t,\lambda)\right)_t = -2i\lambda^2\sigma_3 C_\pm(t,\lambda).$$

A particular solution for these equations is $C_{\pm}(t,\lambda) = e^{-2i\lambda^2\sigma_3 t}$. After substituting this expression of $C_{\pm}(t,\lambda)$ in (2.2.46) and making some rearrangement, we get

$$\partial_t \left(\Psi_{\pm}(x,t,\lambda)\right) = 2i\lambda^2 \Psi_{\pm}(x,t,\lambda)\sigma_3 + V(x,t,\lambda)\Psi_{\pm}(x,t,\lambda). \tag{2.2.47}$$

Now, we can use this differential equation to understand how the scattering data $\mathbb{S}[u_0(x)]$ evolve in time. By differentiating the time-dependent version of (2.2.19) with respect to t, we have

$$\begin{aligned}
\partial_t S(t,\lambda) &= \Psi_-(x,t,\lambda)^{-1}\partial_t(\Psi_+(x,t,\lambda)) \\
&\qquad - \Psi_-(x,t,\lambda)^{-1}\partial_t(\Psi_-(x,t,\lambda))\Psi_-(x,t,\lambda)^{-1}\Psi_+(x,t,\lambda) \\
&= i\lambda^2 \left[S(t,\lambda), \sigma_3\right].
\end{aligned}$$

This implies

$$S(t,\lambda) = e^{-i\lambda\sigma_3 t} S(0,\lambda) e^{i\lambda\sigma_3 t}.$$

Elementwise, we have

$$s_{11}(t,\lambda) = s_{11}(0,\lambda), \quad s_{22}(t,\lambda) = s_{22}(0,\lambda), \tag{2.2.48}$$

$$s_{12}(t,\lambda) = s_{12}(0,\lambda)e^{-2i\lambda^2 t}, \quad s_{21}(t,\lambda) = s_{21}(0,\lambda)e^{2i\lambda^2 t}. \tag{2.2.49}$$

A direct calculation gives

$$r(t,\lambda) = r(0,\lambda)e^{2i\lambda^2 t}, \quad \bar{r}(t,\lambda) = \bar{r}(0,\lambda)e^{-2i\lambda^2 t}. \tag{2.2.50}$$

A key observation to make is that, $s_{22}(t,\lambda)$ has the same zeros as $s_{22}(0,\lambda) \equiv s_{22}(\lambda)$. Thus, the potential $u(x,t)$ is again generic. Let λ_j be the zeros of the scattering coefficient $s_{22}(t,\lambda)$ in $\mathbb{C}^+$. From the time-dependent version of Eq. (2.2.22), we have that

$$\Psi_-^{(1)}(x,t,\lambda_j) = \gamma(t,\lambda_j)\Psi_+^{(2)}(x,t,\lambda_j), \tag{2.2.51}$$

where $\gamma(t,\lambda_j)$ is the proportionality constant. It follows from (2.2.47)

$$\partial_t \Psi_-^{(1)} = i\lambda^2 \Psi_-^{(1)} + V\Psi_-^{(1)}, \quad \partial_t \Psi_+^{(2)} = i\lambda^2 \Psi_+^{(2)} + V\Psi_+^{(2)}. \tag{2.2.52}$$

The time derivative of Eq. (2.2.51) gives

$$\begin{aligned}\partial_t \Psi_-^{(1)}(x,t,\lambda_j) &= \partial_t\gamma(t,\lambda_j)\Psi_+^{(2)}(x,t,\lambda_j) + \gamma(t,\lambda_j)\partial_t\Psi_+^{(2)}(x,t,\lambda_j)\\ &= \partial_t\gamma(t,\lambda_j)\Psi_+^{(2)}(x,t,\lambda_j)\\ &\quad + \gamma(t,\lambda_j)\left(i\lambda_j^2\Psi_+^{(2)}(x,t,\lambda_j) + V(x,t,\lambda_j)\Psi_+^{(2)}(x,t,\lambda_j)\right)\end{aligned}$$

Comparing this with the first differential equation in (2.2.52) evaluated at λ_j, and using (2.2.51) we obtain

$$\partial_t\gamma(t,\lambda_j) = 2i\lambda^2\gamma(t,\lambda_j).$$

This implies that

$$\gamma(t,\lambda_j) = \gamma(0,\lambda_j)e^{2i\lambda_j^2 t}.$$

A direct calculation gives the following time evolution for the norming constants

$$c(t,\lambda_j) = c(0,\lambda_j)e^{2i\lambda_j^2 t}. \tag{2.2.53}$$

Summary: The inverse scattering transform can be summarized as follows: apply the map $\mathbb{S}$ to $u_0(x) \in \mathcal{S}(\mathbb{R})$, evolve the scattering data in time using formulae (2.2.50) and (2.2.53), and then apply the map $\mathbb{P}$ to obtain the solution $u(x,t)$ at time $t > 0$ of the NLS equation (1.1.1) with $u(x,t=0) = u_0(x)$.

2.2.4 Reflectionless potentials: Multisoliton solutions

Let $\kappa = -1$. Assume that the reflection coefficient $r(t,\lambda)$ is identically zero. In this case, the solution to the Riemann-Hilbert problem with poles can be written as

$$m(x,t,\lambda) = \mathbb{I} + \sum_{n=1}^{N}\frac{\operatorname*{Res}_{\lambda=\lambda_n} m(x,t,\lambda)}{\lambda-\lambda_n} + \sum_{n=1}^{N}\frac{\operatorname*{Res}_{\lambda=\lambda_n^*} m(x,t,\lambda)}{\lambda-\lambda_n^*}. \tag{2.2.54}$$

Recall that, using (2.2.44), one has

$$\operatorname*{Res}_{\lambda=\lambda_n} m(x,t,\lambda) = \left[c(\lambda_n)e^{2i\theta(\lambda_n)}\Phi_+^{(2)}(x,t,\lambda_n), 0\right], \tag{2.2.55}$$

$$\operatorname*{Res}_{\lambda=\lambda_n^*} m(x,t,\lambda) = \left[0, -c(\lambda_n)^* e^{-2i\theta(\lambda_n^*)}\Phi_+^{(1)}(x,t,\lambda_n^*)\right]. \tag{2.2.56}$$

Given $\Phi_\pm^{(1)}$, we denote by $\Phi_\pm^{(11)}$ and $\Phi_\pm^{(21)}$ its first and second entries, respectively. Given $\Phi_\pm^{(2)}$, we denote by $\Phi_\pm^{(12)}$ and $\Phi_\pm^{(22)}$ its first and second entries, respectively.

The time-dependent version of the reconstruction formula (2.2.45) is

$$u(x,t) = -2i \sum_{n=1}^{N} c(\lambda_n)^* e^{-2i\theta(\lambda_n^*)} \Phi_+^{(11)}(x,t,\lambda_n^*).$$

In the next lines, we will try to express $\Phi_+^{(11)}(x,t,\lambda_n^*)$ in terms of the scattering data. One can evaluate the second column of (2.2.54), taking into consideration (2.2.44), at $\lambda = \lambda_n$, obtaining

$$\Phi_+^{(2)}(x,t,\lambda_n) = e_2 - \sum_{k=1}^{N} \frac{c(\lambda_k)^* e^{-2i\theta(\lambda_k^*)}}{\lambda_n - \lambda_k^*} \Phi_+^{(1)}(x,t,\lambda_k^*), \tag{2.2.57}$$

for $n = 1, \dots, N$. One can do the same with the first column at λ_n^* to obtain

$$\Phi_+^{(1)}(x,t,\lambda_n^*) = e_1 + \sum_{k=1}^{N} \frac{c(\lambda_k) e^{2i\theta(\lambda_k)}}{\lambda_n^* - \lambda_k} \Phi_+^{(2)}(x,t,\lambda_k), \tag{2.2.58}$$

for $n = 1, \dots, N$. The first entry of (2.2.57) is

$$\Phi_+^{(12)}(x,t,\lambda_j) = - \sum_{k=1}^{N} \frac{c(\lambda_k)^* e^{-2i\theta(\lambda_k^*)}}{\lambda_j - \lambda_k^*} \Phi_+^{(11)}(x,t,\lambda_k^*), \quad j = 1, \dots, N,$$

and, the first entry of (2.2.58) is

$$\Phi_+^{(11)}(x,t,\lambda_n^*) = 1 + \sum_{j=1}^{N} \frac{c(\lambda_j) e^{2i\theta(\lambda_j)}}{\lambda_n^* - \lambda_j} \Phi_+^{(12)}(x,t,\lambda_j), \quad n = 1, \dots, N.$$

Hence, for $n = 1, \dots, N$, one obtains

$$\begin{aligned}
\Phi_+^{(11)}(x,t,\lambda_n^*) &= 1 + \sum_{j=1}^{N} \frac{c(\lambda_j) e^{2i\theta(\lambda_j)}}{\lambda_n^* - \lambda_j} \Phi_+^{(12)}(x,t,\lambda_j) \\
&= 1 - \sum_{j=1}^{N} \frac{c(\lambda_j) e^{2i\theta(\lambda_j)}}{\lambda_n^* - \lambda_j} \left[\sum_{k=1}^{N} \frac{c(\lambda_k)^* e^{-2i\theta(\lambda_k^*)}}{\lambda_j - \lambda_k^*} \Phi_+^{(11)}(x,t,\lambda_k^*) \right] \\
&= 1 - \sum_{k=1}^{N} \left[c(\lambda_k)^* e^{-2i\theta(\lambda_k^*)} \sum_{j=1}^{N} \frac{c(\lambda_j) e^{2i\theta(\lambda_j)}}{(\lambda_n^* - \lambda_j)(\lambda_j - \lambda_k^*)} \right] \Phi_+^{(11)}(x,t,\lambda_k^*).
\end{aligned}$$

Introduce,

$$X = (X_1, \dots, X_N)^T, \quad B = (B_1, \dots, B_N)^T, \quad M = I + (A_{n,k})_{1\le n,k\le N}$$

where

$$X_n = \Phi_+^{(11)}(x,t,\lambda_n^*), \quad B_n = 1,$$

$$A_{n,k} = c(\lambda_k)^* e^{-2i\theta(\lambda_k^*)} \sum_{j=1}^{N} \frac{c(\lambda_j)e^{2i\theta(\lambda_j)}}{(\lambda_n^* - \lambda_j)(\lambda_j - \lambda_k^*)}.$$

The above algebraic system takes the following form

$$MX = B$$

The solution of the system is given by $X_n = \det M_n^{ext}/\det M$ for $n = 1, \dots, N$, where

$$M_n^{ext} = (M_1, \dots, \dots, M_{n-1}, B, M_{n+1}, \dots, M_N).$$

After substituting X in the reconstruction formula (2.2.45), one obtains the pure N-soliton solutions as

$$u(x,t) = 2i\frac{\det M^{inc}}{\det M} \tag{2.2.59}$$

where $M^{inc} = \begin{pmatrix} 0 & H \\ B & M \end{pmatrix}$ with $H = -\left(c(\lambda_1)^* e^{-2i\theta(\lambda_1^*)}, \dots, c(\lambda_n)^* e^{-2i\theta(\lambda_N^*)}\right)^T$.

Consider the case $N = 1$. Let $\lambda_1 = \frac{V+iA}{2}$ with $A > 0$. We have

$$\begin{aligned}
u(x,t) &= 2i\frac{\det\begin{pmatrix} 0 & -c_1^* e^{-2i\theta(\lambda_1^*)} \\ 1 & 1 + A_{11}\end{pmatrix}}{1 + A_{11}} \\
&= iA^2 \frac{2c_1^* e^{-2i\theta(\lambda_1^*)}}{A^2 + |c_1|^2 e^{-2(Ax+2AVt)}} \\
&= iA^2 \frac{2c_1^* e^{-i\left(Vx+(V^2-A^2)t\right)}}{A^2 e^{(Ax+2AVt)} + |c_1|^2 e^{-(Ax+2AVt)}}.
\end{aligned}$$

Let $c_1 = Ae^{i\xi+\xi_0}$, we obtain

$$\begin{aligned}
u(x,t) &= iA\frac{e^{-i\left(Vx+(V^2-A^2)t\right)-i\xi}}{e^{A(x+2Vt)-\xi_0} + e^{-(A(x+2Vt)-\xi_0)}} \\
&= A\,\mathrm{sech}(A(x+2Vt)-\xi_0)e^{-i\left(Vx+(V^2-A^2)t+\xi-\frac{\pi}{2}\right)}.
\end{aligned}$$

This is exactly the four-parameter solution (1.2.4) upon a suitable choice of parameters.

2.3 Non zero boundary conditions at infinity

In this section, we will describe the inverse scattering transform to solve an initial-value problem for the NLS equation with NZBCs (1.1.2). Unlike the ZBCs case, the inverse scattering transform in the defocusing case ($\kappa = 1$) is different from the one in the focusing case ($\kappa = -1$). For this thesis, we will only describe the focusing case. We will follow closely the presentation in Biondini & Kovačič (2014). Interested readers can see Faddeev & Takhtajan (2007) for details about the defocusing case.

Consider the BCs (1.1.2) and take $\kappa = -1$. Recall that it is better to introduce the following change $u(x,t) \mapsto u(x,t)e^{2iq_0^2 t}$. We have seen that this changes the shape of the NLS equation (1.1.1) to (1.1.3). As we mentioned in Chapter 1, this change allows the BCs (1.1.2) to become time-independent (1.1.4). We can also see that the Lax pair (U, V) takes the following form:

$$U(x,t,\lambda) = \begin{pmatrix} -i\lambda & u(x,t) \\ -u^*(x,t) & i\lambda \end{pmatrix} \equiv -i\lambda\sigma_3 + Q(x,t), \tag{2.3.1}$$

$$V(x,t,\lambda) = -2i\lambda^2\sigma_3 + 2\lambda Q(x,t) - iQ_x(x,t)\sigma_3 - i\left(Q^2(x,t) + q_0^2\right)\sigma_3. \tag{2.3.2}$$

In the sequel, we will consider the auxiliary system (2.1.1) and (2.1.2) with this new Lax pair.

2.3.1 Direct problem

Consider the initial data $u(x, t = 0) = u_0(x)$ such that $u_0(x) - u_\pm$ as $x \to \pm\infty$. We will make this precise in the sequel. It is useful to write (2.1.1) in the following way

$$\Psi_x(x,\lambda) = U_\pm(\lambda)\Psi(x,\lambda) + \Delta Q_\pm(x)\Psi(x,\lambda), \tag{2.3.3}$$

where

$$Q_\pm = \begin{pmatrix} 0 & u_\pm \\ -u_\pm^* & 0 \end{pmatrix}, \quad U_\pm(\lambda) = -i\lambda\sigma_3 + Q_\pm, \quad \Delta Q_\pm(x) = Q(x) - Q_\pm.$$

Note that $\Delta Q_\pm(x) \to 0$ as $x \to \pm\infty$. We will follow the same construction as in the case of ZBCs. If $q_0 = 0$, the IST we are about to describe will be the same as in Section 2.2. The eigenfunctions of the scattering problem (2.3.3) are asymptotic to solutions of

$$\Psi_x(x,\lambda) = U_\pm(\lambda)\Psi(x,\lambda). \tag{2.3.4}$$

Unlike in the case of ZBCs, the matrices $U_\pm(\lambda)$ are not diagonal.

Eigenvalues, Riemann surface and uniformization variable The first step towards solving the asymptotic problem (2.3.4) is to diagonalise the matrix function $U_\pm(\lambda)$. A direct calculation shows that the eigenvalues of $U_\pm(\lambda)$ are doubly-branched complex-valued functions of λ and are given by $\pm i\sqrt{\lambda^2+q_0^2}$. The branch points are given by the roots of $\sqrt{\lambda^2+q_0^2}$, that is, at $\lambda = \pm iq_0$. To analyse these two functions, we need to make them single-valued. One way of doing this is to introduce the *two-sheeted Riemann surface*[1] defined by

$$k^2 = \lambda^2 + q_0^2.$$

Explicitly, let $\lambda + iq_0 = r_1 e^{i\theta_1}$ and $\lambda - iq_0 = r_2 e^{i\theta_2}$ with $-\frac{\pi}{2} \leq \theta_{1,2} < \frac{3\pi}{2}$. So, we have $k(\lambda) = (r_1 r_2)^{1/2} e^{i(\theta_1+\theta_2)/2 + in\pi}$, where $n = 0$ corresponds to the first Riemann sheet ($\mathbb{C}_1$) and $n = 1$ to the second sheet ($\mathbb{C}_2$). Then, the discontinuity appears along the imaginary axis between $-iq_0$ and iq_0. That is, the *branch cut* is the segment $i[-q_0, q_0]$. Along the real axis, we set $k(\lambda) = \pm sign(\lambda)\sqrt{\lambda^2+q_0^2}$ where the sign $+$ and $-$ correspond to the first and second sheet, respectively. The $sign(\lambda)$ signals that we have chosen the principal real square root.

To facilitate the implementation of the IST in this case, one can introduce the so-called *uniformization variable*, say z, as follows

$$z \equiv z(\lambda) = \lambda + k(\lambda). \tag{2.3.5}$$

This conformal map can be inverted to give:

$$\lambda(z) = \frac{1}{2}\left(z - \frac{q_0^2}{z}\right), \quad k(z) = \frac{1}{2}\left(z + \frac{q_0^2}{z}\right). \tag{2.3.6}$$

Let C_0 be the circle of radius q_0 in the z-complex plane. Then the transformation (2.3.5) has the following important properties:

[1]A surface made of two copies of the complex plane glued in a specific fashion.

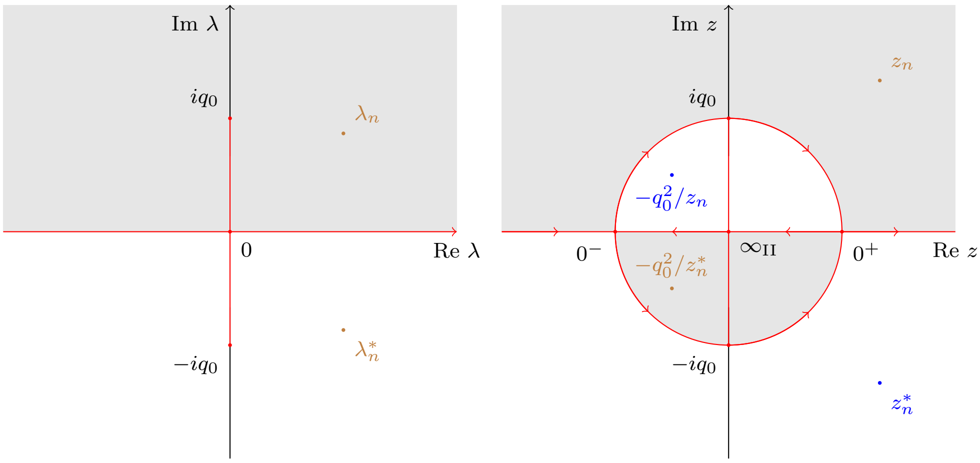

Figure 2.1: Left: The first copy of the Riemann surface, showing the branch cut in red (on the imaginary axis) and the region where Im$k > 0$ is in grey. Right: Showing the regions $D^{\pm}$ and indicating the zeros of $s_{22}(z)$ (brown) and $s_{11}(z)$ (blue). Also showing the orientation of the contour for the Riemann-Hilbert problem in the inverse problem.

Prop 1. On either sheet, the segment $i[-q_0, q_0]$ is mapped to the circle C_0. In particular, $i[0, q_0]$ on $\mathbb{C}_1$ (resp. $\mathbb{C}_2$) is mapped onto the part in the first (resp. second) quadrant of the complex z-plane, $i[-q_0, 0]$ of $\mathbb{C}_1$ (resp. $\mathbb{C}_2$) is mapped onto the part in the third (resp. fourth) quadrant of the z-complex plane;

Prop 2. $\mathbb{C}_1$ is mapped onto the exterior of C_0 and $\mathbb{C}_2$ is mapped onto the interior of C_0;

Prop 3. The limit $\lambda \to \infty$ corresponds to $z \to \infty$ in the first sheet and $z \to 0$ in the second sheet.

In the remaining part of this section, we will express all dependence on λ and k in terms of z using relations (2.3.6). Define

$$D^+ = \left\{z \in \mathbb{C} : \left(|z|^2 - q_0^2\right) \operatorname{Im} z > 0\right\}, \; D^- = \left\{z \in \mathbb{C} : \left(|z|^2 - q_0^2\right) \operatorname{Im} z < 0\right\}. \quad (2.3.7)$$

These domains are shown in Figure 2.1 below. We have the following equivalence: $z \in D^+$ is the same as Im$(k) > 0$, and $z \in D^-$ is equivalent to Im$(k) < 0$.

Eigenvector matrices A direct calculation shows that we can write the eigenvector matrices associated with eigenvalue matrices $-ik(z)\sigma_3$ as

$$E_\pm(z) = \mathbb{I} - \frac{i}{z}\sigma_3 Q_\pm.$$

The determinant of these matrices is

$$\zeta(z) := \det E_\pm(z) = 1 + q_0^2/z^2.$$

We can clearly see that the eigenvector matrices are not invertible at $z = \pm iq_0$, since $\zeta(\pm iq_0) = 0$. In addition to this, there is also another singularity at $z = 0$. The point $z = 0$ will not cause any issue with the construction of the direct problem because as we will see it does not belong to the continuous spectrum. Therefore, we will simply ignore it. The matrices $U_\pm(z)$ can be written in the following form

$$U_\pm(z) = E_\pm(z)\,(-ik(z)\sigma_3)\,E_\pm^{-1}(z), \qquad z \neq \pm iq_0.$$

Due to this diagonalisation of $U_\pm(z)$, we can write down particular solutions of the asymptotic scattering problem (2.3.4): $E_\pm(z)e^{-ik(z)\sigma_3 x}$. Similar to the ZBCs case, we will attempt to construct eigenfunctions for the scattering problem (2.3.3) such that they behave asymptotically as solutions of (2.3.4) when z is an element of the continuous spectrum of the operator $\mathcal{L}$. The values of λ such that $k(\lambda)$ is real-valued constitutes the continuous spectrum. That is $\Sigma_\lambda \equiv \mathbb{R} \cup i[-q_0, q_0]$ is the continuous spectrum in the λ complex plane. Owing to the properties of the transformation $\lambda \mapsto z$, we have that the continuous spectrum in the λ complex plane is mapped to $\Sigma_z \equiv \big(\mathbb{R}\backslash\{0\}\big) \cup C_0$ in the z complex plane. In the sequel, we will drop the index λ or z on Σ. Everything will be clear by the context.

Jost solutions Let $z \in \Sigma$. The *Jost solutions*, denoted $\mu_\pm(x,z)$, are defined as eigenfunctions of the scattering problem (2.3.3) such that

$$\lim_{x\to\pm\infty} \mu_\pm(x,z)e^{ik(z)x\sigma_3} = E_\pm(z). \tag{2.3.8}$$

As in the case of ZBCs, it is convenient to introduce the following functions

$$Y_\pm(x,z) = \mu_\pm(x,z)e^{ik(z)x\sigma_3}.$$

The new eigenfunctions are solutions of the following Volterra integral equations

$$Y_+(x,z) = E_+(z) - \int_x^\infty E_+(z)e^{ik(y-x)\sigma_3}E_+^{-1}(z)\Delta Q_+(y)Y_+(y,z)e^{-ik(y-x)\sigma_3}, \quad (2.3.9)$$

$$Y_-(x,z) = E_-(z) + \int_{-\infty}^x E_-(z)e^{ik(y-x)\sigma_3}E_-^{-1}(z)\Delta Q_-(y)Y_-(y,z)e^{-ik(y-x)\sigma_3}. \quad (2.3.10)$$

Following the same convention as in the case of ZBCs, we denote by $\mu_\pm^{(1)}(x,z)$ and $\mu_\pm^{(2)}(x,z)$ the first and second column of $\mu_\pm(x,z)$, respectively. The same notations are used for $Y_\pm$. Set

$$\Sigma_0 = \Sigma \backslash \{\pm iq_0\}.$$

Let $A = (a_{ij})_{1\leq i,j\leq 2}$ be a 2×2 matrix. Define

$$\|A\|_\star := \max_j \sum_{i=1}^2 |a_{ij}|. \quad (2.3.11)$$

A direct calculation shows that (2.3.11) defines a *matrix norm*[1] in a space of 2×2 matrices. Consider a 2×1 vector f with complex entries. The l^1-norm of f is defined by

$$\|f\|_1 := |f_1| + |f_2|.$$

It can be shown (Horn & Johnson, 2012, pages 344-345) that

$$\|Af\|_1 \leq \|A\|_\star \|f\|_1, \quad (2.3.12)$$

holds for any 2×2 matrix A and 2×1 column vector f.

For some $a, b \in \mathbb{R}$ such that $a < b$, we define

$$\mathcal{P}_+ = [a, +\infty), \quad \mathcal{P}_- = (-\infty, b].$$

Proposition 2.17. *Let $u_0(x) - u_\pm \in L^1(\mathcal{P}_\pm)$. Then the integral equations (2.3.9)-(2.3.10) have unique solutions $Y_\pm(x,z)$, and they are uniformly bounded on $\mathcal{P}_\pm$ for each $z \in \Sigma_0$. In particular, $\mu_\pm(x,z)$ are unique and uniformly bounded solutions of the scattering problem in (2.3.3) such that (2.3.8) holds for each $z \in \Sigma_0$. In addition, if $(1+|x|)^2(u_0(x) - u_\pm) \in L^1(\mathcal{P}_\pm)$, then we reach the same conclusions at $z = \pm iq_0$.*

[1] In addition to the usual properties of a norm, we have $\|AB\|_\star \leq \|A\|_\star \|B\|_\star$ for any 2×2 matrices A and B.

Proof: The following proof uses ideas from Biondini & Kovačič (2014). We will prove this in detail for the first column of $Y_{-}^{(1)}(x,z)$. The analysis for the other columns is similar. Set

$$w(x,z) = E_{-}^{-1}(z)Y_{-}^{(1)}(x,z).$$

It follows that $w(x,z)$ satisfies the integral equation

$$w(x,z) = e_1 + \int_{-\infty}^{x} G(y-x,z)\Delta Q_{-}(y)E_{-}(z)w(y,z)\mathrm{d}y, \tag{2.3.13}$$

where

$$G(s,z) = \operatorname{diag}\left(1, e^{-2iks}\right) E_{-}^{-1}(z).$$

Now, we introduce a Neumann series representation for $w(x,z)$:

$$w(x,z) = \sum_{n=0}^{\infty} w_n(x,z), \tag{2.3.14}$$

with

$$w_0(x,z) = e_1, \quad w_{n+1}(x,z) = \int_{-\infty}^{x} C(x,y,z)w_n(y,z)\mathrm{d}y,$$

where $C(x,y,z) = G(y-x,z)\Delta Q_{-}(y)E_{-}(z)$ and integer $n \geq 0$. From the property (2.3.12), we have

$$\|w_{n+1}(x,z)\|_1 \leq \int_{-\infty}^{x} \|C(x,y,z)\|_{\star}\|w_n(x,z)\|_1\mathrm{d}y.$$

We can use the matrix norm defined above to compute

$$\|E_{\pm}(z)\|_{\star} = 1 + \frac{q_0}{|z|}, \quad \|E_{\pm}^{-1}(z)\|_{\star} = \left(1 + \frac{q_0}{|z|}\right)/|\zeta(z)|.$$

Assume that $z \in \Sigma_0$. From the properties of the map defined in (2.3.6), we must have $k \in \mathbb{R}\backslash\{0\}$. Thus, we have

$$\begin{aligned} \|C(x,y,z)\|_{\star} &\leq \|\operatorname{diag}\left(1, e^{-2ik(y-x)}\right)\|_{\star}\|E_{-}^{-1}(z)\|_{\star}\|\Delta Q_{-}(y)\|_{\star}\|E_{-}(z)\|_{\star} \\ &\leq 2|u_0(y) - u_{-}|c(z), \end{aligned} \tag{2.3.15}$$

where

$$c(z) = \left(1 + \frac{q_0}{|z|}\right)^2/|\zeta(z)| = \frac{(|z| + q_0)^2}{|z^2 + q_0^2|}. \tag{2.3.16}$$

It is clear that $c(z)$ is bounded on $\mathbb{R}\backslash\{0\}$. When z is an element of C_0, the situation is slightly more complicated because the function $c(z) \to \infty$ as $z \to \pm iq_0$. We can first restrict the domain of definition of $c(z)$. Let $\varepsilon > 0$ be given. Define $B_\varepsilon(z_0) = \{z \in \mathbb{C} : |z - z_0| < \varepsilon q_0\}$ and $C_{0,\varepsilon} = C_0 \cap \big[B_\varepsilon(iq_0) \cup B_\varepsilon(-iq_0)\big]$. Let us consider $z \in C_0\backslash C_{0,\varepsilon}$, we have the following straightforward manipulations

$$\begin{aligned} c(z) &= \frac{(|z| + q_0)^2}{|z^2 + q_0^2|} && (2.3.17)\\ &= \frac{4q_0^2}{|z^2 + q_0^2|} && (2.3.18)\\ &\leq 4q_0^2 \left| \frac{1}{2iq_0} \left(\frac{1}{z - iq_0} - \frac{1}{z + iq_0} \right) \right| && (2.3.19)\\ &\leq 2q_0 \left(\frac{1}{|z - iq_0|} + \frac{1}{|z + iq_0|} \right) \leq 2q_0 \left(\frac{1}{q_0} + \frac{1}{\varepsilon q_0} \right) = 2 + 2/\varepsilon. && (2.3.20) \end{aligned}$$

We can see that for $z \in C_0\backslash C_{0,\varepsilon}$, the function $c(z)$ is bounded above by

$$c_\varepsilon = 2 + 2/\varepsilon. \tag{2.3.21}$$

Under these conditions, the Neumann series defined above is absolutely convergent. Indeed, for all positive integers j, we claim

$$\|w_j(x, z)\|_1 \leq \frac{M^j(x)}{j!}, \tag{2.3.22}$$

where

$$M(x) = 2c_\varepsilon \int_{-\infty}^{x} |u_0(y) - u_-|\mathrm{d}y.$$

The proof of this claim is similar to the one in the case of ZBCs (Proposition 2.2).

If $u_0(x) - u_- \in L^1(\mathcal{P}_-)$ then $M(x)$ is finite. Thus by comparison with the exponential series $\exp\big(M(x)\big)$, the Neumann series (2.3.14) converges absolutely for each $z \in \big(\mathbb{R}\backslash\{0\}\big) \cup \big(C_0\backslash C_{0,\varepsilon}\big)$. One can replace x with b in the expression of $M(x)$ to obtain a uniform convergence with respect to $x \in \mathcal{P}_-$ and $z \in \big(\mathbb{R}\backslash\{0\}\big) \cup \big(C_0\backslash C_{0,\varepsilon}\big)$.

We can apply similar arguments as in the proof Proposition 2.2 to obtain the uniqueness of $w(x, z)$. Therefore, for each $z \in \big(\mathbb{R}\backslash\{0\}\big) \cup \big(C_0\backslash C_{0,\varepsilon}\big)$, $\mu_-^{(1)}(x, z)$ is the unique solution of the differential equation (2.3.3) such that

$$e^{ik(z)x}\mu_-^{(1)}(x, z) \to \left(1, \frac{iu_-^*}{z}\right)^T, \quad x \to -\infty.$$

Now, we will discuss what happens in the neighbourhood of the branch points $z = \pm iq_0$. Despite the fact that the eigenvector matrices $E_\pm(z)$ are not invertible at $z = \pm iq_0$, a direct calculation shows

$$\lim_{z\to\pm iq_0} E_\pm(z)e^{ik(y-x)\sigma_3}E_\pm^{-1}(z) = \mathbb{I} - (y-x)(Q_\pm \pm q_0\sigma_3). \tag{2.3.23}$$

Thus, the first column of (2.3.9) evaluated at $z = \pm iq_0$ reads

$$Y_-^{(1)}(x,\pm iq_0) = \begin{pmatrix} 1 \\ \pm e^{-i\theta_-} \end{pmatrix} + \int_{-\infty}^x J_-(x,y)Y_-^{(1)}(y,\pm iq_0)\mathrm{d}y,$$

where $J_\pm(x,y) = \left[\mathbb{I} - (y-x)(Q_- \pm q_0\sigma_3)\right]\Delta Q_-(y)$ and $\theta_- = \arg u_-$. The last step is to find an estimate for $J_\pm(x,y)$:

$$\begin{aligned}\|J_\pm(x,y)\|_\star &\le (\|\mathbb{I}\|_\star + |y-x|\|Q_- \pm q_0\sigma_3\|_\star)\,\|\Delta Q_-(y)\|_\star\\ &\le (1+2q_0|y-x|)\,|u_0(y)-u_-|,\\ &\le C\,(1+|x|+|y|)\,|u_0(y)-u_-|,\\ &\le C\,(1+|x|)\,(1+|y|)|u_0(y)-u_-|,\end{aligned}$$

where $C = 2$ if $q_0 \le 1$ and $C = 2q_0$ if $q_0 > 1$. Again, for convenience, set

$$w(x,\pm iq_0) = Y_-^{(1)}(x,\pm iq_0).$$

We can again consider the Neumann series representation for $w(x,\pm iq_0)$ as in (2.3.14) with

$$w_0(x,\pm iq_0) = \begin{pmatrix} 1 \\ \pm e^{-i\theta_-} \end{pmatrix}, \quad w_{n+1}(x,\pm iq_0) = \int_{-\infty}^x J_-(x,y)w_n(y,\pm iq_0)\mathrm{d}y.$$

The equivalent of (2.3.22) is given by

$$\begin{aligned}\|w_j(x,\pm iq_0)\|_1 &\le 2(1+|x|)\tfrac{M^j(x)}{j!},\\ M(x) &= C\textstyle\int_{-\infty}^x(1+|y|)^2|u_0(y)-u_-|\mathrm{d}y.\end{aligned}$$

Thus we can replicate the same argument as above to obtain that $Y_-(x,\pm iq_0)$ is well defined and it is given by an absolute and uniformly convergent whenever $(1+|x|)^2(u_0(x)-u_-)$ belongs to $L^1(\mathcal{P}_-)$. Finally, $\mu_-^{(1)}(x,\pm iq_0)$ is well-defined solution of (2.3.3) such that

$$\mu_-^{(1)}(x,\pm iq_0) \to \left(1,\pm e^{-i\theta_-}\right)^T, \quad x\to-\infty.$$

This completes the proof. ■

Remark 2.18.

1. *In general, one cannot replace b in the above proof with ∞. Because if $u_0(x) - u_- \in L^1(\mathbb{R})$, thus $u_0(x) - u_-$ will decay at both infinity, therefore one must have that $u_- = u_+$.*

2. *The additional condition that we impose on the potential $u_0(x)$ to obtain regularity at the branch points $z = \pm iq_0$ is by no means optimal. However, for the purpose of this thesis, this condition is enough. We refer interested readers to recent work done on this topic* Demontis *et al.* (2013) *and* Demontis *et al.* (2014).

Lemma 2.19 (Analytic continuation). *Assume that $u_0(x) - u_\pm \in L^1(\mathcal{P}_\pm)$. Let $Y_-(x,z)$ and $Y_+(x,z)$ be the solutions for the integral equations* (2.3.9)-(2.3.10), *respectively. Fix $x \in \mathcal{P}_\pm$, then $Y_\mp^{(1)}(x,\cdot)$ and $Y_\pm^{(2)}(x,\cdot)$ are continuous on $D^\pm \cup \Sigma_0$ and have an analytical continuation on $D^\pm$. In addition, we have*

1. *$\mu_-^{(1)}(x,\cdot)$ and $\mu_+^{(2)}(x,\cdot)$ are continuous on $D^+ \cup \Sigma_0$ and analytic on D^+ with the property*

$$e^{-ikx}\mu_-^{(1)}(x,z) = \begin{cases} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \mathcal{O}(z^{-1}), & \textit{as } z \to \infty, \\ \begin{pmatrix} 0 \\ \frac{-iu_-^*}{z} \end{pmatrix} + \mathcal{O}(1), & \textit{as } z \to 0, \end{cases} \qquad e^{ikx}\mu_+^{(2)}(x,z) = \begin{cases} \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \mathcal{O}(z^{-1}), & \textit{as } z \to \infty, \\ \begin{pmatrix} \frac{-iu_+}{z} \\ 0 \end{pmatrix} + \mathcal{O}(1), & \textit{as } z \to 0. \end{cases}$$

2. *$\mu_+^{(1)}(x,\cdot)$ and $\mu_-^{(2)}(x,\cdot)$ are continuous on $D^- \cup \Sigma_0$ and analytic on D^- with the property*

$$e^{-ikx}\mu_+^{(1)}(x,z) = \begin{cases} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \mathcal{O}(z^{-1}), & \textit{as } z \to \infty, \\ \begin{pmatrix} 0 \\ \frac{-iu_+^*}{z} \end{pmatrix} + \mathcal{O}(1), & \textit{as } z \to 0, \end{cases} \qquad e^{ikx}\mu_-^{(2)}(x,z) = \begin{cases} \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \mathcal{O}(z^{-1}), & \textit{as } z \to \infty, \\ \begin{pmatrix} \frac{-iu_-}{z} \\ 0 \end{pmatrix} + \mathcal{O}(1), & \textit{as } z \to 0. \end{cases}$$

Note that the continuity and analyticity of the Jost solutions at the branch require the extra condition on $u_0(x) - u_\pm$; see Proposition 2.17.

<u>Proof:</u> We will prove these results in detail for $Y_-^{(1)}(x,z)$. The analysis for the other columns will be similar. The idea is to reproduce similar arguments as in

the proof of Proposition 2.17. We set $w(x,z) = E_-^{-1}(z)Y_-(x,z)e_1$. Similarly, it follows that $w(x,z)$, will be the solution of the integral equation (2.3.13). Thus, we can introduce its Neumann series as in (2.3.14). To prove that this series is actually convergent, one can attempt to reproduce (2.3.15). Unlike in the case above, $k(z)$ does not assume real values when z is off the continuous spectrum. Therefore extra conditions need to be imposed to obtain a useful estimate for $C(x,y,z)$. It turns out that if we consider $z \in D^-$, we can recover (2.3.15). Recall that D^- is given by the grey regions in Fig. 2.1. Let $z \in D^-$, we have that

$$\begin{aligned}\|C(x,y,z)\|_\star &\leq \|\operatorname{diag}\left(1, e^{-2ik(y-x)}\right)\|_\star \|E_-^{-1}(z)\|_\star \|\Delta Q_-(y)\|_\star \|E_-(z)\|_\star \\ &\leq 2|u_0(y) - u_-|c(z),\end{aligned}$$

where $c(z)$ is given as in (2.3.16). As in the above proof, we would like to bound the function $c(z)$ above by a quantity that does not depend on z. Given $\varepsilon > 0$, we introduce the following domain

$$D_\varepsilon^- = D^- \backslash \left[B_\varepsilon(iq_0) \cup B_\varepsilon(-iq_0)\right].$$

Let $z \in D_\varepsilon^-$. If $|z| < q_0$, one can repeat the calculations (2.3.17)-(2.3.20) to obtain that the function $c(z)$ is bounded by c_ε given in (2.3.21). Note that one needs to replace the equality in (2.3.18) by strict inequality. If $|z| > q_0$, we first observe the following symmetry

$$c(q_0^2/z) = c(z).$$

Set $\tau = q_0^2/z$. We see that in this case $|\tau| < q_0$. Therefore, one can again repeat the arguments in (2.3.17)-(2.3.20) to get that $c(\tau)$ is bounded above by c_ε. In turn, this proves that $c(z)$ is also bounded above by c_ε when $|z| > q_0$. Thus in both cases, we managed to bound $c(z)$ by the same constant value independent from z. From this, we use similar arguments as in the proof of Proposition 2.17 to prove that the Neumann series defining $w(x,z)$ will be absolutely and uniformly convergent on $\mathcal{P}_-$ for all $z \in D_\varepsilon^-$. The continuity and analyticity properties follow from the fact that the Neumann series converges uniformly and that each term of the series is continuous and analytic. The conclusions on $\mu_-^{(1)}(x,z)$ follow immediately.

To deduce the asymptotic behaviour, as $z \to \infty$ and $z \to 0$, we use the well-known *Wentzel-Kramers-Brillouin* (WKB) asymptotic expansion of $e^{-ikx}\mu_-^{(1)}(x,z)$.

Let us start with the behaviour as $z \to \infty$. We seek the following expansion

$$e^{-ikx}\mu_-^{(1)}(x,z) = \sum_{j=0}^{n} \frac{f_j(x)}{z^j} + \mathcal{O}\left(z^{-(n+1)}\right), \quad z \to \infty.$$

where $f_j(x)$ are to be determined. We only need the leading term of this series. Substituting this series into the equation (2.1.1), we obtain after some manipulations

$$\begin{aligned}\sum_{j=0}^{n} \frac{(f_j(x))_x}{z^j} = &- \frac{i}{2}\big(\sigma_3 f_0(x) - f_0(x)\big)z - \frac{i}{2}\big(\sigma_3 f_1(x) - f_1(x)\big) \\ &+ Q(x)f_0(x) + \sum_{j=1}^{n}\left[\frac{iq_0^2}{2}\big(\sigma_3 f_{j-1}(x) + f_{j-1}(x)\big)\right. \\ &\left. - \frac{i}{2}\big(\sigma_3 f_{j+1}(x) - f_{j+1}(x)\big) + Q(x)f_j(x)\right]\frac{1}{z^j}. \end{aligned} \tag{2.3.24}$$

Note that we expressed the equation (2.1.1) in terms of z to get the above expression. By matching the powers of z, we get from the coefficient of z and the independent term that

$$f_0(x) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad f_1(x) = \begin{pmatrix} i\int_{-\infty}^{x}(q_0^2 - |u(y)|^2)\mathrm{d}y \\ -iu^*(x) \end{pmatrix}.$$

This gives the asymptotic behaviour as $z \to \infty$. Note that we used the value of $\mu_-^{(1)}(x,z)$ at $x = -\infty$ to evaluate the constant and to integrate in the process of computing f_0 and f_1. For the case $z \to 0$, one needs to consider the following series expansion

$$e^{-ikx}\mu_-^{(1)}(x,z) = \sum_{j=-1}^{n} z^j g_j(x) + \mathcal{O}(z^{n+1}), \quad z \to 0,$$

where $g_j(x)$ are to be determined. Using similar arguments as in the case of $z \to \infty$, one gets

$$g_{-1}(x) = \begin{pmatrix} 0 \\ -iu_-^* \end{pmatrix}, \quad g_0(x) = \begin{pmatrix} \frac{i|u(x)|^2}{2} \\ u_-^* - \frac{iu^*(x)|u(x)|^2}{2} \end{pmatrix}.$$

This completes the proof. ■

Scattering coefficients Since $U_\pm$ is traceless, using Abel's theorem, one deduces

$$\det \mu_\pm(x,z) = \zeta(z), \quad z \in \Sigma.$$

This means that $\mu_-(x,z)$ and $\mu_+(x,z)$ are two *fundamental matrices* of the scattering problem (2.3.3) for $z \in \Sigma_0$. Therefore, they must be related as

$$\mu_+(x,z) = \mu_-(x,z)S(z), \quad z \in \Sigma_0, \tag{2.3.25}$$

where the matrix $S(z) = (s_{ij}(z))_{1\le i,j\le 2}$ is called the *scattering matrix* associated to $u_0(x)$. The entries of the scattering matrix $S(z)$ are complex-valued functions defined on Σ and they are referred to as *scattering coefficients* associated with $u_0(x)$. Notice that we still obtain that the scattering matrix associated with $u_0(x)$ is unimodular as in the ZBCs case

$$\det S(z) = 1, \quad z \in \Sigma_0.$$

Proposition 2.20. *Let $u_0(x) - u_\pm$ be an element of $L^1(\mathcal{P}_\pm)$. Then the scattering coefficient $s_{22}(z)$ (respectively, $s_{11}(z)$) is continuous on $D^+ \cup \Sigma_0$ and has an analytic continuation on D^+ (respectively, is continuous on $D^- \cup \Sigma_0$ and has an analytic continuation on D^-). Moreover, the scattering coefficients $s_{12}(z)$ and $s_{21}(z)$ are continuous functions on Σ_0 but do not have, in general, analytic continuations in any region of the z complex plane.*

Proof: The columns of $\mu_-(x,z) = \mu_+(x,z)S(z)^{-1}$ are given by

$$\begin{cases} \mu_-^{(1)}(x,z) = s_{22}(z)\mu_+^{(1)}(x,z) - s_{21}(z)\mu_+^{(2)}(x,z), \\ \mu_-^{(2)}(x,z) = s_{11}(z)\mu_+^{(2)}(x,z) - s_{12}(z)\mu_+^{(1)}(x,z). \end{cases} \tag{2.3.26}$$

By calculating the determinant of $\left(\mu_-^{(1)}(x,z), \mu_+^{(2)}(x,z)\right)$, one obtains

$$s_{22}(z) = \frac{1}{\zeta(z)} \det\left(\mu_-^{(1)}(x,z), \mu_+^{(2)}(x,z)\right). \tag{2.3.27}$$

Similarly, one gets

$$s_{11}(z) = \frac{1}{\zeta(z)} \det\left(\mu_+^{(1)}(x,z), \mu_-^{(2)}(x,z)\right). \tag{2.3.28}$$

The results for $s_{11}(z)$ and $s_{22}(z)$ follow from Lemma 2.19. Similar arguments are used to obtain the results for $s_{12}(z)$ and $s_{21}(z)$. ∎

We can see from the above result that the scattering coefficients have poles at the branch points, and this can be calculated explicitly; see Biondini & Kovačič (2014).

Combining the expressions of $s_{12}(z)$ and $s_{21}(z)$ in terms of the Jost solutions, (2.3.27)-(2.3.28), and the asymptotic behaviours given in Lemma 2.19, we obtain

$$S(z) = \mathbb{I} + \mathcal{O}\left(z^{-1}\right) \quad z \to \infty,$$

and

$$S(z) = \operatorname{diag}(u_-/u_+, u_+/u_-) + \mathcal{O}\left(1\right) \quad z \to 0. \tag{2.3.29}$$

Lemma 2.21 (Riemann surface symmetry)**.** *Let $\mu(x,t,z)$ be a solution of the time-dependent scattering problem (2.3.3). Then $\mu(x,t,-q_0^2/z)$ solves the same equation. In other words,*

$$\mu(x,t,-q_0^2/z) = \mu(x,t,z)M(t,z), \tag{2.3.30}$$

where $M(t,z)$ is a constant matrix with respect to x.

Proof: The proof follows obviously from the fact that $\lambda(-q_0^2/z) = \lambda(z)$. ∎

Let z be a complex number such that $|z| > q_0$, i.e. z, is located outside the circle C_0. Thus, we clearly see that $\left|\frac{-q_0^2}{z}\right| < q_0$, that is $-q_0^2/z$ is located within the circle C_0. Recall that the first copy $(\mathbb{C}_\mathrm{I})$ of the Riemann surface is mapped into the inside of C_0 and the second copy $(\mathbb{C}_\mathrm{II})$ onto the outside of C_0. This means that

$$z \mapsto -q_0^2/z \Longleftrightarrow \lambda \mapsto \lambda, \quad k \mapsto -k,$$

connects eigenfunctions associated with the spectral parameter on both copies. Combining this with the NLS symmetry in Lemma 2.1, we have the following result.

Corollary 2.22. *Consider $u(x) - u_\pm \in L^1(\mathcal{P}_\pm)$. Then the scattering coefficients satisfy*

$$s_{11}(z) = s_{22}^*(z^*), \quad z \in \Sigma, \quad s_{12}(z) = -s_{21}^*(z^*), \quad z \in \Sigma, \tag{2.3.31}$$

$$s_{11}(z) = \frac{u_-}{u_+} s_{22}(-q_0^2/z), \quad z \in \Sigma, \quad s_{12}(z) = \frac{u_+}{u_-^*} s_{21}(-q_0^2/z), \quad z \in \Sigma. \tag{2.3.32}$$

Proof: Suppose that $z \in \Sigma$. A straightforward calculation will show that the Jost solutions $\mu_\pm(x,z)$ satisfy the following two symmetries

$$\mu_\pm(x,z) = (i\sigma_2)\mu_\pm^*(x,z^*)(i\sigma_2)^{-1}, \quad \mu_\pm(x,z) = \frac{i}{z}\mu_\pm(x,-q_0^2/z)\sigma_3 Q_\pm, \quad z \in \Sigma. \tag{2.3.33}$$

These symmetries can be written in terms of the scattering matrix as

$$S(z) = (i\sigma_2)S^*(z^*)(i\sigma_2)^{-1}, \quad S(-q_0^2/z) = \sigma_3 Q_- S(z)(\sigma_3 Q_+)^{-1}, \quad z \in \Sigma. \tag{2.3.34}$$

Elementwise, the first equation in (2.3.34) gives

$$s_{11}(z) = s_{22}^*(z^*), \quad s_{12}(z) = -s_{21}^*(z^*), \quad z \in \Sigma. \tag{2.3.35}$$

Similarly, the second symmetry in (2.3.34) gives

$$s_{11}(z) = \frac{u_-}{u_+}s_{22}(-q_0^2/z), \quad s_{12}(z) = \frac{u_+}{u_-^*}s_{21}(-q_0^2/z), \quad z \in \Sigma. \tag{2.3.36}$$

This completes the proof. ∎

Unlike in the case of ZBCs, the continuous spectrum, in this case, is not just $\mathbb{R}$, it contains a region in the complex plane. This explains why we need to keep track of the complex conjugate in the symmetry between $s_{12}(z)$ and $s_{21}(z)$.

As in the ZBCs, we say that $z \in \mathbb{C}\backslash\Sigma$ is an *eigenvalue* for the operator $\mathcal{L}$ if its eigenfunction is an element of $L^2(\mathbb{R},\mathbb{C}^2)$. We denote by $\mathcal{K}^\pm$ the set of all $z \in \mathbb{C}\backslash\Sigma$ with such a property. The *discrete spectrum* for the operator $\mathcal{L}$, in this case, is

$$\mathcal{K} = \mathcal{K}^+ \cup \mathcal{K}^-.$$

Owing to the relation (2.3.27), one can easily deduce that the set of zeros for the scattering coefficient $s_{22}(z)$ coincides with $\mathcal{K}^+$. The same correspondence can be drawn between $\mathcal{K}^-$ and the set of zeros for $s_{11}(z)$ using (2.3.28).

Let $u_0(x)$ be such that $u_0 - u_\pm$ be an element of $L^1(\mathcal{P}_\pm)$. Then, in the remaining part of this section, we assume the following:

- The scattering coefficients $s_{11}(z)$ and $s_{22}(z)$ do not vanish on Σ;
- $\mathcal{K}^-$ and $\mathcal{K}^+$ have finite number of elements.

We will follow the terminology used in the ZBCs case: potentials that admit the above properties of the scattering coefficients $s_{11}(z)$ and $s_{22}(z)$ will be called *generic*.

Definition 2.23. *The* reflection coefficients, *denoted* $r(z)$ *and* $\tilde{r}(z)$, *are complex-valued functions defined as*

$$r: \begin{array}{ccc} \Sigma & \longrightarrow & \mathbb{C} \\ z & \longmapsto & \frac{s_{21}(z)}{s_{22}(z)} \end{array}, \qquad \bar{r}: \begin{array}{ccc} \Sigma & \longrightarrow & \mathbb{C} \\ z & \longmapsto & \frac{s_{12}(z)}{s_{11}(z)} \end{array}$$

Norming constants Let N be a positive integer. Assume that $u_0(x)$ is a generic potential. For each $1 \leq n \leq N$, let $z_n \in D^+ \cup \mathbb{C}^+$ be a zero of $s_{22}(z)$. That is $s_{22}(z_n) = 0$. Then by taking into consideration the first symmetry in (2.3.32), we see that $-u_0^2/z_n^*$ is also a zero of $s_{22}(\lambda)$. This means that

$$\mathcal{K}^+ = \{z_1, z_2, \ldots, z_N, -u_0^2/z_1^*, -u_0^2/z_2^*, \ldots, -u_0^2/z_N^*\}.$$

From the first relation in (2.3.31), we have

$$\mathcal{K}^- = \{z_1^*, z_2^*, \ldots, z_N^*, -u_0^2/z_1, -u_0^2/z_2, \ldots, -u_0^2/z_N\}.$$

Set

$$\xi_n = z_n, \quad \xi_{n+N} = -u_0^2/z_n^*, \quad n = 1, \ldots, N.$$

Thus, Eq. (2.3.27) implies

$$\mu_-^{(1)}(x, \xi_n) = \gamma(\xi_n)\mu_+^{(2)}(x, \xi_n), \tag{2.3.37}$$

where $\gamma(\xi_n)$ is the proportionality constant. The norming constant associated to $\gamma(\xi_n)$ is given by

$$c(\xi_n) = \frac{\gamma(\xi_n)}{s_{22}'(\xi_n)}.$$

We will use both depending on our needs. Similarly, the first equation in Eq. (2.3.28) implies

$$\mu_-^{(2)}(x, \xi_n^*) = \gamma(\xi_n^*)\mu_+^{(1)}(x, \xi_n^*). \tag{2.3.38}$$

In turn, we have

$$c(\xi_n^*) = \frac{\gamma(\xi_n^*)}{s_{11}'(\xi_n^*)}.$$

The zeroes and norming constants together form the so-called *discrete scattering data.*

Explicitly, the second relation in (2.3.33) gives

$$\mu_\pm^{(1)}(x, z) = \frac{iu_\pm^*}{z}\mu_\pm^{(2)}(x, -u_0^2/z), \quad \mu_\pm^{(2)}(x, z) = \frac{iu_\pm}{z}\mu_\pm^{(1)}(x, -u_0^2/z), \quad z \in \Sigma. \tag{2.3.39}$$

Replacing Eq. (2.3.39) in (2.3.37) and (2.3.38), we get

$$\mu_-^{(1)}(x, -u_0^2/z_n^*) = (u_+^*/u_-)\gamma(z_n^*)\mu_+^{(2)}(x, -u_0^2/z_n^*),$$
$$\mu_-^{(2)}(x, -u_0^2/z_n) = (u_+/u_-^*)\gamma(z_n)\mu_+^{(1)}(x, -u_0^2/z_n),$$

that is

$$\gamma(-u_0^2/z_n^*) = (u_+^*/u_-)\gamma(z_n^*) \text{ and } \gamma(-u_0^2/z_n) = (u_+/u_-^*)\gamma(z_n). \tag{2.3.40}$$

Using the first relation in (2.3.33) and (2.3.40), one can easily prove that

$$\gamma(\xi_n^*) = -\gamma(\xi_n)^* \text{ and } c(\xi_n^*) = -c(\xi_n)^*.$$

Note that, differentiating the first relation in (2.3.36) and using (2.3.35), one obtains

$$s_{22}'(-u_0^2/z_n^*) = (z_n^*/u_0)^2(u_-/u_+)(s_{22}'(z_n))^*.$$

Hence, we get

$$c(-u_0^2/z_n^*) = -(u_0/z_n^*)^2(u_+^*/u_+)(c(z_n))^*.$$

Using the same argument as in the ZBCs case, the scattering coefficient $s_{22}(z)$ has the following explicit form

$$s_{22}(z) = \prod_{n=1}^{2N} \frac{(z-\xi_n)}{(z-\xi_n^*)} \exp\left(-\frac{1}{2\pi i}\int_\Sigma \frac{\log\left(1 + r(\zeta)r^*(\zeta^*)\right)}{\zeta - z}\mathrm{d}\zeta\right), \quad z \in D^+. \tag{2.3.41}$$

Note that the scattering coefficient $s_{11}(z)$ admits a similar expression. Taking the limit of this equation as $z \to 0$, and combine the result with the limits in (2.3.29) one obtains the so-called *theta condition*

$$\arg\left(\frac{u_-}{u_+}\right) = 4\sum_{n=1}^{N} \arg(z_n) + \frac{1}{2\pi}\int_\Sigma \frac{\log\left(1 + r(\zeta)r^*(\zeta^*)\right)}{\zeta}\mathrm{d}\zeta. \tag{2.3.42}$$

This gives the phase difference at both infinities.

Now, we can define the direct map. We will use the same notation as the one we defined in the case of ZBCs.

Definition 2.24. *Consider a generic potential $u_0(x)$ such that $u_0 - u_\pm \in L^1(\mathcal{P}_\pm)$. Let $\mathbb{S}$ be the map that associates to $u_0(x)$ its scattering data, namely*

$$\mathbb{S} : \{u_0(x)\} \mapsto \left\{r(z), (z_n, c(z_n))_{1\le j\le N}\right\}$$

where $r(z), z_n$ and $c(z_n)$ are defined as above.

2.3.2 Inverse problem

Now, we will discuss the inverse problem. As in the case of ZBCs, we state the following general results, see Biondini & Kovačič (2014).

Theorem 2.25 (RHP with NZBCs)**.** *Let r be a function defined on Σ and $(z_n, c(z_n)) \in (D^+ \cap \mathbb{C}^+) \times (\mathbb{C}\backslash\{0\})$, for each $n = 1, \ldots, N$, be given such that the following normalised Riemann-Hilbert problem*

- **Analyticity.** *$m(x,t,z)$ is analytic in $\mathbb{C}\backslash(\Sigma \cup \mathcal{K})$;*
- **Jump condition.** *It has continuous boundary values $m^{\pm}(x,t,s) = \lim_{z\to s} m(x,t,z)$, $s \in \Sigma$, satisfying the* jump condition

$$m^+(x,t,s) = m^-(x,t,s)v(x,t,s), \qquad s \in \Sigma, \tag{2.3.43}$$

where the jump matrix *is given by*

$$v(x,t,s) = \begin{pmatrix} 1+|r(s)|^2 & -r(s)^* e^{-2i\theta(s^*)} \\ -r(s)e^{2i\theta(s)} & 1 \end{pmatrix}, \quad \theta(s) = k(s)(x + 2\lambda(s)t);$$

- **Residues.** *$m(x,t,z)$ has simple poles at ξ_n, ξ_n^* for $n = 1, \ldots, N$, and the residues are given by*

$$\operatorname*{Res}_{z=\xi_n} m(x,t,z) = \lim_{z\to\xi_n} m(x,t,z) \begin{pmatrix} 0 & 0 \\ c(\xi_n)e^{2i\theta(x,t,\xi_n)} & 0 \end{pmatrix}, \tag{2.3.44}$$

$$\operatorname*{Res}_{z=\xi_n^*} m(x,t,z) = \lim_{z\to\xi_n^*} m(x,t,z) \begin{pmatrix} 0 & -c(\xi_n)^* e^{-2i\theta(\xi_n^*)} \\ 0 & 0 \end{pmatrix}; \tag{2.3.45}$$

- **Normalisation.**

$$m(x,t,z) = \mathbb{I} + \mathcal{O}(z^{-1}), \quad z \to \infty$$
$$m(x,t,z) = (i/z)\sigma_3 Q_+ + \mathcal{O}(1), \quad z \to 0,$$

has a unique solution.

Now, let us consider $u_0(x)$ such that $u_0 - u_\pm \in L^1(\mathcal{P}_\pm)$ with the map $\mathbb{S}$ be given as in Definition 2.24. It follows from (2.3.25) that

$$Y_+(x,z) = Y_-(x,z)e^{-ik\sigma_3 x}S(z)e^{ik\sigma_3 x}, \quad z \in \Sigma. \tag{2.3.46}$$

By regrouping the term with an analytic continuation into the same region of the complex plane, one gets

$$m^+(x,z) = m^-(x,z)v(x,z), \quad z \in \Sigma,$$

where

$$m^+(x,z) = \left(\frac{Y_-^{(1)}(x,z)}{s_{22}(z)}, Y_+^{(2)}(x,z) \right), \quad m^-(x,z) = \left(Y_+^{(1)}(x,z), \frac{Y_-^{(2)}(x,z)}{s_{11}(z)} \right),$$

$$v(x,z) := \begin{pmatrix} 1+|r(z)|^2 & -r(z)^* e^{-2i\theta(z^*)} \\ -r(z)e^{2i\theta(z)} & 1 \end{pmatrix},$$

Define

$$m(x,z) = \begin{cases} m^+(x,z), & z \in D^+, \\ m^-(x,z), & z \in D^-. \end{cases} \tag{2.3.47}$$

Lemma 2.26. *The* 2×2 *matrix functions* $m(x,z)$ *defined in* (2.3.47) *solves the Riemann-Hilbert problem with NZBCs at time* $t = 0$*. Moreover, the map*

$$\mathbb{P} : \{r(z), (z_n, c(z_n)_{1 \le n \le N})\} \mapsto \{u_0(x)\}$$

defined by

$$u_0(x) = -i \lim_{z \to \infty} z \left(m(x,z) \right)_{12}, \tag{2.3.48}$$

is the inverse to $\mathbb{S}$.

2.3.3 Time evolution

Using the same arguments as in the case of ZBCs, one gets the following time evolution of the scattering data by

$$r(t,z) = r(z)e^{4ik\lambda t}, \quad \tilde{r}(t,z) = \tilde{r}(z)e^{-4ik\lambda t},$$

$$c(t,z_n) = c(z_n)e^{4ik(z_n)\lambda(z_n)t}, \quad c(t,z_n^*) = c(z_n^*)e^{-4ik(z_n^*)\lambda(z_n^*)t}.$$

2.3.4 Reflectionless potentials

Assume that the reflection coefficient is identically zero. In this case, the solution of the above RHP can be written as

$$m(x,t,z) = \mathbb{I} + (i/z)\sigma_3 Q_+ + \sum_{n=1}^{2N} \frac{\operatorname*{Res}_{z=\xi_n} m(x,t,z)}{z-\xi_n} + \sum_{n=1}^{2N} \frac{\operatorname*{Res}_{z=\xi_n^*} m(x,t,z)}{z-\xi_n^*}.$$

Recall that, using (2.2.44), one has

$$\operatorname*{Res}_{z=\xi_n} m(x,t,z) = \left[c(\xi_n)e^{2i\theta(\xi_n)}Y_+^{(2)}(x,t,\xi_n), 0\right],$$
$$\operatorname*{Res}_{z=\xi_n^*} m(x,t,z) = \left[0, -c(\xi_n)^* e^{-2i\theta(\xi_n^*)}Y_+^{(1)}(x,t,\xi_n^*)\right].$$

Given $Y_\pm^{(1)}$, we denote by $Y_\pm^{(11)}$ and $Y_\pm^{(21)}$ its first and second entries, respectively. Given $\Phi_\pm^{(2)}$, we denote by $Y_\pm^{(12)}$ and $Y_\pm^{(22)}$ its first and second entries, respectively.

The time-dependent version of the reconstruction formula, we have

$$u(x,t) = u_+ - i\sum_{n=1}^{N} c(\lambda_n)^* e^{-2i\theta(\lambda_n^*)}Y_+^{(11)}(x,t,\lambda_n^*). \tag{2.3.49}$$

One can evaluate the second column of the above at $z=\xi_n$, obtaining

$$Y_+^{(2)}(x,t,\xi_n) = \begin{pmatrix} \frac{i}{\xi_n}u_+ \\ 1 \end{pmatrix} - \sum_{k=1}^{2N} \frac{c(\xi_k)^* e^{-2i\theta(\xi_k^*)}}{\xi_n - \xi_k^*} Y_+^{(1)}(x,t,\xi_k^*),$$

for $n = 1,\dots,2N$. In the same way, one evaluates the first column at ξ_n^* to obtain

$$Y_+^{(1)}(x,t,\xi_n^*) = \begin{pmatrix} 1 \\ \frac{i}{\xi_n^*}u_+^* \end{pmatrix} + \sum_{k=1}^{2N} \frac{c(\xi_k)e^{2i\theta(\xi_k)}}{\xi_n^* - \xi_k} Y_+^{(2)}(x,t,\xi_k),$$

for $n = 1,\dots,2N$. Note that one needs only the first component of the eigenfunction to recover the potential. Therefore, we get

$$Y_+^{(12)}(x,t,\xi_j) = \frac{i}{\xi_j}u_+ - \sum_{k=1}^{2N} \frac{c(\xi_k)^* e^{-2i\theta(\xi_k^*)}}{\xi_j - \xi_k^*} Y_+^{(11)}(x,t,\xi_k^*),\ j = 1,\dots,2N,$$

$$Y_+^{(11)}(x,t,\xi_n^*) = 1 + \sum_{j=1}^{2N} \frac{c(\xi_j)e^{2i\theta(\xi_j)}}{\xi_n^* - \xi_j} Y_+^{(12)}(x,t,\xi_j),\ n = 1,\dots,2N.$$

Hence, for $n = 1, \ldots, 2N$, one obtains

$$\begin{aligned} Y_+^{(11)}(x,t,\xi_n^*) &= 1 + \sum_{j=1}^{2N} \frac{c(\xi_j)e^{2i\theta(\xi_j)}}{\xi_n^* - \xi_j} Y_+^{(12)}(x,t,\xi_j) \\ &= 1 + \sum_{j=1}^{2N} \frac{c(\xi_j)e^{2i\theta(\xi_j)}}{\xi_n^* - \xi_j} \left[\frac{i}{\xi_j} u_+ - \sum_{k=1}^{2N} \frac{c(\xi_k)^* e^{-2i\theta(\xi_k^*)}}{\xi_j - \xi_k^*} Y_+^{(11)}(x,t,\xi_k^*) \right] \\ &= 1 + iu_+ \sum_{j=1}^{2N} \frac{c(\xi_j)e^{2i\theta(\xi_j)}}{\xi_j(\xi_n^* - \xi_j)} - \sum_{k=1}^{2N} \left[c(\xi_k)^* e^{-2i\theta(\xi_k^*)} \sum_{j=1}^{2N} \frac{c(\xi_j)e^{2i\theta(\xi_j)}}{(\xi_n^* - \xi_j)(\xi_j - \xi_k^*)} \right] Y_+^{(11)}(x,t,\xi_k^*). \end{aligned}$$

Introduce,

$$X = (X_1, \ldots, X_{2N})^T, \quad B = (B_1, \ldots, B_{2N})^T, \quad M = I + (A_{n,k})_{1 \le n,k \le 2N},$$

where

$$X_n = Y_+^{(11)}(x,t,\xi_n^*), \quad B_n = 1 + iu_+ \sum_{j=1}^{2N} \frac{c(\xi_j)e^{2i\theta(\xi_j)}}{\xi_j(\xi_n^* - \xi_j)},$$

$$A_{n,k} = c(\xi_k)^* e^{-2i\theta(\xi_k^*)} \sum_{j=1}^{2N} \frac{c(\xi_j)e^{2i\theta(\xi_j)}}{(\xi_n^* - \xi_j)(\xi_j - \xi_k^*)}.$$

The above algebraic system takes the following form

$$MX = B.$$

The solution of the system is given by $X_n = \det M_n^{ext} / \det M$ for $n = 1, \ldots, 2N$, where

$$M_n^{ext} = (M_1, \ldots, \ldots, M_{n-1}, B, M_{n+1}, \ldots, M_{2N}).$$

After substituting X in the reconstruction formula (2.3.49), one obtains

$$u(x,t) = u_+ + i \frac{\det M^{inc}}{\det M}, \tag{2.3.50}$$

where

$$M^{inc} = \begin{pmatrix} 0 & H \\ B & M \end{pmatrix}$$

with

$$H = (h_1, \ldots, h_{2N})^T = \left(c(\xi_1^*) e^{-2i\theta(\xi_1^*)}, \ldots, c(\xi_{2N}^*) e^{-2i\theta(\xi_{2N}^*)} \right)^T.$$

Chapter 3

Integrable boundary conditions

3.1 Characterisation of integrable boundary conditions

The study of initial-boundary value problems (IBVPs) for nonlinear PDEs that can be solved by the IST goes back to the 70s. Recall that the IST (see, Chapter 2) is used to solve a class of nonlinear PDEs[1] given on the full-line, that is $-\infty < x < \infty$. Motivated by the similitude between the IST and the Fourier transform [see, Ablowitz, Kaup, Newell & Segur (1974)], Ablowitz & Segur (1975) studied nonlinear integrable PDEs on the half-line by constructing an odd/even extension of the potential to the full-line. This extension allowed them to use the well-developed IST machinery to solve the Cauchy problem. An even extension led to solutions that satisfy Neumann boundary conditions (BCs) at $x = 0$, while an odd extension led to Dirichlet BCs. Note that for this approach to be successful, the nonlinear integrable PDEs of interest must have an even linearized dispersion relation. For instance, the linear version of the NLS equation (1.1.1), that is,

$$iu_t + u_{xx} = 0,$$

admits $w = k^2$ as dispersion relation, where k is the *wave-number* and w the *wave frequency*.

An important step towards a rigorous characterisation of BCs that can pre-

[1] Nonlinear PDEs that admit a Lax pair representation.

serve the integrability of a given classical system was made in Sklyanin (1987). It is worth mentioning that interesting universal nonlinear equations from mathematical physics, for instance, the NLS equation (1.1.1) or the KdV equation (2.0.1), admit a Lax pair formulation[1] but they can also be seen as an infinite-dimensional Hamiltonian system, see for example Faddeev & Takhtajan (2007). Sklyanin used both the Hamiltonian and the Lax pair formulation of integrable classical systems to lay out his approach. In the context of this work, we will only focus on the Lax pair side of his formulation, which will be illustrated in the case of the NLS equation (1.1.1).

Let $U(x, t, \lambda)$ and $V(x, t, \lambda)$ be the Lax pair of the NLS equation given in (2.1.3) and (2.1.4), respectively. The *time-dependent* version of the central equation in Sklyanin (1987) is given by the following *zero curvature boundary condition*

$$\partial_t K(t, \lambda) = V(0, t, -\lambda)K(t, \lambda) - K(t, \lambda)V(0, t, \lambda), \tag{3.1.1}$$

where $K(t, \lambda)$ is an unknown 2×2 matrix and referred to as *reflection matrix*. Solutions $K(t, \lambda)$ of (3.1.1) produce *integrable* BCs at the origin $x = 0$ for the NLS equation. By *integrable* we mean that we can use equation (3.1.1) to construct an infinite number of conserved quantities; see the examples below. Originally, Sklyanin (1987) only considered the time-independent version of the reflection matrix K, i.e. the LHS of equation (3.1.1) was zero. The time-dependent version of Sklynin's equation given in (3.1.1) was introduced in Bowcock, Corrigan, Dorey & Rietdijk (1995).

Let $\Psi(x, t, \lambda)$ be a matrix solution for the auxiliary system (2.1.1)-(2.1.2). The zero curvature boundary condition (3.1.1) means that

$$\Psi(0, t, -\lambda) = K(t, \lambda)\Psi(0, t, \lambda). \tag{3.1.2}$$

Definition 3.1. *Boundary conditions are said to be* time-dependent *if their reflection matrix satisfying* (3.1.1) *is time-dependent. Otherwise, they are said to be* time-independent.

To construct conserved quantities in the presence of integrable BCs on the half-line we can use the same approach traditionally adopted on the full-line. We start by recalling the construction of conserved quantities on the full-line; see for example Caudrelier (2008) for more detail.

[1]We saw this in the case of the NLS equation in Chapter 2.

Conserved quantities on the full-line Consider a vector-valued solution

$$\psi(x,t,\lambda) = \begin{pmatrix} \psi_1(x,t,\lambda) \\ \psi_2(x,t,\lambda) \end{pmatrix}$$

of the auxiliary system (2.1.1)-(2.1.2) associated to a potential in $\mathcal{S}(\mathbb{R})$. Define

$$\Gamma(x,t,\lambda) = \psi_2(x,t,\lambda)\psi_1^{-1}(x,t,\lambda).$$

For convenience, we will drop out the arguments x, t, λ. A direct calculation from (2.1.1) yields the following Riccati equation for Γ

$$\Gamma_x = 2i\lambda\Gamma - u^* - u\Gamma^2. \tag{3.1.3}$$

For convenience, we rewrite the matrix V defined in (2.1.4) as

$$V = \begin{pmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{pmatrix}.$$

Since we are in the case of ZBCs, we have

$$\operatorname{diag}\big(V_{11}, V_{22}\big) \to -2i\lambda^2\sigma_3, \quad V_{12}, V_{21} \to 0, \qquad \text{as } x \to \pm\infty. \tag{3.1.4}$$

Entries (11) and (12) of the zero curvature condition (2.1.5) for the NLS equation are given by

$$uV_{21} + u^*V_{12} - (V_{11})_x = 0, \quad u_t - (V_{12})_x - 2i\lambda V_{12} + 2uV_{22} = 0, \tag{3.1.5}$$

respectively. From (2.1.2), it follows that

$$\Gamma_t = V_{21} - 2V_{11}\Gamma - V_{12}\Gamma^2. \tag{3.1.6}$$

Recall that an expression of the form

$$D_t + F_x = 0$$

is called *conservation law*. The quantity D is known as the *local conserved density*, and F is called the *flux*. Every conservation law defines, under a suitable choice of boundary conditions for $F(t)$, the *conservation of an integral* of D:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_a^b D(x)\mathrm{d}x = F(a) - F(b),$$

where a and b represent the boundaries of the domain of integration. As we will shortly see, we can also have $a = -\infty$ and $b = +\infty$.

One has the following straightforward manipulations

$$\begin{aligned}(u\Gamma)_t &= u_t\Gamma + u\Gamma_t \\ &= ((V_{12})_x + 2i\lambda V_{12} + 2uV_{11})\Gamma + u(V_{21} - 2V_{11}\Gamma - V_{12}\Gamma^2), \text{ see (3.1.5) and (3.1.6)}, \\ &= (V_{12})_x\Gamma + (\Gamma_x + u^* + u\Gamma^2)V_{12} + uV_{21} - uV_{12}\Gamma^2 \\ &= (V_{12}\Gamma)_x + u^*V_{12} + uV_{21} \\ &= (V_{12}\Gamma + V_{11})_x, \text{ see (3.1.5)}.\end{aligned}$$

Thus we have obtained a conservation law for the NLS equation (1.1.1). Under the assumptions (3.1.4), we obtain that

$$I = \int_{-\infty}^{+\infty} u\Gamma \mathrm{d}x \tag{3.1.7}$$

is conserved in time. This integral is referred to as a *generating function* for conserved quantities. Note that Γ admits the following asymptotic expansion

$$\Gamma = \sum_{n=1}^{\infty} \frac{\Gamma_n}{(2i\lambda)^n}, \tag{3.1.8}$$

where

$$\Gamma_1 = -u^*, \quad \Gamma_{n+1} = (\Gamma_n)_x + u\sum_{k=1}^{n-1}\Gamma_k\Gamma_{k-1}, \quad n \geq 1. \tag{3.1.9}$$

Expressions in (3.1.9) are obtained by replacing the series expansion of Γ in the Riccati equation (3.1.3). Therefore, the terms of this series define conserved quantities, and we have an infinite number of them

$$I_n = \int_{-\infty}^{+\infty} u\Gamma_n \mathrm{d}x, \quad n \geq 1. \tag{3.1.10}$$

There is a correspondence between the existence of the above infinite number of conserved quantities with the complete integrability, in Liouville's sense, of the NLS equation (1.1.1) seen as a Hamiltonian system Faddeev & Takhtajan (2007). This is why the NLS equation (1.1.1) is called integrable.

We will now discuss two interesting examples of integrable BCs at $x = 0$, and we will also illustrate the construction of their conserved quantities. We restrict ourselves to the half-line. This means that every other object that involves the potentials will also be restricted on the half-line, for instance, the Lax pair (U, V) and the auxiliary system associated with it.

Examples of integrable BCs In the case of time-independent version of equation (3.1.1), one can only obtain a family of diagonal reflection matrices Sklyanin (1987)

$$K(t, \lambda) = \lambda \sigma_3 + i\alpha \mathbb{I}, \quad \alpha \in \mathbb{R}, \tag{3.1.11}$$

which leads to the well-known Robin boundary conditions

$$u_x(0, t) + 2\alpha u(0, t) = 0, \tag{3.1.12}$$

where $u(x, t)$ is solution to the NLS equation (1.1.1). Thus, following the terminology introduced in Definition 3.1, Robin BCs are time-independent. The construction of an infinite number of conserved quantities in the case of Robin BCs (3.1.12) was addressed in Caudrelier & Zhang (2012). Since we are dealing with equations defined on the half-line, it is clear that the generating function (3.1.7) will no longer be conserved in this case because we need to take into account the contribution from the boundary. Caudrelier & Zhang (2012) found that the correct generating function, in this case, is given by

$$I(t, \lambda) = \frac{1}{2} \int_0^\infty u(x, t)(\Gamma(x, t, \lambda) - \Gamma(x, t, -\lambda)) \, \mathrm{d}x \,, \tag{3.1.13}$$

where $\Gamma = \psi_2 \psi_1^{-1}$ and $\psi_{1,2}$ are the first and second components of the vector solution ψ satisfying the auxiliary system (2.1.1)-(2.1.2). Notice that now Γ is defined only on the half-line. From this we can follow (3.1.8)-(3.1.10) to write down all the conserved quantities.

Another example is the following. Let $u(x, t)$ be the solution of the focusing NLS equation (1.1.1). Consider the reflection matrix defined as

$$K(t, \lambda) = \frac{1}{(2\lambda - \beta)^2 + \alpha^2} \left(-4\lambda^2 - 4i\lambda H(t) + \alpha^2 + \beta^2 \right) , \tag{3.1.14}$$

with

$$H(t) = \begin{pmatrix} \pm\sqrt{\alpha^2 - |u(0, t)|^2} & u(0, t) \\ u^*(0, t) & \mp\sqrt{\alpha^2 - |u(0, t)|^2} \end{pmatrix} , \tag{3.1.15}$$

where α and β are real parameters characterizing the BCs. The normalisation of $K(t,\lambda)$ is chosen so that $K^{-1}(t,\lambda) = K(t,-\lambda)$. Without loss of generality, we can fix $\alpha, \beta > 0$. With this choice of the reflection matrix K, equation (3.1.1) is equivalent to the following BCs at $x = 0$

$$iu_t \quad = \quad (\alpha^2+\beta^2)u - 2|u|^2 u \pm 2u_x\sqrt{\alpha^2-|u|^2}. \tag{3.1.16}$$

Note that the BCs (3.1.16) can equivalently be written as follows, by continuing u and its derivatives to $x \to 0$ and using equation (1.1.1),

$$u_{xx} + (\alpha^2+\beta^2)u \pm 2u_x\sqrt{\alpha^2-|u|^2} = 0. \tag{3.1.17}$$

The choice of sign in (3.1.16) corresponds to the ones in (3.1.15). Equation (3.1.16) defines time-dependent integrable boundary conditions at $x = 0$ for the NLS equation (1.1.1).

Boundary conditions (3.1.17) correspond to the third boundary condition studied in Khabibullin (1993) by a completely different method (the symmetry approach). Note that the reflection matrix (3.1.14) was derived first in Zambon (2014), again using a different method (the Bäcklund transformation approach to integrable boundaries and defects). The BCs (3.1.16) represent the continuous limit of those found in Caudrelier & Crampé (2019), in the same way as the NLS equation is the continuous limit of the Ablowitz-Ladik model.

We now turn our attention to the construction of conserved quantities to justify the fact that these BCs are integrable. This was explained in detail in Caudrelier, Crampe & Dibaya (2022). The generating function is given by

$$\mathcal{I}(t,\lambda) = I(t,\lambda) - \mathcal{K}(t,\lambda)\,,$$

where

$$\mathcal{K}(t,\lambda) = \frac{1}{2}\ln\big(K_{11}(t,\lambda) + K_{12}(t,\lambda)\Gamma(0,t,\lambda)\big), \tag{3.1.18}$$

and $I(t,\lambda)$ is defined as in (3.1.13). Indeed, a direct calculation using the auxiliary system (2.1.1)-(2.1.2) yields

$$\partial_t(u\Gamma) = \partial_x(V_{11} + V_{12}\Gamma)\,. \tag{3.1.19}$$

This can be used for $\Gamma(x,t,\lambda)$ and for $\Gamma(x,t,-\lambda)$ to yield

$$\begin{aligned}\partial_t I(t,\lambda) =& \frac{1}{2}\big(- (V_{11}(0,t,\lambda) + V_{12}(0,t,\lambda)\Gamma(0,t,\lambda)) \\ & + (V_{11}(0,t,-\lambda) + V_{12}(0,t,-\lambda)\Gamma(0,t,-\lambda))\big)\,.\end{aligned}$$

We now use (3.1.2) to obtain

$$\Gamma(0,t,-\lambda) = (K_{21}(t,\lambda) + K_{22}(t,\lambda)\Gamma(0,t,\lambda))(K_{11}(t,\lambda) + K_{12}(t,\lambda)\Gamma(0,t,\lambda))^{-1}$$

and we use (3.1.1) to eliminate $V_{11}(0,t,-\lambda)$ and $V_{12}(0,t,-\lambda)$. We get, after some cancellations

$$\begin{aligned}\partial_t I(t,\lambda) =& \frac{1}{2}\Big(\partial_t K_{11}(t,\lambda) + \partial_t K_{12}(t,\lambda)\Gamma(0,t,\lambda)\Big)(K_{11}(t,\lambda) + K_{12}(t,\lambda)\Gamma(0,t,\lambda))^{-1} \\ &+ \frac{1}{2}K_{12}(t,\lambda)\Big(V_{21}(0,t,\lambda) - 2\lambda V_{11}(0,t,\lambda) \\ &- V_{12}(0,t,\lambda)\Gamma^2(0,t,\lambda)\Big)\Big(K_{11}(t,\lambda) + K_{12}(t,\lambda)\Gamma(0,t,\lambda)\Big)^{-1}.\end{aligned}$$

It remains to note that (2.1.4) implies the following Riccati equation in time for Γ

$$\partial_t\Gamma(0,t,\lambda) = V_{21}(0,t,\lambda) - 2\lambda V_{11}(0,t,\lambda) - V_{12}(0,t,\lambda)\Gamma^2(0,t,\lambda).$$

With this, we deduce

$$\partial_t I(t,\lambda) = \frac{1}{2}\partial_t \ln\left(K_{11}(t,\lambda) + K_{12}(t,\lambda)\Gamma(0,t,\lambda)\right).$$

This shows that $\partial_t I(t,\lambda) \neq 0$ but leads naturally to introduce $\mathcal{K}(t,\lambda)$ as in (3.1.18) and

$$\partial_t \mathcal{I}(t,\lambda) = \partial_t\left(I(t,\lambda) - \mathcal{K}(t,\lambda)\right) = 0\,, \tag{3.1.20}$$

which shows the announced result.

We can then construct all conserved quantities following the procedure described above. Therefore we have proven that the time-dependent BCs (3.1.16) are indeed integrable.

Let us remark that for Robin boundary conditions, since K is diagonal and time-independent (see Eq. (3.1.11)), $K_{12}(t,\lambda) = 0$ and $K_{11}(t,\lambda) = \lambda$, the previous equation simplifies and shows $I(t,\lambda)$ is the generating function for an infinite number of conserved quantities without needing $\mathcal{K}(t,\lambda)$. However, for the time-dependent boundary conditions we are discussing here, this is not the case and $\mathcal{K}(t,\lambda)$ is indeed time-dependent and exactly compensates for the loss of conservation in time of $I(t,\lambda)$. We will illustrate this point with a concrete example in subsection 4.1.3. Integrability holds for the system "half-line+boundary" while the half-line only can be thought of as being an open system coupled to a boundary that acts as a reservoir.

3.2 Solution methods

Sklyanin's approach to integrable boundary conditions did not say anything about how to construct solutions. Solution methods to IBVPs for integrable PDEs with integrable boundary conditions developed rather independently. However, we will see that there is a profound connection between the solution methods and Sklyanin's approach to integrable boundary conditions.

We have already mentioned the work by Ablowitz & Segur (1975), which can be seen as a first step towards solving IBVPs for nonlinear integrable PDEs. The second important step forward was made in the original work by Khabibullin (1991).

In his work, Khabibullin (1991) puts forward the use of Bäcklund-Darboux matrix to map IBVPs on the half-line to Cauchy problems of the same nonlinear PDE. In this way, one can use the IST to solve the Cauchy problem on the full-line and the solution to this will satisfy automatically certain boundary conditions at $x = 0$. Khabibullin (1991) ideas were successfully implemented by Bikbaev & Tarasov (1991) to solve the IBVP for the NLS equation (1.1.1) with Robin BCs (3.1.12). This led to several subsequent interesting results; see for instance, Tarasov (1991), Biondini & Hwang (2009), Caudrelier & Zhang (2012), Biondini & Hwang (2009) and Biondini & Bui (2012). In Biondini & Hwang (2009), this method was referred to as *nonlinear mirror image*; we will keep their terminology in this work as well. Recently, Deift & Park (2011) have studied rigorously the nonlinear mirror method using the language of Riemann-Hilbert problems.

Motivated by some limitations of the nonlinear mirror image method, for instance, the difficulty in dealing with dispersion relations of odd degree, as in the Korteweg-de Vries equation, Fokas and several co-workers developed another solution method. This method is called the unified transform or Fokas method, which does not rely on mapping the problem to a full line problem. Instead, the idea is to perform the simultaneous spectral analysis of both parts of the auxiliary problem associated with the Lax pair of the given integrable PDE on the half-line.

In the next two subsections, we will discuss the nonlinear mirror image method. Afterwards, we will briefly outline the unified transform as it is not used in our work. For both methods, we will only consider the focusing case of the NLS equation (1.1.1), that is $\kappa = -1$.

3.2.1 Bäcklund transformation approach to BCs

Let $\Psi(x,t,\lambda)$ be any fundamental solution of (2.1.1) and (2.1.2) associated to a potential $Q(x,t)$ on the full-line. Define

$$\widetilde{\Psi}(x,t,\lambda) = L(x,t,\lambda)\Psi(x,t,\lambda),$$

where $L(x,t,\lambda)$ is a 2×2 matrix function. A direct calculation shows that the new eigenfuction $\widetilde{\Psi}$ satisfies

$$\begin{cases} \widetilde{\Psi}_x = \widetilde{U}\widetilde{\Psi} := \left(-i\lambda\sigma_3 + \widetilde{Q}\right)\widetilde{\Psi}, \\ \widetilde{\Psi}_t = \widetilde{V}\widetilde{\Psi} := \left(-2i\lambda^2\sigma_3 + 2\lambda\widetilde{Q} - i\widetilde{Q}_x\sigma_3 - i\widetilde{Q}^2\sigma_3\right)\widetilde{\Psi}, \end{cases} \tag{3.2.1}$$

where $\widetilde{Q}$ is a 2×2 off-diagonal matrix with similar structure as Q, if and only if, the matrix function $L(x,t,\lambda)$ is a solution of the following differential equations

$$L_x = \widetilde{U}L - LU\,, \quad L_t = \widetilde{V}L - LV. \tag{3.2.2}$$

From the zero curvature condition of (3.2.1), we see that the (12) entry of $\widetilde{Q}$ solves the NLS equation (1.1.1). This means the matrix L obtained as solution of the differential equations in (3.2.2) induces a *Bäcklund transformation* for the NLS equation or equations of the type (2.1.1). We can write that

$$Q(x,t) \xrightarrow{L} \widetilde{Q}(x,t).$$

The matrix function L is known as the *Bäcklund/Darboux matrix.*

The key observation made in Khabibullin (1991) was that if

$$\widetilde{U}(x,t,\lambda) = -U(-x,t,-\lambda), \quad \widetilde{V}(x,t,\lambda) = V(-x,t,-\lambda), \tag{3.2.3}$$

then it follows from the second equation in (3.2.2) that the solution to the NLS equation satisfies some boundary conditions at $x=0$. Note that to implement successfully these ideas, one needs to construct an appropriate Bäcklund matrix L that can lead to (3.2.3). This was achieved for the first time in Bikbaev & Tarasov (1991) for the NLS equation with Robin BCs, and it is essentially what the nonlinear mirror image method is about. We will review this in detail in the next subsection.

In Caudrelier & Crampé (2019), it was observed that under the assumption (3.2.3), the second equation (3.2.2) is exactly the Sklyanin's criterion for integrable boundary conditions (3.1.1) if one takes

$$L(0,t,\lambda) = K(t,\lambda), \tag{3.2.4}$$

where $K(t,\lambda)$ is the solution of the time-dependent version of Sklyanin's equation (3.1.1). It is worth stressing again that Sklyanin (1987) original equation did not involve time-dependent reflection matrices, that is $\partial_t K(t,\lambda) = 0$. Thus his approach seemed at first different from the one initiated by Khabibullin (1991).

3.2.2 Nonlinear mirror image method for time-independent BCs

In this section, we will construct a Bäcklund matrix L that will allow us to generate Robin BCs (3.1.12) at the origin. In light of the connection made in (3.2.4), we must have

$$L(0,t,\lambda) = K(\lambda) = \lambda\sigma_3 + i\alpha\mathbb{I}.$$

To construct such L, we will fix the time parameter at $t = 0$, and then obtain it as the solution of the first differential equation in (3.2.2). Afterwards, the time evolution of L must be compatible with the second equation in (3.2.2) in order to obtain a useful Bäcklund transformation.

Let $u(x)$ be a complex-valued function on $\mathbb{R}$, and define

$$Q(x) = \begin{pmatrix} 0 & u(x) \\ -u^*(x) & 0 \end{pmatrix}. \tag{3.2.5}$$

We may have the case $u(x)$ defined on $\mathbb{R}^\pm$. This matrix $Q(x)$ can be seen as the one defined in (2.1.3) at a fixed time, say $t = 0$. Thus, we will sometimes refer to it or its entries as potentials. In what follows, we will use $u(x)$ and $Q(x)$ interchangeably. Let $A(x)$ be a matrix function, we say that $A(x)$ belongs to $\mathcal{S}(\mathbb{R})$ $\left[\mathcal{S}(\mathbb{R}_\pm)\right]$ if all its entries are elements of $\mathcal{S}(\mathbb{R})$ $\left[\mathcal{S}(\mathbb{R}_\pm)\right]$. In the sequel, unless otherwise stated, every time we consider a matrix or vector solution of (2.1.1), it should be associated with the potential above. The results we present in this section can be found in Bikbaev & Tarasov (1991) or Deift & Park (2011). We have also enhanced some proofs following Caudrelier, Crampe & Dibaya (2022).

Consider the following 2×2 ordinary differential equation (ODE)

$$\begin{cases} P_x = (-Q + i[\sigma_3, P\sigma_3])P - PQ, \\ P_0 \equiv P(0) = i\alpha, \quad \alpha \in \mathbb{R}. \end{cases} \tag{3.2.6}$$

Lemma 3.2. *If $Q(x) \in \mathcal{S}(\mathbb{R})$, the ODE (3.2.6) has a unique solution on $\mathbb{R}$. In particular, if $Q(x) \in \mathcal{S}(\mathbb{R}_\pm)$, (3.2.6) admits a unique solution on $\mathbb{R}_\pm$.*

Proof: Assume that $Q(x) \in \mathcal{S}(\mathbb{R})$. Let $\Psi_0(x,\lambda) = \left(\Psi_0^{(1)}(x,\lambda), \Psi_0^{(2)}(x,\lambda)\right)$ be the solution of (2.1.1) such that $\Psi_0(0,\lambda) = \mathbb{1}$. The differential equation in (3.2.6) can be rewritten in a matrix commutator form as

$$(P\sigma_3)_x = [-Q + i\sigma_3 P\sigma_3, P\sigma_3]\,.$$

It is convenient to work with $P_1(x) = P(x)\sigma_3$, that is

$$(P_1)_x = [-Q + i\sigma_3 P_1, P_1]\,. \tag{3.2.7}$$

Obviously, we have $P_{1_0} \equiv P_1(0) = i\alpha\sigma_3$. We seek a solution of the form

$$P_1(x) = H(x)P_{1_0}(H(x))^{-1}, \tag{3.2.8}$$

for some invertible matrix $H(x)$ such that $H(0) = \mathbb{1}$. Substituting (3.2.8) into (3.2.7), we get

$$(P_1)_x = [H_x H^{-1}, P_1].$$

Thus, we have

$$[M(x), HP_{1_0}H^{-1}] = 0,$$

where

$$M(x) = H_x H^{-1} - \left(-Q + i\sigma_3 HP_{1_0}H^{-1}\right).$$

In turn, this implies that $H(x)^{-1}M(x)H(x) \equiv D(x)$ is a diagonal matrix. The matrix H is not uniquely defined and it is always possible to consider the transformation $H \mapsto Hh$ where h is an invertible diagonal matrix without changing

P_1. We use this freedom to choose h such that $h_x = -Dh$ and set $\varphi = Hh$, with the conclusion that

$$P_1(x) = \varphi(x) P_{1_0} \varphi(x)^{-1}$$

where φ is a nonsingular (or fundamental) solution of

$$\varphi_x = i\sigma_3 \varphi P_{1_0} - Q\varphi. \tag{3.2.9}$$

Writing $\varphi = (\varphi_1, \varphi_2)$ where $\varphi_{1,2}$ are the column vectors of φ, we see that

$$\varphi_{1x}(x) = (-i\lambda_0\sigma_3 - Q)\varphi_1\,, \quad \varphi_{2x}(x) = (-i\lambda_0^*\sigma_3 - Q)\varphi_2\,, \quad \lambda_0 = -i\alpha.$$

This means that

$$\varphi(x) = (\varphi_1(x), \varphi_2(x)) = \sigma_3 \left(\Psi_0^{(1)}(x,\lambda_0), \Psi_0^{(2)}(x,\lambda_0^*) \right).$$

Recall that $\sigma_{-1} = -i\sigma_2$. Since $\Psi_0(x,\lambda)$ is a solution of (2.1.1), it satisfies the symmetry in (2.1.7):

$$i\sigma_2 \Psi_0(x,\lambda^*)^* (i\sigma_2)^{-1} = \Psi_0(x,\lambda), \tag{3.2.10}$$

for any λ. Note that we have taken into consideration the normalisation of $\Psi_0(x,\lambda)$ at $x = 0$. The second column of equation (3.2.10) evaluated at λ_0^* reads

$$\Psi_0^{(2)}(x,\lambda_0^*) = -i\sigma_2 \Psi_0^{(1)}(x,\lambda_0)^*.$$

Thus $P_1(x)$ is given by $P_1(x) = \varphi(x) P_{1_0} \varphi(x)^{-1}$, where the matrix $\varphi(x)$ can be written as

$$\varphi(x) = \begin{pmatrix} \xi_1(x) & -\xi_2(x)^* \\ -\xi_2(x) & -\xi_1(x)^* \end{pmatrix},$$

where $(\xi_1(x), \xi_2(x))^T = \Psi_0^{(1)}(x,\lambda_0)$. In turn, we have

$$P(x) = \frac{i\alpha}{|\xi_1(x)|^2 + |\xi_2(x)|^2} \begin{pmatrix} |\xi_1(x)|^2 - |\xi_2(x)|^2 & 2\xi_1(x)\xi_2(x)^* \\ -2\xi_1(x)^*\xi_2(x) & -\left(|\xi_1(x)|^2 - |\xi_2(x)|^2\right) \end{pmatrix}. \tag{3.2.11}$$

Since $\det \varphi(x) \neq 0$ for all $x \in \mathbb{R}$, then $P(x)$ defines the global solution for (3.2.6). The case $Q \in \mathcal{S}(\mathbb{R}_\pm)$ is similar. ∎

Lemma 3.3. *Let $Q(x) \in \mathcal{S}(\mathbb{R})$ and $P(x)$ be the solution of (3.2.6). Then $P(x) \to i\beta_\pm$ as $x \to \pm\infty$ such that $\beta_\pm^2 = \alpha^2$.*

Proof: Consider $\alpha > 0$. The analysis for the case $\alpha < 0$ is similar. Let $\lambda \in \mathbb{C}^+$. The differential equation (2.1.1) admits a non-unique 2×1 solution $g(x,\lambda)$ on $\mathbb{R}_-$ such that

$$g(x,\lambda) = e^{i\lambda x} \left[r(x,\lambda), v(x,\lambda) \right]^T,$$

where $[r(x,\lambda), v(x,\lambda)]^T$ is a column vector that goes to $e_2 = (0,1)^T$ as $x \to -\infty$. For any $u(x) \in \mathcal{S}(\mathbb{R})$, fix $x_0 < 0$ such that $\|u\|_{L^1(-\infty,x_0]} < 1/2$. Note that $r(x,\lambda)$ and $v(x,\lambda)$ satisfy the following differential equations

$$r_x + 2i\lambda r = uv, \quad v_x = -u^* r.$$

The trick here is to fix $r(x,\lambda)$ and $v(x,\lambda)$ at two different points of $(-\infty, x_0]$:

$$r(x,\lambda) = -\int_x^{x_0} e^{2i\lambda(y-x)} u(y) v(y,\lambda) \mathrm{d}y, \quad v(x,\lambda) = 1 - \int_{-\infty}^{x} u^*(y) r(y,\lambda) \mathrm{d}y, \quad x \leq x_0.$$

Now, we can replace the second equation into the first and change the order of integration:

$$\int_{-\infty}^{x} \int_{y}^{x_0} e^{2i\lambda(z-y)} u^*(y) u(z) v(z,\lambda) \mathrm{d}z \mathrm{d}y = \int_{-\infty}^{x} \int_{-\infty}^{z} e^{2i\lambda(z-y)} u^*(y) u(z) v(z,\lambda) \mathrm{d}y \mathrm{d}z.$$

This leads to the following Volterra integral equation for $v(x,\lambda)$

$$v(x,\lambda) = 1 + \int_{-\infty}^{x} K(z,\lambda) v(z,\lambda) \mathrm{d}z,$$

where the kernel $K(z,\lambda)$ is given by

$$K(z,\lambda) = u(z) \int_{-\infty}^{z} e^{2i\lambda(z-y)} u^*(y) \mathrm{d}y.$$

Thus, one has

$$|K(z,\lambda)| \leq \frac{|u(z)|}{2}.$$

From this point, one can introduce the Neumann series of $v(x,\lambda)$ as we did in the proof of Proposition 2.2. Thus, we have proved that the Neumann series is actually absolutely and uniformly convergent on $(-\infty, x_0]$, which we can extend to $\mathbb{R}_-$.

Recall that $\lambda \in \mathbb{C}^+$. It is clear that $\Psi_-^{(1)}(x,\lambda)$ and $g(x,\lambda)$ are linearly independent, where $\Psi_-^{(1)}(x,\lambda)$ is the first column the Jost matrix solution $\Psi_-(x,\lambda)$; see Chapter 2. Note that since $\alpha > 0$, we can write

$$\Psi_0^{(1)}(x, i\alpha) = c_1 \Psi_-^{(1)}(x, i\alpha) + c_2 g(x, i\alpha), \tag{3.2.12}$$

where c_1, c_2 are constants. If the second component of $\Psi_-^{(1)}(0, i\alpha)$ vanishes, then we must have $c_2 = 0$. Thus $\Psi_0^{(1)}(x, i\alpha) = c_1\Psi_-^{(1)}(x, i\alpha)$, that is, $\frac{\xi_2(x)}{\xi_1(x)} \to 0$ as $x \to -\infty$. In turn, we have

$$P(x) \to i\alpha, \qquad \text{as } x \to -\infty.$$

However, if the second component of $\Psi_-^{(1)}(0, i\alpha)$ does not vanish, we have that $\Psi_0^{(1)}(x, i\alpha) \sim e^{-\alpha x}\left[r(x, i\alpha), v(x, i\alpha)\right]^T$. Again, we get

$$P(x) \to -i\alpha, \qquad \text{as } x \to -\infty.$$

The above calculations can be summarized as follows: $P(x) \to i\beta_-$ as $x \to -\infty$, where $\beta_- = \alpha$ when the second component of $\Psi_-^{(1)}(0, i\alpha)$ vanishes and $\beta_- = -\alpha$ when the second component of $\Psi_-^{(1)}(0, i\alpha)$ does not vanish.

When it comes to the limit as $x \to +\infty$, one needs to consider solutions on $\mathbb{R}_+$. Equation (2.1.1) admits the following solution on $\mathbb{R}_+$

$$h(x, \lambda) = e^{-i\lambda x}\left[\bar{r}(x, \lambda), \bar{v}(x, \lambda)\right]^T,$$

such that $\left[\bar{r}(x, \lambda), \bar{v}(x, \lambda)\right]^T$ is a column vector that goes to e_1 as $x \to +\infty$. The construction of $h(x, \lambda)$ can be done as the one of $g(x, \lambda)$ above. Using the same argument as above leads to:

$$P(x) \to i\beta_+, \qquad \text{as } x \to +\infty,$$

where $\beta_+ = -\alpha$ when the second component of $\Psi_+^{(2)}(0, i\alpha)$ vanishes and $\beta_+ = \alpha$ when the second component of $\Psi_+^{(2)}(0, i\alpha)$ does not vanish. Note that $\Psi_+^{(2)}(x, \lambda)$ is the second column of the Jost matrix solution $\Psi_+(x, \lambda)$. ■

Potential transformation. Given a potential $Q(x)$ as above, let $P(x) = (p_{ij})_{1\le i,j\le 2}$ be the solution of (3.2.6). Set

$$\tilde{u}(x) = -u(x) - 2ip_{12}(x) = -u(x) + \frac{4\alpha\xi_1(x)\xi_2(x)^*}{|\xi_1(x)|^2 + |\xi_2(x)|^2}.$$

From the explicit expression of $P(x)$ given in (3.2.11), one has $p_{12}(x) = p_{21}^*(x)$. Thus, we have

$$\widetilde{Q}(x) \equiv \begin{pmatrix} 0 & \tilde{u}(x) \\ -\tilde{u}^*(x) & 0 \end{pmatrix} = -Q + i[\sigma_3, P(x)\sigma_3]. \tag{3.2.13}$$

Note that from the above result, the new potential $\widetilde{Q}(x)$ is also a Schwartz function. We define the following Bäcklund matrix

$$L(x,\lambda) = \lambda\sigma_3 + P(x). \tag{3.2.14}$$

The differential equation satisfied by $P(x)$ is equivalent to $L(x,\lambda)$ solving the familiar gauge transformation equation

$$L_x(x,\lambda) = \widetilde{U}(x,\lambda)L(x,\lambda) - L(x,\lambda)U(x,\lambda)\,, \tag{3.2.15}$$

where $\widetilde{U}(x,\lambda) = -i\lambda\sigma_3 + \widetilde{Q}(x)$. Note that this is the time-independent version of the first equation in (3.2.2). In turn, this ensures that if $\Psi(x,\lambda)$ is a solution of (2.1.1), and we define

$$\widetilde{\Psi}(x,\lambda) := L(x,\lambda)\Psi(x,\lambda)\,, \tag{3.2.16}$$

then $\widetilde{\Psi}(x,\lambda)$ solves

$$\widetilde{\Psi}_x(x,t) = \widetilde{U}(x,\lambda)\widetilde{\Psi}(x,\lambda). \tag{3.2.17}$$

We introduce the following important definition.

Definition 3.4. *Let $Q(x)$ be complex-valued and defined on $\mathbb{R}$ as above. The map*

$$L_\alpha : Q \longmapsto \widetilde{Q} = L_\alpha[Q],$$

is called the Bäcklund transformation (BT) of $Q(x)$ with respect to α. If $Q(x)$ is defined on $\mathbb{R}_\pm$, the map $L_\alpha^\pm : Q \mapsto \widetilde{Q} = L_\alpha^\pm[Q]$ is called the Bäcklund transformation of $Q(x)$ with respect to α. We will use the same terminology and notation at the level of the entries $\widetilde{u}(x)$ and $u(x)$.

We will now study some properties of these maps.

Lemma 3.5 (Deift & Park (2011))**.**

1. *If $Q(x) \in \mathcal{S}(\mathbb{R})$ then $\mathcal{R}L_\alpha\mathcal{R}L_\alpha[Q] = Q$,*

2. *If $Q(x) \in \mathcal{S}(\mathbb{R}^\pm)$ then $\mathcal{R}L_\alpha^\pm[Q](x) \in \mathcal{S}(\mathbb{R}^\mp)$ and $\mathcal{R}L_\alpha^\mp\mathcal{R}L_\alpha^\pm[Q] = Q$.*

where $\mathcal{R}[Q](x) \equiv -Q(-x)$.

Proof: Let $Q(x) \in \mathcal{S}(\mathbb{R})$ and $P(x)$ be the solution of (3.2.6). Set

$$Q_1(x) = -\widetilde{Q}(-x) = \mathcal{R}L_\alpha[Q](x), \quad P_1(x) = \sigma_3 P(-x)\sigma_3.$$

We have $Q(-x) = -\widetilde{Q}(-x) + i[\sigma_3, P(-x)\sigma_3] = -\left(-Q_1(x) + i[\sigma_3, P_1(x)\sigma_3]\right)$. The matrix $P(x)$ admits the symmetry $P(x) = -\sigma_3 P^\dagger(x)\sigma_3$.[1] A direct calculation shows that

$$\left(P_1(x)\right)_x = \left(-Q_1(x) + i[\sigma_3, P_1(x)\sigma_3]\right) P_1(x) - P_1(x)Q_1(x)$$

Taking into consideration the fact that $P_1(0) = P(0) = i\alpha \mathbb{I}$, we conclude that $\widetilde{Q}_1(x) = -Q_1(x) + i[\sigma_3, P_1(x)\sigma_3] = -Q(-x)$ which means $\mathcal{R}L_\alpha[Q_1](x) = Q(x)$ and proves the first point of the Lemma. The second point is proven similarly. ■

Corollary 3.6. *The maps $L_\alpha : \mathcal{S}(\mathbb{R}) \to \mathcal{S}(\mathbb{R})$ and $L_\alpha^\pm : \mathcal{S}(\mathbb{R}_\pm) \to \mathcal{S}(\mathbb{R}_\pm)$ are bijections.*

Proof: The proof follows from the above result. ■

Let u be a function defined on $\mathbb{R}$, then we have

$$\left(L_\alpha[u]\right)|_{\mathbb{R}_\pm} = L_\alpha^\pm\left[u|_{\mathbb{R}_\pm}\right]. \tag{3.2.18}$$

Scattering data transformation. As we have seen, the new potential $\widetilde{u}(x) = L_\alpha[u](x)$ is also in the same functional space as the original one, which is the space of Schwartz functions $\mathcal{S}(\mathbb{R})$. Therefore, one can construct Jost solutions, scattering data associated with $\widetilde{u}(x)$ as described in the previous chapter. We refer to every object linked to $\widetilde{u}(x)$ with a tilde. Recall that if $u(x)$ is a generic potential, that is its scattering coefficient $s_{22}(\lambda)$ admits a finite number of simple zeros in $\mathbb{C}^+$. We used $\mathcal{Z}_+$, see Section 2.3, to denote the set of all those simple zeros. In the next lemma, we describe the relationship between the scattering data associated with $u(x) \in \mathcal{S}(\mathbb{R})$ and the scattering data associated with $\widetilde{u}(x)$.

Lemma 3.7. *The relation between the scattering matrix $S(\lambda)$ associated with $u(x)$ and the scattering matrix $\widetilde{S}(\lambda)$ associated with $\widetilde{u}(x)$ reads*

$$\widetilde{S}(\lambda) = \left(\lambda\sigma_3 + i\beta_- \mathbb{I}\right) S(\lambda) \left(\lambda\sigma_3 + i\beta_+ \mathbb{I}\right)^{-1}. \tag{3.2.19}$$

[1] Given a matrix A, $A^\dagger$ is defined as the conjugate transpose of A.

Explicitly, for the scattering coefficients, one gets

$$\widetilde{s}_{21}(\lambda) = -\frac{\lambda - i\beta_-}{\lambda + i\beta_+} s_{21}(\lambda), \quad \lambda \in \mathbb{R}, \tag{3.2.20}$$

$$\widetilde{s}_{22}(\lambda) = \frac{\lambda - i\beta_-}{\lambda - i\beta_+} s_{22}(\lambda), \quad \lambda \in \mathbb{C}^+ \setminus \{i\beta_+\}, \text{ if } \beta_+ > 0. \tag{3.2.21}$$

If $i|\alpha|$ is not a zero of $s_{22}(\lambda)$, then $\widetilde{u}$ is a generic potential if u is generic. Thus, we have either $\widetilde{\mathcal{Z}}_+ = \mathcal{Z}_+$ or $\widetilde{\mathcal{Z}}_+ = \mathcal{Z}_+ \cup \{i|\alpha|\}$. In both cases we have

$$\widetilde{\gamma}(\lambda_k) = -\frac{\lambda_k - i\beta_+}{\lambda_k + i\beta_-} \gamma(\lambda_k), \quad \lambda_k \in Z_+; \tag{3.2.22}$$

and, in the second case, we have

$$\widetilde{\gamma}(i|\alpha|) = \begin{cases} \frac{\Psi_-^{(11)}(0,i\alpha)}{\Psi_+^{(21)}(0,i\alpha)}, & \text{if } \alpha > 0, \\ \frac{\Psi_-^{(12)}(0,-i\alpha)}{\Psi_+^{(22)}(0,-i\alpha)}, & \text{if } \alpha < 0. \end{cases} \tag{3.2.23}$$

<u>Proof:</u> Relation (3.2.16) in terms of Jost solutions takes the form

$$\widetilde{\Psi}_\pm(x,\lambda) = L(x,\lambda)\Psi_\pm(x,\lambda)L_{\pm\infty}^{-1}, \tag{3.2.24}$$

where

$$L_{\pm\infty}(\lambda) = \lambda\sigma_3 + i\beta_\pm \mathbb{I}. \tag{3.2.25}$$

Using $\widetilde{S}(\lambda) = \widetilde{\Psi}_-(x,\lambda)^{-1}\widetilde{\Psi}_+(x,\lambda)$ and $S(\lambda) = \Psi_-(x,\lambda)^{-1}\Psi_+(x,\lambda)$, one gets relation (3.2.19). In turn, we have the relations on the scattering coefficients for all $\lambda \in \mathbb{R}$. The extension of the relation between $s_{22}(\lambda)$ and $\widetilde{s}_{22}(\lambda)$ to $\mathbb{C}^+ \setminus \{i\beta_+\}$ follows immediately. Assume that $s_{22}(i|\alpha|) \neq 0$ and $u(x)$ is a generic potential. This means that for each $\lambda_k \in \mathcal{Z}_+$, one has $s_{22}(\lambda) = \frac{\lambda-\lambda_k}{\lambda-\lambda_k^*} g(\lambda)$, where $g(\lambda_k) \neq 0$. Equivalently, we can write relation (3.2.21) as $\widetilde{s}_{22}(\lambda) = \frac{\lambda-\lambda_k}{\lambda-\lambda_k^*}\widetilde{g}(\lambda)$, where $\widetilde{g}(\lambda) = \frac{\lambda-i\beta_-}{\lambda-i\beta_+} g(\lambda)$. Since $\widetilde{g}(\lambda_k) \neq 0$, we conclude that $\mathcal{Z}_+ \subseteq \widetilde{\mathcal{Z}}_+$. A direct calculation shows that $\widetilde{s}_{22}(i|\alpha|) = 0$ if and only if $\beta_+ = -|\alpha|$ and $\beta_- = |\alpha|$. Thus, $\widetilde{\mathcal{Z}}_+ = \mathcal{Z}_+$ if $\beta_+ = -|\alpha|$ and $\beta_- = -|\alpha|$ and $\widetilde{\mathcal{Z}}_+ = \mathcal{Z}_+ \cup \{i|\alpha|\}$ if $\beta_+ = -|\alpha|$ and $\beta_- = |\alpha|$. This means that $\widetilde{u}(x)$ is also a generic potential. Taking into consideration the analytic continuation of the columns of Jost solutions, one obtains from relation (3.2.24)

$$\widetilde{\Psi}_-^{(1)}(x,\lambda_k) = \frac{L(x,\lambda_k)}{\lambda_k + i\beta_-}\Psi_-^{(1)}(x,\lambda_k), \quad \widetilde{\Psi}_+^{(2)}(x,\lambda_k) = -\frac{L(x,\lambda_k)}{\lambda_k - i\beta_+}\Psi_+^{(2)}(x,\lambda_k). \tag{3.2.26}$$

Thus, we have

$$\begin{aligned}\widetilde{\Psi}_-^{(1)}(x,\lambda_k) &= \widetilde{\gamma}(\lambda_k)\widetilde{\Psi}_+^{(2)}(x,\lambda_k)\\ &= -\frac{\widetilde{\gamma}(\lambda_k)}{\lambda_k - i\beta_+}L(x,\lambda_k)\Psi_+^{(2)}(x,\lambda_k)\\ &= -\frac{\widetilde{\gamma}(\lambda_k)}{\gamma(\lambda_k)(\lambda_k - i\beta_+)}L(x,\lambda_k)\Psi_-^{(1)}(x,\lambda_k).\end{aligned}$$

Comparing this with the first equation in (3.2.26) gives relation (3.2.22). At $\lambda = i|\alpha|$, we have

$$\begin{aligned}L(x,i|\alpha|)\Psi_-^{(1)}(x,i|\alpha|) = \widetilde{\Psi}_-^{(1)}(x,i|\alpha|) &= \widetilde{\gamma}(i|\alpha|)\widetilde{\Psi}_+^{(2)}(x,i|\alpha|)\\ &= \widetilde{\gamma}(i|\alpha|)L(x,i|\alpha|)\Psi_+^{(2)}(x,i|\alpha|).\end{aligned}$$

Evaluating this relation at $x = 0$, gives

$$\begin{pmatrix}\alpha+|\alpha| & 0\\ 0 & \alpha-|\alpha|\end{pmatrix}\Psi_-^{(1)}(0,i|\alpha|) = \widetilde{\gamma}(i|\alpha|)\begin{pmatrix}\alpha+|\alpha| & 0\\ 0 & \alpha-|\alpha|\end{pmatrix}\Psi_+^{(2)}(0,i|\alpha|).$$

Relation (3.2.23) follows from the above. ■

Time evolution The construction of a Bäcklund transformation is useful if it is compatible with the time evolution of the PDE of interest. Consider $Q(x,t) \in \mathcal{S}(\mathbb{R})$ subject to $U_t - V_x + [U,V] = 0$. For each $t \geq 0$, construct $P(x,t)$ as the solution of (3.2.6), and hence also the corresponding $L(x,t,\lambda)$ which then satisfies (3.2.15). In line with Definition 3.4, define then the new potential $\widetilde{Q}(x,t) = -Q(x,t) + i[\sigma_3, P(x,t)\sigma_3]$, for each $t \geq 0$. Call it the Bäcklund transformation of $Q(x,t)$ with respect to α and write $\widetilde{Q}(x,t) = L_\alpha[Q](x,t)$. Then, the following well-known result shows that the new potential also satisfies the NLS equation if and only if L satisfies the t-part of the gauge transformation equation. Specifically, we have

Lemma 3.8. *The following equivalence holds:*

$$\widetilde{U}_t - \widetilde{V}_x + [\widetilde{U},\widetilde{V}] = 0 \Longleftrightarrow L_t(x,t,\lambda) = \widetilde{V}(x,t,\lambda)L(x,t,\lambda) - L(x,t,\lambda)V(x,t,\lambda)$$

where $\widetilde{V}$ is given by replacing Q by $\widetilde{Q}$ in (2.1.2).

Proof: Indeed, we start by proving the implication from the left to the right by assuming that L satisfies the t-part of the gauge transformation equation. Since L also satisfies (3.2.15), the compatibility $L_{xt} = L_{tx}$ yields

$$\left(\widetilde{U}_t - \widetilde{V}_x + [\widetilde{U}, \widetilde{V}]\right) L = L\left(U_t - V_x + [U, V]\right) = 0\,.$$

Hence, $\widetilde{U}_t - \widetilde{V}_x + [\widetilde{U}, \widetilde{V}] = 0$. Conversely, assume that $\widetilde{U}_t - \widetilde{V}_x + [\widetilde{U}, \widetilde{V}] = 0$. Set $\Delta = L_t - \widetilde{V}L + LV$. An explicit calculation gives

$$\Delta = P_t + i\left(\widetilde{Q}_x\sigma_3 + \widetilde{Q}^2\sigma_3\right)P - iP\left(Q_x\sigma_3 + Q^2\sigma_3\right)\,, \tag{3.2.27}$$

which shows that Δ does not depend on λ. Now,

$$0 = \left(\widetilde{U}_t - \widetilde{V}_x + [\widetilde{U}, \widetilde{V}]\right) L = i\lambda[\sigma_3, \Delta] + \Delta_x - \widetilde{Q}\Delta + Q\Delta.$$

Since Δ does not depend on λ, the last equation gives us $[\sigma_3, \Delta] = 0$ and $\Delta_x = \widetilde{Q}\Delta - Q\Delta$. The former equation means that Δ is diagonal. The latter, as a consequence, implies that Δ is constant with respect to x since the term on the right-hand side is off-diagonal. Thus, we can evaluate the constant value of Δ using (3.2.27) at $x = 0$. Since $P(0, t, \lambda) = i\alpha$, we find $\Delta = 0$ as desired. ■

We have now constructed a Bäcklund matrix $L(x, t, \lambda)$ that satisfies both equations in (3.2.2) such that its value at $t = 0 = x$ is given as in (3.1.11). We will now describe how one can make use of this theory to solve IBVPs for the focusing NLS equation (1.1.1) on the half-line with Robin BCs (3.1.12).

We start by introducing the following important concept.

Definition 3.9. *Let $u(x)$ be an element of $\mathcal{S}(\mathbb{R}_+)$, we denote by $u^{ext}(x)$ the Bäcklund extension of $u(x)$ to $\mathbb{R}$ with respect to $\alpha \in \mathbb{R}$, defined by*

$$u^{ext}(x) = \begin{cases} u(x), & x \geq 0, \\ \mathcal{R}L_\alpha^+[u](x), & x < 0. \end{cases} \tag{3.2.28}$$

Note that we can also define the Bäcklund extension with respect to α of a function defined on $\mathbb{R}_-$ or $\mathbb{R}_+$. We denote by $Q^{ext}(x)$ the off-diagonal matrix defined as Q in (3.2.5) with u replaced by its Bäcklund extension.

Definition 3.10. *Let $u(x)$ be an element of $\mathcal{S}(\mathbb{R})$, we say that $u(x)$ is α-symmetric if*

$$u = \mathcal{R}L_\alpha[u]. \tag{3.2.29}$$

A similar definition works for the matrix potential $Q(x)$.

We will sometimes refer to this α-symmetric condition as the *folding condition*. Assume that a function $u(x) \in \mathcal{S}(\mathbb{R})$ is α-symmetric. Set

$$q(x) = u(x), \quad x \geq 0.$$

Thus, we have $\mathcal{R}L_\alpha^+[q](x) = \mathcal{R}L_\alpha^+[u](x) = u(x)$. The last equality is possible because $u(x)$ is α-symmetric. Therefore, one has

$$q^{ext}(x) = u(x).$$

Condition (3.2.29) is the equivalent of (3.2.3). We gathered two interesting properties of an α-symmetric function in the following result.

Lemma 3.11. *Let $u(x)$ be an element of $\mathcal{S}(\mathbb{R})$. If $u(x)$ is α-symmetric, then*

$$\beta_- = \beta_+ \equiv \beta.$$

Moreover, the scattering coefficient $s_{22}(\lambda)$ does not vanish at $\lambda = i|\alpha|$.

<u>Proof:</u> Assume that $u(x)$ is α-symmetric. This implies $\widetilde{U}(x,\lambda) = -U(-x,-\lambda)$. As a result, we have

$$\widetilde{\Psi}(-x,-\lambda) = \Psi(x,\lambda)M(\lambda) \tag{3.2.30}$$

for some matrix $M(\lambda)$. Using the explicit value of $L(x,\lambda)$ at $x = 0$, we get

$$L(-x,-z)L(x,z) = M(z)M(-z),$$

where $M(z)M(-z) = -(\lambda^2 + \alpha^2)$. In turn, this means that $P(x) = \sigma_3 P(-x)\sigma_3$. Therefore, we obtain $\beta_+ = \beta_-$.

Let us assume towards contradiction that $s_{22}(i|\alpha|) = 0$. Consider the case $\beta = -|\alpha|$. Owing to relation (3.2.30), we have

$$\widetilde{\Psi}_\mp(-x,-\lambda) = \Psi_\pm(x,\lambda). \tag{3.2.31}$$

Combining this equation with relation (3.2.24), we get

$$\Psi_{\mp}^{(1)}(-x,-\lambda) = \frac{L(x,\lambda)}{\lambda+i\beta}\Psi_{\pm}^{(1)}(x,\lambda), \quad \Psi_{\mp}^{(2)}(-x,-\lambda) = -\frac{L(x,\lambda)}{\lambda-i\beta}\Psi_{\pm}^{(2)}(x,\lambda). \tag{3.2.32}$$

Since $i|\alpha|$ is a simple zero of $s_{22}(\lambda)$, we have $\Psi_{-}^{(1)}(x,i|\alpha|) = \gamma\Psi_{+}^{(2)}(x,i|\alpha|)$ for some non-zero constant γ. Therefore, one has

$$\begin{aligned}
\Psi_{+}^{(2)}(0,i|\alpha|) &= -i\sigma_2\Psi_{+}^{(1)}(0,-i|\alpha|)^*, \text{ from the second equation in (2.2.28) for } \kappa=-1,\\
&= -i\sigma_2\left[\frac{L(0,-i|\alpha|)}{-2i|\alpha|}\Psi_{-}^{(1)}(0,i|\alpha|)\right]^*, \text{ from the first equation in (3.2.32)},\\
&= -i\sigma_2\left[\frac{|\alpha|\sigma_3-\alpha}{2|\alpha|}\Psi_{-}^{(1)}(0,i|\alpha|)\right]^*\\
&= \begin{pmatrix} 0 & |\alpha|+\alpha \\ |\alpha|-\alpha & 0\end{pmatrix}\frac{\gamma^*}{2|\alpha|}\Psi_{+}^{(2)}(0,i|\alpha|)^*
\end{aligned}$$

Thus if $\alpha>0$, we should get $\Psi_{+}^{(2)}(0,i|\alpha|)=0$, which is a contradiction. We reach the same conclusion if $\alpha<0$. The case $\gamma=|\alpha|$ is done in a similar way using the second equation in (3.2.32). ■

Lemma 3.12. *The Bäcklund extension of a function $u(x)\in\mathcal{S}(\mathbb{R})$ is always α-symmetric.*

Proof: The proof follows from (3.2.18) and Lemma 3.5. ■

Lemma 3.13. *Let $u(x)$ be a continuously differentiable function on $\mathbb{R}$. If $u(x)$ is α-symmetric, then at $x=0$ it satisfies Robin BCs (3.1.12).*

Proof: The result follows from the second equation in (3.2.2). ■

There is an alternative proof to the above result based solely on the first equation in (3.2.2). This can be found in Deift & Park (2011).

Lemma 3.14. *Let $u(x)$ be an element of $\mathcal{C}^2(\mathbb{R}_+)$ and $u_x(0^+)+2\alpha u(0)=0$,*[1] *then the Bäcklund extension of u belongs to $\mathcal{C}^2(\mathbb{R})$.*

[1]Note that $u_x(0^+)$ stands for the right-side derivative with respect to x of $u(x)$ at $x=0$.

Proof: Assume that u is an element of $\mathcal{C}^2(\mathbb{R}_+)$ such that $u_x(0^+) + 2\alpha u(0) = 0$. For convenience, let us write this in a matrix form:

$$Q(x) \in \mathcal{C}^2(\mathbb{R}_+), \quad Q_x(0^+) + 2\alpha Q(0) = 0.$$

It follows from the definition that $L_\alpha^+[Q](x)$ belongs to $\mathcal{C}(\mathbb{R}_+)$. Thus, we have that $\mathcal{R}L_\alpha^+[Q](x)$ is in $\mathcal{C}(\mathbb{R}_-)$. As a consequence, we have that the Bäcklund extension $Q^{ext}(x)$ is $\mathcal{C}^2$ on $\mathbb{R}\backslash\{0\}$. We have the following straightforward calculation

$$\begin{aligned} Q^{ext}(0^-) - Q^{ext}(0^+) &= \mathcal{R}L_\alpha^+[Q](0) - Q(0) \\ &= Q(0) - i[\sigma_3, P(0)\sigma_3] - Q(0) = Q(0) - Q(0) = 0. \end{aligned}$$

This means that $Q^{ext}(x)$ is continuous at $x = 0$, that is $Q^{ext}(0^+) = Q^{ext}(0^-)$. We also have the following

$$\begin{aligned} Q_x^{ext}(0^+) - Q_x^{ext}(0^-) &= \widetilde{Q}_x(0^-) - Q_x(0^+) \\ &= -2Q_x(0^+) + i[\sigma_3, P_x(0^+)\sigma_3] \\ &= -2Q_x(0^+) + i[\sigma_3, -2i\alpha Q(0)\sigma_3] = -2\left(Q_x(0^+) + 2\alpha Q(0)\right) = 0. \end{aligned}$$

This means that the Bäcklund extension is also differentiable at $x = 0$. Thus, using similar calculations, one shows that $\partial_x^2 Q^{ext}(0^+) = \partial_x^2 Q^{ext}(0^-)$. This concludes the proof. ∎

Lemma 3.15. *Let $Q(x)$ be an element of $\mathcal{S}(\mathbb{R})$ and $\Psi(x,\lambda)$ a 2×2 invertible solution of (2.1.1). Then*

$$S(\lambda) = \lim_{x\to\infty} e^{-i\lambda\sigma_3 x}\Psi(-x,\lambda)\Psi^{-1}(x,\lambda)e^{-i\lambda\sigma_3 x}. \tag{3.2.33}$$

Proof: A proof can be found in (Deift & Park, 2011, Lemma 4.27). ∎

Proposition 3.16. *Let $u(x) \in \mathcal{S}(\mathbb{R})$ be a generic potential and p, s positive integers. Then $u(x)$ is α-symmetric if and only if the following symmetries hold*

$$s_{22}^*(-\lambda^*) = s_{22}(\lambda), \quad \lambda \in \mathbb{C}^+, \quad s_{21}(-\lambda) = \frac{\lambda - i\beta}{\lambda + i\beta} s_{21}(\lambda), \quad \lambda \in \mathbb{R}, \tag{3.2.34}$$

the zeros of $s_{22}(\lambda)$ *are composed of* p *pairs* $(\lambda_k, -\lambda_k^*)$, $k = 1, \ldots, p$ *and* s *self-symmetric zeros* $\lambda_k = i\omega_k \in i\mathbb{R}^+$, $k = 1, \ldots, s$*; their norming constants satisfy the symmetry relation*

$$\lambda_k \neq \pm i\beta \quad \text{and} \quad \gamma(\lambda_k)\gamma^*(-\lambda_k^*) = \frac{\lambda_k + i\beta}{\lambda_k - i\beta}, \; k = 1, \ldots, 2p+s, \quad \beta = (-1)^{2p+s}\alpha. \tag{3.2.35}$$

<u>Proof:</u> Assume that $u(x)$ is α-symmetric. It follows from (3.2.31) that $\widetilde{S}(\lambda) = S^{-1}(-\lambda)$. Thus in terms of its entries, we obtain $\widetilde{s}_{22}(\lambda) = s_{11}(-\lambda)$ and $\widetilde{s}_{21}(\lambda) = -s_{21}(-\lambda)$. Combining this with (3.2.20) and (3.2.21), we obtain the symmetries in (3.2.34). From the first relation (3.2.34), we see that if λ_k is a zero of $s_{22}(\lambda)$ then so is $-\lambda_k^*$. Taking into consideration the analytic continuation of the Jost solutions

$$\widetilde{\Psi}_+^{(1)}(-x, -\lambda) = \Psi_-^{(1)}(x, \lambda), \quad \widetilde{\Psi}_-^{(2)}(-x, -\lambda) = \Psi_+^{(2)}(x, \lambda), \quad \lambda \in \mathbb{C}^+. \tag{3.2.36}$$

Recall that if $\lambda_k \neq \pm i\beta$ is a zero of $s_{22}(\lambda)$, we have $\widetilde{\Psi}_-^{(1)}(x, \lambda_k) = \widetilde{\gamma}(\lambda_k)\widetilde{\Psi}_+^{(2)}(x, \lambda_k)$ and $\Psi_-^{(1)}(x, \lambda_k) = \gamma(\lambda_k)\Psi_+^{(2)}(x, \lambda_k)$, hence

$$\begin{aligned}
\widetilde{\Psi}_-^{(1)}(x, \lambda_k) &= \widetilde{\gamma}(\lambda_k)\widetilde{\Psi}_+^{(2)}(x, \lambda_k) \\
&= \widetilde{\gamma}(\lambda_k)\widetilde{\Psi}_-^{(2)}(-x, -\lambda_k), \text{ used the second equation (3.2.36)}, \\
&= \widetilde{\gamma}(\lambda_k)\left[-(i\sigma_2)\Psi_-^{(1)}(-x, -\lambda_k^*)^*\right], \text{ used the first equation in (2.2.28)}, \\
&= -\gamma(-\lambda_k^*)^*\widetilde{\gamma}(\lambda_k)\left[(i\sigma_2)\Psi_+^{(2)}(-x, -\lambda_k^*)^*\right] \\
&= -\gamma(-\lambda_k^*)^*\widetilde{\gamma}(\lambda_k)\Psi_+^{(1)}(-x, -\lambda_k) \text{ used the second equation in (2.2.28)}, \\
&= -\gamma(-\lambda_k^*)^*\widetilde{\gamma}(\lambda_k)\Psi_-^{(1)}(x, \lambda_k), \text{ used the first equation (3.2.36)}.
\end{aligned}$$

This means that $\widetilde{\gamma}(\lambda_k) = -1/\gamma(-\lambda_k^*)^*$. Combining this with (3.2.22), we obtain the symmetry in (3.2.35). Let us consider the eigenfunction $\Psi_0(x, \lambda)$ of the scattering problem (2.1.1) such that $\Psi_0(0, \lambda) = \mathbb{I}$. Owing to the α-symmetric property of the potential $u(x)$, we have

$$\widetilde{\Psi}_0(-x, -\lambda) = L(-x, -\lambda)\Psi_0(-x, -\lambda) = \Psi_0(x, \lambda)M(\lambda) \tag{3.2.37}$$

for some matrix $M(\lambda)$. Evaluating this equation at $x = 0 = \lambda$, one obtains $M(0) = i\alpha\mathbb{I}$. Hence, using (3.2.37), one gets

$$S(0) = \lim_{x \to \infty} \Psi_0(-x, 0)\Psi_0(x, 0)^{-1} = \lim_{x \to \infty} i\alpha \left(P(-x)\right)^{-1} = \frac{\alpha}{\beta}\mathbb{I}.$$

Evaluating relation (3.2.19) at $\lambda = 0$ and using $\widetilde{S}(\lambda) = S(-\lambda)^{-1}$, yields $S(0) = s_{22}(0)\mathbb{I}$ which means $s_{22}(0) = \frac{\alpha}{\beta}$. Computing $\lim\limits_{\substack{\lambda\to 0\\ \mathrm{Im}\lambda>0}} s_{22}(\lambda)$ from the formula (2.2.36) and using $|s_{21}(\lambda)| = |s_{21}(-\lambda)|$, one obtains $s_{22}(0) = \prod\limits_{k=1}^{2p+s} \dfrac{\lambda_k}{\lambda_k^*}$. Combined this with $s_{22}(\lambda) = s_{22}^*(-\lambda^*)$, one deduces $s_{22}(0) = (-1)^{2p+s}$. Hence, one has $\beta = (-1)^{2p+s}\alpha$. The rest of this proof follows ideas as the one in Deift & Park (2011). ■

We now summarise the nonlinear mirror image strategy, as proposed in Bikbaev & Tarasov (1991):

$$\begin{cases} iu_t + u_{xx} + 2|u|^2 u = 0 \quad \text{for } x \geq 0, \quad t \geq 0, \\ u(x,0) = u_0(x) \in \mathcal{S}(\mathbb{R}_+), \quad \text{(initial condition)}, \\ u_x(0,t) + 2\alpha u(0,t) = 0\,, \quad t \geq 0, \quad \alpha \in \mathbb{R}\backslash\{0\}, \quad \text{(Robin BCs)}. \end{cases} \tag{3.2.38}$$

Starting from the initial condition $u_0(x) \in \mathcal{S}(\mathbb{R}^+)$ satisfying $(u_0)_x + 2\alpha u_0 = 0$ at $x = 0$, construct its Bäcklund transformation $\widetilde{u}_0(x) = L_\alpha^+[u_0](x)$ and introduce an extension $u_0^{ext}(x)$ to the full-line as in (3.2.28). Then

(i) It follows from Lemmas 3.5 and 3.12 that $u_0^{ext}(x)$ satisfies condition (3.2.29) and Robin boundary condition upon setting $\beta = (-1)^{2p+s}\alpha$;

(ii) The extension $u_0^{ext}(x)$ provides a valid initial condition to implement the inverse scattering method on $\mathbb{R}$ in order to obtain the solution $u^{ext}(x,t)$ for $t \geq 0$. The compatibility of symmetries (3.2.34)-(3.2.35) with the time evolution $s_{22}(t,\lambda) = s_{22}(\lambda)$, $s_{21}(t,\lambda) = s_{21}(\lambda)e^{2i\lambda^2 t}$ and $\gamma(t,\lambda_k) = \gamma(\lambda_k)e^{2i\lambda_k^2 t}$ known from IST, ensures that the condition $\widetilde{u}^{ext}(x,t) = -u^{ext}(-x,t)$ now holds for all $t \geq 0$. As a consequence, so does the boundary condition $u_x^{ext}(0,t) + 2\alpha u^{ext}(0,t) = 0$ for all $t \geq 0$. The desired solution of the IBVP for the NLS equation with Robin BCs is simply obtained by taking $u(x,t) = u^{ext}(x,t)|_{\mathbb{R}_+}$.

3.2.3 Fokas approach to integrable BCs

A detailed discussion of the results we present here can be found in Fokas (2008). For convenience, we will use similar notations as in Fokas (2008).

Let $u_0(x)$ be an element of $\mathcal{S}(\mathbb{R}_+)$. Assume that we have two smooth functions $g_0(t)$ and $g_1(t)$. Set

$$Q_0(x) = \begin{pmatrix} 0 & u_0(x) \\ \kappa u_0(x)^* & 0 \end{pmatrix},$$

$$\overline{Q}_0(t,\lambda) = 2\lambda \begin{pmatrix} 0 & g_0(t) \\ \kappa g_0(t)^* & 0 \end{pmatrix} - i \begin{pmatrix} 0 & g_1(t) \\ \kappa g_1(t)^* & 0 \end{pmatrix} \sigma_3 - i\kappa |g_0(t)|^2 \sigma_3.$$

The following differential equations have unique solutions

$$\omega_x + 2i\lambda \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \omega = Q_0(x)\omega, \quad 0 < x < \infty, \quad \lambda \in \mathbb{C}_+, \quad \lim_{x\to\infty} \omega(x,\lambda) = e_2; \tag{3.2.39}$$

$$\phi_t + 4i\lambda^2 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \phi = \overline{Q}_0(t,\lambda)\phi, \quad t > 0, \quad \lambda \in \mathbb{C}, \quad \phi(0,\lambda) = e_2. \tag{3.2.40}$$

Let us denote by $\omega(x,\lambda)$ and $\phi(t,\lambda)$ the solutions to the equations (3.2.39) and (3.2.40), respectively. Let T be a finite time, the maps

$$\{u_0(x)\} \mapsto \{a(\lambda), b(\lambda)\}, \quad \{g_0(t), g_1(t)\} \mapsto \{A(\lambda), B(\lambda)\}, \tag{3.2.41}$$

defined as

$$\begin{pmatrix} b(\lambda) \\ a(\lambda) \end{pmatrix} = \omega(0,\lambda), \quad \begin{pmatrix} B(\lambda) \\ A(\lambda) \end{pmatrix} = \phi(T,\lambda),$$

are *invertible*. We refer to $a(\lambda), b(\lambda), A(\lambda)$ and $B(\lambda)$ as *scattering coefficients*. It is important to mention that to have an effective invertibility of the above maps one needs additional conditions on the scattering functions. For the first map, when $\kappa = -1$, one needs to assume that the scattering coefficient $a(\lambda)$ has N_1 simple zeros, denoted λ_j for $j = 1, \ldots, N_1$, in the second quadrant of the λ-complex plane. Note that $a(\lambda)$ can also have a finite number of simple zeros in the first quadrant, but they are not necessary to define effectively the inverse map. Regarding the second map, introduce the following function

$$d(\lambda) = a(\lambda)A(\lambda^*)^* - \kappa b(\lambda)B(\lambda^*)^*.$$

When $\kappa = -1$, one needs to assume that $d(\lambda)$ has N_2 simple zeros, that we denote k_j for $j = 1, \ldots, N_2$, in the second quadrant of the same complex plane.

Moreover, we assume that the zeros of $a(\lambda)$ and $d(\lambda)$ do not coincide. To connect with the theory in Chapter 2, we want the initial condition $u_0(x)$ and the smooth functions $g_0(t), g_1(t)$ to be generic. As in the case of the IST method, when $\kappa = 1$, the scattering coefficient $a(\lambda)$ cannot admit zeros for the same reason. However, the situation is a bit different for $d(\lambda)$, thus we will require that in this case, $d(\lambda)$ should not have any zeros.

The scattering coefficients have the following properties

- $a(\lambda)$ and $b(\lambda)$ are well-defined and analytic in $\mathbb{C}^+$ and continuous and bounded $\mathbb{C}^+ \cup \mathbb{R}$. Moreover,

$$a(\lambda) = 1 + \mathcal{O}(\lambda^{-1}), \quad b(\lambda) = \mathcal{O}(\lambda^{-1}), \quad \lambda \to \infty.$$

- $A(\lambda)$ and $B(\lambda)$ are entire functions that are bounded in the interior of the first and third quadrants. We also have

$$A(\lambda) = 1 + \mathcal{O}(\lambda^{-1}), \quad B(\lambda) = \mathcal{O}(\lambda^{-1}), \quad \lambda \to \infty.$$

At this point, we can draw a comparison between the unified transform and the IST method that we discussed in detail in Chapter 2. The first map in (3.2.41) can be seen as the direct map in the case of the IST method on the full line: we have mapped the initial condition to a set of scattering data. So, in the "spectral space", the information about the initial condition is encoded in the spectral functions $a(\lambda)$ and $b(\lambda)$. However, the second map in (3.2.41) is completely different when compared to the IST method on the full line. This is normal because the unified transform is tackling IBVP on the half-line, therefore one needs to deal with what is happening at the boundary. In this case, $A(\lambda)$ and $B(\lambda)$, carry the information related to the boundary conditions in the spectral space.

We can use this information to construct the solution to the NLS equation (1.1.1) on the half-line with suitable initial and boundary conditions. Given $u_0(x)$ in $\mathcal{S}(\mathbb{R}_+)$, suppose that there exist smooth functions $g_0(t)$ and $g_1(t)$ such that $g_0(0) = u_0(0) = g_1(0)$. Define the maps in (3.2.41). Then the so-called *global relation*

$$a(\lambda)B(\lambda) - b(\lambda)A(\lambda) = 0, \quad \arg\lambda \in [0, \pi] \tag{3.2.42}$$

is satisfied if and only if

$$u(x,t) = 2i \lim_{\lambda\to\infty} \lambda \left(m(x,t,\lambda)\right)_{12}$$

solves the NLS equation (1.1.1) with

$$u(x,0) = u_0(x), \quad u(0,t) = g_0(t), \quad u_x(0,t) = g_1(t),$$

where $m(x,t,\lambda)$ is the 2×2 solution to the following normalised RH problem

- **Analyticity.** $m(x,t,\lambda)$ is analytic in $\mathbb{C}\setminus\left(\mathbb{R}\cup i\mathbb{R}\cup\{\lambda_1,\ldots,\lambda_{N_1},k_1,\ldots,k_{N_2}\}\right)$;

- **Jump condition.** $m(x,t,\lambda)$ satisfies the *jump condition*

$$m_-(x,t,\lambda) = m_+(x,t,\lambda)v(x,t,\lambda), \quad \lambda\in\mathbb{R}\cup i\mathbb{R},$$

  with

$$m(x,t,\lambda) = \begin{cases} m_+(x,t,\lambda), & \arg\lambda\in(0,\frac{\pi}{2})\cup(\pi,\frac{3\pi}{2}),\\ m_-(x,t,\lambda), & \arg\lambda\in(\frac{\pi}{2},\pi)\cup(\frac{3\pi}{2},2\pi),\end{cases}$$

  where the *jump matrix* $v(x,t,\lambda)$ is given by $v_1,\ldots,v_4$ for $\arg\lambda = \frac{\pi}{2},\pi,\frac{3\pi}{2}$, and 2π, respectively, with

$$v_1 = \begin{pmatrix} 1 & 0\\ \Gamma(\lambda)e^{2i\theta(\lambda)} & 1\end{pmatrix}, \quad v_3 = \begin{pmatrix} 1 & \kappa\Gamma(\lambda^*)^* e^{-2i\theta(\lambda)}\\ 0 & 1\end{pmatrix},$$
$$v_4 = \begin{pmatrix} 1 & -\varrho(\lambda)e^{-2i\theta(\lambda)}\\ \kappa\varrho(\lambda)e^{2i\theta(\lambda)} & 1-\kappa|\varrho(\lambda)|^2\end{pmatrix},$$
$$v_2 = v_3v_4^{-1}v_1 = \begin{pmatrix} 1-\kappa|\varrho(\lambda)-\kappa\Gamma(\lambda)^*|^2 & \left(\varrho(\lambda)-\kappa\Gamma(\lambda)^*\right)e^{-2i\theta(\lambda)}\\ -\kappa\left(\varrho(\lambda)-\kappa\Gamma(\lambda)^*\right)e^{2i\theta(\lambda)} & 1\end{pmatrix},$$

  where

$$\varrho(\lambda) = \frac{b(\lambda)}{a(\lambda)^*}; \quad \lambda\in\mathbb{R}, \quad \Gamma(\lambda) = \frac{\kappa B(\lambda^*)^*}{a(\lambda)d(\lambda)}, \quad \lambda\in\mathbb{R}\cup i\mathbb{R}.$$

- **Residues.** $m(x,t,\lambda)$ has simple poles at zeros of $a(\lambda)$ and $d(\lambda)$ and the residues at those poles are given by

$$\underset{\lambda=\lambda_j}{\mathrm{Res}}\, m(x,t,\lambda) = \lim_{\lambda\to\lambda_j} m(x,t,\lambda)\begin{pmatrix} 0 & 0 \\ \frac{e^{2i\theta(\lambda_j)}}{a'(\lambda_j)b(\lambda_j)} & 0 \end{pmatrix},$$

$$\underset{\lambda=\lambda_j^*}{\mathrm{Res}}\, m(x,t,\lambda) = \lim_{\lambda\to\lambda_j^*} m(x,t,\lambda)\begin{pmatrix} 0 & \frac{\kappa e^{-2i\theta(\lambda_j)}}{(a'(\lambda_j)b(\lambda_j))^*} \\ 0 & 0 \end{pmatrix},$$

$$\underset{\lambda=k_j}{\mathrm{Res}}\, m(x,t,\lambda) = \lim_{\lambda\to k_j} m(x,t,\lambda)\begin{pmatrix} 0 & 0 \\ \frac{\kappa B(k_j^*)^* e^{2i\theta(k_j)}}{a(k_j)d'(k_j)} & 0 \end{pmatrix},$$

$$\underset{\lambda=k_j^*}{\mathrm{Res}}\, m(x,t,\lambda) = \lim_{\lambda\to k_j^*} m(x,t,\lambda)\begin{pmatrix} 0 & \frac{B(k_j^*)e^{-2i\theta(k_j)}}{(a(k_j)d'(k_j))^*} \\ 0 & 0 \end{pmatrix};$$

- **Normalisation.** $m(x,\lambda) = \mathbb{I} + \mathcal{O}(\lambda^{-1})$ as $\lambda \to \infty$.

Recall that the boundary conditions are encoded in the spectral functions $A(\lambda)$ and $B(\lambda)$, and the initial condition is linked with $a(\lambda)$ and $b(\lambda)$. From the above RH problem, we know that the solution to the NLS equation is expressed in terms of the spectral data, i.e. $a(\lambda), b(\lambda), A(\lambda)$ and $B(\lambda)$.

To effectively use the unified transform approach to solving IBVP for the NLS equation, one needs to use the global relation (3.2.42) to express the unknown boundary in terms of the given data. A classical example of this situation is the following: Consider an IBVP for the NLS equation (1.1.1) on the half-line with Dirichlet BC, i.e. we have $u(x,0) = u_0(x)$ and $u(0,t) = g_0(t)$. We cannot write down the solution because $g_1(t)$ (in the spectral space we have, $A(\lambda)$ and $B(\lambda)$) is unknown. However, In this case, we can use the global relation to express $A(\lambda)$ and $B(\lambda)$ in terms of $u_0(x)$ and $u(0,t)$. In other words, $u_x(0,t) = g_1(t)$ in terms of u_0 and $u(0,t)$. This is known as Dirichlet to Neumann map.

Note that it is not always possible to solve the global relation for $A(\lambda)$ and $B(\lambda)$ for any boundary conditions. However, the class of boundary conditions for which one can solve the global relation for the unknown data is known as *linearizable boundary conditions*. Dirichlet, Neumann and Robin boundary conditions are all linearizable. The general equation that generates these so-called linearizable boundary conditions is given by

$$V(0,t,\nu(\lambda))N(\lambda) - N(\lambda)V(0,t,\lambda) = 0, \tag{3.2.43}$$

where $\nu(\lambda)$ is a function of λ, $N(\lambda)$ is an unknown 2×2 matrix. In the case of the NLS equation (1.1.1), one has that $\nu(\lambda) = -\lambda$. Thus Fokas criterion (3.2.43) to obtain boundary conditions that preserve the Livouille integrability for the NLS equation coincides with the time-independent version of Sklyanin's condition (3.1.1).

We have seen that Sklyanin's condition characterises integrable boundary conditions that appear in the implementation of both the nonlinear mirror image and unified transform in the case of time-independent BCs. In Biondini, Fokas & Shepelsky (2014), it was shown that the nonlinear mirror image method that we described in Subsection 3.2.2 is equivalent to the unified transform presented above in the case of Robin boundary conditions (3.1.12).

We saw in Section 3.1 that the reflection matrix $K(t, \lambda)$ (3.1.14) generates interesting and complicated time-dependent BCs (3.1.16) within Sklyianin's framework. An interesting question that one might ask is: Can we apply the nonlinear mirror image method and the unified transform to solve IBVPs for the NLS equation (1.1.1) with BCs (3.1.16) on the half-line?

We have a partial answer to this question. Yes, we can apply the nonlinear mirror image method to solve IBVPs for the NLS equation with BCs (3.1.16) and ZBCs on the half-line. We describe this in the next chapter.

Chapter 4

New implementation of the nonlinear mirror image method

This chapter contains two sections. In Section 4.1, we implement the nonlinear mirror image to solve IBVPs for the NLS equation (1.1.1) with integrable time-dependent BCs (3.1.16) at $x = 0$ and zero boundary conditions at infinity. The difficulty arising from having such time-dependent boundary conditions at $x = 0$ is overcome by changing the viewpoint of the method and fixing the Bäcklund transformation at infinity. We present two classes of solutions. One is very similar to the case of Robin boundary conditions, see Subsection 3.2.2, and the second is a unique feature of the BCs (3.1.16). The content of this section appeared in Caudrelier, Crampe & Dibaya (2022). In Section 4.2, we present the nonlinear mirror image method to solve IBVPs for the NLS equation (1.1.1) with Robin BCs (3.1.12) and non-zero boundary conditions at infinity. This is an ongoing joint work with Dr Vincent Caudrelier. We have interesting partial results to discuss, but also some issues left to solve.

Note that in both sections, we restrict our analysis to the focusing NLS equation (1.1.1), that is $\kappa = -1$.

4.1 Nonlinear mirror image for time-dependent BCs

4.1.1 Bäcklund matrix normalised at ∞

In Subsection 3.2.2, we constructed the Bäcklund matrix L as the solution to the first equation in (3.2.2) at time $t = 0$. Since we wanted to generate Robin BCs, we took the value of L at $x = 0$ as equal to the time-independent reflection matrix $K(\lambda)$, given in (3.1.11). To ensure that the Bäcklund matrix L will indeed generate another solution to the NLS equation (1.1.1), it must evolve in time with respect to the second equation in (3.2.2).

When it comes to time-dependent BCs (3.1.16), the above procedure turns out to be difficult because the value of a Bäcklund matrix, say B, that generates the BCs (3.1.16) is no longer time-independent at $x = 0$; see (3.1.14)-(3.1.15). Therefore, we need a different strategy to go around this difficulty. The idea is to construct the Bäcklund matrix B as the product of two Bäcklund matrices of type (3.2.14) and to fix the boundary value of B at $x = \infty$, which is time-independent even in our case, as opposed to the value at $x = 0$. This discussion suggests that we need to first review the properties of Bäcklund matrix of type (3.2.14) in detail in a way that is tailored to our needs.

Let $u(x)$ be a complex-valued function defined on $\mathbb{R}$. The function $u(x)$ can also be defined on $\mathbb{R}^\pm$. Let $Q(x)$ be given as in Subsection 3.2.2. We will use similar notations and conventions as in Subsection 3.2.2.

Consider the following 2×2 ODE

$$\begin{cases} P_x = \frac{i\rho}{2}[\sigma_3, P] + (-Q + i[\sigma_3, P\sigma_3])P - QP, & \rho \in \mathbb{R}, \\ \lim\limits_{x \to +\infty} P(x) = \dfrac{i\gamma_+}{2}\mathbb{I}, & \gamma_+ \in \mathbb{R}\backslash\{0\}. \end{cases} \tag{4.1.1}$$

This ODE is the analogue of (3.2.6). In this case, we have included an extra parameter ρ to take into consideration both parameters that appear in the BCs (3.1.16), see (4.1.22) below.

Lemma 4.1. *Let $Q(x) \in \mathcal{S}(\mathbb{R})$, the ODE (4.1.1) has a (unique) solution $P(x)$ with the properties:*

(a) *$P(x)$ has a diagonal asymptotic at $-\infty$,*

$$\lim_{x\to-\infty} P(x) = \frac{i\gamma_-}{2}\mathbb{I}, \quad \gamma_-^2 = \gamma_+^2.$$

(b) $[\sigma_3, P(x)\sigma_3] \in \mathcal{S}(\mathbb{R})$.

The proof for this result uses similar ideas to the proofs of Lemmas 3.2 and 3.3. However, for the reader's convenience, we include it in Appendix B.1 and it contains the details we need concerning the various cases $\gamma_- = \pm\gamma_+$. We summarise these various cases here: Let $Q(x)$ be as in the above lemma and $S(\lambda) = (s_{ij}(\lambda))_{1\le i,j\le 2}$ be the scattering matrix associated to it:

a. If $\gamma_+ < 0$ and $\lambda_+ = -\frac{\rho+i\gamma_+}{2}$ is not a simple zero of $s_{22}(\lambda)$ then

$$\begin{cases} \gamma_- = \gamma_+\,, & \text{if } \mu_2 = 0 \text{ in (B.1.10)}\,, \\ \gamma_- = -\gamma_+\,, & \text{if } \mu_2 \neq 0 \text{ in (B.1.10)}\,. \end{cases}$$

The first case $\mu_2 = 0$ is rarely mentioned in the literature since it is rather "useless" from the point of view of the dressing method: it does not create a new zero for $s_{22}(\lambda)$, see formula (4.1.8) below. However, it does allow for the case $\gamma_- = \gamma_+$ when $\gamma_+ < 0$ and $\lambda_+ = -\frac{\rho+i\gamma_+}{2}$ is not a simple zero of $s_{22}(\lambda)$, a case that cannot be overlooked in the construction of the mirror image approach for the time-dependent BCs in Subsection 4.1.2.

b. If $\gamma_+ < 0$ and λ_+ is a simple zero of $s_{22}(\lambda)$ then $\gamma_- = -\gamma_+$.

c. If $\gamma_+ > 0$ and λ_+ is not a simple zero of $s_{11}(\lambda)$ then $\gamma_- = \gamma_+$.

d. If $\gamma_+ > 0$ and λ_+ is a simple zero of $s_{11}(\lambda)$ then $\gamma_- = -\gamma_+$.

Case a. shows a small subtlety related to our approach of fixing $P(x)$ by its limit at ∞. The freedom in μ_2 indicates that $P(x)$ is not uniquely determined when $\gamma_+ < 0$ and $u(x)$ is such that λ_+ is not a simple zero of $s_{22}(\lambda)$. In general, one would also need to specify whether $\mu_2 = 0$ or not. As we explained, if the goal was to create a soliton on a given background solution $u(x)$, one would naturally choose $\mu_2 \neq 0$. This freedom will not be a problem for the application of the Bäcklund transformation to the half-line problem. The additional symmetry coming from the folding of $u(x)$ will fix uniquely the structure of $s_{22}(\lambda)$ relative to whether $\gamma_- = \gamma_+$ or $\gamma_- = -\gamma_+$. With this in mind, we proceed with the fact that $P(x)$ can be constructed uniquely as in the above proposition (fixing μ_2 as required if we are in case a.).

Transformation of the potential Given a potential $Q(x)$ as above, we get $P(x)$ as the solution of the differential equation (4.1.1). Set

$$\widetilde{Q}(x) \equiv \begin{pmatrix} 0 & \widetilde{u}(x) \\ -\widetilde{u}^*(x) & 0 \end{pmatrix} = -Q(x) + i[\sigma_3, P(x)\sigma_3]. \tag{4.1.2}$$

It follows from Lemma 4.1 that $\widetilde{Q}(x)$ is also a Schwartz function. We define the following Bäcklund matrix

$$\mathcal{L}(x,\lambda) = \left(\lambda + \frac{\rho}{2}\right)\sigma_3 + P(x)\,, \quad \rho \in \mathbb{R}\,. \tag{4.1.3}$$

The differential equation satisfied by $P(x)$ is equivalent to $\mathcal{L}(x,\lambda)$ solving the expected differential equation

$$\mathcal{L}_x(x,\lambda) = \widetilde{U}(x,\lambda)\mathcal{L}(x,\lambda) - \mathcal{L}(x,\lambda)U(x,\lambda)\,, \tag{4.1.4}$$

where $\widetilde{U}(x,\lambda) = -i\lambda\sigma_3 + \widetilde{Q}(x)$. As, we saw in Subsection 3.2.2, the eigenfunction defined by

$$\widetilde{\Psi}(x,\lambda) := \mathcal{L}(x,\lambda)\Psi(x,\lambda)\,, \tag{4.1.5}$$

will solve $\widetilde{\Psi}_x(x,t) = \widetilde{U}(x,\lambda)\widetilde{\Psi}(x,\lambda)$ if $\Psi(x,\lambda)$ is a solution of the x-part of Lax pair.

As in Subsection 3.2.2, we introduce the following definition.

Definition 4.2. *The map*

$$\begin{array}{rccc} \mathcal{L}_{\rho,\gamma_+}: & \mathcal{S}(\mathbb{R}) & \longrightarrow & \mathcal{S}(\mathbb{R}) \\ & Q & \longmapsto & \widetilde{Q} = \mathcal{L}_{\rho,\gamma_+}[Q], \end{array}$$

is called the Bäcklund transformation (BT) of $Q(x)$ with respect to (ρ,γ_+). We will use the same terminology and notation at the level of the entries $\widetilde{u}(x)$ and $u(x)$.

The content of Lemma 4.1 can be restricted to the half-line $\mathbb{R}^+$ with $Q(x) \in \mathcal{S}(\mathbb{R}^+)$. This gives a matrix $\mathcal{L}(x,\lambda)$ as in (4.1.3) and defined on $\mathbb{R}^+$. We denote the corresponding map as $\mathcal{L}^+_{\rho,\gamma+} : Q \mapsto \widetilde{Q} = \mathcal{L}^+_{\rho,\gamma+}[Q]$. Similarly, we can restrict to $\mathbb{R}^-$, with $Q(x) \in \mathcal{S}(\mathbb{R}^-)$ but with the understanding that we could fix $P(x)$ at $-\infty$, that is $P(x) \to \frac{i\gamma}{2}\mathbb{I}$ as $x \to -\infty$, $\gamma \in \mathbb{R}\backslash\{0\}$. The corresponding Bäcklund transformation of $Q(x)$ $(x < 0)$ with respect to (ρ, γ) will be denoted by the map $\mathcal{L}^-_{\rho,\gamma} : Q \mapsto \widetilde{Q} = \mathcal{L}^-_{\rho,\gamma}[Q]$.

Lemma 4.3 (Deift & Park (2011))**.**

1. *If* $Q(x) \in \mathcal{S}(\mathbb{R})$ *then* $\mathcal{R}\mathcal{L}_{-\rho,\gamma-}\mathcal{R}\mathcal{L}_{\rho,\gamma+}[Q] = Q$,
2. *If* $Q(x) \in \mathcal{S}(\mathbb{R}^\pm)$ *then*

$$\mathcal{R}\mathcal{L}^\pm_{\rho,\gamma\pm}[Q](x) \in \mathcal{S}(\mathbb{R}^\mp),\ \mathcal{R}\mathcal{L}^-_{-\rho,\gamma-}\mathcal{R}\mathcal{L}^+_{\rho,\gamma+}[Q] = Q, \quad \mathcal{R}\mathcal{L}^+_{-\rho,\gamma+}\mathcal{R}\mathcal{L}^-_{\rho,\gamma-}[Q] = Q,$$

where $\mathcal{R}Q(x) \equiv -Q(-x)$.

Proof: The strategy of this proof is similar to the one given in Lemma 3.5. So, we highlight the main differences. Assume $Q(x) \in \mathcal{S}(\mathbb{R})$ and let $P(x)$ be the solution of (4.1.1). Define

$$Q_2(x) = -\widetilde{Q}(-x) = \mathcal{R}\mathcal{L}_{\rho,\gamma+}[Q](x), \quad P_2(x) = \sigma_3 P(-x)\sigma_3.$$

We have $Q(-x) = -\widetilde{Q}(-x) + i[\sigma_3, P(-x)\sigma_3] = -\left(-Q_2(x) + i[\sigma_3, P_2(x)\sigma_3]\right)$. The matrix $P(x)$ admits the symmetry $P(x) = -\sigma_3 P^\dagger(x)\sigma_3$. A direct calculation shows that

$$(P_2(x))_x = -\frac{i\rho}{2}[\sigma_3, P_2(x)] + \left(-Q_2(x) + i[\sigma_3, P_2(x)\sigma_3]\right)P_2(x) - P_2(x)Q_2(x)$$

Taking into consideration the fact that $\lim\limits_{x\to+\infty} P_2(x) = \lim\limits_{x\to-\infty} P(x) = \dfrac{i\gamma_-}{2}\mathbb{I}$, we conclude that

$$\widetilde{Q}_2(x) = -Q_2(x) + i[\sigma_3, P_2(x)\sigma_3] = -Q(-x)$$

which means $\mathcal{R}\mathcal{L}_{-\rho,\gamma-}[Q_2](x) = Q(x)$ and proves the first point of the Lemma. The second point is proven similarly. ∎

Corollary 4.4. *The maps $\mathcal{L}_{\rho,\gamma_+} : \mathcal{S}(\mathbb{R}) \to \mathcal{S}(\mathbb{R})$ and $\mathcal{L}^{\pm}_{\rho,\gamma_\pm} : \mathcal{S}(\mathbb{R}^\pm) \to \mathcal{S}(\mathbb{R}^\pm)$ are bijections.*

Proof: It follows from Lemma 4.3. ■

Note that if $Q(x) \in \mathcal{S}(\mathbb{R})$, then

$$\left(\mathcal{L}_{\rho,\gamma_+}[Q](x)\right)|_{\mathbb{R}^\pm} = L^{\pm}_{\rho,\gamma_\pm}\left[Q|_{\mathbb{R}^\pm}\right](x).$$

Scattering data transformation Here we review the relation between the scattering data associated to $u(x) \in \mathcal{S}(\mathbb{R})$ and the data associated to $\widetilde{u}(x) = \mathcal{L}_{\rho,\gamma_+}[u](x)$.

Lemma 4.5. *The relation between the scattering matrix $S(\lambda)$ associated to $u(x)$ and the scattering matrix $\widetilde{S}(\lambda)$ associated to $\widetilde{u}(x)$ reads*

$$\widetilde{S}(\lambda) = \left((2\lambda+\rho)\sigma_3 + i\gamma_- \mathbb{I}\right) S(\lambda) \left((2\lambda+\rho)\sigma_3 + i\gamma_+ \mathbb{I}\right)^{-1}. \tag{4.1.6}$$

Explicitly, for the scattering coefficients, one gets

$$\widetilde{s}_{21}(\lambda) = -\frac{2\lambda+\rho-i\gamma_-}{2\lambda+\rho+i\gamma_+} s_{21}(\lambda), \quad \lambda \in \mathbb{R}, \tag{4.1.7}$$

$$\widetilde{s}_{22}(\lambda) = \frac{2\lambda+\rho-i\gamma_-}{2\lambda+\rho-i\gamma_+} s_{22}(\lambda), \quad \lambda \in \mathbb{C}^+ \setminus \left\{\frac{-\rho+i\gamma_+}{2}\right\}, \text{ if } \gamma_+ > 0. \tag{4.1.8}$$

In the case, $\gamma_- = \gamma_+$, $\widetilde{u}(x)$ is a generic potential if $u(x)$ is also generic. Thus we have $\widetilde{\mathcal{Z}}_+ = \mathcal{Z}_+$. The norming constants are

$$\widetilde{\gamma}(\lambda_k) = -\frac{2\lambda_k+\rho-i\gamma_+}{2\lambda_k+\rho+i\gamma_-} \gamma(\lambda_k), \quad \lambda_k \in \mathcal{Z}_+. \tag{4.1.9}$$

Proof: The proof is similar to the one given in Lemma 3.7. ■

Remark 4.6. *For Robin BCs, the case $\gamma_- = -\gamma_+$ is not necessary since the folding condition will set $\gamma_- = \gamma_+$. We will see that in the case of time-dependent BCs (3.1.16), the folding condition will not rule out the case $\gamma_- = -\gamma_+$. It is this case that gives rise to an interesting class of soliton solutions; see Subsection 4.1.3.*

Proposition 4.7. *Let $u(x) \in \mathcal{S}(\mathbb{R})$ and $P(x)$ be a solution of (4.1.1). The properties of $P(x)$ imply that we have the following useful explicit representation of $\mathcal{L}$ in terms of u and $\widetilde{u} = \mathcal{L}_{\rho,\gamma_+}[u]$,*

$$\mathcal{L}(x,\lambda) = \left(\lambda + \frac{\rho}{2}\right)\sigma_3 + P(x)\,, \quad P(x) = \frac{i}{2}\begin{pmatrix} \epsilon(x)\sqrt{\gamma_+^2 - |\widetilde{u}+u|^2} & \widetilde{u}+u \\ -(\widetilde{u}+u)^* & \epsilon(x)\sqrt{\gamma_+^2 - |\widetilde{u}+u|^2} \end{pmatrix}, \tag{4.1.10}$$

where $\epsilon(x)$ is a sign function completely determined by γ_+ and u. By construction, we have

$$\gamma_+^2 - |\widetilde{u}(x) + u(x)|^2 \geq 0\,, \quad \forall x \in \mathbb{R}\,.$$

Finally,

$$\mathcal{L}^{-1}(x,\lambda) = \sigma_3 \frac{\left(\lambda + \frac{\rho}{2}\right)\sigma_3 - P(x)}{\left(\lambda + \frac{\rho}{2}\right)^2 + \frac{\gamma_+^2}{4}}\sigma_3. \tag{4.1.11}$$

Proof: This result was given e.g. in Caudrelier (2008) but we give here more details, especially on the function $\epsilon(x)$ which takes value ± 1. The starting point is the construction of $P(x)$ as in Appendix B.1, see (B.1.4), which gives

$$P(x) = \begin{pmatrix} C(x) & D(x) \\ D^*(x) & C(x) \end{pmatrix}, \quad C(x) = \frac{i\gamma_+}{2}\frac{|\xi_1(x)|^2 - |\xi_2(x)|^2}{|\xi_1(x)|^2 + |\xi_2(x)|^2}, \quad D(x) = -\frac{i\gamma_+}{2}\frac{\xi_2(x)^*\xi_1(x)}{|\xi_1(x)|^2 + |\xi_2(x)|^2}.$$

A direct calculation then gives that $\det P(x) = -\frac{\gamma_+^2}{4}\mathbb{I}$ so that $C(x)^2 = -\frac{\gamma_+^2}{4} + |D(x)|^2$. Since $C^*(x) = -C(x)$, we deduce $|C(x)|^2 = \frac{\gamma_+^2}{4} - |D(x)|^2 \geq 0$. Next, formula (4.1.2) gives $D(x) = \frac{i}{2}(u(x) + \widetilde{u}(x))$. Hence, we have

$$\gamma_+^2 - |u(x) + \widetilde{u}(x)|^2 \geq 0\,.$$

Combining everything, we have $C(x) = \frac{i\epsilon(x)}{2}\sqrt{\gamma_+^2 - |u(x) + \widetilde{u}(x)|^2}$, where $\epsilon(x)^2 = 1$ gives the sign in front of the square root. We can determine its value by comparing the two expressions for $C(x)$, yielding

$$\epsilon(x)|\gamma_+|\sqrt{1 - \frac{1}{\gamma_+^2}|u(x) + \widetilde{u}(x)|^2} = \gamma_+ \frac{|\xi_1(x)|^2 - |\xi_2(x)|^2}{|\xi_1(x)|^2 + |\xi_2(x)|^2}\,.$$

The sign of the expression on the LHS is of course $\epsilon(x)$ by construction. The sign of the expression on the RHS is the product of the sign of γ_+ and that of $|\xi_1(x)|^2 - |\xi_2(x)|^2$. The latter is completely determined by $u(x)$. Finally, (4.1.11) is a consequence of the fact that $(P(x)\sigma_3)^2 = -\frac{\gamma_+^2}{4}\mathbb{I}$ as is checked directly. ■

Time evolution As we mentioned in Section 3.2.2, a Bäcklund transformation is useful if it is compatible with the time evolution of the PDE of interest. Consider $Q(x,t) \in \mathcal{S}(\mathbb{R})$ subject to $U_t - V_x + [U,V] = 0$. For each $t \geq 0$, construct $P(x,t)$ as the solution of (4.1.1), and hence also the corresponding $\mathcal{L}(x,t,\lambda)$ which then satisfies (4.1.4). In line with Definition 4.2, define then the new potential $\widetilde{Q}(x,t) = -Q(x,t) + i[\sigma_3, P(x,t)\sigma_3]$, for each $t \geq 0$. Call it the Bäcklund transformation of $Q(x,t)$ via (ρ, γ_+) and write $\widetilde{Q}(x,t) = \mathcal{L}_{\rho,\gamma_+}[Q](x,t)$. Then, the following well known result shows that the new potential also satisfies NLS if and only if $\mathcal{L}$ satisfies the t-part of the gauge transformation equation. Specifically, as proved in Lemma 3.8, the following equivalence holds

$$\widetilde{U}_t - \widetilde{V}_x + [\widetilde{U}, \widetilde{V}] = 0 \Longleftrightarrow \mathcal{L}_t(x,t,\lambda) = \widetilde{V}(x,t,\lambda)\mathcal{L}(x,t,\lambda) - \mathcal{L}(x,t,\lambda)V(x,t,\lambda), \tag{4.1.12}$$

where $\widetilde{V}(x,t,\lambda)$ is given by replacing $Q(x,t)$ by $\widetilde{Q}(x,t)$ in (2.1.4). Now, we have obtained a Bäcklund matrix $\mathcal{L}(x,t,\lambda)$ that satisfies both equations in (3.2.2) with the appropriate $\widetilde{U}$ and $\widetilde{V}$. This means that it generates a Bäcklund transformation $\widetilde{u}(x,t) = \mathcal{L}_{\rho,\gamma_+}[u](x,t)$ that solves the NLS equation (1.1.1).

We now discuss the connection between $\mathcal{L}$ above and L constructed in Subsection 3.2.2. In the next result, we show that if we require the potential $u(x) \in \mathcal{S}(\mathbb{R})$ to satisfy relation (3.2.29) with $\widetilde{u}(x) = \mathcal{L}_{\rho,\gamma_+}[u](x)$, then the Bäcklund matrix $\mathcal{L}$ will have a diagonal form at $x = 0$ similar to (3.1.11), upon a correct choice of the parameter $\gamma_\pm$, and the parameter ρ must be zero. This means that it will also generate Robin BCs.

Lemma 4.8. *Let $Q(x) \in \mathcal{S}(\mathbb{R})$ be a generic potential. Let $\mathcal{L}(x,\lambda)$ be given as in (4.1.3) where $P(x)$ solves (4.1.1). If $\widetilde{Q}(x) = \mathcal{L}_{\rho,\gamma_+}[Q](x)$ is such that*

$$\widetilde{Q}(x) = -Q(-x) \tag{4.1.13}$$

holds, then

$$\gamma_- = \gamma_+ \equiv \gamma\,, \quad \rho = 0\,, \quad \text{and} \quad \mathcal{L}(0,\lambda) = \lambda\sigma_3 + \frac{i(-1)^N\gamma}{2}\mathbb{I}\,.$$

where N is the number of simple zeros for the scattering coefficient $s_{22}(\lambda)$.

<u>Proof:</u> If (4.1.13) is satisfied then $\widetilde{U}(-x,-\lambda) = -U(x,\lambda)$. The Jost solutions are then related by

$$\widetilde{\Psi}_\pm(-x,-\lambda) = \Psi_\mp(x,\lambda). \tag{4.1.14}$$

From Lemma 4.1, Eqs. (4.1.5) and (4.1.14), one then deduces

$$\Psi_\pm(x,\lambda)=\mathcal{L}(x,\lambda)^{-1}\Psi_\mp(-x,-\lambda)\left(\left(\lambda+\frac{\rho}{2}\right)\sigma_3+\frac{i\gamma_\pm}{2}\mathbb{I}\right),\quad \lambda\in\mathbb{R}.$$

It follows that

$$\begin{aligned}\Psi_+(x,\lambda)=&\mathcal{L}(x,\lambda)^{-1}\mathcal{L}(-x,-\lambda)^{-1}\Psi_+(x,\lambda)\Bigg(\left(-\lambda+\frac{\rho}{2}\right)\sigma_3\\&+\frac{i\gamma_-}{2}\mathbb{I}\Bigg)\left(\left(\lambda+\frac{\rho}{2}\right)\sigma_3+\frac{i\gamma_+}{2}\mathbb{I}\right),\end{aligned}$$

which we rewrite as

$$\begin{aligned}\Psi_+(x,\lambda)^{-1}\mathcal{L}(-x,-\lambda)\mathcal{L}(x,\lambda)\Psi_+(x,\lambda)=&\Bigg(\left(-\lambda+\frac{\rho}{2}\right)\sigma_3\\&+\frac{i\gamma_-}{2}\mathbb{I}\Bigg)\left(\left(\lambda+\frac{\rho}{2}\right)\sigma_3+\frac{i\gamma_+}{2}\mathbb{I}\right).\end{aligned}$$

In fact, the latter relation is a consequence of the more general property that

$$\Psi(x,\lambda)^{-1}\mathcal{L}(-x,-\lambda)\mathcal{L}(x,\lambda)\Psi(x,\lambda)$$

is independent of x for any fundamental solution of $\Psi_x=U\Psi$, as a direct calculation shows. In particular, we must have

$$\left(\left(-\lambda+\frac{\rho}{2}\right)\sigma_3+\frac{i\gamma_-}{2}\mathbb{I}\right)\left(\left(\lambda+\frac{\rho}{2}\right)\sigma_3+\frac{i\gamma_+}{2}\mathbb{I}\right)=\Psi_+(0,\lambda)^{-1}\mathcal{L}(0,-\lambda)\mathcal{L}(0,\lambda)\Psi_+(0,\lambda).$$

From (4.1.2), we have that $P(0)$ is diagonal when $\widetilde{Q}(x)+Q(-x)=0$ and from the properties of $P(x)$ established in Lemma 4.1, we see that $P(0)=\frac{i\nu}{2}\mathbb{I}$ with $\nu^2=\gamma_+^2$. Therefore, we obtain the condition

$$2i\lambda(\gamma_--\gamma_+)\sigma_3+i\rho(\gamma_-+\gamma_+)\sigma_3-\gamma_-\gamma_+\mathbb{I}=\Psi_+(0,\lambda)^{-1}(2i\rho\nu\sigma_3)\Psi_+(0,\lambda)-\nu^2\mathbb{I}.$$

Taking the trace and using $\nu^2=\gamma_+^2$ we deduce $\gamma_+=\gamma_-\equiv\gamma$. Another consequence of (4.1.14) is that $\widetilde{S}(\lambda)=S^{-1}(-\lambda)$ so in particular $\widetilde{s}_{21}(\lambda)=-s_{21}(-\lambda)$ and $\widetilde{s}_{22}(\lambda)=s_{22}^*(-\lambda)$. Combined with (4.1.7) and (4.1.8), we obtain $s_{21}(-\lambda)=\frac{2\lambda+\rho-i\gamma}{2\lambda+\rho+i\gamma}s_{21}(\lambda)$ and $s_{22}^*(-\lambda^*)=s_{22}(\lambda)$. The symmetry for $s_{21}(\lambda)$ is consistent if and only if $(\rho-i\gamma)^2=(\rho+i\gamma)^2$. Since $\gamma\neq0$ this implies $\rho=0$. For any fundamental solution $\Psi(x,\lambda)$, using the fact that $\widetilde{U}(-x,-\lambda)=-U(x,\lambda)$, we deduce $\widetilde{\Psi}(-x,-\lambda)=\Psi(x,\lambda)M(\lambda)$ for some matrix $M(\lambda)$. Evaluating the latter equation at $x = 0 = \lambda$ and using (4.1.5), one obtains $M(0) = \frac{i\nu}{2}$. Hence, using (3.2.33), one gets

$$S(0) = \lim_{x\to\infty} \Psi(-x,0)\Psi(x,0)^{-1} = \frac{\gamma}{\nu}.$$

Evaluating relation (4.1.6) at $\lambda = 0$ and using $\widetilde{S}(\lambda) = S(-\lambda)^{-1}$, one obtains

$$S(0) = s_{22}(0)\mathbb{I}$$

which means $s_{22}(0) = \frac{\gamma}{\nu}$. Computing $\lim\limits_{\substack{\lambda\to 0\\ \mathrm{Im}\lambda>0}} s_{22}(\lambda)$ from (2.2.36) and using $|s_{21}(\lambda)| = |s_{21}(-\lambda)|$, one obtains $s_{22}(0) = \prod\limits_{k=1}^{N} \frac{\lambda_k}{\lambda_k^*}$. Combined this with $s_{22}(\lambda) = s_{22}^*(-\lambda^*)$, one gets $s_{22}(0) = (-1)^N$. Hence, one has $\nu = (-1)^N\gamma$. This completes the proof.

■

This lemma shows that the Bäcklund matrix $\mathcal{L}(x,\lambda)$ we constructed in Lemma 4.1 has a diagonal value at $x = 0$ which is equal to $K(\lambda)$ upon setting $\gamma_+ = \gamma_- \equiv \gamma$ to $(-1)^N 2\alpha$. Therefore, in the case of Robin BCs (3.1.12), fixing the boundary value of the Bäcklund matrix at $x = 0$ or as $x \to \infty$ does not make any difference. One can obtain the characterisation of the folding condition (4.1.13) as in Proposition 3.16 and then apply the nonlinear mirror image algorithm to solve (3.2.38).

4.1.2 Bäcklund matrix for time-dependent BCs

In this section, we will use the Bäcklund matrix $\mathcal{L}$ constructed in Subsection 4.1.1 as the building block of the Bäcklund matrix, say B, that will allow us to implement the nonlinear mirror image method to solve IBVPs for the focusing NLS equation (1.1.1) with the time-dependent BCs (3.1.16). Recall that the time-dependent BCs (3.1.16) are equivalent to Sklyanin's equation for the reflection matrix $K(t,\lambda)$ given by (3.1.14)-(3.1.15). Owing to the connection established by (3.2.4), the Bäcklund matrix B must be defined such that under the conditions (3.2.3) we have

$$B(0,t,\lambda) = K(t,\lambda). \tag{4.1.15}$$

Fix $t = 0$. The construction of B follows the following three steps:

Step 1 Consider a potential $Q(x) \in \mathcal{S}(\mathbb{R})$. Let $\mathcal{L}_1(x,\lambda)$ be a Bäcklund matrix that produces $\widehat{Q}(x)$ from $Q(x)$, that is

$$Q(x) \xrightarrow{\mathcal{L}_1} \widehat{Q}(x).$$

We know that $\mathcal{L}_1(x,\lambda)$ must satisfy the following differential equation

$$\partial_x \mathcal{L}_1(x,\lambda) = \widehat{U}(x,\lambda)\mathcal{L}_1(x,\lambda) - \mathcal{L}_1(x,\lambda)U(x,\lambda),$$

where $\widehat{U}(x,\lambda) = -i\lambda\sigma_3 + \widehat{Q}(x)$.

Step 2 Starting from $\widehat{Q}(x)$ constructed in **Step 1**, we can consider the second Bäcklund matrix $\mathcal{L}_2(x,\lambda)$ yielding $-\widehat{Q}(-x)$ from $\widehat{Q}(x)$, that is

$$\widehat{Q}(x) \xrightarrow{\mathcal{L}_2} -\widehat{Q}(-x).$$

The Bäcklund matrix $\mathcal{L}_2$ is solution of

$$\partial_x \mathcal{L}_2(x,\lambda) = -\widehat{U}(-x,-\lambda)\mathcal{L}_2(x,\lambda) - \mathcal{L}_2(x,\lambda)\widehat{U}(x,\lambda)\,.$$

In this case, the Bäcklund matrix $\mathcal{L}_2$ is obtained explicitly as

$$\mathcal{L}_2(x,\lambda) = \widehat{\Psi}(-x,-\lambda)C_2(\lambda)\widehat{\Psi}^{-1}(x,\lambda)$$

for some matrix $C_2(\lambda)$ independent of x and where $\widehat{\Psi}(x,\lambda)$ is a solution of $\widehat{\Psi}_x(x,\lambda) = \widehat{U}(x,\lambda)\widehat{\Psi}(x,\lambda)$[1].

Step 3 Finally, we consider a third transformation $\mathcal{L}_3$ producing $\widetilde{Q}(x)$ from $-\widehat{Q}(-x)$, we write

$$-\widehat{Q}(-x) \xrightarrow{\mathcal{L}_3} \widetilde{Q}(x).$$

Again, the Bäcklund matrix is obtained as the solution of

$$\partial_x \mathcal{L}_3(x,\lambda) = \widetilde{U}(x,\lambda)\mathcal{L}_3(x,\lambda) + \mathcal{L}_3(x,\lambda)\widehat{U}(-x,-\lambda),$$

where $\widetilde{U}(x,\lambda) = -i\lambda\sigma_3 + \widetilde{Q}(x)$.

[1] The exact normalisation would come from fixing a boundary condition but is not relevant for our argument.

Note that in each step we considered Bäcklund matrix of the form (4.1.3). We choose the Bäcklund matrix B to be defined by

$$B(x,\lambda) := \mathcal{L}_3\mathcal{L}_2\mathcal{L}_1(x,\lambda).$$

It is clear that B produces $\widetilde{Q}$ from Q

$$Q(x) \xrightarrow{B} \widetilde{Q}(x),$$

and it satisfies the appropriate differential equation

$$B_x(x,\lambda) = \widetilde{U}(x,\lambda)B(x,\lambda) - B(x,\lambda)U(x,\lambda)\,. \tag{4.1.16}$$

We make two other assumptions. First, we assume that $\mathcal{L}_1 \equiv \mathcal{L}$. This means that, see Lemma 4.1,

$$\lim_{x\to\pm\infty} \mathcal{L}_1(x,\lambda) = \left(\lambda + \frac{\rho}{2}\right)\sigma_3 + \frac{i\gamma_\pm}{2}\mathbb{I}.$$

The second assumption is

$$\lim_{x\to\infty} B(x,\lambda) = h(\lambda)\big[(-2\lambda+\rho)\sigma_3 - i\gamma_-\big]\big[(2\lambda+\rho)\sigma_3 + i\gamma_+\big], \tag{4.1.17}$$

where

$$h(\lambda) = \frac{1}{(2\lambda-\rho)^2 + \gamma_+^2}.$$

Lemma 4.9. *Let $Q(x) \in \mathcal{S}(\mathbb{R})$. Then*

$$\widetilde{Q}(x) = -Q(-x) \tag{4.1.18}$$

holds if and only if $\widehat{Q}(x)$ is odd. Under this condition, we have

$$B(x,\lambda) = \mathcal{L}_1^{-1}(-x,-\lambda)\mathcal{L}_1(x,\lambda), \tag{4.1.19}$$

and

$$B(0,\lambda) = K(0,\lambda)$$

where $K(t,\lambda)$ is given by (3.1.14)-(3.1.15).

Proof: A direct calculation gives

$$\begin{aligned}\widetilde{U}(x,\lambda)+U(-x,-\lambda)=&\mathcal{L}_1^{-1}(-x,-\lambda)\Big(\partial_x\big(\mathcal{L}_1(-x,-\lambda)\mathcal{L}_3(x,\lambda)\big)\\&-\Big[\mathcal{L}_1(-x,-\lambda)\mathcal{L}_3(x,\lambda),\widehat{U}(-x,-\lambda)\Big]\Big)\mathcal{L}_3^{-1}(x,\lambda).\end{aligned}$$

Hence the folding condition $\widetilde{U}(x,\lambda)+U(-x,-\lambda)=0$ is equivalent to

$$\partial_x\big(\mathcal{L}_1(-x,-\lambda)\mathcal{L}_3(x,\lambda)\big)=\Big[\mathcal{L}_1(-x,-\lambda)\mathcal{L}_3(x,\lambda),\widehat{U}(-x,-\lambda)\Big]\,,$$

which in turn is equivalent to

$$\mathcal{L}_1(-x,-\lambda)\mathcal{L}_3(x,\lambda)=\widehat{\Psi}(-x,-\lambda)C_3(\lambda)\widehat{\Psi}^{-1}(-x,-\lambda)\,,$$

for some matrix $C_3(\lambda)$. Using this to eliminate $\mathcal{L}_3$ and recalling the expression of $\mathcal{L}_2$, we obtain

$$B(x,\lambda)=\mathcal{L}_1^{-1}(-x,-\lambda)\widehat{\Psi}(-x,-\lambda)C_3(\lambda)C_2(\lambda)\widehat{\Psi}^{-1}(x,\lambda)\mathcal{L}_1(x,\lambda).$$

Choosing for definiteness the Jost solution $\widehat{\Psi}_+(x,\lambda)$, we obtain that

$$\lim_{x\to\infty}\widehat{\Psi}_+(-x,-\lambda)C_3(\lambda)C_2(\lambda)\widehat{\Psi}_+^{-1}(x,\lambda)=\mathbb{I}\,,$$

which yields $C_3(\lambda)C_2(\lambda)=\widehat{S}^{-1}(-\lambda)$. Summarising and recalling that

$$\widehat{\Psi}_+(-x,-\lambda)\widehat{S}^{-1}(-\lambda)=\widehat{\Psi}_-(-x,-\lambda),$$

we have obtained that the folding symmetry is equivalent to

$$B(x,\lambda)=\mathcal{L}_1^{-1}(-x,-\lambda)\widehat{\Psi}_-(-x,-\lambda)\widehat{\Psi}_+^{-1}(x,\lambda)\mathcal{L}_1(x,\lambda)\,.$$

This suggests that of all the possible potentials $Q(x)$, those that are such that

$$\widehat{Q}(x)=-\widehat{Q}(-x)$$

will fulfil the desired conditions. Indeed, in that special case of (4.1.14), we have $\widehat{\Psi}_-(-x,-\lambda)\widehat{\Psi}_+^{-1}(x,\lambda)=\mathbb{I}$ and we have (4.1.19). It remains to check that such a B can satisfy (4.1.15) to ensure that it is the right candidate with the required properties. Using Proposition 4.7, which gives

$$\mathcal{L}_1(x,\lambda)=\left(\lambda+\frac{\rho}{2}\right)\sigma_3+\frac{i}{2}\begin{pmatrix}\epsilon_1(x)\sqrt{\gamma_+^2-|\widehat{u}+u|^2} & \widehat{u}+u\\ -(\widehat{u}+u)^* & \epsilon_1(x)\sqrt{\gamma_+^2-|\widehat{u}+u|^2}\end{pmatrix},$$

a direct calculation shows, recalling that we work under the assumption $\widehat{u}(x) = -\widehat{u}(-x)$,

$$\begin{aligned} B(0,\lambda) &= \mathcal{L}_1^{-1}(0,-\lambda)\mathcal{L}_1(0,\lambda) \\ &= \frac{1}{(2\lambda-\rho)^2+\gamma_+^2}\left(-4\lambda^2\mathbb{I} - 4i\lambda H + \gamma_+^2 + \rho^2\right), \end{aligned} \tag{4.1.20}$$

with

$$H = \begin{pmatrix} \epsilon_1(0)\sqrt{\gamma_+^2 - |u|^2(0)} & u(0) \\ u^*(0) & -\epsilon_1(0)\sqrt{\gamma_+^2 - |u|^2(0)} \end{pmatrix}. \tag{4.1.21}$$

This is the desired result, see (3.1.14), if we set $\rho = \beta$ and $\gamma_+^2 = \alpha^2$. ∎

Recall that condition (4.1.18) is the equivalent of the "α-symmetric" (3.2.29) property. The above lemma has shown that under the assumption that (4.1.18) holds, the Bäcklund transformation B is given as the product $\mathcal{L}_1^{-1}(-x,-\lambda)\mathcal{L}_1(x,\lambda)$ to realise the Bäcklund transformation $Q \mapsto \widetilde{Q}$ where $\mathcal{L}_1(x,\lambda)$ realises the map $Q \mapsto \widehat{Q} = \mathcal{L}_{1\rho,\gamma_+}[Q]$ with $\widehat{Q}$ being an odd function.

In the rest of this section, we fix

$$\rho = \beta\,, \quad \gamma_+ = \varepsilon\alpha\,, \quad \varepsilon = \pm 1\,. \tag{4.1.22}$$

The compatibility of this whole construction with the desired time evolution must be established. Since $B(x,\lambda)$ is constructed entirely on $\mathcal{L}_1(x,\lambda)$ (composed as in (4.1.19)), this step is ensured by the equivalence (4.1.12) and the setup explained before it. Specifically

Proposition 4.10. *For each $t \geq 0$, let $u(x,t) \in \mathcal{S}(\mathbb{R})$ be a given solution of the focusing NLS equation (1.1.1), and $\mathcal{L}_1(x,t,\lambda)$ be as in (4.1.3) with $P(x,t)$ constructed as in Lemma 4.1. Suppose $\widehat{u}(x,t) = \mathcal{L}_{1\rho,\gamma_+}[u](x,t)$ is an odd function in x. Then, with $\rho = \beta$, $\gamma_+ = \varepsilon\alpha$,*

$$B(0,t,\lambda) = K(t,\lambda)\,,$$

and

$$K_t(t,\lambda) = V(0,t,-\lambda)K(t,\lambda) - K(t,\lambda)V(0,t,\lambda)\,.$$

In other words, we have that $u(x,t)$ satisfies BCs (3.1.16) at $x = 0$.

Proof: The equivalence (4.1.12) applied to $B(x,t,\lambda)$, combined with the fact that $\widetilde{V}(x,t,\lambda) = V(x,t,-\lambda)$ (since $\widehat{u}(x,t) = -\widehat{u}(-x,t)$ is equivalent to $\widetilde{u}(x,t) = -u(-x,t)$), yields

$$B_t(0,t,\lambda) = V(0,t,-\lambda)B(0,t,\lambda) - B(0,t,\lambda)V(0,t,\lambda)\,.$$

It remains to show that $B(0,t,\lambda) = K(t,\lambda)$. This is exactly the same calculation leading to (4.1.20)-(4.1.21) but with the time dependence included. For completeness, here are the main steps. A direct calculation yields

$$\begin{aligned} B(x,t,\lambda) =& h(\lambda)\big[-4\lambda^2\mathbb{I} - 4\lambda\sigma_3\left(P(x,t)+P(-x,t)\right) \\ &+ 2\rho\sigma_3\left(P(x,t)-P(-x,t)\right) - 4\sigma_3P(x,t)\sigma_3P(-x,t) + \rho^2\big], \end{aligned} \tag{4.1.23}$$

so

$$B(0,t,\lambda) = h(\lambda)\left[-4\lambda^2 - 8\lambda\sigma_3P(0,t) - 4(\sigma_3P(0,t))^2 + \rho^2\right]\,.$$

For each $t \geq 0$, $P(x,t)$ has the properties given in Proposition 4.7 so, recalling that $\widehat{u}(x,t)$ is an odd function in x (so $\widehat{u}(0,t) = 0$), we get

$$\sigma_3P(0,t) = \frac{i}{2}\begin{pmatrix} \epsilon(0,t)\sqrt{\gamma_+^2 - |u|^2(0,t)} & u(0,t) \\ u^*(0,t) & -\epsilon(0,t)\sqrt{\gamma_+^2 - |u|^2(0,t)} \end{pmatrix}\,.$$

This also gives $(\sigma_3P(0,t))^2 = -\frac{\gamma_+^2}{4}\mathbb{I}$, completing the proof. ∎

We have constructed a Bäcklund matrix B that generates the BCs (3.1.16) at the origin. As we have done previously, our next task is to use this machinery to solve IBVPs for again the focusing NLS equation (1.1.1) with BCs (3.1.16) on the half-line. This brings up the concept of Bäcklund extension; see Definition 3.9. Let $u(x)$ be an element of $\mathcal{S}(\mathbb{R}_+)$. We define the Bäcklund extension of $u(x)$ by

$$u^{ext}(x) = \begin{cases} u(x)\,, & x \geq 0\,, \\ -\widetilde{u}(-x)\,, & x < 0\,. \end{cases} \tag{4.1.24}$$

By construction, this extension satisfies the equivalent of "α-symmetric" condition. The only technical point is its smoothness at $x = 0$. As in the Robin case, the BCs ensure continuity of the Bäcklund extension, its first derivative and its second derivative automatically. In the present case we have the additional results that all higher odd order derivative are also continuous. The continuity of the even ones could be ensured in principle by imposing higher order boundary conditions in the spirit of Appendix D in Bikbaev & Tarasov (1991). In fact, the recent work Zhang (2021) gives an account on such higher boundary conditions which are compatible with NLS. We do not elaborate further on this here and simply give the following partial result on this issue.

Lemma 4.11 (Smoothness of the Bäcklund extension)**.** *Let $Q(x)$ be an element of $\mathcal{S}(\mathbb{R}_+)$. The Bäcklund extension of $Q(x)$ is an element of $\mathcal{C}^2(\mathbb{R})$*[1] *and its odd derivatives to all orders are continuous.*

Proof: By definition, $Q^{ext}(x)$ is an element of $\mathcal{S}\left(\mathbb{R}\backslash\{0\}\right)$. We only need to check its smoothness properties at $x=0$. For convenience, let us write (4.1.23) for short as $B(x,\lambda)=h(\lambda)\left[-4\lambda^2+\lambda B_1(x)+B_2(x)\right]$, where $B_1(x)=-4(\sigma_3P(x)+\sigma_3P(-x))$ and $B_2(x)=2\rho(\sigma_3P(x)-\sigma_3P(-x))-4\sigma_3P(x)\sigma_3P(-x)-\rho^2$. From (4.1.16) it follows that

$$\widetilde{Q}(x)=Q+\frac{i}{4}[B_1,\sigma_3],\quad B_{1x}=i[B_2,\sigma_3]+\widetilde{Q}B_1-B_1Q,\quad B_{2x}=\widetilde{Q}B_2-B_2Q.$$

A direct calculation shows that $Q^{ext}(0^+)-Q^{ext}(0^-)=2Q(0)+2i[\sigma_3,\sigma_3P(0)]=0$, which means that $Q^{ext}(x)$ is continuous at $x=0$. Using the explicit expression of $B_1(x)$ above, we deduce that $\partial_{2n+1}B_1(0)=0$, for all positive integer n. Therefore, all $\partial_{2n+1}Q^{ext}(x)$ exist and are continuous at $x=0$. Finally,

$$\begin{aligned}
Q^{ext}_{xx}(0^+)-Q^{ext}_{xx}(0^-)=&Q_{xx}(0)+\widetilde{Q}_{xx}(0)\\
=&2Q_{xx}(0)+\frac{i}{2}[B_{1xx}(0),\sigma_3]\\
=&2\begin{pmatrix}0 & u_{xx}+(\alpha^2+\beta^2)u\\ -u^*_{xx}-(\alpha^2+\beta^2)u^* & 0\end{pmatrix}\\
&+2\begin{pmatrix}0 & -2\varepsilon(0)u_x\Lambda\\ 2\varepsilon(0)u^*_x\Lambda & 0\end{pmatrix}\\
=&0,
\end{aligned}$$

where $\Lambda=\sqrt{\alpha^2-|u|^2(0)}$, on account of the fact that we assume that u satisfies (3.1.17). ■

These smoothness properties are compatible with time evolution by construction.

[1] A matrix function belongs to $\mathcal{C}^2(\mathbb{R})$ if all its entries are also elements of $\mathcal{C}^2(\mathbb{R})$.

Characterisation of the folding condition In this sub-section, we will characterise the symmetry properties of the scattering data of a potential $Q(x)$ satisfying the condition we have just discussed, that is $\widehat{Q} = \mathcal{L}_{1\rho,\gamma_+}[Q]$ is an odd function. Unlike the Robin case, this condition does not impose $\gamma_+ = \gamma_-$ and we have to consider the two cases $\gamma_+ = \pm\gamma_-$. We proceed in several steps, concentrating on the continuous data first.

Proposition 4.12. *Let $Q(x) \in \mathcal{S}(\mathbb{R})$ be such that $\widehat{Q} = \mathcal{L}_{1\rho,\gamma_+}[Q]$ is an odd function. Then, its scattering data satisfies*

$$S^{-1}(-\lambda) = \mathcal{B}(\lambda)\, S(\lambda)\, \mathcal{B}(-\lambda)\,, \quad \mathcal{B}(\lambda) = \begin{pmatrix} \frac{2\lambda+\beta+i\gamma_-}{-2\lambda+\beta+i\gamma_+} & 0 \\ 0 & \frac{2\lambda+\beta-i\gamma_-}{-2\lambda+\beta-i\gamma_+} \end{pmatrix}. \tag{4.1.25}$$

Explicitly, if $\gamma_+ = \gamma_- = \varepsilon\alpha$, we get

$$s_{22}(-\lambda) = s_{22}^*(\lambda^*)\,, \quad s_{21}(-\lambda) = -\frac{2\lambda+\beta-i\varepsilon\alpha}{2\lambda+\beta+i\varepsilon\alpha}\,\frac{2\lambda-\beta-i\varepsilon\alpha}{2\lambda-\beta+i\varepsilon\alpha}\, s_{21}(\lambda)\,. \tag{4.1.26}$$

If $\gamma_+ = -\gamma_- = \varepsilon\alpha$, we get

$$s_{22}(-\lambda) = \frac{2\lambda+\beta-i\varepsilon\alpha}{2\lambda+\beta+i\varepsilon\alpha}\,\frac{2\lambda-\beta+i\varepsilon\alpha}{2\lambda-\beta-i\varepsilon\alpha}\, s_{22}^*(\lambda^*)\,, \quad s_{21}(-\lambda) = -s_{21}(\lambda)\,. \tag{4.1.27}$$

<u>Proof:</u> We have the following relation between Jost solutions

$$\begin{cases} \widehat{\Psi}_\pm(x,\lambda) = \mathcal{L}_1(x,\lambda)\Psi_\pm(x,\lambda)\,\mathcal{L}_{1\pm}^{-1}(\lambda), \\ \mathcal{L}_{1\pm}(\lambda) = \lim\limits_{x\to\pm\infty} \mathcal{L}_1(x,\lambda) = \left(\lambda + \frac{\rho}{2}\right)\sigma_3 + \frac{i\gamma_\pm}{2}\,\mathbb{I}. \end{cases} \tag{4.1.28}$$

It implies in particular that $\widehat{S}(\lambda) = \mathcal{L}_{1_-}(\lambda) S(\lambda) \mathcal{L}_{1_+}^{-1}(\lambda)$. Since $\widehat{U}(-x,-\lambda) = -\widehat{U}(x,\lambda)$, we also have

$$\widehat{\Psi}_\pm(-x,-\lambda) = \widehat{\Psi}_\mp(x,\lambda)\,. \tag{4.1.29}$$

This implies in particular that $\widehat{S}^{-1}(-\lambda) = \widehat{S}(\lambda)$. Combining these two results, and recalling that $\rho = \beta$ and $\gamma_+ = \varepsilon\alpha$, yields (4.1.25) with $\mathcal{B}(\lambda) = \mathcal{L}_{1_+}^{-1}(-\lambda)\mathcal{L}_{1_-}(\lambda)$ as desired. ∎

These symmetries have consequences for the possible discrete data. We gather the result in the following Proposition.

Proposition 4.13. *Let $Q(x) \in \mathcal{S}(\mathbb{R})$ be a generic potential such that $\widehat{Q} = \mathcal{L}_{1\rho,\gamma_+}[Q]$ is an odd function.*

1. *If $\gamma_+ = \gamma_- = \varepsilon\alpha$:*

   - *The zeros of $s_{22}(\lambda)$ are composed of p pairs $(\lambda_k, -\lambda_k^*)$, $k = 1,\dots,p$ and s self-symmetric zeros* Biondini & Bui (2012) *$\lambda_k = i\sigma_k \in i\mathbb{R}^+$, $k = 1,\dots,s$. The number s of self-symmetric zeros is necessarily even. Explicitly,*

$$s_{22}(\lambda) = \prod_{j=1}^{p} \frac{\lambda-\lambda_j}{\lambda-\lambda_j^*}\,\frac{\lambda+\lambda_j^*}{\lambda+\lambda_j}\,\prod_{k=1}^{s}\frac{\lambda-i\sigma_k}{\lambda+i\sigma_k}\; e^{-\frac{1}{2\pi i}\int_{\mathbb{R}}\frac{\log\left(1+|r(z)|^2\right)}{z-\lambda}\mathrm{d}z}. \tag{4.1.30}$$

   - *The related norming constants satisfy the symmetry relation*

$$\gamma^*(-\lambda_k^*)\gamma(\lambda_k) = -\frac{2\lambda_k-\beta+i\varepsilon\alpha}{2\lambda_k-\beta-i\varepsilon\alpha}\,\frac{2\lambda_k+\beta+i\varepsilon\alpha}{2\lambda_k+\beta-i\varepsilon\alpha}, \tag{4.1.31}$$

   $k = 1,\dots,2p+s$.

2. *If $\gamma_+ = -\gamma_- = \varepsilon\alpha$:*

   - *The zeros of $s_{22}(\lambda)$ include $\frac{-\beta+i\alpha}{2}$ when $\varepsilon = 1$ or $\frac{\beta+i\alpha}{2}$ when $\varepsilon = -1$, and p pairs $(\lambda_k, -\lambda_k^*)$, $k = 1,\dots,p$. There are no self-symmetric zeros. Explicitly,*

$$s_{22}(\lambda) = \begin{cases} \dfrac{2\lambda+\beta-i\alpha}{2\lambda+\beta+i\alpha}\displaystyle\prod_{j=1}^{p}\frac{\lambda-\lambda_j}{\lambda-\lambda_j^*}\,\frac{\lambda+\lambda_j^*}{\lambda+\lambda_j}\; e^{-\frac{1}{2\pi i}\int_{\mathbb{R}}\frac{\log\left(1+|r(z)|^2\right)}{z-\lambda}\mathrm{d}z}, & \varepsilon = 1, \\ \dfrac{2\lambda-\beta-i\alpha}{2\lambda-\beta+i\alpha}\displaystyle\prod_{j=1}^{p}\frac{\lambda-\lambda_j}{\lambda-\lambda_j^*}\,\frac{\lambda+\lambda_j^*}{\lambda+\lambda_j}\; e^{-\frac{1}{2\pi i}\int_{\mathbb{R}}\frac{\log\left(1+|r(z)|^2\right)}{z-\lambda}\mathrm{d}z}, & \varepsilon = -1, \end{cases} \tag{4.1.32}$$

   - *The related norming constants satisfy the symmetry relation*

$$\gamma^*(-\lambda_k^*)\gamma(\lambda_k) = -1\,, \quad k = 1,\dots,p\,. \tag{4.1.33}$$

Proof: We first derive the properties of the zeros of $s_{22}(\lambda)$ in each case and will provide proof for the relation of the norming constants at the end as it can be done in one go for both cases. In the case $\gamma_+ = \gamma_-$, relation (4.1.26) implies that if λ_k is a zero of $s_{22}(\lambda)$ then so is $-\lambda_k^*$ and the first claim follows. Note that none of these zeros can be equal to $(\beta+i\alpha)/2$ in view of the summary given after Lemma 4.1. Using (4.1.28) and (4.1.29), we obtain

$$\Psi_+(-x,-\lambda) = B(x,\lambda)\Psi_-(x,\lambda)\mathcal{B}^{-1}(\lambda) \tag{4.1.34}$$

where $B(x,\lambda)$ is given as in (4.1.19) and $\mathcal{B}(\lambda)$ as in (4.1.25). This implies $\Psi_-(0,0)S(0) = B(0,0)\Psi_-(0,0)\mathcal{B}^{-1}(0)$ that is, since $B(0,0) = \mathbb{I}$ and $\mathcal{B}(0) = \mathbb{I}$ when $\gamma_+ = \gamma_-$, $S(0) = \mathbb{I}$. Thus $s_{22}(0) = 1$. But from (4.1.30), we have $s_{22}(0) = (-1)^{2p+s}$ so s must be even.

In the case $\gamma_+ = -\gamma_-$, relation (4.1.27) is rephrased by introducing

$$A(\lambda) = \begin{cases} \dfrac{2\lambda+\beta+i\alpha}{2\lambda+\beta-i\alpha}\, s_{22}(\lambda)\,, & \varepsilon = 1\,, \\ \dfrac{2\lambda-\beta+i\alpha}{2\lambda-\beta-i\alpha}\, s_{22}(\lambda)\,, & \varepsilon = -1\,, \end{cases}$$

in terms of which it reads $A(-\lambda) = A^*(\lambda^*)$. This is the same relation as dealt with before so we deduce that the zeros of $s_{22}(\lambda)$ come in pairs or in singlets of purely imaginary numbers and (4.1.32) follows. We use again (4.1.34) to deduce

$$\Psi_-(0,0)S(0) = B(0,0)\Psi_-(0,0)\mathcal{B}^{-1}(0)$$

but this time, since $\gamma_+ = -\gamma_- = \varepsilon\alpha$, $\mathcal{B}^{-1}(0) = \begin{pmatrix} \frac{\beta+i\varepsilon\alpha}{\beta-i\varepsilon\alpha} & 0 \\ 0 & \frac{\beta-i\varepsilon\alpha}{\beta+i\varepsilon\alpha} \end{pmatrix}$. Therefore, $s_{22}(0) = \frac{\beta-i\varepsilon\alpha}{\beta+i\varepsilon\alpha}$. Comparing with (4.1.32) which gives $s_{22}(0) = (-1)^{2p+s}\frac{\beta-i\varepsilon\alpha}{\beta+i\varepsilon\alpha}$, we deduce again that s is even. In fact $s=0$ because (4.1.33) (whose proof is next) would implies $|\gamma(i\sigma_k)|^2 = -1$, a contradiction.

We turn to the proof of the relation on the norming constants for which we don't need to distinguish the cases. Eq. (4.1.34) gives the following relations between the column vectors of $\Psi_\pm$

$$\Psi_+^{(1)}(-x,-\lambda) = B(x,\lambda)\Psi_-^{(1)}(x,\lambda)\frac{-2\lambda+\rho+i\gamma_+}{2\lambda+\rho+i\gamma_-}\,, \tag{4.1.35}$$

$$\Psi_+^{(2)}(-x,-\lambda) = B(x,\lambda)\Psi_-^{(2)}(x,\lambda)\frac{-2\lambda+\rho-i\gamma_+}{2\lambda+\rho-i\gamma_-}\,. \tag{4.1.36}$$

Evaluating (4.1.35) at $\lambda = \lambda_k$ and recalling that $\Psi_-^{(1)}(x,\lambda_k) = \gamma(\lambda_k)\Psi_+^{(2)}(x,\lambda_k)$ we get

$$\Psi_+^{(1)}(-x,-\lambda_k) = \gamma(\lambda_k)B(x,\lambda_k)\Psi_+^{(2)}(x,\lambda_k)\frac{-2\lambda_k+\rho+i\gamma_+}{2\lambda_k+\rho+i\gamma_-}\,.$$

Now, evaluating (4.1.36) at $\lambda = -\lambda_k$ and using it to eliminate $\Psi_+^{(2)}(x,\lambda_k)$ yields

$$\Psi_+^{(1)}(-x,-\lambda_k) = \gamma(\lambda_k)\underbrace{B(x,\lambda_k)B(-x,-\lambda_k)}_{\mathbb{I}}\Psi_-^{(2)}(-x,-\lambda_k)\frac{2\lambda_k+\rho-i\gamma_+}{-2\lambda_k+\rho-i\gamma_-}\frac{-2\lambda_k+\rho+i\gamma_+}{2\lambda_k+\rho+i\gamma_-}\,.$$

Recalling the NLS symmetry (2.1.7) (when $\kappa = -1$) which gives

$$\Psi_-^{(1)}(x,\lambda) = i\sigma_2 \Psi_-^{(2)}(x,\lambda^*)^*, \quad \Psi_+^{(2)}(x,\lambda) = -i\sigma_2 \Psi_+^{(1)}(x,\lambda^*)^*, \quad \lambda \in \mathbb{C}^+.$$

Note that $\sigma_{-1} = -i\sigma_2$; see (2.1.6). These symmetries together with

$$\Psi_-^{(1)*}(-x,-\lambda_k^*) = \gamma^*(-\lambda_k^*)\Psi_+^{(2)*}(-x,-\lambda_k^*)$$

give

$$\Psi_+^{(2)*}(-x,-\lambda_k^*) = -\gamma(\lambda_k)\gamma^*(-\lambda_k^*)\frac{2\lambda_k+\rho-i\gamma_+}{-2\lambda_k+\rho-i\gamma_-}\frac{-2\lambda_k+\rho+i\gamma_+}{2\lambda_k+\rho+i\gamma_-}\Psi_+^{(2)*}(-x,-\lambda_k^*)\,.$$

The result follows by spelling out the two cases $\gamma_+ = \gamma_-$ or $\gamma_+ = -\gamma_-$ and substituting $\rho = \beta$ and $\gamma_+ = \varepsilon\alpha$. ■

The main novelty is the first fraction in (4.1.32) which shows the presence of a single, not purely imaginary, zero. In the absence of any other zero, that is $n = 0$, and assuming the pure soliton case that is the reflection coefficient $r(\lambda) = 0$, we will see in the next section that this term gives rise to an emitted or absorbed soliton at the boundary.

Remark 4.14. *In expression* (4.1.32)*, we could ask what happens for instance if one zero λ_k is equal to $(-\beta+i\alpha)/2$. A short calculation shows that $s_{22}(\lambda)$ would then contain the factor $\left(\frac{2\lambda+\beta-i\alpha}{2\lambda+\beta+i\alpha}\right)^2\frac{2\lambda-\beta-i\alpha}{2\lambda-\beta+i\alpha}$. In that case, $(-\beta+i\alpha)/2$ would be a double zero which takes us beyond our working hypothesis of generic potentials.*

We now discuss the converse of Propositions 4.12 and 4.13. The short argument in Bikbaev & Tarasov (1991) uses the known one-to-one correspondence in IST between a generic potential of the type we consider in this work and its scattering data. We can invoke the same result here and conclude that a potential u is such that $\widehat{u}$ is odd (equivalently $\widetilde{u}(x) = -u(-x)$) if and only if its scattering data satisfies the symmetries of Propositions 4.12 and 4.13. In the case $\gamma_+ = \gamma_-$, we also present in Appendix B.2 a direct (but long) proof along the lines of that given in Deift & Park (2011) which uses Riemann-Hilbert problems techniques. It illustrates the main differences between the present case and the Robin case detailed in Deift & Park (2011). In particular, the use of a two-step construction mimicking the construction of $B(x,\lambda)$ from $\mathcal{L}_1(x,\lambda)$ is detailed.

Proposition 4.15. *Let $Q(x) \in \mathcal{S}(\mathbb{R})$ be such that its scattering data satisfies the symmetries of Propositions 4.12 and 4.13. Then $\widetilde{u}(x) = -u(-x)$ holds.*

The result that the symmetries on the scattering data are compatible with the time evolution also holds by the same reasoning that B is the composition of two Bäcklund transformations constructed on $\mathcal{L}_1$.

4.1.3 Soliton solutions

Solutions and discussions in this section follow closely the ones in Caudrelier, Crampe & Dibaya (2022).

Absorption/emission of one soliton by/from the boundary. The simplest new solution that our results predict is the case where one soliton can disappear or appear at the boundary. To our knowledge, this is the first time that such an exact solution for such a phenomenon is computed. The following one-soliton solution, defined for $x, t \geq 0$,

$$u(x,t) = \alpha e^{i\phi}\, e^{i(\alpha^2-\beta^2)t+i\varepsilon\beta x}\, \mathrm{sech}(\alpha(x - x_0 - 2\varepsilon\beta t)\,, \quad \varepsilon = \pm 1\,, \tag{4.1.37}$$

satisfies the focusing NLS equation[1] and the boundary condition (3.1.16). The parameters ϕ and x_0 are the arbitrary phase and position shifts. We note that the velocity $\pm\beta$ and amplitude α are controlled by the boundary parameters. The sign ε in (4.1.37) corresponds to the sign in (4.1.22). For $x_0 > 0$ and $\varepsilon = -1$ in (4.1.37), the soliton disappears from the half-line $x > 0$ after a time $t \sim \frac{x_0}{2\beta}$. For $x_0 < 0$ and $\varepsilon = 1$ in (4.1.37), the soliton appears on the half-line $x > 0$ after a time $t \sim -\frac{x_0}{2\beta}$. As mentioned above, this solution is generated by the first fraction in (4.1.32). This is a new class of solutions allowed by the time-dependent boundary conditions (3.1.16). It breaks the intuition developed so far by the nonlinear mirror image method in that such a single soliton being emitted or absorbed has no mirror counterpart. The bulk density is defined by

$$N(t) = \int_0^\infty |u(x,t)|^2\, \mathrm{d}x\,. \tag{4.1.38}$$

[1]We encountered this on the full-line in Chapter 2 with generic parameters.

For the solution (4.1.37), a direct calculation yields

$$N(t) = \alpha + \alpha \tanh(\alpha(x_0 + 2\varepsilon\beta t))\,. \tag{4.1.39}$$

It is clear that this is a time-dependent quantity. Its time behaviour is consistent with the picture of the one soliton being emitted or absorbed by the boundary. By this we mean, that the value of $N(t)$ as $t \to \pm\infty$ interpolates between 2α (soliton present on the half-line) and 0 (soliton absent on the half-line).

Note that our formula (3.1.20) tells us that the combination[1]

$$\mathcal{I}_1 = I_1 - \mathcal{K}_1 = \int_0^\infty |u(x,t)|^2 \mathrm{d}x \pm \sqrt{\alpha^2 - |u(0,t)|^2} \tag{4.1.40}$$

will be conserved in time. Noticing that the $\pm$ sign in front of the square root is equal to[2] $-\mathrm{sign}(x_0 + 2\varepsilon\beta t)$ we have $\pm\sqrt{\alpha^2 - |u(0,t)|^2} = -\alpha\tanh(\alpha(x_0 + 2\varepsilon\beta t))$, hence the result.

Multisoliton solutions

A special case of the results presented above contains those in Grüner (2020) that were obtained by dressing. Specifically, the formulas for the position and phase shifts presented in Remark 2 of Grüner (2020) can be obtained from the formulas in Proposition 4.13, in the case $\gamma_+ = \gamma_-$, with some standard algebraic manipulations. Their explicit form in general is not crucial for our purposes. We will see an example below. In structure, these are the same as the ones originally given in Biondini & Hwang (2009) for the Robin case. The essential difference accounting for the presence of different boundary conditions is the appearance of the function $\frac{2\lambda+\beta-i\varepsilon\alpha}{2\lambda+\beta+i\varepsilon\alpha}\frac{2\lambda-\beta-i\varepsilon\alpha}{2\lambda-\beta+i\varepsilon\alpha}$ which replaces the function $\frac{2\lambda-i\alpha}{2\lambda+i\alpha}$ characteristic of the Robin case. However, the new case $\gamma_+ = -\gamma_-$ has not been seen before by the method of Grüner (2020).

Two solitons reflected. In the case $\gamma_+ = \gamma_-$, we can apply the results presented above to compute a two-soliton solution on the half-line being reflected by the boundary at $x = 0$. It suffices to use the four-soliton solution of NLS on

[1] Here and in the next formula, we have dropped irrelevant constants related to the normalisation of the matrix K.

[2] This can be found by checking the boundary condition (3.1.17) for solution (4.1.37).

the full line, recalled in Subsection 2.2.4, with the constraints given in Proposition 4.13: the discrete data satisfies $\lambda_3 = -\lambda_1^*$, $\lambda_4 = -\lambda_2^*$ and the associated norming constants are linked by, for $k = 1, 2$:

$$c(\lambda_{k+2})^* = \frac{-1}{c(\lambda_k)s_{22}'(\lambda_k)s_{22}'(-\lambda_k^*)^*}\,\frac{2\lambda_k - \beta + i\varepsilon\alpha}{2\lambda_k - \beta - i\varepsilon\alpha}\,\frac{2\lambda_k + \beta + i\varepsilon\alpha}{2\lambda_k + \beta - i\varepsilon\alpha}\;. \tag{4.1.41}$$

Fig. 4.1 shows two plots of two-soliton solutions for different choices of the parameters and both are reflected by the boundary. Of course, such pictures will

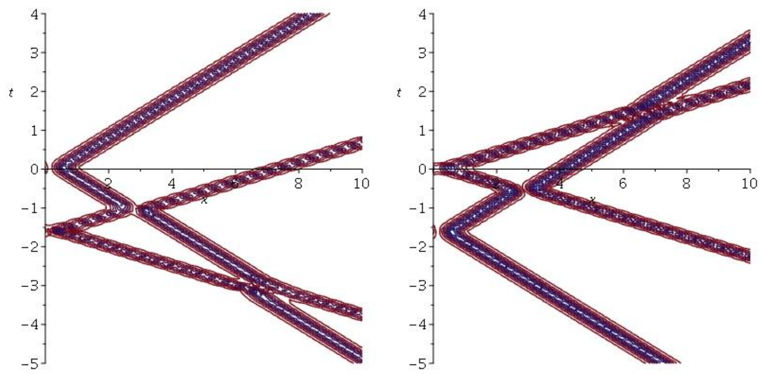

Figure 4.1: 2D-contour plots of $|u(x,t)|$ corresponding to two solitons reflected with time-dependent BCs (3.1.16) for $\alpha = 2$ and $\beta = 1$. The same zeros $\lambda_1 = 1+2i$ and $\lambda_2 = (1+5i)/2$ are used for both plots and with $\varepsilon = 1$ in (4.1.41). The norming constants are $c(\lambda_1) = -4e^{-20}$, $c(\lambda_2) = 5e^5$ on the left and $c(\lambda_1) = -4e^4$, $c(\lambda_2) = 5e^{-10}$ on the right.

look very familiar to the reader accustomed to solitons reflections in the Robin case. The point is to offer a visual appreciation of the integrability of the time-dependent case. Between the two plots, the only parameter that we changed is the position shift ξ_1, ξ_2 of each soliton. This is the analogue with a boundary of the well-known property that solitons undergo elastic collisions whose order is irrelevant to the final result of position and phase shifts. The plots in Fig. 4.1 are to be compared with the graphical representation of the (quantum) reflection equation in Fig. 4.2 which is well known in quantum integrable systems. The main reason to mention this is that, in the multicomponent case, soliton collisions among themselves and with a boundary are related to the set-theoretical Yang-Baxter and reflection equations and provide examples of Yang-Baxter and reflection maps, see Caudrelier & Zhang (2014) and references therein. The extension of such ideas to the present time-dependent BCs is an interesting open problem.

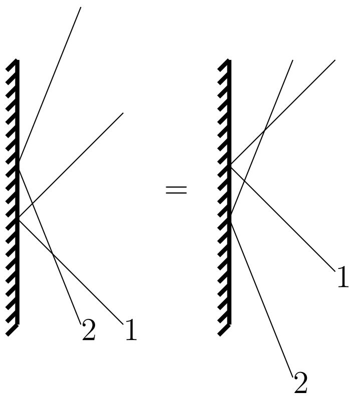

Figure 4.2: Line representation of the (quantum) reflection equation depicting a two-particle process being factorised into two possible successions of particle-particle interactions and particle-boundary interactions. The consistency of the two possibilities which must yield the same physical scattering matrix requires the reflection equation.

One soliton reflected and one absorbed. The previous result with two-soliton reflected on the half-line is similar to the ones obtained for the Robin boundary condition and corresponds to the case $\gamma_+ = \gamma_-$. We now turn to the new possibility offered by the time-dependent case that is $\gamma_+ = -\gamma_-$. A completely new type of solution is then possible, as mentioned previously: one soliton may be absorbed/emitted by the boundary. To illustrate this type of solution, in this paragraph, we focus on the case when one soliton is absorbed and one is reflected. This is obtained from a three-soliton solution of NLS on the full line, recalled in Subsection 2.2.4, with $s_{22}(\lambda)$ given by (4.1.32) with $p = 1$ (and $r(\lambda) = 0$). This means that two of the three zeros/norming constants are required to obey the symmetry relations of Proposition 4.13 part 2. The third zero, say λ_0 is the special zero involving the boundary parameters α and β: $\lambda_0 = -\beta/2 + i\alpha/2$ or $\lambda_0 = \beta/2 + i\alpha/2$. The sign of the real part determines whether the soliton travels towards or away from the boundary. The norming constants associated

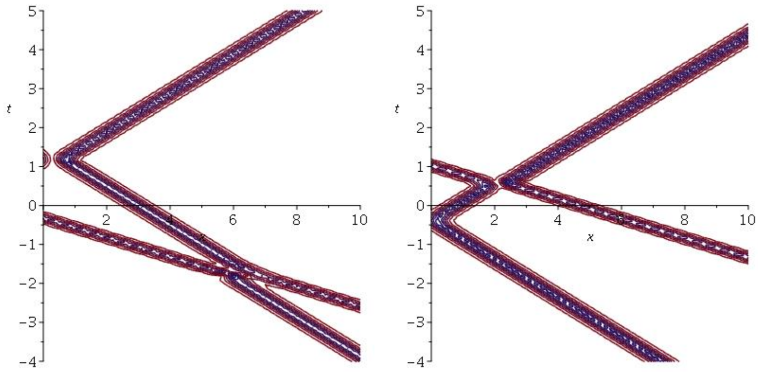

Figure 4.3: 2D-contour plots of $|u(x,t)|$ corresponding to two solitons, one reflected and one absorbed with time-dependent BCs (3.1.16) for $\alpha = 4$, $\beta = 2$. The same zeros $\lambda_0 = 1+2i$ and $\lambda_1 = (1+5i)/2$ are used for both plots. The norming constants are $c(\lambda_0) = 1,\ c(\lambda_1) = 5e^{15}$ on the left and $c(\lambda_0) = e^{20},\ c(\lambda_1) = 5$ on the right.

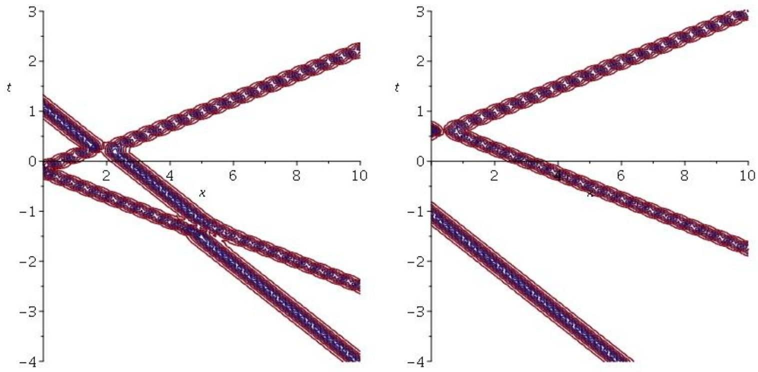

Figure 4.4: 2D-contour plots of $|u(x,t)|$ corresponding to two solitons, one reflected and one absorbed with time-dependent BCs (3.1.16) for $\alpha = 6$, $\beta = 1$. The same zeros $\lambda_0 = (1+6i)/2$ and $\lambda_1 = (2+5i)/2$ are used for both plots. The norming constants are $c(\lambda_0) = 4e^{16},\ c(\lambda_1) = 5$ on the left and $c(\lambda_0) = 4e^{-8},\ c(\lambda_1) = 5e^{15}$ on the right.

with the reflected soliton are linked by (see (4.1.33))

$$c(\lambda_2) = \frac{-1}{c(\lambda_1)^* \, s_{22}'(\lambda_1)^* \, s_{22}'(-\lambda_1^*)} \,, \tag{4.1.42}$$

whereas the norming constant $c(\lambda_0)$ is free.

Fig. 4.3 and Fig. 4.4 show such plots for different choices of parameters, to mimic the situation in Fig. 4.1, but with the essential difference that one of the two incoming solitons is absorbed by the boundary, while the other is reflected as before.

Fig. 4.5 shows the line representation of the equations underpinning the phenomenon of Fig. 4.3 and Fig. 4.4, in the same way as Fig. 4.2 does for Fig. 4.1. Somewhat intriguingly, if we interpret the absorption (or emission) of the single soliton as a type of transmission into the boundary (or to the mirror half-line), these equations correspond to (quantum) reflection-transmission equations, see e.g. Caudrelier (2005); Caudrelier *et al.* (2005) and references therein. This puzzling observation deserves further investigation beyond the scope of this work.

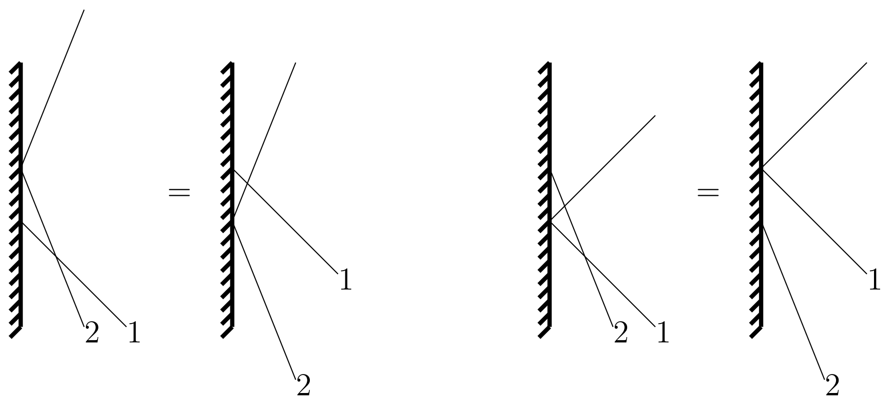

Figure 4.5: Line representation of two-soliton solutions when one is absorbed.

One soliton reflected and one emitted. A solution with one soliton reflected and one emitted can simply be obtained from the previous solution by changing the sign of the velocity of the absorbed soliton in the previous paragraph, that is, the sign of the real part of λ_0. Graphs for such solutions are easily obtained by reversing the time flow, $t \to -t$, in the figures of the previous paragraph.

4.2 Nonlinear mirror image method for Robin BCs with NZBCs

In this section, we discuss partial results obtained when applying the nonlinear mirror image method to solve IBVPs for the focusing NLS equation with Robin BCs (3.1.12) with non-zero boundary conditions at the infinity.

For some $a, b \in \mathbb{R}$ such that $a < b$, we define

$$\mathcal{P}_+ = [a, +\infty), \quad \mathcal{P}_- = (-\infty, b].$$

Consider a function $u(x)$ such that $u(x) \to u_\pm$ as $x \to \pm\infty$ with $|u_\pm| = q_0 \neq 0$. Recall that the uniformization variable z is defined by

$$z(\lambda) = \lambda + k(\lambda),$$

where $k(\lambda)$ is doubly-branched function given as

$$k^2 = \lambda^2 + q_0^2.$$

Consider the following ODE

$$\begin{cases} P_x = \big(Q + i[\sigma_3, P]\big)P - PQ, \\ P_0 \equiv P(0) = i\alpha\sigma_3, \quad \alpha \in \mathbb{R}\backslash[-q_0, q_0]. \end{cases} \tag{4.2.1}$$

Lemma 4.16. *If $u(x) - u_\pm$ belongs to $L^1(\mathcal{P}_\pm)$ then the ODE (4.2.1) has a unique solution.*

The proof for this result is similar to the one of Lemma 3.2. It leads to the following unique solution

$$P(x) = \frac{i\alpha}{|\xi_1(x)|^2 + |\xi_2(x)|^2} \begin{pmatrix} |\xi_1(x)|^2 - |\xi_2(x)|^2 & 2\xi_1(x)\xi_2^*(x) \\ 2\xi_1^*(x)\xi_2(x) & -(|\xi_1(x)|^2 - |\xi_2(x)|^2) \end{pmatrix},$$

where $(\xi_1(x), \xi_2(x))^T = \Psi^0(x, z(\lambda_0))$, with $\Psi^0(x, z)$ a vector-valued solution of (2.3.3) such that $\Psi^0(0, z) = e_1$.

Lemma 4.17. *If $u(x) - u_\pm$ belongs to $L^1(\mathcal{P}_\pm)$ and $P(x)$ is the solution of (4.2.1), then $P(x)$ has the following asymtotics*

$$P(x) \to P_\pm = i\big(r_\pm \mathbb{I} - Q_\pm\big)\sigma_3, \qquad \text{as } x \to \pm\infty,$$

such that $r_\pm^2 = \alpha^2 - q_0^2$.

The proof of this result is given in Appendix B.3.

As we saw in Subsections 3.2.2 and 4.1.1, the matrix function defined by

$$\mathbb{L}(x,z) = \lambda(z) + P(x), \tag{4.2.2}$$

solves

$$\mathbb{L}_x(x,z) = \widetilde{U}(x,z)\mathbb{L}(x,z) - \mathbb{L}(x,z)U(x,z)\,, \tag{4.2.3}$$

where $\widetilde{U}(x,z) = -i\lambda(z)\sigma_3 + \widetilde{Q}(x)$ with $\widetilde{Q}(x)$ given by

$$\begin{pmatrix} 0 & \widetilde{u}(x) \\ -\widetilde{u}^*(x) & 0 \end{pmatrix} \equiv \widetilde{Q}(x) := Q(x) + i[\sigma_3, P(x)]. \tag{4.2.4}$$

We also know that in this case, the eigenfunction $\widetilde{\Psi}(x,z)$ defined as

$$\widetilde{\Psi}(x,z) := \mathbb{L}(x,z)\Psi(x,z)\,, \tag{4.2.5}$$

will solve $\widetilde{\Psi}_x(x,t) = \widetilde{U}(x,z)\widetilde{\Psi}(x,z)$ if $\Psi(x,z)$ is a solution of (2.3.3).

The Bäcklund matrix $\mathbb{L}$ induces a BT on the potentials. We have the following definition.

Definition 4.18. *Let $Q(x)$ be complex-valued and defined on $\mathbb{R}$ as above. The map*

$$\mathbb{L}_\alpha : Q \longmapsto \widetilde{Q} = \mathbb{L}_\alpha[Q],$$

is called the Bäcklund transformation of $Q(x)$ with respect to α. If $Q(x)$ is defined on $\mathbb{R}_\pm$, the map $\mathbb{L}^\pm_\alpha : Q \mapsto \widetilde{Q} = \mathbb{L}^\pm_\alpha[Q]$ is called the Bäcklund transformation of $Q(x)$ with respect to α. We will use the same terminology and notation at the level of the entries $\widetilde{u}(x)$ and $u(x)$.

From the above lemma, it follows that the new potential $\widetilde{u}(x)$ has the following asymptotic behaviour

$$\widetilde{u}(x) \to -u_\pm, \qquad \text{as } x \to \pm\infty.$$

This means that the Bäcklund transformation $\widetilde{u}(x)$ belongs to the same functional space as the initial potential $u(x)$; we have

$$\widetilde{U}_\pm(z) = \sigma_3 U_\pm(z)\sigma_3.$$

Recall that $U_{\pm} = -i\lambda(z)\sigma_3 + Q_{\pm}$. Since these matrices have common eigenvalues, the appropriate domain to study solutions of $\widetilde{\Psi} = \widetilde{U}\widetilde{\Psi}$ is the same two-sheeted Riemann surface defined in Section 2.3. Note that the new eigenvector matrices are simply given by

$$\widetilde{E}_{\pm}(z) = \sigma_3 E_{\pm}(z)\sigma_3,$$

where $E_{\pm} = \mathbb{I} - \frac{i}{z}\sigma_3 Q_{\pm}$. Finally, the Jost solutions $\widetilde{\Psi}_{\pm}(x,z)$ based on $\widetilde{u}(x)$ are uniquely defined by

$$\lim_{x\to\pm\infty} \widetilde{\Psi}(x,z)e^{ik(z)\sigma_3 x} = \sigma_3 E_{\pm}(z)\sigma_3, \quad z \in \Sigma.$$

One can prove a similar result to Lemma 3.5. This will allow us to state that the map defined in the above definition is a bijection.

Set

$$\mathbb{L}_{\pm}(z) := \lim_{x\to\pm\infty} \mathbb{L}(x,z) = \lambda(z) + P_{\pm}.$$

Recall that if $u(x)$ is a generic potential, the scattering coefficient $s_{22}(\lambda)$ associated to it has a finite number of simple zeros in D^+. We denote the set of these simple zeros by $\mathcal{K}_+$. We will now discuss the relationship between the scattering data associated to $u(x)$ and the ones associated with the new potential $\widetilde{u}(x)$.

Lemma 4.19. *Consider $u(x)$ and $P(x)$ as in Lemma 4.16. Let $S(z)$ and $\widetilde{S}(z)$, $z \in \Sigma$, be the scattering matrices associated to $u(x)$ and its Bäcklund transformation $\widetilde{u}(x)$. Then*

$$\widetilde{S}(z) = M_-^{-1}(z)S(z)M_+(z), \quad z \in \Sigma, \tag{4.2.6}$$

where

$$M_{\pm}(z) = E_{\pm}^{-1}(z)\mathbb{L}_{\pm}^{-1}(z)\sigma_3 E_{\pm}(z) = 2z\left(z^2 + 2ir_{\pm}z\sigma_3 + q_0^2\right)^{-1}\sigma_3$$

Elementwise, we have

$$\widetilde{s}_{22}(z) = \frac{g_+(z^*)^*}{g_-(z^*)^*}s_{22}(z), \quad \widetilde{s}_{21}(z) = -\frac{g_+(z^*)^*}{g_-(z)}s_{21}(z), \quad z \in \Sigma, \tag{4.2.7}$$

where

$$g_{\pm}(z) = (z - \tau_{\pm})\left(z + q_0^2/\tau_{\pm}^*\right),$$

with $\tau_\pm = -i(\alpha + r_\pm)$. *In addition, if* $\tau_\pm$ *and* $-q_0^2/\tau_\pm^*$ *are not zeros of* s_{22}, *then* $\widetilde{u}(x)$ *is a generic potential if* $u(x)$ *is generic. Thus, we have* $\widetilde{\mathcal{K}}_+ = \mathcal{K}_+$. *The norming constants associated to zeros of* $\widetilde{s}_{22}(z)$ *are given by*

$$\widetilde{\gamma}(\xi_n) = -\frac{g_+(\xi_n^*)^*}{g_-(\xi_n)}\gamma(\xi_n), \tag{4.2.8}$$

where $\xi_n = z_n$ *and* $\xi_{n+N} = -q_0^2/z_n^*$ *for* $n = 1, \ldots, N$.

Proof: The proof is similar to the one given in Lemma 3.7. ■

For reference, note that

$$M_\pm(z) = \begin{pmatrix} \frac{2z}{g_\pm(z)} & 0 \\ 0 & \frac{2z}{-g_\pm(z^*)^*} \end{pmatrix}, \quad g_\pm(z) \equiv (z - \tau_\pm)\left(z + q_0^2/\tau_\pm^*\right) = z^2 + 2ir_\pm z + q_0^2.$$

A Bäcklund transformation is useful if it is compatible with the time evolution of the PDE of interest, which is the NLS equation in our case. We can repeat similar arguments as in Lemma 3.8 to show that the BT $\widetilde{u}(x,t)$ will indeed satisfy the NLS equation if and only if the Bäcklund matrix $\mathbb{L}$ evolves in time with respect to

$$\mathbb{L}_t = \widetilde{V}(x,t,z)\mathbb{L}(x,t,z) - \mathbb{L}(x,t,z)V(x,t,z),$$

where $\widetilde{V}$ is defined accordingly.

To make use of the above construction, we need to introduce the notion of "α-symmetry" or folding condition that will enable us to produce Robin BCs from the Bäcklund matrix $\mathbb{L}$ constructed above. We have:

Definition 4.20 (α-symmetric property)**.** *Let* $u(x)$ *be as in Lemma 4.16. We say that* $u(x)$ *is* α*-symmetric if*

$$\widetilde{u}(x) = u(-x). \tag{4.2.9}$$

Lemma 4.21. *Let* $u(x)$ *be as in Lemma 4.16. If* $u(x)$ *is* α*-symmetric then* $r = r_- = r_+$ *and* $u_- = -u_+$.

Proof: Assume that $u(x)$ is α-symmetric. This implies $\widetilde{U}(x,z) = -\sigma_3 U(-x,-z)\sigma_3$. As a result, we have

$$\widetilde{\Psi}(-x,-z) = \Psi(x,z)M(z) \tag{4.2.10}$$

for some matrix $M(z)$. We have

$$\mathbb{L}(-x,-z)\mathbb{L}(x,z) = M(z)M(-z),$$

where $M(z)M(-z) = -(\lambda(z)^2+\alpha^2)$. In turn, this means that $P(x) = \sigma_3 P(-x)\sigma_3$. Therefore,

$$P_- = \sigma_3 P_+ \sigma_3 \Longleftrightarrow r_- = r_+ \text{ and } u_- = -u_+.$$

■

Recall that $\tau_\pm = -i(\alpha + r_\pm)$ and $g_\pm(z) = (z-\tau_\pm)\left(z + u_0^2/\tau_\pm^*\right)$. As a consequence of the above lemma, we set

$$\tau \equiv \tau_- = \tau_+, \quad g(z) \equiv g_+(z) = g_-(z).$$

For reference, note that

$$g(-z_n^*)^* = g(z_n), \quad g(-z_n) = g(z_n^*)^*.$$

Lemma 4.22. *If $Q(x)$ is α-symmetric and continuously differentiable, then $Q(x)$ satisfies Robin boundary condition*

$$Q_x(0,t) + 2\alpha Q(0,t) = 0.$$

<u>Proof:</u> Note that $\widetilde{Q}(x) = -Q(x) + i[\sigma_3, P(x)\sigma_3]$. It follows that $\widetilde{Q}(0) = -Q(0)$.

$$\begin{aligned}\widetilde{Q}_x(0) - Q_x(0) &= -Q_x(0) + i[\sigma_3, P_x(0)\sigma_3] - Q_x(0)\\ &= -2Q_x(0) + i[\sigma_3, P_x(0)\sigma_3]\\ &= -2Q_x(0) + i[\sigma_3, -2i\alpha Q(0)\sigma_3]\\ &= -2\left(Q_x(0) + 2\alpha Q(0)\right)\end{aligned}$$

From $\widetilde{Q}(x) = -Q(-x)$, it follows that $\widetilde{Q}_x(0) = Q_x(0)$. Therefore, one has $Q_x(0) + 2\alpha Q(0) = 0$. ■

Theorem 4.23. *Let $u(x)$ be a generic potential such that $u(x) - u_\pm \in L^1(\mathcal{P}_\pm)$. If $u(x)$ is α-symmetric, then we have*

$$s_{22}(-z^*)^* = s_{22}(z), \quad s_{22}(u_0^2/z) = \tfrac{u_+}{u_-}s_{22}(z), \quad z \in D^+ \cup \Sigma,$$
$$s_{21}(-z) = \tfrac{g(z^*)^*}{g(z)}s_{21}(z), \quad z \in \Sigma.$$

The simple zeros of $s_{22}(\lambda)$ *are composed of* p *quartets* $(z_j, -z_j^*, -q_0^2/z_j^*, q_0^2/z_j)$, $j = 1, \dots, p$ *and* s *pairs of self-symmetric zeros* $i\omega_j, q_0^2/i\omega_j \in i\mathbb{R}$, $j = 1, \dots, s$; *their norming constants satisfy the symmetry relation*

$$\gamma(z_j)\gamma^*(-z_j^*) = \frac{g(z_j)}{g(z_j^*)^*}, \quad \gamma(z_j)\gamma(u_0^2/z_j) = -\frac{u_-^*}{u_+}\frac{g(z_j)}{g(z_j^*)^*}, \tag{4.2.11}$$

where $\xi_j \neq \tau, \xi_{j+2(2p+s)} \neq -\frac{q_0^2}{\tau^*}$ *for* $j = 1, \dots, 2p+s$.

For reference, we have

$$C(-z_j^*) = -\left[\frac{g(z_j)}{g(z_j^*)^*}\frac{1}{C(z_j)}\frac{1}{(s_{22}'(z_j))^2}\right]^*, \quad C(u_0^2/z_j) = \left(\frac{q_0^4}{z_j^2}\frac{1}{u_+^2}\right)\frac{g(z_j)}{g(z_j^*)^*}\frac{1}{C(z_j)}\frac{1}{(s_{22}'(z_j))^2}.$$

The proof of this result is provided in Appendix B.4.

What we have achieved. As we can see from the content of this section, we are able to do the following:

- To construct a Bäcklund matrix $\mathbb{L}(x, z)$ defined in (4.2.2) where $P(x)$ is the solution of (4.2.1).
- Under the folding condition (4.2.9):
  - To prove in Lemma 4.21 that the diagonal entries of $P_\pm$ coincide, that is $r_- = r_+$, and the off-diagonal entries are the same up to a sign, that is $u_- = -u_+$. If $u_\pm = 0$, i.e. we are in the case of ZBCs at infinity, the result in Lemma 4.21 coincides with the one in Lemma 4.8.
  - To prove in Lemma 4.22 that we actually obtain Robin BCs (3.1.12).
- To characterise the folding condition in terms of the scattering data; see Theorem 4.23.

Despite this achievement, we were unable to construct solutions on the half-line for the focusing NLS equation (1.1.1) that satisfy Robin BCs at $x = 0$ and non-zero boundary conditions at infinity.

What we can do next. We would like to investigate closely the role that the theta condition (2.3.42) can play in this construction.

Finally, we would like to mention that, in Tarasov (1991), the nonlinear mirror image for the defocusing NLS equation with Robin BCs and non-zero boundary at infinity was implemented. They found very similar symmetries to the ones we obtained in Theorem 4.23. However, there is almost no explication of the derivation of their symmetries or any explanation of the theta condition in their case.

Appendix A

Miscellaneous

In this appendix, we will introduce the general notions related to functional spaces we used in this work. The content of this appendix can be found in Brezis (2011) and Walter (1974).

A complex or real vector space X is called a *normed space* if to each of its elements, we can associate a positive real number, denoted $\|x\|_X$ and called the norm of x, such that the following properties hold:

(a) $\|x\|_X \geq 0$ for all x in X.

(b) $\|x\|_X = 0$ implies $x = 0$ if x belongs to X.

(c) $\|\alpha x\|_X = |\alpha| \|x\|_X$ if x belongs to X and α a real or complex number.

(d) $\|x + y\|_X \leq \|x\|_X + \|y\|_X$ for all x, y in X.

We say that $\| \cdot \|_X$ defines a seminorm on X if all the above properties hold but the second.

A complex vector space X is called an *inner product space* if to each pair of its elements x and y, we can associate a complex number, denoted $(x, y)_X$ and called *scalar or inner product* of x and y, such that the following properties hold:

(a) $(x, x)_X \geq 0$ for all x in X.

(b) $(x, x)_X = 0$ only if $x = 0$.

(c) $(\alpha x, y)_X = \alpha (x, y)_X$ if x, y belong to X and α a complex number.

(d) $(x+y,z)_X = (x,z)_X + (y,z)_X$ if x, y, and z belong to X.

(e) $(y,x)_X = (y,x)_X^*$.

Every inner product on a vector space X defines a norm on X, given by

$$\|x\|_X = \sqrt{(x,x)_X}.$$

A.1 L^p spaces

Let p be an element of $[1,\infty)$ and d a positive integer. Consider $\Omega \subseteq \mathbb{R}^d$. We denote by $L^p(\Omega)$ the space of functions $f : \Omega \subseteq \mathbb{R}^d \to \overline{\mathbb{R}}$ such that

$$\int_\Omega |f|^p \mathrm{d}x < \infty.$$ [1]

The following defines a norm on $L^p(\Omega)$

$$\|f\|_{L^p(\Omega)} = \left(\int_\Omega |f|^p\right)^{1/p}.$$

We also define

$$\|f\|_{L^\infty(\Omega)} = \text{ess sup}|f|,$$

where the *essential supremum* of a measurable function g is defined as the minimal $c \in [-\infty,\infty]$ such that $g(x) \le c$ a.e., that is,

$$\text{ess sup g} = \inf\{c \in [-\infty,\infty] : |\{x : g(x) > c\}| = 0\}.$$

A complex-valued function belongs to $L^p(\Omega)$ if its real and imaginary parts belong to $L^p(\Omega)$, for $1 \le p \le \infty$. Elements of $L^1(\Omega)$ are called *absolutely integrable functions on* Ω.

Theorem A.1. *Consider an integer $d \ge 1$. Let Ω be an open subset of $\mathbb{R}^d$. Suppose that $f : \Omega \times \mathbb{R}^d \to \mathbb{C}$ and that $f(x,\cdot) : \mathbb{R}^d \to \mathbb{C}$ is absolutely integrable for each $x \in \Omega$. Let $F(x) = \int_{\mathbb{R}^d} f(x,y)dy$.*

[1] In addition to this condition, these functions must measurable. The same remark applies to functions in L^∞.

(a) *Suppose that there exists $g \in L^1(\mathbb{R}^d)$ such that $|f(x,y)| \le g(y)$ for all x, y. If $\lim_{x\to x_0} f(x,y) = f(x_0,y)$ for every y, then*

$$\lim_{x\to x_0} F(x) = F(x_0).$$

In particular, if $f(\cdot, y)$ is continuous for each y, then F is continuous.

(b) *Suppose that $\partial_{x_j} f(x,y)$ exists for all x, y and that there exists $g \in L^1(\mathbb{R}^d)$ such that $|\partial_{x_j} f(x,y)| \le g(y)$ for all x. Then $\partial_{x_j} F(x)$ exists and is given by*

$$\partial_{x_j} F(x) = \int_{\mathbb{R}^d} \partial_{x_j} f(x,y)\, dy$$

Definition A.2. *Let $(X, (\cdot,\cdot))$ be an inner product space, and let $L : X \to X$ be a linear operator.*

- *The adjoint of L is the unique linear operator $L^* : X \to X$ that satisfies*

$$(Lf, g) = (f, L^* g)\,, \qquad \text{for all } f, g \in X.$$

- *The operator L is said to be self-adjoint if it is its own adjoint.*

Consider $\Omega \subseteq \mathbb{R}$. Let $f : \Omega \to \mathbb{C}^2$ be a function such that $f = (f_1, f_2)^T$. We say that $f \in L^2(\Omega, \mathbb{C}^2)$ if

$$\|f\|_2^2 := \int_\Omega \left(|f_1|^2 + |f_2|^2\right) \mathrm{d}x < \infty.$$

It is known that this space is a complex vector space. The following relation

$$(f,g)_2 := \int_\Omega \left(f_1(x) g_1^*(x) + f_2(x) g_2^*(x)\right) \mathrm{d}x,$$

defines an inner product space structure on $L^2(\Omega, \mathbb{C}^2)$.

Lemma A.3. *If $\kappa = 1$, then the linear operator $\mathcal{L}$ defined in (2.1.9) is self-adjoint on $L^2(\mathbb{R}, \mathbb{C}^2)$.*

Proof: A calculation yields the following

$$(\mathcal{L} f, g)_2 = i \int_{\mathbb{R}} \left((f_{1_x} - u f_2) g_1^* + (u^* f_1 - f_{2_x}) g_2^*\right) \mathrm{d}x.$$

The proof follows from simple integration by parts. ∎

A.2 Schwartz space

Let j and d be positive integers. We denote by $\mathcal{C}^j(\mathbb{R}^d)$ the space of complex-valued functions that are *j-times continuously differentiable* on $\mathbb{R}^d$. We denote by $\mathcal{C}^\infty(\mathbb{R}^d)$ the space of complex-valued functions that are *infinitely differentiable* on $\mathbb{R}^d$, and defined as

$$\mathcal{C}^\infty(\mathbb{R}^d) = \bigcap_{j=0}^{\infty} \mathcal{C}^j(\mathbb{R}^d).$$

If $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}^d$ and $x = (x_1, \ldots, x_d)$, we define

$$\partial_x^\alpha = \partial_{x_1}^{\alpha_1} \partial_{x_2}^{\alpha_2} \cdots \partial_{x_d}^{\alpha_d}, \quad x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_d^{\alpha_d}.$$

Define

$$\mathcal{S}(\mathbb{R}^d) = \left\{ f \in \mathcal{C}^\infty(\mathbb{R}^d) \mid \sup_{x \in \mathbb{R}^d} |x^\alpha \partial_x^\beta f(x)| < \infty, \, \forall \alpha, \beta \in \mathbb{N}^d \right\}.$$

$\mathcal{S}(\mathbb{R}^d)$ is called *Schwartz space* and its element are called *Schwartz functions*. They can be understood as functions whose derivatives (including the function itself) decay faster than any power of $|x|$ as $|x| \to \infty$. For this reason, elements of $\mathcal{S}(\mathbb{R}^d)$ are sometimes referred to as *rapidly decaying* functions. For all $\alpha, \beta \in \mathbb{N}^d$, the function $\| \cdot \|_{\alpha,\beta} : \mathcal{S}(\mathbb{R}^d) \to \mathbb{R}_+$ such that

$$\|f\|_{\alpha,\beta} := \sup_{x \in \mathbb{R}^d} |x^\alpha \partial_x^\beta f(x)|.$$

makes $\mathcal{S}(\mathbb{R}^d)$ a semi-normed space. The topology on $\mathcal{S}(\mathbb{R}^d)$ is generated by this family of semi-norms.

It is straightforward to deduce from the definition of $\mathcal{S}(\mathbb{R}^d)$ to deduce that its elements will satisfy the following: for all positive integer N and $\alpha \in \mathbb{N}^d$ there exists a constant $C >$ such that

$$|\partial_x^\alpha f(x)| \leq C\big(1 + |x|\big)^{-N}, \quad x \in \mathbb{R}^d.$$

This property can be used to prove that $\mathcal{S}(\mathbb{R}^d) \subset L^p(\mathbb{R}^d)$, for all $1 \leq p < \infty$. One can even prove that $\mathcal{S}(\mathbb{R}^d)$ is in fact a *dense subspace* of $L^p(\mathbb{R}^d)$, for $1 \leq p < \infty$.

Appendix B

Proofs of Chapter 4

B.1 Proof of Lemma 4.1

It is convenient to work with $P_1(x) = P(x)\sigma_3$ which satisfies

$$P_{1x} = \left[\frac{i\rho}{2}\sigma_3 - Q + i\sigma_3 P_1, P_1\right]. \tag{B.1.1}$$

We seek a solution of the form

$$P_1(x) = \varphi_1(x)\left(\frac{i\gamma_+}{2}\sigma_3\right)\varphi_1(x)^{-1}$$

for some invertible matrix $\varphi_1(x)$. Substituting the ansatz into (B.1.1) yields

$$\left[\varphi_{1x}(x)\varphi_1(x)^{-1}, \varphi_1(x)\left(\frac{i\gamma_+}{2}\sigma_3\right)\varphi_1(x)^{-1}\right] = \left[\frac{i\rho}{2}\sigma_3 - Q + \right.$$
$$\left. i\sigma_3\varphi_1(x)\left(\frac{i\gamma_+}{2}\sigma_3\right)\varphi_1(x)^{-1}, \varphi_1(x)\left(\frac{i\gamma_+}{2}\sigma_3\right)\varphi_1(x)^{-1}\right],$$

which means that

$$\varphi_{1x}(x)\varphi_1(x)^{-1} - \left(\frac{i\rho}{2}\sigma_3 - Q + i\sigma_3\varphi_1(x)\left(\frac{i\gamma_+}{2}\sigma_3\right)\varphi_1(x)^{-1}\right) = M(x)$$

with

$$\left[M(x), \varphi_1(x)\left(\frac{i\gamma_+}{2}\sigma_3\right)\varphi_1(x)^{-1}\right] = 0.$$

In turn, this implies that $\varphi_1(x)^{-1}M(x)\varphi_1(x)$ is a diagonal matrix, which we denote by $D(x)$. The matrix φ_1 is not uniquely defined and it is always possible to consider the transformation $\varphi_1 \mapsto \varphi_1 h$ where h is an invertible diagonal matrix without changing P_1. We use this freedom to choose h such that $h_x = -Dh$ and set $\varphi = \varphi_1 h$, with the conclusion that

$$P_1(x) = \varphi(x)\left(\frac{i\gamma_+}{2}\sigma_3\right)\varphi(x)^{-1}$$

where φ is a nonsingular (or fundamental) solution of

$$\varphi_x(x) = \left(\frac{i\rho}{2}\sigma_3 - Q\right)\varphi + i\sigma_3\varphi(x)\left(\frac{i\gamma_+}{2}\sigma_3\right). \tag{B.1.2}$$

Writing $\varphi = (\varphi_1, \varphi_2)$ where $\varphi_{1,2}$ are the column vectors of φ, we see that

$$\varphi_{1x}(x) = (-i\lambda_+\sigma_3 - Q)\varphi_1\,, \quad \varphi_{2x}(x) = (-i\lambda_+^*\sigma_3 - Q)\varphi_2\,, \quad \lambda_+ = -\frac{\rho + i\gamma_+}{2}\,.$$

We impose the standard conditions

$$\lim_{x\to+\infty}\varphi_1(x)e^{i\lambda_+ x} = e_1\,, \quad \lim_{x\to+\infty}\varphi_2(x)e^{-i\lambda_+^* x} = e_2\,. \tag{B.1.3}$$

Equation (B.1.2) is unchanged under the transformation $\varphi(x) \mapsto (i\sigma_2)\varphi^*(x)(i\sigma_2)^{-1}$ where σ_2 is the second Pauli matrix. Hence, in general there exists a nonsingular matrix C such that $\varphi(x) = (i\sigma_2)\varphi^*(x)(i\sigma_2)^{-1}C$. Taking into account (B.1.3), we find $C = \mathbb{I}$, so $\varphi_2(x) = -i\sigma_2\varphi_1^*(x)$, and hence the matrix $\varphi(x)$ can be written as

$$\varphi(x) = \begin{pmatrix} \xi_1(x) & -\xi_2(x)^* \\ \xi_2(x) & \xi_1(x)^* \end{pmatrix}.$$

Thus, $P_1(x)$ takes the following form

$$P_1(x) = \frac{i\gamma_+}{2(|\xi_1(x)|^2 + |\xi_2(x)|^2)}\begin{pmatrix} |\xi_1(x)|^2 - |\xi_2(x)|^2 & 2\xi_2(x)^*\xi_1(x) \\ 2\xi_1(x)^*\xi_2(x) & -\left(|\xi_1(x)|^2 - |\xi_2(x)|^2\right) \end{pmatrix}. \tag{B.1.4}$$

Therefore, it is enough to solve for φ_1 which we recall is a solution of the linear problem

$$\varphi_{1x} = (-i\lambda_+\sigma_3 - Q)\varphi_1\,, \tag{B.1.5}$$

with the condition $\lim\limits_{x\to+\infty}\varphi_1(x)e^{i\lambda_+ x} = e_1$. The rest of the construction of $P_1(x)$ and its properties hinges on the following important standard result, see e.g. (Coddington & Levinson, 1955, pp. 104-105). Equation (B.1.5) admits two fundamental solutions $\chi^\pm(x)$ satisfying

$$\lim_{x\to\pm\infty}\chi^\pm(x)e^{i\lambda_+ x\sigma_3} = \mathbb{I}\,.$$

Note that they are not necessarily unique. Also, we point out that their relationship with the columns of Jost solutions is key in the following. So we need to distinguish the cases $\gamma_+ > 0$ and $\gamma_+ < 0$. In Lemma 2.4, we proved that for $\lambda \in \mathbb{C}^\pm$,

$$e^{i\lambda x}\Psi_{\mp}^{(1)}(x,\lambda) \to e_1, \quad x \to \mp\infty,\ e^{-i\lambda x}\Psi_{\pm}^{(2)}(x,\lambda) \to e_2, \quad x \to \pm\infty, \tag{B.1.6}$$

exponentially with rate $\mathrm{Im}(\lambda)$ and $-\mathrm{Im}(\lambda)$, respectively.

Using similar arguments as in Proposition 2.2, Lemma 2.4, and Remark 2.8, we can prove that

$$e^{-i\lambda x}\Psi_{+}^{(2)}(x,\lambda) = \begin{pmatrix} 0 \\ s_{22}(\lambda) \end{pmatrix} + \mathcal{O}(1), \quad x \to -\infty \text{ and } \lambda \in \mathbb{C}^+. \tag{B.1.7}$$

$$e^{i\lambda x}\Psi_{+}^{(1)}(x,\lambda) = \begin{pmatrix} s_{11}(\lambda) \\ 0 \end{pmatrix} + \mathcal{O}(1), \quad x \to -\infty \text{ and } \lambda \in \mathbb{C}^-, \tag{B.1.8}$$

$$e^{-i\lambda x}\Psi_{-}^{(2)}(x,\lambda) = \begin{pmatrix} 0 \\ s_{11}(\lambda) \end{pmatrix} + \mathcal{O}(1), \quad x \to +\infty \text{ and } \lambda \in \mathbb{C}^-, \tag{B.1.9}$$

Case $\gamma_+ < 0$: In this case $\lambda_+ \in \mathbb{C}^+$ and it could be a zero of $s_{22}(\lambda)$ or not. Suppose first that $s_{22}(\lambda_+) \neq 0$. Then we know that $\Psi_{-}^{(1)}(x,\lambda_+)$ and $\Psi_{+}^{(2)}(x,\lambda_+)$ are linearly independent. From (2.2.22) and the asymptotic behaviour in (2.2.26), (B.1.6), and (B.1.7), we see that we can take

$$X(x) = \sigma_3\Big(\Psi_{-}^{(1)}(x,\lambda_+)/s_{22}(\lambda_+), \Psi_{+}^{(2)}(x,\lambda_+)\Big)$$

as a fundamental matrix for (B.1.5). Hence, we have

$$\varphi_1(x) = X(x)\begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} \tag{B.1.10}$$

for some constants μ_1 and μ_2. Condition (B.1.3) yields $\mu_1 = 1$ but μ_2 is free. As $x \to -\infty$, we have $\varphi_1(x) \sim \begin{pmatrix} e^{-i\lambda_+ x}/s_{22}(\lambda_+) \\ -\mu_2 s_{22}(\lambda_+)e^{i\lambda_+ x} \end{pmatrix}$. There are two sub-cases, either $\mu_2 \neq 0$ or $\mu_2 = 0$. Assume that $\mu_2 \neq 0$. Then, $\frac{\xi_1(x)}{\xi_2(x)} \to 0$ as $x \to -\infty$ which leads to

$$\lim_{x\to-\infty} P_1(x) = \frac{-i\gamma_+}{2}\sigma_3\,.$$

If $\mu_2 = 0$, then
$$\lim_{x\to-\infty} P_1(x) = \frac{i\gamma_+}{2}\sigma_3\,.$$
Suppose now that $s_{22}(\lambda_+) = 0$. Then $\Psi_-^{(1)}(x,\lambda_+)$ and $\Psi_+^{(2)}(x,\lambda_+)$ are no longer linearly independent. Our strategy is to use the fundamental solution $\chi^-(x)$ to exhibit a convenient fundamental matrix that we will use to determine $\varphi_1(x)$. On the one hand, we know that $\varphi_1(x) = \chi^+(x)\begin{pmatrix}1\\ c\end{pmatrix}$ for some constant c and that $\chi^+(x) = \chi^-(x)C$ for some constant invertible matrix C. Hence $\varphi_1(x) = \chi^-(x)\begin{pmatrix}\alpha\\ \beta\end{pmatrix}$ for some constants α, β. Finally, we also have that $\sigma_3\Psi_-^{(1)}(x,\lambda_+) = \chi^-(x)\begin{pmatrix}1\\ d\end{pmatrix}$ for some constant d. Hence, let us define $Y(x) = \left(\sigma_3\Psi_-^{(1)}(x,\lambda_+), \chi_2^-(x)\right)$ where $\chi_2^-(x)$ is the second column vector of $\chi^-(x)$. This is also a fundamental matrix since $Y(x) = \chi^-(x)\begin{pmatrix}1 & 0\\ d & 1\end{pmatrix}$ and it satisfies
$$\lim_{x\to-\infty} Y(x)e^{i\lambda_+ x\sigma_3} = \mathbb{1}\,.$$
Putting everything together, we have $\varphi_1(x) = Y(x)\begin{pmatrix}\alpha\\ \delta\end{pmatrix}$ where $\delta = \beta - \alpha d$ is some constant. We now show that $\delta \neq 0$ necessarily. Since $s_{22}(\lambda_+) = 0$, we know that $\Psi_-^{(1)}(x,\lambda_+) = \gamma\Psi_+^{(2)}(x,\lambda_+)$ so that $Y(x) = (\gamma\sigma_3\Psi_+^{(2)}(x,\lambda_+), \chi_2^-(x))$. Finally, we also have $\chi_2^-(x) = \chi^+(x)\begin{pmatrix}\mu\\ \nu\end{pmatrix}$ for some constants μ, ν. Hence,
$$\begin{aligned}
\lim_{x\to+\infty}\varphi_1(x)e^{i\lambda_+ x} &= \lim_{x\to+\infty}\left(\gamma\sigma_3\Psi_+^{(2)}(x,\lambda_+), \chi^+(x)\begin{pmatrix}\mu\\ \nu\end{pmatrix}\right)\begin{pmatrix}\alpha\\ \delta\end{pmatrix}e^{i\lambda_+ x}\\
&= \lim_{x\to+\infty}\left(\gamma\sigma_3\Psi_+^{(2)}(x,\lambda_+)e^{-i\lambda_+ x}, \chi^+(x)e^{i\lambda_+ x\sigma_3}\begin{pmatrix}\mu\\ \nu e^{2i\lambda_+ x}\end{pmatrix}\right)\begin{pmatrix}\alpha e^{2i\lambda_+ x}\\ \delta\end{pmatrix}\\
&= \begin{pmatrix}0 & \mu\\ -\gamma & 0\end{pmatrix}\begin{pmatrix}0\\ \delta\end{pmatrix}\,. \qquad \text{(B.1.11)}
\end{aligned}$$
Comparing with (B.1.3), we obtain $\mu\delta = 1$ thus showing that $\delta \neq 0$. Therefore, going back to $\varphi_1(x) = Y(x)\begin{pmatrix}\alpha\\ \delta\end{pmatrix}$ with $Y(x) = (\sigma_3\Psi_-^{(1)}(x,\lambda_+), \chi_2^-(x))$, we obtain

that $\varphi_1(x) \sim \begin{pmatrix} \alpha e^{-i\lambda_+ x} \\ \delta e^{i\lambda_+ x} \end{pmatrix}$ as $x \to -\infty$, with $\delta \neq 0$. Hence, $\frac{\xi_1(x)}{\xi_2(x)} \to 0$ as $x \to -\infty$ which leads to

$$\lim_{x\to-\infty} P_1(x) = -\frac{i\gamma_+}{2}\sigma_3\,.$$

Case $\gamma_+ > 0$: In that case $\lambda_+ \in \mathbb{C}^-$ and it could be a zero of $s_{11}(\lambda)$ or not, or equivalently, λ_+^* could be a zero of $s_{22}(\lambda)$ or not. We follow a similar strategy as for the previous case but the change of sign in γ_+ yields a major difference: here $\varphi_1(x) = \sigma_3\Psi_+^{(1)}(x,\lambda_+)$. Indeed, in general we have

$$\varphi_1(x) = \chi^+(x)\begin{pmatrix}\nu_1\\ \nu_2\end{pmatrix}\,,\quad \Psi_+^{(1)}(x,\lambda_+) = \sigma_3\chi^+(x)\begin{pmatrix}\tau_1\\ \tau_2\end{pmatrix}\,,$$

for some constants ν_1, ν_2, τ_1 and τ_2. Imposing (B.1.3) and the asymptotic of $\Psi_+^{(1)}(x,\lambda)$ as in (B.1.6) requires $\nu_1 = 1 = \tau_1$ and $\nu_2 = 0 = \tau_2$. Suppose first that $s_{11}(\lambda_+) \neq 0$, then we can use (B.1.8) and deduce that

$$\lim_{x\to-\infty} P_1(x) = \frac{i\gamma_+}{2}\sigma_3\,.$$

Suppose now that $s_{11}(\lambda_+) = 0$ so that $\Psi_+^{(1)}(x,\lambda_+)$ and $\Psi_-^{(2)}(x,\lambda_+)$ are no longer linearly independent and $\Psi_+^{(1)}(x,\lambda_+) = \gamma'\Psi_-^{(2)}(x,\lambda_+)$ for some nonzero constant γ'. We can use the asymptotic behaviour of $\Psi_-^{(2)}(x,\lambda)$ given in (B.1.6) to conclude that

$$\lim_{x\to-\infty} P_1(x) = -\frac{i\gamma_+}{2}\sigma_3\,.$$

This concludes the proof of part (a). For part (b), we can adapt the proof of (Terng & Uhlenbeck, 1998, Theorem 6.6) whose main points are as follows. From the construction of $\varphi_1(x)$, in all cases, we have either $\xi_1(x)/\xi_2(x)$ or $\xi_2(x)/\xi_1(x)$ tends to 0 exponentially as $e^{\mp|\gamma_+|x}$ as $x \to \pm\infty$. Hence, from (B.1.4), we see that $[\sigma_3, P_1(x)]$ decays exponentially as $x \to \pm\infty$. We can see that in the equation

$$P_{1x} = \frac{i\rho}{2}[\sigma_3, P_1] + i[\sigma_3, P_1]P_1 - [Q, P_1], \tag{B.1.12}$$

the first two terms on the right-hand side decay exponentially while the third term has the same decay as Q which is assumed to be in $\mathcal{S}(\mathbb{R})$. Thus P_{1x} has the same decay as Q. Hence, by repeated differentiation of (B.1.1), we obtain that P_{1x} has the same decay properties as Q at $\pm\infty$ and therefore belongs to $\mathcal{S}(\mathbb{R})$.

B.2 Proof of Proposition 4.15

We consider the case $\gamma_- = \gamma_+$ and denote this common value by γ instead of $\varepsilon\alpha$ for convenience in this proof. Similarly we keep ρ instead of β. Let us assume that symmetries (4.1.26) and (4.1.31) are satisfied. We will adapt the strategy proposed in Deift & Park (2011) to show that $\widetilde{u}(x) = -u(-x)$ holds. Consider $m(x,\lambda) = (m_1(x,\lambda), m_2(x,\lambda))$ the solution to a normalized RHP with jump matrix given by

$$v(x,\lambda) := \begin{pmatrix} 1+|r(\lambda)|^2 & -r^*(\lambda)e^{-2i\lambda x} \\ -r(\lambda)e^{2i\lambda x} & 1 \end{pmatrix}.$$

Consider two functions $q_1(x)$ and $E_1(x)$ defined as

$$\begin{cases} q_1(x) = \sigma_3 \left(m_1\left(x, \widehat{\lambda}\right), m_2\left(x, \widehat{\lambda}^*\right)\right), \\ E_1(x) = q_1(x)\sigma_3 \left(\frac{i\gamma}{2}\right) q_1(x)^{-1}\sigma_3, \\ \widehat{\lambda} = -\frac{\rho+i\gamma}{2}. \end{cases}$$

Set

$$\widehat{m}(x,\lambda) \equiv \begin{cases} Z(x,\lambda)m(x,\lambda)z(\lambda)^{-1}, & \lambda \in \mathbb{C}^+ \setminus \left\{\frac{-\rho+i|\gamma|}{2}\right\}, \\ Z(x,\lambda)m(x,\lambda)z(\lambda)^{-1}, & \lambda \in \mathbb{C}^- \setminus \left\{\frac{-\rho-i|\gamma|}{2}\right\}, \end{cases}$$

where $Z(x,\lambda) = \left(\lambda + \frac{\rho}{2}\right)\sigma_3 + E_1(x)$ and $z(\lambda) = \left(\lambda + \frac{\rho}{2}\right)\sigma_3 + \frac{i\gamma}{2}\mathbb{1}$. Consider other two functions $q_2(x)$ and $E_2(x)$

$$\begin{cases} q_2(x) = \left(\widehat{m}_1\left(x, -\widehat{\lambda}\right), \widehat{m}_2\left(x, -\widehat{\lambda}^*\right)\right), \\ E_2(x) = \sigma_3 q_2(x) \left(\frac{i\gamma}{2}\right) \sigma_3 q_2(x)^{-1}. \end{cases}$$

For convenience, we set

$$a(\lambda) = s_{22}(\lambda).$$

Let us define the following matrix function

$$\widetilde{m}(x,\lambda) \equiv \begin{cases} (i\sigma_2)W(x,\lambda)\widehat{m}(x,\lambda)w(\lambda)^{-1}a(\lambda)^{\sigma_3}(i\sigma_2)^{-1}, & \lambda \in \mathbb{C}^+ \setminus \left\{\frac{\rho+i|\gamma|}{2}\right\}, \\ (i\sigma_2)W(x,\lambda)\widehat{m}(x,\lambda)w(\lambda)^{-1}a^*(\lambda^*)^{-\sigma_3}(i\sigma_2)^{-1}, & \lambda \in \mathbb{C}^- \setminus \left\{\frac{\rho-i|\gamma|}{2}\right\}, \end{cases} \tag{B.2.1}$$

where $W(x,\lambda) = \left(-\lambda + \frac{\rho}{2}\right)\sigma_3 - E_2(x)$ and $w(\lambda) = \left(-\lambda + \frac{\rho}{2}\right)\sigma_3 - \frac{i\gamma}{2}\mathbb{1}$.

We claim:

$$m(x,\lambda) = \widetilde{m}^*(-x,-\lambda^*). \tag{B.2.2}$$

We have the following straightforward computation

$$
\begin{aligned}
(\widetilde{m}_-)^{-1}\,\widetilde{m}_+(x,\lambda) &= (i\sigma_2)a^*(\lambda)^{\sigma_3}z(\lambda)w(\lambda)v(x,\lambda)\,(z(\lambda)w(\lambda))^{-1}\,a(\lambda)^{\sigma_3}(i\sigma_2)^{-1} \\
&= (i\sigma_2)\begin{pmatrix} (1+|r(\lambda)|^2)|a(\lambda)|^2 & -\frac{2\lambda+\rho+i\gamma}{2\lambda-\rho+i\gamma}\frac{2\lambda+\rho-i\gamma}{2\lambda-\rho-i\gamma}\frac{a^*(\lambda)}{a(\lambda)}r^*(\lambda)e^{-2i\lambda x} \\ -\frac{2\lambda-\rho+i\gamma}{2\lambda+\rho+i\gamma}\frac{2\lambda-\rho-i\gamma}{2\lambda+\rho-i\gamma}\frac{a(\lambda)}{a^*(\lambda)}r(\lambda)e^{2i\lambda x} & \frac{1}{|a(\lambda)|^2} \end{pmatrix}(i\sigma_2)^{-1} \\
&= \begin{pmatrix} 1+|r(-\lambda)|^2 & -e^{2i\lambda x}r(-\lambda) \\ -e^{-2i\lambda x}r^*(-\lambda) & 1 \end{pmatrix}.
\end{aligned}
$$

Set $\overline{m}(x,\lambda) \equiv \widetilde{m}^*(-x,-\lambda^*)$. Then,

$$
(\overline{m}_-)^{-1}\,\overline{m}_+(x,\lambda) = \left((\widetilde{m}_-)^{-1}\,\widetilde{m}_+(-x,-\lambda)\right)^* = v(x,\lambda).
$$

When $a(\lambda)$ admits a finite number of simple zeros $\lambda_k \in \mathbb{C}^+$,

$$
\begin{aligned}
\operatorname*{Res}_{\lambda=\lambda_k} m(x,\lambda) &= \lim_{\lambda\to\lambda_k}(\lambda-\lambda_k)m(x,\lambda) \\
&= \lim_{\lambda\to\lambda_k} m(x,\lambda)\begin{pmatrix} 0 & 0 \\ c(\lambda_k)e^{2i\lambda_k x} & 0 \end{pmatrix}.
\end{aligned}
$$

Equivalently,

$$
\lim_{\lambda\to\lambda_k} a(\lambda)m_1(x,\lambda) = \gamma(\lambda_k)e^{2i\lambda_k x}\mu_2(x,\lambda_k).
$$

For $\overline{m}(x,\lambda)$, we have

$$
\begin{aligned}
\left(\operatorname*{Res}_{\lambda=-\lambda_k^*}\overline{m}(-x,\lambda)\right)^* &= \lim_{\lambda\to-\lambda_k^*}\left[(\lambda-(-\lambda_k^*))\overline{m}(-x,\lambda)\right]^* \\
&= \lim_{\lambda\to\lambda_k} -(\lambda-\lambda_k)\overline{m}^*(-x,-\lambda^*) \\
&= -\lim_{\lambda\to\lambda_k}\Big[(i\sigma_2)(WZ)(x,\lambda)(\lambda-\lambda_k)\times \\
&\qquad m(x,\lambda)a(\lambda)^{\sigma_3}\left(z(\lambda)w(\lambda)\right)^{-1}(i\sigma_2)^{-1}\Big] \\
&= -(i\sigma_2)(WZ)(x,\lambda_k)\begin{bmatrix} 0 & \dfrac{\mu_2(x,\lambda_k)}{a'(\lambda_k)} \end{bmatrix}\left((zw)(\lambda_k)\right)^{-1}(i\sigma_2)^{-1} \\
&= 4(i\sigma_2)(WZ)(x,\lambda_k)\times \\
&\qquad \begin{bmatrix} \dfrac{1}{2\lambda_k+\rho-i\gamma}\dfrac{1}{2\lambda_k-\rho-i\gamma}\dfrac{1}{a'(\lambda_k)}m_2(x,\lambda_k) & 0 \end{bmatrix},
\end{aligned}
\tag{B.2.3}
$$

Thus at $-\lambda_k^*$, the second column of $\overline{m}(x,\lambda)$ is analytic, and the first column has a simple pole. Similarly, at $-\lambda_k$ the first column of $\overline{m}(x,\lambda)$ is analytic, and the second column has a simple pole. On the other hand, we have

$$
\begin{aligned}
\lim_{\lambda\to-\lambda_k^*}\left[\overline{m}(-x,\lambda)\begin{pmatrix}0 & 0\\ e^{-2i\lambda_k^* x}\overline{c}(-\lambda_k^*) & 0\end{pmatrix}\right]^* &= \lim_{\lambda\to\lambda_k}\Bigg[\overline{m}^*(-x,-\lambda^*)\times\\
&\qquad \begin{pmatrix}0 & 0\\ e^{2i\lambda_k x}\overline{c}^*(-\lambda_k^*) & 0\end{pmatrix}\Bigg]\\
&= -\lim_{\lambda\to\lambda_k}\Bigg[(i\sigma_2)(WZ)(x,\lambda)m(x,\lambda)\times\\
&\qquad a(\lambda)^{\sigma_3}\left[(zw)(\lambda)\right]^{-1}\times\\
&\qquad \begin{pmatrix}e^{2i\lambda_k x}\overline{c}^*(-\lambda_k^*) & 0\\ 0 & 0\end{pmatrix}\Bigg]\\
&= 4(i\sigma_2)(WZ)(x,\lambda_k)\Bigg[\frac{\gamma(\lambda_k)}{2\lambda_k+\rho+i\gamma}\times\\
&\qquad \frac{\overline{c}^*(-\lambda_k^*)}{2\lambda_k-\rho+i\gamma}m_2(x,\lambda_k)\quad 0\Bigg],
\end{aligned}
\tag{B.2.4}
$$

Note that $\overline{c}(\lambda_k)$ stands for the discrete data that appears in the normalised RHP for $\overline{m}(x,\lambda)$. Compare (B.2.3) and (B.2.4), using (4.1.31) and the first equation in (4.1.26), one gets

$$
\begin{aligned}
\overline{c}(-\lambda_k^*) &= \left[\frac{2\lambda_k-\rho+i\gamma}{2\lambda_k-\rho-i\gamma}\frac{2\lambda_k+\rho+i\gamma}{2\lambda_k+\rho-i\gamma}\frac{1}{\gamma(\lambda_k)}\frac{1}{a'(\lambda_k)}\right]^*\\
&= -\frac{\gamma(-\lambda_k^*)}{(a'(\lambda_k))^*} = \frac{\gamma(-\lambda_k^*)}{a'(-\lambda_k^*)}\\
&= c(-\lambda_k^*).
\end{aligned}
$$

Hence, it follows that $\overline{c}(\lambda_k) = c(\lambda_k)$. Let us discuss what happens at $\lambda = \widehat{\lambda}, \widehat{\lambda}^*, -\widehat{\lambda}, -\widehat{\lambda}^*$:

$$
\begin{aligned}
\widehat{m}(x,\lambda) &= Z(x,\lambda)m(x,\lambda)z(\lambda)^{-1}\\
&= q_1(x)\sigma_3\begin{pmatrix}\left[(\sigma_3 q_1(x))^{-1}m\right]_{11} & -\frac{2\lambda+\rho+i\gamma}{2\lambda+\rho-i\gamma}\left[(\sigma_3 q_1(x))^{-1}m\right]_{12}\\ -\frac{2\lambda+\rho-i\gamma}{2\lambda+\rho+i\gamma}\left[(\sigma_3 q_1(x))^{-1}m\right]_{21} & \left[(\sigma_3 q_1(x))^{-1}m\right]_{22}\end{pmatrix},
\end{aligned}
$$

Note that $[(\sigma_3 q_1(x))^{-1}m(x,\lambda)]_{21} = 0$ and $[(\sigma_3 q_1(x))^{-1}m(x,\lambda)]_{12} = 0$ at $\widehat{\lambda}$ and $\widehat{\lambda}^*$, respectively. This means that $\widehat{m}(x,\lambda)$ does not have any pole at $\widehat{\lambda}$ and $\widehat{\lambda}^*$. Again, we have

$$W(x,\lambda)\widehat{m}(x,\lambda)w(\lambda)^{-1} = \sigma_3 q_2(x)\times \begin{pmatrix} \left[\sigma_3 q_2(x)^{-1}\widehat{m}\right]_{11} & -\frac{2\lambda-\rho+i\gamma}{2\lambda-\rho-i\gamma}\left[\sigma_3 q_2(x)^{-1}\widehat{m}\right]_{12} \\ -\frac{2\lambda-\rho-i\gamma}{2\lambda-\rho+i\gamma}\left[\sigma_3 q_2(x)^{-1}\widehat{m}\right]_{21} & \left[\sigma_3 q_2(x)^{-1}\widehat{m}\right]_{22} \end{pmatrix}.$$

Note that $\left[\sigma_3 q_2(x)^{-1}\widehat{m}\right]_{21}(x,-\widehat{\lambda}) = 0$ and $\left[\sigma_3 q_2(x)^{-1}\widehat{m}\right]_{12}(x,-\widehat{\lambda}^*) = 0$. Combined with the above discussion, we see that $\widetilde{m}^*(-x,-\lambda^*)$ does not have extra poles at $\widehat{\lambda}, \widehat{\lambda}^*, -\widehat{\lambda}, -\widehat{\lambda}^*$. Therefore, we have proved the above claim. In terms of vector columns, Eq (B.2.2) is:
For $\lambda \in \mathbb{C}^+$

$$\begin{cases} m_1(x,\lambda) = \frac{(i\sigma_2)(W\widehat{m}_2)^*(-x,-\lambda^*)}{a^*(-\lambda^*)\left(-\left(\lambda+\frac{\rho}{2}\right)+\frac{i\gamma}{2}\right)}, \\ m_2(x,\lambda) = \frac{-a^*(-\lambda^*)}{\lambda+\frac{\rho}{2}+\frac{i\gamma}{2}}(i\sigma_2)(W\widehat{m}_1)^*(-x,-\lambda^*), \end{cases} \tag{B.2.5}$$

and, for $\lambda \in \mathbb{C}^-$

$$\begin{cases} m_1(x,\lambda) = \frac{a(-\lambda)}{-\left(\lambda+\frac{\rho}{2}\right)+\frac{i\gamma}{2}}(i\sigma_2)(W\widehat{m}_2)^*(-x,-\lambda^*), \\ m_2(x,\lambda) = \frac{-1}{a(-\lambda)\left(\lambda+\frac{\rho}{2}+\frac{i\gamma}{2}\right)}(i\sigma_2)(W\widehat{m}_1)^*(-x,-\lambda^*). \end{cases} \tag{B.2.6}$$

Assume that $\gamma < 0$, so $\widehat{\lambda} \in \mathbb{C}^+$. Using

$$(i\sigma_2)\widehat{m}^*(x,\lambda^*)(i\sigma_2)^{-1} = \widehat{m}(x,\lambda),$$
$$(i\sigma_2)W^*(x,\lambda^*)(i\sigma_2)^{-1} = W(x,\lambda),$$

it follows from the first Eq. in (B.2.5) and the second in (B.2.6)

$$m_1\left(x,\widehat{\lambda}\right) = \frac{1}{i\gamma a\left(\widehat{\lambda}\right)} W\left(-x,-\widehat{\lambda}\right)\widehat{m}_1(-x,-\widehat{\lambda}),$$
$$m_2\left(x,\widehat{\lambda}^*\right) = \frac{1}{i\gamma a\left(-\widehat{\lambda}^*\right)} W\left(-x,-\widehat{\lambda}^*\right)\widehat{m}_2\left(-x,-\widehat{\lambda}^*\right).$$

A direct calculation shows that

$$W(-x,-\widehat{\lambda}) = \sigma_3 q_2(-x)\begin{pmatrix} -i\gamma & 0 \\ 0 & 0 \end{pmatrix}\sigma_3 q_2(-x)^{-1},$$
$$W(-x,-\widehat{\lambda}^*) = \sigma_3 q_2(-x)\begin{pmatrix} 0 & 0 \\ 0 & -i\gamma \end{pmatrix}\sigma_3 q_2(-x)^{-1},$$

which implies

$$\mu_1\left(x,\widehat{\lambda}\right)=\frac{1}{a\left(\widehat{\lambda}\right)}\sigma_3 q_2(-x)\sigma_3\begin{pmatrix}1\\0\end{pmatrix},$$

$$\mu_2\left(x,\widehat{\lambda}^*\right)=-\frac{1}{a\left(-\widehat{\lambda}^*\right)}\sigma_3 q_2(-x)\sigma_3\begin{pmatrix}0\\1\end{pmatrix}.$$

Therefore, one has

$$\begin{aligned}q_1(x)&=\sigma_3\left(\mu_1\left(x,\widehat{\lambda}\right),\mu_2\left(x,\widehat{\lambda}^*\right)\right)\\&=-q_2(-x)\begin{pmatrix}\frac{1}{a\left(\widehat{\lambda}\right)} & 0\\ 0 & \frac{1}{a\left(-\widehat{\lambda}^*\right)}\end{pmatrix}\sigma_3,\end{aligned}$$

from which a direct calculation shows that

$$E_1(x)=\sigma_3 E_2(-x)\sigma_3.$$

From the direct scattering problem, we know that $\frac{\Psi_-^{(1)}(x,\widehat{\lambda})}{a(\widehat{\lambda})}=e^{-i\widehat{\lambda}x}\mu_1\left(x,\widehat{\lambda}\right)$. Hence, since $a\left(\widehat{\lambda}\right)\neq 0$ and $\gamma_+=\gamma_-$, from (B.1.10) (the constant $\mu_2=0$) we have

$$\begin{aligned}\varphi(x)&=\sigma_3\left(\frac{\Psi_-^{(1)}\left(x,\widehat{\lambda}\right)}{a\left(\widehat{\lambda}\right)},\frac{-(i\sigma_2)\Psi_-^{(1)^*}\left(x,\widehat{\lambda}\right)}{a^*\left(\widehat{\lambda}\right)}\right)\\&=\sigma_3\left(e^{-i\widehat{\lambda}x}\mu_1\left(x,\widehat{\lambda}\right),-e^{i\widehat{\lambda}^*x}(i\sigma_2)\mu_1^*\left(x,\widehat{\lambda}\right)\right)\\&=q_1(x)\begin{pmatrix}e^{-i\widehat{\lambda}x} & 0\\ 0 & e^{i\widehat{\lambda}^*x}\end{pmatrix}.\end{aligned}$$

It follows that,

$$\begin{aligned}P(x)&=\varphi(x)\left(\frac{i\gamma}{2}\right)\sigma_3\varphi(x)^{-1}\sigma_3\\&=q_1(x)\left(\frac{i\gamma}{2}\right)\sigma_3 q_1(x)^{-1}\sigma_3\\&=E_1(x).\end{aligned}$$

The case $\gamma > 0$ will lead to the same conclusion, one needs to use the other equations in (B.2.5) and (B.2.6) instead. Let

$$m(x,\lambda) = \mathbb{I} + \frac{m(x)}{\lambda} + \mathcal{O}\left(\lambda^{-2}\right), \quad a(\lambda) = 1 + \frac{a_1}{\lambda} + \mathcal{O}\left(\lambda^{-2}\right)$$

be the asymptotic expansions as $\lambda \to \infty$. As $\widetilde{m}(x,\lambda) = m^*(-x,-\lambda^*)$, it follows from (B.2.1) that

$$-m^*(-x) = (i\sigma_2)\left(m(x) + (a_1-\rho)\sigma_3 + \sigma_3(P(-x)+P(x))\right)(i\sigma_2)^{-1}$$

Again, using the symmetries

$$m(x,\lambda) = (i\sigma_2)m^*(x,\lambda^*)(i\sigma_2)^{-1}, \quad P(x) = (i\sigma_2)P^*(x)(i\sigma_2)^{-1},$$

we see that $m_{12}(x) = -m_{21}^*(x)$ and $P_{12}(x) = -P_{21}^*(x)$. Then, it follows that

$$\begin{aligned}
u(x) &= 2im_{12}(x) \\
&= -2i\left[(i\sigma_2)\left(m(-x) + a_1\sigma_3 - \rho\sigma_3 + \sigma_3 P(x) + \sigma_3 P(-x)\right)(i\sigma_2)^{-1}\right]_{12}^* \\
&= 2i\left[m(-x) + P(-x) + P(x)\right]_{21}^* \\
&= 2im_{21}^*(-x) + \left[P^*(x) + P^*(-x)\right]_{21} \\
&= -2im_{12}(-x) - 2i\left[P(x) + P(-x)\right]_{12} \\
&= -\left(u(-x) + 2i\left[P(x) + P(-x)\right]_{12}\right) \\
&= -\left(u(-x) + 2i\left[P(x) + P(-x)\right]_{12}\right) = -\widetilde{u}(-x).
\end{aligned}$$

■

B.3 Proof of Lemma 4.17

Let $\lambda_0 = -i\alpha$. We have by definition that (see Section 2.3)

$$z(\lambda_0) = \lambda_0 + k(\lambda_0) = -i\alpha - i\kappa\sqrt{\alpha^2 - q_0^2},$$

where $\kappa = \operatorname{sign}(\alpha)\varepsilon$ and ε is 1 on $\mathbb{C}_1$, -1 on $\mathbb{C}_2$.

Let $|\alpha| > q_0$. In Section 2.3, we defined $D^\pm$ as

$$D^+ = \left\{z \in \mathbb{C} : \left(|z|^2 - q_0^2\right)\operatorname{Im}z > 0\right\} \text{ and } D^- = \left\{z \in \mathbb{C} : \left(|z|^2 - q_0^2\right)\operatorname{Im}z < 0\right\}.$$

Recall that $k(\lambda_0)$ can be either $-i\sqrt{\alpha^2 - q_0^2}$ or $i\sqrt{\alpha^2 - q_0^2}$. So, we distinguish two sub-cases:

1. $\varepsilon = 1$ and $\alpha > q_0$ or $\varepsilon = -1$ and $\alpha < -q_0$, which implies that $z(\lambda_0) \in D^-$.

2. $\varepsilon = 1$ and $\alpha < -q_0$ or $\varepsilon = -1$ and $\alpha > q_0$, which implies that $z(\lambda_0) \in D^+$.

Consider the first sub-case, which means $z(\lambda_0) \in D^-$. Equation (2.3.3) admits a non-unique solution $g(x,z)$ on $\mathbb{R}^+$ such that

$$g(x,z) = e^{ik(z)x}\left[\begin{pmatrix} -\frac{iu_+}{z} \\ 1 \end{pmatrix} + h(x,z)\right],$$

where $h(x,z)$ is a column vector that goes to zero as $x \to \infty$ for $z(\lambda) \in D^- \cup \Sigma$. Recall that Σ is defined as the union of the real axis with a circle of radius q_0 centred at the origin. Indeed, we can show by a direct calculation that the following integral equation solves (2.3.3)

$$\begin{aligned} g(x,z) = e^{ikx}\begin{pmatrix} -\frac{iu_+}{z} \\ 1 \end{pmatrix} &+ \int_a^x E_+(z)\begin{pmatrix} e^{-ik(x-y)} & 0 \\ 0 & 0 \end{pmatrix} E_+(z)^{-1}\Delta Q_+(y) g(y,z)dy \\ &- \int_x^\infty E_+(z)\begin{pmatrix} 0 & 0 \\ 0 & e^{ik(x-y)} \end{pmatrix} E_+(z)^{-1}\Delta Q_+(y) g(y,z)dy, \end{aligned}$$

where $\Delta Q_+(x) = Q(x) - Q_+$ and $E_\pm(z) = \mathbb{I} - \frac{i}{z}\sigma_3 Q_\pm$. By using the above integral representation of $g(x,z)$, we can show using similar arguments as in Proposition 2.17 that $g(x,z)$ is well defined on $\mathbb{R}^+$ for $z \in D^- \cup \Sigma$.

In this setting, $\Psi_+^{(1)}(x, z(\lambda_0))$ and $g(x, z(\lambda_0))$ are linearly independent, then we can write

$$\Psi^0(x, z(\lambda_0)) = c_1\Psi_+^{(1)}(x, z(\lambda_0)) + c_2 g(x, z(\lambda_0)).$$

If the second component of $\Psi_+^{(1)}(x, z(\lambda_0))$ does not vanish at $x = 0$, we should have that $(\xi_1(x), \xi_2(x))^T \sim e^{ik(\lambda_0)x}\begin{pmatrix} -\frac{iu_+}{z(\lambda_0)} \\ 1 \end{pmatrix}$ as $x \to \infty$. Hence, $\frac{\xi_1(x)}{\xi_2(x)} \to -\frac{iu_+}{z(\lambda_0)}$ as $x \to \infty$ which implies that

$$P_1(x) \to \frac{i\alpha}{|z(\lambda_0)|^2 + u_0^2}\begin{pmatrix} q_0^2 - |z(\lambda_0)|^2 & 2iu_+ z(\lambda_0)^* \\ -2iu_+^* z(\lambda_0) & -(q_0^2 - |z(\lambda_0)|^2) \end{pmatrix} = -i\left(\sqrt{\alpha^2 - q_0^2}\,\mathbb{I} + Q_+\right)\sigma_3.$$

If the second component of $\Psi_+^{(1)}(x, z(\lambda_0))$ vanishes at $x = 0$, then $c_2 = 0$ that is $(\xi_1(x), \xi_2(x))^T = c_1\Psi_+^{(1)}(x, z(\lambda_0))$. Recall that $\Psi_+^{(1)}(x, z(\lambda_0)) \sim e^{-ik(\lambda_0)x}\begin{pmatrix} 1 \\ -\frac{iu_+^*}{z(\lambda_0)} \end{pmatrix}$

as $x \to \infty$, hence $\frac{\xi_2(x)}{\xi_1(x)} \to -\frac{iu_+^*}{z(\lambda_0)}$ as $x \to \infty$. As a result, as $x \to \infty$, we have

$$P_1(x) \to \frac{i\alpha}{|z(\lambda_0)|^2 + q_0^2} \begin{pmatrix} |z(\lambda_0)|^2 - q_0^2 & -2iu_+ z(\lambda_0) \\ 2iu_+^* z(\lambda_0)^* & -(|z(\lambda_0)|^2 - q_0^2) \end{pmatrix} = i\left(\sqrt{\alpha^2 - q_0^2}\mathbb{I} - Q_+\right)\sigma_3.$$

We summarise what we have done so far:

$$P_1(x) \to i\left(r_+\mathbb{I} - Q_+\right)\sigma_3 \quad \text{as } x \to +\infty$$

such that $r_+ = -\sqrt{\alpha^2 - q_0^2}$ if the second component of $\Psi_+^{(1)}(0, z(\lambda_0))$ does not vanish, and $r_+ = \sqrt{\alpha^2 - q_0^2}$ if the second component of $\Psi_+^{(1)}(0, z(\lambda_0))$ does vanish.

To extract the asymptotic behaviour of $P_1(x)$ at $-\infty$, we need the following column vector solution of (2.3.3) on $\mathbb{R}^-$:

$$f(x, z) = e^{-ikx}\left[\begin{pmatrix} 1 \\ -\frac{iu_-^*}{z} \end{pmatrix} + v(x, z)\right],$$

where $v(x, z)$ is a column vector that goes to zero as $x \to -\infty$. This vector solution can be obtained rigorously using a similar type of argument as for $g(x, z)$ above. In this case, $f(x, z(\lambda_0))$ and $\Psi_-^{(2)}(x, z(\lambda_0))$ are linearly independent. Hence,

$$\Psi^0(x, z(\lambda_0)) = c_1 f(x, z(\lambda_0)) + c_2\Psi_-^{(2)}(x, z(\lambda_0)).$$

If the second component of $\Psi_-^{(2)}(x, z(\lambda_0))$ does not vanish at $x = 0$, we should have that $(\xi_1(x), \xi_2(x))^T \sim e^{-ik(\lambda_0)x}\begin{pmatrix} 1 \\ -\frac{iu_-^*}{z(\lambda_0)} \end{pmatrix}$. Hence, $\frac{\xi_2(x)}{\xi_1(x)} \to -\frac{iu_-^*}{z(\lambda_0)}$ as $x \to -\infty$ which implies that

$$P_1(x) \to i\left(\sqrt{\alpha^2 - q_0^2} - Q_-\mathbb{I}\right)\sigma_3. \tag{B.3.1}$$

If the second component of $\Psi_-^{(2)}(x, z(\lambda_0))$ vanishes at $x = 0$, then $c_1 = 0$ that is $(\xi_1(x), \xi_2(x))^T = c_2\Psi_-^{(2)}(x, z(\lambda_0))$. Again, recall that $\Psi_-^{(2)}(x, z) \sim e^{ikx}\begin{pmatrix} -\frac{iu_-}{z} \\ 1 \end{pmatrix}$ as $x \to -\infty$. This implies that $\frac{\xi_1(x)}{\xi_2(x)} \to -\frac{iu_-}{z(\lambda_0)}$ as $x \to -\infty$. As a result, we have

$$P_1(x) \to -i\left(\sqrt{\alpha^2 - q_0^2}\mathbb{I} + Q_-\right)\sigma_3.$$

Again, we summarise what we have done so far:

$$P_1(x) \to i\left(r_-\mathbb{I} - Q_-\right)\sigma_3 \quad \text{as } x \to -\infty$$

such that $r_- = \sqrt{\alpha^2 - q_0^2}$ if the second component of $\Psi_-^{(2)}(0, z(\lambda_0))$ does not vanish, and $r_- = \sqrt{\alpha^2 - q_0^2}$ if the second component of $\Psi_-^{(2)}(0, z(\lambda_0))$ does vanish. This concludes the proof.

B.4 Proof of Lemma 4.23

Assume that $u(x)$ is α-symmetric. We prove relations given in (4.2.11). Using Lemma 4.21, equation (4.2.10) in terms of Jost solutions becomes

$$\widetilde{\Psi}_{\mp}(-x, -z) = \sigma_3 \Psi_{\pm}(x, z). \tag{B.4.1}$$

This implies that

$$\widetilde{S}(z) = S^{-1}(-z) \Longrightarrow \widetilde{s}_{22}(z) = s_{11}(-z) \text{ and } \widetilde{s}_{21}(z) = -s_{21}(-z).$$

Combining these symmetries, relations (2.3.35) and (4.2.7) one obtains the first two relations in (4.2.11). Equation (B.4.1) together with

$$\tilde{\Psi}_{\pm}(x, z) = L(x, z)\Psi_{\pm}(x, z)M_{\pm}(z), \quad z \in \Sigma,$$

give

$$\Psi_{\pm}(x, z) = L(-x, -z)\Psi_{\mp}(-x, -z)M_{\pm}(-z), \quad z \in \Sigma.$$

Taking into consideration the analytic continuations of columns of Jost solutions, for $z \in D^+$, we have

$$\Psi_-^{(1)}(x, z) = -\tfrac{1}{g(-z)} 2zL(-x, -z)\Psi_+^{(1)}(-x, -z), \tag{B.4.2}$$

$$\Psi_+^{(2)}(x, z) = \tfrac{1}{g(-z^*)^*} 2zL(-x, -z)\Psi_-^{(2)}(-x, -z). \tag{B.4.3}$$

In terms of column vectors, the first symmetry in (2.3.33) gives

$$\Psi_-^{(1)}(x, z) = -(i\sigma_2)\Psi_-^{(2)}(x, z^*)^*, \quad \Psi_+^{(2)}(x, z) = (i\sigma_2)\Psi_+^{(1)}(x, z^*)^*, \quad z \in D^+.$$

Using these relations one can show that

$$\begin{aligned}\Psi_-^{(1)}(x, z_n) &= \gamma(z_n)\Psi_+^{(2)}(x, z_n) \\ &= -\frac{\gamma(z_n)\gamma^*(-z_n^*)}{g(-z_n^*)^*} 2z_n L(-x, -z_n)\Psi_+^{(1)}(-x, -z_n).\end{aligned}$$

Hence comparing this with the first equation in (B.4.2) evaluated at z_n, we obtain

$$\gamma(z_n)\gamma^*(-z_n^*) = \frac{g(-z_n^*)^*}{g(-z_n)} = \frac{g(z_n)}{g(z_n^*)^*},$$

since $g(-z_n^*)^* = g(z_n)$ and $g(-z_n) = g(z_n^*)^*$.

$$\begin{aligned}
\Psi_-^{(1)}(x, u_0^2/z_n) &= \gamma(u_0^2/z_n)\Psi_+^{(2)}(x, u_0^2/z_n) \\
&= \frac{\gamma(q_0^2/z_n)}{g(-q_0^2/z_n^*)^*}\left(2\frac{u_0^2}{z_n}\right) L(-x, -q_0^2/z_n)\Psi_-^{(2)}(-x, -q_0^2/z_n) \\
&= \left(\frac{u_+}{u_-^*}\right)\frac{\gamma(q_0^2/z_n)\gamma(z_n)}{g(-q_0^2/z_n^*)^*}\left(2\frac{u_0^2}{z_n}\right) L(-x, -q_0^2/z_n)\Psi_+^{(1)}(-x, -q_0^2/z_n)
\end{aligned}$$

Hence comparing this with the first equation in (B.4.2) evaluated at u_0^2/z_n, we get

$$\gamma(u_0^2/z_n)\gamma(z_n) = -\left(\frac{u_-^*}{u_+}\right)\frac{g(-q_0^2/z_n^*)^*}{g(-q_0^2/z_n)} = -\left(\frac{u_-^*}{u_+}\right)\frac{g(z_n)}{g(z_n^*)^*},$$

since $g(-q_0^2/z_n^*)^* = (u_0^2/z_n^2)g(z_n)$ and $g(-q_0^2/z_n) = (u_0^2/z_n^2)g(z_n^*)^*$.

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