A Gutzwiller Trace formula for Dirac Operators on a Stationary Spacetime

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... To all who have taught me

Declaration

To the best of my knowledge, I declare that the research work in this thesis is original and my own work, except where either cited or commonly known. The work presented in Chapter 3 is identical to the joint paper with Alexander Strohmaier "On microlocalization and the construction of Feynman propagators for normally hyperbolic operators", arXiv:2012.09767 [math.AP] (accepted in *Communications in Analysis and Geometry*) with a few added background materials. Chapter 4 comprise a slightly expository version of the single-authored article "A Gutzwiller Trace formula for Dirac Operators on a Stationary Spacetime" arXiv:2109.09219 [math.AP] (accepted in *The Journal of Geometric Analysis*).

Abstract

A Duistermaat-Guillemin-Gutzwiller trace formula for Dirac-type operators on a globally hyperbolic spatially compact stationary spacetime is achieved by generalising the recent construction by Strohmaier and Zelditch [Adv. Math. **376**, 107434 (2021)] to a vector bundle setting. We have analysed the spectrum of the Lie derivative with respect to a global timelike Killing vector field on the solution space of the Dirac equation and found that it consists of discrete real eigenvalues. The distributional trace of the time evolution operator has singularities at the periods of induced Killing flow on the manifold of lightlike geodesics. This gives rise to the Weyl law asymptotic at the vanishing period. A pivotal technical ingredient to prove these results is the Feynman propagator. In order to obtain a Fourier integral description of this propagator, we have generalised the classic work of Duistermaat and Hörmander [Acta Math. **128**, 183 (1972)] on distinguished parametrices for normally hyperbolic operators on a globally hyperbolic spacetime by propounding their microlocalisation theorem to a bundle setting. As a by-product of these analyses, another proof of the existence of Hadamard bisolutions for a normally hyperbolic operator (resp. Dirac-type operator) is reported.

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Chapter 1

Introduction

In 1966, Kac [1] asked in a rather leisurely fashion:

"Can you hear the shape of a drum?"

This is arguably the quintessential question in spectral geometry. The Gutzwiller trace formula is one of the paramount tools used to investigate such a beautiful interplay between geometry and the spectral world in the sense explained below.

1.1 Trace formulae

1.1.1 A prelude

In order to understand the notion of a trace formula precisely, let us consider the simplest non-trivial differential operator: the derivative

$$D := -i\frac{d}{dx} \tag{1.1}$$

acting on smooth functions $C^\infty(\mathbb{S})$ on a circle $\mathbb{S}$ of length T. Since $C^\infty(\mathbb{S})$ is in a one-to-one correspondence with T-periodic functions on $\mathbb{R}$, one defines

$$L^2(\mathbb{S}) := \left\{ u \in C^\infty(\mathbb{R}) \mid \forall x \in \mathbb{R} : u(x+T) = u(x), \int_0^T \bar{u}(x)\, u(x)\, dx < \infty \right\}. \tag{1.2}$$

It is straightforward to compute the spectrum of D:

$$\operatorname{Spec} D = \{\lambda_n \mid n \in \mathbb{Z}\}, \quad \lambda_n := n\omega, \quad \omega := \frac{2\pi}{T} \tag{1.3}$$

corresponding to the orthonormal eigenfunctions $\phi_n := e^{i\lambda_n x}/\sqrt{T}$. For any Schwartz function $\rho \in \mathcal{S}(\mathbb{R})$ on $\mathbb{R}$ and for all $T > 0$, the **Poisson summation formula** reads (see e.g. [2, Thm. 1.2])

$$\sum_{n \in \mathbb{Z}} \rho(\lambda_n) = T \sum_{k \in \mathbb{Z}} (\mathcal{F}^{-1}\rho)(kT), \tag{1.4}$$

where $\mathcal{F}^{-1}(\rho)$ denotes the inverse Fourier transformation of ρ. One observes that the left-hand side of this identity entirely consists of integral multiple of the fundamental angular frequency ω on $\mathbb{S}$, whereas the right-hand side involves only the length T of $\mathbb{S}$. In other words, *the Poisson summation formula associates the spectral data with a geometric quantity*. As a natural suspicion, we may ask whether such a connection is something specific to the simple operator (D) and/or the geometry ($\mathbb{S}$) we have considered. However, this suspicion does not hold. For instance, the Poisson summation formula holds for a Laplacian on a torus (see e.g. [2, Thm. 1.3]). Thus, one may wonder whether there are some deeper reasons behind such a bridge between the spectral data and the geometric world.

In order to gain a systematic understanding, let us consider the half-density bundle (see Section 2.2.1) $\Omega^{1/2}\Sigma \to \Sigma$ and a (positive-definite) hermitian vector bundle ($\mathscr{E}_\Sigma \to \Sigma, \langle \cdot | \cdot \rangle$), over a $d-1$-dimensional *compact* manifold Σ without boundary and let P be an *elliptic* (see Definition 2.3.7), *symmetric, first-order* pseudodifferential operator (see Definition 2.3.2) on $\mathscr{E}_\Sigma \otimes \Omega^{1/2}\Sigma$. It is well-known that the spectrum of P consists of discrete real eigenvalues $\lambda_n^\pm$ and is not necessarily semi-bounded; each $\lambda_n^\pm$ is isolated and has a finite multiplicity. Here, $\lambda_n^\pm$ are enumerated in increasing-order in the sense that $\ldots < \lambda_2^- < \lambda_1^- < 0 =: \lambda_0 < \lambda_1^+ < \lambda_2^+ < \ldots$ and with multiplicities. In general, P is not a trace-class operator but its suitable functions can be. For instance, the most important operators in geometric analysis, such as, the Laplace-Beltrami operator (see (3.11)), the d'Alembertian (see Example 3.2.2), the Schrödinger operator, and the Dirac operator (see Definition 3.4.1) are not trace-class yet their suitable functions are trace-class. We opt for an operator of the following form, originally due to Levitan [3] and to Avakumovič [4] (see also [5])

$$\forall t \in \mathbb{R} : U_t := \mathrm{e}^{-\mathrm{i}tP}. \tag{1.5}$$

This is by no means a canonical choice; see Table 1.1 for some other possibilities. However, adopting (1.5) over the other options offers the following two advantages [5, p. 196] (see also [6, p. XXV]):

U_t	t	P	Partial Differential Equation (PDE)	Method
$\frac{1}{t-P}$	$\mathbb{R} \setminus \operatorname{Spec} P$	-	$(t-P)U_t = I$	Resolvent
P^{-t}	$\mathbb{C}$	-	-	ζ-function
e^{-tP}	$\mathbb{R}_+$	Δ	Heat eq. $(\partial_t + \Delta)u = 0$	Hypoelliptic PDE
$\mathrm{e}^{-\mathrm{i}tP}$	$\mathbb{R}$	$\pm\sqrt{\Delta}$	Half-wave eq. $(-\mathrm{i}\,\partial_t \pm \sqrt{\Delta})u = 0$	Hyperbolic PDE
	$\mathbb{R}$	H	Schrödinger eq. $(-\mathrm{i}\,\partial_t - H)u = 0$	Parabolic PDE
	$\mathbb{R}$	H_D	Dirac eq. $Du = 0$	Hyperbolic PDE

Table 1.1: Different choices of U_t with corresponding domains of t and feasible P (see e.g. [5, 6]. Here Δ, H, H_D, D are the Laplace-Beltrami operator, the non-relativistic Hamiltonian, the Dirac-Hamiltonian, and the Dirac-type operator, respectively.

- U_t is the solution operator to the Cauchy problem of the half-wave equation. That is, given an initial data $k \in H^1(\Sigma; \mathscr{E}_\Sigma \otimes \Omega^{1/2}\Sigma)$, one has

$$(-\mathrm{i}\,\partial_t + P)u = 0 \Rightarrow u = U_t(k), \tag{1.6}$$

where $H^1(\cdot)$ denotes the Sobolev space (see Section 3.3.1). We can even consider P of order $m \geq 2$ and replace P in the preceding equation by $|P|^{1/m}$ without destroying its hyperbolic nature whilst $-\mathrm{i}^m\,\partial_t^m + P$ is no more hyperbolic for $m > 2$. This, in fact, shows our motivation to consider only first-order operators without loss of generality. The primary reason in favour of the strictly hyperbolic operator $-\mathrm{i}\,\partial_t + P$ compared to a hypoelliptic or a parabolic one is the *finite propagation speed of singularities* of U_t.

- The Weyl eigenvalue counting function of P

$$\mathsf{N}^+(\lambda) := \#\{n \,|\, \lambda_n^+ \leq \lambda^+\}, \quad \mathsf{N}^-(\lambda) := \#\{n \,|\, \lambda_n^- \geq \lambda^-\} \tag{1.7}$$

is related to the distributional trace (as precised below) $\operatorname{Tr} U_t$ via the distributional Fourier transform (see e.g. [7, Chap. 3]):

$$\frac{\mathrm{d}\mathsf{N}^\pm}{\mathrm{d}\lambda} = \mathcal{F}_{t\mapsto\lambda}^{-1}(\operatorname{Tr} U_t). \tag{1.8}$$

Therefore, the information contained in U_t can be proficiently transferred to the spectral function by means of the Tauberian arguments.

Our choice comes with the caveat that U_t is not a trace-class operator as, for instance, U_0 is the identity operator. Thus, $\operatorname{Tr} U_t$ makes sense only as a distribution on $\mathbb{R}$:

$$\operatorname{Tr} U_t : \mathcal{S}(\mathbb{R}) \to \mathbb{C}, \; \rho \mapsto \operatorname{Tr} U_\rho := \operatorname{Tr}\left(\int_\mathbb{R} U_t \,\mathcal{F}^{-1}(\rho)\,\mathrm{d}t\right), \tag{1.9a}$$

$$U_\rho u := \left(\int_\mathbb{R} U_t \,\mathcal{F}^{-1}(\rho)\,\mathrm{d}t\right)u := \int_\mathbb{R} U_t \,\mathcal{F}^{-1}(\rho)\,u\,\mathrm{d}t, \tag{1.9b}$$

$$(\mathcal{F}^{-1}\rho)(t) = \frac{1}{2\pi}\int_\mathbb{R} \mathrm{e}^{\mathrm{i}t\lambda}\rho(\lambda)\,\mathrm{d}\lambda, \tag{1.9c}$$

for any $u \in C_c^\infty(\Sigma; \mathscr{E}_\Sigma \otimes \Omega^{1/2}\Sigma)$, where the rightmost integral in (1.9b) is defined as a vector-valued Riemann integral. Furthermore, it is a 1-parameter group of unitary operators (see Section 4.6) on $L^2(\Sigma; \mathscr{E}_\Sigma \otimes \Omega^{1/2}\Sigma)$ and a bounded operator on $H^s(\Sigma; \mathscr{E}_\Sigma \otimes \Omega^{1/2}\Sigma)$, uniformly in $t \in \mathbb{R}$ and for any $s \in \mathbb{R}$ (see [5, Sec. 3] for the scalar, and, for instance, [8, §20] for an elaboration).

Therefore, it makes sense to talk about the distributional trace of U_t on $L^2(\Sigma; \mathscr{E}_\Sigma \otimes \Omega^{1/2}\Sigma)$ and by the functional calculus $\operatorname{Tr} U_t = \sum_n \exp(-\mathrm{i}\,t\lambda_n^\pm)$. Equivalently, we can compute it deploying the Schwartz kernel U_t of U_t. Putting both together, we arrive at the *quintessential feature of trace formulae* (see e.g. [9, Intro.])

$$\sum_n \mathrm{e}^{-\mathrm{i}t\lambda_n^\pm} = \operatorname{Tr} U_t = \int_\Sigma \operatorname{tr}\left(\mathsf{U}_t(x,x)\right), \tag{1.10}$$

where $U_t(x, x)$ is the restriction of the Schwartz kernel $U_t(x, y)$ of U_t to the diagonal in $\Sigma \times \Sigma$ and tr is the endomorphism trace (see (3.8)). This unveils the underpinning of the bridge connecting the spectral data of P to the geometric world where this operator acts on, as glimpsed in the Poisson summation formula (1.4) where $P = D$ and $\Sigma = \mathbb{S}$.

The geometry ($\mathbb{S}$) considered in (1.4) was simple enough to compute the spectrum and the Schwartz kernel explicitly, and hence one has an identity. But, this is an atypical situation in a generic compact manifold. Therefore, one can, at best, approximate U_t employing the theory of hyperbolic[1] differential equations, usually by constructing para-metrices (see Definition 3.2.14), and then obtain the Tauberian part in a straightforward fashion. This leads to the notion of asymptotic trace formulae, where one investigates large eigenvalues only by shifting the test function $\rho(\cdot) \mapsto \rho_\mu(\cdot) := \rho(\cdot - \mu)$ by an amount μ and letting $\mu \to \infty$. In particular, if $\{\phi_n\}$ are orthonormal eigensections of P on $\mathscr{E}_\Sigma$, then (see Section 2.3.4 for the precise meaning of an anti-dual bundle $\bar{\mathscr{E}}_\Sigma^* \to \Sigma$)

$$U_\rho(x, y) = \int_{t \in \mathbb{R}} \sum_{n \in \mathbb{N}} e^{-i t \lambda_n^\pm} (\bar{\phi}_n^* \boxtimes \phi_n)(x, y) \, (\mathcal{F}^{-1} \rho)(t) \, dt, \qquad (1.11)$$

where the sum converges in $C^\infty\big(\Sigma \times \Sigma; (\bar{\mathscr{E}}_\Sigma^* \boxtimes \mathscr{E}_\Sigma) \otimes \Omega^{1/2}(\Sigma \times \Sigma)\big)$ (see e.g. [11, p. 133] for the scalar version), and the *asymptotic trace formula* reads

$$\mathrm{Tr}(U_{\rho_\mu}) = \int_\Sigma \mathrm{tr}\big(U_{\rho_\mu}(x, x)\big), \quad \text{as} \quad \mu \to \pm\infty. \qquad (1.12)$$

The field of asymptotic (semi-classical) trace formulae stems from the study of Green's operator for a Schrödinger operator in the limit of vanishing Planck's constant by Gutzwill-er [12] (see also [13, Ch. 17]). This seminal work is not entirely rigorous; see for instance, the expositions [2, 14] for a scrutinised discussion of his original idea. A number of mathematically diligent proofs [10, 15–22] have been reported since then. Amongst these, Chazarain [16, 17] (for Laplace-Beltrami operator on a closed Riemannian manifold) and Duistermaat-Guillemin [10] (for a scalar and positive P with other assumptions as in the on going discussion) deployed the global theory of Fourier integral operators to derive the complete singularity structure of the wave-trace. We refer to the monographs [23, Chap. XXIX], [24, Chap. 1], [25, Chap. 11] for details.

1.1.2 Duistermaat-Guillemin-Gutzwiller trace formula

To obtain a concrete understanding, let Δ be the Laplacian (in Geometers' convention) on a $d - 1$-dimensional closed Riemannian manifold (Σ, h). It is well-known that its

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[1]The situation is opposite for the non-hyperbolic methods (resolvent, ζ-function, heat-kernel) where the PDE part is simpler but the Tauberian part always fails to provide a decent remainder estimation. In fact, the *best* remainder estimations have been achieved employing hyperbolic methods [5, 10] (see also the monograph [6] and references therein).

spectrum Spec Δ comprises discrete eigenvalues λ_n^2 and that the counting function $\mathsf{N}_\Delta(\lambda)$ satisfies the celebrated Weyl law [5]

$$\mathsf{N}_\Delta(\lambda) = \left(\frac{\lambda}{2\pi}\right)^{d-1} \mathrm{vol}(\bar{\mathbb{B}}^{d-1})\, \mathrm{vol}(\Sigma), \quad \text{as} \quad \lambda \to \infty, \tag{1.13}$$

where $\mathrm{vol}(\bar{\mathbb{B}}^{d-1})$ is the volume of the closed unit ball $\bar{\mathbb{B}}^{d-1}$ in $\mathbb{R}^{d-1}$. As aforementioned, the counting measure $\mathrm{d}\mathsf{N}_\Delta/\mathrm{d}\lambda$ is related to the distributional trace $\mathrm{Tr}\, U_t$ via the Fourier transform (1.8), where

$$\forall t \in \mathbb{R} : U_t := \mathrm{e}^{-\mathrm{i}t\sqrt{\Delta}} \tag{1.14}$$

is the solution map to the Cauchy problem of the half-wave operator $-\mathrm{i}\,\partial_t + \sqrt{\Delta}$. Moreover, the singularity analysis of $\mathrm{Tr}\, U_t$ at $t = 0$ gives rise to the half-wave trace invariants and its singular support is contained in the set of periods of periodic geodesics γ on Σ. Furthermore, $\mathrm{Tr}\, U_t$ admits a singularity expansion around the non-zero periods $t = T \neq 0$ (under some technical assumptions related to the clean intersection (see Definition A.1.15) condition between certain manifolds) [10, 16] and its leading-order term can be expressed by means of the fundamental periods and eigenvalues of the Poincaré maps of γ's [10, Thm. 4.5].

In their seminal work, Duistermaat and Guillemin [10] have actually considered $U_t := \mathrm{e}^{-\mathrm{i}t\sqrt[m]{P}}$ where P is a positive, elliptic, symmetric pseudodifferential operator of order $m > 0$ on a compact boundaryless manifold Σ.

The Gutzwiller trace formula is structurally similar to that of Duistermaat-Guillemin but the asymptotic is in the limit of vanishing Planck's constant. To be precise, take $\Sigma = \mathbb{R}^{d-1}$ and $\mathscr{E}$ a trivial $\mathbb{R}$-line bundle for simplicity. Let $H := \hbar^2\Delta + V$ be the non-relativistic Hamiltonian so that $U_{t,\hbar} := \mathrm{e}^{-\mathrm{i}tH/\hbar}$, where $\hbar$ is some real parameter and V is any appropriate smooth potential on $\mathbb{R}^{d-1}$. If $\hbar$ is small enough then the spectrum of H on $L^2(\mathbb{R}^{d-1})$ is discrete and there exists an orthonormal basis $\{\phi_n\}$ for $L^2(\mathbb{R}^{d-1})$ of eigenfunctions of H. Let $\mathsf{E}_{n,\hbar}$ be the associated eigenvalues of H. Then one seeks asymptotic expansion of $\sum_n \rho(\mathsf{E}_{n,\hbar} - \mathsf{E}/\hbar)$ in the limit $\hbar \to 0$. Physically, $\hbar$ is identified with the reduced Planck's constant so that $\hbar \to 0$ can be reckoned as the quantum-to-classical transition and hence the name semi-classical trace formula (see e.g. [2, Sec. 2]). In this thesis, however, we have only pursued along the tenet of the Duistermaat-Guillemin trace formula and set $\hbar = 1$ throughout our exploration. Note, $\mathrm{e}^{-\mathrm{i}t\Delta}$ is *not* a trace-class operator since, for instance, on a circle its singular support is the entire real axis [10, p. 46].

1.1.3 Sandoval trace formula

Sandoval has generalised the Duistermaat-Guillemin trace formula for a Dirac-type operator $\hat{D}$ on a hermitian vector bundle $(\mathscr{E}_\Sigma \to \Sigma, \langle\cdot|\cdot\rangle)$ over a closed Riemannian manifold (Σ, h) by studying

$$\forall t \in \mathbb{R} : \hat{U}_t := \mathrm{e}^{-\mathrm{i}\,t\hat{D}}. \tag{1.15}$$

Arguably, $\hat{D}$ is the most fundamental first-order differential operator in geometric analysis and it differs from a Laplace-Beltrami operator Δ on $\mathscr{E}_\Sigma$ in that its spectrum $\mathrm{Spec}\,\hat{D}$ is unbounded from both below and above whereas $\mathrm{Spec}\,\Delta$ is bounded from the below (semi-bounded). Sandoval's investigation results in a straightforward generalisation of the Weyl law [26, (5)] and Dirac-wave trace invariants [26, Thm. 2.2]. But, genuine bundle features show up in terms of the holonomy group in the singularity analysis of $\mathrm{Tr}\,U_{T\neq 0}(\hat{D})$ [26, Thm. 2.8]. However, there is an ad-hoc choice regarding the connection used to deduce the holonomy group as elucidated below.

As a matter of fact, the Weitzenböck connection (see (3.126)) $\hat{\nabla}$ determined by $\hat{D}$ and h does not automatically induce a Clifford connection albeit such a connection always exists (see Remark 3.4.3) and the corresponding operator is usually called a *compatible Dirac-type operator* $\check{D}$. They differ only by a smooth term $W := \hat{D} - \check{D} \in C^\infty(\Sigma; \mathrm{End}\,\mathscr{E})$. To compute the principal symbol $\sigma_{\hat{U}_t}$ of $\hat{U}_t$, Sandoval used the parallel transporter $\check{\mathcal{T}}$ with respect to the Weitzenböck connection $\check{\nabla}$ of $\check{D}$ instead of $\hat{\mathcal{T}}$ corresponding to $\hat{\nabla}$ [26, Prop. 5.3]. Moreover, she employed the trivialisation characterised by the vanishing subprincipal symbol $\sigma^{\mathrm{sub}}_{\check{D}^2}$ of $\check{D}^2$ [26, Cor. 5.11]. As a consequence, $\check{\mathcal{T}}$ appearing in expression of $\sigma_{\hat{U}_t}$ is somewhat expedient and the splitting of $\hat{\mathcal{T}}$ in terms of $\check{\mathcal{T}}$ and the average value[2] of W along the geodesic $[0, t] \ni s \mapsto \gamma(s) \in \mathbb{S}^*\Sigma$ is rather ad-hoc.

The objective of our study is twofold. First, we have relaxed all the aforementioned ad-hoc considerations to compute $\sigma_{\hat{U}_t}$. Next, one notices that $\hat{U}_t$ solves the initial value problem of the differential equation

$$(-\mathrm{i}\,\partial_t + \hat{D})u = 0, \quad u(t_0) = k \tag{1.16}$$

on a d-dimensional product manifold (in the context of general relativity, this is known as the *ultrastatic spacetime*; see Remark 4.2.6 (e))

$$(\mathscr{M}, \mathsf{g}) := (\mathbb{R} \times \Sigma, \mathrm{d}t^2 - \mathsf{h}) \tag{1.17}$$

admitting (at least) a global timelike Killing vector field $Z := \partial_t$ so that the external parameter t in $\hat{U}_t$ can be considered as the canonical global time-coordinate on $\mathbb{R}$. But, $-\mathrm{i}\,\partial_t + \hat{D}$ does not respect general relativistic covariance and so it seems more natural to instead work with the Lorentzian Dirac-type operator D. Thus the study is devoted to investigating a Lorentzian generalisation of Sandoval's trace formula.

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[2]See [26, (16)] for the precise expression.

1.1.4   Riemannian trace formulae: A relativistic viewpoint

The Riemannian to Lorentzian transition is conceptually non-trivial because the solution map to the Cauchy problem of the Dirac equation

$$Du = 0, \quad u \restriction \Sigma = k \tag{1.18}$$

on a globally hyperbolic spacetime (see Definition 3.1.3) is no longer of the form (1.15), since the Dirac-Hamiltonian (the counterpart of $\hat{D}$) becomes time-dependent. In order to contemplate what should be looked for in a Lorentzian setting, we revisit the product scenario from a relativistic viewpoint. One reckons that the solutions of (1.16) are of the following form

$$\psi_n^{\pm} := e^{-i t \lambda_n^{\pm}} \phi_n^{\pm}, \tag{1.19}$$

where $\phi_n^{\pm}$ are the eigensections of $\hat{D}$ corresponding to the eigenvalues $\lambda_n^{\pm}$. But

$$Z\psi_n^{\pm} = -i \lambda_n^{\pm} \psi_n^{\pm} \tag{1.20}$$

and consequently,

$$\operatorname{Tr}_{L^2(\Sigma;\mathscr{E})} \hat{U}_t = \sum_{\lambda_n^{\pm} \in \operatorname{Spec} \hat{D}} e^{-i t \lambda_n^{\pm}} = \operatorname{Tr}_{\ker(-i \partial_t + \hat{D})} e^{tZ}. \tag{1.21}$$

In other words, the *trace of the Dirac-wave group $\hat{U}_t$ on a closed Riemannian manifold is equivalent to the trace of the flow e^{tZ} induced by the global timelike Killing vector field Z of an ultrastatic spacetime $\mathscr{M}$, acting on the kernel $\ker(-i \partial_t + \hat{D})$ of the half-wave operator $-i \partial_t + \hat{D}$.* This is essentially a straightforward generalisation of the observation originally due to Strohmaier and Zelditch [27, Sec. 10.1] in the context of a scalar wave operator. Thus, the deep results on Riemannian spectral geometry as mentioned in Section 1.1.2 and 1.1.3 (and many more as available in the compendia [23, 24, 28–30], for instance) are essentially achieved by investigating the solution operator of the half-wave equation on an ultrastatic spacetime with a compact Cauchy hypersurface (see Definition 3.1.4) without boundary.

It is, therefore, evident that a global timelike Killing flow (see (4.3)) Ξ on a Lorentzian manifold plays a pivotal role and the most general class of spacetimes admitting such a flow is known as the stationary spacetime (see Definition 4.2.1) $(\mathscr{M}, g, \Xi)$. Physically speaking, these spacetimes admit a *canonical flow of time* but unlike the ultrastatic case there is *no* preferred time coordinate. They are interesting because a number of exact solutions of Einstein's equation (Schwarzschild and Kerr black holes) belong to this class. We demand the global hyperbolicity condition on $(\mathscr{M}, g, \Xi)$ to ensure the well-posed Cauchy problem for (1.18). Therefore, the object of our study is the distributional trace $\operatorname{Tr} U_t$ (see (4.21)) of the *time evolution operator* U_t on a compact (to guarantee discrete eigenvalues) Cauchy hypersurface $\Sigma \subset \mathscr{M}$ without boundary. In addition to the conceptual issues, our computational techniques (see Section 4.3.3 for details) are different

from those used by Sandoval, primarily due to the fact that the Dirac Hamiltonian H_D (see (4.36)) on a stationary spacetime is *not* of Dirac-type in contrast to $\hat{D}$. Furthermore, we have computed σ_{U_t} (see Lemma 4.7.3 and 4.7.4) using the Weitzenböck connection induced by D in order to avoid the aforementioned expedient choice in Sandoval's work, as explained in the comment after the statement of the foremost Theorem 4.3.6.

As a quintessential relativistic operator, it furthermore allows to study the semi-classical limit of quantum field theoretic observables interacting with classical gravity[3]. Whilst spin (see Example 3.4.4)- or spin^c-Dirac operators are desirable from a physics point of view, we prefer to work on more general Dirac-type operators because they do not enforce any topological restrictions on $\mathscr{M}$[4] unlike the spin[5]-Dirac operators.

We would like to close this subsection by mentioning that this is not the only liter-ature on Lorentzian trace formulae rather the first step towards this direction has been taken up for the d'Alembertian operator (see Example 3.2.2) on the same spacetime ge-ometry as in this thesis by Strohmaier and Zelditch [27]. We are going to describe their work briefly in the next section. Therefore, the thesis (see Section 4.3.2 for the primary results) can be viewed as the Lorentzian generalisation of Sandoval's work propounding the framework of Strohmaier-Zelditch into a bundle setting. Recently, McCormick [32] has also extended Strohmaier-Zelditch trace formula for the vector d'Alembertian on a spatially compact globally hyperbolic stationary spacetime equipped with the Kaluza-Klein metric by employing a few results of this investigation.

1.1.5 Strohmaier-Zelditch trace formula

In the set-up of a globally hyperbolic stationary spacetime $(\mathscr{M}, g, \Xi)$ with a compact Cauchy hypersurface (Σ, h), Strohmaier-Zelditch [27] have considered $\Box$, the d'Alembert operator with a smooth potential invariant under the induced Killing flow Ξ_s^*, so that the spectral problem can be stated as

$$\Box \psi_n = 0, \quad Z\psi_n = \lambda_n \psi_n, \tag{1.22}$$

where λ_n are the eigenvalues of the (timelike) complete Killing vector field Z (see Re-mark 4.2.6 (a)) on $\mathscr{M}$ and ψ_n are the joint eigenfunctions of $\Box$ and Z. This eigenvalue problem is equivalent to that of an operator pencil of certain[6] form which entails that the eigenvalue problem cannot be interpreted for some operator on Σ. Furthermore, it reduces to the classical eigenvalue problem $(\Delta - \lambda^2)\phi_n = 0$ only when the spacetime

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[3]We do not require Einstein field equation to be satisfied.

[4]Global existence of a Lorentz metric depends on the topology of $\mathscr{M}$. In particular, such a metric exists in all non-compact manifolds and compact manifolds with vanishing Euler characteristics.

[5]A spin-structure always exists locally but its global existence depends on some higher orientability property of the base manifold. For instance, spin (resp. spin^c) structure exists if and only if the second (resp. the third integral) Stiefel–Whitney class of $\mathscr{M}$ vanishes [31].

[6]See [27, (29)] for the precise form.

metric g is of the product form (1.17), where Δ is the Laplacian on Σ.

In the relativistic setting, the classical phase space is the conic symplectic manifold $\mathcal{N}$ of lightlike (co)geodesics on the cotangent bundle $T^*\mathcal{M}$ of $\mathcal{M}$ in contrast to the cotangent bundle $T^*\Sigma \to \Sigma$ over a Cauchy hypersurface Σ, as expected from the non-relativistic scenario (see Section 4.4.2 for details). The trace $\mathrm{Tr}\, U_t$ of the quantum dynamics, governed by the time-evolution operator U_t of $\Box$, is given by (cf. (1.21))

$$\mathrm{Tr}\, U_t = \mathrm{Tr}_{\ker \Box}\, \mathrm{e}^{tZ}. \tag{1.23}$$

Then, tailoring Duistermaat-Guillemin [10] framework in this Lorentzian set-up, they have achieved asymptotic expressions for $\mathrm{Tr}\, U_{t=0}$ and $\mathrm{Tr}\, U_{T\neq 0}$, and have subsequently obtained a Lorentzian Weyl law [27] (see also [33] and the review [34]).

1.2 Feynman propagators

On a globally hyperbolic spacetime, the Cauchy problem for any normally hyperbolic operator (see Definition 3.2.1) $\Box$ is well-posed. This implies the existence of unique advanced and retarded fundamental solutions (propagators) (see Definition 3.2.10 and Remark 3.2.13). The properties of these fundamental solutions are extremely important for the understanding of classical wave propagation, such as the electromagnetic waves. They appear naturally because of causality: the retarded fundamental solution propagates to the future, whereas the advanced fundamental solution propagates to the past. In quantum field theory, the appearance of time-ordering (see Appendix C.3) and the enforcement of positivity of energy have led to the development of another type of fundamental solution; the Feynman propagator. It propagates *positive energy solutions to the future* and *negative energy solutions to the past*, and thus combines causality with the notion of positivity of energy (see Example C.3.2). The explicit formulae of Lorentz invariant Feynman propagators for normally hyperbolic operators in Minkowski spacetime is available in the standard textbooks of quantum field theory. As recalled in Example C.3.2, the usual construction employs the Fourier transform. However, on a generic globally hyperbolic spacetime there is *neither a Fourier transform nor any reasonable notion of energy in the absence of a global timelike Killing vector field*. A priori, it is, therefore unclear what a Feynman propagator should be.

In the theory of partial differential equations, the notion of a parametrix (see Definition 3.2.14) is often useful in the first stage on the construction of a true fundamental solution. A parametrix, per se, is an inverse of the operator $\Box$ modulo smoothing operators. Parametrices are considered equivalent if they differ by smoothing operators. It was a deep insight of Duistermaat and Hörmander [35] that there is a well-defined notion of Feynman parametrices and these parametrices are unique up to smoothing operators. In other words, they are *unique as parametrices*. In fact, the notion of distinguished parametrices for any scalar pseudodifferential operator of real principal type

(see Definition 2.2.10) is given in their seminal article where they identified a geometric notion of pseudoconvexity which allows one to prove uniqueness of such parametrices. Roughly speaking, despite not not having any notion of energy due to the lack of a global timelike Killing vector field, there is still a microlocal notion of positivity of energy and a corresponding flow on the cotangent bundle. This microlocal notion can be used to characterise Feynman parametrices (see Definition 3.2.17): they are distinguished by their wavefront sets rather than the support properties.

Feynman parametrices play an extremely important role in quantum field theory on curved spacetimes, and the theory has actually been developed to a certain extent first in the physics literature. Canonical quantisation of linear fields can be done in two stages. In the first step (see Appendix C.1), one constructs a field algebra from the space of solutions of the respective equation of motion and next (see Appendix C.2), some quantum state is required to construct a Hilbert space representation of this field algebra. In Minkowski spacetime these two steps are usually combined into one owing to the existence of the vacuum state (see Example C.2.2), while it is more fruitful to separate them in curved spacetimes. The first step can be done without any problems, exactly the same way as in Minkowski spacetime but the second step necessitates a notion of a reasonable state. It has been realised that a state compels certain conditions in order to perform the usual operations in perturbative quantum field theory [36–38]. One of the identified conditions is a restriction of the type of singularity that one obtains from the state, the so-called Hadamard condition (Definition C.2.3). Although Duistermaat and Hörmander were certainly aware of the developments in physics, it was realised only much later by Radzikowski [39] that the expectation values of the time-ordered products with respect to states satisfying the Hadamard condition are Feynman propagators. In fact, the construction of a Hadamard state is equivalent to the construction of a Feynman propagator that satisfies a certain (see Proposition 3.3.2) positivity property. The fact that this positivity property holds for parametrices was already shown by Duistermaat and Hörmander. This property for parametrices implies the existence of such Feynman propagators and hence of Hadamard states (see [40]).

Apart from its applications in physics, Feynman parametrices also play an enormously important role in mathematics. For instance, they appear in the context of Lorentzian index theory as inverses modulo compact operators of the Dirac operator with Atyiah-Patodi-Singer boundary conditions [41], local index theorem [42], and asymptotically static spacetimes [43]. These concepts also arise in Vasy's treatment of asymptotically hyperbolic problems, see for example [44] and references therein, where even a nonlinear problem is discussed in this context. Moreover, the notion of Feynman propagator also comes up naturally in the Lorentzian generalisation of the Duistermaat-Guillemin-Gutzwiller trace formula as observed first in [27].

So far, Duistermaat and Hörmander's construction of distinguished parametrices for scalar pseudodifferential operators of real principal type has not been used much in the

physics literature probably due to its great generality and the associated complex notation. Their main idea is, however, compelling and simple. The operator can be conjugated microlocally to a vector field and it is therefore sufficient to construct a parametrix for the operator of differentiation on the real line. There are two distinguished fundamental solutions for the operator of differentiation. A choice of parametrix for each connected component of the characteristic set of the operator results in a distinguished parametrix. Thus, there are 2^N distinguished parametrices for such a pseudodifferential operator if N is the number of connected components of its characteristic set. For the wave operator, this gives 4 distinguished parametrices in dimensions $d \geq 3$ and 16 distinghuished parametrices in $d = 2$.

The aim of this investigation is threefold. First, we would like to revise and simplify the construction of Duistermaat-Hörmander [35] in the special case of a normally hyperbolic operator. The second aim is to fill a gap in the literature: microlocalisation and the corresponding construction of distinguished parametrices in Duistermaat-Hörmander is covered in the literature only for scalar operators. Several constructions in index theory, in trace formulae, and also in physics require the existence and uniqueness of Feynman parametrices for operators acting on vector bundles. It is known that most of the constructions carry over to the case of any normally hyperbolic operator, since its principal symbol is a scalar. There are, however, also important differences that appear on the level of subprincipal symbols (see Definition 2.3.4). In this study, we would like to give a precise statement (see Theorems 3.3.10 and 3.3.13) of microlocalisation for these class of geometric operators. We then provide a detailed construction (Theorem 3.3.1) of Feynman parametrices for vector bundles with complete proofs for their uniqueness and discuss the effect of curvature of the bundle connection. We show (see Proposition 3.3.2) that the construction can be carried out in such a way that the above mentioned positivity property holds, provided that the vector bundle has a hermitian inner product with respect to which the operator is formally selfadjoint. Subsequently, we promote the Feynman parametrices to the Feynman propagators (see Theorem 3.3.3) by employing the well-posed Cauchy problem for a normally hyperbolic operator on a globally hyperbolic spacetime. Third and finally, for any Dirac-type operator, we give (see Theorem 3.4.8) a much more direct construction of Feynman propagators satisfying a positivity property. We also discuss some consequences, including the usual propagation of singularity theorem (see Theorem 3.3.16). These are well known to hold for very general operators on vector bundles. For instance, the propagation of polarisation sets has been proven by Dencker using microlocalisation in the matrix setting [45].

As explained above (and see (C.16)), the construction of Feynman propagators satisfying the positivity property is equivalent to the construction of Hadamard states. There are several constructions of Hadamard states, even in the analytic category [46]. Amongst the methods to construct them, there are direct ones using singularity expansions employing the Lorentzian distance function (the so-called Hadamard expansion) [40, 47–52], spectral methods that rely on frequency splitting (deformation method)

[53, 54], pseudodifferential methods [55–60] (see also the monograph [61]), hologra-phy [62–65] (see also the exposition [66]), and even global methods [44, 67–69] (see also the review [70]). Many of these constructions can and have been generalised to the bundle case. Each comes with their advantages and disadvantages.

1.3   Organisation

The thesis consists of three chapters and three appendices. Chapter 2 essentially contains the background materials required for the thesis, while our genuine contributions have been placed in Chapters 3 and 4. Appendices A and B primarily supplement Chapter 2 and substantially Chapter 4, whereas Chapter 3 is complemented by Appendix C.

Fourier integral operators are the prime technical tool used in this thesis. Specifically, the symbolic viewpoint has been utilised heavily. We have begun Chapter 2 with the local theory to motivate the global formulation. To make the local-to-global transition smooth, first the naive theory of Fourier integral operators on Euclidean spaces has been given in a pedagogical fashion, and then an invariant local formalism has been presented. This characterisation paves the way to generalise the theory of Lagrangian distributions on manifolds and subsequently on vector bundles. Besides reviewing the standard formu-lae, we have proven bundle version of a few theorems usually available only for the scalar case. Although, these proofs follow the standard ones with minor modifications, they are not easy to find in the existing literature in a systematic way.

The subject matter of Chapter 3 is the Feynman propagators for any normally hy-perbolic operator and any Dirac-type operator on a globally hyperbolic spacetime. To keep the discourse self-contained, we start with explaining the necessary backgrounds on Lorentzian geometry and on these operators. Our primary findings are presented afterwards in Section 3.3.1, 3.4.4, and 3.5.6. In order to derive these results, we have introduced the notion of a P-compatible connection (see Definition 3.3.6) induced by the subprincipal symbol of a pseudodifferential operator P. This enables us to formulate (see Theorem 3.3.10 and Remark 3.3.12) the microlocalisation of a pseudodifferential operator of real-principal type on a vector bundle. Consequently, normally hyperbolic operators are microlocalised (see Theorem 3.3.13).

The trace of the time-flow on the kernel of any Dirac-type operator on a globally hyperbolic stationary spacetime has been investigated in Chapter 4. This chapter is com-prised adequate background on the geometry of stationary spacetimes followed by our primary findings in Section 4.3.2.

Appendix A summarises the composition of vector bundle-valued half-densities on homogeneous canonical relations and relevant materials on conic symplectic geometry.

A write up on vector bundle-valued polyhomogeneous symbol class on a canonical relation has been offered in Appendix B.

Appendix C contains a non-technical prelude to quantum field theory in curved space-time in the algebraic approach and a technical introduction to algebraic quantum states. To relate this approach with the standard physics literature, we have used the Minkowski vacuum as an elucidating example. This also motivates the demand of imposing the Hadamard condition as a necessary criterion to select a physical state in curved space-time.

1.4 Conventions

Throughout the thesis, a vector bundle $\mathscr{E} \to M$ means a smooth $\mathbb{C}$-vector bundle over a Hausdorff, second countable $d \in \mathbb{N}$-dimensional topological space M furnished with an equivalence class of smooth atlases. We use the notation $\mathring{\mathscr{E}}$ to symbolise the zero-section removed part of $\mathscr{E}$. Occasionally, $\mathscr{E}$ is endowed with a non-degenerate sesquilinear form $(\cdot|\cdot)$ which is assumed to be anti-linear in its first argument. By a hermitian form $\langle\cdot|\cdot\rangle$ on $\mathscr{E}$, we mean a positive-definite $(\cdot|\cdot)$. A Lorentzian manifold and its special case - a globally hyperbolic manifold, both are denoted by $(\mathscr{M}, \mathrm{g})$ with metric signature $+ - \ldots -$, and $d := \dim \mathscr{M} \geq 2$.

Let $\Omega^{1/2} \to M$ be the bundle of half-densities over M. We denote the vector spaces of smooth and compactly supported smooth half-densities on $\mathscr{E}$ by $C^\infty(M; \mathscr{E} \otimes \Omega^{1/2})$ and $C_c^\infty(M; \mathscr{E} \otimes \Omega^{1/2})$, and endow them with the Fréchet space and the inductive limit topologies, respectively. The vector spaces of (compactly supported) distributional half-densities $(\mathscr{E}'(M; \mathscr{E} \otimes \Omega^{1/2}) := \big(C^\infty(M; \mathscr{E}^* \otimes \Omega^{1/2})\big)'$ resp.) $\mathscr{D}'(M; \mathscr{E} \otimes \Omega^{1/2}) := \big(C_c^\infty(M; \mathscr{E}^* \otimes \Omega^{1/2})\big)'$ on $\mathscr{E}$ are defined by the topological duals of $(C^\infty(M; \mathscr{E}^* \otimes \Omega^{1/2})$ resp.) $C_c^\infty(M; \mathscr{E}^* \otimes \Omega^{1/2})$, which are equipped with the weak $*$-topologies induced by the topologies of $(C^\infty(M; \mathscr{E}^* \otimes \Omega^{1/2})$ resp.) $C_c^\infty(M; \mathscr{E}^* \otimes \Omega^{1/2})$, where $\mathscr{E}^* \to M$ is the dual bundle of $\mathscr{E}$ (see e.g. [71, p. 307], [72, pp. 24-25, 146] for details of these spaces). On an oriented Lorentzian manifold $(\mathscr{M}, \mathrm{g})$, we use the natural Lorentzian volume element to identify $C^\infty(\mathscr{M}; \mathscr{E}) = C^\infty(\mathscr{M}; \mathscr{E} \otimes \Omega^{1/2}\mathscr{M})$ and $\mathscr{D}'(\mathscr{M}; \mathscr{E}) = \mathscr{D}'(\mathscr{M}; \mathscr{E} \otimes \Omega^{1/2}\mathscr{M})$. Additionally, one defines the space $C_{sc}^\infty(\mathscr{M}; \mathscr{E})$ of spatially compact smooth sections of $\mathscr{E}$ as the set of all $u \in C^\infty(\mathscr{M}; \mathscr{E})$ for which there exists a compact subset K of $\mathscr{M}$ such that $\operatorname{supp} u \subset J(K)$ where $J(K) := J^+(K) \cup J^-(K)$ and $J^\pm(K)$ are the causal future(past) of K; as a vector space $C_{sc}^\infty(\mathscr{M}; \mathscr{E}) \subset C^\infty(\mathscr{M}; \mathscr{E})$. Echoing this spirit $C^\infty(M)$ denotes the set of all $\mathbb{C}$-valued smooth functions on M.

The Fourier transform $\mathcal{F}(u)$ of any $u \in L^1(\mathbb{R}^d, \mathrm{d}x)$ is defined by

$$\mathcal{F}(u) := \int_{\mathbb{R}^d} \mathrm{e}^{-\mathrm{i}x\cdot\theta} u \, \mathrm{d}x, \quad \text{and} \quad u = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \mathrm{e}^{\mathrm{i}x\cdot\theta} \mathcal{F}(u) \, \mathrm{d}\theta, \tag{1.24}$$

whenever $\mathcal{F}(u) \in L^1\big(\mathbb{R}^d, \mathrm{d}\theta/(2\pi)^d\big)$, and here $\cdot$ is either the Euclidean or the Minkowski inner product depending on the context. If $T : C_c^\infty(N; \mathscr{F} \otimes \Omega^{1/2}N) \to \mathcal{D}'(M; \mathscr{E} \otimes \Omega^{1/2}M)$ is an integral operator then its Schwartz kernel is denoted by T.

The symbol $\sim$ stands for the asymptotic summation and only the polyhomogeneous symbol class S^m is utilised in this thesis so that $\Psi\mathrm{DO}^m$ (resp. I^m) is the set of pseudodifferential operators (resp. Lagrangian distributions) having polyhomogeneous total symbol. We have used the notation $S^{m-[m']} := S^m/S^{m-m'}$ to represent the quotient space (and similarly $\Psi\mathrm{DO}^{m-[m']}, I^{m-[m']}$, etc.). We have succinctly written $u \equiv v$ to mean $u - v \in C^\infty(M; \mathscr{E} \otimes \Omega^{1/2})$ for any $u, v \in \mathcal{D}'(M; \mathscr{E} \otimes \Omega^{1/2})$.

Chapter 2

Fourier Integral Operators

The standard theory of Lagrangian distributions on a vector bundle is recalled in this chapter. In order to motivate the global formalism we begin with the local theory of partial differential operators and generalise in ascending order, i.e., pseudodifferential operators are followed by Fourier integral operators. The formulation of pseudodifferential (and Fourier integral) operators on a manifold has been used as an intermediate step to complement the bundle setting.

2.1 Local theory of Fourier integral operators

2.1.1 Differential operators

Let U be an open subset of a $d \in \mathbb{N}$-dimensional Euclidean space $\mathbb{R}^d$. A partial differential operator L of order (at most) $m \in \mathbb{R}$ with smooth coefficients on U can be expressed as

$$L = \sum_{|\alpha| \leq m} f_\alpha \, \mathrm{D}^\alpha : C^\infty(U) \to C^\infty(U), \tag{2.1}$$

for some $f_\alpha \in C^\infty(U)$ and here the multiindex notation (see Appendix B) has been used. Employing Fourier transform $\mathcal{F}(u)$ of $u \in C_c^\infty(U)$ one obtains

$$(Lu)(x) \quad = \quad \int_U \mathsf{L}(x, y) \, u(y) \, \mathrm{d}y, \tag{2.2}$$

$$\mathsf{L}(x, y) \quad := \quad \int_{\mathbb{R}^d} \mathrm{e}^{\mathrm{i}(x-y) \cdot \theta} \mathsf{l}(x, \theta) \frac{\mathrm{d}\theta}{(2\pi)^d}, \tag{2.3}$$

$$\mathsf{l}(x, \theta) \quad := \quad \sum_{|\alpha| \leq m} f_\alpha \theta^\alpha, \tag{2.4}$$

where L and l are called the **Schwartz kernel** and the **total symbol** of L, respectively. The leading order term

$$l(x, \theta) := \sum_{|\alpha|=m} f_\alpha \theta^\alpha \tag{2.5}$$

of the total symbol is called the **principal symbol** of L. Let us denote the set of all partial differential operators of order most $m \in \mathbb{R}$ on U by $\mathrm{PDO}^m(U)$.

One observes that the total symbol

$$\sigma^{\mathrm{tot}} : \mathrm{PDO}^m(U) \to \mathbb{P}_m(U \times \mathbb{R}^d), \quad L \mapsto \sigma_L^{\mathrm{tot}} := \mathsf{l} \tag{2.6'}$$

is an isomorphism between $\mathrm{PDO}^m(U)$ and the set $\mathbb{P}_m(U \times \mathbb{R}^d)$ of all smooth functions on $U \times \mathbb{R}^d$ which are polynomials of degree at most m in $\mathbb{R}^d$. This isomorphism tells that *knowing the total symbol* l *up to smooth terms is equivalent to the knowledge of the operator* L *modulo smooth* $\mathrm{End}\left(C^\infty(U)\right)$, and hence, a sharper version of (2.6') is

$$\sigma^{\mathrm{tot}} : \mathrm{PDO}^{m-[\infty]}(U) \to \mathbb{P}_{m-[\infty]}(\mathrm{T}^*U), \quad [L] \mapsto \sigma_{[L]}^{\mathrm{tot}} := [\mathsf{l}], \tag{2.6}$$

where $\mathrm{PDO}^{m-[\infty]}(U)$ means $\mathrm{PDO}^m(U)$ modulo smooth operators on U. One also observes that changing next to the leading-order terms in (2.1) does not change the principal symbol:

$$\sigma : \mathrm{PDO}^{m-[1]}(U) \to \mathbb{P}_{m-[1]}(\mathrm{T}^*U), \quad [L] \mapsto \sigma_{[L]} := [l], \tag{2.7}$$

where $\mathrm{PDO}^{m-[1]} := \mathrm{PDO}^m/\mathrm{PDO}^{m-1}$. Note that *neither* l *nor* L is well-behaved under the change of coordinates whilst l is. More precisely, let us consider a diffeomorphism

$$\kappa : U \to V, \ x \mapsto y := \kappa(x) \tag{2.8}$$

between two open subsets $U, V \subset \mathbb{R}^d$ whose cotangent lift is

$$\mathrm{T}^*\kappa : \mathrm{T}^*V \to \mathrm{T}^*U, \ (y, \eta) \mapsto (x, \xi) := \mathrm{T}^*\kappa\,(y, \eta) := \left(\kappa^{-1}(y), \mathrm{T}^*_{\kappa^{-1}(y)}\kappa\,(\eta) \right), \tag{2.9}$$

where $\mathrm{T}^*_y V \ni \eta \mapsto \mathrm{T}^*_{\kappa^{-1}(y)}\kappa\,(\eta) := \eta \circ \mathrm{d}_x \kappa \in \mathrm{T}^*_x U$ and $\mathrm{d}_x \kappa : \mathrm{T}_x U \to \mathrm{T}_{\kappa(x)} V$ is the tangent mapping of κ. Hence one has the induced mapping

$$(\mathrm{T}^*\kappa)^* : C^\infty(\mathrm{T}^*U) \to C^\infty(\mathrm{T}^*V), \ a \mapsto \left((\mathrm{T}^*\kappa)^* a \right)(y, \eta) := a\left(\mathrm{T}^*\kappa\,(y, \eta) \right). \tag{2.10}$$

Then the chain rule

$$\mathrm{D}_x^\alpha \mapsto \mathrm{d}_x \kappa\,(\mathrm{D}_x^\alpha) = \left(\frac{\partial \kappa^{j_1}}{\partial x^1} \frac{\partial}{\mathrm{i}\,\partial y^{j_1}} \right)^{\alpha_1} \cdots \left(\frac{\partial \kappa^{j_d}}{\partial x^d} \frac{\partial}{\mathrm{i}\,\partial y^{j_d}} \right)^{\alpha_d} \tag{2.11}$$

entails that the transformed total symbol $\mathsf{l} \mapsto (\mathrm{T}^*\kappa)^*\mathsf{l}$ ends up with messy tangled terms involving higher derivatives of κ, whilst the principal symbol transforms invariantly:

$$\left((\mathrm{T}^*\kappa)^* l \right)(y, \eta) = \sum_{|\alpha|=m} \left((\kappa^{-1})^* f_\alpha \right)(y) \left(\left((\kappa^{-1})^* \frac{\partial \kappa^{j_1}}{\partial x^1} \right)(y)\, \eta_{j_1} \right)^{\alpha_1} \cdots \left(\left((\kappa^{-1})^* \frac{\partial \kappa^{j_d}}{\partial x^d} \right)(y)\, \eta_{j_d} \right)^{\alpha_d}. \tag{2.12}$$

In other words, the following diagrams commute for any $L_U \in \mathrm{PDO}^m(U)$, $L_V \in \mathrm{PDO}^m(V)$:

$$
\begin{array}{ccc}
C^\infty(U) & \xrightarrow{\ L_U\ } & C^\infty(U) \\
{\kappa^*}\big\uparrow & & \big\uparrow{\kappa^*} \\
C^\infty(V) & \xrightarrow{\ L_V\ } & C^\infty(V)
\end{array}
\qquad\qquad
\begin{array}{ccc}
\mathrm{PDO}^{m-[1]}(U) & \xrightarrow{\ \sigma_{[L_U]}\ } & S^{m-[1]}(\mathrm{T}^*U) \\
{\boldsymbol\kappa}\big\downarrow & & \big\downarrow{(\mathrm{T}^*\kappa)^*} \\
\mathrm{PDO}^{m-[1]}(V) & \xrightarrow{\ \sigma_{[L_V]}\ } & S^{m-[1]}(\mathrm{T}^*V)
\end{array}
$$

Figure 2.1: Transformation of a partial differential operator and its principal symbol under the diffeomorphism $\kappa := (2.8)$. In the right diagram, $\mathrm{PDO}^m(U) \ni L_U \mapsto L_V := \boldsymbol\kappa(L_U) := (\kappa^*)^{-1} \circ L_U \circ \kappa^* \in \mathrm{PDO}^m(V)$, and all other symbols are as defined in the ongoing discussion.

In order to achieve an intrinsic characterisation of these notions, we observe that [74] (see also [75], [76, pp. 151-152])

$$
\mathrm{e}^{-\mathrm{i}\lambda x\cdot\xi} L(\mathrm{e}^{\mathrm{i}\lambda x\cdot\xi}) = \sum_{|\alpha|\le m} f_\alpha \xi^\alpha \lambda^\alpha, \quad \forall\lambda \in \mathbb{R}_+ \tag{2.13}
$$

is a polynomial in λ of degree m for any $L \in \mathrm{PDO}^m(U)$. Conversely, any continuous linear operator satisfying the preceding property is a differential operator of order m and can be expressed as (2.2). Evidently, hence

$$
\sigma_L^{\mathrm{tot}}(x,\xi) := \mathrm{e}^{-\mathrm{i}x\cdot\xi} L(\mathrm{e}^{\mathrm{i}x\cdot\xi}), \qquad \sigma_L(x,\xi) := \lim_{\lambda\to\infty} \lambda^{-m} \mathrm{e}^{-\mathrm{i}\lambda x\cdot\xi} L(\mathrm{e}^{\mathrm{i}\lambda x\cdot\xi}). \tag{2.14}
$$

2.1.2 Pseudodifferential operators

The essential notion of a pseudodifferential operator is to consider an operator of the form (2.2) whose Schwartz kernel is also singular at the diagonal yet generalises a differential operator in the sense that it does not respect supports and that the total symbol (2.5) of a differential operator is replaced by a wider class of functions. To be precise, a **pseudodifferential operator** P on U of **order** (at most) $m \in \mathbb{R}$ is a continuous linear map (see e.g. [77, p. 69])

$$
P : C_{\mathrm{c}}^\infty(U) \to C^\infty(U), \ u \mapsto (Pu)(x) := \int_U \mathsf{P}(x,y)\, u(y)\, \mathrm{d}y \tag{2.15}
$$

whose Schwartz kernel $\mathsf{P} \in \mathcal{D}'(U \times U)$ is of the form

$$
\mathsf{P}(x,y) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \mathrm{e}^{\mathrm{i}(x-y)\cdot\theta} \mathsf{p}(x,y;\theta)\, \mathrm{d}\theta, \tag{2.16}
$$

for some element p in the symbol class (see Definition B.1.6) $S^m(U \times U \times \mathbb{R}^d) \supsetneq \mathbb{P}_m(U \times U \times \mathbb{R}^d)$. Note, the preceding expression is an oscillatory integral and thus it must be understood in a distributional sense. We denote the set of all such operators by $\Psi\mathrm{DO}^m(U)$.

Recall, the set of **smoothing operators** $\Psi\mathrm{DO}^{-\infty}(U)$ is defined as the set of all linear continuous operators $C_\mathrm{c}^\infty(U) \to C^\infty(U)$ whose Schwartz kernels are smooth.

Naturally, we are now tempted to have the notions of the total symbol and the principal symbol, for which, let us introduce a subset of $\Psi\mathrm{DO}^m(U)$ which respects support.

Definition 2.1.1. Let $U \subset \mathbb{R}^d$ be an open set and $\Psi\mathrm{DO}^m(U)$ the set of all pseudodifferential operators on U of order (at most) $m \in \mathbb{R}$. An element $P \in \Psi\mathrm{DO}^m(U)$ is called **properly supported** if both projections $\mathrm{supp}\,\mathsf{P} \to U, U$ from the support of its Schwartz kernel P in $U \times U$ to U are proper maps, i.e., for every compact set $K \subset U$ there is a compact set $\tilde{K} \subset U$ such that [78, p. 103] (see also [77, Def. 18.1.21])

$$\mathrm{supp}\,u \subset K \Rightarrow \mathrm{supp}(Pu) \subset \tilde{K}, \quad u \restriction \tilde{K} = 0 \Rightarrow (Pu) \restriction K = 0. \tag{2.17}$$

Analogously, $\mathsf{p} \in S^m(U \times U \times \mathbb{R}^d)$ is called properly supported if the projectors $\mathrm{clo}\big(\mathrm{Pr}(\mathrm{supp}\,\mathsf{p})\big) \to U, U$ are proper, where $\mathrm{Pr}\,\mathsf{p}$ projects p onto $U \times U$ and clo denotes the closure.

A properly supported pseudodifferential operator can be defined with restricting supports, i.e., $P : C_\mathrm{c}^\infty(U) \to C_\mathrm{c}^\infty(U), P : \mathcal{E}'(U) \to \mathcal{E}'(U)$ and moreover $P : C^\infty(U) \to C^\infty(U), P : \mathcal{D}'(U) \to \mathcal{D}'(U)$. These operators are useful because

Proposition 2.1.2. *Every pseudodifferential operator of order m can be written as a combination of an m-order properly supported pseudodifferential operator and a smoothing operator [78, p. 103].*

The concept of the total symbol of a pseudodifferential operator is formalised as follows.

Theorem 2.1.3. *As in the terminologies of Definition 2.1.1, any properly supported $P \in \Psi\mathrm{DO}^m(U)$ can be uniquely expressed as [78, Thm. 2.1.1]*

$$(Pu)(x) = \int_{\mathbb{R}^d} \mathrm{e}^{\mathrm{i}(x-y)\cdot\xi} \sigma_P^\mathrm{tot}(x,\xi)\, u(y)\, \mathrm{d}y\, \frac{\mathrm{d}\xi}{(2\pi)^d}, \tag{2.18}$$

where the total symbol $\sigma_P^\mathrm{tot}(x,\xi) := \mathrm{e}^{-\mathrm{i}\,x\cdot\xi} P(\mathrm{e}^{\mathrm{i}\,\bullet\cdot\xi})(x)$ is given by means of the following asymptotic summation (see Definition B.1.4) formula

$$\sigma_P^\mathrm{tot}(x,\xi) \sim \sum_{\alpha \in \mathbb{N}_0^d} \frac{1}{\alpha!} (\partial_\xi^\alpha \mathrm{D}_y^\alpha \mathsf{p})(x,x,\xi), \tag{2.19}$$

where p as in (2.16) is an element of $S^m(U \times U \times \mathbb{R}^d)$.

Conversely, given any proper $\mathsf{p} \in S^m(U \times U \times \mathbb{R}^d)$, every operator of the form (2.18) with total symbol (2.19) is in the set of properly supported $\Psi\mathrm{DO}^m(U)$.

Since a generic pseudodifferential operator P differs from its properly supported part $\tilde{P}$ only by a smoothing operator, one sets

$$\sigma^{\mathrm{tot}} : \Psi\mathrm{DO}^{m-[\infty]}(U) \to S^{m-[\infty]}(\mathrm{T}^*U), \quad [P] \mapsto \sigma^{\mathrm{tot}}_{[P]} := \sigma^{\mathrm{tot}}_{\tilde{P}}. \tag{2.20}$$

The preceding expression of $\sigma^{\mathrm{tot}}_{[P]}$ then exhibits that (2.4) holds for pseudodifferential operators in the asymptotic sense. Furthermore, this expression paves the way to define the principal symbol $\sigma_{\tilde{P}}$ of $\tilde{P}$, which is precisely the *leading order homogeneous term* in (2.19). Clearly, it does not see smooth terms, and so σ_P is identified with it.

Definition 2.1.4. As in the terminologies of Definition 2.1.1, let $S^m(\mathrm{T}^*U)$ be the space of symbols on the cotangent bundle T^*U over U. Then, the **principal symbol** σ_P of $P \in \Psi\mathrm{DO}^m(U)$ is defined by the isomorphism [78, p. 110]

$$\sigma \; : \; \Psi\mathrm{DO}^{m-[1]}(U) \to S^{m-[1]}(\mathrm{T}^*U), \; [P] \mapsto$$
$$\sigma_{[P]} := \text{homogeneous term of degree } m \text{ in (2.19)} \mod S^{m-1}(\mathrm{T}^*U). \tag{2.21}$$

In terms of transformation properties, σ^{tot}_P and σ_P resemble those of a differential operator. More precisely, under the action of $\kappa := (2.8)$, the total symbol of any properly supported $P \in \Psi\mathrm{DO}^m(U)$ transforms like [78, (2.1.14)] (see also [77, Thm. 18.1.17])

$$\big((\mathrm{T}^*\kappa)^*\sigma^{\mathrm{tot}}_P\big)(y,\eta) \sim \sum_{|\alpha| \leq m} \frac{1}{\alpha!} \partial^\alpha_\xi \mathsf{p}(x,\xi) \, \mathrm{D}^\alpha_{\tilde{x}} e^{\mathrm{i}\,\rho(x,\tilde{x})\cdot(\mathrm{T}^*_x\kappa)^{-1}(\xi)}\Big|_{\tilde{x}=x} \tag{2.22}$$

where $(x,\xi) := \big(\kappa^{-1}(y), \mathrm{T}^*_x\kappa\,\eta\big)$ and $\rho(x,\tilde{x}) := \kappa(\tilde{x}) - \kappa(x) - (\tilde{x}-x)\mathrm{d}_x\kappa$.

Remark 2.1.5. The leading-homogeneous term p in (2.22) satisfies (see e.g. [77, p. 83])

$$\big((\mathrm{T}^*\kappa)^*p\big)(y,\eta) = p\big(x, \mathrm{T}^*_x\kappa\,\eta\big). \tag{2.23}$$

It is then evident that the principal symbol is an invariantly defined homogeneous function on the punctured cotangent bundle $\dot{\mathrm{T}}^*U$ in contrast to the total symbol. In other words, Figure 2.1 holds for pseudodifferential operators as well.

So far, we have specified a (properly supported) pseudodifferential operator by its Schwartz kernel (2.18) which has been given as an oscillatory integral whose integrand is $e^{\mathrm{i}(x-y)\cdot\xi}\sigma^{\mathrm{tot}}_P(x,\xi)$. But, neither the exponent nor the total symbol are invariant under a diffeomorphism, and so it is unforeseeable whether the Schwartz kernel has an invariant meaning. So, we ought to look for an intrinsic characterisation of such class distributions. It is well-recognised that the singularity structure is a characteristic feature of a distribution. But, our purpose cannot be served by the singular support[1] as it is not diffeomorphism invariant. Thus, one introduces the **wavefront set** [79], [78, (2.5.2)]

$$\mathrm{WF}\,u := \bigcap_{\substack{P \in \Psi\mathrm{DO}^0(U) \\ Pu \in C^\infty(U)}} \mathrm{Char}\,P, \quad \mathrm{Char}\,P := \{(x_0,\xi^0) \in \dot{\mathrm{T}}^*U \mid \sigma_P(x_0,\xi^0) = 0\}. \tag{2.24}$$

---

[1]Recall, the **singular support** singsupp u of a distribution $u \in \mathcal{D}'(U)$ is the set of all points in U having no open neighbourhood to which the restriction of u is a smooth function (see e.g. [76, Def. 2.2.3].

of a distribution $u \in \mathcal{D}'(U)$, where the intersection runs over all properly supported P and Char P is called the **characteristic** of P. Evidently, a wavefront set transforms covariantly under a diffeomorphism. Equivalently, $\dot{\mathrm{T}}^*U \ni (x_0, \xi^0) \notin \mathrm{WF}\, u$ if and only if *there exists a compactly supported smooth function f on U non-vanishing at $x_0 \in U$ such that the Fourier transform of $(\mathcal{F}(fu))(\xi)$ is rapidly decreasing in a conic neighbourhood of ξ^0* [78, Prop. 2.5.5]. Note, wavefront set is a generalisation of the singular support because the latter is the base projection of the former [78, Thm. 2.5.3]. Speaking differently, singsupp u *only provides the location of the singularities whilst* $\mathrm{WF}\, u$ *furnishes the cotangent vectors causing them, on top of their locations.* If $\mathrm{A} \in \mathcal{D}'(U \times V)$ is a bidistribution where V is an open subset of any Euclidean space, then its **twisted wavefront set** is given by [78, Thm. 2.5.14]

$$\mathrm{WF}'\, \mathrm{A} := \{(x, \xi; y, -\eta) \in \dot{\mathrm{T}}^*U \times \dot{\mathrm{T}}^*V | (x, \xi; y, \eta) \in \mathrm{WF}\, \mathrm{A}\} \tag{2.25}$$

Details of wavefront set is available, for instance, in [76, 80, 81].

It follows that the wavefront set of the Schwartz kernel of a pseudodifferential operator [78, p. 124] (see also [77, Thm. 18.1.16, 18.1.26]):

$$\mathrm{WF}\, \mathrm{P} \subseteq \{(x, \xi; x, -\xi) \in \dot{\mathrm{T}}^*U \times \dot{\mathrm{T}}^*U\} = (\varDelta U)^{\perp *} \tag{2.26}$$

is contained in the conormal bundle $(\varDelta U)^{\perp *}$ of the diagonal embedding

$$\varDelta : U \to U \times U, \ x \mapsto \varDelta(x) := (x, x). \tag{2.27}$$

By projecting $\mathrm{T}^*U \times \mathrm{T}^*U \to \mathrm{T}^*U$ on the first factor, $(\varDelta U)^{\perp *}$ can be identified with T^*U and

$$\mathrm{WF}\, P = \mathrm{ES}\, P := \{(x, \xi) \in \dot{\mathrm{T}}^*U \mid (x, \xi; x, \xi) \in \mathrm{WF}'\, \mathrm{P}\}, \tag{2.28}$$

is called the **wavefront set**, also known as the **essential support of a pseudodifferential operator** P. This is the *smallest conic set such that P is of order $-\infty$ in* $\dot{\mathrm{T}}^*U \setminus \mathrm{WF}\, P$. In particular, if P is properly supported then $\dot{\mathrm{T}}^*U \setminus \mathrm{WF}\, P$ is the largest open conic set where σ_P^{tot} (2.19) is rapidly decreasing [78, Prop. 2.5.8].

Given a $P \in \Psi\mathrm{DO}^m(U)$, there exists a unique formally dual operator $P^* \in \Psi\mathrm{DO}^m(U)$ such that

$$P^* : \mathcal{E}'(U) \to \mathcal{D}'(U), \ u \mapsto (P^*u)(\phi) := u(P\phi) \tag{2.29}$$

for any $\phi \in C_c^\infty(U)$. Pseudodifferential operators are *microlocal*, meaning that

$$\mathrm{WF}(Pu) \subseteq \mathrm{WF}\, P \cap \mathrm{WF}\, u \Rightarrow \mathrm{singsupp}(Pu) \subseteq \mathrm{singsupp}\, u \tag{2.30}$$

for any $u \in \mathcal{E}'(U)$. Elements of $\Psi\mathrm{DO}^{-\infty}(U)$ are called smoothing operators due to the fact that they map $\mathcal{E}'(U) \to C^\infty(U)$. We remark that, by the Peetre theorem [82] - *every linear and local operator is a partial differential operator L*:

$$\forall u \in \mathcal{D}'(U) : \mathrm{supp}(Lu) \subseteq \mathrm{supp}\, u. \tag{2.31}$$

Thus, pseudodifferential operators are pseudolocal (actually microlocal) generalisation of differential operators in the sense that they respect the singular support (actually wavefront set) instead of the support.

With the precise formulation (2.26) of singularities of P, one furthermore notices that $(\Delta U)^{\perp *}$ is generated by the non-degenerate phase function (see Definition A.1.8) $\varphi := (x - y) \cdot \xi$, as explained in Example A.1.11. In fact, it is the conormal bundle rather the oscillatory integral (2.16) that characterises P invariantly [78] (for details, see e.g. [83], [84, Sec. 2.3], [85, Sec. VIII.1]).

Definition 2.1.6. As in the terminologies of Definition 2.1.4, let $(\Delta U)^{\perp *}$ be the conormal bundle of the diagonal embedding (2.27). Then the space $I^m\big(U \times U, (\Delta U)^{\perp *}\big)$ of **conormal distribution of degree** (at most) m is defined as the set of all bidistributions $P \in \mathcal{D}'(U \times U)$ on U of the form [78, (2.4.1)]

$$P(\phi \otimes u) = \int_{\mathbb{R}^d} \int_U \int_U e^{i(x-y)\cdot\xi} p(x, y, \xi)\, \phi(x)\, u(y)\, \mathrm{d}x\, \mathrm{d}y \frac{\mathrm{d}\xi}{(2\pi)^d}, \qquad (2.32)$$

where $u, \phi \in C_c^\infty(U)$ and $p \in S^m(U \times U \times \mathbb{R}^d)$.

Then, $P \in \Psi DO^m(U)$ is invariantly defined as the integral operator (2.15) whose Schwartz kernel $P \in I^m\big(U \times U, (\Delta U)^{\perp *}\big)$.

Example 2.1.7. Let Δ be the Laplacian on an open set $U \subset \mathbb{R}^d$. Then $\sqrt[n]{\Delta} \in \Psi DO^{2/n}(U)$.

2.1.3 Fourier integral operators

As observed in the last two sections, the partial differential operators are local whereas the pseudodifferential operators are microlocal. Fourier integral operators are a vast generalisation of the latter in the sense that it maps a function on a set to a distribution on another set by allowing a general phase function. Let $U \subset \mathbb{R}^{d_U}$ and $V \subset \mathbb{R}^{d_V}$ be open sets. Then, a **Fourier integral operator of order** (at most) $m \in \mathbb{R}$ is the continuous linear map [78, Sec. 1.4] (see also [23, Prop. 25.1.5'] and, e.g. [85, Sec. VI.2])

$$A : C_c^\infty(V) \to \mathcal{D}'(U), \ v \mapsto (Av)(x) := \int_V A(x, y)\, v(y)\, \mathrm{d}y \qquad (2.33)$$

whose Schwartz kernel A is given by the oscillatory integral of the form

$$A(x, y) = (2\pi)^{-(d_U + d_V + 2n - 2e)/4} \int_{\mathbb{R}^n} e^{i\,\varphi(x, y; \theta)} a(x, y; \theta)\, \mathrm{d}\theta \qquad (2.34)$$

where φ is a clean phase function with excess e on $U \times V \times \dot{\mathbb{R}}^n$ (see Definition A.1.8), $a \in S^{m+(d_U + d_V - 2n - 2e)/4}(U \times V \times \mathbb{R}^n)$ and $\mathrm{d}\theta$ is the Lebesgue measure on $\mathbb{R}^n$.

Being an oscillatory integral, (2.34) must be understood as a *formal expression* that does not pointwise make sense rather it defines a bidistribution $\mathcal{D}'(U \times V)$ in the sense that [78, Sec. 1.2] (see also, e.g. [84, Sec. 2.2-2.3], [8, Sec. 1.1-1.3] for details)

$$(Av)(\phi) := \mathsf{A}(\phi \otimes v) = (2\pi)^{-(d_U + d_V + 2n - 2e)/4} \int_{\mathbb{R}^n} \int_V \int_U e^{i\,\varphi(x,y;\theta)} \mathsf{a}(x, y; \theta)\, \phi(x)\, v(y)\, \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}\theta \tag{2.35}$$

for any $\phi \in C_c^\infty(U)$. The precise meaning of this expression is as follows [78, Thm. 1.4.1]:

(i) If φ has no critical point as a function of $(x, y; \theta)$ then the oscillatory integral (2.35) exists and it is a continuous bilinear form for the C_c^k-topologies on v and ϕ whenever

$$m - k < -n. \tag{2.36}$$

When (2.36) is valid then one obtains a continuous linear map

$$A : C_c^k(V) \to \mathcal{D}'^k(U) \tag{2.37}$$

whose Schwartz kernel $\mathsf{A} \in \mathcal{D}'^k(U \times V)$ is given by the oscillatory integral

$$\mathsf{A}(u) = \int_{\mathbb{R}^n} \int_V \int_U e^{i\,\varphi(x,y;\theta)} a(x, y; \theta)\, u(x, y)\, \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}\theta, \quad \forall u \in C_c^\infty(U \times V). \tag{2.38}$$

(ii) If, for each fixed x, φ has no critical point $(y_0; \theta^0)$ then (2.34) is defined as an oscillatory integral. One obtains a continuous map

$$A : C_c^k(V) \to C(U) \tag{2.39}$$

when (2.36) is valid. By differentiation under the integral sign, it follows that A is also a continuous map from $C_c^k(V)$ to $C^j(U)$ provided

$$m + n + j < k. \tag{2.40}$$

(iii) If, for each fixed y, φ has no critical point $(x_0; \theta^0)$ then the adjoint of A has the properties listed in (ii), so

$$A : \mathcal{E}'^j(V) \to \mathcal{D}'^k(U) \tag{2.41}$$

is a continuous map when (2.40) is fulfilled. In particular, A defines a continuous map from $\mathcal{E}'(V)$ to $\mathcal{D}'(U)$.

(iv) Let $\mathscr{R}$ be the open set of all $(x, y) \in U \times V$ such that $\varphi(x, y; \theta)$ has no critical point as a function of θ. Then the oscillatory integral

$$\mathsf{A}(x, y) = (2\pi)^{-(d_U + d_V + 2n - 2e)/4} \int_{\mathbb{R}^n} e^{i\,\varphi(x,y;\theta)} a(x, y; \theta)\, \mathrm{d}\theta \tag{2.42}$$

defines a function in $C^\infty(\mathscr{R})$ which is equal to the distribution (2.38) in $\mathscr{R}$. If $\mathscr{R} = U \times V$, it follows that A is an integral operator with the smooth Schwartz kernel $\mathsf{A} \in C^\infty(U \times V)$, so

$$A : \mathcal{E}'(V) \to C^\infty(U) \tag{2.43}$$

is a continuous map, called the **smoothing operator** from V to U.

As described for the kernel of a pseudodifferential operator, we would like to characterise the oscillatory integral A in an invariant fashion starting with the observation that its wavefront set [78, Thm. 3.2.6]

$$\text{WF}\,A \subseteq C', \quad C' := \left\{ (x_0, \mathrm{d}_{x_0}\varphi; y_0, -\mathrm{d}_{y_0}\varphi) \in \dot{\mathrm{T}}^*U \times \dot{\mathrm{T}}^*V \,|\, (\mathrm{grad}_\theta\,\varphi)(x_0, y_0; \theta^0) = 0 \right\} \quad (2.44)$$

is contained in the homogeneous twisted canonical relation (Definition A.1.12) C' from $\dot{\mathrm{T}}^*V$ to $\dot{\mathrm{T}}^*U$ which is closed in $\dot{\mathrm{T}}^*(U \times V)$. The Schwartz kernel of a Fourier integral operator is then intrinsically formalised as,

Definition 2.1.8. Let $U \subset \mathbb{R}^{d_U}, V \subset \mathbb{R}^{d_V}$ be open sets and $C \subset \dot{\mathrm{T}}^*U \times \dot{\mathrm{T}}^*V$ a homogeneous canonical relation which is closed in $\dot{\mathrm{T}}^*(U \times V)$. Then the space $I^m(U \times V, C')$ of **Lagrangian distributions of order** (at most) $m \in \mathbb{R}$ is defined as the set of all bidistributions $A \in \mathcal{D}'(U \times V)$ on $U \times V$ of the form (2.35) [78, Def. 3.2.2].

Therefore, *a Fourier integral operator A associated with a closed homogeneous canonical relation $C \subset \dot{\mathrm{T}}^*U \times \dot{\mathrm{T}}^*V$ is defined by the integral operator* (2.33) *whose Schwartz A kernel is an element of $I^m(U \times V, C')$.* We denote the set of all such operators by $\text{FIO}^m(V \to U, C')$. Note, given an oscillatory integral representation of A of the form (2.34), the order μ of homogeneity of a is related to the order m of A by (see e.g. [84, (2.4.22)])

$$m = \mu + \frac{n}{2} - \frac{d_U + d_V}{4}. \quad (2.45)$$

The apparently counter-intuitive $(2\pi)^{-(d_U + d_V + 2n - 2e)/4}$ factor in (2.35) has been chosen in order to match the $(2\pi)^{-d}$ factor in (2.16) when $d_U = d_V = d = n$ and $e = 0$. As expected, $I^m(U \times U, (\Delta U)^{\perp *})$ is a special case of $I^m(U \times V, C')$ when C is given by the graph of the identity homogeneous symplectomorphism on U, globally generated by the particular non-degenerate phase function $\varphi := (x - y) \cdot \xi$ (as elucidated in Example A.1.11). Besides $C^\infty(U \times V) \subset I^{-\infty}(U \times V, C')$ (see e.g. [85, Prop. 3.2 (p. 439)]), i.e., the kernel of a smoothing operator (2.43) can be thought as an element of $I^{-\infty}(U \times V, C')$.

In order to define the principal symbol of a Fourier integral operator, one cannot directly look for an isomorphism between $\text{FIO}^m(V \to U, C')$ and the respective symbol class, as done for pseudodifferential operators (Definition 2.1.4) because there is no analog of (2.19) for the total symbol of a Fourier integral operator due to the fact that a canonical relation is more intricate than the conormal bundle $(\Delta U)^{\perp *}$. Since a pseudodifferential operator is characterised by its Schwartz kernel and $(\Delta U)^{\perp *} \cong \mathrm{T}^*U$, its principal symbol isomorphism map (2.21) is equivalent to [78, Thm. 2.4.2]

$$I^{m-[1]}(U \times U, (\Delta U)^{\perp *}) \cong S((\Delta U)^{\perp *}). \quad (2.46)$$

This paves the right direction to address the principal symbol of a Fourier integral operator. In other words, one pursues an isomorphism between $I^m(U \times V, C')$ and respective symbol space on C. It turns out that one actually requires to consider half-density (see Section 2.2.1 and Remark 2.2.8 for further motivation)-valued symbols on C of order $m + (d_U + d_V)/4$ in order to define an invariant principal symbol [78, Thm. 3.2.1].

Definition 2.1.9. As in the terminologies of Definition 2.1.8, suppose that $\Omega^{1/2}C, \mathbb{M} \to C$ are the half-density bundle (see Appendix A.1.6) and the Keller-Maslov bundle (see Definition A.1.19) over C, respectively, and that $S^{m+(d_U+d_V)/4-[1]}(C; \mathbb{M} \otimes \Omega^{1/2}C)$ is the space (see (B.14)) of $\mathbb{M} \otimes \Omega^{1/2}C$-valued symbols on C. Then, the **principal symbol** of a Lagrangian distribution is defined by the isomorphism [10, (7.6)] (see also [23, Prop. 25.1.5'])

$$\sigma \ : \ I^{m-[1]}(U \times V, C') \to S^{m+\frac{d_U+d_V}{4}-[1]}(C; \mathbb{M} \otimes \Omega^{1/2}C),$$

$$[A] \ \mapsto \ \sigma_{[A]}(x, \xi; y, \eta) := \int_{\mathfrak{C}_{\xi,\eta}} d\theta'' a(x, y; \theta', \theta'') \mathbb{m} \otimes |dv_C|^{\frac{1}{2}} \mod S^{m+\frac{d_U+d_V}{4}-1}(\cdot), \quad (2.47)$$

where $I^{m-[1]} := I^m/I^{m-1}$, a is the top-order homogeneous term of a in (2.34), $\mathbb{m}$ and $\sqrt{|dv_C|}$ are sections of $\mathbb{M}$ and $\Omega^{1/2}C$, respectively. For each $(x, \xi; y, \eta) \in C$, φ is a representative of the stable equivalence class (see Remark A.1.9) of clean phase functions with excess e (see Definition A.1.8) for C and

$$\mathfrak{C}_{\xi,\eta} := \{(x, y; \theta) \in (\mathrm{grad}_\theta \varphi)^{-1}(0) \mid d_x\varphi := \xi, d_y\varphi =: \eta\} \quad (2.48)$$

is the e-dimensional fibre over the corresponding Lagrangian fibration (see (A.33)). Here, $d\theta''$ is the Lebesgue measure on $\dot{\mathbb{R}}^e$ and the variable θ'' is defined by the splitting[2] $\dot{\mathbb{R}}^n \in \theta = (\theta', \theta'') \in \dot{\mathbb{R}}^{n-e} \times \dot{\mathbb{R}}^e$ such that the projection $\mathfrak{C}_{\xi,\eta} \ni (x, y; \theta', \theta'') \mapsto \theta'' \in \mathbb{R}^e$ has a bijective differential so that, for a fixed $\theta'' = \mathrm{cst}$, $\varphi(x, y; \theta', \mathrm{cst})$ is non-degenerate.

Explicit expressions of $\mathbb{m}$ (see Appendix A.1.4) and $\sqrt{|dv_C|}$ (see Appendix A.1.6) in terms of φ entail [23, (25.1.4)', p. 15]

$$\sigma_A(x, \xi; y, \eta) = \sqrt{|d\xi||d\eta|} \int_{\mathfrak{C}} a(x, y; \theta', \theta'') \frac{e^{i\pi/4 \, \mathrm{sgn}(\mathrm{Hess}_{x,y;\theta'} \varphi)}}{\sqrt{|\det(\mathrm{Hess}_{x,y;\theta'} \varphi)|}} d\theta'', \quad (2.49)$$

where $\mathrm{Hess}_{x,y;\theta'} \varphi$ is the Hessian matrix (see (3.18)) of $\varphi(x, y; \theta', \mathrm{cst})$ with respect to $(x, y; \theta') \in \mathfrak{C}_{\xi,\eta}$:

$$\mathrm{Hess}_{x,y;\theta'} \varphi = \begin{pmatrix} \dfrac{\partial^2 \varphi}{\partial x^i \partial x^{i\grave{}}} & \dfrac{\partial^2 \varphi}{\partial x^i \partial y^j} & \dfrac{\partial^2 \varphi}{\partial x^i \partial \theta'_k} \\[2mm] \dfrac{\partial^2 \varphi}{\partial y^j \partial x^i} & \dfrac{\partial^2 \varphi}{\partial y^j \partial y^{j\grave{}}} & \dfrac{\partial^2 \varphi}{\partial y^j \partial \theta'_k} \\[2mm] \dfrac{\partial^2 \varphi}{\partial \theta'_k \partial x^i} & \dfrac{\partial^2 \varphi}{\partial \theta'_k \partial y^j} & \dfrac{\partial^2 \varphi}{\partial \theta'_k \partial \theta'_{k\grave{}}} \end{pmatrix} \quad (2.50)$$

for $i, i\grave{} = 1, \ldots, d_U$; $j, j\grave{} = 1, \ldots, d_V$; $k, k\grave{} = 1, \ldots, n-e$. Here, $d\xi \, d\eta$ is the Lebesgue measure on C at $(x, \xi; y, \eta)$.

---

[2]Such splitting is always possible due to the Thom splitting (also known as the parametrised Morse) lemma (see e.g. [77, App. C. 6], [86, p. 52]).

If C is parametrised by a non-degenerate phase function φ, i.e., $e = 0$, then the θ-variable is no longer required to be split up and so the expression of σ_A simplifies to [78, Thm. 3.2.5] (see also [23, Prop. 25.1.5] and (A.68))

$$\sigma_{[A]}(x, \xi; y, \eta) := a(x, y; \theta) \frac{e^{i\pi/4 \operatorname{sgn}(\operatorname{Hess} \varphi)}}{\sqrt{|\det(\operatorname{Hess} \varphi)|}} \sqrt{|d\xi||d\eta|} \mod S^{m + \frac{d_U + d_V}{4} - 1}(\cdot),$$

$$\operatorname{Hess} \varphi = \begin{pmatrix} \dfrac{\partial^2 \varphi}{\partial x^i \partial x^{i`}} & \dfrac{\partial^2 \varphi}{\partial x^i \partial y^j} & \dfrac{\partial^2 \varphi}{\partial x^i \partial \theta_k} \\ \dfrac{\partial^2 \varphi}{\partial y^j \partial x^i} & \dfrac{\partial^2 \varphi}{\partial y^j \partial y^{j`}} & \dfrac{\partial^2 \varphi}{\partial y^j \partial \theta_k} \\ \dfrac{\partial^2 \varphi}{\partial \theta_k \partial x^i} & \dfrac{\partial^2 \varphi}{\partial \theta_k \partial y^j} & \dfrac{\partial^2 \varphi}{\partial \theta_k \partial \theta_{k`}} \end{pmatrix}. \tag{2.51}$$

Remark 2.1.10. The half-density σ_A is invariant under any diffeomorphism apart from the Maslov factor of absolute value 1 [23, (25.1.14)].

Example 2.1.11. Suppose that $U \subset \mathbb{R}^d$ and $V \subset \mathbb{R}^n$ are open sets and that $\kappa \in C^\infty(U, V)$. Then, the definition of pullback

$$\kappa^* : C_c^\infty(V) \to C^\infty(U), \; u \mapsto \kappa^* u := u \circ \kappa \tag{2.52}$$

entails (see e.g. [84, p. 37], [87, Sec. 5.25])

$$(\kappa^* u)(x) = u(\kappa(x)) = \delta_{\kappa(x)}(u) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \int_V e^{i(\kappa(x) - y) \cdot \eta} u(y) \, dy \, d\eta. \tag{2.53}$$

That is, κ^* is a Fourier integral operator whose Schwartz kernel is given by

$$\mathsf{K} := (2\pi)^{-(n - d + d + 3n)/4} \int_{\mathbb{R}^n} e^{i\varphi} d\eta, \quad \varphi := (\kappa(x) - y) \cdot \eta \tag{2.54}$$

where the phase function φ is non-degenerate whose fibre-critical set (see (A.18)) is

$$\mathscr{C} := \{(x_0, y_0; \eta^0) \in U \times V \times \dot{\mathbb{R}}^n \mid y_0 = \kappa(x_0)\}. \tag{2.55}$$

Thereby, κ^* is of order ((2.45)) $\frac{n - d}{4}$ and is associated with the canonical relation (see (A.33))

$$C := \{(x, \xi; y, -\eta) \in T^*U \times \dot{T}^*V \mid y = \kappa(x), \xi = d_x^* \kappa(\eta)\}, \tag{2.56}$$

where $d_x^* \kappa : T_{\kappa(x)}^* V \to T_x^* U$ is the algebraic dual of the tangent mapping $d_x \kappa : T_x U \to T_{\kappa(x)} V$ of κ. Furthermore, its principal symbol is given by (cf. (2.51))

$$\sigma_{\mathsf{K}}(x, \xi; y, \eta) := \frac{\sqrt{|d\xi||d\eta|}}{(2\pi)^{\frac{n-d}{4}}} \frac{e^{i\pi/4 \operatorname{sgn}(\operatorname{Hess} \varphi)}}{|\det(\operatorname{Hess} \varphi)|^{\frac{1}{2}}}, \quad \operatorname{Hess} \varphi = \begin{pmatrix} \dfrac{d^2 \kappa^j}{dx^i dx^{i`}} \eta_j & 0 & \dfrac{d\kappa^j}{dx^i} \\ 0 & 0 & -1 \\ \dfrac{d\kappa^j}{dx^i} & -1 & 0 \end{pmatrix}, \tag{2.57}$$

where we have used $\kappa(x) = (\kappa^1(x), \ldots, \kappa^n(x))$.

In addition, if κ is a diffeomorphism then we can consider its cotangent lift $\mathrm{T}^*\kappa$, and thus $C' = \left\{ \left(\mathrm{T}^*\kappa\,(y,\eta); (y,\eta) \right) \right\}$ is the graph of the induced symplectomorphism

$$\mathrm{T}^*\kappa : \dot{\mathrm{T}}^*V \to \dot{\mathrm{T}}^*U, \ (y,\eta) \mapsto \varkappa(y,\eta) := \left(\kappa^{-1}(y), \mathrm{T}^*_x\kappa\,(\eta) \right). \tag{2.58}$$

A careful look at C in the preceding example reveals that C is *not* homogeneous because ξ in (2.56) is not necessarily non-zero even if η is. As a consequence, $\mathrm{Hess}\,\varphi$ in σ_κ may vanish. This entails that κ^* is *not* a *homogeneous* Fourier integral operator. The assumption of homogeneity in this context ensures that a Fourier integral operator (and its dual) maps smooth functions to *smooth functions* rather than distributions (see e.g. [23, (25.2.1)]). We will discuss a plausible solution of this problem for the restriction map (a special case of the preceding example) in a bundle setting in Example 2.3.12.

2.2   Fourier Integral Operators on manifolds

We would like to extend our preceding discussion on manifolds in this section. As pseudodifferential operators and their generalisation - Fourier integral operators are defined as integral operators by means of invariantly defined Schwartz kernels, it follows that those kernels must be integrable, at least in their second variable. However, there is no invariant way to integrate a function on a manifold. Thus, we are compelled to work with densities instead of functions and particularly, half-densities turn out to be convenient.

2.2.1   Distributional half-densities

Densities

We recall that an $s \in \mathbb{C}$-**density** on a d-dimensional real vector space $\mathscr{V}$ is a function $\mu : \mathscr{V}^d \to \mathbb{C}$ such that (see e.g. [25, Sec. 6.1-6.3], [85, pp. 374-377], [88, pp. 428-432])

$$\mu(Tv_1, \dots, Tv_d) = |\det T|^s \mu(v_1, \dots, v_d) \tag{2.59}$$

for any linear transformation $T : \mathscr{V} \to \mathscr{V}$ and for all $(v_1, \dots, v_d) \in \mathscr{V}^d$. Such a quantity *always exists* and the set of all such densities is denoted by $\Omega^s(\mathscr{V})$. It is a 1-dimensional vector space over $\mathbb{C}$.

The s-**density bundle** $\Omega^s M \to M$ over a manifold M is defined by

$$\forall s \in \mathbb{C} : \ \Omega^s M := \bigsqcup_{x \in M} \Omega^s(\mathrm{T}_x M), \tag{2.60}$$

which is a (smooth) *trivialisable* $\mathbb{C}$-line bundle, i.e., for each s, $\Omega^s(\mathrm{T}_x M) \cong \mathbb{C}$ but in a non-canonical fashion. A section of $\Omega^s M$ are called an s-**density** on M; $s = 1$-density is simply called a density. Let $(U, (x^i))$ be a local chart around $x \in M$ and $\left(\mathrm{T}_x U, (\partial/\partial x^i) \right)$

(resp. $\left(\mathrm{T}_x^* U, (\mathrm{d}x^i)\right)$) the induced natural chart for the tangent bundle $\mathrm{T}M \to M$ (resp. cotangent bundle $\mathrm{T}^*M \to M$). Then

$$|\mathrm{d}x^1(x) \wedge \ldots \wedge \mathrm{d}x^d(x)|^s\left(\frac{\partial}{\partial x^1}(x) \wedge \ldots \wedge \frac{\partial}{\partial x^d}(x)\right) = 1 \qquad (2.61)$$

defines a canonical local section $|\mathrm{d}x|^s := |\mathrm{d}x^1 \wedge \ldots \wedge \mathrm{d}x^d|^s \in C^\infty(U; \Omega^s M)$, where $\partial/\partial x^1(x) \wedge \ldots \wedge \partial/\partial x^d(x)$ is the generator of the line $\dot{\Lambda}^d \mathrm{T}_x M$ defined by local coordinates. $|\mathrm{d}x|^s$ defines a nowhere-vanishing section of $\Omega^s M$ by allowing x to vary in the preceding equation. Any (compactly supported) smooth s-density μ on M can locally be expressed as

$$\mu = f|\mathrm{d}x|^s \qquad (2.62)$$

for a unique $\mathbb{C}$-valued (compactly supported) smooth function $f = \mu(\partial/\partial x^1 \wedge \ldots \wedge \partial/\partial x^d)$ on U. A density μ on M is said to be **positive** if its value at each point x of M has that property: $\mu(x)(X_1, \ldots, X_d) > 0$ for any linearly independent d-tuple of tangent vectors $(x; X_1, \ldots, X_d)$ on M. Note, *a smooth manifold M always admits a smooth positive density on M and the set of positive densities on M is an open subset of ΩM whose intersection with each fibre is convex*.

Example 2.2.1. (a) Let $M := \mathbb{R}^d$ with standard coordinates $(x^1, \ldots, x^d)$ and f be a smooth function on $\mathbb{R}^d$. A canonical trivialisation of the density bundle $\Omega\mathbb{R}^d \to \mathbb{R}^d$ is provided by Lebesgue measure $\mathrm{d}x$. Then the most general smooth s-density on $\mathbb{R}^d$ is given by $\mu := f|\mathrm{d}x|^\alpha$ and it is positive whenever f is so.

(b) Let M be an oriented manifold, i.e., it admits a volume form $\mathrm{d}v$. Then, every smooth function f on M defines a smooth s-density $\mu := f|\mathrm{d}v|^s$ on M, where $|\mathrm{d}v|$ is a positive density, defined by $|\mathrm{d}v|_x := |\mathrm{d}v(x)|$ for each $x \in M$.

(c) Let (M, σ) be a $2d$-dimensional symplectic manifold (see Section A.1.1) and $f \in C^\infty(\mathrm{M})$. M is always orientable because it admits a smooth canonical volume form, the **Liouville form**

$$\mathrm{d}v := (-1)^{d(d-1)/2}\frac{\sigma^d}{d!}, \quad \sigma^d := \sigma \wedge \ldots \wedge \sigma \qquad (2.63)$$

and thus any smooth s-density on M is given by $\mu := f|\mathrm{d}v|^s$. In particular, the cotangent bundle $\mathrm{T}^*M \to M$ over a d-dimensional manifold M is a symplectic manifold and its Liouville form is given by (see Example A.1.2)

$$\mathrm{d}v \restriction U = \mathrm{d}x^1 \wedge \ldots \wedge \mathrm{d}x^d \wedge \mathrm{d}\xi_1 \wedge \ldots \wedge \mathrm{d}\xi_d = \mathrm{d}x\,\mathrm{d}\xi \qquad (2.64)$$

in the Darboux chart $\left(U, (x^i, \xi_i)\right)$. Thus, any smooth s-density on T^*M is globally (resp. locally) given by $\mu = f|\mathrm{d}v|^s$ (resp. $\mu = f(|\mathrm{d}x||\mathrm{d}\xi|)^s$).

(d) Let $(\mathcal{M}, \mathrm{g})$ be a d-dimensional Lorentzian manifold (see Section 3.1). At each $x \in \mathcal{M}$, $(\mathcal{M}, \mathrm{g})$ admits a *canonical smooth volume element*, the **Lorentzian volume element** dv_g, induced by the Lorentzian metric g (see e.g. [89, p. 195]):

$$\mathrm{dv}_\mathrm{g}(x)(X_1, \ldots, X_d) := \det\big(\mathrm{g}_x(X_n, e_i)\big), \tag{2.65}$$

where $X_n \in \mathrm{T}_x\mathcal{M}$ and $\{e_i\}$ is an orthogonal basis for $\mathrm{T}_x\mathcal{M}$. In terms of a local oriented coordinate chart $(U, (x^i))$ on $\mathcal{M}$, this canonical volume element reads

$$\mathrm{dv}_\mathrm{g}(x) = \sqrt{|\det \mathrm{g}|}\, \mathrm{d}x^1 \wedge \ldots \mathrm{d}x^d. \tag{2.66}$$

Thus, every smooth s-density μ on $\mathcal{M}$ is locally given by $\mu(x) = f(x)|\mathrm{d}x|^s$ for a unique $f \in C^\infty(U)$. However, dv_g *might not exists globally* in contrast to μ. The (global) *Lorentzian volume form exists if and only if $\mathcal{M}$ is orientable* and then one identifies a smooth function on $\mathcal{M}$ with a smooth s-density on $\mathcal{M}$ employing the Lorentzian s-density $|\mathrm{dv}_\mathrm{g}|^s := |\det \mathrm{g}|^{s/2}|\mathrm{d}x|^s$:

$$C^\infty(\mathcal{M}) \ni f \mapsto f|\mathrm{dv}_\mathrm{g}|^s \in C^\infty(\mathcal{M}; \Omega^s\mathcal{M}). \tag{2.67}$$

(e) A (Σ, ι) semi-Riemannian hypersurface of an orientable semi-Riemannian manifold $(\mathcal{M}, \mathrm{g})$ is orientable if and only if it admits a smooth unit normal vector field N. In particular, if c is a value of $f \in C^\infty(\mathcal{M}, \mathbb{R})$ then $\Sigma := f^{-1}(c)$ is a semi-Riemannian hypersurface of $\mathcal{M}$ if and only if $\mathrm{g}(\mathrm{grad}\, f, \mathrm{grad}\, f) \lessgtr 0$ on Σ. In this case

$$\mathrm{N} := \frac{\mathrm{grad}\, f}{\|\mathrm{grad}\, f\|_\mathrm{g}} \tag{2.68}$$

is a (unique up to $\pm$) unit normal vector field along Σ. Thus Σ is orientable whose orientation is determined by N and its volume form is given by (see e.g. [89, Lem. 7.8, Prop. 4.17], [88, Prop. 15.21])

$$\mathrm{dv}_\Sigma = \iota^*(\mathrm{N} \lrcorner \mathrm{dv}_\mathcal{M}), \tag{2.69}$$

where $\lrcorner$ is the interior multiplication.

To illustrate this, let us consider the standard d-sphere $\mathbb{S}^d$. The inclusion $\iota : \mathbb{S}^d \hookrightarrow \mathbb{R}^{d+1}$ is a smooth embedding and the vector field $\mathrm{N} := x^i \partial/\partial x^i$ is nowhere tangent to $\mathbb{S}^d$ and thus N induces a density on $\mathbb{S}^d$ via the preceding equation. In particular, $\mathrm{d}x = \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3$ is a volume form of $\mathbb{R}^3$ and then the preceding equation entails that $|x^1\, \mathrm{d}x^2 \wedge \mathrm{d}x^3 + x^2\, \mathrm{d}x^3 \wedge \mathrm{d}x^1 + x^3\, \mathrm{d}x^1 \wedge \mathrm{d}x^2|$ is the induced density on $\mathbb{S}^2$, where $(x^1)^1 + (x^2)^2 + (x^3)^2 = 1$ on $\mathbb{S}^2$ (see e.g. [88, Exm. 15.22, Ex. 16 − 9]).

In what follows, it is required to consider homogeneous (half-)densities on a punctured cotangent bundle and on its conic Lagrangian subsmanifolds. These are examples (see Example A.1.2, A.1.11) of a conic symplectic manifold $(\mathcal{M}, \mathrm{m}_\lambda)$ (see Definition A.1.1 for a precise formulation). A half-density μ on $\mathcal{M}$ is called **homogeneous** resp. **positively homogeneous** of degree $k \in \mathbb{R}$ if

$$\forall \lambda \in \dot{\mathbb{R}} \quad (\text{resp. } \mathbb{R}_+) : \mathrm{m}_\lambda^* \mu = \lambda^k \mu. \tag{2.70}$$

Example 2.2.2. (a) The standard s-density (Example 2.2.1 (a)) $|\mathrm{d}\theta|^s$ on $\mathring{\mathbb{R}}^n$ is homogeneous of degree sn (see e.g. [23, p. 13], [84, (2.1.13)]).

(b) As in Example 2.2.1 (c), the conormal bundle (see Example A.1.7) $(\Delta M)^\perp$ of the diagonal in $M \times M$ is isomorphic to T^*M. Hence it has a natural half-density $|\mathrm{d}x|^{1/2}|\mathrm{d}\xi|^{1/2}$ of degree $\dim(M \times M)/4 = d/2$ (see e.g. [77, p. 104]).

(c) The density $|\mathrm{d}v_C|$ on a canonical relation C as constructed in Appendix A.1.6 and used in Definition 2.1.9 is of degree n because the degree of homogeneity of the Lebesgue density (see (A.65)) on the fibre-critical manifold is $d_M + d_N$ whereas that of the determinant in (A.65) is $d_M + d_N - n$ (see e.g. [11, p. 123]).

Distributional densities

The pointwise product between a s-density and $1-s$ density is a 1-density which permits us to define the following bilinear form (see e.g. [9, Sec. 4.2], [84, Sec. 1.1])

$$\langle \cdot, \cdot \rangle : C^\infty(M; \Omega^s M) \times C_c^\infty(M; \Omega^{1-s} M) \to \mathbb{C}, \ (\mu, \phi) \mapsto \langle \mu, \phi \rangle := \int_M \mu(x)\, \phi(x). \quad (2.71)$$

This form is continuous because $|\langle \mu, \phi \rangle| \leq \|\mu\|_{L^1(M;\sqrt{\Omega})} \|\phi\|_\infty$. In this thesis only $s = 1/2$ will be used unless stated otherwise. Thus, we define the space $\mathcal{D}'(M; \Omega^{1/2}M) := \big(C_c^\infty(M; \Omega^{1/2}M)\big)'$ of **distributional half-densities** on M as the *topological dual* of $C_c^\infty(M; \Omega^{1/2}M)$, i.e., the space of all $\mathbb{C}$-valued continuous $\mathbb{C}$-linear functionals on $C_c^\infty(M; \Omega^{1/2}M)$. Throughout the thesis, $\mathcal{D}'(M; \Omega^{1/2}M)$ will be endowed with the weak $*$-topology. Note, one can define distributions $\mathcal{D}'(M)$ on M by patching up distributions $\mathcal{D}'(U_\alpha)$ on local charts $\{(U_\alpha, \kappa_\alpha)\}_\alpha$ for M, where $\mathcal{D}'(U_\alpha)$ is obtained by pushing-forward the distributions $\mathcal{D}'\big(\kappa_\alpha(U_\alpha); \Omega\mathbb{R}^d\big)$. But, $\mathcal{D}(M; \Omega) \neq \mathcal{D}'(M)$ (see e.g. [76, Sec. 6.3], [9, Sec. 4.2.2]).

2.2.2 Pseudodifferential operators

To put forward the concept of a conormal distribution on a manifold, one can choose a local chart where the invariant Definition 2.1.6 applies and then, pullback the local pieces, glue together and prove chosen coordinate chart independence. However, it turns out that there is an intrinsic way to define this object which requires to introduce the notion of a Besov space as described below.

Let $1 \leq \mathfrak{p} \leq \infty$ and $m \in \mathbb{R}$. The **Besov space** $\mathfrak{p}H^m(\mathbb{R}^d)$ is the Banach space of all temperate distributions $u \in \mathcal{S}'(\mathbb{R}^d)$ on a Euclidean space $\mathbb{R}^d$ such that $\mathcal{F}(u) \in L^2_{\mathrm{loc}}(\mathbb{R}^d)$ and the norm (for $\mathfrak{p} \neq \infty$) $\mathfrak{p}\|u\|_m := \big(\sum_i \|(2\pi)^{-d}\int_{U_i} e^{ix\cdot\xi}(\mathcal{F}u)(\xi)\,\mathrm{d}\xi\|_m^{\mathfrak{p}}\big)^{1/\mathfrak{p}}$ is finite; in case $\mathfrak{p} = \infty$ one sets $\infty\|u\|_m := \big(\sup_i \|(2\pi)^{-d}\int_{U_i} e^{ix\cdot\xi}(\mathcal{F}u)(\xi)\,\mathrm{d}\xi\|_m^{\mathfrak{p}}\big)^{1/\mathfrak{p}}$. Here, $\|\cdot\|_s$ denotes the Sobolev norm and $U_i := \{\xi \in \mathbb{R}^d \mid R_{i-1} < |\xi| < R_i\}, R_0 := 0, R_i := 2^{i-1}$ for all $i > 0$ (see

e.g. [77, Def. B.1.1]). If $U \subset \mathbb{R}^d$ is an open subset, then the **Besov space** $^{\mathfrak{p}}H_{\text{loc}}^s(U)$ is defined as the set of all distributions $u \in \mathcal{D}'(U)$ such that $\phi u \in {}^{\mathfrak{p}}H^s(\mathbb{R}^d)$ for every $\phi \in C_c^\infty(U)$. More generally, on a manifold M, the **Besov space** $^{\mathfrak{p}}H_{\text{loc}}^s(M)$ is defined as the set of all distributions $u \in \mathcal{D}'(M)$ such that $(\kappa^{-1})^* u \in {}^{\mathfrak{p}}H_{\text{loc}}^s(\kappa(U))$ for every smooth coordinate chart (U, κ) for M. The topology is defined by the seminorms $u \mapsto {}^{\mathfrak{p}}\|\phi(\kappa^{-1})^* u\|_s$ where ϕ is an arbitrary element in $C_c^\infty(\kappa(U))$ (see e.g. [77, p. 475]).

Definition 2.2.3. Let $\Omega^{1/2} \to M$ be the half-density bundle over a d-dimensional manifold M and $S \subset M$ a closed submanifold. Then the space $I^m(M, S; \Omega^{1/2})$ of **distributional half-densities** on M, **conormal** with respect to S and of **degree** (at most) $m \in \mathbb{R}$, is defined as the set of all distributional half-densities $u \in \mathcal{D}'(M; \Omega^{1/2})$ such that $L_1 \dots L_N u \in {}^\infty H_{\text{loc}}^{-m-d/4}(M; \Omega^{1/2})$ for all $N \in \mathbb{N}_0$ and for all $L_i \in \text{PDO}^1(M; \Omega^{1/2})$ with smooth coefficients tangential to S. Here $^\infty H_{\text{loc}}^{-m-d/4}(M; \Omega^{1/2})$ denotes the Besov space and the topology is the weakest one which makes the maps $u \mapsto L_1 \dots L_N u \in {}^\infty H_{\text{loc}}^{-m-d/4}(M; \Omega^{1/2})$ continuous [77, Def. 18.2.6].

Definition 2.2.4. Let $\Omega^{1/2} \to M$ be the bundle of half-densities over a manifold M and $m \in \mathbb{R}$. A **pseudodifferential operator** P on M of **order** (at most) m is a continuous linear mapping (see e.g [77, p. 100])

$$P : C_c^\infty(M; \Omega^{1/2}) \to C^\infty(M; \Omega^{1/2}), \quad u \mapsto (Pu)(x) := \int_M \mathsf{P}(x, y)\, u(y) \qquad (2.72)$$

whose Schwartz kernel P is an element in the space of conormal distributions $I^m(M \times M, (\Delta M)^{\perp *}; \Omega^{1/2}(M \times M))$ where $\Delta : M \to M \times M$ is the diagonal embedding. The set of all such operators is denoted by $\Psi\text{DO}^m(M; \Omega^{1/2})$.

The abstract definition implies that a pseudodifferential operator P on a manifold M is locally a collection $\{P_\alpha\}_\alpha$ of pseudodifferential operators $P_\alpha \in \Psi\text{DO}^m(\kappa_\alpha(U_\alpha))$ on local charts $\{(U_\alpha, \kappa_\alpha)\}_\alpha$ for M. To see this assertion, let $\sqrt{|\mu|}$ be an arbitrary but fixed half-density on M and let $\hat{\mathsf{M}}, \check{\mathsf{M}} \in \text{End}(C_c^\infty(U_\alpha))$ be defined as multiplications by $\hat{\chi}, \check{\chi} \in C_c^\infty(U_\alpha)$, respectively. Then (see e.g. [77, p. 85], [90, Sec. 7.3]),

$$P_\alpha := P_{\kappa_\alpha} := (\kappa_\alpha^*)^{-1} \circ P \restriction U_\alpha \circ \kappa_\alpha^* \in \Psi\text{DO}^m(\kappa_\alpha(U_\alpha)),$$
$$P \restriction U_\alpha := \hat{\mathsf{M}} \circ P \circ \check{\mathsf{M}} \in \Psi\text{DO}^m(U_\alpha) \qquad (2.73)$$

is a local representative of P with respect to the local charts and the chosen half-density. By Proposition 2.1.2, the Schwartz kernel $\mathsf{P}_\alpha(\boldsymbol{x}, \boldsymbol{y})\sqrt{|\mu_\alpha(\boldsymbol{x})|} \otimes \sqrt{|\mu_\alpha(\boldsymbol{y})|}$ of P_α must be of the form

$$\mathsf{P}_\alpha(\boldsymbol{x}, \boldsymbol{y}) = \int_{\mathbb{R}^d} e^{i(\boldsymbol{x}-\boldsymbol{y})\cdot\boldsymbol{\xi}} \sigma_{P_\alpha}^{\text{tot}}(\boldsymbol{x}, \boldsymbol{\xi}) \frac{d\boldsymbol{\xi}}{(2\pi)^d} \quad \text{mod } C^\infty(\kappa_\alpha(U_\alpha) \times \kappa_\alpha(U_\alpha)) \qquad (2.74)$$

and vice-versa. Here, the total symbol $\sigma_{P_\alpha}^{\text{tot}}(\boldsymbol{x}, \boldsymbol{\xi})$ is given by (2.19) and $\mu_\alpha(\boldsymbol{x})$ is the Euclidean representative (cf. (2.62) and Example 2.2.1(a)) of the chosen density at $\boldsymbol{x} :=$

$\kappa_\alpha(x) \in \kappa_\alpha(U_\alpha)$. In general, it is necessary to have sufficiently many local patches U_α for M so that the products $U_\alpha \times U_\alpha$ form an atlas for $M \times M$. But P is off-diagonally smooth, and thus, it suffices to consider just an atlas $\{(U_\alpha, \kappa_\alpha)\}_\alpha$ for M.

Remark 2.2.5. Let $\dot{T}^*M \to M$ be the punctured cotangent bundle over a manifold M and let $\{(U_\alpha, \kappa_\alpha)\}_\alpha$ (resp. $\{(\mathcal{U}_\alpha, \varkappa_\alpha)\}_\alpha$) be an atlas (resp. a homogeneous symplectic atlas (see Theorem A.1.4 and Example A.1.2)) for M (resp. $\dot{T}^*M$) such that

$$\varkappa_\alpha : \mathcal{U}_\alpha \to \kappa_\alpha(U_\alpha) \times \dot{\mathbb{R}}^d. \tag{2.75}$$

The preceding map allows us to pullback $\sigma_{P_\alpha}^{\text{tot}}$ to obtain $\sigma_{P \upharpoonright U_\alpha}^{\text{tot}} \in S^m(\mathcal{U}_\alpha)$ (see Appendix B.2). But, $\sigma_{P_\alpha}^{\text{tot}}$ does not behave nicely under a coordinate transformation as observed in (2.22) and thus it is impossible to patch up the local isomorphisms $\Psi\text{DO}^{m-[\infty]}(U_\alpha) \cong S^{m-[\infty]}(\mathcal{U}_\alpha)$ to obtain a global one.

Analogously, $\sigma_{P \upharpoonright U_\alpha}$ is simply obtained by pulling back the leading order homogeneous term σ_{P_α} in $\sigma_{P_\alpha}^{\text{tot}}$:

$$\sigma : \Psi\text{DO}^m(U_\alpha) \to S^m(\mathcal{U}_\alpha), \ P \upharpoonright U_\alpha \mapsto \sigma_{P \upharpoonright U_\alpha} := \varkappa_\alpha^* \sigma_{P_\alpha}. \tag{2.76}$$

In contrast to $\sigma_{[P]}^{\text{tot}}$, $\sigma_{[P]}$ can be promoted to the global isomorphism $\Psi\text{DO}^{m-[1]}(M) \cong S^{m-[1]}(T^*M)$ due to the fact that $\sigma_{P \upharpoonright U_\alpha}$ is an invariantly defined homogeneous function on $\mathcal{U}_\alpha$ as Remarked in 2.1.5 (see e.g. [77, pp. 85-86]).

Definition 2.2.6. As in the terminologies of Definition 2.2.4, let $S^m(T^*M)$ be the space of half-density-valued symbols (see Appendix B.2) on the cotangent bundle T^*M over M. Then, the **principal symbol** σ_P of $P \in \Psi\text{DO}^m(M; \Omega^{1/2})$ is defined by the isomorphism (see e.g. [77, (18.1.29)′])

$$\sigma : \Psi\text{DO}^{m-[1]}(M; \Omega^{1/2}) \to S^{m-[1]}(T^*M), \tag{2.77}$$

where $\Psi\text{DO}^{m-[1]} := \Psi\text{DO}^m / \Psi\text{DO}^{m-1}$. In a homogeneous symplectic chart (as in Remark 2.2.5) $\{(\mathcal{U}_\alpha, \varkappa_\alpha)\}_\alpha$ for $\dot{T}^*M$, the preceding isomorphism is given by

$$[P] \mapsto \sigma_{[P]}(x, \xi) := \varkappa_\alpha^*(\sigma_{[P_\alpha]}) |dv_{T^*M}(x, \xi)|^{\frac{1}{2}} \mod S^{m-1}(\cdot), \tag{2.78}$$

where dv_{T^*M} is the natural volume (Example 2.2.1(c)) form on T^*M and $\sigma_{[P_\alpha]}$ is the principal symbol of a Euclidean representative $P_\alpha := (2.73)$ of P.

At this point one may wonder whether the next-to-leading order term in $\sigma_{[P]}^{\text{tot}}$ has some invariant meaning. Unfortunately, this naive expectation does not go well, rather the following combination works.

Definition 2.2.7. As in the terminologies of Definitions 2.2.4 and 2.2.6, the **subprincipal symbol** σ_P^{sub} of $P \in \Psi\mathrm{DO}^m(M; \Omega^{1/2})$ is defined by the isomorphism [35, (5.2.8)] (see also [77, Thm. 18.1.33])

$$
\sigma^{\mathrm{sub}} \ : \ \Psi\mathrm{DO}^{m-[2]}(M; \Omega^{1/2}) \to S^{m-1-[1]}(\mathrm{T}^*M),
$$

$$
[P] \mapsto \sigma_{[P]}^{\mathrm{sub}}(x, \xi) := p_{m-1}|\mathrm{dv}_{\mathrm{T}^*M}(x, \xi)|^{\frac{1}{2}} + \frac{\mathrm{i}}{2}\frac{\partial^2 \sigma_P}{\partial x^i \partial \xi_i}(x, \xi) \quad \mathrm{mod}\ S^{m-2}(\cdot), (2.79)
$$

where p_{m-1} is the next-to-leading order term in the expression (cf. (2.19)) of the total symbol of P, pulled back to $\mathcal{U}_\alpha$.

Remark 2.2.8. $\sigma_{P\upharpoonright U}^{\mathrm{sub}}$ transforms as [35, Prop. 5.2.1] (see also [77, (18.1.33′)]):

$$
\big((\mathrm{T}^*\kappa)^* \sigma_P^{\mathrm{sub}}\big)(\boldsymbol{x}, \boldsymbol{\xi}) = \sigma_{P_\kappa}^{\mathrm{sub}}\big(\boldsymbol{x}, \mathrm{T}_x^*\kappa\,\boldsymbol{\xi}\big) \tag{2.80}
$$

under a coordinate change $U \ni x \mapsto \boldsymbol{x} := \kappa(x) \in \kappa(U)$. Thus it is an *invariantly defined homogeneous function* on $\dot{\mathrm{T}}^*M$ (cf. Remark 2.1.5).

We will compute subprincipal symbol of $P \in \Psi\mathrm{DO}^m(\mathcal{M})$ on a Lorentzian manifold $(\mathcal{M}, \mathrm{g})$ in Chapter 3. Then, one has the *canonical* Lorentzian volume form dv_{g} (Example 2.2.1 (d)) so that $\sigma_P = p\sqrt{|\mathrm{dv}_{\mathrm{g}}|}$, and σ_P^{sub} is given by (see e.g. [24, Rem. 2.1.10])

$$
\sigma_P^{\mathrm{sub}}(x, \xi) = p_{m-1}(x, \xi) + \frac{\mathrm{i}}{2}\frac{\partial^2 p}{\partial x^i \partial \xi_i}(x, \xi) + \frac{\mathrm{i}}{2}\Gamma_{ji}^j(x)\frac{\partial p}{\partial \xi_i}(\xi) \quad \mathrm{mod}\ S^{m-2}(\dot{\mathrm{T}}^*\mathcal{M}) \tag{2.81}
$$

where p, p_{m-1} are considered as functions and Γ_{ji}^j are the contracted Christoffel symbols on $\mathcal{M}$. Therefore, *subprincipal symbol is independent of local coordinates* but *depends on the chosen half-density*.

Figure 2.2: Transformation of a pseudodifferential operator, its principal symbol and subprincipal symbol under a coordinate chart (U, κ).

We now introduce the following terminologies required to define a special class of pseudodifferential operators which is going to play a pivotal role in this thesis.

Definition 2.2.9. Let M be a manifold and H a Hamiltonian function on its cotangent bundle T*M. A **bicharacteristic** (strip) of H through a point $(x_0, \xi^0) \in$ T*M is the integral curve $\gamma : \mathbb{R} \supseteq \mathbb{I} \to$ T*M of the Hamiltonian vector field X_H (see (A.3)) with initial data $\gamma(s_0) := (x_0, \xi^0)$. If $H(x_0, \xi^0) = 0$ then γ is called the **null bicharacteristic** (strip) of H. The projection $c : \mathbb{I} \to M$ of γ on M is called the **bicharacteristic** (curve).

Definition 2.2.10. As in the terminologies of Definition 2.2.4, any properly supported $P \in \Psi\text{DO}^m(M; \Omega^{1/2})$ is of **real principal type** on a manifold M if P has a real homogeneous principal symbol p of order m and no complete null bicharacteristic (strip) of p stays over a compact set in M [35, Def. 6.3.2].

Remark 2.2.11. The non-trapping property for the bicharacteristics of p locally means that dp is not proportional to the Liouville form (see Example A.1.2) θ of T*M at any $(x_0, \xi^0) \in \text{Char}\, P$. Equivalently, this can be rephrased as that X_p and the radial vector field (see Definition A.8) are linearly independent at every $(x_0, \xi^0) \in \text{Char}\, P$:

$$\forall (x_0, \xi^0) \in \text{Char}\, P : X_p(x_0, \xi^0) \neq 0. \tag{2.82}$$

This property implies that Char P is a *smooth, closed conic hypersurface of dimension* $2d - 1$ (see e.g. [23, p. 54], [85, Sec. VIII.7]).

2.2.3 Lagrangian distributions

To define a Lagrangian distribution on M one observes that $I^m(M, S; \Omega^{1/2})$ is the largest subspace of the Besov space (introduced in Section 2.2.2) $^\infty H_{\text{loc}}^{-m-d/4}(M; \Omega^{1/2})$ which is invariant under the action of all first-order partial differential operators tangent to S. Moreover, any first-order pseudodifferential operator having vanishing principal symbol on $S^{\perp *}$ keeps $I^m(M, S; \Omega^{1/2})$ invariant [77, Thm. 18.2.12]. Since conormal bundles are a particular case of Lagrangian submanifolds (see Example A.1.7), the last fact is used to define Lagrangian distributions.

Definition 2.2.12. Let $\Omega^{1/2}, \dot{\text{T}}^*M \to M$ be the bundle of half-densities and the punctured cotangent bundle over a d-dimensional manifold M and $\Lambda \subset \dot{\text{T}}^*M$ a closed conic Lagrangian submanifold (see Definition A.1.5). Then the space $I^m(M, \Lambda; \Omega^{1/2})$ of **Lagrangian distributional half-densities** on M, of **order** (at most) $m \in \mathbb{R}$, is defined as the set of all distributional half-densities $u \in \mathcal{D}'(M; \Omega^{1/2})$ such that $P_1 \ldots P_N u \in {}^\infty H_{\text{loc}}^{-m-d/4}(M; \Omega^{1/2})$ for all $N \in \mathbb{N}_0$ and for all properly supported $P_i \in \Psi\text{DO}^1(M; \Omega^{1/2})$ having vanishing principal symbols on Λ, where $^\infty H_{\text{loc}}^{-m-d/4}(M; \Omega^{1/2})$ denotes the Besov space [23, Def. 25.1.1].

In pursuance of utilising $I^m(M, \Lambda; \Omega^{1/2})$ for the Schwartz kernel of a Fourier integral operator, M has to be replaced by a product manifold $M \times N$ where N is an arbitrary manifold and then the role of Λ will be played by a canonical relation (see Definition A.1.12).

Definition 2.2.13. Let M, N be manifolds endowed with the bundles of half-densities $\Omega^{1/2}M \to M, \Omega^{1/2}N \to N$. Suppose that $\dot{T}^*M$ (resp. $\dot{T}^*N$) is the punctured cotangent bundle of M (resp. N) and that C is a homogeneous canonical relation form $\dot{T}^*N$ to $\dot{T}^*M$ which is closed in $\dot{T}^*(M \times N)$. A linear continuous operator [23, Def. 25.2.1]

$$A : C_c^\infty(N; \Omega^{1/2}N) \to \mathcal{D}'(M; \Omega^{1/2}M), \ v \mapsto (Av)(x) := \int_N A(x, y)\, v(y) \qquad (2.83)$$

whose Schwartz kernel A belongs to the Lagrangian distribution $I^m(M \times N, C'; \Omega^{1/2}(M \times N))$, is called the **Fourier integral operator of order** (at most) $m \in \mathbb{R}$ from compactly supported half-densities on N to distributional half-densities on M **associated with the canonical relation** C. We will denote set of all such operators by $\mathrm{FIO}^m(\Omega^{1/2}N \to N, \Omega^{1/2}M \to M; C')$.

Similar to the pseudodifferential operators, Fourier integral operators can also be written as a collection of local Fourier integral operators on atlases for M and N. But their localisation is intricate because a generic canonical relation is more complicated than the conormal bundle $(\Delta M)^{\perp *}$ and the requisite to localise in covectors together with their base points makes our previous localisation by cutoff functions inadequate. We localise an element of $I^m(M \times N, C'; \Omega^{1/2}(M \times N))$ by making use of

Lemma 2.2.14. *As in the terminologies of Definition 2.2.12, if $u \in I^m(M, \Lambda; \Omega^{1/2})$ then*

$$\mathrm{WF}\, u \subset \Lambda \qquad (2.84)$$

and $Pu \in I^m(M, \Lambda; \Omega^{1/2})$ if $P \in \Psi\mathrm{DO}^0(M; \Omega^{1/2})$. Conversely, $u \in I^m(M, \Lambda; \Omega^{1/2})$ if for every $(x, \xi) \in \dot{T}^*M$ one can find properly supported $Q \in \Psi\mathrm{DO}^0(M; \Omega^{1/2})$ non-characteristic at (x, ξ) such that $Qu \in I^m(M, \Lambda; \Omega^{1/2})$ [23, Lem. 25.1.2].

Therefore, appropriate pseudodifferential operators play the same role for Lagrangian distributions that cutoff functions (cf. (2.73)) do for the conormal bundle $(\Delta M)^{\perp *}$ and the preceding lemma reduces the investigation of $A \in I^m(M \times N, C'; \Omega^{1/2}(M \times N))$ to the case where WF A is contained in a small closed conic neighbourhood of $(x, \xi; y, \eta) \in C'$ and $(x, y) \in \mathrm{supp}\, A$. More generally, given a homogeneous canonical relation C of the open cone $\mathfrak{U} \times \mathfrak{B} \subset \dot{T}^*M \times \dot{T}^*N$, one says that $A \in I^m(M \times N, C'; \Omega^{1/2}(M \times N))$ at $(x, \xi; y, \eta) \in \mathfrak{U} \times \mathfrak{B}$ if there is an open conic neighbourhood $\mathcal{U} \times \mathcal{V}$ of $(x, \xi; y, \eta)$ contained in $\mathfrak{U} \times \mathfrak{B}$ such that $(P \otimes Q)A \in I^m(M \times N, C'; \Omega^{1/2}(M \times N))$ for all properly supported $P \in \Psi\mathrm{DO}^0(M; \Omega^{1/2}M)$ (resp. $Q \in \Psi\mathrm{DO}^0(N; \Omega^{1/2}N)$) having $\mathrm{WF}\, P \subset \mathrm{clo}(\mathcal{U})$ (resp. $\mathrm{WF}\, Q \subset \mathrm{clo}(\mathcal{V})$). In fact, it suffices to know this for some such $P \otimes Q$ non-characteristic at $(x, \xi; y, \eta)$ [23, p. 5].

In order to get a local description of A, let $\{(U_\alpha, \kappa_\alpha)\}_\alpha$ (resp. $\{(V_\beta, \rho_\beta)\}_\beta$) be atlas for M (resp. N). Since $\dot{T}^*M$ and $\dot{T}^*N$ are conic symplectic manifolds (see Definition A.1.1 and Example A.1.2), there are homogeneous atlas $\{(\mathcal{U}_\alpha, \varkappa_\alpha)\}_\alpha$ (resp. $\{(\mathcal{V}_\beta, \varrho_\beta)\}_\beta$) for $\dot{T}^*M$ (resp. $\dot{T}^*N$) such that

$$\varkappa_\alpha : \mathcal{U}_\alpha \to \kappa_\alpha(U_\alpha) \times \dot{\mathbb{R}}^{d_M}, \qquad \varrho_\beta : \mathcal{V}_\beta \to \rho_\beta(V_\beta) \times \dot{\mathbb{R}}^{d_N}, \qquad (2.85)$$

by the homogeneous Darboux Theorem A.1.4. For each $(x, \xi; y, \eta) \in C'$, suppose that $[\varphi_{\alpha,\beta}]$ is an element of the stable equivalence class of phase functions for C' with excess $e_{\alpha,\beta}$ (see Definition A.1.8 and Remark A.1.9), defined on an open conic neighbourhood of $\big(\boldsymbol{x} := \kappa_\alpha(x), \boldsymbol{y} := \rho_\beta(y), \theta\big)$ in $\kappa_\alpha(U_\alpha) \times \rho_\beta(V_\beta) \times \dot{\mathbb{R}}^{n_{\alpha,\beta}}$ for some $n_{\alpha,\beta} \in \mathbb{N}$, and that

$$\mathscr{C}_{\alpha,\beta} := \{(\boldsymbol{x}_0, \boldsymbol{y}_0; \theta^0) \in \kappa_\alpha(U_\alpha) \times \rho_\beta(V_\beta) \times \dot{\mathbb{R}}^{n_{\alpha,\beta}} | \operatorname{grad}_\theta \varphi_{\alpha,\beta}(\boldsymbol{x}_0, \boldsymbol{y}_0; \theta^0) = 0\} \qquad (2.86)$$

is the fibre-critical manifold of $\varphi_{\alpha,\beta}$. Then (see (A.33))

$$C'_{\alpha,\beta} := \{(\boldsymbol{x}, \boldsymbol{\xi}; \boldsymbol{y}, \boldsymbol{\eta}) \in \varkappa_\alpha(\mathcal{U}_\alpha) \times \varrho_\beta(\mathcal{V}_\beta) | (\boldsymbol{x}, \boldsymbol{y}; \theta) \in \mathscr{C}_{\alpha,\beta}, \boldsymbol{\xi} = \mathrm{d}_{\boldsymbol{x}} \varphi_{\alpha,\beta}, \boldsymbol{\eta} = \mathrm{d}_{\boldsymbol{y}} \varphi_{\alpha,\beta})\} \quad (2.87)$$

is a Euclidean representative of C' in a conic neighbourhood $\mathcal{U}_\alpha \times \mathcal{V}_\beta$ of $(x, \xi; y, \eta) \in C'$. Hence, locally

$$\mathsf{A} = \sum_{\alpha,\beta} (\kappa_\alpha \times \rho_\beta)_*^{-1} \mathsf{A}_{\alpha,\beta} \qquad (2.88)$$

where $\operatorname{supp} \mathsf{A}_{\alpha,\beta}$ is locally finite and module $C^\infty\big(\kappa_\alpha(U_\alpha) \times \rho_\beta(V_\beta)\big)$ is of the form [78, Def. 3.2.2], [23, Prop. 25.1.5']:

$$\mathsf{A}_{\alpha,\beta}(\boldsymbol{x}, \boldsymbol{y}) \equiv (2\pi)^{-(d_M + d_N + 2n_{\alpha,\beta} - 2e_{\alpha,\beta})/4} \int_{\mathbb{R}^{n_{\alpha,\beta}}} \mathrm{e}^{\mathrm{i}\,\varphi_{\alpha,\beta}(\boldsymbol{x}, \boldsymbol{y}, \theta)} \mathsf{a}_{\alpha,\beta}(\boldsymbol{x}, \boldsymbol{y}; \theta) \,\mathrm{d}\theta. \qquad (2.89)$$

Here, $\equiv$ means modulo smoothing kernels, $\mathsf{a}_{\alpha,\beta} \in S^{m+(d_M + d_N - 2n_{\alpha,\beta} - 2e_{\alpha,\beta})/4}(\mathbb{R}^{d_M} \times \mathbb{R}^{d_N} \times \mathbb{R}^{n_{\alpha,\beta}})$ having support in the interior of a sufficiently small conic neighbourhood of $\mathscr{C}_{\alpha,\beta}$ contained in the domain of definition of $\varphi_{\alpha,\beta}$ and $\mathrm{d}\theta$ is the Lebesgue measure at $\theta \in \dot{\mathbb{R}}^{n_{\alpha,\beta}}$.

Since the principal symbol of a scalar Lagrangian distribution has good transformation property under a diffeomorphism, one defines σ_A in a local chart to identify with $\sigma_{\mathsf{A}_{\alpha,\beta}}$ by enjoying the independence with respect to the chosen coordinates.

Definition 2.2.15. As in the terminologies of Definition 2.2.12, let $\Omega^{1/2}C, \mathbb{M} \to C$ be the half-density bundle (see Appendix A.1.6) and the Keller-Maslov bundle (see Definition A.1.19) over C, respectively, and $S^{m+(d_M+d_N)/4-[1]}(C; \mathbb{M} \otimes \Omega^{1/2}C)$ the space (see (B.14)) of $\mathbb{M} \otimes \Omega^{1/2}C$-valued symbols on C. Then the **principal symbol** of a Lagrangian distribution is defined by the isomorphism [10, (7.6)] (see also [23, Prop. 25.1.5'])

$$\sigma : I^{m-[1]}\big(M \times N, C'; \Omega^{1/2}(M \times N)\big) \to S^{m+\frac{d_M+d_N}{4}-[1]}(C; \mathbb{M} \otimes \Omega^{1/2}C), \qquad (2.90)$$

where $I^{m-[1]} := I^m/I^{m-1}$. In a homogeneous symplectic chart $(\mathcal{U}_\alpha \times \mathcal{V}_\beta, \varkappa_\alpha \times \varrho_\beta)$ for C' (see e.g. [77, Thm. 21.2.8]) induced from those $\{(\mathcal{U}_\alpha, \varkappa_\alpha)\}_\alpha$ (resp. $\{(\mathcal{V}_\beta, \varrho_\beta)\}_\beta$) for $\dot{\mathrm{T}}^*M$ (resp. $\dot{\mathrm{T}}^*N$) with (2.85), the preceding isomorphism is given by

$$[\mathsf{A}] \mapsto \sigma_{[\mathsf{A}]}(x, \xi; y, \eta) := (\varkappa_\alpha \times \varrho_\beta)^* \left(\int_{\mathbb{C}} \mathrm{d}\theta'' a_{\alpha,\beta}(\boldsymbol{x}, \boldsymbol{y}; \theta', \theta'') \right) \mathrm{m}(x, \xi; y, \eta) \otimes$$
$$\sqrt{|\mathrm{d}\mathrm{v}_C(x, \xi; y, \eta)|} \mod S^{m+\frac{d_U+d_V}{4}-1}(\cdot), \qquad (2.91)$$

where $a_{\alpha,\beta}$ is the top-order homogeneous term of $\mathsf{a}_{\alpha,\beta}$ in (2.89) and all other symbols are as in Definition 2.1.9.

Example 2.2.16. Let (Σ, ι_Σ) be an n-dimensional immersed submanifold of a d- dimensional manifold M and $\Omega^{1/2}M \to M$ the bundle of half-densities over M. Then the pullback

$$\iota_\Sigma^* : C_c^\infty(M; \Omega^{1/2}M) \to C^\infty(\Sigma; \Omega^{1/2}M_\Sigma) \tag{2.92}$$

of the injective immersion $\iota_\Sigma : \Sigma \hookrightarrow M$ is called the **restriction map**. It is known that Σ admits an adapted atlas, i.e., an atlas whose charts are induced from the charts of M. In other words, for each $x' \in \Sigma$, there exist a chart (V, ρ) for Σ at x' and a chart (U, κ) for M at $\iota_\Sigma(x')$ such that $\iota_\Sigma(V) \subset U$ and

$$\kappa \circ \iota_\Sigma \circ \rho^{-1}(\boldsymbol{x}') = (0, \dots, 0, \boldsymbol{x}^{d-n+1}, \dots, \boldsymbol{x}^d) \tag{2.93}$$

is a chart for Σ. We read off from as a special case of Example 2.1.11 that ι_Σ^* is a Fourier integral operator whose Schwartz kernel is given by

$$\begin{aligned}
\mathsf{R}_\Sigma(\boldsymbol{x}', \boldsymbol{y}) &:= (2\pi)^{-(d-n+n+3d)/4} \int_{\mathbb{R}^d} e^{i\varphi_\Sigma} d\boldsymbol{\eta}, \\
\varphi_\Sigma &:= (\boldsymbol{x}^i - \boldsymbol{y}^i)\boldsymbol{\eta}_i - \boldsymbol{y}^j\boldsymbol{\eta}_j, \quad i = \mathrm{codim}\,\Sigma + 1, \dots, d, \ j = 1, \dots, \mathrm{codim}\,\Sigma, \tag{2.94}
\end{aligned}$$

where $\mathrm{codim}\,\Sigma := \dim M - \dim \Sigma$ is the codimension of Σ. The phase function φ_Σ is non-degenerate and thus (see e.g. [84, (2.4.4), (5.1.2)])

$$\begin{aligned}
\mathsf{R}_\Sigma &\in I^{\mathrm{codim}\,\Sigma/4}\big(\Sigma \times M, \Lambda_\Sigma'; \Omega^{1/2}(M_\Sigma \times M)\big), \tag{2.95a} \\
\Lambda_\Sigma' &:= \{(x', \xi'; x, \xi) \in \mathrm{T}^*M_\Sigma \times \dot{\mathrm{T}}^*M \mid x = \iota_\Sigma(x'), \xi' = \xi \upharpoonright \mathrm{T}_{x'}M_\Sigma\}, \tag{2.95b} \\
\sigma_{\mathsf{R}_\Sigma} &:= (2\pi)^{-\mathrm{codim}\,\Sigma/4}\, \mathbb{1}_{\Omega^{1/2}(M_\Sigma \times M)} \sqrt{|\mathrm{dv}_{\Lambda_\Sigma}|} \otimes \mathbb{1}, \tag{2.95c}
\end{aligned}$$

where $|\mathrm{dv}_\Lambda|$ is the density on Λ_Σ and $\mathbb{1}$ is a section of the Keller-Maslov bundle $\mathbb{L} \to \Lambda_\Sigma$. The canonical symplectic form σ on T^*M is given by (see Example A.1.2) $\sigma = dx^i \wedge d\xi_i + dx^j \wedge d\xi_j$ in the adapted coordinates and so $|\mathrm{dv}_\Lambda| = |dx' \wedge d\xi|$ is the induced density on Λ_Σ. $\mathbb{L}$ comprises global constant sections (cf. Example A.1.21) constructed from φ_Σ (see e.g. [91]).

As observed in Example 2.1.11, scrutinising Λ_Σ in the preceding example, one perceives that Λ_Σ is *not* homogeneous because elements $(y', 0; \iota_\Sigma(y'), \eta)$ occur in $\Lambda_\Sigma := (2.95b)$ whenever $\eta \in \Sigma^{\perp*}_{\iota_\Sigma(y')}$ where $\Sigma^{\perp*}$ is the conormal bundle (see Example A.1.7). Therefore, ι_Σ^* is *not* a *homogeneous* Fourier integral operator. We will present a solution to this issue for the bundle version of the restriction operator in Example 2.3.12.

2.3 Fourier Integral Operators on vector bundles

We will now describe the foremost technical tool of this thesis equipped with the preparations done in the last two sections. Let us begin with some rudimentary facts about integral operators acting on vector bundles.

2.3.1 Integral operators

Let $\mathscr{E}, \Omega^{1/2}M \to M$ (resp. $\mathscr{F}, \Omega^{1/2}N \to N$) be a vector bundle and bundle of half-densities over a d_M (resp. d_N)-dimensional manifold M (resp. N). Recall, the set $\mathcal{D}'(M; \mathscr{E} \otimes \Omega^{1/2}M)$ of **half-density valued distributions on** $\mathscr{E}$ is the $\Omega^{1/2}M$-valued $\mathbb{C}$-linear functionals on $C_c^\infty(M; \mathscr{E}^* \otimes \Omega^{1/2}M)$, and if $\mathrm{pr}_M : M \times N \to M, \mathrm{pr}_N : M \times N \to N$ are the Cartesian projectors then the **exterior tensor bundle** between $\mathscr{E}$ and $\mathscr{F}$ is defined by

$$\mathscr{E} \boxtimes \mathscr{F} := \mathrm{pr}_M^* \mathscr{E} \otimes \mathrm{pr}_N^* \mathscr{F} \to M \times N. \tag{2.96}$$

Suppose that $\{(U_\alpha, \kappa_\alpha)\}_\alpha$ (resp. $\{(V_\beta, \rho_\beta)\}_\beta$) is an atlas for M (resp. N) such that $\mathscr{E}$ (resp. $\mathscr{F}$) admits a local trivialisation over U_α (resp. V_β) and that $\{(U_\alpha, \chi_\alpha)\}_\alpha$ (resp. $\{(V_\beta, \tau_\beta)\}_\beta$) is a bundle-atlas for $\mathscr{E}$ (resp. $\mathscr{F}^*$). Let $|\mathrm{d}x|$ (resp. $|\mathrm{d}y|$) be an arbitrary but fixed nowhere-vanishing global section (Section 2.2.1) of $\Omega^{1/2}M$ (resp. $\Omega^{1/2}N$) and let (E_r) (resp. (F^k)) be a frame for $\mathscr{E}$ (resp. $\mathscr{F}$). Locally, $\phi \in C_c^\infty(M; \mathscr{E} \otimes \Omega^{1/2}M), \psi \in C_c^\infty(N; \mathscr{F} \otimes \Omega^{1/2}N), \Upsilon \in C_c^\infty(M \times N; (\mathscr{E}^* \boxtimes \mathscr{F}) \otimes \Omega^{1/2}(M \times N)), u \in \mathcal{D}'(M; \mathscr{E} \otimes \Omega^{1/2}M), v \in \mathcal{D}'(N; \mathscr{F} \otimes \Omega^{1/2}N), \mathsf{T} \in \mathcal{D}'(M \times N, (\mathscr{E} \boxtimes \mathscr{F}^*) \otimes \Omega^{1/2}(M \times N))$ can be expressed as (see e.g. [9, Sec. 4.2], [72, pp. 148-150])

$$
\begin{align}
\phi &= \phi^r \mathrm{E}_r \otimes \sqrt{|\mathrm{d}x|}, \quad \phi^r \in C_c^\infty(U_\alpha), \tag{2.97a}\\
\psi &= \psi^k \mathrm{F}_k \otimes \sqrt{|\mathrm{d}y|}, \quad \psi^k \in C_c^\infty(V_\beta), \tag{2.97b}\\
\Upsilon &= \Upsilon_r^k (\mathrm{E}^r \boxtimes \mathrm{F}_k) \otimes \sqrt{|\mathrm{d}x||\mathrm{d}y|}, \quad \Upsilon_r^k \in C_c^\infty(U_\alpha \times V_\beta), \tag{2.97c}\\
u &= u^r \mathrm{E}_r \otimes \sqrt{|\mathrm{d}x|}, \quad u^r \in \mathcal{D}'(U_\alpha), \tag{2.97d}\\
v &= v^k \mathrm{F}_k \otimes \sqrt{|\mathrm{d}y|}, \quad v^k \in \mathcal{D}'(V_\beta), \tag{2.97e}\\
\mathsf{T} &= \mathsf{T}_k^r (\mathrm{E}_r \boxtimes \mathrm{F}^k) \otimes \sqrt{|\mathrm{d}x||\mathrm{d}y|}, \quad \mathsf{T}_k^r \in \mathcal{D}'(U_\alpha \times V_\beta), \tag{2.97f}
\end{align}
$$

where $r = 1, \ldots, \mathrm{rk}\,\mathscr{E}$ and $k = 1, \ldots, \mathrm{rk}\,\mathscr{F}$. Thus, one has the bijections depending on the choices of coordinate-charts and bundle-charts

$$C^\infty(U_\alpha; \mathscr{E}_\alpha \otimes \Omega_\alpha^{1/2}) \cong \left(C^\infty(U_\alpha; \Omega_\alpha^{1/2})\right)_{\mathrm{rk}\,\mathscr{E} \times 1} \cong \left(C^\infty(\kappa_\alpha(U_\alpha))\right)_{\mathrm{rk}\,\mathscr{E} \times 1}, \tag{2.98a}$$

$$\mathcal{D}'(U_\alpha; \mathscr{E}_\alpha \otimes \Omega_\alpha^{1/2}) \cong \left(\mathcal{D}'(U_\alpha; \Omega_\alpha^{1/2})\right)_{\mathrm{rk}\,\mathscr{E} \times 1} \cong \left(\mathcal{D}'(\kappa_\alpha(U_\alpha))\right)_{\mathrm{rk}\,\mathscr{E} \times 1}, \tag{2.98b}$$

$$C^\infty(V_\beta; \mathscr{F}_\beta \otimes \Omega_\beta^{1/2}) \cong \left(C^\infty(V_\beta; \Omega_\beta^{1/2})\right)_{\mathrm{rk}\,\mathscr{F} \times 1} \cong \left(C^\infty(\rho_\beta(V_\beta))\right)_{\mathrm{rk}\,\mathscr{F} \times 1}, \tag{2.98c}$$

$$\mathcal{D}'(V_\beta; \mathscr{F}_\beta \otimes \Omega_\beta^{1/2}) \cong \left(\mathcal{D}'(V_\beta; \Omega_\beta^{1/2})\right)_{\mathrm{rk}\,\mathscr{F} \times 1} \cong \left(\mathcal{D}'(\rho_\beta(V_\beta))\right)_{\mathrm{rk}\,\mathscr{E} \times 1}. \tag{2.98d}$$

According to the Schwartz kernel theorem (see e.g. [23, p. 93]), given a bidistribution $\mathsf{T} \in \mathcal{D}'(M \times N, (\mathscr{E} \boxtimes \mathscr{F}^*) \otimes \Omega^{1/2}(M \times N))$, one has a unique continuous linear operator

$$T : C_c^\infty(N; \mathscr{F} \otimes \Omega^{1/2}N) \to \mathcal{D}'(M; \mathscr{E} \otimes \Omega^{1/2}M), \quad v \mapsto (Tv)(x) := \int_N \mathsf{T}(x, y)\, v(y) \tag{2.99}$$

and vice-versa. One defines the restricted operator

$$T_\beta^\alpha := \left(T(\psi \upharpoonright V_\beta)\right) \upharpoonright U_\alpha : C_c^\infty(V_\beta; \mathscr{F}_\beta \otimes \Omega_\beta^{1/2}) \to \mathcal{D}'(U_\alpha; \mathscr{E}_\alpha \otimes \Omega_\alpha^{1/2}), \tag{2.100}$$

which can be identified with a $\operatorname{rk}\mathscr{E} \times \operatorname{rk}\mathscr{F}$-matrix of scalar operators (see e.g. [9, Sec. 4.3.1])

$$T_{\beta,k}^{\alpha,r} := \chi_{\alpha*} \circ T_\beta^\alpha \circ \tau_\beta^* : C_c^\infty(V_\beta; \Omega_\beta^{1/2}) \to \mathcal{D}'(U_\alpha; \Omega_\alpha^{1/2}). \qquad (2.101)$$

This can be further boiled down to the level of Euclidean spaces

$$T_{\beta,k,j}^{\alpha,r,i} := \kappa_{\alpha*} \circ T_{\beta,k}^{\alpha,r} \circ \rho_\beta^* : C_c^\infty\big(\rho_\beta(V_\beta)\big) \to \mathcal{D}'\big(\kappa_\alpha(U_\alpha)\big), \qquad (2.102)$$

as depicted in the commutative diagram in Fig. 2.3.

$$
\begin{array}{ccc}
C_c^\infty(V_\beta; \mathscr{F}_\beta \otimes \Omega_\beta^{1/2}) & \xrightarrow{\quad T_\beta^\alpha \quad} & \mathcal{D}'(U_\alpha; \mathscr{E}_\alpha \otimes \Omega_\alpha^{1/2}) \\
\Big\uparrow \tau_\beta^* & & \Big\downarrow \chi_{\alpha*} \\
C_c^\infty(V_\beta; \Omega_\beta^{1/2}) & \xrightarrow{\quad T_{\beta,k}^{\alpha,r} \quad} & \mathcal{D}'(U_\alpha; \Omega_\alpha^{1/2}) \\
\Big\uparrow \rho_\beta^* & & \Big\downarrow \kappa_{\alpha*} \\
C_c^\infty\big(\rho_\beta(V_\beta)\big) & \xrightarrow{\quad T_{\beta,k,j}^{\alpha,r,i} \quad} & \mathcal{D}'\big(\kappa_\alpha(U_\alpha)\big)
\end{array}
$$

Figure 2.3: Euclidean representative of an integral operator acting on vector bundles.

Therefore, we can always locally express an integral operator acting on half-density-valued vector bundles as a system of scalar integral operators acting on functions on subsets of Euclidean spaces:

$$T \leftrightarrow \big(T_{\beta,k,j}^{\alpha,r,i}\big), \quad r = 1, \dots, \operatorname{rk}\mathscr{E}; k = 1, \dots, \operatorname{rk}\mathscr{F}; i = 1, \dots, d_M; j = 1, \dots, d_N. \qquad (2.103)$$

2.3.2   Pseudodifferential operators

The notion of the Besov space $^\infty H_{\mathrm{loc}}^{-m-d/4}(M; \Omega^{1/2})$ introduced in Section 2.2.2 generalises for a vector bundle $\mathscr{E} \to M$, $^\infty H_{\mathrm{loc}}^{-m-d/4}(M; \mathscr{E} \otimes \Omega^{1/2})$, in a straightforward way.

Definition 2.3.1. Let $\mathscr{E}, \Omega^{1/2} \to M$ be a vector bundle and the half-density bundle over a d-dimensional manifold M, respectively, and S a closed submanifold of M. Then the space $I^m(M, S; \mathscr{E} \otimes \Omega^{1/2})$ of **half-density-valued distributions of $\mathscr{E}$, conormal** with respect to S and of **degree** (at most) $m \in \mathbb{R}$, is defined as the set of all half-density-valued distributions $u \in \mathcal{D}'(M; \mathscr{E} \otimes \Omega^{1/2})$ on $\mathscr{E}$ such that $L_1 \dots L_N u$ belongs in the Besov space $^\infty H_{\mathrm{loc}}^{-m-d/4}(M; \mathscr{E} \otimes \Omega^{1/2})$ for all $N \in \mathbb{N}_0$ and for all $L_i \in \mathrm{PDO}^1(M; \mathscr{E} \otimes \Omega^{1/2})$ with smooth coefficients tangential to S. Here the topology is the weakest one which makes the maps $u \mapsto L_1 \dots L_N u \in {}^\infty H_{\mathrm{loc}}^{-m-d/4}(M; \mathscr{E} \otimes \Omega^{1/2})$ continuous [77, Def. 18.2.6].

Let $\mathscr{F} \to M$ be a vector bundle over a manifold M. We make use of the natural vector bundle isomorphism

$$\mathrm{Hom}(\mathscr{E}, \mathscr{F}) \cong \mathscr{E}^* \otimes \mathscr{F} \to M \tag{2.104}$$

in the definition below.

Definition 2.3.2. Let $\mathscr{E}, \mathscr{F}, \Omega^{1/2} \to M$ be two vector bundles and the half-density bundle over a manifold M, respectively. A **pseudodifferential operator** P from $\mathscr{E}$ to $\mathscr{F}$ of **order** (at most) $m \in \mathbb{R}$ is a continuous linear map (see e.g. [77, p. 100])

$$P : C_{\mathrm{c}}^{\infty}(M; \mathscr{E} \otimes \Omega^{1/2}) \to C^{\infty}(M; \mathscr{F} \otimes \Omega^{1/2}), \quad u \mapsto (Pu)(x) := \int_M \mathsf{P}(x, y)\, u(y)$$

whose Schwartz kernel P belongs to the conormal distribution $I^m\big(M \times M, (\Delta M)^{\perp*}; \mathrm{Hom}(\mathscr{E}, \mathscr{F}) \otimes \Omega^{1/2}(M \times M)\big)$ where $\Delta : M \to M \times M$ is the diagonal embedding. We denote the set of all such operators by $\Psi\mathrm{DO}^m(M; \mathscr{E} \otimes \Omega^{1/2}, \mathscr{F} \otimes \Omega^{1/2})$.

As inscribed in Section 2.3.1, locally P (resp. its Schwartz kernel P) are in a bijection with a matrix of properly supported scalar pseudodifferential operators (P_r^k) (resp. scalar kernels (P_r^k)) on respective Euclidean spaces up to a smoothing operator (resp. kernel); cf. Proposition 2.1.2 (for details, see, for instance [77, Def. 18.1.32], [9, Sec. 1.5.3]):

$$
\begin{aligned}
P &\leftrightarrow (P_r^k)_{\mathrm{rk}\,\mathscr{F} \times \mathrm{rk}\,\mathscr{E}} \quad \mod (\Psi_r^k)_{\mathrm{rk}\,\mathscr{F} \times \mathrm{rk}\,\mathscr{E}}, \quad \Psi_r^k \in \Psi\mathrm{DO}^{-\infty}(\kappa_\alpha(U_\alpha)), \\
\mathsf{P} &\leftrightarrow \big(\mathsf{P}_r^k\big)_{\mathrm{rk}\,\mathscr{F} \times \mathrm{rk}\,\mathscr{E}} \quad \mod \big(\Psi_r^k\big)_{\mathrm{rk}\,\mathscr{F} \times \mathrm{rk}\,\mathscr{E}}, \quad \Psi_r^k \in I^{-\infty}\big(\kappa_\alpha(U_\alpha) \times \kappa_\alpha(U_\alpha)\big)
\end{aligned} \tag{2.105}
$$

where (cf. Theorem 2.1.3)

$$\mathsf{P}_r^k(\boldsymbol{x}, \boldsymbol{y}) = \int_{\mathbb{R}^d} \frac{\mathrm{d}\boldsymbol{\xi}}{(2\pi)^d} \mathrm{e}^{\mathrm{i}(\boldsymbol{x}-\boldsymbol{y})\cdot\boldsymbol{\xi}} \sigma_{P_r^k}^{\mathrm{tot}}(\boldsymbol{x}, \boldsymbol{\xi}), \quad \sigma_{P_r^k}^{\mathrm{tot}}(\boldsymbol{x}, \boldsymbol{\xi}) \sim \sum_{\alpha \in \mathbb{N}_0^d} \frac{1}{\alpha!} (\partial_{\boldsymbol{\xi}}^\alpha \mathrm{D}_{\boldsymbol{y}}^\alpha \mathsf{p}_r^k)(\boldsymbol{x}, \boldsymbol{x}, \boldsymbol{\xi}) \tag{2.106}$$

for some proper $\mathsf{p}_r^k \in S^m\big(\kappa_\alpha(U_\alpha) \times \kappa_\alpha(U_\alpha) \times \mathbb{R}^d\big)$ and $\mathrm{d}\boldsymbol{x}$ is the Lebesgue measure at each $\boldsymbol{x} := \kappa(x)$. Therefore, one sets

$$\sigma_{[P]}^{\mathrm{tot}}(x, \xi) \equiv \sigma_{\left[\left(P_r^k\right)_{\mathrm{rk}\,\mathscr{F} \times \mathrm{rk}\,\mathscr{E}}\right]}^{\mathrm{tot}}(\boldsymbol{x}, \boldsymbol{\xi}) \equiv \big(\sigma_{[P_r^k]}^{\mathrm{tot}}(\boldsymbol{x}, \boldsymbol{\xi})\big)_{\mathrm{rk}\,\mathscr{F} \times \mathrm{rk}\,\mathscr{E}} \tag{2.107}$$

for the total symbol of P.

Definition 2.3.3. As in the terminologies of Definitions 2.3.2 and 2.2.6, let $\pi : \mathrm{T}^*M \to M$ be the cotangent bundle over M and $S^m\big(\mathrm{T}^*M, \mathrm{Hom}(\mathscr{E}, \mathscr{F})\big)$ the symbol space of $\mathrm{Hom}(\mathscr{E}, \mathscr{F})$-valued half-densities (see Appendix B.3) on T^*M. Then the **principal symbol** is defined by the isomorphism (see e.g. [77, p. 92])

$$\sigma : \Psi\mathrm{DO}^{m-[1]}(M; \mathscr{E} \otimes \Omega^{1/2}, \mathscr{F} \otimes \Omega^{1/2}) \to S^{m-[1]}\big(\mathrm{T}^*M, \mathrm{Hom}(\mathscr{E}, \mathscr{F})\big), \tag{2.108}$$

where $\Psi\mathrm{DO}^{m-[1]} := \Psi\mathrm{DO}^m / \Psi\mathrm{DO}^{m-1}$. The preceding map is locally given by

$$[P] \mapsto \sigma_{[P]}(x, \xi) := \big(\sigma_{[P_r^k]}\big)_{\mathrm{rk}\,\mathscr{F} \times \mathrm{rk}\,\mathscr{E}} \sqrt{|\mathrm{dv}_{\mathrm{T}^*M}(x, \xi)|} \quad \mod S^{m-[1]}(\cdot), \tag{2.109}$$

where $\sigma_{[P_r^k]}$ is the principal symbol of a Euclidean representative P_r^k of P in any homogeneous symplectic chart for $\dot{\mathrm{T}}^*M$ and any bundle-charts for $\mathscr{E}, \mathscr{F}$.

Definition 2.3.4. As in the terminologies of Definitions 2.3.2, 2.3.3 and 2.2.7, the **sub-principal symbol** of $P \in \Psi\mathrm{DO}^m(M; \mathscr{E} \otimes \Omega^{1/2}, \mathscr{F} \otimes \Omega^{1/2})$ is defined by the isomorphism [77, Thm. 18.1.33])

$$
\begin{aligned}
\sigma^{\mathrm{sub}} \quad : \quad & \Psi\mathrm{DO}^{m-[2]}(M; \mathscr{E} \otimes \Omega^{1/2}, \mathscr{F} \otimes \Omega^{1/2}) \to S^{m-[2]}(\mathrm{T}^*M, \mathrm{Hom}(\mathscr{E}, \mathscr{F})), \\
& [P] \mapsto \sigma^{\mathrm{sub}}_{[P]}(x, \xi) := \left(\sigma^{\mathrm{sub}}_{[P^k_r]}\right)_{\mathrm{rk}\,\mathscr{F} \times \mathrm{rk}\,\mathscr{E}},
\end{aligned}
\tag{2.110}
$$

where $\sigma^{\mathrm{sub}}_{[P^k_r]}$ is the subprincipal symbol of P^k_r.

Remark 2.3.5. The principal symbol is *independent* of the choice of trivialisations of $\mathscr{E}, \mathscr{F}$ but the subprincipal symbol depends on the choice which has been explored in-depth in Section 3.3.2. Therefore, combining with Remark 2.2.8 we summarise that the subprincipal symbol is *independent of local coordinates* but *depends on the chosen half-density and bundle chart*.

In order to purvey the notion of real principal type pseudodifferential operators in the bundle setting, let $\mathscr{G} \to M$ be a vector bundle. Then there is the natural bilinear mapping (see e.g. [90, Lem. 8.3.6], [9, p. 85])

$$
\begin{aligned}
\cdot \circ \cdot \quad : \quad & S^m(\mathrm{T}^*M, \mathrm{Hom}(\mathscr{E}, \mathscr{G})) \times S^{m'}(\mathrm{T}^*M, \mathrm{Hom}(\mathscr{G}, \mathscr{F})) \to S^{m+m'}(\mathrm{T}^*M, \mathrm{Hom}(\mathscr{E}, \mathscr{F})), \\
& (p, p') \mapsto (p' \circ p)(x, \xi) := p'(x, \xi)(p(x, \xi)).
\end{aligned}
\tag{2.111}
$$

Besides, we note that $S^0(\mathrm{T}^*M, \mathrm{Hom}(\mathscr{E}, \mathscr{F}))$ has a distinguished element I given by the identity $\mathbb{1}_{\mathrm{Hom}(\mathscr{E}, \mathscr{F})} = \mathbb{1}_{\mathrm{Hom}(\pi^*\mathscr{E}, \pi^*\mathscr{F})}$ homomorphism $(\pi^*\mathscr{E})_{(x,\xi)} \ni p \mapsto p \in (\pi^*\mathscr{F})_{(x,\xi)}$:

$$
\mathrm{T}^*_x M \ni (x, \xi) \mapsto \mathrm{I}(x, \xi) := \mathbb{1}_{\mathrm{Hom}(\mathscr{E}, \mathscr{F})_{(x,\xi)}} \in \mathrm{Hom}(\pi^*\mathscr{E}, \pi^*\mathscr{F}).
\tag{2.112}
$$

Definition 2.3.6. As in the terminologies of Definitions 2.3.2 and 2.3.3, $P \in \Psi\mathrm{DO}^m(M; \mathscr{E} \otimes \Omega^{1/2}, \mathscr{F} \otimes \Omega^{1/2})$ having principal symbol p is of **real principal type** at $(x_0, \xi^0) \in \dot{\mathrm{T}}^*M$, if there exists a symbol $\tilde{p} \in S^0(\mathcal{U}, (\pi^*\mathrm{Hom}(\mathscr{E}, \mathscr{F}))_{\mathcal{U}})$ such that [45, Def. 3.1]

$$
(\tilde{p} \circ p)(x, \xi) = q\, \mathbb{1}_{\pi^*\mathrm{Hom}(\mathscr{E}, \mathscr{F})_{(x,\xi)}}
\tag{2.113}
$$

for any (x, ξ) in an open conic neighbourhood $\mathcal{U}$ of (x_0, ξ^0) and where q is the principal symbol of a scalar real principal type operator $Q \in \Psi\mathrm{DO}^m(M; \Omega^{1/2})$. We say that P is of real principal type in $\dot{\mathrm{T}}^*M$ if it is so at every $(x_0, \xi^0) \in \dot{\mathrm{T}}^*M$.

The Euclidean formulation (2.24) of the wavefront set generalises in bundle setting in a straightforward way:

$$
\forall u \in \mathcal{D}'(M; \mathscr{E} \otimes \Omega^{1/2}) : \mathrm{WF}\, u := \bigcap_{\substack{P \in \Psi\mathrm{DO}^m(M; \mathscr{E} \otimes \Omega^{1/2}) \\ Pu \in C^\infty(M; \mathscr{E} \otimes \Omega^{1/2})}} \mathrm{Char}\, P
\tag{2.114}
$$

except the characterisation of $\mathrm{Char}\, P$ used in (2.24) is required to generalise as follows.

Definition 2.3.7. As in the terminologies of Definitions 2.3.2 and 2.3.3, let $P \in \Psi\text{DO}^m(M; \mathscr{E} \otimes \Omega^{1/2}, \mathscr{F} \otimes \Omega^{1/2})$ whose principal symbol is denoted by p. P is said to be **non-characteristic** at some point $(x_0, \xi^0) \in \dot{\mathrm{T}}^*M$ if $p \circ \tilde{p} - \mathbb{1} \in S^{-1}(\mathcal{U}, (\pi^*\text{Hom}(\mathscr{E}, \mathscr{F}))_{\mathcal{U}})$ in a conic neighbourhood $\mathcal{U}$ of (x_0, ξ^0) for some $\tilde{p} \in S^m(\mathcal{U}, (\pi^*\text{Hom}(\mathscr{E}, \mathscr{F}))_{\mathcal{U}})$. The set of all characteristic points of P is denoted by $\text{Char } P$ (see e.g. [77, Def. 18.1.25], [90, Sec. 8.4]).

This definition implies that the microlocal ellipticity condition is equivalent to

$$\forall (x, \xi) \in \mathcal{U} : \det p(x, \xi) \neq 0. \tag{2.115}$$

It also follows that

$$\text{WF } u = \bigcup_{r=1,\ldots,\text{rk } \mathscr{E}} \text{WF}(u^r) \tag{2.116}$$

where $u^r \in \mathcal{D}'(U)$ is a component of u with respect to a trivialisation of $\mathscr{E}$ over $U \subset M$ for a chosen half-density.

To illustrate these concepts we present the wave operator as an example (see [39, Prop. 4.2-4.4] for the scalar case together with [92, Prop. 2.1], [93, Sec. 2]).

Example 2.3.8. Let $\square$ be a normally hyperbolic operator (see Definition 3.2.1) on a vector bundle $\mathscr{E} \to \mathscr{M}$ over a globally hyperbolic spacetime (see Definition 3.1.3) $(\mathscr{M}, \text{g})$. By definition, the principal symbol is given by the spacetime metric g^{-1} on the cotangent bundle $\mathrm{T}^*\mathscr{M}$ times the identity endomorphism. Thus the condition for being a real principal type differential operator on $\mathscr{E}$ is satisfied trivially because the scalar wave operator (see Example 3.2.2) on $(\mathscr{M}, \text{g})$ is of real principal type. The characteristic set of $\square$ is the (co)lightcone bundle (see Section 3.1.1) $\text{Char } \square = \dot{\mathrm{T}}^*_0 \mathscr{M} \to \mathscr{M}$ over $\mathscr{M}$ and the integral curves of the vector field X_{g} generated by the Hamiltonian g^{-1} are the geodesic strips in the punctured cotangent bundle $\mathrm{T}^*\mathscr{M}$ whose projections on $\mathscr{M}$ are the geodesics.

2.3.3 Lagrangian distributions

The Lagrangian distribution on a vector bundle, its principal symbol and their composition are some of the indispensable concepts involved in this thesis. Definition 2.2.12 generalises to half-density-valued sections of a vector bundle in a straightforward fashion yet presented for the sake of completeness.

Definition 2.3.9. Let $\mathscr{E}, \Omega^{1/2}, \dot{\mathrm{T}}^*M \to M$ be a vector bundle, the half-density bundle, and the punctured cotangent bundle over a d-dimensional manifold M, respectively, and $\Lambda \subset \dot{\mathrm{T}}^*M$ a closed conic Lagrangian submanifold (see Definition A.1.5). Then the **space** $I^m(M, \Lambda; \mathscr{E} \otimes \Omega^{1/2})$ of **half-density-valued Lagrangian distributional sections** of $\mathscr{E}$, of **order** (at most) $m \in \mathbb{R}$, is defined as the set of all half-density-valued distributions $u \in \mathcal{D}'(M; \mathscr{E} \otimes \Omega^{1/2})$ such that $P_1 \ldots P_N u \in {}^\infty H_{\text{loc}}^{-m-d/4}(M; \mathscr{E} \otimes \Omega^{1/2})$ for all $N \in \mathbb{N}_0$ and for all properly supported $P_i \in \Psi\text{DO}^1(M; \mathscr{E} \otimes \Omega^{1/2})$ having vanishing principal symbols on Λ, where ${}^\infty H_{\text{loc}}^{-m-d/4}(M; \mathscr{E} \otimes \Omega^{1/2})$ denotes the Besov space (Section 2.3.2) [23, Def. 25.1.1].

Let $\mathscr{F}, \Omega^{1/2}N \to N$ be a vector bundle and the half-density bundle over a manifold N. On top of the replacements mentioned after Definition 2.2.12, we are compelled to substitute $\mathscr{E} \otimes \Omega^{1/2}M \to M$ by $(\mathscr{E} \boxtimes \mathscr{F}^*) \otimes \Omega^{1/2}(M \times N) \to M \times N$ in order to define a Fourier integral operator from $\mathscr{F} \otimes \Omega^{1/2}N$ to $\mathscr{E} \otimes \Omega^{1/2}M$. One makes use of the natural bundle isomorphism

$$\mathrm{Hom}(\mathscr{F}, \mathscr{E}) \cong \mathscr{E} \boxtimes \mathscr{F}^* \to M \times N, \tag{2.117}$$

where the homomorphism bundle $\mathrm{Hom}(\mathscr{F}, \mathscr{E}) \to M \times N$ is the vector bundle whose fibres $\mathrm{Hom}(\mathscr{F}, \mathscr{E})_{(x,y)} = \mathrm{Hom}(\mathscr{F}_y, \mathscr{E}_x)$ over $(x, y) \in M \times N$ are $\mathbb{C}$-linear maps from the vector space $\mathscr{F}_y$ to the vector space $\mathscr{E}_x$.

Definition 2.3.10. Let $\mathscr{E}, \Omega^{1/2}M, \dot{\mathrm{T}}^*M \to M$ (resp. $\mathscr{F}, \Omega^{1/2}N, \dot{\mathrm{T}}^*N \to N$) be a vector bundle, the half-density bundle, and the punctured cotangent bundle over a d_M (resp. d_N)-dimensional manifold M (resp. N) and C a homogeneous canonical relation (see Definition A.1.12) from $\dot{\mathrm{T}}^*N$ to $\dot{\mathrm{T}}^*M$ of M, which is closed in $\dot{\mathrm{T}}^*(M \times N)$. A linear continuous operator [23, Def. 25.2.1]

$$A : C_c^\infty(N; \mathscr{F} \otimes \Omega^{1/2}N) \to \mathcal{D}'(M; \mathscr{E} \otimes \Omega^{1/2}M), \ v \mapsto (Av)(x) := \int_N \mathsf{A}(x, y)\, v(y) \tag{2.118}$$

whose Schwartz kernel A is an element of the Lagrangian distribution $I^m(M \times N, C'; \mathrm{Hom}(\mathscr{F}, \mathscr{E}) \otimes \Omega^{1/2}(M \times N)$ is called a **Fourier integral operator** from $\mathscr{F}$ to $\mathscr{E}$, of **order** (at most) $m \in \mathbb{R}$ associated with the **canonical relation** C. We will denote the set of all such operators by $\mathrm{FIO}^m(\mathscr{F} \otimes \Omega^{1/2}N \to N, \mathscr{E} \otimes \Omega^{1/2}M \to M; C')$.

The localisation (Lemma 2.2.14) of Lagrangian distributions on a manifold holds true in the bundle scenario as well [23, Lem. 25.1.2]. Following the demonstration in Section 2.3.1, it is evident that $\mathsf{A} \in I^m(M \times N, C'; \mathrm{Hom}(\mathscr{F}, \mathscr{E}) \otimes \Omega^{1/2}(M \times N))$ can be written as a matrix of scalar Lagrangian distributions on respective Euclidean spaces

$$\mathsf{A} \leftrightarrow (\mathsf{A}_k^r)_{\mathrm{rk}\,\mathscr{E} \times \mathrm{rk}\,\mathscr{F}} \in \left(I^m(\kappa_\alpha(U_\alpha) \times \rho_\beta(V_\beta); C_\varphi')\right)_{\mathrm{rk}\,\mathscr{E} \times \mathrm{rk}\,\mathscr{F}} \tag{2.119}$$

where $C_\varphi' := C_{\alpha,\beta}' := (2.87)$ is given by means of a phase function $\varphi := \varphi_{\alpha,\beta}$ with excess $e := e_{\alpha,\beta}$ as elucidated in Section 2.2. A_k^r is of the form (2.89) [23, Prop. 25.1.5']

$$\mathsf{A}_k^r(\boldsymbol{x}, \boldsymbol{y}) = (2\pi)^{-(d_M + d_N + 2n - 2e)/4} \int_{\mathbb{R}^n} e^{i\varphi(\boldsymbol{x}, \boldsymbol{y}; \theta)} \mathsf{a}_k^r(\boldsymbol{x}, \boldsymbol{y}; \theta)\, d\theta \quad \mathrm{mod}\ C^\infty(\kappa_\alpha(U_\alpha) \times \rho_\beta(V_\beta)) \tag{2.120}$$

where $\mathsf{a}_k^r \in S^{m+(d_M + d_N - 2n - 2e)/4}(\mathbb{R}^{d_M} \times \mathbb{R}^{d_N} \times \mathbb{R}^n)$ having support in the interior of a sufficiently small conic neighbourhood of the fibre-critical manifold (2.86) of φ contained in the domain of definition of φ.

Definition 2.3.11. As in the terminologies of Definition 2.3.10, let $\widetilde{\mathrm{Hom}}(\mathscr{F}, \mathscr{E}), \Omega^{1/2}C, \mathbb{M} \to C$ be the pullback of $\mathrm{Hom}(\mathscr{F}, \mathscr{E})$ to $\dot{\mathrm{T}}^*M \times \dot{\mathrm{T}}^*N$ restricted to C (see (B.20)), the half-density bundle (see Appendix A.1.6) and the Keller-Maslov bundle (see Definition A.1.19) over

C, respectively, and $S^{m+(d_M+d_N)/4-[1]}\big(C; \mathbb{M}\otimes\Omega^{1/2}C\otimes\widetilde{\mathrm{Hom}}(\mathscr{F},\mathscr{E})\big)$ the space (see (B.14)) of $\mathbb{M}\otimes\Omega^{1/2}C\otimes\widetilde{\mathrm{Hom}}(\mathscr{F},\mathscr{E})$-valued symbols on C. Then the **principal symbol** of a Lagrangian distribution is defined by the isomorphism [23, Prop. 25.1.5']))

$$\sigma : I^{m-[1]}\big(M{\times}N, C'; \mathrm{Hom}(\mathscr{F},\mathscr{E})\otimes\Omega^{1/2}(M{\times}N)\big) \to S^{m+\frac{d_M+d_N}{4}-[1]}\big(C; \mathbb{M}\otimes\Omega^{1/2}C\otimes\widetilde{\mathrm{Hom}}(\mathscr{F},\mathscr{E})\big), \tag{2.121}$$

where $I^{m-[1]} := I^m/I^{m-1}$. The preceding map is locally given by

$$[A] \mapsto \sigma_{[A]}(x,\xi; y,\eta) \;:=\; \left(\int_{\mathbb{C}} \mathrm{d}\theta'' a_k^r(\boldsymbol{x},\boldsymbol{y}; \theta',\theta'')\right)_{\mathrm{rk}\,\mathscr{E}\times\mathrm{rk}\,\mathscr{F}} \mathrm{m}(x,\xi; y,\eta) \otimes$$
$$\sqrt{|\mathrm{d}v_C(x,\xi; y,\eta)|} \quad \mathrm{mod}\; S^{m+\frac{d_M+d_N}{4}-1}(\cdot), \tag{2.122}$$

where a_k^r is the top-order homogeneous term of a_k^r in (2.120) and all other symbols are as in Definition 2.1.9.

2.3.4 Adjoint of a Fourier integral operator

To describe the adjoint of a Fourier integral operator, let us recall some standard notions from linear algebra. Let $\mathscr{U}$ and $\mathscr{V}$ be $\mathbb{C}$-vector spaces. Then the set $\mathrm{Hom}(\mathscr{U},\mathscr{V})$ of $\mathbb{C}$-linear maps $T : \mathscr{U} \to \mathscr{V}$ is a vector space and $\mathrm{End}\,\mathscr{V} := \mathrm{Hom}(\mathscr{V},\mathscr{V})$ is an unital algebra over $\mathbb{C}$ under the composition of endomorphisms. If $\mathscr{V}^* := \mathrm{Hom}(\mathscr{V},\mathbb{C})$ is the algebraic dual of $\mathscr{V}$ then the set of algebraic dual maps $T^* \in \mathrm{Hom}(\mathscr{V}^*,\mathscr{U}^*)$ is defined by $(T^*\psi)(u) = \psi(Tu)$ for all $\psi \in \mathscr{V}^*, u \in \mathscr{U}$. The conjugate of $\mathscr{V}$ is the complex vector space $\bar{\mathscr{V}} := \{\bar{v}\,|\,v \in \mathscr{V}\}$ with an antilinear isomorphism $C : \mathscr{V} \ni v \mapsto \bar{v} \in \bar{\mathscr{V}}$ where, for any given $v \in \mathscr{V}$, $\mathscr{V}^* \ni \psi \mapsto \bar{v}(\psi) := \overline{\psi(v)} \in \mathbb{C}$. In other words, $\mathscr{V}$ is identical to $\bar{\mathscr{V}} := \overline{\mathrm{Iso}}(\mathscr{V}^*,\mathbb{C})$ as a *set but not as a vector space* because the latter is equipped with the scalar multiplication $\lambda \bullet v := \bar{\lambda}v$ in contrast to $\lambda \cdot v := \lambda v$ for the former for any $\lambda \in \mathbb{C}$. If $\mathscr{V}$ is equipped with a bilinear form $\langle\cdot,\cdot\rangle$ then an intertwiner $\mathscr{B} \in \mathrm{Hom}(\mathscr{V},\mathscr{V}^*)$ is determined by $(\mathscr{B}v)(u) := \langle v,u\rangle$ for any $u \in \mathscr{V}$. Analogously, for a given sesquilinear form $(\cdot|\cdot)$ on $\mathscr{V}$, another intertwiner $\mathscr{A} := \bar{\mathscr{B}} \circ C : \mathscr{V} \to \bar{\mathscr{V}}^*$ is characterised by $(\mathscr{A}v)(\bar{u}) := (u|v)$. One has the identifications $(\bar{\mathscr{V}})^* = \overline{\mathscr{V}^*} = \bar{\mathscr{V}}^*$ and $\bar{\bar{\mathscr{V}}} = \mathscr{V}$.

Since $\mathscr{E}$ is a vector bundle, the above description applies fibrewise $\mathscr{E}_x$ and one has the (anti)dual $(\bar{\mathscr{E}}^* := \sqcup_{x\in M}\bar{\mathscr{E}}_x^* \to M)\mathscr{E}^* := \sqcup_{x\in M}\mathscr{E}_x^* \to M$ bundles over a manifold M (see e.g. [72, p. 22]). A Fourier integral operator (cf (2.39))

$$A : C_c^\infty(N; \mathscr{F}\otimes\Omega^{1/2}N) \to C^\infty(M; \mathscr{E}\otimes\Omega^{1/2}M) \tag{2.123}$$

associated with a canonical relation C, has a unique formal adjoint in the following sense. If C^{-1} denotes the inverse relation of C obtained by interchanging T^*M and T^*N then there exists a unique Fourier integral operator

$$\bar{A}^* : C_c^\infty(M; \bar{\mathscr{E}}^*\otimes\Omega^{1/2}M) \to C^\infty(N; \bar{\mathscr{F}}^*\otimes\Omega^{1/2}N) \tag{2.124}$$

associated with C^{-1} such that [23, Thm. 25.2.2]

$$\int_N (\bar{A}^*\phi)(y)\, v(y) := \int_M (Av)(x)\, \phi(x) \Leftrightarrow \bar{A}^*(v \boxtimes \phi) := A(\phi \boxtimes v) \tag{2.125}$$

for any $\phi \in C_c^\infty(M; \bar{\mathscr{E}}^* \otimes \Omega^{1/2}M)$ and any $v \in C_c^\infty(N; \mathscr{F} \otimes \Omega^{1/2}N)$. Its Schwartz kernel and principal symbol are given by

$$\bar{A}^* \in I^m\big(N \times M, C^{-1\prime}; \mathrm{Hom}(\bar{\mathscr{E}}^*, \bar{\mathscr{F}}^*) \otimes \Omega^{1/2}(N \times M)\big), \tag{2.126a}$$

$$\sigma_{[\bar{A}^*]} = \mathsf{s}^*(\overline{\sigma_{[A]}}^*) \in S^{m + \frac{d_M + d_N}{4} - [1]}\big(C^{-1}; \Omega^{1/2}C^{-1} \otimes \mathbb{M}C^{-1} \otimes \widetilde{\mathrm{Hom}}(\bar{\mathscr{E}}^*, \bar{\mathscr{F}}^*)\big), \tag{2.126b}$$

where $\mathsf{s} : N \times M \to M \times N$ is the interchanging (Schwartz) map and $\mathbb{M}C^{-1} \to C^{-1}$ is the Keller-Maslov bundle over C^{-1}.

Locally $\bar{A}^*(y, x)$ is given by

$$A_r^k(y, x) \equiv (2\pi)^{-(d_M + d_N + 2n - 2e)/4} \int_{\mathbb{R}^n} e^{-i\,\varphi(y,x;\theta)} \bar{a}_r^k(y, x; \theta)\, d\theta, \tag{2.127a}$$

$$\sigma_{A_r^k}(y, \eta; x, \xi) = \sqrt{|d\eta|\,|d\xi|} \int_{\mathsf{s}(\mathbb{C})} \bar{a}_r^k(y, x; \theta', \theta'') \frac{e^{-i\pi/4\,\mathrm{sgn}(\mathrm{Hess}_{y,x;\theta'}\,\varphi)}}{\sqrt{|\det(\mathrm{Hess}_{y,x;\theta'}\,\varphi)|}} d\theta'' \tag{2.127b}$$

whenever A is given by (2.120) and $\mathsf{s}(\mathbb{C})$ is defined by interchanging x and y in (2.48).

2.3.5   Algebra of Fourier integral operators

A necessary assumption for the product (composition) of two Fourier integral operators to be well-defined is that the first operator must be properly supported. Then the defined composition may, however, still fail to be a Fourier integral operator. For instance, the composition of two canonical relations does not necessarily have the required properties to define another Fourier integral operator. To ensure a well-defined product that is again a Fourier integral operator, we are obliged to assume that their Schwartz kernels (Lagrangian distributions) are *properly supported* and the composition of canonical relations is *clean, proper* (see Definition A.1.18) and *connected* [94, Sec. 4.1], [10, Thm. 5.4]. Given a vector bundle $\mathscr{G} \to O$ over a manifold O besides $\mathscr{E} \to M$ and $\mathscr{F} \to N$, if $A \in I^m\big(M \times O, C'; \mathrm{Hom}(\mathscr{G}, \mathscr{E}) \otimes \Omega^{1/2}(M \times O)\big)$ and $B \in I^{m'}\big(O \times N, \Lambda'; \mathrm{Hom}(\mathscr{F}, \mathscr{G}) \otimes \Omega^{1/2}(O \times N)\big)$ with the required restrictions, then [23, Thm. 25.2.3]

$$AB := A \circ B \in I^{m+m'+e/2}\big(M \times N, (C \circ \Lambda)'; \mathrm{Hom}(\mathscr{F}, \mathscr{E}) \otimes \Omega^{1/2}(M \times N)\big), \tag{2.128}$$

where e is the excess (see Definition A.1.18 and (A.40)) of the clean composition $C \circ_e \Lambda$ and its principal symbol is given by

$$\sigma_{AB} = \sigma_A \diamond_e \sigma_B. \tag{2.129}$$

The precise meaning of the right hand side of this equation demands a few new terminologies which are detailed in Appendix A.2.3.

The space of Fourier integral operators is not an algebra unless the canonical relation is symmetric and transitive [95, Ex. 1 and comment on it in the succeeding page]. In case the composition of the canonical relation C by itself is clean with excess zero, proper and connected the space of Fourier integral operators $\left(I^{\bullet}(M \times M, C'; \mathrm{Hom}(\mathscr{E}, \mathscr{E}) \otimes \Omega^{1/2}(M \times M)), \circ\right)$ is an *associative graded algebra* over the field $\mathbb{C}$. The algebra (2.129) of principal symbols constitute an algebra which is commutative only the scalar case but non-commutative in general. We remark that the product formula (2.129) becomes simpler (see (2.156)) when both C and Λ are local canonical graphs and then the product is always defined as detailed later in Section 2.3.6.

Example 2.3.12. Let $\mathscr{E}, \Omega^{1/2}M \to M$ be a vector bundle and the half-density bundle over a manifold M, respectively, and $\iota_{\Sigma} : \Sigma \hookrightarrow M$ an immersed submanifold. Given a fibrewise isomorphism $\hat{\iota}_{\Sigma} : \mathscr{E}_{\Sigma_{x'}} \cong \mathscr{E}_{\iota_{\Sigma}(x')}$, the **restriction map**

$$\iota_{\Sigma}^* : C_{\mathrm{c}}^{\infty}(M; \mathscr{E} \otimes \Omega^{1/2}M) \to C^{\infty}(\Sigma; \mathscr{E}_{\Sigma} \otimes \Omega^{1/2}M_{\Sigma}) \tag{2.130}$$

is defined by the pullback of the morphism $(\iota_{\Sigma}, \hat{\iota}_{\Sigma})$. This boils down to Example 2.2.16 after trivialising $\mathscr{E}$ and thus its Schwartz kernel R_{Σ} is a Lagrangian distribution of order $\mathrm{codim}\,\Sigma/4$ associated with the canonical relation Λ'_{Σ} defined in (2.95b). As commented after Example 2.2.16, one discerns that $\Lambda_{\Sigma} \subset \mathrm{T}^*M_{\Sigma} \times \dot{\mathrm{T}}^*M$ is *not* a homogeneous canonical relation because it is not guaranteed that restriction of all elements in $\dot{\mathrm{T}}^*M$ to Σ is non-zero. Geometrically speaking, the complication arises due to the elements of the conormal bundle (see Example A.1.7) $\Sigma^{\perp*} \subset \mathrm{T}^*M$. To circumvent this difficulty, one temporarily introduces a cutoff function χ on $\Sigma^{\perp*}$ and sets [91, Sec. 5.2]

$$\mathsf{R}_{\chi}(x', y) := \int_{\mathbb{R}^d} \mathrm{e}^{\mathrm{i}\,\varphi_{\Sigma}}\left(1 - \chi(y, \eta)\right)\frac{\mathrm{d}\eta}{(2\pi)^d}, \quad \varphi_{\Sigma} := (\iota_{\Sigma}(x') - y) \cdot \eta \tag{2.131}$$

so that such elements do not occur in the support of $1 - \chi$. Then the respective operator ι_{χ}^* is a homogeneous Fourier integral operator of order $\mathrm{codim}\,\Sigma/4$ associated with the homogeneous canonical relation $\Lambda_{\chi} \subset \dot{\mathrm{T}}^*M_{\Sigma} \times \dot{\mathrm{T}}^*M$. Putting altogether:

$$\mathsf{R}_{\chi} \in I^{\mathrm{codim}\,\Sigma/4}\left(\Sigma \times M, \Lambda'_{\chi}; \mathrm{Hom}(\mathscr{E}, \mathscr{E}_{\Sigma}) \otimes \Omega^{1/2}(M_{\Sigma} \times M)\right), \tag{2.132a}$$

$$\Lambda'_{\chi} := \{(x', \xi'; x, \xi) \in \dot{\mathrm{T}}^*M_{\Sigma} \times \dot{\mathrm{T}}^*M \mid x = \iota_{\Sigma}(x'), \xi' = \xi \upharpoonright \mathrm{T}_{x'}M_{\Sigma}\}, \tag{2.132b}$$

$$\sigma_{\mathsf{R}_{\chi}}|_{\mathrm{supp}(1-\chi)}(x', \xi'; x, -\xi) := \frac{\sqrt{|\mathrm{dv}_{\Lambda}(x', \xi'; x, -\xi)|}}{(2\pi)^{\mathrm{codim}\,\Sigma/4}} \mathbb{1}_{\mathrm{Hom}(\mathscr{E}, \mathscr{E}_{\Sigma}) \otimes \Omega^{1/2}(M_{\Sigma} \times M)} \otimes \mathbb{l}(x', \xi'; x, -\xi), \tag{2.132c}$$

where $|\mathrm{dv}_{\Lambda}|$ is the density on Λ_{χ} and $\mathbb{l}$ is a section of the Keller-Maslov bundle $\mathbb{L} \to \Lambda_{\chi}$ as in Example 2.2.16.

Equivalently, one can say that ι_Σ^* is not a homogeneous Fourier integral operator due to plausible occurrence of elements of $\Sigma^{\perp*}$ in its canonical relation and the cutoff above can be emulated by composing with a $P \in \Psi\mathrm{DO}^0(M; \mathscr{E} \otimes \Omega^{1/2}M)$ with ES $P \cap \dot{\Sigma}^{\perp*} = \emptyset$. The principal symbol of R_Σ is then $(2\pi)^{-\operatorname{codim}\Sigma/4}\, \mathbb{1}\, |\mathrm{d}v_\Lambda|^{1/2} \otimes \mathbb{1}$ over WF P (see e.g. [27, Lem. 8.3] for the scalar version).

Any distribution on $\mathscr{E}$ can be restricted to $\mathscr{E}_\Sigma$ when its wavefront set is disjoint with $\Sigma^{\perp*}$. In particular, if Λ is such that the wavefront set of every $u \in I^m(M, \Lambda; \mathscr{E} \otimes \Omega^{1/2}M)$ is disjoint from the conormal bundle $\Sigma^{\perp*}$ of $\Sigma \subset \mathscr{M}$ then the restriction operator can be extended to a sequentially continuous linear operator

$$\iota_\chi^* : I^m(M, \Lambda; \mathscr{E} \otimes \Omega^{1/2}M) \to I^{m+\operatorname{codim}\Sigma/4}(\Sigma, \Lambda|_{\mathrm{T}M_\Sigma}; \mathscr{E}_\Sigma \otimes \Omega^{1/2}\Sigma), \ u \mapsto \iota_\chi^* u,$$
$$\sigma_{\iota_\chi^* u}(x', \xi') \asymp (2\pi)^{-\operatorname{codim}\Sigma/4}\sigma_u(x', \xi'), \tag{2.133a}$$

where the last equation is in the sense of modulo Keller-Maslov contribution.

Products of operators with vanishing principal symbol

By far, we have learned to compute the principal symbol of the product of Lagrangian distributions given those of individual distributions. An innate question in this context is that what if any of the principal symbols vanishes identically. We are going to address a special case of this situation: product between a pseudodifferential operator and a Lagrangian distribution whenever the principal symbol of the former vanishes. This is motivated by its direct applicability at some later parts of this thesis. It turns out that the Lie derivative plays an essential role in this regard, so we spell out a few rudimentary formulae.

$$
\begin{array}{ccc}
\pi^*\mathscr{E} \xrightarrow{\ \cong\ } \mathscr{E} & \qquad & C^\infty(\mathrm{T}^*M; \pi^*\mathscr{E}) \xrightarrow{\ \nabla^{\mathrm{Hom}(\pi^*\mathscr{E}, \pi^*\mathscr{E})}_{X_p}\ } C^\infty(\mathrm{T}^*M; \pi^*\mathscr{E}) \\
\downarrow \qquad\qquad \downarrow & & \pi^* \uparrow \qquad\qquad\qquad \downarrow (\pi^*)^{-1} \\
\mathrm{T}^*M \xrightarrow{\ \pi\ } M & & C^\infty(M; \mathscr{E} \otimes \Omega^{1/2}) \xrightarrow[\ \nabla^{\mathscr{E}}_X\ ]{} C^\infty(M; \mathscr{E} \otimes \Omega^{1/2})
\end{array}
$$

Figure 2.4: Pullback of a vector bundle $\mathscr{E} \to M$ via the canonical projection $\pi : \mathrm{T}^*M \to M$ and the P-compatible connection on $\mathscr{E}$.

Let $(\mathscr{E} \to M, \nabla^{\mathscr{E}}_X)$ be a vector bundle over a manifold M, equipped with a covariant derivative $\nabla^{\mathscr{E}}_X$ with respect to *some* vector field X on M and let $\Xi_s : M \to M$ be the flow of X. This covariant derivative induces a parallel transport map

$$\hat{\Xi}_s(x) : \mathscr{E}_x \to \mathscr{E}_{\Xi_s(x)} \tag{2.134}$$

along the integral curve $c_x(s) = \Xi_s(x)$ of X passing through $x \in M$. Then the **Lie derivative** $\pounds_X$ along X is defined by

$$\pounds_X u := \lim_{s \to 0} \frac{\hat{\Xi}_{-s}\big(u(\Xi_s(x))\big) - u(x)}{s} := \frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0} \Xi_s^* u, \tag{2.135}$$

where

$$\Xi_s^* : C^\infty(M;\mathscr{E}) \to C^\infty(M;\mathscr{E}), \ u \mapsto \Xi_s^* u := \hat{\Xi}_{-s} \circ u \circ \Xi_s \qquad (2.136)$$

is the pullback map on $\mathscr{E}$ induced by Ξ_s and the linear isomorphism $\hat{\Xi}_s$. For convenience, we set

$$L_X := -i\pounds_X : C^\infty(M;\mathscr{E}) \to C^\infty(M;\mathscr{E}) \qquad (2.137)$$

and call this first-order differential, the *Lie derivative* as well. Locally, $X = X^i \partial/\partial x^i$, $C^\infty(M;\mathscr{E} \otimes \Omega^{1/2}) \ni u = u^r E_r \otimes \sqrt{|dx|}$ and thus the Leibniz rule entails (see e.g. [96, p. 130], [23, (25.2.11)], [85, (2.60) (p. 377)])

$$\pounds_X u = \left(X^i \frac{\partial u^r}{\partial x^i} + \frac{1}{2} \operatorname{div}(X) u^r \right) E_r \otimes \sqrt{|dx|} + u^r \pounds_X(E_r) \otimes \sqrt{|dx|}. \qquad (2.138)$$

In what follows, we consider $P \in \Psi DO^m(M;\mathscr{E} \otimes \Omega^{1/2})$ having a scalar principal symbol (Definition 2.3.3) p and subprincipal symbol (Definition 2.3.4) σ_P^{sub}. Let X_p be the Hamiltonian vector field (see (A.3)) generated by p and γ its integral curve (cf. Definition 2.2.9) on the cotangent bundle $\pi : T^*M \to M$. These data enables us to define a natural covariant derivative in a geometric fashion as detailed in Section 3.3.2. The so-called "P-compatible covariant derivative" (see Definition 3.3.6) $\nabla_{X_p}^{\operatorname{Hom}(\pi^*\mathscr{E}, \pi^*\mathscr{E})}$ is a covariant derivative on the homomorphism bundle $\operatorname{Hom}(\pi^*\mathscr{E}, \pi^*\mathscr{E}) \to T^*M$ over T^*M along γ. The projection c of γ on M is the integral curve of the vector field $X := d\pi(X_p)$ on M induced by X_p, where $d\pi : T(T^*M) \to TM$ is the tangent map of π. Thus we have a covariant derivative $\nabla_X^{\mathscr{E}}$ on $\mathscr{E}$ along c, as depicted in Figure 2.4.

The following result is closely related to [26, Thm. 5.9].

Theorem 2.3.13. *As in the terminologies of Definition 2.3.10, let* $A \in I^m\big(M \times N, C'; \operatorname{Hom}(\mathscr{F}, \mathscr{E}) \otimes \Omega^{1/2}(M \times N)\big)$ *and* a *the principal symbol of* A. *Suppose that* $P \in \Psi DO^{m'}(M;\mathscr{E} \otimes \Omega^{1/2}M)$ *is properly supported with a scalar principal symbol* p *and a subprincipal symbol* σ_P^{sub}, *and that* p *vanishes identically on the projection of* C *in* $\dot{T}^*M$. *Then,* $PA \in I^{m+m'-1}\big(M \times N, C'; \operatorname{Hom}(\mathscr{F}, \mathscr{E}) \otimes \Omega^{1/2}(M \times N)\big)$ *and its principal symbol is*

$$\sigma_{PA} = \left(L_{X_p} + \sigma_P^{\text{sub}} \right) a, \qquad (2.139)$$

where L_{X_p} *is* $-i$ *times the Lie derivative along the Hamiltonian vector field* X_p *of* p, *lifted to a function on* $\dot{T}^*M \times \dot{T}^*N$, *so* X_p *is tangential to* C.

Proof. Since the principal symbol is locally defined (Definition 2.3.11), our strategy is to make use of the partition of unity to boil down the statements in the level of a matrix of scalar operators as demonstrated in Section 2.3.2, 2.3.3 and then employ the scalar-version of this statement [35, Thm. 5.3.1] (see also, e.g. [23, Thm. 25.2.4], [85, pp. 451-454]). The differences in the proof between the scalar and the bundle versions are essentially bookkeeping, yet we provide the details for completeness. Locally, the transition functions of $\mathbb{M}$ are constant (see Section A.1.4) so the Maslov factor can be

ignored while computing the Lie derivative. Moreover, a is a matrix-valued density (Definition 2.3.11) on C' and thus the last term in (2.138) does not contribute. It is always possible to choose a local coordinate $(x^i, \xi_i; y^j, \eta_j)$ around $(x, \xi; y, \eta) \in \dot{T}^*M \times \dot{T}^*N$ such that (see (A.34)-(A.36)) $x^i \xi_i + y^j \eta_j - H$ is a non-degenerate phase function for C, where H is a smooth positively homogeneous $\mathbb{R}$-valued function of degree 1 on an open conic neighbourhood of $(\xi, \eta) \in \dot{T}^*_x M \times \dot{T}^*_y N$. Then (cf. (2.120) and see e.g. [23, Lem. 25.2.5]),

$$A^l_k(y, z) = (2\pi)^{-3(d_M + d_N)/4} \int_{\mathbb{R}^{d_M + d_N}} e^{i(\eta \cdot y + \zeta \cdot z - H(\eta, \zeta))} a^l_k(\eta, \zeta) \, d\eta \, d\zeta, \qquad (2.140)$$

where $a^l_k \in S^{m - (d_M + d_N)/4}(\mathbb{R}^{d_M + d_N})$ having support in a conic neighbourhood of $(y = \partial H / \partial \eta, z = \partial H / \partial \zeta)$. For any $v \in C^\infty_c(N; \mathscr{F} \otimes \Omega^{1/2} N)$, by definition of Fourier transform:

$$
\begin{aligned}
\left(\mathcal{F}(A^l_k v^k)\right)(\xi) &= \int_{\mathbb{R}^{d_M}} dy \, e^{-i y \cdot \xi} (A^l_k v^k)(y) \\
&= \int_{\mathbb{R}^{d_M}} dy \, e^{-i y \cdot \xi} \int_{\mathbb{R}^{d_N}} A^l_k(y, z) \, v^k(z) \, dz \\
&= (2\pi)^{-3(d_M + d_N)/4} \int_{\mathbb{R}^{d_M}} dy \int_{\mathbb{R}^{d_N}} dz \int_{\mathbb{R}^{d_M + d_N}} d\eta \, d\zeta \\
&\quad e^{i y \cdot (\eta - \xi) + i z \cdot \zeta - i H(\eta, \zeta)} a^l_k(\eta, \zeta) \, v^k(z) \\
&= (2\pi)^{d_M - \frac{3}{4}(d_M + d_N)} \int_{\mathbb{R}^{d_N}} e^{-i H(\xi, \zeta)} a^l_k(\xi, \zeta) \, (\mathcal{F} v^k)(-\zeta) \, d\zeta, \quad (2.141)
\end{aligned}
$$

where we have made use of the Fubini's theorem for oscillatory integrals [78, (1.2.4)]. Since P is properly supported, by an application of Theorem 2.1.3 we obtain

$$(P^r_l A^l_k v^k)(x) = \frac{1}{(2\pi)^{\frac{3}{4}(d_M + d_N)}} \int_{\mathbb{R}^{d_M}} d\xi \int_{\mathbb{R}^{d_N}} d\zeta \, e^{i x \cdot \xi - i H(\xi, \zeta)} (\sigma^{tot}_P)^r_l(x, \xi) \, a^l_k(\xi, \zeta)(\mathcal{F} v^k)(-\zeta), \tag{2.142}$$

where $(\sigma^{tot}_P)^r_l$ is the total symbol of P in the chosen coordinate-charts and bundle-charts. The preceding equation entails that the Schwartz kernel of PA is

$$(P^r_l A^l_k)(x, y) = (2\pi)^{-\frac{3}{4}(d_M + d_N)} \int_{\mathbb{R}^{d_M}} d\xi \int_{\mathbb{R}^{d_N}} d\eta \, e^{i(x \cdot \xi + y \cdot \eta - H)} (\sigma^{tot}_P)^r_l(x, \xi) \, a^l_k(\xi, \eta). \tag{2.143}$$

One can decompose $(\sigma^{tot}_P)^r_l = p^r_l + \tilde{p}^r_l$ as a combination of its principal part p^r_l and the remainder terms $\tilde{p}^r_l$ (cf. Definition B.1.6). We have assumed that $p = 0$ on the projection of C on $\dot{T}^*M$. By means of Taylor's series one can write

$$p^r_l(x, \xi) = \left(x^i - \frac{\partial H}{\partial \xi_i}\right) f^r_{l,i}(x, \xi, \eta) = \frac{\partial(x \cdot \xi + y \cdot \eta - H)}{\partial \xi_i} f^r_{l,i}(x, \xi, \eta), \tag{2.144}$$

where $f^r_{l,i}$ is homogeneous of degree m' with respect to (ξ, η), given by the mean value theorem:

$$f^r_{l,i}(x, \xi, \eta) = \int_0^1 \frac{\partial p^r_l}{\partial x^i}\left(\lambda x + (1 - \lambda)\frac{\partial H}{\partial \xi}, \xi\right) d\lambda. \tag{2.145}$$

In particular, $f^r_{l,i} \restriction C = (\partial_x p^r_l)(\partial_\xi H, \xi)$. It is always possible to assume that a vanishes in a neighbourhood of 0 and then an integration by parts gives for PA:

$$
\begin{aligned}
P^r_l \mathsf{A}^l_k &= (2\pi)^{-\frac{3}{4}(d_M + d_N)} \int_{\mathbb{R}^{d_M + d_N}} e^{i(x\cdot\xi + y\cdot\eta - H)} \, (\tilde{\mathsf{p}}^r_l + p^r_l)\mathsf{a}^l_k \, d\xi \, d\eta \\
&= (2\pi)^{-\frac{3}{4}(d_M + d_N)} \int_{\mathbb{R}^{d_M + d_N}} e^{i(x\cdot\xi + y\cdot\eta - H)} \left(\tilde{\mathsf{p}}^r_l \mathsf{a}^l_k - \mathrm{D}_{\xi_i}(f^r_{l,i}\mathsf{a}^l_k) \right) d\xi \, d\eta. \quad (2.146)
\end{aligned}
$$

Since $a^l_k = \mathsf{a}^l_k(\xi, \eta)\sqrt{|d\xi| \, |d\eta|}$ is the principal symbol of A, hence

$$
\begin{aligned}
(\sigma_{P\mathrm{A}})^r_k &= \left(\tilde{\mathsf{p}}^r_l a^l_k - \mathrm{D}_{\xi_i}(f^r_{l,i}a^l_k) \right)_{x = \partial H/\partial\xi} \sqrt{|d\xi| \, |d\eta|} \\
&= \left(\tilde{\mathsf{p}}^r_l a^l_k - \mathrm{D}_{\xi_i}(f^r_{l,i})a^l_k - f^r_{l,i}\mathrm{D}_{\xi_i}(a^l_k) \right)_{x = \partial H/\partial\xi} \sqrt{|d\xi| \, |d\eta|} \quad (2.147)
\end{aligned}
$$

whenever (ξ, η) are taken as coordinates on C. We will suppress the bundle indices from afterwords as there is no scope of confusion.

By definition, $X_p = \partial p/\partial\xi_i \partial/\partial x^i - \partial p/\partial x^i \partial/\partial\xi_i$. So, Char P is a hypersurface near (x, ξ) and X_p spans the symplectic-orthogonal (in $\dot{\mathrm{T}}^*M$) of its tangent space. Now we pullback $p(x, \xi)$ via the projector $\dot{\mathrm{T}}^*M \times \dot{\mathrm{T}}^*N \to \dot{\mathrm{T}}^*M$ to consider it as a function $p(x, \xi; y, \eta)$ on $\dot{\mathrm{T}}^*M \times \dot{\mathrm{T}}^*N$. The restriction of this function $p(x, \xi; y, \eta)$ on C' is only a function of (ξ, η) as $x^i = \partial H/\partial\xi_i$ and $y^j = \partial H/\partial\eta_j$ on C'. Thus, X_p on C must be of the form

$$
-\frac{\partial}{\partial x^i}(p(x, \xi))\frac{\partial}{\partial\xi_i} = -f_i(x, \xi, \eta)\frac{\partial}{\partial\xi_i}, \quad x = \frac{\partial H}{\partial\xi}, \quad (2.148)
$$

in our chosen parametrisation of C and it is pointwise tangent to C. We compute

$$
\begin{aligned}
\mathrm{L}_{X_p} a &= -f_i\left(\frac{\partial H}{\partial\xi}, \xi, \eta\right)\frac{\partial a}{i\,\partial\xi_i} - \frac{1}{2}\frac{\partial}{i\,\partial\xi_i}\left(f_i\left(\frac{\partial H}{\partial\xi}, \xi, \eta\right)\right)a \\
&= -f_i\left(\frac{\partial H}{\partial\xi}, \xi, \eta\right)\mathrm{D}_{\xi_i}(a) - (\mathrm{D}_{\xi_i}f_i)\left(\frac{\partial H}{\partial\xi}, \xi, \eta\right)a \\
&\quad + (\mathrm{D}_{\xi_i}f_i)\left(\frac{\partial H}{\partial\xi}, \xi, \eta\right)a + \frac{i}{2}\frac{\partial}{\partial\xi_i}\left(f_i\left(\frac{\partial H}{\partial\xi}, \xi, \eta\right)\right)a \\
&= -f_i\left(\frac{\partial H}{\partial\xi}, \xi, \eta\right)\mathrm{D}_{\xi_i}(a) - (\mathrm{D}_{\xi_i}f_i)\left(\frac{\partial H}{\partial\xi}, \xi, \eta\right)a \\
&\quad - i\frac{\partial f_i}{\partial\xi_i}\left(\frac{\partial H}{\partial\xi}, \xi, \eta\right)a + \frac{i}{2}\frac{\partial f_i}{\partial\xi_i}\left(\frac{\partial H}{\partial\xi}, \xi, \eta\right)a + \frac{i}{2}\frac{\partial^2 H}{\partial\xi_i\partial\xi_j}\frac{\partial f_i}{\partial x^j}\left(\frac{\partial H}{\partial\xi}, \xi, \eta\right)a \\
&= -f_i\left(\frac{\partial H}{\partial\xi}, \xi, \eta\right)\mathrm{D}_{\xi_i}(a) - (\mathrm{D}_{\xi_i}f_i)\left(\frac{\partial H}{\partial\xi}, \xi, \eta\right)a \\
&\quad - \frac{i}{2}\left(\frac{\partial f_i}{\partial\xi_i}\left(\frac{\partial H}{\partial\xi}, \xi, \eta\right) - \frac{\partial^2 H}{\partial\xi_i\partial\xi_j}\frac{\partial f_i}{\partial x^j}\left(\frac{\partial H}{\partial\xi}, \xi, \eta\right)\right)a \\
&= -f_i\left(\frac{\partial H}{\partial\xi}, \xi, \eta\right)\mathrm{D}_{\xi_i}(a) - (\mathrm{D}_{\xi_i}f_i)\left(\frac{\partial H}{\partial\xi}, \xi, \eta\right)a - \frac{i}{2}\frac{\partial^2 p}{\partial x^j\partial\xi_j}\left(\frac{\partial H}{\partial\xi}, \xi\right) \quad (2.149)
\end{aligned}
$$

and insert this into the hindmost expression of σ_{PA} to reach our goal:

$$\sigma_{PA} = \left(\mathrm{L}_{X_p} + \mathsf{p}_{m'-1}\left(\frac{\partial H}{\partial \boldsymbol{\xi}}, \boldsymbol{\xi}\right) + \frac{\mathrm{i}}{2}\frac{\partial^2 p}{\partial \boldsymbol{x}^j \boldsymbol{\xi}_j}\left(\frac{\partial H}{\partial \boldsymbol{\xi}}, \boldsymbol{\xi}\right)\right) a + \ldots = (\mathrm{L}_{X_p} + \sigma_P^{\mathrm{sub}}) a \quad \mathrm{mod}\ S^{m+m'-2}(\ldots). \tag{2.150}$$

Here $\mathsf{p}_{m'-1}$ is the pullback of the next-to-leading order term in the asymptotic expression of the total symbol of P to C' and we have used Definition 2.3.4 together with

$$\frac{\partial^2 p}{\partial \boldsymbol{x}^j \boldsymbol{\xi}_j}(\boldsymbol{x}, \boldsymbol{\xi}) = \frac{\partial f_j}{\partial \boldsymbol{\xi}_j}(\boldsymbol{x}, \boldsymbol{\xi}, \boldsymbol{\eta}) - \frac{\partial^2 H}{\partial \boldsymbol{\xi}_j \partial \boldsymbol{\xi}_i}(\boldsymbol{\xi}, \boldsymbol{\eta})\frac{\partial f_i}{\partial \boldsymbol{x}^j}(\boldsymbol{x}, \boldsymbol{\xi}, \boldsymbol{\eta}) + \left(\boldsymbol{x}^i - \frac{\partial H}{\partial \boldsymbol{\xi}_i}(\boldsymbol{\xi}, \boldsymbol{\eta})\right)\frac{\partial f_i}{\partial \boldsymbol{x}^j \partial \boldsymbol{\xi}_j}(\boldsymbol{x}, \boldsymbol{\xi}, \boldsymbol{\eta}) \tag{2.151}$$

when evaluated at $\boldsymbol{x} = \partial H/\partial \boldsymbol{\xi}$.   ∎

2.3.6 Lagrangian distributions defined by symplectomorphisms

The simplest canonical relation $\Gamma' \subset \dot{\mathrm{T}}^*M \times \dot{\mathrm{T}}^*N$ is that defined as the twisted graph Γ' of a homogeneous symplectomorphism (see (A.1) and (A.7)) from $\dot{\mathrm{T}}^*N$ to $\dot{\mathrm{T}}^*M$, which enforces $d := \dim M = \dim N$ (see Example A.1.10). In this case, it is always possible to choose coordinates (y_0^i) on the image of $y_0 \in N$ under some local chart such that (see e.g. [23, Prop. 25.3.3])

$$\Gamma_\varphi = \{(\boldsymbol{x}_0, \mathrm{d}_{\boldsymbol{x}_0}\varphi; \mathrm{grad}_{\boldsymbol{\eta}^0}\varphi, \boldsymbol{\eta}^0)\}, \qquad \det\frac{\partial^2 \varphi}{\partial \boldsymbol{x} \partial \boldsymbol{\eta}}(\boldsymbol{x}_0, \boldsymbol{\eta}^0) \neq 0 \tag{2.152}$$

in an open conic neighbourhood of $(\boldsymbol{x}_0, \boldsymbol{\xi}^0; y_0, \boldsymbol{\eta}^0) \in (\mathbb{R}^d \times \dot{\mathbb{R}}^d) \times (\mathbb{R}^d \times \dot{\mathbb{R}}^d)$ making use of a non-degenerate $\mathbb{R}$-valued phase function $\varphi(\boldsymbol{x}, \boldsymbol{\eta})$ on an open conic neighbourhood of $(\boldsymbol{x}_0, \boldsymbol{\eta}^0)$.

This class of Lagrangian submanifolds results in the simplest class of Fourier integral operators A via the corresponding space of Lagrangian distributions $\mathrm{A} \in I^m\big(M \times N, \Gamma'; \mathrm{Hom}(\mathscr{F}, \mathscr{E}) \otimes \Omega^{1/2}(M \times N)\big)$ whose Euclidean representative is of the form [78, pp. 169-173] (see e.g. [23, (25.3.2)'], [97, (65.25)])

$$\mathrm{A}_k^r(\boldsymbol{x}, \boldsymbol{y}) \equiv \int_{\mathbb{R}^d} \mathrm{e}^{\mathrm{i}(\varphi(\boldsymbol{x},\boldsymbol{\eta}) - \boldsymbol{y}\cdot\boldsymbol{\eta})} \mathsf{a}_k^r(\boldsymbol{x}, \boldsymbol{y}; \boldsymbol{\eta})\frac{\mathrm{d}\boldsymbol{\eta}}{(2\pi)^d}, \tag{2.153}$$

where the total symbol a_k^r is of homogeneous of degree m, having support in the interior of a small conic neighbourhood of $(\boldsymbol{x}_0, \boldsymbol{\eta}^0)$ contained in the domain of definition of φ.

In order to describe the principal symbol σ_A of A we note that the natural half-density dv_Γ (see (A.70)) on Γ can be factored out from $\sigma_\mathrm{A}(x, \xi; y, \eta) \in S^{m+(d+d)/4}\big(\Gamma; \mathbb{M} \otimes \widetilde{\mathrm{Hom}}(\mathscr{F}, \mathscr{E}) \otimes \Omega^{1/2}\Gamma\big)$ so that the order $m + (d + d)/4$ of the half-density-valued principal symbol is reduced to m as $\Omega^{1/2}\Gamma$ is of order $d/2$. Therefore, the principal symbol map

$$\sigma : I^{m-[1]}\big(M \times N, \Gamma'; \Omega^{1/2}(M \times N) \otimes \mathrm{Hom}(\mathscr{F}, \mathscr{E})\big) \to S^{m-[1]}\big(\Gamma; \mathbb{M} \otimes \widetilde{\mathrm{Hom}}(\mathscr{F}, \mathscr{E})\big) \tag{2.154}$$

is locally given by (see e.g. [23, p. 27])

$$\sigma_{A_k^r}(\boldsymbol{x}, \boldsymbol{\eta}) := a_k^r\left(\boldsymbol{x}, \frac{\partial \varphi}{\partial \boldsymbol{\eta}}; \boldsymbol{\eta}\right) \left|\det \frac{\partial^2 \varphi}{\partial \boldsymbol{x} \partial \boldsymbol{\eta}}\right|^{-1/2} \mathbb{m} \mod S^{m-1}(\cdot), \tag{2.155}$$

where the Keller-Maslov bundle $\mathbb{M}$ is trivialised by the phase function in (2.153).

We remark that the composition $AB \in I^{m+m'}\left(M \times N, (\Gamma \circ \Lambda)'; \text{Hom}(\mathscr{F}, \mathscr{E}) \otimes \Omega^{1/2}(M \times N)\right)$ between $A \in I^m\left(M \times O, \Gamma', \text{Hom}(\mathscr{G}, \mathscr{E}) \otimes \Omega^{1/2}(M \times O)\right)$ and a properly supported $B \in I^{m'}\left(O \times N, \Lambda'; \text{Hom}(\mathscr{F}, \mathscr{G}) \otimes \Omega^{1/2}(O \times N)\right)$ is always well-defined, where $\Gamma \circ \Lambda$ is the graph of the composition of symplectomorphisms $\dot{T}^*N \ni (y, \eta) \mapsto (z, \zeta) \mapsto (x, \xi) \in \dot{T}^*M$. Furthermore, the expression of σ_{AB} simplifies quite a bit compared to (2.129) and it is given by the standard composition of homomorphisms [78, p. 180] (see also, e.g. [97, Thm. 65.7], [85, (6.11) (p. 465)])

$$\sigma_{AB}(x, \xi; y, \eta) = \sum_{(z,\zeta) \mid (x,\xi;z,\zeta) \in \Gamma, (z,\zeta;y,\eta) \in \Lambda} \sigma_A(x, \xi; z, \zeta) \left(\sigma_B(z, \zeta; y, \eta)\right). \tag{2.156}$$

If the respective vector bundles are hermitian then the algebra of Lagrangian distributions is a $*$-algebra.

Parametrices and elliptic operators

We will now introduce the notion of ellipticity [35, p. 186] for a Fourier integral operator and show that an approximate inverse always exists for such an operator.

Definition 2.3.14. As in the terminologies of Definitions 2.3.10 and 2.3.11, and (2.154), let Γ be the graph of a homogeneous symplectomorphism from $\dot{T}^*N$ to $\dot{T}^*M$. Then, a Lagrangian distribution $A \in I^m\left(M \times N, \Gamma'; \text{Hom}(\mathscr{F}, \mathscr{E}) \otimes \Omega^{1/2}(M \times N)\right)$ is called **non-characteristic** at $(x_0, \xi^0; y_0, \eta^0) \in \Gamma$ if its principal symbol has an inverse $\in S^{-m}\left(\Gamma; \mathbb{M}^{-1} \otimes \widetilde{\text{Hom}}(\mathscr{E}, \mathscr{F})\right)$ in a conic neighbourhood of $(x_0, \xi^0; y_0, \eta^0)$. A is called **elliptic** if it is non-characteristic at every point of Γ. The complement of non-characteristic points is called the **characteristic set** Char A of A [23, Def. 25.3.4].

We note that (2.153) and (2.155) imply that the non-characteristic points belong to WF′ A. If Γ^{-1} is also a graph and $A \in I^m\left(M \times N, \Gamma'; \text{Hom}(\mathscr{F}, \mathscr{E}) \otimes \Omega^{1/2}(M \times N)\right)$ is elliptic and properly supported, then A has a **unique parametrix** $E \in I^{-m}\left(N \times M, \Gamma^{-1'}; \text{Hom}(\mathscr{E}, \mathscr{F}) \otimes \Omega^{1/2}(N \times M)\right)$, i.e.,

$$EA - I_{\mathscr{F}} \in \Psi\text{DO}^{-\infty}(N; \mathscr{F} \otimes \Omega^{1/2}N), \qquad AE - I_{\mathscr{E}} \in \Psi\text{DO}^{-\infty}(M; \mathscr{E} \otimes \Omega^{1/2}M). \tag{2.157}$$

A proof of the microlocal version of this claim is going to be presented shortly after one devises a variant of Lagrangian distributions where the closedness assumption on the Lagrangian submanifold has been relaxed.

Definition 2.3.15. As in the terminologies of Definition 2.3.9, let $C \subset \dot{\mathrm{T}}^*M \times \dot{\mathrm{T}}^*N$ be a homogeneous canonical relation which is not necessarily closed in $\dot{\mathrm{T}}^*(M \times N)$ and $\mathcal{K} \subset C$ a conic subset which is closed in $\dot{\mathrm{T}}^*(M \times N)$. By $I^m\big(M \times N, \mathcal{K}'; \mathrm{Hom}(\mathscr{F}, \mathscr{E}) \otimes \Omega^{1/2}(M \times N)\big)$, one denotes the set of all matrices with entries (2.120) together with the additional condition that, for each α, β as in (2.89), the restriction of $\mathrm{a}_{\alpha,\beta;k}^{\ r}$ to some conic neighbourhood in $\mathbb{R}^{d_M} \times \mathbb{R}^{d_N} \times \mathbb{R}^{n_{\alpha,\beta}}$ of the pullback of $C \setminus \mathcal{K}$ by the Lagrangian fibration (see (A.33)) $\jmath : \mathscr{C}_{\alpha,\beta} := (2.86) \to C'_{\alpha,\beta} := (2.87)$, is in the class $S^{-\infty}(\cdot)$.

As in the terminologies of Definition 2.3.11, $S^m\big(\mathcal{K}, \mathbb{M} \otimes \Omega^{1/2}C \otimes \widetilde{\mathrm{Hom}}(\mathscr{F}, \mathscr{E})\big)$ denotes the set of all $a \in S^m\big(C, \mathbb{M} \otimes \Omega^{1/2}C \otimes \widetilde{\mathrm{Hom}}(\mathscr{F}, \mathscr{E})\big)$ such that $a \in S^{-\infty}\big(C \setminus \mathcal{K}, \mathbb{M} \otimes \Omega^{1/2}C \otimes \widetilde{\mathrm{Hom}}(\mathscr{F}, \mathscr{E})\big)$ [35, p. 187].

Note, the definition depends on both $\mathcal{K}$ and C yet suppressed to lighten the notion. The analogue of the principal symbol isomorphism (Definition 2.3.11) reads [35, p. 187]

$$I^{m-[1]}\big(M \times N, \mathcal{K}'; \mathrm{Hom}(\mathscr{F}, \mathscr{E}) \otimes \Omega^{1/2}(M \times N)\big) \cong S^{m + \frac{d_M + d_N}{4} - [1]}\big(\mathcal{K}, \mathbb{M} \otimes \Omega^{1/2}C \otimes \widetilde{\mathrm{Hom}}(\mathscr{F}, \mathscr{E})\big).$$
$$(2.158)$$

Theorem 2.3.16. *As in the terminologies of Definitions 2.3.10 and 2.3.15, suppose that Γ is the graph of a homogeneous symplectomorphism $\varkappa$ from an open conic subset $\mathcal{V} \subset \dot{\mathrm{T}}^*N$ into $\dot{\mathrm{T}}^*M$ and that $\mathcal{K} \subset \Gamma$ is a conic subset which is closed in $\dot{\mathrm{T}}^*(M \times N)$. Then, for any conic subset $\mathcal{U} \subset \mathcal{V}$ such that $\mathcal{U}$ (resp. $\varkappa(\mathcal{U})$) is closed in $\dot{\mathrm{T}}^*N$ (resp. $\dot{\mathrm{T}}^*M$), if $\mathrm{A} \in I^m\big(M \times N, \mathcal{K}'; \mathrm{Hom}(\mathscr{F}, \mathscr{E}) \otimes \Omega^{1/2}(M \times N)\big)$ is non-characteristic on $\{(\varkappa(\mathcal{U}), \mathcal{U})\} \subset \Gamma$, then there exists an elliptic $\mathrm{E} \in I^{-m}\big(N \times M, \mathcal{K}^{-1'}; \mathrm{Hom}(\mathscr{E}, \mathscr{F}) \otimes \Omega^{1/2}(N \times M)\big)$ such that*

$$\mathrm{ES}(EA - I_{\mathscr{F}}) \cap \mathcal{U} = \emptyset, \qquad \mathrm{ES}(AE - I_{\mathscr{E}}) \cap \varkappa(\mathcal{U}) = \emptyset, \qquad (2.159)$$

where $I_{\mathscr{E}} \in \Psi\mathrm{DO}^0(M; \mathscr{E} \otimes \Omega^{1/2}M)$ and $I_{\mathscr{F}} \in \Psi\mathrm{DO}^0(N; \mathscr{F} \otimes \Omega^{1/2}N)$ are identity operators. The parametrix E is unique in the sense that $(\mathcal{U}, \varkappa(\mathcal{U})) \not\subset \mathrm{WF}'(\mathrm{E} - \mathrm{E}')$ for any other parametrix E' of A.

Proof. The arguments used to prove the scalar-version [35, Prop. 5.1.2] (see also, e.g. [84, Thm. 4.2.5]) flows over the bundle case albeit provided here for completeness. Let a be a principal symbol of A in the sense of (2.158). By hypotheses, there exists a $b(\cdot) \in S^{-m}\big(\mathcal{K}, \mathbb{M}^{-1} \otimes \widetilde{\mathrm{Hom}}(\mathscr{E}, \mathscr{F})\big)$ such that $ba = \sigma_{I_{\mathscr{F}}}$ in a conic neighbourhood of $\varDelta\mathcal{U}$. By utilising appropriate microlocal partition of unities[3] Ψ (resp. Φ) subordinated to $\mathcal{U}$ (resp. $\varkappa(\mathcal{U})$), one chooses $b = 0$ outside of a sufficiently small conic neighbourhood of $\{(\varkappa(\mathcal{U}), \mathcal{U})\}$ where $\mathcal{U}$ is identified with $\varDelta\mathcal{U}$ (via projection). Then we have

---

[3]As in the terminologies of Definition 2.3.2, a family $\{\Psi_\alpha\}_\alpha$ of $\Psi_\alpha \in \Psi\mathrm{DO}^0(M; \mathscr{E} \otimes \Omega^{1/2})$ is called a **microlocal partition of unity** if the supports of the Schwartz kernels of Ψ_α's are locally finite, $\mathrm{WF}(\Psi_\alpha) \subset \mathcal{U}_\alpha$ and $\sum \Psi_\alpha = I$. Here, $(\mathcal{U}_\alpha)_\alpha$ is an open cover of the cosphere bundle $\mathbb{S}^*M$ such that both $\Omega^{1/2}$ and $\mathscr{E}$ admit trivialisations over its projection on M. A microlocal partition of unity always exists (see e.g. [98, Lem. 6.10, 6.11] for a scalar version).

a properly supported $B_0 \in I^{-m}\big(N \times M, \mathcal{K}^{-1\prime}; \mathrm{Hom}(\mathcal{E}, \mathcal{F}) \otimes \Omega^{1/2}(N \times M)\big)$ whose principal symbol is $\sigma_\Psi b \sigma_\Phi \equiv b$, and by the composition of Lagrangian distributions ((2.128) and (2.129)), there exists a properly supported $R \in \Psi\mathrm{DO}^{-1}(N; \mathcal{F} \otimes \Omega^{1/2}N)$ such that $B_0 A = I_\mathcal{F} - R$. Next, we want to invert $I_\mathcal{F} - R$ by making use of the Neumann series: $(I_\mathcal{F} - R)^{-1} = \sum_{k \in \mathbb{N}_0} R^k$. Set $B_k := R^k B_0 \in I^{-m-k}\big(N \times M, \mathcal{K}^{-1\prime}; \mathrm{Hom}(\mathcal{E}, \mathcal{F}) \otimes \Omega^{1/2}(N \times M)\big)$ and then

$$I_\mathcal{F} = (I_\mathcal{F} - R) \sum_{k=0}^{\infty} R^k = \sum_{k=0}^{N-1} R^k (I_\mathcal{F} - R) + \sum_{k=N}^{\infty} R^k (I_\mathcal{F} - R) = \sum_{k=0}^{N-1} B_k A + R^N. \qquad (2.160)$$

Let E be defined by the asymptotic summation: $\mathsf{E} :\sim \sum_{k \in \mathbb{N}_0} B_k$, then inserting the last equation one obtains

$$EA - I_\mathcal{F} = \left(E - \sum_{k=0}^{N-1} B_k \right) A - R^N \sim \sum_{k=N}^{\infty} B_k A - R^N \in \Psi\mathrm{DO}^{-N}(N; \mathcal{F} \otimes \Omega^{1/2}N) \qquad (2.161)$$

for every $N \in \mathbb{N}$, which in turn proves the first part of the theorem as a right parametrix can be constructed analogously.

To prove the uniqueness, we suppose that E' is another right parametrix for A. Then

$$\big\{ (\mathcal{U}, \varkappa(\mathcal{U})) \big\} \not\subset \mathrm{WF}'\, \mathsf{E} = \mathrm{WF}'(EAE') = \mathrm{WF}'\, \mathsf{E}' \qquad (2.162)$$

and similarly for the left parametrix. $\blacksquare$

Egorov's theorem

We are now going to present a key theorem for microlocalisation (see Theorem 3.3.10) related to the constructions presented in this section.

Theorem 2.3.17 (Egorov's theorem). *As in the terminologies of Definition 2.3.10, let Γ be the graph of a homogeneous symplectomorphism $\varkappa : \dot{\mathrm{T}}^*N \to \dot{\mathrm{T}}^*M$. Suppose that* $\mathsf{A} \in I^m\big(M \times N, \Gamma'; \mathrm{Hom}(\mathcal{F}, \mathcal{E}) \otimes \Omega^{1/2}(M \times N)\big), \mathsf{B} \in I^{-m}\big(N \times M, \Gamma^{-1\prime}; \mathrm{Hom}(\mathcal{E}, \mathcal{F}) \otimes \Omega^{1/2}(N \times M)\big)$ *and that* $P \in \Psi\mathrm{DO}^{m'}(M; \mathcal{E} \otimes \Omega^{1/2}M)$ *having a scalar principal symbol σ_P. Then,* $\mathsf{B}P\mathsf{A} \in \Psi\mathrm{DO}^{m'}(N; \mathcal{F} \otimes \Omega^{1/2}N)$ *is properly supported whose principal symbol is given by*

$$\sigma_{\mathsf{B}P\mathsf{A}} = \sigma_{\mathsf{B}\mathsf{A}}(\varkappa^* \sigma_P), \qquad (2.163)$$

where $\varkappa^* : C^\infty\big(\dot{\mathrm{T}}^*M, \mathrm{Hom}(\mathcal{E}, \mathcal{E})\big) \to C^\infty\big(\dot{\mathrm{T}}^*N, \mathrm{Hom}(\mathcal{F}, \mathcal{F})\big)$ *is the pullback map via $\varkappa$.*

Proof. We will use the same strategy employed for proving the scalar version [23, Thm. 25.3.5]. By repeated applications of the composition of Lagrangian distributions, we have $P\mathsf{A} \in I^{m+m'}\big(M \times N, \Gamma'; \mathrm{Hom}(\mathcal{F}, \mathcal{E}) \otimes \Omega^{1/2}(M \times N)\big)$ and $\mathsf{B}P\mathsf{A} \in \Psi\mathrm{DO}^{m'}(N; \mathcal{F} \otimes \Omega^{1/2}N)$. In order to compute $\sigma_{P\mathsf{A}}$, one lifts σ_P to Γ via the pullbacks of the projector $\dot{\mathrm{T}}^*M \times \dot{\mathrm{T}}^*N \to \dot{\mathrm{T}}^*M$ followed by the inclusion $\Gamma \hookrightarrow \dot{\mathrm{T}}^*M \times \dot{\mathrm{T}}^*N$. Then $\sigma_{P\mathsf{A}} = \sigma_P \sigma_\mathsf{A}$. Equivalently, one

can lift $\varkappa^*\sigma_P$ to $\dot{\mathrm{T}}^*M \times \dot{\mathrm{T}}^*N$ via the projector $\dot{\mathrm{T}}^*M \times \dot{\mathrm{T}}^*N \to \dot{\mathrm{T}}^*N$ and then consider it as a homomorphism on Γ as before. Thus $PA - AR \in I^{m+m'-1}\big(M \times N, \Gamma'; \mathrm{Hom}(\mathscr{F}, \mathscr{E}) \otimes \Omega^{1/2}(M \times N)\big)$ if $R \in \Psi\mathrm{DO}^{m'}(N; \mathscr{F} \otimes \Omega^{1/2}N)$ having the principal symbol $\varkappa^*\sigma_P$. As σ_P is scalar, therefore $BPA - BAR \in \Psi\mathrm{DO}^{m'-1}(N; \mathscr{F} \otimes \Omega^{1/2}N)$ which entails the claim.   ∎

2.4   Literature

Historically, the theory *pseudodifferential operator*s was developed from the theory of singular integral operators by Joseph J. KOHN and Louis NIRENBERG [99] and enriched, in particular, by Lars HÖRMANDER [74].

*Fourier integral operator*s are an enormous generalisation of pseudodifferential operators, originated from the investigation of the singularities of solutions of hyperbolic differential equations by Peter LAX [100] and in the study of asymptotic analysis in the context of geometric optics and quantum mechanics by Victor MASLOV [101, Chap. 8]. Their somewhat local formulation was later systematically developed and globalised in two seminal articles by Lars HÖRMANDER [78] and by Johannes J. DUISTERMAAT and Lars HÖRMANDER [35] for scalar operators while the bundle version is available in the fourth volume of Hörmander's treatise [23].

Egorov's theorem is named after Yu. V. EGOROV [102] who proved it at first. Since then various generalisation of this result has been reported sporadically, e.g. [45], [103, Thm. 3.2], [104, Thm. 1.7], [105, Thm. 6.1], [106, Prop. 3.3], [58, Prop. A.3]. Theorem 2.3.17 is slightly general than the statements available in literature to the best of our knowledge.

We refer, for instance, the monographs [8, 77, 85, 107] and the expository lecture notes [87, 90, 98] for an in-depth discussion on pseudodifferential operators and microlocal analysis. For Fourier integral operators, the textbooks [11, 84, 85] are referred for details.

Chapter 3

Feynman Propagators

The subject matter of this chapter is a microlocal construction of Feynman propagators for normally hyperbolic operators and for Dirac-type operators on a globally hyperbolic spacetime. These results are shown by constructing Feynman parametrices and by employing the well-posed Cauchy problem. A vector bundle generalisation of the microlocalisation of a pseudodifferential operator of real principal type is achieved in order to give the microlocal construction of Feynman parametrices.

3.1 Lorentzian manifolds

By a **Lorentzian manifold** $(\mathcal{M}, \mathrm{g})$ we mean a Hausdorff, second countable, connected, and smooth $d \geq 2$-dimensional manifold $\mathcal{M}$ endowed with a smooth Lorentzian metric g of signature $(+, - \ldots -)$. The global existence of Lorentzian metrics depends on the topology of $\mathcal{M}$: *they always exist on any non-compact $\mathcal{M}$, whereas a compact $\mathcal{M}$ admits a Lorentzian metric if and only if its Euler characteristic vanishes.* Hausdorff condition together with the existence of a Lorentzian metric entails that $\mathcal{M}$ is paracompact.

Since g is a (symmetric) non-degenerate bilinear map $\mathrm{g} : C^\infty(\mathcal{M}; \mathrm{T}\mathcal{M}) \times C^\infty(\mathcal{M}; \mathrm{T}\mathcal{M}) \to C^\infty(\mathcal{M}, \mathbb{R})$, it induces a unique vector bundle (linear) isomorphism $C^\infty(\mathcal{M}; \mathrm{T}\mathcal{M}) \ni X \mapsto X^\flat \in C^\infty(\mathcal{M}; \mathrm{T}^*\mathcal{M})$ defined pointwise via $X_x^\flat(\cdot) := \mathrm{g}_x(X_x, \cdot)$ for any $x \in \mathcal{M}$. Moreover, it induces another unique symmetric, non-degenerate, indefinite bilinear form g^{-1} on $\mathrm{T}^*\mathcal{M}$, pointwise given by $\mathrm{g}_x^{-1}(\xi_x, \eta_x) := \mathrm{g}_x(\xi_x^\sharp, \eta_x^\sharp)$ where $\sharp : C^\infty(\mathcal{M}, \mathrm{T}^*\mathcal{M}) \to C^\infty(\mathcal{M}; \mathrm{T}\mathcal{M})$ is the inverse of $\flat$.

3.1.1 Causal structure

The causal structure is one of the drastic differences between a Lorentzian and a Riemannian manifold. The signature of g allows one to make the following classification.

Definition 3.1.1. A cotangent vector $\xi \in T_x^*\mathcal{M}$ on any point x in a Lorentzian manifold $(\mathcal{M}, g)$ is called **timelike** if $g_x^{-1}(\xi, \xi) > 0$, **spacelike** if $g_x^{-1}(\xi, \xi) < 0$ or $\xi = 0$, **lightlike** if $g_x^{-1}(\xi, \xi) = 0$ whenever $\xi \neq 0$, and **causal** if ξ is either timelike or lightlike.

The attribution of any covector according to this classification is called its causal character. The set $\dot{T}_{0,x}^*\mathcal{M} \subset \dot{T}_x^*\mathcal{M}$ of all lightlike covectors at $x \in \mathcal{M}$ is called the **lightcone** at x and it is an inner product space. One notices that, if $\xi, \eta \in \dot{T}_x^*\mathcal{M}$ are timelike then $g_x^{-1}(\xi, \eta) \lessgtr 0$. Thus, at each $x \in \mathcal{M}$, the cotangent vector space $\dot{T}_x^*\mathcal{M}$ has two possible open convex cones of timelike covectors whose boundaries consist of the lightlike covectors, and there is no intrinsic way to distinguish one from another albeit it is crucial to do so for some purpose. $\dot{T}_x^*\mathcal{M}$ is called **time-oriented** if one makes a choice (termed *future*) for one of these cones; the other cone is then termed as *past*. Whether such a designation of future and past is possible in a continuous fashion as x varies over $\mathcal{M}$ gives the notion of time-orientability of $(\mathcal{M}, g)$, which is logically independent of the topological orientation of $\mathcal{M}$. If $\mathfrak{t}$ is a smooth function on $\mathcal{M}$ that assigns to each $x \in \mathcal{M}$ either a future or equivalently a past cone, then it is called a **time-orientation** of $(\mathcal{M}, g)$. If $\mathcal{M}$ admits a time-orientation then $(\mathcal{M}, g)$ is said to be time-orientable.

Definition 3.1.2. A time-oriented and orientated Lorentzian manifold is called a **spacetime** $(\mathcal{M}, g)$.

Note, a Lorentzian manifold is time-orientable if and only if it admits a smooth (highly non-unique) *global timelike vector field X*. Let us pick a time-orientation once and for all such that X is future-directed at all points and then any causal $\xi \in \dot{T}_x^*\mathcal{M}$ is future directed if and only if $g_x^{-1}(\xi, X^\flat) > 0$. Therefore, we have a consistent notion of future and past of any point in spacetime. To be precise, any differentiable curve $c : \mathbb{I} \to \mathcal{M}$ is called spacelike (resp. future/past directed causal) if its tangent $\mathrm{d}c/\mathrm{d}s$ is spacelike (resp. future/past directed causal), where $\mathbb{I} \subset \mathbb{R}$ is an interval and $s \in \mathbb{I}$.

We now introduce the causal structure of spacetime $(\mathcal{M}, g)$. If $x, y \in \mathcal{M}$, then

- $x \rhd y$ means that there is a future directed timelike curve in $\mathcal{M}$ from x to y;
- $x \unrhd y$ means that either there is a future directed causal curve in $\mathcal{M}$ from x to y, or $x = y$.

The **causal future** $J^+(x) := \{y \in \mathcal{M} \mid x \unrhd y\}$ of a point $x \in \mathcal{M}$ is defined as the set of all points that can be reached by a future directed causal curve in $\mathcal{M}$ emanating from x and x itself. Analogously, $J^-(x) := \{y \in \mathcal{M} \mid y \unrhd x\}$. Then the causal future (past) of a set $K \subset \mathcal{M}$ is defined by

$$J^\pm(K) := \bigcup_{x \in K} J^\pm(x) \tag{3.1}$$

and one also uses the notation

$$J(K) := J^+(K) \cup J^-(K). \tag{3.2}$$

If one replaces causal curves with timelike curves then it yields **chronological** future/past.

3.1.2 Globally hyperbolic spacetimes

A generic spacetime is too general to consider as it may contain pathological mathematical and physical properties. For instance, the initial-value problem for a wave equation is not well-posed on an arbitrary spacetime. Also, there may exist closed causal curves favouring the "grandfather paradox". Therefore, we look for a subset of spacetimes excluding all such undesirable possibilities, as introduced below.

Definition 3.1.3. A spacetime $(\mathcal{M}, \mathrm{g})$ is called **globally hyperbolic** if and only if it is causal and the diamonds $J^+(x) \cap J^-(y)$ are compact for all $x, y \in \mathcal{M}$.

In what follows, any globally hyperbolic spacetime admits a specific global orthogonal splitting. Such a characterisation is of utmost importance for our thesis and we introduce the following terminologies in order to formulate the characterisation.

Definition 3.1.4. A **Cauchy hypersurface** Σ in a Lorentzian manifold $(\mathcal{M}, \mathrm{g})$ is a subset $\Sigma \subset \mathcal{M}$ which is met exactly once by every inextensible timelike curve.

Definition 3.1.5. Any $\mathrm{t} \in C^\infty(\mathcal{M}, \mathbb{R})$ on a spacetime $(\mathcal{M}, \mathrm{g})$ is called a

- **time function** if t is strictly increasing along all future-directed causal curves;

- **temporal function** if t is smooth having a future-directed timelike gradient;

A (time and/or temporal) function is called **Cauchy** if its level sets $\Sigma_t := \mathrm{t}^{-1}(t)$ are Cauchy hypersurfaces for any $t \in \mathbb{R}$.

We now inscribe the global orthogonal splitting of a globally hyperbolic spacetime following the survey article [108, Thm. 3.78].

Theorem 3.1.6 (Geroch [109]-Bernal-Sánchez [110] splitting theorem). *Let $(\mathcal{M}, \mathrm{g})$ be a spacetime. It is globally hyperbolic if and only if it admits a smooth spacelike Cauchy hypersurface Σ.*

In this case, it allows a Cauchy temporal function t and, thus, it is isometrically diffeomorphic to the smooth product manifold

$$(\mathcal{M}, \mathrm{g}) \cong (\mathbb{R} \times \Sigma, \beta \, \mathrm{d}t^2 - \mathrm{h}_t), \tag{3.3}$$

where $\beta \in C^\infty(\mathbb{R} \times \Sigma, \mathbb{R}_+)$ is the lapse function, $t \in C^\infty(\mathbb{R} \times \Sigma, \mathbb{R})$ is the natural projection, each level set Σ_t of $t = \mathrm{cst}$ is a spacelike Cauchy hypersurface, and h_t is a Riemannian metric on each Σ_t, which varies smoothly with t.

Moreover, if Σ is a prescribed topological Cauchy hypersurface then there exists a smooth Cauchy function t such that Σ is one of its level sets ($\Sigma = \Sigma_0$). In addition, if Σ is

- *acausal then t becomes a smooth Cauchy time function;*

- *spacelike then* t *can be modified to obtain a Cauchy temporal function on* $\mathscr{M}$ *such that* $\Sigma = \mathrm{t}^{-1}(0)$.

Furthermore, if $S \subset \mathscr{M}$ *is a compact achronal set then it can be extended to a Cauchy hypersurface. Additionally, if* S *is acausal and a smooth spacelike submanifold with boundary then it can be extended to a spacelike Cauchy hypersurface* $\Sigma \supset S$.

Remark 3.1.7. Globally hyperbolic spacetimes $(\mathscr{M}, \mathrm{g})$ can always be foliated by a Cauchy hypersurface Σ, but the foliation is *non-unique*. Necessarily, Σ is a closed subset and an *embedded* topological submanifold of codimension 1.

Example 3.1.8. Let $\mathbb{I} \subset \mathbb{R}$ be open and let (Σ, h) be a connected Riemannian manifold. Then $(\mathscr{M} := \mathbb{I} \times \Sigma, \mathrm{g} := \mathrm{d}t^2 - f^2 \mathrm{h})$ is a globally hyperbolic spacetime if and only if (Σ, h) is complete, where $f \in C^\infty(\mathbb{I}, \mathbb{R}_+)$ (see e.g. [73, Lem. A.5.14]).

Physically speaking, our understanding of the physical universe is currently best explained by the "ΛCDM model" where the spacetime geometry is given by the Robertson-Walker spacetime (see e.g. [111, Sec. 5.3], [89, Def. 7, p. 343]). This so-called standard model of cosmology describes, for instance, the cosmic microwave background, expansion of the universe and the cosmological Redshift pretty accurately (see e.g. [112]). However, the standard model has a few limitations and the de Sitter spacetime (see e.g. [111, Sec. 5.2], [89, p. 229]) offers a solution of some of these open problems. The preceding example encompasses both these spacetimes together with, of course, the Minkowski spacetime. Besides, a number of black hole solutions of Einstein equation are globally hyperbolic, e.g., the interior and exterior of Schwarzschild spacetime, parts of regions of the Reissner–Nordström and the Kerr spacetimes. There are, of course, solutions of Einstein equation which are not globally hyperbolic at all. One of such spacetimes received huge attention in the context of superstring theory is the anti-de Sitter solution (see e.g. [111, Sec. 5.2], [89, p. 229]). Roughly speaking, globally hyperbolic spacetimes are interesting because they offer prediction (or retrodiction) of the entire future (or past) history of the universe from conditions at an instant of time.

Remark 3.1.9. If $(\mathscr{M}, \mathrm{g})$ is a globally hyperbolic spacetime then the set $\dot{\mathrm{T}}^*_0 \mathscr{M} \to \mathscr{M}$ of lightcones (lightlike geodesics on $\mathrm{T}^* \mathscr{M}$) has a smooth bundle structure (see e.g. [92]).

3.2   Normally hyperbolic operators and parametrices

3.2.1   Normally hyperbolic operators (NHOs)

Definition 3.2.1. Let $\mathscr{E} \to \mathscr{M}$ be a vector bundle over a Lorentzian manifold $(\mathscr{M}, \mathrm{g})$. A second-order linear differential operator (see e.g. [113, Def. 3.1], [73, Sec. 1.5])

$$\Box : C^\infty(\mathscr{M}; \mathscr{E}) \to C^\infty(\mathscr{M}; \mathscr{E}) \tag{3.4}$$

is called a **normally hyperbolic operator** if its principal symbol

$$\sigma_\square(x,\xi) := g_x^{-1}(\xi,\xi) \; \mathbb{1}_{(\text{End}\,\mathscr{E})_x}. \tag{3.5}$$

is given by the spacetime metric g^{-1} on the cotangent bundle $T^*\mathscr{M}$ for all $(x,\xi) \in C^\infty(\mathscr{M}; T^*\mathscr{M})$.

In a local coordinate chart $\big(U,(x^i)\big)$ of $\mathscr{M}$, after trivialising $\mathscr{E}$, $\square$ is given by

$$\square \restriction U = -g^{ij}\frac{\partial^2}{\partial x^i \partial x^j} + A^i\frac{\partial}{i\,\partial x^i} + B, \quad i,j = 1,\dots,d, \tag{3.6}$$

where $A^i \in C^\infty(U; TU \otimes \text{End}\,\mathscr{E}_U), B \in C^\infty(U, \text{End}\,\mathscr{E}_U)$.

Normally hyperbolic operators arise naturally in the context of geometric analysis and quantum field theories in curved spacetime, and we list below a few.

Example 3.2.2. Let $\mathscr{E} \to \mathscr{M}$ be a trivial-$\mathbb{C}$-line bundle, that is, its sections are just $\mathbb{C}$-valued smooth functions $C^\infty(\mathscr{M})$ on $(\mathscr{M},g)$. The **covariant Klein-Gordon operator** is defined by (see e.g. [113, Sec. 3.1, Exm. 1], [73, Exm. 1.5.1])

$$\square := -\,\text{div}\circ\text{grad} -m^2 - \lambda\,\text{ric} : C^\infty(\mathscr{M}) \to C^\infty(\mathscr{M}), \tag{3.7}$$

where $m^2 \in \mathbb{R}_+$ is a parameter, physically interpreted as the mass-squared of a linear Klein-Gordon field, λ is a coupling, ric is the Ricci scalar of $\mathscr{M}$, grad $f := g^\sharp(df)$ is the gradient of a function $f \in C(\mathscr{M})$, and $\text{div}(X) := \text{tr}(\nabla^{LC}X)$ is the divergence of a vector field $X \in C(\mathscr{M}; T\mathscr{M})$. Here ∇^{LC} is the Levi-Civita connection of $\mathscr{M}$ and the **endomorphism trace**

$$\text{tr} : \text{End}\,\mathscr{E} \to \mathbb{C} \tag{3.8}$$

is the finite-dimensional trace on the fibres $\mathscr{E}_x$ of any vector bundle $\mathscr{E} \to M$ over a generic manifold M, defined by the *composition of the canonical isomorphism* $\text{End}(\mathscr{E}_x) \cong \mathscr{E}_x^* \otimes \mathscr{E}_x$ *followed by the contraction mapping* $\mathscr{E}_x^* \otimes \mathscr{E}_x \to \mathbb{C}$.

The particular case of the preceding expression of $\square$ when $m,\lambda = 0$ is called the **d'Alembertian** or the **relativistic wave operator** which is locally given by

$$\square = -g^{ij}\nabla_i^{LC}\nabla_j^{LC}, \quad i,j = 1,\dots,d. \tag{3.9}$$

In case $(\mathscr{M},g) := (\mathbb{R}\times\Sigma, dt^2 - h)$ is a ultrastatic manifold then

$$\square = -\frac{\partial^2}{\partial t^2} - \Delta \tag{3.10}$$

is known as the **wave operator of a Riemannian manifold** (Σ, h) where t is the natural coordinate on $\mathbb{R}$ and Δ is the Laplace-Beltrami operator of (Σ, h) in the geometers' convention:

$$\Delta := -\,\text{div}_h\circ\text{grad} = -\frac{1}{\sqrt{\det h}}\frac{\partial}{\partial x^i}\Big(h^{ij}\sqrt{\det h}\frac{\partial}{\partial x^j}\Big), \quad i,j = 2,\dots,d. \tag{3.11}$$

Example 3.2.3. Suppose that $\mathscr{E} \to \mathscr{M}$ be a trivial-$\mathbb{C}$-line bundle over a $d \geq 3$-dimensional Lorentzian manifold $(\mathscr{M}, \mathrm{g})$ and that $\Box$ resp. ric are the d'Alembertian resp. the Ricci scalar of $(\mathscr{M}, \mathrm{g})$. The **Yamabe operator** is defined by (see e.g. [113, Sec. 3.1, Exm. 2], [73, Def. 3.5.9])

$$\mathrm{Y} := \Box + \frac{d-2}{4(d-1)} \, \mathrm{ric} : C^\infty(\mathscr{M}) \to C^\infty(\mathscr{M}). \tag{3.12}$$

Example 3.2.4. Let $\mathscr{E} := \wedge^k \mathrm{T}^* \mathscr{M}$ be the bundle of k-forms over an oriented Lorentzian manifold $(\mathscr{M}, \mathrm{g})$. The **Hodge-d'Alembert operator** is defined by (see e.g. [113, Sec. 3.1, Exm. 3], [73, Exm. 1.5.3])

$$\Box := \mathrm{d}^* \circ \mathrm{d} + \mathrm{d} \circ \mathrm{d}^* : C^\infty\big(\mathscr{M}; \wedge^k \mathrm{T}^* \mathscr{M}\big) \to C^\infty\big(\mathscr{M}; \wedge^k \mathrm{T}^* \mathscr{M}\big), \tag{3.13}$$

where d and d^* are the exterior and the coexterior differentials, respectively.

Example 3.2.5. Let $(\mathscr{E} \to \mathscr{M}, \nabla^{\mathscr{E}})$ be a vector bundle equipped with a connection $\nabla^{\mathscr{E}}$ and $(\mathrm{T}^* \mathscr{M} \to \mathscr{M}, \nabla^{\mathrm{LC}})$ is the cotangent bundle endowed with the Levi-CIvita connection ∇^{LC} over a Lorentzian manifold $(\mathscr{M}, \mathrm{g})$. Then we have an induced connection

$$\nabla^{\mathrm{LC} \otimes \mathscr{E}} := \nabla^{\mathrm{LC}} \otimes \mathbb{1}_{\mathscr{E}} + \mathbb{1}_{\mathrm{T}^* \mathscr{M}} \otimes \nabla^{\mathscr{E}} \tag{3.14}$$

on $\mathrm{T}^* \mathscr{M} \otimes \mathscr{E}$. The **connection d'Alembertian** is defined by the composition of the following three maps (see e.g. [73, Exm. 1.5.2])

$$C^\infty(\mathscr{M}; \mathscr{E}) \xrightarrow{\nabla^{\mathscr{E}}} C^\infty(\mathscr{M}; \mathrm{T}^* \mathscr{M} \otimes \mathscr{E}) \xrightarrow{\nabla^{\mathrm{LC} \otimes \mathscr{E}}} C^\infty(\mathscr{M}; \mathrm{T}^* \mathscr{M} \otimes \mathrm{T}^* \mathscr{M} \otimes \mathscr{E}) \xrightarrow{\mathrm{tr}_{\mathrm{g}} \otimes I} C^\infty(\mathscr{M}; \mathscr{E}) \tag{3.15}$$

where $\mathrm{tr}_{\mathrm{g}} : C^\infty(\mathscr{M}; \mathrm{T}^* \mathscr{M} \otimes \mathrm{T}^* \mathscr{M}) \to C^\infty(\mathscr{M}, \mathbb{R})$ denotes the **metric trace**:

$$\mathrm{tr}_{\mathrm{g}}\big((x, \xi) \otimes (x, \eta)\big) := \mathrm{g}_x^{-1}(\xi, \eta). \tag{3.16}$$

The minus of a connection d'Alembertian,

$$\Box := -\mathrm{tr}_{\mathrm{g}}\big(\nabla^{\mathrm{LC} \otimes \mathscr{E}} \circ \nabla^{\mathscr{E}}\big) \tag{3.17}$$

is a normally hyperbolic operator.

One observes that the scalar d'Alembertian $(3.9) - \mathrm{tr}_{\mathrm{g}}(\nabla^{\mathrm{LC}} \circ \mathrm{d}u)$ is a special case of the connection d'Alembertian when u is a section of any trivial $\mathbb{C}$-line bundle. Note that the first two composition of (3.15)

$$\mathrm{Hess} := \nabla^{\mathrm{LC} \otimes \mathscr{E}} \circ \nabla^{\mathscr{E}} : C^\infty(\mathscr{M}; \mathscr{E}) \to C^\infty(\mathscr{M}; \mathrm{T}^* \mathscr{M} \otimes \mathrm{T}^* \mathscr{M} \otimes \mathscr{E}) \tag{3.18}$$

is called the **Hessian** and the corresponding differential operator $\mathrm{Hess}_{X,Y} = \nabla_X^{\mathscr{E}} \nabla_Y^{\mathscr{E}} - \nabla_{\nabla_X^{\mathrm{LC}} Y}^{\mathscr{E}}$ is called the **second covariant derivative** on $\mathscr{E}$, where X, Y are any tangent vectors on $\mathscr{M}$ (see e.g. [114, p. 66]).

Example 3.2.6. Let $\left(\mathscr{E} \to \mathscr{M}, (\cdot|\cdot), \nabla^{\mathscr{E}}\right)$ be a vector bundle equipped with a sesquilinear form $(\cdot|\cdot)$ and a connection $\nabla^{\mathscr{E}}$ whose dual is $\nabla^{\mathscr{E}*}$, over a Lorentzian manifold $(\mathscr{M}, g)$. The **Bochner-d'Alembertian**, defined by

$$\Box := \nabla^{\mathscr{E}*} \circ \nabla^{\mathscr{E}} : C^{\infty}(\mathscr{M}; \mathscr{E}) \to C^{\infty}(\mathscr{M}; \mathscr{E}), \tag{3.19}$$

differs from the corresponding connection d'Alembertian only by a minus sign: $\nabla^{\mathscr{E}*} \circ \nabla^{\mathscr{E}} = -\operatorname{tr}_{g}\left(\nabla^{\mathrm{LC} \otimes \mathscr{E}} \circ \nabla^{\mathscr{E}}\right)$ (see e.g. [114, Def. 2.4]). In any orthonormal frame $\{e_i\}$ of T$\mathscr{M}$, it is given by

$$\Box = -g^{ij}\left(\nabla^{\mathscr{E}}_{e_i}\nabla^{\mathscr{E}}_{e_j} - \Gamma^{\mathrm{LC}\,k}_{ij}\nabla^{\mathscr{E}}_{e_k}\right), \quad i, j, k = 1, \dots, k, \tag{3.20}$$

where Γ^{LC} is the Levi-Civita connection 1-form with respect to e_i: $\nabla^{\mathrm{LC}}_{e_i}e_j = \Gamma^{\mathrm{LC}\,k}_{ij}e_k$.

There are some operators which are not normally hyperbolic yet can be related with a normally hyperbolic one with additional constraints.

Example 3.2.7. The **Proca operator** acting on covectors ξ on $\mathscr{M}$ is defined as

$$P := d^* \circ d + m^2 : C^{\infty}(\mathscr{M}; T^*\mathscr{M}) \to C^{\infty}(\mathscr{M}; T^*\mathscr{M}), \quad m \in \dot{\mathbb{R}}, \tag{3.21}$$

which is not normally hyperbolic. Nevertheless, this is equivalent to the normally hyperbolic operator $d^* \circ d + d \circ d + m^2$ whenever the **Lorenz constraint** $d^*\xi = 0$ is imposed.

It is evident from the preceding examples that $\Box$ differs from some d'Alembertian only by a smooth term. This is actually true in general.

Remark 3.2.8. By the Weitzenböck formula, given a normally hyperbolic operator $\Box$ on any vector bundle $\mathscr{E} \to \mathscr{M}$ over a Lorentzian manifold $(\mathscr{M}, g)$, there exists a *unique* bundle connection $\nabla^{\mathscr{E}}$, called the **Weitzenböck connection** and a *unique* potential $V \in C^{\infty}(\mathscr{M}; \operatorname{End}\mathscr{E})$ such that (see e.g. [113, Prop. 3.1], [73, Lem. 1.5.5, 1.5.6]):

$$\Box = -\operatorname{tr}_{g}\left(\nabla^{\mathrm{LC} \otimes \mathscr{E}} \circ \nabla^{\mathscr{E}}\right) + V. \tag{3.22}$$

Thus, in an orthonormal tangent frame $\{\partial/\partial x^i\}$ and a bundle frame $\{E_r\}$, a straightforward calculation using (3.20) and $\nabla^{\mathscr{E}}_i = \partial_i + \Gamma_i$, Γ being the Weitzenböck-connection 1-form with respect to ∂_i: $\nabla^{\mathscr{E}}_i E_r = \Gamma^{r'}_{ir}E_{r'}$ entails

$$(\Box u)^r = g^{ij}\left(-\frac{\partial^2 u^r}{\partial x^i \partial x^j} - 2\Gamma^r_{ir'}\frac{\partial u^{r'}}{\partial x^j} + \Gamma^{\mathrm{LC}\,k}_{ij}\frac{\partial u^r}{\partial x^k} - \left(\frac{\partial \Gamma^r_{jr'}}{\partial x^i} - \Gamma^r_{ir''}\Gamma^{r''}_{jr'} + \Gamma^{\mathrm{LC},k}_{ij}\Gamma^r_{kr'}\right)u^{r'}\right) + Vu^r, \tag{3.23}$$

where $i, j, k = 1, \dots, d$ and $r, r', r'' = 1, \dots, \operatorname{rk}\mathscr{E}$, and Γ^{LC} is the Levi-Civita connection 1-form with respect to ∂_i. Therefore, the subprincipal symbol of any normally hyperbolic operator is given by

$$
\begin{aligned}
\sigma^{\mathrm{sub}}_{\Box}(x, \xi) \overset{(2.81)}{=}\ & -2\,\mathrm{i}\,g^{ij}\Gamma_i\xi_j + \mathrm{i}\,g^{ij}\Gamma^{\mathrm{LC}\,k}_{ij}\xi_k + \frac{\mathrm{i}}{2}\frac{\partial^2(g^{\mu\nu}\xi_\mu\xi_\nu)}{\partial x^i \partial \xi_i} + \frac{\mathrm{i}}{2}\Gamma^{\mathrm{LC}\,j}_{ji}\frac{\partial g^{\mu\nu}\xi_\mu\xi_\nu}{\partial \xi_i} \\
=\ & -2\,\mathrm{i}\,g^{ij}\Gamma_i\xi_j + \mathrm{i}\left(g^{ij}\Gamma^{\mathrm{LC}\,k}_{ij} + \frac{\partial g^{ik}}{\partial x^i} + \Gamma^{\mathrm{LC}\,j}_{ji}g^{ik}\right)\xi_k \\
=\ & -2\,\mathrm{i}\,g^{ij}\Gamma_i\xi_j, \tag{3.24}
\end{aligned}
$$

because of the identities (see e.g. [115, (3.4.9)])

$$\mathrm{g}^{ij}\Gamma^{\mathrm{LC}\,k}_{\ ij} = -\frac{1}{\sqrt{|\det \mathrm{g}|}}\frac{\partial(\sqrt{|\det \mathrm{g}|}\mathrm{g}^{ik})}{\partial x^i}, \quad \Gamma^{\mathrm{LC}\,j}_{\ ji} = \frac{1}{2}\frac{1}{|\det \mathrm{g}|}\frac{\partial|\det \mathrm{g}|}{\partial x^i}. \tag{3.25}$$

If $\mathscr{E}$ carries a bundle metric $(\cdot|\cdot)$ such that $\Box$ is symmetric then $\nabla^{\mathscr{E}}$ is compatible with $(\cdot|\cdot)$ as well. We do not, however, assume this unless stated otherwise.

3.2.2   Green's operators

Definition 3.2.9. Let $\mathscr{E} \to \mathscr{M}$ be a vector bundle over a time-oriented Lorentzian manifold $(\mathscr{M}, \mathrm{g})$ and L a differential operator on $\mathscr{E}$. A **Green's operator** $G^{\times}$ for L is a linear mapping (see e.g. [73, Def. 3.4.1])

$$G^{\times} : C^{\infty}_{\mathrm{c}}(\mathscr{M}; \mathscr{E}) \to C^{\infty}(\mathscr{M}; \mathscr{E}) \mid L \circ G^{\times} = I, \ G^{\times} \circ L|_{C^{\infty}_{\mathrm{c}}(\mathscr{M};\mathscr{E})} = I. \tag{3.26}$$

A Green's operator is called

- **retarded** G^{ret} if $\mathrm{supp}(G^{\mathrm{ret}}u) \subset J^{+}(\mathrm{supp}\, u)$ and

- **advanced** G^{adv} if $\mathrm{supp}(G^{\mathrm{adv}}u) \subset J^{-}(\mathrm{supp}\, u)$,

where $J^{\pm}(K)$ denotes the causal future/past of a set $K \subset \mathscr{M}$. The **Jordan-Lichnerowicz-Pauli** or the **causal** Green's operator is defined as the antisymmetric combination

$$G := G^{\mathrm{ret}} - G^{\mathrm{adv}}. \tag{3.27}$$

In physics, Green's operators are usually called *propagators* and we will use this terminology as well. Propagators are closely related to the concept of fundamental solutions.

Definition 3.2.10. As in the terminologies of Definition 3.2.9, a **fundamental solution** of L at any $x \in \mathscr{M}$ is an $\mathscr{E}^{*}_{x}$-valued distribution on $\mathscr{E}$ (see e.g. [73, Def. 2.1.1], [72, Def. 3.1])

$$F^{\times}_{x} : C^{\infty}_{\mathrm{c}}(\mathscr{M}; \mathscr{E}^{*}) \to \mathscr{E}^{*}_{x}, \ \phi \mapsto F^{\times}_{x}(L^{*}\phi) = \phi(x) \tag{3.28}$$

where L^{*} is the formal adjoint of L. In other words, $F^{\times}_{x} \in \mathscr{D}'(\mathscr{M}; \mathscr{E}; \mathscr{E}^{*}_{x})$ such that

$$LF^{\times}_{x} = \delta_{x}. \tag{3.29}$$

A fundamental solution is called **retarded** F^{ret} if $\mathrm{supp}\, F^{\mathrm{ret}}_{x} \subset J^{+}(x)$ and **advanced** F^{adv} if $\mathrm{supp}\, F^{\mathrm{adv}}_{x} \subset J^{-}(x)$. The antisymmetric combination of the preceding two defines the **Jordan-Lichnerowicz-Pauli** or the **causal** fundamental solution $F := F^{\mathrm{ret}} - F^{\mathrm{adv}}$.

To explain these concepts, let us look at the simplest non-trivial differential operator, the partial derivative.

Example 3.2.11. Let $x \in \mathbb{R}^d$ and we write it as $x = (x^1, x')$ where $x^1 \in \mathbb{R}$ and $x' = (x^2, \ldots, x^d) \in \mathbb{R}^{d-1}$. Since the distributional derivative of Heaviside step function[1] Θ_{x_0} is the Dirac delta distribution δ_{x_0} concentrated at x_0, the advanced, retarded, and causal fundamental solutions of $D_1 := -i\,\partial/\partial x^1$ are

$$
\begin{align}
F_1^{\text{adv}}(x) &= -i\,\Theta(-x^1) \otimes \delta(x'), \tag{3.30a} \\
F_1^{\text{ret}}(x) &= -i\,\Theta(x^1) \otimes \delta(x'), \tag{3.30b} \\
F_1(x') &= -i\,\delta(x'). \tag{3.30c}
\end{align}
$$

Their Schwartz kernels are given by

$$
\begin{align}
\mathsf{F}_1^{\text{adv}}(x-y) &= -i\,\Theta(y^1-x^1) \otimes \delta(x'-y'), \tag{3.31a} \\
\mathsf{F}_1^{\text{ret}}(x-y) &= -i\,\Theta(x^1-y^1) \otimes \delta(x'-y'), \tag{3.31b} \\
\mathsf{F}_1(x'-y') &= -i\,\delta(x'-y') = \frac{-i}{(2\pi)^{d-1}} \int_{\mathbb{R}^{d-1}} e^{i(x^2-y^2)\theta_2+\ldots+i(x^d-y^d)\theta_d}\, d\theta_2 \ldots d\theta_d. \tag{3.31c}
\end{align}
$$

One reads off the following facts from the Fourier integral representations of the preceding fundamental solutions.

Proposition 3.2.12. *As in the terminologies of Example 3.2.11, let χ be a smooth function on $\mathbb{R}^d \times \mathbb{R}^d$ vanishing near the diagonal. Then [35, Prop. 6.1.2] (see also [23, Prop. 26.1.2])*

$$
\begin{align}
\text{WF}'\,\mathsf{F}_1^{\text{adv,ret}} &= \Delta\dot{\mathrm{T}}^*\mathbb{R}^d \bigcup C_1^{\text{adv,ret}}, \tag{3.32} \\
C_1^{\text{adv,ret}} &:= \{(x,\xi;y,\eta) \in C_1 \,|\, x^1 \lessgtr y^1\}, \tag{3.33} \\
C_1 &:= \{(x,\xi;y,\eta) \in \mathrm{T}^*\mathbb{R}^d \times \mathrm{T}^*\mathbb{R}^d \,|\, x'=y', \xi'=\eta' \neq 0, \xi_1 = 0 = \eta_1\}, \tag{3.34} \\
\mathsf{F}_1, \chi\mathsf{F}_1^{\text{adv,ret}} &\in I^{-1/2}(\mathbb{R}^d \times \mathbb{R}^d, C_1'), \tag{3.35}
\end{align}
$$

where $\Delta\dot{\mathrm{T}}^*\mathbb{R}^d$ is the diagonal in $\dot{\mathrm{T}}^*\mathbb{R}^d \times \dot{\mathrm{T}}^*\mathbb{R}^d$ and $I^{-1/2}(\mathbb{R}^d \times \mathbb{R}^d, C_1')$ is the Lagrangian distribution (Definition 2.1.8) on $\mathbb{R}^d \times \mathbb{R}^d$ associated with the canonical relation C_1.

Suppose that $F_x^{\text{ret,adv}}$ is a family of retarded and advanced fundamental solutions for the adjoint $\square^*$ operator of a normally hyperbolic operator $\square$ on a vector bundle $\mathscr{E} \to \mathscr{M}$ over a time-oriented Lorentzian manifold $(\mathscr{M}, g)$ and that $F_x^{\text{ret,adv}}$ depends smoothly on $x \in \mathscr{M}$ in the sense that $x \mapsto F_x^{\text{ret,adv}}(\phi)$ is smooth for any $\phi \in C_c^\infty(\mathscr{M}; \mathscr{E}^*)$. If it satisfies the differential equation $\square(F_\cdot^{\text{ret,adv}}\phi) = \phi$ then (see e.g. [73, Prop. 3.4.2])

$$
(G^{\text{adv,ret}}\phi)(x) := F_x^{\text{ret,adv}}(\phi) \tag{3.36}
$$

defines advanced resp. retarded Green's operators for $\square$.

---

[1] We recall that it is defined as $\Theta_{x_0}(x) := 1$ for $x > x_0$ and $\Theta_{x_0}(x) := 0$ for $x < x_0$.

Remark 3.2.13. It is a classical result that any normally hyperbolic operator $\square$ on a vector bundle $\mathscr{E} \to \mathscr{M}$ over a globally hyperbolic spacetime $(\mathscr{M}, \mathrm{g})$ admits *unique retarded and advanced Green's operators* (see e.g. [73, Cor. 3.4.3], [72, Prop. 4.1, Rem. 4.3 b]):

$$G^{\mathrm{adv,ret}} : C_{\mathrm{c}}^{\infty}(\mathscr{M}; \mathscr{E}) \to C_{\mathrm{sc}}^{\infty}(\mathscr{M}; \mathscr{E}) \tag{3.37}$$

Furthermore, the Green's operators $(G^{\mathrm{adv,ret}})^*$ for the formal adjoint $\square^*$ of $\square$ are related to $G^{\mathrm{adv,ret}}$ by (see e.g. [73, Lem. 3.4.4])

$$\forall \phi \in C_{\mathrm{c}}^{\infty}(\mathscr{M}; \mathscr{E}^*), \forall u \in C_{\mathrm{c}}^{\infty}(\mathscr{M}; \mathscr{E}) : \left((G^{\mathrm{adv,ret}})^* \phi\right)(u) = \phi\left(G^{\mathrm{ret,adv}} u\right). \tag{3.38}$$

Hence, the causal propagator

$$G := G^{\mathrm{ret}} - G^{\mathrm{adv}} : C_{\mathrm{c}}^{\infty}(\mathscr{M}; \mathscr{E}) \to C_{\mathrm{sc}}^{\infty}(\mathscr{M}; \mathscr{E}) \tag{3.39}$$

satisfies

$$(G^* \phi)(u) = -\phi(Gu). \tag{3.40}$$

A natural question in this context is, *are there more Green's operators for a given $\square$?* In what follows, the answer leads to a special kind of propagators which are uniquely characterised in terms of their wavefront sets instead of their supports, in contrast to the advanced and retarded propagators.

3.2.3   Parametrices

In order to investigate the preceding question, it is useful to introduce a weaker notion of Green's operator, as defined below.

Definition 3.2.14. Let $\mathscr{E}, \Omega^{1/2} \to M$ be a vector bundle resp. the half-density bundle over a manifold M and P a properly supported pseudodifferential operator (Definition 2.3.2 and Definition 2.1.1) on $\Omega^{1/2}$-valued sections of $\mathscr{E}$. A **right parametrix** of P is a continuous operator

$$E_{\mathrm{R}} : C_{\mathrm{c}}^{\infty}(M; \mathscr{E} \otimes \Omega^{1/2}) \to C^{\infty}(M; \mathscr{E} \otimes \Omega^{1/2}) \quad | \quad PE_{\mathrm{R}} = I + R, \tag{3.41}$$

where I is the identity pseudodifferential operator and R is a smoothing operator. Similarly, a **left parametrix** is defined by $E_{\mathrm{L}} P = I + L$ where L is a smoothing operator and one says E is a parametrix of P when it is both a right and a left parametrix.

Now, let $\mathscr{E} \to \mathscr{M}$ be a vector bundle over a globally hyperbolic spacetime $(\mathscr{M}, \mathrm{g})$ and let E be a parametrix of a normally hyperbolic operator $\square$ on $\mathscr{E}$:

$$\square E \equiv I, \quad E\left(\square \upharpoonright C_{\mathrm{c}}^{\infty}(\mathscr{M}; \mathscr{E})\right) \equiv I, \tag{3.42}$$

where $\equiv$ means modulo smoothing operators. Remark 3.2.13 entails that $\square$ admits at least two parametrices: the advanced E^{adv} and the retarded E^{ret}. We would like to know whether there are more parametrices. This question has been investigated in a great detail by Duistermaat and Hörmander [35, Sec. 6.5, 6.6] for any scalar pseudodifferential operator of real principal type (Definition 2.2.10). To translate their analysis in the present context, we revisit Example 2.3.8 and consider the following relation.

Definition 3.2.15. Let $\dot{\mathrm{T}}^*_0 \mathscr{M} \to \mathscr{M}$ be the bundle of lightlike covectors over a globally hyperbolic spacetime $(\mathscr{M}, \mathrm{g})$. The **geodesic relation** C is defined as the set of all covector pairs in the product manifold $\dot{\mathrm{T}}^*_0 \mathscr{M} \times \dot{\mathrm{T}}^*_0 \mathscr{M}$ lying on the same geodesic on $\dot{\mathrm{T}}^*_0 \mathscr{M}$ [35, (6.5.2)] (see also [23, (26.1.8)]):

$$C := \left\{ (x, \xi; y, \eta) \in \dot{\mathrm{T}}^*_0 \mathscr{M} \times \dot{\mathrm{T}}^*_0 \mathscr{M} \mid \exists! s \in \mathbb{R} : (x, \xi) = \Phi_s(y, \eta) \right\}. \tag{3.43}$$

The **forward** (resp. **backward**) **geodesic relations** $C^\pm$ are defined as the set of all $(x, \xi; y, \eta)$ such that (x, ξ) lies after (resp. before) (y, η) on a geodesic:

$$C^\pm := \left\{ (x, \xi; y, \eta) \in \dot{\mathrm{T}}^*_0 \mathscr{M} \times \dot{\mathrm{T}}^*_0 \mathscr{M} \mid \exists! s \in \mathbb{R}_{\geqslant 0} : (x, \xi) = \Phi_s(y, \eta) \right\}. \tag{3.44}$$

Here, Φ_s is the geodesic flow on the cotangent bundle and $s \in \mathbb{R}$ is the flow-parameter.

Remark 3.2.16. The geodesic relation is a conic Lagrangian submanifold (see Definition A.1.5) of $\dot{\mathrm{T}}^*_0 \mathscr{M} \times \dot{\mathrm{T}}^*_0 \mathscr{M}$ which is closed in $\dot{\mathrm{T}}^*_0 (\mathscr{M} \times \mathscr{M})$ and consequently a homogeneous canonical relation (see Definition A.1.12) from $\dot{\mathrm{T}}^*_0 \mathscr{M}$ to $\dot{\mathrm{T}}^*_0 \mathscr{M}$ [35, Prop. 6.5.2] (see also [23, Prop. 26.1.13]).

With these terminologies in hand, the wavefront sets of the Schwartz kernels $\mathsf{E}^{\mathrm{adv}}$, $\mathsf{E}^{\mathrm{ret}}$ are given by

$$\begin{aligned} \mathrm{WF}' \, \mathsf{E}^{\mathrm{adv}} &\subset \Delta \dot{\mathrm{T}}^* \mathscr{M} \cup C^{\mathrm{adv}}, \quad C^{\mathrm{adv}} := \{ (x, \xi, y, \eta) \in C \mid x \in J^-(y) \}, \tag{3.45} \\ \mathrm{WF}' \, \mathsf{E}^{\mathrm{ret}} &\subset \Delta \dot{\mathrm{T}}^* \mathscr{M} \cup C^{\mathrm{ret}}, \quad C^{\mathrm{ret}} := \{ (x, \xi, y, \eta) \in C \mid x \in J^+(y) \}. \tag{3.46} \end{aligned}$$

Duistermaat-Hörmander's [35, p. 218] groundbreaking analysis shows that, if $d \geq 3$ then there are two more parametrices of $\square$ whose wavefront sets are off-diagonally given by the forward and backward geodesic relations. These parametrices are called the **distinguished parametrices** in pure mathematics, whereas they are known as the Feynman parametrices in the theoretical physics literature.

Definition 3.2.17. Let $\mathscr{E} \to \mathscr{M}$ be a vector bundle over a globally hyperbolic spacetime $(\mathscr{M}, \mathrm{g})$ and $\square$ a normally hyperbolic operator on $\mathscr{E}$. The **Feynman** E^{Fyn} and the **anti-Feynman** E^{aFyn} parametrices of $\square$ are those parametrices of $\square$ whose Schwartz kernels satisfy [35, p. 229]

$$\mathrm{WF}' \, \mathsf{E}^{\mathrm{Fyn}} \subset \Delta \dot{\mathrm{T}}^* \mathscr{M} \cup C^+, \qquad \mathrm{WF}' \, \mathsf{E}^{\mathrm{aFyn}} \subset \Delta \dot{\mathrm{T}}^* \mathscr{M} \cup C^-, \tag{3.47}$$

where $\Delta \dot{\mathrm{T}}^* \mathscr{M}$ is the diagonal to the punctured cotangent manifold $\dot{\mathrm{T}}^* \mathscr{M} \times \dot{\mathrm{T}}^* \mathscr{M}$ and $C^\pm$ are the forward (resp. backward) geodesic relations.

Therefore, given $\square$ on $d \geq 3$, one has at most 4 independent parametrices E^{adv}, E^{ret}, E^{Fyn}, E^{aFyn}. Evidently, the hindmost two are characterised by their wavefront sets rather supports.

Remark 3.2.18. There are 16 parametrices of $\square$ in $d = 2$ because $\dot{\mathrm{T}}^*_0 \mathscr{M}$ has 4 connected components, whereas $\dot{\mathrm{T}}^*_{0,\pm} \mathscr{M}$ are the two connected components of $\dot{\mathrm{T}}^*_0 \mathscr{M}$ in $d \geq 3$.
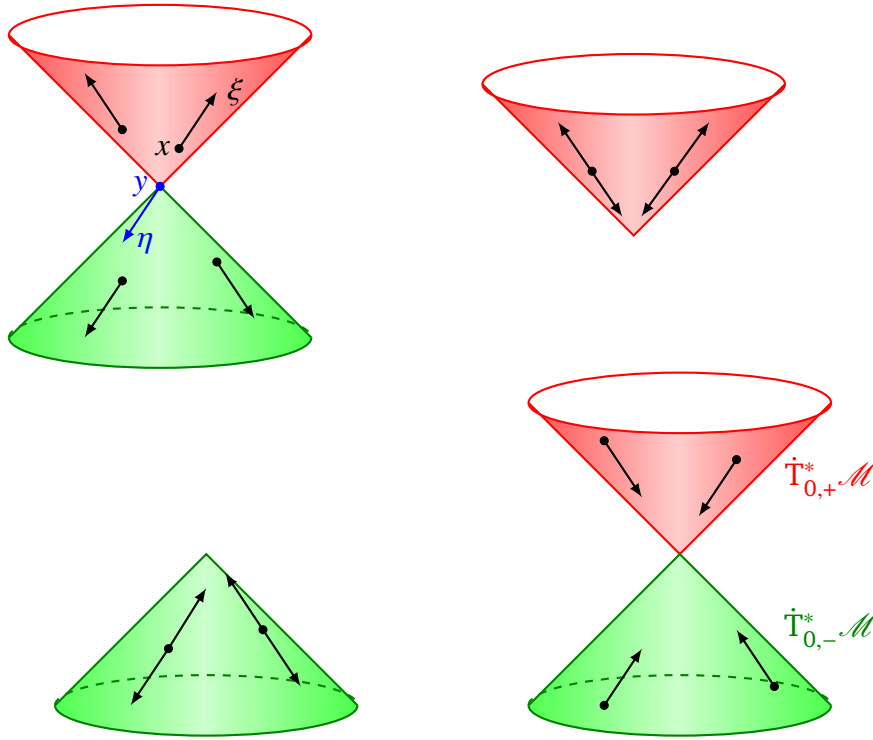
Figure 3.1: Schematic illustration of $C^+, C^{\mathrm{ret}}, C^{\mathrm{adv}}, C^-$ in spacetime dimensions $d \geq 3$. In these diagrams y is fixed at the vertices of the cones and only the directions of ξ are shown while those of η being left undisplayed for simplicity except for the top left one (redrawn from [39, Fig. 1]).

3.3 Feynman propagators for a NHO

3.3.1 Primary results

To begin with, we provide a bundle version of the classic result by Duistermaat-Hörmander [35, Thm. 6.5.3] (see also [23, Thm. 26.1.14]) on the existence and uniqueness of Feynman parametrices of a normally hyperbolic operator. Recall that $H^s_{\mathrm{loc}}(M; \mathscr{E} \otimes \Omega^{1/2})$ is the space of $\Omega^{1/2}$-valued sections of $\mathscr{E}$ that are locally in the Sobolev space H^s with respect to any smooth chart and any smooth bundle chart. The space of sections in $H^s_{\mathrm{loc}}(M; \mathscr{E} \otimes \Omega^{1/2})$ of compact support is denoted by $H^s_{\mathrm{c}}(M; \mathscr{E} \otimes \Omega^{1/2})$. As usual, the space $H^s_{\mathrm{loc}}(M; \mathscr{E} \otimes \Omega^{1/2})$ is equipped with the locally convex topology of convergence locally in $H^s(M; \mathscr{E} \otimes \Omega^{1/2})$. The space $H^s_{\mathrm{c}}(M; \mathscr{E} \otimes \Omega^{1/2})$ is the union $\bigcup_{K \Subset M} H^s_{\mathrm{c}}(K; \mathscr{E} \otimes \Omega^{1/2})$ and it is equipped with the inductive limit topology, where the union runs over all compact subset K of M. For details, we refer, for instance [77, App. B1].

Theorem 3.3.1 (Existence and uniqueness of Feynman parametrices)**.** *Let $\mathscr{E} \to \mathscr{M}$ be a vector bundle over a globally hyperbolic spacetime $(\mathscr{M}, \mathrm{g})$ and $\Box$ a normally hyperbolic op-*

erator on $\mathscr{E}$. Then, there exist unique[2] Feynman E^{Fyn} and anti-Feynman E^{aFyn} parametrices (given by (3.96)) of $\square$. Moreover, for every $k \in \mathbb{R}$, E^{Fyn} and E^{aFyn} extend to continuous maps from $H_c^k(\mathscr{M}; \mathscr{E})$ to $H_{\mathrm{loc}}^{k+1}(\mathscr{M}; \mathscr{E})$ and the difference of their Schwartz kernels

$$\mathsf{E}^{\mathrm{Fyn}} - \mathsf{E}^{\mathrm{aFyn}} \in I^{-3/2}\big(\mathscr{M} \times \mathscr{M}, C'; \mathrm{Hom}(\mathscr{E}, \mathscr{E})\big), \tag{3.48}$$

where $I^{-3/2}(\ldots)$ is the space of Lagrangian distributions (Definition 2.3.9) associated to the geodesic relation C (Definition 3.2.15). Furthermore, $\mathsf{E}^{\mathrm{Fyn}} - \mathsf{E}^{\mathrm{aFyn}}$ is non-characteristic (Definition 2.3.14) at every point of C.

A special case of this result for the massive Klein-Gordon operator (Example 3.2.2) was given by Radzikowski [39, Prop. 4.2-4.4] as a direct consequence of [35, Thm. 6.5.3]. Employing the distinguished global phase function approach of Fourier integrals operators [116], Capoferri *et al.* [69, Thm. 5.2] have constructed these parametrices for scalar wave operators with time-independent smooth potential in spatially compact globally hyperbolic ultrastatic spacetime. Recently, Lewandowski [40, Prop. 3.5] has published a direct construction for a real vector bundle utilising the Hadamard series expansion along with the presentation of [73]. In contrast to these literature, our proof is purely microlocal as in the original treatment [35, Thm. 6.5.3]. This only requires a bundle version of microlocalisation, as developed in due course (Theorem 3.3.10, 3.3.13) which is along the lines of Dencker's [45] proof of propagation of the polarization sets albeit our presentation is more geometric.

The Feynman parametrix E^{Fyn} can be turned into a Green's operator G^{Fyn} - the Feynman propagator for $\square$. As elucidated in Appendix C, a Feynman propagator is intimately related to a quantum state and hence this is equivalent to construct a Hadamard 2-point distribution provided certain positivity[3] property (as described below) is satisfied. This remarkable property was first observed by Duistermaat-Hörmander [35, Thm. 6.6.2] for scalar operators. Namely, that one can modify E^{Fyn} by a smoothing operator R such that

$$\hat{W} + R := -\mathrm{i}(E^{\mathrm{Fyn}} - G^{\mathrm{adv}}) + R \tag{3.49}$$

is non-negative in the sense that $(\hat{\mathsf{W}} + \mathsf{R})(\bar{u} \otimes u) \geq 0$ for any $u \in C_c^\infty(\mathscr{M})$. This means, of course, that there exists a Feynman parametrix that has this property with $R = 0$. Thus, there exists a Feynman parametrix that satisfies $\hat{W} \geq 0$ in the above sense. Unlike functions, there is no notion of a positive operator on a vector bundle unless the bundle is hermitian. Hence, we equip $\mathscr{E}$ with a hermitian form and generalise this positivity property in the case of vector bundles.

Proposition 3.3.2. *Let $\mathscr{E} \to \mathscr{M}$ be a vector bundle over a globally hyperbolic spacetime $(\mathscr{M}, \mathrm{g})$ and $\square$ a normally hyperbolic operator on $\mathscr{E}$. We assume that $\mathscr{E}$ is endowed with a (non-degenerate) sesquilinear form $(\cdot|\cdot) \in C^\infty(\bar{\mathscr{E}}^* \otimes \mathscr{E})$ such that $\square$ is symmetric. Then, there*

---

[2]In the sense of parametrices, i.e., modulo smoothing operators.

[3]This is related to the positivity of Wightman distributions in quantum field theory.

exists a Feynman parametrix E^{Fyn} of $\Box$ such that $\hat{W} := -\mathrm{i}(E^{\mathrm{Fyn}} - G^{\mathrm{adv}})$ is symmetric, where G^{adv} is the advanced propagator for $\Box$. Additionally, if $(\cdot|\cdot)$ is hermitian (positive-definite) then $\hat{W}$ can be chosen non-negative.

Duistermaat-Hörmander have proven their version by deploying a refined microlocalisation of scalar pseudodifferential operators [35, Lem. 6.6.4]. We provide such a microlocalisation for normally hyperbolic operators in (3.103).

Finally, we turn the Feynman parametrix E^{Fyn} into a Feynman propagator G^{Fyn} utilising the well-posedness of Cauchy problem for $\Box$ on a globally hyperbolic spacetime, which in turn gives the existence of Hadamard bisolutions of $\Box$.

Theorem 3.3.3 (Existence of Feynman propagators). *As in the setup of Proposition 3.3.2, there exists a Feynman propagator G^{Fyn} for $\Box$ on $(\mathscr{E} \to \mathscr{M}, (\cdot|\cdot))$ such that $W := -\mathrm{i}(G^{\mathrm{Fyn}} - G^{\mathrm{adv}})$ is symmetric with respect to the sesquilinear form $(\cdot|\cdot)$. In addition, if $(\cdot|\cdot)$ is hermitian then G^{Fyn} can be chosen such that $\mathsf{W}(\bar{u}^* \otimes u) \geq 0$ for any $u \in C_c^\infty(\mathscr{M}; \mathscr{E})$. Thus, W defines a Hadamard state. Here $C^\infty(\mathscr{M}; \mathscr{E}) \ni u \mapsto \bar{u}^* \in C^\infty(\mathscr{M}; \bar{\mathscr{E}}^*)$ is the fibrewise linear mapping induced by $(\cdot|\cdot)$.*

Since Duistermaat-Hörmander [35, p. 229] have considered a much wider class of operators - scalar pseudodifferential operators of real principal type, the pivotal step of determining the appropriate smoothing operators required to obtain Feynman propagators from respective parametrices was, however, left open. This indeterminacy can be fixed in various ways on special spacetimes even in the absence of the timelike Killing vector field. Such constructions have appeared in the literature on microlocal analysis. For example, Gell-Redman *et al*. [44, Thm. 3.6] have treated the scalar wave operator in spaces with non-trapping Lorentzian scattering matrices. Vasy has constructed Feynman propagators by making assumptions on global dynamics for (i) scalar symmetric operators with real principal symbol in closed manifolds [67, Thm. 1], and in spaces of Lorentzian scattering matrices for (ii) wave operators in Melrose's b-pseudodifferential algebraic framework [67, Thm. 7] and (iii) Klein-Gordon operators in Melrose's scattering pseudodifferential algebraic formalism [67, Thm. 10, 12]. His idea is to identify the appropriate spaces where these operators are invertible and then define the Feynman propagators as the inverse of those operators satisfying the required properties - a generalisation of Feynman's original (see Example C.3.2 and the comment afterward) "$\pm \mathrm{i}\,\epsilon$" prescription. As a consequence, he has also achieved respective positivity property for Feynman parametrices [67, Cor. 5, 9, 11, 13]. A special case of Theorem 3.3.3 for the Klein-Gordon operator minimally coupled to a static electromagnetic potential on a static spacetime has been proven by Dereziński and Siemssen [68, Thm. 7.7]. In the spirit of the limiting absorption principle, they have shown that the Feynman propagator can be considered as the boundary value of the resolvent of the Klein-Gordon operator. Employing Proposition 3.3.2 and ideas from [58, Sec. 3.3], Lewandowski [40, Thm. 4.3] has recently given a construction of Hadamard states on a Riemannian vector bundle.

In order to proof these assertions, we need some preparation, as presented in the next two subsections. Some of the results of the following sections are interesting in a broader context, so we will work on a vector bundle $\mathscr{E} \to M$ over a generic manifold M for greater generality and come back to the context of a globally hyperbolic spacetime $(\mathscr{M}, g)$ afterward.

3.3.2 Connection 1-forms and subprincipal symbols

Let $\mathscr{E}, \Omega^{1/2} \to M$ be a vector bundle resp. the half-density bundle, over a manifold M. This has been observed that the subprincipal symbol (Definition 2.3.4) σ_P^{sub} of $P \in \Psi\mathrm{DO}^m(M; \mathscr{E} \otimes \Omega^{1/2}), m \in \mathbb{R}$ with scalar principal symbol p transforms like a partial[4]- connection 1-form under change of bundle charts (see e.g. [106, Prop. 3.1]). This is perhaps not surprising given that the subprincipal symbol appears as a constant term in transport equations. Connections that are naturally defined from transport equations have appeared first in the work of Dencker on propagation of polarisation sets. They are often referred to as the *Dencker connections* [45, p. 365-366] in the mathematical physics literature. For the Maxwell system, Dencker found that this connection equals the Levi-Civita connection. For the spin-Dirac operator (see Example 3.4.4), it was verified, for example, in [56] that the Dencker connection is indeed the spin-connection. In this section, we will systematically explain the precise relation between a geometrically defined connection on $\mathscr{E}$ and the parallel transporter induced by σ_P^{sub}. As already Remarked 2.3.5, σ_P^{sub} depends on the choice of trivialisation of $\mathscr{E}$ and $\Omega^{1/2}$. We are going to see below that under a change of bundle frames, this matrix will transform like a connection 1-form along the Hamiltonian vector field X_p in the bundle $\pi^*\mathscr{E} \to \dot{\mathrm{T}}^*M$ where $\pi : \dot{\mathrm{T}}^*M \to M$ is the punctured cotangent bundle of M.

To begin with, let us summarise the transformation properties of the subprincipal symbol. In case P has a scalar principal symbol p and $Q \in \Psi\mathrm{DO}^{m'}(M; \mathscr{E} \otimes \Omega^{1/2}), m' \in \mathbb{R}$ properly supported with principal symbol q, we have the following multiplication formula [10, (1.4)] (see also, e.g. [24, Prop. 2.1.13])

$$
\begin{aligned}
\sigma_{PQ}^{\mathrm{sub}} &= \sigma_{\Sigma_l P_l^r Q_k^l}^{\mathrm{sub}} = \sum_l \sigma_{P_l^r Q_k^l}^{\mathrm{sub}} \overset{[10,\,(1.4)]}{=} \sum_l \sigma_{P_l^r}^{\mathrm{sub}} q_k^l + p_l^r \sigma_{Q_k^l}^{\mathrm{sub}} + \frac{1}{2\mathrm{i}} \{p_l^r, q_k^l\} \\
&= \sigma_P^{\mathrm{sub}} q + p \, \sigma_Q^{\mathrm{sub}} + \frac{1}{2\mathrm{i}} \{p, q\},
\end{aligned} \tag{3.50}
$$

where we have used the linearity of σ^{sub} in the second line and $\{\cdot, \cdot\}$ is the Poisson bracket (see (A.5)). Consequently, we also have [10, (1.3)]

$$
\sigma_{P^k}^{\mathrm{sub}} = k \, p^{k-1} \, \sigma_P^{\mathrm{sub}} \tag{3.51}
$$

---

[4]We recall that the **partial connection** $\nabla^{\mathscr{E}}$ on a vector bundle $\mathscr{E} \to M$ over a manifold M with respect to a foliation N of M is defined by the covariant derivative $\nabla_X^{\mathscr{E}}$ satisfying the standard properties for any vector field X on N (see e.g. [117, p. 24]. Note, not all partial connections arise as restrictions of some "full" connection, for instance, the Bott connection associated with any involutive distribution of any manifold.

for any $k \in \mathbb{N}$. In addition, if P is elliptic (Definition 2.3.7) and E is a parametrix (Definition 3.2.14) of P then

$$\sigma_E^{\mathrm{sub}} = -\, p^{-2}\, \sigma_P^{\mathrm{sub}}, \tag{3.52}$$

and in this sense the above formula also holds for any negative integer k.

We will now show that the subprincipal symbol indeed has the claimed transformation property under a change of bundle charts. In fact, we show a microlocal version of this statement (see also [118]).

Proposition 3.3.4. *Let $\Omega^{1/2} \to M$ be the half-density bundle over a manifold M and let $m, m' \in \mathbb{R}, N \in \mathbb{N}$. Assume $P \in \Psi\mathrm{DO}^m(M, \mathbb{C}^{N \times N} \otimes \Omega^{1/2})$ having a scalar principal symbol p. Suppose that $Q \in \Psi\mathrm{DO}^{m'}(M, \mathbb{C}^{N \times N} \otimes \Omega^{1/2})$ is non-characteristic at some $(x_0, \xi^0) \in \dot{\mathrm{T}}^*M$ and that E is a microlocal parametrix of Q in an open conic neighbourhood $\mathcal{U}$ of (x_0, ξ^0). Then the subprincipal symbol of EPQ is*

$$\forall (x, \xi) \in \mathcal{U} : \sigma_{EPQ}^{\mathrm{sub}}(x, \xi) = (q^{-1}\sigma_P^{\mathrm{sub}}q)(x, \xi) - \mathrm{i}\left(q^{-1}X_p(q)\right)(x, \xi), \tag{3.53}$$

where q is the principal symbol of Q and X_p is the Hamiltonian vector field generated by p.

Proof. Since P has a scalar principal symbol, the order of the commutator $[P, Q]_-$ is $m + m' - 1$ and the principal symbol of EPQ is microlocally that of P. These imply that the order of $E[P, Q]_-$ is $m - 1$ and that $(x_0, \xi^0) \notin \mathrm{ES}(EPQ - P - E[P, Q]_-)$. Thus, deploying the order,

$$\sigma_{EPQ}^{\mathrm{sub}} = \sigma_P^{\mathrm{sub}} + \sigma_{E[P,Q]_-} \tag{3.54}$$

on $\mathcal{U}$. It remains to compute the principal symbol of $[P, Q]_-$. Since the principal symbol of P is scalar-valued, we obtain the commutator relation (see e.g. [106, (14)])

$$\sigma_{[P,Q]_-} = -\mathrm{i}\{p, q\} + [\sigma_P^{\mathrm{sub}}, q]_-, \tag{3.55}$$

as a consequence of the multiplication formula (3.50). Therefore, on $\mathcal{U}$, we get $\sigma_{Q^{-1}[P,Q]_-} = q^{-1}\sigma_{[P,Q]_-} = -\mathrm{i}\, q^{-1}\{p, q\} + q^{-1}\sigma_P^{\mathrm{sub}}q - \sigma_P^{\mathrm{sub}}$ and the result follows. ∎

As a simple consequence of the above product formula we also have

Proposition 3.3.5. *As in the terminologies of Proposition 3.3.4, let Q has a scalar principal symbol q and vanishing subprincipal symbol. If P has a scalar principal symbol p and a subprincipal symbol σ_P^{sub}, then*

$$\sigma_{QPQ}^{\mathrm{sub}} = q^2 \sigma_P^{\mathrm{sub}}. \tag{3.56}$$

With these preparations in hand, we introduce the following notion of a connection 1-form induced by the subprincipal symbol of a pseudodifferential operator.

Definition 3.3.6. Let $(\mathscr{E}, \mathscr{F} \to M; \nabla^{\mathscr{E}}, \nabla^{\mathscr{F}})$ be two vector bundles over a manifold M, equipped with their connections $\nabla^{\mathscr{E}}, \nabla^{\mathscr{F}}$ and let $\Omega^{1/2} \to M$ be the half-density bundle. A connection $\nabla^{\mathscr{E} \otimes \mathscr{F}}$ on the tensor product bundle $\mathscr{E} \otimes \mathscr{F} \to M$ will be called $P \in \Psi\mathrm{DO}^m(M; \mathscr{E} \otimes \Omega^{1/2}, \mathscr{F} \otimes \Omega^{1/2})$-**compatible** if and only if

$$\forall (x, \xi) \in \operatorname{Char} P : \Gamma\big((\mathrm{d}_x \pi) X_P\big)(x, \xi) = \mathrm{i}\, \sigma_P^{\mathrm{sub}}(x, \xi), \tag{3.57}$$

where σ_P^{sub} resp. $\operatorname{Char} P$ are the subprincipal symbol resp. the characteristic set of P, Γ is the connection 1-form of $\nabla^{\mathscr{E} \otimes \mathscr{F}}$ and $\pi : \dot{\mathrm{T}}^*M \to M$ is the punctured cotangent bundle. In other words, $\nabla^{\mathscr{E} \otimes \mathscr{F}}$ indices the covariant derivative

$$\nabla_{X_P}^{\pi^* \mathrm{Hom}(\mathscr{E}, \mathscr{F})} = X_P + \Gamma\big((\mathrm{d}\pi) X_P\big) \tag{3.58}$$

on the bundle $\pi^* \mathrm{Hom}(\mathscr{E}, \mathscr{F}) \to \dot{\mathrm{T}}^*M$ along the Hamiltonian vector field X_P generated by the principal symbol p of P.

This definition makes sense because both quantities, Γ and σ_P^{sub} have the same transformation law under change of bundle frame. Hence $X_P + \mathrm{i}\, \sigma_P^{\mathrm{sub}}$ has an invariant meaning.

Proposition 3.3.7. *Let $\mathscr{E}, \Omega^{1/2} \to M$ be a vector bundle resp. the half-density bundle over a manifold M and $m, m' \in \mathbb{R}, N \in \mathbb{N}$. Suppose that $P \in \Psi\mathrm{DO}^m(M; \mathscr{E} \otimes \Omega^{1/2})$ has a scalar principal symbol p and that $\mathscr{E}$ admits a P-compatible connection $\nabla^{\mathscr{E}}$. Assume that $Q \in \Psi\mathrm{DO}^{m'}(M, \mathbb{C}^{N \times N} \otimes \Omega^{1/2})$ having a scalar principal symbol q and vanishing subprincipal symbol. Then $\nabla^{\mathscr{E}}$ is QPQ-compatible.*

Proof. We fix any bundle frame to check this. Then $X_{qpq} = \{qpq, \cdot\} = \{q^2 p, \cdot\} = q^2\{p, \cdot\} + 2pq\{q, \cdot\}$. When restricted to $\operatorname{Char} P$ this equals $q^2\{p, \cdot\} = q^2 X_P$. Hence, on $\operatorname{Char} P$ we have

$$\nabla_{X_{qpq}}^{\pi^* \mathrm{Hom}(\mathscr{E}, \mathscr{E})} - X_{qpq} = q^2\left(\nabla_{X_P}^{\pi^* \mathrm{Hom}(\mathscr{E}, \mathscr{E})} - X_P\right) = q^2 \sigma_P^{\mathrm{sub}} = \sigma_{QPQ}^{\mathrm{sub}}. \tag{3.59}$$

∎

The primary observation is now that the Weitzenböck connection (Remark 3.2.8) defined by a normally hyperbolic operator $\square$ is compatible with $\square$ in the above sense.

Theorem 3.3.8. *Let $\mathscr{E} \to \mathscr{M}$ be a vector bundle over a Lorentzian manifold $(\mathscr{M}, \mathrm{g})$ and $\square$ a normally hyperbolic operator on $\mathscr{E}$. If $\nabla^{\mathscr{E}}$ is the corresponding Weitzenböck connection then it is $\square$-compatible.*

Proof. Since the computation is local, we choose a coordinate basis $\{(x^i, \partial_i)\}$ of $\mathrm{T}_x\mathscr{M}$ to compute the Hamiltonian vector field X_{g} generated by the principal symbol g^{-1} of $\square$:

$$X_{\mathrm{g}}(x, \xi) = \frac{\partial(\mathrm{g}^{\mu\nu}\xi_\mu\xi_\nu)}{\partial \xi_i} \frac{\partial}{\partial x^i} - \frac{\partial(\mathrm{g}^{\mu\nu}\xi_\mu\xi_\nu)}{\partial x^i} \frac{\partial}{\partial \xi_i} = 2g^{ij}\xi_j \frac{\partial}{\partial x^i} - \frac{\partial \mathrm{g}^{\mu\nu}}{\partial x^i}\xi_\mu\xi_\nu \frac{\partial}{\partial \xi_i}. \tag{3.60}$$

Hence, $(\mathrm{d}_x \pi) X_{\mathrm{g}} = 2g^{ij}\xi_j \partial_i$ and picking any bundle frame $\{E_r\}$, we can see that

$$\Gamma\big((\mathrm{d}_x \pi) X_{\mathrm{g}}\big)(x, \xi) = 2g^{ij}\Gamma_i\xi_j \overset{(3.24)}{=} \mathrm{i}\, \sigma_\square^{\mathrm{sub}}(x, \xi). \tag{3.61}$$

∎

A straightforward application of this theorem and Proposition 3.3.7 yields

Corollary 3.3.9. *As in the terminologies of Theorem 3.3.8, let $N \in \mathbb{N}$. Suppose that $Q \in$ $\Psi\mathrm{DO}^{-1/2}(\mathcal{M}, \mathbb{C}^{N \times N})$ having a scalar principal symbol and vanishing subprincipal symbol. Then $\nabla^{\mathscr{E}}$ is $Q \square Q$-compatible, i.e. the Weitzenböck connection is compatible with the first-order operator $Q \square Q$.*

3.3.3   Microlocal conjugate of a NHO

Two pseudodifferential operators are called microlocally conjugate if they can be conjugated to one another by an elliptic Fourier integral operator once they have been appropriately localised in cotangent space. The key point is that under some natural assumptions any first-order pseudodifferential operator can be microlocally conjugated to a vector field. This is originally due to Duistermat and Hörmander [35, Prop. 6.1.4] for scalar operators of real principal type (Definition 2.2.10), which has been extended to vector bundles by Dencker [45] who formulated microlocal conjugate of a system of classical pseudodifferential operators locally of real principal type (Definition 2.3.6). More precisely, Dencker transformed the system of operators to a scalar pseudodifferential operator P (modulo smoothing operators) with vanishing subprincipal symbol by conjugating with a system of elliptic pseudodifferential operators. Then, he depicted the microlocal conjugation of P with $\mathrm{D}_1 := -\mathrm{i}\, \mathbb{1}_{\mathbb{C}^{k \times k}}\, \partial_1$ by an elliptic Fourier integral operator associated with a symplectomorphism connecting σ_P to σ_{D_1}.

In this section we will explain microlocalisation in an intrinsic geometric language and formulate Dencker's result in a slightly more general form. We are going to show that any "appropriate"(as precisely given below) first-order pseudodifferential operator $P \in \Psi\mathrm{DO}^m(M; \mathscr{E} \otimes \Omega^{1/2})$ having a scalar principal symbol is microlocally conjugate to D_1 in the sense of (3.63c) in the theorem below.

Theorem 3.3.10 (Microlocalisation). *Let $\mathscr{E}, \Omega^{1/2} \to M$ be a vector bundle and the bundle of half-densities over a d-dimensional manifold M, respectively. Suppose that $P \in$ $\Psi\mathrm{DO}^1(M; \mathscr{E} \otimes \Omega^{1/2})$ is a properly supported pseudodifferential operator on $\mathscr{E}$ with real scalar principal symbol p such that*

(a) $p(x_0, \xi^0) = 0$ for some element (x_0, ξ^0) in the punctured cotangent bundle $\dot{\mathrm{T}}^*M$ of M;

(b) the Hamiltonian vector field X_p of p and the radial direction are linearly independent at (x_0, ξ^0).

Then for any $m \in \mathbb{R}$, there exist

(i) a homogeneous symplectomorphism $\varkappa$ from an open conic neighbourhood $\tilde{\mathcal{U}}$ of $(0, \eta_1\, \mathrm{d}y^1)$ in $\dot{\mathrm{T}}^*\mathbb{R}^d$ to an open conic coordinate chart $(\mathcal{U}, (x^i, \xi_i))$ of (x_0, ξ^0) in $\dot{\mathrm{T}}^*M$ such that

$$\varkappa^* p = \xi_1 \mathbb{1}_{\mathrm{Hom}(\mathscr{E}, \mathscr{E})}; \tag{3.62}$$

(ii) *properly supported Lagrangian distributions* $B \in I^m\big(M \times \mathbb{R}^d, \Gamma'; \mathrm{Hom}(\mathbb{C}^k, \mathscr{E}) \otimes \Omega^{1/2}(M \times \mathbb{R}^d)\big)$ *and* $\tilde{B} \in I^{-m}\big(\mathbb{R}^d \times M, \Gamma^{-1\prime}; \mathrm{Hom}(\mathscr{E}, \mathbb{C}^k) \otimes \Omega^{1/2}(\mathbb{R}^d \times M)\big)$ *so that* $B\tilde{B}, \tilde{B}B$ *both are zero-order pseudodifferential operators and*

$$\mathrm{WF}'\, B \subset \mathcal{U}_{(x_0, \xi^0; 0, \eta_1 dy^1)}, \qquad \mathrm{WF}'\, \tilde{B} \subset \tilde{\mathcal{U}}_{(0, \eta_1 dy^1; x_0, \xi^0)}, \tag{3.63a}$$

$$(x_0, \xi^0) \notin \mathrm{ES}(B\tilde{B} - I_{\mathscr{E}}), \qquad (0, \eta_1 dy^1) \notin \mathrm{ES}(\tilde{B}B - I), \tag{3.63b}$$

$$(x_0, \xi^0) \notin \mathrm{ES}(BD_1\tilde{B} - P), \qquad (0, \eta_1 dy^1) \notin \mathrm{ES}(\tilde{B}PB - D_1), \tag{3.63c}$$

where $k := \mathrm{rk}\, \mathscr{E}$, Γ *is the graph of* $\varkappa$, $D_1 := -\mathrm{i}\, \mathbb{1}_{\mathbb{C}^{k \times k}}\, \partial/\partial y^1 : C_c^\infty(\mathbb{R}^d, \mathbb{C}^k) \to C_c^\infty(\mathbb{R}^d, \mathbb{C}^k)$, *and* $\mathcal{U}_{(x_0, \xi^0; 0, \eta_1 dy^1)}$ *resp.* $\tilde{\mathcal{U}}_{(0, \eta_1 dy^1; x_0, \xi^0)}$ *are small conic neighbourhoods of* $(x_0, \xi^0; 0, \eta_1 dy^1) \in \dot{\mathrm{T}}^*M \times \dot{\mathrm{T}}^*\mathbb{R}^d$ *resp.* $(0, \eta_1 dy^1; x_0, \xi^0) \in \dot{\mathrm{T}}^*\mathbb{R}^d \times \dot{\mathrm{T}}^*M$.

In additional, if there are more structures on $\mathscr{E}$ *then the Fourier integral operators* $B, \tilde{B}$ *can be simplified further as follows.*

(A) *If* $\mathscr{E} \to M$ *is endowed with a sesquilinear form* $(\cdot|\cdot)$ *with respect to which* P *is symmetric, then* $\tilde{B}$ *can be chosen as the formal adjoint of* B *provided that* $\mathbb{C}^k$ *is endowed with a standard sesquilinear form of the same signature as* $(\cdot|\cdot)$.

(B) *If* $\mathscr{E} \to M$ *is equipped with a P-compatible connection* $\nabla^{\mathscr{E}}$ *then the principal symbols of* B *resp.* $\tilde{B}$ *can be chosen* $\mathbb{1}$ *near* $(x_0, \xi^0; 0, \eta_1 dy^1)$ *resp.* $(0, \eta_1 dy^1; x_0, \xi^0)$ *with respect to a frame that is parallel along* X_p.

(C) *On the vector bundle* $\big(\mathscr{E} \to M, (\cdot|\cdot), \nabla^{\mathscr{E}}\big)$ *where* $\nabla^{\mathscr{E}}$ *is a P-compatible connection and* P *is symmetric with respect to the sesquilinear form* $(\cdot|\cdot)$, *we can choose* B *such that its principal symbol equals* $\mathbb{1}$ *near* $(x_0, \xi^0; 0, \eta_1 dy^1)$ *with respect to a frame that is unitary and parallel along* X_p, *and* $\tilde{B} = B^*$.

A schematic of this notion has been portrayed in Figure 3.2.

Figure 3.2: A schematic diagram of microlocalisation. The diagram on the right commutes in a microlocal sense (3.63c) and the map $\hat{\varkappa}$ is defined by $\hat{\varkappa}(\cdot) = \sigma_B(\cdot)\sigma_{\tilde{B}}$ with all other symbols as defined in Theorem 3.3.10.

Proof. We will prove the proposition imitating the strategy used for the scalar version [35, Prop. 6.1.4, Lem. 6.6.4] (see also [23, Prop. 26.1.3]). The existence of $\varkappa$ satisfying (3.62) is guaranteed by the homogeneous Darboux Theorem A.1.4 which prerequisites our hypotheses (a) and (b). Suppose that $b \in S^m\big(\Gamma; \mathbb{M} \otimes \widetilde{\mathrm{Hom}}(\mathbb{C}^k, \mathscr{E})\big)$ has an inverse

in a conic neighbourhood of $(x_0, \xi^0; 0, \eta_1 \mathrm{d}y^1) \in \Gamma$. Then we can obtain a properly supported $\mathrm{B}_1 \in I^m\big(M \times \mathbb{R}^d, \Gamma'; \mathrm{Hom}(\mathbb{C}^k, \mathscr{E}) \otimes \Omega^{1/2}(M \times \mathbb{R}^d)\big)$ such that $\mathrm{WF}' \mathrm{B}_1 \subset \mathcal{U}_{(x_0, \xi^0; 0, \eta_1 \mathrm{d}y^1)}$ and B_1 is non-characteristic (Definition 2.3.14) at $(x_0, \xi^0; 0, \eta_1 \mathrm{d}y^1)$ from the construction given in Section 2.3.6.

By Theorem 2.3.16, there exists a unique microlocal parametrix $\tilde{\mathrm{B}}_1 \in I^{-m}\big(\mathbb{R}^d \times M, \Gamma^{-1\prime};$ $\mathrm{Hom}(\mathscr{E}, \mathbb{C}^k) \otimes \Omega^{1/2}(\mathbb{R}^d \times M)\big)$ such that (3.63b) is satisfied . Since B_1 and $\tilde{\mathrm{B}}_1$ have reciprocal principal symbols to each other on $\mathcal{U}_{(x_0, \xi^0; 0, \eta_1 \mathrm{d}y^1)}$, $\tilde{\mathrm{B}}_1 P \mathrm{B}_1$ has the principal symbol $\eta_1 \, \mathbb{1}_{\mathbb{C}^{k \times k}}$ on $\mathcal{U}'$ due to (3.62) and the Egorov Theorem 2.3.17. Furthermore

$$(0, \eta_1 \mathrm{d}y^1) \notin \mathrm{ES}(\tilde{\mathrm{B}}_1 P \mathrm{B}_1 - \mathrm{D}_1 - Q) \tag{3.64}$$

for some $Q \in \Psi\mathrm{DO}^0(\mathbb{R}^d, \mathbb{C}^{k \times k})$.

To find the operators B and $\tilde{B}$ with the claimed properties, we construct properly supported elliptic $B_2, \tilde{B}_2 \in \Psi\mathrm{DO}^0(\mathbb{R}^d, \mathbb{C}^{k \times k})$ such that

$$\tilde{B}_2 B_2 \equiv I \Leftrightarrow \tilde{B}_2 B_2 = I \quad \mathrm{mod} \ \Psi\mathrm{DO}^{-\infty}(\mathbb{R}^d, \mathbb{C}^{k \times k}), \tag{3.65}$$

$$\tilde{B}_2 (\mathrm{D}_1 + Q) B_2 \equiv \mathrm{D}_1 \Leftrightarrow \tilde{B}_2 (\mathrm{D}_1 + Q) B_2 = \mathrm{D}_1 \quad \mathrm{mod} \ \Psi\mathrm{DO}^{-\infty}(\mathbb{R}^d, \mathbb{C}^{k \times k}), \tag{3.66}$$

and one sets $B := \mathrm{B}_1 B_2$ and $\tilde{B} := \tilde{B}_2 \tilde{\mathrm{B}}_1$. Then (3.65), (3.66) and (3.64) imply

$$(0, \eta_1 \mathrm{d}y^1) \notin \mathrm{ES}\big(\tilde{B}_2 (\tilde{\mathrm{B}}_1 \mathrm{B}_1 - I) B_2\big) = \mathrm{ES}(\tilde{B} B - I), \tag{3.67}$$

$$(0, \eta_1 \mathrm{d}y^1) \notin \mathrm{ES}\big(\tilde{B}_2 (\tilde{\mathrm{B}}_1 P \mathrm{B}_1 - \mathrm{D}_1 - Q) B_2\big) = \mathrm{ES}(\tilde{B} P B - \mathrm{D}_1), \tag{3.68}$$

which proves the second half of (3.63b) and (3.63c), and the first half follows immediately after we multiply from left and right by B and by $\tilde{B}$.

It therefore remains to construct B_2 and $\tilde{B}_2$ such that (3.65) and (3.66) hold. By the existence of a parametrix (see e.g. [77, Thm. 18.1.24]), for every elliptic $B_2 \in \Psi\mathrm{DO}^0(\mathbb{R}^d, \mathbb{C}^{k \times k})$ there exists $\tilde{B}_2 \in \Psi\mathrm{DO}^0(\mathbb{R}^d, \mathbb{C}^{k \times k})$ such that $\tilde{B}_2 B_2 - I$ and $B_2 \tilde{B}_2 - I$ are smooth. Multiplying (3.66) by B_2 from the left we arrive at the equivalent condition for (3.66) that

$$[\mathrm{D}_1, B_2]_- + Q B_2 \equiv 0 \tag{3.66$'$}$$

for some elliptic B_2.

We will now construct a solution of (3.66$'$) order by order, starting with the principal symbol. The principal symbol of (3.66$'$) vanishes provided

$$-\mathrm{i}\left\{\sigma_{\mathrm{D}_1}, \sigma_{B_2}\right\} + \sigma_Q \sigma_{B_2} = 0, \tag{3.69}$$

as the the subprincipal symbol of D_1 vanishes; cf. (3.55). If q is the principal symbol of Q, then the preceding equation yields

$$\frac{\partial b_0}{\partial y^1} = -\mathrm{i}\, q b_0 \tag{3.70}$$

for the principal symbol b_0 of B_2. This is a first-order differential equation, hence a unique solution exists given the initial condition $b_0(y^1 = 0, \cdot) = \mathbb{1}_{\mathbb{C}^{k \times k}}$ and this solution depends smoothly on q. By construction $b_0(y, \eta)$ is homogeneous of degree zero and $\det(b_0(y, \eta))$ is non-vanishing. Therefore b_0^{-1} exists and is homogeneous of degree zero. Defining a properly supported $B_{2,0} \in \Psi\mathrm{DO}^0(\mathbb{R}^d, \mathbb{C}^{k \times k})$ with homogeneous principal symbol b_0, we now successively construct $B_{2,k} \in \Psi\mathrm{DO}^{-k}(\mathbb{R}^d, \mathbb{C}^{k \times k})$ so that, for every $n \in \mathbb{N}$:

$$[D_1, B_{2,0} + \ldots + B_{2,n}]_- + Q(B_{2,0} + \ldots + B_{2,n}) = R_{n+1} \in \Psi\mathrm{DO}^{-(n+1)}(\mathbb{R}^d, \mathbb{C}^{k \times k}). \qquad (3.71)$$

This is equivalent to the corresponding principal symbols b_n of $B_{2,n}$ and r_n of R_n of degree $-n$, to satisfy

$$-\mathrm{i}\frac{\partial b_n}{\partial y^1} + q b_n = -r_n. \qquad (3.72)$$

This equation can be solved by the Duhamel principle and the solution reads

$$b_n(y, \eta) = -\mathrm{i}\, b_0(y, \eta) \int_0^{y^1} b_0^{-1}(t, y^2, \ldots, y^d; \eta)\, r_n(t, y^2, \ldots, y^d; \eta)\, \mathrm{d}t. \qquad (3.73)$$

Then, using asymptotic summation (see Definition B.1.4) of the symbols of $B_{2,n}$, we can now construct an operator B_2 satisfying (3.66′).

The case when P is symmetric:
We are now going to prove that the operators B and $\tilde{B}$ can be chosen microlocally unitary in case P is symmetric with respect to $(\cdot|\cdot)$. We endow the space $C_c^\infty(\mathbb{R}^d, \mathbb{C}^k)$ with a standard sesquilinear scalar product of the same signature as $(\cdot|\cdot)$. The operator D_1 is then symmetric. Furthermore, we make the choice $m = 0$. Acting with $B \in I^0(M \times \mathbb{R}^d, \Gamma'; \mathrm{Hom}(\mathbb{C}^k, \mathscr{E}) \otimes \Omega^{1/2}(M \times \mathbb{R}^d))$ from the left of (3.63c) gives the equivalent microlocal conjugate relation

$$(x_0, \xi^0; 0, \eta_1 \mathrm{d}y^1) \notin \mathrm{WF}'(PB - BD_1) \qquad (3.74)$$

between P and D_1. Taking its adjoint we have $(0, \eta_1 \mathrm{d}y^1; x_0, \xi^0) \notin \mathrm{WF}'(B^*P - D_1 B^*)$ where $B^* \in I^0(\mathbb{R}^d \times M, \Gamma^{-1\prime}; \mathrm{Hom}(\mathscr{E}^*, \mathbb{C}^k) \otimes \Omega^{1/2}(\mathbb{R}^d \times M))$ and consequently

$$(0, \eta_1 \mathrm{d}y^1) \notin \mathrm{ES}(B^*PB - D_1 B^*B), \qquad (3.75a)$$
$$(0, \eta_1 \mathrm{d}y^1) \notin \mathrm{ES}(B^*PB - B^*BD_1). \qquad (3.75b)$$

Thus $(0, \eta_1 \mathrm{d}y^1) \notin \mathrm{ES}\,[B^*B, D_1]_-$ and B^*B is non-characteristic at $(0, \eta_1 \mathrm{d}y^1)$. Since B^*B is a pseudodifferential operator and $\mathrm{i}D_1$ is differentiation with respect to y^1, the total symbol of the commutator $[B^*B, D_1]_-$ is the $-\mathrm{i}$ times the y^1-derivative of the total symbol of B^*B. Thus, on the the level of symbols, the above implies that the total symbol of B^*B is the sum of a term independent of y^1 and a term that is rapidly decaying in a conic neighbourhood $\mathcal{U}'$ of $(0, \eta_1 \mathrm{d}y^1)$. It is bounded below on $\mathcal{U}'$ because of the ellipticity of B^*B there. By Proposition 3.3.11 (presented immediately after this proof), one can find

a properly supported symmetric $\Psi \in \Psi\mathrm{DO}^0(U', \mathbb{C}^{k \times k})$ whose principal symbol is the same as that of B^*B and such that

$$(0, \eta_1 \mathrm{d}y^1) \notin \operatorname{Char} \Psi, \tag{3.76a}$$

$$(0, \eta_1 \mathrm{d}y^1) \notin \operatorname{ES}(\Psi^*\Psi - B^*B), \tag{3.76b}$$

$$(0, \eta_1 \mathrm{d}y^1) \notin \operatorname{ES}[\Psi, \mathrm{D}_1]_-, \tag{3.76c}$$

where U' is the projection of $\mathcal{U}'$ on $\mathbb{R}^d$. Note that Proposition 3.3.11 (stated for operators acting on the same bundle) can be used only in a local trivialisation of $\mathscr{E}$ by an orthonormal frame that identifies the fibre of $\mathscr{E}$ with $\mathbb{C}^k$ in such a way that the sesquilinear forms are identified. It is at this stage that we must require the *sesquilinear forms to have the same signature*.

Here the last property follows from the fact that the construction of the full symbol of Ψ in Proposition 3.3.11 involves only multiplication and asymptotic summation of symbols. The property of a symbol being a sum of two terms, one independent of y_1 and another rapidly decaying in a conic neighborhood of $\mathcal{U}'$, is preserved under these operations. The full symbol of Ψ is therefore also of this form.

The ellipticity (3.76a) entails a unique microlocal parametrix Φ for Ψ. In other words, there exists a properly supported $\Phi \in \Psi\mathrm{DO}^0(U', \mathbb{C}^{k \times k})$ such that

$$(0, \eta_1 \mathrm{d}y^1) \notin \operatorname{Char} \Phi, \tag{3.77a}$$

$$(0, \eta_1 \mathrm{d}y^1) \notin \operatorname{ES}(\Phi\Psi - I) \Leftrightarrow (0, \eta_1 \mathrm{d}y^1) \notin \operatorname{ES}(\Psi\Phi - I). \tag{3.77b}$$

Note, (3.77a), (3.76b) and (3.76c) imply that

$$(0, \eta_1 \mathrm{d}y^1) \notin \operatorname{ES}(\Phi^*\Psi^*\Psi\Phi - \Phi^*B^*B\Phi), \tag{3.78a}$$

$$(0, \eta_1 \mathrm{d}y^1) \notin \operatorname{ES}(\Phi[\Psi, \mathrm{D}_1]_-\Phi). \tag{3.78b}$$

This, accounting (3.77b) entails that

$$(0, \eta_1 \mathrm{d}y^1) \notin \operatorname{ES}\left(I - (B\Phi)^*B\Phi\right), \tag{3.79a}$$

$$(0, \eta_1 \mathrm{d}y^1) \notin \operatorname{ES}[\Phi, \mathrm{D}_1]_-, \tag{3.79b}$$

which completes the proof since $(x_0, \xi^0; 0, \eta_1 \mathrm{d}y^1) \notin \operatorname{WF}'(B[\mathrm{D}_1, \Phi]_-)$ and therefore with $B` := B\Phi$ we have $(x_0, \xi^0; 0, \eta_1 \mathrm{d}y^1) \notin \operatorname{WF}'(PB` - B`\mathrm{D}_1)$.

The case of connection P-compatibility:
We assume that $\nabla^{\mathscr{E}}$ is P-compatible. Since the construction of the symbols of B and $\tilde{B}$ are local, we can fix a local frame and local coordinates. We will reduce the general situation to the case when the subprincipal symbol of P vanishes near (x_0, ξ^0).

(i) *The case of vanishing subprincipal symbol*: We assume that P is given in local coordinates with respect to some bundle frame and that in this frame $\sigma_P^{\mathrm{sub}} = 0$ near (x_0, ξ^0). Let $\tilde{P}$ be a scalar operator whose principal symbol is p and subprincipal symbol vanishes. Using Weyl-quantisation (see e.g. [77, Sec. 18.5]), B_1 and $\tilde{B}_1$ can be constructed in such a way that Egorov's theorem holds up to the subprincipal symbol level [119, Thm. 1]. We can choose these operators in such a way that their principal symbols are constant $\mathbb{1}$ and that the subprincipal symbol of $\tilde{B}_1 \tilde{P} B_1$ (resp. $B_1 \mathrm{D}_1 \tilde{B}_1$) vanishes near $(0, \eta_1 \mathrm{d} y^1)$ (resp. (x_0, ξ^0)). We will now use the same construction as before starting with B_1 and $\tilde{B}_1$. Then the principal symbol q of the remainder term Q vanishes. The construction of B_2 and $\tilde{B}_2$ then yields operators with total symbols that are scalar and constant principal symbols equal to $\mathbb{1}$. We conclude that in case the subprincipal symbol of P vanishes near (x_0, ξ^0) and $B, \tilde{B}$ can be chosen as scalar operators. The principal symbols that are constant along the flow lines of X_p and $X_{\sigma_{\mathrm{D}_1}}$.

(ii) *The case of non-vanishing subprincipal symbol*: We will microlocally transform P to a scalar pseudodifferential operator $\tilde{P}$. To be specific, for any properly supported $\tilde{P} \in \Psi\mathrm{DO}^1(M; \Omega^{1/2})$, we want to have a $\hat{B} \in \Psi\mathrm{DO}^0(M; \mathscr{E} \otimes \Omega^{1/2})$ such that $\hat{B}$ is non-characteristic at (x_0, ξ^0) and $(x_0, \xi^0) \notin \mathrm{ES}(P\hat{B} - \hat{B}\tilde{P}I)$. We construct this operator locally and hence fix a local frame of $\mathscr{E}$ near the point x_0. In the pullbacked bundle $\pi^*\mathscr{E} \to \dot{\mathrm{T}}^*M$ where $\pi : \dot{\mathrm{T}}^*M \to M$, we can also construct a local frame that is parallel along X_p with respect to $\nabla_{X_p}^{\pi^*\mathscr{E}}$. This local parallel frame is constructed by choosing a local transverse to X_p and use the original frame on this transverse. Parallel transport along the flow lines of X_p then gives the desired frame. The change of frame matrix from the original frame to the parallel frame is then a local section b of $\mathrm{Hom}(\pi^*\mathscr{E}, \pi^*\mathscr{E}) \to \dot{\mathrm{T}}^*M$. By construction, this frame is homogeneous of degree zero. We now choose an elliptic zero-order pseudodifferential operator $\hat{B}$ whose principal symbol $\sigma_{\hat{B}}$ equals b on $\mathcal{U}$. Let $\check{B}$ be a parametrix of $\hat{B}$. The subprincipal symbol of $\check{B}P\hat{B}$ is equal to $-\mathrm{i} X_p \sigma_{\hat{B}} + [\sigma_P^{\mathrm{sub}}, \sigma_{\hat{B}}]_-$ on $\mathcal{U}$, by an application of (3.55). By P-compatibility, this is exactly the formula for the connection 1-form in the parallel local bundle frame and it therefore vanishes. Proposition 3.3.4 implies that this is precisely the formula for the subprincipal symbol of $\check{B}P\hat{B}$ on $\mathcal{U}$ and hence, $\tilde{P} \equiv \check{B}P\hat{B}$ has vanishing subprincipal symbol on $\mathcal{U}$.

This reduces the problem to the case of vanishing subprincipal symbol discussed in the hindmost paragraph and let B_3 and $\tilde{B}_3$ are the corresponding scalar Fourier integral operators. This means B and $\tilde{B}$ are of the form $B := \hat{B}B_3$ and $\tilde{B} := \tilde{B}_3\check{B}$. Since the principal symbol of $\hat{B}$ is the transition function to a parallel frame and B_3 is a scalar operator, these imply that the principal symbols of B and $\tilde{B}$ can be chosen $\mathbb{1}$ with respect to a parallel frame along X_p.

The case of symmetric P with connection P-compatibility:
Finally, suppose that P is symmetric with respect to $(\cdot | \cdot)$ and that $\nabla^{\mathscr{E}}$ is P-compatible. The

above construction of $\hat{B}$ can be repeated using an orthonormal frame of $\mathscr{E}$ and another orthonormal frame of $\pi^*\mathscr{E}$ that is parallel along X_p. One obtains an operator that has principal symbol $\mathbb{1}$ near (x_0, ξ^0) with respect to the parallel orthonormal frame. Now one repeats the construction of Ψ as before and note that the principal symbol can be kept as $\mathbb{1}$ in that way to turn B into a microlocally unitary operator in the sense that $\tilde{B} = B^*$ with the desired properties.   ∎

As a supplement of the preceding proof we present a variant of Hörmander's square root construction [78, Prop. 2.2.2] for vector bundles.

Proposition 3.3.11. *Let* $\left(\mathscr{E} \to M, (\cdot|\cdot)\right)$ *be a vector bundle over a manifold M, endowed with a sesquilinear form $(\cdot|\cdot)$, and $\Omega^{1/2} \to M$ the half-density bundle over M. Suppose that $P \in \Psi\mathrm{DO}^m(M; \mathscr{E} \otimes \Omega^{1/2})$ is a properly supported pseudodifferential operator of order $m \in \mathbb{R}_+$, symmetric with respect to $(\cdot|\cdot)$, elliptic in a conic neighbourhood $\mathcal{U}$ of $(x_0, \xi^0) \notin \mathrm{ES}\, P$ in the punctured cotangent bundle $\dot{\mathrm{T}}^*M$ of M, and that its principal symbol is given by*

$$p = q^* q \tag{3.80}$$

for some $q \in C^\infty(\mathcal{U}, \mathrm{End}\,\mathscr{E})$. Then, one can find a properly supported symmetric (with respect to $(\cdot|\cdot)$) $Q \in \Psi\mathrm{DO}^{m/2}(U; \mathscr{E} \otimes \Omega^{1/2})$ such that

$$(x_0, \xi^0) \notin \mathrm{ES}(P - Q^*Q), \qquad (x_0, \xi^0) \notin \mathrm{ES}\, Q, \tag{3.81}$$

where U is the base projection of $\mathcal{U}$.

Proof. The hypothesis on p implies that q is homogeneous of degree $m/2$ which in turns entails that $q \in S^{m/2}(\mathcal{U}, \mathrm{End}\,\mathscr{E})$; see Definition B.1.6 and Appendix B.3. We define a properly supported $Q_0 \in \Psi\mathrm{DO}^{m/2}(U; \mathscr{E} \otimes \Omega^{1/2})$ whose principal symbol is q. Without lose of generality Q_0 can be taken symmetric, otherwise one can just replace Q_0 by $(Q_0 + Q_0^*)/2$ without changing the principal symbol. Then $P - Q_0^*Q_0 \in \Psi\mathrm{DO}^{m-1}(U; \mathscr{E} \otimes \Omega^{1/2})$.

Now we are left with estimations for lower order terms and will show that it is always possible to obtain properly supported and symmetric $Q_k \in \Psi\mathrm{DO}^{m/2-k}(U; \mathscr{E} \otimes \Omega^{1/2})$ for all $k \in \mathbb{N}$, such that

$$R_k := P - (Q_0 + \ldots + Q_k)^*(Q_0 + \ldots + Q_k) \in \Psi\mathrm{DO}^{m-1-k}(U; \mathscr{E} \otimes \Omega^{1/2}). \tag{3.82}$$

We proceed inductively. Clearly, $k = 1$ has been checked. Observe that, if Q_ks have been chosen accordingly, then

$$P - |Q_0 + \ldots + Q_k|^2 = R_k - (Q_k^*Q_0 + Q_0^*Q_k) - Q_k^*Q_k + \ldots \in \Psi\mathrm{DO}^{m-1-k}(U; \mathscr{E} \otimes \Omega^{1/2}). \tag{3.83}$$

Since R_k is symmetric, the principal symbol of $R_k - R_k^*$ is $2\,\mathrm{i}\,\mathfrak{I}(\sigma_{R_k})$ modulo $S^{m-2-k}(\mathcal{U}, \mathrm{End}\,\mathscr{E})$. Then the desired operators B_k are achieved if we set

$$\forall (x, \xi) \in \mathcal{U} : \sigma_{R_k}(x, \xi) = \sigma_{Q_k}^* \circ q\,(x, \xi) + q^* \circ \sigma_{Q_k}\,(x, \xi), \tag{3.84}$$

that is, in the region where $q(x, \hat{\xi})$ is invertible in $\mathrm{End}\,\mathscr{E}_x$ for each $x \in U$ and $\hat{\xi} := \xi/\|\xi\| \in \mathbb{S}^{\mathrm{rk}(\mathscr{E})-1}$. Finally, the result entails by constructing the total symbol of Q in the sense of asymptotic summation; in other words: $Q :\sim \sum_{k=0}^\infty Q_k$.   ∎

Remark 3.3.12. The hypotheses of the preceding theorem are satisfied by any pseudodifferential operator Q of real principal type by Remark 2.2.11, and hence Theorem 3.3.10 also holds for Q by a redefinition of B.

We will now show microlocalisation of a normally hyperbolic operator $\square$ as a consequence of the preceding result by replacing the generic manifold M by a Lorentzian (not necessarily globally hyperbolic) spacetime $(\mathcal{M}, \mathrm{g})$. In particular, the role of the characteristic points (x_0, ξ^0) of P in Theorem 3.3.10 (a) will be played by any lightlike covector on $\mathcal{M}$ and thus, the bicharacteristics of p are actually the lightlike geodesics in $\mathrm{T}^*\mathcal{M}$.

Theorem 3.3.13. *Let $\mathcal{E} \to \mathcal{M}$ be a vector bundle of rank k over a d-dimensional Lorentzian manifold $(\mathcal{M}, \mathrm{g})$ and $\square : C_{\mathrm{c}}^{\infty}(\mathcal{M}; \mathcal{E}) \to C_{\mathrm{c}}^{\infty}(\mathcal{M}; \mathcal{E})$ a normally hyperbolic operator. Denote by $\nabla^{\mathcal{E}}$, the Weitzenböck connection associated with $\square$ and by Γ, the graph of a homogeneous symplectomorphism from an open conic neighbourhood $\mathcal{U}'$ of $(0, \eta_1 \mathrm{d}y^1)$ in $\dot{\mathrm{T}}^*\mathbb{R}^d$ to an open conic coordinate chart $(\mathcal{U}, (x^i, \xi_i))$ centered at any lightlike covector (x_0, ξ^0) on $\mathcal{M}$. Then for any $m \in \mathbb{R}$, one can find properly supported Lagrangian distributions $A \in I^{-1/2+m}(\mathcal{M} \times \mathbb{R}^d, \Gamma'; \mathrm{Hom}(\mathbb{C}^k, \mathcal{E}))$ and $\tilde{A} \in I^{-1/2-m}(\mathbb{R}^d \times \mathcal{M}, \Gamma^{-1\prime}; \mathrm{Hom}(\mathcal{E}, \mathbb{C}^k))$ so that $A\tilde{A}, \tilde{A}A$ both are pseudodifferential operators of order -1 and*

$$\mathrm{WF}'\, A \subset \mathcal{U}_{(x_0, \xi^0; 0, \eta_1 \mathrm{d}y^1)}, \qquad \mathrm{WF}'\, \tilde{A} \subset \mathcal{U}'_{(0, \eta_1 \mathrm{d}y^1; x_0, \xi^0)}, \tag{3.85a}$$

$$(x_0, \xi^0) \notin \mathrm{ES}(A\tilde{A} - I_{\mathcal{E}}), \qquad (0, \eta_1 \mathrm{d}y^1) \notin \mathrm{ES}(\tilde{A}A - I), \tag{3.85b}$$

$$(x_0, \xi^0) \notin \mathrm{ES}(A\mathrm{D}_1\tilde{A} - \square), \qquad (0, \eta_1 \mathrm{d}y^1) \notin \mathrm{ES}(\tilde{A}\square A - \mathrm{D}_1). \tag{3.85c}$$

Here $\mathrm{D}_1 = -\mathrm{i}\, \mathbb{1}_{\mathbb{C}^{k \times k}}\, \partial/\partial y^1 : C_{\mathrm{c}}^{\infty}(\mathbb{R}^d, \mathbb{C}^k) \to C_{\mathrm{c}}^{\infty}(\mathbb{R}^d, \mathbb{C}^k)$ and $\mathcal{U}_{(x_0, \xi^0; 0, \eta_1 \mathrm{d}y^1)}$ resp. $\mathcal{U}'_{(0, \eta_1 \mathrm{d}y^1; x_0, \xi^0)}$ are small conic neighbourhoods of $(x_0, \xi^0; 0, \eta_1 \mathrm{d}y^1) \in \dot{\mathrm{T}}^*\mathcal{M} \times \dot{\mathrm{T}}^*\mathbb{R}^n$ resp. $(0, \eta_1 \mathrm{d}y^1; x_0, \xi^0) \in \dot{T}^*\mathbb{R}^n \times \dot{\mathrm{T}}^*\mathcal{M}$.

In addition, if $\mathcal{E} \to \mathcal{M}$ is endowed with a sesquilinear form $(\cdot|\cdot)$ with respect to which $\square$ is symmetric, then A can be chosen as a scalar operator with respect to a unitary bundle frame that is parallel along the geodesic flow and microlocally $A = \tilde{A}^*$ if $\mathbb{C}^k$ is endowed with a standard sesquilinear form of the same signature as $(\cdot|\cdot)$.

Proof. The strategy is, as usual, to reduce $\square$ to a first-order operator so that Theorem 3.3.10 can be applied. To do so, choose a properly supported elliptic symmetric pseudodifferential operator L on $\mathcal{E}$ of order $-1/2$ having a scalar principal symbol l and vanishing subprincipal symbol. Such an operator always exists, as it can be constructed locally and then patched to a global operator using a suitable partition of unity. Then $L\square L$ is a first-order pseudodifferential operator whose principal symbol $l^2(x, \xi)\, \mathrm{g}_x^{-1}(\xi, \xi)\, \mathbb{1}_{\mathrm{End}\,\mathcal{E}}$ vanishes on (x_0, ξ^0). By Theorem 3.3.8 and Corollary 3.3.9, ∇^E is compatible with both $\square$ and $L\square L$, respectively. Therefore the hypotheses of Theorem 3.3.10 are satisfied and the conclusion entails by putting $A := L\mathrm{B}, \tilde{A} := \tilde{\mathrm{B}}L$ where $\mathrm{B}, \tilde{\mathrm{B}}$ are the Lagrangian distributions as constructed in Theorem 3.3.10. In case $\square$ is symmetric, we choose L symmetric as well. ∎

Remark 3.3.14. We have microlocally conjugated $\square$ to D_1 around a lightlike covector in the preceding theorem. It is also possible to refine the construction as follows. Let $\mathbb{I}$ be a compact interval in $\mathbb{R}$ and $\gamma : \mathbb{I} \to \dot{T}^*M$ an integral curve of X_p:

$$p \circ \gamma = 0. \tag{3.86}$$

If the composition of γ and the projection $\dot{T}^*M \to \dot{T}^*M/\mathbb{R}_+$ is injective, then one can find a conic neighbourhood $\mathcal{V}'$ of $\mathbb{I} \times \{(0, \eta_1 dy^1)\}$ and a smooth homogeneous symplectomorphism ϱ from $\mathcal{V}'$ to an open conic neighbourhood $\varrho(\mathcal{V}') \subset \dot{T}^*M$ of $\gamma(\mathbb{I})$ such that [23, Prop. 26.1.6]

$$\varrho\big(\mathbb{I} \times \{(0, \eta_1 dy^1)\}\big) = \gamma(\mathbb{I}), \qquad \varrho^* p = \xi_1 \, \mathbb{1}_{\mathrm{End}\,\mathcal{E}} \,. \tag{3.87}$$

Imitating the proof of Theorem 3.3.10, one can microlocalise P to D_1 on $\gamma(\mathbb{I})$ (see [23, Prop. 26.1.3']) for the scalar version).

As a consequence, if Γ is the graph of a homogeneous symplectomorphism from a conic neighbourhood of $\mathbb{I} \times \{(0, \eta_1 dy^1)\}$ in $\dot{T}^*\mathbb{R}^d$ to a conic neighbourhood of lightlike geodesic $\gamma(\mathbb{I})$ in $\dot{T}^*\mathcal{M}$, then the proof of Theorem 3.3.13 shows that, for any $m \in \mathbb{R}$, there exists Lagrangian distributions $A \in I^{-1/2+m}\big(\mathcal{M} \times \mathbb{R}^d, \Gamma'; \mathrm{Hom}(\mathbb{C}^k, \mathcal{E})\big)$ and $\tilde{A} \in I^{-1/2-m}\big(\mathbb{R}^n \times \mathcal{M}, \Gamma^{-1'}; \mathrm{Hom}(\mathcal{E}, \mathbb{C}^N)\big)$ so that $A\tilde{A}, \tilde{A}A$ both are pseudodifferential operators of order -1 and

$$\mathrm{WF}'\, A \subset \mathcal{V}_\Gamma, \qquad \mathrm{WF}'\, \tilde{A} \subset \mathcal{V}'_{\Gamma^{-1}}, \tag{3.88a}$$

$$\gamma(\mathbb{I}) \cap \mathrm{ES}(A\tilde{A} - I_E) = \emptyset, \qquad \mathbb{I} \times \{(0, \eta_1 dy^1)\} \cap \mathrm{ES}(\tilde{A}A - I) = \emptyset, \tag{3.88b}$$

$$\gamma(\mathbb{I}) \cap \mathrm{ES}(AD_1\tilde{A} - \square) = \emptyset, \qquad \mathbb{I} \times \{(0, \eta_1 dy^1)\} \cap \mathrm{ES}(\tilde{A}\square A - D_1) = \emptyset, \tag{3.88c}$$

where $\mathcal{V}_\Gamma, \mathcal{V}'_{\Gamma^{-1}}$ are small conic neighbourhoods of Γ restricted to $\gamma(\mathbb{I})$ and its inverse, respectively.

We close this section by a simple application of Theorem 3.3.10 to derive a bundle version of Hörmander's propagation of singularity theorem [79], [35, Thm. 6.1.1']. Since

$$C^\infty(M; \mathcal{E} \otimes \Omega^{1/2}) = \bigcap_{s \in \mathbb{R}} H^s_{\mathrm{loc}}(M; \mathcal{E} \otimes \Omega^{1/2}), \tag{3.89}$$

such a refinement of the usual notion of (smooth) wavefront set is captured by the Sobolev wavefront set.

Definition 3.3.15. Let $\mathcal{E}, \Omega^{1/2} \to M$ be a vector bundle resp. the half-density bundle over a manifold M and $s \in \mathbb{R}$. The **Sobolev wavefront set** $\mathrm{WF}^s\, u$ of a distribution $u \in \mathcal{D}'(M; \mathcal{E} \otimes \Omega^{1/2})$ relative to Sobolev space $H^s_{\mathrm{loc}}(M; \mathcal{E} \otimes \Omega^{1/2})$ is defined by [35, p. 201]

$$\mathrm{WF}\, u := \bigcap_{\substack{P \in \Psi\mathrm{DO}^0(M; \mathcal{E} \otimes \Omega^{1/2}) \\ Pu \in H^s_{\mathrm{loc}}(M; \mathcal{E} \otimes \Omega^{1/2})}} \mathrm{Char}\, P = \bigcap_{\substack{P \in \Psi\mathrm{DO}^s(M; \mathcal{E} \otimes \Omega^{1/2}) \\ Pu \in L^2_{\mathrm{loc}}(M; \mathcal{E} \otimes \Omega^{1/2})}} \mathrm{Char}\, P \tag{3.90}$$

where $\mathrm{Char}\, P$ is the characteristic set of P and the intersection is over all properly supported P.

Locally this means, for any open set $U \subset \mathbb{R}^d$, $(x_0, \xi^0) \notin \mathrm{WF}^s u$ is not in the Sobolev wavefront set of a distribution $u \in \mathcal{D}'(M; \mathscr{E} \otimes \Omega^{1/2})$ if and only if there exists a compactly supported section f on U, non-vanishing at $x_0 \in U$ such that the weighted Fourier transform $(1 + |\xi|^2)^s |\mathcal{F}(fu)(\xi)|^2$ is integrable in a conic neighbourhood of ξ^0. An accumulation of important properties of this finer wavefront set is available, for example, in [120, App. B].

Hörmander's theorem has been generalised for system of pseudodifferential operators by Taylor [107, Thm. 4.1 (p. 135), 2.2 (p. 154)] (for the Sobolev wavefront set) and by Dencker [45, Thm. 4.2] (for the polarisation set). Our result, as stated below is similar to that of Taylor but formulated in a geometric fashion and proved in a different way employing microlocalisation on vector bundles developed in Theorem 3.3.10.

Theorem 3.3.16 (Propagation of Sobolev regularity)**.** *Let $\mathscr{E}, \Omega^{1/2} \to M$ be a vector bundle resp. the half-density bundle over a manifold M and $u \in \mathcal{D}'(M; \mathscr{E} \otimes \Omega^{1/2})$. Suppose that, for some $m \in \mathbb{R}$, $\Psi \in \Psi\mathrm{DO}^m(M; \mathscr{E} \otimes \Omega^{1/2})$ satisfies the hypotheses (a) and (b) of Theorem 3.3.10 and that $\mathcal{I}$ is an interval on an integral curve of the Hamiltonian vector field generated by the principal symbol of Ψ such that $\mathcal{I} \cap \mathrm{WF}^s(\Psi u) = \emptyset$. Then either $\mathcal{I} \cap \mathrm{WF}^{s+m-1} u = \emptyset$ or $\mathcal{I} \subset \mathrm{WF}^{s+m-1} u$.*

Proof. Once Theorem 3.3.10 is at our disposal, the rest of the proof is the same as its scalar version [35, Thm. 6.1.1'] (see also [23, Thm. 26.1.4]). For completeness we give the details here. By conjugating Ψ with appropriate Fourier integral operators we reduce the statement to the case $m = 1$. In other words, one can just consider P as in Theorem 3.3.10 instead of Ψ to conclude the assertion by utilising the Sobolev continuity properties of pseudodifferential (see e.g. [77, p. 92]) and Fourier integral operators (see e.g. [23, Cor. 25.3.2]). Via microlocalisation (Theorem 3.3.10), the analysis further boils down to $P = D_1, s = 0$ and $(x_0, \xi^0) = (0, \eta_1 dy^1)$ by choosing the order of the Fourier integral operator B in Theorem 3.3.10 equals to $-s$. Then the claim follows from the form of the advanced and retarded fundamental solutions (Example 3.2.11) of D_1 and as these map from $L^2_c(\mathbb{R}^d, \mathbb{C}^k)$ to $L^2_{\mathrm{loc}}(\mathbb{R}^d, \mathbb{C}^k)$ (cf. Proposition 3.2.12). ∎

By Remark 3.3.12, the preceding theorem can be restated as

Theorem 3.3.17. *Let $\mathscr{E}, \Omega^{1/2} \to M$ be a vector bundle resp. the half-density bundle over a manifold M and $u \in \mathcal{D}'(M; \mathscr{E} \otimes \Omega^{1/2})$. Suppose that P is an m-order pseudodifferential operator on $\mathscr{E} \otimes \Omega^{1/2}$ of real principal type and that $\mathcal{I}$ is an interval on an integral curve of the Hamiltonian vector field generated by q as in Definition 2.3.6 such that $\mathcal{I} \cap \mathrm{WF}^s(Pu) = \emptyset$. Then either $\mathcal{I} \cap \mathrm{WF}^{s+m-1} u = \emptyset$ or $\mathcal{I} \subset \mathrm{WF}^{s+m-1} u$.*

3.3.4 Proof of Theorem 3.3.1

To begin with, we show that the Feynman parametrix is unique, if it exists.

Uniqueness of the Feynman parametrix

Suppose that E_{L} (resp. E_{R}) is a left (resp. right) Feynman parametrix of $\Box$, i.e., the off-diagonal contribution of $\mathrm{WF}'(E_{\mathrm{L}}, E_{\mathrm{R}})$ is given by C^+; cf. Definition 3.2.17. To prove that $E_{\mathrm{L}} - E_{\mathrm{R}}$ is a smoothing operator, we would like to argue that $E_{\mathrm{L}} \Box E_{\mathrm{R}}$ is congruent both to E_{L} and to E_{R} modulo smoothing operators. But E_{L} and E_{R} are not properly supported which makes this a non-trivial task. To circumvent this difficulty, one employs the fact that $E_{\mathrm{L}} Q E_{\mathrm{R}}$ is defined when Q is a pseudodifferential operator having Schwartz kernel of compact support in $\mathscr{M} \times \mathscr{M}$ and then $Q : \mathcal{D}'(\mathscr{M}; \mathscr{E}) \to \mathcal{E}'(\mathscr{M}; \mathscr{E})$.

If $(x, \xi; y, \eta) \in \mathrm{WF}'(E_{\mathrm{L}} Q E_{\mathrm{R}})$ but $(x, \xi), (y, \eta)$ are in the complement of $\mathrm{ES}\, Q$, then $(x, \xi; z, \zeta), (z, \zeta; y, \eta) \in C^+$ for some $(z, \zeta) \in \mathrm{ES}\, Q$. This follows from the behaviour of wavefront sets under composition of kernels, as available in, for instance [76, Theorem 8.2.10], and the fact that $\mathrm{WF}'\, Q \subset \Delta \dot{\mathrm{T}}^*\mathscr{M}$. This entails that $(x, \xi), (y, \eta)$ and (z, ζ) are on the same lightlike geodesic (strip) $\gamma(s)$ with (z, ζ) in between the other two points.

Since $\mathscr{M}$ is assumed to be globally hyperbolic, $J(K)$ is compact for any compact $K \subset \mathscr{M}$ [111, Corollary of Prop. 6.6.1]. Therefore, if the projections of endpoints of γ on $\mathscr{M}$ are in K then the projection $c(s)$ of $\gamma(s)$ on $\mathscr{M}$ stays over $J(K)$, i.e., $c(s) \in J(K)$, as shown in Figure 3.3. Consequently, if $\mathrm{singsupp}\, Q \cap J(K) = \emptyset$ then $\mathrm{WF}'(E_{\mathrm{L}} Q E_{\mathrm{R}})$ cannot have any point over $K \times K$. Let $\chi \in C_c^\infty(\mathscr{M}; \mathscr{E})$ such that χ is identically 1 on $J(K)$ and define M_χ be the corresponding multiplication operator. We observe that $[\mathrm{M}_\chi, \Box]_-$ vanishes identically on $J(K)$. Thus $\mathrm{WF}'(E_{\mathrm{L}}[\mathrm{M}_\chi, \Box]_- E_{\mathrm{R}})$ contains no point over $K \times K$ and so it true for

$$
\begin{aligned}
(\dot{\mathrm{T}}^* K \times \dot{\mathrm{T}}^* K) \cap \mathrm{WF}'(E_{\mathrm{L}} - E_{\mathrm{R}}) \ &= \ (\dot{\mathrm{T}}^* K \times \dot{\mathrm{T}}^* K) \cap \mathrm{WF}'(E_{\mathrm{L}} \mathrm{M}_\chi \Box E_{\mathrm{R}} - E_{\mathrm{L}} \Box \mathrm{M}_\chi E_{\mathrm{R}}) \\
&= \ (\dot{\mathrm{T}}^* K \times \dot{\mathrm{T}}^* K) \cap \mathrm{WF}'(E_{\mathrm{L}}[\mathrm{M}_\chi, \Box]_- E_{\mathrm{R}}) \\
&= \ \emptyset.
\end{aligned}
\tag{3.91}
$$

Since K is arbitrary, we conclude that $E_{\mathrm{L}} - E_{\mathrm{R}}$ is a smoothing operator and similarly for the anti-Feynman parametrix.
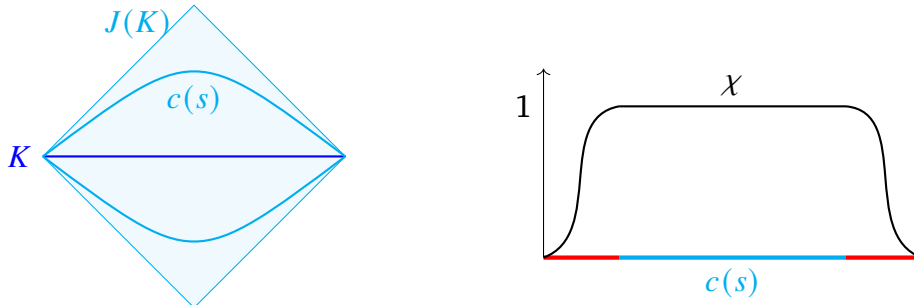

Figure 3.3: A schematic visualisation of a consequence of the global hyperbolicity of spacetime. Here time flows from left to right.

Construction of the Feynman parametrix

Notice that $\Box E_R = I + R$ is equivalent to $(E_R)^*\Box^* = I + R^*$ and, since the adjoint $\Box^*$ of $\Box$ is also a normally hyperbolic operator, the existence of left parametrices with the listed properties in this theorem follows from the existence of right parametrices of $\Box$. It is therefore sufficient to construct a right parametrix of $\Box$ with the required regularities.

Lemma 3.3.18. *As in the terminologies of Theorem 3.3.13: $\Box$ and D_1 are microlocally conjugate to each other, A, Ã are the conjugating Lagrangian distributions with $m = 0$, and $\mathcal{U}$ is the conic neighbourhood of any lightlike covector (x_0, ξ^0) on $\mathcal{M}$. Suppose that $\mathsf{F}_1^{\mathrm{ret,adv}}$ are the Schwartz kernels of the retarded and advanced fundamental solutions of D_1 and that $\chi \in C_c^\infty(\mathbb{R}^d \times \mathbb{R}^d, \mathbb{R})$ is identically 1 in a neighbourhood of the diagonal and vanishes outside another sufficiently small neighbourhood of the diagonal. If $T \in \Psi\mathrm{DO}^0(\mathcal{M}; \mathcal{E})$ with $\mathrm{ES}\, T \subset \mathcal{U}$ and $\mathsf{F}^\pm := \mathsf{A}(\chi\mathsf{F}_1^{\mathrm{ret,adv}})\tilde{\mathsf{A}}T$, then*

$$\mathrm{WF}'\,\mathsf{F}^\pm \subset \varDelta\dot{\mathrm{T}}^*\mathcal{M} \cup C^\pm, \tag{3.92a}$$

$$\Box\mathsf{F}^\pm = T + \mathsf{R}^\pm, \quad \mathsf{R}^\pm \in I^{-3/2}\big(\mathcal{M} \times \mathcal{M}, C^{\pm\prime}; \mathrm{Hom}(\mathcal{E}, \mathcal{E})\big), \tag{3.92b}$$

$$\mathsf{F}^+ - \mathsf{F}^- \in I^{-3/2}\big(\mathcal{M} \times \mathcal{M}, C'; \mathrm{Hom}(\mathcal{E}, \mathcal{E})\big). \tag{3.92c}$$

Proof. The properties (3.32) and (3.35) immediately entail (3.92a) and (3.92c). To show (3.92b), by definition:

$$\Box\mathsf{F}^\pm = (\Box\mathsf{A} - \mathsf{A}D_1)(\chi\mathsf{F}_1^{\mathrm{ret,adv}})\tilde{\mathsf{A}}T + \mathsf{A}D_1(\chi\mathsf{F}_1^{\mathrm{ret,adv}})\tilde{\mathsf{A}}T. \tag{3.93}$$

Since $\Box$ and D_1 are microlocal conjugates to one another, (3.85c) can be re-expressed as

$$(x_0, \xi^0; 0, \eta_1 \mathrm{d}y^1) \notin \mathrm{WF}'(\Box\mathsf{A} - \mathsf{A}D_1). \tag{3.85c'}$$

Then, there is a conic neighbourhood $\mathcal{V}$ of $(0, \eta_1 \mathrm{d}y^1)$ such that $(\Box\mathsf{A} - \mathsf{A}D_1)v$ is smooth for any $v \in \mathcal{D}'(\mathbb{R}^d, \mathbb{C}^k)$ when $\mathrm{WF}\, v \subset \mathcal{V}$. Since $\mathrm{WF}'(\chi\mathsf{F}_1^{\mathrm{ret,adv}})$ can be made arbitrary close to the diagonal in $\dot{\mathrm{T}}^*\mathbb{R}^d \times \dot{\mathrm{T}}^*\mathbb{R}^d$ by choosing the support of χ close to the diagonal in $\mathbb{R}^d \times \mathbb{R}^d$, we can pick an appropriate χ and a conic neighbourhood $\mathcal{W}$ of $(0, \eta_1 \mathrm{d}y^1)$ such that $\mathrm{WF}\big((\chi\mathsf{F}_1^{\mathrm{ret,adv}})v\big) \subset \mathcal{V}$ provided $\mathrm{WF}\, v \subset \mathcal{W}$. If $\mathrm{ES}\, T \subset \varkappa(\mathcal{W})$ then the first term in the right-hand side of (3.93) is smooth and we are left with the last term of that equation. By definition: $D_1(\chi\mathsf{F}_1^{\mathrm{ret,adv}}) = I + \tilde{\mathsf{R}}^\pm$ where $\tilde{\mathsf{R}}^\pm := D_{x^1}\big(\chi(x, y)\big)\mathsf{F}_1^{\mathrm{adv,ret}} \in I^{-1/2}(\mathbb{R}^d \times \mathbb{R}^d, C_1'; \mathbb{C}^{k \times k})$ (cf. Proposition 3.2.12). Since $\mathsf{A}\tilde{\mathsf{A}}T - T = (\mathsf{A}\tilde{\mathsf{A}} - I)T$ is smooth as long as $\mathrm{ES}\, T$ is sufficiently close to (x_0, ξ^0), it follows that $\Box\mathsf{F}^\pm - T = \mathsf{R}^\pm$ where $\mathsf{R}^\pm - \mathsf{A}\tilde{\mathsf{R}}^\pm\tilde{\mathsf{A}}T$ is smooth, which concludes the proof. ∎

To end the proof of Theorem 3.3.1, we choose a locally finite covering $\{\mathcal{U}_\alpha\}$ of $\dot{\mathrm{T}}^*\mathcal{M}$ by open cones $\mathcal{U}_\alpha$ such that Lemma 3.3.18 is applicable when $\mathrm{ES}\, T_\alpha \subset \mathcal{U}_\alpha$. Denoting the projection of $\mathcal{U}_\alpha$ on $\mathcal{M}$ by U_α and picking $\mathcal{U}_\alpha$ so that U_α are also locally finite, we set

$$I = \sum_\alpha T_\alpha, \quad \mathrm{ES}\, T_\alpha \subset \mathcal{U}_\alpha, \quad \mathrm{supp}\, \mathsf{T}_\alpha \subset U_\alpha \times U_\alpha. \tag{3.94}$$

Then for every α, $\mathsf{A}(\chi\mathsf{F}_1^{\text{ret,adv}})\tilde{\mathsf{A}}T_\alpha$ can be chosen according to the Lemma 3.3.18 with $\text{supp}\left(\mathsf{A}(\chi\mathsf{F}_1^{\text{ret,adv}})\tilde{\mathsf{A}}T_\alpha\right) \subset U_\alpha \times U_\alpha$, and hence the sum

$$\mathsf{F}^\pm := \sum_\alpha \mathsf{A}(\chi\mathsf{F}_1^{\text{ret,adv}})\tilde{\mathsf{A}}T_\alpha \tag{3.95}$$

is well-defined which satisfies the claimed properties (3.47) and (3.48) in the statement of Theorem 3.3.1. Note that, if χ is taken as a function of $y - z$ then $\chi\mathsf{F}_1^{\text{ret,adv}}$ is a convolution by a measure of compact support and therefore it continuously maps $H_{\text{c}}^s(\mathbb{R}^d, \mathbb{C}^k)$ to $H_{\text{c}}^s(\mathbb{R}^d, \mathbb{C}^k)$. By the mapping properties of pseudodifferential and Fourier integral operators, [78, Thm. 4.3.1] (see also [23, Cor. 25.3.2]), the factors $A, \tilde{A}, T_\alpha$ are continuous maps from H_{c}^s to H_{c}^s. Since the sum of kernels is locally finite this shows that $E^{\text{Fyn,aFyn}}$ continuously maps $H_{\text{c}}^s(\mathcal{M}; \mathcal{E})$ into $H_{\text{loc}}^s(\mathcal{M}; \mathcal{E})$ for all $s \in \mathbb{R}$.

Until now, we have just shown that $\square\mathsf{F}^\pm = I + \mathsf{R}^\pm$. By Lemma 3.3.19 (stated and proven just after the on going proof), we can choose $\mathsf{G}^\pm \in I^{-3/2}(\mathcal{M} \times \mathcal{M}, C^{\pm\prime}; \text{Hom}(\mathcal{E}, \mathcal{E}))$ so that $\square\mathsf{G}^\pm - \mathsf{R}^\pm$ is smooth. Moreover, $G^\pm$ extend to continuous mappings from $H_{\text{c}}^s(\mathcal{M}; \mathcal{E})$ to $H_{\text{loc}}^s(\mathcal{M}; \mathcal{E})$ for every $s \in \mathbb{R}$. This follows from mapping properties of Fourier integral operators, for example, Theorem 25.3.8 in [23], bearing in mind that the corank of the symplectic form σ_Γ (see (A.70)) on Γ is two. Since locally a Fourier integral operator on a bundle is a matrix of scalar Fourier integral operators (Section 2.3.3), this theorem can be applied here directly. Hence,

$$E^{\text{Fyn,aFyn}} := F^\pm - G^\pm \tag{3.96}$$

is a right parametrix which has this continuity property.

Furthermore, the construction shows that $\mathsf{F}^+ - \mathsf{F}^-$ and thus $\mathsf{E}^{\text{Fyn}} - \mathsf{E}^{\text{aFyn}}$ is non-characteristic on $\varDelta\dot{\mathrm{T}}_0^*\mathcal{M}$, as can be seen from the integral representation (Example 3.2.11) of F_1 and wavefront set properties of $\mathsf{A}, \tilde{\mathsf{A}}$. Since $\square(\mathsf{E}^{\text{Fyn}} - \mathsf{E}^{\text{aFyn}})$ is smooth, it follows that its principal symbol satisfies a first-order homogeneous differential equation along the lightlike geodesic strip by Theorem 2.3.13 and Remark 3.2.8. Therefore $\mathsf{E}^{\text{Fyn}} - \mathsf{E}^{\text{aFyn}}$ is non-characteristic everywhere: $\text{WF}'(\mathsf{E}^{\text{Fyn}} - \mathsf{E}^{\text{aFyn}}) = C$ and then it can be concluded that $\text{WF}'\,\mathsf{E}^{\text{Fyn,aFyn}} \supset C^\pm$ because of $\text{WF}'\,\mathsf{E}^{\text{Fyn,aFyn}} \subset \varDelta\dot{\mathrm{T}}^*\mathcal{M} \cup C^\pm$. Finally, the fact

$$\varDelta\dot{\mathrm{T}}^*\mathcal{M} = \text{WF}'\,\mathsf{I} = \text{WF}'(\square\mathsf{E}^{\text{Fyn,aFyn}}) \subset \text{WF}'\,\mathsf{E}^{\text{Fyn,aFyn}} \tag{3.97}$$

together with Lemma 3.3.19 complete the proof.

Lemma 3.3.19. *Let $\mathcal{E} \to \mathcal{M}$ be a vector bundle over a Lorentzian manifold $(\mathcal{M}, \mathrm{g})$ and $\square$ a normally hyperbolic operator on $\mathcal{E}$. Suppose that $C^\pm$ are the forward/backward geodesic relations. If $\mathsf{B} \in I^m(\mathcal{M} \times \mathcal{M}, C^{\pm\prime}; \text{Hom}(\mathcal{E}, \mathcal{E}))$ then one can find a properly supported $\mathsf{A} \in I^m(\mathcal{M} \times \mathcal{M}, C^{\pm\prime}; \text{Hom}(\mathcal{E}, \mathcal{E}))$ such that $\square\mathsf{A} - \mathsf{B}$ is smoothing.*

Proof. The assertion has been proven for scalar pseudodifferential operators [35, Thm. 5.3.2] (see also [23, Lem. 26.1.16]) which we tailor for normally hyperbolic operators. Let A_0 be a properly supported Lagrangian distribution of order m on $\mathcal{M} \times \mathcal{M}$ whose principal symbol is a_0 and that of B is b. Since the principal symbol of $\Box$ vanishes on lightlike covectors, Theorem 2.3.13 implies that $\Box A_0 - B$ is of order $m - 1$ on lightlike covectors on $\mathcal{M}$, provided

$$\left(-\,\mathrm{i}\, \pounds_{X_{\mathrm{g}}} + \sigma_{\Box}^{\mathrm{sub}} \right) a_0 = b \tag{3.98}$$

is satisfied. Identifying half-densities with functions according to Remark 2.2.1 (d) and making use of the fact that the Keller-Maslov bundle $\mathbb{M}$ is a trivial complex vector bundle (see Defining A.1.19), we write (cf. (B.10)) $a_0 = a_0 \mathbb{m}$ and $b = b \mathbb{m}$ where $\mathbb{m}$ is a non-vanishing section of $\mathbb{M}$ and a_0, b are scalar symbols of degree m. The preceding equation then reads $(-\,\mathrm{i}\, X_{\mathrm{g}} + \sigma_{\Box}^{\mathrm{sub}}) a_0 = b$ by (2.138). In other words, inserting (3.60) and (3.24) yields

$$\left(-\frac{\partial \mathrm{g}^{\mu\nu}}{\partial x^i} \xi_\mu \xi_\nu \frac{\partial}{\partial \xi_i} + 2\mathrm{g}^{ij}\xi_j \frac{\partial}{\partial x^i} + 2\mathrm{g}^{ij}\Gamma_i^i \xi_j \right) a_0 = b. \tag{3.99}$$

This equation has a unique solution for a given initial condition and the essential support of the solution is contained in that of b, that is $\mathrm{WF}'\, A_0 \subset C^\pm$. Now the conclusion follows from applying the argument iteratively to construct Lagrangian distributions A_k of order $m - k$ such that $\Box(A_0 + \ldots + A_k) - B \in I^{m-k-1}\big(\mathcal{M} \times \mathcal{M}, C^{\pm\prime}; \mathrm{Hom}(\mathscr{E}, \mathscr{E})\big)$ and setting $A :\sim A_0 + \ldots + A_k + \ldots$. ∎

3.3.5 Proof of Proposition 3.3.2

Taking the adjoint $\mathsf{E}^{\mathrm{Fyn}*}\Box \equiv I$ of the Feynman parametrix $\mathsf{E}^{\mathrm{Fyn}}$ we find that $\mathrm{WF}'\, \mathsf{E}^{\mathrm{Fyn}*} \subset \varDelta\,\dot{\mathrm{T}}^*\mathcal{M} \cup C^-$ by Definition 3.2.17. Thus $\mathsf{E}^{\mathrm{Fyn}*} - \mathsf{E}^{\mathrm{aFyn}}$ is smooth and so is $(\mathsf{E}^{\mathrm{Fyn}} - \mathsf{G}^{\mathrm{adv}})(\mathsf{E}^{\mathrm{Fyn}} - \mathsf{G}^{\mathrm{adv}})^*$. Since $\mathrm{WF}'\big(-\,\mathrm{i}(\mathsf{E}^{\mathrm{Fyn}} - \mathsf{G}^{\mathrm{adv}})\big) \cap \mathrm{WF}'\big(\mathrm{i}(\mathsf{E}^{\mathrm{Fyn}} - \mathsf{G}^{\mathrm{adv}})^*\big) = \emptyset$, each of these is in $C^\infty\big(\mathcal{M} \times \mathcal{M}; \mathrm{Hom}(\mathscr{E}, \mathscr{E})\big)$. Therefore $-\,\mathrm{i}(\mathsf{E}^{\mathrm{Fyn}} - \mathsf{G}^{\mathrm{adv}})$ can be normalised so that it is symmetric.

To address the positivity, first consider the operator D_1 as in Section 3.3.3 and observe that $\mathrm{i}\,F_1$ is positive as evident from Example 3.2.11:

$$\mathrm{i}\,\mathsf{F}_1(\bar{u}^* \otimes u) \geq 0. \tag{3.100}$$

To account $\hat{W}$, one notes that the Lagrangian distributions A and A^* in Theorem 3.3.13 preserve wave front sets and thus, one obtains $(x_0, \xi^0; 0, \eta_1 \mathrm{d}y^1) \notin \mathrm{WF}'(\hat{W} - AF_1 A^*)$ from parametrix differences. But it is not sufficient to construct a positive $\hat{W}$ on a conic neighbourhood in $\dot{\mathrm{T}}^*\mathcal{M} \times \dot{\mathrm{T}}^*\mathcal{M}$ of the inverse image in C of a neighbourhood of any point in $\dot{\mathrm{T}}_0^*\mathcal{M}/C$. This compels a refined version microlocalisation and hence respective homogeneous Darboux theorem.

The space of lightlike (unparametrised) geodesic strips is a $2d-3$-dimensional smooth conic manifold which can be naturally identified with the cotangent unit sphere bundle $\mathbb{S}^*\Sigma$ of any Cauchy hypersurface Σ with respect to the induced metric h on Σ [93]. We choose L, an elliptic symmetric operator of order $-1/2$ with scalar principal symbol and vanishing subprincipal symbol as in Theorem 3.3.13. One can always choose the principal symbol of L such that $X_{L\Box L}$ is a complete vector field [35, Thm. 6.4.3, p. 234] (see also [23, Lem. 26.1.11, 26.1.12] and [27, Prop. 2.1]). Since the bundle of forward/backward lightlike covectors $\dot{\mathrm{T}}^*_{0,\pm}\mathscr{M}$ are connected components of the lightcone bundle $\dot{\mathrm{T}}^*_0\mathscr{M}$, there exists a homogeneous diffeomorphism $\dot{\mathrm{T}}^*_0\mathscr{M} \to (\mathbb{S}^*\Sigma \oplus \mathbb{S}^*\Sigma) \times \mathbb{R} \times \mathbb{R}_+$ such that $X_{L\Box L}$ is mapped to the vector field $\partial/\partial s$ when the variables in $\mathbb{S}^*\Sigma \times \mathbb{R} \times \mathbb{R}_+$ are denoted by $(\cdot, s, \cdot)$. In this setting, a refined Darboux theorem yields that, for every lightlike covector (x_0, ξ^0) on $\mathscr{M}$, there is an open conic coordinate chart $\big(\mathcal{U}, (x^i, \xi_i)\big)$ and a homogeneous symplectomorphism

$$\kappa : \mathcal{U} \to \mathcal{U}', \ (x,\xi) \mapsto \kappa(x,\xi) := \big(y^1(x,\xi), \ldots, y^d(x,\xi); \eta_1(x,\xi), \ldots, \eta_d(x,\xi)\big), \quad (3.101)$$

bijectively mapping $\mathcal{U}$ to an open conic neighbourhood $\mathcal{U}'$ of $\kappa(x_0, \xi^0) := (0, \eta_1 \mathrm{d}y^1)$ in $\dot{\mathrm{T}}^*\mathbb{R}^d$ such that

$$\big(\kappa^{-1}\big)^* p = \xi_1 \, \mathbb{1}_{\mathrm{End}\,\mathscr{E}}, \quad (3.102)$$

$\mathrm{Char}\,\mathrm{D}_1 := \{(y,\eta) \in \mathbb{R}^d \times \dot{\mathbb{R}}^d \mid y^1 = 0\}$ is symmetric with respect to the plane $y^1 = 0$ and convex in the direction of y^1-axis, and $\mathcal{U}' \cap \mathrm{Char}\,\mathrm{D}_1$ is invariant under the translation along the y^1-axis. Since $\mathrm{d}\kappa^{-1}(X_{L^*\Box L}) = \mathbb{1}_{\mathbb{C}^{k\times k}}\, \partial/\partial y^1$, it follows that $\mathcal{U} \cap \mathrm{Char}\,\Box$ is invariant under $X_\Box$ [35, Lem. 6.6.3].

Suppose that $\mathcal{V}_1, \mathcal{V}_2$ are closed conic neighbourhoods of (x_0, ξ^0) such that $\mathcal{V}_1 \subset \mathcal{U}$ and $\mathcal{V}_2 \subset \mathrm{int}(\mathcal{V}_1)$ while $\mathcal{V}_\iota \cap \dot{\mathrm{T}}^*_0\mathscr{M}, \iota = 1, 2$ are invariant under the geodesic flow and that $\mathcal{V}'_\iota := \kappa(\mathcal{V}_\iota)$ are convex in the y^1-direction and symmetric about the plane $y^1 = 0$. Remark 3.3.14 then entails that there exists a properly supported Lagrangian distribution $\mathsf{A} \in I^{-1/2}\big(\mathscr{M} \times \mathbb{R}^d, \Gamma'_1; \mathrm{Hom}(\mathbb{C}^k, \mathscr{E})\big)$ so that A^*A and AA^* both are pseudodifferential operators of order -1 and

$$\mathrm{Char}\,\mathsf{A} \cap \Gamma'_2 = \emptyset, \quad (3.103\mathrm{a})$$

$$\mathrm{ES}(A^*A - I) \cap \mathcal{V}'_2 = \emptyset \Leftrightarrow \mathrm{ES}(AA^* - I_\mathscr{E}) \cap \mathcal{V}_2 = \emptyset, \quad (3.103\mathrm{b})$$

$$\mathrm{WF}'(\Box\mathsf{A} - \mathsf{A}\mathrm{D}_1) \cap \Gamma_2 = \emptyset, \quad (3.103\mathrm{c})$$

where Γ_1 and Γ_2 are the graphs of the restriction of κ to $\mathcal{V}_1$ and to $\mathcal{V}_2$, respectively.

It follows from Theorem 3.3.1 and (3.103) that we can microlocally conjugate $\hat{\mathsf{W}}$ to $\mathrm{i}\,\mathsf{F}_1$ (see [35, p. 237] for the scalar version):

$$\mathrm{WF}'(\hat{\mathsf{W}} - \mathsf{A}\,\mathrm{i}\,\mathsf{F}_1\mathsf{A}^*) \cap (\mathcal{V}_2 \times \mathcal{V}_2) = \emptyset, \quad \mathcal{V}_2 \cap \dot{\mathrm{T}}^*_0\mathscr{M} \subset \{\dot{\mathrm{T}}^*_{0,\pm}\mathscr{M}\}. \quad (3.104)$$

Lemma 3.3.20. *For a normally hyperbolic operator $\square$ as in Proposition 3.3.2 and Lagrangian distributions A given by (3.103), let $\Psi \in \Psi\mathrm{DO}^0(\mathcal{M}; \mathcal{E})$ such that $\mathrm{ES}\,[\Psi, \square]_- \cap \dot{\mathrm{T}}_0^*\mathcal{M} = \emptyset, \mathrm{ES}\,\Psi \subset \mathcal{V}_2$ and $\dot{\mathrm{T}}_0^*\mathcal{M} \cap \mathcal{V}_2 \subset \dot{\mathrm{T}}_{0,\pm}^*\mathcal{M}$. Then*

$$\hat{\mathsf{W}}\Psi\Psi^* \equiv \Psi\mathsf{A}\,\mathrm{i}\,\mathsf{F}_1\mathsf{A}^*\Psi^* \Leftrightarrow \hat{\mathsf{W}}\Psi\Psi^* = \Psi\mathsf{A}\,\mathrm{i}\,\mathsf{F}_1\mathsf{A}^*\Psi^* \quad \mathrm{mod}\ C^\infty(\mathcal{M}\times\mathcal{M}; \mathrm{Hom}(\mathcal{E}, \mathcal{E})). \quad (3.105)$$

Proof. The idea of the proof is to use microlocalisation to reduce this to the special case D_1. To reach the conclusion our main task is then to compute the relevant commutators which is straightforward and exactly analogous to the scalar counterpart [35, Lem. 6.6.5]. So, we only sketch the main steps for completeness. The ellipticity (3.103a) allows one to relate any properly supported $\Psi \in \Psi\mathrm{DO}^0(\mathcal{M}; \mathcal{E})$ and $\Psi' \in \Psi\mathrm{DO}^0(\mathbb{R}^d, \mathbb{C}^{k\times k})$ with $\mathrm{ES}\,\Psi \subset \mathcal{V}_2$ and $\mathrm{ES}\,\Psi' \subset \mathcal{V}_2'$ by

$$\Psi' = A^*\Psi A \quad \mathrm{mod}\ \Psi\mathrm{DO}^{-\infty}(\mathbb{R}^d, \mathbb{C}^{k\times k}), \quad \Psi = A\Psi'A^* \quad \mathrm{mod}\ \Psi\mathrm{DO}^{-\infty}(\mathcal{M}; \mathcal{E}). \quad (3.106)$$

A direct computation shows $\mathrm{ES}\,[\Psi', \mathsf{D}_1]_- \cap \mathrm{Char}\,\mathsf{D}_1 = \emptyset$ which implies that the derivative of the symbol of Ψ' with respect to y^1 is of order $-\infty$ in a neighbourhood of $\mathrm{Char}\,\mathsf{D}_1$. Denoting the convolution by the Dirac measure at $(h, 0, \ldots, 0)$ by τ_h, we can rewrite F_1 as $\mathsf{F}_1 = \int \tau_h \mathrm{d}h$ and observe that $\tau_h\Psi' - \Psi'\tau_h$ is of order $-\infty$ in a conic neighbourhood of $\mathrm{Char}\,\mathsf{D}_1$. Since $\mathrm{Char}\,\mathsf{D}_1$ is invariant under translation in the y^1-direction, one has

$$\forall v \in \mathcal{E}'(\mathbb{R}^d, \mathbb{C}^k) : \mathrm{WF}([\tau_h, \Psi']_- v) \cap \mathrm{Char}\,\mathsf{D}_1 = \emptyset. \quad (3.107)$$

The result follows after the integration with respect to h and a few straightforward algebraic manipulations. ∎

We observe that $\Psi\mathsf{A}\,\mathrm{i}\,\mathsf{F}_1\mathsf{A}^*\Psi^*$ is non-negative by (3.100) since AA^* is non-negative with respect to the hermitian form on $\mathcal{E}$. Thus the only thing left to conclude Proposition 3.3.2 is that the identity can be expressed as a sum of operators of the form $\Psi\Psi^*$ discussed in Lemma 3.3.20, provided the prerequisites in this lemma are satisfied. Aiming to show this, we consider a closed conic neighbourhood $\mathcal{V}_3$ of any lightlike covector (x_0, ξ^0) where $\mathcal{V}_3 \subset \mathrm{int}(\mathcal{V}_2)$ and $\mathcal{V}_3 \cap \dot{\mathrm{T}}_0^*\mathcal{M}$ is invariant under the geodesic flow. One can prove that there exists $\Psi \in \Psi\mathrm{DO}^0(\mathcal{M}; \mathcal{E})$ with $\mathrm{ES}\,\Psi \subset \mathcal{V}_2$ such that Ψ has a non-negative principal symbol equals to $\mathbb{1}$ on $\mathcal{V}_3$ and $\mathrm{ES}\,[\Psi, \square]_- \cap \dot{\mathrm{T}}_0^*\mathcal{M} = \emptyset$ [35, Lem. 6.6.6]. In other words, the hypotheses in Lemma 3.3.20 can be satisfied for Ψ. Taking a suitable cover of $\dot{\mathrm{T}}_0^*\mathcal{M}$ one can use these operators to construct a family of operators $\Psi_\alpha \in \Psi\mathrm{DO}^0(\mathcal{M}; \mathcal{E})$ satisfying

$$\mathrm{ES}\,\Psi_\alpha \bigcap \dot{\mathrm{T}}_0^*\mathcal{M} \subset \dot{\mathrm{T}}_{0,\pm}^*\mathcal{M}, \quad (3.108a)$$

$$\mathrm{ES}\left(I - \sum_\alpha \Psi_\alpha\Psi_\alpha^*\right) \bigcap \dot{\mathrm{T}}_{0,\pm}^*\mathcal{M} = \emptyset, \quad (3.108b)$$

$$\mathrm{ES}\,[\square, \Psi_\alpha]_- \bigcap \dot{\mathrm{T}}_0^*\mathcal{M} = \emptyset. \quad (3.108c)$$

The construction of such a family that provides a microlocal partition of unity (see the Footnote 2.3.6 in the proof of Theorem 2.3.16 for this concept) is carried out in detail in [35, p. 238]. The proof follows the usual strategy of summing the operators and multiplying from left and right by the root of a parametrix. One can verify that the hypotheses in Lemma 3.3.20 are preserved under this construction. To finalise the proof of Proposition 3.3.2 we simply write

$$\hat{W} = \sum_\alpha \hat{W} \Psi_\alpha \Psi_\alpha^* \equiv \sum_\alpha \Psi_\alpha A_\alpha \, i \, F_1 A_\alpha^* \Psi_\alpha^*, \tag{3.109}$$

which terminates the proof as every term on the right hand side of (3.109) is non-negative with respect to the hermitian form on $\mathscr{E}$.

3.3.6   Feynman propagators from Feynman parametrices

We postpone to proof Theorem 3.3.3 in order to make the following observations. Let us denote a point in $\mathscr{M}$ as $(t,x) \in \mathbb{R} \times \Sigma$. Then we can define the distributions $f_1 \otimes \delta_\Sigma$, $f_0 \otimes \delta_\Sigma' \in \mathcal{E}'(\mathscr{M};\mathscr{E})$ by

$$f_1 \otimes \delta_\Sigma(\phi) := \int_\Sigma \phi(x) \, f_1(x) \, dv_h(x), \quad f_0 \otimes \delta_\Sigma'(\phi) := \int_\Sigma (\partial_N \phi)(x) \, f_0(x) \, dv_h(x), \tag{3.110}$$

where $\phi \in C_c^\infty(\mathscr{M};\bar{\mathscr{E}}^*)$, N is a future directed unit normal vector field on $\mathscr{M}$ along Σ, and dv_h is the Riemannian volume form. We denote by $\mathcal{W}_\Sigma \subset \mathcal{E}'(\mathscr{M};\mathscr{E})$, the span of the set of distributions of this form. By a duality argument both retarded and advanced Green's operators extend to continuous maps $\mathcal{E}'(\mathscr{M};\mathscr{E}) \to \mathcal{D}'(\mathscr{M};\mathscr{E})$, and if G is the causal propagator and $f := f_0 \otimes \delta_\Sigma' + f_1 \otimes \delta_\Sigma$, then $u = G(f)$ is a smooth solution of $\Box u = 0$ with Cauchy data (f_0, f_1) on Σ (see e.g. [73, Lem 3.2.2], [121, Thm. 3.20]).

Assume that $w\Box$ and $\bar{\Box}^* w$ are smooth for any bidistribution $w \in \mathcal{D}'(\mathscr{M} \times \mathscr{M}; \mathscr{E} \boxtimes \bar{\mathscr{E}}^*)$. Let us denote by $\mathcal{E}'_{\Sigma^{\perp *}}(\mathscr{M};\mathscr{E})$, the set of compactly supported distributions with wavefront set contained in the conormal bundle (see Example A.1.7) $\Sigma^{\perp *}$ of Σ. This space is endowed with a natural locally convex topology [78, p. 125] with respect to which $C_c^\infty(\mathscr{M};\mathscr{E})$ is sequentially dense in $\mathcal{E}'_{\Sigma^{\perp *}}(\mathscr{M};\mathscr{E})$ (see e.g. [76, Thm. 8.2.3] or the exposition [80, Sec. 4.3.1] for details). The bidistribution w can be defined as a sequentially continuous bilinear form on $\mathcal{E}'_{\Sigma^{\perp *}}(\mathscr{M};\mathscr{E})$. Since $\mathcal{W}_\Sigma \subset \mathcal{E}'_{\Sigma^{\perp *}}(\mathscr{M};\mathscr{E})$, the form w is then also defined on $\mathcal{W}_\Sigma$.

Lemma 3.3.21. *Let $(\mathscr{E} \to \mathscr{M}, (\cdot|\cdot))$ be a hermitian vector bundle over a globally hyperbolic manifold (M,g) whose Cauchy hypersurface is Σ. Suppose that $\Box$ is a normally hyperbolic operator on $\mathscr{E}$ and that $w \in \mathcal{D}'(\mathscr{M} \times \mathscr{M}, \mathscr{E} \boxtimes \bar{\mathscr{E}}^*)$ is a bisolution of $\Box$. Then, w is non-negative with respect to $(\cdot|\cdot)$ if and only if $w(\bar{f}^* \otimes f) \geq 0$ for any $f \in \mathcal{W}_\Sigma$, where $\mathcal{W}_\Sigma$ is the span of the distribution Cauchy data (3.110) of $\Box$.*

Proof. We begin with the fact that w extends to a separately sequentially continuous bilinear form on $\mathcal{E}'_{\Sigma^{\perp *}}(\mathcal{M}; \bar{\mathcal{E}}^*) \times \mathcal{E}'_{\Sigma^{\perp *}}(\mathcal{M}; \mathcal{E})$ as it is a bisolution; let us denote the extension by the same letter. As this form is a bisolution it vanishes on $\bar{\Box}^* C^\infty_c(\mathcal{M}; \bar{\mathcal{E}}^*) \times C^\infty_c(\mathcal{M}; \mathcal{E})$ and also on $C^\infty_c(\mathcal{M}; \bar{\mathcal{E}}^*) \times \Box C^\infty_c(\mathcal{M}; \mathcal{E})$. By sequential continuity and sequential density of C^∞_c in $\mathcal{E}'_{\Sigma^{\perp *}}$, it follows that w vanishes on $\bar{\Box}^* \mathcal{E}'_{\Sigma^{\perp *}}(\mathcal{M}; \bar{\mathcal{E}}^*) \times \mathcal{E}'_{\Sigma^{\perp *}}(\mathcal{M}; \mathcal{E})$ and on $\mathcal{E}'_{\Sigma^{\perp *}}(\mathcal{M}; \bar{\mathcal{E}}^*) \times \Box \mathcal{E}'_{\Sigma^{\perp *}}(\mathcal{M}; \mathcal{E})$. We will show that any $u \in C^\infty_c(\mathcal{M}; \mathcal{E})$ can be written as $u = f + \Box \phi$ for some $f \in \mathcal{W}_\Sigma$ and some $\phi \in \mathcal{E}'_{\Sigma^{\perp *}}(\mathcal{M}; \mathcal{E})$, and the analogous statement holds for the conjugate-dual bundle and the conjugate-dual normally hyperbolic operator. This obviously implies the statement.

To see that this is indeed the case, observe that $\Box(Gu) = 0$ and therefore, if (f_0, f_1) is the Cauchy data of $G(u)$ on Σ and f the corresponding element in $\mathcal{W}_\Sigma$, then we have $G(u) = G(f)$. This implies that

$$G(u - f) = 0 \Leftrightarrow G^{\text{ret}}(u - f) = G^{\text{adv}}(u - f). \tag{3.111}$$

Thus $\phi := G^{\text{ret}}(u - f)$ must be a compactly supported distribution because $\operatorname{supp} \phi = \operatorname{supp}\big(G^{\text{ret}}(u - f)\big) \cap \operatorname{supp}\big(G^{\text{adv}}(u - f)\big) \subset J^+\big(\operatorname{supp}(u - f)\big) \cap J^-\big(\operatorname{supp}(u - f)\big)$ is compact due to the global hyperbolicity of $\mathcal{M}$. By propagation of singularities (Theorem 3.3.16) this distribution must be again in $\mathcal{E}'_{\Sigma^{\perp *}}(\mathcal{M}; \mathcal{E})$ and we have $u - f = \Box \phi$. The same proof works for the bundle $\bar{\mathcal{E}}^*$ and the operator $\bar{\Box}^*$. ∎

Finally, we are prepared to show the existence of Feynman propagators of $\Box$. Moreover, if there exists a hermitian form on $\mathcal{E}$ then Feynman propagators can be chosen non-negative with respect to this form and hence implying existence of 2-point bidistributions corresponding to Hadamard states.

Proof of Theorem 3.3.3. Let $\hat{W}$ be the parametrix of $\Box$ constructed in Proposition 3.3.2. Since the wavefront set of $\hat{W}$ does not intersect with the conormal bundle $\Sigma^{\perp *}$ of any Cauchy hypersurface Σ in $\mathcal{M}$, we fix a Σ and restrict $\hat{W}$ and its normal derivative to create distributional Cauchy data on $\Sigma \times \Sigma$. Suppose that $\mathcal{E}$ is endowed with a hermitian form $(\cdot | \cdot)$ such that $\hat{W}$ is non-negative with respect to $(\cdot | \cdot)$. Then the restriction $\hat{W}_\Sigma$ of $\hat{W}$ on Σ is non-negative as well. Next we construct $W \in \mathcal{D}'(\mathcal{M} \times \mathcal{M}; \mathcal{E} \boxtimes \bar{\mathcal{E}}^*)$ employing the solution operator of $\Box$ with the same distributional Cauchy data as $\hat{W}$ on $\Sigma \times \Sigma$. This will be a bisolution of $\Box$ with the property

$$\big(\forall u \in \mathcal{E}'_{\Sigma^{\perp *}}(\mathcal{M}; \mathcal{E}) : W(\bar{u}^* \otimes u) \geq 0\big) \Leftrightarrow W \geq 0, \tag{3.112}$$

by the sequential density of $C^\infty_c(\Sigma; \mathcal{E}_\Sigma)$ in $\mathcal{E}'_{\Sigma^{\perp *}}(\mathcal{M}; \mathcal{E})$ and Lemma 3.3.21. Note that $R = W - \hat{W}$ is smooth since it solves the inhomogeneous problem with zero Cauchy data, which entails that

$$G^{\text{Fyn}} := E^{\text{Fyn}} + i R \tag{3.113}$$

is the Feynman propagator with the required positivity property. ∎

3.3.7 Causal propagator

In this section we will inscribe the causal propagator

$$G := G^{\mathrm{ret}} - G^{\mathrm{adv}} : C^\infty_c(\mathcal{M}; \mathcal{E}) \to C^\infty_{\mathrm{sc}}(\mathcal{M}; \mathcal{E}) \tag{3.114}$$

of a normally hyperbolic operator $\square$ on a vector bundle $\mathcal{E} \to \mathcal{M}$ over a globally hyperbolic spacetime as a Fourier integral operator. The result has been proven by Duistermaat-Hörmander [35, Thm. 6.6.1] for any scalar pseudodifferential operator of real principal type, but a systematic derivation for the bundle version is not so easy to find.

Remark 3.3.22. There is a natural density dv_C on the geodesic relation C and hence on the forward/backward geodesic relations $C^\pm$, as originally due to Duistermaat and Hörmander [35, p. 230] for a generic manifold M, which simplifies considerably for a globally hyperbolic spacetime $(\mathcal{M}, \mathrm{g})$ as reported by Strohmaier and Zelditch [27, (52), Rem. 7.1]. By Definition 3.2.15: for each $(x, \xi; y, \eta) \in C$ there is a unique $s \in \mathbb{R}$ such that $(x, \xi) = \Phi_s(y, \eta)$ so that C can be identified with an open subset of $\mathbb{R} \times \dot{\mathrm{T}}^*_0\mathcal{M}$, where Φ_s is the geodesic flow on $\mathrm{T}^*\mathcal{M}$ and $s \in \mathbb{R}$ is the flow parameter. $\dot{\mathrm{T}}^*_0\mathcal{M}$ is a conic contact manifold where the Hamiltonian

$$H_{\mathrm{g}} : C^\infty(\mathcal{M}; \mathrm{T}^*\mathcal{M}) \to \mathbb{R}, \ (x, \xi) \mapsto H_{\mathrm{g}}(x, \xi) := \frac{1}{2}\mathrm{g}^{-1}_x(\xi, \xi) \tag{3.115}$$

induced by the spacetime metric g vanishes identically and its Hamiltonian reduction is the the space of scaled-lightlike geodesics $\mathcal{N}$ [122, pp. 10-12], [123, Thm. 2.1]. Following Strohmaier-Zelditch, we have used the prefactor $1/2$ in the preceding definition so that the relativistic Hamiltonian flow is identical to the geodesic flow. Denoting by $\tilde{s}$, the dilation parameter on $\dot{\mathrm{T}}^*_0\mathcal{M}$, the natural half-density on C is given by

$$\sqrt{|\mathrm{dv}_C|} := \sqrt{|\mathrm{d}s|} \otimes \sqrt{|\mathrm{d}\tilde{s}|} \otimes \sqrt{|\mathrm{dv}_{\mathcal{N}}|}. \tag{3.116}$$

Note, this density differs from that by Duistermaat-Hörmander by a factor of 2 because they used the Hamiltonian flow of g^{-1} to parametrise $\mathcal{N}$, in contrast to the flow of the Hamiltonian vector field $X_{\mathrm{g}/2}$ generated by H_{g}. Moreover,

$$\pounds_{X_{\mathrm{g}}}\mathrm{dv}_C = 0. \tag{3.117}$$

The densities on $C^\pm$ follow from the fact $\dot{\mathrm{T}}^*_0\mathcal{M} = \dot{\mathrm{T}}^*_{0,+}\mathcal{M} \sqcup \dot{\mathrm{T}}^*_{0,-}\mathcal{M}$ in $d \geq 3$.

   Employing Theorem 3.3.1, Definition 3.3.6, Remark 3.3.22 and Theorem 2.3.13, we imitate the proof for the scalar version [35, Thm. 6.6.1] to obtain

Theorem 3.3.23. *Let $\mathcal{E} \to \mathcal{M}$ resp. $\pi : \dot{\mathrm{T}}^*\mathcal{M} \to \mathcal{M}$ be a vector bundle resp. the punctured cotangent bundle over a globally hyperbolic spacetime $(\mathcal{M}, \mathrm{g})$ and $\square$ a normally hyperbolic*

operator on $\mathscr{E}$ whose principal symbol is the spacetime metric g^{-1} on $\mathrm{T}^*\mathscr{M}$. Then, the Schwartz kernel G of the causal propagator for $\square$ is the Lagrangian distribution

$$\mathsf{G} \in I^{-3/2}\big(\mathscr{M} \times \mathscr{M}, C'; \mathrm{Hom}(\mathscr{E}, \mathscr{E})\big), \tag{3.118a}$$

$$\sigma_{\mathsf{G}} = \frac{\mathrm{i}}{2}\sqrt{2\pi|\mathrm{d}\mathrm{v}_C|} \otimes \mathbbm{m}w, \tag{3.118b}$$

$$\mathrm{Char}\,\mathsf{G} \cap C = \emptyset, \tag{3.118c}$$

where $\mathrm{d}\mathrm{v}_C$ is the natural volume form on the geodesic relation C, $\mathbbm{m}$ is a section of the Keller-Maslov bundle $\mathbb{M} \to C$ (as constructed in [35, pp. 231-232]), and w is the unique element of $C^\infty\big(C; \pi^*\mathrm{Hom}(\mathscr{E}, \mathscr{E})_C\big)$ that is diagonally the identity endomorphism and off-diagonally covariantly constant

$$\nabla^{\pi^*\mathrm{Hom}(\mathscr{E},\mathscr{E})}_{X_{\mathrm{g}/2}}w = 0 \tag{3.119}$$

with respect to the $\square$-compatible Weitzenböck covariant derivative $\nabla^{\pi^*\mathrm{Hom}(\mathscr{E},\mathscr{E})}_{X_{\mathrm{g}/2}}$ along the geodesic vector field $X_{\mathrm{g}/2}$.

Remark 3.3.24. By Definition 3.3.6, X_{g} acts on $C^\infty\big(\mathrm{T}^*\mathscr{M}; \pi^*\mathrm{Hom}(\mathscr{E}, \mathscr{E})\big)$. Thus (3.119) must be interpreted in the sense of the induced Hamiltonian vector field on C by the vector field $(X_{\mathrm{g}}, 0)$ on $(\mathrm{T}^*\mathscr{M} \times \mathrm{T}^*\mathscr{M})$ [35, Rem. 1, p. 216] (see also [23, Rem. 1, p. 69]).

3.4 Feynman propagators for a Dirac operator

In this section we will extend our analysis for Dirac-type operators. These are the first-order differential operators whose squares are normally hyperbolic, and arguably the most fundamental first-order operators in geometric analysis as studied in the context of the celebrated Atiyah-Singer index theorem.

3.4.1 Dirac-type operators

Definition 3.4.1. Let $\mathscr{E} \to \mathscr{M}$ be a vector bundle over a Lorentzian manifold $(\mathscr{M}, \mathrm{g})$. A **Dirac-type operator** is a first-order differential operator

$$D : C^\infty(\mathscr{M}; \mathscr{E}) \to C^\infty(\mathscr{M}; \mathscr{E}) \tag{3.120}$$

whose principal symbol σ_D satisfies the Clifford relation (see e.g. [124, Def. 2.18])

$$\sigma_D(x, \xi)^2 = \mathrm{g}_x^{-1}(\xi, \xi)\,\mathbbm{1}_{\mathrm{End}\,\mathscr{E}_x} \tag{3.121}$$

for all $(x, \xi) \in C^\infty(\mathscr{M}; \mathrm{T}^*\mathscr{M})$.

In other words, D^2 is a normally hyperbolic operator and by the polarisation identity (see e.g. [124, Rem. 2.19])

$$[\sigma_D(x, \xi), \sigma_D(x, \eta)]_+ = 2\,\mathrm{g}_x^{-1}(\xi, \eta)\,\mathbbm{1}_{\mathrm{End}\,\mathscr{E}_x} \tag{3.122}$$

for any $(x, \xi), (x, \eta) \in C^\infty(\mathcal{M}; T^*\mathcal{M})$ and where $[\cdot, \cdot]_+$ is the anticommutator bracket. The principal symbol σ_D defines a Clifford action of $T^*\mathcal{M}$ on $\mathscr{E}$ by

$$\forall f \in C^\infty(\mathcal{M}) : \sigma_D(\mathrm{d}f) := \mathrm{i}\,[D, f]_-, \tag{3.123}$$

which is independent of any chosen element $f \in C^\infty(\mathcal{M})$. This turns $\mathscr{E} \to \mathcal{M}$ into a bundle of Clifford modules $(\mathscr{E} \to \mathcal{M}, \sigma_D)$ over $(\mathcal{M}, g)$. Furthermore, the hindmost equation yields the Leibniz rule for D:

$$D(fu) = \sigma_D(\mathrm{d}f)\,(u) + f D(u) \tag{3.124}$$

for any $f \in C^\infty(\mathcal{M})$ and any $u \in C^\infty(\mathcal{M}; \mathscr{E})$. The Clifford mapping (3.122) defines the (pointwise) Clifford multiplication

$$\mathsf{c} : C^\infty(\mathcal{M}; T^*\mathcal{M} \otimes \mathscr{E}) \to C^\infty(\mathcal{M}; \mathscr{E}), \ (x, \xi) \otimes u \mapsto \mathsf{c}\big((x, \xi) \otimes u\big) := \sigma_D(x, \xi)\,(u). \tag{3.125}$$

Assumption 3.4.2. We endow the vector bundle $\mathscr{E} \to \mathcal{M}$ over a Lorentzian manifold $(\mathcal{M}, g)$ with a sesquilinear form $(\cdot|\cdot)$ such that D is symmetric.

Given any D, there exists a unique $(\cdot|\cdot)$-compatible Weitzenböck connection $\nabla^\mathscr{E}$ on $\mathscr{E}$ and a unique potential $V \in C^\infty(\mathcal{M}; \mathrm{End}\,\mathscr{E})$ such that

$$D^2 = -\mathrm{tr}_g(\nabla^{\mathrm{LC}\otimes\mathscr{E}} \circ \nabla^\mathscr{E}) + V, \tag{3.126}$$

by the Weitzenböck formula (Remark 3.2.8).

The most general Dirac-type operator on the Clifford module bundle $\big(\mathscr{E} \to \mathcal{M}, (\cdot|\cdot), \sigma_D\big)$ then differs from the composition of the following two maps

$$C^\infty(\mathcal{M}; \mathscr{E}) \xrightarrow{\nabla^\mathscr{E}} C^\infty(\mathcal{M}; T^*\mathcal{M} \otimes \mathscr{E}) \xrightarrow{\mathsf{c}} C^\infty(\mathcal{M}; \mathscr{E}) \tag{3.127}$$

only by a zero-order term, i.e.,

$$D = -\mathrm{i}\,\mathsf{c} \circ \nabla^\mathscr{E} + U, \tag{3.128}$$

where the potential term $U \in C^\infty(\mathcal{M}; \mathrm{End}\,\mathscr{E})$ is defined by the preceding equation. Note, Assumption 3.4.2 entails that U and $-\mathrm{i}\,\nabla_X^\mathscr{E}$ are symmetric with respect to $(\cdot|\cdot)$ provided that X is divergence free.

The Weitzenböck connection $\nabla^\mathscr{E}$ induces a connection $\nabla^{\mathrm{Hom}(\mathscr{E},\mathscr{E})}$ on the vector bundle $\mathrm{Hom}(\mathscr{E}, \mathscr{E}) \to \mathcal{M}$ by

$$\nabla_X^{\mathrm{Hom}(\mathscr{E},\mathscr{E})}\big(\sigma_D(x, \xi)\big)(u) := \nabla_X^\mathscr{E}\big(\sigma_D(x, \xi)\,(u)\big) - \sigma_D(x, \xi)\big(\nabla_X^\mathscr{E}(u)\big) \tag{3.129}$$

for all $X \in T_x\mathcal{M}$ and all $\xi \in T_x^*\mathcal{M}$ at any $x \in \mathcal{M}$. In general, $\nabla^{\mathrm{Hom}(\mathscr{E},\mathscr{E})}$ is *not* a Clifford connection because using the Levi-Civita connection we have

$$\begin{aligned}
X\big(g_x^{-1}(\xi, \eta)\big) &= g_x^{-1}\big(\nabla_X^{\mathrm{LC}}\xi, \eta\big) + g_x^{-1}\big(\xi, \nabla_X^{\mathrm{LC}}\eta\big) \\
&= \frac{1}{2}\big[\sigma_D\big(x, \nabla_X^{\mathrm{LC}}\xi\big), \sigma_D(x, \eta)\big]_+ + \frac{1}{2}\big[\sigma_D(x, \xi), \sigma_D\big(x, \nabla_X^{\mathrm{LC}}\eta\big)\big]_+, \tag{3.130}
\end{aligned}$$

and then by taking the covariant derivative of (3.122) we obtain

$$\left[\nabla_X^{\mathrm{Hom}(\mathscr{E},\mathscr{E})}\big(\sigma_D(x,\xi)\big) - \sigma_D\big(x,\nabla_X^{\mathrm{LC}}\xi\big), \sigma_D(x,\eta)\right]_+ = \left[\nabla_X^{\mathrm{Hom}(\mathscr{E},\mathscr{E})}\big(\sigma_D(x,\eta)\big) - \sigma_D\big(x,\nabla_X^{\mathrm{LC}}\eta\big), \sigma_D(x,\xi)\right]_+$$

(3.131)

for all $\eta \in \mathrm{T}_x^*\mathscr{M}$ and others as before. In particular, we can always locally choose an orthonormal frame $\{e_i\}$ and its coframe $\{\varepsilon^i\}$ on $\mathscr{M}$ and express

$$D = -\mathrm{i}\,\slashed{\nabla}^{\mathscr{E}} + U, \quad \slashed{\nabla}^{\mathscr{E}} := \sigma_D(\varepsilon^i)\nabla_{e_i}^{\mathscr{E}}, \tag{3.132}$$

where the Feynman-slash notation $\slashed{\nabla}^{\mathscr{E}}$ has been used, i.e., $\nabla^{\mathscr{E}}$ is composed with Clifford multiplication and then traced over. A straightforward computation yields

$$D^2 = -\slashed{\nabla}^{\mathscr{E}}\slashed{\nabla}^{\mathscr{E}} - \mathrm{i}\,\slashed{\nabla}^{\mathscr{E}}(U) + U^2 - \left(\slashed{\nabla}^{\mathrm{Hom}(\mathscr{E},\mathscr{E})}\big(\sigma_D(\varepsilon^j)\big) + \mathrm{i}\,U\sigma_D(\varepsilon^j) + \mathrm{i}\,\sigma_D(\varepsilon^j)U\right)\nabla_{e_j}^{\mathscr{E}}. \tag{3.133}$$

But

$$
\begin{aligned}
\slashed{\nabla}^{\mathscr{E}}\slashed{\nabla}^{\mathscr{E}} &= \frac{1}{2}\big(\slashed{\nabla}^{\mathscr{E}}\slashed{\nabla}^{\mathscr{E}} + \slashed{\nabla}^{\mathscr{E}}\slashed{\nabla}^{\mathscr{E}}\big) \\
&= \frac{1}{2}\big(\sigma_D(\varepsilon^i)\,\sigma_D(\varepsilon^j)(\nabla_{e_i}^{\mathscr{E}}\nabla_{e_j}^{\mathscr{E}} - \Gamma_{ij}^k\nabla_{e_k}^{\mathscr{E}}) + \sigma_D(\varepsilon^j)\,\sigma_D(\varepsilon^i)(\nabla_{e_j}^{\mathscr{E}}\nabla_{e_i}^{\mathscr{E}} - \Gamma_{ji}^k\nabla_{e_k}^{\mathscr{E}})\big) + \slashed{\Gamma}^k\nabla_{e_k}^{\mathscr{E}} \\
&= \frac{1}{2}\big(\sigma_D(\varepsilon^i)\,\sigma_D(\varepsilon^j)\,\mathrm{Hess}_{e_i,e_j}^{\mathscr{E}} + \sigma_D(\varepsilon^j)\,\sigma_D(\varepsilon^i)\,\mathrm{Hess}_{e_j,e_i}^{\mathscr{E}}\big) + \slashed{\Gamma}^k\nabla_{e_k}^{\mathscr{E}} \\
&= \frac{1}{2}\big(\sigma_D(\varepsilon^i)\,\sigma_D(\varepsilon^j)(\mathrm{Hess}_{e_j,e_i}^{\mathscr{E}} + \mathscr{R}_{e_i e_j}) + \sigma_D(\varepsilon^j)\,\sigma_D(\varepsilon^i)\,\mathrm{Hess}_{e_j,e_i}^{\mathscr{E}}\big) + \slashed{\Gamma}^k\nabla_{e_k}^{\mathscr{E}} \\
&= \mathrm{g}^{ij}\,\mathrm{Hess}_{e_j,e_i}^{\mathscr{E}} + \frac{1}{2}\mathscr{R} + \slashed{\Gamma}^k\nabla_{e_k}^{\mathscr{E}},
\end{aligned}
$$

(3.134)

where $\mathscr{R}$ resp. Γ are the curvature resp. connection 1-form of $\nabla^{\mathscr{E}}$. Here, we have used (3.20), (3.18), (3.122) and $\mathscr{R}_{X,Y} = \mathrm{Hess}_{X,Y} - \mathrm{Hess}_{Y,X}$. Inserting the preceding equation into the last equation for D^2 yields

$$
\begin{aligned}
D^2 &= -\mathrm{tr}_{\mathrm{g}}(\nabla^{\mathrm{LC}\otimes\mathscr{E}} \circ \nabla^{\mathscr{E}}) + \mathfrak{R} + U^2 - \mathrm{i}\,\slashed{\nabla}^{\mathscr{E}}(U) \\
&\quad -\left(\slashed{\nabla}^{\mathrm{Hom}(\mathscr{E},\mathscr{E})}\big(\sigma_D(\varepsilon^j)\big) + \slashed{\Gamma}^j + \mathrm{i}\,U\sigma_D(\varepsilon^j) + \mathrm{i}\,\sigma_D(\varepsilon^j)U\right)\nabla_{e_j}^{\mathscr{E}},
\end{aligned}
$$

(3.135)

where

$$\mathfrak{R} : C^\infty(\mathscr{M};\mathscr{E}) \to C^\infty(\mathscr{M};\mathscr{E}), \ u \mapsto \mathfrak{R}u := -\frac{1}{2}\big(\sigma_D(X^\flat)\,\sigma_D(Y^\flat)\,\mathscr{R}_{X,Y}\big)u. \tag{3.136}$$

is the **Weitzenböck curvature** of $\nabla^{\mathscr{E}}$. Since $\nabla^{\mathscr{E}}$ is a Weitzenböck connection, comparing with (3.126) we equate the coefficients of $\nabla_{e_j}^{\mathscr{E}}$ in the foregoing equation of D^2 to zero to obtain the covariant derivative of σ_D:

$$\slashed{\nabla}^{\mathrm{Hom}(\mathscr{E},\mathscr{E})}\big(\sigma_D(\varepsilon^j)\big) = -\slashed{\Gamma} - \big[U, \sigma_D(\varepsilon^j)\big]_+, \quad j = 1,\ldots,d. \tag{3.137}$$

Remark 3.4.3. A Clifford connection $\tilde{\nabla}^{\mathrm{Hom}(\mathscr{E},\mathscr{E})}$, characterised by

$$\forall (x,X) \in C^\infty(\mathscr{M}; \mathrm{T}\mathscr{M}), \forall (x,\eta) \in C^\infty(\mathscr{M}; \mathrm{T}^*\mathscr{M}) : \left[\tilde{\nabla}_X^{\mathrm{Hom}(\mathscr{E},\mathscr{E})}, \sigma_D(x,\eta) \right]_- = \sigma_D\big(x, \nabla_X^{\mathrm{LC}}\eta\big), \tag{3.138}$$

always exists and then the Clifford module bundle $(\mathscr{E} \to \mathscr{M}, \sigma_D, \tilde{\nabla}^{\mathscr{E}})$ is called the **bundle of Dirac modules**, where the most general Dirac-type operator has the form

$$\tilde{D} := -\mathrm{i}\, \mathsf{c} \circ \tilde{\nabla}^{\mathscr{E}} + \tilde{U} \tag{3.139}$$

for all $\tilde{U} \in C^\infty(\mathscr{M}; \mathrm{End}\,\mathscr{E})$ such that (cf. (3.137))

$$[\tilde{U}, \sigma_D(\cdot)]_- = 0. \tag{3.140}$$

The operator $\tilde{D}$ has the same principal symbol as D so that they differ only by a smooth term. Since $\tilde{\nabla}$ is compatible with the Clifford multiplication (3.125), $\tilde{D}$ is sometimes called **compatible Dirac-type operator**.

Example 3.4.4. Let $(\mathscr{M}, \mathrm{g})$ be a d-dimensional spin-spacetime, i.e., a spacetime $(\mathscr{M}, \mathrm{g})$ whose tangent bundle admits a spin-structure. This means, there exists a pair $(\mathscr{P}, \Theta)$ where $(\mathscr{P}, \mathrm{Spin}_0, \mathscr{M})$ is a principal $\mathrm{Spin}_0(1, d-1)$-bundle and $\Theta : \mathscr{P} \to \mathscr{Q}$ is a twofold covering of the bundle $(\mathscr{Q}, \mathrm{SO}_0, \mathscr{M})$ of positively oriented and time-oriented tangent frames, such that the following diagram commutes [125] (see also, e.g. [126])

Figure 3.4: Lorentzian spin-structure on a spacetime. The horizontal lines are the group actions on the principal bundles.

Here, $\mathrm{Spin}_0(1, d-1)$ resp. $\mathrm{SO}_0(1, d-1)$ are the connected components to the identity of the Lorentzian spin group $\mathrm{Spin}(1, d-1)$ resp. the special orthogonal group $\mathrm{SO}(1, d-1)$ and $\mathrm{Ad} : \mathrm{Spin}_0(1, d-1) \to \mathrm{SO}_0(1, d-1)$ is the twofold covering map.

A spin structure always exists locally but its global existence depends on the topology of $\mathscr{M}$. A necessary and sufficient condition is that the *second Stieffel-Whitney class of $\mathscr{M}$ vanishes* [31] (Riemannian signature); see [125, 127–130] for the Lorentzian setting. Under this topological restriction, the number of inequivalent spin structures on $\mathscr{M}$ is equal to the number of elements in the cohomology $\mathrm{H}^1(\mathscr{M}; \mathbb{Z}_2)$ [31, Lem.] (Riemannian signature; see [131] for a 4-dimensional spacetime). Then, for a fixed spin-structure $(\mathscr{P}, \Theta)$, the **spinor bundle** is defined as the associated vector bundle

$$\mathscr{S} := \mathscr{P} \times_{\tilde{\gamma}} \mathcal{S}_d, \tag{3.141}$$

where $\mathcal{S}_d$ is the complex spinor module. When d is *even*, $\dim_{\mathbb{C}} \mathcal{S}_d = 2^{d/2}$, the spinor module decomposes $\mathcal{S}_d = \mathcal{S}_d^+ \oplus \mathcal{S}_d^-$ into the submodules $\mathcal{S}_d^{\pm}$ of positive(negative) chirality and

$$\tilde{\gamma} = \tilde{\gamma}^+ \oplus \tilde{\gamma}^- : \mathrm{Spin}_0(1, d-1) \to \mathrm{Aut}\, \mathcal{S}_d^+ \times \mathrm{Aut}\, \mathcal{S}_d^- \subset \mathrm{Aut}\, \mathcal{S}_d \qquad (3.142)$$

is the *spinor representation*. For an *odd d*, $\dim_{\mathbb{C}} \mathcal{S}_d = 2^{(d-1)/2}$ and

$$\tilde{\gamma} : \mathrm{Spin}_0(1, d-1) \to \mathrm{Aut}\, \mathcal{S}_d \qquad (3.143)$$

is the *spinor representation*. Sections of a spinor bundle are called the **spinors**. In other words, for each $x \in \mathcal{M}$, the fibre $\mathscr{S}_x$ of the spinor bundle $\mathscr{S}$ over x consists of equivalence classes of pairs $[p, s]$ subject to the free left $\mathrm{Spin}_0(1, d-1)$-group action on $\mathscr{P} \times \mathcal{S}_d$:

$$\forall p \in \mathscr{P}, \forall u \in \mathcal{S}_d, \forall g \in \mathrm{Spin}_0(1, d-1) : g \bullet [p, u] = [pg^{-1}, \tilde{\gamma}(g)\, u]. \qquad (3.144)$$

Since $\mathrm{T}\mathcal{M}$ sets inside $\mathrm{Spin}_0(1, d-1)$, $\tilde{\gamma}$ induces an $\mathrm{End}\, \mathcal{S}_d$-valued map γ on $\mathrm{T}\mathcal{M}$:

$$\gamma(x, X)^2 := \mathrm{g}_x(X, X)\; \mathbb{1}_{\mathrm{End}\, \mathcal{S}_d}. \qquad (3.145)$$

This enables us to define the Clifford multiplication

$$\mathrm{c} : C^\infty(\mathcal{M}; \mathrm{T}\mathcal{M} \otimes \mathscr{S}) \to C^\infty(\mathcal{M}; \mathscr{S}),\; \mathrm{c}\big([\Theta(p), X] \otimes [p, s]\big) := [p, \gamma(X)\, s], \qquad (3.146)$$

where we have expressed $\mathrm{T}\mathcal{M} = \mathcal{Q} \times_\rho \mathbb{R}^d$ as a real vector bundle associated to $\mathcal{Q}$ by the standard representation ρ of $\mathrm{SO}_0(1, d-1)$ on $\mathbb{R}^d$.

Employing the Levi-Civita connection 1-from Γ^{LC} on $\mathcal{Q}$, we define the spinor connection 1-form $\Gamma^{\mathscr{S}} := \mathrm{Ad}_*^{-1} \circ \Theta^*(\Gamma^{\mathrm{LC}})$ which induces a connection $\nabla^{\mathscr{S}}$ on $\mathscr{S}$. This spin connection is a *metric* and *Clifford* connection. Moreover, it leaves the positive-negative chiral splitting invariant and satisfies the Leibniz rule. The **massive spin-Dirac operator** is defined by [125, Def. 3.1]

$$D := -\mathrm{i}\, \mathrm{c} \circ \nabla^{\mathscr{S}} + \mathrm{m} : C^\infty(\mathcal{M}; \mathscr{S}) \to C^\infty(\mathcal{M}; \mathscr{S}), \qquad (3.147)$$

where m is a parameter which can be interpreted as the physical mass of a spinor in appropriate spacetime geometry.

Remark 3.4.5. If $(\mathcal{M}, \mathrm{g})$ is a 4-dimensional non-compact spacetime then the sufficient and necessary condition to admit a spin-structure entails the existence of a global system of orthonormal tetrads [127, Thm.]. In particular, a number of exact solutions of Einstein equation, e.g., the Schwartzschild, the Robertson-Walker, the Gödel, the plane waves, and any *globally hyperbolic spacetime* admit at least a spin-structure [128]. Furthermore, the spinor bundle is trivial yet there may exists inequivalent spin-structures $(\mathscr{P}, \Theta)$ (given by number of elements in $\mathrm{H}^1(\mathcal{M}; \mathbb{Z}_2)$) encoded in Θ [131].

3.4.2   Green's operators

Since D^2 is normally hyperbolic, it admits unique advanced G^{adv} and retarded G^{ret} Green's operators on any globally hyperbolic spacetime $(\mathscr{M}, \mathrm{g})$ [132, Thm. 1]. Hence

$$F^{\mathrm{adv,ret}} := DG^{\mathrm{adv,ret}} : C_{\mathrm{c}}^{\infty}(\mathscr{M};\mathscr{E}) \to C_{\mathrm{sc}}^{\infty}(\mathscr{M};\mathscr{E}) \tag{3.148}$$

are the unique advanced resp. retarded Green's operators for D. As before,

$$F := F^{\mathrm{ret}} - F^{\mathrm{adv}} : C_{\mathrm{c}}^{\infty}(\mathscr{M};\mathscr{E}) \to C_{\mathrm{sc}}^{\infty}(\mathscr{M};\mathscr{E}) \tag{3.149}$$

defines the Pauli-Jordan-Lichnerowicz Green's operator, also known as the *causal propagator* for D. Assumption 3.4.2 and Remark 3.2.13 entail that

$$\forall u, v \in C_{\mathrm{c}}^{\infty}(\mathscr{M};\mathscr{E}) : (F^{\mathrm{adv,ret}}u|v) = (u|F^{\mathrm{ret,adv}}v), \quad (Fu|v) = -(u|Fv). \tag{3.150}$$

Lemma 3.4.6. *Let $\mathscr{E} \to \mathscr{M}$ (resp. $\pi : \dot{\mathrm{T}}^*\mathscr{M} \to \mathscr{M}$) be a vector bundle (resp. the punctured cotangent bundle) over a globally hyperbolic spacetime $(\mathscr{M}, \mathrm{g})$ and D a Dirac-type operator on $\mathscr{E}$ whose principal symbol is denoted by σ_D. The Schwartz kernel F of the causal propagator of D is then*

$$\mathsf{F} \in I^{-1/2}\big(\mathscr{M} \times \mathscr{M}, C'; \mathrm{Hom}(\mathscr{E}, \mathscr{E})\big), \tag{3.151a}$$

$$\sigma_{\mathsf{F}} = \mathrm{i}\sqrt{\frac{\pi}{2}}\,\sigma_D \circ w\sqrt{|\mathrm{dv}_C|} \otimes \mathbb{1}, \tag{3.151b}$$

where dv_C is the natural volume form on the geodesic relation (Definition 3.2.15) C, $\mathbb{1}$ is a section of the Keller-Maslov bundle $\mathbb{L}_C \to C$ (as constructed in [35, pp. 231-232]), and w is the unique element of $C^{\infty}\big(C; \pi^*\mathrm{Hom}(\mathscr{E}, \mathscr{E})_C\big)$ that is diagonally the identity endomorphism and off-diagonally covariantly constant

$$\nabla_{X_{\mathrm{g}/2}}^{\pi^*\mathrm{Hom}(\mathscr{E},\mathscr{E})} w = 0 \tag{3.152}$$

with respect to the D^2-compatible Weitzenböck covariant derivative (Definition 3.3.6) $\nabla_{X_{\mathrm{g}/2}}^{\pi^*\mathrm{Hom}(\mathscr{E},\mathscr{E})}$ along the geodesic vector field $X_{\mathrm{g}/2}$ (Remark 3.3.24).

If $(\mathscr{E} \to \mathscr{M}, \sigma_D, \tilde{\nabla}^{\mathscr{E}})$ is a bundle of Dirac modules with the corresponding Dirac-type operator $\tilde{D}$ then the preceding holds with the replacement of $\nabla^{\pi^*\mathrm{Hom}(\mathscr{E},\mathscr{E})}$ in (3.152) by the $\tilde{D}^2$-compatible connection $\tilde{\nabla}^{\pi^*\mathrm{Hom}(\mathscr{E},\mathscr{E})}$ induced by the Clifford connection $\tilde{\nabla}^{\mathscr{E}}$.

Proof. This follows from Theorem 3.3.23 and the facts that D is a first-order differential operator, $\mathrm{Char}\, D = \dot{\mathrm{T}}_0^*\mathscr{M}$, and $\mathsf{F}(x, \cdot) = D_x\mathsf{G}(x, \cdot)$. ∎

3.4.3 Cauchy problem

Since D^2 is a normally hyperbolic operator and $(\mathcal{M}, \mathrm{g})$ is a globally hyperbolic spacetime, the Cauchy problem for D is well-posed [132, Thm. 2]. In other words, for an arbitrary but fixed Cauchy hypersurface $\Sigma_t \subset \mathcal{M}$, the mapping

$$\mathcal{R}_t : \ker D \to C_c^\infty(\Sigma_t; \mathscr{E}_{\Sigma_t}), \ u \mapsto \mathcal{R}_t(u) := u \restriction \Sigma_t, \quad \mathrm{supp}\, u \subset J\big(\mathrm{supp}(u \restriction \Sigma_t)\big) \quad (3.153)$$

is a homeomorphism, where J is defined by (3.2). In order to endow $\ker D$ with a hermitian inner product, we make use of the following

Assumption 3.4.7. Given an arbitrary but fixed future-directed unit normal covector field $(\cdot, \zeta)$ on a globally hyperbolic spacetime $(\mathcal{M}, \mathrm{g})$ along any Cauchy hypersurface Σ,

$$\langle \cdot | \cdot \rangle := \big(\sigma_D(\cdot, \zeta) \cdot \big| \cdot \big) \quad (3.154)$$

is a fibrewise hermitian form on the bundle of Clifford modules $\big(\mathscr{E} \to \mathcal{M}, \sigma_D, (\cdot|\cdot)\big)$. The hermitian form $\langle \cdot | \cdot \rangle$ depends on the choice of $(\cdot, \zeta)$.

By global hyperbolicity (Section 3.1.2) of $(\mathcal{M}, \mathrm{g})$, there exists a (non-unique) global Cauchy temporal function t such that each Cauchy hypersurface $\Sigma_t := \mathsf{t}^{-1}(t)$ for any $t \in \mathbb{R}$ is a level set of t. Then

$$\zeta := \frac{\mathrm{d}\mathsf{t}}{\|\mathrm{d}\mathsf{t}\|} \quad (3.155)$$

is a unit normal covector field on $\mathcal{M}$ along Σ_t and we choose the time-orientation such that ζ is future-directed.

Employing the preceding topological isomorphism $\mathcal{R}_t$, $\ker D$ can be equipped with a hermitian inner product (see e.g. [124, Lem. 3.17]):

$$\langle u | v \rangle := \int_\Sigma \Big(\mathsf{c}\big((x, \zeta) \otimes (u \restriction \Sigma)\big)\big| v \restriction \Sigma\Big)_x \mathrm{d}v_\mathsf{h}(x), \quad (3.156)$$

where $\mathrm{d}v_\mathsf{h}$ is the Riemannian volume element on Σ. We remark that $\langle \cdot | \cdot \rangle$ is independent of the chosen Cauchy hypersurface Σ due to the Green-Stokes formula (see (3.166)). Therefore, $(\mathscr{E} \to \mathcal{M}, \langle \cdot | \cdot \rangle)$ is a *hermitian vector bundle*, of course *depending* on $(\cdot, \zeta)$, but *independent* of chosen Σ.

We infer from the preceding equation that, given an initial data $k \in C_c^\infty(\Sigma; \mathscr{E}_\Sigma)$, any smooth solution u of the Dirac equation (1.18) can be expressed as (see [133, Prop. 2.4 (b)] for a spin Dirac operator)

$$u = -\mathrm{i}\, F \circ (\iota_\Sigma^*)^{-1}\, \mathsf{c}(\zeta \otimes k), \quad (3.157)$$

where the restriction operator

$$\iota_\Sigma^* : C_c^\infty(\mathcal{M}; \mathscr{E}) \to C_c^\infty(\Sigma; \mathscr{E}_\Sigma) \quad (3.158)$$

has been discussed elaborately in Example 2.3.12 and we have taken into account Remark 3.1.7. All the maps in the exact complex

$$0 \to C_c^\infty(\mathcal{M};\mathscr{E}) \xrightarrow{D} C_c^\infty(\mathcal{M};\mathscr{E}) \xrightarrow{F} C_{sc}^\infty(\mathcal{M};\mathscr{E}) \xrightarrow{D} C_{sc}^\infty(\mathcal{M};\mathscr{E}) \qquad (3.159)$$

are sequentially continuous as a consequence of D being a local operator and the following exact complex being sequentially continuous (see e.g. [73, Prop. 3.4.8])

$$0 \to C_c^\infty(\mathcal{M};\mathscr{E}) \xrightarrow{D^2} C_c^\infty(\mathcal{M};\mathscr{E}) \xrightarrow{G} C_{sc}^\infty(\mathcal{M};\mathscr{E}) \xrightarrow{D^2} C_{sc}^\infty(\mathcal{M};\mathscr{E}). \qquad (3.160)$$

3.4.4 Primary result

Since D^2 is a normally hyperbolic operator and the first hypothesis in Theorem 3.3.3 is satisfied by Assumption 3.4.2, one obtains Feynman propagator F^{Fyn} for D immediately by setting

$$F^{\mathrm{Fyn}} := DG^{\mathrm{Fyn}}, \qquad (3.161)$$

where G^{Fyn} is the Feynman propagator for D^2. Since $\mathrm{Char}\, D = \dot{\mathrm{T}}_0^*\mathcal{M}$, one has

$$\mathrm{WF}'(\mathsf{F}^{\mathrm{adv,ret,Fyn,aFyn}}) = \mathrm{WF}'(\mathsf{G}^{\mathrm{adv,ret,Fyn,aFyn}}) = \varDelta\dot{\mathrm{T}}^*\mathcal{M} \bigcup C^{\mathrm{adv,ret,+,-}}, \qquad (3.162)$$

where the canonical relations are defined in (3.45) and in Definition 3.2.15. Nevertheless, the existence of Hadamard states cannot be deduced due to the positivity issue as $(\cdot|\cdot)$ is *not* definite. One may wish to consider symmetricness of D with respect to some hermitian form $\langle\cdot|\cdot\rangle$ so that the hypotheses of Theorem 3.3.3 are satisfied in such a way that the existence of Hadamard states can be concluded. The natural inner product on spinors on a spacetime is not positive-definite rather that is *indefinite* and therefore the positivity property of Feynman propagators for a spin-Dirac operator (Example 3.4.4) cannot be inferred directly deploying Theorem 3.3.3. In other words, the requirement of symmetricness with respect to $\langle\cdot|\cdot\rangle$ turns out to be too restrictive to encompass all Dirac-type operators on globally hyperbolic manifolds (see Section 3.5.5 and 3.5.6 for concrete examples) and so we refrain to impose this condition.

Theorem 3.4.8. *Let $\mathscr{E} \to \mathcal{M}$ be a vector bundle over a globally hyperbolic spacetime $(\mathcal{M}, \mathrm{g})$, endowed with a non-degenerate sesquilinear form $(\cdot|\cdot)$ and D a Dirac-type operator on $\mathscr{E}$ whose principal symbol is given by σ_D. Suppose that $(\mathscr{E} \to \mathcal{M}, \sigma_D, (\cdot|\cdot))$ is a bundle of Clifford modules over $(\mathcal{M}, \mathrm{g})$ and that D is symmetric with respect to $(\cdot|\cdot)$. Then, there exists a Feynman propagator F^{Fyn} for D such that $W := -\mathrm{i}(F^{\mathrm{Fyn}} - F^{\mathrm{adv}})$ is symmetric, where F^{adv} is the advanced propagator for D. In addition, if there exists a hermitian form $\langle\cdot|\cdot\rangle$ on $\mathscr{E}$ in the sense of Assumption 3.4.7, then F^{Fyn} can be chosen such that W is non-negative with respect to $(\cdot|\cdot)$ and hence defines a Hadamard bisolution of D.*

Proof. To begin with, recall that $\operatorname{Char} D^2 = \dot{\mathrm{T}}_0^*\mathscr{M}$ has only two connected components in three or higher spacetime dimensions[5]: the forward lightcone $\dot{\mathrm{T}}_{0,+}^*\mathscr{M}$ and the backward lightcone $\dot{\mathrm{T}}_{0,-}^*\mathscr{M}$. Therefore, C has four different orientations[6] [35, p. 218] (see also [39, p. 540]): $(C^+, C^-), (C^-, C^+), (C^{\mathrm{adv}}, C^{\mathrm{ret}}), (C^{\mathrm{ret}}, C^{\mathrm{adv}})$ corresponding to $\{\dot{\mathrm{T}}_0^*\mathscr{M}, \dot{\mathrm{T}}_{0,+}^*\mathscr{M}, \dot{\mathrm{T}}_{0,-}^*\mathscr{M}, \emptyset\}$. Th -is paves a way to have a (non-unique) microlocal decomposition of the causal propagator G for D^2 as

$$\mathsf{G} \equiv \mathsf{E}^+ - \mathsf{E}^-, \tag{3.163a}$$

$$\mathsf{E}^\pm D^2 \equiv D^2 \mathsf{E}^\pm \equiv 0, \tag{3.163b}$$

$$\mathsf{E}^\pm \in I^{-3/2}\big(\mathscr{M} \times \mathscr{M}, C^{\pm\prime}; \operatorname{Hom}(\mathscr{E}, \mathscr{E})\big), \tag{3.163c}$$

where $\equiv$ means modulo smoothing kernels. Clearly, this is a slightly stronger version of Theorem 3.3.1, which can be proven exactly in the same fashion; cf Theorem 3.3.23.

To achieve the analog of the proceeding results for D, we remember $\operatorname{Char} D = \operatorname{Char} D^2$ and decompose F as $\mathsf{S}^+ + \mathsf{S}^-$ where

$$\mathsf{S}^\pm := \sum_\alpha Q_\alpha^\pm \mathsf{F} Q_\alpha^\pm \tag{3.164}$$

and $(Q_\alpha^\pm)^* Q_\alpha^\pm$ are microlocal partition of unity such that $DQ_\alpha^\pm \equiv Q_\alpha^\pm D$ and $\mathrm{ES}\, Q_\alpha^\pm = \varDelta\,\dot{\mathrm{T}}_{0,\pm}^*\mathscr{M}$. Consequently,

$$\mathsf{F} \equiv \mathsf{S}^+ + \mathsf{S}^-, \tag{3.165a}$$

$$\mathsf{S}^\pm D \equiv D\mathsf{S}^\pm \equiv 0, \tag{3.165b}$$

$$\mathsf{S}^\pm \in I^{-1/2}\big(\mathscr{M} \times \mathscr{M}, C^{\pm\prime}; \operatorname{Hom}(\mathscr{E}, \mathscr{E})\big), \tag{3.165c}$$

by an application of (2.128). The operators $Q_\alpha^\pm$ can be constructed by choosing any $\mathsf{q}_\alpha^\pm \in S^{0-[\infty]}\big(\dot{\mathrm{T}}^*\mathscr{M}, \operatorname{Hom}(\mathscr{E}, \mathscr{E})\big)$ as its left total symbol such that $\operatorname{esssupp} \mathsf{q}_\alpha^\pm = \dot{\mathrm{T}}_{0,\pm}^*\mathscr{M}$. In other words $\mathsf{q}_\alpha^\pm$ is of order $-\infty$ near $y \in \mathscr{M}$ and in a conic neighbourhood of η for all (y, η) in the complement of $\operatorname{esssupp} \mathsf{q}^\pm$. Furthermore, we pick $\operatorname{supp} \mathsf{q}_\alpha^\pm$ slightly away from the projection of $\mathrm{ES}\, Q_\alpha^\pm$ on M so that $[D, Q_\alpha^\pm]_-$ is smooth.

Lemma 3.4.9. *As in the terminologies of Theorem 3.4.8, let F be the causal propagator for D. Then $\mathrm{i}\, F$ is non-negative with respect to $(\cdot|\cdot)$.*

Proof. Let $u, v \in C^\infty(\mathscr{M}; \mathscr{E})$ having compact $\operatorname{supp} u \cap \operatorname{supp} v$ and let $K \subset \mathscr{M}$ be a compact set having smooth boundary ∂K with an outward unit normal covector field ζ and volume element dv. Then, the Green-Stokes formula yields (see e.g. [134, Prop. 9.1, (p. 178)]), [133, (1.7)])

$$\int_{x \in K} \big((Du|v)_x - (u|Dv)_x\big)\mathrm{dv}_{\mathrm{g}}(x) = -\mathrm{i} \int_{x' \in \partial K} \big(\sigma_D(\cdot, \zeta)\, u|v\big)_{x'}\mathrm{dv}(x'), \tag{3.166}$$

---

[5] In two dimensions, $\dot{\mathrm{T}}_0^*\mathscr{M}$ has four connected components; cf. Remark 3.2.18.
[6] By an orientation of the geodesic relation C, it is meant that any splitting of $C \setminus \varDelta\dot{\mathrm{T}}_0^*\mathscr{M}$ into a disjoint union of open $C^1, C^2 \subset C$ which are inverse relations [35, p. 218].

where dv_{g} is the Lorentzian volume (Remark 2.2.1 (d)) form on $\mathcal{M}$. Setting $u := F^{\mathrm{ret}}w$ and $v := Fw$ for any $w \in C_c^\infty(\mathcal{M}; \mathcal{E})$, the above formula entails $(w | \mathrm{i}\, Fw) \geq 0$ due to the compactness of $\mathrm{supp}(F^{\mathrm{ret}}u) \subset J^+(\mathrm{supp}\, u)$.   ∎

Therefore, $(u | \mathrm{i}\, S^\pm u) \geq 0$ by (3.165a). Note, $C^\pm = C \cap (\dot{\mathrm{T}}_{0,+}^* \mathcal{M} \times \dot{\mathrm{T}}_{0,+}^* \mathcal{M})$. Inspecting the (twisted) wavefront sets of F^{Fyn} and F^{adv} as before in the proof of Proposition 3.3.2, we have the analogue of Proposition 3.3.2 for Dirac-type operators:

$$(u | \hat{W}u) \geq 0, \quad \hat{W} := \mathrm{i}(F^{\mathrm{adv}} - F^{\mathrm{Fyn}}). \tag{3.167}$$

As in the proof of Theorem 3.3.3, in order to turn $\hat{W}$ into an exact distributional bisolution W, we are going to employ the well-posedness of the Cauchy problem for D on a globally hyperbolic spacetime [132, Thm. 2]. For any $k \in C_c^\infty(\Sigma; \mathcal{E}_\Sigma)$, we define distributional Cauchy data $k \otimes \delta_\Sigma$ by

$$k \otimes \delta_\Sigma(\phi) := \int_{x \in \Sigma} \phi(x)\, \mathsf{c}(\zeta \otimes k)_x \, \mathrm{dv_h}(x) \tag{3.168}$$

for any $\phi \in C^\infty(\mathcal{M}; \mathcal{E}^*)$. By $\mathcal{Y}_\Sigma \subset \mathcal{E}'_{\Sigma^{\perp *}}(\mathcal{M}; \mathcal{E})$, we denote the set of distributions of the above form where $\mathcal{E}'_{\Sigma^{\perp *}}(\mathcal{M}; \mathcal{E})$ is the set of compactly supported distributions whose wavefront set is contained in the conormal bundle $\Sigma^{\perp *}$. Any bidistribution $\mathrm{w} \in \mathcal{D}'(\mathcal{M} \times \mathcal{M}; \mathcal{E} \boxtimes \bar{\mathcal{E}}^*)$ such that $\mathrm{w}D$ and $D\mathrm{w}$ are smooth in distributional sense, can be defined as a sequential continuous bilinear form on $\mathcal{Y}_\Sigma$.

Lemma 3.4.10. *As in the terminologies of Theorem 3.4.8, suppose that* $\mathrm{w} \in \mathcal{D}'(\mathcal{M} \times \mathcal{M}; \mathcal{E} \boxtimes \bar{\mathcal{E}}^*)$ *is a bisolution of D. Then $(u|\mathrm{w}u) \geq 0$ if and only if $\mathrm{w}(\bar{u}^* \otimes u) \geq 0$ for all $u \in \mathcal{Y}_\Sigma$, where $\mathcal{Y}_\Sigma$ is the span of the distributional Cauchy data (3.168) of D.*

Proof. The proof carries over its analogue Lemma 3.3.21 after the obvious replacement of $D^2(= \square)$, its Cauchy data (3.110), and causal propagator G, by D, its Cauchy data (3.168), and causal propagator F, respectively.   ∎

                                                                                          ∎

3.5 Examples

We list below several operators arising in the context of quantum field theories in a curved spacetime and discuss how they fit into our framework.

3.5.1 Covariant Klein-Gordon operator

As described in Example 3.2.2, the natural inner product on $C^\infty(\mathcal{M})$ is positive-definite with respect to which $\square$ is symmetric. Therefore, Theorem 3.3.3 applies and positive Feynman propagators can be constructed using this method.

3.5.2 Connection d'Alembert operator

A Feynman propagate exists for □ (Example 3.2.5) by Theorem 3.3.3. The positivity issue depends on the positivity of the inner product with respect to which □ is symmetric.

3.5.3 Hodge-d'Alembert operator

A Feynman propagator exists for this operator defined in Example 3.2.4 by Theorem 3.3.3. Since the natural inner product with respect to which this operator is symmetric, is not positive-definite unless $k = 0$ or $k = d$, we cannot conclude non-negativity in a straightforward way. For example, for $k = 1$ non-negativity is only expected on a subset as is usual for gauge theories.

3.5.4 Proca operator

As in the terminologies of Example 3.2.7, a Feynman propagator for $d^*d + dd^* + m^2$ can be constructed by this method and the positivity issue is same as in Section 3.5.3.

3.5.5 Twisted spin-Dirac operator

Let $(\mathscr{S} \to \mathscr{M}, \nabla^{\mathscr{S}})$ resp. $(\mathscr{E} \to \mathscr{M}, \nabla^{\mathscr{E}})$ be a spinor (resp. vector) bundle endowed with the spin $\nabla^{\mathscr{S}}$ (resp. bundle $\nabla^{\mathscr{E}}$) connection, over a globally hyperbolic spin-spacetime $(\mathscr{M}, g)$ (Example 3.4.4 and Remark 3.4.5). These two connections induce another connection $\nabla := \nabla^{\mathscr{S}} \otimes \mathbb{1}_{\mathscr{E}} + \mathbb{1}_{\mathscr{S}} \otimes \nabla^{\mathscr{E}}$ on the twisted spinor bundle $\mathscr{S} \otimes \mathscr{E} \to \mathscr{M}$ and the **twisted Dirac operator** is defined by (see e.g. [125] for details)

$$\displaystyle \not{D} := -\mathrm{i}\, \gamma \circ g^{-1} \circ \nabla : C^\infty(\mathscr{M}; \mathscr{S} \otimes \mathscr{E}) \to C^\infty(\mathscr{M}, \mathscr{S} \otimes \mathscr{E}), \qquad (3.169)$$

where $\gamma : C^\infty(\mathscr{M}; \mathrm{T}\mathscr{M}) \to \operatorname{End}\mathscr{S}$ is the Clifford mapping such that

$$\forall (x, X), (x, Y) \in C^\infty(\mathscr{M}; \mathrm{T}\mathscr{M}) : \gamma(X)\,\gamma(Y) + \gamma(Y)\,\gamma(X) = 2\mathrm{g}_x(X, Y)\, \mathbb{1}_{\operatorname{End}\mathscr{S}}, \qquad (3.170)$$

and the Clifford multiplication is given by

$$C^\infty(\mathscr{M}; \mathrm{T}\mathscr{M} \otimes \mathscr{S} \otimes \mathscr{E}) \ni (x, X) \otimes \psi \otimes u \mapsto (\gamma(X)\psi) \otimes u \in C^\infty(\mathscr{M}; \mathscr{S} \otimes \mathscr{E}). \qquad (3.171)$$

The Schrödinger [135]-Lichnwerowicz [136] formula entails (see e.g. [132, Exm. 1])

$$\not{D}^2 = \nabla^* \nabla + \frac{\mathfrak{R}}{4} + \mathfrak{F} \qquad (3.172)$$

where $\mathfrak{R}$ is the scalar curvature of $\mathscr{M}$ and $\mathfrak{F}$ is the Clifford multiplied curvature of $\nabla^{\mathscr{E}}$.

Clearly, $\not{D}^2$ is a normally hyperbolic operator and thus $\not{D}$ is of Dirac-type. This formula also shows that the *Weitzenböck connection is the twisted spin-connection ∇ on the twisted*

spinor bundle and therefore the induced connection on $\mathrm{Hom}(\mathscr{S} \otimes \mathscr{E}, \mathscr{S} \otimes \mathscr{E}) \to \dot{\mathrm{T}}^*\mathscr{M}$ is to be used for microlocalisation. Since $\not{D}$ is symmetric with respect to the natural inner product $(\cdot | \cdot)$ on $\mathscr{S} \otimes \mathscr{E}$, a Feynman propagator for $\not{D}$ exists by Theorem 3.3.3 yet the existence of Hadamard states cannot be concluded due to the fact that $(\cdot | \cdot)$ is not positive-definite unless $(\mathscr{M}, \mathrm{g})$ is a Riemannian manifold. However, $(\mathscr{M}, \mathrm{g})$ is globally hyperbolic and thus, there exist a smooth spacelike Cauchy hypersurfaces Σ and a future-directed unit vector field N on $\mathscr{M}$ normal to Σ so that $\gamma(\mathrm{N})\cdot$ defines the Clifford multiplication to achieve the hermitian form $\langle \cdot | \cdot \rangle$ (cf. Assumption 3.4.7). Then, Theorem 3.4.8 is applicable and we have Feynman propagators with the desired positivity with respect to $(\cdot | \cdot)$.

3.5.6   Rarita-Schwinger operator

As in the terminologies of Example 3.4.4, let $\mathscr{S} \to \mathscr{M}$ be a spinor bundle over a spin-spacetime $(\mathscr{M}, \mathrm{g})$ and the Clifford multiplication is given by $\mathrm{T}^*\mathscr{M} \otimes \mathscr{S} \ni (x, \xi) \otimes u \mapsto \gamma(\mathrm{g}^\sharp_x(\xi))u \in \mathscr{S}$. Then, we have the representation theoretic splitting

$$\mathrm{T}^*\mathscr{M} \otimes \mathscr{S} = \iota(\mathscr{S}) \oplus \mathscr{S}^{3/2}, \tag{3.173}$$

where $\mathscr{S}^{3/2}$ is defined as the kernel of the Clifford multiplication and the embedding ι of $\mathscr{S}$ into $\mathrm{T}^*\mathscr{M} \otimes \mathscr{S}$ is locally defined by

$$\iota(u) := -\frac{1}{d} e_i \otimes \gamma(e^i)u, \tag{3.174}$$

where $\{e_i\}$ is an orthonormal basis of $\mathrm{T}\mathscr{M}$. Suppose that

$$\not{D} := \mathrm{i}(\mathbb{1} \otimes \gamma) \circ \nabla \tag{3.175}$$

is the twisted Dirac operator on $\mathrm{T}^*\mathscr{M} \otimes \mathscr{S}$, then the **Rarita-Schwinger operator** is defined as [124, Def. 2.25]

$$\mathrm{R} := (I - \iota \circ \gamma) \circ \not{D} : C^\infty(\mathscr{M}; \mathscr{S}^{3/2}) \to C^\infty(\mathscr{M}; \mathscr{S}^{3/2}). \tag{3.176}$$

The characteristic set $\mathrm{Char}\,\mathrm{R}$ of R coincides with the set of lightlike covectors in dimensions $d \geq 3$ (see e.g. [124, Lem. 2.26]) and R is a symmetric differential operator whose *Cauchy problem is well-posed* when $(\mathscr{M}, \mathrm{g})$ is globally hyperbolic albeit R^2 is *not* a normally hyperbolic operator (see e.g [124, Rem. 2.27]). Moreover, a hermitian form (4.17) does *not* exist in $d \geq 3$ [124, Example 3.16].

Originally, Rarita and Schwinger (in Minkowski spacetime) considered the twisted Dirac operator $\not{D}$ restricted to $\mathscr{S}^{3/2}$ but not projected back to $\mathscr{S}^{3/2}$ [137, (1)] (see also [138, Sec. 2], [139, Prop. 2.7] for Riemannian and [124, Rem. 2.28] for Lorentzian spin manifolds), that is,

$$\not{D} \upharpoonright C^\infty(\mathscr{M}; \mathscr{S}^{3/2}) : C^\infty(\mathscr{M}; \mathscr{S}^{3/2}) \to C^\infty(\mathscr{M}; \mathrm{T}^*\mathscr{M} \otimes \mathscr{S}), \tag{3.177}$$

in order to ensure the correct number of propagating degrees of freedom for spin-3/2 fields (see, for instance, the reviews [140, 141] for physical motivation and different approaches used in Physics literature). The corresponding Rarita-Schwinger operator is then an overdetermined system and this constrained system limits possible curvatures of the spacetime [142] (see also, e.g. [138, p. 856], [143]):

$$\left(\text{Ric} - \frac{\text{ric}}{d}\text{g} \right)^* u = 0, \tag{3.178}$$

where Ric resp. ric are the Ricci tensor resp. Ricci scalar curvature of $\mathcal{M}$. In other words, the corresponding Rarita-Schwinger field exists only in *Einstein spin manifolds* in contrast to constraint Hodge-d'Alembert operator (Section 3.5.3) and Proca operator (Section 3.5.4) discussed earlier. However, the Rarita-Schwinger operator corresponding to the restricted twisted spin-Dirac operator (3.177) does *not admit a Green's operator* [124, Rem. 2.28].

Since the Cauchy problem for R is well-posed on a globally hyperbolic spacetime $(\mathcal{M}, \text{g})$, the Rarita-Schwinger operator (3.176) admits unique advanced S^{adv} and retarded S^{ret} Green's operators. Moreover, it also admits Feynman S^{Fyn} and anti-Feynman S^{aFyn} Green's operators as $\text{Char}\,\text{R} = \dot{\text{T}}_0^*\mathcal{M}$ (entailing four possible orientations of the geodesic relation). These can be constructed by assuming the construction of the causal Green's operator $S := S^{\text{ret}} - S^{\text{adv}}$ and following the same strategy (cf. the proof of Theorem 3.4.8) as in the case of a Dirac-type operator. However, the positivity issue is non-conclusive.

3.5.7 Higher spin operators

The straightforward attempts to generalise Dirac operator on Minkowski spacetime for arbitrary spin [144] in curved spacetimes[7] leads to difficulties (see e.g [145, p. 324] for a panoramic view and the reviews [140, 141]). A crucial advancement came through Buchdahl operator (in Riemannian manifold) [146] (for a Lorentzian formulation, see [124, Exam. 2.24]) whose square turns out to be a normally hyperbolic operator [124, p. 8], yet the minimum coupling principle seems to be violated and a "by hand" proposal is required in the original idea of Buchdahl. These minor imperfections were cured by Wünsch [147], by Illege [148, 149] and by Illege and Schimming [145] for the massive case deploying the 2-spinor formalism in 4-dimensional curved spacetimes. In particular, the square of Buchdahl operator (as modified by Wünsch and Illge) is a normally hyperbolic operator on certain twisted bundles. In contrast, there are a few open questions for the massless case [150] (see also the reviews [140, 141] for the contemporary status and other formulations used in Physics literature). Hence, the existence of a Feynman propagator is evident either by Theorem 3.3.3 or Theorem 3.4.8 but the issue of positivity is non-conclusive at this stage.

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[7]Not necessarily be globally hyperbolic.

3.6   Literature

The *Lorentzian geometry of spacetime* is quite well-studied and presented in great detail, for instance, in the monographs [89, 111, 115, 151, 152]. The notion of global hyperbolicity was introduced by Jean LERAY [153] to ensure the well-posedness of the Cauchy problem. Later, this class of spacetimes was emphasized in the context of Einstein's general relativity notably by Roger PENROSE [154] by means of his cosmic censorship conjecture (see e.g. [115, pp. 200 - 209, 299 - 308]). The characterisation of a globally hyperbolic manifold in terms of a Cauchy hypersurface is originally due to Robert GEROCH [109] who proved "a topological version" of Theorem 3.1.6. His topological splitting theorem has been sharpened in a series of four articles by Antonio BERNAL and Miguel SÁNCHEZ [110] who, in particular, have achieved an orthogonal smooth splitting in the level of spacetime metric. We refer the survey articles [108, 155] for details.

Fundamental solutions of a normally hyperbolic operator have been constructed in great generality by Jacques HADAMARD [156] and by Marcel RIESZ [157]. Contemporary expositions include, for example [72, 73, 158].

The *Feynman propagator* was introduced by Richard FEYNMAN [159] propounding the idea of Ernst STÜCKELBERG [160] that "particles propagate forward in time whereas antiparticles propagate backward in time" in the context of quantum field theory by means of the so-called time-ordered product (see Appendix C.3). His original definition was not rigorous and can only be understood well in Minkowski spacetime as recalled in Example C.3.2. The mathematical underpinning was given by Johannes J. DUISTERMAAT and Lars HÖRMANDER [35, Sec. 6.6]. In fact, the notion of distinguished parametrices for scalar pseudodifferential operators of real principal type was introduced in their seminal article where they have identified a geometric notion of pseudoconvexity which allows to prove the uniqueness of such parametrices. Although they were motivated and certainly aware of the developments in physics, it was realised only much later by Maciej RADZIKOWSKI [39] that the expectation value of the time-ordered product with respect to a state satisfying the Hadamard condition is the Feynman propagator.

The *spin-Dirac operator* is named after Paul DIRAC [161] who introduced this operator in the context of quantum field theory to describe the dynamics of an electron in Minkowski spacetime. The bundle language was not developed back then and so a geometric formulation of Dirac operator was not available until the precise formulation of spin-structure. Subsequently, Dirac-type operators appear in the work of Michael ATIYAH, Vijay PATODI and Isadore SINGER [162] in their celebrated index theorem. We refer [125, 126] for a discourse on Lorentzian spin-geometry and [114, 163] for a detailed account of Riemannian Dirac-type operators.

Chapter 4

Gutzwiller trace formula

The Duistermaat-Guillemin trace formula of the time-flow on the kernel of a Dirac operator on a spatially compact globally hyperbolic stationary spacetime has been derived in this chapter.

4.1 Trace of a smoothing operator

We recall that a **trace** on an associative algebra $(\mathscr{A}, \cdot)$ over $\mathbb{C}$ is a linear functional

$$\mathrm{Tr} : \mathscr{A} \to \mathbb{C} \quad \text{such that} \quad \forall A, B \in \mathscr{A} : \mathrm{Tr}([A, B]_-) = 0, \tag{4.1}$$

where $[A, B]_- := A \cdot B - B \cdot A$ is the commutator defined by the product $\cdot$ structure on $\mathscr{A}$. In particular, let $\mathscr{E}_\Sigma \to \Sigma$ be a vector bundle over a compact manifold Σ without boundary. Then $\Psi\mathrm{DO}^{-\infty}(\Sigma; \mathscr{E}_\Sigma \otimes \Omega^{1/2}\Sigma)$ is a Fréchet algebra of smoothing operators and

$$\mathrm{Tr} : \Psi\mathrm{DO}^{-\infty}(\Sigma; \mathscr{E}_\Sigma \otimes \Omega^{1/2}\Sigma) \to \mathbb{C}, \ P \mapsto \mathrm{Tr}\,P := \int_M \mathrm{tr}\left(\mathsf{P}(x, x)\right), \tag{4.2}$$

where $\mathsf{P}(x, x)$ is the restriction of the Schwartz kernel P of P to the diagonal in $\Sigma \times \Sigma$ and tr is the endomorphism trace defined by (3.8). The traciality of this trace is the Fubini's theorem and this trace is the unique trace on $\Psi\mathrm{DO}^{-\infty}(\Sigma; \mathscr{E}_\Sigma \otimes \Omega^{1/2}\Sigma)$ (see e.g. [9, Sec. 1.1.7, 4.3.2]).

4.2 Stationary spacetimes

Definition 4.2.1. A spacetime (Definition 3.1.2) $(\mathscr{M}, \mathrm{g})$ is called **stationary** $(\mathscr{M}, \mathrm{g}, \Xi)$ if it admits a global timelike Killing flow Ξ.

A Killing flow $\Xi_s : \mathscr{M} \to \mathscr{M}$, per se, keeps the spacetime metric g invariant:

$$\Xi_s^{\mathrm{T}^*\mathscr{M} \times \mathrm{T}^*\mathscr{M}} \mathrm{g} = 0 \Leftrightarrow \pounds_Z \mathrm{g} = 0 \tag{4.3}$$

for any $s \in \mathbb{R}$, where Z is the infinitesimal generator of Ξ_s, called the *Killing vector field*. In other words, the Killing flow Ξ_s is a *spacetime isometry*. In local coordinates (x^i) on $\mathcal{M}$, $Z = Z^i \partial_i$, and the preceding equation entails the **Killing's equation**:

$$\nabla^{\text{LC}}_i Z_j + \nabla^{\text{LC}}_j Z_i = 0, \quad i, j = 1, \dots, d, \tag{4.4}$$

where ∇^{LC}_X is the Levi-Civita covariant derivative of $(\mathcal{M}, g)$.

Example 4.2.2. Let $(\mathcal{M}, g) = (\mathbb{R}^4, \eta)$ be the 4-dimensional Minkowski spacetime. The set of all isometries of $\mathbb{R}^4$ consists of translations and Lorentz-transformations, and it has a group structure - the *Poincaré group*. In inertial coordinates $(x^i) = (t, x, y, z)$:

- $(Z^i) = (0, 0, 0, 1)$ is the Killing vector field generating the translations in the z-direction.

- $(Z^i) = (0, 0, -z, y)$ is the Killing vector field generating the rotations around the x-axis.

- $(Z^i) = (x, t, 0, 0)$ is the Killing vector field generating the boosts along the x-axis.

The Killing flow is physically interpreted as the *flow of time* which allows a canonical $1 + (d - 1)$ dimensional non-unique topological (resp. local geometric) decomposition of $\mathcal{M}$ (resp. g). This can be achieved either by the projection formalism [164] (see also, e.g. [165, 166]) or by the Kaluza-Klein reduction (see e.g. [152, Sec. XIV.2]). In what follows, we can (and at some point, will) consider a slightly specialised version of stationary spacetimes.

Definition 4.2.3. Let (Σ, h) be a Riemannian manifold. A **standard stationary spacetime** is defined as the triplet $(\mathcal{M}, g, Z)$ where (see e.g. [155, (7.1)], [167, Def. 3.3])

$$\mathcal{M} := \mathbb{R} \times \Sigma \tag{4.5}$$

is the spacetime manifold admitting a complete and timelike Killing vector field Z and a local coordinate chart $(U, (t, x^i))$ such that $Z \restriction U = \partial_t$ and the spacetime metric

$$g \restriction U := \beta^2 \, dt^2 - h_{ij}(dx^i + \alpha^i \, dt)(dx^j + \alpha^j \, dt), \quad i, j = 2, \dots, d, \tag{4.6}$$

where α is the shift vector field and β is the lapse function, both independent of t.

Not all standard stationary spacetimes are globally hyperbolic (Definition 3.1.3) and as argued in Section 3.1.2, we are only interested in those which are. Our desired class is achieved pertaining to the Fermat metric (see e.g. [155, Sec. 7] and references therein)

$$f := \frac{1}{\beta^2 - \|\alpha\|_h^2} h_{ij}\alpha^i dx^j + \sqrt{\frac{1}{\beta^2 - \|\alpha\|_h^2} h_{ij} dx^i dx^j + \left(\frac{h_{ij}\alpha^i dx^j}{\beta^2 - \|\alpha\|_h^2}\right)^2} \tag{4.7}$$

on Σ for a given splitting (4.6). This metric induces a (not necessarily symmetric) distance d_f on Σ which in turn induces forward $\bar{\mathbb{B}}^+(x, r) := \{y \in \mathcal{M} \mid d_f(x, y) \leq r\}$ and backward $\bar{\mathbb{B}}^-(x, r) := \{y \in \mathcal{M} \mid d_f(y, x) \leq r\}$ closed balls. Analogously, one has forward (backward) Cauchy sequences and forward (backward) completeness. Moreover, one defines the closed symmetrised balls $\bar{\mathbb{B}}_s(x, r)$ corresponding to the symmetrised distance $(d_f(x, y) + d_f(y, x))/2$. In terms of these terminologies, the analysis of [168, Thm. 4.3b, 4.4, Cor. 5] can be summarised as

Theorem 4.2.4. *The following properties are equivalent for a standard stationary spacetime* $(\mathcal{M}, g, Z)$ *(see e.g., the reviews [155, Thm. 7.2], [167, Thm. 3.1]).*

- $(\mathcal{M}, g, Z)$ *is globally hyperbolic.*

- *The closed symmetrised balls* $\bar{\mathbb{B}}_s$ *of the Fermat metric (4.7) associated to one (and then to any) standard stationary splitting of* $(\mathcal{M}, g)$ *are compact.*

Furthermore, the slices associated to a standard stationary splitting are Cauchy hypersurfaces if and only if the Fermat metric associated to that splitting is both forward and backward complete.

A sufficient condition is given by

Corollary 4.2.5. *Let* $(\mathcal{M}, g, Z)$ *be a standard stationary spacetime. If* h *is complete and* d_0 *is the distance function on* (Σ, h) *to some (and then any) fixed point* $x_0 \in \Sigma$ *such that*

$$\sup\left\{ \frac{\|\alpha\|_h}{d_0}(x), \frac{\sqrt{\beta^2 - \|\alpha\|_h^2}}{d_0}(x) \mid x \in \Sigma, d_0(x) > 1 \right\} < \infty, \qquad (4.8)$$

then $(\mathcal{M}, g, Z)$ *is globally hyperbolic and all the slices* Σ_t *are Cauchy hypersurfaces.*

In particular, the preceding finite bound holds if α *is sublinear:* $\|\alpha\|_h < \text{cst}_1 d_0(x) + \text{cst}_2$ *and* β *is subquadratic:* $\beta(x) < \text{cst}_3 d_0(x) + \text{cst}_4$ *for some constants (see e.g. [155, Cor. 7.1]).*

Remark 4.2.6. As in the terminologies of Definition 4.2.1 and 4.2.3:

(a) The completeness means that the Killing flow is globally defined

$$\Xi : \mathbb{R} \times \mathcal{M} \to \mathcal{M}, \ (s, x) \mapsto \Xi(s, x) =: \Xi_s(x), \quad d\Xi(s, x; \partial_s, 0) = Z(\Xi(s, x)), \quad (4.9)$$

where Z is the global timelike Killing vector field, locally given by ∂_s. We remark that some authors define stationary spacetimes without imposing the completeness assumption but this is crucial for the thesis.

(b) If one of the timelike Killing vector fields of a globally hyperbolic stationary spacetime is complete then the spacetime is a standard stationary one [169, Thm. 2.3].

(c) The splitting (4.6) is not unique.

(d) A globally hyperbolic stationary spacetime is called **spatially compact** if and only if its Cauchy hypersurface is compact. Such spacetimes are *geodesically complete* [170, Lem. 1.1].

(e) On a generic $d > 2$-dimensional (standard) stationary spacetime the Killing vector field $(\partial_t =)Z$ is not orthogonal to Σ_t because the corresponding 1-form $Z^\flat$ does not satisfy the *hypersurface-orthogonality* condition:

$$Z^\flat \wedge \mathrm{d}Z^\flat \neq 0. \tag{4.10}$$

In other words, the orthogonal geometric distribution of Z is non-involutive. Physically this means that the neighbouring orbits of Z can twist around each other. In $d = 2$ this cannot happen, i.e., every Killing vector field is at least locally hypersurface-orthogonal because there is no freedom to rotate. If one imposes the condition that Σ_t is orthogonal to the orbits of the spacetime isometry, then $(\mathcal{M}, \mathrm{g}, \Xi)$ is called a **static spacetime** and one has a canonical non-unique *global time-coordinate t*. In this case, α vanishes identically so that there is no $\mathrm{d}t\,\mathrm{d}x^i$-type cross terms in (4.6). Additionally, if we demand that Z has a constant norm, then a static spacetime is called an **ultrastatic spacetime**. This enforces β to be the identity function (see e.g. [151, pp. 120-122] [115, (B.3.6) and App. C.3]).

(f) On a d-dimensional Lorentzian manifold, there can be at most $d(d+1)/2$ linearly independent Killing vector fields. In our physical universe, currently $d = 4$. Hence, it can admit maximum 10 Killing vector fields and this maximally symmetric spacetime is known as the *de Sitter spacetime*.

4.3 Gutzwiller trace formula for Dirac operators

In order to get an intuition of the Gutzwiller trace formula for Dirac-type operators on a stationary spacetime we begin with the simplest case.

4.3.1 Poisson summation formula on $\mathbb{S}$

Since $\mathrm{H}^1(\mathbb{S}; \mathbb{Z}_2) = \mathbb{Z}_2$ (cf. Example 3.4.4), a circle $\mathbb{S}$ admits *two inequivalent spin-structures* as described below (see e.g. [171, Sec. 3]).

- **Trivial spin-structure**: One uses $\mathrm{SO}(1) = \{1\}$ and $\mathrm{Spin}(1) = \mathbb{Z}_2 = \{+1, -1\}$. Then the frame bundle $(\mathcal{Q}, \mathrm{SO}(1), \mathbb{S})$ is trivial and so is the spin-structure

$$\mathscr{P}_{\mathrm{triv}} = \mathbb{S} \times \mathrm{Spin}(1), \quad \mathscr{S}_{\mathrm{triv}} := \mathscr{P}_{\mathrm{triv}} \times \mathbb{C}, \tag{4.11}$$

where the double covering Θ is the projection of the first factor. Hence, the spinors on $\mathbb{S}$ are simply ($\mathbb{C}$-valued) functions on $\mathbb{S}$ which are identified with T-periodic functions on $\mathbb{R}$:

$$C^\infty(\mathbb{S}; \mathscr{S}_{\mathrm{triv}}) = C^\infty(\mathbb{S}) = \{u \in C^\infty(\mathbb{R}) \mid \forall T > 0 : u(x + T) = u(x)\}. \tag{4.12}$$

The massless spin-Dirac operator on $\mathbb{S}$ is nothing but the derivative operator D_x:

$$\slashed{D}_{\mathbb{S}} := -\,\mathrm{i}\,\frac{\mathrm{d}}{\mathrm{d}x} = D_x. \tag{4.13}$$

Therefore $\operatorname{Spec} \slashed{D}_{\mathbb{S}} = \operatorname{Spec} D_x$ and the Poisson summation formula (1.4) is essentially the Poisson summation formula for $\slashed{D}_{\mathbb{S}}$ on $\mathbb{S}$ for the trivial spin-structure.

- **Non-trivial spin-structure**: In this case

$$\mathscr{P}_{\mathrm{ntriv}} = \big([0, T] \times \operatorname{Spin}(1)\big)/\!\sim, \quad \forall T > 0, \tag{4.14}$$

where $\sim$ identifies 0 with T while it interchanges two elements of $\operatorname{Spin}(1) = \{+1, -1\}$. Thus, sections of $\mathscr{S}_{\mathrm{ntriv}}$ are no longer $C^\infty(\mathbb{S})$ rather they are anti-periodic:

$$C^\infty(\mathbb{S}; \mathscr{S}_{\mathrm{ntriv}}) = \{u \in C^\infty(\mathbb{R}) \mid \forall T > 0 : u(x + T) = -u(x)\}. \tag{4.15}$$

In contrast to (1.3), the spectrum reads

$$\operatorname{Spec} \slashed{D}_{\mathbb{S}} = \{\lambda_n | n \in \mathbb{Z}\}, \quad \lambda_n := \left(n + \frac{1}{2}\right)\omega, \quad \omega := \frac{2\pi}{T} \tag{4.16}$$

corresponding to the orthonormal eigenspinors $\phi_n := \mathrm{e}^{\mathrm{i}\lambda_n x}/\sqrt{T}$. One can obtain a Poisson summation formula for non-trivial spin-structure analogously and it is structurally similar to that of trivial spin-structure albeit not exactly the same.

In relativistic language, the Poisson summation formulae for trivial and non-trivial spin-structures on $\mathbb{S}$ describe an exact trace formulae $\operatorname{Tr} U_t$ for $U_t := \mathrm{e}^{-\mathrm{i}t\slashed{D}_{\mathbb{S}}}$ on the product spin-manifold $(\mathbb{R} \times \mathbb{S}, \mathrm{d}t^2 - \mathrm{d}x^2)$. The thesis is devoted to an enormous generalisation of this result in the sense that $\mathbb{S}$ in the preceding examples is replaced by a compact Cauchy hypersurface Σ and the role the product metric $\mathrm{d}t^2 - \mathrm{d}x^2$ is played by a stationary spacetime metric (cf. Definition 4.2.1, 4.2.3 and Remark 4.2.6 (b)).

4.3.2 Primary results

Let $\mathscr{E} \to \mathscr{M}$ be a vector bundle over a $d \geq 2$ dimensional spatially compact globally hyperbolic stationary spacetime (Section 4.2) $(\mathscr{M}, \mathrm{g}, \Xi)$. That is, the spacetime manifold $\mathscr{M}$ is homeomorphic to the product manifold $\mathbb{R} \times \Sigma$ where $\Sigma \subset M$ is a spacelike and compact Cauchy hypersurface, and $(\mathscr{M}, \mathrm{g})$ admits a *complete* timelike Killing vector field Z whose flow is Ξ (Remark 4.2.6 (a)).

Assumption 4.3.1. We consider Dirac-type operators (Definition 3.4.1) D on a vector bundle $\mathscr{E} \to \mathscr{M}$ over a globally hyperbolic spacetime $(\mathscr{M}, \mathrm{g}, \varXi)$, subjected to the following assumptions:

(a) $\mathscr{E}$ is endowed with a sesquilinear form $(\cdot|\cdot)$ invariant under the Killing flow $\varXi_s^*$ such that D is symmetric;

(b) Given an arbitrary but fixed future-directed unit normal covector field $(\cdot, \zeta)$ on $\mathscr{M}$ along any Cauchy hypersurface $\varSigma$,

$$\langle\cdot|\cdot\rangle := \left(\sigma_D(\cdot, \zeta) \cdot \big| \cdot\right) \tag{4.17}$$

is a fibrewise hermitian form on the bundle of Clifford modules $(\mathscr{E} \to \mathscr{M}, \sigma_D, (\cdot|\cdot))$, where σ_D is the principal symbol of D;

(c) D commutes with the induced Killing flow $\varXi_s^*$ on $\mathscr{E}$ for all $s \in \mathbb{R}$.

Regarding the second assumption, we recall the comments after Assumption 3.4.7. In the present setting, we use Z to fix the time-orientation and note that on a static spacetime, one can choose the Killing covector field $(\partial_t)^\flat$ (up to normalisation) for ζ (see Remark 4.2.6 (e) for details).

As noted before, the Clifford module bundle $(\mathscr{E} \to M, \sigma_D, (\cdot|\cdot))$ is naturally furnished with the unique Weitzenböck connection $\nabla^{\mathscr{E}}$ ((3.126)) induced by D and g. This connection induces a parallel transport map $\hat{\varXi}_s : \mathscr{E}. \to \mathscr{E}_{\varXi_s(\cdot)}$ along the spacetime isometry $\varXi_s(\cdot)$. Subsequently, the following diagram commutes:

$$
\begin{array}{ccccc}
\mathscr{E} & \xrightarrow{\ \hat{\varXi}_s\ } & \varXi_s^*\mathscr{E} & \xrightarrow{\ \cong\ } & \mathscr{E} \\
\downarrow & & \downarrow & & \downarrow \\
\mathscr{M} & \xrightarrow{\ \mathrm{id}\ } & \mathscr{M} & \xrightarrow{\ \varXi_s\ } & \mathscr{M}
\end{array}
$$

Figure 4.1: Pullback $\varXi_s^*\mathscr{E}$ of a bundle of Clifford modules $(\mathscr{E} \to M, (\cdot|\cdot), \sigma_D)$ over a stationary spacetime $(\mathscr{M}, \mathrm{g}, \varXi)$ via a spacetime isometry $\varXi_s : \mathscr{M} \to \mathscr{M}$.

Let $\pounds_Z$ be the Lie derivative (given by (2.135)) on $\mathscr{E}$ with respect to the Killing vector field Z and

$$\mathrm{L} := -\mathrm{i}\pounds_Z : C^\infty(\mathscr{M}; \mathscr{E}) \to C^\infty(\mathscr{M}; \mathscr{E}) \tag{4.18}$$

so that the induced Killing flow on $C^\infty(\mathscr{M}; \mathscr{E})$ is expressed as $\varXi_s^* = \mathrm{e}^{\mathrm{i}\, s\, \mathrm{L}}$. By hypotheses,

$$[\mathrm{L}, D]_- = 0. \tag{4.19}$$

Thus, on $\ker D$, the eigensections ψ_n of L are the joint eigensections:

$$D\psi_n = 0, \qquad \mathrm{L}\,\psi_n = \lambda_n\psi_n. \tag{4.20}$$

We equip $\ker D$ with the hermitian inner product $\langle\cdot|\cdot\rangle$ to have the Hilbert space $(\ker D, \langle\cdot|\cdot\rangle)$.

Theorem 4.3.2. *Under Assumption [4.3.1](#), the spectrum of* L *(4.18) on the Hilbert space* $\mathscr{H} := (\ker D, \langle\cdot|\cdot\rangle := (3.156))$ *is purely discrete and comprises infinitely many real eigenvalues that grow polynomially and accumulate at* $\pm\infty$.

Thus, we can restrict our attention entirely to smooth sections $C^\infty(\mathscr{M};\mathscr{E})$ of $\mathscr{E}$ owing to the elliptic regularity and $\mathrm{tr}_{\mathscr{H}}\, \mathrm{e}^{\mathrm{i}\,t\,\mathrm{L}}$ makes sense as a distribution with (cf. (1.21)) the identification

$$\mathrm{Tr}\, U_t = \mathrm{tr}_{\mathscr{H}}\, \varXi_t^*, \tag{4.21}$$

where U_t is the time evolution operator of the Dirac equation (1.18).

Since the classical dynamics is governed by the principal symbol σ_D of D, its characteristic set $\mathrm{Char}\, D$ can be viewed as a classical limit of $\ker D$ as per se geometric quantisation. Thus our classical phase space is the lightcone bundle $\dot{\mathrm{T}}_0^*\mathscr{M} \to \mathscr{M}$ over $(\mathscr{M}, \mathrm{g}, \varXi)$ where the metric-Hamiltonian H_{g} (3.115) vanishes and the reduced phase space is the conic symplectic manifold $\mathscr{N}$ of scaled-lightlike geodesic strips [122] (see also [123]). Being a globally hyperbolic spacetime, $\mathscr{M}$ does not admit any closed timelike geodesic. Therefore, the notion of Lorentzian analogue of periodic trajectories is defined by means of the spacetime isometry $\varXi_t$ induced (reduced) symplectic flow $\varXi_t^{\mathscr{N}}$ on $\mathscr{N}$, whose Hamiltonian is given by [27, Lem. 1.1]

$$H : \mathscr{N} \to \mathbb{R}, \ \gamma \mapsto H(\gamma) := \mathrm{g}\Big(\frac{\mathrm{d}c}{\mathrm{d}s}, Z\Big), \tag{4.22}$$

where $\mathbb{R} \ni s \mapsto c(s) \in \mathscr{M}$ is any lightlike geodesic on $\mathscr{M}$ and the value $\mathrm{g}(\mathrm{d}c/\mathrm{d}s, Z)$ is independent of the cotangent lift γ of c. This Hamiltonian is positive as Z is timelike. For any $\mathrm{E} \in \mathbb{R}_+$, we denote the constant E-energy surface by

$$\mathscr{N}_{\mathrm{E}} := \{\gamma \in \mathscr{N} \mid H(\gamma) = \mathrm{E}\}. \tag{4.23}$$

The set of **periods** resp. **periodic lightlike geodesics** of $\varXi_s^{\mathscr{N}}$ are then given by [27, (6)]

$$\mathscr{P} \ := \ \{T \in \mathbb{R}_+ \mid \exists \gamma \in \mathscr{N} : \varXi_T^{\mathscr{N}}(\gamma) = \gamma\}, \tag{4.24a}$$

$$\mathscr{P}_T \ := \ \{\gamma \in \mathscr{N} \mid \varXi_T^{\mathscr{N}}(\gamma) = \gamma\}. \tag{4.24b}$$

Recall, the set of all lengths of periodic geodesics on a manifold counted with multiplicities is called the *length spectrum* of the manifold. Our first finding is that $\mathrm{Tr}\, U_t$ determines the Lorentzian length spectrum of $\mathscr{N}$ in the sense described below.

Proposition 4.3.3. *Let* $\mathscr{E} \to \mathscr{M}$ *be a vector bundle over a spatially compact globally hyperbolic stationary spacetime* $(\mathscr{M}, \mathrm{g}, \varXi)$, *endowed with a sesquilinear form* $(\cdot|\cdot)$, *and* D *a Dirac-type operator on* $\mathscr{E}$ *whose principal symbol is* σ_D *so that* $(\mathscr{E} \to \mathscr{M}, \sigma_D, (\cdot|\cdot))$ *is a bundle of Clifford modules over* $(\mathscr{M}, \mathrm{g}, \varXi)$. *If* U_t *is the time evolution operator of* D *then under Assumption [4.3.1](#),* $\mathrm{Tr}\, U_t$ *is a distribution on* $\mathbb{R}$ *and its singular support*

$$\mathrm{singsupp}\, \mathrm{Tr}\, U_t \subset \{0\} \cup \mathscr{P}, \tag{4.25}$$

where $\mathscr{P}$ *is the set of periods of the induced Killing flow* $\varXi_T^{\mathscr{N}}$ *on the manifold of scaled-lightlike geodesics* $\mathscr{N}$, *given by* (4.24a).

Lagrangian distributions (Section 2.3.3) offer the elegant characterisation of $\mathrm{Tr}\, U_t$, as stated below.

Theorem 4.3.4. *As in the set-up of Proposition 4.3.3, $\mathrm{Tr}\, U_T$ is the Lagrangian distribution $I^{d-7/4}(\mathbb{R}, \Lambda_T)$ where $d := \dim \mathcal{M}$ and*

$$\Lambda_T := \{(T, \tau) \in \mathbb{R} \times \mathbb{R}_- \mid \exists \gamma \in \mathcal{N} : \Xi_T^{\mathcal{N}}(\gamma) = \gamma, \tau = -H(\gamma)\}, \tag{4.26}$$

is the Lagrangian manifold Λ_T, where the Hamiltonian H on $\mathcal{N}$ is given by (4.22). Furthermore,

$$(\mathrm{Tr}\, U_0)(t) = u_0(t) + v_0(t), \tag{4.27}$$

where v_0 is a distribution on $\mathbb{R}$ which is smooth in the vicinity of $t = 0$ and u_0 is a Lagrangian distribution admitting the following singularity expansion around $t = 0$:

$$u_0(t) \quad \sim \quad r(d-1)\frac{\mathrm{vol}(\mathcal{N}_{H \leq 1})}{(2\pi)^{d-1}}\mu_{d-1}(t) + \mathsf{c}_2\, \mu_{d-2}(t) + \dots,$$

$$r \quad := \quad \mathrm{rk}\,\mathcal{E} \int_{\Sigma} \mathrm{g}_x^{-1}(\eta, \zeta)\, \mathrm{dv_h}(x),$$

$$\mu_{d-k}(t) \quad := \quad \int_{\mathbb{R}_{\geq 0}} \mathrm{e}^{-\mathrm{i}t\tau}\tau^{d-1-k}\mathrm{d}\tau, \quad k = 1, 2, \dots. \tag{4.28}$$

Here, μ_{d-k} is a distribution on $\mathbb{R}$ given by the preceding oscillatory integral, η is any lightlike covector on $\mathcal{M}$ and ζ as in Assumption 4.3.1, both restricted to a Cauchy hypersurface Σ of $\mathcal{M}$, c_2 is some constant (Dirac-wave trace invariant), and $\mathrm{vol}\,\mathcal{N}_{H \leq 1}$ is the volume of $\mathcal{N}_{H \leq 1}$.

As remarked in 4.2.6 (b), $(\mathcal{M}, \mathrm{g}, \Xi)$ can be considered as a (globally hyperbolic) *standard* stationary spacetime since Z is complete. Then, we can use the spacetime metric (4.6) without loss of generalities and the volume of $\mathcal{N}_{H \leq 1}$ is given by [27, (15)]

$$\mathrm{vol}\,\mathcal{N}_{H \leq 1} = \mathrm{vol}(\mathbb{B}^{d-1}) \int_{\Sigma} \beta(x)\big(\beta^2(x) - \mathrm{h}_x(\alpha, \alpha)\big)^{-d/2}\mathrm{dv_h}(x), \tag{4.29}$$

where $\mathrm{vol}(\mathbb{B}^{d-1})$ is the volume of the unit ball $\mathbb{B}^{d-1} \subset \mathbb{R}^{d-1}$, and β and α are the lapse function and shift vector field, respectively, appearing in spacetime metric (4.6).

Since L has discrete eigenvalues, let us introduce its eigenvalue counting function[1]

$$\mathsf{N}(\lambda) := \#\{n \in \mathbb{N} \mid 0 \leq \lambda_n \leq \lambda\}. \tag{4.30}$$

Corollary 4.3.5 (Weyl law)**.** *As in the terminologies of Theorems 4.3.2 and 4.3.4, the Weyl eigenvalue counting function of L has the asymptotics*

$$\mathsf{N}(\lambda) = \left(\frac{\lambda}{2\pi}\right)^{d-1} r\, \mathrm{vol}(\mathcal{N}_{H \leq 1}) + O(\lambda^{d-2}), \quad \text{as} \quad \lambda \to \infty. \tag{4.31}$$

---

[1]One can define it with negative eigenvalues as well.

In order to describe $\operatorname{Tr} U_T$ for the non-trivial periods $T \neq 0$, we let $\nabla^{\pi^*\operatorname{Hom}(\mathcal{E},\mathcal{E})}_{X_{\mathrm{g}/2}}$ be the D^2-compatible (Weitzenböck) covariant derivative (Definition 3.3.6) with respect to the geodesic vector field $X_{\mathrm{g}/2}$ on the bundle $\pi^*\operatorname{Hom}(\mathcal{E},\mathcal{E}) \to \dot{\mathrm{T}}^*\mathcal{M}$ where $\pi : \mathrm{T}^*\mathcal{M} \to \mathcal{M}$ is the cotangent bundle. By $\mathrm{Hol} := C^\infty_{\mathrm{Hol}}\big(\dot{\mathrm{T}}^*\mathcal{M}; \pi^*\operatorname{Hom}(\mathcal{E},\mathcal{E})\big)$ we will denote the *set of all sections of $\pi^*\operatorname{Hom}(\mathcal{E},\mathcal{E})$ invariant under the holonomy group of the parallel transporter $\mathcal{T}$ with respect to $\nabla^{\pi^*\operatorname{Hom}(\mathcal{E},\mathcal{E})}_{X_{\mathrm{g}/2}}$.*

We suppose P_γ is the (linearised) Poincaré map of a periodic geodesic γ (see e.g. [96, Sec. 7.1], [27]). Then γ is called **non-degenerate** if 1 is not an eigenvalue of P_γ.

With all these devises, the precise characterisation of $\operatorname{Tr} U_{T \neq 0}$ can be stated as follows.

Theorem 4.3.6. *As in the set-ups of Proposition 4.3.3 and Theorem 4.3.4: if the periods T are discrete and the set $\mathcal{P}_T := $ (4.24b) of periodic lightlike geodesics of $\Xi^{\mathcal{N}}_T$ is a finite union of non-degenerate periodic orbits γ, then*

$$(\operatorname{Tr} U_T)(t) = \sum_{\gamma \in \mathcal{P}_T} u_\gamma(t) + v_T(t), \tag{4.32}$$

where v_T is a distribution on $\mathbb{R}$ that is smooth in the vicinity of $t = T$ and $u_\gamma(t)$'s are Lagrangian distributions having singularities at $t = T_\gamma$ with the asymptotic expansion

$$\lim_{t \to T}(t - T_\gamma)u_\gamma(t) \quad \sim \quad \frac{1}{2\pi} \int_{\mathfrak{F}_{T_\gamma}} \operatorname{tr}\Big(\sigma_D(\gamma)\,\mathcal{T}_{\Xi^{\mathcal{N}}_{T_\gamma}(\gamma)}\,\sigma_D(x,\zeta)\Big) \frac{e^{-\mathrm{i}\pi\mathfrak{m}(\gamma)/2}|\mathrm{d}T_\gamma|}{\sqrt{|\det(I - \mathrm{P}_\gamma)|}} \nu + \dots,$$
$$\nu \quad := \quad \int_{\mathbb{R}_{\geq 0}} e^{-\mathrm{i}(t-T)\tau}\Big(\frac{\tau}{2\pi}\Big)^{d-2}\mathrm{d}\tau, \tag{4.33}$$

where $\mathfrak{m}(\gamma)$ resp. P_γ are the Maslov index resp. the (linearised) Poincaré map of γ, $\mathcal{T}_\gamma$ is an element of the holonomy group Hol_γ with respect to the D^2-compatible Weitzenböck connection at base point $\gamma \in \mathcal{N}$ whose projection on $\mathcal{M}$ is x, and

$$\mathfrak{F}_{T_\gamma} := \Big\{\gamma \in \mathcal{N} \mid \tau = -H(\gamma), \Xi^{\mathcal{N}}_{T_\gamma}(\gamma) = \gamma\Big\} \tag{4.34}$$

is the fixed point set of $\Xi^{\mathcal{N}}_{T_\gamma}$.

At this point we would like to comment on the issue of the parallel transporter raised at the introduction (Section 1.1.3) and reproduce Sandoval's result [26, Thm. 2.8] as a special case of this theorem. We set $\alpha = 0$ and $\beta = 1$ in (4.6) so that the Dirac equation (1.18) on the ultrastatic spacetime becomes

$$-\mathrm{i}\,\partial_t u = H_D u, \quad H_D := \mathsf{c}(\mathrm{d}t)\big(\hat{D} - U\big), \tag{4.35}$$

where H_D is the Dirac Hamiltonian, $\mathsf{c} = $ (3.125) is the Clifford multiplication, and U is the potential in (3.128) allowed by the most general Dirac-type operator. Sandoval

plumped for $U = 0$ and used the parallel transporter $\check{\mathcal{T}}$ corresponding to $\check{D}$ instead of $\hat{D}$ and hence $\check{\mathcal{T}}$ together with a "suitable average" of $W := \hat{D} - \check{D}$ showed up under the tr in (4.33) in her work. However, no such ad-hoc choices are necessary in our formulation at all, rather $\mathcal{T}$ naturally induces the parallel transport $\hat{\mathcal{T}}$ for the preceding choices.

Note, the Maslov index $\mathfrak{m}(\gamma)$ is essentially the Conley-Zehnder index of γ [172] and the factor $e^{-i\pi\mathfrak{m}(\gamma)/2}$ is often known as the Maslov factor.

In the future, several generalisations have been planned. For instance, we intend to address the spectral asymptotics on stationary black holes and explore the semi-classical regime. We also wish to extend the study for Hodge-d'Alambertians and connect with interesting applications on relativistic quantum chaos on curved spacetimes.

4.3.3   Proof strategy and novelty

We divide the description into several steps to give a panorama view.

U_t as a Fourier integral operator

This pivotal idea was originally due to Duistermaat and Guillemin [10] who worked it out (modulo smoothing operators) in the context of scalar half-wave operators on an ultrastatic (Remark 4.2.6 (e)) spacetime. We, however, have not followed their approach directly, instead have expressed U_t in terms of the Killing flow Ξ_t^* and the causal propagator $F = $ (3.149) of D by propounding Strohmaier and Zelditch's work on d'Alembertian [27]. More precisely, there exists unique advanced and retarded Green's operators (Section 3.4.2) for D owing to the global hyperbolicity of $(\mathcal{M}, g)$ and hence F together with the restriction operator ι_Σ^* (Example 2.3.12) pave the way to construct the Cauchy restriction operator $\mathcal{R}$ as described in Section 3.4.3. By Assumption 4.3.1 (c), the time flow determines the time evolution of Cauchy data. Then the combination of these facts allows us to express U_t as a Lagrangian distribution as inscribed in Lemma 4.7.3.
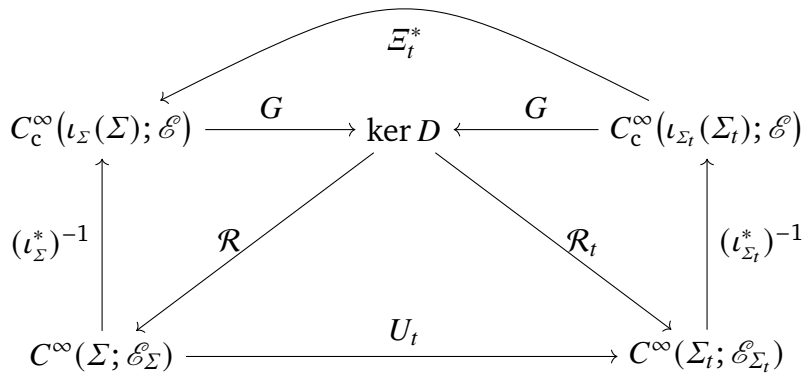

Figure 4.2: The time evolution map U_t in terms of the causal propagator F for D and the induced Killing flow Ξ_t^*. Here, $\mathcal{R}_t$ denotes the Cauchy restriction map and $\iota_{\Sigma_t}^*$ the restriction map.

In order to describe U_t as a Fourier integral operator one requires to obtain such a description of F and ι_Σ^*. Both are well-known for the scalar case and the bundle generalisation is straightforward for the latter. But the former demands an intricate treatment due to possible bundle curvature, which has been spelled out in Lemma 3.4.6. In fact, this is precisely from where the expedient choice (as discussed in Section 1.1.3) in Sandoval's work stems. We have computed σ_F in Lemma 3.4.6 deploying the D^2-compatible Weitzenböck connection in an intrinsically geometric fashion. As a consequence, it solves the relevant leading-order transport equation concisely from where the holonomy group shows up elegantly. Furthermore, it closes the ad-hoc consideration in Sandoval's analysis as explained after Theorem 4.3.6.

In contemporary of this work (after the submission of an arXiv-preprint based on this work), Capoferri and Murro [173, Thm. 1.1] have obtained an oscillatory integral representation of U_t (modulo smoothing operators) for the reduced massless Dirac equation (in the sense of Definition 3.5 of their paper) on a 4-dimensional spatially compact globally hyperbolic spin-spacetime using the global phase function approach [116] of Fourier integral operators. An antecedent of this analysis can be traced back to Capoferri and Vassiliev [174] who have constructed $U_t(\check{D}) = (1.15)$ (modulo smoothing operators) as a summation of two invariantly defined oscillatory integrals, global in space and in time, with distinguished complex-valued phase functions, when $\check{D}$ is a massless spin-Dirac operator on a 3-dimensional closed Riemannian manifold.

It is worthwhile to mention that one cannot deploy the algorithm used in [173, 174] in a straightforward way to derive the Fourier integral description (Lemma 4.7.3) of U_t in our setting. This is primarily because the Dirac Hamiltonian

$$H_D := -\mathrm{i}\, \frac{\mathsf{c}(\mathrm{d}t)\big(\mathsf{h}_{ij}\mathsf{c}(\mathrm{d}x^i)\nabla^{\mathscr{E}j} + U\big) + \mathsf{h}_{ij}\alpha^i \nabla^{\mathscr{E}j}}{\beta^2 - \|\alpha\|_{\mathsf{h}}^2 - \mathsf{h}_{ij}\alpha^i \mathsf{c}(\mathrm{d}t)\mathsf{c}(\mathrm{d}x^j)} \tag{4.36}$$

on a stationary spacetime is *not* of Dirac-type (albeit can be written as a first-order elliptic operator) which is one of the key assumptions of the hindmost literatures.

Principal symbol of $\mathrm{Tr}\, U_t$

If $\mathscr{U}_t(x, y) := \mathsf{U}_t(x, y)\,\mathrm{d}t \otimes \sqrt{|\mathrm{d}v_{\mathsf{h}}(x)|} \otimes \sqrt{|\mathrm{d}v_{\mathsf{h}}(y)|}$ denotes the Schwartz kernel of U_t then one considers the smoothed-out operator U_ρ (cf. (1.9))

$$\mathsf{U}_\rho := \int_{\mathbb{R}} \mathsf{U}_t\, \mathcal{F}^{-1}(\rho)\,\mathrm{d}t, \quad (\mathcal{F}^{-1}\rho)(t) = \frac{1}{2\pi} \int_{\mathrm{Spec}\, \mathsf{L}} \mathrm{e}^{\mathrm{i}t\lambda}\rho(\lambda)\,\mathrm{d}\lambda \tag{4.37}$$

for any Schwartz function $\rho \in \mathcal{S}(\mathbb{R})$ on $\mathbb{R}$ such that $\mathrm{supp}(\mathcal{F}^{-1}\rho)$ is compact. Let $\mathsf{U}_\rho(x, x)$ be the diagonal embedding of $\mathsf{U}_\rho(x, y)$. The distributional trace $\mathrm{Tr}\, U_t$ is then obtained by

$$\mathrm{Tr}\, U_\rho := \int_\Sigma \mathrm{tr}\,\big(\mathsf{U}_\rho(x, x)\big)\,\mathrm{d}v_{\mathsf{h}}(x), \tag{4.38}$$

where tr is the endomorphism trace (3.8).

In order to compute $\sigma_{\mathrm{Tr}\, U_t}$ we employ the bundle generalisation of Duistermaat and Guillemin's [10] idea, due to Sandoval [26]. One notices that the mapping $\mathscr{U}_t(x, y) \mapsto U_t(x, x)\, \mathrm{dv}_\mathrm{h}(x)$ can be viewed as the pullback

$$\Delta^* : C_\mathrm{c}^\infty\Big(\mathbb{R} \times \Sigma \times \Sigma; \Omega\mathbb{R} \boxtimes \big(\mathrm{Hom}(\mathscr{E}_\Sigma, \mathscr{E}_\Sigma) \otimes \Omega^{1/2}(\Sigma \times \Sigma)\big)\Big) \to C_\mathrm{c}^\infty\big(\mathbb{R} \times \Sigma; \Omega\mathbb{R} \boxtimes (\mathrm{End}\, \mathscr{E}_\Sigma \otimes \Omega\Sigma)\big) \tag{4.39}$$

of $\mathscr{U}_t$ via the diagonal embedding

$$\Delta : \mathbb{R} \times \Sigma \to \mathbb{R} \times \Sigma \times \Sigma \tag{4.40}$$

and the fibrewise isomorphism $\hat{\Delta} \in C^\infty\big(\Sigma; \mathrm{Iso}(\mathscr{E}_\Sigma, \mathscr{E}_\Sigma)\big)$ induced by the Weitzenböck connection $\nabla^{\mathscr{E}}$ restricted to Σ. It follows (cf. Example 2.1.11) that Δ^* is a Fourier integral operator of order $(d-1)/4$ associated to the canonical relation $C_{\Delta^*} := \{(t, \tau; x, \xi + \eta; t, \tau; x, \xi; x, \eta) \in \dot{\mathrm{T}}^*\mathbb{R} \times \mathrm{T}^*\mathscr{M}_\Sigma \times \dot{\mathrm{T}}^*\mathbb{R} \times \dot{\mathrm{T}}^*\mathscr{M}_\Sigma \times \dot{\mathrm{T}}^*\mathscr{M}_\Sigma\}$ [10, (1.20)]. Hence, for a fixed t, $\mathrm{tr}(\Delta^*\mathscr{U}_t)$ is density on Σ which can be integrated. The integration over Σ is the pushforward (see e.g. [175, pp. 103-104])

$$\pi_* : C_\mathrm{c}^\infty(\mathbb{R} \times \Sigma; \Omega\mathbb{R} \boxtimes \Omega\Sigma) \to C^\infty(\mathbb{R}; \Omega\mathbb{R}), \quad \mathrm{tr}(\Delta^*\mathscr{U}_t) \mapsto \pi_*\big(\mathrm{tr}(\Delta^*\mathscr{U}_t)\big) \tag{4.41}$$

of the Cartesian projection

$$\pi : \mathbb{R} \times \Sigma \to \mathbb{R}, \tag{4.42}$$

which is also a Fourier integral operator of order $1/2 - (d-1)/4$ associated to the canonical relation $C_{\pi_*} := \{(t, \tau; t, \tau; x, 0) \in \dot{\mathrm{T}}^*\mathbb{R} \times \dot{\mathrm{T}}^*\mathbb{R} \times \mathrm{T}^*\mathscr{M}_\Sigma\}$ [10, (1.22)]. Therefore

$$\mathrm{Tr}\, U_t = \pi_* \circ \mathrm{tr}(\Delta^*\mathscr{U}_t). \tag{4.43}$$

Alternatively, one can also integrate $\Delta^*\mathscr{U}_t$ over Σ so that $\pi_*(\Delta^*\mathscr{U}_t)$ is an $\mathrm{End}\,\mathscr{E}_\Sigma$-valued density on $\mathbb{R}$ by reckoning $\pi_* : C_\mathrm{c}^\infty\big(\mathbb{R} \times \Sigma; \Omega\mathbb{R} \boxtimes (\mathrm{End}\,\mathscr{E}_\Sigma \otimes \Omega\Sigma)\big) \to C^\infty(\mathbb{R}; \Omega_\mathbb{R} \boxtimes \mathrm{End}\,\mathscr{E}_\Sigma)$. Then taking the endomorphism trace one arrives at

$$\mathrm{Tr}\, U_t = \mathrm{tr}(\pi_* \circ \Delta^*\mathscr{U}_t). \tag{4.44}$$

Hence, in this sense

$$\pi_* \circ \mathrm{tr} \circ \Delta^* = \mathrm{tr} \circ \pi_* \circ \Delta^* \tag{4.45}$$

and one utilises the clean intersection (see Definition A.1.18) between $\pi_* \circ \Delta^*$ and $\mathscr{U}_t$ to compute the principal symbol of $\mathrm{Tr}\, U_t$.

Spectral theory

Albeit finite energy solutions of Dirac equation do not live in L^2 sections of $\mathscr{E}$, $\ker D$ can be naturally given a hermitian structure by equipping with a Killing flow invariant hermitian form $\langle \cdot | \cdot \rangle$ owing to Assumption 4.3.1 (b). It is noteworthy that the non-definiteness of $(\cdot | \cdot)$ is a characteristic feature of any Lorentzian spin-manifold: a positive-definite $(\cdot | \cdot)$ invariant under the spin group only exists for the Riemannian case.

Weyl law

There are a number of approaches to deriving the Weyl law (see e.g. the review [28]). Amongst those, we will use the Fourier-Tauberian argument [5] (see also, e.g. [24, App. B], [176] and references therein). The key idea is to relate the Weyl counting function with $\operatorname{Tr} U_t$ via the distributional Fourier transform (1.8) $\mathrm{d}N/\mathrm{d}\lambda = \mathcal{F}^{-1}_{t \mapsto \lambda}(\operatorname{Tr} U_t)$ and compute the right-hand side using the express for $\operatorname{Tr} U_0$.

4.4 Dirac operator on a stationary spacetime

4.4.1 Lie derivative

On a stationary spacetime $(\mathcal{M}, \mathrm{g}, \Xi)$, the cotangent lift $\mathrm{T}^* \Xi_s$ of the spacetime isometry $\Xi_s : \mathcal{M} \to \mathcal{M}$ naturally induces a $\mathbb{R}$ action on the cotangent bundle $\mathrm{T}^* \mathcal{M}$: $\mathrm{T}^* \Xi(s; y, \eta) := (\mathrm{T}^*_x \Xi_s)\eta$. Furthermore, the pullback (restoring densities for clarity)

$$\Xi^* : C^\infty_c(\mathcal{M}; \mathcal{E} \otimes \Omega^{1/2}\mathcal{M}) \to C^\infty(\mathbb{R} \times \mathcal{M}; \Omega\mathbb{R} \boxtimes (\mathcal{E} \otimes \Omega^{1/2}\mathcal{M})) \qquad (4.46)$$

via the Killing flow $\Xi : \mathbb{R} \times \mathcal{M} \to \mathcal{M}$ and the fibrewise isomorphism $\hat{\Xi}_s \in C^\infty(\mathcal{M}; \mathrm{Iso}(\mathcal{E}, \mathcal{E}))$ as depicted in Figure 4.1, is a Fourier integral operator whose Schwartz kernel is given by (cf. Example 2.1.11)

$$\Xi \in I^{-1/4}\Big(\mathbb{R} \times \mathcal{M} \times \mathcal{M}, \Gamma'; \Omega\mathbb{R} \boxtimes \big(\mathrm{Hom}(\mathcal{E}, \mathcal{E}) \otimes \Omega^{1/2}(\mathcal{M} \times \mathcal{M})\big)\Big), \qquad (4.47\mathrm{a})$$

$$\Gamma := \big\{(s, \tau; x, \xi; y, -\eta) \in \dot{\mathrm{T}}^*\mathbb{R} \times \dot{\mathrm{T}}^*\mathcal{M} \times \dot{\mathrm{T}}^*\mathcal{M} \mid \tau = -\xi(Z), (x, \xi) = (\mathrm{T}^*_x \Xi_s)(y, \eta)\big\}, \qquad (4.47\mathrm{b})$$

$$\sigma_\Xi := (2\pi)^{1/4}\, \mathbb{1}_{\mathrm{Hom}(\mathcal{E}, \mathcal{E})}\, \sqrt{|\mathrm{d}v_\Gamma|} \otimes \mathbb{I}, \qquad (4.47\mathrm{c})$$

where $\mathrm{d}v_\Gamma$ is the volume form on the homogeneous canonical relation $\Gamma \subset \dot{\mathrm{T}}^*(\mathbb{R} \times \mathcal{M}) \times \dot{\mathrm{T}}^*\mathcal{M}$ and $\mathbb{I}$ is a section of the Keller-Maslov bundle $\mathbb{L}_\Gamma \to \Gamma$ over Γ, constructed as below. One observes that Γ is the graph of $\mathrm{T}^* \Xi_s$ and at $s = 0$, Γ is essentially the conormal bundle $\Gamma_0 := \{(0, \tau)\} \times (\Delta \dot{\mathrm{T}}^*\mathcal{M})'$. Since $\dot{\mathrm{T}}^*\mathcal{M}$ is a symplectic manifold (see Example A.1.2), it admits the global volume induced by the canonical symplectic form on $\dot{\mathrm{T}}^*\mathcal{M}$. Then $\mathrm{d}v_{\Gamma_0}$ is obtained via the pullback of the projector $\mathrm{Pr} : \Gamma_0 \to \mathbb{R} \times \dot{\mathrm{T}}^*\mathcal{M}$, which is invariant under the flow Ψ_s of the Hamiltonian vector field generated by the extended Hamiltonian $\tau + \xi(Z)$ and given by $\mathrm{d}v_\Gamma = \mathrm{d}s \otimes \mathrm{d}x \wedge \mathrm{d}\xi$ in the parametrisation (4.47b). To construct $\mathbb{L}_\Gamma$, we re-use the fact that Ψ_s sweeps out $\{(0, \tau)\} \times (\Delta \dot{\mathrm{T}}^*M)'$ to Γ, and hence $\mathbb{L}_\Gamma$ is constructed by parallelly transporting the sections of $\mathbb{L}_0$ along the orbits of Ψ_s, where $\mathbb{L}_0$ the Keller-Maslov bundle (see Example A.1.21) over Γ_0 consists of a global constant section.

More generally, Ξ^* can be extended to a sequentially continuous linear map on $I^m(\mathcal{M}, \mathcal{L}; \mathcal{E})$ by

$$\Xi^* : I^m(\mathcal{M}, \mathcal{L}; \mathcal{E}) \to I^{m-1/4}(\mathbb{R} \times \mathcal{M}, \Gamma' \circ \mathcal{L}; \mathcal{E}), \qquad \sigma_{\Xi^* u} \asymp \sigma_\Xi \diamond \sigma_u \qquad (4.48)$$

for any $u \in I^m(\mathscr{M}, \mathscr{L}; \mathscr{E})$. The composition $\diamond$ of principal symbols is presented in detail in Appendix A.2.3 and the equation of $\sigma_{\Xi^* u}$ is in the sense of modulo Keller-Maslov part.

The induced Killing flow Ξ_s^* paves the way to define the Lie derivative $\pounds_Z$ (2.135) on $\mathscr{E}$ with respect to the Killing vector field Z. Note, $\pounds_Z$ is essentially a generalisation of the Lichnerowicz spinor Lie derivative [136, Sec. 6] for stationary spacetimes when one does not necessarily have a spin-structure.

4.4.2 Classical dynamics

The primary tenet of the semiclassical analysis is to connect the relativistic trace formula with its classical dynamics. In non-relativistic mechanics, the cotangent bundle $T^*\Sigma$ models the classical phase space, whereas the Hilbert-space quantum dynamics takes place in $L^2(\Sigma)$. The naive expectation of using this pair or the pair $(T^*\mathscr{M}, L^2(\mathscr{M}; \mathscr{E}))$ does not work because the former depends on the choice of Cauchy hypersurface $\Sigma \subset \mathscr{M}$ and for the latter pair, the (Killing flow invariant) sesquilinear form $(\cdot|\cdot)$ on $\mathscr{E}$ (Assumption 4.3.1 (a)) does not induce any L^2-norm. One can, of course, choose an arbitrary hermitian form in order to have a L^2 space on $\mathscr{E}$, but then this L^2-space will depend on the particular choice of the hermitian form as $\mathscr{M}$ is non-compact. In pursuance of defining the correct classical dynamics, one notes $\operatorname{Char} D$ is the lightcone bundle $\dot{T}_0^*\mathscr{M}$. Since $\operatorname{Char} D$ and $\operatorname{Char} \square$ are identical, the classical dynamics in this case coincides with that in the Strohmaier-Zelditch trace formula (Section 1.1.5) and hence we adopt their formulae. The metric-Hamiltonian H_{g} is a homogeneous function of degree 2 in the cotangent fibres. Referring to this as the dilation, let $\mathbb{E}$ be the Euler vector field which is the generator of this action. On $\dot{T}_0^*\mathscr{M}$, clearly H_{g} vanishes and we have $[\mathbb{E}, X_{\mathsf{g}/2}]_- = X_{\mathsf{g}/2}$ where $X_{\mathsf{g}/2}$ is the Hamiltonian vector field of H_{g}. Then the Hamiltonian reduction of $\dot{T}_0^*\mathscr{M}$ is the manifold of scaled-lightlike geodesics $\mathscr{N}$. That is, if $(\mathscr{M}, \mathsf{g})$ is geodesically complete then $\mathscr{N}$ is the quotient of $\dot{T}_0^*\mathscr{M}$ by the $\mathbb{R}$-group action generated by $X_{\mathsf{g}/2}$. Similarly, by taking the quotient of $\mathscr{N}$ by the $\mathbb{R}_+$-group action generated by $\mathbb{E}$ we obtain the space of unparametrised-lightlike geodesics $\tilde{\mathscr{N}}$. If $(\mathscr{M}, \mathsf{g})$ is a spatially compact globally hyperbolic spacetime then $\tilde{\mathscr{N}}$ is a *conic compact contact manifold* whose symplectisation is the conic symplectic manifold $\mathscr{N}$ induced from the conic contact manifold $\dot{T}_0^*\mathscr{M}$ [122, pp. 10-12], [123, Thm. 2.1].

We remark that $\mathscr{N}$ is defined *invariantly* and [27, Prop. 2.1] (see also [92, 93])

$$\varsigma : \mathscr{N} \to \dot{T}^*\Sigma \tag{4.49}$$

is a homogeneous symplectomorphism. Furthermore, the geodesic flow on any spatially compact globally hyperbolic spacetime $(\mathscr{M}, \mathsf{g})$ does not necessarily have to be complete. For instance, $\mathscr{M} := (-1, 1) \times \Sigma$ with the spacetime metric $\mathsf{g} := \mathrm{d}t^2 - \mathsf{h}$ where (Σ, h) is a compact Riemannian manifold [27, Rem. 2.2].

As announced in Theorem 4.3.4 and 4.3.6, periodic geodesics γ on $\mathcal{N}$ play a vital role in our investigation. Employing the symplectomorphism (4.49), these can be considered periodic geodesics on $\dot{\mathrm{T}}^*\Sigma$ as well, which matches better with our intuitive expectation. However, $\dot{\mathrm{T}}^*\Sigma$ *depends* heavily on the *choice of Cauchy hypersurface Σ* and as discussed in Section 4.2, there is no canonical choice for Σ on a (globally hyperbolic) stationary spacetime $(\mathcal{M}, \mathrm{g}, \Xi)$. Hence, $\mathcal{N}$ offers the natural and invariant manifold to describe the classical dynamics in contrast to $\dot{\mathrm{T}}^*\Sigma$. Recall that a geodesic $c : \mathbb{R} \to \mathbb{R} \times \Sigma$ in $\mathcal{M}$ can be expressed as $c(s) = (t(s), c_\Sigma(s))$, where c_Σ is the base projection of $\varsigma(\gamma(s))$. A periodic geodesic c in $\mathcal{M}$ with period T then means

$$\forall s \in \mathbb{R} : c_\Sigma(s + T) = c_\Sigma(s). \tag{4.50}$$

We note that the set of T-periodic curves on $\mathcal{N}$ is precisely the length spectrum of (Σ, h) for an ultrastatic spacetime. For details, see, for instance [177–180].
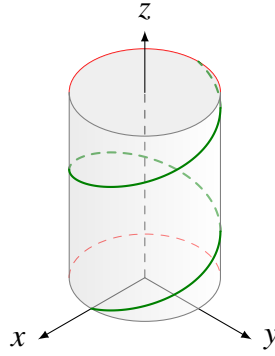
Figure 4.3: A periodic geodesic on $(\mathbb{R} \times \mathbb{S}, \mathrm{d}t^2 - \mathrm{d}x^2)$.

4.4.3 Time evolution operator

On a spatially compact globally hyperbolic spacetime this mapping is defined by

$$U_{t',t} := \mathcal{R}_t \circ (\mathcal{R}_{t'})^{-1} : C^\infty(\Sigma_{t'}; \mathscr{E}_{\Sigma_{t'}}) \to C^\infty(\Sigma_t; \mathscr{E}_{\Sigma_t}), \tag{4.51}$$

which is a homeomorphism and extends to a unitary operator (denoted by the same symbol) on the space of square integrable sections on Cauchy hypersurfaces, i.e.,

$$U_{t',t} : L^2(\Sigma_{t'}; \mathscr{E}_{\Sigma_{t'}}) \to L^2(\Sigma_t; \mathscr{E}_{\Sigma_t}) \tag{4.52}$$

is an isometry.

4.5 Trace formula

In this section, we will work on the set-up in Section 4.3.2. That is, $\Sigma \subset \mathcal{M}$ is an embedded submanifold and $\iota_\Sigma : \Sigma \hookrightarrow \mathcal{M}$ is proper (Remark 3.1.7). As a consequence, $\iota_\Sigma^* : C_{\mathrm{c}}^\infty(\mathcal{M}; \mathscr{E}) \to C_{\mathrm{c}}^\infty(\Sigma; \mathscr{E}_\Sigma) = C^\infty(\Sigma; \mathscr{E}_\Sigma)$.

Theorem 4.5.1. *As in the terminologies of Theorem 4.3.4, let* F *be the Schwartz kernel of the causal propagator for* D. *The smoothed-out time evolution operator* $U_\rho = (4.37)$ *is a trace-class operator on the Hilbert space* $\mathscr{H} := (\ker D,\ (3.156))$ *and its trace is given by*

$$\operatorname{Tr} U_\rho = -\mathrm{i} \int_\Sigma \operatorname{tr} \int_\mathbb{R} \big(\varXi_{-t}^* \circ \mathsf{F} \circ (\iota_\Sigma^*)^{-1}\, \sigma_D(\cdot, \zeta) \big)(x, y)\, (\mathscr{F}^{-1}\rho)(t)\, \mathrm{d}t|_{x=y}\, \mathrm{d}v_\mathsf{h}(x), \qquad (4.53)$$

where $\varXi_t^* : C_\mathrm{c}^\infty(\mathscr{M}; \mathscr{E}) \to C_\mathrm{c}^\infty(\mathscr{M}; \mathscr{E})$ *is the induced Killing flow (Figure 4.1) and* $\iota_\Sigma^* : C_\mathrm{c}^\infty(\mathscr{M}; \mathscr{E}) \to C^\infty(\Sigma; \mathscr{E}_\Sigma)$ *is the restriction map. In this thesis,* $\varXi_{-t}^*\mathsf{F}$ *is meant to be the pullback of* $\mathsf{F}(\cdot,)$ *in the first argument via* $\varXi_{-t}$.

Proof. By Assumption 4.3.1, the hermitian form (3.156), the retarded F^ret and advanced F^adv propagators, all are preserved under the action of $\varXi_t^*$. Hence F is preserved as well. In other words, if $u \in \ker D$ having initial data on some Σ then $\varXi_t^* u \in \ker D$ having Cauchy data on some Σ_t, which means that the time flow $\varXi_t$ induces time evolution of Cauchy data. Let us now choose an arbitrary but fixed Σ. This picks a global time coordinate t on $\mathscr{M}$ and then the generator of the Killing flow $\varXi_t$ is given by ∂_t. Thus, U_t is identified with the evolution of Cauchy data via the induced Killing flow (see Figure 4.2 for a schematic illustration):

$$U_t = \mathcal{R} \circ \varXi_{-t}^* \circ \mathcal{R}^{-1}. \qquad (4.54)$$

We read off $\mathcal{R}^{-1} = -\mathrm{i}\, F \circ (\iota_\Sigma^*)^{-1} \sigma_D(\cdot, \zeta)$ from (3.157) and observe that the twisted wave-front set (Lemma 3.4.6) of F contains only lightlike covectors. Therefore, integration over t results in a smooth Schwartz kernel $\mathsf{U}_\rho(x, y)$ and the expression of $\operatorname{Tr} U_\rho$ entails from (4.38).   ∎

4.6   Spectral theory of L on $\ker D$

Recall that, a **strongly continuous one-parameter unitary group** is a family $\{U_t | t \in \mathbb{R}\}$ of unitaries U_t on a Hilbert space $\mathcal{H}$ such that

$$\forall t, s \in \mathbb{R} : U_t U_s = U_{t+s}, \qquad (4.55a)$$

$$\forall u \in \mathcal{H}, \forall t \in \mathbb{R} : \lim_{h \to 0} U_{t+h} u = U_t u. \qquad (4.55b)$$

In particular, this implies that

$$U_{-t} = U_t^{-1} = U_t^*. \qquad (4.56)$$

By Stone's theorem (see e.g. [181, Thm. VIII.8]), every U_t has a unique generator A, i.e., $U_t = \mathrm{e}^{-\mathrm{i}tA}$. If

$$U_\rho := \int_\mathbb{R} \mathrm{e}^{-\mathrm{i}tA} (\mathscr{F}^{-1}\rho)(t)\, \mathrm{d}t \qquad (4.57)$$

is a compact operator for any $\rho \in \mathcal{S}(\mathbb{R})$ on $\mathbb{R}$ such that $\operatorname{supp}(\mathscr{F}^{-1}\rho)$ is compact, then the spectrum of A is discrete and consists of eigenvalues λ_n of finite algebraic multiplicities m_n. Moreover, whenever U_ρ is trace-class then

$$\operatorname{Tr} U_\rho = \sum_n m_n \rho(\lambda_n). \qquad (4.58)$$

Proof of Theorem 4.3.2. Since D commutes with induced Killing flow Ξ_t^* for all $t \in \mathbb{R}$ and U_t is the time evolution operator of D, the discreteness of $\mathrm{Spec}\, L$ and the polynomial growth of eigenvalues λ_n follow from the fact that $\mathsf{U}_\rho = (4.37)$ is a trace-class operator for $\rho \in \mathcal{S}(\mathbb{R})$ such that $\mathrm{supp}(\mathcal{F}^{-1}\rho)$ is compact. All λ_n are real as a consequence of the selfadjointness of L on $\ker D$. ∎

4.7 Proof of the main theorems

In order to implement the strategy outlined in Section 4.3.3, we recall that that

$$\pi_* \circ \Delta^* : C_c^\infty\big(\mathbb{R} \times \Sigma \times \Sigma; \mathrm{Hom}(\mathscr{E}_\Sigma, \mathscr{E}_\Sigma)\big) \to C^\infty(\mathbb{R}; \mathrm{End}\,\mathscr{E}_\Sigma) \tag{4.59}$$

is a zero-order Fourier integral operator whose Schwartz kernel K is that of an identity map [26, Lem. 5.2] (see also [10, Lem. 6.2, 6.3]):

$$\mathsf{K} \in I^0\Big(\mathbb{R} \times \mathbb{R} \times \Sigma \times \Sigma, C_{\pi_* \circ \Delta^*}; \mathrm{Hom}\big(\mathrm{Hom}(\mathscr{E}_\Sigma, \mathscr{E}_\Sigma), \mathrm{End}\,\mathscr{E}_\Sigma\big)\Big), \tag{4.60a}$$

$$C_{\pi_* \circ \Delta^*} = \big(\Delta(\mathbb{R} \times \Sigma)\big)^{\perp *}, \tag{4.60b}$$

$$\sigma_\mathsf{K} = \Pi^*\big(|\mathrm{d}t \wedge \mathrm{d}\tau \wedge \mathrm{d}x \wedge \mathrm{d}\xi|^{1/2}\big)\, \mathbb{1}\,. \tag{4.60c}$$

Here the canonical relation $C_\mathsf{K} = \{(t, \tau; t, \tau; x, \xi; x, -\xi) \in \dot{\mathrm{T}}^*\mathbb{R} \times \dot{\mathrm{T}}^*\mathbb{R} \times \dot{\mathrm{T}}^*\Sigma \times \dot{\mathrm{T}}^*\Sigma\}$ of $\pi_* \circ \Delta^*$ has been identified with the cornormal bundle $\big(\Delta(\mathbb{R} \times \Sigma)\big)^{\perp *}$ to the diagonal in $\mathbb{R} \times \Sigma \times \mathbb{R} \times \Sigma$, Π is the projector $C_{\pi_* \Delta^*} \ni (t, \tau; x, \xi; x, -\xi; t, -\tau) \mapsto (t, \tau; x, \xi) \in \dot{\mathrm{T}}^*\mathbb{R} \times \dot{\mathrm{T}}^*\Sigma$, and the Keller-Maslov bundle $\mathbb{T} \to \big(\Delta(\mathbb{R} \times \Sigma)\big)^{\perp *}$ is trivial (see Example A.1.21).

Theorem 4.5.1 implies that $\mathrm{Tr}\, U_t$ exists as a distribution in $\mathcal{D}'(\mathbb{R})$ and it can be re-expressed as

$$\mathrm{Tr}\, U_t = \int_\Sigma \mathrm{tr}\, \big((\Xi_{-t}^* \mathsf{F})(x, y)\, \sigma_D(y, \zeta)\big)\big|_{x=y}\, \mathrm{dv_h}(x). \tag{4.61}$$

Since the geodesic relation (Definition 3.2.15) is disjoint with the conormal bundle $\Sigma^{\perp *}$, restriction of $\Xi_{-t}^* \mathsf{F}$ is well-defined. Then applying (4.44), the preceding equation can be written as

$$\mathrm{Tr}\, U_t = \mathrm{tr}\, \Big(\pi_* \circ \Delta^* \circ (\iota_x^* \boxtimes \iota_y^*)\big((\Xi_{-t}^* \mathsf{F})(x, y)\, \sigma_D(y, \zeta)\big)\Big) \tag{4.62}$$

for a fixed but arbitrary $t \in \mathbb{R}$ and any $x, y \in \mathscr{M}$. Hence, our task is to compute the principal symbol of

$$\mathsf{F}_t := (\iota_x^* \boxtimes \iota_y^*)\big((\Xi_{-t}^* \mathsf{F})(x, y)\big). \tag{4.63}$$

We have worked out ι_Σ^* in Example 2.3.12, so let us begin by describing $\Xi_{-t}^* \mathsf{F}$ as a Lagrangian distribution. To begin with, one notes that w appearing in the expression of σ_F satisfies [35, Thm. 6.6.1]

$$(X_{\mathsf{g}^\pm/2} \pm \mathrm{i}\, \sigma_{D^2,\pm}^{\mathrm{sub}})w = 0, \tag{4.64}$$

by Lemma 3.4.6 and Definition 3.3.6, where $\mathsf{g}^\pm$ resp. $\sigma_{\Box,\pm}^{\mathrm{sub}}$ are the lifts of g resp. $\sigma_\Box^{\mathrm{sub}}$ to $\dot{\mathrm{T}}^*\mathscr{M} \times \dot{\mathrm{T}}^*\mathscr{M}$ via the projections on the first resp. second copies of $\dot{\mathrm{T}}^*\mathscr{M}$. This means

that $\mathrm{i}(2\pi)^{3/4}\sigma_D \circ w\sqrt{|\mathrm{d}t|} \otimes \sqrt{|\mathrm{d}v_C|}/2$ is the principal symbol of $\Xi^*\mathsf{F}$ on each C^+ and C^-. Employing (4.47), (4.48), Lemma 3.4.6, and Appendix A.2.3, it is then straightforward to obtain

Lemma 4.7.1. *Suppose that* $(\mathscr{E} \to \mathscr{M}, \sigma_D)$ *is a bundle of Clifford modules over a globally hyperbolic stationary spacetime* $(\mathscr{M}, \mathrm{g}, \Xi)$ *and that* F *is the Schwartz kernel of the causal propagator of any Dirac-type operator* D *on* $\mathscr{E}$. *Then the pullback (with respect to the first argument) of* F *by the Killing flow,*

$$\Xi^*\mathsf{F} \in I^{-3/4}\big(\mathbb{R} \times \mathscr{M} \times \mathscr{M}, \Gamma' \circ C; \mathrm{Hom}(\mathscr{E}, \mathscr{E})\big), \tag{4.65a}$$

$$\Gamma' \circ C = \big\{ (t, -\xi(Z); x, \xi; y, \eta) \in \dot{\mathrm{T}}^*\mathbb{R} \times \dot{\mathrm{T}}_0^*\mathscr{M} \times \dot{\mathrm{T}}_0^*\mathscr{M} \mid (x, \xi) = \mathrm{T}^*\Xi_t \circ \Phi_s(y, \eta) \big\}, \tag{4.65b}$$

$$\sigma_{\Xi^*\mathsf{F}} \asymp \mathrm{i}(2\pi)^{\frac{3}{4}}\sigma_D \circ w\big(\mathrm{T}^*\Xi_t \circ \Phi_s(y, \eta), (y, \eta)\big) \, |\mathrm{d}t|^{\frac{1}{2}} \otimes \big|\mathrm{d}v_C\big(\mathrm{T}^*\Xi_t \circ \Phi_s(y, \eta), (y, \eta)\big)\big|^{\frac{1}{2}}, \tag{4.65c}$$

where the principal symbol is modulo Keller-Maslov part, Z is the infinitesimal generator of Ξ and all other symbols are as defined in Lemma 3.4.6 and (4.47b).

Next, we compute the restriction of $\Xi^*\mathsf{F}$ on $\Sigma_x \times \Sigma_y$ by an application of Example 2.3.12.

Lemma 4.7.2. *As in the terminologies of Lemma 4.7.1, let Σ be a Cauchy hypersurface of $\mathscr{M}$. Then the distribution (4.63)*

$$\mathsf{F}_t \;\in\; I^{-1/4}\big(\mathbb{R} \times \Sigma \times \Sigma, C_t; \mathrm{Hom}(\mathscr{E}_\Sigma, \mathscr{E}_\Sigma))\big), \tag{4.66a}$$

$$C_t \;:=\; \big\{ \big((t, -\xi(Z)), (x, \xi)|_{\mathrm{T}\Sigma}, (y, \eta)|_{\mathrm{T}\Sigma}\big) \in \dot{\mathrm{T}}^*\mathbb{R} \times \mathrm{T}^*\mathscr{M}_\Sigma \times \mathrm{T}^*\mathscr{M}_\Sigma \mid$$
$$(x, \xi) = \mathrm{T}^*\Xi_t \circ \Phi_s(y, \eta) \big\}, \tag{4.66b}$$

$$\sigma_{\mathsf{F}_t} \;\asymp\; \mathrm{i}(2\pi)^{\frac{1}{4}}\sigma_D \circ w\big(\mathrm{T}^*\Xi_t \circ \Phi_s(y, \eta)|_{\mathrm{T}\Sigma}, (y, \eta)|_{\mathrm{T}\Sigma}\big)$$
$$|\mathrm{d}t|^{\frac{1}{2}} \otimes \big|\mathrm{d}v_C\big(\mathrm{T}^*\Xi_t \circ \Phi_s(y, \eta)|_{\mathrm{T}\Sigma}, (y, \eta)|_{\mathrm{T}\Sigma}\big)\big|^{\frac{1}{2}}, \tag{4.66c}$$

where the principal symbol is modulo the Keller-Maslov part.

Proof. The first two assertions are immediate from (2.132a), (2.132b), (4.65a) and (4.65b). To compute the principal symbol one notes that $\mathrm{d}v_{\dot{\mathrm{T}}_0^*\mathscr{M}}(y, \eta)$ induces a volume element $\mathrm{d}v_{\dot{\mathrm{T}}_0^*\mathscr{M}_\Sigma}(y, \eta) := \mathrm{d}v_{\dot{\mathrm{T}}_0^*\mathscr{M}}/\mathrm{d}y^1$ on $\dot{\mathrm{T}}_0^*\mathscr{M}_\Sigma$ when Σ is parametrised by $y^1 = \mathrm{cst}$ and then $\mathrm{d}v_{\dot{\mathrm{T}}_0^*\mathscr{M}_\Sigma}(y, \eta) = \mathrm{d}\eta_1 \wedge \mathrm{d}y' \wedge \mathrm{d}\eta'$ in the adapted coordinates $(y^1 = \mathrm{cst}, y', \eta_1, \eta')$ on $\mathrm{T}^*\mathscr{M}$. Finally, the claim follows from (2.132c), (4.65c), (2.132c) and transporting (y, η) to (x, ξ) by the geodesic flow and the Killing flow. $\blacksquare$

If the composition $C'_{\pi_*\Delta^*} \circ C_t$ is clean then we can compute $\mathrm{Tr}\, U_t$ by the standard composition of Fourier integral operators. But, in general,

$$C'_{\pi_*\Delta^*} \circ C_t = \big\{ (t, \tau) \in \mathbb{R} \times \dot{\mathbb{R}} \mid \tau = -\xi(Z), (x, \xi)|_{\mathrm{T}\Sigma} = \big(\mathrm{T}^*\Xi_{-t} \circ \Phi_t(y, \eta)\big)|_{\mathrm{T}\Sigma} \big\} \tag{4.67}$$

may not be clean because (see Appendix A.1.3: (A.38), (A.44), and (A.45) for the symbol $\star$ and for further details) the fibres of $C'_{\pi_*\Delta^*} \star C_t \to C'_{\pi_*\Delta^*} \circ C_t$ can be identified with

the fibres over $\tau \in \dot{\mathrm{T}}_t^*\mathbb{R}$, i.e., the set $\{\mathfrak{F}_t\} := (4.34)$ of periodic geodesics, where we have used (4.22) and (4.49). Then, even if $\{\mathfrak{F}_t\}$ happens to be manifolds, the chances of $\dim \mathfrak{F}_t$ is a constant for all t is very low; for instance, if all the orbits of $\Xi_t^{\mathcal{N}}$ are periodic with the same period $t = T$, then the composition is clean (see e.g. [20, p. 289]). We remark that this, however, is not an issue for the trivial period ($T = 0$) and the assumptions made on classical dynamics in Theorem 4.3.6 ensures a clean intersection in the case of non-trivial periods ($T \neq 0$).

Let us record for the future computations that

$$\begin{aligned}
&\dim \mathscr{M} = d, \dim(\mathrm{T}^*\mathscr{M}) = 2d, \dim(\dot{\mathrm{T}}_0^*\mathscr{M}) = 2d - 1, \\
&\dim \mathscr{N} = 2d - 2, \dim \tilde{\mathscr{N}} = 2d - 3, \\
&\dim \Sigma = d - 1, \dim(\mathrm{T}^*\Sigma) = 2d - 2 = \dim(\dot{\mathrm{T}}_0^*\mathscr{M}_\Sigma).
\end{aligned} \tag{4.68}$$

4.7.1 Principal symbol of $\mathrm{Tr}\, U_t$ at $t = 0$

We begin with the trivial periodic orbits where a big singularity is expected as U_t reduces to an identity operator in this situation. To describe $\sigma_{\mathrm{Tr}\, U_0}$, it is useful to have the notion of the symplectic residue [182, Def. 6.1], introduced by Guillemin to derive Weyl's law in the context of Weyl algebra quantising a conic symplectic manifold. By construction (cf. (4.22)), H^{1-d} is a homogeneous function of degree $1 - d$. Then its symplectic residue is defined by

$$\mathrm{res}\, H^{1-d} := \int_{\tilde{\mathscr{N}}} \vartheta_{H^{1-d}}, \quad \vartheta := \mathbb{E} \lrcorner \mathrm{dv}_{\mathscr{N}}, \tag{4.69}$$

where $\lrcorner$ denotes the interior multiplication by the Euler vector field (Section 4.4.2) $\mathbb{E}$ on $\mathscr{N}$ and $\vartheta_{H^{1-d}}$ is the pull-back of $H^{1-d}\vartheta$ on $\mathscr{N}$. Homogeneity of H entails (cf. [182, Proof of Lemma 6.3])

$$\mathrm{res}\, H^{1-d} = (d - 1)\, \mathrm{vol}\, \mathscr{N}_{H \leq 1}. \tag{4.70}$$

On a globally hyperbolic standard stationary spacetime, $\mathrm{vol}\, \mathscr{N}_{H \leq 1}$ has been computed by Strohmaier-Zelditch [27, (15)] and it is given by (4.29).

Lemma 4.7.3. *As in the terminologies of Theorem 4.5.1 (and hence Theorem 4.3.4 as well),*

$$\mathsf{U}_t \in I^{-1/4}\big(\mathbb{R} \times \Sigma \times \Sigma, C_t; \mathrm{Hom}(\mathscr{E}_\Sigma, \mathscr{E}_\Sigma)\big) \tag{4.71}$$

associated with the canonical relation $C_t := (4.66\mathrm{b})$ and, modulo Keller-Maslov part, its principal symbol is given by

$$\sigma_{\mathsf{U}_t}(x, \xi; y, \eta) \asymp \sigma_{\mathsf{F}_t}(x, \xi; y, \eta)\, \sigma_D(y, \eta), \tag{4.72}$$

where σ_{F_t} is given by (4.66c).

Furthermore, $\operatorname{Tr} U_t$ *is a Lagrangian distribution on* $\mathbb{R}$ *of order* $d - 7/4$ *associated with the Lagrangian submanifold* Λ_T *and its principal symbol at* $T = 0$ *is given by*

$$\sigma_{\operatorname{Tr} U_0}(\tau) \asymp r \frac{d-1}{(2\pi)^{d-1}} \operatorname{vol}(\mathscr{N}_{H\le 1}) |\tau|^{d-2} \sqrt{|\mathrm{d}\tau|}. \tag{4.73}$$

Proof. The first assertion simply follows from $U_t(x, y) = F_t(x, y)\, \sigma_D(y, \zeta)$. We then use identification of $\mathscr{N}$ with $\dot{\mathrm{T}}_0^* \mathscr{M}_\Sigma$ so that $|\mathrm{dv}_{\mathscr{N}}|$ is identified with $|\mathrm{dv}_{\dot{\mathrm{T}}_0^* \mathscr{M}_\Sigma}|$. If E is a regular value of H then $\mathscr{N}_E$ is a codimension one $X_H|_{\mathscr{N}_E}$ invariant embedded submanifold of $\mathscr{N}$, which inherits a natural volume form $\mathrm{dv}_{\mathscr{N}_E}$ from $\mathrm{dv}_{\mathscr{N}}$, invariant under the action of $X_H|_{\mathscr{N}_E}$ [96, Thm. 3.4.12]:

$$\mathrm{dv}_{\mathscr{N}} = \mathrm{dv}_{\mathscr{N}_E} \wedge \frac{\mathrm{d}H}{\|\operatorname{grad} H\|} \Leftrightarrow \mathrm{dv}_{\mathscr{N}_E}(\ldots) = \mathrm{dv}_{\mathscr{N}}\left(\frac{\operatorname{grad} H}{\|\operatorname{grad} H\|}, \ldots\right). \tag{4.74}$$

One observes that at $T = 0$, $\mathrm{T}^* \varXi_0\big(\varPhi_0(y, \eta)\big) = (y, \eta)$ and $C = \varDelta \dot{\mathrm{T}}_0^* \mathscr{M}$ so that $C_0' = \big\{(0, -\xi(Z))\big\} \times (\varDelta \dot{\mathrm{T}}_0^* \mathscr{M})|_{\mathrm{T}\Sigma \times \mathrm{T}\Sigma}$ and $\big|\mathrm{dv}_C(y, \eta; y, \eta)|_{\mathrm{T}\Sigma \times \mathrm{T}\Sigma}\big|^{1/2} = |\mathrm{dv}_{\mathscr{N}}(y, \eta)|$. The excess of $C'_{\pi_* \varDelta^*} \circ C_0$ is (see Appendix (A.43))

$$e = \dim \mathfrak{F}_0 = 2d - 3. \tag{4.75}$$

Hence, by the composition of Lagrangian distributions, $\operatorname{Tr} U_0 \in I^{d-7/4}(\mathbb{R}, \Lambda_T)$. One observes [10, (6.6)] that $\varphi(t, \tau)$ defined by $(t - T)\tau$ (resp. 0) for $\tau < 0$ (resp. $\tau \ge 0$), is a phase function for Λ_T. Obviously, φ is not smooth around $\tau = 0$ but the required modification will only reflect by some smooth terms in the oscillatory integral described below, which we suppress notationally for brevity. Any element of this Lagrangian distribution can be written as scalar multiples $c_1, c_2, \ldots$ of an oscillatory integral of the form (cf. (2.34))

$$(\operatorname{Tr} U_0)(t) = (2\pi)^{-3/4}(2\pi)^{d-1} \int_{\mathbb{R}_{\ge 0}} e^{-\mathrm{i}t\tau}(c_1 \tau^{d-2} + c_2 \tau^{d-3} + \ldots)\mathrm{d}\tau, \tag{4.76}$$

because $\sigma_{\operatorname{Tr} U_0}$ must be of order $d-2$ (cf. Definition 2.1.9) as entailed by the order of $\operatorname{Tr} U_0$.

Computation of $\sigma_{\operatorname{Tr} U_0}$ involves the following four steps as detailed in Section A.2.3. Briefly speaking, first one obtains the tensor product $\sigma_K \otimes \sigma_{U_0}$ followed by intersecting with the diagonal. Then we integrate over $\mathfrak{F}_0$ and take tr of the hindmost quantity. Putting all these together and comparing with the last expression of $\operatorname{Tr} U_0$ we obtain

$$\sigma_{\operatorname{Tr} U_0}(0, \tau) = r \frac{\operatorname{res}(H^{1-d})}{(2\pi)^{d-1}} |\tau|^{d-2} \sqrt{|\mathrm{d}\tau|}, \tag{4.77}$$

where we have used Lemma 4.7.2 and Lemma 3.4.6 together with the identity

$$\operatorname{tr}\big(\sigma_D(y, \eta)\, \sigma_D(y, \zeta)\big) = \operatorname{tr} \frac{\sigma_D(y, \eta)\, \sigma_D(y, \zeta) + \sigma_D(y, \zeta)\, \sigma_D(y, \eta)}{2} = \mathrm{g}_y^{-1}(\eta, \zeta)\, \operatorname{rk}(\mathscr{E}) \tag{4.78}$$

using the cyclicity of trace and (3.122).   ∎

4.7.2   Principal symbol of $\mathrm{Tr}\,U_t$ at $t = T \in \mathcal{P}$

This is the scenario corresponding to the non-trivial periodic orbits when $C_{\pi_*\Delta^*} \circ C_T$ is clean or equivalently the set of fixed points $\mathfrak{F}_T$ is clean. Recall, the fixed point sets of $\Xi_T^{\mathcal{N}}$ are the union of periodic orbits γ of $\Xi_T^{\mathcal{N}}$ which depends substantially on manifold $(\mathcal{M}, \mathrm{g})$ and we are only concerned about non-degenerate orbits as briefly defined below (details are available in [10, Sec. 4]). Let σ be the symplectic form on $\mathrm{T}^*\mathcal{M}$ (see Example A.1.2). The induced Killing flow $\Xi_s^{\mathcal{N}}$ preserves the level sets $\{H(c) = \tau\}$ since $\Xi_s^{\mathcal{N}}$ is the Hamiltonian $H = (4.22)$ flow on $\mathcal{N}$. Then its restriction to $\mathcal{V} := \mathrm{T}_c\{H = \mathrm{E}\}$ for any E satisfies $\sigma|_{\mathcal{V}}(X_H, \cdot) = 0$ and by the *non-degeneracy* [27, p. 44] of γ, it is meant that the *nullspace of $\sigma|_{\mathcal{V}}$ is spanned by X_H*. This condition implies that $\ker(I - \mathrm{d}_\gamma \Xi_s^{\mathcal{V}}) = \mathbb{R}X_H$ and $\mathrm{img}(I - \mathrm{d}_\gamma \Xi_s^{\mathcal{V}}) = \mathcal{V}/\mathbb{R}X_H$ where $\mathrm{d}_\gamma \Xi_s^{\mathcal{N}} : \mathcal{V} \to \mathcal{V}$ and $\ker(I - \mathrm{d}_\gamma \Xi_s^{\mathcal{N}})$ is to be understood as the kernel of $I - \mathrm{d}_\gamma \Xi_s^{\mathcal{N}}$ on $\mathcal{V}$. In this setting, the linearised Poincaré map P_γ is defined as

$$I - \mathrm{P}_\gamma : \mathcal{V}/\ker(I - \mathrm{d}_\gamma \Xi_s^{\mathcal{N}}) \to \mathcal{V}/\ker(I - \mathrm{d}_\gamma \Xi_s^{\mathcal{N}}) \tag{4.79}$$

the linear symplectic quotient map induced by $I - \mathrm{d}_\gamma \Xi_s^{\mathcal{N}}$.

Lemma 4.7.4. *As in the terminologies of Theorem 4.3.6, the principal symbol of $\mathrm{Tr}\,U_t$ at $t = T \in \mathcal{P}$ is*

$$\sigma_{\mathrm{Tr}\,U_T}(\tau) = \frac{\tau^{d-2}}{(2\pi)^{d-1}} \int_{\mathfrak{F}_T} \mathrm{tr}\left(\sigma_D(\gamma)\, \mathcal{T}_{\Xi_T^{\mathcal{N}}(\gamma)} \sigma_D(x, \zeta)\right) \frac{\mathrm{e}^{-\mathrm{i}\pi\mathfrak{m}(\gamma)/2}|\mathrm{d}T| \otimes \sqrt{|\mathrm{d}\tau|}}{\sqrt{|\det(I - \mathrm{P}_\gamma)|}}. \tag{4.80}$$

Proof. We begin with the fact that non-degenerate γ belongs to a 2-dimensional cylinder, transversally intersecting the energy hypersurfaces [96, p. 576] which implies that one can use $\tau = -H(\gamma)$ as a coordinate for fibre over (T, τ). Then the half-density valued density at (T, τ) is given by [10, pp. 60 - 61] (see also, e.g. [25, Thm. 6.1.1])

$$|\mathrm{d}T| \otimes \sqrt{\frac{|\mathrm{d}\tau|}{|\det(I - \mathrm{P}_\gamma)|}} \tag{4.81}$$

where $|\mathrm{d}T|$ is the density on γ induced by the flow.

To compute $\sigma_{\mathrm{Tr}\,U_T}$, we use identifications $\mathcal{N} = \dot{\mathrm{T}}_0^*\mathcal{M}_\Sigma$, $|\mathrm{dv}_{\mathcal{N}}| = |\mathrm{dv}_{\dot{\mathrm{T}}_0^*\mathcal{M}_\Sigma}|$ once again. For $T \neq 0$, C is no more given by $\Delta \dot{\mathrm{T}}_0^*\mathcal{M}$ and so we read off the off-diagonal expression of σ_F from Lemma 3.4.6 to obtain

$$\sigma_{\mathrm{U}_T}(x, \xi; y, \eta) \asymp (2\pi)^{\frac{1}{4}} \mathrm{i}\, \sigma_D(x, \xi)\, w(x, \xi; y, \eta)|\mathrm{d}t|^{\frac{1}{2}} \otimes \left|\mathrm{dv}_C(x, \xi; y, \eta)\right|^{\frac{1}{2}} \tag{4.82}$$

modulo the Keller-Maslov contribution, where $(x, \xi) = \mathrm{T}^*\Xi_t \circ \Phi_s(y`, \eta`)|_{\mathrm{T}\Sigma}$, $(y, \eta) = (y`, \eta`)|_{\mathrm{T}\Sigma}$ for $(y`, \eta`) \in \dot{\mathrm{T}}_0^*\mathcal{M}$. As before, $\sigma_{\mathrm{Tr}\,U_T}$ is computed employing (4.44) by performing the four steps used in Lemma 4.7.3. These yield the claimed expression

$$\sigma_{\mathrm{Tr}\,U_T}(T, \tau) \asymp \frac{\tau^{d-2}}{(2\pi)^{d-1}} \int_{\mathfrak{F}_T} \mathrm{tr}\left(\sigma_D(\gamma)\, \mathcal{T}_{\Xi_T^{\mathcal{N}}(\gamma)} \sigma_D(x, \zeta)\right) \frac{|\mathrm{d}T| \otimes \sqrt{|\mathrm{d}\tau|}}{\sqrt{|\det(I - \mathrm{P}_\gamma)|}} \tag{4.83}$$

modulo the contribution coming from the Keller-Maslov line bundle $\mathbb{M} \to \Lambda_T$.

Thus, we are left with the computation of the $\mathbb{M}$-part which has been calculated in [10, (6.16)] (see also [172, Sec. 3]). For completeness, we briefly outline the main steps. First, one constructs $\mathbb{L} \to C_T$ from $\mathbb{L}_C, \mathbb{L}_\Gamma$ and $\mathbb{L}_\Lambda$ following the procedure detailed in Section A.1.4. Let φ resp. ϕ be generating functions of $C_{\pi_* \Delta^*}$ resp. C_T. Then $\psi := \varphi + \phi$ locally generate Λ_T. Suppose that we partition $[0,1]$ as $0 = s_0 < s_1 < \ldots < s_N = 1$ such that $\{\mathcal{L}_\alpha\}_{k=0,\ldots,N}$ be a conic covering of Λ_T around $\gamma(s_\alpha)$ and that $\{\psi_\alpha\}$ corresponding generating functions. Then, following the steps explained in Section A.1.5 one obtains that $e^{-i\pi \mathfrak{m}(\gamma)/2}$ is the Keller-Maslov contribution.   ∎

4.8 Proof of Weyl law

As shown in Theorem 4.3.4 and 4.3.6 that $\operatorname{Tr} U_t$ has singularities at $t = 0, T$. Our aim is to only cut the $t = 0$ singularity. In order to do so, one introduces a Schwartz function $\chi(\lambda)$ on $\mathbb{R}$ such that (see e.g. [11, p. 133], [8, Sec. 21.1-3], [183, Thm. 7.5])

(i) $\chi(\lambda) > 0$ for all λ;

(ii) $(\mathcal{F}\chi)(0) = 1$;

(iii) $(\mathcal{F}\chi)(t) = (\mathcal{F}\chi)(-t)$;

(iv) $\operatorname{supp}(\mathcal{F}_{\lambda \mapsto t}\chi) \subset (-\varepsilon, \varepsilon)$, where $\varepsilon > 0$ is sufficiently small.

Then employing expression of $\operatorname{Tr} U_0$ from Theorem 4.3.4 and the aforementioned properties of χ, we obtain

$$
\begin{aligned}
\mathcal{F}_{t \mapsto \lambda}^{-1}\big(\mathcal{F}_{\cdot \mapsto t}(\chi) \operatorname{Tr} U_t\big) \ &\approx \ \int \mathrm{d}\tau \, \delta(\tau - \lambda) r \frac{\operatorname{res}(H^{1-d})}{(2\pi)^{d-1}} \tau^{d-2} + \ldots \\
&= \ r(d-1) \frac{\operatorname{vol}(\mathcal{N}_{H \le 1})}{(2\pi)^{d-1}} \lambda^{d-2} + \ldots,
\end{aligned}
\tag{4.84}
$$

where $f \approx g$ means that $f/g \to 1$ as $t \to 0$ for any functions $f(t), g(t)$. That means that

$$
(\chi * \mathrm{d}\mathsf{N})(\lambda) = r(d-1) \frac{\operatorname{vol}(\mathcal{N}_{H \le 1})}{(2\pi)^{d-1}} \lambda^{d-2} + O(\lambda^{d-3}),
\tag{4.85}
$$

where $*$ denotes the convolution. Employing

$$
(\chi * \mathsf{N})(\lambda) = \int_{-\infty}^{\lambda} (\chi * \mathrm{d}\mathsf{N})(\mu) \, \mathrm{d}\mu = r \frac{\operatorname{vol}(\mathcal{N}_{H \le 1})}{(2\pi)^{d-1}} \lambda^{d-1} + O(\lambda^{d-2}).
\tag{4.86}
$$

The Weyl law entails from

$$
\mathsf{N}(\lambda) = (\mathsf{N} * \chi)(\lambda) - \int \big(\mathsf{N}(\lambda - \mu) - \mathsf{N}(\lambda)\big) \chi(\mu) \, \mathrm{d}\mu
\tag{4.87}
$$

together with the facts $\mathsf{N}(\lambda - \mu) - \mathsf{N}(\lambda) \lessgtr \langle \mu \rangle^{d-1} \langle \lambda \rangle^{d-2}$ and $\int \chi(\mu) \, \mathrm{d}\mu = 1$.

4.9 Literature

The idea of the asymptotic trace formula was originally due to Martin GUTZWILLER [12] in the context of quantum mechanics, which was given a rigour mathematical underpinning by Johannes J. DUISTERMAAT and Victor W. GUILLEMIN [10]. We refer to the references mentioned before Section 1.1.2 for important chronological steps in between and, for instance, the survey article [184] for a historical detour.

Jens BOLTE and Stefan KEPPELER [185, 186] have generalised Gutzwiller's work for spin Dirac operators on Minkowski spacetime (see also the elucidating articles [187, 188] and references therein for chronological developments). In contemporary to their reports, mathematical rigour analysis has been reported by Mary R. SANDOVAL [26] promoting the Duistermaat-Guillemin framework for Dirac-type operators on a closed Riemannian manifold. Her result on Dirac-wave-trace invariants for trivial periods is closely related to the evaluation of the residues of the eta-invariant (by Branson and Gilkey [163]) and the behaviour of eigenfunctions in the high energy limit (by Jakobson and Strohmaier [106]). In the particular case of a massless spin-Dirac operator on a closed 3-Riemannian manifold, Capoferri and Vassiliev [174] have computed the third local Weyl coefficients by advancing the framework by Chervova *et al.* [189] (see also the review by Avetisyan *et al.* [29]) for asymptotic spectral analysis of a general elliptic first-order system; see [189, Sec. 11] and [7, Chap. 3] for a bibliographic overview. An in-depth investigation of different spectral coefficients of a selfadjoint Laplace-type operator on a hermitian vector bundle over a closed Riemannian manifold has been performed by Li and Strohmaier [30]. In particular, they have obtained the relevant coefficients for Dirac-type operators with a more general bundle endomorphism than in Sandoval [26] (identity endomorphism) and in Branson-Gilkey [163] (scalar endomorphism), and bridged the results by Chervova *et al.* [189] (and their follow up works as mentioned in [30]) with the known heat-trace invariants.

The primary and common ingredient of [26, 174, 185, 186] is to determine the time evolution operator (modulo smoothing operators) by solving the transport equations order by order. Bolte-Keppeler have also identified terms responsible for spin-magnetic and spin-orbit interactions in the semiclassical expression. However, they have finally considered regularised truncated time evolution operator by introducing an energy localisation to deal with the continuous spectrum of Dirac Hamiltonian arising due to the non-compact Minkowski spacetime. Such restrictions were absent in the hindmost references as they considered closed Riemannian manifolds and utilised the full power of Fourier integral operator theory in contrast to Bolte-Keppeler who have worked with oscillatory integrals. A novel feature in the lines of research by Vassiliev and his collaborators [29, 189] is the second term of the Weyl law (see the references cited in these papers for earlier works). Analogous results have been achieved by Li-Strohmaier [30] employing a different spectral analysis. Specifically, the global phase function approach of Fourier integral operators has been deployed in [174] to construct (modulo smooth-

ing operators) the solution operator of a massless spin-Dirac operator on a 3-dimensional closed Riemannian manifold. They have also provided a closed formula for the principal symbol and an algorithm for computing the subprincipal symbol of this operator.

A general relativistic generalisation of Duistermaat-Guillemin-Gutzwiller trace formula has been initiated by Strohmaier and Zelditch [27, 33] (see also the review [34]) who have studied the d'Alembertian on a globally hyperbolic spatially compact stationary spacetime as briefed in Section 1.1.5. Their crucial step was to set up a relativistic description of the classical and the quantum dynamics and advance the celebrated Duistermaat-Guillemin [10] framework accordingly. In particular, they have expressed the time evolution operator by means of the causal propagator of d'Alembertian and Killing flow, and computed its principal symbol by utilising the symbolic calculus of Fourier integral operators based on Duistermaat and Hörmander's [35] classic work of distinguished parametrices. Subsequently, they employed Guillemin's symplectic residue [182] approach at trivial period and tailored the Duistermaat-Guillemin computation for the non-trivial periods. Consequently, the Weyl law in the space of light-like geodesics has been reported by them using the standard Fourier-Tauberian argument. Apart from this investigation, their work has been generalised in a bundle setting for a d'Alembertian on a globally hyperbolic stationary Kaluza-Klein spacetime by Mc-Cormick [32] who utilised some technical results of this thesis.

Appendix A

Canonical Relations & Compositions

In this chapter we will first briefly recall some standard notions of conic symplectic geometry and then review canonical relations, their clean intersection, and the composition of bundle-valued densities on the them.

A.1 Canonical relations

A.1.1 Preliminaries on conic symplectic geometry

We recall that a **symplectic manifold** is a pair (M, σ) where M is a smooth manifold and σ is a smooth, non-degenerate, closed 2-form. Symplectic manifolds must be even dimensional and are always *orientable* because they admit the **Liouville form** dv_M defined in (2.63). If $(M, \sigma_{d_M}), (N, \sigma_N)$ are two symplectic manifolds and $\varkappa : N \to M$ is a smooth map (resp. diffeomorphism - in this case $\dim M = \dim N$) with

$$\varkappa^* \sigma_M = \sigma_N \tag{A.1}$$

then $\varkappa$ is called a **symplectic map** (resp. **symplectomorphism**). The latter is known as a *canonical transformation* in Physics literature. If $\dim M = \dim N$ then any symplectic map $\varkappa$ is a volume preserving $\varkappa^* v_M = v_N$ local diffeomorphism. Stated differently, the total volume of a symplectic manifold is a global symplectic invariant. By the Darboux theorem, *every $2d$-dimensional symplectic manifold (M, σ) is locally symplectomorphic to* $(\mathbb{R}^{2d}, \sigma_0)$ where σ_0 is the standard symplectic form on $\mathbb{R}^{2d}$. This implies that

$$\sigma \restriction U = dx^i \wedge d\xi_i, i = 1, \dots, d, \quad v_M \restriction U = dx^1 \wedge \dots \wedge dx^d \wedge d\xi_1 \wedge \dots \wedge d\xi_d = dx d\xi \tag{A.2}$$

on any symplectic chart $(U, (x^i, \xi_i))$ for (M, σ).

We recall σ induces a vertical vector bundle isomorphism $\sigma^\flat : TM \to T^*M$ whose inverse is denoted by $\sigma^\sharp : T^*M \to TM$. Let $f \in C^\infty(M, \mathbb{R})$. Then the vector field

$$X_f := (df)^\sharp \tag{A.3}$$

is called the **Hamiltonian vector field** generated by f, which is locally given by

$$X_f(x,\xi) = \frac{\partial f}{\partial \xi_i}(x,\xi)\frac{\partial}{\partial x^i}(x,\xi) - \frac{\partial f}{\partial x^i}(x,\xi)\frac{\partial}{\partial \xi_i}(x,\xi). \tag{A.4}$$

The **Poisson bracket** is defined by

$$\{f,g\} := \sigma(X_f, X_g) = X_f(g) \tag{A.5}$$

for any $f, g \in C^\infty(\mathsf{M}, \mathbb{R})$. Locally it reads

$$\{f,g\}(x,\xi) = \frac{\partial f}{\partial \xi_i}(x,\xi)\frac{\partial g}{\partial x^i}(x,\xi) - \frac{\partial f}{\partial x^i}(x,\xi)\frac{\partial g}{\partial \xi_i}(x,\xi). \tag{A.6}$$

Definition A.1.1. A d-dimensional manifold $(\mathcal{M}, \mathsf{m}_\lambda)$ is called **conic** if there is a given smooth map $\mathbb{R}_+ \times \mathcal{M} \ni (\lambda, x) \mapsto \mathsf{m}_\lambda(x) := \mathsf{m}(\lambda, x) \in \mathcal{M}$ and for every $x \in \mathcal{M}$, there is an open neighbourhood U with $\mathsf{m}_{\mathbb{R}_+}(U) = U$ and a diffeomorphism $\kappa : U \to \mathbb{U}$ such that $\lambda\kappa = \kappa\,\mathsf{m}_\lambda$, where $\mathbb{U}$ is an open cone in $\dot{\mathbb{R}}^d$.

If a conic manifold $(\mathsf{M}, \mathsf{m}_\lambda)$ is also symplectic (M, σ) with the property

$$\mathsf{m}_\lambda^* \sigma = \lambda\sigma, \tag{A.7}$$

then $(\mathsf{M}, \mathsf{m}, \sigma)$ is called a **conic symplectic manifold** (see e.g. [77, Definition 21.1.8]).

Since $\mathsf{m}_\lambda \circ \mathsf{m}_{\lambda'} = \mathsf{m}_{\lambda\lambda'}$ holds true in $\dot{\mathbb{R}}^d$, succinctly speaking, a conic manifold $(\mathcal{M}, \mathsf{m})$ is a manifold endowed with a *free, proper,* and *smooth $\mathbb{R}_+$-multiplicative group action* m on $\mathcal{M}$ (see e.g. [84, Def. 2.1.1]). If $f \in C^\infty(\mathcal{M})$ then a **radial vector field** is defined by

$$\mathscr{X}(f) := \frac{\mathrm{d}(\mathsf{m}_\lambda^* f)}{\mathrm{d}\lambda}(1) \tag{A.8}$$

and locally this means $\mathscr{X} = \xi_i \partial/\partial\xi_i$.

Example A.1.2. If M is a d-dimensional manifold then its cotangent bundle T^*M can naturally be seen as a symplectic manifold because it carries a *canonical* 1-*form*, also known as the **Liouville 1-form** θ whose negative differential is **canonical symplectic form** $\sigma := -\mathrm{d}\theta$ on T^*M. In symplectic coordinates (x^i, ξ_i) on T^*M,

$$\theta = \xi_i\,\mathrm{d}x^i, \quad \sigma = \mathrm{d}x^i \wedge \mathrm{d}\xi_i. \tag{A.9}$$

Whilst (T^*M, σ) is a symplectic manifold, the punctured cotangent bundle $(\dot{\mathrm{T}}^*M, \sigma)$ is naturally a conic symplectic manifold where

$$\mathsf{m}_\lambda(x,\xi) := (x, \lambda\xi), \quad \kappa(x,\xi) := \big((|\xi_1| + \ldots + |\xi_d|)(x^1, \ldots, x^d), (\xi_1, \ldots, \xi_d)\big). \tag{A.10}$$

Example A.1.3. Let $\mathscr{V} \to M$ be a $\mathbb{R}$-vector bundle of rank n over a manifold M. Then $(\dot{\mathscr{V}}, \mathsf{m}_\lambda)$ is a conic manifold with respect to the multiplication m_λ defined as all dilations in its fibres:

$$\mathsf{m}_\lambda : \dot{\mathscr{V}}_x \to \dot{\mathscr{V}}_x, \ v(x) \mapsto \mathsf{m}_\lambda\big(v(x)\big) := (\lambda v)(x) \tag{A.11}$$

for any $\lambda \in \mathbb{R}_+$. Locally $\mathscr{V} \cong U \times \mathbb{R}^n$ and $\dot{\mathscr{V}} \cong U \times \dot{\mathbb{R}}^n$ for some open set $U \subset M$ (see e.g. [84, p. 27]). $\dot{\mathscr{V}}$ being a conic then locally translates

$$(x; \theta) \in U \times \dot{\mathbb{R}}^n \Rightarrow \mathsf{m}_\lambda(x; \theta) = (x; \lambda\theta) \in U \times \dot{\mathbb{R}}^n. \tag{A.12}$$

A set $\mathcal{U} := \{(x; \theta) \in U \times \mathbb{R}^n\}$ is called a **conic subset** of $U \times \mathbb{R}^n$ if it is stable under all dilation (A.12): $(x; \theta) \mapsto (x; \lambda\theta)$. Thereby, a positively homogeneous function a of degree s on $\mathcal{U}$ means (cf. Definition A.1.1)

$$a(x; \lambda\theta) = \lambda^s a(x; \theta). \tag{A.13}$$

Theorem A.1.4 (Homogeneous Darboux Theorem). *Let* $(\mathsf{M}, \mathsf{m}_\lambda, \sigma)$ *be a* $2d$-*dimensional conic symplectic manifold. Suppose that I and J are subsets of $\{1, \ldots, d\}$ and that $(x^i)_{i \in I}$ and $(\xi_j)_{j \in J}$ are smooth functions on a conic neighbourhood of any $(x, \xi) \in \mathsf{M}$ such that [35, Prop. 6.1.3] (see also [23, Thm. 21.1.9])*

 (a) x^i and ξ_j are homogeneous of degree 0 and 1, respectively,

 (b) x^i Poisson commutes with x^j for all $i, j \in I$ and so does ξ_i with ξ_j for all $i, j \in J$ whilst $\{\xi_j, x^i\} = \delta^i_j$ for all $i \in I$ and for all $j \in J$,

 (c) the Hamiltonian vector fields X_{x^i}, X_{ξ_j} and the radial vector field $\mathscr{X}$ are linearly independent at (x, ξ),

 (d) there are arbitrary real numbers $\boldsymbol{x}^i$ and $\boldsymbol{\xi}_j$ so that $x^i(x, \xi) = \boldsymbol{x}^i$ and $\xi_j(x, \xi) = \boldsymbol{\xi}_j$ with the assumption that $\boldsymbol{\xi}_k \neq 0$ for some $k \notin I$.

Then there exists smooth functions $(x^{i'})_{i' \notin I}$ and $(\xi_{j'})_{j' \notin J}$ in a conic neighbourhood of (x, ξ) such that (a), (b) and (d) hold true for indices running from 1 to d, and (c) remains valid if $i' \neq k$. In particular, $(x^i; \xi_i)$ defines a homogeneous[1] symplectomorphism

$$\varkappa : \mathcal{U} \to \varkappa(\mathcal{U}), \qquad \mathsf{m}_\lambda^* \varkappa = \lambda \varkappa \tag{A.14}$$

of degree 1 from an open conic neighbourhood $\mathcal{U}$ of (x, ξ) in M to an open conic neighbourhood $\varkappa(\mathcal{U})$ of $(\boldsymbol{x}, \boldsymbol{\xi}) \in \dot{\mathrm{T}}^*\mathbb{R}^d$.

---

[1]Cf. (A.7).

A.1.2   Lagrangian submanifolds

Definition A.1.5. Let (M, σ) be a symplectic manifold. An immersion $\iota : \Lambda \hookrightarrow M$ is called **Lagrangian** if and only if $\iota^*\sigma = 0$ and $\dim \Lambda = \dim M/2$.

Example A.1.6. Let M be a manifold. Then the zero covector and the image $\xi(M)$ of an exact 1-form ξ on M are Lagrangian submanifolds of T^*M.

Example A.1.7. Let (Σ, ι) be a submanifold of a manifold M. The quotient vector bundle $\Sigma^\perp := T M_\Sigma / T\Sigma$ is called the **normal bundle** whose algebraic dual

$$\Sigma^{\perp*} := T^*M_\Sigma / T^*\Sigma, \quad T^*M_\Sigma := T^*M \restriction \Sigma \tag{A.15}$$

is called the **conormal bundle** of (Σ, ι). It is a $\mathbb{R}$-vector bundle over Σ of $\operatorname{rk} \Sigma^{\perp*} = \operatorname{codim} \Sigma$ so $\dim \Sigma^{\perp*} = \dim M$, and is naturally isomorphic to the annihilator of $d\iota$ in $T M_\Sigma$, which is a Lagrangian submanifold of T^*M (see e.g. [190, p. 374]).

If we assume that (Σ, ι) is embedded then its conormal bundle is characterised as

$$\Sigma^{\perp*} := \big\{ (y, \eta) \in T^*M \mid y \in \Sigma, \eta \restriction T_y M_\Sigma = 0 \big\}. \tag{A.16}$$

Note, $\dot{\Sigma}^{\perp*} \subset \dot{T}^*M$ is a conic Lagrangian manifold (see e.g. [84, Prop. 3.7.2]).

For any exact 1-form ξ on a manifold M, we can write $\xi = d\varphi$ for some $\varphi \in C^\infty(M, \mathbb{R})$. The φ is an example of the so-called generating function for the Lagrangian submanifold $\xi(M) = d\varphi(M)$. This is an archetypal result for any Lagrangian immersion of a generic symplectic manifold because σ vanishes identically on Λ and so the Poincaré lemma implies that locally $\sigma = d\vartheta$ for some potential ϑ on M. Applying this lemma once again yields an open subset $\mathcal{L} \subset \Lambda$ and a $\varphi \in C^\infty(\mathcal{L}, \mathbb{R})$ depending on ϑ, such that $(\iota^*\vartheta) \restriction \mathcal{L} = d\varphi$. If we consider the particular class of Lagrangian immersions Λ of $\pi : T^*M \to M$ then the set $\Sigma(\Lambda)$ of all critical points of

$$\Pi : \pi \circ \iota : \Lambda \to M, \ (x, \xi) \mapsto \Pi(x, \xi) := x \tag{A.17}$$

is called the **singular set** and its projection on M is called the **caustic** of Λ. $\Sigma(\Lambda)$ comprises the points where Λ is not transversal to cotangent fibration and in its complement Λ can be always locally represented as the graph of an exact 1-form $d\varphi$ defined on some open subset of M. We will see below that this construction can be carried over all Λ by introducing some auxiliary parameter $\theta \in \mathbb{R}^n$ (see e.g. [20], [190, Rem. 8.8.2]).

The primary objective of introducing this specific class of submanifolds is to complement the theory of Fourier integral operators (Section 2.1.3, 2.2, 2.3.3) where we only require the *conic* Lagrangian manifolds Λ. We remark that such Lagrangian manifolds are precisely those where the canonical 1-form θ vanishes identically and vice-versa [78, (3.1.3)] (see also, e.g. [84, Prop. 3.7.1]).

We are now going to systematically describe the parametrisation of a conic Lagrangian manifold $\Lambda \subset \dot{T}^*M$ by means of a generating function, also known as a phase function.

Definition A.1.8. Let M be a d-dimensional manifold and $\mathcal{U} \subset M \times \dot{\mathbb{R}}^n$ an open conic set where n is not necessarily equal to d. A $\varphi \in C^\infty(\mathcal{U}, \mathbb{R})$ is called a **clean phase function** with **excess** $0 \leq e \leq n$ if [10, Sec. 7] (see also, e.g. [77, Def. 21.2.15], [20, Def. 1])

(a) it is positive-homogeneous of degree 1 in the $\dot{\mathbb{R}}^n$-variable θ,

(b) its differential $\mathrm{d}\varphi \neq 0$ is non-vanishing,

(c) its **fibre-critical set**

$$\mathscr{C} := (\mathrm{grad}_\theta\, \varphi)^{-1}(0) := \left\{ (x_0; \theta^0) \in \mathcal{U} \,\middle|\, \left(\frac{\partial \varphi}{\partial \theta_k}\right)_{k=1,\ldots,n}(x_0; \theta^0) = 0 \right\} \qquad (A.18)$$

is a $d + e$-dimensional smooth conic submanifold of $M \times \dot{\mathbb{R}}^n$, whose tangent space is

$$\mathrm{T}_{(x;\theta)}\mathscr{C} := \ker\left(\mathrm{d}\left(\frac{\partial \varphi}{\partial \theta_k}\right)_{k=1,\ldots,n}(x; \theta) \right). \qquad (A.19)$$

If $e = 0$ then φ is called a **non-degenerate phase function**.

Given a clean φ, the differentials $\mathrm{d}(\partial\varphi/\partial\theta_k)$ at any point of $\mathscr{C}$ span a linear space of dimension $n - e$ and the map [10, Lem. 7.1]

$$J_\varphi : \mathscr{C} \to \Lambda_\varphi \subset \dot{\mathrm{T}}^*M, \quad (x; \theta) \mapsto J(x; \theta) := (x, \mathrm{d}_x\varphi) \qquad (A.20)$$

is locally a homogeneous *fibration* of fibre dimension e and its image is an *immersed conic Lagrangian submanifold* Λ_φ. If $e = 0$ then $(\mathrm{d}(\partial\varphi/\partial\theta_1), \ldots, \mathrm{d}(\partial\varphi/\partial\theta_n))$ are linearly independent at $(x; \theta) \in \mathscr{C}$ and (A.20) is a homogeneous *immersion* [78, p. 134] (see also, e.g. [20]).

We remark that $\Lambda \subset \dot{\mathrm{T}}^*M$ is a conic Lagrangian manifold if and only if, there exists a conic neighbourhood $\mathcal{L}$ for every $(x, \xi) \in \Lambda$ such that

$$\Lambda \cap \mathcal{L} = \Lambda_\varphi \Leftrightarrow \{(x, \xi)\} = \{(x, \mathrm{d}_x\varphi)\} \qquad (A.21)$$

for some non-degenerate phase function φ on $\mathcal{L}$ (see e.g. [84, Prop. 3.7.3], [191, Thm., p. 26]). In this situation, we say that φ **locally generates** Λ (see e.g. [190, p. 668]). Such a local generating function always exists and has a canonical form, i.e., *every (conic) Lagrangian submanifold is locally generated by a non-degenerate phase function*. For any $(x, \xi) \in \Lambda$, local coordinates (x^i) at $x \in M$ can be chosen such that with the corresponding homogeneous symplectic coordinates (x^i, ξ_i) on $\dot{\mathrm{T}}^*M$, the Lagrangian plane $\xi_i = \mathrm{cst}$ through (x, ξ) is transversal to Λ. Furthermore, there exists a non-degenerate phase function φ in an open conic subset of $M \times \dot{\mathbb{R}}^d$ having the unique form [78, Thm. 3.1.3] (see also, e.g. [77, Thm. 21.2.16, 21.2.18], [85, Prop. 1.4, Cor. 1.1, Rem 1.1 (pp. 418-419)])

$$\varphi = x^i \xi_i - H, \quad \text{so that} \quad \Lambda_\varphi = \Lambda_H = \left\{ \frac{\partial H}{\partial \xi_i}, \xi_i \right\}, \qquad (A.22)$$

where H is a smooth positively homogeneous $\mathbb{R}$-valued function of degree 1 on an open conic neighbourhood of $\xi \in \dot{T}^*_x M$. We remark that the homogeneous immersion (A.20) is the identity map in this case.

The fascinating existence result immediately provokes

- Are local phase functions unique?

- Is it possible to achieve a global parametrisation of a Lagrange manifold utilising a phase function?

- How the number n of fibre variables are chosen?

The first question rises the notion of an equivalence class of phase functions. Technically speaking, let $\tilde{\mathcal{U}} \subset M \times \dot{\mathbb{R}}^{\tilde{n}}$ be an open conic set such that there exists a diffeomorphism

$$v : \mathcal{U} \to \tilde{\mathcal{U}}, \ (x; \theta) \mapsto v(x; \theta) := (x; \tilde{\theta}), \tag{A.23}$$

where $\tilde{\theta}$ is a smooth function of $(x; \theta)$, which is homogeneous with respect to θ of degree 1 and fibre-preserving $\widetilde{\mathrm{Pr}} \circ v = \mathrm{Pr}$ where $\mathrm{Pr}, \widetilde{\mathrm{Pr}} : \mathcal{U}, \tilde{\mathcal{U}} \to M$ are projection operators. Then φ is said to be (locally) **equivalent** to a phase function $\tilde{\varphi}$ on $\tilde{\mathcal{U}}$ if [78, p. 134] (see also, e.g. [191, p. 27])

$$v^* \tilde{\varphi} = \varphi. \tag{A.24}$$

Given any $(x_0, \xi^0) \in \Lambda$ we would like to know the necessary and sufficient conditions for the existence of equivalent phase functions. Let φ and $\tilde{\varphi}$ be phase functions for Λ with the same excess e on some open conic subsets $U \times \dot{\mathbb{R}}^n$ and $\tilde{U} \times \dot{\mathbb{R}}^{\tilde{n}}$, respectively, such that $x_0 \in U \subset M$. Then φ and $\tilde{\varphi}$ are *equivalent* on any open conic neighbourhoods $\mathcal{U}$ and $\tilde{\mathcal{U}}$ of x in $U \times \dot{\mathbb{R}}^n$ and $\tilde{U} \times \dot{\mathbb{R}}^{\tilde{n}}$, respectively, if and only if [85, Prop. 1.5 (p. 421)] (see also [20, Prop. 1])

$$\mathscr{C}_\varphi \ni (x_0; \theta^0) \mapsto (x_0, \mathrm{d}_{x_0} \varphi) \ = \ (x_0, \mathrm{d}_{x_0} \tilde{\varphi}) \hookleftarrow (x_0; \tilde{\theta}) \in \mathscr{C}_{\tilde{\varphi}}, \tag{A.25a}$$

$$n \ = \ \tilde{n}, \tag{A.25b}$$

$$\mathrm{sgn}\left(\mathrm{Hess}_\theta \, \varphi \, (x_0; \theta^0)\right) \ = \ \mathrm{sgn}\left(\mathrm{Hess}_{\tilde{\theta}} \, \varphi \, (x_0; \tilde{\theta}^0)\right), \tag{A.25c}$$

where $\mathscr{C}_\varphi$ resp. $\mathscr{C}_{\tilde{\varphi}}$ are fibre-critical manifolds of φ resp. $\tilde{\varphi}$ and sgn denotes the signature of the Hessian ((3.18)) matrix $\mathrm{Hess}_\theta \, \varphi$ of φ with respect to its fibre-variable $\theta \in \dot{\mathbb{R}}^n$. These constraints can be alternatively given an operational viewpoint.

Remark A.1.9. Let $\mathfrak{M}(\Lambda)$ be the class of equivalent phase functions for Λ. Given $\varphi \in \mathfrak{M}(\Lambda)$, the following operations produce another element of $\mathfrak{M}(\Lambda)$ [86, p. 48] (see also, e.g. [190, p. 671]):

(i) *Addition*: $\tilde{\varphi} := \varphi + \lambda \in \mathfrak{M}(\Lambda)$ for any $\lambda \in \mathbb{R}$;

(ii) *Composition*: $\tilde{\varphi} = $ (A.24) $\in \mathfrak{M}(\Lambda)$ for any fibre-preserving diffeomorphism (A.23);

(iii) *Suspension*: For any non-degenerate bilinear form q on $\dot{\mathbb{R}}^{\tilde{n}}$, $\tilde{\varphi} := \mathrm{Pr}^*_{\tilde{\mathcal{U}}} \varphi + \mathrm{Pr}^*_{\dot{\mathbb{R}}^{\tilde{n}}} q$ on $\tilde{\mathcal{U}} := U \times \dot{\mathbb{R}}^{\tilde{n}}$ is an element of $\mathfrak{M}(\Lambda)$.

(iv) *Restriction*: If φ is a phase function on $M \times \dot{\mathbb{R}}^n$ then $\tilde{\varphi} := \varphi \upharpoonright \tilde{\mathcal{U}} \in \mathfrak{M}(\Lambda)$ where $\tilde{\mathcal{U}} \subset M \times \dot{\mathbb{R}}^n$ is an open conic subset containing the preimage of $J_\varphi(x, \xi)$.

These four operations generate an equivalence relation among phase functions called **stable equivalence**. So, we conclude that *a phase function is not unique yet it is so up to stable equivalence*.

The parametrisation of a Lagrangian submanifold by means of an equivalent class of phase functions cannot be performed globally due to topological restrictions. When M is compact, these obstructions require cohomological and k-theoretic language to formulate, which are somehow tangential to the subject matter, so we refrain those precise formulae and refer [192, 193] together with the earlier references cited therein. The non-compact case is still an open issue to the best of our knowledge. Notwithstanding, a global parametrisation can be achieved, for instance, by an equivalence class of $\mathbb{C}$-valued non-degenerate phase functions in the particular case whenever conic Lagrangian submanifolds are given by the graphs of homogeneous symplectomorphisms [116, Lem. 1.2, 1.7].

Finally, the fibre-dimension is bounded from below by [85, Prop. 1.3, p. 417]

$$n \geq d - \mathrm{rk}(d_{(x,\xi)}\Pi) + e \tag{A.26}$$

for any $(x, \xi) \in \Lambda$, where $d\Pi$ is the differential of (A.17).

If (M, σ_M) and (N, σ_N) are two symplectic manifolds then their product manifold $M \times N$ can be naturally equipped with the symplectic forms

$$\sigma_M \boxplus \sigma_N := \mathrm{pr}^*_M \sigma_M + \mathrm{pr}^*_N \sigma_N, \quad \sigma_M \boxminus \sigma_N := \mathrm{pr}^*_M \sigma_M - \mathrm{pr}^*_N \sigma_N, \tag{A.27}$$

where $\mathrm{pr}_M, \mathrm{pr}_N : M \times N \to M, N$ are the Cartesian projectors. In symplectic coordinates $(x^i, \xi_i; y^j, \eta_j)$ for $M \times N$, these read $dx^i \wedge d\xi_i \boxplus dy^j \wedge d\eta_j$ resp. $dx^i \wedge d\xi_i \boxminus dy^j \wedge d\eta_j$.

Example A.1.10. Let $(M, \sigma_M), (N, \sigma_N)$ be $2d$-dimensional symplectic manifolds and $\varkappa : N \to M$ a symplectomorphism. Then its graph

$$\Gamma = \left\{ \left(\varkappa(y, \eta); (y, \eta) \right) | (y, \eta) \in N \right\} \subset (M \times N, \sigma_M \boxminus \sigma_N) \tag{A.28}$$

is a Lagrangian submanifold whereas the twisted graph $\Gamma' = \left\{ \left(\varkappa(y, \eta); (y, -\eta) \right) \right\}$ is a Lagrangian submanifold with respect to the product symplectic form $\sigma_M \boxplus \sigma_N$. Note, *any diffeomorphism $\varkappa$ is a symplectomorphism if and only if its graph $\Gamma \subset (M \times N, \sigma_M \boxminus \sigma_N)$ is a Lagrangian submanifold*.

A special case of the preceding example for the identity symplectomorphism has some interesting consequences as presented below.

Example A.1.11. Let M be a manifold whose punctured cotangent bundle $(\dot{\mathrm{T}}^*M, \sigma_M)$ is considered as a conic symplectic manifold as in Example A.1.2 and let id : $M \to M$ be the identity diffeomorphism. Then its cotangent lift

$$\mathrm{T}^* \,\mathrm{id} : \dot{\mathrm{T}}^*M \to \dot{\mathrm{T}}^*M, \ (x, \xi) \mapsto (\mathrm{T}_x^* \mathrm{id})(\xi)\,(X) := \xi\big(\mathrm{d}_x \mathrm{id}\,(X)\big) = \xi(X) \qquad (A.29)$$

for any $X \in \mathrm{T}_x M$, is the identity homogeneous symplectomorphism preserving the Liouville form. Its twisted graph

$$\Gamma' = \{(x, \xi; x, -\xi) \in \dot{\mathrm{T}}^*M \times \dot{\mathrm{T}}^*M\} = (\Delta_M M)^{\perp*} = (\Delta \dot{\mathrm{T}}^*M)' \subset (\dot{\mathrm{T}}^*M \times \dot{\mathrm{T}}^*M, \sigma_M \boxplus \sigma_M) \quad (A.30)$$

is a Lagrangian submanifold and it is nothing but the conormal bundle (Example A.1.7) $(\Delta_M M)^{\perp*}$ of the diagonal embedding $\Delta_M : M \to M \times M$. Equivalently, this is the twisted diagonal embedding $(\Delta \dot{\mathrm{T}}^*M)' \hookrightarrow \dot{\mathrm{T}}^*M \times \dot{\mathrm{T}}^*M \cong \dot{\mathrm{T}}^*(M \times M)$ of the product punctured cotangent bundle. Γ' parametrisation by the non-degenerate phase function $\varphi := (x^i - y^i)\xi_i$ for any coordinates (x^i) (resp. (y^i)) around x (resp. y) in M and $\xi \in \dot{\mathbb{R}}^d$ (see e.g. [85, Exm. 5.2, 5.3 (pp. 455-456)]).

Example A.1.10 exhibits that the standard recipe to construct a Lagrangian submanifold $\Lambda \subset (\mathsf{M} \times \mathsf{N}, \sigma_\mathsf{M} \boxminus \sigma_\mathsf{N})$ as the graph of a symplectomorphism only works provided that $\dim \mathsf{M} = \dim \mathsf{N}$ which cannot be guaranteed a priori and relaxing this condition leads to the notion of a canonical relation.

Definition A.1.12. If $(\mathsf{M}, \sigma_\mathsf{M})$ and $(\mathsf{N}, \sigma_\mathsf{N})$ are two symplectic manifolds then an immersed Lagrangian submanifold (C, ι) of $(\mathsf{M} \times \mathsf{N}, \sigma_\mathsf{M} \boxminus \sigma_\mathsf{N} := (A.27))$ is called a **canonical relation** from N to M. If M, N, and C all are conic, then we call the relation **homogeneous** [78, Def. 4.1.2] (see also [77, Def. 21.2.12]).

Example A.1.13. Let $(\mathsf{M}, \sigma_\mathsf{M})$ be a symplectic manifold and let $(\mathrm{T}^*\mathbb{R} \times \mathsf{M}, \sigma_\mathbb{R} \boxplus \sigma_\mathsf{M}, H)$ be a time-dependent Hamiltonian system. The twisted graph (see e.g. [190, pp. 441 - 442], [20, Sec. 4.2] for details)

$$\Gamma' = \{(t, \tau; x, \xi; y, -\eta) \in \mathrm{T}^*\mathbb{R} \times \mathsf{M} \times \mathsf{M} \mid (x, \xi) = \Phi_t(y, \eta), \tau + H(x, \xi) = 0\} \qquad (A.31)$$

of H-flow Φ_t which is assumed to be complete, is a canonical relation of $\mathrm{T}^*\mathbb{R} \times \mathsf{M} \times \mathsf{M}$.

Remark A.1.14. By Definition A.1.12, $C \subset \mathsf{M} \times \mathsf{N}$ is a Lagrangian submanifold with respect to $\sigma_\mathsf{M} \boxminus \sigma_\mathsf{N}$, then it follows that the twisted canonical relation $C' \subset \mathsf{M} \times \mathsf{N}$ is a Lagrangian submanifold with respect to the product symplectic form $\sigma_\mathsf{M} \boxplus \sigma_\mathsf{N}$.

Let C be a homogeneous canonical relation from $\pi_N : \dot{\mathrm{T}}^*N \to N$ to $\pi_M : \dot{\mathrm{T}}^*M \to M$. Being a Lagrangian submanifold, it can also be uniquely parametrised by the stable equivalence class $[\varphi]$ of phase functions φ defined on an open conic subset of $M \times N$ as described below. For each $(x_0, \xi^0; y_0, \eta^0) \in C$, there exists a homogeneous symplectic coordinate neighbourhood $\mathcal{U} \times \mathcal{V}$ on $\dot{\mathrm{T}}^*M \times \dot{\mathrm{T}}^*N$ and a clean phase function φ on $\pi_M(\mathcal{U}) \times \pi_N(\mathcal{V}) \times \dot{\mathbb{R}}^n$ such that

$$\mathscr{C}_\varphi := \{(x_0, y_0; \theta^0) \in \pi_M(\mathcal{U}) \times \pi_N(\mathcal{V}) \times \dot{\mathbb{R}}^n \mid \mathrm{grad}_\theta \, \varphi \, (x_0, y_0; \theta^0) = 0\} \subset M \times N \times \dot{\mathbb{R}}^n \quad (A.32)$$

is a $d_M + d_N + e$-dimensional submanifold for some $\theta^0 \in \dot{\mathbb{R}}^n$. Then, C is locally obtained by the homogeneous fibration

$$J_\varphi : \mathscr{C}_\varphi \to C_\varphi, \ (x_0, y_0; \theta^0) \mapsto J(x_0, y_0; \theta^0) := (x_0, \mathrm{d}_{x_0}\varphi; y_0, \mathrm{d}_{y_0}\varphi) \tag{A.33}$$

of fibre-dimension e. Obviously, one can opt $e = 0$ so that the preceding mapping is a homogeneous immersion. Alternatively, the non-degenerate phase function

$$\psi := x_0^i \xi_i^0 + y_0^j \eta_j^0 - H \tag{A.34}$$

for C can be chosen on any homogeneous symplectic chart $\left(\mathcal{U} \times \mathcal{V}, (x_0^i, \xi_i^0; y_0^j, \eta_j^0)\right)$ where H is a smooth positively homogeneous $\mathbb{R}$-valued function of degree 1 on an open conic neighbourhood of $(\xi^0, \eta^0) \in \dot{\mathrm{T}}^*_{x_0}M \times \dot{\mathrm{T}}^*_{y_0}N$. The corresponding homogeneous immersion

$$J_H : \mathscr{C}_H \to C_H, \ (x_0, \xi^0; y_0, \eta^0) \mapsto \left(\frac{\partial H}{\partial \xi_i^0}, \xi_i^0; \frac{\partial H}{\partial \eta_j^0}, \eta_j^0\right) \tag{A.35}$$

is then actually the identity map. Complying with Remark A.1.14, this means

$$C'_\varphi = \{(x_0, \mathrm{d}_{x_0}\varphi; y_0, -\mathrm{d}_{y_0}\varphi)\} \quad \text{and} \quad C'_H = \left\{\left(\frac{\partial H}{\partial \xi_i^0}, \xi_i^0; \frac{\partial H}{\partial \eta_j^0}, -\eta_j^0\right)\right\}. \tag{A.36}$$

are canonical relations with respect to $\sigma_M \boxplus \sigma_N$ (see e.g. [77, (21.2.9)]).

A.1.3 Clean composition

A relation maps a set to another set, thus, if S is a subset of a symplectic manifold $(\mathrm{O}, \sigma_\mathrm{O})$ and $C \subset (\mathrm{M} \times \mathrm{O}, \sigma_M \boxminus \sigma_\mathrm{O})$ is a canonical relation then $C(S) = \{\xi \in \mathrm{M} \mid (\xi, \zeta) \in C, \text{for some } \zeta \in S\}$ can be expressed as $C(S) = \mathrm{pr}_\mathrm{M}\left(C \cap \mathrm{pr}_\mathrm{O}^{-1} S\right)$ where $\mathrm{pr}_\mathrm{M}, \mathrm{pr}_\mathrm{O} : \mathrm{M} \times \mathrm{O} \to \mathrm{M}, \mathrm{O}$ are the Cartesian projectors. The composition of C with another canonical relation $\Lambda \subset (\mathrm{O} \times \mathrm{N}, \sigma_\mathrm{O} \boxminus \sigma_\mathrm{N})$ then reads (see e.g. [77, p. 290])

$$C \circ \Lambda := \{(\xi, \eta) \in \mathrm{M} \times \mathrm{N} \mid (\xi, \zeta) \in C, (\zeta, \eta) \in \Lambda, \text{for some } \zeta \in \mathrm{O}\} = \Pi(C \star \Lambda), \tag{A.37}$$

where we have used

$$C \star \Lambda := (C \times \Lambda) \bigcap (\mathrm{M} \times \varDelta\mathrm{O} \times \mathrm{N}) \tag{A.38}$$

and the Cartesian projector $\Pi : \mathrm{M} \times \varDelta\mathrm{O} \times \mathrm{N} \to \mathrm{M} \times \mathrm{N}$.

But, in general, $C \circ \Lambda$ is *not* a manifold and one requires to know "how" $C \times \Lambda$ intersects with $\mathrm{M} \times \varDelta\mathrm{O} \times \mathrm{N}$ for which we introduce the terminology of clean intersection (see e.g. [10, Sec. 5]).

Definition A.1.15. Let S and Σ be two submanifolds of a manifold M. If their intersection $S \cap \Sigma$ is a submanifold satisfying (see e.g. [77, Def. C.3.2], [86, Def. 5.9])

$$\forall x \in S \cap \Sigma : \mathrm{T}_x(S \cap \Sigma) = \mathrm{T}_x S \cap \mathrm{T}_x \Sigma \tag{A.39}$$

then the intersection is said to be **clean**, denoted by $S \pitchfork_e \Sigma$, with **excess** $e \in \mathbb{N}_0$ given by

$$e := \operatorname{codim} S + \operatorname{codim} \Sigma - \operatorname{codim}(S \cap \Sigma). \tag{A.40}$$

If $e = 0$ then S and Σ are said to intersect **transversally**, symbolically $S \pitchfork \Sigma$.

The criterion that $S \pitchfork \Sigma$ at any $x \in S \cap \Sigma$ can be restarted as

$$\mathrm{T}_x S + \mathrm{T}_x \Sigma = \mathrm{T}_x M \Leftrightarrow S_x^\perp \cap \Sigma_x^\perp = \{0\}. \tag{A.41}$$

Example A.1.16. Let $(x, y, z) \in \mathbb{R}^3$. Then $\{z = 0\} \pitchfork \{z = x\}$ but $\{z = 0\} \not\pitchfork \{z = xy\}$.

Intuitively speaking, $S \pitchfork \Sigma$ means they intersect "as little as possible". Thus, $S \cap \Sigma = \emptyset$ is trivially $S \pitchfork \Sigma$.

Example A.1.17. The x-axis intersects with the y-axis in $\mathbb{R}^3$ cleanly with $e = 2 + 2 - 3 = 1$. In $\mathbb{R}^2$, the x-axis and $\{y = x^2\}$ do not intersect cleanly.

If $(C \times \Lambda) \pitchfork_e (\mathsf{M} \times \Delta \mathsf{O} \times \mathsf{N})$ then (see e.g. [77, Thm. 21.2.14], [86, Prop. 5.28])

$$\operatorname{rk} \Pi = \frac{\dim \mathsf{M} + \dim \mathsf{N}}{2} \tag{A.42}$$

and the range $C \circ \Lambda$ of Π is locally an *immersed Lagrangian submanifold* of $(\mathsf{M} \times \mathsf{N}, \sigma_\mathsf{M} \boxminus \sigma_\mathsf{N})$. Furthermore, if U is an open set in $\mathsf{M} \times \mathsf{N}$ and $\Pi \upharpoonright \Pi^{-1}(U)$ is *proper* then $\Pi(\Pi^{-1}U)$ is an immersed *closed* Lagrangian submanifold of U and $\Pi^{-1}U$ is a fibre space with *compact fibres* over it (see e.g. [77, Rem. p. 291]).

Definition A.1.18. Let $(\mathsf{M}, \sigma_\mathsf{M})$, $(\mathsf{O}, \sigma_\mathsf{O})$ and $(\mathsf{N}, \sigma_\mathsf{N})$ be symplectic manifolds and let $C \subset (\mathsf{M} \times \mathsf{O}, \sigma_\mathsf{M} \boxminus \sigma_\mathsf{O})$ and $\Lambda \subset (\mathsf{O} \times \mathsf{N}, \sigma_\mathsf{O} \boxminus \sigma_\mathsf{N})$ be closed canonical relations. Then C and Λ is said to have a **clean composition** $C \circ_e \Lambda$ **with excess** $e \in \mathbb{N}_0$ if $C \times \Lambda$ and $\mathsf{M} \times \Delta \mathsf{O} \times \mathsf{N}$ intersects cleanly with excess e. The composition is called **proper** if the Cartesian projector Π : $\mathsf{M} \times \Delta \mathsf{O} \times \mathsf{N} \to \mathsf{M} \times \mathsf{N}$ is proper (see e.g. [77, Thm. 21.2.14], [85, Def. 5.2 (p. 458)]).

We remark that (see e.g. [25, Thm. 4.2.2])

$$e = \dim \mathfrak{F} \tag{A.43}$$

where

$$\begin{aligned}
\mathfrak{F} &:= \{\mathfrak{F}_{(\xi,\eta)} \mid (\xi, \eta) \in C \circ_e \Lambda\}, \\
\mathfrak{F}_{(\xi,\eta)} &:= \{(\xi, \zeta, \zeta, \eta) \in C \star \Lambda \mid \Pi(\xi, \zeta, \zeta, \eta) = (\xi, \eta) \in C \circ_e \Lambda\}
\end{aligned} \tag{A.44}$$

is the set of compact connected fibres $\mathfrak{F}_{\xi,\eta}$ of the smooth fibration

$$\Pi : C \star \Lambda \to C \circ_e \Lambda. \tag{A.45}$$

In terms of generating functions, if $\varphi(x, z; \theta)$ and $\phi(z, y; \vartheta)$ are non-degenerate phase functions for homogeneous canonical relations $C \subset \dot{\mathrm{T}}^*M \times \dot{\mathrm{T}}^*O$ and $\Lambda \subset \dot{\mathrm{T}}^*O \times \dot{\mathrm{T}}^*N$ in open conic neighbourhoods of $(x_0, z_0; \theta^0) \in M \times O \times \dot{\mathbb{R}}^{n_M}$ and $(z_0, y_0; \vartheta^0) \in O \times N \times \dot{\mathbb{R}}^{n_N}$ such that $(x_0, z_0; \theta^0) \in \mathscr{C}_\varphi$ and $(z_0, y_0; \vartheta^0) \in \mathscr{C}_\phi$, respectively, with $\mathrm{d}_z \varphi (x_0, z_0; \theta^0) + \mathrm{d}_z \phi (z_0, y_0; \vartheta^0) = 0$, then (see e.g. [77, Prop. 21.2.19])

$$\psi(x, z, y; \theta, \vartheta) := \varphi(x, z; \theta) + \phi(z, y; \vartheta) \tag{A.46}$$

is a clean phase function with excess e parametrising $C \circ_e \Lambda$. In addition, if ψ is non-degenerate in $(z'; \theta', \vartheta')$ then the e variables $(z''; \theta'', \vartheta'')$ parametrise $\mathfrak{F}$; the meaning of the splitting $z = (z', z'')$ has been explained in Definition 2.1.9.

A.1.4 Keller-Maslov bundle

Given a Lagrangian submanifold Λ, one can associate a number of bundle structures over Λ such that these bundles encode information about how the local pieces of Λ generated by phase functions are glued together.

Definition A.1.19. Let $\dot{\mathrm{T}}^*M \to M$ be the punctured cotangent bundle over a manifold M and $\Lambda \subset \dot{\mathrm{T}}^*M$ a conic Lagrangian submanifold. The **Keller-Maslov bundle** $\mathbb{M} \to \Lambda$ over Λ is a complex line bundle obtained from some principal bundle with structure group $\mathbb{Z}/4\mathbb{Z}$, which is trivial as a vector bundle [78, p. 148].

Remark A.1.20. Having a discrete structure group, $\mathbb{M}$ carries a natural flat connection with holonomy in $\mathbb{Z}/4\mathbb{Z}$ (see e.g. [86, p. 56]).

We are now going to present a construction of $\mathbb{M}$ which is an integral part of the principal symbol (Definition 2.3.11) of a Lagrangian distribution and consequently has played a pivotal part in the non-zero periodic contribution (Theorem 4.3.6) of the Duistermaat-Guillemin-Gutzwiller trace formula. Let φ be a *non-degenerate* phase function for Λ. Then employing the Hessian of φ one has the integer-valued map: $\mathscr{C}_\varphi \ni (x; \theta) \mapsto \mathrm{sgn} \left(\mathrm{Hess}_\theta \, \varphi \, (x; \theta) \right) \in \mathbb{Z}$. Note, this function can be somewhat discontinuous as the Hessian can be singular at some elements of $\mathscr{C}_\varphi$. We now consider an open conic (Leray[2]-) covering $\{\mathcal{L}_\alpha\}$ of Λ, indexed by a countable set with the corresponding non-degenerate phase functions φ_α and the fibre-critical manifolds $\mathscr{C}_\alpha$. It follows from (A.25) that the mapping [78, (3.2.15)] (see also, e.g. [20], [25, Sec. 5. 13], [190, pp. 686-687])

$$\mathfrak{s}_{\alpha\beta} : \mathscr{C}_\alpha \cap \mathscr{C}_\beta \to \mathbb{Z}, \quad (x; \theta^\alpha, \theta^\beta) \mapsto$$
$$\mathfrak{s}_{\alpha\beta}(x; \theta^\alpha, \theta^\beta) := \frac{1}{2} \Big(\mathrm{sgn} \left(\mathrm{Hess}_\theta \, \varphi_\beta \, (x; \theta^\beta) \right) - n_\beta - \mathrm{sgn} \left(\mathrm{Hess}_\theta \, \varphi_\alpha \, (x; \theta^\alpha) \right) + n_\alpha \Big) \tag{A.47}$$

---

[2]This means that finite intersection of $\mathcal{L}_\alpha$'s are either empty or diffeomorphic to the open ball.

is constant for all connected intersections $\mathscr{C}_\alpha \cap \mathscr{C}_\beta$. Thereby,

$$g_{\alpha\beta} := e^{i/2\pi \mathfrak{s}_{\alpha\beta} \circ J_{\alpha\beta}^{-1}} : \mathcal{L}_\alpha \cap \mathcal{L}_\beta \to \dot{\mathbb{C}} \tag{A.48}$$

clearly satisfies the cocycle property: $g_{\alpha\beta'} g_{\beta'\beta} = g_{\alpha\beta}$ whenever $\mathcal{L}_\alpha \cap \mathcal{L}_{\beta'} \cap \mathcal{L}_\beta \neq \emptyset$ with $|g_{\alpha\beta}| = 1$. Hence, the collection of non-zero complex numbers $\{g_{\alpha\beta}\}$ for all α and β such that Λ_α and Λ_β are not disjoint, defines our transition function and the global construction of $\mathbb{M}$ is achieved by taking the disjoint union

$$\mathbb{M} := \bigsqcup_\alpha (\mathcal{L}_\alpha \times \mathbb{C})/\sim \tag{A.49}$$

modulo the equivalence relation

$$\big((x, \xi^\alpha), c_\alpha\big) \sim \big((x, \xi^\beta), c_\beta\big) \Leftrightarrow (x, \xi^\alpha) = (x, \xi^\beta) \in \mathcal{L}_\alpha \cap \mathcal{L}_\beta, c_\alpha = g_{\alpha\beta} c_\beta. \tag{A.50}$$

The constant functions from $\mathcal{L}_\alpha$ to $1_\mathbb{C}$ form flat local sections of $\mathbb{M}$:

$$\mathsf{e}_\alpha : \mathcal{L}_\alpha \to \mathbb{M}, \ (x, \xi) \mapsto \big[\big((x, \xi), 1_\mathbb{C}\big)\big] \tag{A.51}$$

and hence manufacture $\mathbb{M}$ into a line bundle with flat connection over Λ. Any section m of $\mathbb{M} \to \mathcal{L}_\alpha$ then can be written as $\mathsf{m} = \sum \mathsf{m}^\alpha \mathsf{e}_\alpha$.

Alternatively, one can define a $\sqrt[8]{1_\mathbb{C}}$-valued mapping

$$\mathsf{m}_\alpha := e^{i/4\pi \, \mathrm{sgn}(\mathrm{Hess}_\theta \, \varphi) \circ J^{-1}} : \mathcal{L}_\alpha \to \dot{\mathbb{C}} \tag{A.52}$$

and declare this as the flat section of $\mathbb{M} \upharpoonright \mathcal{L}_\alpha$. Then $\mathbb{M}$ is constructed employing transition functions $g_{\alpha\beta}$ valued in the structure group $\mathbb{Z}/4\mathbb{Z} = \{\pm 1, \pm i\}$, which are locally constant on each disjoint covering of Λ and absolute value 1 so that $\mathsf{m}_\alpha = g_{\alpha\beta} \mathsf{m}_\beta$.

Since φ is unique only up to the stable equivalence, the above construction must be examined under the four constituent operations mentioned in Remark A.1.9. Clearly $\mathrm{sgn}(\mathrm{Hess}_\theta \, \varphi_\alpha)$ is invariant under all those except suspension for which $\mathrm{sgn}(\mathrm{Hess}_\theta \, \varphi_\alpha) = \mathrm{sgn}(\mathrm{Hess}_{\tilde{\theta}} \, \tilde{\varphi}_\alpha) + \mathrm{sgn} \, \mathsf{q}$ - ensuring the construction for the equivalence class $[\varphi_\alpha]$ rather than just a representative φ_α.

So far, the construction has been given for a non-degenerate phase function but it can be well adopted for a clean phase function $\psi(x; \theta)$ on $U \times \dot{\mathbb{R}}^n$ as discussed below. Let $\mathrm{rk}\big(\mathrm{Hess}_\theta \, \psi\,(x_0; \theta^0)\big) = e$ for some $0 < e \leq n$ on the fibre-critical manifold $\mathscr{C}_\psi \ni (x_0; \theta^0)$. We can split (cf. Definition 2.1.9) $\dot{\mathbb{R}}^n \ni \theta = (\theta', \theta'') \in \dot{\mathbb{R}}^{n-e} \times \dot{\mathbb{R}}^e$ the fibre-variable θ in two groups and perform a linear change of θ-coordinates such that the matrix $\mathrm{Hess}_{\theta'} \, \psi\,(x_0; \theta'^0, \theta''^0)$ is non-degenerate for a fixed $\theta''^0 \in \dot{\mathbb{R}}^e$ and that the fibration [10] (see also, e.g. [25, Thm. 5.8.2, p. 119, 128])

$$\varsigma : U \times \dot{\mathbb{R}}^n \to U \times \dot{\mathbb{R}}^e, \ (x; \theta', \theta'') \mapsto (x; \theta'') \tag{A.53}$$

produces a non-degenerate phase function $\phi(x; \theta', \theta''^0)$ on $U \times \dot{\mathbb{R}}^{n-e}$ for Λ:

$$\psi = \varsigma^* \phi. \tag{A.54}$$

Consequently, the θ-variable can be reduced so that $\mathrm{Hess}_{\theta''} \psi(x_0; \theta'^0, \theta''^0) = 0$. Evidently, $\mathscr{C}_\psi = \varsigma^{-1} \mathscr{C}_\phi$ and the signature map $\mathrm{sgn}\left(\mathrm{Hess}_\theta \psi(x_0; \theta^0)\right)$ is simply the pullback of the signature map $\mathscr{C}_\varphi \to \mathbb{Z}$ associated with ϕ. In addition, the immersion $J_\phi : \mathscr{C}_\varphi \to \Lambda$ lifts to a fibre-preserving fibration $J_\psi : \mathscr{C}_\psi \to \Lambda$ and hence the construction of $\mathbb{M} \to \Lambda$ by means of ψ proceeds analogously.

Example A.1.21 (Keller-Maslov bundle $\mathbb{L}_0 \to (\varDelta M)^{\perp*}$ over the conormal bundle $(\varDelta M)^{\perp*}$)**.** As in the terminologies of Example A.1.11, the non-degenerate phase function $\varphi := (x^i - y^i)\xi_i$ parametrises $(\varDelta M)^{\perp*}$ globally. Then $\mathrm{Hess}_\xi \varphi = 0$ and thus we have a global transition function $g_{\alpha\beta} = 1$ resp. a global constant section $\mathrm{m}_\alpha = 1$ by (A.48) resp. (A.52).

A.1.5 Maslov index

The exponent in (A.52) has a beautiful topological interpretation for which let us recall some notions from symplectic topology. Let $(\mathscr{V}, \sigma)$ be a $2d$-dimensional real symplectic vector space and we choose a σ-compatible complex structure J on $\mathscr{V}$. A choice of J defines a positive-definite hermitian form $(\cdot|\cdot) := \sigma(\cdot, \mathsf{J}\cdot) + \mathrm{i}\,\sigma(\cdot, \cdot)$ on the complex vector space $\mathscr{V}$. Then the set $\mathscr{L}(\mathscr{V})$ of all Lagrangian subspaces of $(\mathscr{V}, \sigma)$ is manifold of dimension $d(d+1)/2$, called the **Lagrange-Graßmann manifold** of $(\mathscr{V}, \sigma)$. Let us denote by $\mathrm{O}(\Lambda)$, the isometry group of a given $\Lambda \in \mathscr{L}(\mathscr{V})$ with respect to the scalar product on Λ induced by $(\cdot|\cdot)$. It follows that $\mathrm{O}(\Lambda) \subset \mathrm{U}(\mathscr{V}) = \mathrm{U}(\mathscr{V}, (\cdot|\cdot))$ as a Lie group (see e.g. [190, Prop. 7.5.3/3]) when one makes the identification $\mathrm{O}(\Lambda) = \{g \in \mathrm{U}(\mathscr{V}) | g\Lambda = \Lambda\}$. Then, the mapping $\mathrm{U}(\mathscr{V}) \ni g \mapsto \det g \in \mathbb{S} \subset \mathbb{C}$ values ± 1 on $\mathrm{O}(\Lambda)$ and every element of a Lagrangian-Graßmannian defines the isomorphism $\mathrm{U}(\mathscr{V})/\mathrm{O}(\Lambda) \ni [g] \mapsto g\Lambda \in \mathscr{L}(\mathscr{V})$ (see e.g. [190, Prop. 7.6.1]). Thus, we have an induced mapping

$$\det{}_\Lambda^2 : \mathscr{L}(\mathscr{V}) \to \mathbb{S} \subset \mathbb{C}. \tag{A.55}$$

Definition A.1.22. Let $\mathscr{L}(\mathscr{V})$ be a Lagrangian Graßmannian of a symplectic vector space $(\mathscr{V}, \sigma)$. The **Maslov index** $\mathfrak{n}(\gamma)$ of a closed curve $\gamma : \mathbb{S} \to \mathscr{L}(\mathscr{V})$ is defined to be the degree of the mapping $\det_\Lambda^2 \circ \gamma : \mathbb{S} \to \mathbb{S}$ where $\det_\Lambda^2$ is defined by (A.55) [194, Sec. 1.3-1.4] (see also, e.g. [190, Def. 7.7.1]).

Proposition A.1.23. *The Maslov index is an integer-valued homotopy invariant quantity and it is independent of the choice of Lagrangian subspace or complex structure (see e.g. [190, Prop. 7.7.2]).*

In order to define this index for closed curves on a Lagrangian immersion, let

$$\chi : \iota^*(\mathrm{T}\mathrm{T}^*M) \to \Lambda \times \mathbb{R}^{2d} \tag{A.56}$$

be a trivialisation of the vertical geometric distribution to $\{0\} \times \mathbb{R}^d$, i.e., $\chi(T_{\iota(x,\xi)} T_x^* M) = \{0\} \times \mathbb{R}^d$ for any $(x,\xi) \in \Lambda$. Then, the image of $d_{(x,\xi)}\iota : T_{(x,\xi)}\Lambda \to T_{\iota(x,\xi)} T^* M$ is identified with a Lagrangian subspace in $\mathbb{R}^{2d}$ via χ and one obtains the following mapping (see e.g. [190, (12.6.3)])

$$I : \Lambda \to \mathscr{L}(\mathbb{R}^{2d}), \ (x,\xi) \mapsto I(x,\xi) := \chi\big(d_{(x,\xi)}\iota\,(T_{(x,\xi)}\Lambda)\big). \tag{A.57}$$

Definition A.1.24. Let Λ be a Lagrangian submanifold of a symplectic manifold of dimension $2d$ and $\mathfrak{n}$ the Maslov index for closed curves in the Lagrangian-Graßmannian $\mathscr{L}(\mathbb{R}^{2d})$ of the symplectic vector space $\mathbb{R}^{2d}$. Then the **Maslov index** $\mathfrak{m}(\gamma)$ of any closed curve γ in Λ is defined by (see e.g. [190, p. 678])

$$\mathfrak{m}(\gamma) := \mathfrak{n}(I \circ \gamma), \tag{A.58}$$

where $\mathfrak{n}$ is the Maslov index of the closed curve $I \circ \gamma$ in $\mathscr{L}(\mathbb{R}^{2d})$ and I is defined by (A.57).

It turns out that $\mathfrak{m}$ is equal to the so-called "Maslov intersection index" [194, Thm. 1.5] (see also, e.g. [190, Prop. 12.6.6]). The formal definition (see e.g. [190, Def. 12.6.5]) and details of the intersection index are tangential to the ongoing discussion so we abstain from that detour and instead remark that *this index is an integer-valued homotopy invariant quantity with fixed end points* [194, Thm. 2.2] and it can be computed by means of phase functions as explained below. For our purpose, it is sufficient to consider the situation when $\gamma : [0,1] \to \Lambda$ is a closed curve with end points in the set $\Sigma_0(\Lambda) \subset \Lambda$ defined by the points where the differential of (A.17) has rank d. Suppose that we have picked a partition of $[0,1]$ by the numbers $0 = s_0 < s_1 < \ldots < \ldots < s_N = 1$ such that there exist open conic subsets $\mathcal{L}_1, \ldots, \mathcal{L}_N$ of Λ so that $\gamma([s_{\alpha-1}, s_\alpha]) \subset \mathcal{L}_\alpha$ and $\mathcal{L}_\alpha \cap \mathcal{L}_\beta$ are connected and that the immersions $\mathcal{L}_\alpha \hookrightarrow \dot{T}^* M$ and (A.20) are equivalent. Then the Maslov index is given by (see e.g. [20, (5)], [190, Cor. 12.6.12])

$$\mathfrak{m}(\gamma) = \sum_{\alpha=1}^{N-1} \mathfrak{s}_{\alpha\alpha+1}. \tag{A.59}$$

This reveals that $\mathbb{M} \to \Lambda$ is essentially defined by the representation $e^{i\pi\mathfrak{m}(\gamma)/2} = i^{\mathfrak{m}(\gamma)}$ of $\pi_1(\Lambda)$. In other words, sections $\mathfrak{m}$ of $\mathbb{M}$ can be identified with functions on the universal cover of Λ satisfying the equivalence relation (see e.g. [195, (1.3.3)])

$$\mathfrak{m}\big(\gamma \cdot (x,\xi)\big) = i^{-\mathfrak{m}(\gamma)}\,\mathfrak{m}(x,\xi). \tag{A.60}$$

We close this section by two simple examples to give an intuitive understanding of the abstract formulae.

Example A.1.25. Let $\mathbb{R}^2$ be equipped with the canonical symplectic structure and (x,ξ) its symplectic coordinates. Suppose that $\Lambda := \{x^2 + \xi^2 = 1\} \hookrightarrow \mathbb{R}^2$ and that its orientation is assumed clockwise. Λ is an embedded Lagrangian submanifold for dimensional reasons and we consider a closed curve $\mathbb{R} \ni s \mapsto \gamma(s)$ running clockwise through Λ exactly

once. An element $\check{s}$ of the flow parameter is called a crossing of γ if $\gamma(\check{s}) \in \Sigma(\Lambda)$ and $\frac{\partial x}{\partial \xi}(\gamma(\check{s})) = 0$ whilst $- \operatorname{sgn} \frac{\partial x}{\partial \xi}$ equals to the signature of the Hessian of the phase function with respect to ξ outside of $\gamma(\check{s})$. Hence, the coorientation points from the lower half-plane to the upper half-plane at the critical point $(-1, 0)$, whereas at $(1, 0)$, the otherwise, yielding $\mathfrak{m}(\gamma) = 2$.

The figure eight immersion in $\mathbb{R}^2$ can be worked out analogously where it turns out that the Maslov index vanishes (see e.g. [190, Exm. 12.6.13] for details).

A.1.6 Natural volume form

Let φ be a non-degenerate phase function for C whose fibre-critical manifold is $\mathscr{C}$. Then the map

$$M \times N \times \dot{\mathbb{R}}^n \ni (x, y; \theta) \mapsto \left(\frac{\partial \varphi}{\partial \theta_1}(x, y; \theta), \ldots, \frac{\partial \varphi}{\partial \theta_n}(x, y; \theta) \right) \in \dot{\mathbb{R}}^n \qquad (A.61)$$

is surjective because

$$\forall (x_0, y_0; \theta^0) \in \mathscr{C} : \operatorname{rk}\left(d\left(\frac{\partial \varphi}{\partial \theta_k} \right)(x_0, y_0; \theta^0) \right) = n, \quad k = 1, \ldots, n, \qquad (A.62)$$

by Definition A.1.8, forbye, $T_{(x_0, y_0; \theta^0)}\mathscr{C} = \ker\left((d\frac{\partial \varphi}{\partial \theta_k})(x_0, y_0; \theta^0) \right)$. Thus, $\mathscr{C}$ can be endowed with the volume form (see e.g. [84, (4.1.1)]):

$$\mathrm{dv}_\mathscr{C}(X, Y) \left(d\frac{\partial \varphi}{\partial \theta_1} \wedge \ldots \wedge d\frac{\partial \varphi}{\partial \theta_n} \right)(Z^1, \ldots, Z^n) = (dx \otimes dy \otimes d\theta)(X, Y; Z) \qquad (A.63)$$

for any vector fields $X, Y, Z = (Z^1, \ldots, Z^n)$ on $M, N, \dot{\mathbb{R}}^n$, respectively. In other words,

$$\mathrm{dv}_\mathscr{C} := \delta(\operatorname{grad}_\theta \varphi) \qquad (A.64)$$

is the quotient of the measure $dx \otimes dy \otimes d\theta$ on $M \times N \times \dot{\mathbb{R}}^n$ by the pullback of the delta-distribution δ in $\dot{\mathbb{R}}^n$ under the mapping (A.61). If $(\lambda_l) = (\lambda_1, \ldots, \lambda_{d_M + d_N})$ denotes arbitrary local coordinates of $\lambda \in \mathscr{C}$ and we extend them to a neighbourhood in $M \times N \times \dot{\mathbb{R}}^n$ then $\mathrm{dv}_\mathscr{C}(\lambda)$ locally[3] means [78, p. 143, Prop. 4.1.3] (see e.g. [11, p. 123])

$$\mathrm{dv}_\mathscr{C}(\lambda) = \left| \det\begin{pmatrix} \dfrac{\partial \lambda_l}{\partial x^i} & \dfrac{\partial \lambda_l}{\partial y^j} & \dfrac{\partial \lambda_l}{\partial \theta_k} \\ \dfrac{\partial^2 \varphi}{\partial x^i \partial \theta_k} & \dfrac{\partial^2 \varphi}{\partial y^j \partial \theta_k} & \dfrac{\partial^2 \varphi}{\partial \theta_k \partial \theta_{k'}} \end{pmatrix} \right|^{-1} |d\lambda|, \qquad (A.65)$$

where $|d\lambda|$ is the Lebesgue density on $\mathscr{C}$. This expression is independent of (λ_l) but depends on the choice of (x^i, y^j) which we keep aside for the moment. Thereby

$$\mathrm{dv}_C := \left((d_J)^{-1} \right)^* \mathrm{dv}_\mathscr{C} \qquad (A.66)$$

---

[3]For instance, see [76, (6.1.1)] for the local definition of pullback of a distribution.

is a natural measure on C defined by transferring $\mathrm{dv}_{\mathscr{C}}$ via the immersion $J := $ (A.33) and hence (A.65) can be identified as the local expression of $\mathrm{dv}_C(\lambda)$ when λ is considered as an element in C. Since φ is non-degenerate one can pick its canonical form $\psi := $ (A.34) so that C is parametrised by (A.35) and $(\lambda_l) = (\xi_i = \partial\psi/\partial x^i, \eta_j = \partial\psi/\partial y^j)$ can be chosen as local coordinates on C, yielding

$$\mathrm{dv}_C(x, \xi; y, \eta) = \left| \det \left(\begin{array}{ccc} \dfrac{\partial^2\psi}{\partial x^i \partial \xi_i} & \dfrac{\partial^2\psi}{\partial y^j \partial \eta_j} & \dfrac{\partial^2\psi}{\partial \xi_i \partial \eta_j} \end{array} \right) \right|^{-1} |\mathrm{d}\xi||\mathrm{d}\eta|. \tag{A.67}$$

Had we insisted a generic φ instead of the canonical form ψ, C would be parametrised by (A.33). Then, one can go for the natural choice of coordinates $(\lambda_l) = (\xi_i = \partial\varphi/\partial x^i, \eta_j = \partial\varphi/\partial y^j)$ on C and obtains

$$\mathrm{dv}_C(x, \xi; y, \eta) = |\det(\mathrm{Hess}\,\varphi)|^{-1}|\mathrm{d}\xi||\mathrm{d}\eta|. \tag{A.68}$$

By far, an arbitrary but fixed non-degenerate phase function φ has been used to construct dv_C. However, as noted before φ is only unique up to the stable equivalence. So we consider another element $\tilde{\varphi} = $ (A.24) of this equivalence class and then the corresponding Jacobian $\mathrm{J}(\tilde{\lambda}, \mathrm{grad}_{\tilde{\theta}}\,\tilde{\varphi}; x, \tilde{\theta})$ is related to that in (A.65) by (see e.g. [85, p. 440])

$$\mathrm{J}(\lambda, \mathrm{grad}_\theta\,\varphi; x, \theta) = \big(\mathrm{J}(\tilde{\theta}; \theta)\big)^2 \mathrm{J}(\tilde{\lambda}, \mathrm{grad}_{\tilde{\theta}}\,\tilde{\varphi}; x, \tilde{\theta}), \tag{A.69}$$

where $\tilde{\lambda} \circ \upsilon = \lambda \in \mathscr{C}_\varphi$ for a fibre-preserving diffeomorphism $\upsilon = $ (A.23).

The situation becomes much simpler when the canonical relation Γ is the graph of a symplectomorphism from $\dot{\mathrm{T}}^*N$ to $\dot{\mathrm{T}}^*M$. In this case Γ is a symplectic manifold with respect to the symplectic form (see e.g. [23, p. 25])

$$\sigma_\Gamma := \mathrm{pr}_M^*\,\sigma_M = \mathrm{pr}_N^*\,\sigma_N \tag{A.70}$$

and we have the corresponding Liouville volume form $\mathrm{v}_\Gamma := $ (2.63).

Therefore, one has the bundle $\Omega^{1/2}C \to C$ of half-densities over a homogeneous canonical relation C, whose local sections are $\sqrt{|\mathrm{dv}_C|}$.

A.2 Composition of densities on canonical relations

A.2.1 Half-densities

To define a half-density on the canonical relation $C \circ_e \Lambda \subset \dot{\mathrm{T}}^*M \times \dot{\mathrm{T}}^*N$ (Definition A.1.18), given half-densities on closed conic canonical relations $C \subset \dot{\mathrm{T}}^*M \times \dot{\mathrm{T}}^*O$ and $\Lambda \subset \dot{\mathrm{T}}^*O \times \dot{\mathrm{T}}^*N$, we note that [10, Lem. 5.2] (see also, e.g. [25, Thm. 7.1.1, (7.4)], [77, Thm. 21.6.7])

$$\Omega^{1/2}_{(x,\xi;z,\zeta)}C \otimes \Omega^{1/2}_{(z,\zeta;y,\eta)}\Lambda \cong \Omega^{1/2}_{(x,\xi;z,\zeta;y,\eta)}\mathfrak{F} \otimes \Omega^{1/2}_{(x,\xi;y,\eta)}(C \circ_e \Lambda), \tag{A.71}$$

where $\mathfrak{F}$ is the set of fibres $\mathfrak{F}_{(x,\xi;y,\eta)} := $ (A.44) of the fibration $\Pi := $ (A.45). This entails that by taking tensor product between half-densities on C and Λ followed by intersecting with the diagonal gives a half-density on $\mathfrak{F}$ times a half-density on $C \circ_e \Lambda$, and therefore, integrating over the fibre $\mathfrak{F}_{(x,\xi;y,\eta)}$ we achieve the desired composition of half-densities on canonical relations [10, p. 64]:

$$\cdot \diamond_e \cdot \ : \ C^\infty\big(C; \Omega^{1/2}C\big) \times C^\infty\big(\Lambda; \Omega^{1/2}\Lambda\big) \to C^\infty\big(C \circ_e \Lambda; \Omega^{1/2}(C \circ_e \Lambda)\big),$$

$$(\mu, \nu) \mapsto \mu \diamond_e \nu\,(x,\xi; y,\eta) := \int_{\mathfrak{F}_{(x,\xi;y,\eta)}} (\mu \otimes \nu)\big(\Pi^{-1}(x,\xi; y,\eta)\big). \quad \text{(A.72)}$$

If $e = 0$ then the isomorphism (A.71) reduces to [78, p. 179] (see also, e.g. [25, (7.3)])

$$\Omega^{1/2}C \otimes \Omega^{1/2}\Lambda \cong \Omega^{1/2}(C \circ \Lambda) \quad \text{(A.73)}$$

and hence (A.72) simply becomes the pointwise evaluation (see e.g. [84, Thm. 4.2.2])

$$\mu \diamond \nu\,(x,\xi; y,\eta) = \sum_{(z,\zeta)\,|\,(x,\xi;z,\zeta)\in C,\,(z,\zeta;y,\eta)\in \Lambda} \mu(x,\xi; z,\zeta)\,\nu(z,\zeta; y,\eta). \quad \text{(A.74)}$$

We call the pairs $(x,\xi), (y,\eta)$ (resp. (z,ζ)) in the preceding equation as the output (rep. input) variables of the composition following the terminology from [27, Lem. 8.4].

A.2.2 Keller-Maslov bundle-valued half-densities

The principal symbol (Definition 2.2.15) of a scalar Lagrangian distribution associated with a canonical relation C is a Keller-Maslov bundle $\mathbb{L} \to C$ valued density on C. Hence, we are compelled to extend the composition (A.72) on sections of $\mathbb{L} \otimes \Omega^{1/2}C$. The first step, thus is to construct the Keller-Maslov bundle $\mathbb{M} \to C \circ_e \Lambda$ given $\mathbb{L} \to C$ and $\mathbb{T} \to \Lambda$. This is achieved by utilising the phase function $\psi := $ (A.46) into (A.52) and (A.48), as it is locally constant on $\mathfrak{F}$. Next, one employs the isomorphism [10, (5.7)] (see also, e.g. [25, (5.29)])

$$\mathbb{L} \boxtimes \mathbb{T} := \Pi_C^*\mathbb{L} \otimes \Pi_\Lambda^*\mathbb{T} \cong \Pi^*\mathbb{M}, \quad \text{(A.75)}$$

where $\Pi_C, \Pi_\Lambda : C \star \Lambda \to C, \Lambda$ are the projections, in order to make the identification

$$(\Pi^*\mathrm{m})(x,\xi; z,\zeta; z,\zeta; y,\eta) = (\Pi_C^*\mathbb{l} \otimes \Pi_\Lambda^*\mathrm{t})(x,\xi; z,\zeta; z,\zeta; y,\eta) \quad \text{(A.76)}$$

for any given sections $\mathbb{l}$ resp. t of $\mathbb{L}$ resp. $\mathbb{T}$ with a section m of $\mathbb{M}$. The desired extension of (A.72) is then

$$C^\infty\big(C; \mathbb{L}\otimes\Omega^{\frac{1}{2}}C\big)\times C^\infty\big(\Lambda; \mathbb{T}\otimes\Omega^{\frac{1}{2}}\Lambda\big) \to C^\infty\big(C\circ_e\Lambda; \mathbb{M}\otimes\Omega^{\frac{1}{2}}(C\circ_e\Lambda)\big), (\mathbb{l}\otimes\mu, \mathrm{t}\otimes\nu) \mapsto \mathrm{m}\otimes(\mu\diamond_e\nu).$$
$$\text{(A.77)}$$

A.2.3   Homomorphism bundle-valued half-densities

Finally, as in the set-up of Section 2.3.5 we consider any vector bundles $\mathscr{E} \to M, \mathscr{G} \to O, \mathscr{F} \to N$ and so $\mathrm{Hom}(\mathscr{G}, \mathscr{E}) \to M{\times}O$ is a homomorphism bundle, forbye, $\widetilde{\mathrm{Hom}}(\mathscr{G}, \mathscr{E}) \to C$ represents the pullback of $\mathrm{Hom}(\mathscr{G}, \mathscr{E})$ to $\mathrm{T}^*(M \times O)$ followed by restriction to C. One then extends (A.77) by tensoring with $\widetilde{\mathrm{Hom}}(\mathscr{G}, \mathscr{E}), \widetilde{\mathrm{Hom}}(\mathscr{F}, \mathscr{G}), \widetilde{\mathrm{Hom}}(\mathscr{F}, \mathscr{E})$, respectively to achieve our final composition rule [23, (25.2.10)]

$$C^\infty\big(C; \mathbb{L} \otimes \Omega^{1/2}C \otimes \widetilde{\mathrm{Hom}}(\mathscr{G}, \mathscr{E})\big) \times C^\infty\big(\Lambda; \mathbb{T} \otimes \Omega^{1/2}\Lambda \otimes \widetilde{\mathrm{Hom}}(\mathscr{F}, \mathscr{G})\big)$$
$$\to C^\infty\big(C \circ_e \Lambda; \mathbb{M} \otimes \Omega^{1/2}(C \circ_e \Lambda) \otimes \widetilde{\mathrm{Hom}}(\mathscr{F}, \mathscr{E})\big),$$
$$(\mathbb{1} \otimes a, \mathbb{t} \otimes b) \mapsto \mathbb{m} \otimes (a \diamond_e b),$$
$$(a \diamond_e b)(x, \xi; y, \eta) := \int_{\mathfrak{F}_{(x,\xi;y,\eta)}} (a \boxtimes b)\big(\Pi^{-1}(x, \xi; y, \eta)\big). \tag{A.78}$$

As before, the hindmost expression reduces to standard composition of homomorphisms when $e = 0$:

$$(a \diamond b)(x, \xi; y, \eta) = \sum_{(z,\zeta) \,|\, (x,\xi;z,\zeta)\in C,\, (z,\zeta;y,\eta)\in\Lambda} a(x, \xi; z, \zeta)\big(b(z, \zeta; y, \eta)\big). \tag{A.79}$$

A.3   Literature

The notions of *Maslov index* [101, §7] and *Lagrangian submanifold* [101, Def. 4.25] are originally due to Victor MASLOV who also studied the local generating functions [101, Thm. 4.20] for a Lagrangian manifold in some special cases. Vladimir ARNOL'D put forwarded the former in the modern mathematical footing [194] while Lars HÖRMANDER generalised the latter ideas by introducing the *canonical relation* [78, Def. 4.1.2] and the *non-degenerate phase function* [78, p. 91]. He gave the systematic route to locally generate a generic Lagrangian manifold by means of a phase function and obtained the necessarily and sufficient conditions [78, Thm. 3.1.16] for two phase functions to parametrise the same Lagrangian manifold. Moreover, he advanced the works by Keller [196] and by Maslov [101, §10] to give complete analytic description of the *Keller-Maslov bundle* [78, Sec. 3.3]. Furthermore, the composition of Keller-Maslov bundle-valued half-densities on *transversally intersecting* canonical relations was achieved by him [78, Thm. 4.2.2]. Subsequently, Johannes DUISTERMAAT and Victor GUILLEMIN envisaged the *clean phase function* [10, p. 71] and propounded Hörmader's composition formula for *cleanly intersecting* canonical relations. In contemporary of these developments a more geometric formulation was offered by Alan WEINSTEIN in a series of papers [94, 191, 197] (see also, e.g. [20]) where the terms *Lagrangian correspondence*, *Morse family* and *stable equivalence* were coined for canonical relation, phase function and equivalent class of phase functions for a given Lagrangian manifold, respectively. An in-depth discussion on these aspects of symplectic geometry is available, for instance, in the textbooks [96, 190] and we refer Hörmader's treatises [23, 77] together with the expositions [25, 84, 85] for the conic symplectic geometry.

Appendix B

Symbols

We accumulate the background materials on a vector bundle-valued classical symbols on a conic manifold as required for this thesis.

Throughout this chapter, we have assumed that $d \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}, n \in \mathbb{N}, m \in \mathbb{R}$ and used multi-indices $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n, \beta = (\beta_1, \ldots, \beta_d) \in \mathbb{N}_0^d$ together with the multi-index notion for partial derivatives:

$$\partial_\theta^\alpha := \frac{\partial^{|\alpha|}}{\partial \theta_1^{\alpha_1} \ldots \partial \theta_n^{\alpha_n}}, \qquad \mathrm{D}_x^\beta := (-\mathrm{i})^{|\beta|} \frac{\partial^{|\beta|}}{\partial (x^1)^{\beta_1} \ldots \partial (x^d)^{\beta_d}} \tag{B.1}$$

where $|\alpha| := |\alpha_1| + \ldots + |\alpha_n|$ and so on for $|\beta|$. Recall, the **Japanese bracket** is defined by

$$\langle \theta \rangle := \sqrt{1 + \theta \cdot \theta}, \tag{B.2}$$

where $\cdot$ is the Euclidean inner product.

B.1 Symbols on a Euclidean space

Definition B.1.1. Let U be an open subset of the Euclidean space $\mathbb{R}^d$. The set $S_{1,0}^m(U \times \mathbb{R}^n)$ of **Kohn-Nirenberg symbols** of **order** (at most) m on $U \times \mathbb{R}^n$ is defined as the set of all $\mathbb{C}$-valued smooth functions $a \in C^\infty(U \times \mathbb{R}^n)$ on $U \times \mathbb{R}^n$ such that, for every compact set $K \subset U$, the estimation (see e.g. [77, (18.1.1)′])

$$\forall x \in K, \forall \theta \in \mathbb{R}^n, \forall \alpha \in \mathbb{N}_0^n, \forall \beta \in \mathbb{N}_0^d : \left| (\mathrm{D}_x^\beta \partial_\theta^\alpha a)(x; \theta) \right| \le \mathrm{cst}_{\alpha,\beta;K} \langle \theta \rangle^{m-|\alpha|} \tag{B.3}$$

holds true for some constant $\mathrm{cst}_{\alpha,\beta;K}$.

One sets

$$S_\bullet^\infty := \bigcup_{m \in \mathbb{R}} S_\bullet^m, \quad S_\bullet^{-\infty} := \bigcap_{m \in \mathbb{R}} S_\bullet^m, \quad S_\bullet^{m-[k]} := S_\bullet^m / S_\bullet^{m-k}. \tag{B.4}$$

Example B.1.2. The set of all positively homogeneous smooth functions ((A.13)) on $U \times \mathbb{R}^n$ of order m is automatically $S_{1,0}^m(U \times \mathbb{R}^n)$.

We equip the set $S_{1,0}^m(U \times \mathbb{R}^n)$ with the seminorm separating points defined as the best constant in (B.3):

$$\|\cdot\|_{m,N} : S_{1,0}^m(U \times \mathbb{R}^n) \to \mathbb{R}_{\geq 0}, \ a \mapsto \|a\|_{m,N} := \sup_{(x;\theta) \in K \times \mathbb{R}^n} \max_{|\alpha|+|\beta| \leq N} \frac{\left|(D_x^\beta \partial_\theta^\alpha a)(x;\theta)\right|}{\langle \theta \rangle^{m-|\alpha|}}, \quad \text{(B.5)}$$

which induces the metric

$$\mathsf{d}_m : S_{1,0}^m(U \times \mathbb{R}^n) \times S_{1,0}^m(U \times \mathbb{R}^n) \to \mathbb{R}_{\geq 0}, \ (a,b) \mapsto \mathsf{d}_m(a,b) := \sum_{N \in \mathbb{N}_0} \frac{1}{2^N} \frac{\|a-b\|_{m,N}}{1 + \|a-b\|_{m,N}}, \quad \text{(B.6)}$$

yielding

Proposition B.1.3. *As in the terminologies of Definition B.1.1,* $\left(S_{1,0}^m(U \times \mathbb{R}^n), \mathsf{d}_m := \text{(B.6)}\right)$ *and* $\left(S_{1,0}^{\pm\infty}(U \times \mathbb{R}^n), \mathsf{d}_{\pm\infty}\right)$ *are Fréchet spaces. Furthermore,* $S_{1,0}^m(U \times \mathbb{R}^n) \hookrightarrow S_{1,0}^{m'}(U \times \mathbb{R}^n)$ *is a continuous inclusion for all* $m \leq m'$, *which is never dense if* $m < m'$ *[78, Prop. 1.1.11].*

The above symbol class is too general for the thesis, and it turns out that the "classical symbol" class, a subclass of $S_{1,0}^m$, is sufficient for our purpose. To introduce this subclass, we recall a notion related to the following completeness property of the space of symbols [198, Thm. 2.7]

Definition B.1.4. As in the terminologies of Definition B.1.1, suppose that $(a_k)_{k \in \mathbb{N}_0} \in S_{1,0}^{m_k}(U \times \mathbb{R}^n)$ with $m_0 > m_1 > \ldots$ and $m_k \to -\infty$ as $k \to \infty$, and that $a \in S_{1,0}^{\mu_0}(U \times \mathbb{R}^n)$ where $\mu_N := \max_{k \geq N} m_k$. Then a is called the **asymptotic summation** of a_k:

$$a \sim \sum_{k \in \mathbb{N}_0} a_k \Leftrightarrow \operatorname{supp} a \subset \bigcup_{k \in \mathbb{N}_0} \operatorname{supp}(a_k), \ \forall N \in \mathbb{N} : \left(a - \sum_{k=0}^{N-1} a_k\right) \in S_{1,0}^{\mu_N}(U \times \mathbb{R}^n). \quad \text{(B.7)}$$

Remark B.1.5. Asymptotic summation always exists. Furthermore, it is unique modulo $S_{1,0}^{-\infty}(U \times \mathbb{R}^n)$ and independent of any rearrangement of the series $\sum a_k$ (see e.g. [77, Prop. 18.1.3]).

Now, our desired symbol class is formulated as follows.

Definition B.1.6. Let U be an open subset of the Euclidean space $\mathbb{R}^d$. The set $S^m(U \times \mathbb{R}^n)$ of **polyhomogeneous symbols** of **order** (at most) m on $U \times \mathbb{R}^n$ is defined as the set of all $a \in S_{1,0}^m(U \times \mathbb{R}^n)$ which can be expressed as (see e.g. [77, Def. 18.1.5])

$$a \sim \sum_{k \in \mathbb{N}_0} a_k \chi_k \quad \text{(B.8)}$$

where, for each k, a_k is positively homogeneous of degree $m - k$ when $|\theta| > 1$, $\sim$ denotes the asymptotic summation and $\chi_k \in C^\infty(\mathbb{R}^n)$ vanishing identically near $\theta = 0$ with $\chi_k = 1$ if $|\theta| \geq 1$.

As a subspace $S^m(U \times \mathbb{R}^n) \subset S^m_{1,0}(U \times \mathbb{R}^n)$ is dense but not closed whereas the residual symbol space $S^{-\infty}_{1,0}(U \times \mathbb{R}^n) = S^{-\infty}(U \times \mathbb{R}^n) \subset S^m(U \times \mathbb{R}^n)$ is closed for any m. Apart from the subspace topology, $S^m(U \times \mathbb{R}^n)$ can be endowed with a seminorm for each a_k and a seminorm (cf. (B.5)) ensuring asymptotic summability

$$\left|(\mathrm{D}^\beta_x \partial^\alpha_\theta (a - \sum_{k=0}^{N-1} a_k))(x;\theta)\right| \leq \mathrm{cst}_{\alpha,\beta;N} \langle \theta \rangle^{\mu_N - |\alpha|}. \tag{B.7'}$$

Thus, $S^m(U \times \mathbb{R}^n)$ is equipped with a countable number of seminorms and it becomes a *Fréchet space* with respect to the metric induced by these seminorms (cf. (B.6)) (see e.g. [87, Sec. 2.16]).

We observe, $\mathrm{D}^\beta_x \partial^\alpha_\theta a_k$ is of degree $m - k - |\alpha|$ entailing $a_k \in S^{m-k}(U \times \mathbb{R}^n)$ provided that $a \in C^\infty(U \times \mathbb{R}^n)$ and vanishes for large x. Therefore, away from $\theta = 0$, attributed by large θ, *polyhomogeneous symbols are essentially smooth functions with boundary*.

In order to formulate the principal symbol (Definition 2.1.9) of a Fourier integral operator we are compelled to extend the notion of symbols on a conic set.

Definition B.1.7. Let $U \subset \mathbb{R}^d$ be an open set. The set $S^m_{1,0}(\mathcal{U})$ of **Kohn-Nirenberg symbols** of **order** (at most) m on a conic set $\mathcal{U} \subset U \times \mathbb{R}^n$ is defined as the set of all $a \in C^\infty(\mathcal{U})$ such that for every compact $\mathcal{K} \subset \mathcal{U}$ the estimation (B.3) holds true in $\{(x;\lambda\theta) \mid (x;\theta) \in \mathcal{K}\}$ where $\lambda \geq 1$ (see e.g. [77, p. 83]).

Remark B.1.8. Definition B.1.7 reproduces Definition B.1.1 by making the choice $\mathcal{K} := K \times \{\theta \mid |\theta| \leq 1\}$ where K is as in the later definition (see e.g. [77, p. 84]). The asymptotic summability carries over $S^m_{1,0}(\mathcal{U})$ and so we can define and topolise the corresponding polyhomogeneous class $S^m(\mathcal{U})$ analogously. Also, by construction:

$$\mathrm{D}^\beta_x : S^m(U \times \mathbb{R}^n) \to S^m(U \times \mathbb{R}^n), \quad \partial^\alpha_\theta : S^m(U \times \mathbb{R}^n) \to S^{m-|\alpha|}(U \times \mathbb{R}^n). \tag{B.9}$$

B.2 Symbols on manifolds

The principal symbol (Definition 2.2.6) and the subprincipal symbol (Definition 2.2.7) of a pseudodifferential operator acting on a half-density bundle (Section 2.2.1) $\Omega^{1/2} \to M$ over a manifold M are invariantly defined homogeneous functions on the punctured cotangent bundle $\dot{\mathrm{T}}^*M$ as Remarked in 2.2.5 and 2.2.8. Moreover, the principal symbol (Definition 2.2.15) of a Lagrangian distribution is a half-density on a homogeneous canonical relation (Definition A.1.12) $C \subset \dot{\mathrm{T}}^*M \times \dot{\mathrm{T}}^*N$ where N is a manifold whose dimension d_N is not necessarily equal to that $d_M := d$ of M. As evident from the local case, it is sufficient to consider only the smooth functions on $\dot{\mathrm{T}}^*M$ for large covectors. Therefore, guided by the fact that $\dot{\mathrm{T}}^*M$ (Example A.1.2) and C both are examples of a conic manifold (Definition A.1.1), together with Remark B.1.8, we are going to describe symbols on the following class of conic manifolds.

Definition B.2.1. Let $(\dot{\mathscr{V}}, \mathrm{m}_\lambda)$ be the conic manifold as described in Example A.1.3. The set $S^m_{1,0}(\dot{\mathscr{V}})$ of **Kohn-Nirenberg symbols** of **order** (at most) m on this conic manifold is defined as the set of all $a \in C^\infty(\dot{\mathscr{V}})$ such that the functions $\lambda^{-m} \mathrm{m}^*_\lambda a$ are uniformly bounded in $C^\infty(\dot{\mathscr{V}})$ when $\lambda \geq 1$ (see e.g. [23, p. 13], [84, Def. 2.1.2]).

One of the advantages of this definition is that it is well applicable for densities. In particular, if μ is a fixed positive half-density on $\dot{\mathscr{V}}$ having degree of homogeneity m' (Definition A.1.1) then every element $\mathsf{a} \in S^m(\dot{\mathscr{V}}; \Omega^{1/2}\dot{\mathscr{V}})$ can be written in the form

$$\mathsf{a} = a\mu \tag{B.10}$$

for some scalar $a \in S^{m-m'}(\dot{\mathscr{V}})$. See Example 2.2.2 for an illustration of this concept.

Equivalently, a bottom-up approach can be adapted for which let us consider an atlas $\{(U_\alpha, \kappa_\alpha)\}_\alpha$ for M and a homogeneous symplectic atlas $\mathcal{A} := \{(\mathcal{U}_\alpha, \varkappa_\alpha)\}_\alpha$ for $\dot{\mathrm{T}}^*M$ such that $\varkappa_\alpha : \mathcal{U}_\alpha \cong \kappa_\alpha(U_\alpha) \times \dot{\mathbb{R}}^d$, as considered in Remark 2.2.5. Then

$$S^m_{1,0}(\mathcal{U}_\alpha) = \left\{ a \in C^\infty(\mathcal{U}_\alpha) \mid \forall (\mathcal{U}_\alpha, \varkappa_\alpha) \in \mathcal{A} : (\varkappa^*_\alpha)^{-1} a \in S^m_{1,0}\big(\kappa_\alpha(U_\alpha) \times \dot{\mathbb{R}}^d\big) \right\} \tag{B.11}$$

and consequently, $S^m_{1,0}(\dot{\mathrm{T}}^*M)$ is constructed utilising the partition of unity. We remark that it is sufficient to satisfy these requirements for an atlas due to

Lemma B.2.2. *Let $U \subset \mathbb{R}^{d_U}, V \subset \mathbb{R}^{d_V}$ be open sets. Suppose that $\mathcal{U} \subset U \times \dot{\mathbb{R}}^{n_U}, \mathcal{V} \subset V \times \dot{\mathbb{R}}^{n_V}$ are open and conic sets, and that $\kappa : \mathcal{U} \to \mathcal{V}$ is a smooth proper map commuting with multiplication by positive scalars in the second variable. If $a \in S^m(V \times \mathbb{R}^{n_V})$ has support in the interior of a compactly based conic subset of $\mathcal{V}$ then $a \circ \kappa \in S^m(U \times \mathbb{R}^{n_U})$ provided the composition is defined as 0 outside $\mathcal{U}$ (see e.g. [77, Lem. 25.1.6], [84, Def. 2.1.2]).*

The asymptotic summability carries over $S^m_{1,0}(\dot{\mathrm{T}}^*M)$ (see e.g. [84, Prop. 2.1.2]) and so we define $S^m(\dot{\mathrm{T}}^*M)$ analogously. It is now straightforward to extend the construction for the product manifold $\dot{\mathrm{T}}^*M \times \dot{\mathrm{T}}^*N \cong \dot{\mathrm{T}}^*(M \times N)$. Given a homogeneous canonical relation (C, ι) of $\dot{\mathrm{T}}^*M \times \dot{\mathrm{T}}^*N$, one achieves

$$S^m(C) := \iota^* S^m(\dot{\mathrm{T}}^*M \times \dot{\mathrm{T}}^*N). \tag{B.12}$$

Since any open conic cover $\{C_{\alpha,\beta}\}$ of C is locally generated as elucidated in Section A.1.2

$$\mathcal{U}_\alpha \times \mathcal{V}_\beta \xrightarrow{\ker \mathrm{grad}_{\xi,\eta} \psi} \mathscr{C}_{\alpha,\beta} \xrightarrow{(A.35)} C_{\alpha,\beta} \tag{B.13}$$

by a non-degenerate phase function ψ (A.34) on a homogeneous symplectic neighbourhood $\mathcal{U}_\alpha \times \mathcal{V}_\beta$ on $\dot{\mathrm{T}}^*M \times \dot{\mathrm{T}}^*N$, locally (B.12) means to obtain $S^m(C_{\alpha,\beta})$ from $S^m(\mathcal{U}_\alpha \times \mathcal{V}_\beta)$ via appropriate pullbacks of the preceding mapping. Finally, we incorporate $\Omega^{1/2}C \otimes \mathbb{M}$-valued symbols $\mathsf{a} \in S^{m+(d_M+d_N)/4}(C; \Omega^{1/2}C \otimes \mathbb{M})$ simply by using (B.10) (see e.g. [77, Def. 18.2.10]):

$$\mathsf{a} = a \, |\mathrm{dv}_C|^{1/2} \otimes \mathrm{m}, \quad a \in S^{m+(d_M+d_N-2n)/4}(C) \tag{B.14}$$

where dv_C is the natural volume form on C, as constructed in Section A.1.6, and m is a section of the Keller-Maslov bundle $\mathbb{M} \to C$ as detailed in Section A.1.4. The former resp. latter are of homogeneous of degree n resp. 0, as explained in Example 2.2.2.

B.3   Symbols on vector bundles

One of the primary tools to address the subject matter of this thesis is the symbol calculus on vector bundles. Loosely speaking, this is achieved by replacing the $\mathbb{C}$-valued smooth maps by a bundle-valued smooth maps in the discussion of Section B.2 yet we spell out the details to supplement the theory of Fourier integral operators on vector bundles.

Definition B.3.1. Let $(\dot{\mathcal{V}}, \mathsf{m}_\lambda)$ be a conic manifold as described in Example A.1.3 and let $\mathbb{C}^{r\times k}$ denotes the set of all r by k matrices over $\mathbb{C}$. The set $S_{1,0}^m(\dot{\mathcal{V}}, \mathbb{C}^{r\times k})$ of $\mathbb{C}^{r\times k}$-**valued Kohn-Nirenberg symbols** on $\dot{\mathcal{V}}$ of **order** m is defined as the set of all $a \in C^\infty(\dot{\mathcal{V}}, \mathbb{C}^{r\times k})$ such that the functions $\lambda^{-m}\mathsf{m}_\lambda^* a$ are uniformly bounded in $C^\infty(\dot{\mathcal{V}}, \mathbb{C}^{r\times k})$ whenever $\lambda \geq 1$.

This means that for any $a \in C^\infty(\mathcal{U}, \mathbb{C}^{r\times k})$, for every compact set $\mathcal{K} \subset \mathcal{U}$, the estimation

$$\left\| (D_x^\beta \partial_\theta^\alpha a)(x; \theta) \right\| \leq \mathrm{cst}_{\alpha,\beta;\mathcal{K};\|\cdot\|} \, \langle \theta \rangle^{m-|\alpha|} \tag{B.15}$$

is valid for all $(x; \theta) \in \{(x; \lambda\theta) | (x; \theta) \in \mathcal{K}\}$ when $\lambda \geq 1$, for any $\alpha \in \mathbb{N}_0^n, \beta \in \mathbb{N}_0^d$ and for some constant $\mathrm{cst}_{\alpha,\beta;\mathcal{K};\|\cdot\|}$. Here $\|\cdot\|$ is some choice of norm on $\mathbb{C}^{r\times k}$ and this particular choice does not matter (see e.g. [9, (1.5.1.3)]).

Since $C^\infty(\mathcal{U}, \mathbb{C}^{r\times k})$ is isomorphic to the $r \times k$ matrix of elements $C^\infty(\mathcal{U})$, we have $S_{1,0}^m(\mathcal{U}, \mathbb{C}^{r\times k}) \cong \left(S_{1,0}^m(\mathcal{U})\right)_{r\times k}$ for any m and any $r, k \in \mathbb{N}$. As in the scalar case, one can equip $S_{1,0}^m(\mathcal{U}, \mathbb{C}^{r\times k})$ with the Fréchet topology and the asymptotic completeness can be shown. Consequently, we obtain the polyhomogeneous symbol class $S^m(\mathcal{U}, \mathbb{C}^{r\times k})$ and (see e.g. [9, Sec. 1.5.2])

$$\forall m \in \mathbb{R}, \forall r, k \in \mathbb{N} : S^m(\mathcal{U}, \mathbb{C}^{r\times k}) \cong \left(S^m(\mathcal{U})\right)_{r\times k}. \tag{B.16}$$

Our quintessential example is the homomorphism bundle $\mathrm{Hom}(\mathscr{F}, \mathscr{E}) \to M \times N$ (Section 2.3.3), given a vector bundle $\mathscr{E} \to M$ (resp. $\mathscr{F} \to N$) over a manifold M (resp. N), where neither $r := \mathrm{rk}\,\mathscr{E}$ and $k := \mathrm{rk}\,\mathscr{F}$ nor d_M and d_N are necessarily equal. Employing the canonical projections $\pi_M : \mathrm{T}^*M \to M, \pi_N : \mathrm{T}^*N \to N$, one has the induced homomorphism bundle $(\pi_M \boxtimes \pi_N)^* \mathrm{Hom}(\mathscr{F}, \mathscr{E}) \to \mathrm{T}^*M \times \mathrm{T}^*N$ and the natural isomorphism

$$C^\infty\left(\mathrm{T}^*M \times \mathrm{T}^*N, \mathrm{Hom}(\mathscr{F}, \mathscr{E})\right) \cong C^\infty\left(\mathrm{T}^*M \times \mathrm{T}^*N; (\pi_M \boxtimes \pi_N)^* \mathrm{Hom}(\mathscr{F}, \mathscr{E})\right), \tag{B.17}$$

which will be often exploited without further comments.

Let $\mathcal{A} := \{(\mathcal{U}_\alpha, \varkappa_\alpha)\}_\alpha$ (resp. $\mathcal{B} := \{(\mathcal{V}_\beta, \varrho_\beta)\}_\beta$) be a homogeneous symplectic atlas for $\dot{\mathrm{T}}^*M$ (resp. $\dot{\mathrm{T}}^*N$) where $(\pi_M \boxtimes \pi_N)^* \mathrm{Hom}(\mathscr{F}, \mathscr{E})$ admits an atlas $\mathfrak{A} := \{(\mathcal{U}_\alpha \times \mathcal{V}_\beta, \vartheta_{\alpha,\beta})\}_{\alpha,\beta}$, i.e.,

$$\vartheta_{\alpha,\beta} : \left((\pi_M \boxtimes \pi_N)^* \mathrm{Hom}(\mathscr{F}, \mathscr{E})\right)_{\alpha,\beta} \to \mathcal{U}_\alpha \times \mathcal{V}_\beta \otimes \mathbb{C}^{r\times k} \tag{B.18}$$

is the induced trivialisation of $(\pi_M \boxtimes \pi_N)^* \operatorname{Hom}(\mathscr{F}, \mathscr{E})$ over $\mathcal{U}_\alpha \times \mathcal{V}_\beta$. Thereby

$$
\begin{aligned}
S^m\Big(\mathcal{U}_\alpha \times \mathcal{V}_\beta; \big((\pi_M \boxtimes \pi_N)^* \operatorname{Hom}(\mathscr{F}, \mathscr{E})\big)_{\alpha,\beta}\Big) \;\; &:= \;\; \Big\{ a \in C^\infty\Big(\big((\pi_M \boxtimes \pi_N)^* \operatorname{Hom}(\mathscr{F}, \mathscr{E})\big)_{\alpha,\beta}\Big) \Big| \\
& \qquad \forall (\mathcal{U}_\alpha \times \mathcal{V}_\beta, \vartheta_{\alpha,\beta}) \in \mathfrak{A} : \\
& \qquad (\vartheta_{\alpha,\beta}^*)^{-1} a \in S^m\big(\mathcal{U}_\alpha \times \mathcal{V}_\beta, \mathbb{C}^{r \times k}\big) \Big\} \\
& \cong \;\; \big(S^m(\mathcal{U}_\alpha \times \mathcal{V}_\beta)\big)_{r \times k}. \qquad\qquad\text{(B.19)}
\end{aligned}
$$

Assembling these local pieces together deploying the partition of unity one then achieves $S^m\big(\dot{\mathrm{T}}^*M \times \dot{\mathrm{T}}^*N, \operatorname{Hom}(\mathscr{F}, \mathscr{E})\big)$.

For a canonical relation $\iota : C \to \dot{\mathrm{T}}^*M \times \dot{\mathrm{T}}^*N$, one sets

$$
\begin{aligned}
S^m\big(C; \widetilde{\operatorname{Hom}}(\mathscr{F}, \mathscr{E})\big) \;\; &:= \;\; \iota^* S^m\big(\mathrm{T}^*M \times \mathrm{T}^*N, \operatorname{Hom}(\mathscr{F}, \mathscr{E})\big), \\
\widetilde{\operatorname{Hom}}(\mathscr{F}, \mathscr{E}) \;\; &:= \;\; \iota^*\big((\pi_M \boxtimes \pi_N)^* \operatorname{Hom}(\mathscr{F}, \mathscr{E})\big). \qquad\qquad\text{(B.20)}
\end{aligned}
$$

To end, $S^m\big(C; \Omega^{1/2}C \otimes \mathbb{M} \otimes \widetilde{\operatorname{Hom}}(\mathscr{F}, \mathscr{E})\big)$ is constructed as in (B.14).

B.4 Literature

The symbol class $S^m_{1,0}$ was named after Joseph J. KOHN and Louis NIRENBERG who introduced the class in their study of pseudodifferential operators [99]. At any rate, this is not the most general possibility, for instance, the Hörmander symbol class [198] is a vast generalisation of it, originated in the investigation of fundamental solutions of hypoelliptic[1] operators of constant strength. We refer the monographs [8, 9, 77] and the expository lecture notes [87, 90] for details.

---

[1]A differential operator P with smooth coefficients is called **hypoelliptic** if the equation $Pu = f$ only has smooth solutions u when f is smooth.

Appendix C

Hadamard States

A bare minimum introduction of Feynman propagators by means of the time-ordered product is presented to bridge the quantum field theoretic approach and the microlocal formalism.

In this chapter we will consider quantum field theory in a classical curved spacetime (Definition 3.1.2), also known as semi-classical gravity in the physics community. According to Einstein's general theory of relativity - the best experimentally tested theory of gravity by far, our universe is some 4-dimensional Lorentzian manifold. On the other hand, quantum field theory in Minkowski spacetime has proven extremely successful by most experimental outcomes. Therefore, semi-classical gravity stems from the urge to extend the standard quantum field theoretic framework to a Lorentzian manifold, in physics terms, to include gravitational effects on quantum fields. It should be possible to derive semi-classical gravity by taking a suitable limit of a more fundamental theory wherein the spacetime metric is treated in accord with the principle of quantum physics. However, this has not been done - except in formal and heuristic ways - simply because no present quantum theory of gravity has been developed to the point where such a well-defined limit can be taken.

Quantum field theory in curved spacetime is expected to provide an *accurate description of quantum phenomena in a regime where the effects of spacetime curvature is significant yet the effects of quantum gravity may be neglected*. In particular, it is anticipated that this formalism should be applicable to the quantum phenomenology in the early universe and near (and inside of) black holes - provided that one does not attempt to describe phenomena occurring so near to the singularities that the curvature reaches some extreme scale and the quantum nature of spacetime metric must be taken into account (see e.g. the review [199] and references therein).

There are numerous ways to define a quantum theory and each approach comes with its merits and demerits; we refer the reviews [199–201] for compendia of quantisation schemes and related aspects. In this thesis, we will use the locally covariant framework [202] as it offers a mathematically precise and conceptually sound formulation of

quantum field theory in curved spacetime. Loosely speaking, the essential idea is to take quantum observables as the fundamental object to define a quantum theory through the association of some appropriate algebra to every extended bounded open region of the spacetime, satisfying *general covariance, Einstein causality* and *time-slice axiom*. These postulates are motivated on physical ground and can be interpreted as follows. The first requirement is a ramification of general relativity which tells that quantum fields should transform "nicely" under any coordinate change of spacetime geometry. Einstein causality ensures finite propagation speed to any physical information and the last demand provides the dynamics of the theory. To formulate all these notions precisely, it requires to introduce quite a bit of terminologies from operator algebra and Lorentzian geometry, which are beyond the scope of this thesis, and is not essential to comprehending the idea of Hadamard states and time-ordered products. Thus, we refer the expositions [199, 201, 203] for interested readers. For us, it is sufficient to keep in mind some unital $*$-algebra $\mathscr{A}$ which satisfies these "postulates". Concrete characterisation of such an algebra actually defines a quantum field theory, as briefly discussed below.

C.1   Algebra of quantum fields

Let $\mathscr{E} \to \mathscr{M}$ be a vector bundle over a globally hyperbolic spacetime (Definition 3.1.3) $(\mathscr{M}, \mathrm{g})$ and let $\bar{\mathscr{E}}^*$ be the conjugate algebraic-dual bundle (Section 2.3.4) of $\mathscr{E}$. A *linear quantum field* is some algebra (called the **field algebra**) $\mathscr{F}(\mathscr{M}; \mathscr{E})$-valued distribution

$$\Phi : C_{\mathrm{c}}^{\infty}(\mathscr{M}; \mathscr{E}) \to \mathscr{F}(\mathscr{M}; \mathscr{E}), \ u \mapsto \Phi(u), \tag{C.1}$$

where $\mathscr{F}(\mathscr{M}; \mathscr{E})$ can be taken as the algebra $\mathscr{C}\mathscr{C}\mathscr{R}(\mathscr{M}; \mathscr{E})$ (resp. $\mathscr{C}\mathscr{A}\mathscr{R}(\mathscr{M}; \mathscr{E})$) of canonical commutation (resp. anticommutation) relations, as defined below. $\mathscr{C}\mathscr{C}\mathscr{R}(\mathscr{M}; \mathscr{E})$ (resp. $\mathscr{C}\mathscr{A}\mathscr{R}(\mathscr{M}; \mathscr{E})$) is a unital $*$ (resp. C^*)-algebra generated over $\mathbb{C}$ by the symbols $\mathbb{1}_- := \mathbb{1}_{\mathscr{C}\mathscr{C}\mathscr{R}}$ (resp. $\mathbb{1}_+ := \mathbb{1}_{\mathscr{C}\mathscr{A}\mathscr{R}}$), $\Phi(u)$ and its conjugate algebraic-dual $\bar{\Phi}^*(\phi)$, modulo the relations

$$\Phi(cu) = c \, \Phi(u), \quad \bar{\Phi}^*(c\phi) = \bar{c} \, \bar{\Phi}^*(\phi), \tag{C.2a}$$

$$\Phi(Lu) = 0 = \bar{\Phi}^*(\bar{L}^*\phi), \tag{C.2b}$$

$$[\Phi([u]), \bar{\Phi}^*([\phi])]_{\mp} = \mathrm{i}\,\hbar\, \mathsf{G}([\phi] \boxtimes [u]) \, \mathbb{1}_{\mp} \tag{C.2c}$$

for any $[u] \in C_{\mathrm{c}}^{\infty}(\mathscr{M}; \mathscr{E})/\mathrm{img}\,(L)$ and $[\phi] \in C_{\mathrm{c}}^{\infty}(\mathscr{M}; \bar{\mathscr{E}}^*)/\mathrm{img}\,(\bar{L}^*)$. Here $\hbar$ is the reduced Planck constant and it is $\hbar = 1$ in the *natural units* which we use throughout the thesis. The brackets $[\cdot, \cdot]_{\mp}$ are the **commutator** and the **anticommutator**, respectively:

$$[\Phi(u), \bar{\Phi}^*(\phi)]_{\mp} := \Phi(u)\, \bar{\Phi}^*(\phi) \mp \bar{\Phi}^*(\phi)\, \Phi(u) \tag{C.3}$$

and $\mathsf{G} \in \mathcal{D}'(\mathscr{M} \times \mathscr{M}; \mathscr{E} \boxtimes \bar{\mathscr{E}}^*)$ is the Schwartz kernel of the causal propagator (Definition 3.2.9) for the differential operator L depending on the specific model of Φ, as listed in Table C.1. Quantum fields are called *Bosonic* (resp. *Fermionic*) if $\mathscr{F}$ is $\mathscr{C}\mathscr{C}\mathscr{R}$ (resp. $\mathscr{C}\mathscr{A}\mathscr{R}$).

Spin of a quantum field	Equation of motion	Quantisation
0	Klein-Gordon (Example 3.2.2)	[204, 205]
1/2	Dirac (Example 3.4.4)	[37, 52, 124, 133, 206]
1		
Abelian massive	Proca (Example 3.2.7)	[207]
Abelian massless	Maxwell	[208–210]
Non-abelian massless	Yang-Mills	[51, 211]
2	Einstein	
3/2	Rarita-Schwinger (Section 3.5.6)	[143]

Table C.1: Models of generally covariant quantum field theories. Although we have not discussed Maxwell equation and Yang-Mills equation in this thesis but they can be related with the Hodge-d'Alembertian (Example 3.2.4) by a suitable "gauge choice".

The field algebra is not necessarily the observable algebra because not all quantum fields are necessarily observables, for example, the Fermi fields, as they do not commute for spacelike separated regions in $\mathscr{M}$. Nevertheless, their appropriate bilinear combinations are observables. We will denote the algebra of observables by $\mathscr{A}(\mathscr{M};\mathscr{E})$. For instance, $\mathscr{CAR} \not\subset \mathscr{A}$ but $\mathscr{CCR} \subset \mathscr{A}$.

C.2 Quantum states and Hadamard condition

Definition C.2.1. Let $\mathscr{E} \to \mathscr{M}$ be a vector bundle over a globally hyperbolic spacetime and $\mathscr{A}(\mathscr{M};\mathscr{E})$ an algebra of quantum observables. An **algebraic state** ω of $\mathscr{A}(\mathscr{M};\mathscr{E})$ is a continuous (in a suitable topology) linear functional $\omega : \mathscr{A}(\mathscr{M};\mathscr{E}) \to \mathbb{C}$ which is positive: $\omega(A^*A) \geq 0$ for all $A \in \mathscr{A}(\mathscr{M};\mathscr{E})$ and normalised: $\omega(\mathbb{1}) = 1$.

The standard Hilbert space states used in physics literature are related to algebraic states via the Gelfand-Najmark-Segal (GNS) theorem - stating that (see e.g. [212, Thm. 2]), given an algebraic state ω of any unital $*$-algebra $\mathscr{A}(\mathscr{M};\mathscr{E})$ there exists a quadruple $(\mathscr{H}_\omega, \mathscr{D}_\omega, \pi_\omega, \Omega_\omega)$ where $\mathscr{D}_\omega$ is a dense subspace of the GNS-Hilbert space $(\mathscr{H}_\omega, \langle \cdot | \cdot \rangle)$, π_ω is an unital algebra representation of $\mathscr{A}(\mathscr{M};\mathscr{E})$ on $\mathscr{D}_\omega$ with the property $\pi_\omega(A^*) \subset (\pi_\omega(A))^\dagger$, $\dagger$ being the adjoint with respect to $\langle \cdot | \cdot \rangle$, and Ω_ω is the GNS-vector such that $\mathscr{D}_\omega = \pi_\omega(\mathscr{A}(\mathscr{M};\mathscr{E}))\Omega_\omega$ and

$$\forall A \in \mathscr{A}(\mathscr{M};\mathscr{E}) : \omega(A) = \langle \Omega_\omega | \pi_\omega(A)\Omega_\omega \rangle. \tag{C.4}$$

By definition, a state is specified by a collection of **Wightman n-point distributions**

$$\mathsf{W}_n : C_c^\infty(\mathscr{M};\mathscr{E})^{\otimes n} \otimes C_c^\infty(\mathscr{M};\bar{\mathscr{E}}^*)^{\otimes n} \to \mathbb{C}, \ u_1 \otimes \ldots \otimes u_n \otimes \phi_1 \otimes \ldots \phi_n \mapsto$$
$$\mathsf{W}_n(u_1, \ldots, u_n, \phi_1, \ldots, \phi_n) := \omega(\Phi(u_1) \ldots \Phi(u_n) \bar{\Phi}^*(\phi_1) \ldots \bar{\Phi}^*(\phi_n)). \tag{C.5}$$

By construction, the field equation (C.2b), the canonical commutation (resp. anticommutation) relation (C.2c), and positivity requirement of a state entail

$$W\big(L(u), \phi\big) = 0 = W\big(u, \bar{L}^*(\phi)\big), \tag{C.6a}$$

$$W(u, \phi) - W(\phi, u) = i\, G(\phi \boxtimes u), \tag{C.6b}$$

$$W(\bar{u}^*, u) \geq 0 \tag{C.6c}$$

for the 2-point Wightman distribution.

If $(\mathcal{M}, \mathrm{g})$ admits a non-trivial isometry group $\mathrm{Iso}\,\mathcal{M}$ preserving the orientation and the time-orientation, then $\mathscr{A}(\mathcal{M}; \mathscr{E})$ induces a representation of $\mathrm{Iso}\,\mathcal{M}$ by elements $\alpha_\Xi \in \mathrm{Aut}\,\big(\mathscr{A}(\mathcal{M}; \mathscr{E})\big)$. As a consequence, one has an induced automorphism α_Ξ^* on the states of $\mathscr{A}(\mathcal{M}; \mathscr{E})$. A state ω is called $\mathrm{Iso}\,\mathcal{M}$-**invariant** if

$$\forall \alpha_\Xi \in \mathrm{Aut}\,\big(\mathscr{A}(\mathcal{M}; \mathscr{E})\big) : \alpha_\Xi^* \omega = \omega. \tag{C.7}$$

Example C.2.2. If $(\mathbb{R}^4, \eta)$ is the 4-dimensional Minkowski spacetime then the *proper orthochronous Poincaré group* $\mathbb{P}_0^\uparrow$ is the spacetime isometry group (Example 4.2.2). Let $\mathscr{E} \to \mathbb{R}^4$ be a trivial $\mathbb{R}$-line bundle and set L in (C.2b) to be the *Klein-Gordon operator*

$$\square := -\frac{\partial^2}{\partial x^2} - \mathrm{m}^2, \quad \mathrm{m}^2 \in \mathbb{R}_+ \tag{C.8}$$

with the mass-squared term m^2 (cf. Example 3.2.2). In physics language, this is called a **massive real scalar quantum field theory on Minkowski spacetime** whose dynamics is governed by the Klein-Gordon equation. Since $(\mathbb{R}^4, \eta)$ is globally hyperbolic, there exist unique retarded G^{ret} and advanced G^{adv} Green's Schwartz kernels for $\square$ and hence the causal propagator is given by (see e.g. [76, Thm. 6.2.3], [80, Sec. 4.3.4]):

$$G_{x-y} = -i(\mathcal{F}\mu_+ - \mathcal{F}\mu_-)(x - y), \tag{C.9a}$$

$$\mu_\pm(\xi) := \theta(\pm\xi_1)\, \delta\big(\eta(\xi, \xi) - \mathrm{m}^2\big), \tag{C.9b}$$

$$G_{x-y}^{\mathrm{ret,adv}} = \pm\theta\big(\pm(x^1 - y^1)\big)\, G_{x-y}, \tag{C.9c}$$

$$G_{x-y}^{\mathrm{adv}} = G_{-(x-y)}^{\mathrm{ret}}, \tag{C.9d}$$

where $\mu_\pm(\xi)$ are the unique (up to constant multiples) measures on the mass-hyperboloids $\{\xi \in \mathbb{R}^4 \mid \eta(\xi, \xi) = \mathrm{m}^2, \xi_1 \gtrless 0\}$ invariant under the proper orthochronous Lorentz group (see e.g. [213, Thm. IX. 33, IX. 37]) and the multiplication by the Heaviside step function θ is well-defined due to the wavefront set (see (C.12b)) of G. In standard physics literature, these measures are expressed as $\int_{\mathbb{R}^4} \delta\big(\eta(\xi, \xi) - \mathrm{m}^2\big)\, \theta(\xi_1 \gtrless 0)\, d\xi/(2\pi)^4$ and preceding equations are presented as the formal[1] expressions (see e.g. [214, Sec. 1-3-1])

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[1]These integrals do not converge, so they must be understood as Fourier transforms of Schwartz distributions.

$$\mathrm{G}^{\mathrm{ret}}_{x-y} = -\mathrm{i}\,\theta(x^1 - y^1) \int_{\mathbb{R}^3} \left(\mathrm{e}^{-\mathrm{i}\,\mathrm{E}(x^1 - y^1)} - \mathrm{e}^{\mathrm{i}\,\mathrm{E}(x^1 - y^1)}\right) \mathrm{e}^{\mathrm{i}\xi'\cdot(x'-y')} \frac{\mathrm{d}\xi'}{(2\pi)^3 2\mathrm{E}}, \quad \text{(C.10a)}$$

$$\mathrm{G}^{\mathrm{adv}}_{x-y} = -\mathrm{i}\,\theta(y^1 - x^1) \int_{\mathbb{R}^3} \left(\mathrm{e}^{-\mathrm{i}\,\mathrm{E}(y^1 - x^1)} - \mathrm{e}^{\mathrm{i}\,\mathrm{E}(y^1 - x^1)}\right) \mathrm{e}^{\mathrm{i}\xi'\cdot(y'-x')} \frac{\mathrm{d}\xi'}{(2\pi)^3 2\mathrm{E}}, \quad \text{(C.10b)}$$

$$\mathrm{G}_{x-y} = -\mathrm{i} \int_{\mathbb{R}^3} \left(\mathrm{e}^{-\mathrm{i}\,\mathrm{E}(x^1 - y^1)} - \mathrm{e}^{\mathrm{i}\,\mathrm{E}(x^1 - y^1)}\right) \mathrm{e}^{\mathrm{i}\xi'\cdot(x'-y')} \frac{\mathrm{d}\xi'}{(2\pi)^3 2\mathrm{E}}, \quad \text{(C.10c)}$$

where we have used the notation $x = (x^1, x')$, $\xi = (\xi_1, \xi') \in \mathbb{R} \times \mathbb{R}^3$ and $\mathrm{E} := \sqrt{\xi' \cdot \xi' + \mathrm{m}^2}$ is the energy where $\cdot$ denotes the Euclidean inner product on $\mathbb{R}^3$.

In the Minkowski spacetime, there exists a unique $\mathbb{P}_0^{\uparrow}$-invariant state, the **vacuum** ω_0 whose 2-point Wightman distribution W_0 is given by (see e.g. [213, Thm. IX. 34])

$$W_0(x - y) = -\mathrm{i}(\mathcal{F}\mu)(x - y) = \int_{\mathbb{R}^3} \mathrm{e}^{-\mathrm{i}\,\mathrm{E}(x^1 - y^1)} \mathrm{e}^{\mathrm{i}\xi'\cdot(x'-y')} \frac{\mathrm{d}\xi'}{(2\pi)^3 2\mathrm{E}}. \quad \text{(C.11)}$$

Evidently, the singularities of the Green's operators and W_0 are (see e.g [80, p. 118])

$$\mathrm{WF}' \, \mathrm{G}^{\mathrm{ret}}_{\mathrm{adv}} = \left\{(x, \xi; x, \xi) \in \dot{\mathrm{T}}^*\mathbb{R}^4 \times \dot{\mathrm{T}}^*\mathbb{R}^4\right\} \bigcup \left\{(x, \xi; y, \xi) \in \dot{\mathbb{R}}^4 \times \dot{\mathbb{R}}^4 \times \dot{\mathbb{R}}^4 \times \dot{\mathbb{R}}^4 \,|\right.$$
$$\left. \eta(x - y, x - y) = 0, x^1 \gtrless y^1, \xi = \lambda(x - y), \lambda \in \mathbb{R}\right\}, \quad \text{(C.12a)}$$

$$\mathrm{WF}' \, \mathrm{G} = \left\{(x, \xi; y, \eta) \in \dot{\mathrm{T}}^*\mathbb{R}^4 \times \dot{\mathrm{T}}^*\mathbb{R}^4 \,|\, \eta(\xi, \xi) = 0, x - y = \lambda\xi, \lambda \in \mathbb{R}\right\}, \quad \text{(C.12b)}$$

$$\mathrm{WF}' \, \mathrm{W}_0 = \left\{(x, \xi; y, \eta) \in \dot{\mathrm{T}}^*\mathbb{R}^4 \times \dot{\mathrm{T}}^*\mathbb{R}^4 \,|\, \eta(\xi, \xi) = 0, x - y = \lambda\xi, \lambda \in \mathbb{R}, \xi_1 > 0\right\}. \quad \text{(C.12c)}$$

However, there is *no analog of a vacuum state* or even any other preferred state in a generic globally hyperbolic spacetime due to the lack of any non-trivial spacetime symmetry. There were numerous attempts to get around this problem and all suffered from significant limitations; see, for instance, the review [201] for a summary. In fact, under a few natural hypotheses, it has been proven that it is impossible to find any preferred state of a non-trivial locally covariant QFT in a globally hyperbolic spacetime [215, Thm. 6.13] (see also the review [216, Thm. 1]). On the other hand, a generic algebraic state can be too singular to perform all the desired operations of a quantum field theory, e.g., defining non-linear observables like the quantum stress-energy tensor. Therefore, we utilise the preceding example as our guiding principle to select a physical state.

Since the tangent space of every spacetime is locally Minkowski and the singularity of a distribution is "locally" determined, we embrace the condition

$$\mathrm{WF}\,\mathrm{W} = \mathrm{WF}\,\mathrm{W}_0 \quad \text{(C.13)}$$

as a plausible criterion to select physical quantum states in any curved spacetime.

Definition C.2.3. As in the terminologies of Definition C.2.1, a state ω of $\mathscr{A}(\mathscr{M}; \mathscr{E})$ is called a **Hadamard state** if its 2-point Wightman distribution has the following wave-front set [39]

$$\mathrm{WF}' \mathrm{W} = \{(x, \xi; y, \eta) \in \dot{\mathrm{T}}_0^* \mathscr{M} \times \dot{\mathrm{T}}_0^* \mathscr{M} \mid \exists! s \in \mathbb{R}_{\geqslant 0} : (x, \xi) = \Phi_s(y, \eta), \xi \rhd 0\}, \qquad \text{(C.14)}$$

where $\dot{\mathrm{T}}_0^* \mathscr{M} \to \mathscr{M}$ is the lightcone bundle over $\mathscr{M}$, Φ_s is the geodesic flow on the cotangent bundle and $\xi \rhd 0$ means that ξ is future-directed[2].

We note that Hadamard states are *necessary* and *sufficient* to define covariant non-linear quantum observables in any globally hyperbolic spacetime [36–38, 217] (see also, the reviews [199, 201]). Therefore, only Hadamard states have been considered in this thesis.

Remark C.2.4. Hadamard states are *never unique* as we have the freedom to add appropriate smooth terms.

C.3   Time-ordered product

In order to construct non-linear quantum observables perturbatively one requires the notion of the time-ordered product between classical observables. Geometrically speaking, a classical field φ is a smooth section of $\mathscr{E}$. A time-ordered product $\mathtt{T}$ is a linear map from local and covariant polynomial expressions in φ to $\mathscr{F}(\mathscr{M}; \mathscr{E})$-valued distribution satisfying a number of postulates. The precise characterisation of those properties requires several new terminologies and, more importantly, the details are not essential to understand the subject matter of the ongoing discussion. Therefore, we abstain from providing the formal definition and refer [37] (see also, the reviews [199, 218] for details with earlier developments). It is sufficient for our purpose to note that

$$\mathtt{T}\big(\varphi(x) \otimes \bar{\varphi}^*(y)\big) = \begin{cases} \Phi_x \bar{\Phi}_y^*, & x \notin J^-(y), \\ \pm \bar{\Phi}_y^* \Phi_x, & x \notin J^+(y) \end{cases}, \qquad \text{(C.15a)}$$

$$\mathtt{T}\big(L\varphi(x) \otimes \bar{\varphi}^*(y)\big) = \mathrm{i}\,\hbar \delta_{x,y} = \mathtt{T}\big(\varphi(x) \otimes \bar{L}^* \bar{\varphi}^*(y)\big), \qquad \text{(C.15b)}$$

$$\mathrm{W}_{x,y}^{\mathrm{T}} := \omega\left(\mathtt{T}\big(\varphi(x) \otimes \bar{\varphi}^*(y)\big)\right), \qquad \text{(C.15c)}$$

$$\mathrm{WF}' \mathrm{W}^{\mathrm{T}} = C^+, \qquad \text{(C.15d)}$$

where W^{T} is called the **time-ordered 2-point Wightman distribution**, C^+ is the forward geodesic relation (Definition 3.2.15) and $\pm$ is for Bosonic (resp. Fermionic) observables. The preceding two equations therefore entail that [39]

$$\mathrm{G}^{\mathrm{Fyn}}(u \boxtimes \phi) := -\,\mathrm{i}\,\mathrm{W}^{\mathrm{T}}(u \boxtimes \phi) \qquad \text{(C.16)}$$

is the Feynman propagator (Definition 3.2.10 and 3.2.17).

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[2]That is, $\xi(X) \geq 0$ for any futue-directed $X \in \mathrm{T}_x \mathscr{M}$.

Remark C.3.1. The Feynman propagator depends on the chosen quantum state in contrast to the retarded, advanced and causal propagators.

To comprehend this new propagator better, let us look again at the simplest case: the Klein-Gordon quantum field theory in Minkowski spacetime.

Example C.3.2. In the terminologies and notations introduced in Example C.2.2: there is a unique Feynman propagator G_0^{Fyn} corresponding to the Minkowski vacuum:

$$G_{0,x-y}^{\mathrm{Fyn}} = -\mathrm{i}\left(\theta(x^1 - y^1)\,(\mathcal{F}\mu_+)(x-y) + \theta(y^1 - x^1)\,(\mathcal{F}\mu_-)(x-y)\right), \tag{C.17}$$

which is presented in the standard physics textbooks by[3] (see e.g. [214, Sec. 1-3-1])

$$G_{0,x-y}^{\mathrm{Fyn}} = -\mathrm{i}\int_{\mathbb{R}^3}\left(\theta(x^1 - y^1)\,\mathrm{e}^{-\mathrm{i}\,\mathrm{E}(x^1-y^1)} + \theta(y^1 - x^1)\,\mathrm{e}^{\mathrm{i}\,\mathrm{E}(x^1-y^1)}\right)\mathrm{e}^{\mathrm{i}\,\xi'\cdot(x'-y')}\frac{\mathrm{d}\xi'}{(2\pi)^3 2\mathrm{E}} \tag{C.18}$$

and its singularity structure is given by (see e.g. [80, p. 118])

$$\begin{aligned}
\mathrm{WF}'\,G_0^{\mathrm{Fyn}} \;=\; & \left\{(x,\xi;x,\xi)\in\dot{\mathrm{T}}^*\mathbb{R}^4\times\dot{\mathrm{T}}^*\mathbb{R}^4\right\}\bigcup\left\{(x,\xi;y,\xi)\in\dot{\mathbb{R}}^4\times\dot{\mathbb{R}}^4\times\dot{\mathbb{R}}^4\times\dot{\mathbb{R}}^4\,|\right. \\
& \left.\eta(x-y,x-y) = 0, \xi = \lambda(x-y), \lambda \gtrless 0\right\}. \tag{C.19}
\end{aligned}$$

These propagators have the following physical interpretation:

- The retarded propagator (C.10a) propagates positive ($\mathrm{e}^{-\mathrm{i}\,\mathrm{E}(x^1-y^1)}$) and negative ($\mathrm{e}^{\mathrm{i}\,\mathrm{E}(x^1-y^1)}$) frequencies forward in time,

- The advanced propagator (C.10b) propagates positive and negative frequencies backward in time,

- The Feynman propagator (C.18) propagates positive frequencies forward and negative frequencies backward in time,

This formal expression of G_0^{Fyn} by means of the Fourier transform of $\lim_{\varepsilon\to 0} 1/(\eta(\xi,\xi) - \mathrm{m}^2 + \mathrm{i}\,\varepsilon)$ is often used in physics textbooks as the starting point to construct the Feynman propagator. However, neither the Fourier transformation nor the notation of a particle are available in an arbitrary globally hyperbolic spacetime. Thus, one cannot use this approach in a generic situation. Nevertheless, time-ordered product is well-defined in any globally hyperbolic spacetime and hence this can be used to define the Feynman propagators satisfactorily. One then needs to prove positivity of G_ω^{Fyn} (cf. Proposition 3.3.2) in order to construct a Hadamard state ω. Note, Feynman propagators are now *no longer unique* as one has the freedom to add suitable smoothing terms without changing C^+. In other words, G_ω^{Fyn} is *unique* only up to those *smoothing terms*, that is, there exists the notion of a *unique Feynman parametrix* E^{Fyn} independent of any chosen ω.

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[3]See Footnote C.2.2

We can also view the one-to-one correspondence between ω and $\mathsf{G}_\omega^{\mathrm{Fyn}}$ the way around. Since, all Hadamard states have the same wavefront set, one can characterise the Feynman propagators in terms of the wavefront set as have been done in Definition 3.2.17. The microlocal viewpoint has been adapted because it offers an intrinsic definition of the Feynman parametrix which has applications beyond quantum field theories as mentioned in Section 1.2.

C.4   Literature

The idea of using the category of *Hadamard states* as an appropriate replacement for the Minkowski vacuum and a crucial structural ingredient of any quantum field theory in curved spacetime was proposed by Bernard KAY [219, Sec. 1.1] and by Rudolf HAAG, Heide NARNHOFER and Ulrich STEIN [220]. Bernard KAY and Robert WALD [205] were the first to characterise these states precisely in terms of the short-distance behaviour of the 2-point Wightman distribution. They had actually defined a local and a global Hadamard conditions. Recently, Valter MORETTI [221] located and closed a technical gap in their global condition. The modern definition of Hadamard states in the microlocal framework is due to Maciej RADZIKOWSKI [39]. Most of the earlier literature on Hadamard states was for scalar quantum field theories and the vector bundle generalisation developed subsequently [49, 51, 56, 206, 222, 223]. We refer, for instance, the exposition [224] for details.

A mathematically precise formalism of the *time-ordered product* in curved spacetime was initiated by Romeo BRUNETTI and Klaus FREDENHAGEN [36], but there were a few ambiguities in their work. Employing the idea of local covariance of quantum observables, Stefan HOLLANDS and Robert WALD have fixed those issues [37] and have proven the existence of time-ordered product for scalar quantum field theory [38]. Later, the seminal result was extended by Stefan HOLLANDS [51] for quantum Yang-Mills theory.

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