How do Teachers of Mathematics Introduce Differential Calculus?

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The candidate confirms that the work submitted is his/her own and that appropriate credit has been given where reference has been made to the work of others.

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Abstract

This thesis reports on the findings of a qualitative study investigating how teachers of mathematics in English schools introduce differentiation in elementary calculus. Understanding how teachers introduce the derivative is crucial to uncovering more meaningful and effective ways for helping students understand differential calculus. The study adopts the commognitive framework and investigates the teaching of the derivative by examining the *word use* and *narratives*, the *visual mediators* and the *routines* in the teachers' pedagogical calculus discourse. Interviews with the teachers and observation of their introductory lessons on the derivative were used to collect qualitative data for the study. For the analysis of the qualitative data, the study introduces a new approach that is described as a *commognitive thematic discourse analysis*, which is a combination of the commognitive framework and thematic analysis. The *commognitive thematic discourse analysis* was then used to deconstruct the teachers' pedagogical calculus discourse on the derivative and to identify the overarching themes in the research data, which are presented as a narrative of the findings. The study found that teachers used multiple visual mediators such as numerical, algebraic and graphical representations in constructing the definition of the derivative. Using dynamic geometry software such as GeoGebra and Autograph enhanced the teachers' construction and substantiation routines in teaching the definition of the derivative. The study also found that teachers were able to construct the definition of the derivative without using the formal definition of limits; instead, the teachers took what is described as a '*quasi-limit'* approach. The study also uncovered some inconsistency and ambiguity with word use and calculus symbolism in the teachers' pedagogical calculus discourse in the transition between *gradient* (for straight line graphs) and the *gradient function* (for curved line graphs). This study is aimed at contributing to research that seeks to understand and improve the teaching of elementary calculus.

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Chapter 1 Introduction

This study is an investigation into the teaching of elementary differential calculus at school, thus, researching the teaching of calculus to pre-university level students. The study explores how teachers of mathematics introduce differential calculus to secondary school students (16 to 18 years) in England. This study adopts a purely qualitative approach to research. In the following sections of this chapter, Section 1.1 explains the background and my motivations for carrying out this study. In Section 1.2 the research topic and context are explained. A definition of calculus, a brief history of the discovery of calculus, the significance of calculus and its place in the mathematics curricula are all explained in Section 1.2. The next Section 1.3 presents a justification for the study and explains the significance of investigating the teachers' pedagogical calculus discourse. Section 1.4 gives the overall research aims and identifies the prima facie questions for the research. Finally, Section 1.5 gives an outline of the thesis.

1.1 Background to the study

My prime motivation for researching the teaching of differential calculus stems from my experience and observations over a period spanning at least two decades, as a scholar, teacher and lecturer in mathematics education. From my career as a teacher of mathematics, I taught elementary calculus at the Advanced level (A-level) to post-16 (high school) students at schools in England. Thus, I developed some knowledge and understanding of the curriculum arrangements (programme of study) and subject specifications for mathematics in the UK. As a lecturer and teacher educator at the university, I taught (worked with) pre-service trainee teachers on the Post Graduate Certificate Education (PGCE) programme. I also worked with in-service teachers of mathematics on the Teaching Advanced Mathematics (TAM[1]) programme. Both programmes, PGCE and TAM, involved teaching mathematics education to the student teachers and supporting their knowledge and skills development by observing the student teachers teach mathematics in their

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[1] TAM course is designed to support the continued professional development of secondary teachers of mathematics in the UK, most particularly, teachers of GCSE Mathematics who wish to train to teach A level Mathematics, although some teachers with A level teaching experience do also enrol to broaden and deepen their mathematics subject knowledge and gain new teaching ideas. The TAM course is organised by the MEI and run-in conjunction with participating centres and universities in the UK. For more information about the TAM course follow the link: https://amsp.org.uk/events/details/5200.

placement schools. Thus, my epistemological stance in this qualitative research is influenced, not just by research on mathematics education, but also by my professional/academic practice and background as a teacher of mathematics and a teacher educator.

During my academic career as a teacher educator, I taught elementary calculus to trainee teachers of mathematics on the PGCE programme. I observed that for many of my student teachers, their understanding of basic calculus concepts was often limited to some disjointed algebraic rules, which they could hardly explain. When asked to explain their understanding of the derivative, it was often the case that many of these student teachers would focus on the algorithmic rules of differentiating functions, the sort of standard rules for differentiation that have been required to pass their A-level mathematics examinations. The trainee teachers on the PGCE programme would have been taught some elementary calculus (at least) at school and/or at university, before enrolling on the PGCE programme. The vast majority of these trainee teachers were postgraduate students, whose undergraduate degrees were in mathematics. Nonetheless, the common denominator here is that all the trainee teachers would have had some A-level mathematics, and calculus is a core component of A-level mathematics in England.

Klein (1908/1932 cited in Winslow and Gronbaek, 2014) had long described what he called a ‘double discontinuity’, referring to a gap in content knowledge between the transition from high school to university, and another gap between the teacher’s university knowledge and knowledge for high school teaching. The first discontinuity results from ways of learning and doing mathematics at school that do not apply to university, and the second discontinuity appears in the application of knowledge learnt from university to high school teaching. The latter can be explained in terms of a gap between being the student role (both at school and university) and the teacher role at high school.

Facing what looked like Klein’s double discontinuity (Klein,1908/1932 cited in Winslow and Gronbaek, 2014), as the teacher educator I was concerned as to how the trainee teachers would explain the derivative to their students if they could not explain the definition of the derivative. I wondered how and why so many students who had gone through A-level mathematics and undergraduate-level calculus would struggle to demonstrate an adequate understanding of differentiation. I wondered if this was an isolated observation or a common problem with calculus. Thus, I undertook a preliminary review of the literature on students’ challenges with calculus.

My preliminary review of literature on calculus education revealed that it was well established from a variety of studies (e.g. Orton, 1983a; 1983b; Tall, 1992; Ferrini-

Mundy and Gaudard, 1992), that students had difficulties with calculus. Many studies in the 1980s (e.g. Dreyfus and Eisenberg, 1983; Even et al., 1988; Monk, 1989; Orton, 1983a; 1983b; Tall and Blackett, 1986; and Vinner, 1983; 1987) suggested that the students' understanding of basic concepts in elementary calculus such as the limit, functions, the derivative, and integrals was inadequate. Calculus has been the subject of much debate and research for at least the last five decades. In a plenary presentation at the International Congress on Mathematical Education (ICME) conference in 1992 in Québec, Tall (1992) gave an extensive summary of the challenges that students encounter in learning calculus. Some of the difficulties in calculus that Tall (1992) talked about had earlier been highlighted by Orton's (1983a; 1983b) study on students' understanding of elementary calculus involving 16 to 22 year-olds. Orton's (1983a; 1983b) study shows that although the students' routine performance on differentiation items was adequate, they lacked adequate intuition or understanding of the derivative concept. Eichler and Erens's (2014) study, which investigated teachers' beliefs towards calculus, found that all the 29 teachers in their study had a shared view on calculus, as a set of rules for students to be memorised and used in solving routine problems.

Similar to the observations I made about my trainee teachers' knowledge and understanding of differential calculus, Berry and Nyman (2003) also report problems with students' understanding of calculus. Berry and Nyman (2003, p.481) report that:

> Our experience is that the vast majority of students in introductory calculus courses do not develop an appreciation of the theoretical concepts or an intuitive 'feel' for the ideas…Successful students go away from the course knowing that they must 'find where the derivative is zero' without really understanding why it is important…Techniques of integration are little more than a 'bag of tricks'.

Berry and Nyman (2003) also worked with mathematics undergraduate students and postgraduate students training to be teachers over several years in the United Kingdom (UK) and the United States of America (USA). They reported similar observations to my experiences with trainee teachers as described above. They described the students' understanding as 'a set of loosely connected actions based on a set of algebraic rules that can be applied in restricted, often artificial, algebraic situations' (Berry and Nyman, 2003, p.482).

My observations about student teachers' difficulties with calculus were, therefore, not an isolated case. There is strong evidence in the literature (e.g. Berry and Nyman, 2003; Dreyfus and Eisenberg, 1983; Even et al., 1988; Monk, 1989; Orton, 1983a; 1983b; Tall and Blackett, 1986; and Vinner, 1983; 1987) spanning over five decades that confirms students' challenges with calculus. The majority of the studies I reviewed focus on the learning of calculus. Of the few studies that looked at the teaching of calculus, I did not come across a study that focused primarily on the introduction of differential calculus at school. This was a gap in research. My study, therefore, sets to explore how teachers of mathematics introduce differential calculus at school to the 16 to 18 age range of students.

1.2 Research context

Calculus is the area of mathematics, which studies how things change. Simply put, calculus is the mathematical study of change. How do we determine the speed of a falling object at an instant in time, for instance, its speed when it hits the ground? Calculus was invented out of studying continuously changing quantities, and answers to such problems are what became known as a derivative. Calculus is broadly divided into differential calculus and integral calculus. Differential calculus also referred to as simply differentiation deals with the rates at which quantities change, finding the slope of a tangent to a curve or derivatives of functions. Integral calculus or simply integration, on the other hand, is concerned with the accumulation of quantities, finding the area under a curve, the volume of a geometric solid or integrals of functions. Simplistically, the process of integration is the inverse of differentiation.

The discovery of calculus around the 1670s by Sir Isaac Newton (1642 - 1727) in England and Gottfried Wilhelm Leibniz (1646 - 1716) a German mathematician and philosopher, was one of the most famous breakthroughs in the history of mathematics. The two men, Sir Isaac Newton (1642 - 1727) and Gottfried Wilhelm Leibniz (1646 - 1716) independently invented calculus (Reyes, 2004; Rosenthal, 1951). Although Leibniz was the first to publish his work on calculus in 1684, both these men deserve equal credit for independently creating calculus. For the rest of their lives, they accused each other of plagiarism. The dispute as to who discovered calculus first led to a rift in the European mathematical community lasting over a century. In spite of the dispute, the world largely adopted Leibniz's calculus symbols, $\frac{dy}{dx}$ for example (Henle and Kleinberg, 1979). Newton's physics principles, which remain sufficient to explain much in physics with excellent accuracy, were borne out of calculus. For example, Newton was trying to understand or make sense of why falling objects would constantly accelerate, i.e. the effect of gravity.

Calculus remains ‘one of the greatest achievements of the human intellect’ (NCTM, 1989; Hughes-Hallett et al., 1994, p.vii). Commenting on the usefulness of calculus, Davidson (1991) said:

> By understanding derivatives, the student has at his or her disposal a very powerful tool for understanding the behaviour of mathematical functions. Importantly, this allows us to optimize functions, which means to find their maximum or minimum values, as well as to determine other valuable qualities describing functions. Real-world applications are endless, but some examples are maximizing profit, minimizing stress, maximizing efficiency, minimizing cost, finding the point of diminishing returns, and determining velocity and acceleration.

Real-world applications of calculus are endless, including fields such as medicine, astronomy, business, economics and statistics. Giving a keynote speech in a meeting on calculus in the USA, the National Academy of Engineering president Robert M. White said the ‘national spotlight is on calculus’ because of the ‘linkage between mathematics and economic growth’. Calculus ‘must become a pump rather than a filter in the pipeline’, (Walsh, 1987, p.749). Calculus is fundamental to the study of mathematical sciences, all sciences: physics, chemistry and biology, and engineering (Douglas, 1986). Indeed, calculus has stood the test of time!

Today, still, calculus remains an extremely important component of the mathematics curriculum, from upper secondary school to college or university in the English education system and many countries around the world. In England, calculus is part of the school mathematics programme for the 16 -18 age range, and this has been the case for over six decades. The mathematics content to be taught for the AS/A level mathematics syllabuses (16 -18 age range) is often outlined in the mathematics teaching specifications that are provided by the examination boards[2], such as AQA, Edexcel and OCR. With a particular focus on the introduction to differentiation, the Edexcel Level 3 General Certificate of Education (GCE) in Mathematics for the Core Mathematics (C1)[3] outlines what students need to learn on differentiation as follows:

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[2] These are awarding bodies authorised by the Office of Qualifications and Examinations Regulation (Ofqual) with setting examinations and awarding qualifications, such as GCSEs and A levels, for students in state schools and colleges across the UK.

[3] C1 – Core Mathematics 1, is the first of four Core Mathematics modules (C1, C2, C3 & C4) that, together with two other modules from Mechanics or Statistics or Decision, make up the General Certificate of Education (GCE) Advanced Level Mathematics Qualification. For further information follow the link: https://nrich.maths.org/6147

> The derivative of $f(x)$ as the gradient of the tangent to the graph of $y = f(x)$ at a point; the gradient of the tangent as a limit; interpretation as a rate of change; second order derivatives. For example, knowledge that $\frac{dy}{dx}$ is the rate of change of y with respect to x. Knowledge of the chain rule is not required. The notation $f'(x)$ may be used.
>
> Differentiation of x^n, and related sums and differences. For example, for $n \neq 1$, the ability to differentiate expressions such as $(2x + 5)(x - 1)$ and $\frac{x^2 + 5x - 3}{3x^{\left(\frac{1}{2}\right)}}$ is expected.
>
> Applications of differentiation to gradients, tangents and normals. Use of differentiation to find equations of tangents and normals at specific points on a curve.
>
> (Pearson Education Limited, 2013, p.23)

In 2016, the Department for Education (DfE) published revised content for teaching mathematics AS (and A) level and the new specification has been taught since September 2017. However, the introductory element to what students need to know about differentiation has not changed, except for the inclusion of differentiation from first principles, for small positive integer powers of x. DfE (2016, p.10):

> Understand and use the derivative of $f(x)$as the gradient of the tangent to the graph of $y = f(x)$ at a general point (x, y); the gradient of the tangent as a limit; interpretation as a rate of change; sketching the gradient function for a given curve; second derivatives; differentiation from first principles for small positive integer powers of x;
> Understand and use the second derivative as the rate of change of gradient.
> Differentiate n^x , for rational values of n, and related constant multiples, sums and differences.

Although calculus is usually taught at A level (17 to 18 year-olds) in England, it is not as formal nor as rigorous as the university-level courses on calculus. Students in England, however, can study some elements of differential calculus at the GCSE level, from an optional qualification, Additional Mathematics.

There are many other countries, apart from England, that teach calculus at school, to students in the 14 to 18 age range. In Singapore, the teaching of calculus starts in upper secondary (Years 9 and 10) as part of the GCE O Level Additional Mathematics syllabus. Much of the calculus that is covered at the GCE O level in Singapore is a preserve for the Advanced level in England (Bressoud et al., 2016). In South Korea, calculus is introduced in the second year of high school (Grade 11) and is highly regarded as an essential part of secondary school mathematics (Bressoud et al., 2016). In France, the concept of limit and the derivative are introduced, but without formal definition, at Grade 11 and 12, just an intuitive approach is used. In Germany, calculus is taught at the Senior High School, though not as formal as at the university level calculus; (Bressoud et al., 2016). In the United States of America, calculus was once a preserve of university courses, but it is now taught in high school as a preliminary course for University Calculus. In the United States, in 2014-15, 25% of high school seniors enrolled on calculus courses in school, and about 75% of all students who eventually study calculus at university, take their first calculus course in high school (Bressoud et al., 2016).

Researching the teaching of calculus at the school level, i.e. pre-university stage is therefore of global relevance to mathematics education research and teachers of mathematics in many countries. My study investigates how teachers of mathematics teach elementary differential calculus, particularly, the introduction of the derivative.

1.3 Significance of the study

This study investigates the teaching of calculus, which may provide suggestive evidence for some of the students' difficulties with calculus (as highlighted in Section 1.1 above). Understanding how teachers teach the derivative and the challenges with teaching elementary differential calculus could provide useful research insights into some of the common students' challenges with calculus.

The study takes a discursive research approach rooted in the theory of commognition that conceptualises mathematics as a form of discourse, that is, a special type of communication with specific ways of saying and doing (Sfard, 2008) (see Chapter 3 for an explanation of the commognitive theoretical framework). Thus, the study offers an alternative research perspective (to most past studies) on researching teachers and their teaching of elementary differential calculus (mathematics) that does not focus on teacher knowledge but the teachers' mathematical discourses (Sfard, 2008), herein referred to as the teachers' pedagogical calculus discourse. Pedagogical calculus discourse in this study refers to the amalgam of the teachers' mathematical and didactical discourse on calculus.

The fundamental challenge for teaching the derivative, i.e. introducing differential calculus, is to construct the limit definition of the derivative and to substantiate the process of differentiation to students. It is often the challenge of explaining the instantaneous rate of change and giving a practical way of calculating it. It is, therefore, not only necessary but important to research, not just the learning, but also the teaching of calculus. This study investigates the teaching of the derivative by examining the calculus language, the calculus symbolism and visual mediators and representations in the teachers' pedagogical calculus discourse. In this study, the term 'representations', which is regarded as a familiar term for the target audience for this research (i.e. the teachers of mathematics), is used to refer to various forms of expressing mathematical objects, such as the geometrical, algebraic and numerical forms of expressing a function, for example.

Researching the teachers' pedagogical calculus discourse means examining their communicative activity, i.e. the teachers' forms of saying and doing in teaching calculus. Calculus is laden with specialised mathematical words, e.g. limit, derivative and word use matters in pedagogical calculus discourse. This study examines the teachers' word use and narratives (Sfard, 2008) in constructing the definition of the derivative, which is crucial for illuminating meaningful language for calculus teaching and uncovering more effective ways of introducing the derivative.

It is also important that this research investigates the teachers' use of symbols in constructing the definition of the derivative. Symbolism can be a useful and powerful communication mediator (Sfard, 2008; Tall, 1994) in teaching differential calculus. However, symbolism is also reported as a source of students' difficulties with calculus (Tall, 1992). Besides, there exists symbolic ambiguity in some calculus symbols, for example, the same symbol x in the gradient formula (for straight line graphs) represents a 'letter as specific unknown value' whereas in the limit definition of the derivative represents a 'letter as variable' (Kuchemann, 1978, p.23). Exploring how teachers use symbolism in introducing the derivative is important to provide research insights into the challenges with calculus symbolism.

Further, it is also important to investigate the representations and visual mediators that teachers use in introducing the derivative, for visual mediators are an integral part of the act of communication in literate mathematics discourse (Sfard, 2008). The representations and the visual mediators (e.g. the algebraic symbolic artefacts and graphical mediators) in the teachers' pedagogical calculus discourse provide the images with which teachers and students can 'identify the object of their talk and coordinate their communication' (Sfard, 2008, p.145). Besides, the use of multiple visual mediators and representations in teaching can 'broaden communicational possibilities' (Sfard, 2008, p.156) and widen learning opportunities for the students.

Overall, researching the teachers' pedagogical calculus discourse to understand how teachers of mathematics introduce the derivative, is crucial to finding more meaningful and effective ways to uncover what students need to know to understand differential calculus.

1.4 Research aims

This study aims to contribute to the existing knowledge base or research on calculus education, and to a growing body of knowledge that seeks to understand and improve the teaching and learning of calculus in schools. It is also aimed at raising awareness of, and drawing mathematics education research's attention to the teaching of differential calculus at the pre-university stage.

The study is intended to inform the teaching of differential calculus. As a teacher of mathematics, a teacher educator and a mathematics researcher, it is my hope and intention that this study would speak, not only to the mathematics education research community but to the teachers of mathematics, too. The study has relevance for teacher education and training, e.g. Initial Teacher Training (ITT) and Continued Professional Development (CPD) programmes. I also hope that this study will be invaluable for my professional learning and development as a researcher, a teacher and a mathematics teacher educator.

This study seeks to understand what is said in the teacher's differential pedagogical calculus discourse, and how it is said; what is used and how it is used; and what is done, and how it is done. The following prima facie research questions guided the initial review of literature. In teaching differential calculus:

- What mathematical language do teachers use and why?
- What mediational means do teachers use and why?
- How do teachers introduce the derivative?

These preliminary questions were instrumental for the initial search for and engaging with literature. The resultant review of literature led to the discovery of the commognitive framework (Sfard, 2008) as an appropriate theoretical framework for the study (see Chapter 3) and to the formulation of the substantive research questions (see Section 4.2 on page 62), that then guided the ultimate design for the research

1.5 Thesis outline

The thesis is divided into 10 chapters as shown in Table 1.1 below. In Chapters 2 and 3 a review of literature for the study is given. Chapter 2 primarily reviews literature on calculus education and research, whereas Chapter 3 explains the conceptual framework for the study - the commognitive theoretical framework (Sfard, 2008), followed by a review of literature on commognitive studies. Chapters 4 and 5 explain the research design, methods of data collection and methods of data analysis used for the study. Three chapters are reporting the findings of the study.

Chapters 6, 7 and 8 present and discuss the findings of the study. Chapter 9 presents a discussion of selected findings and a critical evaluation of the commognitive methodology applied in this study. Finally, Chapter 10 presents the research conclusions, summarising the main findings and their implications for practice and research, highlights the main research contributions, explains the limitations of the research, and makes recommendations for future research.

experiences ranging between three and ten years.

Table 1.1 Thesis outline

Introduction	Chapter 1: Introduction
Literature review & theoretical framework	Chapter 2: Literature review Chapter 3: Commognitive theoretical framework
Methodology	Chapter 4: Research design Chapter 5: Qualitative data analysis
Findings & Discussion	Chapter 6: Mathematical language for calculus teaching Chapter 7: Symbolism for calculus teaching Chapter 8: Pedagogies on the derivative Chapter 9: Discussion
Conclusion	Chapter 10: Conclusions and implications

Chapter 2 Literature review

2.1 Introduction

My research is an investigation of the teaching of elementary differential calculus through the lens of the commognitive framework (Sfard, 2008). Thus, there are two broad strands to my literature review: a review of literature on calculus (presented in Chapter 2) and a review of literature on commognitive analyses (presented in Chapter 3). The review of literature on calculus (see Sections 2.2 to 2.4) will explore some historical and curriculum aspects of calculus education, define the derivative and analyse past research on the derivative and calculus education. The review of literature on commognitive analyses (see Section 3.6) will examine commognitive studies on teaching and learning calculus. More broadly, the latter review will include literature on discursive studies that have applied the commognitive framework in their analyses of teaching and learning. Research insights from such commognitive analyses could inform my broader interpretation of the commognition theory, my analysis of teaching and learning and the discussion of the findings of my study. For a more in-depth explanation and application of the commognition theory to this study, see Chapter 3.

2.2 Calculus education

Calculus is seen as a gatekeeper to the STEM subjects, i.e. science, technology, engineering, and mathematics at higher education and university. Reporting in the Mathematical Association of America (MAA)'s National Study of College Calculus of 2015, Bressoud et al. (2015) identify the calculus entry requirement into STEM disciplines as a huge barrier, for too many students, to pursue further study or careers that require calculus and mathematics.

In the UK, since the Jeffrey Report of 1944, school mathematics education in England saw numerous attempts to introduce elementary differentiation and integration to the 14 – 16 age range. The Jeffery Report of 1944 proposed that introducing calculus early would be beneficial to the more able 14 -16 year-olds. There was a strong case to introduce calculus to such students before the age of 16, the time at which some of them could decide not to study mathematics any further (Orton, 1986).

A review of the mathematics national curricula (NC) from 1989 to 2013 for the 14 – 16 age range (Key stage 4/GCSEs) shows that simple functions and drawing of graphs, rates of change in terms of speed, velocity, acceleration, ratio and gradient (slope) of straight-line graphs have been part of the content to be covered. Thus, by the age of 16, pupils would have covered in some varying degrees, or at least been introduced to, some preliminary calculus ideas. In 2014, a new mathematics curriculum for the 14-16 age range or Key stage 4 was launched. This is the first mathematics national curriculum in recent years to apply the Jeffery Report of 1944 recommendation to introduce some elements of calculus to students of the14-16 age range. Although it does not mention calculus by the term, it is very clear that it allows for the teaching of elements of differentiation. The new NC programme of study DfE (2014, p.3) specifies the mathematical content that should be taught to all pupils, in standard type; and the additional mathematical content to be taught to more highly attaining pupils, in {braces}. Below are three extracts from the 2014 Mathematics programmes of study: key stage 4 of the National Curriculum in England (DfE, 2014, p.9, bold in original):

> Where appropriate, interpret simple expressions as functions with inputs and outputs; {**interpret the reverse process as the 'inverse function'; interpret the succession of two functions as a 'composite function'**} (p.7).
> {**Calculate or estimate gradients of graphs and areas under graphs (including quadratic and other non-linear graphs) and interpret results in cases such as distance-time graphs, velocity-time graphs and graphs in financial contexts**} (p.8).
> {**Interpret the gradient at a point on a curve as the instantaneous rate of change; apply the concepts of instantaneous and average rate of change (gradients of tangents and chords) in numerical, algebraic and graphical contexts**} (p.9).

The teaching of functions and the functional notation (Lagrange's notation) would lay a foundation for calculus, for example, Lagrange's notation is used with calculus symbolism such as $f'(x)$. Estimating gradients with curves and applications to distance-time, velocity-time graphs would form the basis upon which the introduction of the concepts of differentiation (rate of change and gradients), and the concepts of integration (area under graphs and curves) could be developed. Nevertheless, within these curricula, there has not been any direct link made (or implied) of any of these topics to post-16 calculus. Whether teachers have been able to identify and link these topics to calculus or whether they have used these topics or related concepts as a basis to introduce and develop the concept of the derivative is a subject for investigation.

In the US, there was an apparent general atmosphere of dissatisfaction and crisis in teaching and learning calculus in the 1980s, which led to The Calculus Reform

Movement in the USA (Tall, 1994; Tucker and Leitzel, 1994) which instigated changes in the teaching of calculus. Even though concerted efforts had been made to highlight the importance of calculus and to draw attention to necessary educational reforms, e.g. the National Research Council report ‘Calculus for a New Century: A Pump not a Filter’ of 1988, and The Calculus Reform Movement, Bressoud et al. (2015, p.v) report that there was very little progress made with calculus education.

> Calculus is still a filter, but until 2010 we knew very little about who takes it, how it is taught, or what makes for effective calculus programs that promote rather than inhibit students’ continuation into successful careers in science and engineering. Existing knowledge on the effects of class size, placement procedures, *use of technology, or pedagogical approaches* [my italics] was either not specific to calculus or of a very local nature.

It was after 2010 that the MAA undertook the first nationwide study on college-level Calculus in the USA, that combined large-scale survey data and in-depth case study analysis. The study investigated ‘who takes Calculus I and why, what their preparation has been, what they experience in the classroom, and how this affects their confidence, enjoyment of mathematics, and intention to persist in the study of mathematics’ (Bressoud et al., 2015, p.(v)). The MAA study also looked at institutional practices that promote the retention of STEM students.

The American situation with calculus education described by Bressoud et al. (2015) above, appears to be similar to the English context. Calculus is yet to be a pump for the STEM disciplines; it is still a filter. There is little evidence of studies focusing on calculus education in the English education system context, i.e. there are not many studies that are specific to calculus education in England. In modern times, there has not been a similar large-scale study specifically on calculus education in the United Kingdom, similar to the Mathematical Association of America (MAA)’s ‘Insights and Recommendations from the MAA National Study of College Calculus’ of 2015. My study, though on a limited scope, seeks to investigate pedagogical approaches including the use of technology specific to elementary differential calculus in English classrooms in England. This study is a contribution to research that seeks to understand and promote calculus education at the preliminary level.

2.3 Defining the derivative

There is a common recognition amongst many mathematics educators and researchers of the difficulties in introducing the concept of the derivative arising from the idea of a limit (Hobbs and Relf, 1997; Tall, 1992; Thompson, 1994; Zandieh, 2000). The derivative is often symbolically defined (represented), as follows:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}, \text{ where } h \neq 0.$$

The derivative $f'(x)$, is a function whose value at any point on the graph of the function $f(x)$ is defined as the limit of a ratio (the difference quotient). There are three aspects of the concept of derivative, the difference quotient ratio ($\frac{f(x+h)-f(x)}{h}$), the limit, and the function $f'(x)$, which Zandieh and Knapp (2006) refer to as the three 'layers' (p.4) of the derivative framework.

Zandieh (2000) proposes a conceptual framework for the concept of derivative, which has two main components: multiple representations and layers of process–object pairs, which relates to Sfard's (1991, 1992) framework about mathematical objects resulting from the reification of processes. The concept of derivative, according to Zandieh and Knapp (2006), can be represented verbally, graphically, symbolically and physically. The derivative can be described, as the instantaneous rate of change or demonstrated, for example, as speed or velocity, acceleration and other similar physical examples of rates of change. In literate mathematical discourse, the derivative can often be presented symbolically, as the limit of the difference quotient ratio, and explained graphically, as the gradient of the tangent to a curve at a given point or as the gradient of the line the curve tends to under magnification by or zooming in (i.e. local straightness). These forms of mediation for the derivative are not mutually exclusive as very often, one can depend on the other, for example, Zandieh and Knapp (2006, p.5):

> One domain such as the graphical representation of the derivative as slope may serve as the source for another domain such as the symbolic difference quotient. In this way students may come to understand the meaning of the difference quotient through their understanding of the derivative in the slope context.

The thinking of the components of the concept of derivative, ratio, limit and function, as 'layers' is drawn from Sfard's (1991, 1992) notions of operational and structural conceptions, where processes can be based on previously reified objects, forming a chain of the process–object pairs (Zandieh, 2000; Zandieh and Knapp, 2006). Students' difficulty with calculus, in particular, differential calculus stems from the very definition of derivative because it requires an understanding of functions, the difference quotient and the notion of limit (Thompson, 1994; Zandieh, 2000). Students have challenges understanding the derivative, which can result from the idea of limits and possible confusion caused by assuming that $h \to 0$ and that $h \neq 0$, but then later on substituting $h = 0$ (Range, 2011). There is extensive research focusing on students' understanding of limits, for example, Tall and Vinner (1981); Williams (1991) and Oehrtman (2009) highlight the problems in students' understanding of limit as a process and limit as a value. Range (2011) explains students' difficulties with understanding the derivative as the limit as $h \rightarrow 0$, rather than evaluating at $h = 0$, for example, gives an example of how most students find it very difficult to understand that the expression $\lim_{h \to 0}(2x + h) = 2x$ for the derivative of the function $y = x^2$. Range (2011) suggests that the use of graphical mediation could be a useful complement to algebraic functions for explaining the limit definition of the derivative to students.

Park (2016) explains four elements of the limit definition of the derivative, the function, the difference quotient $\frac{f(x+h)-f(x)}{h}$, the limit, and derivative. Each of the four components of the limit definition of the derivative can be seen as both process and object (Gray and Tall, 1994; Sfard,1992; Zandieh, 2000; Zandieh and Knapp, 2006). For example, a function can represent a process of mapping each element of a domain to one, and only one, element of the range, but a function can also be seen as an object, the relation itself. The difference quotient could be, on one hand, a process comparing respective changes in the dependant variable x and the independent variable y, and an object on the other, the ratio. The limit of the difference quotient as h approaches zero can represent a process, and an object, the limiting value. The derivative can be seen as a process of computing or determining many successive values for the difference quotient as h approaches zero, and as the product of this process, the derivative as a function (Park, 2016). Such dualism inherent with derivative or the limit definition of the derivative is a source of potential challenges for students (Zandieh, 2000; Oehrtman et al., 2008).

Park (2016) explains how the symbol $f'(x) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$ can be considered as a process and as an object, and this duality is often seen as a source of difficulty for realising the derivative symbol as an object (Sfard, 2008), upon which other processes (e.g. of the components parts to the definition of derivative) can be operated.

It took mathematicians more than two centuries to fully understand limits, infinitesimals, and differential calculus, which students today are expected to understand in a very short space of time.

2.4 Educational research on the derivative

The ongoing discussion and debate surrounding the teaching and students' learning of calculus at school or college can be traced back to the beginning of the 20th century. Questions that were raised as part of a debate about the teaching programmes for elementary calculus in 'public and secondary schools' (Godfrey, 1914, p.233) are equally applicable to the present-day debate. Here is an extract of the questions proposed by the International Commission on Mathematical Teaching, as reported by (Godfrey, 1914, p.233-234, my italics):

> How is the pupil *introduced to the ideas of the Differential* and Integral Calculus?
> (a) Does he receive a preliminary training in the lower classes of the school, based on the study of appropriate simple functions and their graphs, so that the matter appears to rise naturally out of the subjects already studied, and not to constitute a supplementary course?
> (b) Is Leibniz's Notation employed? If not, what symbols are used for the differential coefficient and integral?
> (c) Which is considered first, the Differential or the Integral Calculus, or are they taught simultaneously?
> (d) Is the integral introduced as the limit of a summation (definite integral), or as primitive function (inverse differential coefficient)? If in both senses, in what order and in what connection with one another are the two points of view considered?
> (e) Is a textbook used?

Certainly, it has been more than a century since these questions were raised, and yet mathematics education researchers are still discussing and debating the same issues with regard to teaching and students' learning of elementary calculus. The review of literature below will consider these questions except question (d) (Godfrey, 1914, p.233) because this study is mainly focusing on differential calculus, although reference to integral calculus will be made indirectly. A simple answer to part (c) is given by Rosenthal (1951, p.75):

> In our courses on calculus, we usually begin with differentiation and then come later to integration. This is entirely justified since differentiation is simpler and easier than integration.

As a matter of tradition, calculus courses teach or introduce differential calculus before integral calculus.

2.4.1 Students' difficulties with calculus

Students' challenges with calculus and the idea of the derivative have long been reported in literature, (e.g. Berry and Nyman, 2003; Oehrtman et al., 2008; Tall (1992). Some studies on calculus, (e.g. Tall (1992); Thompson, 1994; Zandieh, 2000) have put the difficulties down to the complexity of the definition, symbolism and representation of the derivative (Park, 2016). According to Orton (1983b), students' difficulties with calculus were in understanding the derivative as a rate of change, which was linked to insufficient understanding of the limit concept, ratio and proportionality. He also found that the students had difficulty interpreting graphical representations for the derivative. Ferrini-Mundy and Graham (1994) found that students had difficulties in relating the symbolic representations and the geometrical representations, even though they could compute derivatives using standard rules for differentiation. Borgen and Manu (2002) report students' challenges with relating the graph of a function and the graph of its gradient function. Baker et al. (2000) report on students' lack of understanding of the derivative as a function. White and Mitchelmore (1996) report of students' difficulties in differential calculus that have to do with understanding variables; a variable was seen as a symbol to be manipulated, and not as representing a varying quantity.

There has been a lot of research highlighting students' difficulties with calculus, as highlighted by Tall (1992). Godfrey (1914) had long before suggested that research on students learning calculus should be judged against or complemented with research which focuses on how the students are 'introduced to the idea of Differential (…) Calculus' (p.233). This calls for the research focus to shift onto the teacher since the responsibility at the point of delivery lies with the teacher. I consider it highly important to investigate the teachers' introductory lessons on differential calculus. Many years of teaching experience have taught me a valuable lesson that the way a teacher introduces a new concept to students has a lasting bearing on the students' understanding of that concept. This understanding will in turn have a knock-on effect on the students' understanding of other related concepts.

Sofronas et al. (2011) carried out a study in the USA on students' understanding of calculus in which they asked a brain trust of 24 nationally recognized authorities in the field of mathematics, and in particular calculus. All the 24 participant experts cited 'student understanding of the derivative as a central concept and or a central skill that is fundamental to deep comprehension of the first-year calculus' (Sofronas et al. 2011, p.136). How the concept of the derivative is introduced to students at upper secondary school needs careful consideration.

Part (a) of Godfrey (1914, p.233) highlights the importance of prior knowledge in the teaching and learning of calculus. ‘Does he[she] receive a preliminary training in the lower classes of the school, based on the study of appropriate simple functions and their graphs, so that the matter appears to rise naturally out of the subjects already studied, and not to constitute a supplementary course?’(Godfrey, 1914, p.233). This calls for prior knowledge as a basis upon which the concept of the derivative can be developed. Teachers need to have an awareness of and pay attention to the prior knowledge required for the learners before they can be exposed to differentiation. Mathematics educators and school mathematics curriculum designers need to be cognisant of the prior knowledge required to build the foundation for teaching and learning differentiation. As part of the Schools Council Project – The Mathematics Curriculum 11 – 16: a Critical Review, Neill and Shuard (1982) produced a book in which they discussed the development from children’s early graphical work to calculus ideas that children encounter before the age of 16 (Torner, 1985).

Godfrey (1914) mentions simple functions and their graphs as preliminary content knowledge for the introduction of ideas of differential calculus. The concept of the derivative is multi-faceted (Roorda et al., 2009; Zandieh, 2000). It requires the teacher and student to understand other related concepts such as function, difference quotient, and limit (Thompson, 1994; Zandieh, 2000). Neill and Shuard (1982) also identified such facets to include the idea of the limit, the rate of change, the instantaneous rate of change, the gradient of a graph, functions and functional notation, notation and language. Kendal and Stacey (2003) included procedures for calculating the rate of change and averages too, whilst Zandieh and Knapp (2006) highlighted the importance of language, words and terms such as slope, gradient, increase, velocity and acceleration.

2.4.2 Multiple representations

Hiebert and Carpenter (1992) argue that making connections between concepts, representations, procedures, and ideas is important for students’ understanding of the concept of the derivative. Whilst using different representations or contexts could be beneficial to the students, Zandieh (2000) and Roorda et al. (2009) further argue for making connections between the multiple representations - graphical, numerical, and symbolical representations.

Calls for changes in the curriculum and a move away from the traditional ways calculus was taught have long been initiated both in the United States and Europe (Habre and Abboud, 2006). The Calculus Reform of the early 1990s emphasised the importance of and the inclusion of multiple representations, i.e. graphical, numerical and algebraic representations of derivatives in textbooks and most universities in the

USA (Bressoud et al., 2016). One notable major outcome of the reforms that followed was the ability to identify and use numerical, graphical and algebraic representations which was aimed at developing teaching guided by the principle of the ‘Rule of Three’ which would allow students to look at the same idea from three different angles, i.e. graphically, numerically, and symbolically (also described as analytically, Hughes-Hallett et al. (1994, p.121). Another outcome was the rise in the use of what Zimmermann (1991) called mathematical visualisation which saw the adoption of visual elements of calculus. He argued that visual thinking, which refers to pictorial and visual forms of representation, was very important for understanding calculus. Further outcomes included formal representations (Tall, 1996) and real data representation which would give the students real situations integrating practical activities in the learning of calculus.

According to Verhoef et al. (2014) and Tall (2010), making sense of calculus calls for a more natural approach that blends together the dynamic embodied visualisation and the corresponding symbolic calculation. For example, the concept of the derivative is all about local straightness, whereby if we zoom in and take a close look at a magnified portion of a curve where the function is differentiable, the curve will look like a straight line. Zandieh (2000) argues for the use of multiple representations and levels of process-object duality, for example, a progression from an understanding of derivative at a point to an understanding of derivative as a function. Other studies on calculus (Kendal and Stacey, 2003; Orton, 1983b) have shown that although many students were able to deal with symbolic representations in elementary differentiation, they were unable to relate to other procedures. For example, Kendal and Stacey (2003) observed that many students could at most make connections between graphical and symbolic representations in calculus, but could not make graphic and numeric connections, or symbolic and numeric relationships. Berry and Nyman (2003) posit that students’ ability to make connections between mathematical ideas is an important indicator of understanding.

Zandieh and Knapp's (2006) study highlights the importance of paying attention to metonymic statements for researchers and teachers of mathematics. Metonymic models, according to Zandieh and Knapp (2006) may serve as an affordance or a constraint for students in learning complex calculus concepts such as the derivative. For example, it takes less energy and time to use a metonymy than to explain an idea or narrative in full, but such a shorthand often becomes ‘more susceptible to misinterpretation’ (p.16).

Zandieh and Knapp (2006) report of a student’s (Alex) inconsistent discourse about the derivative as resulting from a rather rigid focus on a graphical interpretation of the tangent line for the ‘derivative picture’. Zandieh and Knapp (2006) acknowledge that teaching the derivative is a complex process, for example, describing the derivative at a point by graphical mediators. Two approaches are often adopted here, secant lines approaching the tangent line at a given point on a curve or zooming in on the point until the curve looks like a straight line. In both approaches, the derivative is the slope of that tangent (straight) line. However, as Zandieh and Knapp (2006) argue, ‘a student who has focused on this image may say that the derivative is (…) the most obvious image or endpoint of this graphical process, the tangent line’ (p.11).

> The slope is implicit in both graphical images, but the tangent line is explicit, visible, and thus more easily remembered. Even without the idea of a limiting process a student may remember a single image, a curve with a line tangent to it, when asked what a derivative is. Again, one might pick up on the tangent line as the explicit, visual representation of the derivative instead of remembering that the derivative is the slope of that line. (Zandieh and Knapp 2006, p.11)

This understanding might explain the metonymic or colloquial utterance that ‘the derivative is the tangent line’, which, as Zandieh and Knapp (2006) explain, is the likely reason for students, ‘like Alex, making metonymic misstatements concerning the derivative as the tangent line’ (p.11).

Furthermore, some studies, e.g. Roorda et al. (2007) have shown that students have difficulty in relating differential ideas learnt across subjects, for example, rate of change procedures from mathematics classes and physics classes (Roorda et al., 2015). Reporting on the work of one high school teacher, Schwalbach and Dosemagen (2000) observed a teacher who used concrete examples from the students’ physics class as a context within which to teach calculus in a mathematics lesson. Scheja et al. (2008) argue that making connections between the properties of graphs of a function and that of its derived function would build a better understanding of the underlying graphical concepts of calculus. Whether students will be able to make connections between different representations depends to a greater extent on how the teacher would use different representations to teach calculus. Ferrini-Mundy and Graham (1991, p.630) report about a student from their study:

> When given a simple limit problem in this format with the appropriate graph displayed, most of the students in the study were able to solve the problem but showed very little geometric understanding. One of the students interviewed claimed "the graph can't help me find an answer." Further probing revealed that the notion of "approaching" was not part of her understanding of limit. She

> saw limit problems as functions to be evaluated and wasn't sure about all the "extra" notation (the arrow, the word "lim").

Nonetheless, an argument for using multiple representations in teaching is that it would appeal to the individual students' preferred ways of learning; and very often, the representations do complement each other. The case reported by Ferrini-Mundy and Graham (1991) above calls for the need to investigate the teacher's practice and gather empirical evidence on how teachers use numerical, graphical and algebraic representations in teaching differential calculus.

2.4.3 Resources

Teachers of mathematics draw on and use a wide range of resources, as well as adapt these resources for purposes of teaching and learning (Adler, 2012). Today, there is an increasing range and wide availability of textual resources such as textbooks, and digital technologies for teaching. The calculus reforms also led to applications-oriented and technology-intensive instruction, which saw the use of sophisticated hand-held calculators and microcomputer graphics packages (Heid, 1988; Ferrini-Mundy and Graham, 1991; Ferrini-Mundy and Gaudard, 1992).

One of the most important questions proposed by the International Commission on Mathematical Teaching back in 1914 was 'Is a textbook used?' (Godfrey, 1914, p.234). Out of the seven popular elementary calculus textbooks cited in Berry and Nyman (2003, p.481-497), only two take a slightly different approach to introducing calculus. Ostebee and Zorn (1997) and Hughes–Hallet (1996) begin with ideas of rates of change, and average velocity before they get to the idea of the tangent line and slope of curves or functions. Ostebee and Zorn (1997) attempt to put the concept of the derivative in a real-life context and then develop the idea graphically. A report by Hobbs and Relf (1997) on the UK textbooks for school calculus shows that the majority of the textbooks concentrate on the algorithmic rules and tricks needed to differentiate and integrate (Berry and Nyman, 2003). Thus, it is very important to investigate how the teacher interacts with the textual and digital resources in teaching calculus. This could be described in terms of 'documentation work' which means 'to work with documents' (drawing from the French term "ingénierie documentaire") (Gueudet and Trouche, 2012, p.24) which encompasses all the interactive ways in which the teacher work with resources.

Reporting on a meeting on Calculus in the USA, Walsh (1987) said there was a strong mood of self-criticism with both the calculus curriculum and the quality of teaching. This is what Walsh (1987, p.749) had to say:

> There appeared to be a consensus that the teaching of calculus has been focused for too long on routine problem solving. New approaches are needed, for example, to come to terms with the use of sophisticated hand-held calculators and computers and, particularly, to give students a better conceptual understanding of the subject

It is clear in this report that technology was seen as a way to solve the problem of students' difficulties with calculus (Orton, 1983a; 1983b; Tall, 1992). A scientific calculator is essentially a simple tool for exploring ideas of calculus (Hobbs and Relf, 1997). Neill and Shuard (1982, p.3):

> It is more valuable for a student first to know from his own experience with a calculator that the derivative of $f(x) = x^2$ at $x = a$ is $2a$, than it is for him to have seen, but not understood, the formal calculations involved in an algebraic treatment.

Neill and Shuard (1982, p.3) argue that although complete understanding is not possible in the early stages, 'a sound intuitive understanding based on a good deal of numerical and graphical experience is possible and should be the aim of the teacher'.

Technology and digital artefacts

Tall et al. (2008), explain the role of technology in teaching and learning calculus, "of all the areas in mathematics, calculus has received the most interest and investment in the use of Technology" (p.207). Tall argues that technology can be instrumental in helping students develop visualization skills and forming visual mental images of calculus concepts (Tall, 1986; 1990; 2003; 2013). More recently, Takaci et al. (2015) carried out a study on modern approaches to teaching calculus- for examining functions and drawing their graphs using a computer-based dynamic imagery program, GeoGebra. Takaci et al. (2015, p.421) reported that:

> GeoGebra can help those students having insufficient knowledge (necessary for solving those tasks) to improve it. We can say that our research shows that the students' learning achievement in examining functions and drawing their graphs is better when they use GeoGebra, working in collaborative groups than without using it. Also, GeoGebra enables creation of effective learning environment for examining functions and drawing their graphs.

However, there has always been scepticism about the role of such technology in the teaching and learning of calculus, as expressed by Ferrini-Mundy and Gaudard (1992, p.58):

> The procedural facility necessary in integration and differentiation will most likely be provided by sophisticated hand-held calculators, while the technical skill previously necessary in areas of curve-sketching will be deferred to accessible microcomputer graphics packages.

In this case, the introduction of technology was viewed as counterproductive for students developing an understanding of calculus. Heid (1988) discusses some arguments for and against the introduction of graphing calculators and special computer software, but her study found that learning calculus with technological tools was no worse than the traditional ways of doing calculus with pen and paper methods. There is a lot of research (e.g. Tall, 1986; 1990; 2003; 2013; Takaci et al., 2015; Walsh, 1987; Hobbs and Relf, 1997; Heid, 1988; Ferrini-Mundy and Graham, 1991; Ferrini-Mundy and Gaudard, 1992) that suggest that graphical calculators and computer-based dynamic imagery software could be used to improve teaching and learning of calculus.

The reformed calculus movement starting in the 1980s in the USA, as well as the academic discussion and debate in research literature on teaching and learning calculus in the recent decades, resulted in an extensive use of technology such as dynamic computer software (Habre and Abound, 2006). Technology was almost believed to be a panacea for the inadequacies in teaching and the perceived students' challenges with understanding calculus. The main strength of technology was seen in its capability of facilitating greater and easier access to numerical, graphical, and symbolic representations of concepts (e.g. Fey, 1989; Goldenberg, 1987; Kaput, 1992; Porzio, 1999; Tall, 2001). The challenges of students' difficulties in calculus (Tall, 1992, 2019) remain; and this is of course a problem for teachers of mathematics.

According to Laborde (2008), there are two levels of 'instrumentation of technology by teachers' (p.1), that is, levels of technology use by teachers. Technology can be used as a tool for carrying out a mathematical activity by the teacher and technology can be used as a tool for teaching mathematics and fostering students' learning (Laborde, 2008). My study is interested in the second level of instrumentation of technology for teaching and fostering students' learning of differential calculus, in particular, the mathematics-specific technologies that are designed primarily to improve the teaching and learning in mathematics, such as dynamic geometry software and computer algebra systems (CAS).

According to Jesso and Kondratieva (2016), some of the advantages of dynamic geometry environment (DGE) is that it can be used to substantiate the limit definition of the derivative with more elaborate functions such as $f(x) = a\sin(bx)$, (where *a* and *b* are real numbers). Further, with both graphical and algebraic windows, a DGE allows the students to see formulas corresponding to the graphs. DGE can be instrumental in constructing the definition of the derivative and substantiating the process of differentiation. How teachers, if ever, use such technologies (e.g. for instructional, visualisation and exploratory activities) in the teaching of calculus, particularly in introducing differential calculus is a focus of my research.

2.4.4 Symbolism

In the literature, symbolic representations and notations in differential calculus are shown as a source of challenges for teachers, let alone the students. Neill and Shuard (1982) point out the problems associated with the notation and language used such as

$$\frac{dy}{dx};\ \frac{\delta y}{\delta x};\ \lim_{\delta x \to 0}\frac{\delta y}{\delta x} = \frac{dy}{dx};\ \ \delta x$$

This is the Leibniz notation, which ‘proves to be almost indispensable in the calculus’ (Tall, 1992, p. 6). If students are to understand differential calculus, the use of such notation must be understood too; otherwise, the notation could be a barrier to learning. Is $\frac{dy}{dx}$ a fraction; is it divisible or is it a single symbol? Do we make the same interpretations of dx in $\frac{dy}{dx}$ as we do $\int f(x)dx$, what is the relation between the dx in these two symbolic representations? How do we explain this $\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dx}$? Can we cancel out the du? $\frac{dy}{dx}$ is obtained from $\frac{\delta y}{\delta x}$ but it does not represent a fraction. $\frac{\delta y}{\delta x}$ is quotient or a fraction, and $\frac{dy}{dx}$ is simply a reminder of $\frac{\delta y}{\delta x}$ (Neill and Shuard, 1982). Whilst it is possible to give meaningful interpretations to these notations, it is important to know that failing to give consistent and coherent meaning to these symbols can cause serious conceptual problems and cognitive conflict (Tall, 1992, p. 6).

Symbolism in mathematics is a source of potential ambiguity since mathematical symbols could be representing either process or concept (Gray and Tall,1991; 19944). The term procept was coined by Gray and Tall (1994) to capture and describe the dualism of the symbol, as process and concept. Tall et al. (1999) assert that procepts are present throughout most of mathematics. According to Gray and Tall (1991), a ‘procept’ is a conception or term which represents an ‘amalgam of a process and a concept’ (p.2). Gray and Tall (1991, p.2) define a procept as ‘the amalgam of process and concept in which process and product are represented by the same symbolism’. However, Gray and Tall (1994) further elaborate their definition to include three components: ‘a *process* that produces a mathematical *object*, and a *symbol* that represents either the process or the object' (p.121). The *symbol* for a procept can evoke either *process* or *concept*.

A *procept* has three constituent elements: the process, the product of the process and the symbolism or notation that denotes the process or the product. For instance, a *symbol* such as 4+3 is a *procept* since it can be seen as the *process of addition* and as the *concept of sum*; and the *symbol* $y = f(x)$ represents the *process of assignment* and the *concept of function* (Gray and Tall, 1994; Tall, 2001). Further examples from calculus procepts include 'the *process of tending to a limit* and the *concept of the value of the limit*, both represented by the same notation such as $\lim_{x \to a} f(x)$nd Tall, 1994,p.2). Similarly, the symbol $\frac{dy}{dx}$ represents both the *process of differentiation* which produces the *concept of derivative* and thus, $\frac{dy}{dx}$ is a calculus procept. Gray and Tall (1991) assert that by 'using the same notation to represent both a process and the product of that process' (p.2), it is possible for anything to 'be a process and an object at the same time' (p.2).

Gray and Tall (1994) argue that the ambiguity with the notion of a procept 'provides a more natural cognitive development which gives enormous power to the [student]' (p.8). The existence of the ambiguity is a learning opportunity that has implications for the teacher. Gray and Tall (1991) argue for the use of notation to represent either a process or product in mathematics and that the ambiguity of symbolism (i.e. a procept) for process and concept is at the 'root of successful mathematical thinking (p.2). To a proceptual thinker, proceptual known facts allow for flexibility and greater fluency in thinking, for example, using known proceptual facts to derive new proceptual known facts, e.g. where the procept $2 + 3 = 5$ may be seen equivalently as $3 + 2 = 5,\ 5 - 3 = 2,\ 5 - 2 = 3$, thus, new proceptual facts. However, Gray and Tall (1991) highlight that:

> What might be a simple combination of proceptual ideas for the [high attaining learners] becomes the coordination of several complex processes for the [low attaining learners], leading to intolerable difficulties and a high probability of failure (p.4).

A proceptual known fact has a 'rich inner structure which may be decomposed and recomposed to produce derived facts' (Gray and Tall, 1991, p.3), thus it is not a merely memorised fact. Tall et al. (1999) identify some links between Gray and Tall (1991; 1994) notion of procepts and the earlier works of Sfard (1989; 1991) about operational and structural mathematics:

> The theory of Sfard (1991) formulates two ways of constructing mathematical conceptions through the complementary notions of operational and structural

> activities as ‘two sides of the same coin’. It is the first of these that involves process-object construction. She makes the important observation that in any given mathematical context, it is usually possible to see both operational and structural elements (Tall et al., 1999, p.233).

This suggests that Tall et al. (1999) see Sfard’s (1991) earlier conceptions of operational activities as constituting both process and object, whereby in Sfard’s (1991) stratification process, operations become reified as objects. Further, Tall et al. (1999) acknowledge that Sfard’s(1992) argument is consistent with Dubinsky's (1986, 1991) characterisation of the encapsulation of process into object, the APOS theory, in which Action is conceptualised as a total Process, then encapsulated as a mental Object, and finally transformed into a mental Schema (i.e. mental structures). Nevertheless, Gray and Tall (1994) argue for their procept theory as a perspective to explain the ambiguity with the duality of procept. For example, they cite the example of functions:

> The case of the function concept, where $f(x)$ in traditional mathematics represents both the process of calculating the value for a specific value of x and the concept of function for general x, is another example where the modern method of conceiving a function as an encapsulated object causes great difficulty (Sfard, 1989). (Tall and Gray, 1994, p.8).

Tall and Gray (1994) concluded: ‘We, therefore, are confident that the notion of procept allows a more insightful analysis of the process of learning mathematics’ (p.8). My study investigates the *teaching* of elementary differential calculus. Symbolism and specialised language or terminology are at the core of calculus discourse. It is, therefore, necessary and important to explore the teachers’ use of calculus symbolism. Symbolism in mathematics and calculus is a source of potential ambiguity because mathematical symbols could be seen to represent either process or concept (Gray and Tall, 1991; 1994).

Further criticism of the inconsistencies in interpreting notations and the use of Leibniz notation comes from Thurston (2000, p.262):

> The derivative f' is defined by specifying its value $f'(x)$ as the well-known limit; $\frac{dy}{dx}$ is then defined to be$f'(x)$. But in $f'(x)$ the x denotes a number. In $\frac{dy}{dx}$ it doesn't. We never write $\frac{dy}{d3}$. And $\frac{dy}{dx}$ is not defined unless y is a function of x. But implicit and parametric differentiation use $\frac{dy}{dx}$ even where y is not a function of x.

The suggestion here is that the function notation is seemingly easier for the students when they are first introduced to the concept of the derivative. However, $\frac{dy}{dx}$ notation is inevitable, ‘it is almost universally employed by users of calculus’ Neill and Shuard (1982, p.15) and should not be delayed too long before it can be introduced to students. Teachers and their students alike need to be familiar with both the Leibniz and function notations. The two notations complement each other in the study of calculus.

Furthermore, the idea of the limit with the associated notation deserves careful consideration in introducing the idea of the derivative. Neill and Shuard (1977) reported that ‘although the idea of a limit, and the notation for limits, lie at the heart of the differential calculus, they receive little attention in the introduction to differentiation in any of the texts reviewed’ (Neill and Shuard, 1977, p.72). It is not uncommon that many students first encounter the idea of the limit when they are introduced to differential calculus.

For many teachers and students alike, doing calculus is equivalent to learning the skill of manipulating symbols and numbers (Hughes-Hallett, 1991). Eichler and Erens's (2014) study, which investigated teachers’ beliefs towards calculus, found that all the 29 teachers in their study had a shared view on calculus, as a set of rules for students to be memorised and use in solving routine problems.

2.5 Conclusion

The discussion and debate surrounding the teaching and students’ learning of calculus at school or college, which goes back to the start of the 20th century still rages on. Since then, there has been a lot of research highlighting students’ difficulties with calculus. To the present day, students’ challenges with calculus still prevail. How calculus is taught matters, more so, how calculus is first introduced to students at school is critical for students’ understanding of calculus. Research that focuses on the teaching of calculus can indeed contribute to the mathematics researchers and educators’ understanding of the students’ challenges with calculus. The teaching of calculus often starts with differential calculus and my study seeks to investigate the teaching of calculus by focusing on the introduction of differentiation at school.

The next chapter will explain the commognitive theoretical framework as the conceptual lens guiding this research. Also, included in the next chapter is a further review of literature on relevant commognitive studies (refer to Section 3.6).

Chapter 3 The Commognitive theoretical framework

3.1 Introduction

For this qualitative research, theoretical constructs from the discursive theory of commognition (Sfard, 2008) were adopted to conceptualise and analyse the teachers' classroom discourse on differential calculus. Miles and Huberman (1994) give a strong rationale for having a conceptual framework for qualitative designs. A conceptual framework, according to Miles and Huberman (1994) 'explains the main things to be studied – the key factors, constructs or variables – and the presumed relationships amongst them' (p.18). Thus, a conceptual framework enables the researcher to notice patterns of behaviours, thus providing the analytical lens and vocabulary to describe the patterns (Ioannou, 2018).

The 'social turn' (Lerman, 2000) marked a turning point in mathematics education research, which saw a shift in research towards the participation perspective to learning, language and social practice. New conceptualisations and theories that explain participation in forms of social practice have since emerged, with more research examining language (e.g. Nardi, 2008; Morgan et al. 2014), classroom discourse (e.g. van Oers, 2001; Yackel and Cobb, 1996), mathematical discourse (Sfard, 2002; 2008) and their 'role in mediation and participation' (Xu and Clarke, 2019, p.129). Nardi (2008) describes words, symbols or diagrams as three 'languages' of mathematics (Nardi, 2008, p.145).

Morgan et al. (2014), for example, in their review of research studies on language and communication in mathematics education (e.g. Arzarello et al., 2009; Bjuland et al., 2008; Maschietto and Bartolini Bussi, 2009; Radford, 2009), concluded that classroom communication involves more than just words and mathematical symbols.

> Thinking about language in mathematics education has broadened from considering primarily either words or mathematical symbolism towards a more comprehensive concern with a range of other means of communication.
> (Morgan et al., 2014, p.844).

Morgan et al. (2014) review found that within mathematics education literature, the term language was used in various settings to refer to words, mathematical symbolism, diagrams, graphs and other non-verbal modes of mathematical communication, including gestures. This wide application of the term language is characteristic of what Sfard (2002; 2007; 2008; 2015) refers to as discourse, which encompasses more than just the common use of the term language.

Since that social turn (Lerman, 2000), the commognitive framework which emphasises the critical role of the social environment in the learning, teaching and doing of mathematics, has become an important theoretical framework in mathematics education research.

Although there are many other theoretical frameworks found in mathematics education research, I found Sfard's (2008) alternative perspective on communicational approach to research that shifts the focus onto analysing the discourses, e.g. mathematical discourse, the most appealing for my research. For example, by adopting the conceptual framework of commognition, I was able to circumvent the very often contentious topic of teacher knowledge by examining the teacher's pedagogical calculus discourse, instead. Further affordances of the commognitive theoretical framework (CTF) (Sfard, 2007; 2008) were that it enabled my research to examine (and get insights into):

i. the objects of instruction within the teachers' pedagogical calculus discourse;
ii. the processes of instruction by which the teachers introduced the concept of the derivative; and
iii. the actions or behaviours of the teachers and observable outcomes of instruction.

I choose the commognitive framework (Sfard 2008) as a befitting and appropriate conceptual framework for my study because my research sought to explore and investigate the teachers' pedagogical calculus discourses: what is said (e.g. word use and narratives) and how; what is used (e.g. communication or visual mediators) and how; and what is done (e.g. routines) and how in teaching elementary differential calculus, in particular, in introducing the concept of the derivative.

The following Sections 3.2 to 3.4 will introduce and explain the main epistemological tenets of the commognitive theoretical framework, the commognitive constructs, the commognitive conflict, and the commognitive terminology and key definitions that are relevant to this study. Section 3.5 argues for the use of the commognitive framework for this research. Section 3.6 is a further review of literature (in addition to Chapter 2) on commognitive studies relevant to this discursive study. It is a critical review of the literature on commognitive studies that have researched teaching and learning through the commognitive framework. Not all the studies reviewed here are specifically about calculus, but I see the analyses made of the word use, narratives, visual mediators and routines in these studies as comparable and transferable to my study. I regard the studies as relevant as they provide a wider knowledge base for my commognitive research into the teachers' pedagogical calculus discourse. Since the studies are based on the theory of commognition, for the convenience of the readers, I decided to put the review of their literature (in Section 3.6) after the commognitive theoretical framework has been explained (in Sections 3.2 - 3.5).

In this chapter, the primary source on commognition theory is Sfard (2008; 2007), but I also draw from other commognitive studies to support my interpretation and explanation of the commognitive framework. Most importantly, I explain my interpretation and application of the commognitive framework to my study.

3.2 The main epistemological tenets of CTF

The key epistemological assumptions that underpin the commognitive theoretical framework include thinking as a form of communication, mathematics as a form of discourse and learning as a change in the discourse of the participants of the community discourse.

3.2.1 Thinking as a form of communication

The CTF is a discursive framework for analysing and interpreting human activity, to understand the 'intricacies of mathematical learning' (Sfard, 2007, p.566). It is built on the premise that 'thinking is a form of communication' (p.565). Commognition encompasses 'both thinking and interpersonal communication' (Sfard, 2007, p.570) and follows rules rooted in historically established customs. Thinking is considered as individualisation of (interpersonal) communication. Thinking is conceptualised as a form of activity of communication with oneself, thus cognition + communication (interpersonal exchanges) = commognition (p. 570). Commognitivists reject any split between thinking and speech or thinking and communication. They see thinking and interpersonal communication as facets of the same phenomenon (Sfard, 2008).

According to the commognitive perspective, communication (as well as learning and development) occur at two levels: first the interpersonal or societal or cultural level and then the intrapersonal level (individualisation). My study explores the teaching of elementary differential calculus, and thus it focuses on the interpersonal level of the communicational actions of the teachers.

3.2.2 Mathematics as a form of discourse

Discourse, from a commognitive perspective, is defined as the communicative activity typical of a certain community (Sfard, 2008). Mathematical discourse is a cultural activity (Sfard, 2008; Kim and Lim, 2017). Sfard (2007) likens commognition to games (Wittgenstein, 1953). There are a vast number of different games, each played according to certain rules and with diverse tools. Likewise, there are many types of communicational activity each characterised and distinguishable by the rules and the mediational means they use, as well as their objects of communication. These distinctive types of communication are what Sfard (2008) calls 'discourses'. Thus, there are different communities of discourse and different types of discourses distinguishable by their objects, the kinds of communication mediators used, and the rules followed by participants of the discourse (Sfard, 2008).

One of the core tenets of the commognitive framework is that different academic disciplines are regarded as specific types of discourse, and in this study, mathematics is seen as a distinct form of discourse (Sfard, 2008). Mathematical discourse refers to acts of communication for constructing and substantiating mathematical meaning according to the shared rules of the discourse itself (Nardi et al., 2014). This study is built on the commognitive premise that mathematics is a specific form of discourse characterised by its own discursive objects, type of words, communication mediators, routines and endorsed narratives. Simply put, mathematics is regarded as a special type of communication with specific ways of saying and doing (Nachlieli and Tabach, 2018). Mathematics is a discourse and calculus is a mathematical discourse, and the teachers' pedagogical calculus discourse is the core unit of analysis in this study.

According to the commognitive perspective, discourses follow rules rooted in historically established customs (Sfard, 2008). Mathematics discourse is, thus, culturally and historically defined, and develops individually and collectively, through distinctive metadiscursive rules and discursive objects (Wing, 2011). There are two categories of *rules* that set discourses apart: the object level and the meta-level rules (also simply referred to as metarules) (Sfard, 2008). The object-level rules are narratives about the features and the behaviour of the objects of a discourse and the metarules are propositions about 'patterns in the activity of the discursants trying to produce and substantiate object-level narratives' (Sfard, 2008, p.201). Both the object-level and metarules in the teachers' pedagogical calculus discourse are of interest in this study.

3.2.3 Mathematical objects as discursive objects

In the commognitive framework, discursive objects are very important tools for thinking and communicating about literate mathematical discourse. Whilst colloquial discourses, often mediated by images of concrete objects that pre-date the discourse (Ryve et al. 2012), are created for 'communication about physical reality, in [literate] mathematical discourses, [discursive] objects are created for the sake of communication' (Sfard, 2008, p.193). Discursive objects are perceptually accessible entities or realisations of a signifier in a discourse. To illustrate this notion of realisations, Sfard (2008) gives an example of what the number ‘5’ could signify to a child. The number ‘5’ is a signifier that could be thought of to mean five fingers, five beads on a string, or a position on a number line; and all these examples (perceptually accessible objects) are *realisations* of the *signifier* ‘5’, which Sfard (2008) describes as the *realisation tree* of the signifier ‘5’.

Sfard (2008) defines signifiers as ‘words or symbols that function as nouns in utterances of the discourse of participants, whereas the term realisation of a signifier S refers to a perceptually accessible object that may be operated upon in the attempt to produce or substantiate narratives about S’ (p.154). The notion of discursive objects is liberating as it ‘effectively allows us to put aside centuries of dispute and unproductive controversy concerning the ontological status of mathematical objects, and importantly, to work in a classroom (once again) with observable phenomena’ (Wing, 2011, p.366).

A mathematical ‘object’ can be defined as ‘a set of realisations’ (Park, 2016, p.399), and the realisations of the discursive objects are visual mediators in the discourse. The visual forms of realisations can be iconic, concrete, gestural or verbal (i.e. written words or algebraic symbols) (Sfard, 2008). These visual forms of realising signifiers are important in mathematical discourse, ‘because what we get in this process is most liable to investigation and manipulation and may thus lead to endorsed narratives in the immediate way’ (Sfard, 2008, p.155). Of particular interest to this study (and of relevance for teaching) is the fact that the same signifier can be realised in different ways by different individuals, and that ‘the same communicational action may refer different interlocutors to different objects’ (Sfard, 2008, p.88). Thus, the mathematical and didactical routines and the visual mediators in the teachers’ pedagogical calculus discourse are of interest to this study.

3.2.4 Learning as participation

Commognitivists replace ‘the metaphor of learning-as-acquisition with learning-as-participation’ (Sfard, 2008, p.92), and argue that learning is based on social foundations, which in turn can be traced back to Vygotsky (1978). The commognitivists break traditions with the behaviourist, cognitivist and acquisitionist perspectives on learning processes (Johnson, 2009). Unlike the cognitivists who view learning as the acquisition of information, storing of knowledge in some mental representations and refining existing mental schemes, the commognitivists view learning as a change in the discourse of the individual participants (e.g. the students) in the discourse (Sfard, 2008).

Mathematical discourse is a ‘historically established activity practised and extended by one generation after another’ (Sfard, 2008, p.203). Teaching mathematics in schools serves to further the continuation of the discourse. Therefore, learning mathematics means joining in this historically and culturally established communicational activity rather than inventing their own new discourse. Thus, to learn mathematics, a student should ‘individualize historically developed, well-established, routines (Nachlieli and Tabach, 2018, p.256). How teachers facilitate this process, i.e. helping students participate in literate calculus discourse is a key focus of this study.

3.2.5 Learning mathematics as a change in discourse

Learning mathematics is seen as individualising mathematical discourse, thus, ‘the process of becoming able to have mathematical communication not only with others but also with oneself’ (Sfard, 2008, p.573). Learning is a change or a shift in discourse, thus, acquiring admission into the endorsed historically established framework of discourses. Such ‘admission into the discourse of those objects requires practises of communication in which an individual’s own framework effectively becomes transformed into the collectively endorsed framework’ (Johnson, 2009, p.384). My study seeks to examine the role of teachers in those communicational activities aimed at promoting participation and admission of newcomers into the historically and collectively established calculus discourse.

Just as there are two categories of rules (object level and metarules) that govern a discourse (see section 3.2.2 above), learning happens at two levels: *object-level learning* and *meta-level learning* (Sfard, 2008). Object level learning involves changes in the student’s existing mathematical objects. It is characterised by the expansion of the existing discourse through new vocabulary, new routines, and new narratives; thus, the participant gets to know better the existing mathematical objects (Sfard, 2008; Tabach and Nachlieli, 2016), e.g. realising that $\frac{dy}{dx}$ and $f'(x)$ are both symbolism for differentiation. Whereas meta-level learning is expressed through a ‘change in meta-rules of the discourse’ (Sfard, 2008, p.573), in which the transformations could be either horizontal or vertical development. Horizontal change involves combining two separate discourses into a new single discourse, e.g. when discourses about algebraic expressions, graphical and numerical approaches are subsumed in the discourse about functions (Tabach and Nachlieli, 2016). Vertical development involves combining the student’s existing mathematical discourse with a new rule, its own meta discourse (Tabach and Nachlieli, 2016), e.g. when a student’s existing discourse about gradient is then combined with discourse about derivatives and limits. Such change in discourse could result from a commognitive conflict (see section 3.3) which can be resolved through meta-level discussions about words use with the teacher. How teachers facilitate learning as participation to promote object level and meta-level learning in calculus discourse is of interest to this study.

The discursive development of individuals, i.e. learning, is explained by changes in the individuals' ways of communicating in each of the four discursive characteristics of word use, visual mediators, endorsed narratives, and routines characteristic of the discourse (Sfard, 2007, 2008). This study will examine these four commognitive constructs in the teachers' pedagogical calculus discourse.

3.3 Commognitive theoretical constructs

Mathematical discourse, according to CTF, is characterised by the following four commognitive constructs: word use, visual mediators, endorsed narratives, and routines (Sfard, 2008). It is therefore logical for a study that seeks to explore the teaching of the derivative to examine the type of words, visual mediators, routines and narratives in the teachers' pedagogical calculus discourse. In the following four subsections, I will define each of these four commognitive constructs and explain their relevance and place in this study.

3.3.1 Word use

Word use simply refers to the type of words used, such as mathematical words and mathematical terminology, like those related to shape and quantities (Nardi et al., 2014), as well as ordinary words used in everyday communication, but with special and specific meanings in mathematics, such as differentiation, limit, point, weight. Sfard (2007) regards word use as 'an all-important matter because, being tantamount to what others call *word meaning,* 'the meaning of a word is its use in language' (Wittgenstein, 1953, p.20), it is responsible to a great extent for how the user sees the world.' (p.571). Elaborating on the commognitive constructs that characterise a discourse, Güçler (2013) identifies three descriptive categories of word use*:* '*colloquial* (talking about mathematical concepts in the everyday sense); *operational* (talking about mathematical concepts as processes or actions); and *objectified* (talking about mathematical concepts as objects or object-like entities) word use' (p.441, italics in original). This study will explore both colloquial and literate mathematical word use in the teachers' pedagogical calculus discourse because calculus *words* (e.g. tangent, slope, gradient, differentiation, derivative, limit) in mathematical discourse signify mathematical objects (Park, 2016). Thus, this study will explore what type of words teachers use in introducing the derivative and examine how they use the words in constructing and substantiating mathematical narratives about the derivative.

Objectification is a key feature in Sfard's (1991, 1992, 2008) commognition framework on word use in mathematical discourse, which results from the process of reification and alienation. According to Sfard (1991) reification is the process through which mathematical operational modes of thinking such as ‘processes, algorithms and actions’ (p.4) are objectified into structural modes of thinking, which sees and treats mathematical concepts and notions (something abstract) like objects (something concrete), i.e. discursive objects. Alienation is a process whereby statements about mathematical modes of thinking (operational or structural) are converted into impersonal discursive forms (Sfard, 2008). Reification and alienation are the processes whereby ‘statements about processes’ are turned into ‘impersonal statements about objects’ (Sfard, 2008, p.63). Objectification, thus, is a means of formalisation utilizing symbolic artefacts, and it increases the practical effectiveness of a discourse and enhances an individual’s capacity for mathematical communication (Sfard, 2008; Güçler, 2013). However, on operational conceptions (focusing on processes) and structural conceptions (focusing on objects), Sfard (1992) emphasises the process of reification and cautions that ‘the fact that a process has been interiorised and condensed into a compact, self-sustained entity, does not mean, by itself, that a person has acquired the ability to think about it structurally. Without reification, her or his approach will remain purely operational’ (p.65). Thus, examining word use can offer insights into the degree of objectification in the teachers’ pedagogical calculus discourse.

3.3.2 Visual mediators

Visual mediators in mathematical discourse are defined as ‘the means with which participants of discourse identify the object of their talk and coordinate their communication’ (Sfard, 2007, p. 571). In my study, visual mediators refer to all the non-verbal means of communication and the visible objects, involving symbolic artefacts like formulae, calculus symbolism, graphs, drawings and diagrams that are created and used in constructing or substantiating mathematical narratives (see Section 3.3.3 below on narratives).

Visual mediators are an essential part of the thinking and communication process in mathematical learning (Sfard, 2008; Presmeg, 2006; Ioannou and Nardi, 2010). Visual mediators, in commognition theory, are viewed and thought of as integral to the act of communication. Thus, visual mediators are seen as integral in the thinking processes; they are not mere auxiliary means representing pre-existing thought (Park, 2016). Since mathematical objects are often intangible, visual mediators as the means by which discursants identify the objects, are an important part of the communication and thinking process. A consideration of links between visual mediators and specialised mathematical words in the calculus discourse is important for my research.

For the analysis of the teaching of the derivative, this study examines the visual mediators in the teachers' pedagogical calculus discourse because visual mediators are the 'providers of the images with which discursants identify the object of their talk and coordinate their communication' (Sfard, 2008, p.145). Visual mediators are central to and critical for the analysis of mathematical literate discourses.

The commognitive framework defines mathematical literate discourses 'as visually mediated by symbolic artefacts and algebraic symbols' (Sfard, 2008, p.146). In this study, the analysis of the teachers' pedagogical calculus discourse will consider various forms of visual mediation, including both iconic and symbolic mediators, in mathematical communication. Exploring the visual mediators in the calculus discourse illuminates the numerical, graphical and algebraic mathematical representations in the teachers' pedagogical calculus discourse. Symbolism, for example, is a core characteristic in literate mathematical discourses and indeed, in calculus discourse.

3.3.3 Narratives

Narratives are sequences of utterances within the discourse 'framed as a description of objects, or of relations between objects or activities with or by objects, and that are subject to *endorsement* or rejection, that is, to being labelled *true* or *false'* (Sfard, 2007, p.572). In mathematical discourse, examples of endorsed narratives include mathematical theories, definitions, proofs and theorems (Sfard, 2008). In this study, the narratives are generally taken to refer to any utterances or propositions, written or spoken (including visual mediators), about discursive mathematical objects. The endorsed narratives are those utterances or propositions that the teachers (or students) consider as true in their calculus and mathematical discourses (Kim and Lim, 2017). In this study, an *utterance* is defined as 'a single, continuous oral [or written] communication of any length by an individual or a group' (Xu and Clarke, 2019, p.135). Thus, simply put, narratives are made up of utterances, and utterances are made up of words.

Mathematical narratives could be seen at either 'object level' or 'metal level' (Sfard, 2008). Object level narratives are about mathematical objects, for example, $\text{If } y = x^2;\ \text{then } \frac{dy}{dx} = 2x$ and the slope (gradient) of a straight line is constant. Meta-level narratives are propositions about the discourse itself, and about the activities of the teachers (and the students), rather than about its objects, which say how mathematics is done, for example, to find the gradient at a point on a curve, find the slope of the tangent to the curve at that particular point. Both object-level and meta-level calculus narratives are subject to analysis in this study.

3.3.4 Routines

According to the commognition theory, a routine is ‘a set of metarules [pattern-defining rules] that describe a repetitive discursive action’ (Sfard, 2008, p.208). Thus, routines are the repetitive patterns in the teachers’ (or learners’) actions, characteristic of mathematical discourse (Sfard, 2007). In this study, routines are defined as the different ways of doing, characteristic of the mathematical and calculus discourse in the teacher’s actions. Thus, a routine describes a process carried out by teachers such as defining, estimating, or proving in the course of constructing or substantiating narratives about mathematics objects (Zayyadi et al., 2019). Mathematical repetitive patterns can be seen in the teachers’ use of mathematical and calculus words and visual mediators in the processes of producing and substantiating narratives (Sfard, 2008) about, for example, the derivative. A routine explains the steps followed by the teachers to construct or substantiate a narrative about the definition of the derivative.

There are two subsets to a routine, the *how* of the routine and *when* of the routine, and these are important considerations for analysing teaching or learning in mathematical discourse. The *how* of a routine is a set of metarules that determine the course of action or procedure, and the *when* of a routine determines the situations in which the course of action or procedure is appropriate (Sfard, 2008, p.208). Wing (2011) explains mathematical understanding in terms of the *how* and *when* of routines:

> Children learning to participate in contemporaneous mathematical discourse have two distinct tasks: learning *how* to carry out a routine and learning *when*. Once a child has mastered both the *how* and the *when* of routines, we would tend to say that she ‘*understands*’ (p.367; my italics).

Sfard (2008) has advice for teaching mathematics:

> School teaching that focuses on the issue of ***how*** routines should be performed to the almost total neglect of the question of ***when*** this performance would be most appropriate, it is more likely to result in the discourse of rituals than of explorations” (p.223).

Teaching that focuses on standard rules of differentiation without adequate substantiation as to why those rules work, is more likely to produce students who cannot apply that knowledge to other situations. Rituals are not aimed at producing new narratives about the discourse. Rituals are discursive actions primarily aimed at pleasing others, for example, for a student to sustain a bond with the teacher (Sfard, 2008). Explorations, however, are aimed at producing new endorsed narratives; new to the student (Sfard, 2008). Thus, an exploration routine is aimed at producing ''historical facts' that are new to the learner or a new 'truth' about mathematical objects' (Nachlieli and Tabach, 2018, p.255). For example, a student that endorses the limit definition of the derivative is endorsing a well-known historical mathematics narrative, a new narrative to the student.

According to Sfard (2008), there are three categories of discursive routines: explorations, deeds and rituals. This study adopts Sfard's (2008) *explorations* for they are aimed at producing endorsed narratives. This study is concerned with how teachers create and substantiate endorsed narratives about mathematical and calculus discursive objects. Exploration routines could either be construction, substantiation or recall (Sfard, 2008). Mathematical routines such as numerical calculations, equation solving, and routines of defining or proving are examples of mathematical explorations (Nachlieli and Tabach, 2018).

Substantiation of a narrative is the process through which a narrative can be endorsed or rejected, for example, the production of proof, which is 'a sequence of endorsed narratives, each of which is deductively inferred from previous ones and the last of which is the narrative that is being endorsed'(Sfard, 2008, p.232).

An example that demonstrates construction routines is given by Viirman (2013). Viirman (2013) elaborates on the construction routine of a definition of a function, which he breaks down into construction by *stipulation,* construction by *exemplar* and construction by *contrast*. Construction by *stipulation* is when the teacher introduces a new concept or new object by means of definition or by stating 'a sufficient and necessary condition for an object to have a certain property' (Nardi et al., 2014, p.191). Construction by *exemplar* is when the teacher introduces a new object by illustrating its properties using an example. Construction by *contrast*, which could also be viewed as construction by *exclusion* is introducing an object using an example carrying a property, which should be excluded. Construction or definition by *exemplar* (Viirman, 2013) should not be confused with *saming* (Sfard, 2008, p.170); *saming* is when the interlocutor presents a series of examples, with the focus on the unifying common property in the examples (Nardi et al., 2014).

Nardi et al. (2014) explain the exploration routines - construction, substantiation and recall (Sfard, 2008) in the analysis of the construction of the function object. Nardi et al. (2014) elaborate on the construction of a definition of function further into 'construction by stipulation, by exemplar and by contrast' (Nardi et al., 2014, p.195).

Similar to Thoma and Nardi's (2016) description of exploration routines, in this study construction routines in the teacher's pedagogical calculus discourse are those aimed at producing new endorsable narratives (for example, teaching the limit definition of the derivative in elementary differential calculus); whilst the substantiation routines are those aimed at endorsing or rejecting previously constructed narratives (for example, differentiation from first principles); and recall routines are simply aimed at remembering endorsed narratives(for example, standard rules of differentiation).
Of particular interest in the analysis of the exploration routines in the teachers' pedagogical calculus discourse would be the study of the *how* of the routine, which explains the procedure or the course of action, and the *when* of the routine, which determines the situations when the procedure is necessary or applicable (Sfard, 2008).

3.4 Commognitive conflict and the learning-teaching agreement

Sfard (2007) associates learning with what she termed *commognitive conflict*, which results from a learner's encounter with new discourse governed by different '*meta-rules* from those according to which a student has been acting so far' (p.574). Commognitive conflict occurs when there are conflicting narratives coming from discourses that differ in their meta-rules, for example, from an expert (teacher) and a novice (student) in the discourse. This means that 'different discursants are acting according to different *meta-rules*' (p.574). An example that illustrates commognitive conflict is found in Nardi et al. (2014) study that reviewed nine commognitive studies. The study notes a commognitive conflict, a discrepancy between the 'lecturer's focus on the production, negotiation and ultimate endorsement of a certain narrative, and the student's focus on eliciting approval for the course of action that would lead to an acceptable solution to the task' (Nardi et al., 2014, p.195). Whereas the lecturer's approach (routine) was explorative, the students' approach (routine) was ritual. Although my study shall be focusing primarily on the teachers' explorative routines, the analysis shall consider commognitive conflict, where possible.

Commognitive conflict should not be confused with the acquisitionists' notion of cognitive conflict. Cognitive conflict, unlike commognitive conflict, arises 'in the encounter between one's belief and the world' (p.574). The notion of commognitive conflict is based on the premise that learning results from social interactions with others (Sfard, 2008). Resolving commognitive conflict results in learning taking place, i.e. change in the discourse of the discursants (Sfard, 2008). The notion of commognitive conflict is of particular relevance to my study, for it is a source of mathematical learning.

A ‘gradual mutual adjusting of discursive ways’ (Sfard, 2008, p.145) by the participants in the discourse is necessary for resolving the commognitive conflict. A resolution is conditional on a successful voluntary and mutual alignment of the discourses of the discursants, which Sfard (2008) describes as a learning–teaching agreement, often resulting in the student accepting and aligning with the discourse of the teacher as the more knowledgeable or the ultimate substantiator (Sfard, 2008). The problem ‘is resolved by choosing one of the two conflicting discourses and abandoning the other’ (Sfard, 2008, p.258).

For meta-level learning to take place, there is a need for the learner’s exposure to the new discourse which will necessitate communicational conflict, ‘one that arises whenever interlocutors differ in their uses of words, in the manner of looking at visual mediators or in the ways they match discursive procedures with problems and situations’ (Sfard, 2015, p.136). Sfard (2015) acknowledges that how to create the unwritten learning-teaching agreement is a question for further empirical research. The notion of commognitive conflict and the learning-teaching agreement add, ‘a valuable new dimension to the discussion of the role of a teacher within dialogic teaching of mathematics (Wing, 2011, p.368), and my study explores how teachers teach the derivative and seeks to understand how they resolve commognitive conflicts in calculus discourse.

3.5 Why the commognitive framework?

The commognitive theory provides a theoretical and analytical perspective (Zayyadi et al., 2019) in understanding how teachers of mathematics teach elementary differential calculus. The commognitive framework was originally developed for the study of thinking and learning (Sfard, 2007). I adopted the commognitive framework for the study of teaching. In this section, I will discuss the application of the commognitive framework for studying teaching. I argue that such a discursive framework allows for the study of, not only the discursive developments of individual students but also the discursive practices of the teachers.

The commognitive theoretical framework has been applied in the analyses of students learning more than it has been applied in the analysis of teaching. Sfard

(2008) does not make a distinctive definition of teaching. A commognitive definition of teaching is given by Tabach and Nachlieli (2016), who takes Sfard's (2008) view that communication is a form of activity and define pedagogy as 'the communicational activity the motive of which is to bring the learners' discourse closer to a canonic discourse' (p.299). Teaching mathematics is a communicational activity aimed at promoting the participation of students in (and admission into) the historically and collectively established mathematics discourse. Resolving the commognitive conflict through the mechanism of the unwritten learning-teaching agreement (Sfard, 2008) means that the teacher's and the learner's roles in a discourse are sympatric and jointly collaborative (Roth and Radford, 2011). Therefore, the applicability of the commognitive framework to the study of teaching lies in the interconnectedness of teaching and learning. Thus, my research adopted the commognition theory as a conceptual framework for the study.

My research explores the *teaching* of elementary differential calculus by examining the discursive patterns in the teachers' pedagogical calculus discourse, and the 'quest for discursive patterns is the gist of commognitive research' (Sfard, 2008, p.200). In my study, the commognitive framework provides the lens for noticing patterns of behaviour and communication in the teachers' pedagogical calculus discourse, and the vocabulary to describe the patterns.

> Mathematical discourses are made distinct by their tools, that is, *words and visual means*, and by the form and outcomes of their processes, that is, the *routines and endorsed narratives* that they produce (Sfard, 2008, p.161).

An analysis of mathematical discourse should focus on the tools, form and outcomes of their process. The *words* and the *visual mediators*, the *routines* and the *endorsed narratives* are what characterise mathematical discourses. These four commognitive theoretical constructs form the core of the conceptual and analytical framework for my study.

> Literate discourses (…) were defined as visually mediated mainly by symbolic artefacts. Along with algebraic symbols, symbolic artefacts include icons, such as conventional or individually designed diagrams, graphs, and other drawings. Students' fluency in this kind of discourse is the goal of school learning (Sfard, 2008, p.146).

Students achieving fluency in mathematical literate discourses is the object of school learning and teaching activities, and the commognition theory offers a framework for analysing not just learning, but teaching forms of doing. By drawing attention to the type of words, symbolic artefacts and algebraic symbols and the visually mediated narratives and routines in the discourse, the commognitive theoretical framework offers a micro-level communicational analysis that ‘captures fine-grained aspects of interactions’ (Nardi et al., 2014, p.185) in the classroom, which generates mathematically rich accounts of data.

I also chose the commognitive theoretical framework for its flexibility in application, which is aptly described by Presmeg (2016) who argues that the commognitive framework has unrealised potential.

> But embracing, as it does, both individual and collective learning of mathematics, and indeed its teaching too, the commognitive theoretical framework still has much unrealized potential to be useful in mathematics education research at all levels (Presmeg, 2016, p.430).

Ioannou (2018) echoes Presmeg (2016) by acknowledging that although CTF has been used extensively within mathematics education, only part of it has so far been operationalised. The breadth and depth of the concepts and principles within CTF allow for a much wider application of the framework in various studies. Its potential as a theoretical framework goes beyond investigating issues of teaching and learning mathematics, to investigating issues of human development.

The CTF is a communicational approach to research that can serve as ‘a conceptual as well as discourse analysis framework’ (Park, 2016, p.396). Other researchers, for example, Park (2013; 2015; 2016) in her studies on derivative has applied the CTF as a conceptual framework as well as an analytical framework. My study investigates how teachers of mathematics introduce differential calculus, i.e. the teaching of the derivative, thus, an analysis of the teachers’ mathematical discourse on the derivative. My study, therefore, applies Sfard’s (2008) communicational perspective as a theoretical lens for research and as a discursive analysis framework for analysing the teacher’s calculus discourse in terms of *word use, visual mediators, narratives* and *routines*.

3.6 Commognitive studies and analyses

In this section, I review the literature that informed my adoption and application of the commognitive framework for my study. There are two main reasons for my review of some of these studies: (i) to draw insights from their commognitive analyses of word use, narratives, communication mediators and routines, and (ii) to examine their findings relevant to my study, e.g. on the teaching and learning of differential calculus. My study explores the teaching of the derivative by examining the teachers’ pedagogical calculus discourse.

Since Sfard's (2008) commognitive framework in mathematics education research, there are many studies (e.g. Ryve et al. (2012), Kim et al. (2012), Ioannou (2012), Güçler (2013; 2016), Park (2013; 2015; 2016), Morgan et al. (2014), (Nardi et al., 2014), Viirman (2014; 2015), Sfard (2015), Heyd-Metzuyanim and Graven (2015), Ng (2015), Tabach and Nachlieli (2016), Presmeg (2016), Wood (2016), Ioannou (2018) and Xu and Clarke (2019)) that have adopted the theory of commognition and applied discursive approaches to researching mathematical discourse and classroom discourse. However, there have been more commognitive studies focusing on students' discourses (i.e. learning) than there are on teaching discourses. For example, Nardi et al. (2014) review nine commognitive studies investigating the learning and teaching of mathematics at the university level. In this review, they investigate discursive shifts in the lecturers' and students' discursive practices, construction routines and resolving of commognitive conflicts. The discursive shifts refer to changes in the mathematical perspectives of the participants in the mathematical discourse, i.e. the lecturers and students. Out of the nine studies that Nardi et al. (2014) reviewed, six studies focus on learning, thus on the discourses of students (e.g. Bar-Tikva, 2009; Kjeldsen and Blomhøj, 2012; Remillard, 2010; Ryve et al., 2013 and Ioannou, 2012), and two concern pedagogical discourses (e.g. Nardi, 2011, Viirman, 2013) and one looks at the interactions between the lecturers and students (e.g. Güçler, 2013). More recently, a study by Heyd-Metzuyamin and Shabtay (2019) examined identity narratives of good mathematics teaching by analysing Exploration Pedagogical Discourse (EPD) (which is aligned to ideas of learning as participation in explorations) and 'the Acquisition Pedagogical Discourse (APD, akin to 'traditional' or 'teacher-centred' instruction' (p.542).

The commognitive studies focusing on mathematical and pedagogical discourses (e.g. Viirman, 2013; Nardi, 2011; Park, 2015) have all looked at the discourses of lecturers, thus at the university level. Although my study focuses on mathematical and pedagogical discourses, it explores the discourse of schoolteachers; it is an investigation of the teaching of elementary differential calculus by schoolteachers.

Viirman (2015) examined the routines of the teaching, the pedagogical routines of 'explanation, motivation and question posing' (p.1167) by university mathematics lecturers teaching functions in first-year mathematics courses in Swedish universities. With a particular focus on the discourse of mathematics teaching, Viirman (2015) makes an extension to Sfard's (2008) types of routines by further categorising routines into mathematical routines (Viirman, 2014) and didactical routines (Viirman, 2015) for producing didactical narratives. This is an important contribution by Viirman (2015) because the *how* of a teacher's mathematical routine is dependent on the teacher's didactical objectives. There is an interplay between a teacher's mathematical routine and didactical routine. Research about didactical routines can provide means for analysing, and vocabulary for describing, the discursive practices of teachers. My analysis shall also consider and evaluate the teachers' didactical routines (Viirman, 2015) as either pedagogical discourses of explorations or acquisition (Heyd-Metzuyanim et al., 2018). In my study, the mathematical and didactical routines of the teachers shall be examined as pedagogies on the derivative, primarily focusing on the exploration routines in the teachers' calculus discourse.

In their study on communication in group work in mathematics education Ryve et al. (2012) look at the commognitive constructs of visual mediators (Sfard, 2008) and technical terms (Mason, 1998; Ryve et al., 2012; Wertsch and Kazak, 2011). By a technical term, Ryve et al. (2012) mean a term that typically belongs to, and has a specific meaning within, a discourse, for example, mathematics discourse. Thus, mathematical technical terms are specialised mathematical words found in literate mathematical discourse. According to Mason (1998) quoted in Ryve et al. (2012), 'each technical term marks a particular way of seeing' (p.252). The type of words used in a discourse, say mathematical discourse in the classroom, could potentially influence how students participate in the discourse, for example, the way the students perceive, talk about, or make sense of a phenomenon (Ryve et al., 2012). Ryve et al. (2012) found that 'the critical evaluation of visual mediators and technical terms, and of links between them, is useful for researchers interested in analysing effective communication and designing environments providing opportunities for students to learn mathematics' (p.497). My study will examine the type of words and the visual mediators in the teachers' pedagogical calculus discourse.

Güçler's (2013) commognitive study looks at the mathematical discourses of both the lecturers and undergraduate students on elementary calculus). Güçler (2013) examines the development of trainees and practising teachers' discourse on calculus. Güçler (2013) examines the teaching of calculus that focuses on definitions of function to college students using the communicational framework of commognition. Güçler's (2013) study focuses mainly on the metalevel rules of discourse as the core element in its conceptual framework. In the study, the instructor explains the meta-rules for the discourse of calculus, and thus meta-level learning was expected. The study found out that explorative discussions on the construction of the definitions of function fostered meta-level learning, as well as object-level learning, which resulted in a change in the meta-rules governing the teachers' discourse on functions, for example, the teachers were observed using the word function whilst performing explorative activities. Güçler's (2013) study highlights some commognitive conflict between the discourses of the lecturers and the students, for example, about the narratives substantiating the limit as a number and limit as a process (Nardi et al., 2014). This study highlights the 'learning as a change in discourse' perspective (Sfard, 2008; 2015), that is, changes in students' discourse on the definitions of function. Unlike Güçler (2013), my study examines the teaching of calculus that focuses on introducing the derivative to school students using elements of the commognition framework. Similar to Güçler (2013), how explorative the teachers' discussions on the construction of the definition of the derivative, is of interest to my research.

Ng (2015), in her study, highlights the learning-as-participation perspective (Sfard, 2008, 2015). Ng (2015) applies elements of the commognitive framework to studying bilingual high school students using dynamic geometry learning about derivatives and antiderivatives. The study investigates bilingual students' verbal discourse about area accumulating functions. The study demonstrates the interdependence of gestures, verbal discourse and dragging with dynamic imagery activities in a dynamic geometry environment. In this study, the students are seen dragging visual objects, and such dragging activities are later used to explain conjectures. This demonstrates a shift in routines from deeds to exploration (Presmeg, 2016). Ng's (2015) study highlights the need for my study to pay attention to the non-linguistic forms of communication in discursive research.

Although Ng's (2015) study adopts the commognitive constructs of word use, visual mediators and routines (Sfard, 2008), the conceptual framework also includes communicational actions such as the use of gestures, dragging, and diagrams. Commenting on the merits and affordances of the commognitive framework in analysing learning activity, Presmeg (2016) praises Ng's (2015) commognitive study: 'the power of the commognitive framework—including gestures and dragging as communicational acts—is amply illustrated in this study of bilingual learners (Presmeg, 2016, p.428). Unlike Ng's (2015) study which focuses on students, my study focuses primarily on teachers. However, just like Ng's (2015), my study seeks to harness 'the power of the commognitive framework' to explore how teachers introduce the derivative. My study will pay attention to the gestures and dragging actions in the teachers' pedagogical calculus discourse, with a particular focus on their use of dynamic geometry.

Park (2013) applied the commognitive framework in the analysis of students' word use and use of visual mediators on the derivative. Park (2013) conducted a survey in which twelve elementary calculus students explained their solution process during interviews. Park (2013) found inconsistencies in the students' use of the word derivative for describing the derivative at a point and for describing the derivative function; most students used the word, 'derivative' for both 'the derivative function' and 'the derivative at a point' (p.624). According to Park (2013) such problematic word use for 'derivative' could be rooted in the English language:

> In English, the relation between 'function' and 'function at a point' is equivalent to the relation between 'the derivative of a function' and 'the derivative at a point.' This equivalency often allows 'derivative' without 'of a function' to be used as 'the derivative of a function' (e.g. 'Is the derivative positive?'). However, in some other languages such as Korean and Japanese, the terms for these two concepts do not share a common word, and thus there is no confusion between the terms (p.624).

The analysis also found students describing 'the derivative as a tangent line', which suggests that the students 'considered the 'derivative' as a point-specific object but also a (linear) function defined on an interval' (p.624). Studies (e.g. Monk, 1994; Tall and Vinner, 1981; Park (2013) on students' learning calculus have shown that moving from (and between) the derivative at a given point on a curve to the derivative of a function is not simple for students. Studies by Tall and Vinner (1981) and Tall (1986) show how students' discourses about the limit on the difference quotient and the tangent lines to a curve, can be inconsistent with mathematical literate discourse, for example, the thinking that 0.99999 . . . never reaches 1, is consistent with their thinking about local straightness and the tangent, for example, that the secant lines (as $h \to 0$) never reach the tangent line (Park, 2013, p.624).

Park (2016) used Sfard's (2008) communicational approach to explore how the derivative is introduced in calculus textbooks. Park (2016) examined the learning and teaching of the derivative at a point and derivative as a function and the connections between them in three widely used USA undergraduate calculus textbooks. Park's (2016) study focused on the realisations of the derivative-at-a-point and realisations of the derivative-of-a-function and analysed how the derivative is realised as a point-specific object and as a function in calculus textbooks, thus, by mathematics experts. Thus, the study investigated the mathematical discourse in written words rather than the spoken words since it is an analysis of textbooks. However, analysing the discourse, as expressed in a textbook, gives insights into the experts' discourse about mathematical objects. The study found some inconsistencies in realisations of the limit process and the limit object, and the derivative process and the derivative object. Park’s (2016) study highlights some ambiguity and inconsistencies in word use within some calculus textbooks, with words as limit and derivative.

Realisations for the derivative as a constant and the derivative as a function were mediated with similar symbolism, which suggest some difficulty with understanding them as different objects (Park, 2016). The most commonly used visual mediators for the derivative were graphs of tangent lines for the derivative at a point and the symbolic - algebraic expression for the derivative of a function. However, Park (2016) observed a disconnect in the way the mediators were used for both situations and argues against the disjoint use of different visual mediators for a process/object, as the constant shift between the various forms of mediations could possibly present challenges to students as newcomers to the mathematical discourse. Thus, one wonders how the learners could possibly make the connections between the mediators if it is not explicit in the textbooks. Park (2016) goes on to argue that:

> For this reason, in the realisation of the derivative, the consistency between the mediation of process and the mediation of the object with one visual mediator, and the explicit transformation across multiple visual mediators that realize the process and object are both important (p.399).

For meta-level learning to take place there is a need for consistency and clarity between the forms of mediation used for the process and the object realisations. The study demonstrates the affordances of a commognitive framework for analysing the discourse in a textbook. Through the commognitive constructs, Park was able to reveal subtle inconsistencies in various presentations of the derivative.

In another study, Park (2015) investigated three calculus instructors’ classroom discourse on the derivative with the commognitive lens, and how they introduced the derivative as a point-specific value and as a function. Park (2015, p.233) made the following four observations:

(a) the instructors frequently used secant lines and the tangent line on the graph of a curve to illustrate the symbolic notation for the derivative at a point without making explicit connections between the graphical illustration and the symbolic notations,
(b) they made a transition from the point-specific view of the derivative to the interval view mainly by changing the literal symbol for a point to a variable rather than addressing how the quantity that the derivative shows, changes over an interval,
(c) they quantified the derivative as a number using functions with limited graphical features, and

(d) they often justified the property of the derivative function with the slope of the tangent line at a point as an indication of the universality of the property.

These findings show that lack of explicit connections between the features of the derivative at a point and as a function. Failure to make such connections would exacerbate students' challenges with the derivative and calculus. Making explicit these aspects and connections through word use and visual mediators with symbolic, graphical, and algebraic notations in these three classrooms would help students to understand, for example, the graphical and symbolic mediation forms for the derivative as a limit. Park (2015) concludes:

> These results showing the instructors' uses of various visual mediators without explicit connections between them, their limited discussion on how the derivative as a function varies, and their dependence on symbolic and algebraic notations, seems related to some well-known student difficulties with the derivative (p.248).

Park's (2015) study focused on the derivative of a function as an object and the transition in the teachers' pedagogy between the derivative at a point and the derivative as a function. The study specifically looked at two research questions: 'In what ways did the instructors [i] address the derivative as a point-specific value? [ii] address the derivative as a function on an interval?' (Park, 2015, p.234), at the post-secondary level in the USA. My study looks at how teachers introduce differential calculus to secondary school students, in English schools in the UK.

Park's (2015) study also highlights how differences in word use in different languages could be an important consideration in the analyses of teachers' classroom discourse. Park suggests that the ambiguity of views between the derivative as a constant and derivative as a function could be coming from colloquial use of the word derivative in English (e.g. 'Is the derivative positive?'). In languages such as French, Japanese, and Korean, the mathematical discourses do not use the same words for 'the derivative at a point' and 'the derivative function' (Park, 2015, p.249). Similarly, Kim and Lim (2017) compare the students' use of the notion and word limit in Korean and English contexts. Unlike in the English language and context, where the word limit is used in both colloquial contexts and mathematical tasks, 'the mathematical word for limit is not commonly used as a colloquial word in Korean' (Kim and Lim, 2017, p.1561).

Similarly, Xu and Clarke's (2019) study highlights the role of spoken mathematics and the need for a consideration of cultural contexts in research on both teaching and learning. Xu and Clarke (2019) examined the effects of classroom dialogue on learning from nine classrooms situated in East Asia and argue that overemphasis on vocal communication could ‘undermine other forms of communication in classrooms (…) that could be generative of student learning’ (p.130). It is important for teaching and commognitive research to pay attention to the role of culture and non-verbal forms of communication in classroom discourse. My study will examine the visual and communication mediators, both vocal and verbal, including word types, narratives, and visual mediators in the teachers’ pedagogical calculus discourse.

Although the studies reviewed here, use the commognitive theoretical framework (Sfard, 2008), they do not always adopt the theory in its entirety nor exclusively adhere to it, and my study is no exception in this regard. This is not an unusual approach in mathematics education research. Presmeg's (2016) review of the application of the commognitive framework by various researchers shows that although many researchers have adopted and applied the commognitive framework in their studies, there is evidence to show some digression from the framework in some parts of their study. For example, Güçler (2013) adopts the commognitive framework without its theoretical entailments and Wood (2016) also adopts the commognitive framework categories of *routines* (i.e. deeds, rituals, and explorations) for data analysis, but calls them *categories of activity* (Presmeg, 2016). In my study, combinations of the graphical, symbolic and numerical visual mediators, for example, in the teachers’ pedagogical calculus discourse are further described as multiple representations.

3.7 Conclusion

Using the commognitive analytic toolkit of word use, visual mediators, narratives and exploration routines, allows for a discursive analysis of the mathematical calculus discourse against the teachers’ pedagogical calculus discourse. Examining the word use and narratives, the mathematical and didactical routines, and the symbolism and visual mediators in the teachers’ pedagogical calculus discourse can help mathematics education researchers to understand how teachers teach calculus. The commognitive framework is a useful tool in communicational research. My study is an analysis of the teachers’ mathematical discourse on calculus. Hence, the commognitive framework (Sfard, 2008) was adopted as the conceptual framework for this study for investigating how teachers introduce differential calculus at school.

The next chapter introduces and explains the research methodology adopted for this study.

Chapter 4 Research design

4.1 Introduction

My research is an exploratory qualitative study of the practice of teaching elementary differential calculus through the commognitive lens. It explores the teaching of the derivative by teachers of mathematics, engaging in pedagogical activities within the social context of their classes, schools and the education community at large. Two methods of collecting data are used: interviews and observations. Qualitative research is adopted for its naturalistic approach that prefers to study people, things and events in their natural settings (Denzin and Lincoln, 2011). Punch (2009) describes qualitative research methods as a 'complex, changing and contested field – a site of multiple methodologies and research practices' (p.115). Thus, qualitative research is very diverse and encompasses an enormous variety. Social scientists have always argued about the best research approach for social research (Denzin and Lincoln, 2011) and education and applied social sciences are hugely varied and complex. The diversity, which has long been a dominant feature of qualitative research often concerns paradigms, designs, approaches to data and data analysis methods (Stake, 2006).

My qualitative study follows the description by Creswell (2013) that qualitative research begins with assumptions (e.g. mathematics is a discourse; teaching and learning are social activities) and adopts a theoretical framework (i.e. the commognitive framework, see Chapter 3) that informs the study of the research problem. Data is collected in its natural settings relative to the objects, people and places under study (i.e. interviews with teachers and observations of their lessons in their classrooms and schools). Data analysis can take both inductive and deductive approaches to establish themes from the content of the data (see Chapter 5 for the qualitative data analysis for this study). The findings and the final written report 'includes the voices of participants, the reflexivity of the researcher, a complex description and interpretation of the problem, and its contribution to the literature or a call for change' (Creswell, 2013, p. 44) (i.e. Chapters 6 to 10).

The design for my research is predominantly interpretivism, which adopts the premise that there exist multiple and subjective realities and meanings or understandings, relative to, and dependant on the situation, context and time, which are co-constructed by the researcher and the participants (Stake, 1995, 2006; Yin, 2014). This ontological and epistemological stance is consistent with the theoretical framework for the study, the commognitive framework (Sfard, 2008), which takes a discursive and interpretative approach to research. From engaging with literature, the commognitive framework (Sfard, 2008) was identified as an appropriate guiding theory for my study (see Chapter 3 for an explanation of the framework).

There are two chapters. Chapters 4 and 5, cover the methodology for this study. Chapter 4 is predominantly on data collection, whereas Chapter 5 focuses more specifically on data analysis. For this qualitative study, a single methodology chapter would have been rather too long. What follows in this Chapter 4, are six sections covering the revised research questions; the pilot study; the data collection methods; the recruitment of participants; the research ethical considerations and a summary of the research design. For the data analysis methods for this study, see Chapter 5.

4.2 Research questions

One of the main notable outcomes of my review of literature was that I was able to review my prime facie research questions (as stated in chapter 1) and align them with the commognitive terminology. The reformulated research questions that then guided the design for this research are given below. Note that the words in italics are the commognitive constructs from the commognition theory (Sfard, 2008).

In teaching differential calculus:

RQ.1 What *word types* and *narratives* do teachers use and why?
RQ.2 What *visual mediators* do teachers use and why?
RQ.3 What mathematical and pedagogical *routines* do teachers use and how?

RQ1 calls for an examination of the teachers' word use, utterances and language on elementary differential calculus. RQ2 is an extension of RQ1 and seeks to investigate the teachers' use of visual mediators and forms of mediation in teaching the derivative. RQ3 supplements RQ1 & RQ2 by drawing attention to the mathematical and pedagogical routines in the teachers' calculus discourse. This study is primarily focusing on the exploration routines, thus, investigating the teachers' various forms of doing in constructing and substantiating narratives about the definition of the derivative, by examining the teachers' word use, utterances and narratives (RQ1) and visual mediators (RQ2) in their pedagogical calculus discourse on elementary differential calculus.

Gathering data for the above research questions call for the need to talk to the teachers and to see them in action too. Data for the study was collected through interviews (more in Section 4.3.1) with teachers of mathematics and observations (more in Section 4.3.2) of their lessons on elementary differential calculus.

4.3 Pilot study

I carried out a pilot study to trial the research design with the following objectives: to test and refine the data collection tools and procedures; to test access and recruitment of participants; and for personal development. The pilot study involved one participant teacher of mathematics. The study applied the data collection methods that had been designed for the study. A pre-lesson interview with the teacher was carried out. This was then followed by an observation of the teacher teaching a lesson on differential calculus. Another interview followed the lesson observation, in which the researcher and the teacher talked about the lesson. The interviews were audio recorded and the lesson was video recorded. Not only was a pilot study necessary, but imperative, as argues Roulston (2016, p.75), to 'subject [my] own participation in research interviews to analytic consideration' as this necessitates reflexivity allowing for a reflection and consideration of my role and actions in the data collection and the research process.

Not all communication, even with interviews, is verbal. At the time of data collection for the pilot study, I did not realise the importance of noting non-verbal communication. For the main study, alongside audio recording the interview, I then planned to take some notes to capture the non-verbal communication or behaviour of the participants.

During the interview, the teacher referred to one of the main textbooks he used for teaching calculus, and he pulled the book off the shelf to show me. With the teacher's consent, I took a picture of the book using my smartphone. The interview was audio recorded and I had not anticipated the need for photography (or videos) for the interviews and so I had not planned for it. However, using my mobile phone was handy and Cohen et al. (2013) agree that a smartphone offers a potentially powerful tool for researchers.

The pilot study exercise made me realise the resource implications in terms of time and training needs. Although the data collected was from just one participant teacher, there was a lot of data from the interview audio recordings and the videos from lesson observations. I realised that I had gathered a substantial amount of data, enough data to write a research paper for publication. I realised how time-consuming was data preparation, processing and analysis in qualitative research. I spent at least a couple of weeks transcribing interview audio data into text, but I still had more audio files to complete. I started coding (open coding) the transcripts from the interview data, but the data analysis process could not be completed before the data collection phase for the main study, given that the pilot study data collection had been done in July. Thus, the pilot study data analysis was rather more of a training exercise for data analysis. Nevertheless, the pilot study methods were instrumental in reshaping the research methodology for the main study.

The pilot study made me realise my training needs in data analysis. Following the pilot study, I undertook some training courses in using Computer-Assisted Qualitative Data Analysis Software (CAQDAS) (Thomas 2013; Roulston, 2014), in particular, NVivo. I had always intended to use traditional methods - manual coding methods for data analysis. The pilot study data made me rethink my plans. Manually coding and analysing data from nine participants (the planned sample for the study) could be overwhelming. Knowing some CAQDAS could turn out to be an invaluable backup plan.

The timing for participant recruitment is a very important issue in this study. Recruitment has to coincide with the autumn term, during which time many teachers tend to cover lessons on calculus. I had planned to carry out the pilot study in the spring term of 2015. However, my ethical review application took a lot of time to put together and that delayed my pilot study. Although the ethical review process for the pilot study was quite time-consuming, it was a successful exercise. The pilot study exercise was very helpful in that it enabled me to adapt the research methodology in light of the lessons from the study.

4.4 Data collection

Gathering data for the study involved talking to individual participant teachers about their teaching of calculus and observing the individual teachers' lessons on introducing differentiation to 16 -18-year-olds at school. Thus, data collection for this study involved the use of both interviews (see Section 4.3.1) and observation (see section 4.3.2). Concurrent with the interview and observations, the researcher made fieldnotes to supplement the audio and video data from the interviews and observations, respectively. The interview would enable the researcher to ask questions and listen to and record the participant teacher's answers in an in-depth manner (Jones, 1985). The observation would enable the researcher to observe and record the teaching activities within the social contexts of the individual teachers' classrooms.

Some critics raise issues about the validity of interview data, such as 'the possibility of interviewer bias and effects, the accuracy of respondents' memories, people's response tendencies, dishonesty, self-deception and social desirability' Punch (2009, p.153). An even more difficult challenge with interview data relates to 'the relationship between what people say, what they do and what they say they do, and the assumption that language is a good indicator of thought and action' Punch (2009, p.153). Fielding (1996) calls for careful research design and planning to counter such technical issues. Thus, a methodological triangulation (Denzin, 2005) was necessary for this research design. Advocates for methodological triangulation (e.g. Thomas, 2013) argue for the need to have alternative kinds of evidence corroborating with each other. There are some critics though, (e.g. Fielding, 1986) who argue that triangulation does not necessarily increase validity or bring objectivity to research. Nonetheless, triangulation, according to Lincoln and Guba (1985), can be seen as a check on data. The methodological triangulation adopted for this study would enable the researcher to understand whether the teachers do what they say they do, and to corroborate the evidence from the interviews and the observations.

What teachers say about their teaching and what they do whilst teaching in the classroom, do not always match up. In other words, *what is taught* is not always the same as *what is to be taught*. From my personal experience in teaching which spans over two decades, both as a teacher and teacher educator, I have observed hundreds of lessons. Many teachers do not necessarily stick to their lesson plans in the actual lessons; the social context within the classroom controls how the teacher conducts the lesson. Various factors can constrain the teaching activity during a lesson in the classroom. It would be necessary, where possible, to discuss the observed lesson with the teacher. There are three phases to data collection in this research design, see Figure 4-2, which shows the sequencing of the phases, a pre-teaching interview (Phase1) is followed by a lesson observation (Phase 2), which is in turn followed by a port-lesson interview (Phase 3) with the teacher.

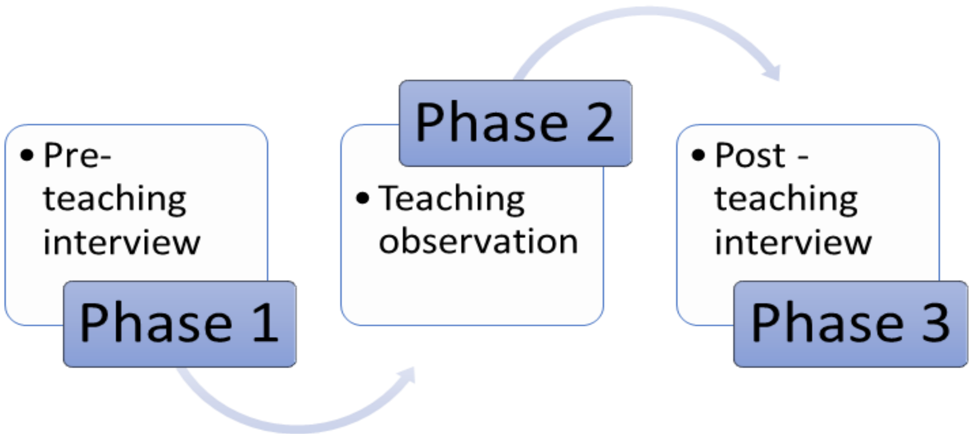


Figure 4.1 Phases of data collection

Observing the teacher in action in the classroom, practically teaching an introduction to differential calculus lesson, would provide some significant data and evidence against which the teacher's teaching claims can be examined. Thus, the data from the lesson observations were complemented with the data from the interviews with the teacher. Combining observational and interview data collection techniques is regarded as a good strategy in qualitative research for it can ‘lead to very rich, and high-quality data’ (Punch, 2009, p.156).

4.4.1 The teacher interviews

The interview is regarded as the most commonly used (Roulston, 2014) and the most powerful data collection instrument (Punch, 2000) in qualitative research. Jones (1985) suggests that to understand other people, to access their perceptions, meanings, definitions of situations and construction of reality, it is best to ask them. Cohen (2018) argues for the flexibility with semi-structured interviews for capturing interviewees’ perspectives, opinions and attitudes. Data collection for this research included the use of the semi-structured interview. According to Thomas (2013), semi-structured interviews allow some standardisation to facilitate conformity between the participants and provide some structure for the interview whilst allowing the interviewee to give elaborated responses and the interviewer freedom to follow up points, as necessary. For this study, two interviews were planned for each participant teacher. The first interview aimed at gathering the participant teachers’ biographical data and background information, and discussing the teacher’s plans for introducing the derivative. The second interview was set to seek more insights into the teachers' approach to teaching the derivative. It was a follow-up interview on the lesson to discuss the observed teaching activity. Teachers are often very busy, and the interviews were scheduled to last for no more than an hour. The interviews were audio recorded. In addition to the audio data, the researcher made field (interview) notes to supplement the audio data.

Pre-teaching (Phase1) semi-structured interviews with the individual participant teachers enabled the gathering of the participant teachers’ biographical data and background information, such as teaching experience, subject knowledge, teaching resources, school context, the school curriculum, and examination boards, their lesson planning and teaching practices. The interview focused primarily on the lesson to be taught, i.e. on the teacher’s plans for introducing the derivative, rather than merely talking about how the teacher teaches calculus in general. An interview schedule was designed for the pre-teaching interview (see Appendix C1). The interview schedule is a framework with a mix of the main questions to be asked, possible follow-up questions and probes. According to Rubin and Rubin (2005), such a mix helps to structure interviews that would ‘elicit depth, detail, vividness, nuances, and richness’ (p.134). Thus, an interview schedule is intended to remind the interviewer of the key issues and aims of the interview, and not to constrict the interviewer; it is not set in stone. The main questions focus on the research problem deriving from the research questions. The probes are meant to help manage or direct the conversation as an encouragement to the interviewee (Thomas, 2013). Follow–up questions are important for getting more depth and understanding about an idea or concept or an issue suggested by the interviewee that is of interest to the research concerns (Rubin and Rubin, 2005). These are usually worked out during the interview and asked on the spot.

A post-teaching (Phase 3) semi-structured interview followed the observed lesson to ask any follow-up questions on some issues arising from the lesson, as well as to capture the teacher’s thoughts and evaluation of their lesson. The observation schedule from Phase 2 served as the interview schedule for this post-teaching interview with the teacher. Thus, it guided the researcher in the follow-up dialogue with the teacher. Again, the post-teaching interviews were audio recorded.

The interviews were carried out face-to-face and audio recorded. As Lincoln and Guba (1985) remind us, there is no consensus in the literature on the best way for recording interview data. Whether audio, video recording or note-taking is the best method for recording the interview, depends on what is the ‘best fit for purpose’ given the type of interview chosen and any practical constraints (Punch, 2009). A disadvantage of using the audio recording for the interviews is that it does not capture non-verbal communication. With a face-to-face interview, the researcher can ‘watch and listen for nuances of their [interviewee] behaviour’ (Thomas, 2013, p.194), which would give the researcher ‘important clues about how they feel about the topic’ Thomas (2013, p.194). In conjunction with the electronic audio recording, note-taking to record such non-verbal clues was used for the interviews.

4.4.2 The lesson observations

Observation is regarded (e.g. Thomas, 2013; Punch, 2009) as one of the most commonly used and important ways of data collection in social sciences and educational research. According to Foster (1996), two main practical issues need clarifying here: approaching observation and recording of observational data. Observation approaches can either be structured or unstructured; participatory or non-participatory. Thomas (2013) and Punch (2009) both point out that irrespective of such distinction often given in literature, some combinations of the two approaches are possible. For example, Thomas (2013) gives an illustration of such combinations, what he calls a ‘continuum observation and participation’ (p.221). Other examples include the framework of Adler and Adler (1994), which describes three membership roles for the observer, and that of Wolcott (1988) which distinguishes between a privileged observer and a limited one.

For this study, the lesson observations followed the pre-teaching interview. The researcher observed the participant teachers in action in their respective classrooms, teaching an introduction to differentiation lesson. The lesson observations were video recorded, to capture data on the teaching activity, the interaction of the teaching and learning material resources, teacher actions, and the curriculum context. The recording was done using a fixed video camera set in the classroom and a wireless microphone connected to the video camera and attached to the teacher. Plans were also in place to use the audio recording of lessons together with some snapshots from the lesson if ever needed. This was meant to cater for some participants, schools and teachers who might not be comfortable with the presence of a fixed video camera recording in the classroom.

For this study, the observation combined elements of unstructured and participatory observation. Unlike structured observation in which the researcher has specific pre-defined behaviours, predetermined categories and classification of data (Punch, 2009; Thomas, 2013), with unstructured observation the researcher would observe and record the actions or behaviour and events as they unfold naturally. Punch (2009) argues that the logic here is that:

> … categories and concepts for describing and analysing the observational data will emerge later in the research, during the analysis, rather than be brought to the research, or imposed on the data, from the start (p.154).

For this study, the observation schedule (see Appendix C2) was a blank template with three columns for recording time, some lesson observation notes, and points for discussion in the post-lesson interview. To complement the video data, the researcher made some fields/lesson notes on the observation schedule for further follow-up in the post-lesson interview. The researcher was guided, directly or subconsciously, by the research questions in identifying the focal points during the lesson observation. The notes on the observation schedule were then used as prompts for the discussion in the post-lesson interview.

Observation need not be seen as one extreme end or the other, the degree of structure or participation could vary, depending on the purposes and context of the research. Unstructured observation involves some participation by the researcher in the social situations in which they are collecting research data, though the degree of participation can vary. According to Burgess (1982), participant observation could involve talking to people, taking notes, watching and anything that helps the researcher to get a deep understanding of the situation.

4.5 Sampling

A ‘purposive sampling’ (Punch, 2009, p.162) approach, guided by the aims and the research questions of the study was applied. Purposive sampling is a non-probability sampling in which the aim is to sample participants strategically, ‘selecting participants based on the kind of information they can provide, usually, because they have the right kind of life experience for the research in question or because they are expert in a certain field’ (Bryman et al., 2021, p.379). In this study, this involved looking for the most appropriate participants who were ‘best-fit’ for the required participants' characteristics for the research project (Bryman et al., 2021). To be eligible, one had to be a teacher of A-level mathematics who was going to teach elementary differential calculus, in particular, the introduction of the derivative. This latter requirement was particularly important so that the researcher could observe the participant teachers teach their introductory lessons on the derivative.

The target population was all teachers of mathematics from colleges or secondary schools teaching A-level mathematics in England, although the research could be done with participants in any country where elementary calculus is taught at school. For convenience and feasibility, the participants for the study were drawn from teachers of A-level mathematics teaching elementary differential calculus in secondary schools or colleges in the north of England. This research project was carried out in England.

The sampling plan was checked out against some of Miles and Huberman's (1994, p.34) six general questions about the qualitative sampling plan:

> Is the sampling *relevant* to your conceptual frame and research questions?
> Will the phenomena you are interested in, *appear*? In principle, can they appear?
> Does your plan enhance *generalizability* of your findings, either through conceptual power or representativeness?
> Can *believable* descriptions and explanations be produced, ones that are true to real life?
> Is the sampling plan *feasible*, in terms of time, money, access to people, and your own work style?
> Is the sampling plan *ethical*, in terms of such issues as informed consent, potential benefits and risks, and the relationship with informants?

The sampling plan was to recruit a total of nine participant teachers with varying teaching experiences. Teaching experience here refers to the number of years the teacher has been teaching A-level mathematics. The plan was for the sample to include three categories depending on the number of years of teaching post-16 mathematics lessons, thus, teachers with less than 3 years, with 3 to 5 years and with more than 5 years of teaching A-level mathematics. For a purely qualitative research design, in which data collection would be done through interviews and lesson observations, a sample of nine participant teachers was considered neither too small nor too big a number.

Headteachers are the gatekeepers to gaining access to their schools and teachers. Headteachers were approached initially via email introducing the research and the research project. At least a hundred emails were sent to secondary schools and 6th form colleges offering A-level mathematics across the north of England. Any positive response was then followed up with an email with the information sheet about the research and consent forms (see Appendix B1). If access was given by the gatekeepers, emails with the information sheets about the research and consent forms were then sent to the teachers inviting them to participate in the research project (see Appendix B2).

Gaining access to schools and recruiting participants was so challenging that not enough teachers were recruited in the first year of data collection. This was a huge drawback to the project as I had to wait for the start of the next school year to collect the required data. I needed to recruit enough teachers willing to participate in my research project before the start of the school year by the beginning of September since differential calculus is one of the early topics to be taught at AS/A level mathematics. Thus, data collection took me two years, and yet it was still proving too difficult to get enough teachers to commit time to participate in my research project.

As a teacher educator (on the PGCE and the Teaching Advanced Mathematics (TAM) programme I was part of a network of teachers and mathematics heads of departments from across England. To recruit more participants, snowball sampling (Bryman et al., 2021) (and purposive sampling) were adopted, whereby I directly reached out to a small number of teachers and mathematics heads of departments (BERA, 2018) and requested that they pass on my research information sheets to eligible teachers within their schools. Where a positive response was received from willing teachers, it was followed up with an email attached with the research information sheet and consent forms to the school gatekeepers (see Appendix B1). Upon receiving access from the gatekeepers, the participant teacher research information sheets and consent forms (see Appendix B2) were then sent to the respective teachers. Eventually, a total of eight teachers were effectively recruited for the research project.

Table 4.1 gives anonymised brief characterisations for each of the eight participants of the study. Two teachers have more than ten years experience of in teaching A-level mathematics; two are novice teachers with less than three years and four have experiences ranging between three and ten years.

Table 4.1 The participants

Participant	Gender	Experience	Qualifications/Training	Highest school qualifications in maths
T1	Male	More than 30 years teaching A-level mathematics	BSc Mathematics degree; and a PGCE (mathematics).	Had done A-level mathematics as a student.
T2	Female	2 years, with 2 years teaching A-level mathematics	BSc Mathematics degree; Master's degree in mathematics; PGCE(mathematics); and TAM[4].	Had done A-level mathematics and A-level Further Mathematics as a student.
T3	Female	14 years; with more than 5 years teaching A-level mathematics	BSc Mathematics degree; PGCE (mathematics); and TAM	Had done A-level mathematics and A-level Further Mathematics as a student.
T4	Male	2 years, with 2 years teaching A-level mathematics	BA Law degree; PGCE (mathematics) through the [5]Teach First programme); and FMSP[6]	Had done A-level mathematics and A-level Further Mathematics as a student.
T5	Male	3 years, with 2 years teaching A-level mathematics.	BA Mathematics and Education degree with QTS	Had done A-level mathematics as a student.
T6	Male	20 years, with 8 years teaching A-level mathematics	BSc Chemistry degree; PGCE (mathematics); and FMSP	Had done A-level mathematics as a student.

[4] Teaching Advanced Mathematics (TAM) is a one-year part-time course run by the MEI in England to support professional development of teachers by training teachers teach Advanced level mathematics.

[5] Teach First Programme offers on job teaching training to graduates. Graduates are placed directly into schools to work and learn how to teach on the job, mostly in challenging schools.

[6] Further Mathematics Support Programme (FMSP) was aimed at improving the teaching of A level Mathematics and Further Mathematics by providing professional development for the teachers and so increase student participation in A level Further Mathematics. The programme (now AMSP) is funded by the government and run by the MEI in England.

T7	Female	At least 5 years, with more than 3 years teaching A-level mathematics	BSc Mathematics degree; PGCE (mathematics)	Had done A-level mathematics as a student.
T8	Male	10 years, with 5 years teaching A-level mathematics.	BSc Mathematics degree; PGCE (mathematics)	Had done A-level mathematics as a student.

4.6 Ethical considerations

My empirical study inevitably carries ethical issues since it is about collecting data from people – the participant teachers in their schools. The responsibility, as O'Leary (2004) reminds us, for upholding the integrity of all aspects of the research process lies with the researcher. In line with the University of Leeds' expectations, ethical considerations should include the psychological health and safety of subject participants, confidentiality and data protection. The ethical considerations as given here have been informed by Miles and Huberman's (1994) general framework of dealing with ethical issues and by BERA's (2018) ethical guidelines for educational research.

Consideration was given to ethical issues that arose in the early stages of the study; issues that arose as the study unfolded; and ethical issues that arose after the study (Punch, 2009). Figure 4.2 summarises the ethical issues that were taken into consideration in designing this study, which was covered in the ethical review forms as approved by the University of Leeds Research Ethics Committee (see Appendix A).

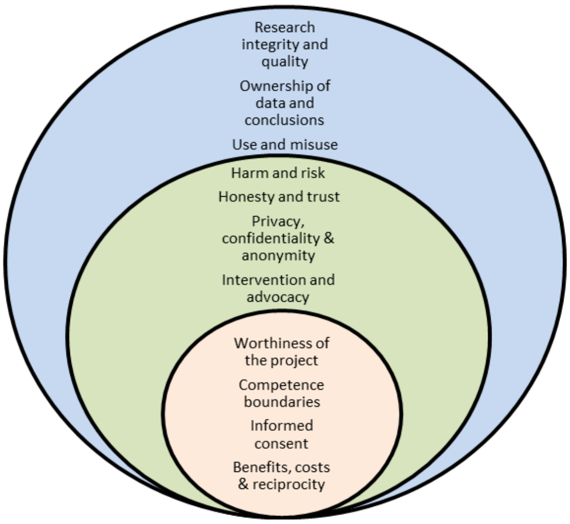


Figure 4.2 Ethical framework

Educational research often interrogates the lived behaviours of others, which implicitly or explicitly brings up issues of power and status. As a teacher educator doing research with teachers, a consideration for the researcher position and researcher effect (BERA, 2018) was made for possible asymmetrical relationships between the researcher- the teacher educator and the subjects - the participant teachers. The researcher made sure that none of the participant teachers was current students of the researcher. The researcher, being a teacher of mathematics by profession, presented himself as a teacher of mathematics researching our practice for teaching the derivative. This struck a common interest with the participant teachers who also felt that not only was it necessary but important to research our teaching of the derivative. All conversations and correspondence between the researcher and the participants were reflective of a professional relationship between teachers. The participant information sheet explained that participation was entirely voluntary and that the participant had a right to withdraw at any stage during the project (see Appendix B2). Upon meeting with the participants, the researcher reiterated the message that participation in the research project was entirely voluntary and reminded the participants that they may withdraw at any time before the end of the study and their data would be deleted (BERA, 2018).

Further, consideration was also made for the balance of risk and harm (BERA, 2018). For some participants being observed teaching may subject them to undue stress caused by the feeling of being watched. So informed consent was obtained before any audio/video recording of the interviews or the observations. The participants were also informed that they could opt for just audio without video if they felt comfortable with the video recording of their teaching.

Interviews, observations and all the data collection tools for this research inevitably have an ethical dimension and for that reason, information sheets and consent forms (Cohen et al., 2013) were prepared for the school gatekeepers, the participant teachers and parents (see Appendices B1, B2 & B3). Informed consent was obtained from all participants and their relevant gatekeepers, where necessary. Children under 16 were not the focus of the research. However, as the research involved observing teachers teaching A-level mathematics lessons, children aged 16-18 were likely to be encountered and could be audio recorded. Students were asked for their verbal consent as a minimum. Although no such requests were made by the schools, any steps required by the school such as asking students' parents' permission would be taken. The information sheets and consent forms for parents were readily available and copies had been given to the gatekeepers. The researcher obtained a DBS check before going into schools and permission from the schools was obtained before audio-recording.

In terms of confidentiality and data protection, the University of Leeds' guidance on research ethics and BERA's (2018) ethical guidelines for educational research, in accordance with the Data Protection Act (1998) was adhered to. Participants were informed that their data will be kept strictly confidential with the researcher and that any publications, reports, lectures or conference presentations given as a result of the research will have their data anonymised so that they will not be able to be identified individually. During the data collection, only information necessary for the research was collected, any data which refers to a subject by name was kept under lock and key, e.g. audio recordings, or in a password-protected folder on the researcher's university secure M-drive for any electronic data. Any data recorded, transcribed or summarised from the originals were made anonymous by using a code number rather than the participant's name. Only the researcher had access to the list of corresponding numbers and names.

Two ethical review applications were made to the University of Leeds Research Ethics Committee. The first was made for the pilot study and clearance was successfully obtained from the University of Leeds Research Ethics Committee (see Appendix A1). The second application was made in preparation for the main Study. This application was almost a duplicate of the pilot study ethics review application, with a few adaptations to reflect on the sample size and dates for data collection. The application received a favourable ethical opinion and clearance was successfully obtained from the University of Leeds Research Ethics Committee (see Appendix A2). Any ethical issues arising from the research were addressed as informed by the

University of Leeds ethical guidelines and following the BERA's (2018) ethical guidelines for educational research.

4.7 Summary of the research design

A methodological triangulation (Denzin, 2005) of the interviews with teachers and the observations of their lessons produced 12 audio files from the pre-teaching interviews and the post-lesson interviews, six video files from the elementary calculus lessons, plus a set of observation notes. See Table 4.2 for the data sets. Eight different teachers originally participated in this study (see Table 4.1). However, the data sets from two of the teachers were incomplete, and so were excluded from the analysis. Table 4.2 is a summary of the research design for the study, listing the research questions, the methods for data collection, the data sets, and the methods for data analysis (more in chapter 5).

Table 4.2 The research design

Research Questions	Data Collection	Data Sets	Data Analysis
In teaching differential calculus: RQ.1 What *word types* and *narratives* do teachers use and why? RQ.2 What *visual mediators* do teachers use and why? RQ.3 What mathematical and pedagogical *routines* do teachers use and how?	**Interviews** with teachers: Pre-teaching interviews Post-teaching interviews **Observations** of teaching and lessons	**12 sets of interview data:** Audio files Interview transcripts **6 sets of observation data:** Video files Video transcripts **Observation notes**	Commognitive & thematic analysis

Chapter 4 has presented the research design covering the recruitment of participants, data collection methods and the ethical considerations for the study. Chapter 5 will present a detailed explanation of the methods for analysing the qualitative data.

Chapter 5 Qualitative data analysis

5.1 Introduction

The diversity of qualitative research means there is a diversity in approaches to qualitative data analysis and so to ensure scholarly rigour, the methods adopted for data analysis ought to be systematic, disciplined and transparent (Coffey and Atkinson, 1996). Data analysis for this qualitative study follows an interpretivist research approach. Thematic analysis, which is a widely used practical and flexible data analysis approach for qualitative research (Braun and Clarke, 2006; Kiger and Varpio, 2020) is adopted. The thematic analysis is undergirded by the epistemological assumptions of commognition (the theory of commognition) (Sfard, 2008), which is the theoretical framework underpinning this study (see chapter 3). There are two main sections in this chapter. Section 5.2 explains this analytical approach, which I shall call *commognitive thematic discourse analysis*. Section 5.3 presents and explains a five-stage, but an iterative process of thematic analysis (Braun and Clarke, 2006; Nowell et al., 2017) that was followed in this study. Further, there are also subsections explaining the inductive and deductive approach to creating themes, transcription and data excerpts, coding, reliability in coding and tables covering coding schemes, themes and exemplar excepts. Given the variety in approaches to qualitative data analysis, the methods applied should be explained and transparent to ensure rigour (Coffey and Atkinson, 1996).

5.2 The analytical approach

The data analysis approach created for this study combines commognitive (Sfard, 2008; Kim et al., 2017) and thematic (Braun and Clarke, 2006; Nowell et al., 2017) analyses; thus a *commognitive thematic discourse analysis*. The analytical approach was applied to identify what the participants talk about, the object of their talk, and what they do with the objects of their talk. It was used to identify and create a set of themes from the research data and to identify narrative excerpts from within the data, as evidence for each theme. The term ‘theme’ in qualitative research refers to some significant patterned response or meaning within the data set in relation to a specific research question (Braun and Clarke, 2006). The commognitive thematic discourse analysis here describes an iterative thematic and a discursive approach to qualitative data analysis.

Braun and Clarke (2006) define thematic analysis as a process for identifying, analysing, and reporting patterns or themes within data, and is a widely used method of data analysis in qualitative research. The beauty of thematic analysis as an approach to qualitative research lies in its flexibility in application and autonomy from any specific research paradigm, which means that thematic analysis can be tailored to the aims, research questions and the conceptual framework of the study (Nowell et al., 2017). The thematic analysis could either be descriptive, at the semantic level, primarily involving identifying patterns and labelling the data; or interpretative, at the latent level, which is about making meaning from the data (Boyatzis, 1998; Thomas, 2013).

At the latent (interpretative) level, thematic analysis overlaps with discourse analysis (Potter and Wetherell, 2001) or rather, according to Braun and Clarke (2006), 'thematic discourse analysis… where broader assumptions, structures and/or meanings are theorised as underpinning what is actually articulated in the data' (p.13). At the latent level, thematic analysis as applied to this study is underpinned by the epistemological assumptions of the commognitive framework (Sfard, 2008) (see Section 3.2 on page 42). The analysis involves identifying and examining the 'underlying ideas, assumptions, and conceptualisations [of the commognitive theoretical framework] – and ideologies - that are theorised as shaping or informing the semantic content of the data' (Braun and Clarke, 2006, p.12). When analysing or interpreting verbal data, be it in speech or written forms, or behavioural data, it is very important to refer to the context, since the same phenomenon can be described in several different ways depending on context. Practically, data analysis at the interpretative level involved relating the general ideas of the commognitive theoretical framework (see Chapter 3 for more information on the theoretical framework for the study), the research questions (see Chapter 4, Section 4.2) and prior literature (see Chapter 2) to the text, to theorise the importance of the themes and their broader meanings, and implications (Patton, 1990).

Mathematics is a discourse, i.e. a form of communication, which is characterised by its *word use*, *visual mediators*, *endorsed narrative* and *routines* (Sfard, 2008). Kim et al. (2017) argue that since the commognition theory encapsulates both cognition and communication, commognitive discourse analysis 'can explain the relationship between interpersonal communication and the cognitive process and how teachers and students move towards a meaningful discourse through participation' (p.448). For this study, the analysis is on the teachers' pedagogical calculus discourse. *Word use* and *endorsed narratives* account for language-dependent elements in the mathematical pedagogical calculus discourse, whilst *visual mediators* in the discourse act as tools for communication. By analysing *routines* we can understand the participants (teachers) behaviours and actions and could get insights into their 'thinking that is not so much strictly related to language' (Kim et al., 2017, p.452).

Commognitive discourse analysis is, therefore, useful in interpreting language and non-language elements of mathematical discourses, and in examining ‘how these two interact and become a discourse’ (Kim et al., 2017, p.452). It is important to point out that the analysis here, is not concerned with syntax, such as the grammatical arrangement of words and phrases in a sentence or the sequencing of sentences, but with the teachers’ pedagogical calculus discourse, e.g. the content of what is said.

To ensure rigour, a systematic approach of constant comparisons at all stages in the analysis process is essential for developing concepts or themes in qualitative data (Punch, 2009). Thomas (2013) describes this iterative nature of the process of qualitative data analysis as a ‘constant comparative method’ (p.235). The data analysis processes were conducted iteratively, although the activities are presented in sequential stages in the process of the thematic analysis presented in Section 5.2.

5.2.1 Deductive or inductive approach

In thematic analysis, themes within data can be created through, either an inductive (bottom-up) approach or a deductive (top-down) theoretical approach (Boyatzis, 1998) or both. The choice between inductive and deductive or ‘theoretical thematic’ (Braun and Clarke, 2006, p.11) analysis can be explained in terms of how and why one is coding the data. Inductive analysis is data-driven, and unlike deductive analysis, the process of coding does not seek to fit the data into pre-existing themes, nor the researcher’s analytic preconceptions (Braun and Clarke, 2006). In an inductive data coding and analysis, themes derive from the content of the data. Whereas in a deductive approach, the researcher comes with predetermined ideas, concepts or topics that they apply in coding and interpreting the data (Braun and Clarke, 2012). For example, coding data for a specific research question lends itself to a more deductive approach (Braun and Clarke, 2006). Note how deductive analysis is described as ‘theoretical’, for example, Boyatzis (1998) describes deductive analysis as a ‘*theoretical* approach’ and Braun and Clarke (2006) p.11) describes it as a ‘*theoretical* thematic’ (p.11) analysis.

In this study, coding and analysis used a combination of inductive and deductive approaches. For a more comprehensive generation of codes and themes, initial coding in this study started inductively on a set of data (i.e. the interview and lesson data transcripts) from one participant teacher, T1. I started by coding data inductively so that what was mapped during the analysis would be closely matched to the content of the data (Braun and Clarke, 2012), thus prioritising participant/data-based meanings over theory-based meanings. Besides, such an open coding of the data would allow for new ideas and themes (that could have been outside of my pre-conceived ideas) to derive from the content of the data themselves (Braun and Clarke, 2012; Patton, 1990). However, researchers ‘cannot free themselves of their theoretical and epistemological commitments, and data are not coded in an epistemological vacuum’ (Braun and Clarke, 2006, p.11), and this is true for my study. Braun and Clarke (2012) further argue that ‘it is impossible to be *purely* inductive, as we always bring something to the data when we analyse it’ (p.3). It would be inevitable that data coding and analysis in this study were influenced, not only by my epistemological commitments and theoretical interest in calculus discourse but also, at least implicitly, by my professional experiences as a teacher of mathematics and as a mathematics teacher educator.

Following the initial inductive coding and analysis of data, the approach to coding and analysing the rest of the data sets progressively became more deductive. As the data coding and analysis progressed, the codes generated from initial coding were subsequently applied to coding new data sets. During this deductive thematic analysis process, the coding process was predominantly driven by my (the researcher) theoretical and analytic interest in the topic, i.e. by the research questions and the conceptual framework for the study (Boyatzis, 1998; Braun and Clarke, 2006; Nowell et al., 2017). This explains the commognitive thematic analytical approach introduced in Section 5.2 above, which is the method applied for analysing the qualitative data in this study.

Even when the approach to data coding and analysis was predominantly deductive, it was not confined to purely deductive means. Braun and Clarke (2012) argue that ‘we rarely *completely* ignore the data themselves when we code for a particular theoretical construct – at the very least, we have to know whether or not it’s worth coding the data for that construct’ (p.3) (italics in original). In reality, ‘deductive and inductive approaches are not necessarily mutually exclusive’ (Campbell et al., 2013, p.314). This argument applies to my study, in which a combination of deductive and inductive approaches to data coding and analysis was applied in deconstructing the teachers’ pedagogical calculus discourse. See Section 5.3 below, for a more in-depth explanation of the generation of the initial codes for this study and how the process then progressed.

5.3 The thematic analysis process

The process of analysing data for this study followed a stepwise procedure as described by Braun and Clarke (2006) but was undergirded by the commognitive theoretical framework (Sfard, 2008). Simply put, thematic analysis describes a process for identifying, analysing, and reporting patterns or themes within qualitative data (Braun and Clarke, 2006; Nowell et al., 2017; Kiger and Varpio, 2020). A five-stage process of thematic analysis is applied and explained in the following subsections:

- 5.3.1 Familiarising with the data
- 5.3.2 Generating initial codes
- 5.3.3 Searching for and reviewing themes
- 5.3.4 Defining and naming themes
- 5.3.5 Producing the report

The activities across all these process stages are interwoven streams interacting with one another, throughout the analysis process (Miles and Huberman, 1994). Thus, the process of thematic analysis here follows an iterative and recursive process.

5.3.1 Familiarising with the data

The first stage in the analysis process was data preparation, which involved transcribing the interviews and lesson observation data, from the audio and video recordings into text. There are six different sets of data included in this analysis as shown in Table 5.1. To anonymise the participants of the study, the data sets were given reference codes as shown in Table 5.1. For example, reference codes for participant teacher 1 (T1) are T1I(i), T1LO and T1I(ii), for the pre-teaching interview, lesson observation and post-teaching interview, respectively. Besides serving for anonymity, these reference codes are used for referencing excerpts in the findings and discussion chapters.

Table 5.1 Data sets and codes

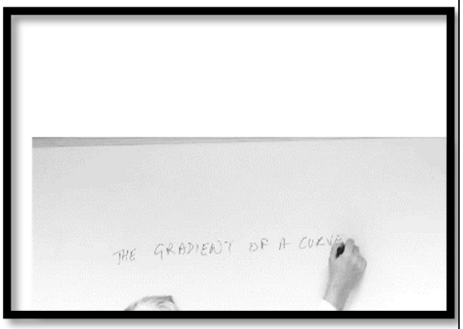
Data Sets	**T1**	**T2**	**T3**	**T4**	**T5**	**T7**
Pre-lesson interview	T1I(i)	T2I(i)	T3I(i)	T4(i)	T5I(i)	T7I(i)
Lesson Observation	T1LO	T2LO	T3LO	T4LO	T5LO	T7LO
Post-lesson interview	T1I(ii)	T2I(ii)	T3I(ii)	T4I(ii)	T5I(ii)	T7I(ii)

The interview audio files and lesson video files were all manually transcribed into unstructured texts by the researcher. Sample transcripts of the interview data and lesson observation data are shown in the two tables below: Table 5.2 and Table 5.3, respectively, taken from the early parts of the pre-teaching interview with T1 and the early part of the observed lesson, T1LO.

Table 5.2 Pre-lesson interview transcript (T1)

1.I.	Can you tell me about your approach to introducing err...I mean you did touch on that earlier, to introducing the concept of the derivative?
2.T.	Well today I am going to ask them what we mean by the gradient of a curve (I: yeah) and I am going to see what sort of answers I will get (I: Umm) Err Um(...) that's what ... I don't even know if I would use the word derivative today (I: Umm). I may well don't use that (I: Yeah). Urr um I might use the word gradient function (I: yeah) we will see (I; yeah). I mean, it might just come out of my head, you know, urr .... but … but what do we mean by gradient? And I will ask them, you know, what do you think the gradient is there... I'm hoping someone will tell me ...urr it's the tangent, and I will say Ok, and I'm going to give them [*looks through some papers*] oh it's here somewhere...I'm gonna give them that graph.
3.I.	Oh yeah, if I can get a copy, copies of these/ (R: there you are)/thanks that's fine.
4.T.	I'm gonna give them that graph (I: Yeah) and I'm going to ask them to draw the tangent and I 'm going to take... see what we get as our measurements.
5.I.	That's very interesting
6.T.	And then we will see what the real one, you know, that's where...I will do that before we even talk about how we ...how we do it (I: yeah) that is the first thing I'll do after we talked about we mean by the gradient of a curve (I: right). That's where it's gonna go and then into the activity of the idea that we're getting ...we're getting to the chord. I will try to demonstrate that if we take the chord near enough, it's an approximation...and I have given them all different points on their table that they have to use (I: Right) …Urr and I will say right what did you get for that one, and see if we can spot a pattern.

Table 5.3 Lesson video transcript from (T1)

Time	Audio	Visual/Comments Respective snapshots from the lesson observation video or brief descriptions of observed behaviours.
0:02:41	T: Ok right so that's what we're gonna be doing anyway for the next few weeks. And I want to pose a problem to you, and the problem is this…	The teacher writes on the white board – “The gradient of a curve”
0:03:09	T: What do we mean by that? That's my first question to you. Now we all know, I hope what is meant by the gradient of a line. Tell me, what do we mean by the gradient of a line? You can tell me in any way you like. *[student: inaudible]* T: Ok, I want a little bit more detail there, but I will buy that. What do you think we mean by gradient of a line? *[student: inaudible]* T: No, I am talking about a line first. *[student: inaudible]* T: I suppose err I asked the question poorly, I think. How do you measure the gradient of a line then? How do you measure the gradient? *[student: inaudible]* T: Right, so my question to you is, what do we mean by the gradient of a curve?	The teacher writing on the white board – “The gradient of a curve”] 

For an explanation of the transcription and presentation of Table 5.2 and Table 5.3, see Section 5.3.1.1, below.

5.3.1.1 Transcription and data excerpts

The transcription could either be everything or selected phases or relevant sections of the interactions, focussing on the content of what is said, rather than the speech patterns of the interaction (Roulston, 2014) between teacher and student. For this study, transcription sought to capture all interactions, i.e. the whole interview audio files or the whole lesson observation video files for each data set. The transcription involved replaying and listening to the interview audio recordings and watching the lesson videos numerous times, noting down initial ideas, and transcribing the audio and video files into text.

Transcribing all the interactions in a data file would take a lot more time than what would be required if selective transcribing had been adopted (Roulston, 2014). However, this was the best option because at the time the researcher had not yet decided on a deductive or inductive approach to data analysis, thus no a priori set of constructs or themes for data analysis had been yet created to inform the selection of what parts of the interactions or sequences could be transcribed. Transcribing everything meant that there would be some information transcribed that will not be used nor relevant for data analysis.

The original transcription sought to capture as many verbal and non-verbal interactions as possible (Roulston, 2014), thus verbatim transcription was used. However, for this study, not all features of the talk were deemed necessary, thus, not much attention was paid to capturing features such as speed of talking or tone of voice, pauses, intonation, or hedges. The main focus was primarily on capturing what was said and done – verbal and non-verbal communication. For the sake of readability, intelligent transcription has been used for excerpts, thus standard UK written English conventions have been applied, for example, grammar and spelling conventions. Thus, unconventional spellings such as *gonna* or *'cause* in the original transcripts would appear as *going to* and *because* in the excerpts. Also, discourse markers or verbal 'gap-fillers' such as, *like, you know, umm*, *uh* have been excluded from the excerpts used in reporting the finding of the study.

There is no one standard layout for transcripts. The layout for the interview data transcript and lesson observation data transcript are shown in Tables 5.2 and Table 5.3 above. The transcript for the video data has a column to add visual data to the text, e.g. snapshots from the video and for possible observer commentary. The transcripts had big margins (especially on paper) on the right-hand side for coding purposes, for writing comments and labels for codes. All the transcripts are numbered in the left-hand margins, numbered by turn, and not by line. The participants' utterances are numbered by turns. Against each numbered turn is an initial to indicate the type of the participant.

Below are a few more points about the structure of the transcripts and the notation used in transcripts, that will most likely feature in the excerpts used in reporting and discussing the findings of the study.

(…) words or part of the utterance left out. These are parts that do not, in my opinion, add value to the excerpt or parts of the data that are not relevant to the point under consideration, which could be not relevant to the theme under discussion.
Missing numbered turns/utterances in excerpts - The original transcription was verbatim, thus the transcription sought to capture all that was said. In-text excerpts, therefore, might leave out some irrelevant texts or words, that will be shown by (…), or numbered turns or utterances that do not add value to the theme or point under consideration.
/ is used to denote overlapping speech; when two discursants speak at the same time, or one interrupts the other speaker.
[ ] commentary by the researcher, for example, to describe the participant's action or gesturing by the participants or the context of the utterance.
Excerpts are indented and font size is reduced by one.

Mathematical symbols – as far as possible in the excerpts, mathematical notation is used in utterances describing mathematical objects, e.g. $y = x^2$ is used instead of ‘y is equal to x squared’.
The transcripts are numbered by turn. Thus, the size (or length) of the numbered utterances shall not be uniform. Some utterances could be a word to a few sentences, whereas other larger utterances could be a paragraph or two.

5.3.1.2 Data excerpts and anonymised codes

Following the ethical principles for participants’ privacy, respect and anonymity, no participants’ names have been used in reporting this study. Instead, the following anonymised code names have been used:

I – Interviewer (the researcher was the only interviewer, and so ‘I’ appears only in the interview transcripts);
T – Teacher (participant teacher and this appears in both interview and lesson transcripts); and
S – Student (this appears only in the lesson transcripts). Where a student's name is mentioned in the teacher’s utterances, it is anonymised in the transcripts with a random initial, for example, Sxxx or Pxxx.

Data sets have been coded for anonymity too. Every excerpt used as evidence in reporting or discussing the findings will have reference to the respective data set. For example, if an excerpt is taken from the data transcripts of participant teacher T3, it will be prefixed with one of the three data code references:

T3I(i) – a reference to the pre-teaching interview with participant teacher T3;
T3LO – reference to the lesson observation of participant teacher T3; or
T3I(ii) – a reference to the post-teaching interview with participant teacher T3.

See Table 5.1, which shows the data sets and their respective reference codes for all the participants. Although, there are several verbal data analysis programmes available for analysing qualitative data, often referred to as Computer Assisted Qualitative Data Analysis Software (CAQDAS), such as NVivo (Thomas 2013), data processing was done manually. I believe by immersing oneself in data processing, the researcher is more likely to make better sense of the findings and, to engage in a more meaningful discussion of the findings.

5.3.2 Generating the initial codes

This is the onset of the ‘data reduction’ (Miles and Huberman, 1994, p.4) process in qualitative analysis, which involves the coding phenomena in the data in a systematic fashion, collating data relevant to each code (Braun and Clarke, 2006).

Data reduction is central to the data analysis process and runs throughout the analysis process; the main activity here is coding. Coding helps the researcher with noticing ideas of interest relevant to the study, and more specifically, to the research questions, and involves systematically organising data into categories by attaching tags, names, colours or marks, abbreviations, or labels – the codes – against pieces of data such as words, utterances, phrases, lines, sentences and paragraphs (Corbin and Strauss, 1990; Punch, 2009; Thomas, 2013).

In literature, there are many different descriptions as well as illustrations of levels of coding and types of coding (Coffey and Atkinson, 1996; Miles and Huberman, 1994; Richards, 2014), but codes can typically be classified into two main types, the low inference descriptive codes and higher inference pattern codes. Miles and Huberman's (1994) *descriptive codes* and *pattern codes* generally equate to what Richards (2014) describes as *topic codes* and *analytic codes*, respectively. *Descriptive codes*, as Punch (2009) explains, are about identifying and labelling what is in the data, whilst *pattern/analytic codes* 'go further, interpreting or interconnecting or conceptualising data' (p.179).

The process of generating initial codes was initially approached inductively using a set of data transcripts for one participant teacher, T1 despite my prior theoretical and epistemological assumptions. Although Braun and Clarke (2006) remind us that data cannot effectively be coded in an epistemological vacuum, other theorists such as Glaser and Strauss (1967) argue that analysing data inductively allows for new ideas and themes to emerge from the data. There was nothing to lose by starting inductively with open coding of data. Using data sets for T1, some words, utterances, phrases, sentences and paragraphs within the transcribed interview (audio) and lesson (video) data were systematically identified and labelled as initial codes.

The open coding on T1's data generated the initial descriptive codes and the initial coding scheme for the data analysis process; see Table 5.6 in Section 5.3.2.2 below. In Table 5.6 there are about 70 descriptive codes listed as nine sets of codes, shown in the columns of the table. This coding scheme was then applied to all the other participant teachers' data, one by one, each time updating the coding scheme in the process until no more new or different codes could be found.

The coding was done manually, although Microsoft Word operations were used for electronic colouring and labelling during the coding process. The identified text and pieces of data were highlighted with some coloured pens and labelling with code names in the right margins of the transcribed scripts but using both hard copies and electronic files of the transcripts. Identifying codes involved paying attention to the statements as well as the actions of the teacher, also considering the structure and context of the teacher's utterances and actions, as far as possible (Charmaz, 2003). It is important to consider the context, for when codes are grouped together, the meaning of the story for which they were initially said, could be lost.

5.3.2.1 Reliability in coding

Not all qualitative researchers use the term reliability or inter-coder reliability, instead, it is the concept of rigour that is considered more important in qualitative research (Merriam, 2009). One of the most authoritative sources in the field of qualitative research is the Sage Handbook of Qualitative Research (Denzin & Lincoln, 2011), which has 43 chapters. Syed and Nelson (2015), noticed that the word reliability or inter-rater agreement does not feature in its index; 'so what, if anything, serves as the parallel concept to reliability for qualitative researchers?' (p.16); it is scholarly rigour. Rigour is more important than inter-rater reliability measures in qualitative research, and it is a product of the researcher, the research context and the research process (Syed and Nelson, 2015). Reliability in coding does not necessarily constitute validity, where it is used, it is a necessary but insufficient consideration for validity.

For this study, rigour derives from, according to Merriam (2009), 'the researcher's presence, the nature of the interaction between researcher and participants, the triangulation of data, the interpretation of perceptions and rich, thick descriptions' (p.165). Putting rigour before inter-coder reliability measures, Syed and Nelson (2015, p.17) argue that:

> The researchers have a deep and intimate knowledge of their participants that goes far beyond the words on paper that tends to be the product of more quantitative approaches. From the standpoint of these researchers, this closeness is what allows for rigour in the interpretative methods and renders trivial the idea that a two-digit coefficient in the Method section as the ultimate sign of rigour.

This is true for my research, which takes an interpretative approach to study the teachers' pedagogical discourse on the derivative, through the lens of the commognitive theoretical framework.

Regardless of some of the criticism against coding reliability checks, I wanted to know if my coding of data transcripts would generally be comparable or consistent, with my supervisors' coding, at least. A check for inter-coder agreement (Campbell et al., 2013) was undertaken involving the primary researcher (me) and my two research supervisors. Any reliability in coding here should be seen as a subjective consensus between my two supervisors and I, and 'not [as] an ultimate decontextualised "truth" that exists outside of the data' (Syed and Nelson, 2015, p.17).

This process was repeated twice using different data transcripts. Appendix D1 shows the interview data transcript that was used for the coding reliability checking exercise. I (primary researcher) coded a selected section of the interview data and produced a list of codes shown in Table 5.4 Coding scheme 1, which were then shared with the other two coders (the supervisors) (see the coding schemes in Section 5.3.2.2 below). A different section of the interview data transcripts i.e. paragraphs 14 to 32 (see Appendix D1) was selected for this task and the coding of the transcript was completed by all three coders, but independently. Upon comparing and discussing the coding, we immediately noticed that we had made quite a few different interpretations and applications for some of the codes. There was, therefore, some ambiguity within some of these codes. The other coders did not quite understand what was represented by the codes because I had not defined or explained the codes to the other coders.

A different transcript was identified i.e. paragraphs 33 to 46 (see Appendix D1) for the second coding reliability checking exercise. I (primary researcher) prepared a set of codes, as shown in Table 5.5 Coding scheme 2. This time I (the primary researcher) explained the codes to the other two coders, before all three coders coded the data transcript, independently. Although not quantitatively rated, the level of agreement in the coding of all the three coders was very high. There were, however, a few areas for further standardisation, that were then discussed and agreed upon by all the three coders. Accordingly, further revisions were then made to the coding scheme.

The degree of agreement for the coding reliability check was not quantitatively rated. Aiming for and settling for coding consensus or agreement, rather than reliability coefficient or percentage measure is a more preferred approach for qualitative studies such as this purely qualitative research. An example is Park (2016), Park does not make a quantitative rating of reliability in her discursive analysis. Instead, to check reliability, Park (2016) asked someone familiar with her study to code a section of the data set. The two researchers then discussed the different codes until reaching an agreement on the coding system.

5.3.2.2 The coding schemes

Three tables, Table 5.4, Table 5.5 and Table 5.6 below, represent subsequent coding schemes developed through the data analysis process for the study. The iterative coding process through the data sets of the study and subsequent review and updating of the coding schemes continued from Table 5.4 through to Table 5.5 and resulted in Table 5.6, which is the main coding scheme for data analysis. These three tables are central to the explanation of the data analysis process for the study, and extensive reference is made to these tables across Sections 5.3.2 to 5.3.5.

Table 5.4 Coding scheme 1

Mediational tools and resources **(MT)**	Mathematical representation **(MR)**	Teaching & learning processes **(P)**	Mathematical terminology & notation **(MTN)**	Teacher knowledge **(TK)**	Curriculum (school) context **(CC)**	**Time**	Miscellaneous
Textbooks Whiteboards Worksheets Computer-Software Computer technology		Visualising Showing Moving	Graph Transformation Rotation Enlargement	Pedagogical Computer technology Teaching experience Learning experience CPD	Educational level Module		Age Past Affect Teacher's belief **(TB)** Teacher attitude

Table 5.5 Coding scheme 2

Mediational tools and resources **(MT)**	Mathematical representations (**MR**)	Teaching & learning processes **(P)**	Mathematical terminology & notation **(MTN)**	Teacher's knowledge & beliefs **(TK) (TB)**	Curriculum (school) context **(CC)**	**Time**	Miscellaneous
Worksheets Computer-Software	Graphical Algebraic	Investigating Demonstrating Drawing Measuring Estimating Questioning Pointing Moving Showing	Tangent Gradient Gradient-function Differentiation Derivative Graph Curve Line Chord Points Pattern Rules	Pedagogical Mathematical Instrumental Relational	Exam boards Curriculum changes Modules Modular course Exams Student demographics Attainment levels	Constraints	History Age Affect Students

Table 5.6 Coding scheme for data analysis

Mediational tools and resources	Mathematical representations	Teaching approach	-ing words	Mathematical terminology & notation	Self-identity & belief statements [Teacher knowledge & beliefs]	Curriculum & Assessment	Time	Miscellaneous
MT	**MR**	**TA**	**-ing**	**MTN**	**BS**	**CA**	**Time**	
Textbooks Whiteboards Worksheets Computer-Software Computer technology	Graphical Algebraic/ symbolic Numerical	Investigating Demonstrating Questioning	Drawing Measuring Estimating Pointing Moving Showing Visualising Sketching Plotting Dragging	Tangent Slope limit Gradient Gradient-function Differentiation Derivative Integration Function, formula, Equation Graph Curve Line Chord Points Transformation Rotation Enlargement Lagrange's notation – $f\ (x)$ Leibnitz notation- $\frac{dy}{dx}$	Pedagogical Mathematical Instrumental Relational Student attributes Teaching experience Learning experience CPD	Exam boards Curriculum changes Modules Modular course Exams Attainment levels Departmental policy Scheme of Work School factors	constraints	History Age Affect Students

Tables 5.4, 5.5 and 5.6 show the coding scheme at three different stages, in progression. The structure of the coding scheme comprises two levels of codes, the *descriptive codes* and the *pattern (analytic) codes*. The *descriptive codes* are those listed in the columns of the coding scheme tables. The *pattern codes* are listed across the coding scheme table, along the first row as headings to the vertical lists of *descriptive codes*. The number of descriptive codes rose three-fold, from 25 (in Table 5.4) to more than 70 (in Tables 5.6), as the coding process progressed through the data sets. There are more than 70 descriptive codes and less than 10 pattern codes in the main coding scheme, Table 5.6. The structure of this coding scheme represents the data reduction process, whereby the initial *descriptive codes* are categorised into data concepts (the *pattern codes*), leading toward the formulation of data themes.

During this coding process, through the three coding schemes (Tables 5.4, 5.5 and 5.6) some notable amendments to the labels for the *pattern codes* were made, and these are explained in the ensuing sections. The pattern code *teaching and learning processes* (P) in the coding schemes Tables 5.4 and 5.5 was later renamed the *-ing words* in the coding scheme Table 5.6. The pattern code *-ing words* here is used to capture and denote the human action process in doing mathematics with artefacts that were observed during the teaching of the derivative, such as zooming, sketching, plotting and dragging. The *-ing words* is a construct borrowed from the '-

ing curriculum' devised by Monaghan (1997) for the action process in doing mathematics such as classifying, drawing, estimating and graphing. Note the contrast between this analytic code *-ing words* and descriptive codes or words such as investigating, demonstrating and questioning, which are separated from the *-ing words* code and assigned a new code, *teaching approach* (TA).

As the coding process progressed, it became difficult and, more and more difficult to categorise some of the teacher's utterances as reflective of the teacher's knowledge or the teacher's beliefs. The pattern codes, *teacher knowledge* (TK) and *teacher beliefs* (TB) in coding schemes Tables 5.4 and 5.5 were later revised and collectively categorised as *belief statements* (BS) in the coding scheme Table 5.6. The *belief statements* (BS) analytic code here, encompasses the self-identity and belief statements about teacher knowledge and beliefs.

What teachers believe about their work has long been a focus of research in mathematics education and social studies. To understand teachers' beliefs, Pajares (1992) argues that researchers must first decide 'what they wish belief to mean and how this meaning will differ from that of similar constructs' (p.308). Similar conceptions such as knowledge, understanding, preferences, meanings, and perspectives could easily be interpreted as teachers' beliefs. Other researchers have also conceptualised teachers' beliefs as a 'system of beliefs' (Eisenhart et al.,1988; Pajares, 1992; Leatham, 2006; Lazim and Abu Osman, 2008). Research on teachers' beliefs has shown that there are often inconsistencies among what is termed teachers' beliefs as well as inconsistencies between the teachers' beliefs and their actions (Leatham, 2006). Leatham (2006) further argues that it is often very difficult for teachers to articulate their beliefs and that researchers' interpretations of those teachers' beliefs are often problematic. In the coding process, it became almost impossible to judge or establish the individual teachers' beliefs about teaching differential calculus; hence pattern/analytic code *belief statements*(BS) was preferred over *teacher beliefs.*

The primary aim of this study is to investigate how teachers teach elementary calculus, not necessarily to study teachers' beliefs. However, it is inevitable in such a study, that beliefs must be inferred. Rokeach (1968) cited in Pajares (1992, p.315) suggested that inference to teachers' beliefs should consider how the evidence of the beliefs is presented: 'belief statements, intentionality to behave in a predisposed manner, and behaviour related to the belief in question'. This suggests that inferences to one's beliefs can be made from one's statements. This study assumes a broad conceptualisation of what constitutes teachers' beliefs, as such, any inference to what teachers say about their teaching of mathematics and calculus is simply coded as teacher's *belief statements*, rather than teacher's *beliefs*.

Another pattern code that changed from the coding of Table 5.4 and 5.5 was *curriculum school context* (CC), which became the *curriculum and assessment* (CA) pattern/analytical code in the coding scheme of Table 5.6. The CA pattern code describes factors such as examinations, testing, curriculum changes, attainment levels and other school factors.

Some pattern codes did not change through the coding process, such as the *mediational tools and resources* (MT), *mathematical representations* (MR) and *Time* codes; they remained the same over the three coding schemes. The *mediational tools and resources* (MT) pattern code, is used to denote artefacts, including digital artefacts whilst the *mathematical representations* (MR) pattern code describes visual mediators and types of mediation including geometric graphical, algebraic symbolic and numeric representations in the teachers' pedagogical calculus discourse. The *Time* code relates to time constraints. During the coding and analysis process, any codes that could not be classified under any of these eight pattern codes, for example, age, history and affect, were classified under the *miscellaneous* code (See Table 5.6).

The coding process continued iteratively and concurrently with the data analysis process until any subsequent coding of the data sets did not reveal nor add any more pattern codes beyond the nine in Table 5.6 coding scheme. At that point, the search for the main themes followed whereby the coding process then predominantly progressed from the descriptive open coding phase to an interpretative one, which involved interconnecting and conceptualising the data.

5.3.3 Searching for and reviewing themes

The coding scheme quickly developed into a dynamic document as the coding process continued iteratively, reviewing the codes and emerging themes against the data. Each iteration introduced some amendments, reassigning and re-categorisations of some of the codes to the various emerging themes. Nonetheless, the search for themes happened concurrently with the generation of initial codes, as explained in the preceding Section 5.3.2. This can be seen in the design of the coding scheme, with its pattern (analytic) codes outlined in the first row and the description codes listed in the columns. This design is similar to what Syed and Nelson (2015) describe as hierarchical coding schemes, 'in which microcodes [description codes] are nested within macrocodes [pattern codes]' (p.8). These patterns codes generated the preliminary themes for the coding system

Although the set of initial descriptive codes were inductively generated, the search for themes and the identification and naming of the nine themes were informed by the theoretical framework for study – the commognitive framework theoretical, the existing literature, and the research questions for the study. The initial broad classifications of pattern codes in the first row of the coding scheme Table 5.6: the *mathematical terminology & notation, mathematical representations, mediational tools and resources, teaching approach and curriculum and assessment* are informed by the researcher's conceptualisation of research questions of the study. Therefore, this study adopted a 'theoretically-driven inductive approach' (Syed and Nelson, 2015, p.7) for the coding process and the development of a dynamic working coding scheme. Some studies have used a similar approach to develop a coding system, for example, in Syed et al. (2011) after inductively generating a large list of initial codes, the researchers then 'searched for themes in the codes informed by the existing literature on the topic' (Syed and Nelson, 2015, p.7).

Unlike the initial open coding which breaks up data into concepts and categories, the reviewing of themes stage combines the data back together by eliminating repetitions and similarities, in a repeated and iterative process. The pattern codes, in the first row of the coding scheme Table 5.6 were reviewed for possible similarities or repetitions, and then collated into potential themes, gathering all data relevant to each potential theme (Braun and Clarke, 2006). The pattern codes of the coding scheme Table 5.6 were collated into six potential themes for the study, namely, language, symbolism, artefacts, representations, routines, and the 'why' factors, displayed in Figure 5.1 thematic map. The thematic map (Figure 5.1) gives a graphical display of how these preliminary themes are linked and connected.

An important decision any researcher must make in the development coding scheme is the number of codes or themes to be used. A large number could allow for more complexity but at the expense of reliability (Syed and Nelson, 2015). According to Campbell et al.(2013), there is no one standard way of deciding on the number of codes or themes. Syed and Nelson (2015) describe the process of developing a coding scheme as an act of 'balancing of parsimony and nuance'(p.8). What matters more is the reliability and usefulness of the data. For this study, a set of anything between five and ten overarching themes was considered appropriate, manageable and useful, given the scope of the study.

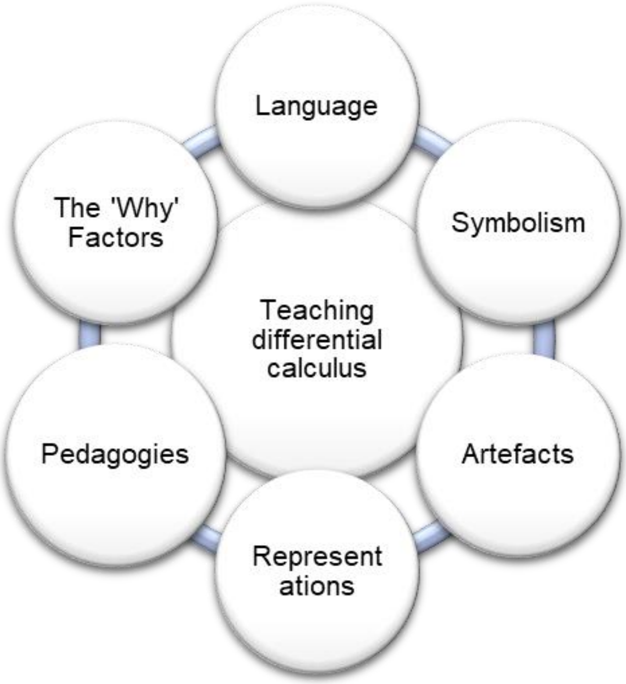


Figure 5.1 Thematic map

The thematic map in Figure 5.1 was borne out of the working coding scheme of Table 5.6. The mathematical terminology and notation (MTN) pattern code resulted in the creation of two separate categories, the *language* theme and the *symbolism* theme. The mediational tools and resources (MT) code became the *artefacts* theme and the mathematical representations (MR) code was simply reframed as the theme of the *representation*. What was previously coded teaching approach (TA) and -ing words were now collectively termed the *pedagogies* theme; and finally, the belief statements (BS) code, curriculum and assessment (CA) code, the time code and miscellaneous codes were all grouped to constitute *the 'why' factors* theme.

As the data analysis process continued iteratively, the themes (coding categories) of Figure 5.1 were subsequently applied back in coding the original data sets 'to ensure appropriate specificity and accuracy, which [would] lead to refinement of the categories [themes]' (Syed and Nelson, 2015, p.8). This iterative approach to coding and data analysis is what Thomas (2013, p.235) described as a 'constant comparative method'. It allowed for a systematic and constant making of comparisons across the codes, themes and between different levels of data analysis (Punch, 2009) and the refining of data continued beyond the generation of the six preliminary themes in Figure 5.1.

5.3.4 Defining and naming themes

The review of preliminary themes as described in Section 5.3.3 above, was then aligned more closely to the general ideas, terminology and descriptions of the commognitive theoretical framework (Sfard, 2008) and the research questions for the study. There are three main research questions, which I shall repeat here to remind the reader. In teaching elementary differential calculus:

RQ.1 What *word types* and *narratives* do teachers use and why?
RQ.2 What *visual mediators* do teachers use and why?
RQ.3 What mathematical and pedagogical *routines* do teachers use and how?

As mentioned earlier in this chapter, the qualitative analysis for this study combines thematic analysis (Braun and Clarke, 2006; Nowell et al., 2017) and commognitive discourse analysis (Sfard, 2008; Kim et al., 2017), thus a combination of a thematic phase and a discursive phase. Whilst the thematic phase helped with identifying what the participants talked about, and the object of their talk, the discursive phase informed by the commognitive interpretive framework (Sfard, 2008) helped to deconstruct what the participant teachers did with the objects of their talk. These two analytic processes were conducted iteratively.

At this stage, the commognitive constructs (i.e. word use, visual mediators, narratives and routines) (Sfard, 2008) were then added to the coding scheme that had so far been generated through the preceding stages of the thematic analysis process, as shown in the overarching themes of Table 5.7. Analysis of the teachers' pedagogical calculus discourse on the derivative focuses on the words and visual mediators**,** routine and endorsed narratives of its processes. The description of the themes in the third column in Table 5.7 is a commognitive characterisation of the respective overarching themes for the analysis system. The description, here, derives from the commognitive constructs of the commognitive theoretical framework. Refer to Chapter 3 for a more in-depth explanation of the commognitive theoretical framework for this study. Note that these overarching themes in Table 5.7 are an elaboration of the thematic map of Figure 5.1.

The processes of searching, reviewing and defining themes involved developing and naming the main categories/themes and their sub-categories/themes, which resulted in six overarching themes presented in Table 5.7 below. The overarching themes have been reconfigured from the analytic/pattern codes of the Table 5.6 coding scheme. Likewise, the subthemes resulted from the reconfiguration of the descriptive codes of the Table 5.6 coding scheme, which involved selecting and re-categorising codes from the coding scheme. Thus, refining 'the specifics of each theme, and the overall story the analysis tells, [in order to] generate clear definitions and names for each theme' (Braun and Clarke, 2006, p.35).

Table 5.7 The overarching themes

Overarching Themes	Subthemes	Description
Mathematical **language** for calculus teaching	Calculus terminology The 'gradient of a curve' The 'gradient' and the gradient function The utterance 'getting closer and closer'	Word use; Narratives
Symbolism in calculus teaching	Calculus symbolism: Leibnitz notation – $\frac{dy}{dx}$; Lagrange's notation – $f'(x)$	Visual mediators
Mathematical **artefacts** for calculus teaching	Digital artefacts Dynamic imagery	Visual mediators
Mathematical **representations** in calculus teaching	Algebraic representations; Graphical representations; Multiple representations	Visual mediators; Narrative
Pedagogies on the derivative	Approximating gradients by drawing tangents; Approximating derivative at a point on a curve, by using secant and tangent lines; From approximating gradients to differentiation	Routines
The '**why**' factors	Teacher's belief statements (self-identity) Time constraints Curriculum and assessment issues	The why factors

Together with their respective sub-themes, these six overarching themes represent the ultimate coding system that was then applied, deductively, in analysing the data transcripts for all the participant teachers in this study. According to Ryan and Bernard (2000); Braun and Clarke (2006); Campbell et al. (2013), a good coding scheme should include a description of each theme, a description of the inclusion and exclusion criteria for that theme, as well as exemplars of excerpts or of units coded as that theme. A description of each of the themes in Table 5.7 above, is given in Table 5.8 below. For illustrative exemplars of units or excerpts identifiable with each of the themes, see Table 5.9 below. The list of exemplar excerpts presented in Table 5.9 is not exhaustive, but illustrative of the data units for each of the themes.

Table 5.8 Describing the themes

Overarching Themes	**Description of the themes**	**Sub-themes**
Mathematical **language** for calculus teaching	Word use; Narratives **Mathematical language for calculus teaching** resulted from the MTN group of codes. This theme applies to RQ1 and represents common and as well as specialised terminology used by the teachers in teaching differential calculus. Alternatively, it describes the commognitive constructs of *word use* and *narratives* in the teachers' pedagogical calculus discourse. The theme has sub-themes that relate to calculus word use (both colloquial and literate discourse) and narratives (both object-level and meta-level narratives). Examples include words such as tangent, gradient, the instantaneous rate of change, 'gradient of a curve', gradient function and the utterance 'getting closer and closer'.	Calculus terminology; The 'gradient of a curve'; tangent; The 'gradient' and gradient function; The utterance 'getting closer and closer'
Symbolism in calculus teaching	Visual mediators **Symbolism in calculus teaching** also derives from the MTN group of codes because the MTN code captured the mathematical symbolism used in calculus discourse. The theme relates to RQ2 which is about visual mediators in the teachers' pedagogical calculus discourse. Mathematical notation in calculus is a form of algebraic mediation and calculus symbols are visual mediators according to the commognitive conceptual framework. Examples of calculus symbolism include the Leibnitz notation – $\frac{dy}{dx}$; Lagrange's notation – $f'(x)$.	Calculus symbolism [Leibnitz notation – $\frac{dy}{dx}$; Lagrange's notation – $f'(x)$]
Mathematical **artefacts** for calculus teaching	Visual mediators **Mathematical artefacts for calculus teaching** stems from the MT code and relates, but not exclusively, to RQ2 and represents the visual mediator, for example, iconic graphical representations of functions and digital artefacts. This theme also relates to RQ3, which seeks to explain how teachers use mathematical artefacts in teaching differential calculus. This theme focuses mainly on digital artefacts or digital technologies used in calculus teaching	Digital artefacts; Dynamic imagery

	such as dynamic imagery. Digital artefacts refer to digital technologies used in calculus teaching such as dynamic imagery tools.	
Mathematical **representations** in calculus teaching	Visual mediators **Mathematical representations in calculus teaching** result from the MR code and relates to RQ2, which specifically seeks to find out about the graphical, symbolic and numerical mathematical representations used by teachers in teaching differential calculus. In terms of the commognitive framework, this theme relates to visual mediators. The theme has three sub-themes, and these are described as: Algebraic mediation for the gradient at any point on a curve refers to symbolic mediation. Graphical and dynamic mediation refers to both static and dynamic graphical imagery. Multiple representations refer to the use of more than one form of mediation and the shifts between forms of mediation.	Algebraic mediation Graphical mediation Multiple representations
Pedagogies on the derivative	Routines **Pedagogies on the derivative** comes from the TA and –ing codes and relates to RQ3, which seeks to explain the type of words and the narratives as well as the visual mediators in the teachers' pedagogical discourse in calculus teaching. In terms of the commognitive framework, this theme relates to mathematical and didactical routines in the teachers' pedagogical calculus discourse. The theme has three sub-themes, and these are described as: Approximating (estimating) gradients by drawing tangents pertains to the use of tangents and curved graphs. Approximating (estimating) the derivative of a curve at a point, by using chords and tangents refers to using a secant line, or chord to estimate the gradient of a tangent at a given point on a curved graph. From approximating gradients to differentiation refers to explanations or illustrations that link calculating gradients to a curved graph at given points in the process of differentiation.	Approximating gradients by drawing tangents; Approximating the derivative of a curve at a point, by using secant and tangent lines; From approximating gradients to differentiation.
The '**why**' factors	**The 'why' factors** combines the BS, CA & Time codes and defines the determinants of the language, artefacts and representations teachers use for	Teacher's belief statements (self-identity);

	calculus teaching. This theme applies to all three research questions. The theme has three sub-themes, and these are: *Teacher's belief statements* (self-identity) describes what could be thought of teachers' beliefs. However, the use of 'belief statements' is a more appropriate way of representing the teachers' statements about their beliefs. *Time constraints* refers to aspects of time for teaching and learning calculus. *Curriculum and assessment issues* refers to any factors related to the school, curriculum and examinations or assessment-related issues. This theme applies to all three research questions. The theme describes the teacher's belief statements (self-identity), which could be thought of as teachers' beliefs. It is very difficult to read what teachers say as representing their beliefs. It is, therefore, more appropriate to refer to statements of what teachers say about their beliefs., as 'belief statements'. Time constraints refers to aspects of time for teaching and learning calculus. Curriculum and assessment issues refers to any factors related to the school, curriculum and examinations or assessment-related issues.	Time constraints; Curriculum and assessment issues

Table 5.9 Exemplar excerpts for the themes

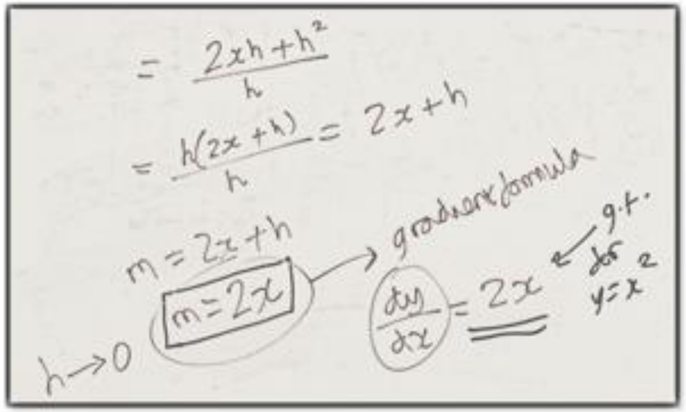
Overarching Themes	**Exemplar excerpts representative of the themes**
Mathematical **language** for calculus teaching: ***Subthemes:*** Calculus terminology; The 'gradient of a curve'; 'Gradient' and gradient function; The utterance 'getting closer and closer'	**T1LO:** 4.T. I want to pose a problem to you, and the problem is this… [*Teacher writing on the white board* – "The gradient of a curve"]. 5.T. What do we mean by that? That's my first question to you. Now we all know, I hope what is meant by the gradient of a line. 7.T. How do you measure the gradient of a line then? How do you measure the gradient? 9.T. Right, so my question to you is what do we mean by *the gradient of a curve*? **T3LO:** 200.T. So, you would use that brilliant idea of using two points. And then if we *move those two points closer and closer*, we *get a better, better approximation* **T7LO:** 104.T. So, if we imagine that the ruler is the tangent at different points. 105.S. Yeah. 106.T. The tangent, the gradient of the tangent will tell us the gradient of the curve at that point. This is what I've just done. I've gone around different points on the curve, and you can see that the slope of the gradient, the ruler is changing isn't it? **T1LO:** 85.T. So, let's make a note of this, [*writing on the board*] If $f(x)$ is x^3, it means $f'(x)$ is $3x^2$. 86.S. What is that dash mean? 87.T. It means the derivative, the gradient function. That's the notation I have used here. 88.S. What does the derivative mean? 89.T. It means the gradient function, the gradient of the curve is $2x$, of x^2. It's not a constant, is it? 90.S. No 91.T. The gradient, a constant? 92.S. No 93.T. It's a function of x 95.T. We call it a gradient function. We call it the derivative. There are other names as well, is that ok?

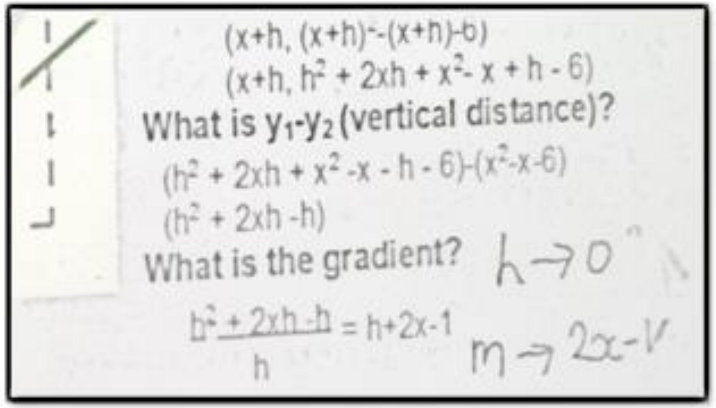
<table>
<tr>
<td>Symbolism in calculus teaching

<u>Subthemes:</u>

Calculus symbolism

[Leibniz notation – $\frac{dy}{dx}$;

Lagrange's notation – $f'(x)$]</td>
<td>T4LO & T2LO


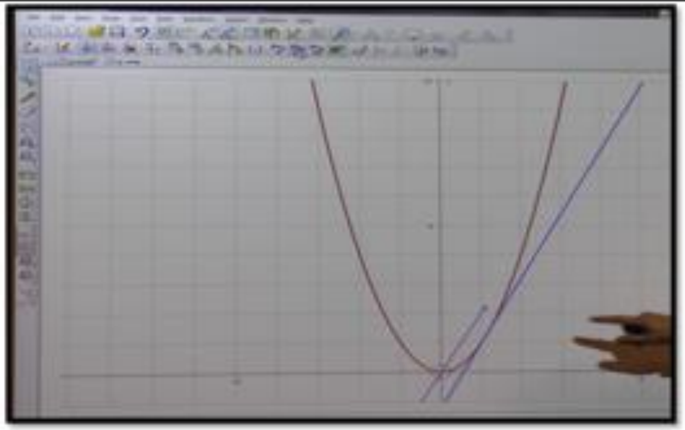


T4LO & T2LO
202.T. That's right because this is a general formula for the gradient. This could tell us the gradient at any point we want. It's not just $m = 10$. The gradient is not just 10 is it? Because Gxxx said right at the start of the lesson that the gradient changes. This is a general formula for the gradient at any point okay. Now we've got, we've got a name for that. We call it the gradient, we call this, we call this the gradient formula, okay. And we've got a special...we've got some special notation for it. Instead of saying $m = 2x$ we right $\frac{dy}{dx} = 2x$. And this is called the gradient formula, okay. So $\frac{dy}{dx} = 2x$, that's the gradient, that's the gradient formula for the curve.</td>
</tr>
<tr>
<td>Mathematical artefacts for calculus teaching

<u>Subthemes:</u>

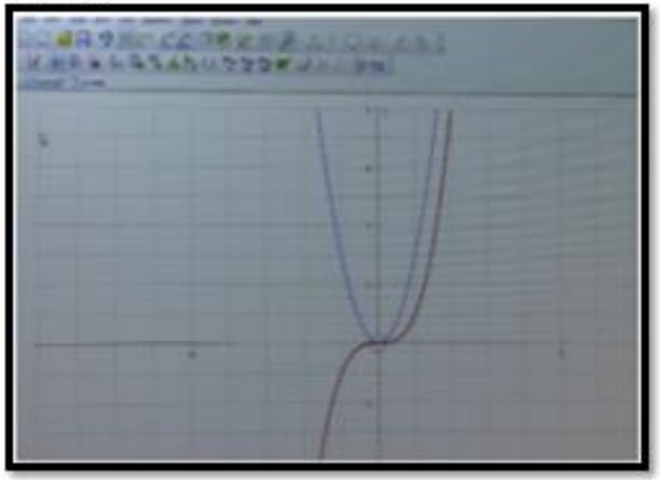
Digital artefacts;

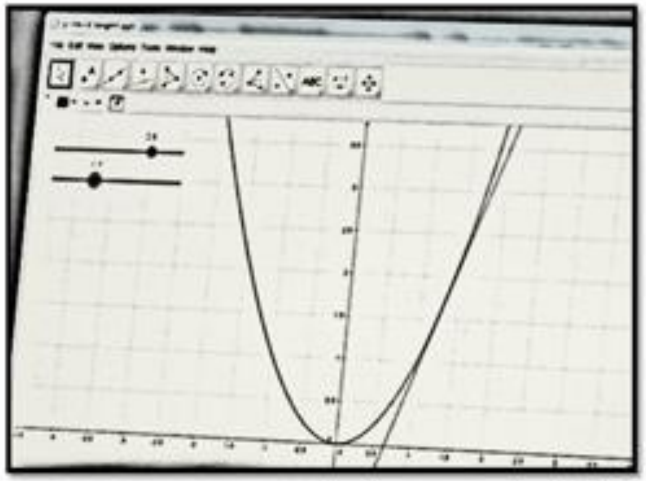
Dynamic imagery</td>
<td>T1LO:
62.T. Let me show you this now. This is a very powerful tool that Autograph has. I am going to show you the graph of the gradient function. What it is going to do is this, it's going to plot the gradient of that curve. It's going to travel down the curve and plot its gradient
63.T. Tell me, what can you tell me about the gradient of the curve to start with, it's going to start from the left-hand side and travel that way? What can you tell me, I don't mean what values there are, what sort of gradients are these? [negative]. They are [negative]/negative, right.</td>
</tr>
</table>



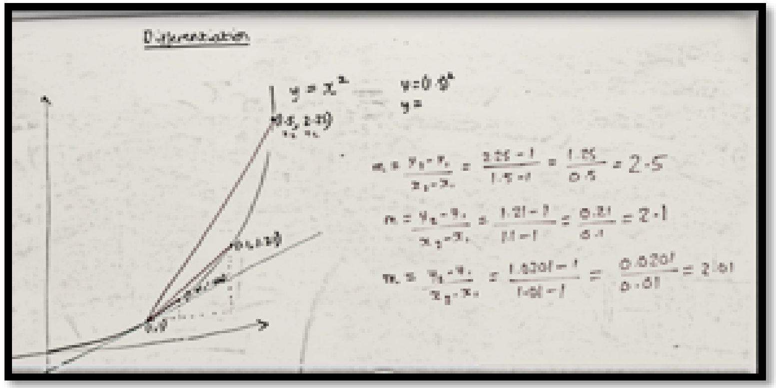
T1LO:

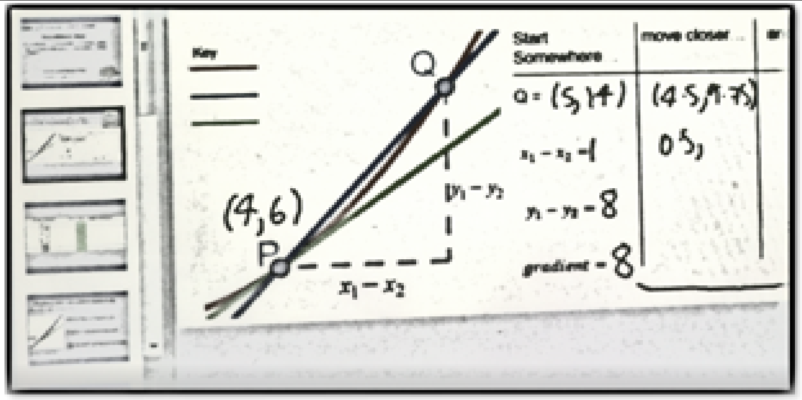
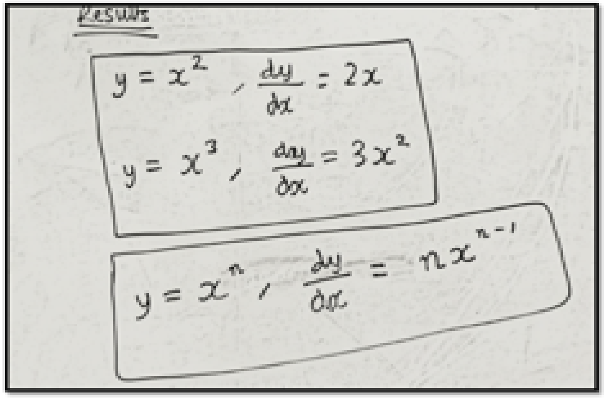
121.S. If that curve like you found out from the red curve [function $f(x) = x^3$], do you find out that the gradient is in that blue curve [gradient function $f'(x) = 3x^2$]?
122.T. Yes/
123.S. But that's a curve, so the gradient changes a lot, doesn't it?
124.T. Yes, that's the whole point the gradient just change; that's exactly the point for a curve the gradient is changing all the time.





T2LO:

	36.T. First of all, I want someone to come over here. I'm going to pick you [laughs]. What I would like you to do is to move these two sliders here with my mouse. See if you can make a tangent. And I want everyone else to figure out what those two sliders mean when he is making it. Yeah, it looks good, it looks good. Right sit back down. What do they mean? What's he moved there to make that tangent? What's going on? Jxxx? 37.S. Is top one got the gradient? And then the bottom one is like; it's transformed like where it is. 38.T. Yeah absolutely. So gradient up there. Second one, not trans …, well, something we can't see? 39.S. y-intercept. 40.T. Yeah, it's the y-intercept, good. So, we're kind of confident we can draw on a tangent. And maybe, especially cause we're on a graph plotter we could find out the equation, the gradient of this tangent which would give us the gradient at a point. Do we agree with that so far? Yeah, okay. **T5I(ii):** 34.T. Yeah, absolutely. At some point last year, we used GeoGebra and I like GeoGebra from a sense that most things that can be modelled even in 3-D you can do in GeoGebra. But in terms of graphing I like to consistently use Desmos now. 35.I. Desmos, yeah. And do the students also use graphing calculators? 36.T. Year 13 so at A2 they use it an awful lot because up until last year every student who came here was given an iPad but sadly this is the first Year 12 cohort where we've not been able to do that for them. So, we don't do it as much. Every, you know when the opportunity arises I do book out ICT suites but with the Year 13s it's just a matter of course that they're there and they just pick up their iPads and get on with it really but with the Year 12s not, not as much, and I wouldn't book an ICT suite out for the sake of using it for five minutes.
Mathematical **representations** in calculus teaching ***Subthemes:*** Algebraic representations Graphical representations;	**T4LO** 

Multiple representations	 **T2LO & T4LO**
Pedagogies on the derivative ***Subthemes:*** Approximating gradients by drawing tangents; Approximating the derivative at a given point on a curve, by using secant and tangent lines;	**T7LO:** 129.T. As h goes towards zero so we're looking at the limit as [*h*] gets smaller and smaller and smaller that point C, remember it was this graph up here *[The teacher pointing at the graph]* that point C gets closer and closer to point B because the triangle's shrinking down and it's becoming much more precise as a measure of gradient. The actual limit as [h] gets really close to zero, the limit of that value getting smaller and smaller is actually the gradient, so it becomes a precise value when h tends towards zero. So this is one way which we can find the gradient of a function, the gradient is found by substituting into this formula here, which won't mean an awful lot to you at the moment but we're going to practise doing this together in one of the questions in a moment. This bit you need to know so I'd highlight this bit for sure. we use the notation $f'(x)$ to stand for the gradient of the function $y = f(x)$. You will use that a lot; it will become like second nature. Find $f'(x)$ means differentiate which means find the gradient. **T4LO:** 136.T. When h gets really, really small, in other words, as h approaches 0…okay when h approaches 0, what do you think happens to the gradient? As h gets really, really small and it approaches 0, what do you think happens, happens to the gradient? 137.S. It approaches 2. 138.T. It approaches 2, brilliant. That's what we were, that's what we were always hoping wasn't it. We were, we were hoping that the gradient was going to be 2 because we sort of knew it would be. And now what Rxxx is saying is as h gets really, really small, well that h is just going to go away, it's going to be 0. So, the gradient we're left with…is 2. And that is differentiation.

From approximating gradients to differentiation.	**T2LO:** 373.T. We know as *h* gets smaller and smaller it goes to 0. So, then our gradient function, let's call it *m* for now for the sake of *it will go to* $2x - 1$ [*my italics*]. Now some people at the start of a lesson were finding that out through the process of *differentiation* which they knew. … We found out the *gradient function* for this [*Teacher pointing to the function*]. It was through something called *first principles*. Something we don't use and asked, one of the girls asked, it isn't tested on but it's really nice to know. I thought it was nice to finish off with something like this [Teacher displays a graph] because I know you've seen this before. We think about the distances there [*Teacher pointing to the graph of the function*], being like $\frac{dy}{dx}$. Then we denote it like that [*Teacher pointing to the graph*].
The '**why**' factors ***Subthemes:*** Teacher's belief statements (self-identity); Time constraints; Curriculum and assessment issues;	**T2I(i):** 106.T. I think the understanding is just as important as the process, the multiplying by the power thing, because I never ever got the connection between the gradient. I remember being at school myself and thinking oh right, that's how I'll find the gradient. I don't know why, why that is again, that's just differentiating. I knew differentiating as a process before I knew it was anything to do with the gradient, and I don't like that I knew that, I'd rather it was other way round. **T1I(i):** 13.T. Normally I give them a more. I make it an investigation over a lesson. I don't have a time for that now. I used to sort of really go to almost like a coursework activity, like we are going to be investigating, work out the rule. I am going to have to lead them a little bit more now to give them a little bit less. I have given them all different points to work on today. Before everyone did everything, but time is an issue, time, time for those things. I have never yet resort to telling them this is the answer, this is how you do it. I have always done something **T5I(ii):** 57.T. It came down to the way that I was taught differentiation and I was shown the algebra and then differentiated, got through the whole of my A level, got to University and sort of within the first few weeks went through the proof for the formula and thought 'I wish I'd be shown really what it was talking about and the reasons why we were doing things'. So, I thought I want to introduce it with both because some approaches just use the between two points and the points getting closer … **T3I(i):** 138.T. I think it's important for students to visualise what is happening. And I could have gone to first principles in the sense that it's still two points and that it would link but it would be a bigger jump. I feel like it's a smaller jump to go to the graphs first. 139. I. Ok. 140.T. And it also keeps linking back cause we, they will have to link differentiation with graphs in the future. So, I think it is useful to be talking about what's happening in a graph. And so, the visual side just gives a different way of viewing the topic, sort of the visual side as well as the algebraic.

	141.I. Right. 142.T. Rather than it feeling more abstract. I think it can feel a bit more abstract if you go straight into the algebra. From speaking to the teachers I've spoken to, one of the teachers misses out doing any graphical investigation and goes straight to the differentiation from first principles. Whereas the other teacher does spend time on working out the gradients of tangents from a graph. And she felt that had a lot of value. **T4LO:** 139.S. I don't understand it! 140.T. *It's difficult, it's really difficult. It's really difficult*. … It's *really complicated* maths. … So, this is really advanced stuff. So, it's fine if you find it difficult. … As long as you understand that all we're doing is, well firstly that *it's really difficult*. 312.T. *It's difficult*, isn't it? 366.T. *It's difficult*, 'isn't it? *It's difficult* stuff. **T4I(ii):** 50.T. I think the concept of adding a general value onto x and considering what happens as that value decreases is quite a hard concept for students to get their head round, especially 'because it's quite novel the idea that h isn't fixed and we have to consider what happens as h changes. I find that a difficult idea to get my head round.

To sum up this section, the process of data analysis progressed from the descriptive to the interpretative level; the latter of which sought to theorise the importance of the themes and their broader meanings, and implications in relation to the research questions, the commognitive theoretical framework and prior literature (Patton, 1990).

5.3.5 Producing the report

According to Miles and Huberman (1994), this is the ‘drawing conclusions’ (p.4) stage, which Roulston (2014) refers to as the ‘interpreting and writing up findings’(p.305) phase. In this phase, explains Roulston (2014, p.305),

> Researchers consider assertions and propositions in light of prior research and theory in order to develop arguments. Researchers develop stories that convey the main ideas developed in data analysis and present data excerpts or stories to support assertions.

The final stage of the data analysis involves selecting ‘vivid and compelling extract examples’ (Braun and Clarke, 2006, p.35), i.e. excerpts from the transcribed text for each theme. The selected excerpts were interpreted and analysed in light of the study’s research questions; the commognitive theoretical framework (Sfard, 2008) and existing research literature. This ‘final opportunity for analysis’ (Braun and Clarke, 2006, p.35) is presented as a scholarly report of the analysis of the data in the findings in Chapters 6 to 8.

Although this is presented as the final stage of the data analysis process, in practice, it happens concurrently with the other stages of analysis. In these stages, data analysis activities are interwoven streams that interact with one another throughout the analysis process (Roulston, 2014; Punch, 2009). Up to this stage, it was necessary to treat the six themes of Table 5.7 (See section 5.3.4 on page 94) separately for coding purposes. At this final stage, the themes were then regrouped to correspond with each of the research questions of the study as shown in Table 5.10 below, which gives an overview of the respective research questions and themes, sub-themes and the respective findings chapters for this study.

Table 5.10 Research questions, themes and linked chapters

Research Questions	**Overarching Themes**	**Sub-themes**	**Findings**
In teaching differential calculus: RQ.1 What *word types* and *narratives* do teachers use and why?	Mathematical language for calculus teaching.	Tangent and instantaneous rate of change The ‘gradient of a curve’; ‘Gradient’ and gradient function; The utterance: ‘getting closer and closer’	Presented and discussed in Chapter 6.

RQ2. What *visual mediators* do teachers use and why?	Symbolism and visual mediators for calculus teaching	Calculus symbolism [Leibnitz notation – $\frac{dy}{dx}$; Lagrange's notation – $f'(x)$] Algebraic, numeric and graphical mediation Digital artefacts and dynamic imagery Visual mediators and multiple representations	Presented and discussed in Chapter 7.
RQ3. What mathematical and pedagogical *routines* do teachers use and how?	Pedagogies on the derivative	Approximating gradients by drawing tangents; Approximating derivative at a given point of a curve, by using secant and tangent lines; From approximating gradients to differentiation.	Presented and discussed in Chapter 8.
RQs 1, 2 & 3 The 'Why' question.	The 'why' factors.	Teacher's belief statements (self-identity); Time constraints; Curriculum and assessment issues.	Presented and discussed across all the findings chapters.

5.4 Conclusion

For reporting the findings in line with the research questions of the study the six themes from Table 5.7 (See Section 5.3.4 on page 94) were further regrouped into four overarching themes as described and shown in Table 5.10 above. The findings under each of the first three overarching themes will be presented in the next three chapters; the 'why' factors are presented and discussed across all the findings chapters.

In the following three findings chapters, evidence illustrative of each of the main overarching themes is presented and examined further, relative to the commognitive framework and the research questions of the study. A range of methods to represent data is used as evidence to support the findings, including brief quotations and excerpts from interview and lesson observation transcripts, descriptions, 'diagrams and visual representations of key concepts; and narratives that represent participants' experiences and perspectives' (Roulston, 2014, p.305). The excerpts in the findings Chapters 6, 7 and 8 will present lived experiences of the participants, along with an in-depth analysis of those excerpts, making a critical appraisal of the findings. The next three chapters report the findings of the study by telling stories about the participant teachers' pedagogical calculus discourse.

Chapter 6 Mathematical language for calculus teaching

6.1 Introduction

This chapter is the first of three chapters reporting the findings of this study, which are henceforth presented according to the overarching themes of the research, namely: Mathematical language for calculus teaching (Chapter 6); Symbolism and visual mediators for calculus teaching (Chapter 7); and Pedagogies on the derivative (Chapter 8).

This chapter will present evidence for and discuss the findings of the research under the mathematical language for calculus teaching theme (see Table 5.7 for the overarching themes). Undergirded by the commognitive theoretical framework, this chapter will report on, and discuss the word use or specialised mathematical terminology and endorsed narratives in the teachers' calculus discourse, as well as ordinary words used in everyday communication, but with special and specific meanings in mathematics that were used by teachers in teaching calculus. Word use and visual mediators are the tools with which the participants of the discourse 'identify the object of their talk and coordinate their communication' (Sfard, 2008, p.145). I will report on visual mediators in Chapter 7.

The excerpts presented for each subtheme in this chapter come from the participants' data transcripts from both the interviews with teachers and the lesson observations on elementary differential calculus. The evidence is drawn from all the participant teachers of the study. For each subtheme, the coding process identified several excerpts from across the participants' data sets. The excerpts presented and discussed for each subtheme are representative and should be seen as illustrative evidence for the findings presented.

Chapter 6 will present and discuss the findings that address the first research question:

In teaching differential calculus, what word types and narratives do teachers use and why?

Calculus discourse is characterised by many specialised mathematical words. Focusing on word use (Sfard, 2008) in the teacher's utterances allows for an analysis of what was said against the intended meaning. In their introductory lessons on differential calculus, teachers in this study used many calculus words, including tangent, the instantaneous rate of change, gradient (slope), derivative, gradient formula, gradient function, differential and differentiation in describing and explaining the derivative at a point and the derived function, $f'(x)$ of a function $f(x)$. The evidence in Sections 6.2 and 6.3 show that there exists some inconsistency in how some calculus words, for example, gradient, tangent and slope, are used in teaching differential calculus. This study found evidence of ambiguity with the use of some calculus keywords in the participant teachers' pedagogical calculus discourse on elementary differential calculus.

What follows are four subsections covering the following sub-themes, namely: tangent and instantaneous rate of change; gradient and 'gradient of a curve'; and the utterance 'getting closer and closer'. This will be followed by a discussion on the findings and a chapter summary.

6.2 Tangent and 'instantaneous rate of change'

This section presents evidence of the teachers' use of specialised calculus words. The evidence indicates some inconsistencies in word use in the teachers' differential calculus discourse. For example, there are three different excerpts presented on the teachers' use of the word tangent and a further two different excerpts are presented on the instantaneous rate of change to illustrate the finding. The other commonly used words were gradient, differentiation, calculus and utterances such as constant gradient and gradient function, The words 'tangent' and 'gradient' were central in all the introductions to differential calculus lessons, and special coverage on gradient and the utterance 'gradient of a curve' is presented in Section 6.2.

There were different definitions and descriptions for the tangent to a curve in the teachers' differential calculus discourse, and to illustrate the variation, three excerpts from different lessons [T5LO], [T1LO] and [T2LO] are presented here. T5 talks about a 'tangent' as going through a point, whilst T1 talks about 'instantaneous direction' as the 'tangent' of the curve, whereas T2 talks about the 'gradient of a tangent' as the 'gradient' of the curve.

The first is an excerpt from T5, which illustrates the construction of the definition for 'tangent', T5LO:

166.T. A tangent, fantastic. Just remind us, what a tangent is?

169.S. Straight

170.T. Straight! Thank you. Yes important, but people might not realise exactly how important. A tangent is a straight line going through…

171.S. that point.

172.T. Anywhere else? No. That point and that point only. Touching the curve only once okay. If I can very accurately draw that line, I can find the gradient on it and that will work absolutely perfectly.

The teacher acknowledges the fact that a tangent is a straight line. However, the teacher's utterances that 'A tangent is a straight line going through…'[170] and 'Touching the curve only once'[172] could imply or possibly be interpreted to mean that a tangent crosses the graph at the point of intersection. However, drawing a tangent at this moment gives an illustration of the 'touching' of the curve. Nonetheless, as much as it is important to pay attention to what is being said, it is also important to consider what is not being said; as can be drawn from the excerpt above.

The second excerpt comes from [T1LO] and the teacher here is using the words 'tangent' and direction' in describing 'gradient'. Using the graph of the function $y = x^2$, T1 talks about the tangent. In [20] - [25] the T1 is defining and explaining 'tangent' by illustrating its key properties with the aid of a graph of the function $y = x^2$. [T1LO]

20. T. Is there anywhere on that curve where you definitely, already know its gradient?
21. S. $x - axis$.

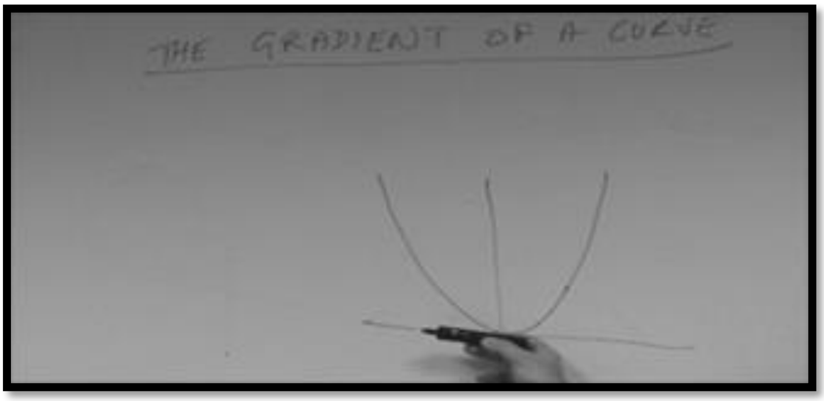


Figure 6.1 A sketch diagram for the graph of $\mathbf{y = x^2}$

22.T. Good, would you all accept that the x-axis is a tangent to the curve? What is the gradient of the $x - axis$?
23.S. Zero.
24.T. Zero. A tangent, you did this in mechanics, is sort of the direction in which you are instantaneously travelling.
25.T The direction in which you're going there [*Teacher pointing at the graph on the board*] is the instantaneous direction, the tangent of the curve.

'Instantaneous direction' [25] is described here as 'the tangent of the curve' [25]. This observation is consistent with the teacher's word use from the pre-lesson interview. I asked T1 about how he was going to approach the lesson, and he said: [T1I(i)]

38.T. Well, today I am going to ask them what we mean by *the gradient of a curve* […] but what do we mean by gradient? And I will ask them, you know, what do you think the *gradient* is there? I'm hoping someone will tell me, urr it's the *tangent*, and I will say Ok.

''… urr it's the *tangent*, …' [38]. Does this mean that the tangent is the gradient, one may ask. In literate mathematical discourse, the slope of the tangent describes the gradient at a given point on a curve. There are two mathematical objects of instruction here, i.e. 'direction' and 'tangent', and 'direction' describes the slope of the tangent, rather than 'tangent'.

In contrast to [T1LO] description of tangent above, consider an example from [T2LO] with regards to the words, 'tangent' and 'gradient'.

40. T. So, we're kind of confident we can draw on a tangent. And maybe, especially because we're on a graph plotter we could find out the equation, the gradient of this tangent which would give us the gradient at a point. Do we agree with that so far?

41. S. Yeah.

What is described as 'direction' in [T1LO, 25], describes 'the gradient' of the tangent, instead. Thus, the utterance 'the tangent of the curve' [T1LO, 25], is inconsistent with the literate mathematical discourse, such as the utterance that 'the gradient of this tangent which would give us the gradient at a point' [T2LO, 40].

There were some inconsistencies in word use with 'instantaneous rate of change' in introducing differential calculus. Two exemplar excerpts are given here to illustrate this finding. In the lesson [T1LO] shown above, the teacher talks of 'instantaneous direction' [25] in describing the tangent to a curve at a given point on the curve. In [T3LO], however, the teacher talks about 'rates of change' and describes differentiation to be 'about the instantaneous rate of change, at that instant' [328]. The teacher used an example of real-life situations to explain differentiation, by playing a video showing aspects of the history of calculus about the founders of calculus (Newton and Leibniz) and the 100m world record holder, Usain Bolt. See Figure 6.2 for the video [T3LO].

326.T. Now I just want to play you a minute of this just to give you a little overview. If you want to, by all means, you can go on to YouTube and watch the whole thing, but these are three people who are very relevant to what we're doing today. [https://www.youtube.com/watch?v=EKvHQc3QEow].

Figure 6.2 The founders of calculus

328.T. But just, that gives you a feel for what we are doing. Hxxx asked me what's the point in this. Why, what is differentiation about? It is about *instantaneous rates of change, at that instant*. So, for Usain Bolt how fast is he going at that moment, maybe as he crosses the finishing line? Not necessarily just over the whole race, but at each *instant*.

So obviously on a curve, we have *instantaneous rates of change*, that is what we're dealing with when we're working with *differentiation*. Now in the next lesson, I'm going to take you further into the algebra side of it, but hopefully, this has given you an overview of what *differentiation* is about and an introduction to *differentiation*.

The teacher in [T3LO] links 'differentiation', 'instantaneous rate of change' and curved line graphs and describes the connection as 'an overview of what differentiation is about and an introduction to differentiation' [328]. The teacher's utterance here 'So obviously on a curve we have… instantaneous *rates* [my italics] of change, that is what we're dealing with when we're working with differentiation'[328] implies that there are 'rates' of change on a curve, unlike on a straight line, where there is a rate of change.

Teachers used the word calculus freely but did not define it. This is exemplified in the excerpt [T3LO] below. When the teacher [T3] said "Hxxx asked me what's the point in this. Why, what is differentiation about?" [328], she was referring to an earlier dialogue with a student during the lesson. In this dialogue, a student asks the teacher why they were learning about gradients [T3LO]:

203.S. Why are we doing this? Like why do we need to do this? Why do/
204.T. Why do we do this topic?
205.S. Yeah.
206.T. Calculus is a really important part of maths. You can use it in so many different ways. Rates of change, so you can look at people's speeds and accelerations when they're running. You can look at rates of change in biology when bacteria are growing. There are lots of different applications.

Notice that the student described it as ‘this’ [203]. It was the teacher who then described ‘this’ to refer to the topic. This suggests the student’s unfamiliarity with the name of the topic, let alone calculus terminology. Calculus is explained as important for calculating rates of changes such as speed and acceleration and bacteria growth in biological studies [206]. The examples given in [T3LO] above describe differential calculus. Calculus as explained here [206] suggests differential calculus. The term calculus appears in the teacher’s utterance [206] was not defined. This was a common occurrence in most lessons observed, in which the term calculus was used. This example shows how the students’ initial experience with calculus here, could be seen as synonymous with differentiation.

6.3 ‘Gradient’ and ‘gradient of a curve’

The word ‘gradient’ was at the core of each lesson, but there are observable differences in how different teachers used this term, and to illustrate the differences in the teachers’ word use on ‘gradient’ and ‘gradient of a curve’, three excerpts from two different lessons, [T1LO] and [T4LO], will be examined. A further five excerpts from four different teachers, [T3LO], [T2LO], [T5LO] and [T4I(ii)], will be examined to illustrate the different (including some inconsistent) ways in which the words ‘constant gradient’ and ‘changing gradients’ were used by the teachers.

The ‘gradient of a curve’, what could that possibly mean? Let us examine these (given below) three excerpts from two different lessons, by two different teachers, in which both teachers are introducing differentiation. T1 in [T1LO] writes on the board ‘The gradient of a curve’ as the title of the lesson, see Figure 6.3 below. Whereas T4 in [T4LO] writes (and asks the students to write in their books) ‘Differentiation’ as the title for the lesson (see Figure 6.5 on page 116).

In the first excerpt [T1LO] below, T1 is introducing the mathematical object of the lesson [T1LO]:

4.T. I want to pose a problem to you, and the problem is this… [*Teacher writing on the whiteboard* – “The gradient of a curve”].
5.T. What do we mean by that? That's my first question to you. Now we all know, I hope what is meant by the gradient of a line.
7.T. How do you measure the gradient of a line then? How do you measure the gradient?
9.T. Right, so my question to you is what do we mean by *the gradient of a curve*?

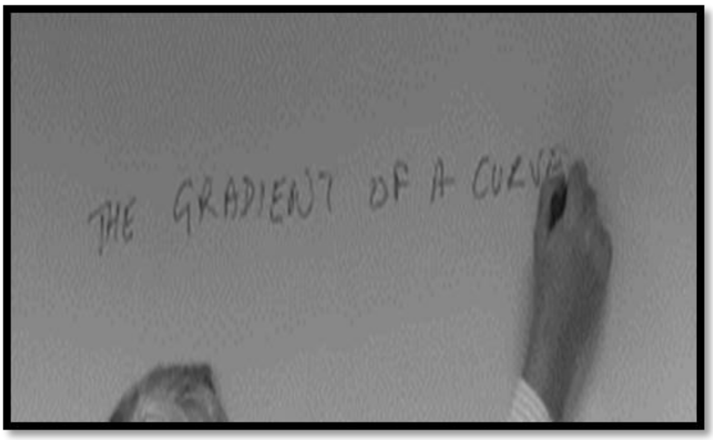


Figure 6.3 The gradient of a curve

The mathematical object of the teacher's discourse is framed here as 'the gradient of a curve'. The teacher begins by asking the 'what' gradient question and then he changed the question to the 'how' to measure the gradient of a [straight] line, and then asked about the 'what' 'gradient of a curve' [9]. The questioning suggests that understanding 'how to' measure the gradient of a line, would lead to understanding 'what is' the 'gradient of a curve'; it does not say 'at a point'. This utterance and the object (discursive), 'gradient of a curve', mediated both verbally [4] and visually [Figure 6.3], is consistent with the teacher's word use in the pre-lesson interview. I asked T1 how he was going to introduce differentiation, and he said: T1I(i)

38.T. Well, today I am going to ask them what we mean by *the gradient of a curve* and I am going to see what sort of answers I will get. I don't even know if I would use the word *derivative* today. I may well don't use that. I might use the word *gradient function*. We will see. I mean, it might just come out of my head, you know, but what do we mean by gradient? And I will ask them, you know, what do you think the *gradient* is there? I'm hoping someone will tell me, urr it's the *tangent*, and I will say Ok.

The teacher's utterance in this excerpt suggests that the teacher is referring to the same mathematical object by the *gradient of a curve*, *derivative,* and *gradient function*. Note that when the teacher talks about the *gradient* at a point, he refers to it as a *tangent*. What is described as the 'gradient of a curve' refers to the 'gradient function' or 'derived function'. This utterance that describes differentiation as 'the gradient of a curve' or 'gradient' as the 'tangent' is ambiguous.

There is an implied association in these utterances, between the gradient of a straight line and the 'gradient of a curve'. The utterance, 'the gradient of a curve' implies the derivative of the function as a constant. Note that the utterances do not specify any particular point on the curve. The graph of $y = x^2$ has infinite points, thus infinite gradients of the tangents to the curve at these infinite points. Thus, the utterance ‘gradient of a curve’ is inconsistent with the literate mathematical discourse about derivatives. Literate mathematical discourse describes the gradient of a tangent at a given point on the curve.

A further look at the lesson, T1LO, shows that the words *gradient* and *tangent* were frequently in the classroom discourse for introducing the derivative. Consider these two object-level narratives regarding gradient from the teacher, T1, during the first lesson on differential calculus.

99.T. The gradient of a curve is not constant, it would depend on x, and it's called the gradient function.
119.T. What I want you to try to understand is that the gradient of a curve is the gradient of a tangent, that I do want you to appreciate, that's important.

These two utterances are contradictory. The teacher’s utterance in [99] that the ‘gradient of a curve is not constant’ is inconsistent with his definition in [119] that ‘the gradient of a curve is the gradient of a tangent’. A tangent here is a straight line, and the gradient of a straight line is, indeed, constant. It follows, therefore, from the utterances above that the *gradient of a curve* [99] cannot be the *gradient of a tangent* [119]. In *literate* mathematical discourse, the gradient function describes the gradient of the tangent *at any point* on the curve. The word use ‘gradient’ in [99] is for the *gradient (or derivative) function*. However, the word use ‘gradient’ in [119] is for the *gradient (or derivative) at a given point*. Unlike the ‘gradient’ in [99], this gradient is not a *function*, but a constant, i.e. a number. At any given point on a curve, the gradient (derivative) of a curve is, in fact, equal to the gradient (derivative) of the tangent to the curve at that given point. What should rather be important about this utterance [119] is in fact what is missing from the utterance, *‘at a given point’*. The utterance in [119] is, therefore, a metonymic statement, a shortened phrase, which if students were to take as given, ‘it would be considered mathematically incorrect, a misstatement’ (Zandieh and Knapp, 2006, p.10).

In the lesson, T1LO, the utterances ‘gradient of a curve’ by the teacher appeared eight times, four times within the first eight minutes, and again four times in the last eight minutes of the 60-minutes lesson. Analysing each of the utterances within its context reveals that by the ‘gradient of a curve’, the teacher was referring to the gradient (or derivative) function. For example, consider the following two episodes. The first one is an extract of the classroom discourse just before the teacher’s narrative in [99] above. The second one is an extract of the classroom discourse immediately after the teacher's utterance in [119] above. Following on the teacher's explanation of the derivative of the function $f(x) = x^3$, which was *visually mediated* through the dynamic imagery of Autograph on the board (see Figure 6-10 below), the following dialogue with a student resulted: T1LO:

88. S. What does the derivative mean?
89. T. It means the gradient function, the gradient of the curve, is $2x$, of x^2. It's not a constant, is it?
90. S. No
91. T. The gradient, a constant?
92. S. No
93. T. It's a function of x.
94. S. Yeah
95. T. We call it a gradient function. We call it the derivative. There are other names as well, is that ok?

There is dualism in the meaning or application of words such as gradient and derivative is illustrated in the excerpt [88-91] from T1LO given in the preceding paragraphs above. By the gradient of the curve, the teacher is referring to the gradient function [89] and note that here 'gradient function' is synonymous with the 'derivative' [95]. Note, however, that *the gradient function*, the *gradient of a curve* [89] and the *gradient* [91] are all referring to the same mathematical object that is exemplified in [95]. An analysis of word use here, reveals that the utterances in [91] and [93] are contradictory; 'the gradient' is indeed a 'constant'! Thus, there are some inconsistencies in word use of calculus terminology.

The mathematical object of the discourse here is the *derivative as a function*, certainly, not the *derivative at a point*. The teacher's frequent use of the specialised mathematical words in his utterances above [89, 91 & 93 & 95] explains what the teacher was referring to by 'gradient of a curve' or derivative. Notice that the teacher followed up this episode with the utterance in [119] above, which implied that the 'gradient of a curve' was constant. This prompted another episode of questions from a student to the teacher. [T1LO]

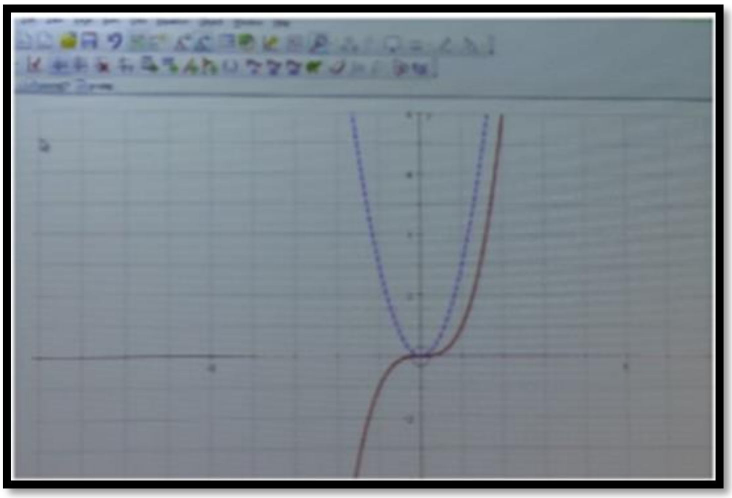


Figure 6.4 Gradient function of $f(x) = x^3$

121.S. If that curve like you found out from the red curve [function $f(x) = x^3$], do you find out that the gradient is in that blue curve [gradient function $f'(x) = 3x^2$]?
122.T. Yes/
123.S. But that's a curve, so the gradient changes a lot, doesn't it?
124.T. Yes, that's the whole point the gradient just changes; that's exactly the point for a curve the gradient is changing all the time.

The student's questions and utterances [121; 123] could suggest that the graphical mediation (showing the graphs of both the function and its gradient function) used by the teacher helped the student in constructing the definition of gradient function.

In a different lesson [T4LO] (see excerpt below), another teacher, T4, is introducing the mathematical object of the lesson, differentiation [T4LO]:

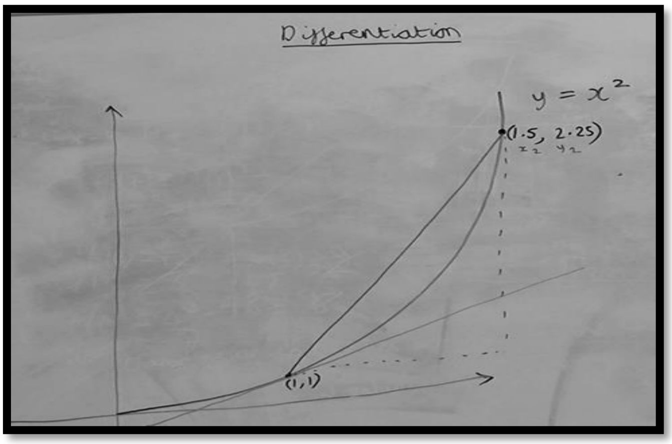


Figure 6.5 Differentiation

7. T. It [*Gradient*] changes at different points. Gxxx could you just describe what happens to the gradient as we move from this point [(1, 1)] to that point [the other point shown on the curve in Figure 6-4]?

8. S.	Yeah, the gradient increases.
9. T.	The gradient increases because the line is getting steeper. So somehow, mathematicians ummed and aahed about this for a while. We needed to come up with a way of working out *the gradient of a curved line*. (...). Can you open your books please and put the title – Differentiation. Okay. Can you put the title – Differentiation?

In contrast to lesson [T1LO] above, here [T4LO] both the calculus words 'gradient' and 'differentiation', are used right from the start of the lesson. Describing the sketch diagram of the $y = x^2$ graph on the board, see Figure 6.5, T4 draws the students' attention to the variability of the derivative of a curved-line graph; thus, the changing 'gradients' at different points [7]. However, similar to [T1LO], the description 'the gradient of a curved line' in the utterance: 'We needed to come up with a way of working out the gradient of a curved line'[9] seems inconsistent with changing or increasing gradients [7 & 8]. What is the word 'gradient' in 'the gradient of a curved line' refer to? The utterance, 'gradient of a curve' is an ambiguous statement, and as we have seen in the two examples [T1LO] and [T4LO], there is evidence of this colloquial word use in introductory lessons on differentiation.

It appears though, that the utterance 'the gradient of a curve' was not just a feature of the introductory part of the lesson, but a common word use. It was not a simple slip of the tongue. The excerpt below from [T4LO], shows this is an advanced stage of the lesson, where the teacher is assigning the students some mathematical problems/questions to solve. T4LO:

565.T.	Okay so Year 12, if a question asks us to find *the gradient of a curve*. What is that clue to do first? If a question asks us to find *the gradient of a curve*, what is that clue for us to do?
[Student answers… 'differentiate']
Differentiate, absolutely. Because it's asking us for the gradient at a particular point you know we need to find the gradient function because that's the formula we're going to use to try and find out the gradient.

In the teacher's utterance [565], the phrase 'gradient of a curve' is repeated twice. Twice, the teacher is asking for a clue [565], to associate with the phrase 'the gradient of a curve'. This excerpt suggests that 'the gradient of a curve' is synonymous with 'differentiation' as explained in the second part of [565]. However, the teacher here goes on to substantiate what he is referring to as 'the gradient of a curve' to mean 'the gradient at a point'.
In an interview [T4I(ii)] with T4 after his lesson on introduction to differentiation, the teacher stated that students' challenges with understanding variables were a contributing factor as to why they find differential calculus difficult. This teacher's understanding of students'

difficulties with learning calculus, could explain the approach as shown in the excerpt [7-9] above.

6.3.1 Gradient and gradient function

Further evidence of inconsistency in word use can be seen in the lessons, T3LO and T2LO. Both the teachers T3 and T2 are describing the same mathematical object, the derivative of a quadratic function $y = x^2$ and $y = x^2 - x - 6$, respectively), but use two different words. T3 describes the derivative of the function $y = x^2$ as: T3LO:

178.T. The gradient is $2x$. Let's just have a look. So $2x$, if we multiply each of these by 2, we seem to get the gradient. So, we're thinking on, if, you can fill your table in now for that first curve. We're thinking the gradient is $2x$.

The utterance ' *gradient is* $2x$*'* [178] seems consistent with the utterance '*gradient of a curve'* [T1LO, 9; T4LO, 565]. The word gradient is used, for example in these three different lessons, to describe what in fact is a gradient function. $2x$ is not the 'gradient' but the 'gradient function' of the function $y = x^2$. In these lessons, the word gradient would describe a constant value for the gradient of a tangent at a given point on the curve. In contrast to T3LO in which $2x$ is described as a gradient, in T2LO the term gradient function is used instead. T2LO:

374.S. So, was the *gradient* $2x - 1$?
375.T. The *gradient function* was $2x - 1$, yeah, for any point x

The student's word use is *gradient* [374], but this teacher immediately responds using the words *gradient function* [375], the latter of which is consistent with literate mathematical discourse. The word gradient seems to have two uses or meanings here. It has been used to refer to the gradient of a tangent at a point on the curve and it has also been used to refer to the gradient function of a function f(x).

Another example, to suggest some inconsistency in word use with the transition from *gradient* (for straight line graphs) to *gradient function* (for curved line graphs) is shown in the excerpt below from T4LO. This excerpt highlights the need for consistency with word use and symbolism in calculus discourse. T4LO:

202.T. This is a *general formula for the gradient at any point* okay? Now we've got a name for that. We call it the gradient, we call this, we call this the *gradient formula*, okay. And we've got a special…we've got some *special notation* for it. Instead of saying $m = 2x$ we write $\frac{dy}{dx} = 2x$ and this is called the *gradient formula*, okay. So $\frac{dy}{dx} = 2x$, that's the gradient, that's the gradient formula for which curve?

Consider these two utterances: (i) "We call it the *gradient*, we call this, we call this the *gradient formula*, okay" [202], and (ii) "So $\frac{dy}{dx} = 2x$, that's the *gradient*, that's the *gradient formula* for which curve?" [202]. Note the word use in both utterances, $\frac{dy}{dx}$ is referred to as the *gradien*t and as the *gradient formula* too. The visual mediation on the board work (See Section 6.4) shows both $\frac{dy}{dx} = 2x$ and $m = 2x$ labelled *gradient function*. Up until this stage, students would have known m to be used to represent the gradient of a straight. The students would have learnt from their GCSE that $y = mx + c$ is the general form for the equation of any straight line, where m represents the gradient (slope) of the line. Thus, $\frac{dy}{dx} = 2x$ and $m = 2x$ would imply that $\frac{dy}{dx}$ and m are signifiers for the same mathematical object, which they are not.

There is also evidence showing teachers using the same word 'gradient' to describe a constant quantity and to describe a variable quantity. To illustrate this inconsistency in word use, five representative excerpts from four different lessons, [T3LO], [T2LO], [T5LO] and [T4I(ii)] will be examined below. Consider, the utterance 'The gradient is $2x$' in the first excerpt below, [T3LO]:

178.T. The gradient is $2x$. Let's just have a look. So $2x$, if we multiply each of these by 2, we seem to get the gradient. So, we're thinking on, if, you can fill your table in now for that first…curve. We're thinking the gradient is $2X$.

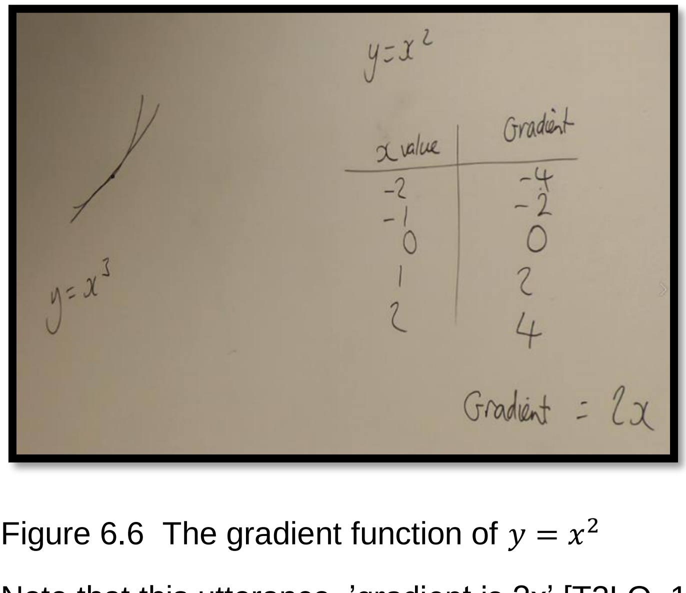


Figure 6.6 The gradient function of $y = x^2$

Note that this utterance, 'gradient is 2x' [T3LO, 178] by T3, is consistent with the utterance 'gradient of a curve' [T1LO, 9; T4LO, 565] by T1 and T4. Figure 6.6 is a visual mediation of the object of the teacher's discourse here. It shows a table of values and a function 'Gradient = $2x$'. This visual mediator is consistent with the utterance in [178], which seems to describe

$2x$ as a constant (uses the word gradient instead of gradient formula) rather than a variable.

Contrast the word use in [T3LO] above with the following [T2LO]. Whilst [T3LO] describes $2x$ as a gradient, T2's utterance 'gradient function is $2x - 1'$ in [T2LO] is endorsable. Both teachers are describing the derivative of y = x^2 and $y = x^2 - x - 6$, respectively. In [T2LO] the teacher talks of gradient function.

374.S. So, was the *gradient* $2x - 1$?
375.T. The *gradient function* was $2x - 1$, yeah, for any point x.

Compare the student and the teacher's utterances on the derivative of $y = x^2 - x - 6$. Note that, like the teacher in [T3LO, 178] above, the student here (in a different lesson) uses the same word 'gradient' [T2LO, 374] to describe what the teacher [T2LO, 375] immediately substantiates as, 'gradient function'.

6.3.2 Gradient and changing gradients

Introducing the derivative developed from gradients of straight-line graphs to investigating curved-line graphs. The need to compare and distinguish between constant gradient and changing gradients was inevitable. There is evidence to suggest some inconsistency in the teachers' utterances on constant gradient and variable gradient. Two excerpts given below demonstrate this inconsistency. These excerpts are from two different lessons by two different teachers. Teacher T3's approach in [T3LO] is to use distance-time graphs, see Figure 6.7 and Figure 6.8, to introduce changing gradients. The other teacher's (T5) approach in [T5LO], was to calculate the gradient of a straight line passing through two given coordinates, see Figure 6.9.

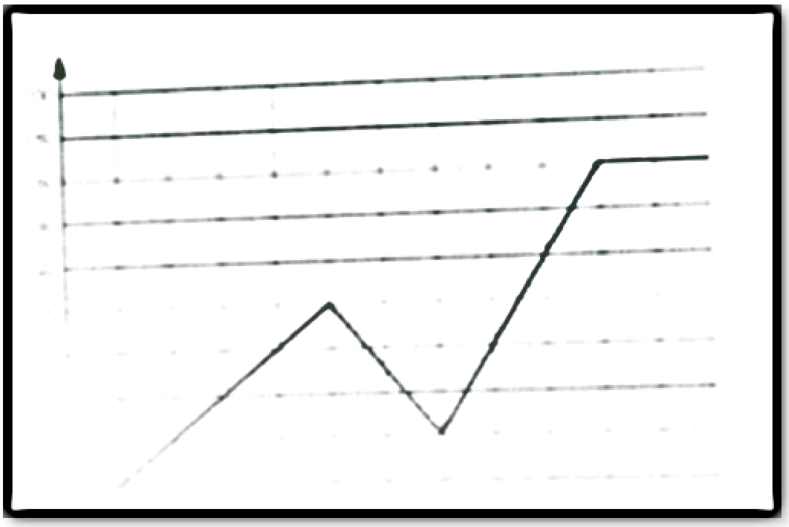


Figure 6.7 Distance-time graph

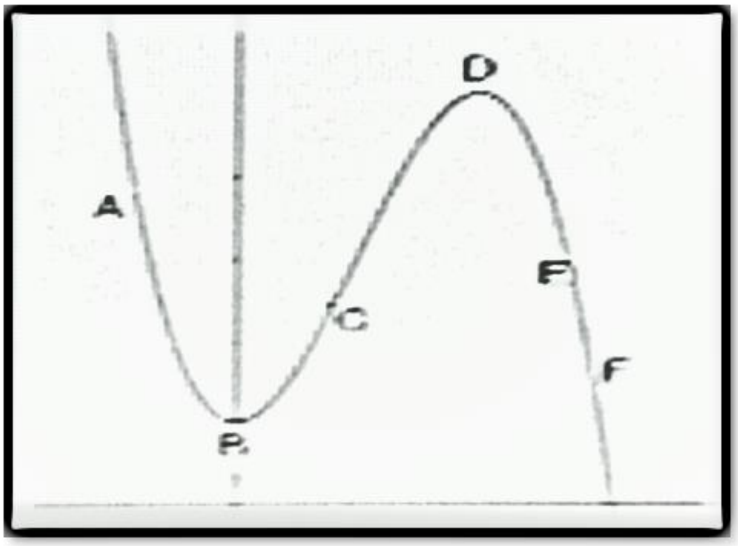


Figure 6.8 Curved-line graph

The teacher [T3LO] presents a straight-line distance-time graph (Figure 6.7) and a curved-line distance-time graph (Figure 6.8). Both the diagrams show different gradients at different points marked on the graphs. The main difference here is that in one diagram there is a straight-line graph, and the other diagram is a curved-line graph. [T3LO]

24.T. Now, what I want us to look at today is to look at finding *the gradient of a curve*. What can anyone tell me about the *gradients of these curves*? [Points to Figure 6.8]. Can anyone tell me anything about those gradients?

27.S. Does it vary?

28.T. They vary good. So, the *gradient on the curve varies*. Did it vary on this last graph? [Points to Figure 6.8].

29.S. Yeah.

30.T It did, but what's the difference between the way that one [Points to Figure 6.7] is changing and this one [Points to Figure 6.8] is changing? Can you tell the difference between the gradient changes on that [Points to Figure 6.7] graph and the gradient changes on this [Points to Figure 6.8] one?

31.S. That [Pointing to Figure 6.8] one is always variable and it's changing.

32.T. Right, so this [Points to Figure 6.7] one stays at a constant gradient for a section and then changes for the gradient for another section. Whereas here [Points to Figure 6.8] it keeps changing, *the gradient keeps changing*.

Note, also, that in this utterance [24], there is yet more evidence of teachers using the words '*the gradient of a curve'*; which has been shown earlier in this Section 6.2 to be an ambiguous statement.

The teacher's utterance [32], shows *changing gradients* on a straight-line graph and *changing gradients* on a curved-line graph. What then, is the difference in *the changes* and what does this mean for introducing differentiation? Note that the teacher describes the curve-line graph as *curves* [Figure 6.8] in "What can anyone tell me about the *gradients of these curves*?" [24]. The teacher acknowledges that "the gradient on the curve *varies*" [28]. The teacher also acknowledges that the *gradients of the lines* in [Figure 6-6] also *vary* [30]. Note the teacher's word use in the excerpt, the words '*vary*' and '*changing'* are being used interchangeably. Although an attempt was made to distinguish between *constant gradient*, which describes the gradient of a straight line and *changing (or variable) gradients* for a curved-line graph, the distinction between *constant gradient* and *variable gradient* is not clear. The teacher's utterances in [28 – 32] together with the visual mediator of the two graphs [Figure 6.7 and Figure 6.8] suggest '*changing' constant gradients* and '*changing' variable gradients*.

In a different lesson, [T5LO], the teacher starts with a review of calculating the gradients of straight lines between two given sets of points, see Figure 6.9 for the examples that were used in the lesson. Using the examples shown in Figure 6.9, the teacher T5 explains the difference between a *constant* gradient and *variable* gradient, [T5LO]:

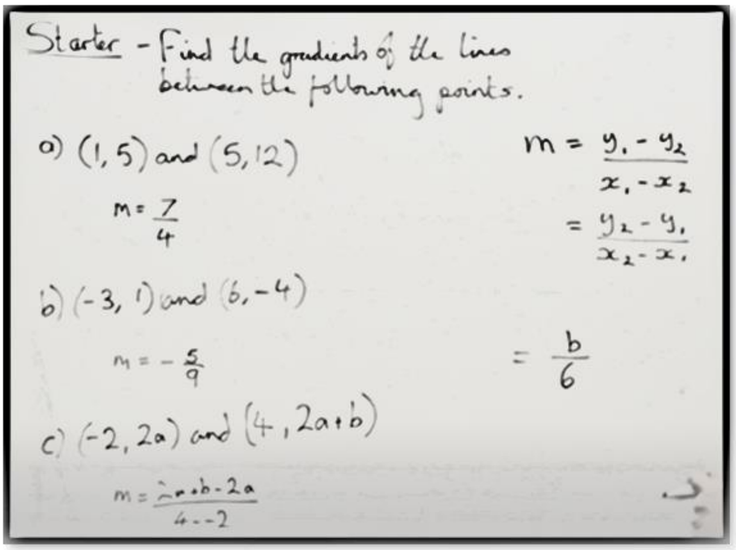


Figure 6.9 Gradient of a straight line

30.T. Okay, what is it then? [Referring to part (c) in Figure 6.9]. If it's not *constant* it must be a …?
31.S. Variable.
32.T. *Variable*, thank you very much Kxxx, okay? This is variable, okay? What about these two gradients? [Referring to part (a) & (b) in Figure 6.9].
33.S. Constant.
34.T. They're *constant*. Okay, they've got constant values, therefore they're constants okay. We're going to be spending a lot of this lesson calculating gradients.

The students were tasked with calculating the gradients of lines between pairs of coordinates. Describing the gradient of a straight line between two points, the teacher says "*Variable*, thank you very much Kxxx, okay? This is variable, okay" [32]. This utterance could potentially be confusing here because the gradient of a straight line is constant. Thus, every straight line passing through the given pairs of points in Figure 6.9 has a constant gradient. Therefore, talking of *variable* gradients here would require further substantiation, because as presented, the utterances are not consistent with the endorsed narratives in mathematical discourses.

Students' difficulty with algebra, in particular, in distinguishing between constants and variables [T4I(ii)] is said to be a factor that contributes to students' difficulty with calculus. In a post-lesson interview with one of the participant teachers [T4I(ii)], he reveals his challenges of introducing differential calculus to his class, and his experiences in teaching Further Mathematics, in which more advanced elements of calculus are covered, [T4I(ii)]:

58. T. I think there's a real issue with the following – students have difficulty firstly distinguishing between *constants* and *variables* in algebra. So even when I

> was teaching FP2 [Further (Pure) Mathematics 2] last year, one of my students still didn't understand the difference between *a, b,* and *c* and *x, y* and *z* really. So, there's that issue. Yeah.

Notably, the teacher remarks "I think there's a real issue with the following – students have difficulty, firstly distinguishing between constants and variables in algebra" [58]. Ambiguity with, or lack of understanding of the distinction between a constant or variable measure, for example, the inconsistency with word use, such as, 'gradient' to refer to both 'constant gradient' and 'gradient function', could potentially make it the learning of differential calculus more challenging.

Talking to teachers in the interviews, there is evidence to suggest that some teachers were hesitant to use the words *differential* or *derivative* in the introduction lesson, even though they actually used these words in their lessons. For example, in T4LO, see how the teacher retracts his use of the word differential in the excerpt below. T4LO:

416.T. And in maths we write the *differentia*l of $f(x)$as $f'(x)$ okay. So, you either call your equation $y = x^3$ in which case the *differential* is, the gradient function is $\frac{dy}{dx}$. But if you choose to call your curve $f(x)$, then your *differential*, your *differential*, sorry your gradient function is $f'(x)$. …

The teacher uses the word 'differential' four times in [416], but he quickly retracts it, and resorts to using the words 'gradient function'. Reflecting on his word use in the post-lesson interview, T4 explains that he had not intended to use the word derivative, even though he was using terminology such as gradient function and differentiation. T4I(ii):

160. I. In the second half, there was a point when you were using the word *differential* and then you'd quickly say the *gradient function* …
161.T. Yeah, I wanted to say *gradient function*.
163.T. Yeah, I kept remembering I was teaching somebody who had only just learnt it and I want to refer to it as the *gradient function*.
167.T. Do you know I'm quite tempted, if I was doing that again, I'd be quite tempted to only ever say *gradient function* and keep using that and then only at a very late stage explain that 'look, if you see *find the derivative* that's just synonymous with *find the gradient function*'. I'd rather them intrinsically think of it as finding the gradient function as a reminder of what the *gradient function*, what *differentiation* tells you about it.

These examples from interviews with teachers (e.g. T1I(i) and T4I(ii)) show that word use is a planning and teaching consideration for the teachers introducing differential calculus. The teachers preferred using the words g*radient function* to *derivative* and *differential* in the introductory lessons. Both the teachers said that they were hesitant to use the words *derivative* and *differential* in the introductory lessons, although they both indeed used these words in their first lesson. T4 was focusing more on the derivative as a function [T4I(ii) - 167], which explains his insistence on using the words gradient function, instead of the word differential or derivative. For whatever other reasons the teachers, T1 and T4, might have, they both thought that they would rather use the terms derivative or differential at a later stage in teaching differential calculus.

6.4 The utterance ‘getting closer and closer’

In their pedagogical calculus discourse, teachers used the utterance ‘getting closer and closer’ in describing what appears to be the concept of limit, which is a key component of the ‘limit definition of the derivative’. The evidence suggests that all the teachers, except one [T7], avoided using the word ‘limit’ in their introduction and defining of the derivative. Instead, teachers used the utterance ‘getting closer and closer’ in explaining what appears to be the ‘limit definition of the derivative’, which I shall from now on refer to as the ’quasi-limit definition of the derivative’. ‘Quasi-limit definition of the derivative’ is a preferred object-level narrative to denote the informal or incomplete nature of the explanation given for the ‘limit definition of the derivative’ (more in Section 8.4 and Section 9.2.1). The concept of limit is a key component of the formal ‘limit definition of the derivative’, and this was missing or deliberately avoided.

In introducing and explaining the derivative, the teachers used two points, e.g. points A and B on a curve (mostly the $y = x^2$ graph) to estimate the slope (gradient) of the tangent at one of the two points, say at point A. The procedure is then to move point B towards point A, in so doing, calculating the slope (gradient) of the successive secant joining point A to point B, until the gap between point A and point B is so infinitesimally small that the secant AB becomes almost the tangent of the curve at point A. This process would describe the tangent as the limit of the secant, but it was the words ‘…closer and closer…’ that were used in explaining the process. Three representative excerpts from [T4LO], [T3LO] and [T5LO] are presented below to illustrate the finding.

In the excerpt [T4LO], T4 is leading an investigation of the slope (gradient) of the tangent to the function $y = x^2$ at the point, $x = 1$, and have computed three different gradients equal to 2.5, 2.1 and 2.01, by using $x = 1.5$, 1.1 and 1.01, respectively. [T4LO]:

75.T. That’s it, good stuff. I think that’s going to be 0.0201 divided by 0.01. What is dividing by 0.01 equivalent to year 12? Gxxx, that’s it, times by a hundred. So, the answer’s going to be 2.01. Right year 12, have a moment to think about our three answers. Think about *what's happening to the value of the gradient as we get closer and closer and closer to our tangent* which is what we want, okay? *Does anyone want to have a look at those three numbers and have a guess at what the, what the true gradient of the curve at (1, 1) might be*? Hxxx?

76.S. 2.
77.T. 2, why is that then?
78.S. Cause it's just *getting closer and closer to*…
79.T. Exactly. *We're getting closer and closer and closer to 2*. *We're getting closer and closer and closer to 2*. So, I think the real answer to this question is going to be the gradient is 2 but we need to try and prove that now. We need to try and prove it, okay?

The phrase 'getting closer and closer' is repeated four times in this short excerpt. The teacher's question here: 'what's happening to the value of the gradient as we get closer and closer and closer to our tangent' draws attention to the limit of the sequence of gradient values. Also, the question 'Does anyone want to have a look at those three numbers and have a guess at what the, what the true gradient of the curve at (1, 1) might be?' [75], draws attention to the idea of approaching the limit value for the gradient at (1, 1) on the graph of the function. In the utterance [79], the phrase 'getting closer and closer' is twice repeated, and the word 'closer' is repeated three times in 'We're getting closer and closer and closer to 2. We're getting closer and closer and closer to 2' [79]. The teacher's utterance [79], confirms that the mathematical object of the teacher's talk, here, is the limit value, although there is no mention of the word 'limit'.

Similarly, T3, also uses the same words 'getting closer and closer' making the point that 'getting closer and closer' would give a better and better approximation, to the slope of the tangent. [T3LO]:

190.T. Probably 12. Right okay, do you want to try maybe on the back, try different points? So, you want to *get closer and closer*. Can use it even down here, *get closer and closer and closer* and try and get to a confident position on it.

193.S. Yeah. So is it basically because they're just getting smaller and/
194.T. Yeah, well it's *a better and better approximation*. Every time you *bring this point closer* it's a better approximation/
195.S. Oh, right okay.
196.T. /to what the tangent would be.
197.S. Oh, so doing it from this point?
198.T. Yeah, because you want the gradient at the point negative 2.
199.S. Oh, right okay.
200.T. So, you would use that brilliant idea of using two points. And then if we *move those two points closer*, we *get a better, better approximation*.

This dialogue [T3LO] between a teacher and students suggests that it is not clear as to what the mathematical object of the teacher's talk is here. What is *getting closer and closer,* and to what, and why? are potential questions that call for clarity here, as can be seen in the student's questions [193; 197]. T3's utterance 'if we *move those two points closer, we get a better, better approximation*' [200] does not seem clear enough to identify the mathematical discursive object here. The utterance in [200] could imply that both points are moving towards each other. Getting closer and closer for a better approximation, evidence in the excerpt above hardly identifies the object of the teacher's talk.

Another teacher [T5] had challenges trying to describe the same mathematical object, using the words '*getting closer and closer together'*. Late in the lesson, the teacher decided to demonstrate what he expected the students to do and how they should have done it. With the aid of visual mediation, see Figure 6.10, the teacher attempts to give further guidance. [T5LO]:

229.T. Yes, please. Okay whatever order that you've done them in, could you just take a very quick look at, at mine? This could be an example of the five coordinates that I chose. So, my original co-ordinate was (0,0). And my other coordinates I've started with $x = 3$ and gradually my x –coordinate has *gotten slightly less and less* towards (0,0).

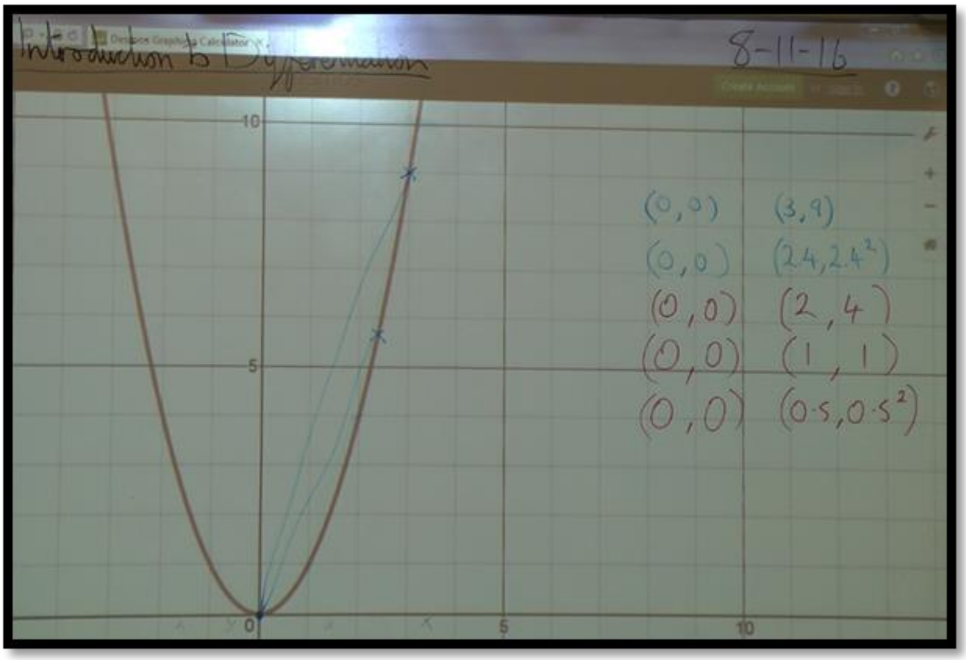


Figure 6.10 Investigating gradients on a curve

229.T. I'm going to ask you in a minute. I'm going to ask every group for their values for the gradients in that order okay? *Getting closer and closer* towards your points so that your two points that you're choosing are *getting closer and closer* okay? So, if you could write your gradients as a list of five gradients for me, please.

It is not clear what the last part of the T5's utterance [229] *'*getting closer and closer towards your points so that your two points that you're choosing are getting closer and closer okay' [229], is referring to. The visual mediation by the graph on the board did (does) not imply both points moving. The 'two points that you are choosing are getting closer and close', how and what point, was not clear. Following the explanation, some students struggled to understand what was required of them, as exemplified by a couple of excerpts given below. The first episode was with one group of students who were 'doing them randomly' [T5LO]:

347.S. Just the random ones. But then like that one goes up by 2, that one goes up by 2.
348.T. Okay and is this, are these *getting closer and closer* to your value every time?
349.S. No, we're just doing them randomly...
350.T. Okay I need them; I need them in order.

Note that the students' utterance in [349] that they were just doing them randomly; this suggests that these students still had not understood the teacher's utterance "Getting closer and closer towards your points so that your two points that you're choosing are getting closer and closer" [229]. In the second episode below, the teacher is talking to a different group of students, [T5LO]:

354.T. But I need them in order *getting closer and closer*. So, choose the $x\ value$ that's furthest away from 2, first.
355.S. Yeah.
356.T. And I need it from one side as well. So, if you're going to go above, I want the one that's furthest away. So, is it 3? So, (3,9) there yeah.

The students had not understood the necessity of ordering the x values and the intended meaning of 'getting closer and closer'. The computation of the gradient (slope) of the tangent line, i.e. the instantaneous rate of change of a function f(x) at a particular point on the curve, requires the computation of the limit of f(x) as x approaches that point. It is this limit that is called the derivative of f(x), and the process of computing the derivative is called differentiation. Teachers avoided using the word 'limit' in their lessons on introducing differentiation but found it difficult to bypass the limit component of the definition of the derivative. Teachers did not use the word 'limit' even where they appeared to be describing the concept. Instead, the utterance 'getting closer and closer' was used repeatedly to describe the process of approaching a point or a limiting value, and thus teachers described a 'quasi-definition of the derivative'.

6.5 Discussion

The evidence, as presented in Chapter 6, (e.g. Section 6.2 and 6.3) shows that there exists some inconsistency in how some calculus words are used in teaching differential calculus. Word use in the teachers' discourse about the derivative (at a point, and as a function) will now be discussed.

Gradient and tangent

This study found evidence of inconsistent word use in the teachers' pedagogical calculus discourse, inconsistent with literate mathematical discourse (presented earlier in Section 6.2 and Section 6.3), examples include the ambiguous utterance 'gradient of a curve' and describing 'gradient' as 'tangent' [T1I(i), 38], or 'instantaneous direction' as 'the tangent of the curve' [T1LO, 25]. These findings in the teacher's use of tangent for the slope of the tangent were found in the students' word use in Ng (2018). The students' utterances in Ng's(2018) study included 'tangent is increasing; tangent line is zero'(p.1183), but it was the slope that was increasing and it was the slope that was zero.

The teacher's utterances in my study could be seen as a slip of the tongue or could be thought of as metonymic statements. This finding also aligns with the findings of Zandieh and Knapp (2006), who reported metonymy use by students in calculus discourse; 'the derivative is the tangent line' and 'the derivative is the change' (p.10). Zandieh and Knapp (2006, p.7) examined the metonymic use of words or signifiers such as derivative and function in the students' calculus discourse. My study, however, reveals metonymic misstatements in the teacher's discourse, e.g. the utterance [T1LO - 25], which describes the 'instantaneous rate of change as the tangent'.

Zandieh and Knapp (2006) describe metonymic misstatement as the use of a 'metonymic short-cut or part of the phrase to stand for the whole phrase' (p.7). Although metonymic misstatements could simply be a slip of the tongue, they could also be an indication of an incorrect understanding of the object at that time, and worse still, if the student believes the metonymic misstatement to be true (Zandieh and Knapp, 2006), for example, if a student believes that the tangent line is the derivative.

As shown in Zandieh and Knapp's (2006) study, such metonymy use 'is not always mathematically valid, for example, the tangent line as the derivative' (Park, 2016, p.398). Zandieh and Knapp (2006):

> Consider the phrase "the derivative is the slope of the tangent line at a point." This phrase can be shortened to "the derivative is the slope," which is appropriate usage in the mathematical community. On the other hand, if a student shortens the phrase to "the derivative is the tangent line," it would be considered mathematically incorrect, a misstatement (p.10).

From my study, both statements describing the gradient as the tangent, and the instantaneous direction as the tangent, are both metonymic misstatements. It is natural to use shortened statements in everyday speech.

These findings suggest that it is important to consider the possible challenges the use of metonymic misstatements can bring, for the students when it is coming from the teacher in a lesson, for example, the teacher describing gradient as tangent (see Section 6.2 page 109). Zandieh and Knapp (2006) report on two metonymic misstatements most commonly used misstatements by students in the interviews they conducted to investigate the students' conception of the concept of derivative. The two misstatements are "the derivative is the tangent line" and "the derivative is the change" (p.10). Eight out of the nine students in a calculus class (who they interviewed three times) made at least one of these two misstatements. Zandieh and Knapp (2006) describe as a less extreme case if the use of a metonymic misstatement comes from a student who usually uses the correct phrase because the student might be focusing on a particular aspect of the concept in that shortened phrase. For example, Zandieh and Knapp (2006, p.10):

> A student might say "the derivative is the tangent line" while referencing a graph with a function and a tangent line drawn. The student may be focusing on the tangent line, because it is visually explicit, even though when questioned they can clarify that it is the slope of the tangent line. At first, this may simply look like a slip of the tongue, but for some students, it appears they may at that moment be mentally focusing on the tangent line as the "picture" of the derivative.

Now, imagine if these metonymic misstatements are coming from the teacher in an introductory lesson on the derivative. To the teacher, it may be a slip of the tongue to say the gradient, direction or derivative is the tangent line, but it might not be seen or understood as such by all the students in the class. Such metonymic misstatements and colloquial word use 'not only impacts the [students'] later use of mathematical words, but also other aspects of mathematical discourse such as endorsed narratives, routines, and visual mediators'(Kim and Lim, 2017, p.1577). It is, therefore, important that research and practitioners pay attention to the language-specific nature of discourse, as it can constrain or promote students' participation in calculus discourse.

The gradient of a curve

The evidence in Section 6.1 and 6.2 show that there is some inconsistency in how some (calculus) words, for example, gradient and slope, are used in teaching differential calculus. In this study, the word 'gradient' is used to describe the 'slope' of a straight line. The word 'slope' describes the rate of change of a function with one independent variable, e.g. $y = f(x)$. Thus, what is often referred to as 'gradient', could more specifically be described as 'slope', i.e. the slope of a straight line or the slope of a tangent line.

For example, in introducing differential calculus, one teacher writes on the board '*The gradient of a curve'* [T1LO] (see Figure 6-3) and the other teacher writes '*Differentiation*' [T4LO] as the titles for their respective first lesson. What does the gradient of a curve mean? Could this be construed to mean differentiation? Scheja et al. (2008) argue that making connections between the properties of graphs of a function and that of its derived function would build a better understanding of the underlying graphical concepts of calculus. Students making connections between different representations depends largely on how the various forms of mediation are presented in the classroom discourse.

The evidence in T1LO [121-124] in Section 6.1 could also suggest that the inconsistent use of the word 'gradient' for both "the gradient function" and "the gradient at a point" prompted the student in [121-124] above to dispute the teacher's utterance in T1LO [119]. [T1LO]

121.S. If that curve like you found out from the red curve [function $f(x) = x^3$], do you find out that the gradient is in that blue curve [gradient function $f'(x) = 3x^2$]?
122.T. Yes/
123.S. But that's a curve, so the gradient changes a lot, doesn't it?
124.T. Yes, that's the whole point the gradient just changes; that's exactly the point for a curve the gradient is changing all the time.

The student utterance in the excerpt [121 – 124] above, on one hand, substantiates the narrative that is visually mediated graphically showing the function $f(x) = x^3$ and its derived function, $f'(x) = 3x^2$. On the other hand, it illuminates the problems with the teacher's metonymic statement that 'the gradient of a curve is the gradient of a tangent' [119]. Also, note the teacher's utterance that "the gradient just changes" [124]; the word 'gradient' here is used as an irregular plural, which is consistent with the colloquial use of the word ‘gradient’, for example, in his utterance ‘the gradient of a curve’ [99; 119]. Zandieh and Knapp (2006) identify another metonymic relationship that is often associated with the use of the word *the derivative*, which often causes difficulties for students in learning differential calculus, the ‘relationship, between a function value and the function itself’ (p.7).

Zandieh and Knapp's (2006) study highlights similar theoretical issues in the metonymic use of calculus words or signifiers such as derivative and function, for example, the dual use of the word derivative to refer to either “the derivative value” or “the derivative function” (p.7). Zandieh and Knapp (2006) note that such use is common in colloquial discourse, but they argue that using such metonymic short-cuts is more likely to cause confusion than using full statements that are consistent with literate discourse. Past research has shown that students struggle with dualism inherent in some calculus words, such as derivative, for example, Zandieh and Knapp (2006) report of students struggling to explain ‘the conflict between calling something a derivative that is a slope or limit value (…) with something that is a graph or equation of a function’ (p.7).

Although it is consistent with literate mathematical discourse to talk of ‘the slope (gradient) of a straight-line graph’, it is rather colloquial or inconsistent with literate mathematical discourse to say, ‘the slope (gradient) of a curved line graph’. Instead, literate mathematical discourse describes the slope (gradient) of the tangent to a curve at a given point.

Gradient and gradient function

Studies on students’ learning calculus (e.g. Monk, 1994; Tall and Vinner, 1981; Park, 2013) have shown that moving from (and between) the derivative at a given point on a curve, to the derivative as a function is not simple for students. The teachers’ calculus and pedagogical discourse in the classroom can facilitate such a transition for the students, by making an explicit distinction between gradient (derivative at a point) and gradient function (derivative as a function). The analysis of students’ discourse on the derivative by Park (2013), also found students describing ‘the derivative as a tangent line’, which suggests that the students ‘considered the ‘derivative’ as a point-specific object but also a (linear) function defined on an interval’ (p.624).

This study found evidence of dualism in the meaning or application of words such as gradient and derivative, e.g. in T1LO [88-91] and T2LO [374-375]; there is dualism in the use of the word gradient, thus, gradient as a constant and gradient as a function. The dualism in the application of the word gradient suggests the use of the word derivative. Replacing the word gradient with the word derivative seems to explain the teachers' use of the word gradient. The inconsistency and the ambiguity with 'the gradient of a curve' would be eliminated by substituting the word *derivative* for *gradient*, thus 'the derivative of a curve'. For example, the derivative of the function $f(x) = x^2$ would mean the derived function of $f(x) = x^2$. The 'gradient is $2x$' would become 'the derivative is $2x$'.

The derivative can be seen as a process of computing or determining many successive values for the difference quotient as h approaches zero, and as the product of this process, the derivative as a function (Park, 2016). Such dualism inherent with the word derivative is a source of potential challenges for students (Zandieh, 2000; Oehrtman et al., 2008). The word derivative has a dual meaning in calculus discourse. The word derivative is sometimes used to refer to the derivative as a constant, i.e. derivative of a function at a given point, and other times it is used to refer to the derivative as a function, i.e. the gradient function of a function (Zandieh and Knapp, 2006; Park, 2016). This dualism was not substantiated in the observed lessons; it was not made explicit for the student. This could mean that students may not be able to understand the dual application of the terminology in calculus. Park (2013) conducted a survey with twelve elementary calculus university students to analyse students' word use and use of visual mediators on the derivative. The study discovered inconsistencies in the students' use of the word derivative for describing the derivative at a point and for describing the derivative function. Park (2013) found that students did not understand the distinction between the *derivative at a point* as a constant and the *derivative* as a function.

The teacher draws attention to the changing *gradients* at different points [7], which is important for understanding the derivative as a co-varying function (Park, 2016). Oehrtman et al. (2008), too, argue for the need to link and match the instantaneous rate of change with the corresponding continuous changes in the independent variable. Making a connection between the changing gradients and the corresponding changing independent variable is non-trivial for students. It is therefore important that teachers make this connection explicit in their instruction on the derivative, more so in introducing the derivative to students, who might be learning about this complex idea for the first time.

As noted by Park (2015) past research has since shown that ‘transitioning between the point-specific and interval views or viewing the derivative function as an object is non-trivial to students’ (p.234). As demonstrated by the T1LO above, there is a need for teachers to carefully consider their use of the word gradient or derivative in their introductory lessons on calculus. It is important to pay attention to the type of words teachers use to refer to the derivative at a point or the derivative as a function, to help students with the transitioning between the two uses of the word ‘the derivative’. Zandieh and Knapp's (2006) study shows that the two most commonly used metonymic misstatements are both associated with the word *derivative:* ‘the derivative is the tangent line’ and ‘the derivative is the change’ (p.10). Kim and Lim (2017) argue that students’ learning is also impacted by ‘the meanings of everyday language in mathematics learning … since some words have significantly different uses in mathematics’ (p.1563). It is important to examine the teachers’ word use the students' difficulties with learning calculus could stem from metonymic misstatements and colloquial word use.

6.6 Summary of findings

The teachers in this study used many calculus words, including tangent, the instantaneous rate of change, gradient(slope), derivative, gradient formula, gradient function, differential and differentiation in describing and explaining the derivative at a point and the derived function, $f'(x)$ of a function $f(x)$. The evidence, as presented in Chapter 6, (e.g. Section 6.2 and 6.3) shows that there exist some inconsistencies in word use with the transition from *gradient* (for straight line graphs) to *gradient function* (for curved line graphs). For example, in T1LO, where the title on the board read *‘The gradient of a curve’*, or in T3LO - the utterance ‘the gradient is *2x*’. Besides, the study found evidence of dualism (and so ambiguity) in the meaning or application of words such as gradient and derivative, e.g. in T1LO [88-91] and T2LO [374-375]; the word gradient and derivative were used to refer to both, gradient as a constant and gradient as a function.

Further inconsistency was found in the use of calculus symbolism, for example, the transition from *gradient* (for straight line graphs) to *gradient function* (for curved line graphs). In particular, in the use of the visual mediator m and $\frac{dy}{dx}$. For example, in [T4LO] (See Section 6.3.1 pages 118-

9), where $\frac{dy}{dx} = 2x$ and $m = 2x$ would imply that $\frac{dy}{dx}$ and m are signifiers for the same mathematical object, which they are not.

The study also found that when substantiating the definition of the derivative, the teachers avoided using the word 'limit' even where they appeared to be describing the limit component of the definition of derivative (see Section 6.4 above). Instead, the utterance 'getting closer and closer' was used repeatedly to describe the process of approaching a point or a limiting value (for more refer to Section 8.4 and Section 9.2.1 for discussion).

Chapter 7 Symbolism and visual mediators for calculus teaching

7.1 Introduction

Chapter 7 is the second of three chapters reporting the findings of the study, which are henceforth presented according to the overarching themes of the research, namely: mathematical language for calculus teaching (Chapter 6); symbolism and visual mediators for calculus teaching (Chapter 7); and pedagogies on the derivative (Chapter 8).

This chapter presents evidence for and discusses the findings of the research under the symbolism and visual mediators for calculus teaching theme (see Table 5.7 for the overarching themes on page 94). The excerpts presented as evidence for the findings and discussed in this chapter (and indeed in the other two findings chapters) are exemplar excerpts from the interview and the lesson observation data transcripts from across all the participant teachers in this research. It is important to remember that in thematic qualitative analyses, the coding process often generates many codes for each theme (Ryan and Bernard, 2000; Braun and Clarke, 2006; Campbell et al., 2013). The exemplar excerpts are presented as representative and illustrative evidence for each subtheme and hence, for the findings of the research.

This chapter presents and discusses the findings that address the second research question of the study.

In teaching differential calculus, what visual mediators do teachers use and why?

Chapter 7 is informed by the commognitive theoretical framework (Sfard, 2008) and presents evidence on calculus symbolism and visual mediators in the teachers' pedagogical calculus discourse. According to the commognitive theoretical perspective, visual mediators could be physical visible objects or symbolic artefacts such as algebraic notation and expressions, tables, graphs, diagrams or drawings that are used to mediate instruction (Sfard, 2008; Ryve et al., 2013). Visual mediators (and word use) are the tools with which the participants of a discourse 'identify the object of their talk and coordinate their communication' (Sfard, 2008, p.145). Sfard (2008) further asserts that visual mediators are an integral part of the act of communication in literate mathematics discourse. Indeed, visual mediators are an integral part of the teachers' pedagogical calculus discourse. Not only is it necessary, therefore, but important in research that seeks to investigate the teachers' pedagogical calculus discourse, to pay special attention to visual mediators and the teachers' acts of mediation in their calculus discourse.

Sfard (2015) describes symbolic artefacts referring to numerals, algebraic expressions and graphs, and that they serve as 'representations of impalpable mathematics objects' (p.132). What follows is a presentation of evidence under three subtheme/ sections: calculus symbolism and algebraic mediation; graphical mediation and digital artefacts; and visual mediators and multiple representations. These subthemes will be followed by a discussion on the findings and a chapter summary. Digital artefacts such as GeoGebra and Autograph were used mainly for graphical mediation, thus the subtheme: graphical mediation and digital artefacts.

7.2 Algebraic mediation and calculus symbolism

In this study, the evidence shows that teachers used both, Leibniz's notation, $\frac{dy}{dx}$ and Langrage's notation, $f'(x)$, but the teachers were cautious not to introduce such calculus symbolism early in the introductory lesson. The evidence also shows the use of the same signifier or symbolic artefact for the derivative at a point and for the derivative as a function, for example, the use of m in T4LO and T2LO.

This section reports on the teachers' use of algebraic symbolic mediation in teaching differentiation. Presented here are examples of the algebraic symbolic representation, as visually mediated, from four different lessons by different teachers, [T4LO], [T2LO], [T1LO] and [T7LO]. T4LO is selected as an example that illustrates the transition from the derivative as a constant to the derivative as a function. In introducing the derivative, unlike all the other lessons that started with $y = x^2$, T2LO started with y = $x^2 - x - 6$, and it is chosen here, as an example to illustrate the algebraic manipulation of the quotient difference as $h \rightarrow 0$ in introducing the derivative. L7LO is the only lesson that used the word *limit*, so it is chosen to illustrate how (if so) the symbolism in this lesson compares to the other lessons. Collectively, this section presents five visual algebraic representations, i.e. board work snapshots, and four excerpts as evidence of the teachers' algebraic symbolic mediation in introducing differentiation.

In the lesson [T4LO], the teacher used the function $y = x^2$ and a sketch diagram of the graph of $y = x^2$, see Figure 7.1 and Figure 7.2 below. Before this stage, the lesson had progressed from computing the gradient of a chord, by using two given points, (x_1, y_1) and (x_2, y_2) with specific numerical coordinates on the graph. The teacher had used the formula $m = \frac{(y_2-y_1)}{(x_2-x_1)}$, where m represents the gradient of a straight line passing through the two given points (x_1, y_1) and (x_2, y_2) for computing the gradients of chords. Figure 7.1 and Figure 7.2 capture the stage when the lesson had moved away from numerical coordinates to general points (x, x^2) and $[(x+h), (x+h)^2]$. Thus, the two snapshots show the algebraic computation for the gradient of a secant line passing through points (x, x^2) and $[(x+h), (x+h)^2]$. These general coordinates are substituted into the formula $m = \frac{(y_2-y_1)}{(x_2-x_1)}$ and what is shown in Figure 7.1 and Figure 7.2 is the algebraic representation of the process of calculating the gradient of the function $y = x^2$ at the point (x, x^2). These two figures show the algebraic manipulations and simplification of the quotient $m = \frac{((\mathrm{x+h})^2-\mathrm{x}^2)}{((\mathrm{x+h})-\mathrm{x})}$ culminating into $m = 2x$.

Note the symbolic mediation $h \to 0$ in Figure 7.2, depicts that as h approaches zero, the gradient $m = 2x$, and this does not explain the gradient as a limit. Note that in the teacher's utterance [192], there is no mention of the word limit. Instead, there are two signifiers in the teacher's utterance, "as h gets closer to 0" and "as this point gets close, close to the original", both describe $h \to 0$ (as *h* approaches *0*). [T4LO]

192.T. Okay. Time to think again guys. The gradient we found is that the gradient is equal to $2x + h$. Same, same question as last time. What happens as *h* gets closer to 0? What happens is, in other words, what happens as this point gets close, close to the original, the original point as *h* gets close to 0? What is the gradient of that line? [*Teacher pointing to the secant on the sketch diagram for the graph of* $y = 2x$]

193.S. $2x$

The student's answer [193] could be seen as evidence that the teacher's narrative was effective in explaining the process of differentiation without resorting to the formal algebraic representation of the definition for differentiation, i.e. the limit definition of the derivative. [T4LO]:

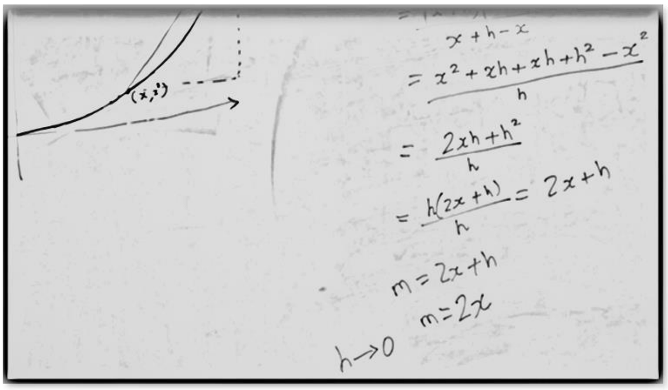


Figure 7.1 The gradient of secant through (x, x^2) and $[(x+h), (x+h)^2]$.

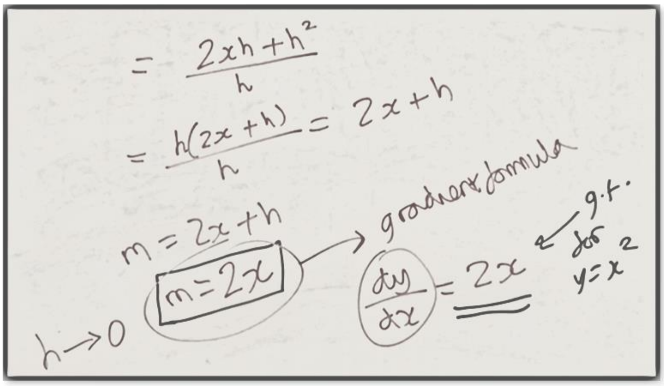


Figure 7.2 From m (gradient) to $\frac{\mathrm{dy}}{\mathrm{dx}}$ (gradient function)

The algebraic symbolic manipulation as shown in [192] in Figure 7.2 is the basis upon which the teacher introduces the derivative of the function $y = x^2$, resulting in these two utterances, "$m = 2x$ is the gradient formula" of $y = x^2$; and "$\frac{dy}{dx} = 2x$ is the gradient function of the function $y = x^2$" [Figure 7.2]. Here, it can be construed that these two utterances imply that $m = \frac{dy}{dx}$.

It was towards the end of the lesson, that the word *differentiation* was linked to $\frac{dy}{dx}$. In this excerpt, the teacher is referring to Figure 7.2, which shows the gradient function of $y = x^2$. Note that according to the teacher's utterances in the excerpt below, [T4LO]:

194.T. Yeah. $m = 2x$. Okay, because as *h* approaches 0, *h* becomes 0 and we're just left with $m = 2x$. Now that looks a bit weird because we've now got a gradient that isn't just a number, okay? What's gone on there? Why have we, usually in GCSE we get $m = 1$ or $m = 3$? Why have we got a bit of algebra for our gradient?

202.T. That's right because this is a general formula for the gradient. This could tell us the gradient at any point we want. It's not just *m* = 10. The gradient's not just 10, is it? Because Gxxx said right at the start of the lesson the gradient changes. This is a *general formula for the gradient at any point,* okay? Now we've got a name for that. We call it the gradient, we call this, we call this the *gradient formula*, okay. And we've got a special…we've got some *special notation* for it. Instead of saying $m = 2x$ we write $\frac{dy}{dx} = 2x$ and this is called the *gradient formula*, okay. So $\frac{dy}{dx} = 2x$, that's the gradient, that's the gradient formula for which curve?

203.S. $y = x^2$

204.T. $y = x^2$. This is always gonna be the gradient formula for the line $y = x^2$, okay and that's it. That's differentiation, finding the gradient formula.

Note in [194, 202] how the teacher [T4LO] moves from m (gradient) to $\frac{dy}{dx}$ (gradient function), this is how the symbolism $\frac{dy}{dx}$ is introduced. The teacher draws the students' attention to the fact that this was no longer the constant (m) gradient, but a variable, what the teacher describes as "a bit of algebra for our gradient" [202]. Note also that differentiation is defined as 'finding the gradient formula' [204], and this description is given after introducing the $\frac{dy}{dx}$ symbolism. In [T4LO] $\frac{dy}{dx}$ was explained simply as "some special notation" [202] that is used instead of the m (a symbolism for gradient). The teacher's utterance here, suggests or implies that $\frac{dy}{dx}$ is the same as m, only a special or different notation for the same representation. Note the utterance: 'we've got some *special notation* for it. Instead of saying $m = 2x$ we write $\frac{dy}{dx} = 2x$ and this is called the *gradient formula*, okay' [202]. Note that what is not done here, could be more important, a substantiation of the difference between $\frac{dy}{dx}$ and m . Consider these two utterances: (i) "We call it the *gradient*, we call this, we call this the *gradient formula*, okay" [202], and (ii) "So $\frac{dy}{dx} = 2x$, that's the *gradient*, that's the *gradient formula* for which curve?" [202]. Note that in both utterances the teacher gives $\frac{dy}{dx}$ two names, i.e. he calls $\frac{dy}{dx}$ *gradien*t and *gradient formula*. See how this is clearly depicted on the board work in Figure 7.2. The teachers write $\frac{dy}{dx} = 2x$ and $m = 2x$, and both are labelled as *gradient function*. The $\frac{dy}{dx}$ differential calculus symbolism was, thus introduced and from this point onwards, the class used the new symbolism. There were no questions from the students nor further explanation from the teacher in this lesson about this new symbolism, $\frac{dy}{dx}$.

In contrast to [T4LO] (and all the other lessons), in which the introduction to differentiation was built on the function y = x^2 , [T2LO] used the function y = $x^2 - x - 6$ instead. In fact, [T2LO] was the only lesson that did not start with the $y = x^2$. Figure 7.3 is a snapshot from the lesson showing the algebraic representation for calculating the gradient of a secant line through the general points $[(x),(x^2-x-6)]$ and $[(x\ +\ h),(x+h)^2-(x+h)-6]$. This was the procedure used to estimate the gradient of the tangent to the graph of the function $\mathrm{y}=x^2-x-6$ at the point $[(x),(x^2-x-6)]$. Similar to [T4LO], the teacher in [T2LO] used the gradient quotient $\frac{(y_2-y_1)}{(x_2-x_1)}$ to compute the gradient estimation for the tangent at $[(x),(x^2-x-6)]$. Similar to [T4LO], note the use of $h\to 0$ in Figure 7.3. However, even though this lesson [T2LO] also uses m for the resultant gradient function, like [T4LO], it does not use the = sign. Note the symbolic representation here is $m\ \to 2x-1$, not $m=2x-1$. [T2LO]:

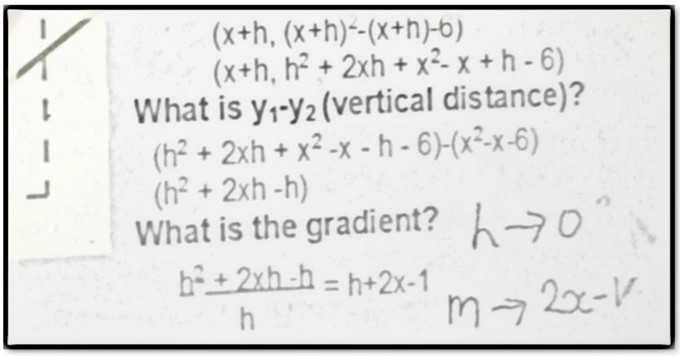


Figure 7.3 The gradient of a secant line

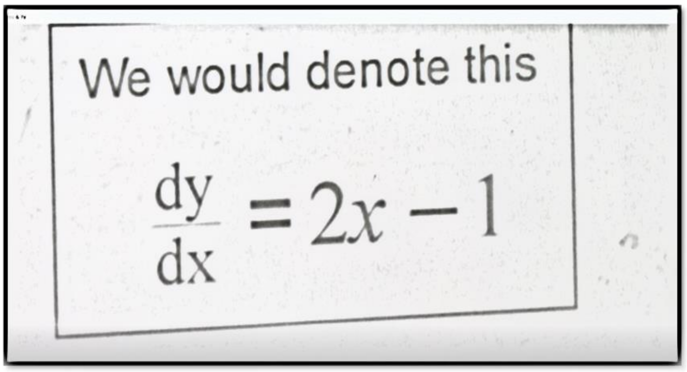


Figure 7.4 The gradient function of $\mathrm{y}=x^2-x-6$

Unlike in [T4LO], the teacher in [T2LO] describes $\frac{dy}{dx}$ as gradient function, rather than gradient, see the excerpt below. In concluding her lesson, the T2 introduces the $\frac{dy}{dx}$ symbolism to describe the gradient function for $y=\ x^2-x-6$. The lesson concludes with the teacher using, and introducing new keywords, ‘gradient function’; and new symbolism, $\frac{dy}{dx}$. The latter was simply introduced through Figure 7.4, which was displayed showing the gradient function for the function y = x^2-x-6 expressed as $\frac{dy}{dx}=2x-1$. The excerpt below [T2LO] shows that there was a minimal explanation given about the new keywords or the new symbolism introduced. [T2LO]:

373.T. We know as h gets smaller and smaller it goes to 0. So, then our gradient function, let's call it m for now for the sake of *it will go to* $2x - 1$ [*my italics*]. Now some people at the start of a lesson were finding that out through the process of *differentiation* which they knew. … We found out the *gradient function* for this [*Teacher pointing to the function*]. It was through something called *first principles*. Something we don't use and asked, one of the girls asked, it isn't tested on but it's really nice to know.
I thought it was nice to finish off with something like this [*Teacher displays Figure 7.4*] because I know you've seen this before. We think about the distances there [*Teacher pointing to the graph of the function*], being like $\frac{dy}{dx}$. Then we denote it like that [*Teacher pointing to Figure 7.4*].

The teacher describes $2x - 1$ as the 'gradient function', not the 'gradient'. Note that even though, m is used in the algebraic representation, the teacher describes it as "… *it will go to* $2x - 1$", which is consistent with the symbolism she displayed in Figure 7.4, that $m \to 2x - 1$. Compare this, to T4LO, Figure 7.3 and Figure 7.4 which says $m = 2x$, instead. The teacher's [T2LO] utterance "as *h* gets smaller and smaller it goes to 0. So, then our gradient function, let's call it m … *will go to* $2x - 1$*"* [373] is consistent with the symbolic representation in Figure 7.3). Note that, similar to [T4LO], m is used to denote gradient function. However, in contrast to [T4LO], see how the teacher [T2LO] substantiates the student's narrative when a student describes $2x - 1$ as the *gradient* [374].

374.S. So, was the gradient $2x - 1$?
375.T. The gradient function was $2x - 1$, yeah, for any point x.

The teacher immediately corrects the student by using the endorsed narrative '*gradient function … for any point* x' [375]. Thus, the point that $\frac{dy}{dx} = 2x - 1$ is the *gradient function* (not gradient) of $y = x^2 - x - 6$ is reinforced. Note, however, that the teacher [T2LO] here explains the gradient function, but not the symbolism, nor was there an attempt to explain its origin here. Even from the students, there were no questions asked about the symbolism or its origins. This, however, was not the case with [T1LO], in which the teacher used Lagrange's differential notation, the $f'(x)$.

Consider the teacher-student dialogue in the excerpt [T1LO] below, [T1LO]:

85.T. So, let's make a note of this, [*writing on the board*] If $f(x)$ is x^3, it means $f'(x)$ is $3x^2$.
86.S. What is that dash mean?
87.T. It means the derivative, the gradient function. That's the notation I have used here.
88.S. What does the derivative mean?
89.T. It means the gradient function, the gradient of the curve is $2x$, of x^2. It's not a constant, is it?
90.S. No

91.T. The gradient, a constant?
92.S. No
93.T. It's a function of x.
95.T. We call it a gradient function. We call it the derivative. There are other names as well, is that ok?

The Lagrange's symbolic representation of the derivative in [T1LO, 86 - 87] by the teacher, opens a dialogue with a student, which then leads to more calculus words. The question in [86] could suggest that the student is having some difficulties with the symbolism in the teacher's utterance [85]. The word use - derivative [87] in response to the student's question about symbolism, demonstrates the use of multiple visual mediators. The teacher switches between visual and vocal mediators, from symbolism [85] to specialised calculus terminology – derivative, gradient function [87]. However, these specialised calculus words appear to have added to the student's difficulty with calculus – the meaning of the derivative [88].

The teacher [T1] reiterates his earlier narrative [87] in [95], linking the words 'derivative' and 'gradient function', and uses a specific example (presumably less complex) to illustrate the properties [89 – 94] of the object of instruction. Note, some contradiction in these utterances [91] and [93]; 'the gradient' is indeed a 'constant'! Once again, as in the lesson [T4LO], note the ambiguity in the utterance: "It means the *gradient function*, the *gradient of the curve* is $2x$, of x^2. It's *not a constant*, is it?" [T1LO, 89], for it does not make a clear distinction between *gradient* and *gradient function*. The teacher's utterance [T1LO, 89] implies that what is termed 'the *gradient of the curve*' is '*gradient function*'. In calculus lessons, specialised symbolism and specialised terminology can be sources of calculus challenges, for both teachers and students, alike.

Unlike [T4LO] and [T2LO], where the introduction to differentiation was built on the gradient quotient $\frac{(y_2-y_1)}{(x_2-x_1)}$ only, the teacher in [T7LO] presented the gradient quotient $\frac{(y_2-y_1)}{(x_2-x_1)}$ and a partial limit definition of the derivative $\frac{f(x+h)-f(x)}{x+h-x}$, see Figure 7.5 below. [T7LO]

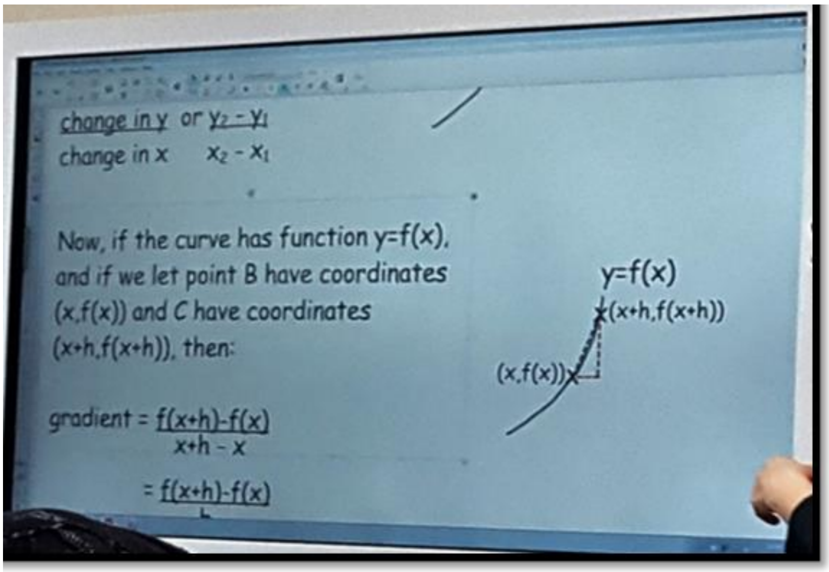


Figure 7.5 The formula for the gradient of a secant line

Lesson T7LO was an exception in that it was the only lesson where the word *limit* was used by the teacher, during a lesson. See the following excerpt in which the teacher is introducing some new symbolism. T7LO

129.T. As h goes towards zero so we're looking at the limit as [h] gets smaller and smaller and smaller, that point C, remember it was this graph up here *[The teacher pointing at the graph in Figure 7.5]* that point C gets closer and closer to point B because the triangle's shrinking down and it's becoming much more precise as a measure of gradient. The actual limit as [h] gets really close to zero, the limit of that value getting smaller and smaller is actually the gradient, so it becomes a precise value when h tends toward zero.
So, this is one way in which we can find the gradient of a function, the gradient is found by substituting into this formula here, which won't mean an awful lot to you at the moment but we're going to practise doing this together in one of the questions in a moment. This bit you need to know so I'd highlight this bit for sure. we use the notation $f'(x)$ to stand for the gradient of the function $y = f(x)$. You will use that a lot; it will become like second nature. Find $f'(x)$ means differentiate which means find the gradient.

The teacher describes the 'gradient' as the limit of that value [quotient in Figure 7.5] "as h gets smaller and smaller and smaller" and "that point C gets closer and closer to point B", i.e. "as h tends towards zero"[129]. The word limit is mentioned three times in the teacher's utterance [129], but it was used here as an everyday word since no definition was specifically given for the word. However, there are at least three observations to point out here.

The first one is the use of the word 'gradient' to refer to both $\frac{(y_2-y_1)}{(x_2-x_1)}$, which had been given as a formula for calculating the slope of straight-line graphs, and ‘gradient’ = $\frac{f(x+h)-f(x)}{x+h-x}$, which had specifically been identified for curved-line graphs. Even the whole utterance [129] talks about the gradient of a function. Thus, there is not a clear distinction, as to what appears to be implied by Figure 7.5, between the gradient for a straight line and gradients on a curve. The second one is that the teacher’s limit narrative as described in the utterances of [129] was not algebraically represented on the board. As can be seen from Figure 7.5, there is no symbolism or representation of the concept of limit.

The third one, unlike with [T4LO] and [T2LO], the ‘gradient’ formula given in Figure 7.5 was not algebraically substantiated in the lesson. The lesson did not use two general points with the given gradient in Figure 7.5 to explain the process of differentiation. Instead, what followed was computing the gradients given numerical coordinates – “the gradient is found by substituting into this formula here, which won’t mean an awful lot to you at the moment but we’re going to practise doing this together in one of the questions in a moment” [129].

Finally, the new symbolism, like with the other teachers’ lessons [T4LO] and [T2LO], was introduced and students were asked to take it as given. However, unlike the other lessons, this lesson [T7LO] did not start by using the m or the $\frac{dy}{dx}$ symbolism for the gradient. Instead, the teacher introduced the Langrage’s notation, “we use the notation $f'(x)$ to stand for the gradient of the function $y = f(x)$”, which she then used to signify differentiation “$f'(x)$ means differentiate …”[129]. No further explanation was given by the teacher for this notation nor were any questions asked by the students.

Summary

In this study, the evidence shows that teachers used both, the Leibniz’s notation, $\frac{dy}{dx}$ and the Langrage’s notation, $f'(x)$. The evidence also suggests that teachers were cautious not to introduce these forms of calculus symbolism early in the introductory lesson. For example, in T3LO, the teacher avoided the use of formal notation, the $\frac{dy}{dx}$ and $f'(x)$ symbolism in the first lesson. In a post-lesson interview, T3 said: T3I(ii):

108.T. I didn’t want to really bring in any notation while they were getting the concepts.
109. I. Yeah.
110.T. So, in my next lesson I’m going to concentrate on notation and proof.

Where the $\frac{dy}{dx}$ and $f'(x)$ symbolism was used, it was towards the end of the lesson, but the notation was barely explained. For example, in T1LO, after the teacher had explained the derivative of the function $y = x^3$, he introduces a new symbolism: T1LO

85.T. So, let's make a note of this, [*writing on the board*] If $f(x)$ is x^3, it means $f'(x)$ is $3x^2$.
86.S. What is that dash mean?
87.T. It means the derivative, the gradient function. That's the notation I have used here.
88.S. What does the derivative mean?

'That's the notation I have used here' [87] was the explanation given here. In a different lesson, T4LO, another teacher introduces a new notation following his explanation for the derivative of the function $y = x^2$; it is simply introduced as a special notation. T4LO:

202.T. And we've got a special…we've got some *special notation* for it. Instead of saying $m = 2x$ we write $\frac{dy}{dx} = 2x$ and this is called the *gradient formula*, okay.

In T4LO, $\frac{dy}{dx}$ was explained simply as "some special notation" [202] that is used instead of the m, which is a symbol usually used to represent the gradient (slope) of a straight line. This utterance implies that $\frac{dy}{dx}$ is the same as m, only a special notation for the same representation. It would, therefore, be necessary to explain the difference in application between $\frac{dy}{dx}$ and m symbolism. Later in the lesson, T4 introduces Lagrange's notation. T4LO:

416.T. Now you might remember there's a second way that mathematicians like to define - y. Does anyone know another way we can write y in maths? Mxxx? $f(x)$. Now that means, let's say if $f(x) = x^3$, that means we need another version of this that involves $f(x)$. And in maths, we write the differential of $f(x)$as $f'(x)$ okay. So, you either call your equation $y = x^3$ in which case the differential is, the gradient function is $\frac{dy}{dx}$. But if you choose to call your curve $f(x)$, then your differential, your differential, sorry your gradient function is $f'(x)$. Either is fine, okay.

The use of m for the gradient function was common in other lessons too. In T2LO, too m is used to represent the gradient function. T2LO:

373.T. We know as *h* gets smaller and smaller; it goes to 0. So, then our gradient function, let's call it *m* for now for the sake of it will go to $2x - 1$.
We think about the distances there [*Pointing to the vertical and horizontal distances on the graph displayed on the board*], being like $\frac{dy}{dx}$. Then we denote it like that [*Displays* $\frac{dy}{dx} = 2x - 1$ *on the board*].

The use of the same signifier or symbolic artefact for the derivative at a point and for the derivative as a function, for example, the use of m in T4LO and

T2LO above, could potentially contribute to students' difficulties with the derivative.

The algebraic symbolic representations such as $\frac{dy}{dx}$ or $f'(x)$ as well as terminology such as *gradient function* and *gradient formula,* often supported by numerical and graphical means, formed the basis for defining and explaining the process of differentiation. In principle, the teachers introduced differentiation as the *limit* of the derivative, but only used a partial representation of the formal limit definition of the derivative. Although all the teachers introduced the process of differentiation, only one used the term limit in the lesson and even then, no formal definition of the concept of limit was given (See Section 9.2.1 on page 218 for a discussion of this point).

7.3 Graphical mediation and digital artefacts

This section presents evidence of the constraints of static imagery and the affordances of digital artefacts for dynamic imagery in exploring tangents to a curve and estimating the gradients at given points on the curve. Graphical mediation took two forms, either static (e.g. sketch diagrams or drawings by hand) or dynamic graphical imagery by means of web-based digital artefacts (also downloadable) of GeoGebra, Autograph and Desmos.

The constraints of the pen and paper method and the affordances of dynamic imagery artefacts of drawing tangents to a given graph are exemplified in excerpts from [T1LO], [T2LO], [T7LO], and [T3LO], which will be examined in this section. The approach of introducing the derivative by drawing tangents to curves relied heavily on iconic mediation, drawings and geometric mediation of functions, but it varied across the different lessons observed. The excerpts from T1LO and T2LO illustrate the iconic and visual mediation by dynamic graphical means through the use of digital artefacts such as Autograph and GeoGebra. In these, [T1LO] and [T2LO], teachers gave the students a pen and paper task, in which the students had to draw a tangent to a given curve at a given point and compute the gradient of the tangent.

In [T1LO] the task was that students had to draw the tangent at the point where $x = 1$ on the graph of $y = x^2$ by eye. [T1LO]

27.T. We'll all get slightly different results, so you're just drawing the tangent by eye. I want you to imagine…

31.T. I can't really do it very well on here [Teacher drawing a tangent to a curve on the board, by free hand]. Don't look at mine, it's been aren't right. I am trying to show you the sort of thing I want you to do.

The students too, found it difficult to draw accurate tangents on graphs by hand, as shown by the range of values that they produced. After collating the students' answers that ranged from 1.4 to 2.6, the teacher said: [T1LO]

54.T. My experience of teaching this, is that most students draw tangents that are too shallow or too steep and that is sort of indicated by the fact that we got more answers that go underneath the 2 and over the 2.

Up to this point, the teacher had used a sketch diagram for the graph of the function $y = x^2$. The teacher then resorts to a more graphical approach to investigate the gradient function of $y = x^2$ and uses digital visual mediation. See Figure 7-6, which shows the graph of the function $y = x^2$, the tangent line having just passed the minimum point and moving up the curve onto the right-hand side, and also the emerging gradient function, simultaneously.

Dynamic imagery is a 'very powerful tool' [T1LO, 62] for graphical mediation in teaching differential calculus. A 'very powerful tool' [62] were the words of T1 as he used Autograph for visual mediation in which he showed a tangent line travelling along the curve of the graph of $y = x^2$. [T1LO]:

62.T. Let me show you this now. This is a very powerful tool that Autograph has. I am going to show you the graph of the gradient function. What it is going to do is this, it's going to plot the gradient of that curve. It's going to travel down the curve and plot its gradient

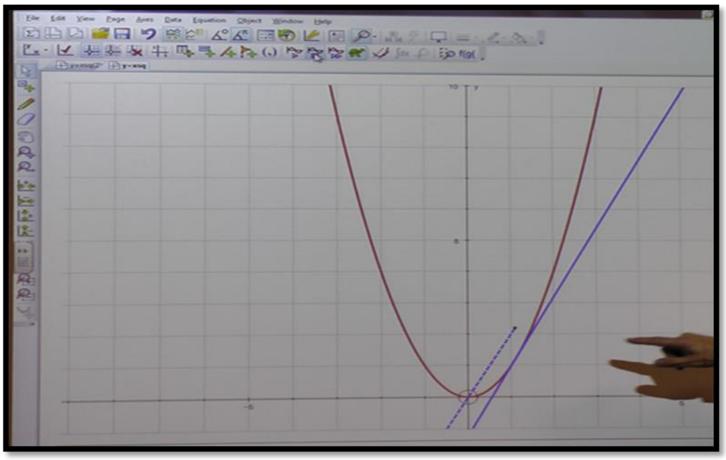


Figure 7.6 $y = x^2$, the tangent and the gradient function

63.T. Tell me what you can tell me about the gradient of the curve to start with, it's going to start from the left-hand side and travel that way. What can you tell me, I don't mean what values there are? What sort of gradients are these? [Students: Negative]. They are negative, right.

65.T. This point down here, which you can't see on the graph, the gradient was minus 1. Here, at this point here [*Referring to the origin*], the gradient is zero. If I continue this line going there [*Referring to the right-hand side*] we get a dotted line.

The teacher is using dynamic imagery to explain the 'process of obtaining the gradient function' [68] using a digital graphing tool, Autograph, which he describes 'very powerful tool' [62]. Dynamic imagery, in Figure 7.6, is used here to show that the gradient varies as the tangent moves. The tangent starts from the left-hand side of the graph of $y = x^2$ and 'travel[s] down' [62] along the curve, passing through the minimum turning point of the graph and continuing to the right-hand side. As the tangent line travels from the left-hand side down the curve to the minimum turning point and over to the right-hand side of the curve, the corresponding gradients are plotted on the same graph, mapping out the gradient function, simultaneously. See Figure 7.7 below, which shows the resultant gradient function $y = 2x$ after the tangent line has passed the view. [T1LO]:

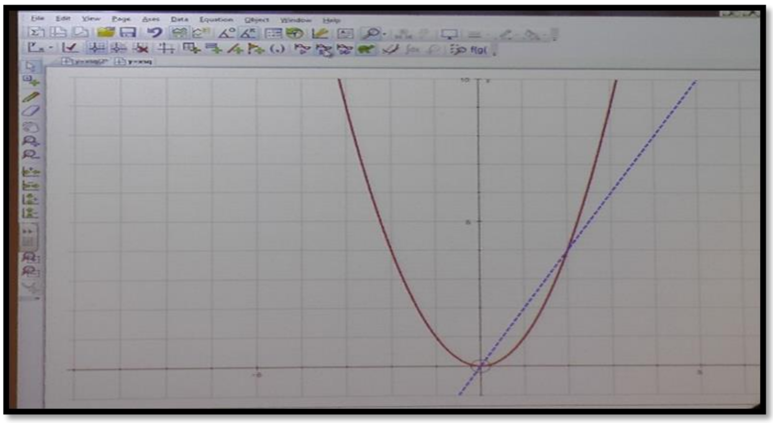


Figure 7.7 The resultant gradient function

67.T. Can anyone look at the equation of that dotted blue line and think? Can you look at that blue dotted line and tell me what its equation is? [S: $2x, y = 2x$]. It is indeed $y = x^2$. How did you know it's $2x$, how did you tell?

[S: Because when you square along...] lovely, that's perfect, that's brilliant, Sxxx!

68.T. The gradient is 2. You are happy it goes to the origin. Now, this $2x$ is called the gradient function of x^2. The process of obtaining the gradient function in mathematics is called differentiating.

Previously, the students had not been able to describe the gradient(s) as a function of x. Instead, they could only give a sequence of values 'you say it goes 2, 4, 6, 8 ...that right?' [61]. The visual mediation in which they could see the tangent travelling along the curve, and at the same time, the gradients being plotted, mapping out a function, enabled students to see the gradients in terms of a function; a graph rather than a sequence of numbers. Note the teacher's word use here - gradient function and equation, and the narrative that 'Now, this $2x$ is called the *gradient function* of x^2' [68]. With the use of digital artefacts, graphical representation enabled the students to deduce the equation of the graph [68], i.e. the gradient function, and differentiation was effectively defined as the process of obtaining the gradient function.

T1LO was the only lesson that showed dynamic imagery of a function $f(x)$ and the graph of its derivative function, $f'(x)$, simultaneously. The evidence suggests that iconic mediation allowed students to see certain features that they could otherwise have missed without the visualisation. This is illustrated by the following excerpt in which a student has made some important observations from Figure 7.8, which shows the graph of $f(x) = x^3$ and its gradient function, T1LO:

121.S. If that curve like you found out from the red curve [function $f(x) = x^3$], do you find out that the gradient is in that blue curve [gradient function $f'(x) = 3x^2$]?
122.T. Yes/
123.S. But that's a curve, so the gradient changes a lot, doesn't it?
124.T. Yes, that's the whole point the gradient just changes; that's exactly the point for a curve the gradient is changing all the time.

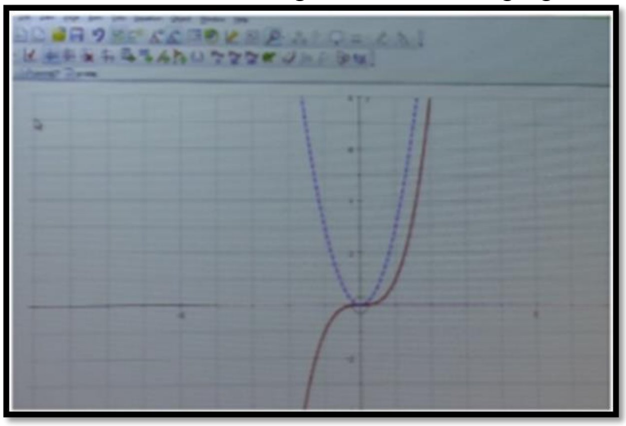


Figure 7.8 The graph of $f(x) = x^3$ and its gradient function

The teacher presented dynamic imagery of the graphs in Figure 7.8, whereby the students could see both graphs mapping out simultaneously. The student's questions and utterances [121; 123] suggest that the graphical mediation (showing the graphs of both the function and its gradient function) used by the teacher helped the student in constructing the definition of gradient function.

Autograph, in particular, the dynamic imagery of Figure 7.6 above demonstrates the changing slopes (gradients) on a curved line graph as opposed to the constant slope (gradient) on a straight-line graph. It provides a dynamic visual graphical mediation of the gradient function $f'(x)$, i.e. $f'(x) = 2x$, of the function $f(x)$, i.e. $f(x) = x^2$ in more ways than what could be possible without it or with static graphical representations. For example, compare the dynamic demonstration above (see Figure 7.6) with the metaphor of an ant travelling along the curve [T1LO] and a ruler moving around the curve [T7LO]. For example, here is an excerpt from earlier in the lesson, T1 introducing the idea of changing gradients of the tangents to a curve at various points along the curve. Here, the teacher was referring to a static visual mediator of a quadratic graph printed on handouts given to the students. [T1LO]:

13.T. Right okay you've now got in front of you the curve $y = x^2$. I want to you imagine you are an ant. Can you imagine what it is like to be an ant and you are literally traveling along the graph?

16.T. What do you think we mean by the gradient of the curve?
17.T. Imagine the ant is there [*pointing at a point on the graph*], would you say the curve is steep there?
18.S. Quite steep, isn't it?

A similar explanation from a different lesson, T7LO, illustrates a moving tangent line by moving a ruler along the curve. T7LO:

104.T. So, if we imagine that the ruler is the tangent at different points.
105.S. Yeah.
106.T. The tangent, the gradient of the tangent will tell us the gradient of the curve at that point. This is what I've just done. I've gone around different points on the curve, and you can see that the slope of the gradient, the ruler is changing, isn't it?

The metaphor of an ant travelling along the curve explanation [T1LO], rests on an individual student's imagination. Its effectiveness depends on the students being able to make the correct realisations for the signified. The placing and moving of a ruler along the curve [T7LO] would be hardly accurate as it is often practically very difficult (if not impossible) to draw tangents to a curve accurately.

Plotting both the function and the gradient function on the same axis as $y = x^2$ and $y = 2x$, if not adequately explained, could be ambiguous to the newcomer to the discourse. The original function represents the $y -$ "$values$" against the $x -$ "$values$". However, the gradient function represents not the y –values, but the gradient values against the x –values. It is important to substantiate and make explicit, this difference in the vertical axis between $f(x)$ and $f'(x)$, for the graph of the original function and the gradient function, respectively. It is possible and reasonable in mathematics to plot several functions (such as $y = f(x)$and $y = f'(x)$) on the same axes, i.e. the $x - y$ coordinate plane, when there are no units associated with the axes. This would be perfectly reasonable to the experienced discussants but may not be so reasonable to the newcomers to the discourse. Teachers should not assume that the learners would simply disregard what could be the units on the axes. It is important that the context in which such representations are used is made explicit for the learners. For example, in mechanics or applied mathematics, the units on an axis specify the nature of the variable, and so it would be very important to use separate vertical axes for f and f' for they indeed measure different variables. Alternative representations could show the two graphs, the $f(x)$ above the $f'(x)$, with the same horizontal $x - axes$ but differently labelled vertical axis.

Another illustration of the affordances of digital artefacts and dynamic imagery is from [T2LO]. In [T2LO] the teacher uses GeoGebra to illustrate a quadratic graph and a tangent to the graph, as shown in Figure 7.9 and then gives the students a handout with the diagram shown in Figure 7.10.
[T2LO]:

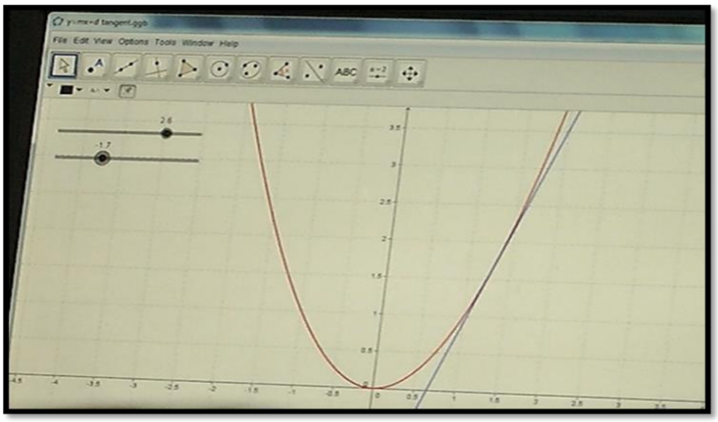


Figure 7.9 GeoGebra: a tangent to a quadratic curve

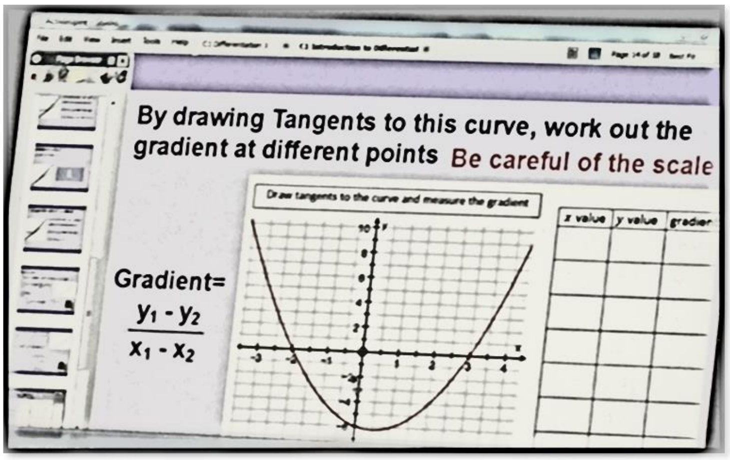


Figure 7.10 The graph of $y = x^2 - x - 6$

44.T. So, pick a point that's definitely on the line, like you can see some. Yeah, there's one there. So then just go for it cause it's going to be an approximation anyway, isn't it? Yeah good. So, the gradient we've got is the y-distance over the x-distance. So, it's like our $\frac{(y_2 - y_1)}{(x_2 - x_1)}$. But we don't have to use that because we can just count.

The students in [T2LO] were finding it difficult to draw the tangents and to work out the gradients. The teacher then tries to guide the students to choose points that would make it easier to compute the value for the gradients, "So, pick a point that's definitely on the line" [44]. In this class, some students had done GCSE (Level 2) Further Mathematics, and so had met the process of differentiation in the past. [T2LO]

53. T. No, we're just drawing on tangents and working out the gradient.
54. S. So, you can work out the gradient by differentiation, can't you?
55. T. Yeah, we can but we're just approximating here.
56. S. Oh right.
57. T. No differentiation yet. In fact, no differentiation for the whole lesson. Not until the next lesson so don't worry about that now. [*The teacher turns to another pair of students*] What are we doing over here? How are we doing?
58. S. I can't do it. I can't.
59. T. Just have a guess. It's only an approximation.

At this point, the teacher turns to the GeoGebra presentation (in Figure 7.9) and demonstrates the approximation for gradient by dragging the tangent and zooming on the point of tangency, by so doing showing the constraints of approximation by drawing tangents by manual or pen and paper means. [T2LO]

93.T. And let's have a look at what may be the issues with the tangent that was drawn here. Oh wow, it's nearly perfect. So, what I'm looking at here [*The teacher zooms in on the graph - Figure 7.9]*, this one doesn't exactly touch. I thought maybe it had gone a bit over, but it doesn't exactly touch. So, it's pretty hard to draw on, basically.

> We already know that it's hard to draw an accurate tangent, pretty much impossible. Let's have a look a little bit closer up [*The teacher moving the GeoGebra sliders on the graph, see Figure 7.9*]. Maybe at this one here and I'll explain what on earth's going on here. So, I've zoomed in on part of our curve. What's happened is I've tried to draw a tangent, but it's gone a little bit over even though Axxx's [*The previous diagram*] was a little bit before[*under*].

The teacher [T2LO] uses the dynamic imagery of GeoGebra to align a straight line as a tangent to the parabola using the sliders by dragging, as depicted in the snapshot of the action shown in Figure 7.9. The teacher then zooms on the graph, at the point of tangency, only to reveal that the straight line had crossed over the curve. The teacher demonstrated the limitations of the drawing of the tangent to a curve as a method for estimating or finding the gradient of a point on a curve. She explains "We already know that it's hard to draw an accurate tangent, pretty much impossible" [93].

Other teachers described the task of drawing a tangent to a curve as 'very difficult' [117], for example, in an excerpt from [T7LO] below. The teacher in [T7LO] avoided the activity altogether. [T7LO]

117.T. So, if I've got a curve, what I could do to work out the gradient at this point B [*Teacher pointing to a point on the graph of* $y = x^2$] is I could draw the tangent to it. The only problem with that is it's very difficult to draw a tangent accurately because it's supposed to just touch the curve at one point, and it needs to be very precise. So, unless we're coming up with an estimate it's an inaccurate way of working out the gradient of a curve at a particular point.

In lessons such as [T7LO], the teachers simply explained how drawing tangents to a curve would give an estimate of the gradient of the curve at the given point but did not set the students for a pen and paper exercise to investigate the activity.

Although in [T3LO] the teacher used GeoGebra, it was in a limited sense. Explaining her planning in an interview with the researcher before the lesson, the T3 said: [T3I(i)]:

42.T. I've introduced a sort of diagram through GeoGebra, but then I've got some worksheets to work out the gradient between two points on a curve.
43. R. Yeah.
44. T. And moving the two points closer together.
45. R. Yeah.
46. T. Trying to predict what the gradient of the tangent would be.

In contrast with the graphical mediation of [T2LO], which had the curve, the secant and the tangent, as well as the use of sliders in the graphics, T3LO used the visual representation shown in Figure 7.11 below.

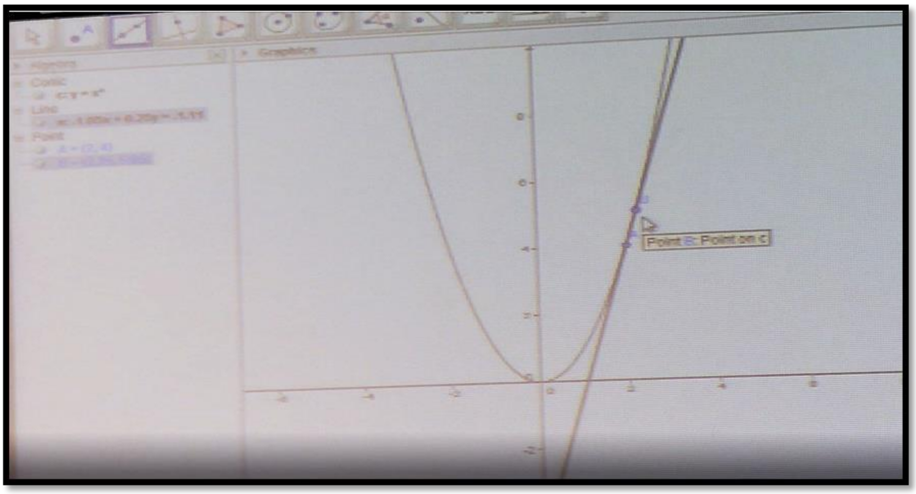


Figure 7.11 GeoGebra: $y = x^2$ and a secant

Although it could show the graph of the function $y = x^2$, the graphical imagery was not visually very clear on the secant (chord). It is difficult to see the secant (chord) from the curve itself. The secant line looks like a tangent line, but it is not a tangent because of the two points marked on the line and the curve. The two points on the curve are meant to illustrate that the points could be joined to find the gradient of the resulting chord. More so, where the graph meets the secant line, it looks more like a straight line than a curve. Visually, this could pose challenges for students, given the expectation that ‘getting closer and closer, should be giving successively better estimates of the gradient at a chosen point. Although digital artefacts were used here [T3LO], they were utilised in a limited sense, given the objective of the instruction.

Talking about his use and students’ use of technology such as Desmos and GeoGebra in teaching introduction to differentiation, T4 thinks that it would be useful for both the teacher and the students to use digital artefacts such as Desmos and GeoGebra in the lesson. T4I(ii)

24.T. I’ve used, I don’t really use technology that much as you can probably tell from my heavily, yeah, all the drawing on the board, but I tried an iPad version of, I think it was Desmos. I haven’t really figured out how I might get the students to use that at the same time as me and make it useful. I find GeoGebra just nice because you can search for the exact slide that you want to explain a particular point and it’s quite versatile like that.

The teacher expresses a favourable or positive opinion on GeoGebra, that he finds it nice, flexible and adaptable, even though he had decided against using it in his lesson. Indeed, there was minimal use of digital artefacts in his lesson. He used hand-drawn sketch graphs on the board to explain and define the derivative. However, he explains that for his plans for the introductory lesson on differentiation, he felt using GeoGebra would complicate his calculus lesson. T4I(ii):

22.T. I mean I considered using GeoGebra and played around with some sliders that altered the chord length on a curve and what I found was what I didn't like complicating it by considering points on either side of the point you wanted to find the gradient at. And I also didn't like ones in which the point didn't, one point didn't stay constant. So, I quite liked setting the one point to (1;1) and then considering only points above (1;1) and what happens as the second point approaches (1;1). I just wanted to keep it, it might just be how I feel about Calculus, but I wanted to keep it as straightforward as possible, keep as many things as the same as possible and only change one particular thing with the *x* co-ordinate and get that *closer and closer* to the point at which we want to know the gradient.

Availability of, access to and ease of use, are all factors that also influence teachers' decisions on the choice and use of digital artefacts. In a post-lesson interview, T5 explains his technology for teaching. The teacher is familiar with and said he used GeoGebra and Desmos with iPads, computer suites and mobile phones with his students. However, the availability of iPads for students and the accessibility of ICT suites are factors that determined the frequency and extent to which the teacher incorporated these technologies in his lessons. T5I(ii)

34.T. Yeah, absolutely. At some point last year, we used GeoGebra, and I like GeoGebra from a sense that most things that can be modelled even in 3-D you can do in GeoGebra. But in terms of graphing, I like to consistently use Desmos now.
35. I. Desmos, yeah. And do the students also use graphing calculators?
36.T. Year 13 so at A2 they use it an awful lot because up until last year every student who came here was given an iPad but sadly this is the first Year 12 cohort where we've not been able to do that for them. So, we don't do it as much. Every, you know when the opportunity arises, I do book out ICT suites but with the Year 13s it's just a matter of course that they're there and they just pick up their iPads and get on with it really but with the Year 12s not, not as much, and I wouldn't book an ICT suite out for the sake of using it for five minutes.

'We don't do it as much' [36], says the teacher because the students no longer received iPads from the school like in the previous years when the teacher would use digital artefacts with the students 'an awful lot'. With his Y12 class, the teacher was no longer using digital artefacts as much as he does with his Year 13 class who have iPads. 'I wouldn't book an ICT suite out for the sake of using it for five minutes' [36], because of limited access to and use of digital artefacts.

Summary

Teaching an introduction to differentiation was visually mediated by the iconic mediators such as the quadratic graphs and by the digital artefacts such as the dynamic imagery of GeoGebra, whose affordances include more accurate graphing of functions, drawing of the tangent utilizing sliders, dragging of variables, and zooming on the graph. Thus, the graphical representations were complemented and supplemented by digital artefacts such as GeoGebra and Autograph, but also by verbal or visual signifiers in the teacher's utterances. Visual mediation employing dynamic imagery digital tools such as GeoGebra and Autograph can be used to supplement and could enhance the traditional static graphical representation in constructing the definition of the derivative.

7.4 Visual mediators and multiple representations

Evidence from across the various lessons shows that in introducing differential calculus, teachers use combinations of, and constant shifts between numerical, algebraic and graphical mediation. To illustrate the findings, five exemplar excerpts from three different lessons, [T4LO], [T2LO] and [T3LO] are presented and examined. They have been chosen because they are representative of the combinations of and shifts between representations witnessed in the teachers' introductory lessons to differentiation.

The first excerpt is from [T4LO] and involves the gradient of a constant, which combines graphical means and algebraic means. At this point in the lesson [T4LO], just before the utterance [481], the teacher had introduced the algebraic symbolism for differentiation, the $\frac{dy}{dx}$ and the standard formula for differentiation, $\frac{dy}{dx} = nx^{n-1}$ for a function $y = x^n$. The class had applied the standard method for differentiation for the function $y = x^2$. [T4LO]

481.T. The gradient at any point. Okay, what about this one then? $y = 4$, okay. Have a think, have a think for a moment, what do you think the gradient function of $y = 4$ would be? You may draw a diagram if you want, that's a hint. But, have a think about what you think the gradient function of $y = 4$, would be. Try drawing a diagram, try thinking, there are a couple, I think there are a couple of ways of thinking about this one. Try and come up with an idea of what you think the gradient function would be.

491.T. Okay Year 12 have a look at this *[Teacher drawing the students' attention to the diagram he's drawing on the board – see Figure 7.12]*. This is what Gxxx's drawn on her axis. And this is my favourite way of working it out. Gxxx started by drawing her axis and then she's drawn the line $y = 4$, okay.

So, that's a line through 4 there. Now the question is, what's the gradient function of that line? What's the gradient function of that line? What's the gradient at any point on that line?

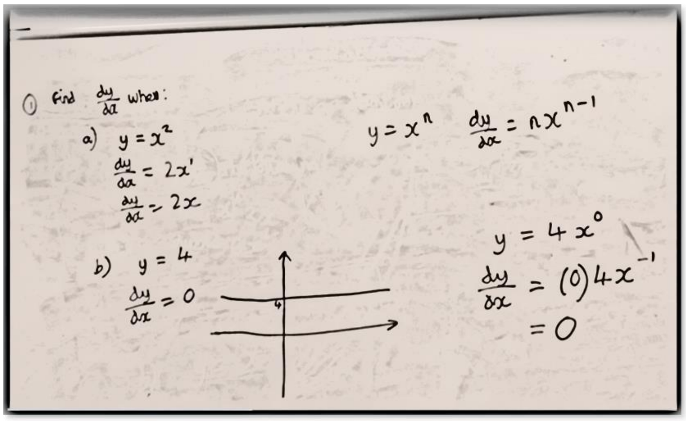


Figure 7.12 The gradient function of a 'constant'

492.S. 0.
493.T. 0, because it's a horizontal line. The gradient function's going to be 0. That's one way of thinking about it. Okay, that's one way of thinking about it. Now, there's a second way of thinking about it which I quite like which uses the formula okay. This one uses the formula. Does anyone have an idea how to use a formula to work out what the gradient function of 4 [$y = 4$] is?

In [481] the teacher T4LO asks students to find the gradient function of $y = 4$. Note that the teacher insists on thinking, '… have a think, what you think the gradient function of $y = 4$ would be' [481]. In this [481] teacher's utterance alone, he mentions the word 'think(ing)' nine times, of which eight of the nine times are directed at the students. Here the students are encouraged to think in multiple ways of solving the problem presented as shown by the teacher's scaffolding instruction to the students in [481], 'try drawing a diagram', 'there are a couple of ways of thinking about this one'.

Shifting the attention to the algebraic approach to solving the problem of finding the gradient function of $y = 4$, the teacher T4LO re-emphasises 'thinking', which is mentioned a further three times in [493]. Here, the teacher directs the students to the standard formula for differentiation that the class had used for differentiating the function $y = x^2$. His question "Does anyone have an idea how to use a formula to work out what the gradient function of 4 [$y = 4$] is?" opened the teacher-student dialogue below, which explains the algebraic way of thinking about the problem. [T4LO]

511.T. Well, what are we missing that we usually have on all the other questions, that we don't have here? We don't have an x. Instead, we've got a number, okay? So, let's assume there's a power of x here *[Teacher pointing next to*

4 in the function $y = 4$] that we can't see. What power would that X be raised to, for it to equal $y = 4$? What power of *x* would work, Mxxx?
512.S. 0.
513.T. 0, okay. Why is that the same as $y = 4$, Gxxx? Why is $4x^0$ the same as $y = 4$?
514.S. x^0 is 1
515.T. That's it!
516.S. And 1 × 4 is 4.
517.T. There you go! So, 4 × 1 is 4. So, $y = 4$. So, we could consider this instead, $4x^0$. Now can someone help me apply the rule now to work out what the gradient function is? What would I do first? Hxxx?
518.S. Times 0 by 4…
519.T. That's right, you times by 0.
520.S. And you'd have 4*x* to the power -1
521.T. That's right. What's that simplified to, Hxxx?
522.S. *x* = 0.
523.T. Exactly and that works doesn't it? So, there seem to be two ways of thinking about it. Either you can think about, well, y = any number will differentiate to 0 because its gradient's always 0. Or you can use a bit of fancy algebra to work out that it's the same as this rule here. Okay, right the next one.

The graphical mediation of the gradient of a constant is substantiated algebraically in [511 - 523] above. By manipulating the algebraic laws of indices, what the teacher describes as using "a bit of fancy algebra to work out…"[523], he was able to demonstrate that the algebraic mediation, gave the same solution to the problem, as the graphical approach. This example, [T4LO], demonstrates the complementary application of graphical and algebraic mediation in substantiating the derivative of a constant function.

The second illustration comes from [T2LO and is a more explicit example of multiple representations. Here the teacher used numerical, graphical and algebraic forms of mediation together, in substantiating the gradient of the tangent to the graph of function $y = x^2 - x - 6$ at point P (4.6). See Figure 7.13 showing the iconic mediator, which is a section of the graph together with the secant and tangent line. Referring to the iconic mediator, the teacher started with a numerical approach – "*I'm going to do this numerically*" [97] - to estimate the gradient of the tangent to the curve at point P (4.6). [T2LO]

97.T. And you're all going to have a particular point. Now you might notice my point was (4,6). … Now I'm going to pick another point, Q, which is definitely on my line, and I'm going to start trying to move it down *[Teachers dragging point Q slightly, towards point P]*. I'm going to do this numerically. Now can anyone tell me a point that this could be? Can someone give me a point Q which could be in that position? Think of an x value. x is 4 there *[Teacher pointing at point P]*. What could x be here [*Teacher pointing at the position of point Q*]? Maybe it could be 5. Let's do 5. What's my y co-ordinate gonna be?

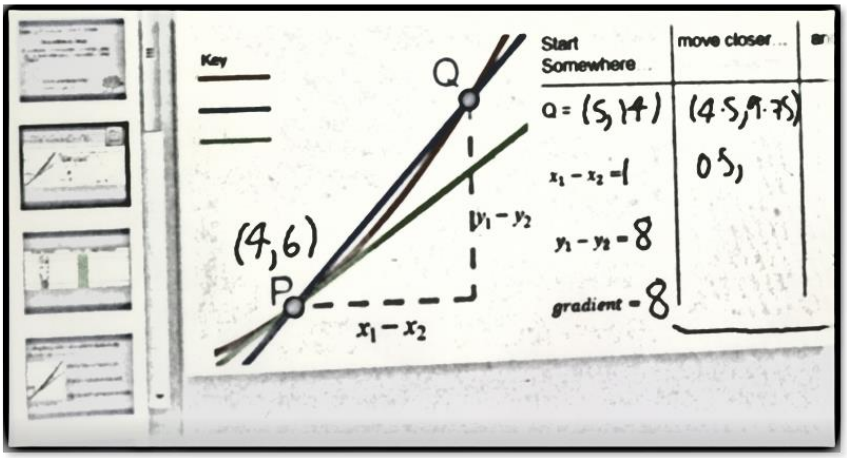


Figure 7.13 Estimating the gradient of the tangent at P(4,6)

107.T. Perfect, 8. So, my gradient there *[Teacher pointing at point P]* is going to be 8/1, which is 8. Then I'm going to think about moving this Q a bit closer. How about instead, I do 4.5 and whatever my answer is? …

This was a numerical approach to estimating the gradient of the function $y = x^2 - x - 6$ at the point P(4, 6). Note in Figure 7.13 that there is a table designed to scaffold and mediate instruction on the numerical approach. Students had been given a printout of the diagram and information printed in Figure 7.13, the graph and the table. The activity for estimating the gradient of the tangent at the point P(4, 6) progressed as shown in the table, starting with $x = 5$, $x = 4.5$, and so forth 'getting closer and closer' to point P(4, 6). Each time, computing the gradients of the secant passing through P and Q and recording the gradients in the table, e.g. for the gradient is 8 for the secant through P(4, 6) and Q(5, 14). Visual mediation here constantly shifts between graphical and numerical representations, back and forth; the two representations here are complementary.

The routine for estimating the gradient of the function $y = x^2 - x - 6$ at the point P (4, 6) progressed onto a more algebraic approach. Here, Q had taken the coordinates $[(x + h), (x + h)^2 - (x + h) - 6]$. Again, the mediation shifts back and forth between the two visual mediators – the graphical and algebraic representations. [T2LO]

326.T. Yeah, so as P and Q get close together, so h tends to, we've seen this before, h tends to 0 because it's going towards 0, isn't it? Then our gradient tends to 7. So, that's what we've done so far. We've had a look graphically approximating it. We've had a look algebraically. And that is true, h will eventually become 0 as they [*P and Q*] become the same point. So, our gradient will be 7.

"We've had a look graphically approximating it. We've had a look algebraically" [326]. Remember, the teacher had started on the same activity with a numerical approach to the investigation - "I'm going to do this numerically" [97]. The teacher [T2LO], in concluding [326] the activity, highlights the multiple approaches combining numerical, graphical and algebraic mediations she had used with the class to estimate the gradient of the tangent at the point P(4, 6) on the curve for the function $y = x^2 - x - 6$.

The third lesson [T3LO] is an example that depicts the numerical, tabular and algebraic visual mediators for the representation of gradient function. Visual mediation here is a combination of the iconic mediation (from tables) and the written symbolic mediation (from the numerals and the algebra), see Figure 7.14, which shows the gradient functions for $y = x^2$ and $y = x^3$, respectively. [T3LO]

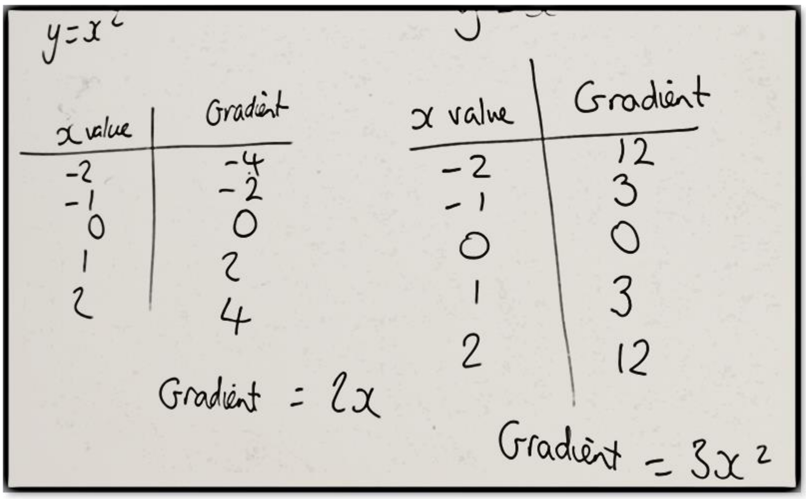


Figure 7.14 Gradient function - numerical and algebraic representations

Two exploratory activities for the gradients of the functions $y = x^2$ and $y = x^3$ at the various points as shown in the tables in Figure 7.14 above, were carried and the respective gradients were recorded. The algebraic representations $2x$ and $3x^2$ for the respective gradient functions for $y = x^2$ and $y = x^3$, are dependent on their respective table of values. Figure 7.14 shows the *x*- values and the corresponding gradients of the curve at the given point. The numerical and the algebraic mediation here, are supplementary, and together they combine to construct the narratives for gradient functions of the functions $y = x^2$ and $y = x^3$. However, what is described as "Gradient = $2x$ and "Gradient = $3x^2$", is in fact the *gradient formula* or *gradient function*. In literate discourses, the gradient function of

$f(x) = x^2$ is $f'(x) = 2x$; and the gradient function of $g(x) = x^3$ is $g'(x) = 3x^2$.

The mediation in [T3LO] is a combination of the iconic mediation (from tables) and the written symbolic mediation (from the numerals and the algebra) (see Figure 7-16). Here, the teacher combines the numerical, tabular and algebraic visual mediators in constructing and substantiating the gradient functions for $y = x^2$ and $y = x^3$, respectively.

The exemplar excerpts from the three lessons [T4LO], [T2LO] and [T3LO] presented above are representative of the combinations of and shifts between forms of mediation witnessed in the teachers' introductory lessons to differentiation. All the teachers used some combinations of numerical, graphical and algebraic mediations in their lessons.

In a post-lesson interview excerpt with T5 below, the teacher explains his lesson and the use of multiple representations. T5I(ii)

18.T. But the best thing that I found was on the Nrich website and that's so the basis of what I did today, which was based on that really. An investigation that started looking at a combination of the graphs and the coordinates so that they had a visual representation and a sort of tangible numerical representation that they could look at as well.

Explaining his reasons for taking an exploratory approach that uses graphical and numerical mediation, T5 referred to his first lesson on differentiation the previous year when he had used some power-points he had found online. T5I(ii)

16.T. And one of the power-points particularly with this lesson just introduced, like you said the Algebra and they introduce the Algebra, so the formality behind the proof of this Algebra was more difficult than they went into at A level.

Similarly, T3 talks of graphical investigation too. Talking about her lesson planning for introducing differentiation, she emphasises the visual representation through graphs. Graphical representation is seen as more visual than algebraic representation, which is described as abstract here. Below is an excerpt from the pre-lesson interview. T3I(i):

138.T. I think it's important for students to visualise what is happening. And I could have gone to first principles in the sense that it's still two points and that it would link but it would be a bigger jump. I feel like it's a smaller jump to go to the graphs first.

139. I. Ok.

140.T. And it also keeps linking back because we, they will have to link differentiation with graphs in the future. So, I think it is useful to be talking about what's happening in a graph. And so, the visual side just gives a different way of viewing the topic, sort of the visual side as well as the algebraic.

141. I. Right.

142.T. Rather than it feeling more abstract. I think it can feel a bit more abstract if you go straight into algebra. From speaking to the teachers, I've spoken to, one of the teachers misses out on doing any graphical investigation and goes straight to the differentiation from first principles. Whereas the other teacher does spend time on working out the gradients of tangents from a graph. And she felt that had a lot of value.

The teacher here considers it more important to start with investigating the gradients of tangents to a graph at given points before taking a rather more algebraic approach to differentiating from first principles. T3 consulted two other teachers as she planned for her lesson. Note that she reports that they were taking different starting points into introducing differentiation, but she favoured a more visual graphical approach.

7.5 Discussion

In this section, a discussion of the findings reported in the preceding three subsections above will be made. The discussion will focus on calculus symbolism, the use of digital artefacts such as dynamic imagery tools with graphical mediation and the use of multiple mediations in the teachers' pedagogical calculus discourse.

Calculus symbolism

Symbolism is part and parcel of teaching differential calculus because symbols are integral to calculus. The use of the same visual mediators or symbolic artefacts for the derivative at a point and for the derivative as a function, for example, the use of m in T4LO and T2LO (see Section 7.2 pages 137 to 141), could potentially contribute to students' difficulties with the derivative. This observation is consistent with the findings from Park's (2016) examination of the calculus discourse of experts as reflected in the three most popular calculus textbooks in the US. Park (2016) found that the 'realisations of both the derivative at a point and the derivative of a function were mediated with nearly identical symbols suggesting a possible difficulty with understanding the difference between them' (p.417). The teachers in this study, certainly do not have difficulty understanding the difference between m and $\frac{dy}{dx}$. However, the evidence in T4LO and T2LO above, suggests that their students could potentially have difficulties understanding the difference between the symbolic artefacts and their applications.

In the lessons observed in this study, the symbolism $\frac{dy}{dx}$ and $f'(x)$ were given to signify the gradient function (or the derivative) of the function, y or $f(x)$, respectively. However, apart from telling students that it is the notation to use for the gradient function, there was no explicit explanation for the meaning or origin of the symbols $f'(x)$ or $\frac{dy}{dx}$, dy and dx. Neither were there any questions from students about the symbolism nor about its origins, apart from one instance in T1LO [86] when a student asked about $f'(x)$. This suggests that the students accepted the calculus symbolism as presented by their teachers, as given facts. Teachers introduced $f'(x)$ and $\frac{dy}{dx}$ symbolism basically as the notation for the gradient function.

According to the commognitive approach, mathematics is a form of discourse (Sfard, 2008) and learning and doing mathematics means becoming capable of participating in the literate discourse (Sfard, 2016). In mathematical discourse, the role of visual mediators is 'fulfilled by symbolic artefacts such as numerals, algebraic expressions and graphs, created specifically to serve as 'representations' of impalpable mathematical objects' (Sfard, 2015, p.132). According to the commognitive approach (Sfard, 2007; 2008) symbolic artefacts, such as the $f'(x)$ and $\frac{dy}{dx}$ in differential calculus, are an integral part to the thinking and communication process in mathematical discourse. Symbolic mediation brings 'generative power' (Sfard, 2008, p.159) to the discourse and offers 'powerful manipulative ability' (Tall. 1992a, p.9). However, calculus symbolism and calculus terminology can be a source of some difficulties with calculus, for both teachers and students, alike. How differential calculus symbolism is introduced matters as students need to learn the symbolism.

$\frac{dy}{dx}$ or $f'(x)$ have been defined as a gradient function, e.g. in T4LO 'the gradient function is $\frac{dy}{dx}$ … your gradient function is $f'(x)$' [417]; in T1LO, $f'(x)$ is defined as 'it means the derivative, the gradient function' [87]; and in T2LO referring to the gradient function the teacher says 'we denote it like this – $\frac{dy}{dx}$. However, in T4LO, not only does the teacher defines $\frac{dy}{dx} = 2x$ as the gradient function for $y = x^2$, but describes it as differentiation, too.

204.T. $y = x^2$. This is always gonna be the gradient formula for the line $y = x^2$, okay and that's it. That's differentiation, finding the gradient formula.

Similarly, in T7LO, $f'(x)$ is defined as standing for gradient (referring to the gradient function) and has been linked to differentiation too. T7LO

129.T. We use the notation $f'(x)$ to stand for the gradient of the function $y = f(x)$. You will use that a lot; it will become like second nature. Find $f'(x)$ means differentiate, which means find the gradient.

T4LO [204] and T7LO [151] show that these symbolic mediators, $\frac{dy}{dx}$ or $f'(x)$ have a dual role in the calculus discourse. On the one hand, $f'(x)$ can be an object narrative for 'the derivative of $f(x)$', and on the other hand, an operational narrative for 'the process of differentiation'. The dual purpose of these calculus symbols, $\frac{dy}{dx}$ or $f'(x)$, if not made explicit, can be a source of confusion for students (Park, 2013). The symbolism $\frac{dy}{dx}$ or $f'(x)$ is an example of what Gray and Tall (1994) call a 'procept' (Tall, 1992b, p.4), a signifier for both the process and product in the same symbolism. Gray and Tall (1994) define procept as 'the amalgam of process and concept' (p.4).

A procept such as $\frac{dy}{dx}$ or $f'(x)$ can call up either a process (finding the derivative) or a concept (Gray and Tall, 1994) or a mathematical object (Sfard, 1992; 2008) or simply put the product of the process in the student's mind. Flexible thinking is required in the face of a procept. A student needs to be able to tell whether the symbolism $\frac{dy}{dx}$ or $f'(x)$ signifying the process of differentiating or the derivative, the product of differentiation. Unless the context makes it explicit, a student may find a procept ambiguous, failing to read whether it is signifying the process or the concept, i.e. the object (Sfard, 1992). The "duality (as process or concept), flexibility (using whichever is appropriate at the time) and ambiguity (not always making it explicit which we are using)" (Tall, 1992a, p.4) in calculus procepts, are a source of challenges for many students and teachers. Given the flexibility and the duality of use of calculus procepts, it is essential that teachers make it explicit enough for students to develop the necessary flexible thinking and understanding to be able to deal with the possible ambiguity of use (Tall, 1992b; Gray and Tall, 1994).

Graphical mediation and dynamic imagery tools

Sfard's (2008) view of visual mediation 'does not distinguish between the static and the dynamic' (Ng, 2018, 1177). Using digital artefacts such as Autograph, GeoGebra and Desmos allowed for a more accurate and dynamic demonstration of the object of instruction in real-time, which meant less time than what would be required when teaching with static graphical imagery. At the click of a button, for example, the teacher in T1LO (refer to Section 7.3 and see Figure 7.6 on page 128) was able to illustrate the relationship between the graph of the function $y = x^2$, its moving tangent line and the gradient function, $y = 2x$ almost instantly; thus, the versatility and the generative power of visual mediation (Tall 1992; Sfard, 2008). However, in all the lessons observed, the students did not use the technology of GeoGebra, Autograph and Desmos that the teachers used, apart from one lesson [T2LO], in which some students were asked to come to the board (interactive whiteboard) to drag objects on the screen, for example, to demonstrate a tangent by dragging a straight-line to a tangency with a graph of a given function.

Dynamic geometry learning and dynamic imagery activities with such programs as GeoGebra, Autograph and Desmos can be instrumental for exploratory learning in calculus. Tall (1986, 1990, 2003, 2013) has long been an advocate for technology in teaching and learning calculus. Tall (2013) argues that the use of digital artefacts can be instrumental in helping students develop visualisation skills and forming visual mental images of calculus concepts. Jesso and Kondratieva (2016) argue that a dynamic geometry environment (DGE) enables for the substantiation of the limit definition of the derivative with more elaborate functions than what would be possible by hand; more complex functions than the commonly used linear or quadratic functions).

Ng's (2015) commognitive study examined bilingual high school students using dynamic geometry to learn about derivatives and antiderivatives. In Ng's study, the students are seen dragging visual objects and later use such dragging activities to explain conjectures (Presmeg, 2016). Ng's (2015) study shows the importance of such activities for it demonstrates the interdependence of gestures, discourse and dragging with dynamic imagery activities in a dynamic geometry environment. In my study, even though dynamic imagery technology is available and accessible to teachers and students alike, not a single teacher set students a task to use any of these digital artefacts.

'Of all the areas in mathematics, calculus has received the most interest and investment in the use of technology' (Tall et al., 2008, p. 207). It is widely accepted that the main strength of technology is its capability of providing greater and easier access to multiple (numerical, graphical, and symbolic) representations of concepts (Tall, 2019; Fey, 1989; Goldenberg, 1987; Kaput, 1992; Porzio, 1999). The challenges of students' difficulties in calculus (Biza, 2017; Winslow and Grobaek, 2014; Tall, 1992, 2019) still remain though; and this is of course a problem for the teachers of mathematics. In this study, the use of DGE such as GeoGebra, Autograph and Desmos in teaching calculus varies amongst the teachers, with some teachers using the technology for drawing and showing the graphs of functions. Where it was put to better use, the teachers used DGE for instructional activities with students in an exploratory way to construct the definition of the derivative.

In this study, where dynamic imagery tools were used most extensively to substantiate the definition of the derivative, e.g. in T2LO, it was used by the teacher (see Section 7.3, pages 151-3). In T2LO, the use of the dynamic imagery with GeoGebra allowed for active processes such as dragging points on the graph or dragging the image around on the interface, as well as zooming on the image or parts of the image. The dragging of points and zooming on the image allowed for a simultaneous demonstration of the secant line getting closer and closer to the tangent line as point Q approaches point P (See Figure 7.9 in Section 7.3 above). The difference in the use of DGE for teaching mathematics can be due to the teachers' different aims when using technology, which could be '(1) visual demonstrations of mathematical facts; (2) experimentations, explorations, and search for new mathematical relations by students' (Jesso and Kondratieva, 2016, p.218).

A study by Takaci et al. (2015) on teaching calculus using a computer-based dynamic imagery program found that the use of GeoGebra had a positive impact on students in examining functions and drawing their graphs. Takaci et al. (2015) report improvements in knowledge and achievement in the students who used GeoGebra. Teachers should allow for and facilitate the use of dynamic imagery tools like GeoGebra by students if such benefits are to be realised. My study has shown evidence that there is a lack of student use of these technologies, which then raises questions for mathematics educators and opens a debate as to why that is the case. The use of digital artefacts can enhance visual mediation in teaching introduction to differentiation, for example, graphical mediation in constructing the definition of the derivative.

Visual mediators and multiple representations

The evidence in this study has shown that the teachers used multiple visual mediators in constructing the definition of the derivative. Teaching the quasi-limit definition of the derivative was visually mediated by written symbols e.g. numerals, algebraic formulae and algebraic symbols, and graphs of functions. Visual mediators are the 'providers of the images with which discursants identify the object of their talk and coordinate their communication' (Sfard, 2008, p.145). The findings of this study chime with other mathematics education researchers (e.g. Verhoef et al., 2014; Tall, 2010), who call for a more natural approach that blends together the dynamic embodied visualisation and the corresponding symbolic calculation in teaching and learning calculus. Sfard (2008) argues that 'the multiplicity of visual realisations broadens communicational possibilities' (p.156) because 'a given narrative may be constructed and substantiated in a number of ways' (p.156).

In this study, teachers talked about taking an exploratory approach to introducing the derivative (e.g. T5 and T3) and explained using graphical, numerical and algebraic mediation. Graphical mediation was seen as more visual than algebraic mediation, the latter was described as abstract (e.g. by T3). T3 argued that 'it's important for students to visualise what is happening' [138], and thus, teachers' introduction to differentiation needs to include a geometric explanation for the differentiation as part of the multiple representations. In this study, multiple representations describe the various forms of mediation present in the teacher's pedagogical calculus discourse. In their study, Ferrini-Mundy and Graham (1991) found that although most students could solve a simple problem about limits, they had little geometric understanding.

> One of the students interviewed claimed "the graph can't help me find an answer." Further probing revealed that the notion of "approaching" was not part of her understanding of limit. She saw limit problems as functions to be evaluated and wasn't sure about all the "extra" notation (the arrow, the word "lim") (Ferrini-Mundy and Graham, 1991, p.630).

This example emphasises the importance of using multiple visual mediators in teaching differential calculus but making clear links between the graphical and symbolic mediation. The student could not make use of the graph nor understand the symbolism, e.g. $\lim_{h \to 0} f(x)$.

In this study, the teachers' choice of lesson resources and activities was influenced by their desire for visual mediation with graphs to complement the usual algebraic mediation in explaining the first principles in differential calculus. This preference for multiple representations that include graphical mediation is consistent with the findings of Kendal and Stacey (2003) who observed that many students in calculus could at most make connections between graphical and symbolic representation but could not make graphic and numeric connections, or symbolic and numeric relationships. Kendal and Stacey's (2003) study stresses the importance of graphical representations for students to make connections with algebraic representations.

None of the teachers in this study went direct to an algebraic symbolism definition of the derivative. In all lessons, geometric (or graphical) representations underpinned all explanations for the introduction to differentiation. Numerical and algebraic explanations were based on graphical representations. Like T3, all the teachers in this study were conscious of the abstract nature of calculus symbolism, in particular, the symbolism for an algebraic construction of the limit definition of the derivative.

In this study, central to the teachers' pedagogical discourse on introducing differential calculus, were multiple visual mediators, but more importantly, constant shifts between the different modes of mediation. This observation is consistent with Zandieh (2000) who argues for using multiple representations in teaching the derivative and process-object duality, for example, to explain the transition from the derivative at a point to the derivative as a function. For introducing differentiation, T3 explains that 'it can feel a bit more abstract if you go straight into the algebra' [142]. It could feel very abstract for the students that are meeting differentiation for the first time to face such symbolism as $\lim_{h \to 0} f(x)$ without geometric nor numerical mediation to complement the instruction.

Further, whilst using different representations or contexts is believed to widen learning opportunities for the students, making connections between the graphical, numerical, and symbolic-algebraic representations is even more important (Roorda et al., 2009; Zandieh, 2000). In her study on the derivative of a function as an object, and the transition in the teachers' pedagogy between the derivative at a point and the derivative as a function, Park (2015) observed:

> These results showing the instructors' uses of various visual mediators without explicit connections between them, their limited discussion on how the derivative as a function varies, and their dependence on symbolic and algebraic notations, seems related to some well-known student difficulties with the derivative (p.248).

Note that the over-reliance of the teachers on the symbolic and algebraic notations, with minimal connections to other forms of mediation such as graphical means, could be behind some of the known students' challenges with the derivative (Park, 2015). The evidence from the study, for example, the three exemplar excerpts from [T4LO], [T3LO] and [T2LO] presented in Section 7.3 is in stark contrast with the findings of Park's (2015) study.

Teachers in this study (e.g. [T4LO], [T3LO] and [T2LO] presented in Section 7.3) used some combinations of visual mediators and there were constant shifts between multiple representations in their introductory lessons to differentiation. For example, in [T4LO], there is an example of the gradient of a constant that combines graphical means and algebraic means, whereas [T2LO] combines numerical, graphical and algebraic means, in which the visual mediation constantly shifts back and forth between graphical and numerical representations. These findings are consistent with Sfard's (2008) assertion that the same signifier may be realized visually in several ways, in different media. Sfard (2008) argues for the multiplicity of visual realisations, 'because each medium has its own discourse that supports its unique set of narratives, the multiplicity of visual realisations broadens communicational possibilities' (p.156). The use of multiple forms of visual mediation can widen learning opportunities for the students.

7.6 Summary of findings

In this study, teachers used both the Leibniz's notation, $\frac{dy}{dx}$ and the Langrage's notation, $f'(x)$. However, the evidence suggests that the teachers were generally cautious with or hesitant in using calculus symbolism such as $\frac{dy}{dx}$ and $f'(x)$ in their first lessons on differential calculus. This study found evidence of didactical ambiguity with some calculus symbolism in the teachers' pedagogical calculus discourse on elementary differential calculus (See Section 7.2). This study revealed evidence of some ambiguity with symbolism used for or between the gradient for a straight line and the gradient function. There is evidence in this study to show that teachers used both m and $\frac{dy}{dx}$ to signify the gradient function, coupled with a lack of adequate substantiation as to the difference between $\frac{dy}{dx}$ and m. The evidence also shows the use of the same signifier or symbolic artefact for the derivative at a point and for the derivative as a function, for example, the use of m in T4LO and T2LO.

In teaching differential calculus, algebraic symbolic artefacts are an important aspect of visual mediation, and so are graphical mediators. Every lesson introducing differentiation in this study started with some iconic graphical mediation. In line with Sfard's (2008) argument for the multiplicity of visual realisations, and consistent with Tall (1992a) for the teaching of calculus, who argues for the need for versatile transitions between representations, graphics, numerics and symbolics (p.9), the teachers in this study used combinations of, and constant shifts between numerical, algebraic and graphical mediation in substantiating and constructing the definition of the derivative.

Chapter 8 Pedagogies on the derivative

8.1 Introduction

This chapter is the third of the three findings chapters of this study, which are henceforth presented according to the overarching themes of the research, namely: mathematical language for calculus teaching (Chapter 6); symbolism and visual mediators for calculus teaching (Chapter 7); and pedagogies on the derivative (Chapter 8).

This chapter will present evidence for and discuss the findings of the research under the pedagogies on the derivative theme (see Table 5.7 for the overarching themes). The evidence is presented in the form of excerpts from both the interview and the lesson observation data transcripts from all the participant teachers in this research. Given the discursive nature of the qualitative analysis of this study, and for practical reasons, it is not possible to report on every individual story of the participant teachers. The exemplar excerpts are representative and illustrative of the evidence for the findings of the research.

Chapter 8 will present and discuss the findings that address the third research question of the study.

In teaching differential calculus: RQ.3 What mathematical and pedagogical *routines* do teachers use and how?

Chapter 8 will report on, and discuss the exploration routines (Sfard, 2008) in the teachers' pedagogical calculus discourse. By pedagogy here, I am referring to the practice as well as the theory of teaching, thus, what teachers do and say and all the processes and strategies in their mathematical and classroom discourse. Pedagogies on the derivative in this study describe the mathematical and the didactical routines (Sfard, 2008; Viirman, 2015) in the teachers' pedagogical calculus discourse.

The commognitive framework provides the lens for noticing patterns of behaviour and communication in the teachers' pedagogical calculus discourse, and for the vocabulary to describe the patterns. By examining the teachers' use of *mathematical words* and *visual mediators* or by paying attention to the processes of 'creating and substantiating *narratives'* (Sfard, 2007, p.572) about the derivative, we can notice the mathematical repetitive patterns in the teachers' actions. The analysis of the pedagogies on the derivative focuses on the exploration routines in the teachers' calculus discourse. Exploration routines are aimed at producing new narratives or substantiating endorsed narratives (Sfard, 2008; Thoma and Nardi, 2016). (See Chapter 3 for more on the types of routines and commognitive theoretical framework).

Chapter 8 reports on the findings of the study, but draws evidence from all the participant teachers, focusing on the teachers' pedagogical calculus discourse from estimating the gradient of a tangent to a curve, progressing to differentiating from first principles using a 'quasi-limit' definition of the derivative (refer to Section 6.4 on page 126) and finally culminating in differentiating polynomials using standard rules of differentiation. I describe the construction of the definition of the derivative by the teachers in this study as a 'quasi-limit' approach because the teachers do not use or explain the word 'limit' in their definition of the derivative. Nor do they explain the conditions for differentiability of the functions in their substantiation of the process of differentiation (more in Section 8.4 and Section 9.2.1).

What follows is a presentation of the evidence excerpts from the interview and lesson observation data under the following four subthemes: approximating gradients by drawing tangents; approximating the derivative at a given point on the curve by using the secant and tangent line; and introducing the derivative: the gradient of the tangent as a limit; and pedagogies on the derivative: The Why factors. These subthemes are followed by a discussion (see Section 8.6) and a chapter summary of the finding.

8.2 Approximating gradients by drawing tangents

In a pre-teaching interview with the researcher, T3 explains her lesson planning for introducing the derivative and the why-factors for her approach, T3I(i):

> 126.T. I want to link, I don't want to come in with a concept that, the students are not familiar with. So, I wanted to just start from somewhere they already know. And they already know about straight lines. We've done a lot of the straight lines recently. They haven't looked at gradients of curves, but we have done curve sketching. So, it's trying to tie that in, to begin with, so that we're just

> talking about gradients and that's a familiar term. … So, I'm hoping that again when we go on to first principles, we're still looking at two points albeit the algebra is more complex, but they are familiar with that. So that should be okay. And then start introducing different types of notations and highlighting different types of notations as, as they get more confident.

The teacher is aware of the possible complexity of different notations and calculus symbolism. T3's approach to the derivative is to base the lesson firmly on what the students already knew (assumed knowledge), straight lines and gradients of straight lines and then develop towards gradients of tangents to a curve at given points. T3's approach is generally representative of the approach taken by all the teachers. Teachers developed their lessons on the gradient of a straight line but focusing on the tangents to curves. By this stage (in the UK), at the AS level, students are expected to be familiar with the concept of a tangent, since it is part of the GCSE mathematics curriculum (e.g. circle geometry at GCSE includes theorems involving tangents).

To illustrate the findings of this study on estimating gradients by drawing tangents, representative evidence from two different lessons, T1LO and T2LO, is presented and examined. These two lessons have been selected because they used different functions and graphs to teach the same idea, i.e. estimating the gradient of a tangent to a curve at a point by drawing tangents. T1LO used the function $y = x^2$, drawing tangents and Autograph. T2LO used a circle and a tangent, then the function $y = x^2 - x - 6$, drawing tangents and GeoGebra. In both lessons, tangency and the gradient of the tangent formed the basis for introducing differentiation.

In T1LO, the teacher introduces the object of the lesson as 'the gradient of the curve' [16] using a metaphor of an ant travelling along the curve; and being at a particular point. T1LO:

13.T. Right okay you've now got in front of you the curve $y = x^2$. I want you to imagine you are an ant. Can you imagine what it is like to be an ant and you are literally travelling along the graph?
16.T. What do you think we mean by the gradient of the curve?
17.T. Imagine the ant is there [*The teacher pointing at the point (1,1)*], would you say the curve is steep there?

The teacher then asks students to draw tangents to the curve at the point (1,1). The teacher demonstrates, in Figure 8.1 below, drawing a tangent to the graph of the function $y = x^2$ at the point (1,1). T1LO:

26.T. Now I want you to locate the point on the graph where X equals one. Can you locate the point $x = 1$? y will also be 1 as well, and I want you to draw with a ruler the tangent, I want the tangent to be as long as you like. Straight line draw you think the tangent is. You're doing this by eye, by no other way, by eye.

27.T. We'll all get slightly different results, so you're just drawing the tangent by eye. I want you to imagine…

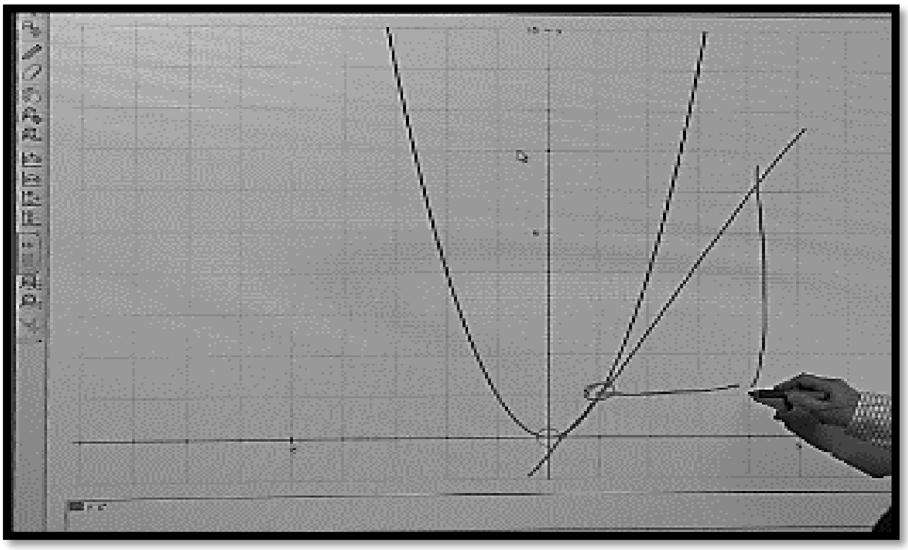


Figure 8.1 Sketching the tangent to the curve of $\mathrm{y} = \mathrm{x}^2$ at $(1,1)$

The teacher [T1LO] emphasises that the students draw the tangent 'by eye' [26], which is mentioned three times in the teacher's utterance in [26-27]. This implies the process of approximating the gradient of the tangent to the curve at the given point. 'We'll all get slightly different results…' [27] implies estimation.

The students are then asked to calculate the gradient of the tangent by measuring the lengths for the base and the height of their triangle using a ruler and then dividing the height by the base. The results of the activity (investigation) were then collected by the teacher and recorded on the board, as shown in Figure 8.2. T1LO:

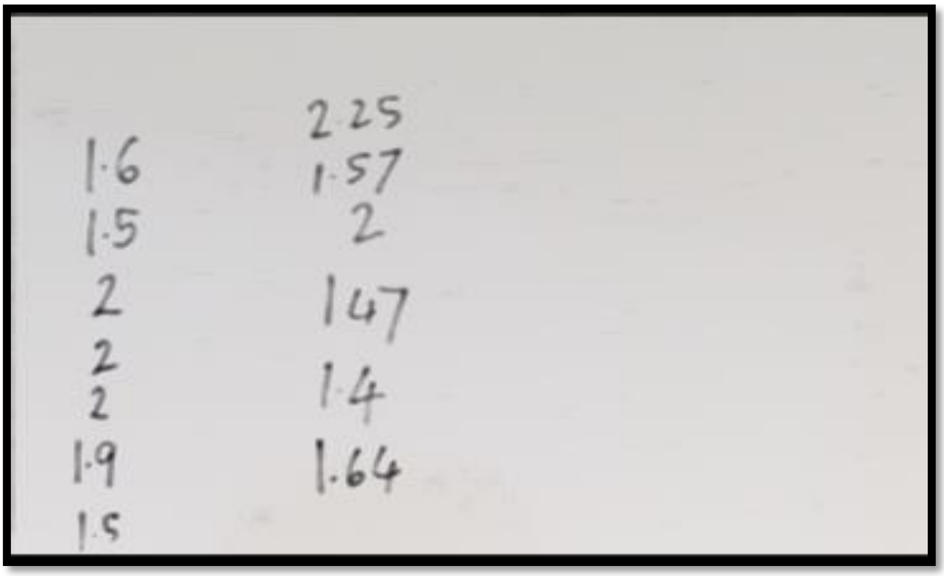


Figure 8.2 Approximate gradients of the tangent to $y = x^2$ at (1.1)

46.T. So, our answers range from 2.6 top, who is the 2.6 person? Who got the 2.6? Down to, I think, was it you, at 1.4? So, we're going to try now to…

47.T. See well, how you would actually do this properly. I'm going to try to show you on this graph here [*Teacher opens an autograph file of* $y = x^2$].

The students' calculation resulted in a range of values for the gradient [46] for the tangent to $y = x^2$ at (1,1) as shown in Figure 8.2. The teacher shifts to a graphical approach by displaying an Autograph dynamic imagery of the graph of $y = x^2$. He then gives a demonstration using a 'free' point, i.e. an ant moving along the curve towards the (1,1). T1LO:

50.T. What do you think it would be if I got the point even nearer, #T? The last point we drew for (1,1) was the point 1.001. If you hadn't gotten that one, it didn't matter. If I had made it 1.000001 which is very close, too close that we can't, we can never see that distance what would you think this gradient would be approaching?

51.S. 2

52.T. Say it again/ [2]/ 2 and in fact 2 is the exact gradient of the tangent at that point. That's what some of you got to.

53.T. I was a bit suspicious about some of those 2s if I will be honest with you, I think some of you might have seen this before. Well done to those who did get 2.

54.T. My experience of teaching this, is that most students draw tangents that are too shallow or too steep and that is sort of indicated by the fact that we got more answers that go underneath the 2 and over the 2.

55.T. It really doesn't matter by the way how good or bad your tangent was that is irrelevant. All I want you to understand is that the gradient of the curve is the gradient of the tangent. So, what can we see, ... at x=1 the gradient of $y = x^2$ is 2?

56.T. But again, what does the gradient of 2 on a curve mean? It means the gradient of that tangent is 2 at that point; y is changing twice as fast as x.

Note that the objective of the activity here was not about the accuracy of the students' sketch diagrams. The teacher explains that it did not matter how accurate the students' tangents were, that some tangents would have been too steep (explained by the values greater than 2) and some tangents would have drawn too shallow (as explained by the values less than 2). The teacher [T1LO] here explains the mathematical object of the activity as 'All I want you to understand is that *the gradient of the curve* is *the gradient of the tangent'* [55]. The teacher's utterance here implies that the '*gradient of the curve'* preludes the *gradient of the tangent*, which appears contradictory to the ensuing sequence of the activity. The gradient of the tangent at the point $(1,1)$ is used as an estimate for the derivative (instantaneous rate of change) of the function $y = x^2$ at the point $(1,1)$of the curve. The teacher explains an interpretation of the gradient of 2 at the point $(1,1)$ in [56].

The teacher, T1, extends the investigation activity by asking the students to find the gradients of the tangent at different points $(2,4)$ and $(3,9)$. The results are recorded on the board, as shown in Figure 8.3, which is a collection of the gradients of the function $y = x^2$ at various points on its graph. He then asks students to conjecture for the gradient at $x = 7$. T1LO:

57.T. Could you do the point (3,9), oh sorry, the point (2, 4). Who else did the point $(2,4)$? What would you expect the gradient here to approach? [S: 4] Right, so [the gradient at] $x = 2$, [for] $y = x^2$ is 4. You should begin to spot a pattern; it's not rocket science.

58.T. Who did the point $(3,9)$? Eh, what do you think...? [S: 6]
$y = x^2$ is 6, and $x = 4\ldots$

59.T. What is it approaching? [S: 8].
Ok, suppose you are given a different point they haven't done.
Suppose I say $x = 7$,

60.T. Without any of these calculations, could you make an intelligent stab on what is the gradient at $x = 7$?
At $x = 1$, the gradient is 2; at $x = 2$ the gradient is …

61.T. At $x = 2$, what do you think the gradient would be? [S: 4]
I was right; you say it goes 2, 4, 6, 8 ...that right?

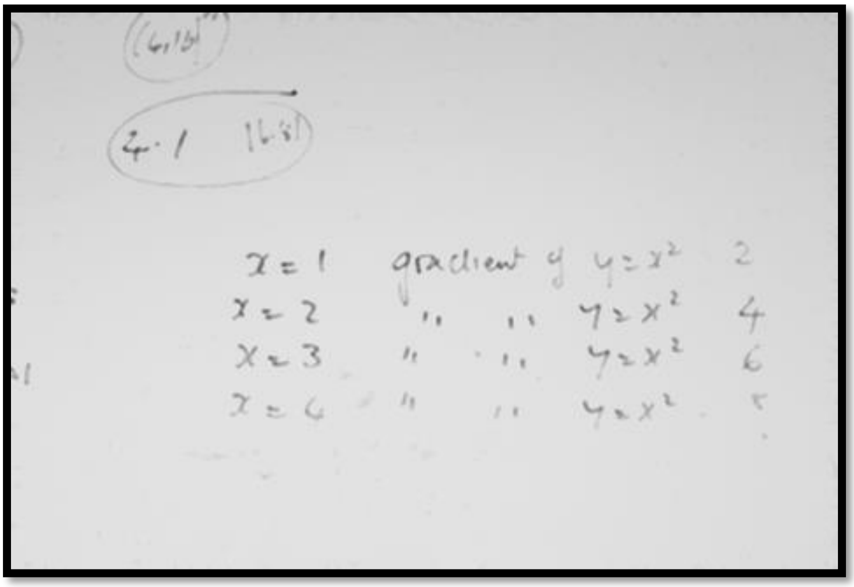


Figure 8.3 Gradient of $\mathrm{y} = \mathrm{x}^2$at various points

The investigation was extended from point (1,1) to other points on the graph of $y = x^2$. However, the teacher did not insist on students using the original method of drawing tangents to the graph at each of these other given points. Instead, the teacher guided the students to deduce a rule, 'Without any of these calculations, could you make an intelligent stab on what is the gradient at $x = 7$?'[60] - from the sequence of the successive gradients – 'you say it goes 2, 4, 6, 8 ...that right'[61]. The teacher immediately follows the pen and paper activity above with a dynamic imagery demonstration of the 'gradient of a curve', shown in Figure 8.4 below. T1LO

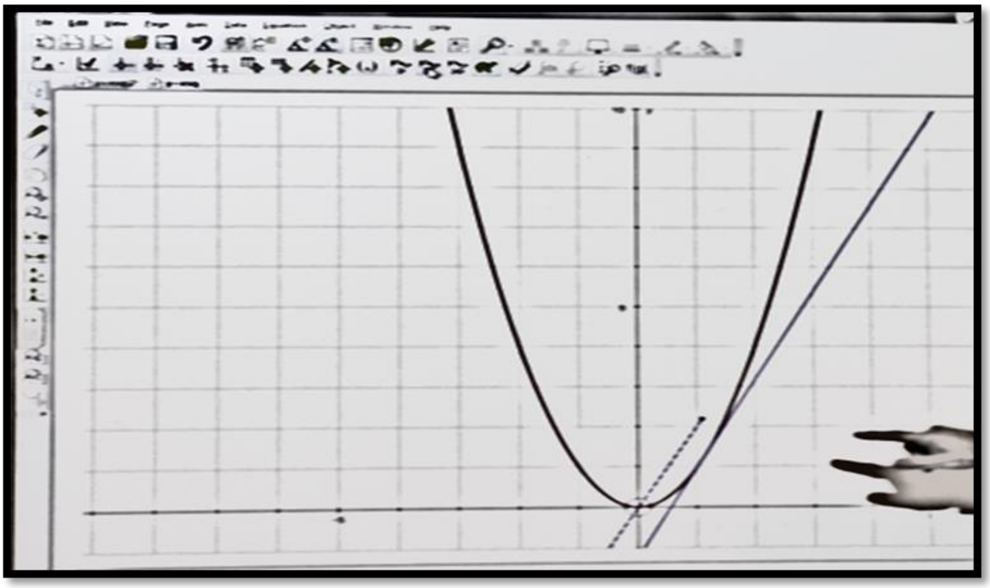


Figure 8.4 Autograph image showing the function $\mathrm{y} = \mathrm{x}^2$, the tangent line and the gradient function

62.T. Let me show you this.... now this is a very powerful tool that Autograph has ... I am going to show you the graph of the gradient function. What it is going to do is this, it's going to plot the gradient of that curve. It's going to travel down the curve and plot its gradient.

63.T. Tell me, what can you tell me about the gradient of the curve to start with, it's going to start from the left-hand side and travel that way? What can you tell me, I don't mean what values there are, what sort of gradients are these? [negative]. They are [negative]/negative, right?

Note T1's word use in utterance [61] – 'gradient function'. This is the first instance the T1 had used this terminology. However, the T1 immediately reverts to using the word 'gradient' [62- 63] to describe what, indeed, is the gradient function. The teacher's narrative here, suggests that 'gradient of a curve' is synonymous with 'gradient function'. For more on word use in teachers' pedagogical calculus discourse see Chapter 6.

Note the teacher's description of the idea of the 'gradient of a curve' by use of a metaphor – "I want you to imagine you are an ant. Can you imagine what it is like to be an ant and *you are literally travelling along the graph*" [13] and the reinforcement by use of dynamic imagery – "it's going to plot the gradient of that curve. *It's going to travel down the curve* and plot its gradient" [62]. The metaphor use here, is symbolic mediation, a form of visual mediation that could not be seen but imagined in the head. Whereas the use of the Autograph imagery is iconic mediation, a form of visual mediation through dynamic imagery. Symbolic and iconic mediators are used as complementary visual mediators in this lesson. The complementary use and shifts in forms of mediation are discussed in Section 7.4.

In contrast to T1LO, in T2LO, the introduction to differentiation started with a diagram showing a tangent to a circle as shown in Figure 8.5, T2LO:

34.T. I really like that. Yeah, I like that thought. So, boys, what Hxxx said is we know this point whatever it is [Pointing at the point of tangency, but the coordinates are not specified]. We also know this point, we know it's (0;0). And as Jxxx said, they are going to be perpendicular, that's one of our circle theorems from GCSE. Could we find, if we wanted, if we knew that point [Point of tangency] could we find the gradient of our tangent?

35.S. Yeah.

36.T. Yeah, we could, couldn't we? We're not going to go into it, but we could do.

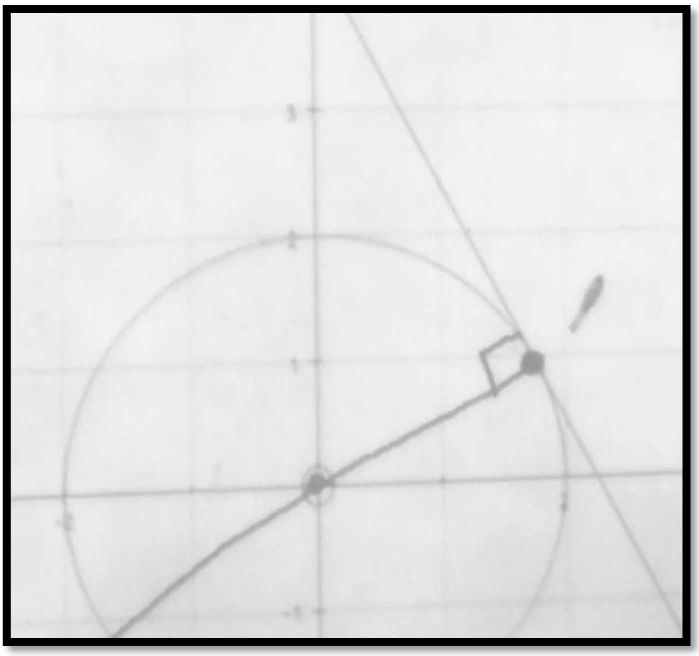


Figure 8.5 Tangent to a circle

The coordinates of the point of intersection of the tangent and the circle in Figure 8.5 are not known, which explains why the gradient of the tangent to the circle was not calculated. It would require calculating the gradient of the radius and then taking the negative reciprocal of that gradient, to get the gradient for the tangent to the circle. This was implied by the teacher's utterance: 'And as Jxxx said, they are going to be perpendicular, that's one of our circle theorems from GCSE' [34]. The evidence here suggests that, and shows how, T2 is building her introduction to differentiation lesson on students' previous learning of tangents, from GCSEs.

Similar to T1LO however, T2 resorts to an interactive GeoGebra file, T2LO:

> 36.T. Yeah, we could, couldn't we? We're not going to go into it, but we could do. What about, right, let's go to this if it will open. Oh, here we go. What about something like this? First of all, I want someone to come over here. I'm going to pick you [laughs]. I had this done to me at University the other day. What I would like you to do is to move these two sliders here Axxx, with my mouse. See if you can make a tangent. And I want everyone else to figure out what those two sliders mean when he is making it. Yeah, it looks good, it looks good. Right sit back down. What do they mean? What's he moved there to make that tangent? What's going on? Jxxx?

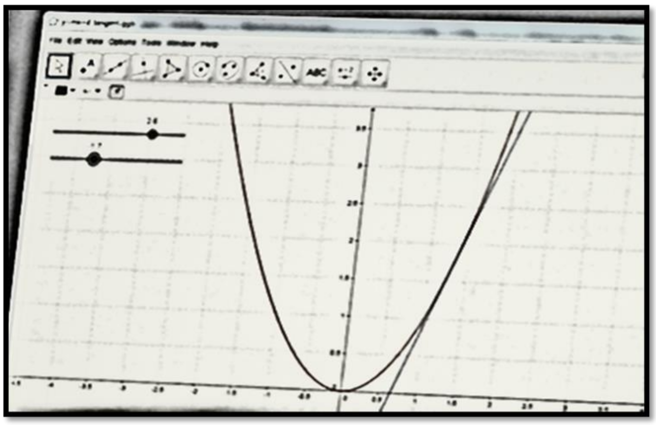


Figure 8.6 GeoGebra imagery of a tangent to a quadratic curve

37.S. Is the top one got the gradient? And then the bottom one is like; it's transformed like where it is.
38.T. Yeah absolutely. So gradient up there. Second one, not trans …, well, something we can't see?
39.S. y-intercept.
40.T. Yeah, it's the y-intercept, good. So, we're kind of confident we can draw on a tangent. And maybe, especially cause we're on a graph plotter we could find out the equation, the gradient of this tangent which would give us the gradient at a point. Do we agree with that so far? Yeah, okay. So maybe have a look for me on this sheet. I've drawn you out a nice curve. I'd like you, I've got some rulers here, to draw some tangents and see if you can work out the gradient at a few points of this curve. So, you draw in tangents. Be careful of the scale. The x-axis is a different scale if you like.

In contrast to T1LO, in T2LO students are asked to interact with the GeoGebra applet graph plotter, which shows some sliders and values on the applet, see Figure 8.6. Once the teacher had demonstrated the tangent to a quadratic graph, she handed out a worksheet, see Figure 8.7, with a quadratic graph upon which the student had to draw tangents, by hand using a pen and a ruler, and measure the gradient. Note that GeoGebra was used for the demonstration on drawing tangents in Figure 8.6, but the students are having to use pen and paper; they are not using GeoGebra.

Similar to lesson T1LO described above, what follows is an activity in which students are calculating the gradients of their drawn tangents to estimate the gradient of the graph at the chosen point of tangency. Further guidance is given as the students start to work out the gradients of their tangents. L2LO:

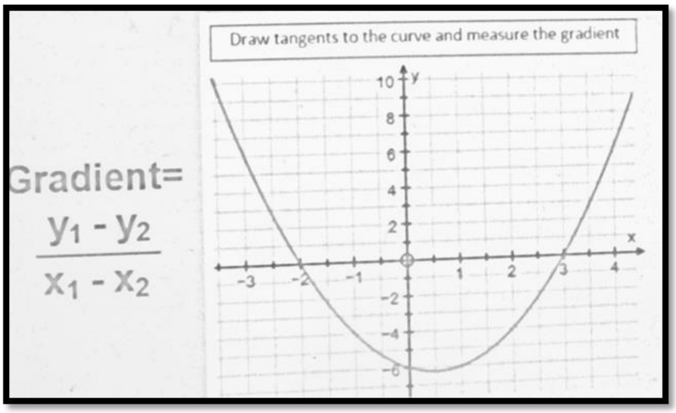


Figure 8.7 Worksheet: Graph of $y = x^2 - x - 6$

42.T. Does anyone need a ruler? Yeah. So, we're not going to do this for long. Pick a few points that are definitely on the curve because there's some that are definitely on them. We can see from where they cross. Maybe check with the person next to you. If you've picked one of the same points, check you've got the same gradient.

The instruction here [42] suggests that the students were expected to select tangency points carefully, for example, by selecting the points that would allow them to count squares for the vertical and horizontal distances, instead of measuring with a ruler. In the excerpt below the teacher is talking to one of the students about the task, T2LO:

67.T. So, what you're doing here is you're assuming that they're at right angles. That's at right angles to the origin. The only, the only reason we assumed that last time, because it was the middle of a circle. If you think about the centre of the circle and the radius, that's at right angles to the tangent, but that isn't. So, all we're doing here is counting along the squares to work out the gradient of that line. Yeah? So again, you're assuming that this is a circle, which it's not. All we're doing is counting. So, we've got a line, if we count how many in the y-direction, yeah?

68.S. Oh, that makes a good point.

The evidence here suggests that the earlier definition of a tangent which was illustrated by means of a circle and a tangent might have confused some students. It appears that students found challenging the task set in Figure 8.7. In the excerpt below, the teacher intervenes in a dialogue in which two students are talking to each other about the task [51-52]. L2LO:

51.S. Can't remember how you do it.
52.S. Is it, are we supposed to be differentiated by now?
53.T. No, no, we're just drawing on tangents and working out the gradient.
54.S. So again, you can work out the gradient by differentiation, can't you?
55.T. Yeah, we can, but we're just approximating here.
56.S. Oh right. Yeah.
57.T. No differentiation yet. In fact, no differentiation for the whole lesson. Not until the next lesson. So, don't worry about that now.

[*Teacher moving on to the next pair of students*] What are we doing over here? How are we doing?

58.S. I can't do it. I can't.

59.T. Just have a guess. It's only an approximation.

Drawing tangents to a curve, measuring and computing the gradients, here clearly proved a difficult task. One student argued for resorting to the methods of differentiation [54], which was turned down by the teacher. The teacher explained to the class that it was important for the student to develop some understanding of the process of differentiation before latching on to the standard methods of differentiation. For a consideration of the teacher's reasons or motivation for their approach to introducing differential calculus, refer to the why-factors in Section 8.5.

8.3 Approximating gradient of a tangent to a curve at a point by using secant and tangent lines

The teachers in this study used some graphical mediation in constructing the definition of the derivative. Although the teachers in this study were observed using the secant and tangent lines for approximating the derivative at a point on the graph of a function (mostly, the graph for $y = x^2$), there are observable contrasting teaching approaches in their construction and substantiation of the definition of the derivative. This study further reveals that in their attempts to define the derivative, teachers did not directly use the word 'limit', instead they all used the utterance '*getting closer and closer'* (See Section 6.3). The data shows that when the teachers used the utterance *getting closer and closer,* they did not always describe the same mathematical object. I give three examples here (see Subsections 8.3.1 to 8.3.3), to illustrate this finding. In Section 8.3.1, the didactical routine focus is on *getting closer and closer* to the 'limiting value' of the gradients of the secant lines, e.g. in T4LO. In Section 8.3.2, the didactical routine focuses on two points *getting closer and closer* 'together', e.g. in T3LO and T5LO. In both Section 8.3.1 and 8.3.2, the teachers used static graphical mediators of hand-drawn sketch diagrams for the graph of the function $y = x^2$. In Section 8.3.3, for example, T2LO, the didactical routine focuses not only on the points or the gradient values but also on the secant line (chord) *getting closer and closer* to the tangent line. T2LO is an illustration of the affordances of dynamic imagery artefacts in constructing and substantiating the definition of the derivative.

8.3.1 The gradients of secants *getting closer and closer* to a particular value

Consider the excerpts below from [T4LO], which is an investigation activity of the slope (gradient) of the tangent to the graph of $y = x^2$ at the point $(1; 1)$. In the excerpt below, the class has just calculated the gradient ($m = 2.5$) for the chord connecting points (1,1) and (1.25, 2.25) on the graph of $y = x^2$, see Figure 8.8, which shows the gradients of the secant lines. Note that the class is using the standard method for calculating the gradient of a straight line – 'the change in y' divided by 'the change in x': $\frac{y2-y1}{x2-x1}$. The title on the board is 'Differentiation' sets the objective of the lesson, thus all the activities of the lesson here constitute differentiation. The teacher T4LO explains:

37.T. In fact, let's think about this for a moment, guys. We've just worked out that this red line *[Chord between (1,1) and (1.25, 2.25)*] has a gradient of 2.5 but remember our aim is to find the gradient of the green line [*the tangent*]. Is our answer too steep or too shallow? Rxxx, go on? Too steep. So, we're going to, this one's a little bit too steep and we, remember we're trying to, we're trying to work out what this gradient is [*tracing or pointing along the tangent line*]. It's a little bit too steep so we're going to choose a point but it's closer, it's closer to that point there [*pointing to the point (1,1)*]. So why don't we choose (1.1) [*the point at which* $x = 1.1$]? Okay, why don't we choose (1.1)? So, it's going to be somewhere closer to here [*marking a point on the curve*]. So, this x-coordinate now is (1.1). And I'll just rub out the dotted lines. Okay Gxxx, how do we work out the y-coordinate when we know that $x = 1.1$?

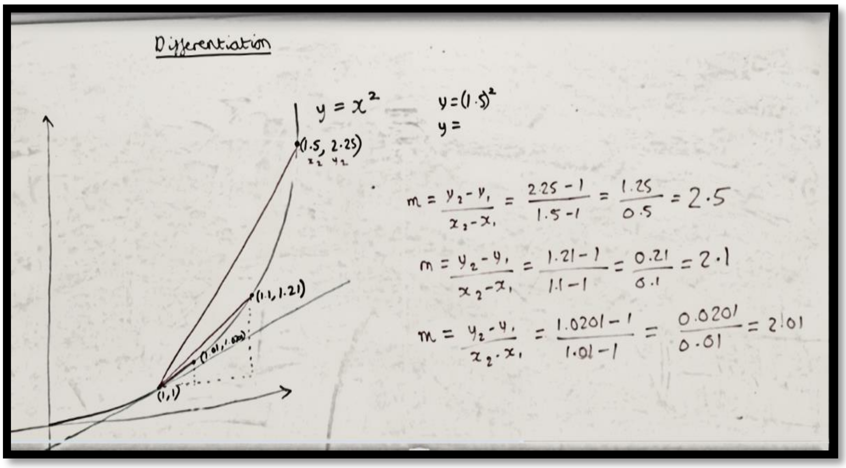


Figure 8.8 Successive gradients of secant lines

Note how the teacher draws the students' attention to the object of the classroom discourse, 'remember our aim is to find the gradient of the green line [*the tangent*]?' [37]. The teacher explains that by calculating the gradient of the chords, they are estimating the gradient of the tangent to the graph at (1,1). The teacher emphasises this point by asking the students to compare the slopes of the chord and the tangent by looking at the diagram in Figure 8-8 and he reinforces the point, 'It's a little bit too steep so we're going to choose a point, but it's closer to… (1,1)' [37].

The process is repeated for points at $x = 1.1$, and $x = 1.01$, thus taking points that are successively closer to the point of tangency, (1,1). See Figure 8.8 for the calculations and graphical representations. Three gradients, $m = 2.5,\ m = 2.1,\ m = 2.01$ in Figure 8.8 were calculated, for $x = 1.25,\ x = 1.1,\ x = 1.01$, respectively. Note the use of m for representing the different gradients of individual chords. Note that throughout this activity of calculating the gradients of the successive chords, see also the two selected excerpts [49] and [65] below, the teacher keeps linking and referring to two lines here – the chord and the tangent. The teacher is emphasising the point of interpreting the gradient of the chord with due reference to the gradient of the tangent. T4LO:

49.T. So, we're going to choose a point a bit closer on because then if we work out the gradient of that, it's going to be a bit closer to our real answer, okay? I just picked 1.1 just cause it's a number I thought of that's a bit, a bit closer to 1, okay? Now this time…this time…we're going to work out this gradient here. And I want you thinking, year 12, about how our estimate is getting closer and closer and closer to the true value.

65.T. Okay, right we're pretty close now. Our lines, this line's getting closer and closer and closer to the green line. Let's pick a point that's even closer, okay? Let's pick somewhere, something that's so incredibly close that it's nearly exactly the same gradient. Let's go for 1.01. And by the way year 12 you can use these, you can use these numbers in your next example, if you want. Alright some, 1.01 squared is going to be 1.02, okay? So, we're now working out the gradient of this.

"Our lines, this line [chord] is getting closer and closer and closer to the green line [tangent]" [65]. Not only does the teacher keeps the students' focus on what is happening with the chord and the tangent but keeps emphasising the point of "*getting closer and closer and closer"* to the 'true' value for the gradient of the tangent at (1,1). "I want you thinking, Year 12, about how our estimate is getting closer and closer and closer to the true value" [49]. Note the teacher's word use in this utterance: "Let's pick somewhere, something that's so incredibly close that it's nearly exactly the same gradient" [65]. This explains the ultimate objective of the activity of selecting points successively getting closer and closer to (1,1), which is to estimate the gradient of the tangent by the gradient of the 'closest' chord.

The teacher asks students to make a conjecture about a value (the 'limit') the gradients of the successive secants were approaching. However, the teacher does not mention the word 'limit', instead, says "Think about what's happening to the value of the gradient as we get closer and closer and closer to our tangent which is what we want"[75]. The teacher's focus here is on the gradient values and the value that are converging to (the limit) 2. T4LO:

75.T. Right, Year 12 have a moment to think about our three answers. Think about what's happening to the value of the gradient as we get closer and closer and closer to our tangent which is what we want, okay? Does anyone want to have a look at those three numbers and have a guess at what the, what the true gradient of the curve at (1,1) might be? Hxxx?
76.S. 2.
77.T. 2, why is that then?
78.S. Cause it's just getting closer and closer to…
79.T. Exactly. We're getting closer and closer and closer to 2. We're getting closer and closer and closer to 2. So, I think the real answer to this question is going to be the gradient is 2 but we need to try and prove that now. We need to try and prove it, okay? Here we go. Here's the clever bit. This is the bit that, this is sort of an idea of what Leibniz and Newton came up with. Can you draw the same axis, maybe on your next page? So, we've still got $y = x^2$. And actually, we're still going to look at, we're still going to consider $x = 1.1$.

Note that the focus here is on the value that the difference quotients converge to, as 'h' approaches zero. The teacher's questions in [75] describe approaching the 'true value' of the gradient values. 'Does anyone want to have a look at those three numbers and have a guess at what the, what the true gradient of the curve at (1; 1) might be? [75]; 'What's happening to the value of the gradient *as we get closer and closer and closer* to our tangent?' [75]. 'We're *getting closer and closer and closer to 2. We're getting closer and closer and closer to 2*' [79]. This clearly explains that the teacher's focus is on the value 2, which is, indeed, the limit value of the gradients of the successive secants. The utterance *getting closer and closer and closer* is used here to describe the idea of the limit.

Note, also, the reference to Leibniz and Newton in the teacher's utterance [79]. The lesson had started with a reference to the history of calculus, in which the teacher introduced the founders of calculus and talked about the 'big argument' about the alleged plagiarism between the two men as to who had discovered the ideas of calculus first.

8.3.2 Two points getting closer and closer together

In contrast to T4LO above, the focus in [T3LO] is on two points on a curve, and the utterance *getting closer and closer* is used to describe the two points getting closer together. T3LO uses a similar approach of using the chord and tangent to estimate the gradient of a tangent at a point on the graph of a quadratic function $y = x^2$. The investigation activity in T3LO builds on students' ideas. The teacher-student dialogue in the excerpt below comes after the class had found it difficult to draw accurate tangents for calculating gradients at given points on a curve. T3LO

64.T. I could work out a point on that curve which might be a useful thing to do to work out a point on that curve. How could I work out the gradient there? If I know that point is (2; 4), how can I work out the gradient with a point? Kxxx?
65.S. Like you just find out like a different point as well.
66.T. Right, so you might find out a different point. What's useful about having two points?
67.S. …
68.T. Right let's have a look [*Referring to Figure 8.9].* So Kxxx suggested…that I have…2 points. So, I've got 2 points on the line. So, I could say I want my gradient here, where I put a point at (2,4) there and then I'll find another point. Let's find a point there. Does that give me the gradient at this point?
69.S. No.
70.S. No.
71.T. It's a really, really good idea. Does it give me the gradient at this point? Kxxx do you want to comment a bit further on it?
72.S. No, it just gives you the gradient of the line between two points.
73.T. Okay. Is it similar to the gradient or is it a long way off of the gradient at that point? Is it near to the gradient?
74.S. Close.
75.T. It's close. What would be better?
76.S. If you get the two closer points.
77.T. Two closer points would be better. Why would two closer points be better?
78.S. Just be more accurate.
79.T. It would be more accurate if you bring those points closer together. Right okay. So, if I would have used [*Referring to Figure 8.9*] this point and that point…it's looking more, more accurate. Yes, Gxxx?

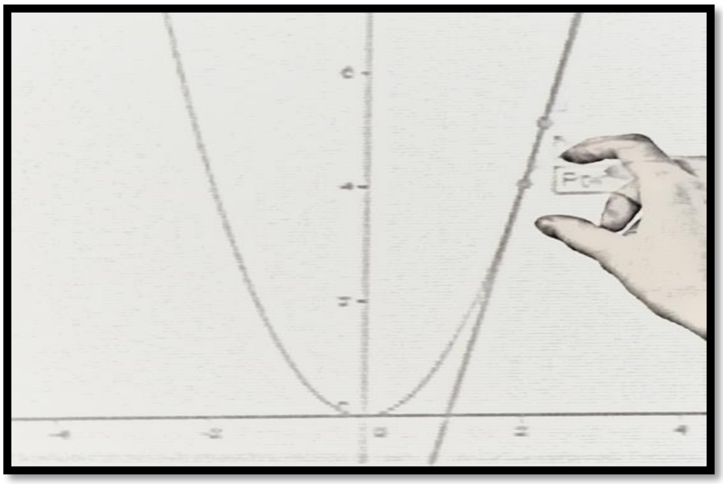


Figure 8.9 Tangent to the graph $y = x^2$

80.S. If you drew a tangent from your point and found the gradient of that.

81.T. Fantastic. If we could draw the tangent to the point. So, if we actually had a graph that we could look at, we could draw a tangent … [*Referring to Figure 8.9*] And we've just decreased that difference between the two points and basically brought the points together and it just touches at one point [*Dragging the two points in Figure 8.9 closer together*]. That would give us a way of working the gradient at that point.

The two points on the curve have been selected rather randomly. Figure 8.10 is a snapshot of a GeoGebra visual image used to illustrate the graph of $y = x^2$ and the tangent. The routine so far suggests moving or dragging the two points closer and closer to each other [79]. Here, the teacher T3LO seems to be making the point of drawing the two points together. There is no mention of a secant line (chord) nor any explicit reference to the relationship, in this case, between the secant and the tangent lines.

The teacher then gives out a worksheet with the graph $y = x^2$, with five different points. The students are working in pairs and each pair is assigned one point. The task set is to estimate the gradient of a curve at a given point by *getting points closer and closer* to each other. What follows are all the instructions given. T3LO

87.T. So, there are five different sheets around the room. They are all the $y = x^2$ graph, but you each have a different point you're looking at. So, some of you are looking at the gradient at point x = 2. Some of you are looking at the gradient of $x = 1$. And I want you to look at 2 points at a time. So, you've been given 2 points that you're going to work out the gradient of that line. And we're going to do exactly as Sxxx mentioned, get the points closer and closer. So, start off with two points and then get them closer and closer together as Sxxx suggested. And see what we notice. So, work on the sheet in your pairs, please.

89.S. Miss, I don't get what we're doing.

90.T. You can use a calculator to help you with the calculation. So, we've got…you know how you said we could find 2 points and join them together and work out the gradient, you've got your 2 points.

The student's utterance 'I don't get what we're doing' [89] could be referring to the utterance *getting two points closer and closer* [87], whether both points move and what would then be the point of tangency. Similar to the student's utterance in [89], a student from T2LO working on a similar task said: T2LO

58. S. I can't do it. I can't.
59. T. Just have a guess. It's only an approximation.

There is anecdotal evidence to suggest that, at least at some point, some students had difficulty with the activity of estimating gradients using tangents and chords.

Further into the lesson [T3LO], T3 reiterates the idea of moving points together. [T3LO]:

200.T. So, you would use that brilliant idea of using two points. And then if we *move those two points closer and closer*, we *get a better, better approximation*.

The focus here is on getting the two points *closer and closer* for a better approximation of the slope of the tangent. However, the utterance [200] could imply both points move towards each other.

Similarly, in a different lesson [T5LO], the focus here is again on pairs of points and computing gradients of tangents at given points on the graph of $y = x^2$. The teacher required students to compute the gradient of a secant line between the two points. The students were asked to draw the graph of $y = x^2$ and then working in pairs, to investigate the gradients of the tangent at each of the points, $x = -2, -1, 1, 2$ on the graph of $y = x^2$. T5LO:

100.T. All I'm going to ask you to do, I'm not going to tell you how to do it for now because there's, I think there are two main ways of doing this in my opinion. Just to give you a tiny little bit of a hint, I'd like you to find the gradients at some points for me, please.
But can I have, Mxxx and Cxxx, could you investigate the gradient where $x = -2$ for me, please? Exxx and Jxxx, at -1. Unlucky you don't get 0, I would like you to investigate 1 for me, please. And then could we have 2 and 3?
I'd like you to investigate the points, okay, the gradients at the point that I've given you. So Oxxx, Cxxx and Mxxx are going to investigate the gradient at this point here [*pointing at the graph*]. Exxx and Jxxx that point there, okay and so on. …

I'm going to give you free rein for now. Okay, what can you tell me about your gradient? Anything at all so far?

The students engaged with the investigation activity, but it was taking rather too long to get to the point the teacher was hoping for, so the teacher intervenes. In the excerpt below, the teacher demonstrates the routine for estimating the gradient of a tangent at a given point on a curve by using two points, T5LO:

172.T. One that I didn't give to anyone then was the (0,0). So that's the one that I'm going to use. I'm going to choose the coordinate (0,0) and I'm going to choose a coordinate somewhere else on the graph. So, the first one that I'm going to choose is I'm going to choose this one up here. Why do you think I've chosen that one specifically rather than say here?

173.S. Because it's got an accurate point.

174.T. Absolutely. We know what it is, we've calculated it. It's a case of those are integer coordinates there of (3,9). Then what I'd do is I'd find the gradient between those 2 points [*Referring to points (0,0) and (3,9) in Figure 8.10*].

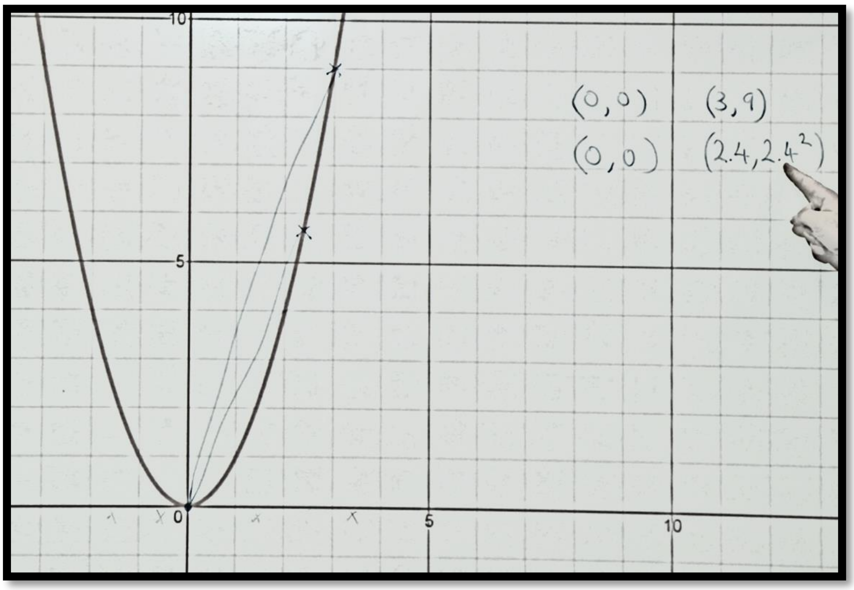


Figure 8.10 Gradients of chords

So, I'd be finding the gradient of this line here [*The hand-drawn line connecting (0,0) to (3,9)*]. Then I want to choose another point somewhere else. This time I am going to challenge myself, okay? I'd like to find one that's just somewhere strange. So, I'd go with something like thereabout, say 2.4. Work out what the value for y, so (2.4, 2.4 squared). And I'll find the gradient between those 2 points. Next, I'll go for quite an easy one again and what about (2,4)? And gradually I'd like you to take the gradient of say 3 or 4 points, each *one getting closer* to your point please, okay? I'd like you to do it for your point though. So, if you guys could do it for minus 2, minus 1, 1, 2 and 3, okay? Does everyone understand what I'm asking them to do? Are you sure?

The graph in Figure 8.10 is drawn using Desmos, which is a dynamic graph plotter, but the dynamic imagery affordances of Desmos were not exploited. Consider the teacher's utterance "And gradually I'd like you to take the gradient of say 3 or 4 points, each one getting closer to your point please, okay" [174]. The implied gradients are indeed the gradients of chords as illustrated in Figure 8.10. As the investigation activity continued, the teacher makes yet another intervention to give further guidance to the students. T5LO:

229.T. Okay whatever order that you've done them in, could you just take a very quick look at, at mine? This could be an example of the five coordinates that I chose. So, my original co-ordinate was (0,0). And my other co-ordinates I've started with $x = 3$ and gradually my $x - coordinate$ has got slightly less and less towards (0,0).

Folks with the negative coordinates, if yours was originally negative (-1, -1), I want you to choose one from up here and you'll approach from this side [*Pointing on the left-hand side of the graph*]. Okay, so your $x - values$ will be gradually increasing towards negative one (-1), okay?

I'm going to ask you in a minute, I'm going to ask every group for their values for the gradients in that order, okay? *Getting closer and closer* towards your points so that your two points that you're choosing are *getting closer and closer,* okay? So, if you could write your gradients as a list of five gradients for me, please.

The teacher's utterance *getting closer and closer* was constantly repeated throughout the lesson as the teacher explained the task to the students. There is evidence [349] to show that even after the teacher's reiteration in [229], some students had not understood the routine nor the object of the teacher's talk, i.e. the object of the classroom discourse. T5LO:

348.T. Okay and is this, are *these getting closer and closer* to your value every time?

349.S. No, we're just doing them randomly.

350.T. Okay I need them; I need them in order.

The evidence here suggests that the utterance *getting closer and closer* was not always clear that it meant gradually moving or dragging one point towards a fixed point, the latter being the point of tangency at which the gradient is required. These students [349] were calculating the gradient of lines connecting some random pairs of points. The teacher (T5) realised that the investigation was not getting the results he was expecting. The ultimate object of the whole activity is captured in the teacher's utterance below, towards the end of the lesson. T5LO:

417.T. Okay so choose points that are *even closer and closer and closer again*. And see if you notice anything about, about that pattern. So, what's the, what have you noticed?

Note that *getting closer and closer* here is describing the points, not the secant line and tangent line, nor the limit value (unlike in (a) above) of the computed gradients. The investigation activity took far more time than what had been anticipated because the majority of the class did not seem to understand what was expected of them.

8.3.3 The secant line getting closer and closer to the tangent

In contrast to the other teachers in Section 8.3.1 and Section 8.3.2 above, T2 uses dynamic graphical imagery and the utterance *getting closer and closer* to explain, not only the points but also the secant line (chord) approaching the tangent line, [T2LO]:

93.T. And we've got two points here P and Q. And they're definitely on our curve. Yeah, we can see that. And they've drawn something called a chord through. We've all heard that word before, a chord. We could kind of guess the gradient of this tangent here at P, this green tangent. Kind of guess it by using the gradient of the blue one. What do you think? Kind of? How could I make that a bit more accurate? So, if I knew the gradient of the blue line because I've got my two points, how can I make it a little bit more accurate? How could I have done this more accurately?

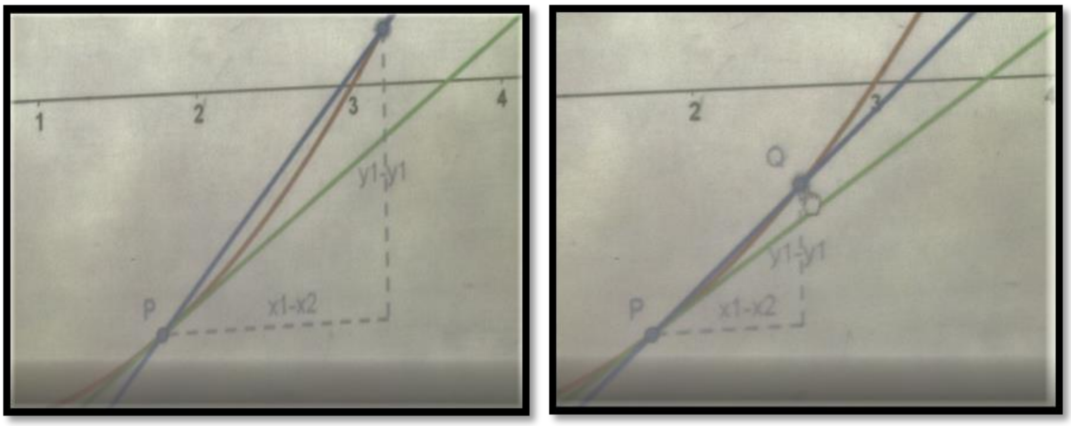


Figure 8.11 Estimating gradient-secant & tangent ((i) & (ii))

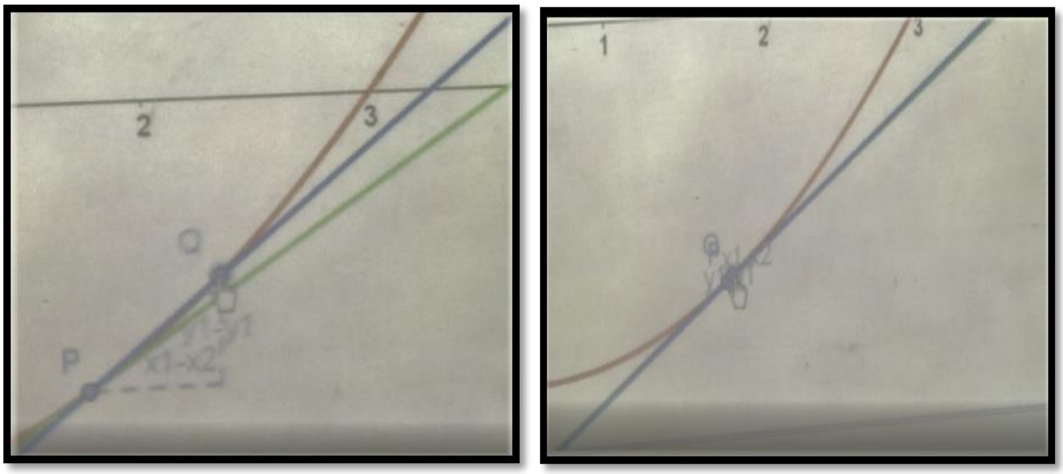


Figure 8.12 Estimating gradient-secant & tangent ((iii) & (iv))

94.S. No?

95.T. How about…if I did this. [Dragging point Q toward point P] More accurate or less accurate? More yeah? Bit more? Bit more? If I get it all the way down there [Point Q reaches point P, see Figure 8.12(iv)] it's pretty much the same thing as a tangent, isn't it? This is a way we can try and use to may be estimate the gradient at the moment. May be even find the gradient, we'll see.

Dynamic graphing imagery of GeoGebra allowed for the dragging of points or lines on the interactive whiteboard interface, as shown in Figures 8.11 and Figure 8.12. As point Q is slowly dragged down along the curve 'getting closer and *closer'* to point P, simultaneously the secant line (chord) slowly rotates 'closer and closer' to the tangent line. As point Q approaches point P, the secant approaches the tangent. In Figure 8.12(iv), the secant line is approaching the tangent line. Note that the focus here is on the secant and tangent line, and *getting closer and closer* here, describes point Q approaching point P and the secant line approaching the tangent line at P. This dynamic graphical imagery as sequenced in Figures 8.11 & 8.12 is an example of the generative power of visual mediation (Sfard, 2008), for it allows for multiple representations.

Following on the above demonstration, the students were then tasked with finding the gradient of the tangent to the graph of the function $y = x^2 - x - 6$ at different given points. They have all been given a worksheet with the graph of $y = x^2 - x - 6$. T2LO:

97. T. So, what I'd want us to do is to really look at finding the gradient of this blue line [*secant line*], this chord. And it's going to help us approximate the gradient at that point there, P. Are we okay so far? All right let me explain how I want you to do that and you're all going to do it a little bit differently. … And you're all going to have a particular point.

Now you might notice my point was (4,6). Noone is going to get that point. And that's going to be here [*marking a point of the graph, Figure 8.13*]. Now

I'm going to pick another point, Q, which is definitely on my line [*graph*]. And I'm going to start trying to move it down.

I'm going to do this numerically. Now can anyone tell me a point that this [*Q*] could be? Thinking about our equation which I haven't really told you about yet, that's our equation up there [$y = x^2 - x - 6$, see *Figure 8.13*].

Can someone give me a point Q which could be in that position [*marked on the graph, Figure 8.13*]? Use your calculator if you want.

Think of an x-value; x is 4 there [*pointing at P*]. What could x be here [*pointing at Q*]? Maybe it could be 5. Let's do 5. What's my y –coordinate going to be? …

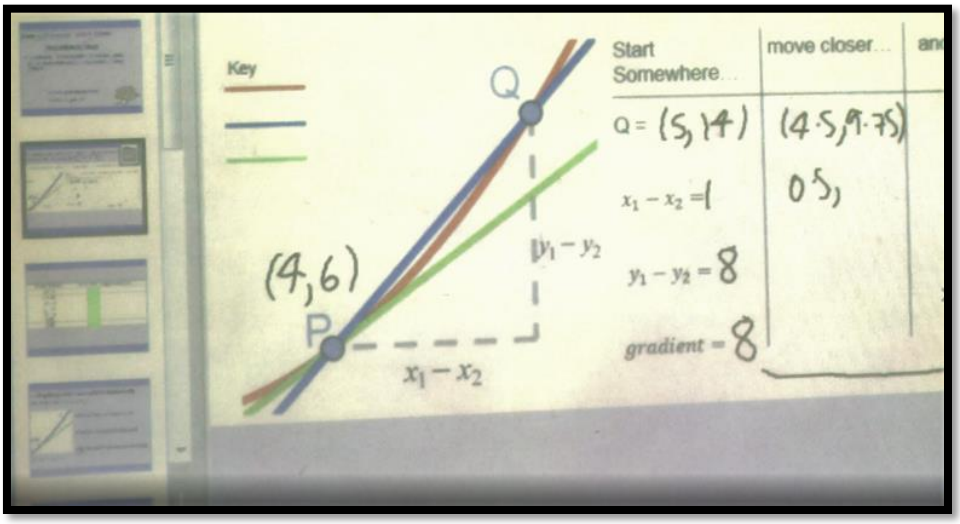


Figure 8.13 Estimating the gradient of the tangent to the graph of $y = x^2 - x - 6$ at (4,6)

101.T. Right, 14 yeah? Do you agree? Right, okay so I'm going to start at that point. I'm then going to think what is, what is this distance here [*Referring to the vertical distance*]?
104.S. It would be 14 minus 6.
105.T. Perfect.
106.S. 8…

The teacher continues to explain the routine for estimating the gradient at point P, by taking successive points for point Q that would gradually draw point Q towards point P, thus the secant line getting closer and closer to the tangent at point P. Note how the teacher asks the students to conjecture [110] thereby guiding them to the targeted anticipated result. T2LO:

107.T. Perfect 8. So, my gradient there [*Filling up the table in Figure 8.13]* is going to be $\frac{8}{1}$ which is 8. Then I'm going to think about moving this Q a bit closer. How about instead, I do 4.5 and whatever my answer is?
109.S. 9.75.
110.T. 9.75, okay. So, I'd then work out my $x_1 - x_2$ which would be 0.5 yeah. We'd do the y [*Pointing at the change in y;* $y_1 - y_2$] that one there and I'd work out the gradient.

Do you think you know what's going on so far? So, if I give you a worksheet with a different point on it, can you try getting closer and closer? And can anyone tell me, thinking about what we just did, what that might be getting closer and closer to as we move closer to the gradient? We use that point for 6, someone has it?

111.S. 7.

112.T. 7. It will get closer and closer to 7. Right, I'm going to stop talking now. Can you move them along the row [*Handing out worksheets*]? Do you all understand what's going on?

The students have different points on which they are having to draw the tangent and work out the gradient by following the same routine as demonstrated by the teacher of 'moving closer and closer'. They have all been given a worksheet with the graph of $y = x^2 - x - 6$. The teacher reiterates the routine to one of the students in the excerpt below. T2LO:

210.S. Am I on the right track?

211.T. Yeah absolutely. So, what you've done is you've started at, started at 2 then you've moved a bit closer to 1.5. Maybe a bit closer, 1.25. Move a bit closer and your gradient will start changing from 2 to a different gradient. And it'll start approaching something.

The teacher's utterance 'it'll start approaching something' [211] suggests looking for the limit of the emerging gradient values. Thus, computing the limit, without using the word limit. The routine of getting the two points closer and closer together, by moving point P closer and closer to point Q, if continued, implies that points P and Q would eventually be at the same point. A student appears to have conjectured this phenomenon. The dialogue below shows that the student had found a problem with the approach. How could the teacher explain the $x_1 - x_2$ approaching zero, not equal to zero divided by zero? T2LO:

107.T. So just keep going with it. Right we're going to stop in about one minute and see how far we've got. Good.

108.S. Miss.

109.T. Yeah.

110.S. What do you think about dividing zero by zero?

111.T. What do I think about it? Well, what do I think about any, dividing anything by zero?

112.S. It's 1 because you divide it by itself. But also if you didn't have any then it would still be 0.

113.T. So, you divide it by itself so it's 1 because you didn't have any. It should never be 1.

114.S. Yeah.

115.T. Yeah, I think dividing anything by 0 is something you should google first of all. Think about that argument.

The didactical routine of using tangents and chords to explain the process of differentiation seems to present conceptual challenges for students and teachers alike. The approach by which to 'keep getting closer and closer and closer' or 'as h approaches zero' appears to lead to zero divided by zero. The answer given [115] to the student's question was not adequately substantiated. The question about $\frac{0}{0}$ is indeed a difficult one to explain. Nonetheless, the teacher was determined to get the students to carry out the investigation which potentially led to such tricky questions. In the excerpt below, here the teacher reiterates and explains the importance for the students to build an understanding of the process of differentiation. T2LO:

246.T. Yeah. Leave it there for now and we'll talk about it. Right, let's stop there. Don't worry if you've not finished. Once again guys this is an investigation to try and develop a bit of understanding about where our differentiation comes from. I know we're not there yet, but it doesn't matter if you've not finished.

Let's have a think once again about what all this meant and what you were doing there when I find this. What you were doing is you had your point P. You had your point P and you picked a point Q and then you made it a bit nearer, a bit nearer, a bit nearer so eventually, if it was ever the same point it would have the same gradient. Does anyone want to tell me what theirs was approaching? Yeah?

There is evidence from this lesson T2LO, to suggest a successful lesson outcome. When the teacher asked, T2LO:

379.T. So, people at the front did, did we understand all that? Is it nice to know where it came from?
380.S. Yeah! Yeah! Yeah!
381.T. Yeah, good.
382.S. Probably the best!
383.T. I wish someone had told me that at A-level, I do!

The acknowledgement from the students was very positive, as highlighted by the student utterance 'Probably the best!' [382]. This was a lesson which made extensive use of dynamic imagery of GeoGebra, whose affordances include more accurate graphing of functions, drawing of the tangent by means of sliders, dragging of variables, and zooming on the graph. This was in a lesson by a teacher [T2] who explained her approach to introducing differentiation as one which aimed at students' understanding of why differentiation works.

Summary

In conclusion to this section, I want to highlight a couple of observations that could raise questions or implications for teaching. The first one relates to estimating the gradient of a tangent at a point on a curve by a consideration of two points, the secant line (or chord) and the tangent line. The teachers repeatedly used the phrase *getting closer and closer* referring to different objects – the gradients, the points, or the chord and the tangent. For example, in the lessons T3LO and T5LO, the teachers do not exclusively refer to the tangent as the goal of '*getting the points closer and closer*' or explicitly mention the secant line; instead, their didactical routines focus primarily on two points. Whereas in T2LO, the teacher makes use of a colour-coded graph showing the curve, the secant line and the tangent, as well as the two points of reference. All three lines, the curve, the secant (chord) and the tangent are explicitly substantiated in the didactical routine.

The second observation relates to how teachers treat the idea of limiting values, e.g. in [T2LO, 110-112] as point Q gets closer and closer to point P, and the successive gradients approach a certain limiting value. This observation pertains to the didactical routine of 'keep getting closer and closer and closer', i.e. estimating the respective gradients of the tangents to the curve at given points, as '$x_1 - x_2$' approaches zero or 'as h approaches zero' or the horizontal distance approaches zero. Does it lead to zero divided by zero? How do teachers explain the idea that 'as h approaches 0', but not equal to zero divided by zero?

8.4 Introducing the derivative: the gradient of the tangent as a limit

All the lessons on introducing differentiation started from the definition of the tangent and then developed onto estimating the gradients of the tangents to the graphs of functions (e.g. $y = x^2, y = x^2 + x - 6 \text{ “}or\text{” } y = x^3$) at given points. Estimating the gradient of the tangent to the graph of a function, $f(x)$ at the point $(x;\ f(x))$ involved using the difference quotient $\frac{f(x+h)-f(x)}{h}$ to compute the gradients of successive secant lines passing through the points $(x;\ f(x))\ and\ (x + h;\ f(x + h))$ as $h \to 0$, i.e. as the point $(x + h;\ f(x + h))$ *gets closer and closer* [italics for emphasis] to the point of tangency $(x;\ f(x))$. Taking the limit of the difference quotient as $h \to 0$ gives the slope of the tangent, which is the derivative of the function at the given point, and this explains the 'the gradient of the tangent as a limit' (Pearson Education Limited, 2013, p.23; DfE, 2016, p.10). Teachers in this study did not substantiate the word 'limit' in constructing the definition of the derivative as the gradient of the tangent as a limit. For an illustrative approach to constructing the definition of the derivative, data excerpts from T4LO are presented below. Further evidence will be drawn from the other teachers, T7, T5 and T2.

Following on from the excerpts [T4LO: 75 - 79] (on page 182 in Section 8.3.1 above), the class concludes that the values of *m*, the gradients of successive chords are converging to 2. Even though the teacher knows the value for the gradient of the tangent to $y = x^2$ at (1,1), he does not confirm the result, instead, he says "So, *I think* the real answer to this question is going to be, the gradient is 2" [79]. The conjectured value for the gradient here is an estimate and so the teacher proposes, "We need to try and prove it" [79]. The activity moves towards a more algebraic mediation, which the teacher describes as 'the clever bit' [79]. Figure 8.14 and the excerpts below give some insights into this utterance. T4LO:

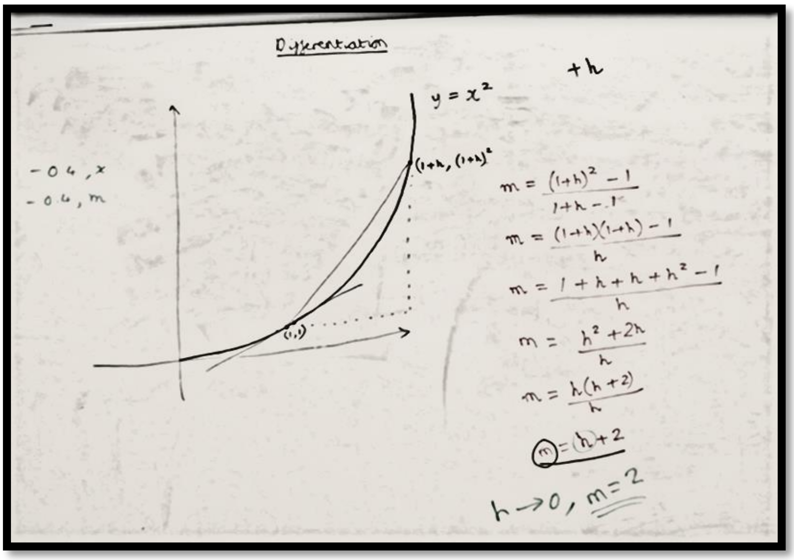


Figure 8.14 The gradient (m) of the tangent to $y = x^2$ at (1,1)

91.T. Alright, so we're mathematicians. We want to consider smaller and smaller numbers. Why don't we just consider a general number, and we can imagine what happens to that number as it gets closer and closer to 0, okay? What I'm going to do is I'm going to imagine adding a number h onto our x-coordinate. Now h can be any size we want but what we're going to think about is what happens when h gets tiny, it gets really, really small, okay? So, we're going to add h to our x –coordinate. What is our new x-coordinate going to be if we add h to it? Mxxx?

92.S. $x + h$.

93.T. $x + h$ brilliant Mxxx! But in this case, we're just going to look at (1,1) first of all. You are right for the next bit. So, actually, first, we're going to look at $1 + h$, okay? That's our first co-ordinate. So, we're adding this number h on. We don't know what it is yet but all we know it's going to be really, really, small and we're going to make it smaller and smaller and smaller. Good stuff Year 12. If the x –co-ordinate is 1 + h, how do we work out what the y-coordinate would be?

94.S. $1 + h$, squared.

95.T. $1 + h$, squared, all in brackets, okay. Nice! This time we're going to do exactly what we did last time. We're still going to work out the gradient. Okay. We're still going to work out that gradient [*drawing the dotted triangle* *onto the graph, see Figure 8.14*], alright. Right this time…Gxxx could you help me substitute the values in this time for, to the formula?

The excerpt above explains the '*clever bit'* [79]. However, note the teacher's word use in these utterances *'as it gets closer and closer to 0*' [91]; *'...when h gets tiny, it gets really, really small*'[91]; '*we know it's going to be really, really small and we're going to make it smaller and smaller and smaller*'[93]. With reference to Figure 8.14, the teacher further explains what happens as h approaches zero (*h* $\rightarrow 0$), and he describes this part as 'the bit of inspiration' [133]: T4LO:

133.T. It [Pointing to point $\{(1+h),(1+h)^2\}$] gets lower and lower and lower and lower until it gets, imagine it gets incredibly close, incredibly close to this point here [*Pointing at point (1,1)*]. Now if we get, if we make h so small that those two points are touching, that will tell us the gradient of the tangent, okay? We're nearly there. This is the bit of inspiration.
134.S. Is h the gradient we're looking for?
135.T. Is h the gradient we're looking for? Actually, no Gxxx, we're looking for what m is. m is going to tell us the gradient, okay? Now here's the, here's the tricky bit.

Again, note the description in [133] '*gets lower and lower and lower and lower until it gets, imagine it gets incredibly close, incredibly close to this point'* and the reference made to the tangent line as $h \rightarrow 0$. Note the teacher's word use in [133], that even though he describes the *two points as touching*, his description is 'if we *make h so small that …';h does not* equal zero and this is a critical point with far-reaching implications. The student's question in [134] explains the importance of word use. The teacher's utterance that 'if we make h so small that those two points are touching, that will tell us the gradient of the tangent' could be construed here to mean that h gives the gradient. The teacher's response, 'Actually, no ….' [135] suggests acknowledgement of this point. '*Now here's the, here's the tricky bit*' [135]. At this point, the calculation on the board, see Figure 8.15 below, has reached $m = 2 + h$. The teacher for the first time writes *h* $\rightarrow 0$ on the board and poses the following question: T4LO:

136.T. When h gets really, really small, in other words, as h approaches 0…okay when h approaches 0, what do you think happens to the gradient? As h gets really, really small and it approaches 0, what do you think happens, happens to the gradient?
137.S. It approaches 2.
138.T. It approaches 2, brilliant. That's what we were, that's what we were always hoping, wasn't it? We were, we were hoping that the gradient was going to be 2 because we sort of knew it would be. And now what Rxxx is saying is as h gets really, really small, well that h is just going to go away, it's going to be 0. So, the gradient we're left with…is 2. And that is differentiation.

As the teacher explains 'as h gets really, really small, well that *h is just going to go away, it's going to be 0*. So, the gradient we're left with…is 2' [138], he circles h on the board, see Figure 8.14. A critical condition here is that $h \neq 0$. Indeed, 'the tricky bit' [135], '*h is just going to go away, it's going to be 0'* [137] did not pass without notice. The teacher, just after explaining that "this is differentiation", a student immediately said that she did not understand it. T4LO:

139.S. I don't understand it!
140.T. It's difficult, it's really difficult. It's really difficult. … So, this is really advanced stuff. So, it's fine if you find it difficult. And what we're going to do is we're going to…I'm going to take you through one, one last step on this and then you're going to have a go at doing exactly the same thing but with x^3.

'It's really difficult' [139]. The teacher was quick to admit that "it is difficult". Note that the phrase is repeated three times in a row. This suggests an acknowledgement by the teacher that students find understanding the explanation for differentiation 'really difficult'. Further, later during the lesson T4LO, the teacher reiterated that differentiating from the first principles was difficult. Not once but several times the teacher acknowledges that the task was difficult, as can be seen in the selected excerpts below. T4LO

287.S. I don't like the way it's set out.
288.T. *It's difficult, it's difficult*, especially the h thing. Thinking about what h means.
289.S. I have to do $a + h$
290.T. I think, just because we want
291.S. and it's a random
292.T a random $+h$ because we want to see what happens as h gets smaller and smaller and smaller. Because you know how like algebra, h can represent any value we want. That means we can think about it representing a smaller and smaller number.

By the h –thing here, the teacher is referring to the h in the quasi-limit definition of the derivative. '*It's difficult, it's difficult'* [288], the teacher is describing differentiation from the first principles. The utterance '*It's difficult'* was repeated several times in the lesson. T4LO

312.T. *It's difficult*, isn't it?
366.T. *It's difficult*, 'isn't it? *It's difficult* stuff.

This study highlights some of the factors that contribute to students' challenges with differential calculus such as the students' difficulties with understanding limits, constants and variables as explained by T4 in the post-lesson interview excerpt below. T4I(ii)

51. I. What do you think makes it difficult though?

52.T. I mean really just that never before have you been told that $m = 2 + h$ and you've never been asked before to ask what happens as h approaches zero. It's not something that, it's not like related to solving equations really, it's something completely new about limits I guess that they'd never come across before.

57. I. Why do you think the students, those students in there or in general, find that idea difficult?

58.T. I think there's a real issue with the following – students have difficulty firstly distinguishing between constants and variables in Algebra. So, even when I was teaching FP2[1] last year one of my students still didn't understand the difference between $a, b,$ and c and x, y and z really. So, there's that issue.

The teacher, T4, explains that the idea of limits is what makes differentiation from first principles really challenging for students. Referring to the gradient function of $y = x^2$, which the class had reached by differentiating from the first principles, $m = 2x + h$, T4 explains that he thought of what then happens as h approaches zero can be daunting to the students. T4 cites two factors here, the idea of h approaching 0, which is related to the concept of the limit, but also explains what he described as a real issue' that 'students have difficulty firstly, distinguishing between constants and variables' [58]. This then calls for careful planning and teaching as reflected in the post-lesson interview with another teacher, T5.

T5 is reflecting on his lesson, T5LO, in which the students' task was to find gradients of tangents at different points on the graph of $y = x^2$ and then to use the set of these gradients to deduce the gradient function for function $y = x^2$. T5I(ii)

89.T. Yeah, it was approaching but we needed to go a little bit further which is why I thought I need to focus everyone on one set of values here because whilst we were looking at 5 different values it was hard for me to get control of exactly what we were choosing. So that's something that I didn't really envisage 'because this is the first time that I've tried teaching it this way.

90.I. Right.

91.T. And when you know about differentiation already it's almost, I don't think I cast my mind back to when I didn't know anything about differentiation. … Once you've been taught you know, once you've been taught the formula because you sort of know what you're looking for. But I can understand from their point of view, they don't know what they're looking for yet, so I just needed to be a little bit more controlled. On reflection, I perhaps should have asked everyone to look at (1;1) and another point that would have given me more information about just the one point to go from.

Teaching arrangements can contribute to the student's challenges with differential calculus. The teacher had given the class five different points for the investigation of the slopes of the tangents at the given points to the curve

[1] Further Mathematics 2 (FP2) is an advanced GCE course]

of $y = x^2$, and the lesson did not go well, prompting the teacher to reconvene the students to focus on just one point. It is very easy for a teacher to overlook some of the inherent complexities that learning calculus presents, especially to students facing calculus for the very first time. This teacher's reflections above show that it is important for the teacher introducing calculus to realise that it could be the students' first encounter with calculus and to plan carefully to facilitate the students' understanding of the topic.

Talking to T4 in an interview after his lesson, in which he had said that differentiation from first principles was 'really difficult', he explains what he thinks makes it difficult. T4I(ii)

52.T. I mean really just that never before have you been told that $m = 2x + h$ and you've never been asked before to ask what happens as h approaches zero. It's not something that, it's not like related to solving equations really, it's something completely new about limits I guess that they'd never come across before.

Other teachers too, acknowledged that learning differentiation from first principles was often challenging for students, for example in T3I(ii) and T4LO, below. In the post-lesson interview with T3I(ii), the teacher explains that she had to extend the time because the students found differentiation from the first principles challenging, so she spent more time on the activity because she wanted the students to understand it. The teacher gives her reflections on the beginning part of the lesson. T3I(ii)

14.T. There were some, I think there was a few still counting squares. So, I think they found it harder, but I think it did, this task definitely sort of got them thinking about the gradient as changing.

15.I. Yeah.

16.T. So that took longer than I thought. Again, for some of my able students, like Bxxx is really able. I think it just took longer than I thought. But I think I slowed down, but then felt like it needed to. They needed to absorb it. And I feel like by the end of the lesson they had a really good understanding.

The teacher explains that it was not just the usual low-attaining students who struggled with the activity, but the high-attaining students too. She adjusted her lesson accordingly to allow her students enough to 'absorb it' [16], as she thinks it is important for the students to get the idea of changing gradients on a curve, i.e. gradient function. Nonetheless, she felt the students had a good understanding of changing gradients on a curve.

Even though the teacher, T4, appreciates the challenges of understanding the 'tricky bit', he explains the main ideas that he wanted the students to understand from the activities. T4LO:

140.T. As long as you understand that all we're doing is, well firstly that it's really difficult. Secondly that all we're doing is we're considering a point that's a little bit further along than our first point [*Pointing at the two points on the graph in Figure 8.15*] and making that smaller and smaller and smaller [*Pointing along the curve, tracing or 'dragging' the upper point down towards (1,1)*] and thinking about the gradient. That's all you need to remember. Just remember it's about that gradient. Okay, finding the gradient at a tangent to a point. Okay, right, now, that's all very well and good. We've found the gradient at that point [*Pointing at (1,1), see Figure 8.15*].

What if we wanted to find the gradient at any point on the line [*curve*]? That's the next step. So, we've found the gradient at one point [*Pointing at (1,1), see Figure 8.15*], that's 2 [*Pointing at* $m = 2$]. We've proved it, brilliant.

Now I'd like to work out what the gradient is at any point we wanted on the line [*curve*], okay? So, you're going to need the same diagram again but we're not going to look at (1,1) now. We're going to think about every single point on the line.

Note that the teacher claims the previous activity to have proved that the gradient of the tangent to the graph of the function $y = x^2$ at the point (1,1) is 2. However, the task now is 'to find the gradient at any point on the line [*curve*]' [140]. It is worth noticing the sequencing of the illustrative examples used by the teacher in the lesson activities this far. The core activity has primarily been the same, '*This time we're going to do exactly what we did last time. We're still going to work out the gradient*' [95]; '*So, you're going to need the same diagram again but we're not going to look at (1,1) now. We're going to think about every single point on the line*' [140], but there is variation in the coordinates; changing one aspect at a time but maintaining the same activity. The first example used purely numerical coordinates. For the second example, when the teacher said, 'we're going to add h to our x-coordinate' [91], and a student said that would be '$x + h$' [92], but the teacher explained that 'So, actually first we're going to look at $1 + h$ okay'. Thus, before moving to use a general (symbolic) representation of the point $x + h$, the teacher wanted to use $1 + h$. Now, the same activity is repeated for the third time, but now using purely general points (x, x^2) and $[(x + h), (x + h)\hat{}2]$ on the same graph of the function $y = x^2$. The result of this third variation of the activity, which the teacher referred to as 'I'm going to take you through one, one last step on this'[140] is captured in Figure 8.15 below. T4LO:

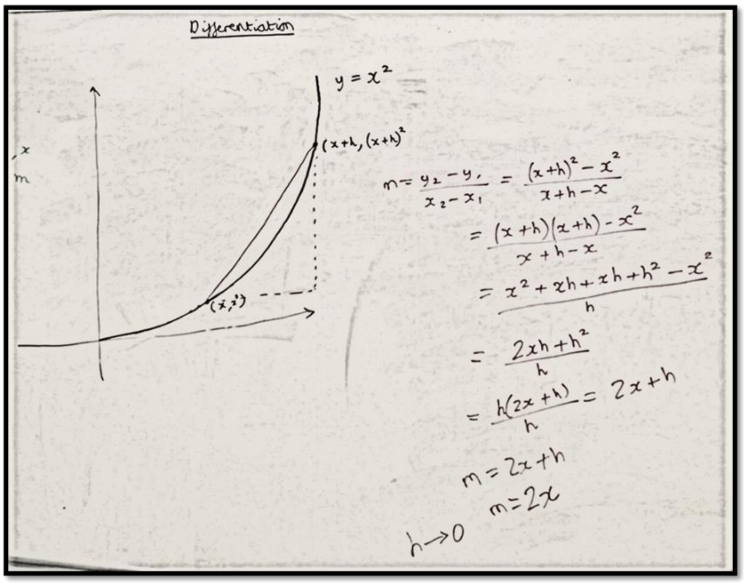


Figure 8.15 Computing the derivative of the function $y = x^2$

The gradients have been $m = 2, m = 2 + h$ for the first and second examples, respectively, but now it turned out as $m = 2x + h$. The same question as before is posed: 'What happens as *h* gets closer to 0?' [192]. T4LO:

192.T. Okay. Time to think again guys. The gradient we found is that the gradient is equal to $2x + h$. Same, same question as last time. What happens as *h* gets closer to 0? What happens? In other words, what happens as this point gets close, close to the original, the original point [*Pointing along the curve, tracing or 'dragging' the upper point down towards* (x, x^2)] as *h* gets close to 0? What is the gradient of that line [*Pointing to the chord in Figure 8.15*]? What do we think? Rxxx?

193.S. *2x.*

194.T. *2x.* Yeah, $m = 2x$. Okay because as h approaches 0, h becomes 0 and we're just left with $m = 2x$. Now that looks a bit weird because we've now got a gradient that isn't just a number, okay? What's going on there? Why have we, usually in GCSE we get $m = 1$ or $m = -3$? Why have we got a bit of algebra for our gradient? Gxxx what do you think?

Again, note the explanation for h→ 0 in the teacher's utterance here. The teacher explains that the gradient of the chord, $m = 2x + h$ becomes $m = 2x$ 'because as h approaches 0, h becomes 0 and we're just left with $m = 2x'$ $[194]$. Up until this point in the lesson, the gradients computed have been represented by m, and the teacher draws attention to the fact that m usually represents constant gradient, where he says 'usually in GCSE we get $m = 1$ or $m = -3$' [194]. Now that *m* = *2x* and not 'just a number' [194], it is at this point in the lesson that the teacher introduces new calculus words and symbols to describe and represent this gradient 'with a bit of algebra' [194], $2x$. The word use and symbolism are captured in Figure 8.16 below and the excerpts below. T4LO:

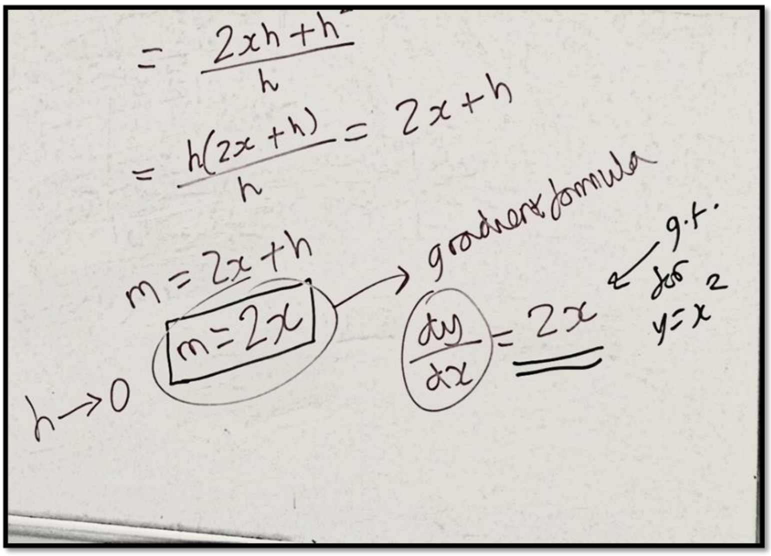


Figure 8.16 Symbolism for derivative

A student answers the teacher's question as to why the gradient was not just a number but has a bit of algebra. T4LO:

201.S. Because it's a general formula.

202.T. That's right because this is a general formula for the gradient. This could tell us the gradient at any point we want. It's not just $m = 10$. The gradient's not just 10, is it? Because Gxxx said right at the start of the lesson the gradient changes. This is a general formula for the gradient at any point, okay?

Now we've got a name for that. We call it the gradient, we call this [*Circling* $m = 2x$ *as shown in Figure 8.16*], we call this the *gradient formula* [*Writing it on the board as shown in Figure 8.16*], okay.

And we've got some special notation for it. Instead of saying $m = 2x$ we write $\frac{dy}{dx} = 2x$. And this [*Pointing at* $\frac{dy}{dx} = 2x$ *in Figure 8.16*] is called the *gradient formula*, okay. So $\frac{dy}{dx} = 2x$, that's the gradient, that's the gradient formula for which curve?

203.S. $y = x^2$

204.T. $y = x^2$! This is always going to be the *gradient formula* for the line $y = x^2$, okay. And that's it. That's *differentiation*, finding the gradient formula.

Note how the teacher adopts the student's ideas in the lesson. He combines the student's word use 'general formula' [201] for the description of $m = 2x$ and an earlier idea from another student that 'gradient changes' and explains $2x$ as 'This is a general formula for the gradient at any point okay' [202]. The teacher's utterance describes the *gradient formula* which is then symbolised by $\frac{dy}{dx}$. The teacher explains $\frac{dy}{dx}$ simply as a 'special notation'[202] for the gradient formula for the function $y = x^2$, and the 'differentiation' is explained and defined as 'finding the gradient formula'[204].

In the excerpt below, the teacher, T7 is describing the derivative at a point C. T7LO:

129.T. As h goes towards zero so we're looking at the limit as [h] gets smaller and smaller and smaller, that point C, remember it was this graph up here *[The teacher pointing a sketch graph on the board]* that point C gets closer and closer to point B because the triangle's shrinking down and it's becoming much more precise as a measure of gradient. The actual limit as [h] gets really close to zero, the limit of that value getting smaller and smaller is actually the gradient, so it becomes a precise value when [h] tends towards zero.

The teacher describes the 'gradient' as the limit of that value (referring to the quotient) [see Figure 7.5 on page 123] "as h gets smaller and smaller and smaller" and "that point C gets closer and closer to point B", i.e. "as h tends towards zero"[129]. The word limit is mentioned three times in the teacher's utterance [129], but it was used here as an everyday word since no definition was specifically given for the word, not as a process or object, anywhere in that lesson.

During the interviews, teachers talked of and used the word 'limit' in their talk, not in their lessons. When talking about their teaching plans and their lessons, it is evident that the teachers planned to use the quasi-limit definition of derivative in introducing and explaining differentiation, even though in the actual lesson, they deliberately avoided using the word limit. For example, talking in an interview before the lesson about her teaching plans for introducing differentiation, T3 talks about the idea of limits and differentiation from the first principles. T3I(i):

134.T. And moving towards $\frac{dy}{dx}$ and the limits as the change in x decreases to zero; so, some first principles.

138.T. And sort of starting with first principles and then moving quickly into actually differentiating using common rules.

T5, talking about the derivative for the function $f(x) = x^2$, in an interview with me, explains: T5I(i)

59.T. I'll start talking about, 'well let's choose a general point (x, y) and another point $[(x+h),\ (x+h)^2]$'. I'll talk about *h* decreasing and that's fine, it requires a knowledge of limits to a certain extent, but no formal ideas of limits but the idea of *h* getting smaller and *eventually becoming zero* [my italics for emphasis]. And out you come with a gradient and then we'll spot patterns.

Note that T5 acknowledges that even though differentiating from first principles 'requires a knowledge of limits to a certain extent', he did not intend to include 'formal ideas of limits but the idea of *h* getting smaller and *eventually becoming zero*' [59]. Reflecting on this lesson in a post-lesson interview, T5 again uses the word 'limit' but he had not used this word in his lesson. In the excerpt, T5 is referring to a class activity investigating the gradient of the tangent to the graph of $f(x) = x^2$ at the point (1;1). T5I(ii)

81.T. And the reason why I wanted to go a little bit further was because it didn't quite show exactly what I was after at that point. It wasn't getting to a limit and the rate at which we were approaching the gradient wasn't yet decreasing sufficiently.
84.I. Okay, so was this then to say these gradients were kind of approaching 2?
85.T. Yeah, yeah it was yeah.

T5 talks about 'h getting smaller and *eventually becoming zero*' [59]. Similarly, T4, explaining the quasi-limit definition of the derivative, says, T4LO

194.T. *2x.* Yeah, $m = 2x$. *Okay because as h approaches 0, h becomes 0* [my italics for emphasis] and we're just left with $m = 2x$.

Also, T7 in explaining the quasi-limit definition of the derivative, for when $h \to 0$, says that '… in fact, it becomes zero' [151]. T7LO

151.T. We're allowed to cancel the h because we can divide the numerator and denominator by h. That h is multiplied by that bracket, so I'm allowed to cancel it off. In this case, it's a bit like saying I've got two lots of 3 divided by 2, I'm allowed in that case to divide by 2 top and bottom. So, I'm left behind with $h + 6$. Now I think, does it ask about what happens as h tends to zero? *When h gets smaller and smaller and gets closer and closer to zero in fact it becomes zero*, what will the gradient turn into?

Note that although the teachers describe h as approaching zero, T5 [59], T4[194] and T7[151], also say that h *becomes zero* [my italics for emphasis]. What does it mean (and what happens) when h gets smaller and smaller and getting closer and closer to zero, i.e. when h approaches zero? Does h 'eventually become zero'? The routine of getting h smaller and smaller, and getting closer and closer to zero (i.e. as h approaches zero) if continued, implies that points $(x, f(x))$and $(x + h, f(x + h))$ would eventually be at the same point.

A student from a different lesson [T2LO] by a different teacher [T2] appears to have conjectured the phenomenon above as suggested by his question to the teacher in the excerpt below. T2LO

110.S. What do you think about dividing 0 by 0?
111.T. What do I think about it? Well, what do I think about any, dividing anything by 0?
112.S. It's 1 because you divide it by itself. But also if you didn't have any then it would still be 0.

113.T. So, you divide it by itself so it's 1 because you didn't have any. It should never be 1.
114.S. Yeah.
115.T. Yeah, I think dividing anything by 0 is something you should 'google' first of all. Think about that argument.

This dialogue suggests that the student had obviously discovered a problem with the explanation. As h continues to approach zero, getting the two points closer and closer and together, implies that the two points could eventually be at the same point. If 'h becomes zero' [T4LO; 194] or 'h getting smaller and *eventually becoming zero*' [T5LO; 56] that would mean that the difference quotient $\frac{f(x+h)-f(x)}{(x+h)-x}$ would be equal to $\frac{0}{0}$, thus undefined. In other words, the derivative of $f(x)$ at $(x, f(x))$ would not exist.

This poses challenges not only to students but to teachers too. T4I(ii)

50.T. I think the concept of adding a general value onto *x* and considering what happens as that value decreases is quite a hard concept for students to get their head around, especially 'because it's quite novel the idea that *h* isn't fixed, and we have to consider what happens as *h* changes. I find that a difficult idea to get my head around.

'I find that [the limit] a difficult idea to get my head round' [50] explains T4 in an interview after his lesson on introducing the derivative. The students and the teacher alike find the idea of 'limit' here, confusing or difficult to comprehend as explained by T4 in the post-lesson interview.

8.5 Pedagogies on the derivative: The why-factors

In all lessons, introducing differentiation started with a consideration of drawing tangents to curves at given points and computing the gradients (slope) of the tangents. The activity gradually developed towards differentiation from first principles, which involves computing the difference quotient, $\frac{f(x+h)-f(x)}{h}$ and finding the limit of that difference quotient as h approaches 0 ($h \to 0$), for example, T2, T4 and T7 (refer to Section 8.3 and Section 8.4 above).

It is necessary to look at the mathematics curriculum which stipulates the differential calculus that ought to be taught to post-16 students studying mathematics at school. The curriculum specifications for the GCE in Mathematics for the Core Mathematics C1 (AS) from 2014 to 2017 state that students need to learn 'the gradient of the tangent as a *limit* [my italics for emphasis]' (Pearson Education Limited, 2013, p.23). However, the syllabus specifications here, unlike the 2017 specifications, do not specify using 'differentiation from first principles' (DfE, 2016, p.10). If the teachers' approach to introducing the derivative can be summarised as teaching differentiation from first principles, what are the teacher's reasons for doing so? (Also, see Section 8.6, on page 212 for further discussion).

There was a common understanding from the teachers that differentiation from first principles was not subject to testing in the examinations. Students asked how differentiation from the first principles would look like exams, for example in T4LO. Teacher [T4] explains that it would not be assessed in the examination. T4LO

285.S. So how would this like be an exam question? What would it be?

286.T. It's, it's not going to strictly be on the exam. It's going to be on the exam for the people starting their GCE A-level next year. But for you, it's just a way for you to start understanding that differentiation is to do with gradient and it gives you the gradient at any point. So, it's quite shocking how many people forget that it's, that it's part of the gradient. Because what you'll see later on is, that we'll learn a way of working out what the gradient function is. But you might forget that it's to do with gradient unless, you know, you learn it this way.

Similarly, from T7LO, the teacher actually assigned students a task to do on differentiating from first principles but explained that it would not be tested upon in the examinations. T7LO

129.T. Okay so time for us to practice some questions. In the book can you turn to Exercise 7a? And you will notice when you turn to that page that these sorts of questions won't be on the exam paper but it's just giving you an understanding of the background behind where differentiation comes from.

Teaching differentiation from principles is about promoting an understanding that differentiation is about gradients at given points and the gradient function. In T2LO, students also asked whether differentiation from first principles was tested in examinations. The teacher [T2] explains that finding the gradient function by differentiating from first principles was not even subject to testing. It is, the teacher argues, important for the students to *understand why* differentiation works. T2LO

373.T. We know as *h* gets smaller and smaller; it goes to 0. So, then our gradient function, let's call it *m* for now for the sake of *it will go to* $2x - 1$[*my italics*]. Now some people at the start of a lesson were finding that out through the process of differentiation which they knew. … We found out the gradient function for this [$y = x^2 + 3x - 6$]. It was through something called *first principles*. Something we don't use and asked. One of the girls asked, it isn't tested on but it's really nice to know.

Similarly, T5 explained to his Year 12 class that the investigation of the gradient formula of a curve, something that the class spent about two hours doing, would not be examined. T5LO

195.T. No, feel free to use your calculator to do this. You won't have a calculator in the exam, but you won't be asked to do this kind of thing in the exam.

Generally, as can be seen in the excepts above, the teachers' core objective in the introductory lessons was to explain why the differentiation rules work, thereby helping students' understanding of differentiation. For example, T2 argues for the importance of explaining the meaning of differentiation as she refers to her experience as a student. In an interview with me, T2 said that she only got to learn the meaning of differentiation past her degree. T2I(i)

106.T. I think the understanding is just as important as the process, the multiplying by the power thing because I never ever got the connection between the gradient. I remember being at school myself and thinking oh right, that's how I'll find the gradient. I don't know why, why that is again, that's just differentiating. I knew differentiating as a process before I knew it was anything to do with the gradient, and I don't like that I knew that, I'd rather it was the other way round.

The experiences of teacher T2 here, are consistent with the explanation by teacher T4 in [286] above. T4 explained that the reason he wanted students to learn differentiation from first principles was so they could 'understand that differentiation is to do with gradient and it gives you the gradient at any point' [286].

In her lesson [T2LO], T2 followed what she had said in the pre-teaching interview [T2I(i)]. She shares with her class, her own experience as a student when she was taught differential calculus. She explains why they were differentiating the function $y = x^2 + 3x - 6$ from first principles: '*It's about understanding why differentiating works'* [89]. In that class (Year 12) there were some students who had come across some differentiation in Further Mathematics[2] in Year 11, but still, the teacher insisted on differentiating from first principles. T2LO

89.T. I've had a few people trying to differentiate. I'll explain to you what this lesson's really about. *It's about understanding why differentiating works*. Now we all know or a lot of us know, who did further maths, how to differentiate in the process of it. But I mean I've spoken to your past teacher.

---

[2] The Further Mathematics (level 2) is a GCSE qualification open as an optional course additional to the compulsory GCSE Mathematics course for 16-year-olds in England. The syllabus included elements of differentiation.

> As far as I know, *you've not learnt why it works.* So, that's what this lesson's about. So, we won't be differentiating at all today.
>
> And the reason why I want you to do it is that no one ever told me till I was way past my degree, why it works. And I really wanted to know. So, we're starting off with this. …

T2's past experiences of learning calculus clearly influenced their approach to teaching differentiation, which focuses on promoting students' understanding of the definition of calculus. This was not an isolated case. In an interview with T5, he explains that he, too, wished had been taught the proof for differentiation, i.e. differentiation from first principles. I asked him what the reasoning behind his approach was to introducing differentiation. T5I(ii)

57.T. It came down to the way that I was taught differentiation and I was shown the algebra and then differentiated, got through the whole of my A level, got to University and sort of within the first few weeks went through the proof for the formula and thought 'I wish I'd be shown really what it was talking about and the reasons why we were doing things'. So, I thought I want to introduce it with both because some approaches just use the between two points and the points getting closer …

Reflecting on their experiences of learning introductory differential calculus at school, teachers explained the importance for students to understand and so be taught the principles behind the process of differentiation. It is for this reason, that they decided to take an investigatory approach in introducing the differentiation, in which they applied forms of differentiation from first principles. As T4 explains below, many people forget or do not have an adequate understanding of differentiation is a process for finding the gradient function of a given function.

286.T. So, it's quite shocking how many people forget that it's, that it's part of the gradient. Because what you'll see later on is, that we'll learn a way of working out what the gradient function is. But you might forget that it's to do with gradient unless, you know, you learn it this way.

This study found that teaching differentiation from first principles is perceived as important and necessary by the teachers even if it is not subject to testing in the course examinations. There is evidence to show that although the students found the investigation activities not easy, they also enjoyed and appreciated understanding why the process of differentiation works. For example, a comment from a student in T2LO said 'Probably the best …' [382]. This was in a lesson by a teacher [T2] who explained her approach to introducing differentiation as one which aimed at students' understanding of why differentiation works. She wanted to give the students an experience that she felt she did not have when she was introduced to calculus at school. T2LO:

379.T. So, people at the front did, did we understand all that? Is it nice to know where it came from?
380.S. Yeah! Yeah! Yeah!
381.T. Yeah, good.
382.S. Probably the best …
383.T. I wish someone had told me that at A-level, I do!

T2 is very satisfied with the lesson outcome as she got a positive confirmation from the students when she asked "Did we understand all that? Is it nice to know where it came from?" [379. The teacher wished she had been taught this part of mathematics when she had her A-levels on the topic; she had not been introduced to differentiation from the first principles. However, it can be said that at least she got an even more rewarding satisfaction from this lesson of her own design and teaching, with such feedback from the students: 'Probably the best'! [382].

T3, reflecting upon her introductory lesson, felt that what had started as a challenging task was eventually a success because she felt the students had a very good understanding by the end of the lesson. T3I(ii)

16.T. So that took longer than I thought. Again, for some of my able students, like Bxxx is really able. I think it just took longer than I thought. But I think I slowed down, but then felt like it needed to. They needed to absorb it. And I feel like by the end of the lesson they had a really good understanding.

Overall, what is remarkable in this study is that although differentiation from first principles was not an assessment requirement, and the limit definition of the derivative was not subject to testing in the course examinations, all the teachers made attempts to define differentiation by teaching differentiation from first principles. Teachers explained that it is important that students are taught not only the standard rules and applying the common rules of differentiation, but also why the rules work. Despite any known or possible complexities with calculus, the definition of the derivative can be taught effectively to post-16 students at school. In some countries, calculus is reserved for later years of education, e.g. university level. In England, calculus is first introduced at school or post-16 college. Based on the findings of this study, it can be argued that complex concepts such as the limit definition of the derivative can indeed be taught in simplified forms that make the mathematical discourse accessible to the students at school.

Formal algebraic proof or representation of differentiation can be very difficult for students at the A level. For example, T5 reflects and talks about his choice of the resources that informed his planning for the introduction to differentiation lesson. T5 wanted an ‘investigation’ approach. He starts talking about some PowerPoint resources he had used for teaching differentiation in the previous year: T5I(ii)

16.T. And one of the power-points particularly with this lesson just introduced, they introduce the Algebra, so the formality behind the proof of this Algebra was more difficult than they went into at A level.

17.I. Okay.

18.T. And then that pretty much stopped the conversation, you just had to differentiate and which I thought was no good. So, I thought well, I want to go through it from an investigatory approach this year, so I narrowed it down to a few sources where I wanted to look at the… But the best thing that I found was on the Nrich website and that’s so the basis of what I did today, which was based on that really. An investigation that started looking at a combination of the graphs and the coordinates so that they had a visual representation and a sort of tangible numerical representation that they could look at as well.

T1 explains his approach to introducing differentiation, that he would rather use an exploratory approach like a coursework project, but he has not been able to do so as much as he would like because of a lack of adequate time. T1I(i)

13.T. Normally I give them more, I make it an investigation over a lesson. I don't have time for that now. I used to sort of really go to almost like a coursework activity like we are going to be investigating, work out the rule. I am going to have to lead them a little bit more now to give them a little bit less. I have given them all different points to work on today. Before everyone did everything, but time is an issue, time, time for those things. I have never yet resorted to telling them this is the answer, this is how you do it. I have always done something

Most teachers talked about using an investigatory approach to introduce and substantiate the process of differentiation. By investigatory approach, the teachers were referring to activities whereby the students would work towards the standard rules for differentiation, rather than starting with a given rule for differentiation. None of the teachers went direct into teaching the standard rules for differentiation. It was clear from the teachers’ explanations that their approaches to introducing differentiation were to start with differentiation from first principles before moving on to using the standard rules.

8.6 Discussion

The discussion here will focus mainly on the why factors and the perception that differentiation from first principles is difficult. Although the school syllabus specifications (for England) for the period leading up to September 2017 state that students need to learn ‘the gradient of the tangent as a limit’ (Pearson Education Limited, 2013, p.23), unlike the 2017 specifications, it does not specify using ‘differentiation from first principles’ (DfE, 2016, p.10).

The why factors

Differentiation from first principles was not subject to examination (see the AS/A level Mathematics Specifications in Pearson Education Limited, 2013, p.23). Thus, in teaching differentiation, the teachers in this study could have started from the standard methods of differentiation. None of the teachers did that. All the teachers took an explorative approach to teaching the definition of the derivative (refer to Sections 8.2 - 8.5 above) by introducing differentiation from the first principles. Unlike the teachers in Heyd-Metzuyanim et al. (2018) study, the teachers’ discourse about doing mathematics valued more exploring mathematical objects and relations than doing mere calculations using the standard methods of differentiation. This study found that the teachers were keen to teach some aspects of differentiation from first principles that would not feature in the assessment. The teachers felt that it was very important for students to be taught the principles behind the process of differentiation before they are introduced to the standard rules or standard algebraic results.

Similar to the study by Heyd-Metzuyamin and Shabtay (2019) that found ‘some relations between teachers’ descriptions of their past as learners, particularly those who described themselves as struggling with mathematics during their school years, and their current pedagogical discourses’ (p.553), the teachers in my study also referred to their past experiences of learning mathematics when asked about their choice for explorative instruction for introducing the derivative. Reflecting on their experiences of learning introductory calculus as students at the A-level, some of the teachers even explained their disappointment that they had not been taught why the rules of differentiation work. This finding is in contrast to the findings from Jennings et al. (2019) who report that several participant-teachers in their study about the Queensland and Australian school curriculum said that ‘the meaning of the limit definition of the derivative was no longer in the syllabus and therefore not taught anymore’ (p.112).

The teacher’s past experiences revealed in this study are indeed not isolated incidences. Evidence of the lack of a more in-depth understanding of the fundamental concepts of calculus has long been reported in the literature. For example, Orton (1983a, 1983b) in his study on students’ understanding of elementary calculus with 16–22-year-olds found that the student lacked understanding of differentiation, and Berry and Nyman (2003) reported similar problems with elementary calculus that they observed from working with undergraduate students and trainee teachers of mathematics. The students’ challenges with calculus could be explained by their experiences with differential calculus when it was first taught or introduced to them. The findings of this study suggest that the problem could be traced back to the teaching of differentiation at school; some teachers explained that they were taught the standard algebraic rules for differentiation without adequate understanding of what was meant by differentiation nor why the rules worked.

‘It’s difficult’ - differentiation from first principles

A recent longitudinal study by Jennings et al. (2019) shows that teachers think that students would find the limit definition of the derivative hard. The study (Jennings et al., 2019) also shows that high school students (Intermediate Mathematics (IM) and Intermediate and Advanced Mathematics (AM) find differentiation from first principles difficult. In the study, the researchers gave the students the definition of the derivative, $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$ and asked them to explain what the definition meant, and also, to determine the derivative $f'(x)$ of the function $f(x) = x^2$. Only 2% of the IM students and 12% of AM explained the definition correctly, and more than half the number of the teachers thought the students would find it hard to explain the definition. On the procedural part of the question, 13% (IM students) and 43% (AM students) were able to use the formula, $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$ to find $f'(x)$, where $f(x) = x^2$. Again, less than half the number of the teachers thought it would be easy for the students to do.

Zandieh and Knapp (2016) acknowledge that teaching the derivative is a complex process, for example, describing the derivative at a point by graphical mediators. There are many factors and examples in literature (e.g. Berry and Nyman, 2003; Oehrtman et al., 2008; Tall, 1993) on what makes calculus difficult to understand for students. Ferrini-Mundy and Graham (1994) report on students’ difficulties in relating the symbolic representations and the geometrical representations, even though the student could compute derivatives using standard rules for differentiation. Teaching the derivative requires the use of multiple representations to include, not only geometric but the numeric and symbolic algebraic forms of mediation, otherwise as Zandieh and Knapp (2006) point out, ‘a student who has focused on this image [geometric representation] may say that the derivative is (…) the most obvious image or endpoint of this graphical process, the tangent line’ (p.11).

In this study, teachers cited some of the factors contributing to students’ difficulties with calculus such as the idea of h approaching 0, which is related to the concept of the limit, but also what T4 described as a real issue –‘that ‘students have difficulty firstly, distinguishing between constants and variables’ [T4I(ii), 58]. The teachers’ observations in this study are consistent with White and Mitchelmore (1996) who reported that students’ difficulties in differential calculus have to do with understanding variables. The students in White and Mitchelmore's (1996) study saw a variable as a symbol to be manipulated, and not as representing a varying quantity.

At that point where (m), $\frac{dy}{dx} = 2x + h$, it seems the explanation that ‘h becomes zero’ T4[194] intuitively makes sense, to give the gradient function $\frac{dy}{dx} = 2x$. However, h cannot become zero ($h \neq 0$). The difference quotient would be undefined if h = 0 and that would render the limit definition of the derivative void, thus the derivative would not exist. The components of the limit definition of the derivative, such as the meanings of limit, the difference quotient and the symbolism present a challenging task for teaching differential calculus (Bos et al., 2019; Zandieh, 2000).

There appears to be a conundrum as far as the interpretation and explanation of ‘as h approaches zero: $h \to 0$’. Does ‘as h approaches zero’ mean that h eventually becomes zero? Or does ‘as h approaches zero’ mean that h will not eventually become zero? Research on students’ understanding of limits (e.g. Tall and Vinner, 1981; Williams, 1991 & 2001; Monaghan (1991); Oehrtman, 2009) highlights the problems in students’ understanding of limit as a process and limit as a value. Studies by Tall and Vinner (1981) and Tall (1986) show how students’ discourses about limits on the difference quotient and the tangent lines to a curve, can be inconsistent with mathematical literate discourse, for example, the thinking that 0.99999 . . . never reaches 1, is consistent with their thinking about local straightness and the tangent, for example, that the secant lines (as $h \to 0$) never reach

the tangent line. This poses challenges not only to students but to teachers too.
'I find that [the limit] a difficult idea to get my head round' [50] explains T4 in an interview after his lesson on introducing the derivative. This complexity evolves around the idea of limits, which is the principle for the limit definition of the derivative. The difficulty in explaining the conundrum just explained seems to explain the seeming avoidance of the word use limit in most of the lessons observed in this study.

This study found that the absence of the word 'limit' in the teachers' discourse in their introductory lessons was not by chance nor mistake, but indeed by design and deliberate. Park (2015) found that teachers used graphical mediators and words such as 'approaching' and 'getting smaller and smaller' when teaching the limit as a process, but used symbolic mediators to equations of functions, and words such as 'the limit is' when teaching the limit as a number (p.242). In this study, teachers repeatedly used phrases such as *getting closer and closer, as h gets closer and closer to zero, as h approaches zero, as h gets smaller and smaller*, in explaining the slope (gradient) of the tangent as a limit, but the word 'limit' was rarely uttered in the lessons, and it was not defined. Past studies have shown the need for careful use of words and utterances that are often used to describe the limit. Monaghan (1991) found that the 'everyday meaning of a limit as a boundary is clearly present' (p.23) in the students' descriptions.

Monaghan (1991) highlights the ambiguities inherent in the four phrases - tends to, approaches, converges, and limits. Monaghan (1991) study found that these four phrases can generate colloquial meanings which could be at odds with the literate mathematical discourse on limits. For example, in the case of functions and sequences, *tends to* and *approaches* were interpreted to mean the same, unlike the *converges to*, but the *limit* was different from all three, as a noun. Asked about the limit of 0.999, one student said:

> Its limit is its final point that it will get to. I think the limit is $0.\dot{9}$ [0.9 recurring] and there again there the limit is 1 but it won't actually get to one, so you can't have 1 as its limit." Monaghan (1991, p.23).

Many students often view limit as a boundary point of a sequence, and Monaghan (1991) argues that 'it must be stressed that students experience very real difficulties in the mystery of this jump to the infinite' (p.24), and 'limits are hard' (Monaghan, 2019, p.131).

Other studies on calculus (e.g. Tall,1992; Thompson, 1994; Zandieh, 2000; Park, 2016; Biza, 2017; Bos et al., 2019), have put the students' difficulties with learning calculus down to the complexity with the definition (e.g. limit) symbolism and multiple representations of the derivative. Orton (1983b) found that students' difficulties were in understanding the derivative as a rate of change and its related limit concept, ratio and proportionality. Borgen and Manu (2002) report students' challenges with relating the graph of a function and the graph of its gradient function, whilst Baker et al. (2000) report of students' lack of understanding of the derivative as a function.

This study has found that teaching differentiation from first principles can be done without using the formal concept of the limit. Monaghan (1993, 2019) argues for the intuitive use of limit ideas in elementary differential calculus. Thus, using the ideas of limits, but without directly explaining or proving them. Monaghan (2019) gives a few reasons for this argument:

> But 'limits are hard' is only one of the grounds of my argument. Another ground is twofold: precalculus students have had very little exposure to limits; an introduction to differentiation gives plenty of opportunity for 'limiting experiences'(p.131).

Monaghan (1993, 2019) suggests that teaching introduction to the derivative could avoid explicit use of 'limit as an object' by talking about the limit of the secant or the limit ratio of the quotient difference. The approach bypasses the formal definition of limit (e.g. the $\varepsilon - \delta$ definitions), thus avoiding getting into technical details of evaluating limits. Further, Monaghan (2019), suggests teaching elementary differential calculus before limits, would afford many opportunities for teachers to talk, intuitively, about limit ideas to their students.

8.7 Summary of findings

The introductory lesson to differential calculus, whose object was to find the gradient of a tangent, started with investigating the gradients of tangents at specific points on a curve, and progressed through differentiation from first principles to a 'quasi-limit' definition of the derivative; 'quasi-limit' since the formal limit definition of the derivative was not given. In principle, the teachers introduced differentiation as the *limit* of the derivative, but only used a partial representation of the formal limit definition of the derivative.

Teachers described constructing the definition of the derivative by differentiating from first principles as very difficult. The teachers also described introducing differentiation by drawing tangents to curves as hard, difficult, and an inaccurate way of working out the gradient of a curve at a particular point. Although differentiation from first principles was not a requirement of the syllabus nor subject to examination, this study found that teachers still wanted to and introduced differentiation from first principles. The teachers explained that it was important to substantiate the definition of the derivate for the students to learn the process of differentiation.

The study also found evidence of teachers motivated to teach differentiation from first principles by their past school experiences whereby the process of differentiation had not been substantiated or taught to them. Although the teachers did not construct the limit definition of the derivative, they were able to substantiate the process of differentiation intuitively, using a quasi-limit definition of the derivative. Differentiation was defined, as the process of finding the gradient formula. The findings and the observations made in this study, though they cannot be generalised, do raise questions for further discussion.

Chapter 9 Discussion

9.1 Introduction

This thesis aims to explore the teaching of elementary differential calculus by examining how teachers of mathematics introduce the derivative. The research adopts the commognitive framework that sees mathematics as a discourse and the study is guided by the following research questions. In teaching differential calculus:

RQ.1 What *word types* and *narratives* do teachers use and why?
RQ.2 What *visual mediators* do teachers use and why?
RQ.3 What mathematical and pedagogical *routines* do teachers use and how?

In this chapter, a selection of the key findings will be discussed since most of the findings of the study have already been discussed in the three previous chapters. Note that the findings from the commognitive thematic discourse analysis of the teachers' pedagogical calculus discourse have been reported in Chapters 6, 7 and 8, structured in accordance with the overarching themes of the study. To each of these three chapters, is a respective discussion of its findings. This arrangement was necessary to give a more coherent and complete commentary on the findings since the findings chapters are relatively very long. Readers are therefore referred to Sections 6.4, 7.4 and 8.5, respectively, for more discussion of the findings from the study with respect to Chapters 6, 7 and 8.

In this chapter, the discussion of findings will revolve mainly around the quasi-limit definition of the derivative and the calculus symbolism in the teachers' pedagogical discourse (See Section 9.2). This discussion chapter will also reflect on the commognitive theoretical framework as applied in the analysis of data in this study (See Section 9.3). Also included in this discussion chapter is a reflection on myself as the researcher in the analysis process (See Section 9.4). Finally, a concluding remark to the discussion chapter.

9.2 Discussion of the key findings

The discussion in this section will focus on the quasi-limit definition of the derivative (with respect to Chapter 8, Section 8.3 and 8.4) and the dualism and ambiguity in calculus symbolism in the teachers' pedagogical calculus discourse (with respect to Chapter 7, Section 7.2).

9.2.1 The quasi-limit definition of the derivative

In this study, the teachers' pedagogies on the derivative, i.e. their approaches to introducing the derivative constitute differentiating from first principles. Differentiation from first principles is the process of finding the gradient (slope) of the tangent to a curve at a given point, i.e. the instantaneous rate of change of a function, $f(x)$. The process is based on using the gradient of the secant line of $f(x)$, given by the difference quotient, $\frac{f(x+h)-f(x)}{h}$ to estimate the gradient of the tangent. As $h \to 0,$ the secant line becomes the tangent to the graph of the function, $f(x)$. In other words, the limit of the gradients of the successive secant lines gives the gradient of the tangent. The gradient of the tangent (that limit) is called the derivative of the function, $f(x)$. The derivative of $f(x)$ (if it exists) is often represented symbolically as,

$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$, where $h \neq 0$.

This formula describes the limit definition of the derivative, and the derivative is indeed the limit. The limit definition of the derivative is multi-faceted (Roorda et al., 2009; Zandieh, 2000) and learning the definition of the derivative requires an understanding of its various components including the function, the symbolism, keywords such as the instantaneous rate of change, the gradient (slope), the tangent, the difference quotient, and the limit (Thompson, 1994; Zandieh, 2000; Zandieh and Knapp, 2006; Biza, 2017). Contrast the limit definition of the derivative with differentiating with standard rules, such as, for a function $f(x) = x^n$, the derivative is given by $f'(x) = nx^{n-1}$.

In principle, finding the derivative, $f'(x)$ of function, $f(x)$ means computing the difference quotient $\frac{f(x+h)-f(x)}{h}$ and finding the limit of that difference quotient as h approaches zero $(h \to 0)$. However, in introducing differentiation, the teachers (for example, T4, T2 and T7) avoided using the word 'limit' and the limit notation in the definition (formula) above. (Refer to Sections 7.2; 8.3 and 8.4 for evidence of this finding). This finding is consistent with the findings of Bos et al. (2019) who report that 'introducing the slope of a curve in a point and the derivative of a function to students is a didactical challenge for teachers' (p.75).

If the derivative is the limit, how can teaching that introduces differentiation bypass the limit? Monaghan (2019) claims that it is so unremarkable that 'people do this and have been doing this for centuries (p.132)'. Faced with a potentially complex routine of constructing and substantiating the limit definition of the derivative of a function, $f(x)$ which (if it exists) is given by $f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$, where $h \neq 0$, the teachers 'paraphrased' this object-level narrative taking out the symbolism $\lim_{h\to 0} f(x)$ and the word limit. The teachers were conscious of the abstract nature of the differential calculus symbolism, in particular, the $\lim_{h\to 0} f(x)$ symbolism. In their calculus discourse during the interviews, most of the teachers used the word 'limit', but not during their teaching. The teachers introduced and explained the process of differentiation to students without the use of the formal definition of limit.

In all the lessons observed in this study, the differentiability of functions was never discussed. A function is differentiable if we can compute its derivative everywhere (at each point) on its curve, that is, it must be continuous on its domain. Note the importance of the stated condition (above), 'if it exists' implies that not all functions are differentiable at every point in their domain. Examples of non-differentiable functions include functions like $f(x) = \frac{1}{x}$ (for it goes to infinity at $x = 0$), $f(x) = \sqrt[3]{x}$ (although continuous, it has a vertical tangent at $x = 0$; thus not all continuous functions are differentiable) and $f(x) = |x|$ (for it has a cusp at $x = 0$, creating a discontinuity in its derivative). In this study, all the teachers' approaches in the limiting sequence did not raise any complications because they worked with graphs of quadratic functions, which meant that the secants approach the tangent in a visually predictable way. Quadratic functions dominate students' early experiences with differentiation and such implicit assumptions about differentiability could also impact students' long-term understanding of the differentiability of functions. However, if the teachers were to work with a general variable x and a general function, differentiability could not be assumed without a defined domain and particular x value. Tall (1992) argues against just 'giving students simple experiences without giving them correspondingly simple long-term conceptions of the concepts being introduced' (p.3). Thus elementary calculus students need to know that not all functions are differentiable, otherwise, the students could draw incorrect or partially correct conclusions about the differentiability of functions. It is therefore necessary and important for teachers to substantiate the property of differentiability in constructing the definition of the derivative.

By implicitly suggesting that all curves are differentiable and by partially applying the limit definition of the derivative, the teachers in this study used an informal limit approach to constructing the definition of the derivative. To describe these teachers' approach, I propose and adopt a new narrative (object level) the 'quasi-limit definition of the derivative' instead of the limit definition of the derivative. Thus, a 'quasi-limit definition of the derivative' (quasi, for it was partial) was used to introduce differential calculus by the teachers.

With the quasi-limit approach, the teachers used words and utterances such as 'as h approaches zero', 'as h gets smaller and smaller', 'as h gets closer and closer to zero', to describe the behaviour of the difference quotient or the secant line approaching the tangent. However, when the teachers used the words *getting closer and closer,* they did not all describe the same mathematical object. Some teachers focused on the limit of the difference quotient, thus, referring to the limit of the slopes of the successive secant lines as $h \to 0$. Others exclusively described two points where the secant line crosses the curve of the function, $f(x)$, thus, focusing on the point $\left(x + h;\ f(x + h)\right)$ *getting closer and closer* to $\left(x;\ f(x)\right)$ as $h \to 0$. Yet, in some lessons, *getting closer and closer* described the secant line *getting closer and closer* to the tangent line, as $h \to 0$, thus, referring to the tangent as the limit of the secant lines. Although some teachers may have assumed that the students would make the connections between the three objects, teachers must make the connections explicit in substantiating the definition of the derivative.

There were some inconsistencies in the interpretations made for 'as h approaches zero', with some teachers saying that h becomes zero. There was no explicit mention to say that h does not equal zero. Learning is a change in discourse and is characterised by discursive changes in the words and visual mediators the students use as they participate in the mathematical discourses (Sfard, 2008). In explaining the quasi-limit definition of the derivative, teachers need to substantiate whether h would equal zero or not, to eliminate what could be an ambiguous case of the difference quotient becoming $\frac{0}{0}$, thus giving a more rounded 'quasi-limit' explanation of the idea of limit. It is also important that teachers define and substantiate the literate words in their pedagogical discourse, such as limit and derivative.

In this study, it was remarkable the teachers taught differentiation from first principles, yet differentiation from first principles was not specified in the syllabus nor subject to testing in the examinations. The teachers argued that it was important for the students to learn, not just the standard methods of differentiation, but to understand why differentiation works. Some teachers explained they had left school without an understanding of why the process of differentiation works. Some said it was later during their studying at university that they first learnt about differentiating from first principles. They explained that because of their past experiences of learning calculus as students, they needed to explain the process of differentiation in their teaching. The teachers explained that it was, therefore, necessary and important to introduce differential calculus from first principles to help students understand the derivative and the meaning of the process of differentiation.

9.2.2 Symbolism for gradient and gradient function; $\boldsymbol{m}$ and $\boldsymbol{\frac{dy}{dx}}$

In this study teachers used both the Leibniz's notation, $\frac{dy}{dx}$ and the Langrage's notation, $f'(x)$, but the study found some ambiguity in the teachers' use of calculus symbolism. This study found some evidence of teachers using the same visual mediators or symbolism for the derivative at a point and for the derivative as a function, in particular, m (for $\frac{dy}{dx}$). For example, T4 describes Leibniz's notation $\frac{dy}{dx}$ simply as "some special notation" [T4LO; 202] (Refer to Section 6.3.1 on page 118) that is used instead of the *m*. This utterance implies that $\frac{dy}{dx}$ is the same as m, only a special notation for the same signifier. The use of m for the gradient function was common in other lessons too, for example, in T2LO, m is used to represent the gradient function, $2x - 1$. Such use of m (e.g. in T4LO and T2LO; refer to Section 7.2 on pages 138-141), which is a symbol usually used to signify the slope of a straight line, to signify the gradient function, is similar to Park's (2016) findings from an examination of the calculus discourse of experts as reflected in the three most popular calculus textbooks in the US. Park (2016) found that the 'realisations of both the derivative at a point and the derivative of a function were mediated with nearly identical symbols suggesting a possible difficulty with understanding the difference between them' (p.417). The teachers in this study, certainly do not have difficulty understanding the difference between m and $\frac{dy}{dx}$. However, the evidence (e.g. in T4LO and T2LO above) suggests that their students could potentially have difficulties understanding the difference between these two symbolic artefacts and their applications.

According to the commognitive theoretical framework (Sfard, 2007; 2008) symbolic artefacts, such as the $f'(x)$ and $\frac{dy}{dx}$ in differential calculus, are an integral part to the thinking and communication process in mathematical discourse. Learning and doing mathematics means becoming capable of participating in the literate discourse (Sfard, 2008; 2016). How differential calculus symbolism is introduced matters if students are to become capable of participating in the calculus discourse. These symbolic artefacts, such as the $f'(x)$ and $\frac{dy}{dx}$, fulfil the role of visual mediators in calculus discourse, to serve as 'representations of impalpable mathematical objects' (Sfard, 2015, p.132).

Calculus symbolism is a useful and powerful communication mediator in calculus discourse. However, teachers need to be conscious of symbolic ambiguity inherent in some of the symbolic artefacts they use in teaching differential calculus, for example, when the same letter x is used to stand for two different things in the straight-line gradient formula $\frac{y_2 - y_1}{x_2 - x_1}$ and in the limit definition of the derivative $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{x+h-x}$. The symbol x in the gradient formula represents a 'letter as specific unknown value' and in the limit definition represents a 'letter as variable' (Kuchemann, 1978, p.23). This symbolic ambiguity, in fact, offers a 'powerful manipulative ability' (Tall, 1992a, p.9), which Sfard (2008, p.159) describes as the 'generative power' of symbolic mediation as it allows the teachers and the students to move between these interpretations and use the symbol as needed, from the particular value to the general value and vice versa. Barwell (2003) in Foster (2011) argues that ambiguity in mathematics is 'an important discursive resource in school mathematics discourse' (p.4). Indeed, ambiguity can be helpful for learning mathematics, in the sense that it presents students with interesting tensions that could be opportunities for discussion and for exploring mathematics further.

This study has shown that the symbolic mediators, $\frac{dy}{dx}$ or $f'(x)$ have a dual role in the calculus discourse. For example, $\frac{dy}{dx}$ or $f'(x)$ have been defined as the gradient function, e.g. in T4LO 'the gradient function is $\frac{dy}{dx}$ … your gradient function is $f'(x)$' [417]; in T1LO $f'(x)$ is defined as 'it means the derivative, the gradient function' [87]; and in T2LO referring to the gradient function the teacher says ‘we denote it like this – $\frac{dy}{dx}$. However, in [T4LO; 204], not only does the teacher defines $\frac{dy}{dx} = 2x$ as the gradient function for $y = x^2$, but describes it as differentiation too. Similarly, in [T7LO; 129], $f'(x)$ is defined as standing for gradient (referring to the gradient function) and is also described to mean differentiation too.

On the one hand, $f'(x)$ or dy/dx can be seen as an object narrative for the derivative of the function $f(x)\ or\ y$, (respectively) and on the other hand, as an operational narrative for the process of differentiation. The symbolism $\frac{dy}{dx}$ or $f'(x)$ is an example of what Gray and Tall (1994) describe as a ‘procept’, which is a signifier for both the process and product in the same symbolism; ‘the amalgam of process and concept’ (p.4). To a student, $\frac{dy}{dx}$ or $f'(x)$ can call up or signify either the process (finding the derivative) or a concept (Gray and Tall, 1994), the product of the process (the derivative). Sfard (2008) argues that such ‘object–process duality of algebraic expressions’ (p.122) makes algebra particularly effective as a tool for enhancing other forms of communication and practical doing in teaching and learning mathematics.

Whilst such duality could be put to good use to facilitate discussion in teaching calculus, if not adequately substantiated, the inherent ambiguity can be a source of students’ challenges with calculus. Sfard (2008) thinks it is outright counterintuitive that ‘a thing [can] be simultaneously a process and this process’s own result’(p.122) and acknowledges that such process-object duality is a source of students’ difficulties and failings with calculus, for example. Calculus symbolism is a well-documented source of students’ difficulties with calculus (Tall, 1992). A student needs to be able to tell whether the symbolism $\frac{dy}{dx}$ or $f'(x)$ is signifying the process of differentiating or the derivative, the product of differentiation. Tall (1992a, p.4) asserts that the ‘duality (as process or concept), flexibility (using whichever is appropriate at the time) and ambiguity (not always making it explicit which we are using)’ in calculus procepts are a source of challenges for many students.

Given the flexibility and the duality of use of calculus procepts, e.g. $\frac{dy}{dx}$ or $f'(x)$, it is essential that teachers make it explicit enough for students to develop the necessary flexible thinking and understanding to be able to deal with the possible ambiguity of use (Tall, 1992b; Gray and Tall, 1994). In this study, the duality in interpretation and application of symbols, such as the letter x , $f'(x)$ and ${}^{dy}/_{dx}$ (in the examples described above) in substantiating the definition of the derivative were not made explicit for the students. There was a lack of adequate clarity with procepts in the teachers' pedagogical calculus discourse. Unless the teacher or the context makes the discursive object (Sfard, 1992) explicit, a student may find a procept ambiguous, failing to read whether it is signifying the process or the product. Indeed, the dual purpose of these calculus symbols, $\frac{dy}{dx}$ or $f'(x)$, if not made explicit, can be a source of confusion for students as reported in Park's (2013) study, students had difficulties with distinguishing between derivative at a point and as a function.

9.3 A reflection on the commognitive theoretical framework

For an explanation of the key tenets and the four commognitive constructs of the theory of commognition, see Chapter 3. Here, I make a brief reflection on the commognitive theoretical framework as applied in the analysis of the data for this research.

9.3.1 Relevance of the commognitive theoretical framework

My research investigates how teachers of mathematics introduce the derivative, by studying teachers' pedagogical calculus discourse. The commognitive theoretical framework tells us that mathematics is a discourse (Sfard, 2008). Mathematics as a discourse is identifiable by its word use, visual mediators, narratives and routines (Sfard, 2008). These four commognitive constructs characterise mathematical discourses, and thus offer a general framework for looking at or researching teaching and learning in mathematics. Although the theory of commognition (Sfard, 2008) was initially developed for the study of learning (Sfard, 2007), I applied aspects of the commognitive framework for the study of teaching, as did other researchers such as Viirman (2015) and Park (2016). Analysing the mathematical discourse of the teacher through the commognitive constructs of word use, visual mediators, routines and narratives allowed for an investigation of how teachers of mathematics introduce differential calculus.

According to the commognitive perspective, learning is defined as a change in the discursive practices of the newcomer (student). Learning entails participation in the community discourse (Sfard, 2008), thus, it becomes clear that 'teaching is also a form of participating in this discourse, only from a different role—that of the leader rather than the learner' (Heyd-Metzuyamin and Shabtay, 2019, p.552). The commognitive definition of learning describes the sympatric nature of teaching and learning in the classroom discourse. Besides, learning results from the commognitive conflict, teaching-learning agreement and the resolving of the commognitive conflict, resulting with the newcomer abandoning their discourse for the expert's (teacher's) narratives (Sfard, 2008; 2015). Hence, the commognitive framework was adapted and applied for the study of teaching elementary calculus at school.

Although my study focuses primarily on the teaching of differential calculus, the discussion of the teachers' classroom discourse refers to research on students learning of calculus too, where necessary. However, any such reference to existing research on students' learning of calculus should not be interpreted to mean a direct causal relationship between the teachers' pedagogical discourse on differential calculus and the student's difficulties with calculus. However, the commognitive perspective on learning sees similarities between individual learning and historical societal discursive development of mathematical discourses (Sfard, 2008). As Park (2015) notes 'the difficulties that past mathematicians had with writing a rigorous definition of the derivative that includes the limit component and works for any x implies that these aspects of the derivative cannot be considered as trivial to today's students' (p.248).

To investigate how teachers of mathematics introduce differential calculus, I needed to find out what the teachers say; what they do; how and why. By focusing on the *keywords (word use) and narratives* (Sfard, 2008) in the teachers' pedagogical calculus discourse, I was able to gather data on what the teachers say. Whereas, a focus on the *visual mediators* (Sfard, 2008) in the teachers' pedagogical calculus discourse allowed for an investigation of what the teachers do and what they use. Further, by focusing on *the how and when of the routines* (Sfard, 2008), I was able to gather data about what the teachers do (and how), in teaching the derivative. I found that the commognitive framework offered well-defined constructs that described the categories for my prima-facie questions above. Thus, a commognitive conceptual framework was established and applied in my study.

Conceptualising mathematics as a form of discourse, that is, a special type of communication with specific ways of saying and doing (Sfard, 2008; Nachlieli and Tabach, 2018), I revised and reframed my prime facie research questions through the lens of the commognitive framework. Thus, the research questions were reformulated as follows: In teaching differential calculus,

RQ.1 What *word types* and *narratives* do teachers use and why?

RQ.2 What *visual mediators* do teachers use and why?

RQ.3 What mathematical and pedagogical *routines* do teachers use and how?

Note that the statement of the research questions of the study is commognitive in the sense that they seek to investigate the word use and narratives, the visual mediators and the routines in the teachers' pedagogical calculus discourse.

Note that within the mathematics discourse as an academic discipline, there are sub-discourses. Others (e.g. Gee, 2014 and Shabtay and Heyd-Metzuyanim, 2017) have attempted to distinguish between *Discourse* (of a community) and *discourse* (of individual interlocutors). For example, Shabtay and Heyd-Metzuyanim (2017) differentiate between discourse, denoting individual teachers' communication and Discourse, denoting pre-existing historically established texts. However, the discrete acts of communication (discourses) of specific people belonging to the community of a Discourse make up the Discourses. Sfard (2008) describes mathematics as a distinct form of discourse but also talks of mathematical discourses. In my study, although I talk of mathematical discourse (i.e. Discourse, according to Shabtay and Heyd-Metzuyanim, 2017), the primary focus is on calculus discourse.

There are many uses of the term pedagogical or pedagogic discourse (e.g. Heyd-Metzuyamin and Shabtay, 2019). My study, similar to that of Heyd-Metzuyamin and Shabtay (2019) is concerned with the content dimension of discourse characterised by word use, visual mediators, narratives and routines (Sfard, 2008) in the teachers' calculus discourse. To capture and encapsulate the content dimension, I propose and use the term the teachers' pedagogical calculus discourse.

9.3.2 Representations and realisations

Sfard (2008) compares what she calls some deceptive similarity between the signifier and the realisations of the signifier, on one hand, and representation and the represented object, on the other.

> The difference is in the implied ontology of the component terms. Whereas in the case of mathematics, representation is to be understood as but a material "incarnation" of a basically intangible abstract entity (mathematical object), realisation belongs to the same

> ontological category as signifier – the category of perceptually accessible entities (Sfard, 2008, p.155).

Mathematical objects are ‘realised with perceptually tangible entities such as words and visual mediators’ (Park, 2016, p.398), which Sfard (2008) refer to as realisations of the signifier. In this study, symbolic-algebraic and geometrical realisations of the signifier derivative, for example, are also described as multiple representations. The representations in my study, are what Park (2016) describes as the visual mediation of realisations (Sfard, 2008) in a communicational approach. My study found that the teachers used multiple symbolic artefacts, namely graphs, numerals and algebraic expressions as representations of mathematical objects (Sfard, 2015) in explaining the derivative. The word ‘representations’ is preferred in my study for it is a more usual term than ‘realisations’ to the target audience for this research, the teachers of mathematics. It is used to refer to various forms of expressing mathematical objects, such as the geometrical, algebraic and numerical forms of expressing, a function, for example. In terms of Sfard’s (2008) commognitive framework, such forms of expression (the representations) could be thought of as realisations of the signifier.

9.3.3 Methods used and data generated

My study took a qualitative approach in which interviews with teachers of mathematics and observations of their introductory lessons on the derivative were used to generate qualitative data on the teachers’ pedagogical calculus discourse. For a detailed explanation of the choice of methods and the data generated, see Chapter 4. There was a good mix of interviews with the teachers and observation of their lessons on introduction to differentiation, but only the first lessons (I will address this latter part in Section 9.4). There was methodological (or data) triangulation from the audio recordings of interviews with seven different teachers, (both pre-teaching and post-teaching interviews) and video recordings of their lessons on elementary differential calculus. The triangulation allowed for the analysis of how teachers talk about the objects of their calculus discourse in and outside of their lessons. The triangulation proved useful in the analysis of the teacher’s pedagogical calculus discourse. By analysing the teachers’ word use in their calculus discourse during the interviews and during their teaching activity, my study found that the teachers’ word use and narratives about some of the discursive objects, e.g. limit, during the interviews (e.g. T4 and T5; refer to Section 8.4 pages 204-206) with the researcher was different in their classroom discourse about the notion of limit. All the teachers, except one (refer to Section 7.2 pages 142-144), did not want to use the word limit in their introductory lessons on differential calculus.

9.3.4 The commognitive thematic discourse analysis

The overall analysis of the qualitative data was undergirded by my conceptualisation of the epistemological tenets of the theory of commognition and its four theoretical constructs of word use, visual mediators, narratives and routine (Sfard, 2008). However, for a systematic approach to analysing the huge amounts of transcribed interview (audio) and lesson observation (video) qualitative data, I adopted and followed a thematic analysis process as described by Braun and Clarke (2006). Thus, the analysis of the qualitative data followed a combination of thematic analysis (Braun and Clarke, 2006) and the commognitive theoretical framework (Sfard, 2008). See Section 5.2 for a detailed description of the process and stages followed.

Faced with the vast amount of qualitative data, there was a need for a systematic approach to qualitative data analysis. I needed a systematic process for identifying, analysing, and reporting patterns or themes within the data; thus, thematic analysis (Braun and Clarke, 2006). Thematic analysis is not a theory, but an established method or an iterative process for analysing qualitative data. I chose thematic analysis for its flexibility and autonomy from any specific research paradigm, which allows thematic analysis to be tailored to the aims, research questions and theoretical framework of the study (Nowell et al., 2017).

The analytical approach to the qualitative data in this study happened at two main levels: the semantic level and the latent level. At the semantic level, thematic analysis is descriptive, primarily involving identifying patterns and labelling the data; whereas at the latent level, analysis is interpretative, and is all about making meaning from the data (Boyatzis, 1998; Thomas, 2013), ‘interpreting or interconnecting or conceptualising data’ (Punch, 2009, p.179). At the latent (interpretative) level, thematic analysis involves identifying and examining the ‘underlying ideas, assumptions, and conceptualisations [of the commognitive theoretical framework] – and ideologies - that are theorised as shaping or informing the semantic content of the data’ (Braun and Clarke, 2006, p.12). It was at this latent level, that the epistemological assumptions of the commognitive framework were applied to the thematic analysis process (Sfard, 2008), and a commognitive thematic analysis was then adopted and applied to the analysis of the data. Note that at the latent (interpretative) level, thematic analysis overlaps with discourse analysis (Potter and Wetherell, 2001), and Braun and Clarke (2006) then describe analysis as 'thematic discourse analysis… where broader assumptions, structures and/or meanings are theorised as underpinning what is actually articulated in the data' (p.13). Hence, the term *commognitive thematic discourse analysis* was used to capture a new analytical approach adapted for my study. Indeed, the commognitive framework can serve as 'a conceptual as well as discourse analysis framework' (Park, 2016, p.396) in research.

The interviews and lesson observations gathered qualitative data. The qualitative analysis of data was in two phases. The first phase of the analysis, i.e. at the semantic level - followed a descriptive thematic process (refer to Section 5.3), which allowed for the categorisation of data into themes. The second phase – i.e. at the latent level - applied the commognitive theoretical constructs to the categories from the thematic analysis process. The themes were then matched with the four commognitive constructs (refer to Table 5.7 on page 94). From this point, subsequent interpretation and explanation of the data were informed by the commognitive framework, as can be seen in reporting the findings of the study in Chapters 6 to 8.

The systematic process of thematic analysis adopted in the commognitive thematic discourse analytical approach, enabled a systematic generation of initial codes, searching for, reviewing and categorisation of codes. The theoretical lens of the four commognitive constructs of word use, visual mediators, narratives and routines allowed (enabled) for defining and naming of the overarching themes, and for producing of the report/ the three findings chapters of the research. Each of the three findings Chapters, 6, 7 and 8 addressed the research questions of the study, RQ1, RQ2 and RQ3, respectively. The analysis and the interpretation of the data excerpts in the findings focus on all the four elements in the teachers' pedagogical calculus discourse; thus, informed by the theory of commognition. The commognitive theory provided a theoretical and analytical perspective for examining and explaining how teachers construct and substantiate the notion of a derivative and the derivative function in introducing differential calculus.

The commognitive constructs of word use and narratives, visual mediators and routines were instrumental in analysing the teachers' pedagogical calculus discourse, particularly, in addressing the *what* and the *how* parts of the research questions, but not the *why* part. By examining the following parts of the research questions of the study, I was able to explore and examine how teachers of mathematics teach differential calculus, through the theoretical lens of the commognitive constructs: *What word types* and *narratives* do teachers use? *What visual mediators* do teachers use? *What* mathematical and pedagogical *routines* do teachers use and *how*? However, an analysis of the four commognitive constructs alone, cannot address the *why* part of the research questions. This shortfall was counteracted by the use of an inductive approach to the initial coding of the data.

Analysing data inductively allows for new ideas and themes to derive from the content of the data (Braun and Clarke, 2012). The *why factors* theme emerged from the inductive approach to coding (Refer to Section 5.2 which explains the inductive and deductive approaches of the analysis) at the semantic level of the thematic analysis process. The *why factors* identify the evidence that would explain the teachers' decisions and choices of keywords, narratives, visual mediators and routines. For example, this study found that teachers wanted to teach differentiation from the first principles although it was not specifically required by the mathematics curriculum nor subject to examination. By focusing on the *why factors*, the study found that the teachers believed it important that differentiation from the first principles was useful for the realisation of the derivative and the learning of differentiation. Another finding of the study explained earlier, is that although the teachers were at liberty to use the word limit in talking about their teaching plans during the pre-teaching interviews, they (deliberately) avoided using the word limit in their classroom calculus discourse. By focusing on the *why factors*, the study found that the teachers believed the notion of the limit to be very difficult for teachers to teach and for students to learn.

9.4 A reflection on self – the researcher

It was inevitable that the coding processes in the data analysis for my study were influenced, at least implicitly, by my professional experiences as a teacher of mathematics and as a mathematics teacher educator. Before I became a teacher educator at the University Leeds in 2012, I had taught elementary calculus to AS and A level students as a teacher of mathematics in schools in England. As a teacher educator, I taught calculus to trainee teachers as part of the mathematics subject content element of the PGCE programme in England. Further, I also taught in-service teachers on the TAM programme; these were teachers already working in various schools, but training to teach post-16 (AS/A level) mathematics. Again, calculus was a significant element of the subject content of the course. Not only was I involved in the university-based teaching activities, but also, travelling to various schools to observe both trainee and in-service teachers teach mathematics, including calculus. I have been a mathematics educator for at least 15 years, and the professional and academic experience, in part, informed my epistemological assumptions and influenced my interpretation and analysis of data in this study.

Braun and Clarke (2006) argue that the data coding process in qualitative research does not happen in an 'epistemological vacuum' and that the researchers 'cannot free themselves of their theoretical and epistemological commitments' (p.11). Overall, the data coding and the analysis in this study were driven by my theoretical interest in the pedagogical calculus discourse, the research questions of the study and the commognitive conceptual framework (Boyatzis, 1998; Braun and Clarke, 2006; Nowell et al., 2017).

Initial coding in this study started inductively, to allow for a more comprehensive generation of codes. Braun and Clarke (2006) argue that such coding does not seek to fit the data into pre-existing themes, or the researcher's analytic preconceptions, is data-driven instead. This open coding allowed for new ideas and themes, for example, *the why factors*, to derive from the data (Braun and Clarke, 2012; Patton, 1990) that were outside of my pre-conceived codes of the four commognitive constructs of word use, visual mediators, narratives and routines. (Refer to Section 5.3 for a more in-depth explanation of the generation of the initial codes for this study and how the process then progressed).

Scholarly rigour is considered more important in qualitative research than the concept of reliability or inter-rater reliability measure (Merriam, 2009; Denzin and Lincoln, 2011; Syed and Nelson, 2015). Rigour is a product of the researcher, the research context and the research process (Syed and Nelson, 2015). For this study, rigour derives from, 'the researcher's presence, the nature of the interaction between researcher and participants, the triangulation of data, the interpretation of perceptions and rich, thick descriptions' (Merriam, 2009, p.165). For this study, the researcher is a teacher of mathematics and a mathematics teacher educator and the participants are teachers of mathematics, which allows for, in Syed and Nelson's (2015) words, 'a deep and intimate knowledge of the participants' (p.17). The research process and data gathering involved both interviews with the teachers and observations of their mathematics lessons, in their natural settings – in their schools and mathematics classrooms. Syed and Nelson (2015) argue that this ‘closeness is what allows for rigour in the interpretative methods’ (p.17). This is true for my study, which takes an interpretative approach to research the teachers’ pedagogical discourse on the derivative, through the lens of the commognitive theoretical framework.

Although reliability in coding does not necessarily constitute validity, it is a necessary consideration for validity. To ensure that my coding of data transcripts was consistent and comparable to other researchers, a check for inter-coder agreement (Campbell et al., 2013) was carried out during the early stages of the coding process, involving my two research supervisors and I (the researcher). After two rounds of independent coding activities (refer to Section 5.3.2.1 on page 85), we had a very strong consensus in our coding; this informed the rest of the coding process. As with interpretative methods, any reliability in coding in my study should be seen as a subjective consensus between my two supervisors and I, and ‘not [as] an ultimate decontextualised “truth” that exists outside of the data’ (Syed and Nelson, 2015, p.17).

9.5 Conclusion

Before I conclude this discussion chapter, a note of self-reflection is in order, about my analysis and evaluation of the teachers’ pedagogical calculus discourse with respect to literate mathematical discourse. I am aware that such interpretations may be understood as subjective or limited to my epistemological stance. I acknowledge this bias and accept that consistency with literate mathematical discourse is a matter of judgement and context; other interpretations may exist too.

Although the findings cannot be generalised, this study draws attention to important questions and findings from this research. The study is wholly qualitative and contributes to research on mathematics education by providing lived experiences of teachers with, and of teaching elementary differential calculus. Following on the foregoing discussion, the conclusions and implications of the findings, and a discussion of the limitations of the study are presented in the next chapter, marking the final chapter for this thesis.

Chapter 10 Conclusions and implications

10.1 Introduction

This concluding chapter presents a summary of the findings and their implications for mathematics education and research, highlights the main contributions to mathematics education and research, discusses the main limitations of the study and makes some recommendations for further research. My research sought to explore the teaching of elementary differential calculus at schools or colleges, by studying the teachers' pedagogical calculus discourse on the derivative. The investigation examined the word use, narratives, and visual mediators such as calculus symbolism in the teachers' calculus and pedagogical discourse, as well as their mathematical and pedagogical, i.e. didactical routines on the derivative.

10.2 Summary and implications of the findings

Here, I summarise what I regard to be the main findings of my research under the following five subheadings: symbolism for gradient and gradient function; graphical mediation with digital artefacts; the quasi-limit definition approach; multiple representations with visual mediators; and inconsistency and ambiguity in calculus word use.

10.2.1 Inconsistency and ambiguity in calculus word use

Learning is highly word-dependent (Kim and Lim, 2017) and word use in differential calculus teaching matters, indeed. My research (see Chapter 6) uncovered some inconsistency with word use in the teachers' pedagogical calculus discourse. My research found evidence of dualism (and so ambiguity) in meaning and the teachers' application of some calculus words such as gradient and derivative, e.g. in T1LO [88-91] and T2LO [374-375] where the word gradient is used to signify both constant gradient and gradient as a function. The word derivative was used to refer to the derivative of a function at a given point, and the gradient function of a function. These findings highlight the implicit ambiguity with such word use and call for teachers to clarify the context of their word use and explain the transition from one use to the other. My research highlights the dual meaning of the word derivative in calculus discourse.

My research has also drawn attention to some ambiguous word use in the teachers' pedagogical calculus discourse, in particular, *the gradient of a curve* [italics for emphasis] (See Section 6.3). Although it is correct to say the ‘gradient of a straight line’, the utterance the ‘gradient of a curve’ is inconsistent with literate mathematics, even though the word *gradient* is characteristic of mathematical calculus discourse. A curve does not have a constant gradient and so, the utterance *gradient of a curve* is an ambiguous object-level narrative. Based on these findings, inconsistent word use of *gradient* could make it difficult for students to ‘appreciate the derivative at a point as a number and the derivative’ as a function (Park, 2013, p.624). My research, therefore, highlights the importance for teachers to pay attention to word use in their calculus discourse, ensuring that a clear distinction is made between gradient at a point and gradient function.

10.2.2 Symbolism for gradient and gradient function

My research (see Chapter 7) found that the calculus notation used in introducing the derivative, e.g. the Leibniz notation $\frac{dy}{dx}$ (for the derivative of the function y) and the Langrage’s notation $f'(x)$ (for the derivative of the function $f(x)$), was not explicitly substantiated by the teachers. On the face of it, this finding would suggest that unsubstantiated calculus symbolism may be a critical factor for students’ difficulties with calculus symbolism, which in turn could imply a long-term impact on the students’ understanding of differential calculus. Beyond the introduction to calculus, the notation becomes standard symbolism. The symbolism becomes part and parcel of the calculus discourse and the process of communication. If the differential calculus symbolism used is not explained at this introduction stage, when will it be explained? Besides, higher-level courses in calculus (may) assume prior knowledge of the differential calculus symbolism. If this happens, it creates a gap in the teaching of, and in the students’ learning experience with differential calculus symbolism. In turn, this could result in yet another gap in students’ understanding of differential calculus.

My research (see Chapter 7) found some inconsistency with symbolism in the teachers’ pedagogical calculus discourse with the transition from *gradient* (for straight line graphs) to *gradient function* (for curved line graphs), in particular, in the use of the visual mediators $\frac{dy}{dx}$ and m to signify the same mathematical object, which they are not (See Section 7.2). The visual mediator m was used to signify both the gradient of a straight line and the gradient function, by the teachers, and this could potentially contribute to students’ difficulties with the derivative. These findings are broadly similar to findings from Park (2016) who reported the use of identical representations for the derivative at a point and the derivative of a function. My research, therefore, highlights the need for teachers to pay attention to calculus symbolism and to explain the difference in application between $\frac{dy}{dx}$ and m symbolism when constructing the definition of the derivative.

10.2.3 The quasi-limit definition approach

My research (see Chapter 8) found that introducing differentiation developed from approximating the gradient of the tangent to a quadratic graph at a given point, through to constructing a 'quasi-limit definition of the derivative'. This study found that teachers were able to construct a definition of the derivative without having to use the formal definition of limit (refer to Sections 8.3 and 8.4). The teachers were able to substantiate the definition of the derivative using the function, the graph of the function, the tangent as the limit of the secant line and the difference quotient for computing the gradients of the successive secant lines as h approaches zero ($h \to 0$). These findings imply that teachers can construct the definition of the derivative and substantiate the process of differentiating without the use of the complex formal definition of limits. These findings suggest that the quasi-limit approach may be instrumental in introducing and explaining differentiation to students without the complexities of the formal definition of limits.

10.2.4 Multiple representations with visual mediators

The evidence (see Chapters 7 and 8) has shown that the teachers in this study used multiple visual mediators and multiple forms of representation in constructing the definition of the derivative and so introducing differentiation. Teaching the quasi-limit definition of the derivative was visually mediated by written symbols e.g. numerals, algebraic formulas and algebraic symbols, and by symbolic artefacts such as diagrams and dynamic graphs of functions by digital artefacts such as Autograph, GeoGebra and Desmos (refer to Sections 7.3; 7.4 and 8.3). These findings resonate with Sfard (2008) who argues for the multiplicity of visual realisations because they 'broaden communicational possibilities' (p.156). These findings imply that the use of multiple visual mediators in constructing and substantiating the definition of the derivative would allow for multiple realisations of the same signifier (Sfard, 2008). The use of numerical, graphical and algebraic representations in teaching elementary differential calculus can also appeal to the individual students' preferred ways of learning.

10.2.5 Construction and substantiation of the derivative with dynamic geometry software

My research (see Chapters 7 and 8) found that construction of the definition of the derivative and the substantiation of differentiation in teaching elementary calculus was enhanced by a supplementary application of dynamic graphical imagery. The dynamic imagery and visualisation affordances of dynamic geometry software were instrumental in mediating instruction in the substantiation of the gradient of the tangent as the limit of the secant, i.e. in explaining or proving the teacher's quasi-limit narrative on the derivative (refer to Sections 7.3 and 8.3). GeoGebra, Autograph and Desmos were used for dynamic geometry and imagery of graphical representations of functions in substantiating the differentiation from first principles narrative and in constructing the definition of the derivative, i.e. the quasi-limit definition of the derivative. These digital artefacts allowed for interactive graphical imagery of dragging points on the graph and zooming in and out on parts of the graphical representations (refer to Section 8.3), which in turn allowed for a dynamic visual graphical demonstration that the slope of the tangent at various points on a curved-line graph constantly changes. For example, Autograph allowed for a demonstration of a dynamic tangent line to the graph of the function $f(x) = x^2$ moving along the curve, and simultaneously, mapping out the graph of the gradient function $f'(x) = 2x$. GeoGebra allowed for a visual mediation of the slope of the tangent line to the graph of a function $f(x)$ at $\left(x;\ f(x)\right)$, as the limit of the slope of the secant line as h gets closer and closer to zero ($h \to 0$). GeoGebra allowed for simultaneous dynamic graphical imagery of the moving point and the rotating secant line, and h getting smaller and smaller and getting closer and closer to zero. These findings suggest that the constraints of static iconic mediators such as the pen and paper graphical representations can be mitigated through the use of digital artefacts, such as dynamic geometry and graphing

10.3 Contributions to mathematics education and research

I should make clear that the findings of my study are restricted to the teachers' pedagogical calculus discourse in general, but particularly on elementary differential calculus. In this section, I would like to highlight what I regard to be the main contributions of my research to the existing body of knowledge on mathematics education and research, such as the new knowledge advanced, a new application of theory and originality of the research.

By focusing on, and examining the teachers' word use, my study revealed some inconsistency and ambiguity in word use in the teachers' pedagogical calculus discourse. Of particular interest is the utterance, '*gradient of a curve*' referring to the gradient function. My study has brought to light that '*gradient of a curve'* is an ambiguous narrative that is indeed inconsistent with literate mathematics. To the best of my knowledge, no previous research has drawn attention to the ambiguity of the utterance, the *gradient of a curve*, or reported its inconsistency with the literate calculus discourse. Thus, my research draws attention to, and raises questions as to, the potential impact of such inconsistent word use on the learning of calculus, given the past research reporting students' challenges with differential calculus.

This study has highlighted an exploration routine for constructing the definition of the derivative which avoids the use of a formal definition of limit, the '*quasi-limit definition of the derivative'*. 'Quasi-limit definition of the derivative' is a new object-level narrative emerging from this study; it is, therefore, a contribution to mathematics education research discourse.

This study has demonstrated an approach to researching teachers and teaching mathematics that does not focus on teacher knowledge but teachers' mathematical discourses (Sfard, 2008), hereby referred to as the teachers' *pedagogical calculus discourse*, a type of mathematical discourse. Pedagogical calculus discourse is a new term coined in this study to refer to the amalgam of the teachers' mathematical and didactical discourse on calculus. Thus, this study set out to investigate the teaching of elementary differential calculus by examining the word types, narratives, visual mediators and exploration routines in the teacher's pedagogical calculus discourse. Researching teachers' pedagogical calculus discourse means examining their communicative activity, i.e. the teachers' forms of saying (word use, narratives, visual mediators) and doing (routines) in teaching calculus, and not the teachers' subject knowledge.

For my study on the teachers' pedagogical discourse on the derivative, I adopted the theory of commognition and developed a conceptual framework for analysing interviews and lessons by focusing on and analysing the four commognitive constructs of word use, visual mediators, narratives and routines in the teachers' calculus discourse. In doing so, I introduced a new application of theory for the analysis of the teachers' pedagogical calculus discourse, the *commognitive thematic discourse analysis*. The commognitive thematic discourse analysis is a new analytic framework that combines the process of thematic analysis and commognitive discourse analysis for analysing qualitative data. Although the methodology of my research was informed by the commognitive theoretical framework, analysing the huge amounts of qualitative data required a systematic approach, and so the process of thematic analysis was effectively combined with the theory of commognition to form the commognitive thematic discourse analytic approach. This creative application of theory resulted in a new analytical framework, i.e. the commognitive thematic discourse analysis, that can be extended to other studies seeking to analyse discourse and qualitative data.

Many studies on calculus have focused on students' learning of calculus reporting students' difficulties with calculus, and not so much on the teachers' teaching of calculus. My study draws attention to teaching, thus contributing to the existing and ongoing research on calculus education. It presents evidence and provides a perspective on some of the questions that teachers, teacher educators and mathematics education researchers might have on calculus education. There seems to be less appetite to investigate the teachers' teaching of calculus. There could be various reasons to explain the limited number of studies that investigate the teachers' teaching of calculus, but as a teacher of mathematics and teacher educator, I was interested to explore the teaching of calculus at schools and colleges.

10.4 Limitations of the research

There are, to my knowledge, three main limitations identifiable with this study, the focus of the study primarily on the teacher, challenges with access to classroom and teachers: and generalisability of a qualitative study given a relatively small sample size in comparison to quantitative studies. I now consider them further.

10.4.1 Teacher data and student data

I should stress that my study has been primarily concerned with teacher data, not student data. Although there was data triangulation from interviews with the teachers and observations of their lessons on elementary differential calculus, only the first introductory lessons were observed. Thus, the findings of my study are restricted to data from the introductory lessons on differentiation. Observations of more lessons were not possible due to limitations in access, time and cost. There is no denying that more observations, for example, of subsequent lessons could provide additional data on the teachers’ pedagogical calculus discourse.

The aim of this study was to investigate the teaching of elementary differential calculus. The study was researching teaching but reference to some student data was inevitable. Although there are some references made to students in this study, the primary focus was on the teacher and the teaching. A focus on teaching and researching the teachers and not primarily on learning would limit any learning claims that can be drawn from this study. Future research, therefore, can look at the impact on student learning of teaching elementary differential calculus. Similar studies in the classroom could focus primarily on the impact on student learning of teaching elementary differentiation from first principles, and the impact of teaching with (and without dynamic) dynamic graphical imager. Furthermore, the studies could include interviews with mathematics graduates about their experiences with calculus at school and their experiences with calculus at the undergraduate level.

10.4.2 Recruitment of participants

It is very difficult to gain access to and consent from schools, teachers, parents and students to carry out research in the classroom, let alone to observe and video-record lessons. Getting schools and teachers willing to participate in this study was very slow and required patience. This was made even more difficult by the fact that there was a limited window within which to collect data. Very often introduction to differentiation is often taught in the early parts of the autumn term, and this meant perfect timing was required. Data collection took two years to complete. The first year’s window passed before I had collected enough data, so I had to plan for a second year-round of data collection, but still, the numbers of willing participants were very small. Changes in the teacher’s programme or disruptions from the weather such as snow days meant that some lessons did not take place and when they could be rescheduled, it was not possible to meet the timetable. Some lessons from two teachers who had agreed to participate were cancelled due to disruption from snow days, even though I had managed to visit the school.

Furthermore, it was often very difficult for teachers to spare time for a face-to-face pre-lesson interview, lesson observation and post-lesson interview.

The study had initially planned for a sample size of nine teachers, a total of eight different teachers finally participated in this study. However, data from two participant teachers were excluded because it was deemed incomplete for the requirements of the study. It turned out that the two lessons, although on differential calculus, were not on the introduction to differentiation. Providing the availability of more resources and time, larger studies could be developed with a similar focus on teaching differential calculus. Given the scope of this study and given the difficulty of gaining access to classrooms, this study provides useful insights into the teaching of introduction to differential calculus, and a basis upon which to develop more longitudinal studies on the teaching of differential calculus and maybe not just the introduction. Sample sizes for qualitative studies are generally small and this is true for this study, which is qualitative research. However, the small scale in sample size is compensated by an in-depth analysis of the data.

10.4.3 Generalisability and transferability of findings

Miles and Huberman (1994) on sampling questions to consider, ask ‘Does your plan enhance *generalisability* of your findings, either through conceptual power or representativeness?’ (p.34). The findings from this study are not generalisable given the small sample size of the research. There is, however, a case for transferability, instead, on the basis of the conceptual power of the study. The hybridisation of the commognition theory and thematic analysis, to form the commognitive thematic discourse analytical framework for analysing the teachers’ pedagogical calculus discourse, can be extended to, and applied in similar qualitative communicational studies investigating teaching and learning on other mathematical discourses.

10.5 Recommendations

This research has expounded evidence of ambiguity and inconsistency with word use and symbolism in the teachers’ pedagogical calculus discourse. On the face of it, it can be argued that my study offers suggestive evidence for factors that contribute to students’ difficulties with calculus. Without further research into the effect on students’ learning of differential calculus of such ambiguity and inconsistency with word use and with calculus symbolism, it will not be possible to attribute or ascertain the impact of these findings on students learning. However, the findings of this study suggest that difficulties with calculus persist for students and teachers alike. Thus, further research should include investigating the teachers' pedagogical calculus discourse together with the students' learning of calculus. Of particular interest following on this study would be more research into the teachers' word use and narratives on differential calculus and its impact on student learning, and also exploring how inconsistency in word use and narratives influence students' learning of differential calculus.

Based on the application of the commognitive framework in this study, i.e. analysing teaching, I would suggest that a framework borne out of, or one that includes the four commognitive constructs, would be an invaluable analytical lens for mathematics teachers and mathematics teacher educators for the purposes of observing and analysing teaching and learning of mathematics. The four commognitive constructs could be applied as a framework, providing a basis and structure, for reflecting on and evaluating mathematics lessons. Thus, I would recommend further research into the commognitive framework, or an adaptation of the theory, for use in mathematics teacher education and teacher development.

One of the main limitations of my research was its primary focus on teacher data, which meant that there was limited student data. Thus, the findings of my research, being based on a relatively small sample of teachers, are only tentative. Nonetheless, I believe that this current study with its methodology, the commognitive thematic discourse analysis, of examining teachers' pedagogical calculus discourse may form a basis for further studies. Further research with a larger sample involving both teachers and students, is needed to substantiate the findings of this study. For example, without further research into the impact of teachers' use of dynamic graphical mediation on students' understanding of differential calculus, it is not possible to generalise the findings from this study on the affordances of dynamic geometry and graphing software. It is important, therefore, for further research to investigate the impact of students' use of dynamic imagery software on students' understanding of the derivative and calculus.

10.6 Concluding remarks

Past research (e.g. Berry and Nyman, 2003; Oehrtman et al., 2008; Tall, 1992; Thompson, 1994; Zandieh, 2000; Park, 2016) report students' challenges with calculus. This study set to investigate the teaching of elementary differential calculus and, so offer a different perspective to research seeking to explain the students' challenges with differential calculus.

Cobb (2009) describes the commognitive framework as having the capacity to attend to 'the macro-level of historically established mathematical discourse, the meso-level of local discourse practices jointly established by the teacher and students (…) and the micro-level of individual students' developing mathematical discourses (p. 207)'. My study has demonstrated that the commognitive framework provides an adaptable conceptual framework and an analytical lens through which we can examine, on a micro-level, how teachers teach mathematics. I would, therefore, refine Cobb's (2009) comment and further argue that the commognitive framework can attend to the micro level, not only of the individual students' developing mathematical discourse but also of the individual teachers' mathematical discourse. Using the commognitive analytic toolkit of word use, visual mediators, narratives and routines enabled the examination of the teachers' mathematical and pedagogical calculus discourse on differential calculus.

As a final remark to this thesis, it is important to remember that in such a purely qualitative research analysis, as with all forms of qualitative analysis, the purpose of the analysis is to offer interpretations about the data; not uncovering truths about the world (Syed and Nelson, 2015). Thus, I do acknowledge and would remind my readers that data analysis in an interpretative (qualitative) study can never be regarded as absolute or complete (Roulston, 2014), as it is subject to various perspectives and different purposes. Whilst the findings from this qualitative research cannot be generalised to represent the population of teachers of mathematics, the value of the research is in learning from the methods, findings, observations, and questions from this research. Readers and other researchers have the option to apply these methods, findings, observations, and questions to other contexts.

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List of Abbreviations

A level	Advanced level
AM	Intermediate and Advanced Mathematics
AQA	Assessment and Qualifications Alliance
AS level	Advanced Subsidiary level
C1	Core Mathematics 1
CAQDAS	Computer Assisted Qualitative Data Analysis Software
CPD	Continuing Professional Development
CTF	Commognitive Theoretical Framework
DfE	Department for Education
FP	Further Mathematics
GCE	General Certificate of Education
GCSE	General Certificate of Secondary Education
ICME	International Congress on Mathematical Education
IM	Intermediate Mathematics
ITT	Initial Teacher Training
MAA	Mathematical Association America
MEI	Mathematics in Education and Industry
NC	National Curriculum
O level	Ordinary Level
OCR	Oxford, Cambridge and RSA
PGCE	Post Graduate Certificate in Education
PME	The International Group for the Psychology of Mathematics Education
RQ	Research Questions
STEM	Science, Technology, Engineering and Mathematics
TAM	Teaching Advanced Mathematics

Appendix A
Research Ethics Considerations

Pilot study ethical review

A.1 Research Ethics Approval Letter [ESSL, Environment and LUBS(AREA) Faculty Research Ethics Committee, University of Leeds] – Pilot Study

Performance, Governance and Operations
Research & Innovation Service
Charles Thackrah Building
101 Clarendon Road
Leeds LS2 9LJ Tel: 0113 343 4873
Email: ResearchEthics@leeds.ac.uk

UNIVERSITY OF LEEDS

Innocent Tasara
School of Education
University of Leeds
Leeds, LS2 9JT

ESSL, Environment and LUBS (AREA) Faculty Research Ethics Committee University of Leeds

4 November 2022

Dear Innocent

Title of study:	**How do teachers of mathematics introduce calculus?**
Ethics reference:	**AREA 14-135**

I am pleased to inform you that the above research application has been reviewed by the ESSL, Environment and LUBS (AREA) Faculty Research Ethics Committee and I can confirm a favourable ethical opinion as of the date of this letter. The following documentation was considered:

Document	Version	Date
AREA 14-135 IT -Ethical_Review_Form_V3.doc	1	22/05/15
AREA 14-135 IT -Information sheet for the Parents.docx	1	22/05/15
AREA 14-135 IT -Information sheet for the participant teacher - interviews and lesson observations - Copy.docx	1	22/05/15
AREA 14-135 IT -Information sheet for the school gatekeepers.docx	1	22/05/15
AREA 14-135 IT -Consent form for the Parents.doc	1	22/05/15
AREA 14-135 IT -Consent form for the participant teacher-interviews and lesson observations.doc	1	22/05/15

AREA 14-135 IT -Consent form for the school gatekeepers.doc	1	22/05/15

Committee members made the following comments about your application:

- This is a carefully considered application which addresses all the main ethical issues.
- The word ‘principal’ is occasionally misspelt as ‘principle’ in section C7 (ii) and in the penultimate paragraph in C20.

Please notify the committee if you intend to make any amendments to the original research as submitted at date of this approval, including changes to recruitment methodology. All changes must receive ethical approval prior to implementation. The amendment form is available at http://ris.leeds.ac.uk/EthicsAmendment.

Please note: You are expected to keep a record of all your approved documentation, as well as documents such as sample consent forms, and other documents relating to the study. This should be kept in your study file, which should be readily available for audit purposes. You will be given a two-week notice period if your project is to be audited. There is a checklist listing examples of documents to be kept which is available at http://ris.leeds.ac.uk/EthicsAudits.

We welcome feedback on your experience of the ethical review process and suggestions for improvement. Please email any comments to ResearchEthics@leeds.ac.uk.

Yours sincerely

Jennifer Blaikie
Senior Research Ethics Administrator, Research & Innovation Service
On behalf of Dr Andrew Evans, Chair, AREA Faculty Research Ethics Committee

CC: Student’s supervisor(s)

A.2 Research Ethics Approval Letter [ESSL, Environment and LUBS(AREA) Faculty Research Ethics Committee, University of Leeds] – Main Study

Main study ethical review

Performance, Governance and Operations
Research & Innovation Service
Charles Thackrah Building
101 Clarendon Road
Leeds LS2 9LJ Tel: 0113 343 4873
Email: ResearchEthics@leeds.ac.uk

UNIVERSITY OF LEEDS

Innocent Tasara
School of Education
University of Leeds
Leeds, LS2 9JT

ESSL, Environment and LUBS (AREA) Faculty Research Ethics Committee
University of Leeds

4 November 2022

Dear Innocent

Title of study: **How do teachers of mathematics introduce calculus?**
Ethics reference: **AREA 14-135 amendment Sept 2015**

I am pleased to inform you that your amendment to the research application listed above has been reviewed by a delegate of the ESSL, Environment and LUBS (AREA) Faculty Research Ethics Committee and I can confirm a favourable ethical opinion as of the date of this letter. The following documentation was considered:

Document	Version	Date
AREA 14-135 Amendment_form.doc	1	08/09/15
AREA 14-135 IT -Ethical_Review_Form_V3.doc	1	08/09/15
AREA 14-135 IT -Low Risk Fieldwork RA form.doc	1	08/09/15
AREA 14-135 IT -Information sheet for the school gatekeepers.docx	1	08/09/15
AREA 14-135 IT -Information sheet for the participant teacher - interviews and lesson observations - Copy.docx	1	08/09/15
AREA 14-135 IT -Information sheet for the Parents.docx	1	08/09/15
AREA 14-135 IT -Consent form for the school gatekeepers.doc	1	08/09/15
AREA 14-135 IT -Consent form for the participant teacher- interviews and lesson observations.doc	1	08/09/15
AREA 14-135 IT -Consent form for the Parents.doc	1	08/09/15
AREA 14-135 IT -Ethical_Review_Form_V3.doc	1	22/05/15

AREA 14-135 IT -Information sheet for the Parents.docx	1	22/05/15
AREA 14-135 IT -Information sheet for the participant teacher - interviews and lesson observations - Copy.docx	1	22/05/15
AREA 14-135 IT -Information sheet for the school gatekeepers.docx	1	22/05/15
AREA 14-135 IT -Consent form for the Parents.doc	1	22/05/15
AREA 14-135 IT -Consent form for the participant teacher- interviews and lesson observations.doc	1	22/05/15
AREA 14-135 IT -Consent form for the school gatekeepers.doc	1	22/05/15

Please notify the committee if you intend to make any further amendments to the original research as submitted at date of this approval as all changes must receive ethical approval prior to implementation. The amendment form is available at http://ris.leeds.ac.uk/EthicsAmendment.

Please note: You are expected to keep a record of all your approved documentation, as well as documents such as sample consent forms, and other documents relating to the study. This should be kept in your study file, which should be readily available for audit purposes. You will be given a two-week notice period if your project is to be audited. There is a checklist listing examples of documents to be kept which is available at http://ris.leeds.ac.uk/EthicsAudits.

We welcome feedback on your experience of the ethical review process and suggestions for improvement. Please email any comments to ResearchEthics@leeds.ac.uk.

Yours sincerely

Jennifer Blaikie
Senior Research Ethics Administrator, Research & Innovation Service
On behalf of Dr Andrew Evans, Chair, AREA Faculty Research Ethics Committee
CC: Student's supervisor(s)
Innocent Tasara
School of Education
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Appendix B
Informed Consent Forms

There are two parts to the Informed Consent Form, Part 1: *Information Sheet* and Part 2: *Certificate of Consent* for each of the following groups- the schools' gatekeepers, the participating teachers and the parents.

B.1 School Gatekeepers

There are two parts to the Informed Consent Form, Part 1: *Information Sheet* and Part 2: *Certificate of Consent* for the school's gatekeepers.

B.1.1 Information Sheet

UNIVERSITY OF LEEDS

School of Education:
Faculty of Education, Social Sciences and Law.

<u>Informed Consent Form: Teaching Observations</u>

Research Project Title: How do teachers of mathematics introduce calculus?
Researcher: Innocent Tasara

This Informed Consent Form has two parts:

- **Information Sheet (to share information about the study with you)**
- **Certificate of Consent (for signatures if you choose to participate)**

<u>Part I: Information Sheet</u>

Introduction
I am Innocent Tasara, a Lecturer in Mathematics Education, Lead Tutor PGCE Secondary Mathematics and a PhD student at the University of Leeds. I am researching into how secondary school teachers of mathematics teach calculus. This study will look into how teachers of mathematics in England teach calculus (post 16), with a particular focus on differentiation. You are being invited to give consent for your school participation in this research project to take place with secondary school teachers of mathematics. Before you decide it is important for you to understand why the research is being done and what it will involve. Please take time to read the following information carefully and discuss it with others if you wish. Please ask if there is anything that is not clear or if you would like more information.

What is the purpose of the research?
The purpose of my research is to investigate aspects of teacher knowledge and practice for calculus teaching with the aim to contribute to the knowledge base for improving the teaching and learning of calculus in secondary schools, as well as the teaching and learning of mathematics teaching (teacher training).

Why have my school been chosen? Do I have to give consent?

Your school has been asked to participate in this research project because you have post 16 mathematics teaching classes. If you do decide to give consent, you will be given a copy of this information sheet to keep (and be asked to sign a consent form).

What will happen if I give consent?
If you give consent, then a teacher of mathematics in your school will be asked to participate in this research project. It is up to the individual teacher to decide whether or not to consent to participating in this research project. If a teacher decides to give consent, he/she will be given a copy of the information sheet for participating teachers to keep (and be asked to sign a consent form). He/she can still withdraw their consent at any time and their data will be deleted.

What will happen if a teacher of mathematics gives consent?
If a teacher gives consent, then he/she will take part in an individual pre-teaching interview, and then be observed teaching a lesson on calculus with one of your post 16 classes, followed by an individual post-teaching interview, all by the researcher. The interviews will only be audio recorded. Only one lesson will be observed, and it will be video and audio recorded. The video is for capturing only the teacher's actions; thus, the camera will be directly focusing on the teacher. Copies of any work produced for the lesson such as lesson plans, worksheets and other resources will also be collected for the research. The interviews will be no more than one hour long and no one else but the researcher will be present unless the participant teacher would like someone else to be there. The information recorded is anonymised and no one else except the researcher will have access to the information documented during the interviews. The interviews will be audio-recorded, but the audio file will have a number rather than the participant's name on it. The audio file will be copied onto the researcher's university computer and saved within a password-protected folder. The files will be destroyed 3 years after the research project has ended.

What type of information will be sought?
Audio recordings of pre-teaching and post-teaching interviews with the teacher; audio and video recordings of the observed lesson will be collected for this research project. The camera will focus primarily on the teacher and not the students. However, as research will involve observing a teacher in a post 16 classroom environment, children aged 16-18 are likely to be encountered and may be audio recorded. Nevertheless, their contributions will not be directly quoted. Lesson-plans and post-lesson evaluations will also be copied, and any other materials produced for your lesson. The email address of the participating teacher will be collected for the purposes of contacting them with a summary of the research results and any correspondence directly related to the research or future research. They will only be contacted by the researcher.

Why is the collection of this information relevant for achieving the research project's objectives?
The audio and video data will be analysed by the researcher to provide useful insights into teachers' knowledge and practice on calculus teaching, insights which would contribute to the knowledge base for improving the teaching and learning of calculus in secondary schools, as well as the teaching and learning of mathematics teaching (teacher training).

What are the possible disadvantages and risks of taking part?
There are no expected disadvantages or risks to taking part. However, a teacher may feel inconvenienced by the presence of a camera in the room, or by the time taken for interviews, though the research has been designed to minimise the amount of time taken.

Benefits

Teachers may benefit directly from the study as they engage in reflective practice about their lesson planning and teaching. The pre-teaching and post-teaching interviews may feed into their lesson evaluation. Participation is likely to help us find out more about how teachers plan lesson, what resources they use and how they teach lessons on calculus.

Reimbursements
The researcher will visit the teachers at their school. The pre-teaching interview, the lesson observation and the post-teaching interview will all take place in your school. Therefore, no cost reimbursements will be necessary.

Will taking part in this project be kept confidential?
All the information that the researcher will collect about the institution and its participant teachers during the course of the research will be kept strictly anonymous. Any information collected about the participants will not be attributable by name. The information will have a number on it instead of their name. If names are mentioned in the audio recordings, any transcription made will be anonymised. Only the researcher will know about that number. This means that the participants will not be able to be identified in any reports or publications.

What will happen to the results of the research project?
The knowledge that the researcher get from this research will be shared with the participants before it is made available to the public. The participant teacher will receive a summary of the results. Results of the research may be published or presented at conferences or in lectures to the public so that other interested people may learn from the research. However, participants will not be able to be identified in any reports, publications, lectures, or conferences.
Contact for further information
If you would like any further information or have any questions at any point during or after the research, you can contact the principal researcher:

Innocent Tasara
(Supervisors: John Monaghan: J.D.Monaghan@education.leeds.ac.uk & Michael Inglis: M.Inglis@leeds.ac.uk)
Lecturer in Mathematics Education
Centre for Studies in Science and Mathematics Education
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University of Leeds, Leeds, LS2 9JT
Office phone: 0113 34 34622
Email I.Tasara@leeds.ac.uk
Office: EC Stoner 8.76

You will be given a full copy of this information sheet to keep.
Thank you for taking the time to read through the information.

B.1.2 Certificate of Consent – school gatekeepers

UNIVERSITY OF LEEDS

School of Education:

Faculty of Education, Social Sciences and Law

<u>Part 2: Certificate of Consent</u>

Consent to take part in the research project: "How do teachers of mathematics introduce calculus?"

	Add your initials next to the statements you agree with
I confirm that I have read and understand the information sheet dated 24/08/15 explaining the above research project and I have had the opportunity to ask questions about the project.	
I understand that my consent for my institution participation is voluntary and that I am free to withdraw my consent at any time without giving any reason and without there being any negative consequences. (If you would like to withdraw, please email the researcher: Innocent Tasara, i.tasara@leeds.ac.uk)	
I give permission for the researcher to have access to participants' responses. I understand that the name of my institution or individual participant teachers will not be linked with the research materials and will not be identified or identifiable in the report(s) or publication(s) that result from the research or in any lecture(s) or conference presentation(s). I understand that responses will be kept strictly confidential.	
I agree for research in the above research project to be carried out and will inform the researcher should my contact details change.	
I understand that a summary of results will be e-mailed (please provide an e-mail address below) to the participant teachers before the final report is shared with others; and that this e-mail address will only be used to contact me about the research. Only the researcher will contact me. I will inform the researcher should the e-mail address change.	

Name	
Position	
Signature	
e-mail address	
Date	
Name of researcher	Innocent Tasara
Signature	
Date	

B.2 Informed Consent Forms: participant teachers

There are two parts to the Informed Consent Form, Part 1: *Information Sheet* and Part 2: *Certificate of Consent* for the participating teachers.

B.2.1 Information Sheet

UNIVERSITY OF LEEDS

School of Education:

Faculty of Education, Social Sciences and Law

Research Project Title: How do teachers of mathematics introduce calculus?
Researcher: Innocent Tasara

This Informed Consent Form has two parts:

- **Information Sheet (to share information about the study with you)**
- **Certificate of Consent (for signatures if you choose to participate)**

Part I: Information Sheet

Introduction
I am Innocent Tasara, a Lecturer in Mathematics Education, Lead Tutor PGCE Secondary Mathematics and a PhD student at the University of Leeds. I am researching into how secondary school teachers of mathematics teach calculus. This study will look into how teachers of mathematics in England teach calculus (post 16), with a particular focus on differentiation. You are being invited to give consent for your participation in this research project to take place with secondary school teachers of mathematics. Before you decide it is important for you to understand why the research is being done and what it will

involve. Please take time to read the following information carefully and discuss it with others if you wish. Please ask if there is anything that is not clear or if you would like more information.

What is the purpose of the research?
The purpose of my research is to investigate aspects of teacher knowledge and practice for calculus teaching with the aim to contribute to the knowledge base for improving the teaching and learning of calculus in secondary schools, as well as the teaching and learning of mathematics teaching (teacher training).

Why have I been chosen? Do I have to give consent?
You have been asked to participate in this research project because you are a teacher of mathematics with post 16 teaching classes. It is up to you to decide whether or not to consent to participate in this research project. If you do decide to give consent, you will be given a copy of this information sheet to keep (and be asked to sign a consent form). You can still withdraw your consent at any time and your data will be deleted. You do not have to give a reason.

What will happen if I give consent?
If you give consent then you will take part in an individual pre-teaching interview, and then observed teaching a lesson on calculus with one of your post 16 classes, followed by an individual post-teaching interview, all by the researcher. The interviews will only be audio recorded. Only one lesson will be observed, and it will be video and audio recorded. The video is for capturing only the teacher's actions; thus, the camera will be directly focusing on the teacher. Copies of any work you produce for the lesson such as lesson plans, worksheets and other resources will also be collected for the research.

The interviews will be approximately one hour long and no one else but the researcher will be present unless you would like someone else to be there. The information recorded is confidential, and no one else except the researcher will have access to the information documented during the interviews. The interviews will be audio-recorded, but the audio file will have a number rather than your name on it. If names are mentioned on the file, any transcription made will be anonymised. The audio file will be copied onto the researcher's university computer and saved within a password-protected folder. The files will be destroyed 3 years after the research project has ended.

What type of information will be sought?
Audio recordings of pre-teaching and post-teaching interviews; audio and video recordings of your lesson and field-notes will be collected for this research project. The video camera will be focused primarily on the teacher and not the students.

Lesson-plans and post-lesson evaluations will also be copied, and any other materials produced for your lesson. Your email addresses will be collected for the purposes of contacting you with a summary of the research results and any correspondence directly related to the research or future research. You will only be contacted by the researcher.

Why is the collection of this information relevant for achieving the research project's objectives?
The audio and video data will be analysed by the researcher to provide useful insights into teachers' knowledge and practice on calculus teaching, insights which would contribute to the knowledge base for improving the teaching and learning of calculus in secondary schools, as well the teaching and learning of mathematics teaching (teacher training).

What are the possible disadvantages and risks of taking part?

There are no expected disadvantages or risks to taking part. However, you may feel inconvenienced by the presence of a camera in the room, or by the time taken for interviews and collecting copies of your work, though the research has been designed to minimise the amount of time taken.

Benefits
You may benefit directly from the study as you engage in reflective practice about your lesson planning and teaching. The pre-teaching and post-teaching interviews may feed into your lesson evaluation. Participation is likely to help us find out more about how teachers plan lessons on calculus, what resources they use and how they teach lessons on calculus

Reimbursements
The researcher will visit you at school. The pre-teaching interview, the lesson observation and the post-teaching interview will all take place in your school. Therefore, no cost reimbursements will be necessary.

Will my taking part in this project be kept confidential?
All the information that the researcher will collect about you during the research will be kept strictly anonymous. You will not be able to be identified in any reports, publications, lectures, or conferences. Any information about you will have a number on it instead of your name. Only the researcher will know what this number is. This means that you will not be able to be identified in any reports or publications.

What will happen to the results of the research project?
The knowledge that the researcher get from this research will be shared with you before it is made available to the public. You will receive a summary of the results. Any information collected about you will not be attributable to you by name. Results of the research may be published or presented at conferences or in lectures to the public so that other interested people may learn from the research. However, participants will not be able to be identified in any reports, publications, lectures, or conferences.

Contact for further information
If you would like any further information or have any questions at any point during or after the research, you can contact the principal researcher:

Innocent Tasara
(Supervisors: John Monaghan: J.D.Monaghan@education.leeds.ac.uk & Michael Inglis: M.Inglis@leeds.ac.uk)
Lecturer in Mathematics Education
Centre for Studies in Science and Mathematics Education
School of Education
University of Leeds, Leeds, LS2 9JT
Office phone: 0113 34 34622
Email I.Tasara@leeds.ac.uk
Office: EC Stoner 8.7

You will be given a full copy of this information sheet to keep.

Thank you for taking the time to read through the information.

B.2.2 Certificate of Consent – participant teachers

UNIVERSITY OF LEEDS

School of Education,
Faculty of Education, Social Sciences and Law.

Part 2: Certificate of Consent

Consent to take part in the research project: "How do teachers of mathematics introduce calculus?"

	Add your initials next to the statements you agree with
I confirm that I have read and understand the information sheet dated 24/08/15 explaining the above research project and I have had the opportunity to ask questions about the project.	
I understand that my consent for participation is voluntary and that I am free to withdraw my consent at any time without giving any reason and without there being any negative consequences. (If you would like to withdraw, please email the researcher: Innocent Tasara, i.tasara@leeds.ac.uk)	
I give permission for the researcher to have access to my anonymised responses. I understand that the name of my institution or individual students will not be linked with the research materials and will not be identified or identifiable in the report(s) or publication(s) that result from the research or in any lecture(s) or conference presentation(s). I understand that responses will be kept strictly confidential.	
I agree for research in the above research project to be carried out and will inform the researcher should my contact details change.	
I understand that a summary of results will be e-mailed (please provide an e-mail address below) to me before the final report is shared with others; and that this e-mail address will only be used to contact me about the research. Only the researcher will contact me. I will inform the researcher should the e-mail address change.	

Name	
Position	
Signature	

e-mail address	
Date	
Name of researcher	Innocent Tasara
Signature	
Date	

B.3 Informed Consent Forms: Parents

There are two parts to the Informed Consent Form, Part 1: *Information Sheet* and Part 2: *Certificate of Consent* for the participating parents.

B.3.1 Information Sheet

UNIVERSITY OF LEEDS

School of Education:
Faculty of Education, Social Sciences and Law.

Informed Consent Form: Teaching Observations

Research Project Title: How do teachers of mathematics introduce calculus?
Researcher: Innocent Tasara

This Informed Consent Form has two parts:

- **Information Sheet (to share information about the study with you)**
- **Certificate of Consent (for signatures if you choose to participate)**

Part I: Information Sheet

Introduction
I am Innocent Tasara, a Lecturer in Mathematics Education, Lead Tutor PGCE Secondary Mathematics and a PhD student at the University of Leeds. I am researching into how secondary school teachers of mathematics teach calculus. This study will look into how teachers of mathematics in England teach calculus (post 16), with a particular focus on differentiation. You are being invited to give consent for your child's participation in this research project to take place with secondary school teachers of mathematics. Before you decide it is important for you to understand why the research is being done and what it will involve. Please take time to read the following information carefully and discuss it with others if you wish. Please ask if there is anything that is not clear or if you would like more information.

What is the purpose of the research?
The purpose of my research is to investigate aspects of teacher knowledge and practice for calculus teaching with the aim to contribute to the knowledge base for improving the

teaching and learning of calculus in secondary schools, as well as the teaching and learning of mathematics teaching (teacher training).

Why has my child been chosen?
This research mainly focuses on the teacher teaching a lesson to a class of post 16 students. Data collected is about the teacher and how he/she teaches calculus. Your child's participation in the research project is by the mere fact that he/she belongs to the class of the participating teacher. For my research to be successful I need to observe participating teachers of mathematics teaching post 16 mathematics lessons. Your school is taking part in this research project, and your child is in the class whose teacher is taking part in this research project.

Do I have to give consent?
It is up to you to decide whether or not to consent to your child participating in this research project. If you do decide to give consent, you will be given a copy of this information sheet to keep (and be asked to sign a consent form). You can still withdraw your consent at any time and any data that may have been collected from your child will be deleted.

What will happen if I give consent?
If you give consent, then your child will be part of the class that will take part in a lesson on calculus to be taught by their teacher. This lesson will be observed by the researcher. Only one lesson will be observed, and it will be video and audio recorded. The video is for capturing only the teacher's actions; thus, the camera will be directly focusing on the teacher. However, as this research will involve observing a teacher in a classroom environment, children (aged 16-18) are likely to be encountered and may be audio recorded. Nevertheless, their contributions will not be directly quoted.

What are the possible disadvantages and risks of taking part?
There are no expected disadvantages or risks to taking part. However, some students may feel a little nervous about the presence of a camera in the classroom.

What are the benefits of taking part?
The audio and video data will be analysed by the researcher to provide useful insights into teachers' knowledge and practice on calculus teaching, insights which would contribute to the knowledge base for improving the teaching and learning of calculus in secondary schools, as well as the teaching and learning of mathematics teaching (teacher training).

Results of the research may be published or presented at conferences or in lectures to the public so that other interested people may learn from the research. However, participants will not be able to be identified in any reports, publications, lectures, or conferences.

Will taking part in this project be kept confidential?
All the information that the researcher will collect about the institution and its participants during the research will be kept strictly anonymous. Any information collected about the participants will not be attributable by name. Such information will have a number on it instead of any participants' names. Only the researcher will know about that number. If names are mentioned in the audio recordings, any transcription made will be anonymised.

This means that the participants will not be able to be identified in any reports or publications.

Contact for further information
If you would like any further information or have any questions at any point during or after the research, you can contact the principal researcher:

Innocent Tasara
(Supervisors: John Monaghan: J.D.Monaghan@education.leeds.ac.uk & Michael Inglis: M.Inglis@leeds.ac.uk)
Lecturer in Mathematics Education
Centre for Studies in Science and Mathematics Education
School of Education
University of Leeds, Leeds, LS2 9JT
Office phone: 0113 34 34622
Email I.Tasara@leeds.ac.uk
Office: EC Stoner 8.76

You will be given a full copy of this information sheet to keep.

Thank you for taking the time to read through the information.

B.3.2 Certificate of Consent - Parents

UNIVERSITY OF LEEDS

School of Education
Faculty of Education, Social Sciences and Law.

Part 2: Certificate of Consent

Consent to take part in the research project: "How do teachers of mathematics introduce calculus?"

	Add your initials next to the statements you agree with
I confirm that I have read and understand the information sheet dated 24/08/15 explaining the above research project and I have had the opportunity to ask questions about the project.	

I understand that my child's participation is voluntary and that I am free to withdraw my child at any time without giving any reason and without there being any negative consequences. If I wish to withdraw my child at any point then I can email the researcher: Innocent Tasara, i.tasara@leeds.ac.uk. If I decide to withdraw my child from this project, then any data collected from them prior to their withdrawal will be destroyed.	
I give permission for the researcher to have access to my child's anonymised responses. I understand that their name will not be linked with the research materials and will not be identified or identifiable in the report(s) or publication(s) that result from the research or in any lecture(s) or conference presentation(s). I understand that their responses will be kept strictly confidential.	
I agree for the data collected from my child to be used in relevant future research in an anonymised form.	
I give permission for my child to take part in the above research project and will inform the lead researcher should our contact details change.	

Name of Participant	
Signature of Parent/Guardian	
Date	
Name of researcher	Innocent Tasara
Signature	
Date	

Appendix C
Data Collection Instruments

C.1 Interview Schedule

	Issue/topic	Interview Questions	Possible follow up questions
1.	Teacher knowledge and experience	Can you tell me about your teaching qualifications and any other training or in-service training for teaching mathematics that you have received? Can you tell me about your experience of teaching post -16 mathematics and calculus?	Did you study A level maths? What was your first degree? For how long?
2	Resources/ Mediational tools	Can you tell me about the resources and technology that you use to plan and teach differentiation/calculus? How is the AS/A level mathematics curriculum structured and delivered in your department?	What textbooks do you use? What digital or web-based resources do you use? Do you use or follow SOW? Which Exam board do you use?
3	Teaching – the concept of the derivative	Can you tell me about your approach to introducing the concept of the derivative? How did you plan the learning activities and the mathematical tasks for the lesson?	What is the reasoning behind your approach? What factors influence your choice? Did you consult any colleagues?
4	Next lesson (Post-lesson focus)	Can you tell me about your plans for the next lesson?	Moving on from the first lesson, what will be the focus of your next lesson? How do you plan to teach that next lesson? Why?

C.2 Observation Schedule

Date:	School Code:	Participant teacher code:
Class context Year Group: Number of students: Female: Male: Lesson Context Period/time: Topic/Content:		

Time	Focus points of the lesson observation	Points for discussion in the post-lesson interview
	• • • • • •	

Appendix D
Pre-teaching Interview Transcript

This is an example of the pre-teaching interview data transcripts. This is the transcript for the interview with T1. This transcript was used for the inter-coder reliability check exercise.

D.1 Interview Transcript [T1I(i)]

I – The Interviewer (Researcher)

R – The Respondent (Teacher)

<u>START</u>

1. I: My first question would be about: can you tell me about, can you tell me about your teaching qualifications, training, and it could be in-service training for teaching maths?
2. R: Well, I have got a maths degree [I: yeah], a 1st class honors maths degree in 1977, and then I did a PGCE a year after in 1978 and I have taught since then.
3. I: So, for how long have you been teaching post 16?
4. R: I was 15 years in 11-18 school [I: OK] and I always taught 6th form. [I: Alright]. So, I have taught 6th continuously form since 1978. But I have been at this college since 1993, so I'm in my 22nd year here/
5. I:/93, Ok/
6. R: but in my previous school for 15 years I always taught A level there and obvious there were juniors as well, but I always had at least one if not two A level classes there.
7. I: Oh, that's ah that's very, very interesting. And umm I think you have answered my second part about for how long you have been doing this. And ah um over the years, when looking at ah err differentiation, has there been any changes in the curriculum or the expectations/
8. R: /oh yes/

9. I:/assessments/

10. R: / yes. When I was a student, we had to be able to prove derivatives from first principles and you would have $f(x) + \partial x$, and we would have to be able to differentiate x or x^3or $\sin x$ even/

11. I: /from first principles/

12. R: /and that was still on the A level course when I first started teaching/ (I; Ok)/

13. R:/so when I first started teaching, I would have taught it in an algebraic way (I: /Um)/ Uh it hasn't been part of the err course now I would say since the mid-80s and things have(..). I haven't actually taught it algebraically (I: Umm) for a long time, though in FP1 they do it a little bit (I: OK), by calling it h (I: using h yeah) and having $x + h$ and letting h approach naught (I: the limit) yeah, but they only do it for err the greatest complexity as a cube (I: right) ah that's all they do, (I: right). So, they do a little bit of it in FP1[door swings open causing some noises] but that's err (...). So that's err I would say last year or the year before was the first year I taught it in that way (I: Um) for a long time. Normally I give them a more (..) I make it an investigation over a lesson. I don't have a time for that now (I: right). And I used to sort of really go to almost like a coursework activity... we are going to investigating...work out the rule., I am going to have to lead them a little bit more now (I: right) to give them a little bit less, I have given them all different points to work on today (I: Um) before they everyone did everything but uh (...)time is an issue, time, time for those things. I have never yet resort to telling them this is the answer this is how you do it (I: right) I have always done something (I: right), I have always done and I will show on autograph the graph of the gradient and show them how that goes, I have got that lined up as well. So that's a little bit about that. (I: Uh

brilliant)

14. I: and uh um in terms of the resources and may be teaching and learning technologies eh that you have, you have used in the past, you just have just mentioned Autograph.
15. R: well that's relatively new, if I say relatively new (I: yeah) it's certainly within, for me that's within the last 5 or 6 years. Err um (..) then it's (...) I just I guess I am fairly old fashion and traditional. I just use whatever board I have got and a pen. Err um I am (..) not (..)really well into computer generated lesson (I: Um) and PowerPoint presentations and things like that (I: Um). If you see other staff there are, we just do different... I just (..) I just wasn't just brought up that way (I: right) and err um (...) I just (..) it is me. I'm, I'm the resource really (I: right)
16. I: and Ur um it's interesting to note that not many teachers actually use technology in teaching. They might use like this smart notebook (R: yes) or PowerPoint just as means of displaying/
17. R: /yes, it's, it's more of medium rather than err ...I know what you mean. They are not... they are not..., it's not it's not doing actual teaching is it? (I: yeah). No, no, it's not!
18. I: So, in that sense, err um it doesn't make much of a difference then?
19. R: I would say it has a little bit in things like transformations of graphs when you can show them this...they can see the graph moving.
20. I: Ok the dynamic (R: yeah), the interactive nature of the (R: yeah) the program, software.
21. R: So, so I would say that there it's an issue they can see the rotation, or they can see the enlargement (I: Um). It's it is abstract otherwise when you are just working out $f(x)$of something/ (I: without the visual)/ without showing yeah. But err

um (...) in terms of what we are doing in general now (I: yeah), I don't think I do use it as a teaching aid.

22. I: Yeah, um Autograph is relatively new (R: yes), probably the last may be ...err about 6 or so years that's when it became available um on the market. I know I know the guy Autograph err um (R: Alright) yeah, yeah, I have been to one of his training sessions. Err and now I am still in touch with him. Every year when they have got like a new uhm software or when they update that they keep sending a copy to me. Even the teachers who are on the TAM course they get free license to use Autograph, but there is also GeoGebra/ (R: yes, we got that now as well). GeoGebra is a free download online (R: yes) and it's very powerful you know; I mean if…if you find time to learn how to use it /
23. R: well, I have had training sessions on it/ (I: yeah)/ but the trouble with me now is that I'm bit long in the tooth, and I struggle to take things in, and this is likely to be my last full time year teaching. (I; Uh) And now I'll retire from full time teaching at the end of this year. I am 60 in June, so I will probably go part time or retire. I am not gonna learn new things now, so you know I'm [I: laughs] just sort of ticking over his arm really. (I: uh ok)
24. I: So, in terms of err books some textbooks, /
25. R: Yeah, err umm we have got those books up there (I: Ok those ones) those are designed for the course, they are ok, (I: yeah) they are adequate. I don't think there is a brilliant book (I: right). I don't think brilliant books have been written (I: right). I don't you can, because what one person thinks is very good another person doesn't like, and err I just use what I think is good at the time (I: Um). If I like the book, I use it if I don't, I use something else. It is as simple as that. And I have other books up there which I dip into ....and I have got thousands of

my own worksheets and things which I have built up over the years (I: yeah).

26. I: I can see Bostock & Chandler there/
27. R: Oh yeah, oh yeah, the old, the old. I got I got even Deccan Potter (I: oh right) and Deccan Potter I used that as a student in 6th form myself/
28. I: /Oh yeah Elementary Analysis, oh yeah, yeah, I used that at the university yeah.
29. R: I refer to ...those are the textbooks I had as a student, not this individual copy but that text (I: yeah, yeah). Further on um those are .../ (I: ah analysis)/ I have got Humphreys and Topping mechanics at home. That's err that's err the mechanics/ (I: Humphrey yeah) That's what I had in lower six, Quadling and Ramsay in Mechanics/ (I: err right Elementary Mechanics)/That's when I was a student. I didn't do any Statistics as a student / (I: Err)/ That was my mechanics book when I was a student. Err, I still dip into things. I have got some GCSE texts which is still quite useful, (I: yeah) good for practice (I: yeah).
30. I: Yeah, very difficult maths in those little GCSEs the old, is it the Oxford ones?
31. R: These ones here?
32. I: oh yeah, those ones, yeah/ (R: Oh yeah) /I remember those [both laughing]
33. I: Umm which exam board do you use here? =
34. R: =AQA (I: AQA)/ (R: yeah)/
35. I: /Is there any reason why they chose AQA or/
36. R: Well, we had OCR for years (I: OK) Err we went

to OCR from JMB (I: Ok) in 1994 (I: Umm) because we wanted to do the modular course (I: right) and there was an immediate upturn in results (I: Yeah) It was fantastic (I: Yeah). We could do modules in November (I: Yeah), March, June. They could re-sit them if they needed to (I: Yeah) and their results were fantastic. Other subjects began to sort of want a piece of this option, and as soon as everything by modular, (I: Yeah) we didn't have that relative advantage (I: No), and there was a notable dip (I: Yeah). Err umm we stuck with OCR really until about 2011 (I: Um) when we thought these papers were getting harder or our students were getting less capable or whatever (I: yeah). We just thought [laughing] the structure of the AQA questions were more accessible to the sort of students we had (I: right). We don't get here that many naturally really gifted (I: right) public school-type students. They are basically down to earth kids (I: Umm). They need teaching carefully/ (I: Umm they work hard)/yeah, they need that structure and err umm... but AQA again isn't trivial (I: Yeah). You know they don't all find it easy; you know they got to know the stuff (I: Yeah). We just, we just need to like..., love the layout of it really, you know what they say? (I: Umm) but it's a matter of, you know, our [inaudible], you know.

37. I: Can you tell me about your approach to introduce err...I mean you did touch on that earlier, to introducing the concept of the derivative?
38. R: Well today I am going to ask them what we mean by the gradient of a curve (I: yeah) and I am going to see what sort of answers I will get (I: Umm) Err

Um(...) that's what ... I don't even know if I would use the word derivative today (I: Umm). I may well don't use that (I: Yeah). Err um I might use the word gradient function (I: yeah) we will see (I; yeah). I mean, it might just come out of my head, you know, err .... but ... but what do we mean by gradient? And I will ask them, you know, what do you think the gradient is there... I'm hoping someone will tell me ...err it's the tangent, and I will say Ok, and I'm going to give them [looks through some papers] oh it's here somewhere...I'm gonna give them that graph.

39. I: Oh yeah, if I can get a copy, copies of these/ (R: there you are)/thanks that's fine.
40. R: I'm gonna give them that graph (I: Yeah) and I'm going to ask them to draw the tangent and I 'm going to take... see what we get as our measurements.
41. I: that's very interesting
42. R: and then we will see what the real one, you know, that's where...I will do that before we even talk about how we ...how we do it (I: yeah) that is the first thing I'll do after we talked about we mean by the gradient of a curve (I: right). That's where it's gonna go and then into the activity of the idea that we are getting ...we getting to the chord. I will try to demonstrate that if we take the chord near enough, it's an approximation...and I have given them all different points on their table that they have to use (I: Right) ...Err and I will say right what did you get for that one, and see if we can spot a pattern.

43. I: So, you are collecting data from/ (R: err yes)/ and then putting it on the board/ (R: yes, yes)/
44. R: ...but I have got 24 different ones (I: right) no one has got ...no two people sitting next to each are doing the same (I: yeah, I get it) [both laughing], and err um I know yeah this is....so what do you think it will be here? (I: yeah) and then I will show them the one on Autograph if we have got time (I: Ok). They will see that the line is the line $2x$. I am not expecting to get much further than that (I: OK). If I do will just ask them what do you think x^3would be? (I: right) Err I would see how far how it goes (I: Umm)
45. I: What is the reasoning behind your approach? because someone would probably approach it in a completely different way...err umm.?
46. R: Err Umm I want them to have at least a feel of what we are trying to do, what differentiation means, rather than just state that right when you start with x^2 you get $2x$, right they will get it, (I: right) but .... what does it mean? (I: yeah) ...and I just want them (..)to have a feel (..) of what it actually means. At the end of the day of course they will just follow the rules, but I would say that's what...that's what...that's what it's about (I: Umm) [noise in the background of students talking]
47. I: What you have just said is exactly what happens with the trainee teachers. [noise in the background of students talking] Well before we looked at the early [inaudible] of differentiation, it was the standard methods, of course, like you said that would come afterwards and that's how we would carry on from...but laying this foundation would give them an understanding of where everything is coming from. I think it's very, very important.
48. R: But the biggest issue we have all the time in this module is time, (I: time), it's time and you're playing

a balancing act all the time about how much depth can you give (..) in the time that you have got. (I: Umm). It's just...it's just that with differentiation, you can't ...I don't ...I don't see how you can start saying, right $y = x^2$, $\frac{dy}{dx} = 2x$... you know... (I: yeah). I certainly won't be using the $\frac{dy}{dx}$, I don't think (I: No). I certainly ... I mean I can't believe I will be using that notation today. (I: right). If I do, I haven't planned to anyway (I: Right). Err umm that will on the ball for tomorrow probably. Err Umm (...) first thing is I want to talk about the assessment [laughs] (I: that's fine).

49. I: That's fine, just one last one (R: yeah) in terms of the factors that are ...you know the factors that you have considered in your choice of resources and your approach... you have mentioned time, I hear that a lot the issue of time (R: Um). and err umm are there any other reasons, other than time?
50. R: that I'm doing this way? (I: Um) (5 seconds silence)
51. I: I am not saying there should be/ (R: No)/ but I'm just trying to...
52. R: Well if I had more time, I think still would do it in the same way (I: yeah) but I would give (...) I would give them more detail and they would have a bit more practice on what we gonna be doing (I: Um). I would give them more points to investigate. I would investigate slightly more complicated curves possibly (I: Right). But I think the approach would essentially still be the same. (I: would be the same). I think I paired the approach down to minimise time, but I think the approach would essentially be the same. (I: yeah umm)
53. I: and err um do you...do you consult colleagues, you know, within the department or do you have like that sharing of /
54. R: /Oh yeah, it's not formal but we do (I: yeah) it. We

all talk about ... I do it this way ...that way err... we all share ideas but it's not (I: no) on a formal basis. I always think if you make things formal, they lose something (I: yeah). You know, you try to formulate action plans and things ...and what does that mean ...you know...but if you actually talk about things you, you realise oh I can't do that...well there you know, but as soon as you...

55. I: Yeah, and err um do you have a scheme of work within the department?
56. R: We do, but it doesn't tell us how we have to teach it, (I: right) if that's what you mean. No, there is no prescribed way of teaching. That is up to us as individual teachers.
57. I: Right, so the teacher is ...you have got that autonomy, so OK I am going to do it in this way/
58. R: /yeah, yeah, I mean you watch us all do it (I: yeah). You might see different things (I: yeah). I suspect you would see that we are not as far different as you would imagine. I suspect we are all roughly very, very similar (I: Um). I am guessing. I think. But we are not governed by having to do it that way.
59. I: Ah thank you very much, thank you very much. Ah ...Oh we have gone 23 minutes, that's fine. I will stop it.
60. R: Shall I have the class in?
61. I: Yeah, yeah, they can come in now, we are absolutely fine now.

END