On a Class of Measures on Configuration Spaces

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Abstract

In this thesis we have explored a new class of measures ν_θ on configuration spaces Γ_X (of countable subsets of Euclidean space $X = \mathbb{R}^d$), obtained as a push-forward of "lattice" Gibbs measure θ on $X^{\mathbb{Z}^d}$. For these measures, we have proved the finiteness of the first and second moments and the integration by parts formula. It has also been proved that the generator of the Dirichlet form of ν_θ satisfies log-Sobolev inequality, which is not typical for measures on configuration spaces. Stochastic dynamics of a particle in random environment distributed according to the measure ν_θ, is presented as an example of possible application of this construction. We consider a toy model of a market, where this stochastic dynamics represents the volatility process of certain European derivative security. We have derived the "Black-Scholes type" pricing partial differential equation for this derivative security.

Contents

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Declaration

I declare that all the work in this thesis is original work, unless otherwise stated.

Introduction

Interest to configuration spaces has grown because of their applications to classical and quantum statistical mechanics, quantum field theory and representation theory.

To fix basic notations, let X be a topological space then the configuration space Γ_X over X is the space of all countable subsets without accumulation points (configurations) of X. Configuration spaces are most often attributed to the study of classical mechanical systems consisting of infinitely many points describing positions of labeled particles. The work in this field gave rise to the study of interacting particle systems initiated by Ruelle and Dobrushin [Rue99, Dob68, Dob69]. We refer the reader to [Geo11] and references there in for some further results in this field. In all these works, distributions of interacting particle systems are described by Gibbs measures on Γ_X.

On the other hand Vershik, Gelfand, Graev [VGG75] used Γ_X (with X a Riemannian manifold) equipped with the Poisson measure in order to construct representation of the group $\mathrm{Diff}_0(X)$ (the group of diffeomorphisms of X with compact support), see also [GSS64, Ism96] and references therein. They have also discussed the construction of quasi-invariant measures over Γ_X.

At the same time, the corresponding representations of the Lie algebra of compactly supported smooth vector fields $\mathrm{Vect}_0(X)$ were constructed and used in *Quantum Field Theory* in [GGPS74], see also [AKR99, GM00].

In [AKR98a, AKR98b], configuration spaces were considered as infinite dimensional manifolds. The development of geometry and analysis on Γ_X required existence of a measure μ on Γ_X which is $Diff_0(X)$-quasi-invariant and satisfies an integration by parts formula. One of the significant results of these papers is the construction of diffusion processes on Γ_X with the help of associated Dirichlet form. We refer the reader to [ADK07, KK02, Kun99] (see also references therein for some further works on analysis and geometry of configuration spaces). Different measures lead to different versions of such analysis (actually corresponding to physical systems defined by these measures).

In [AKR98a, AKR98b] this programme has been realized for Poisson and certain class of Gibbs measures on Γ_X. In [BD09, BD10, BD11], authors have considered the case of Poisson and Gibbs cluster measures (using a special projection construction).

In the present work we explore the projection construction proposed in [BD09] and use its version in order to study a completely different class of measures on configuration spaces Γ_X, obtained as a push-forward of "lattice" Gibbs measure in $X^{\mathbb{Z}^d}$ (throughout this work X represents a d-dimensional Euclidean space $\mathbb{R}^d$). These measures present interesting properties, including the *Log-Sobolev inequality*, which is not typical for measures on Γ_X (note that neither Poisson nor Gibbs measure on Γ_X satisfy Log-Sobolev inequality).

In Chapter 2 we introduce the push-forward construction of measures on configuration spaces. We start with the case of finite configurations. For $n \in \mathbb{Z}_+$, consider the space $X^{(n)} = \{A \subset X, |A| = n\}$ of n-point subsets of X, where $|\cdot|$ denotes the cardinality of A. The space $X^{(n)}$ is called the space on n-point configurations in X. Let us also consider the space defined by

$$\widetilde{X^n} = \{(x_1, x_2, \cdots, x_n) \in X^n; \quad x_i \neq x_j, \quad \forall\, i \neq j\}.$$

We can identify $X^{(n)}$ with the quotient space $\widetilde{X^n}/S_n$ where S_n is the symmetric group acting on $\widetilde{X^n}$ by permutations of the coordinates. Consider the natural projection map $p : \widetilde{X^n} \to \widetilde{X^n}/S_n = X^{(n)}$. Now, given a probability measure θ on $\widetilde{X^n}$, we can define a measure ν_θ on $X^{(n)}$ by the formula

$$\nu_\theta := p^*\theta, \quad \text{that is,} \quad \nu_\theta(A) = \theta(p^{-1}(A)), \quad A \subset \widetilde{X^n}.$$

We show that this construction cannot be directly extended to the case of infinite configuration spaces (because the so obtained measure will be in general concentrated on space of the configurations with accumulation points). Therefore we give a modification of the projection construction above, using a special map, $p : X^{\mathbb{Z}^d} \to \Gamma_X$ given by the formula

$$p(\mathbf{x}) = \{x_k + \alpha(k)\}_{k \in \mathbb{Z}^d},$$

where $\alpha(k) = |k|^{d-1}k$ and $\mathbb{Z}^d$ is the d-dimensional integer lattice. Our next goal is to construct a class of measures on Γ_X, using this map. For this we use a class of probability measures θ on $X^{\mathbb{Z}^d}$ which, (a) have "off-diagonal" support (b) are translation invariant with respect to the lattice shift $\mathbb{Z}^d$ and (c) have finite moments. The main example of such measures is given by Gibbs measures on $X^{\mathbb{Z}^d}$. We introduce the framework of Gelfand triple for $X^{\mathbb{Z}^d}$ and discuss main properties of Gibbs measures on $X^{\mathbb{Z}^d}$ (mainly following [AKR95]). Then we introduce the corresponding push-forward measures $\nu_\theta = p^*\theta$ on Γ_X. We prove the finiteness of their moments and that they are supported on Γ_X.

Chapter 3 addresses the *integration by parts* (IBP) formula, first for measures on $X^{\mathbb{Z}^d}$ and then for measures on Γ_X. We start with recalling the IBP formula for general probability measures in $X^{\mathbb{Z}^d}$ with examples of Gaussian and Gibbs measures. The first main result of this chapter is the extension of the IBP formula for a special class of vector fields $\widehat{v} : X^{\mathbb{Z}^d} \to X^{\mathbb{Z}^d}$, defined as

$$\widehat{v}_k(\mathbf{x}) \;=\; v(x_k + \alpha(k))_{k\in\mathbb{Z}^d},$$

where $v \in Vect_0(X)$. Then we prove the main result of this chapter that is the IBP formula for the push-forward measure ν_θ on Γ_X.

The main aim in Chapter 4 is to discuss the *Log-Sobolev inequality* for the push-forward ν_θ measures on Γ_X. We start with collecting some background material on Log-Sobolev inequality, giving examples and some known criteria for Log-Sobolev inequality. Then we state and prove the main result, that is the Log-Sobolev inequality for the push-forward measure ν_θ on Γ_X such that θ satisfies Log-Sobolev inequality on $X^{\mathbb{Z}^d}$.

In the last chapter we discuss an example of possible application of the constructed measure. We consider stochastic dynamics of a particle in a random environment, described by the measure ν_θ on Γ_X and discuss conditions for regularity of such dynamics. Then we give a "mathematical economics" interpretation of this construction. We discuss a toy model in which this moving particle represents a "traveling trade agent" and generates a stochastic volatility process in a stock market. For this model, we derive the "Black-Scholes type" pricing PDE.

Chapter 1

Preliminaries

1.1 Measures on Hilbert spaces

Cylinder Measures and Week Distributions

In this section we mainly follow [Sko74] to explain the construction of generalized measures on Hilbert spaces. Let H be a real separable Hilbert space with norm $|\cdot|$, scalar product $(\cdot, \cdot)$ and is equipped with Borel σ-algebra $\mathcal{B}(H)$. Let μ be a probability measure on $(H, \mathcal{B}(H))$. A standard way of describing μ is to first define it on a family of "elementary" sets (cylinder sets) and then extend it to the minimal σ-algebra containing these sets. Let us denote by $\mathcal{F}(H)$ the family of all finite dimensional subspaces of H. Let us consider $K \in \mathcal{F}(H)$ and let $\mathcal{B}(K)$ be the Borel σ-algebra on K.

Let $\mathcal{O}_K : H \to K$ be the orthogonal projection operator. For any set $A \in \mathcal{B}(K)$, the cylinder set with base A, is defined as

$$\mathcal{O}_K^{-1}(A) := \{x \in H : \mathcal{O}_K(x) \in A\} \tag{1.1.1}$$

Let $\mathcal{C}_K(H)$ be the collection of all cylinder sets with base in $\mathcal{B}(K)$. It is a σ-algebra on H. Let us denote the union of all σ-algebras $\mathcal{C}_K(H)$ by $\mathcal{C}(H)$ i.e.

$$\bigcup_K \mathcal{C}_K(H) = \mathcal{C}(H).$$

It can be shown that σ-closure of $\mathcal{C}(H)$ coincides with $\mathcal{B}(H)$ [Sko74, Ch.1].

Let μ be a probability measure on $(H, \mathcal{B}(H))$. For any $K \in \mathcal{F}(H)$, let us define the measure μ_K on $\mathcal{B}(K)$, by the formula

$$\mu_K(A) = \mu\big(\mathcal{O}_K^{-1}(A)\big).$$

The measure μ_K is called the projection of measure μ onto the subspace K. The collection of all such projections μ_K, $K \in \mathcal{F}(H)$, is called the system of finite dimensional distributions of measure μ.

Let us consider $K_1, K_2 \in \mathcal{F}(H)$ such that $K_1 \subset K_2$. Let $\mathcal{B}(K_1)$ and $\mathcal{B}(K_2)$ be the Borel σ-algebras defined on them and consider $A_1 \in \mathcal{B}(K_1)$ and

$$A_2 = \mathcal{O}_{K_1}^{-1}(A_1) \cap K_2 \in \mathcal{B}(K_2).$$

We define the consistency condition by the formula

$$\mu_{K_1}(A_1) = \mu_{K_2}(A_2). \tag{1.1.2}$$

A family of measures $\mu' = \{\mu_K\}$ where each μ_K is a probability measure on $\mathcal{B}(K)$, $K \in \mathcal{F}(H)$, is called a *weak distribution* if it satisfies consistency condition.

It is clear that the system of finite dimensional projections of any probability measure μ on H satisfies the consistency conditions (1.1.2) and is therefore a weak distribution. In general we know that not every weak distribution defines a measure on H, that is, coincides with the system of finite dimensional distributions of some probability measure μ on H. The condition that a weak distribution must satisfy in order to correspond a measure is given in the following lemma.

Lemma 1.1.1. *Let B_r be the ball centered at zero and radius r, in H. The weak distribution $\{\mu_K\}$ is generated by some measure μ on $(H, \mathcal{B}(H))$ if and only if for every $\varepsilon > 0$ there exists $b > 0$ such that for all $K \in \mathcal{F}(H)$,*

$$\mu_K(B_r \cap K) \geqslant 1 - \varepsilon, \quad r > b.$$

Definition 1.1.2. *A function $\phi : H \to \mathbb{R}$ is called cylinder if there exists a finite dimensional subspace $K \subset H$ such that ϕ is $\mathcal{C}_K(H)$ measurable. Every cylinder function has the form*

$$\phi(x) = \phi_K(\mathcal{O}_K(x)), \tag{1.1.3}$$

for some $K \in \mathcal{F}(H)$ and $\mathcal{B}(K)$ measurable function $\phi_K : K \to \mathbb{R}$.

Let $\mu' = \{\mu_K\}$ be a weak distribution. Then for an arbitrary non-negative cylinder function $\phi(x)$ we can define its integral with respect to the weak distribution μ' by the formula

$$\int_H \phi(x)\mu'(dx) = \int_K \phi_K(x)\mu_K(dx), \tag{1.1.4}$$

where ϕ_K is as in (1.1.3). Observe that the representation given in (1.1.3) is not unique. So it must be shown that the expression on right hand side of (1.1.4) does not depend on choice of K. Let $K_1, K_2 \in \mathcal{F}(H)$ such that $K_1 \subset K_2$ and

$$\phi(x) = \phi_{K_2}(\mathcal{O}_{K_2}(x)) = \phi_{K_1}(\mathcal{O}_{K_1}(x)), \quad x \in H.$$

Then for $x \in K_2$

$$\phi_{K_1}(\mathcal{O}_{K_1}(x)) = \phi_{K_2}(x).$$

Hence

$$\int\limits_{K_2} \phi_{K_2}(x)\mu_{K_2}(dx) = \int\limits_{K_1} \phi_{K_1}(x)\mu_{K_1}(dx),$$

because of the consistency condition (1.1.2). This implies that the right hand side of (1.1.4) is independent of choice of K so the integral on the left hand side is well defined. Lemma 1.1.1 can also be stated in the form of integrals.

Lemma 1.1.3. *The weak distribution* $\{\mu_K\}$ *is generated by some measure* μ *on* $(H, \mathcal{B}(H))$ *if and only if*

$$\lim_{\epsilon \downarrow 0} \int exp\{-\epsilon(x,x)\}\, \mu'(dx) = 1.$$

Let us give an example of a weak distribution which does not correspond to any measure on H.

Example 1.1.4. For any finite dimensional $K \subset H$, define a measure μ_K by the formula

$$\mu_K(A) = \alpha_K \int\limits_A \exp\left(-\frac{1}{2}(x,x)\right) m_K(dx), \quad A \in \mathcal{B}(K), \tag{1.1.5}$$

where $m_K(dx)$ is the Lebesgue measure associated with a Euclidean structure on K, which is generated by the Hilbert structure of H. Observe that $m_K(dx)$ does not depend on the particular choice of orthonormal basis in K. Let us set $\alpha_K = (2\pi)^{-n}$, where $2n$ is the dimension of K.

Let us show that the collection $\mu' = \{\mu_K\}$ is a weak distribution. Let K_1 and K_2 be two finite dimensional subspaces of H such that $K_1 \subset K_2$ and let K' is orthogonal to K_1 and $K_2 = K_1 + K'$. Then for $A_1 \in \mathcal{B}(K_1)$ and $A_2 = \mathcal{O}_{K_1}^{-1}(A_1) \cap K_2 \in \mathcal{B}(K_2)$

we can write

$$\mu_{K_2}(A_2) = \alpha_{K_2} \int\limits_{A_2} \exp\left(-\frac{1}{2}(x,x)\right) m_{K_2}(dx)$$

$$= \alpha_{K_1}\alpha_K \int\limits_{A_1} \exp\left(-\frac{1}{2}(x_1,x_1)\right) m_{K_1}(dx) \int\limits_{K} \exp\left(-\frac{1}{2}(x,x)\right) m_{K_2}(dx).$$

Using the fact that

$$\alpha_K \int\limits_{K} \exp\left(-\frac{1}{2}(x,x)\right) m_K(dx) = 1 \quad \text{for any} \quad K,$$

we get

$$\mu_{K_2}(A_2) = \mu_{K_1}(A_1).$$

It proves that the consistency condition is satisfied. So that the family $\{\mu_K\}$ forms a weak distribution on H. Next step is to prove that (from Lemma 1.1.3)

$$\lim_{\epsilon \downarrow 0} \int \phi_\epsilon(x)\, \mu'(dx) \neq 1,$$

where $\phi_\epsilon(x) = exp\{-\epsilon(x,x)\}$ and μ' is the finite dimensional distributions under consideration. We start with the following function. For an arbitrary $\epsilon > 0$, and $K \in \mathcal{F}(H)$ let us consider the cylinder function

$$\phi_{K,\epsilon}(x) := \exp\Big(-\epsilon(\mathcal{O}_K(x), \mathcal{O}_K(x))\Big).$$

For this function we have

$$\int \phi_{K,\epsilon}(x)\mu'(dx) = \alpha_K \int \exp\left(-\frac{1}{2}(1+2\epsilon)(x,x)\right) m_K(dx) = (1+2\epsilon)^{-n/2},$$

where $2n$ is the dimension of K. Let us consider an increasing sequence of sets $\{K_n\}_{n\in\mathbb{N}} \subset \mathcal{F}(H)$ such that $\bigcup\limits_n K_n$ is dense in H. Now we approximate the integral of $\phi_\epsilon(x)$ with the help of the sequence of functions $\phi_{K_n,\epsilon}(x)$ given by the formula

$$\phi_{K_n,\epsilon}(x) = \exp\big(-\epsilon(\mathcal{O}_{K_n}(x), \mathcal{O}_{K_n}(x))\big),$$

where $\mathcal{O}_{K_n}$ is projection operator onto K_n. Then we have

$$\int \phi_{K_n,\epsilon}(x)\mu'(dx) = (1+2\epsilon)^{-\frac{n}{2}}.$$

Observe that for an arbitrary sequence K_n we have $\phi_{K_n,\epsilon}(x) \downarrow \phi_\epsilon(x)$ as $n \to \infty$. So we obtain,

$$\begin{aligned}\lim_{\epsilon\to 0}\int \phi_\epsilon(x)\mu'(dx) &= \lim_{\epsilon\to 0}\left\{\lim_{n\to\infty}\int \phi_{K_n,\epsilon}(x)\mu'(dx)\right\} \\ &= \lim_{\epsilon\to 0}\left\{\lim_{n\to\infty}(1+2\epsilon)^{-\frac{n}{2}}\right\} = 0,\end{aligned} \tag{1.1.6}$$

Therefore finite dimensional distributions μ_K defined by (1.1.5) do not correspond a measure.

Similar to the case of measures on finite dimensional spaces, any probability measure μ on H can be defined by its *characteristic functional*.

Definition 1.1.5. ***Characteristic functional***
Let μ be a probability measure on $(H, \mathcal{B}(H))$. Consider a function $\phi_z : H \to \mathbb{C}$ given by the formula $\phi_z(x) := \exp\{\imath(z,x)\}$, $z \in H$. The characteristic functional ψ of the measure μ is defined as

$$\psi(z) = \int \phi_z(x)\,\mu(dx)\,, \quad z \in H. \tag{1.1.7}$$

Observe that ϕ_z is bounded and $\mathcal{B}(H)$-measurable, so $\psi(z) < \infty$ for all $z \in H$.

The characteristic functional has the following properties:

1. $\psi(0) = 1$.
2. $\psi : H \to \mathbb{C}$ is continuous.
3. ψ is positive definite, in the sense that, for any $N \in \mathbb{N}$ an arbitrary set $z_1, \cdots, z_N \in H$

   $$\sum_{i,j=1}^{N} \psi(z_i - z_j)\alpha_i\bar{\alpha}_j \geqslant 0$$

   for all $\alpha_1, \cdots, \alpha_N \in \mathbb{C}$.

In the infinite dimensional case, the properties $(1)-(3)$ do not guarantee that ψ is a characteristic functional of some measure on H. The Minlos-Sazonov theorem gives necessary and sufficient conditions. To state this theorem we need the notion of trace-class operators.

Definition 1.1.6. ***Trace-class operator***
A symmetric operator $\mathbb{T}: H \to H$ is called a trace-class operator if for any orthonormal basis $\{g_i\}_{i\in\mathbb{N}}$ of H, it satisfies the condition $Tr(\mathbb{T}) := \sum_i (\mathbb{T}g_i, g_i) < \infty$, where the series converges absolutely. The trace $Tr(\mathbb{T})$ is independent of the choice of the orthonormal basis.

Theorem 1.1.7. *[Minlos-Sazonov Theorem] A complex valued function $\psi(z)$, defined on H, is the characteristic functional of a normalized measure on $(H, \mathcal{B}(H))$ if and only if it satisfies conditions $(1)-(3)$, and for any $\varepsilon > 0$ there exists a symmetric, positive and trace-class operator $\mathbb{T}_\varepsilon$ such that $Re(\psi(0) - \psi(z)) < \epsilon$ when $(\mathbb{T}_\varepsilon z, z) < 1$.*

Gaussian Measures on Hilbert Spaces

A very important class of measures on Hilbert spaces is given by Gaussian measures. There are variety of ways in literature to define Gaussian measure. We give the following definition.

Definition 1.1.8. *A measure $\eta_{\alpha,\mathbb{A}}$ on $(H, \mathcal{B}(H))$ is called Gaussian measure with mean vector α and covariance operator $\mathbb{A}$ if its characteristics functional has the form*

$$\eta_{\alpha,\mathbb{A}}(\omega) = e^{\imath\langle\alpha,\omega\rangle - \frac{1}{2}\langle\mathbb{A}\omega,\omega\rangle}, \quad \omega \in H. \tag{1.1.8}$$

Here $\alpha \in H$ and $\mathbb{A}$ is a symmetric, bounded and non-negative operator in H.

Theorem 1.1.9. *(See e.g. [DF91]) The weak distributions defined by the* (1.1.8) *is σ-additive on H (that is, it defines a measure on H) if and only if, the operator $\mathbb{A}$ is of trace-class.*

Observe that Example 1.1.4 is concerned with a Gaussian measure with correlation operator $\mathbb{A} = Id$. Thus it is not concentrated on H. In fact, for general $\mathbb{A}$ it is always possible to construct a bigger space (superset of H) such that, the measure corresponding to the finite dimensional distributions μ_K, is concentrated on it. This will be discussed in next section.

Generalized Measures in Hilbert Spaces

If a weak distribution on H satisfies conditions of Minlos-Sazonov theorem then it corresponds some measure on $(H, \mathcal{B}(H))$. Lemma 1.1.3 and Example 1.1.4 show that this is not always the case. If such measure fails to exist then the weak distribution is generated by a so-called generalized measure constructed on some extension of H. The theory of rigged Hilbert spaces, developed by I. Gelfand [GSS64], is used to construct suitable extensions of H. The measures generated by weak distributions in H, concentrate on these extensions. These are called *generalized measures* on H. Let H_0 be a Hilbert space and let $\langle \cdot, \cdot \rangle_0$ be the inner product and $\| \cdot \|_0$ be the norm defined on it. We use the same subscript for elements of that space, for example, x_0, y_0 denote elements of H_0. Let $\mathbb{A} : H_0 \to H_0$ be a bounded, linear, symmetric, positive operator. Let us define another inner product on H_0 by the formula

$$\langle x_0, y_0 \rangle_- = \langle \mathbb{A}x_0, y_0 \rangle_0 \tag{1.1.9}$$

and the norm $\| \cdot \|_-$ defined by the formula

$$\| x_0 \|_-^2 = \langle \mathbb{A}x_0, x_0 \rangle_0.$$

Let us denote the completion of H_0 in this norm by $H_-^{\mathbb{A}}$. To keep our notations simple, we ignore $\mathbb{A}$ in $H_-^{\mathbb{A}}$ for now and denote it as H_-. We will use $H_-^{\mathbb{A}}$ where necessary. By construction H_0 is everywhere dense in H_- that is $H_0 \subset H_-$. The positivity of the operator $\mathbb{A}$ implies that, there exists $\mathbb{A}^{1/2}$ such that $\mathbb{A} = \mathbb{A}^{1/2}\mathbb{A}^{1/2}$. We will use the notation $\mathbb{A}^{-1/2} := (\mathbb{A}^{1/2})^{-1}$. Let us denote the domain of the operator $\mathbb{A}^{-1/2}$ by H_+. The continuity and positivity of $\mathbb{A}^{1/2}$ implies that H_+ is dense in H_0. It can be equipped with the scalar product

$$\langle\, x_+, y_+ \,\rangle_+ = \langle\, \mathbb{A}^{-1} x_+, y_+ \,\rangle_0.$$

The operator $\mathbb{A}$ can be extended to H_- using the continuity argument. Therefore we have the relations

$$\mathbb{A}^{1/2} H_- = H_0\,, \quad \mathbb{A}^{1/2} H_0 = H_+\,, \quad \mathbb{A} H_- = H_+ \tag{1.1.10}$$

Thus the space H_- can be identified with the dual of H_+ in the inner product of H_0, see e.g. [BK95]. It gives the triple $H_+ \subset H_0 \subset H_-$, referred as *Rigged Hilbert space* or *Gelfand triple.*

Lemma 1.1.10. *If a symmetric, positive operator $\mathbb{A}$ is trace-class in H_0 then it can be extended to a symmetric, positive and trace-class operator in H_-.*

Proof. By definition of scalar product in H_- we have

$$\langle \mathbb{A} x_-, y_- \rangle_- = \langle \mathbb{A} x_-, \mathbb{A} y_- \rangle_0.$$

Thus, we can write

$$\langle \mathbb{A} x_-, y_- \rangle_- = \langle \mathbb{A} x_-, \mathbb{A} y_- \rangle_0 = \langle x_-, \mathbb{A} y_- \rangle_-$$

and

$$\langle \mathbb{A} x_-, x_- \rangle_- = \langle \mathbb{A} x_-, \mathbb{A} x_- \rangle_0 \geqslant 0.$$

It proves that $\mathbb{A}$ is symmetric and positive in H_{-}. Now we prove that $\mathbb{A}$ is trace-class in H_{-}. Let us consider that $\mathbb{A}$ admits a system of eigenvectors λ_i that forms an orthonormal basis $\{e_i\}$ in H_0, that is

$$\lambda_i = \langle \mathbb{A} e_i, e_i \rangle, \quad \text{where} \quad \langle e_i, e_i \rangle = 1.$$

We will use the notation $Tr(\mathbb{A})_0$ for the trace of $\mathbb{A}$ in H_0. We have assumed that $\mathbb{A}$ is trace-class in H_0 so we have

$$Tr(\mathbb{A})_0 = \sum_{i=1}^{\infty} \lambda_i < \infty.$$

Let us set

$$k_i = \frac{e_i}{\sqrt{\lambda_i}}.$$

The family $\{k_i\}$ forms an orthonormal basis in H_{-}. The trace of $\mathbb{A}$ in H_{-} is given by

$$\begin{aligned} Tr(\mathbb{A})_{-} &= \sum_{i=1}^{\infty} \langle \mathbb{A}\, k_i \,,\, k_i \rangle_{-} = \sum_{i=1}^{\infty} \langle \mathbb{A}\, k_i \,,\, \mathbb{A}\, k_i \rangle_0 \\ &= \sum_{i=1}^{\infty} \langle \mathbb{A}^2\, k_i \,,\, k_i \rangle_0 = \sum_{i=1}^{\infty} \frac{1}{\lambda_i} \langle \mathbb{A}^2\, e_i \,,\, e_i \rangle_0 \\ &= \sum_{i=1}^{\infty} \lambda_i = Tr(\mathbb{A})_0 < \infty. \end{aligned}$$

■

Let μ be a probability measure on H_{-} and consider that its characteristic functional is given by the formula

$$\psi_{-}(x_{-}) = \int_{H_{-}} e^{\imath \langle x_{-}, y_{-} \rangle_{-}} \mu(dx_{-}).$$

Define a functional ψ on H_{+} by the formula

$$\psi(x_{+}) = \int_{H_{-}} e^{\imath \langle x_{+}, x_{-} \rangle_0} \mu(dx_{-}).$$

From (1.1.10) we know that ψ_- can be expressed in terms of ψ. Indeed, we have

$$\psi_-(x_-) = \psi(\mathbb{A}x_-)$$

Thus ψ characterizes the measure μ. It turns out that any positive definite, continuous functional on H_0 defines a measure on certain extension of H_0. We have the following theorem.

Theorem 1.1.11. *Let ψ be a continuous, positive definite functional on H_0 such that $\psi(0) = 1$ and assume that operator $\mathbb{A}$ is of trace-class in H_0. Define $\psi_- : H_-^{\mathbb{A}} \to \mathbb{C}$ by the formula*

$$\psi_-(x_-) = \psi(\mathbb{A}x_-) \tag{1.1.11}$$

Then ψ_- is the characteristic functional of some measure on H_-.

Proof. We need to check that ψ_- satisfies following conditions:

1. $\psi_-(0) = 1$,
2. $\psi_- : H_-^{\mathbb{A}} \to \mathbb{C}$ is continuous,
3. ψ_- is positive definite, in the sense that, for an arbitrary set $x_-^1, \cdots, x_-^N \in H_-$ we have
$$\sum_{i,j=1}^{N} \psi(x_-^i - x_-^j)\alpha_i\bar{\alpha}_j \geqslant 0$$
for all $\alpha_1, \cdots, \alpha_N \in \mathbb{C}$,
4. For every $\epsilon > 0$ there exists a trace-class operator $\mathbb{A}_\epsilon$ in H_- such that $Re(\psi_-(0) - \psi_-(x_-)) \leqslant \epsilon$ when $\langle \mathbb{A}_\epsilon x_-, x_- \rangle \leqslant 1$.

The first condition is obviously satisfied because $\psi(0) = 1$.

We know that ψ is a continuous functional and $\mathbb{A}$ is a continuous operator so by virtue of the relation (1.1.11) ψ_- is also a continuous functional.

To prove positive definiteness let us proceed as follows. For an arbitrary set $x_+^1, \cdots, x_+^N \in H_+$, we know from positive definiteness of ψ that,

$$\sum_{i,j=1}^{N} \psi(x_+^i - x_+^j)\beta_i \bar{\beta}_j \geqslant 0.$$

Setting $x_+^k = \mathbb{A}\, x_-^k$, $k = 1, \cdots, N$, we can write

$$\sum_{i,j=1}^{N} \psi(\mathbb{A}(x_-^i - x_-^j))\beta_i \bar{\beta}_j \geqslant 0, \tag{1.1.12}$$

which implies that

$$\sum_{i,j=1}^{N} \psi_-(x_-^i - x_-^j)\beta_i \bar{\beta}_j \geqslant 0,$$

which completes the proof of third condition.

Now for the last condition let us start with continuity of $\psi(x_+)$, which implies that, for each $\epsilon > 0$ there exists a $\delta > 0$ such that,

$$Re(\psi(0) - \psi(x_+)) \leqslant \epsilon, \quad \text{when} \quad \langle x_+, x_+\rangle_0 < \delta.$$

We know that $\mathbb{A}x_- \in H_+$, therefore for $x_- \in H_-$ we have

$$Re(\psi(\mathbb{A}0) - \psi(\mathbb{A}x_-)) \leqslant \epsilon, \quad \text{when} \quad \langle \mathbb{A}x_-, \mathbb{A}x_-\rangle_0 \leqslant \delta.$$

Or

$$Re(\psi_-(0) - \psi_-(x_-)) \leqslant \epsilon, \quad \text{when} \quad \langle \frac{1}{\delta}\mathbb{A}x_-, x_-\rangle_0 \leqslant 1.$$

Let us denote $\mathbb{A}_\epsilon = \frac{1}{\delta}\mathbb{A}$. From Lemma 1.1.10 we know that $\mathbb{A}$ is a trace-class operator in H_- which implies that $\mathbb{A}_\epsilon$ is a trace-class operators in H_-. It completes the proof. ■

The measure corresponding to the characteristic functional $\psi_-(x_-)$ is called the generalized measure on H_0.

1.2 Configuration Spaces

Definitions, Main Notations and Structures

In this section, we collect some known facts about configuration spaces of Euclidean spaces, following [AKR98a, AKR98b, BD09].

Finite Configuration Spaces

Let $X = \mathbb{R}^d$ be a d-dimensional Euclidean space. Let $\mathcal{B}(X)$ denote the collection of Borel sets in X and $\mathcal{B}_b(X)$ the collection of bounded sets in $\mathcal{B}(X)$. Let $X^n = X \times X \times \cdots \times X$ be the Cartesian product of n copies of X with the corresponding Borel σ-algebra $\mathcal{B}(X^n)$ defined on it. We define, for each $n \in \mathbb{Z}_+$, the space $X^{(n)}$ of n-point configurations (n-point subsets) in X, that is,

$$X^{(n)} = \{\xi \subset X, \mid \xi \mid = n\}, \quad n \in \mathbb{Z}_+ , \tag{1.2.1}$$

where $\mid A \mid$ denotes cardinality of the set A. Similarly, for each $\Lambda \in \mathcal{B}_b(X)$ we can define $X_\Lambda^{(n)}$ by the formula

$$X_\Lambda^{(n)} = \{\eta \subset \Lambda, \mid \eta \mid = n\}, \quad n \in \mathbb{Z}_+.$$

We also define the space

$$\widetilde{X^n} = \{(x_1, x_2, \cdots, x_n) \in X^n; \quad x_i \neq x_j, \quad \forall\, i \neq j\}. \tag{1.2.2}$$

Let us consider the map $\mathbf{p} : \widetilde{X^n} \longrightarrow X^{(n)}$ defined by the formula

$$\mathbf{p}\,(x_1, x_2, \cdots, x_n) = \{x_1, x_2, \cdots, x_n\}. \tag{1.2.3}$$

Observe that the space $X^{(n)}$ can be identified with the quotient space $\widetilde{X^n}/S_n$ where S_n is the symmetric group acting on $\widetilde{X^n}$ by permutations of the coordinates. Under this identification, the map $\mathbf{p}$ coincides with the canonical projection

$$\widetilde{X^n} \to \widetilde{X^n}/S_n.$$

We equip $X^{(n)}$ with the topology induced by the map $\mathbf{p}$. The corresponding σ-algebra $\mathcal{B}(X^{(n)})$ coincides with the σ-algebra generated by the mappings $\mathfrak{q}_\Lambda : X^{(n)} \to \mathbb{Z}_+$ defined for every $\Lambda \in \mathcal{B}_b(X)$ by the formula

$$\mathfrak{q}_\Lambda(\xi) = |\, \xi \cap \Lambda \,|. \tag{1.2.4}$$

Next we introduce the space of finite configurations X_0 as the union

$$X_0 = \bigcup_{n \in \mathbb{Z}_+} X^{(n)}. \tag{1.2.5}$$

It is equipped with the topology of disjoint union of topologies and the corresponding Borel σ-algebra is denoted by $\mathcal{B}(X_0)$.

Infinite Configuration Spaces

A configuration space over X, denoted by Γ_X, is defined as the set of all locally finite subsets (configurations) in X:

$$\Gamma_X := \{\gamma \subset X, |\gamma \cap K| < \infty \quad \text{for each } K \in \mathcal{B}_b(X)\}.$$

Here $|A|$ denotes the cardinality of set A. We can identify each $\gamma \in \Gamma_X$ with a Radon measure

$$\gamma = \sum_{x \in \gamma} \delta_x,$$

on X. Here δ_x denotes the Dirac measure at x. Thus, Γ_X becomes a subset of the set $\mathcal{M}_0(X)$ of all Radon measures on X. Recall that $\mathcal{M}_0(X)$ has a standard topology called the vague topology. It induces the relative topology $\mathcal{O}(\Gamma_X)$ on Γ_X that is the minimal topology with respect to which each mapping of the form

$$\Gamma_X \ni \gamma \longmapsto \langle f, \gamma \rangle := \sum_{x\in\gamma} f(x) \in \mathbb{R}, \qquad f \in C_0(X), \tag{1.2.6}$$

is continuous. Here $C_0(X)$ denotes the set of all continuous functions on X with compact support. The topology $\mathcal{O}(\Gamma_X)$ is separable and completely metrizable, see e.g. [KMM78]. Let us denote the corresponding Borel σ-algebra by $\mathcal{B}(\Gamma_X)$. The σ-algebra $\mathcal{B}(\Gamma_X)$ coincides with the σ-algebra generated by the mappings of the form

$$\mathfrak{q}_\Lambda : \Gamma_X \to \mathbb{Z}_+ \quad \text{such that} \quad \mathfrak{q}_\Lambda(\gamma) := |\, \gamma \cap \Lambda \,|,$$

that is

$$\mathcal{B}(\Gamma_X) = \sigma(\mathfrak{q}_\Lambda : \Lambda \in \mathcal{B}_b(X)).$$

For every $\Lambda \in \mathcal{B}_b(X)$ let us define the configuration space Γ_Λ by the formula

$$\Gamma_\Lambda = \{\gamma \in \Gamma_X : \gamma \subset \Lambda\}.$$

It is obvious that,

$$\Gamma_\Lambda = \bigcup_{n\in\mathbb{Z}_+} X_\Lambda^{(n)}.$$

It can be shown [Oba87] that the restriction mappings $\mathfrak{h}_\Lambda : \Gamma_X \to \Gamma_\Lambda$ defined by

$$\mathfrak{h}_\Lambda(\gamma) = \gamma \cap \Lambda\,, \quad \Lambda \in \mathcal{B}_b(X),$$

are $\mathcal{B}(\Gamma_X)/\mathcal{B}(\Gamma_\Lambda)$-measurable. Let ν be a positive measure on $\mathcal{B}(\Gamma_X)$. Note that the family of measures $\{\nu^\Lambda\}$ defined by

$$\nu^\Lambda := \nu \circ \mathfrak{h}_\Lambda^{-1}\,, \quad \Lambda \in \mathcal{B}_b(X),$$

is consistent, that is, for all $\Lambda_1, \Lambda_2 \in \mathcal{B}_b(X)$ such that $\Lambda_1 \subset \Lambda_2$

$$\mathfrak{h}^*_{\Lambda_2,\Lambda_1}\left(\nu^{\Lambda_2}\right) = \nu^{\Lambda_2} \circ \mathfrak{h}^{-1}_{\Lambda_2,\Lambda_1} = \nu^{\Lambda_1}.$$

Conversely, by a version of Kolmogorov's extension theorem for projective limit spaces, the consistent family of measures $\{\nu^\Lambda\}$ defines a unique measure ν' on Γ_X such that

$$\nu^\Lambda = \mathfrak{h}_\Lambda^* \, \nu' \,, \quad \Lambda \in \mathcal{B}_b(X).$$

see e.g. [Par67].

Poisson Measures

Let μ be a non-atomic Radon measure on the measure space $(X, \mathcal{B}(X))$, that is, for all $\Lambda \in \mathcal{B}_b(X)$ we have $\mu(\Lambda) < \infty$. Let $\widehat{\mu} = \mu \otimes \mu \otimes \cdots \otimes \mu$ be the product measure on X^n defined by the formula

$$\widehat{\mu}_n(A) = \prod_{i=1}^{n} \mu(A_i); \quad A = A_1 \times A_2 \times \cdots \times A_n, \qquad \forall \, A_i \in \mathcal{B}(X). \tag{1.2.7}$$

The product measure $\widehat{\mu}$ defines a finite measure on $\widetilde{X^n}$ and we can define its image measure ν_n on $X^{(n)}$ under the map $\mathbf{p}_n : \widetilde{X^n} \to X^n$ by the formula

$$\nu_n(A) = \mathbf{p}_n^* \, \widehat{\mu}_n(A) = \widehat{\mu}_n \, \left(\mathbf{p}_n^{-1}(A)\right), \quad \forall \, A \subset X^{(n)}. \tag{1.2.8}$$

That is, ν_n is the push-forward measure of $\widehat{\mu}$ under the map $\mathbf{p}_n$.

The *Lebesgue-Poisson Measure* Π_μ on Γ_Λ with the intensity measure μ is defined by the formula

$$\Pi_\mu := \sum_{n=0}^{\infty} \frac{1}{n!} \, \nu_n.$$

The measure Π_μ is a finite measure on $\mathcal{B}(\Gamma_\Lambda)$ and for all $\Lambda \in \mathcal{B}_b(X)$ we have

$$\Pi_\mu(\Gamma_\Lambda) = e^{\mu(\Lambda)}.$$

Hence, we can define the probability measure $\pi_{\mu,\Lambda}$ on $\mathcal{B}(\Gamma_\Lambda)$ by the formula

$$\pi_{\mu,\Lambda} := e^{-\mu(\Lambda)} \Pi_\mu. \tag{1.2.9}$$

For sets $B_1, \cdots, B_k \in \mathcal{B}(X)$ and $n_1, \cdots, n_k \in \mathbb{Z}_+$, let us define the cylinder sets $C^{n_1,\cdots,n_k}_{B_1,\cdots,B_k}$ by the formula

$$C^{n_1,\cdots,n_k}_{B_1,\cdots,B_k} = \{\gamma \in X_0 \; : \; |\,\gamma \cap B_i\,| = n_i \, , \quad i = 1, \cdots, k\}.$$

Note that, for pairwise disjoint sets $B_1, \cdots, B_k$ the measure $\pi_{\mu,\Lambda}$ satisfies the following property

$$\pi_{\mu,\Lambda}\left(C^{n_1,\cdots,n_k}_{B_1,\cdots,B_k}\right) = \prod_{i=1}^{M} \frac{\mu\,(B_i)^{n_i} \; e^{-\mu(B_i)}}{n_i\,!} \, .$$

It implies that for sets B_i the values $|\,\gamma \cap B_i\,|$ are mutually independent random variables with mean values $\mu(B_i)$ on the probability space $(\Gamma_\Lambda, \mathcal{B}(\Gamma_\Lambda), \pi_{\mu,\Lambda})$. Using definition of the measure $\Pi_{\mu,\Lambda}$ and expression for Laplace transform of $\pi_{\mu,\Lambda}$, the consistency property of the family $\{\pi_{\mu,\Lambda} \; : \; \Lambda \in \mathcal{B}_b(X)\}$ can be proved (See e.g. [Oba87]). Hence, by a version of Kolmogorov's extension theorem we can obtain a unique probability measure π_μ on $\mathcal{B}(\Gamma_X)$ such that

$$\pi_{\mu,\Lambda} = p^*_\Lambda \, \pi_\mu \; , \quad \Lambda \in \mathcal{B}_b(X). \tag{1.2.10}$$

The measure π_μ is called the Poisson measure on $\mathcal{B}(\Gamma_X)$ with intensity measure μ. We follow a standard procedure to compute Laplace transform of the measure π_μ. Let us consider the function $f \in C_0(X)$ and let $\gamma_\Lambda := \gamma \cap \Lambda$ for every $\Lambda \in \mathcal{B}_b(X)$. For $\Lambda = supp(f)$, we can write

$$\langle f, \gamma \rangle = \langle f, \gamma_\Lambda \rangle \, , \quad \gamma \in \Gamma_X,$$

and therefore

$$\int\limits_{\Gamma_X} e^{\langle f,\gamma \rangle} \pi_\mu(d\gamma) = \int\limits_{X_0} e^{\langle f,\gamma_\Lambda \rangle} \pi_{\mu,\Lambda}(d\gamma_\Lambda).$$

Using definition of $\pi_{\mu,\Lambda}$ we get

$$e^{-\mu(\Lambda)}\sum_{n=0}^{\infty}\frac{1}{n!}\int\limits_{X_0}\exp\left(\sum_{k=0}^{n}f(x_k)\right)\mu(dx_1)\cdots\mu(dx_n)$$

$$=e^{-\mu(\Lambda)}\sum_{n=0}^{\infty}\frac{1}{n!}\left(\int\limits_{\Lambda}e^{f(x)}\mu(dx)\right)^n$$

$$=\exp\left(\int\limits_{X}\left(e^{f(x)}-1\right)\mu(dx)\right).$$

Hence for all $f\in C_0(X)$, the Laplace transform of the measure π_μ can be written as

$$L_{\pi_\mu}(f):=\int\limits_{\Gamma_X}e^{\langle f,\gamma\rangle}\pi_\mu(d\gamma)=\exp\left(\int\limits_{X}\left(e^{f(x)}-1\right)\mu(dx)\right). \tag{1.2.11}$$

Differentiable Functions and Vector Fields

For each point $x\in X=\mathbb{R}^d$, let us denote the tangent space at that point by T_xX and the associated tangent bundle would be denoted by the space $T(X)=\bigcup_{x\in X}T_xX$. The gradient on X is denoted by ∇. Following [AKR98a], we define the tangent space of the configuration space Γ_X at $\gamma\in\Gamma_X$ as the Hilbert space

$$T_\gamma\Gamma_X:=L^2(X\to TX;d\gamma),$$

or equivalently

$$T_\gamma\Gamma_X=\bigoplus_{x\in\gamma}T_xX.$$

The scalar product in $T_\gamma\Gamma_X$ is denoted by $\langle\cdot,\cdot\rangle_\gamma$.

A vector field U over Γ_X is a mapping

$$\Gamma_X\ni\gamma\mapsto U(\gamma)=(U(\gamma)_x)_{x\in\gamma}\in T_\gamma\Gamma_X.$$

Thus, for vector fields U_1, U_2 over Γ_X we have

$$\langle U_1(\gamma), U_2(\gamma)\rangle_\gamma = \sum_{x\in\gamma} U_1(\gamma)_x \bullet U_2(\gamma)_x\,, \qquad \gamma \in \Gamma_X.$$

For $\gamma \in \Gamma_X$ and $x \in \gamma$, denote by $\mathcal{O}_{\gamma,x}$ an arbitrary open neighborhood of x in X such that $\mathcal{O}_{\gamma,x} \cap \gamma = x$. For any measurable function $F : \Gamma_X \to \mathbb{R}$, define the function $F_x(\gamma, \centerdot) : \mathcal{O}_{\gamma,x} \to \mathbb{R}$ by

$$F_x(\gamma, y) := F((\gamma \setminus x) \cup y),$$

and set

$$\nabla_x F(\gamma) := \nabla F_x(\gamma, y)|_{y=x}\,, \qquad x \in X,$$

provided $F_x(\gamma, \centerdot)$ is differentiable at x. In what follows we will use the following notations; $C_0^\infty(X)$ for the set of all C^∞-functions on X with compact support, $C_b^\infty(X)$ for the set of all C^∞-functions in X with bounded derivatives and $Vect_0(X)$ for the space of compactly supported smooth vector fields on X.

Definition 1.2.1. *Denote by $\mathcal{F}C(\Gamma_X)$ the class of functions on Γ_X of the form*

$$F(\gamma) = f\big(\langle\phi_1, \gamma\rangle, \cdots, \langle\phi_k, \gamma\rangle\big)\,, \qquad \gamma \in \Gamma_X, \tag{1.2.12}$$

where $k \in \mathbb{N}$, $f \in C_b^\infty(\mathbb{R}^k)$, and $\phi_1, \cdots, \phi_k \in C_0^\infty(X)$.

Each $F \in \mathcal{F}C(\Gamma_X)$ is local, that is, there is a compact set $K \subset X$, which depends on F such that $F(\gamma) = F(\gamma_K)$ for all $\gamma \in \Gamma_X$. Thus, for a fixed γ there are only finitely many non-zero derivatives $\nabla_x F(\gamma)$.

For a function $F \in \mathcal{F}C(\Gamma_X)$, its Γ-gradient $\nabla^\Gamma F$ is defined as follows:

$$\nabla^\Gamma F(\gamma) := (\nabla_x F(\gamma))_{x\in\gamma} \in T_\gamma\Gamma_X, \qquad \gamma \in \Gamma_X. \tag{1.2.13}$$

The directional derivative of F along a vector field U is given by

$$\nabla_U^\Gamma F(\gamma) := \langle\nabla^\Gamma F(\gamma), U(\gamma)\rangle_\gamma = \sum_{x\in\gamma} \nabla_x F(\gamma) \bullet U(\gamma)_x\,, \quad \gamma \in \Gamma_X \tag{1.2.14}$$

Note that the sum on the right-hand side contains only finitely many non-zero terms. Further let $\mathcal{FU}(\Gamma_X)$ be the class of cylindrical vector fields U on Γ_X of the form

$$U(\gamma)_x = \sum_{i=1}^{k} A_i(\gamma)u_i(x) \in T_xX\,, \qquad x \in X, \tag{1.2.15}$$

where $A_i \in \mathcal{F}C(\Gamma_X)$ and $u_i \in Vect_0(X)$, and $i = 1, \cdots, k$ ($k \in \mathbb{N}$).
Any vector field $u \in Vect_0(X)$ generates a constant vector field U on Γ_X defined by $V(\gamma)_x := u(x)$. We shall preserve the notation u for it. Thus,

$$\nabla_u^\Gamma F(\gamma) = \sum_{x\in\gamma} \nabla_x F(\gamma) \cdot u(x)\,, \qquad \gamma \in \Gamma_X \tag{1.2.16}$$

Integration by Parts Formula

Now we give the integration by parts formula for the Poisson measure π_μ on Γ_X. Let us start with the notion of logarithmic derivatives for a measure on X. Let us consider a measure μ on X which is absolutely continuous with respect to the Lebesgue measure and has density $\rho > 0$. The logarithmic derivative is given by

$$X \ni x \mapsto \beta^\mu(x) := \frac{\nabla^X \rho(x)}{\rho(x)} \in T_xX. \tag{1.2.17}$$

Here ∇^X is the gradient on X. For all $v \in Vect_0(X)$ and $f \in C_0^\infty(X)$ we define the directional derivative $\nabla_v^X f(x)$ of $f(x)$ by the formula

$$(\nabla_v^X f_1)(x) := \langle \nabla^X f_1(x), v(x)\rangle_{T_xX}.$$

Therefore, using (1.2.17), for $f_1, f_2 \in C_0^\infty(X)$ we can write

$$\int_X (\nabla_v^X f_1(x) f_2(x)\mu(dx) = -\int_X f_1(x)(\nabla_v^X f_2)(x)\mu(dx) - \int_X f_1(x)f_2(x)\beta_v^\mu(x)\mu(dx),$$

where β_v^μ is called the *vector logarithmic derivative* of the measure μ along the vector v and is given by

$$\beta_v^\mu(x) := \langle \beta^\mu(x), v(x)\rangle_{T_xX} + div^X v(x), \tag{1.2.18}$$

where div^X is the divergence on X. The following result is proved in [AKR98a]

Theorem 1.2.2. *For the Poisson measure π_μ, for all $f, g \in \mathcal{F}C(\Gamma_X)$ and for any $v \in Vect_0(X)$, the following integration by parts formula holds:*

$$\int_{\Gamma_X} \left(\nabla_v^\Gamma f\right)(\gamma)\, g(\gamma)\, \pi_\mu(d\gamma) \tag{1.2.19}$$

$$= -\int_{\Gamma_X} f(\gamma)\, \nabla_v^\Gamma g(\gamma)\, \pi_\mu(d\gamma) - \int_{\Gamma_X} f(\gamma)\, g(\gamma)\, \beta_v^{\pi_\mu}(\gamma)\, \pi_\mu(d\gamma). \tag{1.2.20}$$

where $\beta_v^{\pi_\mu}$ is called the vector logarithmic derivative *of π_μ and is given by the formula*

$$\beta_v^{\pi_\mu}(\gamma) := \langle \beta_v^\mu, \gamma \rangle = \int_X \left[\langle \beta^\mu(x), v(x) \rangle_{T_x X} + div^X v(x)\right] \gamma(dx), \tag{1.2.21}$$

where β_μ is as given in (1.2.17).

Chapter 2

A class of Measures on Configuration Spaces

2.1 Push-Forward Construction of Measures on Configuration Spaces

2.1.1 Measures on Finite Configuration Spaces

Consider the n-point configuration space $X^{(n)}$. We can use the projection map $\mathbf{p} : \widetilde{X^n} \longrightarrow X^{(n)}$ (cf. (1.2.3)) in order to construct a probability measure on $X^{(n)}$.

Indeed for any probability measure θ on $\widetilde{X^n}$, we can define the push-forward measure ν on $X^{(n)}$ by the formula

$$\nu(A) = \mathbf{p}^* \theta(A) = \theta(\mathbf{p}^{-1}(A)), \quad A \in \mathcal{B}(X^{(n)}). \tag{2.1.1}$$

The measure ν is a probability measure on $X^{(n)}$ because θ is probability measure and the map $\mathbf{p}$ is measurable. A simple example of ν can be constructed as follows: Let μ be a probability measure on X which is absolutely continuous with respect to the Lebesgue measure m on $\mathbb{R}$,

$$\mu(dx) = m(x)dx,$$

where the density $m(x)$ is continuous. Let us set

$$\begin{aligned}\theta(dx_1, \cdots, dx_n) \equiv \widehat{\mu}(dx) &:= \mathop{\times}_{i=1}^{n} \mu(dx_i) \\ &= m(x_1) \cdots m(x_n) dx_1 \cdots dx_n. \end{aligned} \tag{2.1.2}$$

$\widehat{\mu}$ has continuous density with respect to the Lebesgue measure and thus $\widehat{\mu}\,(\widetilde{X^n}) = 1$. Thus ν is a probability measure on $X^{(n)}$. Let us consider a random configuration $\gamma \in X^{(n)}$ distributed according to ν.

Lemma 2.1.1. *Let $B \in \mathcal{B}(X)$. The average number of points of γ in B is given by the formula*

$$\boldsymbol{E}_\nu\left(\#(\gamma \cap B)\right) = n\mu(B), \tag{2.1.3}$$

where $\boldsymbol{E}_\nu$ is expectation with respect to measure ν.

Proof. We first prove that, for any $f \in C_0(X)$ the following equality holds:

$$\int_{X^{(n)}} \langle f, \gamma \rangle \nu(d\gamma) = n \int_{\mathbb{R}^d} f(x) \mu(dx). \tag{2.1.4}$$

Using the definition of the push-forward measure ν we can write the left hand side of (2.1.4) as:

$$\int_{X^{(n)}} \langle f, \gamma \rangle \nu(d\gamma) = \int_{X^n} \langle f, p(\mathbf{x}) \rangle \, \widehat{\mu}(d\,\mathbf{x}).$$

Further using the definition of the product measure $\widehat{\mu}$ (formula (1.2.7)) we obtain:

$$\int_{X^n} \langle f, p(\mathbf{x}) \rangle \, \widehat{\mu}(d\,\mathbf{x}) = \int_{X^n} \left[\sum_{k\,=\,1}^{n} \; f(x_k) \right] \mu(dx_1)\mu(dx_2)\cdots\mu(dx_n)$$

$$= n \int_{\mathbb{R}^d} \; f(x)\mu(dx).$$

Observe that,

$$\boldsymbol{E}_\nu(\#(\gamma \cap B)) = \int_{X^{(n)}} \langle \mathbf{1}_B, \; \gamma \rangle \, \nu(d\gamma),$$

where $\mathbf{1}_B$ is the indicator function of set B and $\boldsymbol{E}_\nu$ represents the expectation w.r.t. the measure ν. Formula (2.1.4) implies that

$$\boldsymbol{E}_\nu(\#(\gamma \cap B)) = n \int_B \mu(dx) = n\mu(B).$$

■

Remark 2.1.2. *Observe that* $\lim_{n \,\to\, \infty} \boldsymbol{E}_\nu(\#(\gamma \cap B)) = \infty$

Let $\Gamma_X^\natural$ be the space of all countable subsets (configurations) in X with accumulation and multiple points, and let $\ddot{\Gamma}_X$ be the space of configurations in X without accumulation but with multiple points. That is, $\ddot{\Gamma}_X$ is the set of all $\mathbb{Z}_+$-valued Radon measures on X.

We can try to directly extend the construction above to the infinite setting in the following way. Consider the infinite product space

$$X^\infty = \mathop{\times}_{k=1}^{\infty} X_k \,, \quad X_k = X \,,$$

and define a map by the formula

$$\mathfrak{p}\left((x_1, \cdots, x_n, \cdots)\right) = \{x_1, \cdots, x_n, \cdots\}.$$

An attempt to use the map $\mathfrak{p}$ in order to define a measure on Γ_X meets several problems first of which is the configuration so formed can have multiple and accumulation points and thus does not belong to Γ_X in general. The way to overcome this difficulty is to use the approach suggested in [VGG75]. That is, we can consider a measurable subset A of X^∞ such that the image of A under $\mathfrak{p}$ is the space $\ddot{\Gamma}_X$ and then prove that (a) the set A has a full measure and (b) the map $p : A \to \ddot{\Gamma}_X$ is a measurable map.
Moreover, we cannot expect this solution to work directly because of the Remark 2.1.2. Indeed, consider the product measure

$$\widehat{\mu}_\infty = \mathop{\times}_{k=1}^{\infty} \mu_k, \quad \mu_k = \mu$$

on X^∞. Similar to the case of finite product,

$$\widehat{\mu}_\infty(\widehat{X}^\infty) = 1,$$

where

$$\widehat{X}^\infty = X^\infty \backslash Diag(X^\infty)$$

and

$$Diag(X^\infty) = \{\mathbf{x} \in X^\infty ; \; x_k = x_j, \quad \text{for some} \quad x_k, x_j \in \mathbb{Z}^d\}$$

Now we can define a measure ν on $\ddot{\Gamma}_X$ by the formula

$$\nu = \mathfrak{p}^* \widehat{\mu}_\infty$$

It is clear however that the average number of points of a random configuration γ distributed according to $\widehat{\mu}_\infty$ in any bounded set $B \subset X$ will be infinite. Indeed, similar to the proof of the Lemma 2.1.1,

$$\mathbb{E}\left(\#(\gamma \cap B)\right) = \sum_{k=1}^{\infty} \mu_k(B) = \lim_{n\to\infty} n\mu(B) = \infty.$$

Thus $\nu = \mathfrak{p}^*\widehat{\mu}_\infty$ is not concentrated on the space Γ_X of locally finite configurations. Therefore, we need to modify the map p.

2.1.2 Push-forward measures on Infinite Configuration Spaces

As we have seen in the previous section that the construction on measure ν, in the previous section, cannot be directly extended to infinite configuration spaces. In order to be able to do that, we need to modify the projection map p.

We can modify the construction above in the following way. Consider the infinite product space

$$X^{\mathbb{Z}^d} = \underset{k\,\in\,\mathbb{Z}^d}{\times} X_k,\; X_k = X$$

where $\mathbb{Z}^d$ is the d-dimensional integer lattice and the product is the Cartesian product of identical copies of X. The elements of $X^{\mathbb{Z}^d}$ will be denoted by $\mathbf{x} = (x_k)_{k\,\in\,\mathbb{Z}^d}\,, x_k \in X$, for any $k \in \mathbb{Z}^d$. For $k = (k_1, \cdots, k_d) \in \mathbb{Z}^d$ we define

$$|k| = \sum_{m=1}^{d} |k_m|.$$

Let θ be a probability measure on $X^{\mathbb{Z}^d}$ and $p:\, X^{\mathbb{Z}^d} \to \Gamma_X^\natural$ be a map defined in the following way:

$$p : \left(x_k\right)_{k\,\in\mathbb{Z}^d} \longmapsto \{x_k + \eta(|k|)\, k\}_{k\in\mathbb{Z}^d}\,, \tag{2.1.5}$$

where $\eta : \mathbb{N} \to \mathbb{N}$ is a function satisfying the estimate

$$\eta(m) \geqslant m^{\frac{n}{2}-1}\,, m \in \mathbb{N} \quad \text{and for some} \quad n \in \mathbb{N}. \tag{2.1.6}$$

We can define the *push-forward measure* $\nu_\theta = p^* \theta$ on $\Gamma_X^\natural$ be the formula

$$\nu_\theta(A) = \theta\left(p^{-1}(A)\right), \quad \forall \quad A \subset \Gamma_X. \tag{2.1.7}$$

The correct choice of constant r in the formula (2.1.6) will guarentee that the measure ν_θ is in fact concentrated on Γ_X. It is known that the series $\sum_{k \in \mathbb{Z}^d} (1+|k|^r)^{-2}$ converges for $r \geqslant d$. In what follows, we set $r = d$ just for simplicity. We set $\eta(m) = m^{d-1}$, so that the map p obtains the form

$$p(\mathbf{x}) = \{x_k + \alpha(k)\}_{k \in \mathbb{Z}^d}, \tag{2.1.8}$$

where

$$\alpha(k) = |k|^{d-1} k \tag{2.1.9}$$

In what follows we study properties of so defined measure ν_θ for some important classes of measures on $X^{\mathbb{Z}^d}$. We restrict ourselves to measures concentrated on certain Hilbert subspaces of $X^{\mathbb{Z}^d}$.

2.2 Translation-Invariant Measures on $X^{\mathbb{Z}^d}$ and their Properties

In this section, we introduce a class of measures on $X^{\mathbb{Z}^d}$, which will be used for construction of push-forward measures on Γ_X.

2.2.1 Gelfand Triple Associated with $X^{\mathbb{Z}^d}$

Let $X_0^{\mathbb{Z}^d}$ be the subspace of $X^{\mathbb{Z}^d}$ which consists of all finite sequences in $X^{\mathbb{Z}^d}$ and is equipped with the norm $\|\cdot\|_0$ generated by the inner product

$$(\mathbf{u}\,,\,\mathbf{v})_0 = \sum_{k \in \mathbb{Z}^d} u_k\, v_k, \qquad \mathbf{u} = (u_k)_{k \in \mathbb{Z}^d}\,,\, \mathbf{v} = (v_k)_{k \in \mathbb{Z}^d} \,\in\, X_0^{\mathbb{Z}^d}.$$

The completion of $X_0^{\mathbb{Z}^d}$ in the norm $\|\cdot\|_0$ is a real Hilbert space which will be denoted by $\mathcal{H}_0 = l_2(X)$. Thus we have

$$\mathcal{H}_0 = l_2(X) = \left\{ \mathbf{x} \in X^{\mathbb{Z}^d}, \, \mathbf{x} = (x_k)_{k\in\mathbb{Z}^d} \quad \text{s.t.} \quad \sum_{k\in\mathbb{Z}^d} |x_k|^2 < \infty \right\}.$$

We now have the following rigging of the Hilbert space $\mathcal{H}_0$:

$$X_0^{\mathbb{Z}^d} \subset \mathcal{H}_0 \subset X^{\mathbb{Z}^d},$$

where the duality pairing of $X_0^{\mathbb{Z}^d}$ and $X^{\mathbb{Z}^d}$ is given by the inner product in $\mathcal{H}_0$:

$$(\mathbf{u}\,,\,\mathbf{w})_0 = \sum_{k\,\in\,\mathbb{Z}^d} u_k\, w_k, \qquad \mathbf{u} \in X_0^{\mathbb{Z}^d}\,,\, \mathbf{w}\,\in\, X^{\mathbb{Z}^d}.$$

Let $(\mathbf{u}, \mathbf{v})_+$ be an inner product on $X_0^{\mathbb{Z}^d}$ defined by the formula

$$(\mathbf{u}, \mathbf{v})_+ := \sum_{k\in\mathbb{Z}^d} u_k\, v_k (1 + |k|^d)^2, \qquad \mathbf{u}, \mathbf{v} \in X_0^{\mathbb{Z}^d}.$$

Let us consider the Hilbert space $\mathcal{H}_+$, which is the completion of $X_0^{\mathbb{Z}^d}$ with respect to the norm $\|\cdot\|_+$ generated by this inner product.
Now let $(\mathbf{x}, \mathbf{y})_-$ be the inner product on $\mathcal{H}_0$ defined by the formula

$$(\mathbf{x}, \mathbf{y})_- = \sum_{k\in\mathbb{Z}^d} x_k\, y_k (1 + |k|^d)^{-2}, \qquad \mathbf{x}, \mathbf{y} \in \mathcal{H}_0.$$

Let $\mathcal{H}_-$ be the completion of $\mathcal{H}_0$ in the norm $\|\cdot\|_-$ which is generated by this inner product. $\mathcal{H}_-$ can be identified in a standard way with the dual space $\mathcal{H}_+^{'}$ using the inner product $(\cdot, \cdot)_0$, see [BK95]. Thus we have constructed the chain of spaces

$$X_0^{\mathbb{Z}^d} \subset \mathcal{H}_+ \subset \mathcal{H}_0 \subset \mathcal{H}_- \subset X^{\mathbb{Z}^d}. \tag{2.2.1}$$

Let K_1 and K_2 be real Hilbert spaces. We denote by $C^k(K_1, K_2)$ the set of all mappings from K_1 to K_2 that are k-times continuously differentiable in the sense of Fréchet (e.g. [BK95]) and by $C_b^k(K_1, K_2)$ the set of all mappings g of the class $C^k(K_1, K_2)$ with global boundedness in the usual operator norms of the derivatives

$$g^{(l)} : K_1 \to \mathcal{L}(K_1, \mathcal{L}(K_1, \cdots, \mathcal{L}(K_1, K_2) \cdots), \quad l = 0, 1, \cdots, k,$$

where $\mathcal{L}(K_1, K_2)$ denotes the space of bounded linear operators from K_1 into K_2. For any function $f \in C^2(\mathcal{H}_-) := C^2(\mathcal{H}_-, \mathbb{R})$ we will identify the derivatives $f'(\mathbf{x}) \in \mathcal{L}(\mathcal{H}_-, \mathbb{R})$ and $f''(\mathbf{x}) \in \mathcal{L}(\mathcal{H}_-, \mathcal{L}(\mathcal{H}_-, \mathbb{R}))$ with the vector $\hat{f}'(\mathbf{x}) \in \mathcal{H}_+$ and the operator $\hat{f}''(\mathbf{x}) \in \mathcal{L}(\mathcal{H}_-, \mathcal{H}_+)$ respectively, by the following formulae:

$$f'(\mathbf{x})\mathbf{y} = (\hat{f}'(\mathbf{x}), \mathbf{y})_0, \quad \mathbf{x}, \mathbf{y} \in \mathcal{H}_-$$
$$(f''(\mathbf{x})\mathbf{y})\mathbf{z} = (\hat{f}''(\mathbf{x})\mathbf{y}, \mathbf{z})_0, \quad \mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathcal{H}_- \tag{2.2.2}$$

Let us denote by $B_{\mathcal{H}_-}(\mathbf{y}, r)$ an open ball in $\mathcal{H}_-$ centered at $\mathbf{y}$ and of radius r. We have the following result.

Lemma 2.2.1. *For any $\mathbf{y} \in \mathcal{H}_-$ and $R \in \mathbb{R}_+$, there exists $N \in \mathbb{N}$ such that for all $\mathbf{x} \in B_{\mathcal{H}_-}(\mathbf{y}, \frac{1}{4})$ and for all $k \in \mathbb{Z}^d$ with $|k| > N$, we have*

$$|x_k + \alpha(k)| > R,$$

where $\alpha(k)$ is defined by the formula (2.1.9).

Proof. For any $\mathbf{y} \in \mathcal{H}_-$ we have

$$\sum_{k \in \mathbb{Z}^d} y_k^2 (1 + |k|^d)^{-2} < \infty,$$

which implies that

$$\varepsilon_k := |y_k|(1 + |k|^d)^{-1} \to 0, \qquad |k| \to \infty.$$

Let us fix $\mathbf{y} \in \mathcal{H}_-$ and choose $N_1 \in \mathbb{N}$ such that for all $k \in \mathbb{Z}^d$ with $|k| > N_1$, we have $\varepsilon_k < \frac{1}{4}$, so that

$$|y_k| = \varepsilon_k (1 + |k|^d) < \frac{1}{4}(1 + |k|^d).$$

For any $\mathbf{x} \in B_{\mathcal{H}_-}(\mathbf{y}, \frac{1}{4})$ and $k \in \mathbb{Z}^d$ with $|k| > N_1$, we have

$$\begin{aligned} |x_k + \alpha(k)| &= |x_k - y_k + y_k + |k|^{d-1}k| \\ &\geqslant |y_k + |k|^{d-1}k| - |x_k - y_k|, \\ &\geqslant |k|^d - |y_k| - \frac{1}{4}(1 + |k|^d) \\ &= |k|^d - \varepsilon_k(1 + |k|^d) - \frac{1}{4}(1 + |k|^d) \\ &\geqslant |k|^d - \frac{1}{2}(1 + |k|^d) = \frac{1}{2}(|k|^d - 1). \end{aligned}$$

Here we have used the estimate $|x_k - y_k|(1 + |k|^d)^{-1} \leqslant \|\mathbf{x} - \mathbf{y}\|_- \leqslant \frac{1}{4}$. Let $N_2 \in \mathbb{N}$ be such that

$$N_2^d \geqslant 2R + 1, \tag{2.2.3}$$

and set $N = \max(N_1, N_2)$. Then for $k \in \mathbb{Z}^d$ with $|k| > N$, we have

$$|x_k + \alpha(k)| > R,$$

as required. ■

Corollary 2.2.2. *For the map p, defined in* (2.1.5)*, we have*

$$p(\mathcal{H}_-) \subset \ddot{\Gamma}_X. \tag{2.2.4}$$

Proof. Let us fix $\mathbf{x} \in \mathcal{H}_-$ and show that the configuration $p(\mathbf{x})$ does not have accumulation points. Let $\Lambda \subset X$ be compact and choose $R \in \mathbb{R}_+$ such that $\Lambda \subset B_X(0, R)$. By Lemma 2.2.1 (with $\mathbf{y} = \mathbf{x}$) there exists $N \in \mathbb{N}$ such that $|x_k + \alpha(k)| > R$ for all $k \in \mathbb{Z}^d$ with $|k| > N$. It implies that $|x_k + \alpha(k)| \notin \Lambda$, and the result follows. ■

We preserve the same notation for the restriction of p on $\mathcal{H}_-$.

Theorem 2.2.3. *The map $p : \mathcal{H}_- \to \ddot{\Gamma}_X$ is continuous.*

Proof. Recall that the topology in $\ddot{\Gamma}_X$ is defined as the weakest topology that makes all mappings $\gamma \mapsto \langle \gamma, f \rangle$, $f \in C_0(X)$, continuous. Therefore it is sufficient to show that for any $f \in C_0(X)$ the map $\mathcal{H}_- \ni \mathbf{x} \longmapsto \langle p(\mathbf{x}), f \rangle$ is continuous.
We fix $f \in C_0(X)$ and choose $R \in \mathbb{R}_+$ such that $\mathrm{supp} f \subset B_X(0, R)$. Let $\mathbf{x} \in \mathcal{H}_-$ be fixed and $\{\mathbf{x}^{(n)}\}_{n=1}^{\infty}$ be a sequence of elements of $\mathcal{H}_-$ that converges to $\mathbf{x}$ in $\mathcal{H}_-$ as $n \to \infty$. Without loss of generality we can assume that $\{\mathbf{x}^{(n)}\}_{n=1}^{\infty} \subset B_X(\mathbf{x}, 1/4)$. Lemma 2.2.1 implies that there exists N such that

$$|x_k^{(n)} + \alpha(k)| > R\,,\ |x_k + \alpha(k)| > R\,,\ |k| > N,$$

which in turn implies that

$$f(x_k^{(n)} + \alpha(k)) = f(x_k + \alpha(k)) = 0\,, for \quad |k| > N.$$

Therefore

$$\begin{aligned} \big|\langle p(\mathbf{x}), f \rangle - \langle p(\mathbf{x}^{(n)}), f \rangle\big| &\leqslant \sum_{k \in \mathbb{Z}^d} \Big| f(x_k^{(n)} + \alpha(k)) - f(x_k + \alpha(k)) \Big| && (2.2.5) \\ &= \sum_{\substack{k \in \mathbb{Z}^d \\ |k| \leqslant N}} \Big| f(x_k^{(n)} + \alpha(k)) - f(x_k + \alpha(k)) \Big| && (2.2.6) \\ &\longrightarrow 0\,, \quad n \to \infty, && (2.2.7) \end{aligned}$$

because $x_k^{(n)} \to x_k$ as $n \to \infty$, and f is continuous. ■

Lemma 2.2.4. *Let μ be a probability measure on X. Then for a bounded Borel set $\Lambda \subset X$ we have*

$$\sum_{k \in \mathbb{Z}^d} \mu(\Lambda - \alpha(k)) < \infty, \tag{2.2.8}$$

where

$$\Lambda - \alpha(k) := \{y - \alpha(k) \ :\ y \in \Lambda\}.$$

Proof. In order to prove that the series in (2.2.8) converges we cover the set Λ with open balls. Let $S(x)$ denote the open ball of radius $\frac{1}{2}$ centered at $x \in \Lambda$. Define the collection $\mathbf{S}$ of all open balls with centers in Λ:

$$\mathbf{S} = \{S(x)\,; \quad x \in \Lambda\}.$$

$\mathbf{S}$ is an open cover for Λ. The compactness of Λ implies that there exists a finite sub-cover $\mathbf{T}$ of $\mathbf{S}$,

$$\mathbf{T} = \{S_i = S(x_i)\,; \quad i = 1, 2, \cdots, n\},\ n \in \mathbb{N}.$$

Let us define sets Λ_i in the following way:

$$\Lambda_i = \Lambda \cap S_i\,, \quad S_i \in \mathbf{T}.$$

Clearly

$$\Lambda = \bigcup_{i\,=\,1}^{n} \Lambda_i \qquad \text{and}$$

$$\Lambda - \alpha(k) = \bigcup_{i\,=\,1}^{n} (\Lambda_i - \alpha(k)). \tag{2.2.9}$$

Now we can write

$$\begin{aligned}\sum_{k\,\in\,\mathbb{Z}^d} \mu(\Lambda - \alpha(k)) &\leq \sum_{k\,\in\,\mathbb{Z}^d} \left(\sum_{i\,=\,1}^{n} \mu(\Lambda_i - \alpha(k)) \right) \\ &= \sum_{i\,=\,1}^{n} \left(\sum_{k\,\in\,\mathbb{Z}^d} \mu(\Lambda_i - \alpha(k)) \right). \end{aligned} \tag{2.2.10}$$

As S_i are open balls of radius $\frac{1}{2}$, for $i = 1, \cdots, n$ we have

$$\big(\Lambda_i - \alpha(k)\big) \cap \big(\Lambda_i - \alpha(m)\big) = \emptyset\,, \text{for any} \quad k \neq m \in \mathbb{Z}^d.$$

Therefore

$$\sum_{k\,\in\,\mathbb{Z}^d} \mu(\Lambda_i - \alpha(k)) = \mu\left(\bigcup_{k\in\mathbb{Z}^d} (\Lambda_i - \alpha(k)) \right) \leqslant \mu(X)$$

Thus

$$\sum_{k \in \mathbb{Z}^d} \mu(\Lambda - \alpha(k)) \leqslant \sum_{i=1}^{n} \mu\left(\bigcup_{k\in\mathbb{Z}^d} (\Lambda_i - \alpha(k))\right)$$

$$\leq \sum_{i=1}^{n} \mu(X) = n\, \mu(X) < \infty.$$

■

Theorem 2.2.5. *Let μ be a probability measure on X, such that*

$$\int_X |y|^S \mu(dy) < \infty, \quad \textit{for any} \quad S \in \mathbb{R}^+.$$

Then, for any bounded Borel set $A \subset X$ and $M, N \in \mathbb{N}$, we have

$$\sum_{k \in \mathbb{Z}^d} \mu(A - \alpha(k))(1 + |k|^M)^N < \infty.$$

Proof. The set A is bounded so, for $a_m, b_m \in X$ we can choose a d-dimensional cube

$$\Lambda = \underset{m=1,\cdots,d}{\times} [a_m, b_m]$$

such that $A \subset \Lambda$. Using arguments similar to the proof of Lemma 2.2.4, we can assume that $b_m - a_m < 1/2$, $m = 1, \cdots, d$. Then the shifted sets $\Lambda - \alpha(k), k \in \mathbb{Z}^d$ are mutually disjoint, that is $(\Lambda - \alpha(k)) \cap (\Lambda + \alpha(r)) = \emptyset$ for $k \neq r \in \mathbb{Z}^d$. We have

$$\sum_{k \in \mathbb{Z}^d} \mu(A - \alpha(k))(1 + |k|^M)^N \leqslant \sum_{k \in \mathbb{Z}^d} \mu(\Lambda - \alpha(k))(1 + |k|^M)^N$$

$$= \sum_{k \in \mathbb{Z}^d} \int_{a_1-\alpha(k)_1}^{b_1-\alpha(k)_1} \cdots \int_{a_d-\alpha(k)_d}^{b_d-\alpha(k)_d} (1 + |k|^M)^N \; \mu(dx).$$

For any $m = 1, \cdots, d$ and for $y = (y_1, \cdots, y_d) \in \Lambda - \alpha(k)$ we have

$$a_m - y_m \leqslant \alpha(k)_m \leqslant b_m - y_m,$$

where $\alpha(k)_m$ is the m-th component of the multi-index $\alpha(k)$. Then

$$|\alpha(k)_m| \leqslant \max\left(|a_m - y_m|, |b_m - y_m| \right),$$

and

$$\begin{aligned}
|k|^d = \sum_{m=1}^{d} |\alpha(k)_m| &\leqslant \max\left(\sum_{m=1}^{d} |a_m - y_m|, \sum_{m=1}^{d} |b_m - y_m| \right) \\
&\leqslant d \, \max\left(\sqrt{\sum_{m=1}^{d} |a_m - y_m|^2}, \sqrt{\sum_{m=1}^{d} |b_m - y_m|^2} \right) \\
&= d \, \max\left(\|a - y\|, \|b - y\| \right).
\end{aligned}$$

Then

$$\begin{aligned}
(1 + |k|^M)^N &\leqslant \left(1 + d^{M/d} \max\left(\|a - y\|^{M/d}, \|b - y\|^{M/d} \right) \right)^N \\
&\leqslant C \left(1 + d^{MN/d} \max\left(\|a - y\|^{MN/d}, \|b - y\|^{MN/d} \right) \right) \\
&\leqslant C \left(1 + d^{MN/d} \left(\|a - y\|^{MN/d} + \|b - y\|^{MN/d} \right) \right),
\end{aligned}$$

For some constant $C > 0$. So we have

$$\begin{aligned}
&\sum_{k \in \mathbb{Z}^d} \mu(A - \alpha(k))(1 + |k|^M)^N \\
&\leqslant \sum_{k \in \mathbb{Z}^d} \int_{a_1 - \alpha(k)_1}^{b_1 - \alpha(k)_1} \cdots \int_{a_d - \alpha(k)_d}^{b_d - \alpha(k)_d} C \left(1 + d^{MN/d} \left(\|a - y\|^{MN/d} + \|b - y\|^{MN/d} \right) \right) \mu(dy) \\
&\leqslant C\mu(X) + d^{MN/d} \int_X \left(\|a - y\|^{MN/d} + \|b - y\|^{MN/d} \right) \mu(dy) < \infty,
\end{aligned}$$

because of our assumption on moments of the measure μ. Thus

$$\sum_{k \in \mathbb{Z}^d} \mu(A - \alpha(k))(1 + |k|^M)^N < \infty,$$

as required. ■

Theorem 2.2.6. *Let μ be a probability measure on X, such that*

$$\int_X |y|^S \mu(dy) < \infty, \quad y \in X, \text{ for any } \quad S \in \mathbb{R}^+.$$

Then, for any bounded Borel set $A \subset X$, and any numbers $p, M, N \in \mathbb{N}$, we have

$$\sum_{k \in \mathbb{Z}^d} \mu(A - \alpha(k))^{1/p}(1 + |k|^M)^N < \infty.$$

Proof. Assume first that $M \geqslant d$ and let $q \in \mathbb{N}$ such that $\dfrac{1}{p} + \dfrac{1}{q} = 1$. We can write

$$\sum_{k \in \mathbb{Z}^d} \mu(A - \alpha(k))^{1/p}(1 + |k|^M)^N \tag{2.2.11}$$

$$= \sum_{k \in \mathbb{Z}^d} \mu(A - \alpha(k))^{1/p}(1 + |k|^M)^N (1 + |k|^M)^{2/q}(1 + |k|^M)^{-2/q}. \tag{2.2.12}$$

Using Holder's Inequality, we get

$$\sum_{k \in \mathbb{Z}^d} \mu(A - \alpha(k))^{1/p}(1 + |k|^M)^N (1 + |k|^M)^{2/q}(1 + |k|^M)^{-2/q}$$

$$\leqslant \left[\sum_{k \in \mathbb{Z}^d} \mu(A - \alpha(k))(1 + |k|^M)^{pN}(1 + |k|^M)^{2p/q}\right]^{1/p} \left[\sum_{k \in \mathbb{Z}^d} (1 + |k|^M)^{-2}\right]^{1/q}$$

$$\leqslant \left[\sum_{k \in \mathbb{Z}^d} \mu(A - \alpha(k))(1 + |k|^M)^{N_1}\right]^{1/p} \left[\sum_{k \in \mathbb{Z}^d} (1 + |k|^M)^{-2}\right]^{1/q}, \tag{2.2.13}$$

where N_1 is the smallest integer such that $N_1 \geqslant (N + 2p)/q$. We have

$$\sum_{k \in \mathbb{Z}^d} \mu(A - \alpha(k))(1 + |k|^M)^{N_1} < \infty.$$

by Theorem 2.2.5 and

$$(1+|k|^M)^{-2} < \infty,$$

because $M \geqslant d$. Observe that, for $M < d$ we have

$$\begin{aligned} &\sum_{k \in \mathbb{Z}^d} \mu(A-\alpha(k))(1+|k|^M)^N \\ &< \sum_{k \in \mathbb{Z}^d} \mu(A-\alpha(k))(1+|k|^d)^N < \infty, \end{aligned} \tag{2.2.14}$$

and the theorem is proved. ■

2.2.2 Main Assumptions and Examples of Measures

Let θ be a Borel probability measure on $\mathcal{H}_-$ satisfying the following conditions:

(1) $\theta\,(Diag(\mathcal{H}_-)) = 0$, where

$$Diag(\mathcal{H}_-) = \{\mathbf{x} \in \mathcal{H}_- \; ; \; \exists \quad k, j \; \in \; \mathbb{Z}^d \;\; s.t. \;\; x_k - x_j \in \mathbb{Z}^d\} \tag{2.2.15}$$

(2) For every $j \in \mathbb{Z}^d$, θ is invariant under the map $S_j \,:\, X^{\mathbb{Z}^d} \longmapsto X^{\mathbb{Z}^d}$ defined by the formula

$$S_j : \left(x_k\right)_{k \in \mathbb{Z}^d} \longmapsto \left(x_{k+j}\right)_{k \in \mathbb{Z}^d}, \tag{2.2.16}$$

that is,

$$S_j^* \, \theta \;=\; \theta \,, \quad j \in \mathbb{Z}^d. \tag{2.2.17}$$

(3) All moments of θ are finite, that is,

$$\int_{\mathcal{H}_-} |x_k|^p \theta(d\mathbf{x}) < \infty \,, \quad p = 1, 2, \cdots, \quad k \; \in \; \mathbb{Z}^d. \tag{2.2.18}$$

Now we present three examples of measures that satisfy conditions $(1)-(3)$ above.

Example: Product Measures

Let μ be a probability measure on X. We assume that the following conditions hold.

(1) All moments of the measure μ are finite, that is

$$\int_X |x|^p \mu(dx) < \infty\,, \quad p = 1, 2, \cdots . \tag{2.2.19}$$

(2) The measure μ is absolutely continuous with respect to the Lebesgue measure on X.

Consider the product measure

$$\theta = \bigotimes_{k \in \mathbb{Z}^d} \mu_k\,, \quad k \in \mathbb{Z}^d\,, \quad \mu_k = \mu. \tag{2.2.20}$$

The measure θ is a probability measure on $X^{\mathbb{Z}^d}$, see e.g [Hal74].

Proposition 2.2.7. *The measure θ is supported on $\mathcal{H}_-$ and satisfies conditions* $(1) - (3)$ *of Section 2.2.2.*

Proof. We first prove that $\theta(\mathcal{H}_-) = 1$. We have

$$\begin{aligned}\int_{X^{\mathbb{Z}^d}} \|\mathbf{x}\|_-^2 \theta(d\mathbf{x}) &= \int_{X^{\mathbb{Z}^d}} \sum_{k \in \mathbb{Z}^d} |x_k|^2 (1 + |k|^d)^{-2} \theta(d\mathbf{x}) \\ &= \sum_{k \in \mathbb{Z}^d} (1 + |k|^d)^{-2} \int_X |x|^2 \mu(dx) < \infty. \end{aligned} \tag{2.2.21}$$

Observe that $\|\mathbf{x}\|_-^2 = \infty$ for $\mathbf{x} \in \mathcal{H}_-^c$, where $\mathcal{H}_-^c = X^{\mathbb{Z}^d} \backslash \mathcal{H}_-$, which together with above formula implies that $\theta(\mathcal{H}_-^c) = 0$. Thus $\theta(\mathcal{H}_-) = 1$.

Now let us verify conditions $(1) - (3)$.

Condition (1): For every $k, j \in \mathbb{Z}^d$, define the set

$$D_{kj} = \{\mathbf{x} \in X^{\mathbb{Z}^d} \,:\, x_k - x_j \in \mathbb{Z}^d\}. \tag{2.2.22}$$

Obviously, $Diag(\mathcal{H}_-) \subset \bigcup_{k,j \in \mathbb{Z}^d} D_{kj}$. Due to the structure of the measure θ we have

$$\theta(D_{kj}) = \mu \otimes \mu(Diag(X^2)), \quad \text{for any } k, j \in \mathbb{Z}^d,$$

where $Diag(X^2) = \{(x, y) \in X^2 \, : \, x - y \in \mathbb{Z}^d\}$. The measure $\mu \otimes \mu$ is absolutely continuous with respect to the Lebesgue measure on X^2, which implies that $\mu \otimes \mu(Diag(X^2)) = 0$. Therefore

$$\theta(Diag(\mathcal{H}_-)) \leqslant \sum_{k,j \in \mathbb{Z}^d} \mu \otimes \mu(Diag(X^2)) = 0.$$

Condition (2): It follows directly from formula (2.2.20) that

$$S_j^* \theta = \bigotimes_{k \in \mathbb{Z}^d} \mu_{k+j} = \theta \,, \quad j \in \mathbb{Z}^d.$$

So condition (2.2.15) is satisfied.

Condition (3): We have

$$\int_{X^{\mathbb{Z}^d}} |x_k|^p \theta(d\mathbf{x}) = \int_X |x|^p \mu(dx) < \infty, \quad \text{for any } k \in \mathbb{Z}^d,$$

because of the condition (2.2.19). ■

Example: Gaussian Measures

Let $\mathbb{A}$ be a bounded, strictly positive, symmetric linear operator in $\mathcal{H}_0$. We assume that $\mathbb{A}$ commutes with the map S_j (defined in (2.2.16)), that is

$$S_j \mathbb{A} = \mathbb{A} S_j \,, \quad j \in \mathbb{Z}^d. \tag{2.2.23}$$

Observe that the map $S_j, j \in \mathbb{Z}^d$ preserves the space $\mathcal{H}_0$. Let θ_0 be the Gaussian measure on $\mathcal{H}_-$ with zero mean and correlation operator $\mathbb{A}^{-1}$. It can be defined by its characteristic functional by the formula

$$\int_{\mathcal{H}_-} \exp(\, i \, (\mathbf{x}, \mathbf{y})_0 \,) \theta_0(d\mathbf{x}) = \exp(-\frac{1}{2} \, (\mathbb{A}^{-1} \mathbf{y}, \mathbf{y})_0 \,), \qquad \mathbf{y} \in \mathcal{H}_+ \,, \tag{2.2.24}$$

see e.g. [BK95, DF91].

Proposition 2.2.8. *The measure θ_0 satisfies Conditions (1) - (3) of Section* (2.2.2).

Proof. **Condition (1)**: It is sufficient to show that $\theta_0(D_{kj}) = 0$ for any $k, j \in \mathbb{Z}^d$, where the set D_{kj} is defined in (2.2.22). We have

$$\theta_0(D_{kj}) = \theta_0^{kj}(Diag(X^2)),$$

where θ_0^{kj} is the projection of θ_0 on the space $X^2 = X_k \times X_j$. It is known that all finite dimensional projections of a Gaussian measure on a Hilbert space are Gaussian (see [DF91]). Thus θ_0^{kj} is a Gaussian measure on X^2, and is therefore absolutely continuous with respect to the Lebesgue measure on X^2, which implies that

$$\theta_0^{kj}(Diag(X^2)) = 0.$$

Condition (2) and (3): Condition (2) follows directly from the formula (2.2.23). It is known ([DF91]) that θ_0 satisfies Condition (3). ■

Example: Gibbs Measures

Let θ_0 be the Gaussian measure defined in the previous section. Consider the block matrix representation of $\mathbb{B}$ in the decomposition

$$X^{\mathbb{Z}^d} = \mathop{\times}_{k \in \mathbb{Z}^d} X_k \,, \quad X_k = X$$
$$\mathbb{B} = (B_{kj})_{k,j \in \mathbb{Z}^d} \,, \tag{2.2.25}$$

where $B_{kj} : X_k \to X_j$ is a linear bounded operator (which can be identified with a $d \times d$ matrix). We assume that $B_{kj} = 0$ if $|k - j| > N_0$ for some $N_0 \in \mathbb{N}$.
$\mathbb{B}$ is strictly positive, that is,

$$\exists\, C_{\mathbb{B}} > 0, \quad \forall\, \mathbf{y} \in \mathcal{H}_-, \qquad \langle \mathbb{B}\mathbf{y}, \mathbf{y} \rangle \geqslant C_{\mathbb{B}} |\mathbf{y}|^2. \tag{2.2.26}$$

Let $\mathcal{P}(t), t \in \mathbb{R}$ be a polynomial of even order, that is

$$\mathcal{P}(t) = a_0 + a_1 t + \cdots + a_{2n} t^{2n}, \qquad a_{2n} > 0, \tag{2.2.27}$$

and set $P(x) = \mathcal{P}(|x|), x \in X$. For any finite $\Lambda \subset \mathbb{Z}^d$ and $\epsilon > 0$ we can define the Gibbsian modification of the measure θ_0 as

$$\theta_\epsilon^\Lambda(d\mathbf{x}) = \frac{1}{\mathbf{M}} \exp\Big[-\epsilon \sum_{k \in \Lambda} P(x_k)\Big] \theta_0(d\mathbf{x}), \tag{2.2.28}$$

where $\mathbf{M}$ is the normalization factor given by

$$\mathbf{M} = \int_{X^{\mathbb{Z}^d}} \exp\Big[-\epsilon \sum_{k \in \Lambda} P(x_k)\Big] \theta_0(d\mathbf{x}).$$

For ϵ sufficiently small, there exists the limit $\theta_\epsilon := lim_{\Lambda \to \mathbb{Z}^d} \theta_\epsilon^\Lambda$ in the sense of week convergence of finite dimensional distributions [MR00]. We will call θ_ϵ the Gibbs measure defined by $\mathbb{B}$, P and ϵ.

Proposition 2.2.9. *The measure θ_ϵ is supported on $\mathcal{H}_-$ and satisfies Conditions (1) - (3) of Section 2.2.2.*

Proof. We know from [AKR95] that θ_ϵ is S_j-invariant for $j \in \mathbb{Z}^d$ and

$$\int_{X^{\mathbb{Z}^d}} |x_k|^p \theta_\epsilon(d\mathbf{x}) < \infty\,, \quad p = 1, 2, \cdots .$$

So Conditions (2) and (3) are satisfied. In particular, we have

$$\int_{X^{\mathbb{Z}^d}} |x_k|^2 \theta_\epsilon(d\mathbf{x}) := M < \infty.$$

Therefore, similar to (2.2.21),

$$\begin{aligned} \int_{X^{\mathbb{Z}^d}} \|\mathbf{x}\|_-^2 \theta_\epsilon(d\mathbf{x}) &= \int_{X^{\mathbb{Z}^d}} \sum_{k \in \mathbb{Z}^d} |x_k|^2 (1 + |k|^d)^{-2} \theta_\epsilon \\ &\leqslant M \sum_{k \in \mathbb{Z}^d} (1 + |k|^d)^{-2} < \infty, \end{aligned} \tag{2.2.29}$$

which implies that $\theta_\epsilon(\mathcal{H}_-) = 1$.

Let us now prove condition (1). To prove this we use Dobrushin-Lanford-Ruelle (DLR) equation for Gibbs measure. For all bounded $\Lambda \subset \mathbb{Z}^d$ and $f \in \mathcal{F}C_b^\infty(X^{\mathbb{Z}^d})$ let us define the measure $\Pi_\Lambda(\mathbf{y}, d\mathbf{x})$ by the formula

$$\int\limits_{X^{\mathbb{Z}^d}} f(\mathbf{x})\Pi_\Lambda(\mathbf{y}, d\mathbf{x}) = \frac{1}{\mathbf{Z}} \int\limits_{X^{\mathbb{Z}^d}} e^{-E_\Lambda(\mathbf{x},\mathbf{y})} f(\mathbf{x}_\Lambda \times \mathbf{y}_{\Lambda^c})\theta(d\mathbf{x}_\Lambda) \tag{2.2.30}$$

where $\mathbf{x}_\Lambda = (x_k)_{k\in\Lambda}$ and

$$E_\Lambda(\mathbf{x}, \mathbf{y}) = \sum_{k\in\Lambda} P(x_k) - \sum_{k,j\in\Lambda} \mathbb{A}(k-j)x_k\, x_j - \sum_{\substack{k\in\Lambda \\ j\in\Lambda^c}} \mathbb{A}(k-j)x_k\, x_j$$

It is known that the measure θ_ϵ satisfies the DLR-equation [Geo11], that is, for any finite $\Lambda, A \in \mathbb{Z}^d$ such that $A \subset \Lambda$ we have

$$\theta_\epsilon(A) = \int\limits_{X^{\mathbb{Z}^d}} \Pi_\Lambda(\mathbf{y}, A)\theta_0(d\mathbf{y}) \tag{2.2.31}$$

Now we can set $A = D_{kj}$ and observe that

$$\Pi_\Lambda(\mathbf{y}, D_{kj}) = 0,$$

for any Λ such that $k, j \ni \Lambda$. Then (2.2.31) implies that $ta_\epsilon(D_{kj}) = 0$. ■

2.3 Support and Finiteness of Moments of Push-Forward Measures

Let us consider a measure θ on $X^{\mathbb{Z}^d}$ which satisfies conditions $(1)-(3)$ of Section 2.2.2. Let us define the push-forward measure ν_θ on $\ddot{\Gamma}_X$ by the formula

$$\nu_\theta(A) = \theta\,\left(p^{-1}(A)\right), \quad \forall \quad A \subset \Gamma_X, \tag{2.3.1}$$

where p is given by the formula $\{x_k + \alpha(k)\}_{k \in \mathbb{Z}^d}$ and $\alpha(k) = |k|^{d-1}k$ (see (2.1.8) for details). Theorem 2.2.3 implies that $p : \mathcal{H}_- \to \ddot{\Gamma}_X$ is measurable (with respect to the Borel σ-algebras $\mathcal{B}(\mathcal{H}_-)$ and $\mathcal{B}(\ddot{\Gamma}_X)$). Then the measure ν_θ is a probability measure of $\ddot{\Gamma}_X$.

Condition (1), stated in Section 2.2.2 (formula (2.2.15)) leads to the following theorem:

Theorem 2.3.1. *Measure ν_θ is supported on the space of configurations without multiple points, that is,*

$$\nu_\theta(\Gamma_X) = 1.$$

Proof. It follows from the definition of the map p that the configuration $p(\mathbf{x}), \mathbf{x} \in \mathcal{H}_-$ has multiple points if and only if $x_k - x_j = \alpha(j) - \alpha(k)$ for some $k, j \in \mathbb{Z}^d$. Observe that $\alpha(j) - \alpha(k) \in \mathbb{Z}^d$. Therefore

$$p^{-1}\left(\ddot{\Gamma}_X \setminus \Gamma_X\right) \subset Diag(\mathcal{H}_-).$$

Thus

$$\nu_\theta\left(\ddot{\Gamma}_X \setminus \Gamma_X\right) = \theta\left(p^{-1}\left(\ddot{\Gamma}_X \setminus \Gamma_X\right)\right) \leqslant \theta\left(\left(Diag(\mathcal{H}_-)\right)\right) = 0,$$

because of Condition (1) of Section 2.2.2. ■

Definition 2.3.2. *We say that a measure ν_θ on Γ_X has finite n-th moments if*

$$\mathfrak{m}^n_{\nu_\theta}(f) := \int_{\Gamma_X} |\langle f, \gamma \rangle|^n \, \nu_\theta(d\gamma) < \infty \quad \textit{for any} \quad f \in C_0^\infty(X).$$

We denote by $\mathcal{M}^n(\Gamma_X)$ the class of all measures on Γ_X with finite n-th moments. Observe that, we have the inclusion

$$\mathcal{M}^m(\Gamma_X) \subset \mathcal{M}^n(\Gamma_X) \quad m < n\,, \quad m, n \in \mathbb{N}. \tag{2.3.2}$$

Indeed, by Holder's inequality,

$$\mathfrak{m}_{\nu_\theta}^m(f) = \int_{\Gamma_X} |\langle f, \gamma\rangle|^m \, \nu_\theta(d\gamma)$$

$$\leqslant \left[\int_{\Gamma_X} |\langle f, \gamma\rangle|^n \, \nu_\theta(d\gamma) \right]^{m/n} = \left(\mathfrak{m}_{\nu_\theta}^n(f)\right)^{m/n}, \tag{2.3.3}$$

which implies the inclusion.

In what follows, we prove that second moments of the measure ν_θ are finite. We introduce the following notations. Let

$$X^{\mathbb{Z}^d \setminus j} \;=\; \mathop{\times}_{\substack{k \in \mathbb{Z}^d \\ k \neq j}} X_k,$$

$$\overset{\vee}{\mathbf{x}}_j \;=\; \left(x_k\right)_{\substack{k \in \mathbb{Z}^d \\ k \neq j}} \quad \in X^{\mathbb{Z}^d \setminus j}, \quad j \in \mathbb{Z}^d. \tag{2.3.4}$$

Any element $\mathbf{x} \in X^{\mathbb{Z}^d}$ can be identified with the pair

$$\mathbf{x} \;=\; \left(\overset{\vee}{\mathbf{x}}_j \,,\, x_j \right), \quad j \in \mathbb{Z}^d.$$

For any $k \in \mathbb{Z}^d$ we introduce the projection θ_k of the measure θ onto $X_k = X$, that is,

$$\theta_k(A) = \int_{X^{\mathbb{Z}^d \setminus k}} \theta\left(d\overset{\vee}{\mathbf{x}}_k \,,\, A \right), \quad A \in \mathcal{B}(X). \tag{2.3.5}$$

Proposition 2.3.3. *For any $k, j \in \mathbb{Z}^d$ we have*

$$\theta_k \;=\; \theta_j. \tag{2.3.6}$$

Proof. Using notations (2.3.4) we can write

$$\int_X f(x)\, \theta_k(dx) = \int_{X_k} f(x_k) \left[\int_{X^{\mathbb{Z}^d \setminus k}} \theta\left(d\overset{\vee}{\mathbf{x}}_k \,,\, dx_k \right) \right]$$

$$= \int_{X^{\mathbb{Z}^d}} f(x_k) \theta(d\mathbf{x}). \tag{2.3.7}$$

The measure θ is invariant under the map S_j for any $j \in \mathbb{Z}^d$ (defined by the formula (2.2.16)), so that (2.3.7) can be rewritten in the form

$$\begin{aligned}\int_{X^{\mathbb{Z}^d}} f(x_k)\,\theta(d\mathbf{x}) &= \int_{X^{\mathbb{Z}^d}} f(x_k)\,S^*_{j-k}\,\theta(d\mathbf{x})\\ &= \int_{X^{\mathbb{Z}^d}} f(S_{j-k}\,x_k)\,\theta(d\mathbf{x})\\ &= \int_{X^{\mathbb{Z}^d}} f(x_j)\,\theta(d\mathbf{x})\\ &= \int_X f(x)\,\theta_j(dx),\end{aligned} \tag{2.3.8}$$

which implies that

$$\theta_k \;=\; \theta_j. \tag{2.3.9}$$

■

In what follows, we will use the notation

$$\theta_k \;= \theta_j \;=\; \theta^{(1)}\,,\; j \in \mathbb{Z}^d \tag{2.3.10}$$

. We have the following result.

Theorem 2.3.4. *We have $\nu_\theta \in \mathcal{M}^2(\Gamma_X)$.*

Proof. For $f \in C_0(X)$, the second moment of the measure ν_θ may be written as

$$\begin{aligned}\mathfrak{m}^2_{\nu_\theta}(f) &= \int_{\Gamma_X} \langle\, f, \gamma\,\rangle^2\,\nu_\theta(d\gamma)\\ &= \int_{X^{\mathbb{Z}^d}} \left(\sum_{k\,\in\,\mathbb{Z}^d} f_k(x_k)\right)^2 \theta(d\mathbf{x}),\end{aligned}$$

where $f_k(x_k) = f(x_k + \alpha(k))$. Thus

$$\begin{aligned}
\mathfrak{m}^2_{\nu_\theta}(f) &= \int\limits_{X^{\mathbb{Z}^d}} \left(\sum_{k,j\,\in\,\mathbb{Z}^d} f_k(x_k) f_j(x_j) \right) \theta(d\mathbf{x}) \\
&\leqslant \int\limits_{X^{\mathbb{Z}^d}} \left(\sum_{k,j\,\in\,\mathbb{Z}^d} |f_k(x_k)|\,|f_j(x_j)| \right) \theta(d\mathbf{x}) \\
&= \sum_{k,j\,\in\,\mathbb{Z}^d} \int\limits_{X^{\mathbb{Z}^d}} |f_k(x_k)|\,|f_j(x_j)|\; \theta(d\mathbf{x}). \qquad (2.3.11)
\end{aligned}$$

Here summation and integration are interchanged using Tonelli's theorem (see e.g. [AE09, Ch.6]). By Cauchy-Schwartz Inequality, we can write

$$\int\limits_{X^{\mathbb{Z}^d}} |f_k(x_k)|\,|f_j(x_j)|\; \theta(d\mathbf{x}) \leqslant \sqrt{\int\limits_{X^{\mathbb{Z}^d}} f_k^2(x_k)\theta(d\mathbf{x}) \int\limits_{X^{\mathbb{Z}^d}} f_j^2(x_j)\theta(d\mathbf{x})}$$

Therefore we have

$$\begin{aligned}
\mathfrak{m}^2_{\nu_\theta}(f) &\leqslant \sum_{k,j\,\in\,\mathbb{Z}^d} \sqrt{\int\limits_{X^{\mathbb{Z}^d}} f_k^2(x_k)\theta(d\mathbf{x}) \int\limits_{X^{\mathbb{Z}^d}} f_j^2(x_j)\theta(d\mathbf{x})} \\
&= \left(\sum_{k\,\in\,\mathbb{Z}^d} \sqrt{\int\limits_{X^{\mathbb{Z}^d}} f_k^2(x_k)\theta(d\mathbf{x})} \right)^2 \\
&= \left(\sum_{k\,\in\,\mathbb{Z}^d} \sqrt{\int\limits_{X} f_k^2(x)\theta^{(1)}(dx)} \right)^2 \\
&\leqslant \max_{x\in X} f^2 \left(\sum_{k\,\in\,\mathbb{Z}^d} \sqrt{\theta^{(1)}\left(\operatorname{supp}(f) - \alpha(k)\right)} \right)^2 \qquad (2.3.12)
\end{aligned}$$

The expression on the right hand side converges by the Theorem 2.2.6 with $\mu = \theta^{(1)}$. It implies that second moment of the measure ν_θ is finite. ■

Corollary 2.3.5. *Due to the inclusion* (2.3.2) *we have* $\nu_\theta \in \mathcal{M}^1(\Gamma_X)$*.*

Remark 2.3.6. *For a fixed bounded set* $\Lambda \in \mathcal{B}(x)$ *the* ν_θ*-average number of elements of a configuration* $\gamma \in \Gamma_X$ *is finite, that is,*

$$\boldsymbol{E}_{\nu_\theta}(\#(\gamma \cap \Lambda)) = \int_{\Gamma_X} \langle \mathbf{1}_\lambda, \gamma\rangle \; \nu_\theta(d\gamma).$$

This together with the Corollary 2.3.5 gives an alternative proof of the fact that ν_θ *is concentrated on* Γ_X*.*

Remark 2.3.7. *The n-th moment of the measure* ν_θ *can be expressed in the following form*

$$\mathfrak{m}^n_{\nu_\theta} = \int_{\Gamma_X} \langle\, f, \gamma\,\rangle^n \; \nu_\theta(d\gamma) = \int_{X^{\mathbb{Z}^d}} \left(\sum_{k \,\in\, \mathbb{Z}^d} f_k(x_k) \right)^n \theta(d\mathbf{x})$$

It can shown by the arguments similar to the proof of Theorem 2.3.4 that $\nu_\theta \in \mathcal{M}^n(\Gamma_X)$*.*

Chapter 3

Integration by Parts Formula for Push-Forward Measures

3.1 Integration by Parts Formula on $X^{\mathbb{Z}^d}$

In this section, we recall main definitions related to the integration by parts (IBP) formula on the space $X^{\mathbb{Z}^d}$ following [AKR95]. Let us denote by $\mathcal{F}C_b^\infty(X^{\mathbb{Z}^d})$ the set of functions $f : X^{\mathbb{Z}^d} \to \mathbb{R}$ of the form

$$f(\mathbf{x}) = f_N(x_{m_1}, \cdots, x_{m_N}), \quad \mathbf{x} \in X^{\mathbb{Z}^d}, \tag{3.1.1}$$

for some $N \in \mathbb{N}$, $m_1, \cdots, m_N \in \mathbb{Z}^d$, and $f_N \in C_b^\infty(X^N)$ (which depend on f). Similarly we can also define the set $\mathcal{F}C_b^k(X^{\mathbb{Z}^d})$ for any $k \in \mathbb{N}$, assuming that $f_N \in C_b^k(X^N)$. For $f \in \mathcal{F}C_b^\infty(X^{\mathbb{Z}^d})$ let us define the gradient $\nabla f(\mathbf{x})$ by the formula

$$X^{\mathbb{Z}^d} \ni \mathbf{x} \longmapsto \nabla f(\mathbf{x}) = (\nabla_k f(\mathbf{x}))_{k \in \mathbb{Z}^d} \in X_0^{\mathbb{Z}^d},$$

where

$$\nabla_k f(\mathbf{x}) = \frac{\partial}{\partial x_k} f_N(x_{m_1}, \cdots, x_{m_N}).$$

Let us introduce the class $\mathfrak{M}(\mathcal{H}_-)$ of all probability measures on $\mathcal{H}_-$ possessing a logarithmic derivative. That is, $\theta \in \mathfrak{M}(\mathcal{H}_-)$ if and only if the following formula holds for any $\phi \in X_0^{\mathbb{Z}^d}$ and $f \in \mathcal{F}C_b^\infty(X^{\mathbb{Z}^d})$:

$$\int_{\mathcal{H}_-} (\nabla f(\mathbf{x}), \phi)_0 \; \theta(d\mathbf{x}) = - \int_{\mathcal{H}_-} f(\mathbf{x}) \, \beta_\theta^\phi(\mathbf{x}) \theta(d\mathbf{x}), \tag{3.1.2}$$

where $\beta_\theta^\phi : \mathcal{H}_- \to \mathbb{R}$ is a measurable function. β_θ^ϕ is called the *logarithmic derivative* of the measure θ in the direction of ϕ. It can be represented in the form

$$\beta_\theta^\phi(\mathbf{x}) = (\beta_\theta(\mathbf{x}), \phi)_0, \tag{3.1.3}$$

for some map $\beta_\theta : \mathcal{H}_- \to \mathcal{H}_-$. The map β_θ is called the *vector logarithmic derivative* of the measure θ. We assume that it satisfies the condition

$$\int_{\mathcal{H}_-} \|\beta_\theta(\mathbf{x})\|_-^4 \; \theta(d\mathbf{x}) < \infty. \tag{3.1.4}$$

Example 3.1.1. *Let θ_0 be the Gaussian measure defined by the formula* (2.2.24)*. The integration by parts formulae* (3.1.2) *and* (3.1.3) *holds for the measure θ_0 with*

$$\beta_\theta^\phi(\mathbf{x}) = \beta_{\theta_0}(\mathbf{x}) := -A\mathbf{x}. \tag{3.1.5}$$

It is known [BK95, DF91] that

$$\int_{\mathcal{H}_-} \|\beta_{\theta_0}(\mathbf{x})\|_-^p \; \theta_0(d\mathbf{x}) < \infty \,, \quad \textit{for any} \quad p = 1, 2, \cdots.$$

Example 3.1.2. *Let θ_ε be the Gibbs measure defined in Section 2.2.2. The integration by parts formula* (3.1.2) *and* (3.1.3) *hold for the measure θ_ε with the* vector logarithmic derivative

$$\beta_\theta^\phi(\mathbf{x}) = \beta_{\theta_\varepsilon}(\mathbf{x}) = \beta_{\theta_0}(\mathbf{x}) + \mathcal{Q}(\mathbf{x}), \qquad \mathbf{x} \in \mathcal{H}_-\,, \tag{3.1.6}$$

and $\mathcal{Q} : \mathcal{H}_- \to \mathcal{H}_-$ is a measurable map having the representation

$$\mathcal{Q}(\mathbf{x}) = (\mathcal{Q}_k(\mathbf{x}))_{k\in\mathbb{Z}^d}, \quad \mathcal{Q}_k(\mathbf{x}) = -P'(x_k) \quad k \in \mathbb{Z}^d\,, \quad \mathbf{x} \in \mathcal{H}_-$$

It is known [AKR95] that

$$\int\limits_{\mathcal{H}_-} \|\beta_{\theta_\varepsilon}(\mathbf{x})\|_-^p \,\theta_\varepsilon(d\mathbf{x}) < \infty \qquad \textit{for any} \quad p = 1, 2, \cdots . \tag{3.1.7}$$

For any $\theta \in \mathfrak{M}(\mathcal{H}_-)$, the IBP formula (3.1.2) *can be extended to non-constant vector fields.*

The following result is known [DF91]:

Theorem 3.1.3. *Let us consider a vector field $V \in C_b^1(\mathcal{H}_-, \mathcal{H}_+)$ (cf. Section 2.2.1) given by its components by $V(\mathbf{x}) = \big(V_k(\mathbf{x})\big)_{k\in\mathbb{Z}^d}$. The integration by parts formula takes the form*

$$\int\limits_{\mathcal{H}_-} (\nabla f(\mathbf{x}), V(\mathbf{x}))_0\, \theta(d\mathbf{x}) = -\int\limits_{\mathcal{H}_-} f(\mathbf{x})\, (\beta_\theta(\mathbf{x})\,,\, V(\mathbf{x}))_0\, \theta(d\mathbf{x})$$

$$- \int\limits_{\mathcal{H}_-} f(\mathbf{x})\, \mathbf{div} V(\mathbf{x}) \theta(d\mathbf{x}), \quad f\,,\, g \in \mathcal{F}C_b^\infty(\mathcal{H}_-), \tag{3.1.8}$$

where

$$\mathbf{div} V(\mathbf{x}) = \mathrm{Tr} V'(\mathbf{x}) = \sum_{k\in\mathbb{Z}^d} div_k\, V_k(\mathbf{x}),$$

and $div_k(V_k)$ is the divergence of $V_k : \mathcal{H}_- \to \mathbb{R}$ with respect to x_k.

Integration by Parts Formula for a Special Class of Vector Fields

In what follows we would like to establish the integration by parts formula for a special class of vector fields on $\mathcal{H}_-$. Let θ be a probability measure on $X^{\mathbb{Z}^d}$ satisfying conditions (1) − (3) of Section 2.2.2. Let $v \in Vect_0(X)$ and define a map $\widehat{v} : X^{\mathbb{Z}^d} \to X^{\mathbb{Z}^d}$ by setting

$$\widehat{v}_k(\mathbf{x}) \;=\; v(x_k + \alpha(k))_{k\in\mathbb{Z}^d},$$

where $\alpha(k) = |k|^{d-1}k$. The following result shows that $\widehat{v}$ generates a vector field on $\mathcal{H}_-$.

Proposition 3.1.4. *We have the following:*

1. $\widehat{v} : \mathcal{H}_- \to X_0^{\mathbb{Z}^d}$ *and* $\int\limits_{\mathcal{H}_-} \|\widehat{v}(\mathbf{x})\|_+^2 \theta(d\mathbf{x}) < \infty,$
2. $\mathbf{div}\,\widehat{v}(\mathbf{x}) < \infty$, $\mathbf{x} \in \mathcal{H}_-$, *and* $\int\limits_{\mathcal{H}_-} |\mathbf{div}\widehat{v}(\mathbf{x})|\,\theta(d\mathbf{x}) < \infty.$

Proof. **(1)**: By the definition of the space $\mathcal{H}_-$, every $\mathbf{x} \in \mathcal{H}_-$ satisfies the estimate

$$\sum_{k\in\mathbb{Z}^d} x_k^2(1 + |k|^d)^{-2} < \infty.$$

This implies that $x_k^2(1 + |k|^d)^{-2} \to 0$ as $|k| \to \infty$, and we have,

$$|x_k| = o(1 + |k|^d).$$

The latter formula together with the triangle inequality and the identity $|\alpha(k)| = |k|^d$ imply that

$$|x_k + \alpha(k)| \geqslant (|k|^d - |x_k|) \to \infty, \quad |k| \to \infty,$$

which in turn implies that, for any $\mathbf{x} \in \mathcal{H}_-$, there exists $N \in \mathbb{N}$ such that for $|k| > N$ we have

$$x_k + \alpha(k) \notin \text{supp}(v), \quad k \in \mathbb{Z}^d.$$

Therefore, for any such k we have

$$\widehat{v}_k(\mathbf{x}) = v(x_k + \alpha(k)) = 0.$$

Therefore, for any $\mathbf{x} \in \mathcal{H}_-$, only finite number of the elements of sequences $\widehat{v}_k(\mathbf{x})$ are not equal to zero, which implies that $\widehat{v}(\mathbf{x}) \in X_0^{\mathbb{Z}^d}$.

Moreover, we can write

$$\int\limits_{\mathcal{H}_-} \|\widehat{v}(\mathbf{x})\|_+^2 \theta(d\mathbf{x}) = \int\limits_{\mathcal{H}_-} \sum_{k \in \mathbb{Z}^d} |v(x_k + \alpha(k))|^2 (1 + |k|^d)^2 \theta(d\mathbf{x}).$$

Using the notation (2.3.10), we have

$$\begin{aligned} &= \sum_{k \in \mathbb{Z}^d} (1 + |k|^d)^2 \int\limits_X |v(x + \alpha(k))|^2 \theta^{(1)}(dx) \\ &= \sup_{x \in X} |v(x)| \sum_{k \in \mathbb{Z}^d} (1 + |k|^d)^2 \theta^{(1)}(B - \alpha(k)). \end{aligned} \tag{3.1.9}$$

Observe that $\sup\limits_{x \in X} |v(x)| < \infty$ because $v \in Vect_0(X)$. Lemma 2.2.4 with $\mu = \theta^{(1)}$ implies that the series in (3.1.9) converges. Thus

$$\int\limits_{\mathcal{H}_-} \|\widehat{v}(\mathbf{x})\|_+^2 \theta(d\mathbf{x}) < \infty.$$

(2): The first part of the statement follows directly from the fact that $\widehat{v}_k(\mathbf{x}) = 0$ for k big enough, so that it is only the finite number of non-zero terms in the right hand side of the equality

$$\mathbf{div}\, \widehat{v}(\mathbf{x}) = \sum_{k \in \mathbb{Z}^d} \text{div}_k \widehat{v}_k(x_k).$$

In order to prove integrability of $\mathbf{div}\widehat{v}$, we can write

$$\begin{aligned}\int\limits_{\mathcal{H}_-} |\mathbf{div}\widehat{v}(\mathbf{x})|\,\theta(d\mathbf{x}) &= \int\limits_{\mathcal{H}_-} \left|\sum_{k\,\in\,\mathbb{Z}^d} div_k\, v(x_k+\alpha(k))\right| \theta(d\mathbf{x}),\\ &= \sum_{k\,\in\,\mathbb{Z}^d} \int\limits_X |div\, v(x+\alpha(k))|\,\theta^{(1)}(dx)\\ &\leqslant \sup_{x\in X} |div\, v(x)| \sum_{k\,\in\,\mathbb{Z}^d} \theta^{(1)}(B-\alpha(k)) < \infty,\end{aligned}$$

by Theorem 2.2.5 with $\mu = \theta^{(1)}$ (cf. proof of Part (1)). ■

In the next theorem, we show that the integration by parts formula (3.1.8) can be extended to the vector field $\widehat{v}$. Observe that we cannot apply Theorem 3.1.3 directly because $\widehat{v} \notin C_b^1(\mathcal{H}_-, \mathcal{H}_+)$, in general.

Theorem 3.1.5. *For the vector field $\widehat{v}(\mathbf{x})$, the integration by parts formula* (3.1.8) *holds, that is,*

$$\int\limits_{\mathcal{H}_-} (\nabla f(\mathbf{x}), \widehat{v}(\mathbf{x}))_0\, \theta(d\mathbf{x}) = -\int\limits_{\mathcal{H}_-} f(\mathbf{x})\, \beta_\theta^{\widehat{v}}(\mathbf{x})\theta(d\mathbf{x}), \quad f \in \mathcal{F}C_b^\infty(X^{\mathbb{Z}^d}) \tag{3.1.10}$$

where the logarithmic derivative $\beta_\theta^{\widehat{v}}(\mathbf{x})$ of the measure θ in the direction of $\widehat{v}$ has the form

$$\beta_\theta^{\widehat{v}}(\mathbf{x}) = (\beta_\theta(\mathbf{x}), \widehat{v}(\mathbf{x}))_0 + \mathbf{div}\widehat{v}(\mathbf{x}). \tag{3.1.11}$$

Moreover, $\beta_\theta^{\widehat{v}} \in L^1(\mathcal{H}_-, \theta)$.

Proof. We will use the following approximation arguments. Define a cut-off vector field $\widehat{v}^{(N)}$ by setting

$$\begin{cases} \widehat{v}_k^{(N)} = \widehat{v}_k \quad , \quad |k| \leqslant N,\\ \widehat{v}_k^{(N)} = 0 \quad , \quad \text{otherwise.}\end{cases}$$

Let us show that $\widehat{v}^{(N)} \to \widehat{v}$ in the space $L^2(\mathcal{H}_- \to \mathcal{H}_+, \theta)$ of square integrable maps from $\mathcal{H}_-$ to $\mathcal{H}_+$. Indeed,

$$\begin{aligned}\int\limits_{\mathcal{H}_-} \|\widehat{v}^{(N)}(\mathbf{x}) - \widehat{v}(\mathbf{x})\|_+^2 \theta(d\mathbf{x}) &= \int\limits_{\mathcal{H}_-} \sum_{\substack{k \in \mathbb{Z}^d \\ |k| > N}} |v(x_k + \alpha(k))|^2 (1 + |k|^d)^2 \theta(d\mathbf{x}) \\ &= \sum_{\substack{k \in \mathbb{Z}^d \\ |k| > N}} (1 + |k|^d)^2 \int\limits_X |v(x + \alpha(k))|^2 \theta^{(1)}(dx) \\ &\leqslant \sup_{x \in X} |v(x)| \sum_{\substack{k \in \mathbb{Z}^d \\ |k| > N}} (1 + |k|^d)^2 \theta^{(1)}(B - \alpha(k)).\end{aligned}$$

We know that

$$\sup_{x \in X} |v(x)| \sum_{k \in \mathbb{Z}^d} (1 + |k|^d)^2 \theta^{(1)}(B - \alpha(k)) < \infty,$$

(cf. proof of Proposition 3.1.4, formula (3.1.9)), which implies that

$$\sup_{x \in X} |v(x)| \sum_{\substack{k \in \mathbb{Z}^d \\ |k| > N}} (1 + |k|^d)^2 \theta^{(1)}(B - \alpha(k)) \longrightarrow 0\,, \quad N \to \infty.$$

Therefore

$$\int\limits_{\mathcal{H}_-} \|\widehat{v}^{(N)}(\mathbf{x}) - \widehat{v}(\mathbf{x})\|_+^2 \theta(d\mathbf{x}) \longrightarrow\ 0, \quad N \to \infty.$$

This convergence implies that

$$(\beta_\theta(\cdot), \widehat{v}^{(N)}(\cdot))_0 \longrightarrow (\beta_\theta(\cdot), \widehat{v}(\cdot))_0\,, \quad N \to \infty, \quad \text{in} \quad L^1(\mathcal{H}_-, \theta).$$

It can be shown by similar arguments that

$$\operatorname{div} \widehat{v}^{(N)} \to \operatorname{div} \widehat{v}\,, \quad N \to \infty\,, \quad \text{in} \quad L^1(\mathcal{H}_-, \theta),$$

and that

$$(\nabla f(\cdot), \widehat{v}^{(N)}(\cdot))_0 \longrightarrow (\nabla f(\cdot), \widehat{v}(\cdot))_0\,, \quad N \to \infty, \quad \text{in} \quad L^1(\mathcal{H}_-, \theta).$$

Thus we have that

$$\beta_\theta^{\widehat{v}^{(N)}} \longrightarrow \beta_\theta^{\widehat{v}}, \quad N \to \infty, \quad \text{in} \quad L^1(\mathcal{H}_-, \theta), \tag{3.1.12}$$

where

$$\beta_\theta^{\widehat{v}^{(N)}}(\mathbf{x}) = \left(\beta_\theta^v(\mathbf{x}), \widehat{v}^{(N)}(\mathbf{x})\right)_0 + \mathbf{div}\widehat{v}^{(N)}(\mathbf{x}).$$

As a corollary of formula (3.1.12) we have that $\beta_\theta^{\widehat{v}} \in L^1(\mathcal{H}_-, \theta)$. It is clear that $\widehat{v}^{(N)} \in C_b^1(\mathcal{H}_-, \mathcal{H}_+)$ and therefore we have the following IBP formula

$$\int\limits_{\mathcal{H}_-} \left(\nabla f(\mathbf{x}), \widehat{v}^{(N)}(\mathbf{x})\right)_0 \theta(d\mathbf{x}) = -\int\limits_{\mathcal{H}_-} f(\mathbf{x})\, \beta_\theta^{\widehat{v}^{(N)}}(\mathbf{x})\theta(d\mathbf{x}), \quad f \in \mathcal{F}C_b^\infty. \tag{3.1.13}$$

The limit transition on both sides of the formula (3.1.13) implies the result of the theorem. ■

Next, we refine the integrability properties of the logarithmic derivative $\beta_\theta^{\widehat{v}}$. Let us introduce the Sobolev space $H^{1,2}(\mathcal{H}_-, \theta)$ as a completion of the space $\mathcal{F}C_b^\infty(X^{\mathbb{Z}^d})$ in the norm $\|\cdot\|_{1,2}$ given by the formula

$$\|\mathbf{h}\|_{1,2}^2 = \int\limits_{\mathcal{H}_-} |\mathbf{h}(\mathbf{x})|^2 \theta(d\mathbf{x}) + \int\limits_{\mathcal{H}_-} \|\nabla \mathbf{h}(\mathbf{x})\|_0^2 \theta(d\mathbf{x}) \tag{3.1.14}$$

Theorem 3.1.6. *The IBP formula* (3.1.10) *holds for any* $f \in H^{1,2}(\mathcal{H}_-, \theta)$.

Proof. We will use the following approximation argument. Let $\{f_n\}_{n=1}^\infty$ be a sequence of elements of $\mathcal{F}C_b(X^{\mathbb{Z}^d})$ that approximates $f \in H^{1,2}(\mathcal{H}_-, \theta)$, that is,

$$\int\limits_{\mathcal{H}_-} |f(\mathbf{x}) - f_n(\mathbf{x})|^2 \theta(dx) \to 0 \quad \text{as} \quad n \to \infty, \tag{3.1.15}$$

and

$$\int\limits_{\mathcal{H}_-} \|\nabla f(\mathbf{x}) - \nabla f_n(\mathbf{x})\|_0^2 \theta(dx) \to 0 \quad \text{as} \quad n \to \infty, \tag{3.1.16}$$

For any f_n , $n = 1, 2, \cdots,$ we have the IBP formula

$$\int\limits_{\mathcal{H}_-} (\nabla f_n(\mathbf{x}), \widehat{v}(\mathbf{x}))_0 \theta(d\mathbf{x}) = -\int\limits_{\mathcal{H}_-} f_n(\mathbf{x})\, \beta_\theta^{\widehat{v}}(\mathbf{x})\theta(d\mathbf{x}). \tag{3.1.17}$$

We can pass to the limit on both sides of the equality (3.1.17). Indeed, we have

$$\int\limits_{\mathcal{H}_-} (\nabla\big(f(\mathbf{x}) - f_n(\mathbf{x})\big), \widehat{v}(\mathbf{x}))_0 \theta(d\mathbf{x})$$

$$\leqslant \sqrt{\int\limits_{\mathcal{H}_-} \|\nabla\left(f(\mathbf{x}) - f_n(\mathbf{x})\right)\|_-^2 \theta(d\mathbf{x}) \int\limits_{\mathcal{H}_-} \|\widehat{v}(\mathbf{x})\|_+^2 \theta(d\mathbf{x})}$$

$$\leqslant \sqrt{\int\limits_{\mathcal{H}_-} \|\nabla\left(f(\mathbf{x}) - f_n(\mathbf{x})\right)\|_0^2 \theta(d\mathbf{x}) \int\limits_{\mathcal{H}_-} \|\widehat{v}(\mathbf{x})\|_+^2 \theta(d\mathbf{x})}. \tag{3.1.18}$$

Thus

$$\int\limits_{\mathcal{H}_-} (\nabla\big(f(\mathbf{x}) - f_n(\mathbf{x})\big), \widehat{v}(\mathbf{x}))_0 \theta(d\mathbf{x}) \longrightarrow 0, \quad N \to \infty, \tag{3.1.19}$$

because of the formula (3.1.16) and part (1) of the Proposition 3.1.4. The limit transition in the right hand side of formula (3.1.17) is justified by (3.1.15) ■

Proposition 3.1.7. *The logarithmic derivative $\beta_\theta^{\widehat{v}}$ belongs to $L^2(\mathcal{H}_-, \theta)$.*

Proof. We need to prove that

$$\int\limits_{\mathcal{H}_-} |\beta_\theta^{\widehat{v}}(\mathbf{x})|^2\, \theta(d\mathbf{x}) < \infty.$$

We have

$$\int\limits_{\mathcal{H}_-} |\beta_\theta^{\widehat{v}}(\mathbf{x})|^2\, \theta(d\mathbf{x}) = \int\limits_{\mathcal{H}_-} \left|\big(\beta_\theta(\mathbf{x}), \widehat{v}(\mathbf{x})\big)_0 + \mathbf{div}\widehat{v}(\mathbf{x})\right|^2\, \theta(d\mathbf{x})$$

$$\leqslant 2\int\limits_{\mathcal{H}_-} \big(\beta_\theta(\mathbf{x}), \widehat{v}(\mathbf{x})\big)_0^2\, \theta(d\mathbf{x}) + 2\int\limits_{\mathcal{H}_-} |\mathbf{div}\widehat{v}(\mathbf{x})|^2\, \theta(d\mathbf{x}). \tag{3.1.20}$$

Let us first prove that the second integral in (3.1.20) is finite. Indeed,

$$\mathbf{div}\widehat{v}(\mathbf{x}) = \sum_{k \in \mathbb{Z}^d} \psi(x_k + \alpha(k)),$$

where $\psi(x) = \mathrm{div} v(x)$. Observe that $\psi \in C_0(X)$. We have

$$\int_{\mathcal{H}_-} |\mathbf{div}\widehat{v}(\mathbf{x})|^2 \, \theta(d\mathbf{x}) = \int_{\mathcal{H}_-} \langle \psi, \gamma \rangle^2 \nu_\theta(d\gamma) < \infty,$$

because $\nu_\theta \in \mathcal{M}^2(\Gamma_X)$ (see Theorem (2.3.4)).

For the first integral in (3.1.20), using Cauchy-Schwartz and Hölder's inequalities we obtain

$$\begin{aligned}\int_{\mathcal{H}_-} \big(\beta_\theta(\mathbf{x}), \widehat{v}(\mathbf{x})\big)_0^2 \, \theta(d\mathbf{x}) &\leqslant \int_{\mathcal{H}_-} \|\beta_\theta(\mathbf{x})\|_-^2 \ \|\widehat{v}(\mathbf{x})\|_+^2 \ \theta(d\mathbf{x}) \\ &\leqslant \sqrt{\int_{\mathcal{H}_-} \|\beta_\theta(\mathbf{x})\|_-^4 \ \theta(d\mathbf{x})} \sqrt{\int_{\mathcal{H}_+} \|\widehat{v}(\mathbf{x})\|_+^4 \ \theta(d\mathbf{x})}.\end{aligned}$$

The first integral in the latter expression is finite by (3.1.4). Let us compute the second integral. We have

$$\begin{aligned}\int_{\mathcal{H}_-} \|\widehat{v}(\mathbf{x})\|_+^4 \ \theta(d\mathbf{x}) &= \int_{\mathcal{H}_-} \left| \sum_{k \in \mathbb{Z}^d} |v(x_k + \alpha(k))|^2 (1 + |k|^d)^2 \right|^2 \theta(d\mathbf{x}) \\ &= \sum_{m,k \in \mathbb{Z}^d} D_k D_m \int_{\mathcal{H}_-} |v(x_k + \alpha(k))|^2 \, |v(x_m + \alpha(m))|^2 \theta(d\mathbf{x}),\end{aligned}$$

where $D_k = (1 + |k|^d)^2$ and $D_m = (1 + |m|^d)^2$. Using Hölder's inequality we obtain

$$\begin{aligned}
&\int_{\mathcal{H}_-} \|\widehat{v}(\mathbf{x})\|_+^4 \; \theta(d\mathbf{x}) \\
&\leqslant \sum_{k,m\in\mathbb{Z}^d} D_k D_m \sqrt{\int_{\mathcal{H}_-} |v(x_k+\alpha(k))|^4 \theta(d\mathbf{x}) \int_{\mathcal{H}_-} |v(x_m+\alpha(m))|^4 \theta(d\mathbf{x})} \\
&\leqslant \sup_{x\in X} |v(x)|^4 \sum_{k,m\in\mathbb{Z}^d} D_k D_m \sqrt{\theta^{(1)}(B-\alpha(k))} \sqrt{\theta^{(1)}(B-\alpha(m))} \\
&\leqslant \sup_{x\in X} \left[\sum_{k\in\mathbb{Z}^d} (1+|k|^d)^2 \sqrt{\theta^{(1)}(B-\alpha(k))} \right]^2 , \qquad (3.1.21)
\end{aligned}$$

where $B = \operatorname{supp}(v)$. The latter expression is finite by Theorem 2.2.6. ■

3.2 Integration by Parts Formula on Γ_X

The aim of this section is to prove an integration by parts formula for the measure ν_θ on Γ_X introduced in Section 2.3. First we need to introduce certain classes of functions on Γ_X. For a function $F : \Gamma_X \to \mathbb{R}$, define the function $\mathcal{I}F := F \circ p$, that is

$$\mathcal{I}F(\mathbf{x}) \;=\; F(p(\mathbf{x})) \, , \; \mathbf{x} \in \mathcal{H}_-, \qquad (3.2.1)$$

where $p : \mathcal{H}_- \to \Gamma_X$ is the projection map defined in (2.1.5). Clearly, $\mathcal{I}F$ is a function on $\mathcal{H}_-$.

Lemma 3.2.1. *The operator $\mathcal{I}$ defined above is an isometry from $L^2(\Gamma_X, \nu_\theta)$ to $L^2(\mathcal{H}_-, \theta)$.*

Proof. By the formula (3.2.1), we have

$$\begin{aligned} \| F \|^2_{L^2(\Gamma_X,\nu_\theta)} &= \int_{\Gamma_X} (F(\gamma))^2 \nu_\theta(d\gamma) \\ &= \int_{\mathcal{H}_-} (\mathcal{I}F(\mathbf{x}))^2 \, \theta(d\mathbf{x}) \\ &= \| \mathcal{I}F \|^2_{L^2(\mathcal{H}_-,\theta)} \end{aligned} \tag{3.2.2}$$

and the result is proved. ■

Remark 3.2.2. *The operator $\mathcal{I}$ is not an isomorphism. Indeed, the function $\mathcal{I}F(\mathbf{x})$ is symmetric with respect to permutations of the components of $\mathbf{x} = (x_k)_{k\in\mathbb{Z}^d}$, which implies that $\mathcal{I} : L^2(\Gamma_X, \nu_\theta) \to L^2(X^{\mathbb{Z}^d}, \theta)$ is not surjective.*

In what follows, we will use the notation $\widehat{F} = \mathcal{I}F$, $F \in L^2(\Gamma_X, \nu_\theta)$.

Lemma 3.2.3. *Let $F \in \mathcal{F}C(\Gamma_X)$. Then we have $\widehat{F} \in H^{1,2}(\mathcal{H}_-, \theta)$.*

Proof. According to the definition of the class $\mathcal{F}C(\Gamma_X)$, F has the following form:

$$F(\gamma) = f\big(\langle \phi_1, \gamma\rangle, \cdots, \langle \phi_m, \gamma\rangle\big),$$

where $m \in \mathbb{N}, \phi_j \in C_0^\infty(X)$ for $j = 1, \cdots, m$ and $f \in C_b^\infty(X^m)$. We can write

$$\begin{aligned} \widehat{F}(\mathbf{x}) &= \mathcal{I}F(\mathbf{x}) = F(p(\mathbf{x})) \\ &= f\big(\langle \phi_1, p(\mathbf{x})\rangle, \cdots, \langle \phi_m, p(\mathbf{x})\rangle\big), \end{aligned} \tag{3.2.3}$$

where

$$\langle \phi_j, p(\mathbf{x})\rangle = \sum_{k\,\in\,\mathbb{Z}^d} \phi_j(x_k + \alpha(k))) \,, \qquad j = 1, \cdots, m \,,$$

and $\alpha(k) = |k|^{d-1}k$. We will use the following notations:

$$\begin{aligned} \widehat{\phi}_j(\mathbf{x}) &:= \langle \phi_j, p(\mathbf{x})\rangle, \\ f\big(\widehat{\phi}_j(\mathbf{x})_{j=1}^m \big) &:= f\big(\widehat{\phi}_1(\mathbf{x}), \cdots, \widehat{\phi}_m(\mathbf{x})\big). \end{aligned}$$

So we have

$$\widehat{\phi}_j(\mathbf{x}) = \sum_{k \in \mathbb{Z}^d} \phi_j(x_k + \alpha(k)))$$

and

$$\widehat{F}(\mathbf{x}) = f\big(\,\widehat{\phi}_j(\mathbf{x})_{j=1}^m\,\big).$$

For any $N \in \mathbb{N}$, let us set

$$\widehat{\phi}_i^N(\mathbf{x}) = \sum_{\substack{k \in \mathbb{Z}^d \\ |k| \leqslant N}} \phi_i(x_k + \alpha(k)),$$

and

$$\widehat{F}^N(\mathbf{x}) = f\big(\,\widehat{\phi}_j^N(\mathbf{x})\,\big)_{j=1}^m.$$

According to formula (3.1.1),

$$\widehat{F}^N \;\in\; \mathcal{F}C_b^\infty(X^{\mathbb{Z}^d}).$$

Formula (3.2.3) implies that

$$\nabla_k \widehat{F}(\mathbf{x}) = \sum_{i=1}^m \partial_i f\big(\,\widehat{\phi}_j(\mathbf{x})\,\big)_{j=1}^m \nabla_k \widehat{\phi}_i(\mathbf{x})\,, \quad k \;\in\; \mathbb{Z}^d, \tag{3.2.4}$$

and

$$\nabla_k \widehat{F}^N(\mathbf{x}) = \sum_{i=1}^m \partial_i f\big(\,\widehat{\phi}_j^N(\mathbf{x})\big)_{j=1}^m \nabla_k \widehat{\phi}_i^N(\mathbf{x})\,, \quad k \;\in\; \mathbb{Z}^d, \tag{3.2.5}$$

where $\partial_i f$ is the $i-th$ partial derivative of f. Observe that the expression (3.2.4) is uniformly bounded, that is, $\exists C \in \mathbb{R}$ such that

$$|\nabla_k \widehat{F}(\mathbf{x})| \leqslant C\,, \quad \text{for all} \quad k \;\in\; \mathbb{Z}^d \quad \text{and} \quad \mathbf{x} \in \mathcal{H}_-.$$

The estimate follows from the fact that $f \in C_b^\infty(\mathbb{R}^m)$ and $\phi_i \in C_0^\infty(X)$.

Let us show that $\widehat{F}^N \to F, \quad N \to \infty$, in the norm of the space $H^{1,2}(\mathcal{H}_-, \theta)$. We have to prove that

$$\int\limits_{\mathcal{H}_-} \left(\left| \widehat{F}^N(\mathbf{x}) - \widehat{F}(\mathbf{x}) \right|^2 + \left\| \nabla \widehat{F}^N(\mathbf{x}) - \nabla \widehat{F}(\mathbf{x}) \right\|_0^2 \right) \theta(d\mathbf{x}) \to 0\,, \quad N \to \infty. \quad (3.2.6)$$

We start with the first term in (3.2.6). We can write

$$\left| \widehat{F}^N(\mathbf{x}) - \widehat{F}(\mathbf{x}) \right|^2 = \left| f\big(\widehat{\phi}_j^N(\mathbf{x}) \big)_{j=1}^m - f\big(\widehat{\phi}_j(\mathbf{x}) \big)_{j=1}^m \right|^2.$$

We know that $f \in C_b^\infty(\mathbb{R}^m)$ and is therefore globally Lipschitz. Thus, there exists a constant $C > 0$ such that

$$\begin{aligned} &\left| f\big(\widehat{\phi}_j^N(\mathbf{x}) \big)_{j=1}^m - f\big(\widehat{\phi}_j(\mathbf{x}) \big)_{j=1}^m \right|^2 \\ &\leqslant C \sum_{i=1}^m \left| \widehat{\phi}_i^N(\mathbf{x}) - \widehat{\phi}_i(\mathbf{x}) \right|^2 \\ &= C \sum_{i=1}^m \Bigg| \sum_{\substack{k \in \mathbb{Z}^d \\ |k| > N}} \phi_i(x_k + \alpha(k)) \Bigg|^2. \end{aligned}$$

So we have

$$\int\limits_{\mathcal{H}_-} \left| \widehat{F}^N(\mathbf{x}) - \widehat{F}(\mathbf{x}) \right|^2 \theta(d\mathbf{x}) \leqslant C \sum_{i=1}^m \int\limits_{\mathcal{H}_-} \Bigg| \sum_{\substack{k \in \mathbb{Z}^d \\ |k| > N}} \phi_i(x_k + \alpha(k)) \Bigg|^2 \theta(d\mathbf{x})$$

Observe that

$$\int\limits_{\mathcal{H}_-} \Bigg| \sum_{k \in \mathbb{Z}^d} \phi(x_k + \alpha(k)) \Bigg|^2 \theta(d\mathbf{x}) = \int\limits_{\Gamma_X} |\langle \phi, \gamma \rangle|^2 \nu_\theta(d\gamma) < \infty,$$

because $\phi_i \in C_0^\infty(X)$, for $i = 1, \cdots, m$ and $\nu_\theta \in \mathcal{M}^2(\Gamma_X)$ (Theorem 2.3.4). This implies that

$$\int\limits_{\mathcal{H}_-} \left| \widehat{F}^N(\mathbf{x}) - \widehat{F}(\mathbf{x}) \right|^2 \theta(d\mathbf{x}) \to 0\,, \quad N \to \infty. \quad (3.2.7)$$

Now let us consider the second integral in (3.2.6). Using formulae (3.2.4) and (3.2.5) we can write

$$\left\| \nabla \widehat{F}^N(\mathbf{x}) - \nabla \widehat{F}(\mathbf{x}) \right\|_0^2 = \sum_{k \in \mathbb{Z}^d} \left| \nabla_k \widehat{F}^N(\mathbf{x}) - \nabla_k \widehat{F}(\mathbf{x}) \right|^2$$

$$
\begin{aligned}
&\leqslant 2\sum_{k\in\mathbb{Z}^d}\sum_{i=1}^{m}\left|\partial_i f\big(\,\widehat{\phi}_j^N(\mathbf{x})\big)_{j=1}^m \nabla_k\widehat{\phi}_i^N(\mathbf{x}) - \partial_i f\big(\,\widehat{\phi}_j(\mathbf{x})\,\big)_{j=1}^m \nabla_k\widehat{\phi}_i(\mathbf{x})\right|^2 \\
&\leqslant 2\sum_{k\in\mathbb{Z}^d}\sum_{i=1}^{m}\Big|\partial_i f\big(\,\widehat{\phi}_j^N(\mathbf{x})\big)_{j=1}^m \nabla_k\widehat{\phi}_i^N(\mathbf{x}) - \partial_i f\big(\,\widehat{\phi}_j(\mathbf{x})\,\big)_{j=1}^m \nabla_k\widehat{\phi}_i(\mathbf{x}) \\
&\qquad + \partial_i f\big(\,\widehat{\phi}_j(\mathbf{x})\,\big)_{j=1}^m \nabla_k\widehat{\phi}_i^N(\mathbf{x}) - \partial_i f\big(\,\widehat{\phi}_j(\mathbf{x})\,\big)_{j=1}^m \nabla_k\widehat{\phi}_i^N(\mathbf{x})\Big|^2 \\
&\leqslant 2\sum_{k\in\mathbb{Z}^d}\sum_{i=1}^{m}\Big|\Big(\partial_i f\big(\,\widehat{\phi}_j^N(\mathbf{x})\big)_{j=1}^m - \partial_i f\big(\,\widehat{\phi}_j(\mathbf{x})\,\big)_{j=1}^m\Big) \nabla_k\widehat{\phi}_i^N(\mathbf{x}) \\
&\qquad + \partial_i f\big(\,\widehat{\phi}_j(\mathbf{x})\,\big)_{j=1}^m \Big(\nabla_k\widehat{\phi}_i^N(\mathbf{x}) - \nabla_k\widehat{\phi}_i(\mathbf{x})\Big)\Big|^2 \\
&\leqslant 4\sum_{k\in\mathbb{Z}^d}\sum_{i=1}^{m}\Big[\Big|\partial_i f\big(\,\widehat{\phi}_j^N(\mathbf{x})\big)_{j=1}^m - \partial_i f\big(\,\widehat{\phi}_j(\mathbf{x})\,\big)_{j=1}^m\Big|^2 \,\Big|\nabla_k\widehat{\phi}_i^N(\mathbf{x})\Big|^2 \\
&\qquad + \Big|\partial_i f\big(\,\widehat{\phi}_j(\mathbf{x})\,\big)_{j=1}^m\Big|^2 \,\Big|\nabla_k\widehat{\phi}_i^N(\mathbf{x}) - \nabla_k\widehat{\phi}_i(\mathbf{x})\Big|^2\Big] \\
&\leqslant 4(a_1(\mathbf{x}) + a_2(\mathbf{x})), \qquad (3.2.8)
\end{aligned}
$$

where

$$
\begin{aligned}
a_1^N(\mathbf{x}) &= 4\sum_{k\in\mathbb{Z}^d}\sum_{i=1}^{m}\Big[\Big|\partial_i f\big(\,\widehat{\phi}_j^N(\mathbf{x})\big)_{j=1}^m - \partial_i f\big(\,\widehat{\phi}_j(\mathbf{x})\,\big)_{j=1}^m\Big|^2 \,\Big|\nabla_k\widehat{\phi}_i^N(\mathbf{x})\Big|^2\Big] \\
a_2^N(\mathbf{x}) &= 4\sum_{k\in\mathbb{Z}^d}\sum_{i=1}^{m}\Big[\Big|\partial_i f\big(\,\widehat{\phi}_j(\mathbf{x})\,\big)_{j=1}^m\Big|^2 \,\Big|\nabla_k\widehat{\phi}_i^N(\mathbf{x}) - \nabla_k\widehat{\phi}_i(\mathbf{x})\Big|^2\Big]. \qquad (3.2.9)
\end{aligned}
$$

We will use the general form of Holder's inequality. In our setting, it can be written as,

$$
\begin{aligned}
&\left[\int_{\mathcal{H}_-}\sum_{i=1}^{m}|\phi_i(\mathbf{x})\psi_i(\mathbf{x})|\theta(d\mathbf{x})\right]^2 \\
&\qquad \leqslant \int_{\mathcal{H}_-}\sum_{i=1}^{m}|\phi_i(\mathbf{x})|^2\theta(d\mathbf{x}) \int_{\mathcal{H}_-}\sum_{i=1}^{m}|\psi_i(\mathbf{x})|^2\theta(d\mathbf{x}), \quad \phi_i\,,\ \psi_i\ \in C_b^\infty(X) \quad (3.2.10)
\end{aligned}
$$

Let us consider $a_1^N(\mathbf{x})$. Using (3.2.10) we obtain

$$\Big[\int_{\mathcal{H}_-} a_1^N(\mathbf{x})\theta(d\mathbf{x})\Big]^2$$
$$\leqslant 4\int_{\mathcal{H}_-}\sum_{i=1}^m \Big|\partial_i f\big(\widehat{\phi}_j^N(\mathbf{x})\big)_{j=1}^m - \partial_i f\big(\widehat{\phi}_j(\mathbf{x})\big)_{j=1}^m\Big|^4 \theta(d\mathbf{x})$$
$$\int_{\mathcal{H}_-}\sum_{i=1}^m\Big[\sum_{k\in\mathbb{Z}^d}\Big|\nabla_k\widehat{\phi}_i^N(\mathbf{x})\Big|^2\Big]^2\theta(d\mathbf{x}). \tag{3.2.11}$$

We know that $f\in C_b^\infty(\mathbb{R}^m)$, therefore the function $\partial_i f$ is globally Lipschitz for any $i=1,\cdots,m$. Thus, there exists a constant $C>0$ such that

$$\sum_{i=1}^m \Big|\partial_i f\big(\widehat{\phi}_j^N(\mathbf{x})\big)_{j=1}^m - \partial_i f\big(\widehat{\phi}_j(\mathbf{x})\big)_{j=1}^m\Big|^4 \tag{3.2.12}$$
$$\leqslant C\sum_{i=1}^m\Big|\widehat{\phi}_i^N(\mathbf{x})-\widehat{\phi}_i(\mathbf{x})\Big|^4 \tag{3.2.13}$$
$$= C\sum_{i=1}^m\Big|\sum_{\substack{k\in\mathbb{Z}^d\\ |k|>N}}\phi_i(x_k+\alpha(k))\Big|^4. \tag{3.2.14}$$

So we have

$$\int_{\mathcal{H}_-}\sum_{i=1}^m \Big|\partial_i f\big(\widehat{\phi}_j^N(\mathbf{x})\big)_{j=1}^m - \partial_i f\big(\widehat{\phi}_j(\mathbf{x})\big)_{j=1}^m\Big|^4 \theta(d\mathbf{x})$$
$$\leqslant C\sum_{i=1}^m\int_{\mathcal{H}_-}\Big|\sum_{\substack{k\in\mathbb{Z}^d\\ |k|>N}}\phi_i(x_k+\alpha(k))\Big|^4\theta(d\mathbf{x})$$

Observe that

$$\int_{\mathcal{H}_-}\Big|\sum_{k\in\mathbb{Z}^d}\phi(x_k+\alpha(k))\Big|^4\theta(d\mathbf{x}) = \int_{\Gamma_X}|\langle\phi,\gamma\rangle|^4\nu_\theta(d\gamma)<\infty,$$

because $\phi_i\in C_0^\infty(X)$, for $i=1,\cdots,m$ and $\nu_\theta\in\mathcal{M}^4(\Gamma_X)$ (Remark 2.3.7). This implies that

$$\int_{\mathcal{H}_-}\sum_{i=1}^m \Big|\partial_i f\big(\widehat{\phi}_j^N(\mathbf{x})\big)_{j=1}^m - \partial_i f\big(\widehat{\phi}_j(\mathbf{x})\big)_{j=1}^m\Big|^4 \theta(d\mathbf{x})\to 0\,,\quad N\to\infty. \tag{3.2.15}$$

Now consider the second integral on the right hand side of the formula (3.2.11). We have

$$\begin{cases} \left|\nabla_k \widehat{\phi}_i^N(\mathbf{x})\right|^2 = |\nabla\phi_i(x_k+\alpha(k))|^2, & |k| \leqslant N, \\ \left|\nabla_k \widehat{\phi}_i^N(\mathbf{x})\right|^2 = 0, & |k| > N. \end{cases}$$

Let us denote

$$\psi_i(\mathbf{x}) = |\nabla\phi_i(x_k+\alpha(k))|^2 .$$

We have

$$\int\limits_{\mathcal{H}_-} \sum_{i=1}^m \left[\sum_{k\in\mathbb{Z}^d} \psi_i(\mathbf{x})\right]^2 \theta(d\mathbf{x}) = \int\limits_{\Gamma_X} |\langle\psi_i,\gamma\rangle|^2 \nu_\theta(d\gamma) < \infty, \tag{3.2.16}$$

because $\psi_i \in C_0^\infty(X)$, for $i = 1, \cdots, m$ and $\nu_\theta \in \mathcal{M}^2(\Gamma_X)$ (Theorem 2.3.4). Combining formulae (3.2.15) and (3.2.16) we obtain that

$$\int\limits_{\mathcal{H}_-} a_1^N(\mathbf{x})\theta(d\mathbf{x}) \to 0 \text{ as } N \to \infty. \tag{3.2.17}$$

Now we consider $a_2^N(\mathbf{x})$. We have

$$\sup_{x\in\mathcal{H}_-} \left|\partial_i f\left(\widehat{\phi}_j(\mathbf{x})\right)_{j=1}^m\right| := C_2 < \infty$$

because $\partial_i f \in C_0^\infty(\mathbb{R}^m)$. By the arguments similar to the proof of convergence (3.2.17) we can write

$$\int\limits_{\mathcal{H}_-} a_2(\mathbf{x})\theta(d\mathbf{x}) = 4C_2 \sum_{i=1}^m \int\limits_{\mathcal{H}_-} \sum_{\substack{k\in\mathbb{Z}^d \\ |k|>N}} \left|\nabla\phi(x_k+\alpha(k))\right|^2 \theta(d\mathbf{x})$$

$$\to 0 \quad \text{as} \quad N\to\infty, \tag{3.2.18}$$

This completes the proof. ■

Let

$$\mathcal{I}^* : L^2(X^{\mathbb{Z}^d},\theta) \to L^2(\Gamma_X,\nu_\theta) \tag{3.2.19}$$

be the adjoint operator of the isometry $\mathcal{I}$. We are now in a position to prove the main result of this section.

Theorem 3.2.4. *Let $v \in Vect_0(X)$ and $F \in \mathcal{F}C(\Gamma_X)$. Then the measure ν_θ on Γ_X given by* (2.3.1) *satisfies the integration by parts formula:*

$$\int_{\Gamma_X} \sum_{x\in\gamma} \nabla_x F(\gamma) \,.\, v(x)\nu_\theta(d\gamma) = \int_{\Gamma_X} F(\gamma)\beta^v_{\nu_\theta}(\gamma)\nu_\theta(d\gamma), \tag{3.2.20}$$

where

$$\beta^v_{\nu_\theta} := \mathcal{I}^* \beta^{\widehat{v}}_\theta \in L^2(\Gamma_X, \nu_\theta). \tag{3.2.21}$$

Proof. The left hand side of the formula (3.2.20) can be written in the form

$$\begin{aligned}\int_{\Gamma_X} \sum_{x\in\gamma} \nabla_x F(\gamma) \,.\, v(x)\nu_\theta(d\gamma) &= \int_{X^{\mathbb{Z}^d}} \sum_{x\in p(\mathbf{x})} \nabla_x F(p(\mathbf{x})) \,.\, v(x)\theta(d\mathbf{x}) \\ &= \int_{X^{\mathbb{Z}^d}} \sum_{k\in\mathbb{Z}^d} \nabla_k \widehat{F}(\mathbf{x}) \,.\, v(x_k + \alpha(k))\theta(d\mathbf{x}) \\ &= \int_{X^{\mathbb{Z}^d}} \Big(\nabla \widehat{F}(\mathbf{x})\,,\, \widehat{v}(\mathbf{x})\Big)_0 \theta(d\mathbf{x}) \end{aligned} \tag{3.2.22}$$

We know from Theorem 3.2.3 that $\widehat{F} \in H^{1,2}(\mathcal{H}_-, \theta)$. Thus we can apply the IBP formula (3.1.10) to get

$$\int_{X^{\mathbb{Z}^d}} \Big(\nabla \widehat{F}(\mathbf{x})\,,\, \widehat{v}(\mathbf{x})\Big)_0 \theta(d\mathbf{x}) = \int_{X^{\mathbb{Z}^d}} \widehat{F}(\mathbf{x})\beta^{\widehat{v}}_\theta(\mathbf{x})\,\theta(d\mathbf{x}) \tag{3.2.23}$$

It has been shown in Proposition 3.1.7 that $\beta^{\widehat{v}}_\theta \in L^2(X^{\mathbb{Z}^d}, \theta)$. Therefore we can rewrite the right hand side of (3.2.23) in the form

$$\int_{X^{\mathbb{Z}^d}} \widehat{F}(\mathbf{x})\beta^{\widehat{v}}_\theta(\mathbf{x})\,\theta(d\mathbf{x}) = \int_{\Gamma_X} F(\gamma)\Big(\mathcal{I}^*\beta^{\widehat{v}}_\theta\Big)(\gamma)\,\nu_\theta(d\gamma). \tag{3.2.24}$$

Combining formulae (3.2.22), (3.2.23) and (3.2.24), we obtain the equality

$$\int_{\Gamma_X} \sum_{x\in\gamma} \nabla_x F(\gamma) \cdot v(x)\nu_\theta(d\gamma) = \int_{\Gamma_X} F(\gamma)\beta^v_{\nu_\theta}(\gamma)\nu_\theta(d\gamma), \tag{3.2.25}$$

where

$$\beta^v_{\nu_\theta} = \mathcal{I}^* \beta^{\widehat{v}}_\theta,$$

and the result follows. ■

Chapter 4

Logorathmic Sobolev Inequality of Push-Forward Measures

4.1 What is Log-Sobolev Inequality

The purpose of this section is to prove the Logarithmic Sobolev Inequality (LSI) for the measure ν_θ. We will derive it using the LSI for the measure θ. We need to introduce suitable framework first.

To introduce the notion of LSI we state some known facts. Let $(Y, \mathcal{B}(Y), \mu)$ be a probability space and $\mathbb{H}$ be a positive self-adjoint operator in $L^2(Y,\mu)$ that is

$$\langle \mathbb{H}f, f\rangle \geqslant 0 \quad \text{for all} \quad f \in \mathcal{D}(\mathbb{H}), \tag{4.1.1}$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product in L^2 and $\mathcal{D}(\mathbb{H})$ denotes the domain of the operator $\mathbb{H}$. We will write L^p for $L^p(Y,\mu)$ in this subsection. Let $T_t := \exp(-t\mathbb{H})$ be the corresponding semigroup in L^2. For $p, q \in [1,\infty]$ we will write

$$\|T_t\|_{q\to p} := sup\left\{\|T_t f\|_p \,:\, f \in L^2 \cap L^q\,, \|f\|_q \leqslant 1\right\},$$

where $\|\cdot\|_q$ denotes the L^q norm. The semigroup T_t is called *contractive* from L^q to L^p if $\|T_t\|_{q\to p} \leqslant 1$. For all $t > 0$ the semigroup T_t is contractive in L^2 if and only if (4.1.1) holds for all $f \in \mathcal{D}(\mathbb{H})$ (See e.g. [Gro75])

The semigroup T_t is called *positivity preserving* if for all Borel measurable functions $f \geqslant 0$ and for all $t > 0$ we have $T_t f \geqslant 0$. The semigroup T_t is *positivity preserving contractive semigroup* if and only if

$$\langle \mathbb{H}f, (f-1)^+\rangle \geqslant 0 \quad \text{for all} \quad f \in \mathcal{D}(\mathbb{H}), \tag{4.1.2}$$

where $f^+(x) = max\{f(x), 0\}$ (See e.g. [Gro75]).

Let μ be the symmetrizing measure for T_t, that is, all operators $T_t, t \geqslant 0$, are symmetric in L^2.

Definition 4.1.1. *Log-Sobolev Inequality*

We say that the measure μ satisfies the Logarithmic Sobolev Inequality with constant $C_{LS} > 0$ iff

$$C_{LS}\,\mu\left(f(\mathbb{H}f)\right) \geqslant \mu(f^2\log(f)) - \|f\|^2\log(\|f\|), \tag{4.1.3}$$

for all $f \in \mathcal{D}(\mathbb{H})$. Here we use the notation

$$\mu(g) = \int g(x)\mu(dx), \qquad g : X \to \mathbb{R}^1.$$

It is proved by Rothaus and Simon [Rot85, Sim76] in their famous *mass gap theorem* that the Logarithmic Sobolev inequality (LSI) implies the Poincaré (or Spectral-gap) inequality with $C_{SG} \geqslant C_{LS}$, that is,

$$\frac{1}{C_{SG}}\,\mu(f(\mathbb{H}f)) \geqslant \mu(f^2) - \|f\|^2,$$

for any $f \in \mathcal{D}(\mathbb{H})$. The Poincaré inequality implies that the spectrum of the operator $\mathbb{H}$ has the gap $(0, C_{SG})$. The following criteria of LSI are known:

Bakry-Emery criterion [BE84]:

Let us consider the function valued bilinear forms associated with the operator $\mathbb{H}$,

$$\Gamma_1(f,g) = \frac{1}{2}\bigg(\mathbb{H}(fg) - f\mathbb{H}g - g\mathbb{H}f\bigg)$$

and

$$\Gamma_2(f,g) = \frac{1}{2}\bigg(\mathbb{H}(\Gamma_1(f,g)) - \Gamma_1(f,\mathbb{H}g) - \Gamma_1(\mathbb{H}f,g).\bigg),$$

where $f, g \in \mathcal{D}(\mathbb{H})$. The forms Γ_1 and Γ_2 are also referred as *carré du champ* and *carré du champ itéré*, respectively. The operator $\mathbb{H}$ satisfies Bakry-Emery condition if there exist a constant C such that

$$\Gamma_2(f,g) \geqslant \frac{1}{C}\Gamma_1(f,g).$$

Let us assume that the semigroup T_t is ergodic, that is,

$$\lim_{t\to\infty} T_t f(w) = \mu(f), \quad \mu - a.s.,$$

for any bounded continuous function f. Then Bakry-Emery criterion states that μ satisfies LSI with $C_{LS} = C$.

Perturbation result of Holley and Stroock [HS87]: Let μ be a probability measure that satisfies LSI with the constant $C_{LS}(\mu)$ and let $\bar{\mu}$ be another probability measure which is absolutely continuous with respect to μ with density $e^{-U(x)}$, that is

$$\bar{\mu}(x) := e^{-U(x)}\mu(x),$$

where $U : \mathbb{R} \to \mathbb{R}$ is a bounded function. Then $\bar{\mu}$ satisfies LSI with the constant

$$C_{LS}(\bar{\mu}) \geqslant C_{LS}(\mu) \cdot e^{-2\mathbf{Osc}\, U},$$

where

$$\mathbf{Osc}\, U = \sup_{x\in X} U(x) - \inf_{x\in X} U(x).$$

Below are the examples of some measures satisfying LSI:

Example 4.1.2. ***Gaussian Measure***

The Gaussian Measure *on* $\mathbb{R}^n$ *satisfies Log Sobolev inequality (see e.g. [Gro75]), with the constant* $C_{LS} = 1$.

Example 4.1.3. ***Product Measure***

Let $\mu_k, k = 1, \cdots, n$ *be the probability measures defined on a Hilbert space* H *such that they satisfy LSI with the constants* $C_{LS}(\mu_k)$ *for* $k = 1, \cdots, n$. *Then the* product measure *given by the formula*

$$\bar{\mu}(\times_{k=1}^{N} dx) := \times_{k=1}^{N} \mu_k(dx_k)$$

satisfies LSI (see e.g. [Gro75]), with the constant

$$C_{LS}\, \bar{\mu} \geqslant \min_{1\leqslant k\leqslant N} C_{LS}(\mu_k).$$

Example 4.1.4. ***Gibbs Measure***

It has been shown in [AKR95] that the Gibbs measure θ_ϵ *on* $X^{\mathbb{Z}^d}$ *given by* (2.2.28), *satisfies LSI with constant* C_{LS}, *provided that the condition given below, in* (4.1.4) *holds. Let* ϵ *be as in* (2.2.28) *and for* $\kappa > 0$ *let us choose* $\kappa_1 \in (0, \kappa_0]$ *such that*

$$\forall\ \epsilon \in [0, \kappa_1]\,, \quad \textit{we have} \quad C_{\mathbb{B}} \geqslant -\epsilon\, 2\, C_p, \tag{4.1.4}$$

where $C_{\mathbb{B}}$ *is as given in* (2.2.26) *and* C_p *is given by*

$$C_p = \inf_{t\in\mathbb{R}} P''(t) > -\infty$$

for the polynomials $P(t)$ *defined in* (2.2.27).

Theorem 4.1.5. *Let* κ_1 *satisfies* (4.1.4). *Then for any* $\epsilon \in [0, \kappa_1]$ *the measure* θ_ϵ *satisfies the log-Sobolev inequality given in* (4.2.9) *with the Sobolev coefficient* $C_{LS} = 2C_{\mathbb{B}}^{-1}$.

Example 4.1.6. *Log-concave measure*
To explain this important result we need definitions of log-concavity *and* ergodicity *and we also recall the framework of rigged Hilbert spaces given in the Section 2.2.1. Let* $\mathcal{X}$ *be a dense linear subset of* $\mathcal{H}_+$ *and let* $\phi, \psi \in \mathcal{X}$. *Let us denote by* $\mathcal{A}(\mathcal{H}_-)$ *the family of measures* μ *on* $\mathcal{H}_-$ *for which the logarithmic derivative* β *exists and is differentiable and square integrable (see Section 3.1 for details). According to the notations introduced in Section 2.2.1 its derivative is identified with the bounded operator* $\beta'(x) \in \mathcal{L}(\mathcal{H}_-, \mathcal{H}_+)$. *The measure* μ *is said to be* uniformly log-concave *if for all* $\phi \in \mathcal{X}$ *there exist a* $c > 0$ *such that*

$$\langle -\beta'(x)\phi, \phi\rangle \; \geqslant \; c|\phi|^2 \quad \mu - a.e.$$

To define ergodicity, let $A \in \mathcal{B}(\mathcal{H}_-)$ *and for all* $z \in \mathcal{X}$ *let us define*

$$A_z = A + z = \{x + z | x \in A\}.$$

We say that A *is* $\mathcal{X}$*-invariant if* $A = A_z$ *for all* $z \in \mathcal{X}$. *Then measure* μ *is said to be* $\mathcal{X}$*-ergodic if either* $\mu(A) = 0$ *or* $\mu(A) = 1$ *for all* $\mathcal{X}$*-invariant sets* A.
In [AKR95], it is proved that measures from the family $\mathcal{A}(\mathcal{H}_-)$ *satisfy LSI if they are uniformly log-concave and* $\mathcal{X}$*-ergodic.*

4.2 Logarithmic Sobolev Inequality on Configuration Spaces

Let θ be a probability measure on $\mathcal{H}_-$ such that $\theta \in \mathfrak{M}(\mathcal{H}_-)$ and it satisfies conditions $(1)-(3)$ of Section 2.2.2. Let us introduce the pre-Dirichlet form $\mathcal{E}_\theta$ on $X^{\mathbb{Z}^d}$:

$$\mathcal{E}_\theta(f,f) = \int\limits_{X^{\mathbb{Z}^d}} \|\nabla f(\mathbf{x})\|_0^2\, \theta(d\mathbf{x}), \tag{4.2.1}$$

where $f \in \mathcal{F}C(X^{\mathbb{Z}^d})$. It has been proved in [AKR95] that $(\mathcal{E}_\theta, \mathcal{F}C(X^{\mathbb{Z}^d}))$ is closable. We denote its closure by $(\mathcal{E}_\theta, D(\mathcal{E}_\theta))$. By the definition, $D(\mathcal{E}_\theta)$ is the completion of $\mathcal{F}C(X^{\mathbb{Z}^d})$ in the norm $\|\cdot\|_{\mathcal{E}_\theta}$ given by the formulae

$$\begin{aligned}\|f\|^2_{\mathcal{E}_\theta} &:= \int\limits_{\mathcal{H}_-} f^2(\mathbf{x})\theta(d\mathbf{x}) + \int\limits_{\mathcal{H}_-} \|\nabla f(\mathbf{x})\|_0^2\theta(d\mathbf{x}) \\ &= \|f\|^2_{H^{1,2}(\mathcal{H}_-,\theta)},\end{aligned} \tag{4.2.2}$$

and therefore

$$D(\mathcal{E}_\theta) = H^{1,2}(\mathcal{H}_-, \theta). \tag{4.2.3}$$

Let us introduce a pre-Dirichlet form $\mathcal{E}_{\nu_\theta}$ associated with the measure ν_θ, defined on functions $F_1, F_2 \in \mathcal{F}C(\Gamma_X) \subset L^2(\Gamma_X, \nu_\theta)$ by the expression

$$\mathcal{E}_{\nu_\theta}(F_1, F_2) = \int\limits_{\Gamma_X} \langle \nabla^\Gamma F_1(\gamma), \nabla^\Gamma F_2(\gamma)\rangle_\gamma \nu_\theta(d\gamma). \tag{4.2.4}$$

Theorem 4.2.1. *We have*

$$\mathcal{I}(D(\mathcal{E}_{\nu_\theta})) \subset D(\mathcal{E}_\theta), \tag{4.2.5}$$

and moreover

$$\mathcal{E}_{\nu_\theta}(F,F) = \mathcal{E}_\theta(\widehat{F}, \widehat{F}), \quad F \in D(\mathcal{E}_{\nu_\theta}), \tag{4.2.6}$$

where $\widehat{F} = \mathcal{I}F = F \circ p$, *see* (3.2.1) *for a detailed construction.*

Proof. It follows from Lemma 3.2.3 and formula (4.2.2) that

$$\mathcal{I}(\mathcal{F}C(\Gamma_X)) \subset D(\mathcal{E}_\theta).$$

By definition $D(\mathcal{E}_{\nu_\theta})$ is the completion of $\mathcal{F}C(\Gamma_X)$ in the norm $\|\cdot\|_{\mathcal{E}_{\nu_\theta}}$, where

$$\|F\|^2_{\mathcal{E}_{\nu_\theta}} := \mathcal{E}_{\nu_\theta}(F,F) + \int\limits_{\Gamma_X} F^2(\gamma)\nu_\theta(d\gamma).$$

Observe that, for $F \in \mathcal{F}C(\Gamma_X)$,

$$\begin{aligned} \|F\|^2_{\mathcal{E}_{\nu_\theta}} &= \mathcal{E}_\theta(\widehat{F},\widehat{F}) + \int\limits_{X^{\mathbb{Z}^d}} \widehat{F}^2(\mathbf{x})\theta(d\mathbf{x}) \\ &=: \|\widehat{F}\|^2_{\mathcal{E}_\theta}. \end{aligned} \tag{4.2.7}$$

Therefore, approximating any $F \in D(\mathcal{E}_{\nu_\theta})$ by a sequence $\{F_n\}_{n=1}^\infty \subset \mathcal{F}C(\Gamma_X)$, we obtain that the sequence $\widehat{F}_n$ converges to an element of $D(\mathcal{E}_\theta)$, and we have $\widehat{F} = \lim_{n\to\infty} \widehat{F}_n \in D(\mathcal{E}_\theta)$. This convergence also implies that

$$\begin{aligned} \mathcal{E}_{\nu_\theta}(F,F) &= \lim_{n\to\infty} \mathcal{E}_{\nu_\theta}(F_n,F_n) \\ &= \lim_{n\to\infty} \mathcal{E}_\theta(\widehat{F}_n,\widehat{F}_n) = \mathcal{E}_\theta(\widehat{F},\widehat{F}). \end{aligned} \tag{4.2.8}$$

■

The LSI for the measure θ takes the form

$$C_{LS}\, \mathcal{E}_\theta(f,f) \geqslant \int\limits_{\mathcal{H}_-} |f(\mathbf{x})|^2\, log|f(\mathbf{x})|\theta(d\mathbf{x}) - \|f\|^2_{L^2(\mathcal{H}_-,\theta)}\, log\|f\|_{L^2(\mathcal{H}_-,\theta)}, \tag{4.2.9}$$

for some constant $C_{LS} > 0$ and any $f \in D(\mathcal{E}_\theta)$.

Theorem 4.2.2. *Let us assume that* θ *satisfies the LSI* (4.2.9)*. Then the measure* ν_θ *satisfies the LSI with the same constant* C_{LS}*, that is, for* $F \in D(\mathcal{E}_{\nu_\theta})$*,*

$$C_{LS}\,\mathcal{E}_{\nu_\theta}(F,F) \geqslant \int\limits_{\Gamma_X} |F(\gamma)|^2\,log|F(\gamma)|\nu_\theta(d\gamma) - \|F\|^2_{L^2(\Gamma_X,\nu_\theta)}\,log\|F\|_{L^2(\Gamma_X,\nu\theta)}, \quad (4.2.10)$$

Proof. We have

$$\begin{aligned} C_{LS}\,\mathcal{E}_{\nu_\theta}(F,F) &= c_{LS}\,\mathcal{E}_\theta(\widehat{F},\widehat{F}) \\ &\geqslant \int\limits_{\mathcal{H}_-} |\widehat{F}(\mathbf{x})|^2\,log|\widehat{F}(\mathbf{x})|\theta(d\mathbf{x}) - \|\widehat{F}\|^2_{L^2(\mathcal{H}_-,\theta)}\,log\|\widehat{F}\|_{L^2(\mathcal{H}_-,\theta)} \\ &= \int\limits_{\Gamma_X} |F(\gamma)|^2\,log|F(\gamma)|\nu_\theta(d\gamma) - \|F\|^2_{L^2(\Gamma_X,\nu_\theta)}\,log\|F\|_{L^2(\Gamma_X,\nu\theta)}, \end{aligned} \quad (4.2.11)$$

because of the formula (4.2.9) and the fact that $\mathcal{I} : L^2(\Gamma_X, \nu_\theta) \to L^2(\mathcal{H}_-, \theta)$ is an isometry. ■

Chapter 5

Motion in Random Media and Stochastic Volatility

5.1 Stochastic Dynamics in Random Environment

The aim of this section is to study random motion of a particle in $X = \mathbb{R}^d$ which interacts with configuration of particles distributed according to the measure ν_θ. The motion is given by a random process satisfying the following stochastic differential equation:

$$dZ(t) = a(Z(t), \gamma)dt + dW(t), \quad z \in X. \tag{5.1.1}$$

Here $W(t)$ is a standard Wiener process in X and the drift coefficient a has the following form

$$a(x, \gamma) = \sum_{y \in \gamma} \zeta(x - y), \tag{5.1.2}$$

for some map $\zeta : X \to X$. We need to establish certain regularity properties of the drift coefficient a, which will guarantee the existence of solution of (5.1.1).

We denote by $C_0^{Lip}(X)$ the class of functions $f : X \to \mathbb{R}$ which satisfy the following conditions:

(i) Lipschitz condition:

$$|f(x) - f(x')| \leqslant C_f |x - x'|,$$

for any $x, x' \in X$ and some constant $C_f > 0$.

(ii) Compact support: There exist $r_f > 0$ such that $\mathrm{supp}(f) \subset B_X(0, r_f)$.

We need the following technical results.

Recall that the map $p : \mathcal{H}_- \to \ddot{\Gamma}_X$ is defined as $p(\mathbf{x}) = \{x_k + \alpha(k)\}_{k \in \mathbb{Z}^d}$ where $\alpha(k) = |k|^{d-1}k$.

Lemma 5.1.1. *For all $\mathbf{y} \in \mathcal{H}_-$ there exists a constant $\mathcal{C} = \mathcal{C}(\mathbf{y})$ such that*

$$\sup_{\mathbf{x} \in B_{\mathcal{H}_-}(\mathbf{y}, \frac{1}{4})} |p(\mathbf{x}) \cap B_X(0, R)| \leqslant \mathcal{C}(\mathbf{y}) R,$$

For all $\mathbf{x} \in B_{\mathcal{H}_-}(\mathbf{y}, \frac{1}{4})$ and $R \geqslant \frac{1}{2}$.

Proof. Let $\mathbf{y} \in \mathcal{H}_-$ be fixed. Similar to the proof of Lemma 2.2.1, for any $\mathbf{x} \in B_{\mathcal{H}_-}(\mathbf{y}, \frac{1}{4})$ there exists a constant $N_1 = N_1(\mathbf{y}) \in \mathbb{N}$ such that $|x_k + \alpha(k)| \geqslant R$ for all $k \in \mathbb{Z}^d$ satisfying the inequality $|k| \geqslant N$ where $N = \max\{N_1, \sqrt[d]{2R+1}\}$.

Then

$$\begin{aligned}|p(\mathbf{x}) \cap B_X(0,R)| &= \#\{k \in \mathbb{Z}^d : |x_k + k| \leqslant R\} \\ &\leqslant \#\{k \in \mathbb{Z}^d : |k| \leqslant N\} = 4\,d\,N \\ &\leqslant 4d \ \max(N_1, 2R+1) \\ &\leqslant 8\,d\,N_1(\mathbf{y})\,(R+\frac{1}{2}) \leqslant 16\,d\,N_1(\mathbf{y})\,R,\end{aligned}$$

provided $R \geqslant \frac{1}{2}$. It proves the statement with $\mathcal{C}(\mathbf{y}) = 16\,d\,N_1(\mathbf{y})$. ■

Lemma 5.1.2. *For ν_θ-a.a. $\gamma \in \Gamma_X$, there exists a constant $\mathcal{C} = \mathcal{C}(\gamma) \in \mathbb{R}$ such that*

$$|\gamma \cap B_X(0,R)| \leqslant \mathcal{C}(\gamma)\,R, \qquad R \geqslant \frac{1}{2}.$$

Proof. It follows from Lemma 5.1.1 that

$$|p(\mathbf{y}) \cap B_X(0,R)| \leqslant \mathcal{C}(y)\,R, \qquad R \geqslant \frac{1}{2}.$$

Recall that the set $p(\mathcal{H}_-)$ has full ν_θ-measure, that is, $\nu_\theta(p(\mathcal{H}_-)) = \theta(\mathcal{H}_-) = 1$. This implies the result with $\mathcal{C}(\gamma) = \mathcal{C}(\mathbf{y})$. ■

Corollary 5.1.3. *For ν_θ-a.a. $\gamma \in \Gamma_X$ and any $R \geqslant \frac{1}{2}$ and $x \in X$ we have*

$$|\gamma \cap B_X(x,R)| \leqslant \mathcal{C}(\gamma)(|x| + R).$$

Proof. The statement follows from Lemma 5.1.2 and the inclusion $B_X(x,R) \subset B_X(0,|x|+R)$. ■

Now we can study regularity properties of the map $a(\cdot,\gamma)$.

Theorem 5.1.4. *Let $\zeta \in C_0^{Lip}(X)$. Then for ν_θ-a.a. $\gamma \in \Gamma_X$ the function $a(\cdot,\gamma)$ is locally Lipschitz with linear growth, that is, it satisfies the following conditions:*

(i) for any $R > 0$, there exists a constant $C_R > 0$ such that

$$|a(x,\gamma) - a(x',\gamma)| \leqslant C_R|x - x'|, \qquad x, x' \in B_X(0,R)$$

(ii) there exist constants $C_1, C_2 > 0$ *such that* $|a(x,\gamma)| \leqslant C_1|x| + C_2$, $x \in X$. *The constants* C_1, C_2, C_R *may depend on* γ.

Proof. (i): Let $x, x' \in B_X(0,R)$, $R > 0$. Then

$$|a(x,\gamma) - a(x',\gamma)| \leqslant \sum_{y\in\gamma} |\zeta(x-y) - \zeta(x'-y)|. \tag{5.1.3}$$

Observe that $\zeta(x-y) = 0$ for any y such that $x - y \notin B_X(0, r_\zeta)$ or equivalently $y \notin B_X(x, r_\zeta)$. In particular, $\zeta(x-y) = 0$ if $x \in B_X(0,R)$ and

$$y \notin \bigcup_{x\in B_X(0,R)} B_X(x, r_\zeta) \subset B_X(0,R'),$$

where $R' = R + r_\zeta$. Therefore the number of non-zero terms in (5.1.3) cannot exceed the number of elements of γ in $B_X(0,R')$, and we have the following inequality:

$$|a(x,\gamma) - a(x',\gamma)| \leqslant \sum_{y\in B_X(0,R')} C_\zeta |(x-y) - (x'-y)|$$

$$\leqslant |\gamma \cap B_X(0,R')| C_\zeta |x - x'| = C_R|x - x'|,$$

where

$$C_R = C(a, R, \gamma) = |\gamma \cap B_X(0, R + r_\zeta)| C_\zeta < \infty,$$

for ν_θ-a.a. $\gamma \in \Gamma$. Thus the local Lipschitz property of $a(\cdot,\gamma)$ is proved.

(ii): We have

$$|a(x,\gamma)| \leqslant \sum_{y\in\gamma} |\zeta(x-y)|.$$

Recall that $\zeta(x-y) = 0$ for $y \notin B_X(x, r_\zeta)$. Moreover it follows from the definition of the class $C_0^{Lip}(X)$ that,

$$\sup_{z\in X} |\zeta(z)| \leqslant C_\zeta r_\zeta.$$

Then similar to the proof of (i), we have,

$$|a(x,\gamma)| \leqslant \sum_{y\in\gamma} |\zeta(x-y)| \leqslant |\gamma \cap B_X(x, r_\zeta)| C_\zeta r_\zeta.$$

Corollary 5.1.3 implies that

$$|\gamma \cap B_X(x, r_\zeta)| \leqslant C_R(\gamma)(|x| + r),$$

for some constant $C_R(\gamma) > 0$, where $r = \max(r_\zeta, \frac{1}{2})$ and the result follows. ■

Theorem 5.1.5. *Let $\zeta \in C_0^{Lip}(X)$. Then, for ν_θ-a.a. $\gamma \in \Gamma_X$, the stochastic differential equation* (5.1.1) *has a unique solution for any initial value $x(0) \in X$ and any time $t \in \mathbb{R}$.*

Proof. Follows from Theorem 5.1.4 and general theory of stochastic differential equations, see e.g. [Øks03]. ■

Our next gaol is to establish certain continuity and smoothness properties of the map $a(\cdot, \gamma)$.

Theorem 5.1.6.
1. *Let $\zeta \in C^1(X, X)$. Then $a(\cdot, \gamma) \in C^1(X, X)$ for ν_θ-a.a. $\gamma \in \Gamma$.*
2. *Let $\zeta \in C^k(X, X)$ for some $k = 1, 2, \cdots$. Then $a(\cdot, \gamma) \in C^k(X, X)$.*

Proof. We proved that for any $R \in \mathbb{R}^+$, any $x \in B_X(0, R)$, there exists $R' \in \mathbb{R}^+$ such that

$$|a(x, \gamma)| = \sum_{y \in \gamma \cap B_X(0, R')} \zeta(x - y),$$

provided ζ has compact support. This sum contains only finite number of non-zero terms (for ν_θ-a.a. $\gamma \in \Gamma$).
This implies that $a(\cdot, \gamma) \in C^1(X, X)$ (resp. $C^k(X, X)$) provided $\zeta(\cdot - y) \in C^1(X, X)$ (resp. $\zeta(\cdot - y) \in C^k(X, X)$) for any $y \in X$. This implies the result of the theorem.

■

5.2 Stochastic Volatility Models

5.2.1 Preliminaries

Derivative securities are contracts with their prices based on the price of another asset called the *primary asset* or *underlaying asset*. In this section we are interested in *European options*. A European option is a contract which can only be exercised at the time of maturity. Let us consider, for example, European call option. It gives its buyer the right, but not the obligation, to buy the the agreed units of the underlaying asset at the predefined time, called the expiration date or maturity date, for a predefined price, called the strike price. Let the strike price be K and let the price of the asset at the expiry time T is S_T then the value of the contract at maturity, that is *pay-off*, can be expressed as

$$h(S_T) = (S_T - K)^+ = \begin{cases} S_T - K & \text{if} \quad S_T > K \\ 0 & \text{if} \quad S_T < K. \end{cases} \tag{5.2.1}$$

If the price at maturity is higher than the strike price than the holder will exercise the option to make a profit. The European options in their standard form, are path independent because the function $h(S_T)$ depends only on the price of the stock at maturity, that is, S_T. The question of derivative pricing involves finding the pricing function $V(t, S_t)$ which gives price at any time t.

Let us consider a filtered probability space $(\Omega, \mathcal{B}(\Omega), \mathcal{F}_t, \mathbb{P})$, where $\{\mathcal{F}_t\}_{t\geqslant 0}$ is the filtration generated by the asset prices up to time $t < T$. Let price of the stock is given by the process S_t satisfying following stochastic differential equation (SDE)

$$dS_t = \mu S_t \, dt + \sigma S_t \, dW_t, \tag{5.2.2}$$

where W_t is the standard Brownian motion, μ and σ are constants representing rate of return and volatility respectively. Black-Scholes-Merton took a root that assures the elimination of risk by adjusting between the risk-less (e.g. bonds, bank accounts) and risky assets (e.g. options). The strategy they use is called *dynamic hedging strategy*, because it allows continuous trading and 'hedging' means reduction in risk. The portfolio of risky and risk-less asset, they consider, assumes following properties; replicating (value of portfolio is almost surely equal to that of the security at time T), self-financing (variations in the value of portfolio are only due the change in prices of assets) and there is no-arbitrage opportunity (to make a profit with no cost). They derived their benchmark Black-Scholes PDE

$$\mathcal{L}_{BS}(V) = 0,$$

where

$$\mathcal{L}_{BS} = \frac{\partial}{\partial t} + \frac{1}{2}\sigma^2 s^2 \frac{\partial^2}{\partial s^2} + r\left(s\frac{\partial}{\partial s} - \cdot\right).$$

This equation holds for $s > 0$ and $t < T$ and is solved backward in time with the final value condition $V(T, s) = h(s)$. The solution of the final value PDE exists and is unique. This solution is called Black-Scholes formula and gives price of the call option given the current price of the stock, time of maturity of the option, the strike price of the option, the volatility of the underlaying asset and the interest rate of the risk-less asset.

The same formula may also be derived with the *equivalent martingale measures* approach. The discounted stock price $\tilde{S}_t = e^{-rt}\, S_t$ satisfies

$$d\tilde{S}_t = (\mu - r)\tilde{S}_t dt + \sigma \tilde{S}_t dW_t, \tag{5.2.3}$$

which implies that the discounted stock price is not a martingale because above expression contains a non-zero drift term if $\mu \neq r$. We can construct another measure $\mathbb{P}^*$, equivalent to $\mathbb{P}$, with respect to which the discounted stock price becomes a martingale and assures no-arbitrage opportunity. The relation between between martingales and no-arbitrage is explained after construction of this measure. This measure is constructed using *Girsanov Theorem*.

Let W_t be the Brownian motion defined on the filtered probability space $(\Omega, \mathcal{B}(\Omega), \mathcal{F}_t, \mathbb{P})$ with $\mathbb{P}(W_0 = 0) = 1$. Let U_t be a $\mathbb{P}$-measurable and $\mathcal{F}_t$-adapted process, satisfying the condition (Novikov condition)

$$\mathbb{E}_{\mathbb{P}} \left[\exp \left(\frac{1}{2} \int_0^T U_t^2 dt \right) \right] < \infty.$$

The stochastic integral

$$\int_0^t U_t dW_t$$

is well defined and is a continuous, local martingale. Let us set

$$M_t = \exp \left[\int_0^t U_s dW_s - \frac{1}{2} \int_0^t S_s^2 ds \right].$$

Then M_t is also a continuous local martingale. Let us define the equivalent martingale measure $\mathbb{P}^*$ by the formula

$$d\mathbb{P}^* = M_T \, d\mathbb{P}.$$

Here $\mathbb{P}^*$ is a probability measure on $(\Omega, \mathcal{B}(\Omega))$.

Theorem 5.2.1 (Girsanov theorem). *(see e.g. [KS91]) Define the process W_t^* by the formula*

$$W_t^* = W_t + \int_0^t U_s ds, \quad 0 \leqslant t < \infty.$$

The process W_t^* is a Brownian motion under the probability measure $\mathbb{P}^*$.

Now we apply Girsanov theorem to change the drift coefficient in (5.2.3) into a non-degenerate diffusion coefficient. Let us re-write (5.2.3) as

$$d\tilde{S}_t = \sigma \, \tilde{S}_t \left[\left(\frac{\mu - r}{\sigma} \right) dt + dW_t \right],$$

Let us set

$$U_t = \varpi := \frac{\mu - r}{\sigma}.$$

In this case we set W_t^* with $U_t = \varpi$ which becomes $W_t^* = W_t + \varpi t$ and then define the probability measure $\mathbb{P}^*$ with $H_T = M_T$ by the formula

$$d\mathbb{P}^* = H_T \mathbb{P}.$$

By Girsanov theorem, the process W_t^* is a Brownian motion under the probability $\mathbb{P}^*$. The discounted price process $\tilde{S}_t$ satisfies

$$d\tilde{S}_t = \sigma\, \tilde{S}_t\, dW_t^*. \tag{5.2.4}$$

Let us consider the portfolio

$$V_t = a_t S_t + b_t e^{rt},$$

where V(t) denotes the value of the portfolio at time t, S_t is the price process for the risky asset (stock) and e^{rt} is the price of risk-less asset (bond) at time t. The pair (a_t, b_t) is called the *trading strategy* and a_t, b_t are adapted processes with respect to the filtration $\{\mathcal{F}_t\}$ and satisfy

$$\mathbb{E}\left\{ \int_0^T a_t^2 dt \right\} < \infty\ , \quad \int_0^T b_t dt < \infty.$$

The self-financing of the portfolio means that the only change in the value is because of the change in the market. It is expressed as

$$dV_t = a_t\, dS_t + r\, b_t\, e^{rt} dt$$

This implies that the discounted value of the portfolio $d\tilde{V}_t = e^{-rt} V_t$ is a martingale under the probability measure $\mathbb{P}^*$ and is a self-financing strategy itself. The proof is simple,

$$d\tilde{V}_t = -re^{-rt} V_t dt + e^{-rt} dV_t$$

using values of V_t and dV_t from above, we get

$$\begin{aligned} d\tilde{V}_t &= -re^{-rt}a_tS_tdt + e^{-rt}a_tdS_t \\ &= a_td(e^{-rt}S_t) \\ &= a_td\tilde{S}_t, \end{aligned} \tag{5.2.5}$$

using (5.2.4), we get

$$d\tilde{V}_t = \sigma\, a_t\tilde{S}_t\, dW_t^*. \tag{5.2.6}$$

Last equation proves that $\tilde{V}_t$ is a martingale with respect to $\mathbb{P}^*$ and $d\tilde{V}_t = a_td\tilde{S}_t$ shows that the portfolio is self-financing.

The relation between no-arbitrage opportunity and the martingale property of the price process (or the value of the portfolio) is very important to elaborate. Let us prove this by contradiction and for that matter let us consider that the trading strategy pair $(a_t, b_t)_{t\geqslant 0}$ is a self-financing but an arbitrage strategy. It means that the value of the portfolio is always higher than the money in the bank, that is,

$$V_T \geqslant e^{rt}V_0, \tag{5.2.7}$$

with

$$\mathbb{P}(V_T \geq e^{rt}V_0) > 0. \tag{5.2.8}$$

But we know from the martingale property that

$$\mathbb{E}^*(V_T) = e^{rT}V_0.$$

Because $\mathbb{P}$ and $\mathbb{P}^*$ are equivalent measures so (5.2.7) and (5.2.8) cannot hold. It completes the proof.

The pay-off of the derivative security is the function of the price of underlying asset at time T. Let us denote that function by $H := h(S_T)$. The portfolio V_t we have considered, will replicate the derivative security if value of portfolio is a.s. equal to the pay-off at the time of maturity T, that is,

$$a_T S_T + b_T e^{rT} = H.$$

As we have proved above that $\tilde{V}_t$ is a martingale under $\mathbb{P}^*$ so we have

$$\tilde{V}_t = \mathbb{E}^* \left\{ \tilde{V}_T \mid \mathcal{F}_t \right\}.$$

Using replicating property and re-introducing the discounting factor, we get

$$\tilde{V}_t = \mathbb{E}^* \left\{ e^{-r(T-t)} H \mid \mathcal{F}_t \right\}.$$

The *Markov property* of the price process S_t says that the expectation with respect to the past $\mathcal{F}_t$ is same as with respect to the process S_t. The value of the portfolio may be written as

$$\tilde{V}_t = \mathbb{E}^* \left\{ e^{-r(T-t)} H(S_T) \mid S_t \right\}.$$

Let $P(t, x)$ represents the price of the derivative security at time t with observed price of the stock $S_t = s$ then the pricing formula becomes

$$P(t, x) = \mathbb{E}^* \left\{ e^{-r(T-t)} H(S_T) \mid S_t = s \right\}.$$

The pricing formula for the price of European derivative security calculated with equivalent martingale measures gives the same value as Black-Scholes formula.

There are many important characteristics of stock return variability observed from Empirical data. First, the implied volatility when plotted against strike price of the asset, gives a convex curve often called “volatility smile” i.e. it changes randomly with jumps of price movements with a tendency to revert to the mean. Second, the volatility of the stock and the spot price are correlated. Wide variety

of research literature is available addressing problem of stock return variability (stochastic volatility models) with volatility satisfying different stochastic processes.
For the purpose of literature review let us fix some notations. Let us consider the following SDE

$$dS_t = \mu S_t dt + f(Y_t) S_t dW_t,$$

where μ is drift, W_t is standard Brownian motion and $f(Y_t)$ gives volatility process. In 1987, [HW87] gave their pioneering work considering non-correlated case of stochastic volatility model with $f(y) = \sqrt{y}$. Later in [Hes93] a closed form solution was presented for the correlated case with Y_t satisfying CIR model. It was further extended to jump, exponential-OU and Lévy processes (for correlated case) in [Sco02, FPS00, CGMY03] and [PSM08]. Recently, [AS09] gave power series solution with volatility satisfying general Itô diffusion process with $f \in C^\infty(\mathbb{R})$.

5.2.2 The Model

Let us consider model of a market with single risky asset S_t with price evolution described by the SDE

$$dS_t = \mu\, S_t\, dt + \sigma_t\, S_t\, dW_t, \tag{5.2.9}$$

where W_t is a 1-dimensional Brownian motion, $\mu \in \mathbb{R}$ is the drift of the asset, and σ_t is a volatility process given by the formula

$$\sigma_t = f(Z_t), \quad t \geqslant 0.$$

Here $f : \mathbb{R}^d \to \mathbb{R}_+$ is a bounded continuous function such that $f(z) \geqslant C > 0, \forall\, z \in \mathbb{R}^d$ and Z_t is a d-dimensional stochastic process given by (5.1.1). We restate (5.1.1) for quick reference

$$dZ_t = a(Z_t, \gamma)dt + dB_t, \quad z \in \mathbb{R}^d, \tag{5.2.10}$$

where B_t is a d-dimensional Brownian motion. We suppose that W_t and B_t are independent. Theorem 5.1.5 implies that solution of equation (5.1.1) exists for any initial value and hence the system of equations (5.2.9) and (5.1.1) has a unique solution for any initial data. We need the following class of functions.

Definition 5.2.2. *We denote by $\mathscr{K}(\mathbb{R}^k)$ for $k \in \mathbb{Z}_+$ the class of continuous and bounded functions $z : \mathbb{R}^k \to \mathbb{R}$ satisfying the following conditions,*

1. *Lipschitz and linear growth conditions, that is,*

$$|z(y) - z(\bar{y})| \leqslant K_1|y - \bar{y}|$$
$$|z(y)| \leqslant K_2(1 + |y|),$$

   where $y, \bar{y} \in \mathbb{R}^k$ and K_1, K_2 are constants.

2. *bound on first and second derivatives, that is*

$$|\partial_y^k z(y)| \leqslant C(1 + |y|^m),$$

   where $\partial_y^k z$ denotes k-th partial derivative of z with respect to y for $k = 1, 2$ and m and C are positive constants.

5.2.3 Pricing Partial Differential Equation

We denote by Ω the space of trajectories of the process $(S_t, Z_t)_{t\geqslant 0}$ and by $\mathbb{P}$ the corresponding distribution which is a probability measure on Ω. Let D be a European derivative security with payoff $h(S_T)$ at time $T > 0$. According to the general approach to pricing theory (see e.g. [FPS00]), the no-arbitrage price D_t of the derivative security D is given by the formula

$$D_t = \mathbb{E}_Q\left(h(S_T)|\mathcal{F}_t\right) e^{-r(T-t)}, \tag{5.2.11}$$

where $(\mathcal{F}_t)_{t\geqslant 0}$ is the filtration generated by the price process S_t, and $\mathbb{Q}$ is an equivalent martingale measure of S_t, that is, a probability measure on the space Ω which is equivalent to $\mathbb{P}$ and such that the process S_t is a $\mathbb{Q}$-martingale.
Let $u(Z_t)$ be a $\mathcal{F}_t$-adapted process defined by the formula

$$u(Z_t) = \frac{\mu - r}{f(Z_t)}, \tag{5.2.12}$$

where r is the risk-free rate of return. To construct the equivalent martingale measures we set

$$M_t = \exp\left[-\int_0^t \{u(Z_\tau)dW_\tau + \chi(S_\tau, Z_\tau)dB_\tau\} - \frac{1}{2}\int_0^t \{u^2(Z_\tau)d\tau + \chi^2(S_\tau, Z_\tau)d\tau\}\right],$$

where $\chi : \mathbb{R}^1 \times \mathbb{R}^d \to \mathbb{R}^d$ is a mapping that satisfies the following condition

$$\mathbb{E}_{\mathbb{P}}\left[\exp\left(\frac{1}{2}\int_0^T \big(\chi(S_\tau, Z_\tau)\big)^2 d\tau\right)\right] < \infty. \tag{5.2.13}$$

Because $f(Z_t) \geqslant C > 0$ therefore $u(Z_t)$ also satisfies a similar condition, that is

$$\mathbb{E}_{\mathbb{P}}\left[\exp\left(\frac{1}{2}\int_0^T u(Z_\tau)^2 d\tau\right)\right] < \infty. \tag{5.2.14}$$

Let us define the processes W_t^* and B_t^* by the formulae

$$W_t^* = W_t + \int_0^t u(Z_\tau)d\tau$$

$$B_t^* = B_t + \int_0^t \chi(S_\tau, Z_\tau)d\tau \tag{5.2.15}$$

and the probability measure $\mathbb{Q}$ on Ω as

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = M_T. \tag{5.2.16}$$

Then by Girsanov theorem the processes B_t^* and W_t^* are Brownian motions under $\mathbb{Q}$. The system of SDEs (5.2.9) and (5.1.1) takes the form

$$dS_t = rS_t dt + f(Z_t)S_t dW_t^* \tag{5.2.17}$$

$$dZ_t = g(t, S_t, Z_t)dt + dB_t^* \tag{5.2.18}$$

where

$$g(t, S_t, Z_t) = a(Z_t) + \chi(S_t, Z_t)$$

Proposition 5.2.3. *The process* $(1+r)^{-t}S_t$ *is a* $\mathbb{Q}$*-martingale.*

Proof. The proof is standard. An application of the Girsanov theorem shows that the process W_t^* is a standard Brownian motion under $\mathbb{Q}$. In particular, equation (5.2.17) implies that $(1+r)^{-t}S_t$ is a martingale. ■

Now we come back to the problem of derivative security D considered at the start of this section. In the light of general theory of derivative pricing we remark that the price process of the derivative security D is given by the formula (5.2.11), where Q is defined by formula (5.2.16). Let us consider the terminal value problem

$$\frac{\partial V}{\partial t} + \frac{1}{2}s^2 f^2(z)\frac{\partial^2 V}{\partial s^2} + \frac{1}{2}\sum_{k=1}^{d}\frac{\partial^2 V}{\partial z_k^2} + r\left(s\frac{\partial V}{\partial s} - V\right) + g(t,s,z)\sum_{k=1}^{d}\frac{\partial V}{\partial z_k} = 0, \tag{5.2.19}$$

together with the terminal condition

$$V(T, s, z) = h(s).$$

Theorem 5.2.4. *Assume that* $f \in \mathscr{K}(\mathbb{R}^d)$, $h \in \mathscr{K}(\mathbb{R})$ *and* $g(t,\cdot) \in \mathscr{K}(\mathbb{R}^{d+1})$ *uniformly in* $t \in [0, T]$ *then the price* D_t *is given by the formula*

$$D_t = V(t, S_t, Z_t),$$

where S_t *and* Z_t *are the stock price and the volatility process respectively and the function* $V(t, s, z)$ *is the solution to terminal value problem* (5.2.19).

Proof. Recall that, because of the Markov property of the process (S_t, Z_t), formula (5.2.11) can be re-written in the form

$$D_t = V(t, S_t, Z_t),$$

where $V(t,s,z) = e^{-r(T-t)}\mathbb{E}_{s,z}\left(h(S_{T-t})\right)$ and $\mathbb{E}_{s,z}$ is the expectation with respect to the solution of (5.2.17) and (5.2.18) with the initial condition

$$S_0 = s, Z_0 = z.$$

Let $\mathcal{U}$ be a differentiable operator defined by the formula

$$\mathcal{U}f(s,z) = \frac{1}{2}s^2 f(z)^2 \frac{\partial^2}{\partial s^2} + \frac{1}{2}\sum_{k=1}^{d}\frac{\partial^2}{\partial z_k^2} + rs\frac{\partial}{\partial s} + g(t,s,z)\sum_{k=1}^{d}\frac{\partial}{\partial z_k}. \tag{5.2.20}$$

Consider the initial value problem

$$\left(\frac{\partial}{\partial t} + \mathcal{U}\right)\widehat{v}(t,s,z) = 0, \tag{5.2.21}$$
$$\widehat{v}(0,s,z) = h(s).$$

Operator $\mathcal{U}$ is the Markov generator of the process (S_t, Z_t) that solves the system (5.2.17) and (5.2.18). Under the conditions assumed on functions f, h and g in the statement of the theorem, there is a unique solution to (5.2.21) given by the formula [Fri75]

$$\widehat{v}(t,s,z) = \mathbb{E}_{s,z}\left(h(S_t)\right). \tag{5.2.22}$$

Making change of time $t \mapsto T - t$ we see that the function

$$v(t,s,z) = \mathbb{E}_{s,z}\left(h(S_{T-t})\right)$$

satisfies the terminal condition problem

$$\frac{\partial v}{\partial t} = \mathcal{U}v\,, \quad v(T,s,z) = h(s) \tag{5.2.23}$$

and thus the pricing function $V(t,s,z)$ satisfies the terminal value problem (5.2.19).

■

List of Notations

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