Structural Characterisations of Hereditary Graph Classes and Algorithmic Consequences

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The candidate confirms that the work submitted is his own, except where work which has formed part of a jointly authored publication has been included. The contribution of the candidate and the other authors to this work has been explicitly indicated below. The candidate confirms that appropriate credit has been given within the thesis where reference has been made to the work of others.

Some parts of the work presented in Chapters 2 and 3 have been published in the following articles:

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The work presented in the first and second sections of Chapter 2 appears in the first publication. I was the main author. The contribution of the other author was that of a primary supervisor.

The first and second sections of Chapter 3 present work from the second and third publications, but the results of Chapter 3 (namely, those presented in Sections 3.3, 3.4 and 3.5) are my own and do not appear in the second or third publications.

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Abstract

A *hole* is a chordless cycle of length at least four, and is *even* or *odd* depending on the parity of its length. Many interesting classes of graphs are defined by excluding (possibly among other graphs) holes of certain lengths. Most famously perhaps is the class of Berge graphs, which are the graphs that contain no odd hole and no complement of an odd hole. A graph is *perfect* if the chromatic number of each of its induced subgraphs is equal to the size of a maximum clique in that subgraph. It was conjectured in the 1960's by Claude Berge that Berge graphs and perfect graphs are equivalent, that is, a graph is perfect if and only if it is Berge. This conjecture was finally resolved by Chudnovsky, Robertson, Seymour and Thomas in 2002, and it is now called the strong perfect graph theorem.

Graphs that do not contain even holes are structurally similar to Berge graphs, and for this reason Conforti, Cornuéjols, Kapoor and Vušković initiated the study of even-hole-free graphs. One of their main results was a decomposition theorem and a recognition algorithm for even-hole-free graphs, and many techniques developed in the pursuit of a decomposition theorem for even-hole-free graphs proved useful in the study of perfect graphs. Indeed, the proof of the strong perfect graph theorem relied on decomposition, and many interesting graph classes have since then been understood from the viewpoint of decomposition.

In this thesis we study several classes of graphs that relate to even-hole-free graphs. First, we focus on β-perfect graphs, which form a subclass of even-hole-free graphs. While it is unknown whether even-hole-free graphs can be coloured in polynomial time, β-perfect graphs can be coloured optimally in polynomial time using the greedy colouring algorithm. The class of β-perfect graphs was introduced in 1996 by Markossian, Gasparian and Reed, and since then several classes of β-perfect graphs have been identified but no forbidden induced subgraph characterisation is known. In this thesis we identify a new class of β-perfect graphs, and we present forbidden induced subgraph characterisations for the class of β-perfect hyperholes and for the class of claw-free β-perfect graphs. We use these characterisations to decide in polynomial time whether a given hyperhole, or more generally a claw-free graph, is β-perfect.

A graph is ℓ-holed (for an integer $\ell \geq 4$) if every one of its holes is of length ℓ. Another focus of the thesis is the class of ℓ-holed graphs. When ℓ is odd, the ℓ-holed graphs form a subclass of even-hole-free graphs. Together with Preissmann, Robin, Sintiari, Trotignon and Vušković we obtained a structure theorem for ℓ-holed graphs

where $\ell \geq 7$. Working independently, Cook and Seymour obtained a structure theorem for the same class of graphs. In this thesis we establish that these two structure theorems are equivalent. Furthermore, we present two recognition algorithms for ℓ-holed graphs for odd $\ell \geq 7$. The first uses the structure theorem of Preissmann, Robin, Sintiari, Trotignon, Vušković and the present author, and relies on decomposition by a new variant of a 2-join called a special 2-join, and the second uses the structure theorem of Cook and Seymour, and relies only on a process of clique cutset decomposition. We also give algorithms that solve in polynomial time the maximum clique and maximum stable set problems for ℓ-holed graphs for odd $\ell \geq 7$.

Finally, we focus on circular-arc graphs. It is a long standing open problem to characterise in terms of forbidden induced subgraphs the class of circular-arc graphs, and even the class of chordal circular-arc graphs. Motivated by a result of Cameron, Chaplick and Hoàng stating that even-hole-free graphs that are pan-free can be decomposed by clique cutsets into circular-arc graphs, we investigate the class of even-hole-free circular-arc graphs. We present a partial characterisation for the class of even-hole-free circular-arc graphs that are not chordal.

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Chapter 1

Introduction

A *graph* is a pair (V, E) of sets where $E \subseteq \binom{V}{2}$. The elements of V are the *vertices* of G and the elements of E are the *edges* of G. In this thesis we deal primarily with *finite* and *simple* graphs, that is, graphs with a finite number of vertices and edges, with at most one edge between any two distinct vertices and with no edge between a vertex and itself.

Many real world problems can naturally be modelled as a graph problem, and as such graph theory sees applications in a wide range of fields, including genomics, electrical engineering, computer science and operations research. Suppose we want to find the shortest route from Leeds to London. By modeling the UK road network as a graph (by letting edges represent roads and vertices represent intersections of roads), the task of finding the shortest route from Leeds to London becomes a problem of finding the shortest path (that is, an alternating sequence of distinct vertices and edges) between two vertices of this graph. This problem of finding the shortest path between two vertices in a graph is well understood and efficient algorithms for solving this problem are known. However, many graph problems that correspond to problems from industry are computationally hard (formally, they are NP-hard, and it is widely believed that NP-hard problems cannot be solved in polynomial time).

We illustrate this by describing an NP-hard graph problem that arises from (a simplified version of) the task of assigning frequencies to cell towers. When making a phone call, a mobile phone connects to a nearby cell tower, which transmits voice and other data between the two parties of the call. Different cell towers broadcast at different frequencies, and these frequencies must be assigned to cell towers in a way that minimises interference. Two nearby cell towers should not broadcast at the same frequency. One way to ensure that there is no interference is to assign to each cell tower a unique frequency, but for practical reasons one is interested in minimising the number of distinct frequencies used. This problem can be modelled as a graph colouring problem. Build a graph by creating a vertex for each cell tower, and link two vertices by an edge if the two corresponding cell towers are within a certain distance (this distance should correspond to the range of transmission of the cell towers). An optimal assignment of frequencies to cell towers can now be found by computing an optimal colouring of this graph; that is, an assignment of colours to the vertices of this graph in such a way that no two adjacent vertices receive the same colour, and such that the number of colours used is as small as possible. Graph colouring is well known to be NP-hard, and therefore it is unlikely that one can find in polynomial time an optimal colouring for any graph.

Towards finding efficient algorithms for colouring and other NP-hard problems, one might restrict the problem to a specific class of graphs instead of the class of all graphs. In this thesis we the study the structure of, and algorithmic consequences for, classes of graphs defined by forbidding certain substructures.

More specifically, our focus will be on so-called hereditary graph classes. Two graphs G and H are *isomorphic* if there exists a bijection $\varphi : V(G) \to V(H)$ such that for all $u, v \in V(G)$, u and v are adjacent in G if and only if $f(u)$ and $f(v)$ are adjacent in H. An *induced subgraph* of a graph G is any graph H with $V(H) \subseteq V(G)$ such that any two vertices $u, v \in V(H)$ are adjacent in H if and only if they are adjacent in G. A class of graphs is *hereditary* if it is closed under isomorphism and under taking induced subgraphs. Many well studied graph classes are hereditary, including forests, planar graphs, k-colourable graphs, and any class of graphs defined by forbidden induced subgraph (that is, graphs that do not contain some fixed set of graphs as induced subgraphs).

For graphs G and H, we say that G *contains* H if some induced subgraph of G is isomorphic to H, and that G is *H-free* if G does not contain H. For a family of graphs $\mathcal{H}$, we say G is $\mathcal{H}$-free if G is H-free for every $H \in \mathcal{H}$. An induced subgraph F of G is *proper* if $V(F) \neq V(G)$. A graph G is a *minimal forbidden induced subgraph* for a hereditary class of graphs $\mathcal{C}$ if G does not belong to $\mathcal{C}$ but every proper induced subgraph of G belongs to $\mathcal{C}$.

A property of hereditary graph classes is that they can be characterised by a list of minimal forbidden induced subgraphs.

Theorem 1.1. *Let $\mathcal{C}$ be a hereditary class of graphs, and let $\mathcal{H}$ be the set of all minimal forbidden induced subgraphs for $\mathcal{C}$. Then $\mathcal{C}$ is exactly the class of $\mathcal{H}$-free graphs.*

Proof. If $G \in \mathcal{C}$, then by the definition of $\mathcal{C}$, every induced subgraph of G also belongs to $\mathcal{C}$, and since no graph in $\mathcal{H}$ belongs to $\mathcal{C}$, it follows that G is $\mathcal{H}$-free. Thus, every graph in $\mathcal{C}$ is $\mathcal{H}$-free. To see that every $\mathcal{H}$-free graph belongs to $\mathcal{C}$, let G be $\mathcal{H}$-free and suppose that $G \notin \mathcal{C}$. Let H be a minimum induced subgraph of G that does not belong to $\mathcal{C}$. Then $H \in \mathcal{H}$, and therefore G is not $\mathcal{H}$-free, a contradiction. Thus, every $\mathcal{H}$-free graph belongs to $\mathcal{C}$. □

For many interesting hereditary graph classes, finding a characterisation in terms of minimal forbidden induced subgraphs is no simple task. For instance, it was conjectured by Claude Berge in the 1960's that for the class of perfect graphs (which we will discuss soon) there are two types of minimal forbidden induced subgraphs: odd holes (a chordless cycle on an odd number of at least five vertices) and odd antiholes (the complement of an odd hole, where the *complement* of a graph G is the graph $\overline{G}$ with vertex set $V(G)$ such that distinct $u, v \in V(G)$ are adjacent in $\overline{G}$ if and only if they are nonadjacent in G). This conjecture received much attention over 40 years, and it was finally proved by Maria Chudnovsky, Neil Robertson, Paul Seymour and Robin Thomas in 2002. In this thesis (in Chapters 2 and 4) we work towards finding characterisations in terms of minimal forbidden induced subgraphs for β-perfect graphs and for a subclass of circular-arc graphs.

A graph is *connected* if there is a path between any two vertices, and is *disconnected* otherwise. A *cutset* is any set C of vertices or edges of a graph G such that the graph obtained from G by removing the vertices and edges from C is disconnected. The use of decomposition has proven very useful in the study of hereditary graph classes. In this thesis we will often be concerned with "decomposition theorems" for hereditary graph classes. A decomposition theorem for a class of graphs $\mathcal{C}$ is any theorem of the form "if G is a graph that belongs to $\mathcal{C}$, then G is 'basic' or G has a certain type of cutset."

For decomposition theorems to be useful, what it means to be "basic" and the types of cutset used must satisfy certain properties. Suppose, for instance, that we have a decomposition theorem for a class of graphs $\mathcal{C}$, and we want to use this decomposition theorem in order to solve the recognition problem for this class. That is, we wish to give an algorithm that decides whether a given graph belongs to the class $\mathcal{C}$. In this case, we need that the basic graphs are "easily" recognisable, and that decomposition by the cutsets used is class preserving.

What we mean by class preserving is the following. When decomposing a graph G by a cutset C, one removes C from G and the resulting graph has several components, from which one forms so-called blocks of decomposition, one for each component, by taking each component and adding vertices or edges. A *clique* is a set of pairwise adjacent vertices, and a *clique cutset* is a cutset that is a clique. To form the blocks of decomposition with respect to a clique cutset, for example, take each component of the graph obtained by removing the cutset, and add back the vertices of the cutset and any edges of the original graph between vertices of the cutset and vertices of the component. A type of cutset is *class preserving* for a class $\mathcal{C}$ provided that a graph G belongs to $\mathcal{C}$ if and only if all the blocks of decomposition of G with respect to any cutset of that type also belong to $\mathcal{C}$.

Each of the next three chapters focuses on a class of graphs that in some way relates to the class of even-hole-free graphs. We mentioned holes informally earlier when talking about perfect graphs, but let us now define them formally. A *cycle* (of *length* k) is a graph C with vertex set $\{x_1, \ldots, x_k\}$ (where $k \geq 3$) and edge set $\{x_1x_2, x_2x_3, \ldots, x_{k-1}x_k, x_kx_1\}$. A *hole* of a graph G is an induced cycle of length at least 4, and a hole is *odd* or *even* depending on the parity of its length. For $k \geq 4$, a *k-hole* is a hole of length k, and we denote such a hole by C_k. A graph is *even-hole-free* if it contains no even hole.

For a graph G, we denote by $\chi(G)$ the *chromatic number* of G, i.e., the minimum number of colours needed to colour the vertices of G so that no two adjacent vertices receive the same colour; and by $\omega(G)$ we denote the size of a largest clique in G. A graph G is *perfect* if $\chi(H) = \omega(H)$ for every induced subgraph H of G. Claude Berge was the first to study perfect graphs, in part motivated by a problem from information theory. Perfect graphs also have connections to linear and integer programming, see e.g. [1]. Many NP-hard graph problems, such as colouring, finding a maximum clique and finding a maximum stable set (a *stable set* is a set of pairwise nonadjacent vertices) can all be solved in polynomial time for perfect graphs. By the strong perfect graph theorem [12], a graph is perfect if and only if it contains no odd hole and no odd antihole. Since a graph that contains no hole of length 4 also contains no antihole of length at least 6, even-hole-free graphs are structurally quite similar to perfect graphs. This observation was the initial motivation for the study of even-hole-free graphs.

Techniques discovered from the study of even-hole-free graphs were used in the context of perfect graphs, and it was a decomposition based approach that ultimately led to the proof of the strong perfect graph theorem. The nature of even-hole-free graphs has been studied through their generalisation to so-called odd-signable graphs. A *signing* of a graph G is an assignment of 0,1 weights to each edge of G. The *weight* of an induced subgraph of G is the sum of the weights of its edges. A graph is *odd-*

signable if it admits a signing in which every triangle and every hole has odd weight. A graph is *even-signable* if it admits a signing in which every triangle has odd weight and every hole has even weight. By considering the signing that assigns weight 1 to every edge, one sees that even-hole-free graphs are odd-signable and odd-hole-free graphs are even-signable.

Odd-signable and even-signable graphs can be characterised by the fact that they do not contain certain types of special graphs called Truemper configurations as induced subgraphs.

A *triangle* is a complete graph on three vertices. A *prism* is a graph made of three vertex-disjoint paths $P_1 = a_1, \dots, b_1$, $P_2 = a_2, \dots, b_2$ and $P_3 = a_3, \dots, b_3$, each of length at least 1, such that $\{a_1, a_2, a_3\}$ and $\{b_1, b_2, b_3\}$ induce triangles, and there are no edges between these three paths except those of the two triangles. A *pyramid* is a graph made of three paths $P_1 = a, \dots, b_1$, $P_2 = a, \dots, b_2$ and $P_3 = a, \dots, b_3$, each of length at least 1, two of which have length at least 2, with $V(P_1) \cap V(P_2) \cap V(P_3) = \{a\}$, and such that $\{b_1, b_2, b_3\}$ induces a triangle and there are no edges between these three paths except those of the triangle and those incident to a. A *theta* is a graph made of three internally vertex-disjoint paths (that is, intersecting possibly only at their ends) $P_1 = a, \dots, b$, $P_2 = a, \dots, b$ and $P_3 = a, \dots, b$ of length at least 2 such that there are no edges between these three paths except the three edges incident to a and the three edges incident to b. We may denote prisms, pyramids and thetas with three paths as above by $3PC(a_1a_2a_3, b_1b_2b_3)$, $3PC(b_1b_2b_3, a)$ and $3PC(a, b)$ respectively. A *three-path-configuration* (or *3PC* for short) is any prism, pyramid or theta. A *wheel* consists of a hole H, called the *rim*, together with an additional vertex x, called the *centre*, that has at least 3 neighbors in the hole; we sometimes denote such a wheel by (H, x). A wheel is *odd* if it contains an odd number of triangles, and is *even* if the centre has an even number of neighbours in the rim. A *Truemper configuration* is any 3PC or wheel (see Figure 1.1). We refer the reader to [52] for a survey on the use of Truemper configurations in the study of hereditary graph classes.

Theorem 1.2 (Conforti, Cornuéjols, Kapoor and Vušković [17]). *A graph is odd-signable if and only if it contains no theta, prism, or even wheel.*

Theorem 1.3 (Conforti, Cornuéjols, Kapoor and Vušković [17]). *A graph is even-signable if and only if it contains no pyramid or odd wheel.*

The first decomposition theorem for even-hole-free graphs was in fact a decomposition theorem for 4-hole-free odd-signable graphs [17], and this led to the first polynomial-time recognition algorithm [18] for the class of even-hole-free graphs (with

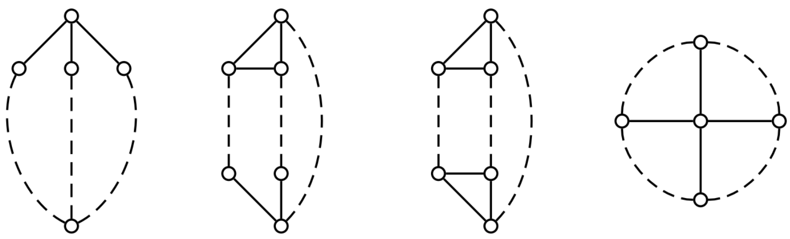


Figure 1.1: From left-to right: a theta, pyramid, prism and wheel. Dashed lines denote paths of length at least 1.

running time of about $\mathcal{O}(n^{40})$ for an n-vertex graph). Chudnovsky, Kawarabayashi and Seymour [10] used a method called cleaning (which is also used in algorithms for recognising perfect graphs) to obtain a faster recognition algorithm for even-hole-free graphs, with running time $\mathcal{O}(n^{31})$. da Silva and Vušković [22] strengthened the decomposition theorem for 4-hole-free odd-signable graphs from [17] and as a consequence obtained a recognition algorithm for even-hole-free graphs with running time $\mathcal{O}(n^{19})$. Chang and Lu [9], also using the decomposition theorem from [22], gave a recognition algorithm for even-hole-free graphs with running time $\mathcal{O}(n^{11})$. The fastest known algorithm to date for recognising even-hole-free graphs runs in $\mathcal{O}(n^{9})$ time and is due to Lai, Lu and Thorup [39]. The class of even-hole-free graphs is still an active object of research. It remains open whether one can solve the colouring and maximum stable set problems in polynomial time for even-hole-free graphs.

Even-hole-free graphs are also of interest due to their connection to β-perfect graphs, which form the focus of Chapter 2. We now outline the contributions of the thesis.

1.1 Contributions of the thesis

Chapter 2: β-perfect graphs

For a graph G, let $\beta(G)$ be the maximum of $\delta(H) + 1$ taken over all induced subgraphs H of G, where $\delta(G)$ denotes the minimum degree of a vertex in G. A graph G is *β-perfect* if $\chi(H) = \beta(H)$ for every induced subgraph H of G. The class of β-perfect graphs is a subclass of the class of even-hole-free graphs. By colouring greedily with respect to an easily-computable ordering of the vertices of a β-perfect graph, one can optimally colour β-perfect graphs in polynomial time (we explain this in more detail in the introduction to Chapter 2). The class of β-perfect graphs was first introduced in 1996 by Markossian, Gasparian and Reed. Since then, a number of classes of graphs have been shown to be β-perfect, but it remains open to characterise the class of β-perfect graphs in terms of forbidden induced subgraphs and to give an algorithm that in polynomial time decides whether a graph is β-perfect.

A *twin wheel* is a wheel whose centre has exactly three neighbours in the hole, and these three neighbours form a path. A *cap* is any graph that consists of a hole together with an additional vertex that has exactly two neighbours in the hole, and these two neighbours are adjacent. In Section 2.1, we prove that (even hole, twin wheel, cap)-free graphs are β-perfect. This generalises a result of Markossian, Gasparian and Reed [42], who showed that (even hole, diamond, cap)-free graphs are β-perfect (where a *diamond* is the graph obtained from the complete graph on four vertices by removing one edge).

A *hyperhole* is any graph G consisting of $k \geq 4$ cliques $X_1, \dots, X_k$ such that (with subscripts modulo k) X_i is complete to $X_{i-1} \cup X_{i+1}$ and anticomplete to $V(G) \setminus (X_{i-1} \cup X_i \cup X_{i+1})$ for all $i \in \{1, \dots, k\}$. In Section 2.2, we give a forbidden induced subgraph characterisation of β-hyperholes, and we use this characterisation to decide in linear time whether a given hyperhole is β-perfect.

The *claw* is the graph that consists of three pairwise nonadjacent vertices that are all adjacent to an additional fourth vertex (in other words, it is the complete bipartite graph $K_{1,3}$). In Section 2.3, we give a forbidden induced subgraph characterisation of claw-free β-perfect graphs. This result relies heavily on a forbidden induced subgraph characterisation of β-perfect "rings", which are a generalisation of hyperholes. Thus, in proving the main result of Section 2.3, we obtain a generalisation of the result of Section 2.2. From our characterisation of claw-free β-perfect graphs, we derive an algorithm that decides in polynomial time whether a given claw-free graph is β-perfect.

Chapter 3: Graphs with all holes the same length

For an integer $\ell \geq 4$, a graph G is *ℓ-holed* if every hole of G is of length ℓ. Chapter 3 is concerned with ℓ-holed graphs. A group consisting of Myriam Preissmann, Cléophée Robin, Ni Luh Dewi Sintiari, Nicolas Trotignon, Kristina Vušković and the present author worked on obtaining a structure theorem for the class of ℓ-holed graphs (where $\ell \geq 7$). It was discovered during this work that another group of researchers, consisting of Linda Cook and Paul Seymour, were working on this same problem at the same time. Both groups independently obtained structure theorems, but came together to submit work as a joint publication [19]. Each structure theorem describes exactly the structure of ℓ-holed graphs that have no clique cutset and no universal vertex (a vertex is *universal* if it is adjacent to every other vertex in the graph). Since piecing together two ℓ-holed graphs along a clique (in other words, reversing a clique cutset) yields another ℓ-holed graph, and similarly so does adding a universal vertex to an ℓ-holed graph, theorems that describe the structure of ℓ-holed graphs that have no clique cutset and no universal vertex tell us how all ℓ-holed graphs may be generated.

In Section 3.1 we present the structure theorem of Cook and Seymour, and in Section 3.2 we present the structure theorem of Preissmann, Robin, Sintiari, Trotignon, Vušković and the present author. The joint publication [19] contains the first structure theorem. Since only one of these structure theorems is to be peer reviewed, it is of interest to establish that the two structure theorems are equivalent; in Section 3.3, we prove that they are indeed equivalent.

In Section 3.4, we introduce a variant of 2-joins called special 2-joins. (2-joins are a type of edge cutset that are used in the decomposition of perfect graphs [12] and even-hole-free graphs [17], for instance.) In Section 3.4.3, we give a clique cutset and special 2-join based decomposition theorem for ℓ-holed graphs (where $\ell \geq 7$ is odd), and we use this decomposition theorem in Section 3.5.2 to decide in polynomial time whether a graph is ℓ-holed for some odd $\ell \geq 7$. In Section 3.5, we give polynomial time algorithms that solve the maximum clique and maximum stable set problems for ℓ-holed graphs when $\ell \geq 7$ is odd, and we present a second recognition algorithm for ℓ-holed graphs when $\ell \geq 7$ is odd which is based on a process of clique cutset decomposition. Finally, we conclude the chapter of ℓ-holed graphs in Section 3.5.4 with a discussion on how one can recognise ℓ-holed graphs without the use of decomposition.

Chapter 4: Even-hole-free circular-arc graphs

Let C be a circle and let $\mathcal{A}$ be a collection of arcs of C. Consider the graph whose vertex set consists of the arcs from $\mathcal{A}$ such that $A, A' \in \mathcal{A}$ are adjacent if and only if $A \cap A' \neq \emptyset$. Any graph constructible in this way is called a *circular-arc graph*. These graphs generalise the well known class of interval graphs, which are the intersection graphs of intervals of the real line. Interval graphs are well understood, in the sense that we have a number of characterisations for the class of interval graphs, among them a forbidden induced subgraph characterisation. Many problems on graphs that are NP-hard in general, such as colouring and finding a maximum clique or stable set, are solvable in polynomial time on interval graphs. However, things are more complicated when one turns to circular-arc graphs. First, there is no known forbidden induced subgraph characterisation of circular-arc graphs, and second, problems such as colouring remain NP-complete when restricted to circular-arc graphs. In [6], even-hole-free graphs that are pan-free (a *pan* is a hole together with a vertex that has exactly one neighbour in the hole) are decomposed by clique cutsets into so-called unit circular-arc graphs, which gives way to a linear-time recognition algorithm and a polynomial-time colouring algorithm for this class. Motivated by this connection between even-hole-free graphs and circular-arc graphs, in Chapter 4 we work towards characterising even-hole-free circular-arc graphs. Since it is a long standing open problem to characterise chordal circular-arc graphs, we restrict our focus to even-hole-free circular-arc graphs that are not chordal. We obtain a partial result in this direction; namely, we give a forbidden induced subgraph characterisation of even-hole-free circular-arc graphs that are not chordal that furthermore do not contain so-called crossing vertices.

1.2 Terminology and notation

Let G be a graph. The vertex set of G is denote by $V(G)$ and the edge set of G by $E(G)$. When referring to an edge $\{x, y\}$ of G, we may simply write xy, and we say that x, y are the *ends* of the edge. For a vertex $x \in V(G)$, we denote by $N_G(x)$ (or simply $N(x)$ when G is clear from context) the set of all neighbours of x in G, and $N_G[x]$ denotes the closed neighbourhood of x in G, i.e., the set $N_G(x) \cup \{x\}$. If $S \subseteq V(G)$, then we denote by $N_S(x)$ the set $N(x) \cap S$, and similarly if S is an induced subgraph of G, then we denote by $N_S(x)$ the set $N(x) \cap V(S)$. For a set $S \subseteq V(G)$, we denote by $N_G(S)$ the set of vertices of G not in S that have a neighbour in S. We denote by $d_G(x)$ (or $d(x)$ if G is clear from context) the *degree* of x in G, i.e., $d_G(x) = |N_G(x)|$. By $\delta(G)$ we denote the minimum degree of a vertex in G, and by $\Delta(G)$ we denote the maximum degree of a vertex in G.

If A and B are disjoint subsets of $V(G)$, then we say that A is *complete* to B if every vertex of A is adjacent to every vertex of B, and that A is *anticomplete* to B if every vertex of A is nonadjacent to every vertex of B. If A is a singleton, say $A = \{a\}$, then we may say that a is complete or anticomplete to B.

A *clique* is a set of pairwise adjacent vertices, and a *stable set* is a set of pairwise nonadjacent vertices. The size of a maximum clique in G is denoted by $\omega(G)$ and the size of a maximum stable set in G is denoted by $\alpha(G)$. A *complete graph* is a graph whose vertex set is a clique. We refer to the complete graph on three vertices as the *triangle*, and denote by K_k the complete graph on k vertices. A graph G is *bipartite* if $V(G)$ admits a partition (A, B) such that every edge of G has one end in A and the other end in B; under these circumstances, the partition (A, B) is referred to as the *bipartition* of G. A bipartite graph with bipartition (A, B) is *complete* if A is complete to B. For integers $m, n \geq 1$ we denote by $K_{m,n}$ the complete bipartite graph with bipartition (A, B) where $|A| = m$ and $|B| = n$.

For a set $S \subseteq V(G)$ we denote by $G[S]$ the subgraph of G induced by S, and by $G \setminus S$ the subgraph of G induced by $V(G) \setminus S$. For two graphs G and H we denote by $G \cup H$ the *disjoint union* of G and H, i.e., the graph with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H)$. For a graph G, the *complement* of G is the graph $\overline{G}$ with vertex set $V(G)$ such that distinct vertices $u, v \in V(G)$ are adjacent in $\overline{G}$ if and only if they are nonadjacent in G.

In this thesis, by a *path* we mean an induced (i.e., chordless) path. If P is a path with $V(P) = \{x_1, \dots, x_k\}$ and edge set $\{x_1x_2, \dots, x_{k-1}x_k\}$, we may write $P = x_1, \dots, x_k$. The two vertices of degree 1 of a path are called its *ends*. If P is a path, say with ends x and y, then P is called an *xy-path*; we denote by P^* the set $V(P) \setminus \{x, y\}$, and call this set the *interior* of P. The *length* of a path is the number of its edges. For vertices x, y of G we denote by $d_G(x, y)$ (or simply $d(x, y)$) the *distance* between x and y, i.e., the length of a shortest xy-path. For an integer $k \geq 1$ we denote by P_k the path on k vertices, and for an integer $k \geq 4$, we denote by C_k the hole on k vertices. A graph is *chordal* if it has no hole.

A graph G is *connected* if between any two distinct vertices u, v of G there is a uv-path, and is *disconnected* otherwise. A graph is *k-connected* if it has more than k vertices and there exists no set of $k - 1$ vertices whose removal yields a disconnected graph. A *component* of a graph G is a maximal connected subgraph of G. A clique C in a graph G is a *clique cutset* if $G \setminus C$ is disconnected. A *tree* is a connected graph with no cycles. The vertices of a tree of degree 1 are called the *leaves* of the tree.

If C is a clique cutset of a graph G and $C_1, \dots, C_k$ are the components of $G \setminus C$, then the *blocks of decomposition* of G with respect to C are the graphs $G[V(C_i) \cup C]$ for $i \in \{1, \dots, k\}$. A *clique cutset decomposition tree* of a graph G is a tree T satisfying the following:

- the root of T is G;
- each non-leaf node H of T has a clique cutset C such that $V(H) \setminus C$ admits a partition (A, B) where A is anticomplete to B in H, and the children of H in T are the graphs $G[A \cup C]$ and $G[B \cup C]$, one of which has no clique cutset and is a leaf of T;
- the leaves of T are induced subgraphs of G that have no clique cutset.

Chapter 2

β-perfect graphs

In 1996, Markossian, Gasparian and Reed introduced in [42] the class of β-perfect graphs. For a graph G, let $\beta(G)$ be the maximum of $\delta(H) + 1$ taken over all induced subgraphs H of G. A graph G is *β-perfect* if $\chi(H) = \beta(H)$ for every induced subgraph H of G. We say that G is *β-imperfect* if G is not β-perfect, and that G is *minimally β-imperfect* if G is β-imperfect but all the proper induced subgraphs of G are β-perfect.

The class of β-perfect graphs is a subclass of the class of even-hole-free graphs:

Lemma 2.1 (Markossian, Gasparian and Reed [42])**.** *If G is a β-perfect graph, then G is even-hole-free.*

Proof. If H is an even hole, then $\chi(H) = 2$ and $\beta(H) = 3$. $\square$

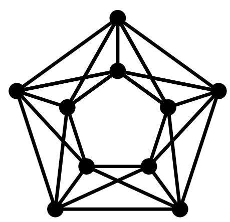


Figure 2.1: An even-hole-free graph that is not β-perfect.

Since there are even-hole-free graphs that are not β-perfect (see Figure 2.1), the class of all β-perfect graphs forms a proper subclass of even-hole-free graphs.

For any graph G, the parameter $\beta(G)$ is an upper bound on $\chi(G)$, as we now show. Let $V(G) = \{v_1, \ldots, v_n\}$ say, and let $v_1, \ldots, v_n$ be any ordering of $V(G)$. Now colour the vertices of G *greedily* with respect to this ordering; that is, consider each vertex among $v_1, \ldots, v_n$ in order, and assign to the vertex v_i the smallest positive integer not already assigned to any neighbour of v_i in $G[\{v_1, \ldots, v_i\}]$. It is easily seen that this algorithm produces a colouring of G that uses at most $\Delta(G) + 1$ colours (and that it runs in time $\mathcal{O}(|V(G)| + |E(G)|)$). Suppose now that $V(G)$ is ordered as $v_1, \ldots, v_n$ so that, for each $i \in \{1, \ldots, n\}$, the vertex v_i is of minimum degree in $G[\{v_1, \ldots, v_i\}]$. Colouring greedily with respect to this ordering produces a colouring of G that uses at most $\beta(G)$ colours. Thus, for any graph G, the parameter $\beta(G)$ is an upper bound on the chromatic number of G. It now follows from the definition of β-perfect that the greedy colouring algorithm, applied in the way just described, produces (in polynomial time) optimal colourings for β-perfect graphs. It is unknown whether even-hole-free graphs in general can be coloured in polynomial time.

Throughout this chapter we will use several known properties of minimal β-imperfect graphs, the first being:

Lemma 2.2. *If G is a minimal β-imperfect graph, then $\beta(G) = \delta(G) + 1$.*

Proof. Let G be a minimal β-imperfect graph and let H be an induced subgraph of G such that $\beta(G) = \delta(H) + 1$. Suppose H is a proper induced subgraph of G. By our choice of H we have that $\beta(H) = \beta(G)$, and therefore, since H is β-perfect, $\chi(H) = \beta(H) = \beta(G)$. But then, since $\chi(G) \geq \chi(H)$, we have that $\chi(G) \geq \beta(G)$, and hence $\chi(G) = \beta(G)$, contradicting our assumption that G is minimally β-perfect. So $H = G$ and $\beta(G) = \delta(G) + 1$. □

For the proof of the next property of minimal β-imperfect graphs, we need the following well-known result of Dirac about chordal graphs. A vertex is *simplicial* if its neighbourhood is a clique, and is a *simplicial extreme* if it is simplicial or has degree 2.

Theorem 2.3 (Dirac [25])**.** *If G is a chordal graph that is not a complete graph, then G contains at least two nonadjacent simplicial vertices.*

Lemma 2.4 (Markossian, Gasparian and Reed [42])**.** *If G is a minimal β-imperfect graph that is not an even hole, then G contains no simplicial extreme.*

Proof. On the contrary, suppose G is a minimal β-imperfect graph that contains a simplicial extreme v. If $N_G(v)$ is a clique, then by Lemma 2.2, $\beta(G) = \delta(G) + 1 \leq d_G(v) + 1 \leq \omega(G) \leq \chi(G)$, and hence $\beta(G) \leq \chi(G)$, a contradiction. So G contains no simplicial vertex, and therefore $d_G(v) = 2$. Furthermore, by Theorem 2.3, G is not chordal, and therefore G contains a hole H. Since G is not an even hole, and since even holes are minimally β-imperfect, it follows that G is even-hole-free, and therefore H is an odd hole. Thus $\chi(H) = 3$, and hence $\chi(G) \geq 3$. By Lemma 2.2, $\beta(G) = \delta(G) + 1 \leq 3 \leq \chi(G)$, a contradiction. □

By the definition of β-perfect, the class of β-perfect graphs is hereditary, and therefore there exists some forbidden induced subgraph characterisation for this class. In the case of perfect graphs, it was conjectured by Berge in 1961 [2], and proved by Chudnovsky, Robertson, Seymour and Thomas in 2002 [12], that a graph is perfect if and only if it contains no odd hole and no complement of an odd hole. At present, no forbidden induced subgraph characterisation for the class of β-perfect graphs is known or has been conjectured, and there exists no known algorithm that decides whether a given graph is β-perfect. However, several classes of graphs defined by forbidden induced subgraphs have been identified as subclasses of the class of β-perfect graphs. We now survey these results.

A *diamond* is any graph obtained by removing exactly one edge from the complete graph on four vertices. A *cap* is any graph consisting of a hole together with an additional vertex whose neighbourhood in the hole consists of two adjacent vertices.

Theorem 2.5 (Markossian, Gasparian and Reed [42]). *If G is an (even hole, diamond, cap)-free graph, then G is β-perfect.*

A *6-cap* is a cap on exactly 6 vertices. de Figueiredo and Vušković generalised Theorem 2.5 with the following.

Theorem 2.6 (de Figueiredo and Vušković [24]). *If G is an (even hole, diamond, 6-cap)-free graph, then G is β-perfect.*

Another subclass of (even hole, diamond)-free graphs shown to be β-perfect is the following. A *net* is the graph on six vertices a, b, c, x, y, z with edge set $\{ab, bc, ac, ax, by, cz\}$.

Theorem 2.7 (Keijsper and Tewes [37]). *If G is an (even hole, diamond, net)-free graph, then G is β-perfect.*

This result was further strengthened by Keijsper and Tewes, who showed that instead of forbidding the diamond it suffices to forbid graphs D_1, D_2, D_4, D_5 and D_6, each of which contains the diamond as a proper induced subgraph, and instead of forbidding the 6-cap it suffices to forbid S_1 and S_2, both of which contain the 6-cap as a proper induced subgraph. (Note that S_1 and S_2 both contain the net as a proper induced subgraph, and therefore the following result also generalises Theorem 2.7.) See Figure 2.2 for depictions of graphs D_1, D_2, D_4, D_5, D_6, S_1 and S_2.

Theorem 2.8 (Keijsper and Tewes [37]). *If G is an (even hole, D_1, D_2, D_4, D_5, D_6, S_1, S_2)-free graph, then G is β-perfect.*

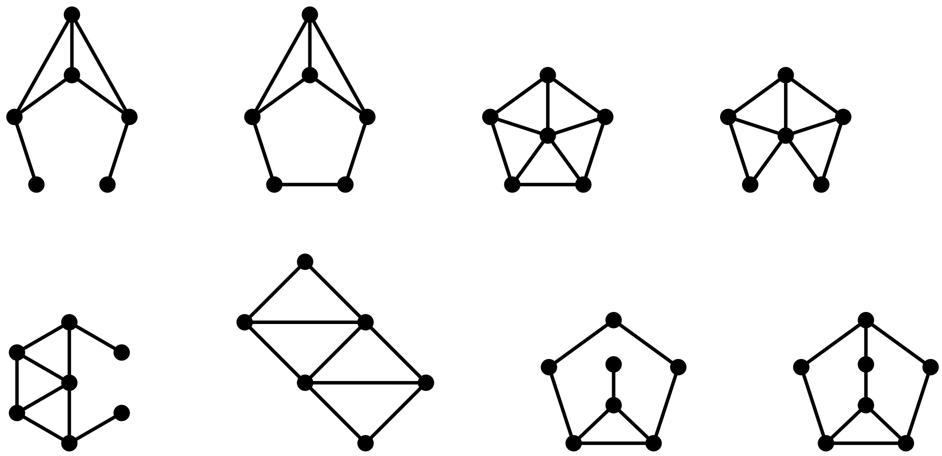


Figure 2.2: From left-to-right and top-to-bottom: the graphs D_1, D_2, D_3, D_4, D_5, D_6, S_1 and S_2 from [37].

The authors of [24] conjectured a strengthening of Theorem 2.6, namely that one need only forbid even holes and the diamond to ensure β-perfection.

In [38], Kloks, Müller and Vušković gave a decomposition theorem for (even hole, diamond)-free graphs, from which they derived the following, thereby answering positively the conjecture of de Figueiredo and Vušković.

Theorem 2.9 (Kloks, Müller and Vušković [38])**.** *If G is an (even hole, diamond)-free graph, then G is β-perfect.*

We conclude our survey of known results with the following. The *claw* is the graph on four vertices that has three pairwise nonadjacent vertices each adjacent to the remaining fourth vertex. (In other words, the claw is the complete bipartite graph $K_{1,3}$.)

Theorem 2.10 (Keijsper and Tewes [37])**.** *If G is a (claw, even hole, D_1, D_2, D_3)-free graph, then G is β-perfect.*

All of the above results, besides Theorem 2.8, that identify classes of β-perfect graphs establish β-perfection by showing that every graph in the class in question has a simplicial extreme. To illustrate this technique, suppose $\mathcal{C}$ is a hereditary class of even-hole-free graphs. Suppose we are able to show that every graph in $\mathcal{C}$ has a simplicial extreme. It follows from Lemma 2.4 that $\mathcal{C}$ contains no minimal β-imperfect graph. Now suppose $G \in \mathcal{C}$ is β-imperfect; then some induced subgraph H of G is minimally β-imperfect, and since $\mathcal{C}$ is hereditary, $H \in \mathcal{C}$, a contradiction. Thus, if $\mathcal{C}$ is a hereditary class of even-hole-free graphs each of which has a simplicial extreme, then every graph in $\mathcal{C}$ is β-perfect. In this way, it follows from Theorem 2.3 that chordal graphs are β-perfect; this fact is used often throughout this chapter, so we state it here:

Theorem 2.11. *Chordal graphs are β-perfect.*

Notice also that all the classes of β-perfect graphs identified by the above results are defined by forbidding possibly among other graphs at least one chordal graph, and consequently, none of these classes contain all chordal graphs. For our first result of this chapter, in Section 2.1, we identify a class of graphs that (1) contains graphs with no simplicial extreme, (2) contains the class of chordal graphs, and (3) contains the class of (even hole, diamond, cap)-free graphs, and we prove that every graph in this class is β-perfect.

Our second result, in Section 2.2, concerns "hyperholes"; these are graphs consisting of cliques arranged in a circular fashion with the property that between any two consecutive cliques there are all possible edges and between two nonconsecutive cliques there are no edges (we give a more formal definition in Section 2.2). We give a complete structural characterisation of β-perfect hyperholes, which we then use to give a linear-time algorithm for deciding whether a hyperhole is β-perfect.

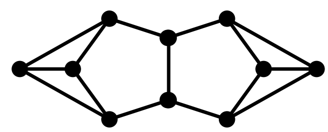


Figure 2.3: A minimal β-imperfect graph that has a clique cutset.

Our third result builds upon the second, but in order to put the final result of this chapter into context, we briefly discuss the use of clique cutset decomposition in obtaining structural characterisations of hereditary graph classes.

Suppose $\mathcal{C}$ is a hereditary class of graphs (that is, one closed under the induced subgraph relation), and consider the task of proving that a graph G belongs to $\mathcal{C}$ if and only if G is $\mathcal{F}$-free, for some family of graphs $\mathcal{F}$. When proving the "if" direction, i.e., that if G is $\mathcal{F}$-free, then G belongs to $\mathcal{C}$, one often supposes for the sake of contradiction that it fails to hold and then considers a minimum counterexample, i.e., a graph that is $\mathcal{F}$-free, does not belong to $\mathcal{C}$, but all its proper induced subgraphs do belong to $\mathcal{C}$. If it can be shown that no such minimum counterexample has a clique cutset, then we may assume that G has no clique cutset, which greatly simplifies any structural analysis.

However, returning to β-perfect graphs, there are minimally β-imperfect graphs with clique cutsets (see Figure 2.3 for an example), so it seems that the above strategy fails when $\mathcal{C}$ is taken to be the class of β-perfect graphs. Having said this, it turns out that certain types of clique cutsets (called "double clique cutsets") do not appear in minimal β-imperfect graphs. Therefore, if for some class of graphs we can show that the existence of a clique cutset implies the existence of a double clique cutset, then we may assume that no minimal β-imperfect graph in the class has a clique cutset. In Section 2.3, we do exactly this for the class of claw-free graphs; we show that a claw-free graph (satisfying a number of additional assumptions) has a double clique cutset whenever it has a clique cutset, and therefore no minimal β-imperfect claw-free graph admits a clique cutset. Then, by a result of Boncompagni, Penev and Vušković, the problem of characterising the class of claw-free β-perfect graphs reduces to the problem of characterising β-perfect "rings" (which are a generalisation of hyperholes, also defined formally later).

In Section 2.3, we characterise β-perfect rings, from which we derive a characterisation of claw-free β-perfect graphs. This result is a generalisation of Theorem 2.10. Using this characterisation, we give an algorithm that decides in polynomial time whether a claw-free graph is β-perfect.

The results from Sections 2.1 and 2.2 appear in [36]. A paper containing the results from Section 2.3 will be submitted for publication.

2.1 β-perfection of (even hole, twin wheel, cap)-free graphs

Recall that a *wheel* consists of a hole, called the *rim*, together with an additional vertex, called the *centre*, that has at least 3 neighbours in the hole. If the centre is complete to the hole, then we say that the wheel is a *universal wheel*, and if its neighbourhood in the hole consists precisely of three consecutive vertices of the hole, then we say that the wheel is a *twin wheel*. A wheel that is neither a universal wheel nor a twin wheel is called a *proper wheel*.

In this section we prove that (even hole, twin wheel, cap)-free graphs are β-perfect. Note that chordal graphs are properly contained in the class of (even hole, twin wheel, cap)-free graphs. Furthermore, since diamond-free graphs do not contain twin wheels, this result generalises Theorem 2.5.

The following well-known characterisation of chordal graphs, and the construction of (cap, 4-hole)-free graphs with a hole and no clique cutset given by Theorem 2.13 are used to prove Lemma 2.14, a decomposition theorem for (even hole, twin wheel, cap)-free graphs.

Theorem 2.12 (Dirac [25])**.** *If G is a chordal graph, then either G is a complete graph or it has a clique cutset.*

Given graphs G and F, we say that G is *obtained by blowing up vertices of F into cliques* provided that there exists a partition $\{X_v\}_{v \in V(F)}$ of $V(G)$ into nonempty cliques such that for all distinct $u, v \in V(F)$, if $uv \in E(F)$ then X_u is complete to X_v in G, and if $uv \notin E(F)$ then X_u is anticomplete to X_v in G.

A vertex v in a graph G is *universal* if v is complete to $V(G) \setminus \{v\}$, and a clique $C \subseteq V(G)$ is *universal* if every vertex in C is a universal vertex of G.

Theorem 2.13 (Cameron, da Silva, Huang and Vušković [7])**.** *Let G be a (cap, 4-hole)-free graph that contains a hole and has no clique cutset. Let F be any maximal induced subgraph of G with at least 3 vertices that is triangle-free and has no clique cutset. Then G is obtained from F by first blowing up vertices of F into cliques, and then adding a universal clique. Furthermore, any graph obtained by this sequence of operations starting from a (triangle, 4-hole)-free graph with at least 3 vertices and no clique cutset is (cap, 4-hole)-free and has no clique cutset.*

Lemma 2.14. *If G is an (even hole, twin wheel, cap)-free graph, then one of the following holds:*

(i) G is a complete graph;

(ii) G consists of a triangle-free graph on at least 3 vertices that has no clique cutset together with a (possibly empty) universal clique; or

(iii) G has a clique cutset.

Proof. Let G be an (even hole, twin wheel, cap)-free graph and assume that (i) and (iii) do not hold. By Theorem 2.12, G contains a hole. Let F be any maximal induced subgraph of G with at least 3 vertices that is triangle-free and has no clique cutset. By Menger's theorem, every vertex v of F is contained in a cycle, and since F is triangle-free, such a cycle contains a hole that contains v. Since G does not contain a twin wheel, Theorem 2.13 implies that (ii) holds, i.e. $V(G) \setminus V(F)$ is a clique that is complete to $V(F)$. $\square$

Lemma 2.15. *If G is a graph whose vertex set can be partitioned into (possibly empty) sets A and B so that:*

- *A is a clique of G;*
- *if $B \neq \emptyset$, then $G[B]$ is a (triangle, even hole)-free graph; and*
- *A is complete to B;*

then G is β-perfect.

Proof. It suffices to show that $\chi(G) = \beta(G)$. Clearly we may assume that $B \neq \emptyset$. By Theorem 2.5, $\chi(G[B]) = \beta(G[B])$. But then $\beta(G) = \beta(G[B]) + |A| = \chi(G[B]) + |A| = \chi(G)$. □

Let $S \subseteq V(G)$ be a clique cutset of G and let $C_1, \dots, C_k$ be the connected components of $G \setminus S$. The *blocks of decomposition* of G with respect to the clique cutset S are graphs $G_i = G[V(C_i) \cup S]$, for $i = 1, \dots, k$. If, for some i, G_i has no clique cutset then G_i is an *extreme block* and S is an *extreme clique cutset.* To complete our proof we will use the following well-known property of clique cutsets.

Lemma 2.16. *If a graph G has a clique cutset, then it has an extreme clique cutset.*

Lemma 2.17 (Markossian, Gasparian and Reed [42])**.** *Let G be a (triangle, even hole)-free graph. Let x be a vertex of G. Then either $\{x\}$ is complete to $V(G) \setminus \{x\}$ or there is some vertex y in $G \setminus N[x]$ such that y has degree at most 2 in G.*

A graph is *k-degenerate* if every one of its subgraphs has a vertex of degree at most k. It is well known that the chromatic number of a k-degenerate graph is at most $k+1$.

Theorem 2.18. *If G is an (even hole, twin wheel, cap)-free graph, then G is β-perfect.*

Proof. Suppose not and let G be a minimally β-imperfect (even hole, twin wheel, cap)-free graph. By Lemma 2.2, $\beta(G) = \delta(G) + 1$.

Lemmas 2.14 and 2.15 together imply that G has a clique cutset. Let K be an extreme clique cutset of G (it exists by Lemma 2.16). Let $C_1, \dots, C_k$ be the connected components of $G \setminus K$, and $G_1, \dots, G_k$ their respective blocks of decomposition. Without loss of generality, let $G_1 = G[V(C_1) \cup K]$ be an extreme block. Since G_1 has no clique cutset, by Lemma 2.14 G_1 is either a complete graph or a 2-connected triangle-free graph together with a universal clique.

If G_1 is a clique, then every vertex of C_1 is a simplicial extreme in G_1 and hence in G, contradicting Lemma 2.4. So $V(G_1)$ may be partitioned into sets A_1 and B_1 such that A_1 is a clique, $G[B_1]$ is 2-connected triangle-free, and A_1 is complete to B_1. Since $G[B_1]$ is 2-connected triangle-free, by Lemma 2.17 B_1 contains 2 nonadjacent distinct vertices y_1 and y_2 that are both of degree 2 in $G[B_1]$. It follows that y_1 and y_2 are both of degree $2 + |A_1|$ in G_1. Without loss of generality, assume that $y_1 \in V(C_1)$. It follows that $d_G(y_1) = 2 + |A_1|$. Since $G[B_1]$ contains an odd hole and is 2-degenerate, $\chi(G[B_1]) = 3$ and so $\chi(G_1) = 3 + |A_1|$. But then

$$\beta(G) = \delta(G) + 1 \leq d_G(y_1) + 1 = 3 + |A_1| = \chi(G_1) \leq \chi(G),$$

and hence $\beta(G) = \chi(G)$, contradicting our assumption that G is minimally β-imperfect.
□

We observe that the class of (even hole, twin wheel, cap)-free graphs can be recognised in polynomial time. In [16] it was shown that (even hole, cap)-free graphs can be recognised in polynomial time. To recognise whether a graph contains a twin wheel it suffices to test for every $\{u, v, x, y\} \subseteq V(G)$ that induces a diamond, with say $uv \notin E(G)$, whether there is a path from u to v in $G \setminus ((N[x] \cup N[y]) \setminus \{u, v\})$.

2.2 β-perfect hyperholes

A *ring* is any graph R whose vertex set can be partitioned into $k \geq 4$ nonempty sets $Y_1, \ldots, Y_k$ such that for all $i \in \{1, \ldots, k\}$ the following hold (where, throughout this chapter, subscripts are to be taken modulo k):

- Y_i is a clique;
- Y_i is anticomplete to $V(R) \setminus (Y_{i-1} \cup Y_i \cup Y_{i+1})$;
- some vertex of Y_i is complete to $Y_{i-1} \cup Y_{i+1}$; and
- for all distinct $y, y' \in Y_i$, $N_R[y] \subseteq N_R[y']$ or $N_R[y'] \subseteq N_R[y]$.

Under these circumstances we say that R is of *length* k and that R is a *k-ring*. Furthermore, R is *even* or *odd* according to the parity of k, and is *long* if $k \geq 5$. We sometimes refer to the sets $Y_1, \ldots, Y_k$ as the *bags* of R, and to $(Y_1, \ldots, Y_k)$ as a *ring partition* of R.

A *hyperhole* is any ring $R = (Y_1, \ldots, Y_k)$ such that, for each $i \in \{1, \ldots, k\}$, Y_i is complete to $Y_{i-1} \cup Y_{i+1}$. The terminology defined for rings applies to hyperholes, so we may speak of even and odd hyperholes, long hyperholes, k-hyperholes, and hyperholes of length k. The graph in Figure 2.1 is a 5-hyperhole. Observe that rings are a generalisation of hyperholes; in this section we consider only rings that are also hyperholes, but in Section 2.3 we will often work with rings that are not hyperholes.

In this section we present a forbidden induced subgraph characterisation for the class of β-perfect hyperholes, and using this characterisation we obtain an algorithm that decides in linear time whether a hyperhole is β-perfect.

We use frequently the following result on the chromatic number of a hyperhole.

Theorem 2.19 (Narayanan and Shende [43])**.** *If H is a hyperhole, then*

$$\chi(H) = \max\left\{\omega(H), \left\lceil \frac{|V(H)|}{\alpha(H)} \right\rceil\right\}.$$

Corollary 2.20. If H is a minimally β-imperfect k-hyperhole with k odd, then

$$|V(H)| \leq \frac{(\beta(H)-1)(k-1)}{2}.$$

Proof. Since k is odd, $\alpha(H) = \frac{k-1}{2}$, and hence by Theorem 2.19, $\chi(H) \geq \frac{2|V(H)|}{k-1}$. Since H is minimally β-imperfect, $\beta(H) > \chi(H)$. It follows that $\beta(H) - 1 \geq \frac{2|V(H)|}{k-1}$. □

The following lemma will be used repeatedly throughout this section.

Lemma 2.21. *Let $H = (X_1, \ldots, X_k)$ be a minimally β-imperfect k-hyperhole. Then for all $i \in \{1, \ldots, k\}$, the following hold:*

(i) if $|X_{i-1}| = |X_{i+1}| = 1$, then $|X_i| \geq \beta(H) - 2$;

(ii) if $|X_i| = 1$, then $|X_{i-2} \cup X_{i-1} \cup X_i| \geq \beta(H)$ and $|X_i \cup X_{i+1} \cup X_{i+2}| \geq \beta(H)$;

(iii) if $|X_i| = |X_{i+1}| = 1$, then $|X_{i+2}| \geq \beta(H) - 2$ and $|X_{i-1}| \geq \beta(H) - 2$.

Proof. Suppose that for some $i \in \{1, \ldots, k\}$, $|X_{i-1}| = |X_{i+1}| = 1$ and $|X_i| \leq \beta(H) - 3$. Fix a vertex $x \in X_i$. Then $d(x) \leq \beta(H) - 2$ and so $\beta(H) \geq d(x) + 2 \geq \delta(H) + 2$, contradicting Lemma 2.2. So (i) holds.

To prove (ii), by symmetry it suffices to prove that if $|X_1| = 1$, then $|X_1 \cup X_2 \cup X_3| \geq \beta(H)$. Suppose that $|X_1| = 1$ but $|X_1 \cup X_2 \cup X_3| \leq \beta(H) - 1$. Fix a vertex $x \in X_2$. Then $d(x) \leq \beta(H) - 2$ and so $\beta(H) \geq d(x) + 2 \geq \delta(H) + 2$, contradicting Lemma 2.2. So (ii) holds.

It follows directly from (ii) that (iii) holds. □

Lemma 2.22. *Let $H = (X_1, \ldots, X_k)$ be a k-hyperhole such that $|X_i| \geq 2$ for all $i \in \{1, \ldots, k\}$. Then H is not β-perfect.*

Proof. If k is even then clearly H is not β-perfect, so we may assume that k is odd, and hence $k \geq 5$. Consider a k-hyperhole $H' = (X'_1, \ldots, X'_k)$ such that for all $i \in \{1, \ldots, k\}$, $|X'_i| = 2$. Clearly $\delta(H') = 5$, and so $\beta(H') \geq 6$. Using Theorem 2.19 we obtain $\chi(H') = \max\{4, \lceil \frac{4k}{k-1} \rceil\}$. Since $\lceil \frac{4k}{k-1} \rceil = 5$ for all $k \geq 5$, $\chi(H') = 5 < \beta(H')$. Therefore H' is not β-perfect. Since we may find an induced subgraph of H that is isomorphic to H', it follows that H is not β-perfect. □

2.2.1 The 5-hyperholes and 7-hyperholes

We begin by characterising β-perfect 5-hyperholes and 7-hyperholes.

Theorem 2.23. *Let $H = (X_1, \ldots, X_5)$ be a 5-hyperhole. Then H is β-perfect if and only if some bag of H is of size 1.*

Proof. If H is β-perfect, then some bag of H is of size 1, for otherwise Lemma 2.22 is contradicted.

Now suppose, without loss of generality, that $|X_1| = 1$ but that H is not β-perfect. Since every induced subgraph of H is either chordal or is a 5-hyperhole, by Theorem 2.11 we may assume that H is minimally β-imperfect. By Lemma 2.21, $|X_1 \cup X_2 \cup X_3| \geq \beta(H)$ and $|X_4 \cup X_5 \cup X_1| \geq \beta(H)$. So $|V(H)| \geq 2\beta(H) - 1 = \frac{4(\beta(H)-1)}{2} + 1$, contradicting Corollary 2.20. □

So the graph in Figure 2.1 is the only minimally β-imperfect 5-hyperhole.

Theorem 2.24. *Let $H = (X_1, \ldots, X_7)$ be a 7-hyperhole. Then H is β-perfect if and only if for some $i \in \{1, \ldots, 7\}$, either $|X_i| = |X_{i+1}| = 1$ or $|X_i| = |X_{i+2}| = 1$.*

Proof. Suppose that H is β-perfect but $(|X_i|, |X_{i+1}|) \neq (1, 1)$ and $(|X_i|, |X_{i+2}|) \neq (1, 1)$ for all $i \in \{1, \ldots, 7\}$. By Lemma 2.22, we may assume without loss of generality that $|X_1| = 1$. We begin by claiming that we may assume that for all $i \in \{1, \ldots, 7\} \setminus \{1, 4\}$, $|X_i| \geq 2$. Suppose that $|X_j| = 1$ for some $j \in \{1, \ldots, 7\} \setminus \{1, 4\}$. From our assumption that $(|X_i|, |X_{i+1}|) \neq (1, 1)$ and $(|X_i|, |X_{i+2}|) \neq (1, 1)$ for all $i \in \{1, \ldots, 7\}$, it follows that $j \in \{4, 5\}$. So, without loss of generality, we may assume that $|X_4| = 1$. It follows from the same assumption that all remaining bags are of size at least 2.

Therefore H contains a hyperhole $H' = (X'_1, \ldots, X'_7)$ such that $|X'_1| = |X'_4| = 1$ and with all remaining bags being of size 2. By Theorem 2.19,

$$\chi(H') = \max\{\omega(H'), \left\lceil \frac{|V(H')|}{3} \right\rceil\} = \max\{4, \lceil \frac{12}{3} \rceil\} = 4.$$

But $\beta(H') \geq \delta(H') + 1 = 5 > \chi(H')$, and hence H' is not β-perfect, contradicting our assumption that H is β-perfect.

Suppose now that $|X_1| = |X_2| = 1$ but that H is not β-perfect. Since every induced subgraph of H is chordal or a 7-hyperhole with two consecutive bags of size 1, by Theorem 2.11 we may assume that H is minimally β-imperfect. By Lemma 2.21, $|X_3| \geq \beta(H) - 2$ and $|X_7| \geq \beta(H) - 2$. If $|X_4| \geq 2$, then $\chi(H) \geq \omega(H) \geq |X_3 \cup X_4| \geq$

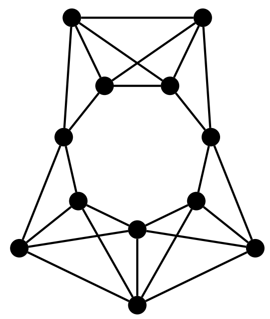


Figure 2.4: A minimally β-imperfect 7-hyperhole.

$\beta(H)$ and so $\chi(H) = \beta(H)$, contradicting our assumption that H is minimally β-imperfect. So $|X_4| = 1$, and by symmetry $|X_6| = 1$. It then follows from Lemma 2.21 that $|X_5| \geq \beta(H)-2$. But now $|V(H)| \geq 3(\beta(H)-2)+4 = \frac{6(\beta(H)-1)}{2}+1$, contradicting Corollary 2.20.

Finally, suppose that $|X_1| = |X_3| = 1$ but that H is not β-perfect. As before, by Theorem 2.11 we may assume that H is minimally β-imperfect. It follows from Lemma 2.21 that $|X_2| \geq \beta(H)-2$, $|X_3 \cup X_4 \cup X_5| \geq \beta(H)$, and $|X_6 \cup X_7 \cup X_1| \geq \beta(H)$. Therefore $|V(H)| \geq 3\beta(H) - 2 = \frac{6(\beta(H)-1)}{2} + 1$, contradicting Corollary 2.20. This completes the proof. □

It follows from Theorem 2.24 that the graph in Figure 2.4 is the only minimally β-imperfect 7-hyperhole.

2.2.2 Odd hyperholes of length at least 9

We now define some terminology. Let $R = (Y_1, \ldots, Y_k)$ be an odd ring. For $i, j, m \in \{1, \ldots, k\}$, the tuple $(Y_i, \ldots, Y_j)$ is a *sequence of m bags of R* if for $\ell \in \{1, \ldots, m\}$, the ℓ-th element of the sequence is $Y_{i+\ell-1}$ (and in particular the m-th element of the sequence is the bag Y_j). A *sector of R* is a sequence $(Y_i, \ldots, Y_j)$ of at least 2 bags such that $|Y_i| = |Y_j| = 1$, and all the other bags in the sequence have size at least 2, and Y_s is complete to Y_{s+1} for each $s \in \{i, \ldots, j-1\}$. We say that Y_i and Y_j are the *end bags* of the sector, and all the other bags are called the *interior bags* of the sector. The *length* of a sector is the number of its interior bags. A sector is an *n-sector*, for an integer $n \geq 0$, if it is of length n. A sector is *safe* if it has length 1 or length at least 3. A *super-sector of R* is a sequence $(Y_i, \ldots, Y_j)$ of at least 5 bags such that $|Y_i| = |Y_{i+1}| = |Y_{j-1}| = |Y_j| = 1$, and for each $h \in \{i+1, \ldots, j-2\}$, $(|Y_h|, |Y_{h+1}|) \neq (1, 1)$, and for each $s \in \{i, \ldots, j-1\}$, Y_s is complete to Y_{s+1}. If $(Y_i, \ldots, Y_j)$ is a super-sector of R then we say that R

contains a super-sector. We say that a super-sector $(Y_i, \ldots, Y_j)$ *contains an n-sector* if some subsequence of $(Y_{i+1}, \ldots, Y_{j-1})$ is an n-sector of R (note that although (Y_i, Y_{i+1}) and (Y_{j-1}, Y_j) are 0-sectors of R, they are, by definition of "contains an n-sector", not contained in the super-sector $(Y_i, \ldots, Y_j)$).

We point out that some of these terms appear in [36] but with slightly different meanings. For instance, we want a "sector" to mean a sequence of bags whose end bags are of size 1, and whose interior bags of size at least 2, with any two consecutive interior bags being complete. When dealing with hyperholes, the latter condition about consecutive interior bags being complete is redundant (and is therefore not present in the definition of "sector" given in [36]), but in Section 2.3 we focus on rings, and there this condition is important.

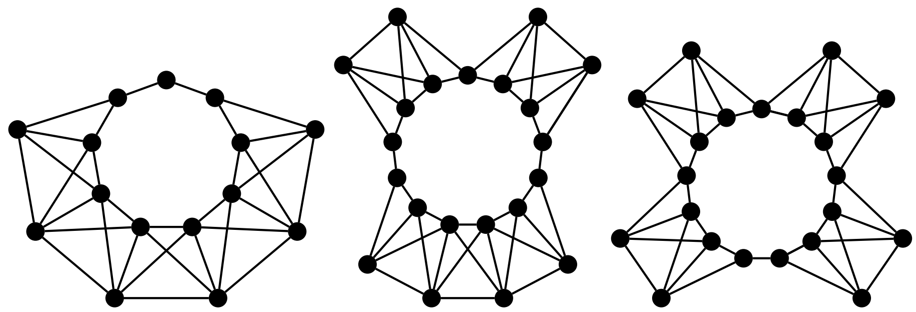


Figure 2.5: From left to right: hyperholes satisfying parts (i), (ii), and (iii) of the definition of a trivial hyperhole.

A k-hyperhole $H = (X_1, \ldots, X_k)$ is *trivial* if at least one of the following holds:

(i) for some $i \in \{1, \ldots, k\}$, $|X_i| = |X_{i+1}| = |X_{i+2}| = 1$;

(ii) H contains a super-sector that contains only 2-sectors;

(iii) H contains exactly one 0-sector, and all its other sectors are of length 2.

See Figure 2.5 for examples of trivial hyperholes. A *nontrivial* hyperhole is a hyperhole that is not trivial.

Lemma 2.25. *Let $H = (X_1, \ldots, X_k)$ be an odd k-hyperhole. If $|X_i| = |X_{i+1}| = 1$ and $|X_{i+2}|, |X_{i+3}| \geq 2$ for some $i \in \{1, \ldots, k\}$, then H is not minimally β-imperfect.*

Proof. We may assume, by symmetry, that $|X_1| = |X_2| = 1$ and $|X_3|, |X_4| \geq 2$. Let x be the vertex in X_2, and suppose that H is minimally β-imperfect. Then, by Lemma 2.2, $\beta(H) = \delta(H)+1$. Therefore $\beta(H) \leq d(x)+1 = |X_3|+2 \leq |X_3|+|X_4| \leq \omega(H) \leq \chi(H)$. But then $\chi(H) = \beta(H)$, a contradiction. $\square$

Lemma 2.26. *If H is a trivial odd hyperhole, then H is β-perfect.*

Proof. Let $H = (X_1, \ldots, X_k)$ be a trivial odd hyperhole, and assume that H is not β-perfect. Since every induced subgraph of H is either chordal or a trivial hyperhole, by Theorem 2.11 we may assume that H is minimally β-imperfect. By Lemma 2.25, H cannot satisfy (ii) or (iii) of the definition of a trivial hyperhole, and hence for some $i \in \{1, \ldots, k\}$, $|X_i| = |X_{i+1}| = |X_{i+2}| = 1$. Let x be the vertex of X_{i+1}. Then $d(x) = 2$, and hence x is a simplicial extreme of H, contradicting Lemma 2.4. $\square$

A *base hyperhole* is any odd hyperhole $H = (X_1, \ldots, X_k)$ such that for all $i \in \{1, \ldots, k\}$: $|X_i| \leq 2$, $(|X_i|, |X_{i+1}|, |X_{i+2}|) \neq (1, 1, 1)$, and $(|X_i|, |X_{i+1}|) \neq (2, 2)$. It follows that every sector of H is of length 0 or 1, and hence every proper induced subgraph of a base hyperhole is either chordal or a trivial hyperhole. Note that if H is a base hyperhole, then $\omega(H) = 3$ and $\beta(H) = 4$. We say that a base hyperhole H is *good* if it has exactly one sector of length 0, and *bad* otherwise. Note that, up to isomorphism, there is only one good base hyperhole of length k. Also, observe that since k is odd, every base hyperhole must have a sector of length 0, and hence bad base hyperholes have at least two sectors of length 0. See Figure 2.6 for examples of base hyperholes.

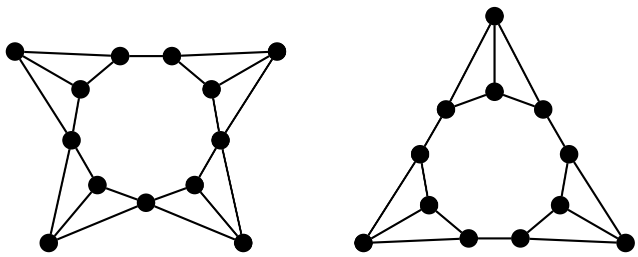


Figure 2.6: The unique (up to isomorphism) good base hyperhole of length 9 (left) and a bad base hyperhole that has three sectors of length 0 (right).

We now characterise β-perfect base hyperholes. First, we prove the following useful lemma on the number of vertices in a base hyperhole.

Lemma 2.27. *Let H be a base hyperhole of length k. The following hold.*

(i) If H is good, then $|V(H)| = \frac{3(k-1)}{2} + 1$.

(ii) If H is bad, then $|V(H)| \leq \frac{3(k-1)}{2}$.

Proof. Suppose that H is good. Then H contains exactly one 0-sector, and all other sectors are of length 1. It follows that H has $\frac{k-1}{2}$ bags of size 2 and $\frac{k-1}{2} + 1$ bags of size 1. Therefore $|V(H)| = \frac{3(k-1)}{2} + 1$, and (i) holds.

Now suppose that H is bad. Let (X_i, X_{i+1}) and (X_j, X_{j+1}) be distinct 0-sectors of H (their existence follows from the definition of a bad base hyperhole). Let m denote the number of bags in the sequence $(X_{i+2}, \ldots, X_{j-1})$, and let m' denote the number of bags in the sequence $(X_{j+2}, \ldots, X_{i-1})$. Since $k = m + m' + 4$, we may assume without loss of generality that m is even and m' is odd. It follows from H being a base hyperhole and from $|X_{i+1}| = |X_j| = 1$ that $|X_{i+2} \cup \cdots \cup X_{j-1}| \leq 2\,(m/2) + m/2 = 3m/2$. Similarly, we can obtain the bound $|X_{j+2} \cup \cdots \cup X_{i-1}| \leq 2\,\lceil m'/2 \rceil + \lfloor m'/2 \rfloor = m' + 1 + (m'-1)/2$. Now, using these bounds together with the fact that $m + m' = k - 4$ and $|X_i \cup X_{i+1} \cup X_j \cup X_{j+1}| = 4$, we obtain

$$\begin{aligned}
|V(H)| &\leq \frac{3m}{2} + m' + \frac{m'-1}{2} + 5 \\
&= \frac{3m}{2} + \frac{3m'}{2} + \frac{9}{2} \\
&= \frac{3(k-4)}{2} + \frac{9}{2} \\
&= \frac{3(k-1)}{2}.
\end{aligned}$$

Therefore (ii) holds. □

Lemma 2.28. *Let $H = (X_1, \ldots, X_k)$ be a base hyperhole. Then H is β-perfect if and only if H is good. Furthermore, if H is bad, then H is minimally β-imperfect.*

Proof. Since H is a base hyperhole, $\omega(H) = 3$ and $\beta(H) = 4$. Suppose that H is bad. Substituting the upper bound on $|V(H)|$ given by Lemma 2.27 (ii) into the equation in Theorem 2.19 (observing that since k is odd, $\alpha(H) = \frac{k-1}{2}$) gives

$$\chi(H) \leq \max\left\{\omega(H), \left\lceil \frac{2\left(\frac{3(k-1)}{2}\right)}{k-1} \right\rceil\right\} = \max\left\{3, \left\lceil \frac{3(k-1)}{k-1} \right\rceil\right\} = 3.$$

Therefore $\chi(H) < \beta(H)$, and hence H is not β-perfect. Since any proper induced subgraph of H is either a chordal graph or a trivial hyperhole, by Theorem 2.11 and Lemma 2.26, every proper induced subgraph of H is β-perfect. Therefore H is minimally β-imperfect.

Now suppose that H is good but not β-perfect. Since every proper induced subgraph of H is chordal or a trivial hyperhole, by Theorem 2.11 and Lemma 2.26, H is minimally

β-imperfect. By Corollary 2.20,

$$\frac{2|V(H)|}{k-1} \leq \beta(H) - 1 = 3,$$

which implies that $|V(H)| \leq \frac{3(k-1)}{2}$. But this contradicts Lemma 2.27 (i). $\square$

Lemma 2.29. *Every nontrivial odd hyperhole contains a base hyperhole.*

Proof. Let $H = (X_1, \ldots, X_k)$ be a nontrivial odd hyperhole. Suppose that at most one bag of H is of size 1. Without loss of generality, we may assume that for $i \in \{2, \ldots, k\}$, $|X_i| \geq 2$. For $i \in \{1, \ldots, k\}$, if i is odd then let X'_i be any one-element subset of X_i, and otherwise let X'_i be any two-element subset of X_i. Then clearly $H' = (X'_1, \ldots, X'_k)$ is a base hyperhole that is contained in H.

So we may assume that there are at least two distinct bags of H that are of size 1, and hence H contains at least 2 sectors. Let $j_1, \ldots, j_t$ be the indices of the bags of H that are of size 1, ordered such that $j_1 < \cdots < j_t$, and let $S_1, \ldots, S_t$ be the sectors of H such that for $i = 1, \ldots, t-1$, $S_i = (X_{j_i}, \ldots, X_{j_{i+1}})$, and $S_t = (X_{j_t}, \ldots, X_{j_1})$.

We construct a hyperhole $H' = (X'_1, \ldots, X'_k)$, where for $i = 1, \ldots, k$, $X'_i \subseteq X_i$, as follows. For $i = 1, \ldots, t$, $X'_{j_i} = X_{j_i}$. For every safe sector S_i, we reduce the interior bags of S_i according to the following rules:

- If the length of S_i is odd, then for $h \in \{j_i + 1, \ldots, j_{i+1} - 1\}$,

$$|X'_h| = \begin{cases} 1 & \text{if } h - j_i \text{ is even,} \\ 2 & \text{otherwise.} \end{cases}$$

- If the length of S_i is even, then $|X'_{j_{i+1}-1}| = 2$, $|X'_{j_{i+1}-2}| = 1$, and for $h \in \{j_i + 1, \ldots, j_{i+1} - 3\}$,

$$|X'_h| = \begin{cases} 1 & \text{if } h - j_i \text{ is even,} \\ 2 & \text{otherwise.} \end{cases}$$

To finish off the construction of H' we now need to reduce all 2-sectors. To do so, we consider the following cases.

Case 1. H has no 0-sector.

For every 2-sector S_i, we ensure that $|X'_{j_i+1}| = 2$ and $|X'_{j_i+2}| = 1$. The resultant hyperhole H' is clearly a base hyperhole.

Case 2. H has exactly one 0-sector.

Since H is nontrivial, it must contain a safe sector. Without loss of generality, let us say that S_1 is the 0-sector and S_s is a safe sector. For every 2-sector S_i we reduce its interior bags according to the following rule:

- If $1 < i < s$, then $|X'_{j_i+1}| = 2$ and $|X'_{j_i+2}| = 1$, and otherwise $|X'_{j_i+1}| = 1$ and $|X'_{j_i+2}| = 2$.

The resultant hyperhole H' is clearly a base hyperhole.

Case 3. H has at least two 0-sectors.

Since H is nontrivial, any two consecutive 0-sectors together with intermediate bags form a super-sector. We consider each super-sector $S = (X_l, \ldots, X_r)$ separately. Since H is nontrivial, S contains a safe sector S_s. Let S_i be a 2-sector contained in S. If S_i is contained in the subsequence $(X_l, \ldots, X_{j_s})$ of S, then $|X'_{j_i+1}| = 2$ and $|X'_{j_i+2}| = 1$, and otherwise $|X'_{j_i+1}| = 1$ and $|X'_{j_i+2}| = 2$. The resultant hyperhole H' is clearly a base hyperhole. □

Lemma 2.30. *Let $H = (X_1, \ldots, X_k)$ be a minimally β-imperfect k-hyperhole. For integers $i, j, m \in \{1, \ldots, k\}$ and with $m \geq 3$ and m odd, let $(X_i, \ldots, X_j)$ be a sequence of m bags of H such that for all $h \in \{i, \ldots, j\}$, $|X_h| = 1$ if $h - i$ is even and $|X_h| \geq 2$ otherwise. Then $|X_i \cup \cdots \cup X_j| \geq \frac{(\beta(H)-1)(m-1)}{2} + 1$.*

Proof. Observe that in the sequence $(X_i, \ldots, X_j)$ there are $\frac{m-1}{2}$ bags of size at least 2 and $\frac{m+1}{2}$ bags of size 1. It now follows from Lemma 2.21 that

$$\begin{aligned}
|X_i \cup \cdots \cup X_j| &\geq (\beta(H) - 2)\left(\frac{m-1}{2}\right) + \frac{m+1}{2} \\
&= \frac{(\beta(H) - 1)\,(m-1)}{2} - \frac{m-1}{2} + \frac{m+1}{2} \\
&= \frac{(\beta(H) - 1)(m-1)}{2} + 1,
\end{aligned}$$

as required. □

Let $H' = (X'_1, \ldots, X'_k)$ be a good base hyperhole with k odd and $k \geq 9$. Without loss of generality, let us assume that $|X'_3| = |X'_4| = 1$. Let A be one of the sets $\{1, 3\}$, $\{1, 6\}$, $\{3, 4\}$, $\{4, 6\}$. We call A a *free set* of H'. A hyperhole $H = (X_1, \ldots, X_k)$ is an *extension* of H' if for all $i \in \{1, \dots, k\}$, $X'_i \subseteq X_i$ and for some free set A of H', for all $i \in \{1, \dots, k\} \setminus A$, $|X'_i| = 1$ if and only if $|X_i| = 1$. (Such a free set A of H' is called a *free set of* H.) We now give a structural characterisation of β-perfect k-hyperholes with k odd and $k \geq 9$.

Lemma 2.31. *Let H be a k-hyperhole with k odd and $k \geq 9$. If H is nontrivial and is not an extension of a good base hyperhole, then H contains a bad base hyperhole.*

Proof. Suppose that H is neither trivial nor an extension of a good base hyperhole. Since H is nontrivial, by Lemma 2.29 it contains a base hyperhole $H' = (X'_1, \dots, X'_k)$ (with $X'_i \subseteq X_i$ for all $i \in \{1, \dots, k\}$). We may assume that H' is good, for otherwise we are done. Without loss of generality we assume that $|X'_3| = |X'_4| = 1$. Since H is not an extension of H', one of the following holds:

- for some even $i \in \{8, \dots, k\}$, $|X_i| \geq 2$;
- $|X_1| \geq 2$ and $|X_4| \geq 2$;
- $|X_3| \geq 2$ and $|X_6| \geq 2$.

First, let us suppose that for some even $i \in \{8, \dots, k\}$, $|X_i| \geq 2$. Consider a hyperhole $H'' = (X''_1, \dots, X''_k)$ such that for all $h \in \{1, \dots, k\}$:

$$|X''_h| = \begin{cases} 1 & \text{if } h \in \{1, 3, i-1, i+1\}, \\ 1 & \text{if } h \in \{4, \dots, i-2\} \cup \{i+2, \dots, k\} \text{ and } h \text{ is even}, \\ 2 & \text{otherwise.} \end{cases}$$

Clearly H'' is a base hyperhole that is contained in H. Furthermore, (X''_{i-2}, X''_{i-1}) and (X''_{i+1}, X''_{i+2}) are two distinct 0-sectors of H'', and hence H'' is bad.

Suppose that $|X_1| \geq 2$ and $|X_4| \geq 2$. Consider a hyperhole $H'' = (X''_1, \dots, X''_k)$ such that for all $h \in \{1, \dots, k\}$:

$$|X''_h| = \begin{cases} 1 & \text{if } h \in \{2, 3, 5, k\}, \\ 1 & \text{if } h \in \{6, \dots, k-1\} \text{ and } h \text{ is even}, \\ 2 & \text{otherwise.} \end{cases}$$

Clearly H'' is a base hyperhole that is contained in H. Furthermore, (X''_2, X''_3) and (X''_5, X''_6) are two distinct 0-sectors of H'', and hence H'' is bad. A symmetric argument may be used for the case where $|X_3| \geq 2$ and $|X_6| \geq 2$. $\square$

Theorem 2.32. *A k-hyperhole H with k odd and $k \geq 9$ is β-perfect if and only if it is trivial or it is an extension of a good base hyperhole.*

Proof. Let $H = (X_1, \ldots, X_k)$ be a k-hyperhole with k odd and $k \geq 9$. By Lemma 2.28 and Lemma 2.31 it follows that if H is β-perfect then it is trivial or an extension of a good base hyperhole.

Now suppose that H is either trivial or is an extension of a good base hyperhole, but H is not β-perfect. By Lemma 2.26, H is an extension of a good base hyperhole $H' = (X'_1, \ldots, X'_k)$. Furthermore, since every induced subgraph of H is either chordal, a trivial hyperhole, or an extension of a good base hyperhole, we may assume that H is minimally β-imperfect. Let A be a free set of H. By symmetry, it suffices to consider the following three cases. In each case, we obtain a lower bound on $|V(H)|$ which contradicts the bound given in Corollary 2.20.

Case 1. $A = \{1, 3\}$.

Applying Lemma 2.30 to the sequence $(X_4, \ldots, X_{k-1})$ gives the lower bound

$$|X_4 \cup \cdots \cup X_{k-1}| \geq (\beta(H) - 1)\left(\frac{k-5}{2}\right) + 1.$$

It follows from Lemma 2.21 applied to X_{k-1}, X_k, X_1 and to X_2, X_3, X_4 that $|X_{k-1} \cup X_k \cup X_1| \geq \beta(H)$ and $|X_2 \cup X_3 \cup X_4| \geq \beta(H)$. Adding together these bounds, and subtracting 2 to account for double counting, we obtain

$$\begin{aligned}|V(H)| &\geq (\beta(H) - 1)\left(\frac{k-5}{2}\right) + 2\beta(H) - 1 \\ &= (\beta(H) - 1)\left(\frac{k-1}{2}\right) - 2\left(\beta(H) - 1\right) + 2\beta(H) - 1 \\ &= (\beta(H) - 1)\left(\frac{k-1}{2}\right) + 1,\end{aligned}$$

contradicting Corollary 2.20. This completes Case 1.

Case 2. $A = \{1, 6\}$.

Using Lemma 2.21, we easily obtain the following bounds: $|X_{k-1} \cup X_k \cup X_1| \geq \beta(H)$, $|X_2 \cup X_3 \cup X_4| \geq \beta(H)$, $|X_3 \cup X_4 \cup X_5| \geq \beta(H)$, and $|X_6 \cup X_7 \cup X_8| \geq \beta(H)$. If $k = 9$, then adding together these bounds and subtracting 3 to account for double counting gives the bound

$$|V(H)| \geq 4\beta(H) - 3 = (\beta(H) - 1)\binom{\frac{k-1}{2}}{2} + 1,$$

contradicting Corollary 2.20. So $k > 9$. By applying Lemma 2.30 to the sequence $(X_8, \ldots, X_{k-1})$, we obtain the additional bound

$$|X_8 \cup \cdots \cup X_{k-1}| \geq (\beta(H) - 1)\left(\frac{k-9}{2}\right) + 1.$$

Adding this to the above bounds, and subtracting 4 to account for double counting, we obtain

$$\begin{aligned}
|V(H)| &\geq (\beta(H) - 1)\left(\frac{k-9}{2}\right) + 4\beta(H) - 3 \\
&= (\beta(H) - 1)\left(\frac{k-1}{2}\right) - 4(\beta(H) - 1) + 4\beta(H) - 3 \\
&= (\beta(H) - 1)\left(\frac{k-1}{2}\right) + 1,
\end{aligned}$$

contradicting Corollary 2.20. This completes Case 2.

Case 3. $A = \{3, 4\}$.

Applying Lemma 2.30 to the sequence $(X_6, \ldots, X_k, X_1)$ gives the lower bound

$$|X_6 \cup \cdots \cup X_k \cup X_1| \geq (\beta(H) - 1)\left(\frac{k-5}{2}\right) + 1.$$

It follows from Lemma 2.21 applied to the sets X_1, X_2, X_3 and to X_4, X_5, X_6 that $|X_1 \cup X_2 \cup X_3| \geq \beta(H)$ and $|X_4 \cup X_5 \cup X_6| \geq \beta(H)$. Adding together these bounds, and subtracting 2 to account for double counting, we obtain

$$\begin{aligned}
|V(H)| &\geq (\beta(H) - 1)\left(\frac{k-5}{2}\right) + 2\beta(H) - 1 \\
&= (\beta(H) - 1)\left(\frac{k-1}{2}\right) - 2(\beta(H) - 1) + 2\beta(H) - 1 \\
&= (\beta(H) - 1)\left(\frac{k-1}{2}\right) + 1,
\end{aligned}$$

contradicting Corollary 2.20. This completes Case 3. $\square$

2.2.3 Forbidden induced subgraph characterisation

By putting together previously obtained results we obtain the following forbidden induced subgraph characterisation of β-perfect hyperholes (in which all the excluded induced subgraphs are minimally β-imperfect). Let H_1 be the graph in Figure 2.1, and let H_2 be the graph in Figure 2.4.

Theorem 2.33. *A hyperhole is β-perfect if and only if it is (even hole, bad base hyperhole, H_1, H_2)-free.*

Proof. Suppose that H is a β-perfect hyperhole. Clearly H must be even-hole-free. It follows from Lemma 2.22 that H is H_1-free, from Theorem 2.24 that H is H_2-free, and from Theorem 2.28 that H does not contain a bad base hyperhole.

Now suppose that $H = (X_1, \ldots, X_k)$ is a k-hyperhole that does not contain an even hole, H_1, H_2, or a bad base hyperhole. In particular, k is odd. If $k = 5$, then it follows from H being H_1-free that some bag of H has size 1. Therefore H is β-perfect by Theorem 2.23. If $k = 7$, then it follows from H being H_2-free that for some $i \in \{1, \ldots, k\}$, either $|X_i| = |X_{i+1}| = 1$, or $|X_i| = |X_{i+2}| = 1$. Therefore, by Theorem 2.24, H is β-perfect. Now suppose that $k \geq 9$. Since H contains no bad base hyperhole, by Lemma 2.31 it is either trivial or an extension of a good base hyperhole, and hence it is β-perfect by Theorem 2.32. $\square$

2.2.4 A recognition algorithm for β-perfect hyperholes

In this section, we give a linear-time algorithm that decides whether an input hyperhole is β-perfect. We observe that in [4] it is shown that hyperholes can be recognised in linear-time, and that a hyperhole partition can be found also in linear-time.

Theorem 2.34. *There is an algorithm with the following specifications:*

> **Input:** A k-hyperhole $H = (X_1, \ldots, X_k)$.
> **Output:** Yes if H is β-perfect, and No otherwise.
> **Running time:** $\mathcal{O}(k)$.

Proof. Consider the following algorithm:

Step 1. If k is even, then return No.

Step 2. If $k = 5$, then check whether some bag of H is of size 1. If so, then return Yes. Otherwise, return No.

Step 3. If $k = 7$, then check whether $|X_i| = |X_{i+1}| = 1$ or $|X_i| = |X_{i+2}| = 1$ for some $i \in \{1, \ldots, k\}$. If so, then return Yes. Otherwise, return No.

Step 4. From now on, we may assume that k is odd and $k \geq 9$. Check whether H is trivial. If so, then return Yes.

Step 5. We may now assume that H is nontrivial. Using the reduction rules given in Lemma 2.29, construct a base hyperhole $H' = (X'_1, \ldots, X'_k)$ such that $V(H') \subseteq V(H)$.

Step 6. Check whether H' is bad. If so, then return No.

Step 7. Check whether H is an extension of H'. If so, then return Yes. Otherwise, return No.

We now prove that the algorithm is correct. Suppose that the algorithm returns Yes when given a k-hyperhole $H = (X_1, \ldots, X_k)$ as input. As a result of Step 1, we may assume that k is odd. If $k = 5$, then by Step 2, $|X_i| = 1$ for some $i \in \{1, \ldots, k\}$. It follows from Theorem 2.23 that H is β-perfect. If $k = 7$, then by Step 3, either $|X_i| = |X_{i+1}| = 1$ or $|X_i| = |X_{i+2}| = 1$ for some $i \in \{1, \ldots, k\}$. It follows from Theorem 2.24 that H is β-perfect. If $k \geq 9$, then by Steps 4 and 7, either H is trivial or H is an extension of a good base hyperhole. It follows from Theorem 2.32 that H is β-perfect.

Suppose now that the algorithm returns No, but that H is β-perfect. Since even holes are not β-perfect, we may assume that k is odd. If $k = 5$, then by Theorem 2.23, $|X_i| = 1$ for some $i \in \{1, \ldots, k\}$. But then the algorithm returns Yes in Step 2. If $k = 7$, then by Theorem 2.24, $|X_i| = |X_{i+1}| = 1$ or $|X_i| = |X_{i+2}| = 1$ for some $i \in \{1, \ldots, k\}$. But then the algorithm returns Yes in Step 3. So we may assume that $k \geq 9$. By Theorem 2.32, H is either trivial or an extension of a good base hyperhole. If H is trivial, then the algorithm returns Yes in Step 4. If H is an extension of a good base hyperhole, then H is an extension of the hyperhole H' constructed in Step 5 of the algorithm. But then the algorithm returns Yes in Step 7. This completes the proof that the algorithm is correct.

Each step of the algorithm can clearly be performed in $\mathcal{O}(k)$ time. □

2.3 Claw-free β-perfect graphs

In this section we give a forbidden induced subgraph characterisation for the class of claw-free β-perfect graphs and present an algorithm which uses this characterisation for deciding in polynomial time whether a claw-free graph is β-perfect.

2.3.1 Minimal β-imperfect graphs

Recall that a *clique cutset* of a graph G is a clique $C \subseteq V(G)$ such that $V(G) \setminus C$ admits a partition (V_1, V_2) where V_1 is anticomplete to V_2. In this case we say that G is the *clique-sum* of $G[V_1 \cup C]$ and $G[V_2 \cup C]$. A *double clique cutset* of a graph G is a clique cutset C such that C admits a partition (C_1, C_2) and $V(G) \setminus C$ admits a partition (V_1, V_2) such that the only edges between $V_1 \cup C_1$ and $V_2 \cup C_2$ are those between C_1 and C_2; in particular, the sets C_1, C_2, V_1 and V_2 are all nonempty. In this case we say that G is the *double clique-sum* of $G[V_1 \cup C]$ and $G[V_2 \cup C]$, and that G *admits a double clique cutset.*

In [13], Chudnovsky and Seymour observe that clique-sums do not necessarily preserve the property of being claw-free, but double clique-sums do. The same phenomenon occurs with respect to the property of β-perfection, that is, β-perfection is not necessarily preserved under the operation of clique-sums (see Figure 2.3), but it is under the operation of double clique-sums. The proof of this fact will use the following; it is well-known.

Lemma 2.35. *Let G be a graph, C a clique cutset of G, and (V_1, V_2) a partition of $V(G) \setminus C$ such that V_1 is anticomplete to V_2. Then $\chi(G) = \max\{\chi(G[V_1 \cup C]), \chi(G[V_2 \cup C])\}$.*

Lemma 2.36. *No minimal β-imperfect graph admits a double clique cutset.*

Proof. On the contrary, suppose G is a minimal β-imperfect graph that admits a double clique cutset C. Let sets C_1, C_2, V_1 and V_2 be as in the definition of double clique cutset, and set $G_1 = G[V_1 \cup C]$ and $G_2 = G[V_2 \cup C]$. By minimality, G_1 and G_2 are β-perfect, and hence $\chi(G_1) = \beta(G_1)$ and $\chi(G_2) = \beta(G_2)$. By Lemma 2.2, $\delta(G) + 1 = \beta(G) > \chi(G) \geq \chi(G_1) = \beta(G_1) \geq \delta(G_1) + 1$, and therefore $\delta(G) > \delta(G_1)$. Similarly, $\delta(G) > \delta(G_2)$. Let $v \in V(G)$ be such that $d_G(v) = \delta(G)$. Since $d_G(u) = d_{G_1}(u)$ for all $u \in V_1 \cup C_1$ and since $\delta(G) > \delta(G_1)$, it follows that $v \in C_2$. By symmetry, it follows that $v \in C_1$, a contradiction. $\square$

We now prove that if a claw-free graph having certain properties has a clique cutset, then it has a double clique cutset.

Lemma 2.37. *Let G be a connected (claw, C_4)-free graph that contains no simplicial vertex. If G admits a clique cutset, then G admits a double clique cutset.*

Proof. Suppose G has a clique cutset. Among all clique cutsets of G, let C be one that minimises $|C|$. Let (V_1, V_2) be a partition of $V(G) \setminus C$ such that V_1 is anticomplete to V_2. By the minimality of C, every vertex in C has a neighbour in both V_1 and V_2, and C is nonempty since G is connected.

(1) For every vertex $c \in C$, both $N(c) \cap V_1$ and $N(c) \cap V_2$ are cliques.

Proof of (1): If a vertex $c \in C$ has two nonadjacent neighbours u and v in V_1, then for any neighbour w of c in V_2, the set $\{c, u, v, w\}$ induces a claw, a contradiction. Therefore $N(c) \cap V_1$ is a clique, and by symmetry so is $N(c) \cap V_2$. This proves (1).

Set $N_1 = N(C) \cap V_1$ and $N_2 = N(C) \cap V_2$.

(2) At least one of N_1 and N_2 is a clique.

Proof of (2): Suppose that neither of N_1 and N_2 is a clique, and let x, y be two nonadjacent vertices of N_1. Fix $u \in N_C(x)$ and $v \in N_C(y)$; by (1), $u \neq v$, and u, y are nonadjacent, and v, x are nonadjacent. If $N_{V_2}(u) \neq N_{V_2}(v)$, then up to symmetry there exists a vertex $w \in N_{V_2}(u) \setminus N(v)$, yielding a claw $G[\{x, u, v, w\}]$, a contradiction. So $N_{V_2}(u) = N_{V_2}(v)$; set $N_2' = N_{V_2}(u) = N_{V_2}(v)$. By (1), N_2' is a clique, so there exists a vertex $z \in N_2 \setminus N_2'$. Fix $w \in N_C(z)$; clearly $w \notin \{u, v\}$. If w, x are adjacent, then $\{w, x, v, z\}$ induces a claw, a contradiction. So w, x are nonadjacent, and by symmetry so are w and y. It follows that there exists some $s \in N_{V_1}(w) \setminus \{x, y\}$, and s is adjacent to u, for otherwise $\{w, s, u, z\}$ induces a claw; and similarly, s is adjacent to v. By (1), s is adjacent to both x and y, but now $\{s, x, y, w\}$ induces a claw, a contradiction. So at least one of N_1 and N_2 is a clique, and this proves (2).

By (2), we may assume N_1 is a clique. Since C is also a clique, and since G is C_4-free (so, in particular, $G[N_1 \cup C]$ is C_4-free), it follows that $G[N_1 \cup C]$ is chordal. If $G[V_1 \cup C]$ is chordal, then by Theorem 2.3, some vertex belonging to V_1 is simplicial in $G[V_1 \cup C]$ and hence in G, a contradiction. Therefore $G[V_1 \cup C]$ is not chordal, so in particular $G[V_1 \cup C] \neq G[N_1 \cup C]$; that is, $V_1 \setminus N_1 \neq \emptyset$.

Let ℓ denote the maximum distance in $G[V_1 \cup C]$ from a vertex of V_1 to C. Since $V_1 \setminus N_1$ is nonempty, $\ell \geq 2$. Set $L_0 = C$ and $L_1 = N_1$, and for each $i \in \{2, \ldots, \ell\}$ let $L_i = N(L_{i-1}) \setminus L_{i-2}$. Observe that $(L_0, \ldots, L_\ell)$ is a partition of $V_1 \cup C$, and that if there is an edge between L_i and L_j, then $|i - j| \leq 1$. Let $k \in \{0, \ldots, \ell\}$ be the smallest integer such that some hole H of $G[V_1 \cup C]$ intersects L_k. (Since no vertex of G is simplicial, $G[V_1 \cup C]$ is not chordal by Theorem 2.3, and hence k is well-defined.) So every hole of $G[V_1 \cup C]$ is a hole of $G[L_k \cup \cdots \cup L_\ell]$, and since $G[N_1 \cup C]$ (or, equivalently, $G[L_0 \cup L_1]$) is chordal, $k \geq 1$.

(3) For every $i \in \{1, \ldots, \ell - 1\}$ and every $v \in L_i$, $N(v) \cap L_{i+1}$ is a clique.

Proof of (3): For otherwise the vertex v together with any one of its neighbours in L_{i-1} and any two of its nonadjacent neighbours in L_{i+1} forms a claw, a contradiction. This proves (3).

(4) For all $i \in \{2, \ldots, \ell\}$, if x, y are two nonadjacent vertices of L_i, then there exists an xy-path of length at least 3 with interior in $L_1 \cup \cdots \cup L_{i-1}$.

Proof of (4): The statement holds for $i = 2$ by (3) together with the fact that every vertex of L_2 has a neighbour in L_1. Let $i > 2$, let x, y be two nonadjacent vertices of L_i, and fix $x' \in N(x) \cap L_{i-1}$ and $y' \in N(y) \cap L_{i-1}$. By (3), x, y' are nonadjacent and y, x' are nonadjacent, so in particular $x' \neq y'$. If x' and y' are adjacent, then $xx'y'y$ is the desired path; and if x' and y' are nonadjacent, then by induction there is an $x'y'$-path of length at least 3 with interior in $L_1 \cup \cdots \cup L_{i-2}$, which together with x and y forms the desired path. This proves (4).

For brevity we introduce the following terminology: if $i \in \{2, \ldots, \ell\}$ and $x, y \in L_i$ are nonadjacent, then an *xy-link* is any xy-path of length at least 3 with interior in $L_1 \cup \cdots \cup L_{i-1}$; at least one such path exists by (4).

(5) For every $i \in \{1, \ldots, k\}$ and every $v \in L_i$, $N(v) \cap L_{i-1}$ is a clique.

Proof of (5): Fix $i \in \{1, \ldots, k\}$ and $v \in L_i$, and suppose $N(v) \cap L_{i-1}$ contains two nonadjacent vertices x and y. Since C and N_1 are cliques, $i \geq 3$. But now any xy-link together with the vertex v forms a hole that intersects L_{i-1}, contrary to the minimality of k. This proves (5).

Recall that H is a hole of G that intersects L_k.

(6) $|V(H) \cap L_k| = 2$, and the two vertices of $V(H) \cap L_k$ are adjacent.

Proof of (6): Suppose $G[V(H) \cap L_k]$ contains a 3-vertex path xyz, and let P be an xz-link. But now the graph induced by $V(P) \cup (V(H) \setminus \{y\})$ contains a hole that intersects L_{k-1}, contrary to the definition of k. So $G[V(H) \cap L_k]$ is P_3-free.

To prove the second statement, it suffices to show that $G[V(H) \cap L_k]$ has only one component, and that this component is of size two. By (3), every component of $G[V(H) \cap L_k]$ contains at least two vertices, and thus it follows from P_3-freeness that every component is of size two. Suppose $G[V(H) \cap L_k]$ has at least two components, let S be any one of its components, and fix $s \in V(S)$. Then there exists a path P from s to some $t \in (V(H) \cap L_k) \setminus V(S)$ with interior in $L_{k+1} \cup \cdots \cup L_\ell$, which together with an st-link forms a hole that contradicts the minimality of k. This proves (6).

In view of (6), let us say $V(H) \cap L_k = \{v, w\}$. Let u be the neighbour of v in H different from w, and u' the neighbour of w in H different from v. Note that $u, u' \in L_{k+1}$, and since G is C_4-free, u is not adjacent to u'.

(7) $N(v) \cap L_{k-1} = N(w) \cap L_{k-1}$.

Proof of (7): For if not, then up to symmetry there exists some $z \in L_{k-1}$ adjacent to v and nonadjacent to w, yielding a claw $G[\{u, v, w, z\}]$. This proves (7).

In view of (7), let $X = N(v) \cap L_{k-1} = N(w) \cap L_{k-1}$ and let $Y = \{y \in L_k : N(y) \cap X \neq \emptyset\}$. So in particular $\{v, w\} \subseteq Y$.

(8) $X \cup Y$ is a clique.

Proof of (8): The set X is a clique by (5). Fix $y \in Y$ and suppose that y is nonadjacent to some $x \in X$. Clearly $y \notin \{v, w\}$. Since v and w are complete to X and $N(y) \cap X \neq \emptyset$, it follows from (3) that y is adjacent to both v and w. By (3), and since uu' is not an edge, we may assume without loss of generality that y is nonadjacent to u. But now $\{u, v, y, x\}$ induces a claw, a contradiction. So X is complete to Y. It now follows from (3) and since $X \neq \emptyset$ that Y is a clique. This proves (8).

Let $Y' = \{y \in Y : N(y) \cap L_{k-1} \neq X\}$. By (8), $X \subsetneq N(y) \cap L_{k-1}$ for every $y \in Y'$.

(9) Y' is anticomplete to L_{k+1}.

Proof of (9): Fix $y \in Y'$ and $z \in N(y) \cap (L_{k-1} \setminus X)$, and suppose that y has a neighbour a in L_{k+1}. By (8), y is complete to $\{v, w\}$. If y is adjacent to u, then $\{y, u, w, z\}$ induces a claw, a contradiction. So by symmetry y is anticomplete to $\{u, u'\}$ and hence $a \notin \{u, u'\}$. If a and v are nonadjacent, then $\{y, a, v, z\}$ induces a claw, a contradiction.

So a is adjacent to v and by symmetry a is adjacent to w. By (3), a is complete to $\{u, u'\}$. But now, since uu' is not an edge, $\{a, u, u', y\}$ induces a claw, a contradiction. So Y' is anticomplete to L_{k+1}. This proves (9).

(10) There is no path P from $L_k \setminus Y$ to $Y \setminus Y'$ with $P^* \subseteq L_{k+1} \cup \cdots \cup L_\ell$. In particular, $L_k \setminus Y$ is anticomplete to $Y \setminus Y'$.

Proof of (10): Suppose there exists a path P with ends $a \in L_k \setminus Y$ and $y \in Y \setminus Y'$ and with (possibly empty) interior in $L_{k+1} \cup \cdots \cup L_\ell$. Fix $x \in N(y) \cap L_{k-1}$ and $z \in N(a) \cap L_{k-1}$. Since $a \notin Y$, a is nonadjacent to x and $z \notin X$, and therefore (since $y \in Y \setminus Y'$) y is nonadjacent to z. But now an xz-link (or the edge xz, if x, z are adjacent) together with P forms a hole that intersects L_{k-1}, a contradiction. This proves (10).

(11) $X \cup Y$ is a clique cutset of G.

Proof of (11): By (8), $X \cup Y$ is a clique. Suppose that $X \cup Y$ is not a clique cutset of G. Observe that, by the existence of $u, u' \in L_{k+1}$, $k < \ell$, and therefore $L_{k+1} \cup \cdots \cup L_\ell \neq \emptyset$. If $L_k = Y$, then it follows from (8) that Y is a clique cutset of G, and therefore so is $X \cup Y$, a contradiction. So $L_k \neq Y$, and hence there exists a path P in $G \setminus (X \cup Y)$ from u to some vertex z in $L_k \setminus Y$ and with interior in $L_{k+1} \cup \cdots \cup L_\ell$. But now $P \cup \{v\}$ contains a path that violates (10). This proves (11).

(12) If F is a component of $G \setminus (X \cup Y)$, then either $N(V(F)) \subseteq Y \setminus Y'$ or $N(V(F)) \subseteq X \cup Y'$.

Proof of (12): Let F be a component of $G \setminus (X \cup Y)$, and suppose there exist vertices $s, t \in V(F)$ such that s has a neighbour $y \in Y \setminus Y'$ and t has a neighbour $y' \in X \cup Y'$. By (9), $t \notin L_{k+1}$, and therefore either $k \geq 2$ and $t \in L_{k-2} \cup L_{k-1} \cup L_k$, or $k = 1$ and $t \in V_2 \cup L_0 \cup L_1$. Since $s \notin X$ and $y \in Y \setminus Y'$, it follows that $s \notin L_{k-1}$. Furthermore, $s \notin L_k$ by (10), and therefore $s \in L_{k+1}$. Since F contains an st-path, there exists a path in $G \setminus (X \cup Y)$ from s to some vertex in $(L_k \cup L_{k-1} \cup L_{k-2}) \setminus (X \cup Y)$. By choosing such a path of minimum length, we obtain a path P from s to some vertex $a \in L_k \setminus Y$ whose interior lies in $L_{k+1} \cup \cdots \cup L_\ell$. But now $P \cup \{y\}$ contains a path that contradicts (10). This proves (12).

Clearly $G \setminus (X \cup Y)$ contains at least one component some vertex of which has a neighbour in $Y \setminus Y'$, and at least one component some vertex of which has a neighbour in $X \cup Y'$. It now follows from (11) and (12) that the set $X \cup Y$, partitioned $(Y \setminus Y', X \cup Y')$, is a double clique cutset of G. □

Lemma 2.38. *No minimal β-imperfect claw-free graph admits a clique cutset.*

Proof. Let G be a minimally β-imperfect claw-free graph that admits a clique cutset. By minimality, G is connected. We may assume that G is not an even hole (since an even hole admits no clique cutset), so G is even-hole-free (and, in particular, C_4-free) by minimality and has no simplicial vertex by Lemma 2.4. But now, by Lemma 2.37, G admits a double clique cutset, contrary to Lemma 2.36. □

2.3.2 Structure of (claw, even hole)-free graphs

In this section we derive from a result of Boncompagni, Penev and Vušković [4] a decomposition theorem for (claw, even hole)-free graphs. We first need the following terminology. We refer the reader to Chapter 1 for definitions of three-path-configurations (thetas, pyramids, prisms) and wheels.

A *component* of G is a maximal connected induced subgraph of G. A graph is *anticonnected* if its complement is connected. An *anticomponent* of G is a maximal anticonnected induced subgraph of G. A component or anticomponent is *trivial* if it has only one vertex, and *nontrivial* otherwise. Since the complements of anticomponents of G are components of $\overline{G}$, between any two anticomponents of G there is every possible edge. Therefore the set of all vertices belonging to trivial anticomponents is a clique.

Lemma 2.39. *If a graph is (claw, even hole)-free, then it is (3PC, proper wheel)-free.*

Proof. Let G be a (claw, even hole)-free graph. Since G is even-hole-free, G is odd-signable, and therefore it follows from Theorem 1.2 that G contains no even wheel, no theta and no prism. A pyramid contains a claw, so G contains no pyramid, and therefore G contains no 3PC. It remains to show that G contains no proper wheel.

Towards a contradiction, suppose G contains a proper wheel W with rim H and centre x. Let C be any component of $G[N_H(x)]$. If C consists of a single vertex, say c, then x is anticomplete to $N_H(c)$, and hence $N_H[c] \cup \{x\}$ induces a claw, a contradiction. So $|V(C)| \geq 2$. If $C = H$, then W is a universal wheel, a contradiction. So $C \neq H$, and therefore C is a path. If $|V(C)| \geq 5$, then any three pairwise nonadjacent vertices of C together with x induce a claw, a contradiction. So each component of $G[N_H(x)]$ is a path on at least 2 and at most 4 vertices.

Suppose $G[N_H(x)]$ contains only one component C. By definition, x has at least 3 neighbours in H, and W is not an even wheel, so C is a path on 3 vertices. But then W is a twin wheel, and hence not a proper wheel, a contradiction. So $G[N_H(x)]$ contains at least two components. Suppose one of them, say C, contains two nonadjacent vertices u and v. Then the vertices u, v, x together with one vertex from any other component of $G[N_H(x)]$ besides C induce a claw, a contradiction. It follows that each component of $G[N_H(x)]$ has exactly 2 vertices. But then W is an even wheel, a contradiction. $\square$

We refer the reader back to Section 2.2 for the definition of a ring and associated terminology.

Lemma 2.40 (Boncompagni, Penev and Vušković [4]). *If R is a ring of length k, then every hole in R is of length k.*

A hole is *long* if it is of length at least 5.

Theorem 2.41 (Boncompagni, Penev and Vušković [4]). *If G is a (3PC, proper wheel)-free graph, then one of the following holds.*

(i) G has exactly one nontrivial anticomponent, and this anticomponent is a long ring;

(ii) G is (long hole, $K_{2,3}$, $\overline{C_6}$)-free;

(iii) $\alpha(G) = 2$, and every anticomponent of G is either a 5-hyperhole or a $(C_5, \overline{C_6})$-free graph;

(iv) G admits a clique cutset.

By specialising Theorem 2.41 to (claw, even hole)-free graphs, we obtain the following decomposition theorem.

Lemma 2.42. *If G is a (claw, even hole)-free graph, then G is a complete graph or an odd ring, or G contains a universal vertex, or G admits a clique cutset.*

Proof. Let G be a (claw, even hole)-free graph. By Theorem 2.12, we may assume that G contains a hole, and since G is even-hole-free, G contains an odd hole, and hence a long hole. By Lemma 2.39, G is (3PC, proper wheel)-free, and hence G satisfies one of (i)-(iv) in the statement of Theorem 2.41. Since G contains a long hole, (ii) does not hold; and if (iv) holds, i.e., if G has a clique cutset, then we are done; so we may assume (iv) does not hold. Therefore (i) or (iii) holds. That is:

- G has exactly one nontrivial anticomponent, and this anticomponent is a long ring; or
- $\alpha(G) = 2$, and every anticomponent of G is either a 5-hyperhole or a $(C_5, \overline{C_6})$-free graph.

In the first case, G contains a universal vertex (if some anticomponent of G is trivial), or G is an odd ring (if no anticomponent of G is trivial), and we are done. So we may assume that $\alpha(G) = 2$ and every anticomponent of G is either a 5-hyperhole or a $(C_5, \overline{C_6})$-free graph. Since $\alpha(G) = 2$, H is of length 5, and therefore H belongs to an anticomponent F of G that is a 5-hyperhole. If $G \setminus F$ contains two nonadjacent vertices, then they together with two nonadjacent vertices of H induce a C_4, a contradiction. So $G \setminus F$ is a clique, and hence G contains a universal vertex if $G \setminus F$ is nonempty, and G is a 5-hyperhole (and therefore an odd ring) otherwise. □

2.3.3 β-perfect rings

In this section we give a forbidden induced subgraph characterisation for the class of β-perfect rings.

We will make use of the following fact about the chromatic number of a ring.

Theorem 2.43 (Maffray, Penev and Vušković [41]). *Let $k \geq 4$ be an integer and let R be a k-ring. Then $\chi(R) = \max\{\chi(H) : H$ is a k-hyperhole in $R\}$.*

Throughout the remainder of the chapter we may use Theorem 2.43 implicitly, i.e., we may write "let H be a hyperhole contained in R such that $\chi(H) = \chi(R)$" without reference to Theorem 2.43.

Small rings

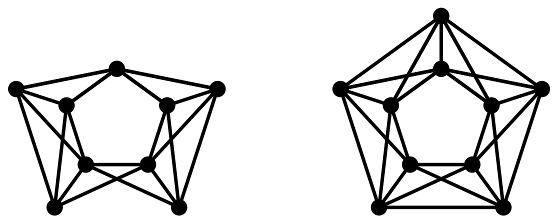


Figure 2.7: The two minimal β-imperfect rings of length 5.

Let R_5 denote the graph on the left of Figure 2.7 and H_5 the graph on the right. We summarise Lemma 2.22 and Theorem 2.23 in the following.

Lemma 2.44. *The following hold:*

- *H_5 is minimally β-imperfect.*
- *A 5-hyperhole $H = (X_1, \dots, X_5)$ is β-perfect if and only if $|X_i| = 1$ for some $i \in \{1, \dots, 5\}$.*

Lemma 2.45. *R_5 is minimally β-imperfect.*

Proof. Let $(Y_1, \dots, Y_5)$ be a ring partition of R_5, say with $|Y_1| = 1$. Let H be a hyperhole contained in R_5 such that $\chi(H) = \chi(R_5)$. Since Y_3 is not complete to Y_4 we see that $Y_3 \cup Y_4 \not\subseteq V(H)$ and therefore $|V(H)| \leq |V(R_5)| - 1$. Now, by Theorem 2.19,

$$\chi(H) \leq \max\left\{\omega(R_5), \left\lceil \frac{|V(R_5)| - 1}{2} \right\rceil\right\} \leq 4,$$

so $\chi(R_5) \leq 4$. It follows that $\chi(R_5) \leq 4 < 5 = \delta(R_5) + 1 \leq \beta(R_5)$ and therefore $\chi(R_5) < \beta(R_5)$.

It remains to prove that every proper induced subgraph of R_5 is β-perfect. To the contrary, suppose some proper induced subgraph R of R_5 is minimally β-imperfect. Let $y_3 \in Y_3$ and $y_4 \in Y_4$ be nonadjacent vertices of R_5. By Lemma 2.11, R is not chordal, so R contains a vertex from each of $Y_1, \dots, Y_5$; and R is not a hyperhole, for otherwise (since $|Y_1| = 1$) it follows from Lemma 2.44 that R is β-perfect, a contradiction. So R is a ring that is not a hyperhole, and therefore R contains both y_3 and y_4. If $Y_3 \cup Y_4 \not\subseteq V(R)$, then one of y_3, y_4 is a simplicial vertex of R, contrary to Lemma 2.4; so $Y_3 \cup Y_4 \subseteq V(R)$. Let y_1 be the unique vertex of Y_1. By Lemma 2.4, $d_R(y_1) \geq 3$ and therefore $Y_2 \subseteq V(R)$ without loss of generality. Since $Y_2 \cup Y_3$ is a clique of R_5, we now have that $Y_2 \cup Y_3$ is also a clique of R. But now, by Lemma 2.2, $\beta(R) = \delta(R) + 1 = 4 = |Y_2 \cup Y_3| = \omega(R) \leq \chi(R)$ and hence $\beta(R) = \chi(R)$, a contradiction. □

Lemma 2.46. *A 5-ring is β-perfect if and only if it is (H_5, R_5)-free.*

Proof. A β-perfect 5-ring is (H_5,R_5)-free by Lemmas 2.44 and 2.45. To prove the converse, let $R = (Y_1, \dots, Y_5)$ be a (H_5, R_5)-free 5-ring. We begin by proving the following two claims.

(1) Let $i \in \{1, \dots, 5\}$, $y_i \in Y_i$ and $y_{i+1} \in Y_{i+1}$. If y_i and y_{i+1} are nonadjacent, then $|N(y_i) \cap Y_{i-1}| = 1$ or $|N(y_{i+1}) \cap Y_{i+2}| = 1$.

Proof of (1): Suppose otherwise. Then, up to symmetry, there exist nonadjacent vertices $y_1 \in Y_1$ and $y_2 \in Y_2$ such that $|N(y_1) \cap Y_5| \geq 2$ and $|N(y_2) \cap Y_3| \geq 2$. For each $i \in \{1, \dots, 5\}$, let x_i be a vertex of Y_i that is complete to $Y_{i-1} \cup Y_{i+1}$. Since y_1 and y_2 are nonadjacent, $x_1 \neq y_1$ and $x_2 \neq y_2$. Fix $y_3 \in (N(y_2) \cap Y_3) \setminus \{x_3\}$ and $y_5 \in (N(y_1) \cap Y_5) \setminus \{x_5\}$. But now the graph induced by $\{x_1, y_1, x_2, y_2, x_3, y_3, x_4, x_5, y_5\}$ is isomorphic to R_5, a contradiction. This completes the proof of (1).

(2) There exists an integer $i \in \{1, \dots, 5\}$ and a vertex $x_i \in Y_i$ that is complete to $Y_{i-1} \cup Y_{i+1}$ such that $Y_i \setminus \{x_i\}$ is anticomplete to $(Y_{i-1} \cup Y_{i+1}) \setminus \{x_{i-1}, x_{i+1}\}$ for some $x_{i-1} \in Y_{i-1}$ and $x_{i+1} \in Y_{i+1}$.

Proof of (2): Suppose otherwise. Then, for every $i \in \{1, \dots, 5\}$, there exist distinct vertices $x_i, y_i \in Y_i$ such that x_i is complete to $Y_{i-1} \cup Y_{i+1}$ and y_i has at least one neighbour in $(Y_{i-1} \cup Y_{i+1}) \setminus \{x_{i-1}, x_{i+1}\}$. Up to symmetry we may assume that y_1 has a neighbour $y_2' \in Y_2 \setminus \{x_2\}$ and y_4 has a neighbour y_5' in $Y_5 \setminus \{x_5\}$. By (1), vertices y_1 and y_5' are adjacent. Since R is H_5-free, y_3 is not adjacent to both y_2' and y_4, so we may assume without loss of generality that y_3 and y_4 are nonadjacent. Since y_4 has at least two neighbours in Y_5, we see by (1) that $|N(y_3) \cap Y_2| = 1$, and in particular y_2' and y_3 are nonadjacent. Now by a symmetric argument applied to y_2' and y_3 it follows that $|N(y_3) \cap Y_4| = 1$, which contradicts our assumption that y_3 has at least one neighbour in $(Y_2 \cup Y_4) \setminus \{x_2, x_4\}$. This completes the proof of (2).

Towards a contradiction, suppose R is not β-perfect. Since every proper induced subgraph of R contains a simplicial vertex or is a 5-ring (that is also (H_5, R_5)-free, thereby satisfying (1) and (2)), it follows from Lemma 2.4 that we may assume R is minimally β-imperfect.

(3) Let $i \in \{1, \dots, 5\}$ and suppose that there exists a vertex $x_i \in Y_i$ that is complete to $Y_{i-1} \cup Y_{i+1}$ such that $Y_i \setminus \{x_i\}$ is anticomplete to $(Y_{i-1} \cup Y_{i+1}) \setminus \{x_{i-1}, x_{i+1}\}$ for some $x_{i-1} \in Y_{i-1}$ and $x_{i+1} \in Y_{i+1}$. Then at least one of Y_{i-1} and Y_{i+1} is of size 1.

Proof of (3): Without loss of generality, suppose $i = 1$, and towards a contradiction suppose that $|Y_2| \geq 2$ and $|Y_5| \geq 2$. Fix vertices $y_2 \in Y_2$ and $y_5 \in Y_5$ such that $N_R[y_2] \subseteq N_R[y_2']$ for every $y_2' \in Y_2$ and $N_R[y_5] \subseteq N_R[y_5']$ for every $y_5' \in Y_5$. Let Y_3' be the subset of Y_3 such that y_2 is complete to Y_3' and anticomplete to $Y_3 \setminus Y_3'$, and similarly let Y_4' be the subset of Y_4 such that y_5 is complete to Y_4' and anticomplete to $Y_4 \setminus Y_4'$. Now, since Y_3' is complete to Y_2 and Y_4' is complete to Y_5, it follows from (1) that Y_3' is complete to Y_4', and therefore $\{x_1\} \cup Y_2 \cup Y_3' \cup Y_4' \cup Y_5$ induces a hyperhole H.

Observe that $N_R(y_2) = \{x_1\} \cup (Y_2 \setminus \{y_2\}) \cup Y_3'$ and $N_R(y_5) = \{x_1\} \cup (Y_5 \setminus \{y_5\}) \cup Y_4'$, and therefore, by Lemma 2.2, the sets $Y_2 \cup Y_3'$ and $Y_4' \cup Y_5$ both have size at least $\beta(R) - 1$. It follows that $|V(H)| \geq 2\beta(R) - 1$, and hence $\chi(H) \geq \beta(R)$ by Theorem 2.19. But now $\chi(R) \geq \beta(R)$, a contradiction. This proves (3).

By (2), and without loss of generality, there is a vertex $x_1 \in Y_1$ that is complete to $Y_2 \cup Y_5$ and $Y_1 \setminus \{x_1\}$ is anticomplete to $(Y_2 \cup Y_5) \setminus \{x_2, x_5\}$. By (3), at least one of Y_2 and Y_5 is of size 1; without loss of generality, suppose $|Y_2| = 1$. Now we may apply (3) with $i = 2$ to get that at least one of Y_1 and Y_3 is of size 1. That is, R has two consecutive bags of size 1; so suppose without loss of generality that $|Y_1| = |Y_2| = 1$, and let x_1 and x_2 be the unique vertices from Y_1 and Y_2 respectively. Observe that $d_R(x_1) = 1 + |Y_5|$ and $d_R(x_2) = 1 + |Y_3|$, and therefore it follows from Lemma 2.2 that $|Y_3| \geq \beta(R) - 2$ and $|Y_5| \geq \beta(R) - 2$. Let x_4 be a vertex of Y_4 that is complete to $Y_3 \cup Y_5$, and consider the hyperhole H induced by $Y_1 \cup Y_2 \cup Y_3 \cup \{y_4\} \cup Y_5$. From the above bounds, we get that $|V(H)| \geq 2\beta(R) - 1$, and therefore it follows from Theorem 2.19 that $\chi(H) \geq \beta(R)$. So $\chi(R) \geq \beta(R)$, but this contradicts the fact that R is minimally β-imperfect. $\square$

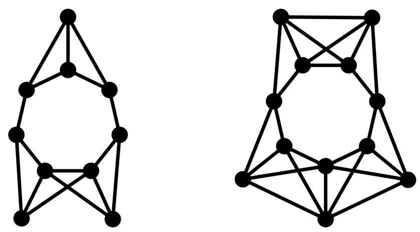


Figure 2.8: The two minimal β-imperfect rings of length 7.

Let R_7 denote the graph on the left of Figure 2.8 and H_7 the graph on the right. From Lemma 2.24 we get the following.

Lemma 2.47. *The following hold:*

- *H_7 is minimally β-imperfect.*
- *A 7-hyperhole $H = (X_1, \ldots, X_7)$ is β-perfect if and only if $|X_i| = |X_{i+1}| = 1$ or $|X_i| = |X_{i+2}| = 1$ for some $i \in \{1, \ldots, 7\}$.*

Lemma 2.48. *R_7 is minimally β-imperfect.*

Proof. Let $(Y_1, \ldots, Y_7)$ be a ring partition of R_7 and assume without loss of generality that Y_5 is not complete to Y_6. Clearly every proper induced subgraph of R_7 contains a vertex of degree at most 2, and therefore, by Lemma 2.4, every proper induced subgraph of R_7 is β-perfect. So it suffices to show that $\chi(R_7) < \beta(R_7)$. Let H be a hyperhole in R such that $\chi(H) = \chi(R)$. Since Y_5 is not complete to Y_6, we have that $Y_5 \cup Y_6 \not\subseteq V(H)$ and hence $|V(H)| \leq |V(R_7)| - 1 = 9$. Now by Theorem 2.19 applied to H we see that $\chi(R_7) = \chi(H) \leq 3 < \beta(R_7) \leq 4$. Therefore R_7 is minimally β-imperfect. □

Lemma 2.49. *A 7-ring is β-perfect if and only if it is (H_7, R_7)-free.*

Proof. A β-perfect 7-ring is (H_7, R_7)-free by Lemmas 2.47 and 2.48. We now prove the converse. Let $R = (Y_1, \dots, Y_7)$ be a (H_7, R_7)-free 7-ring. The following fact is an immediate consequence of R being R_7-free.

(1) For all $i \in \{1, \dots, 7\}$, if $|Y_i| \geq 2$, then Y_{i+3} is complete to Y_{i+4}.

We now establish the following.

(2) There exists an integer $i \in \{1, \dots, 7\}$ such that $|Y_i| = |Y_{i+1}| = 1$ or $|Y_i| = |Y_{i+2}| = 1$.

Proof of (2). Suppose otherwise. If each of $Y_1, \dots, Y_7$ has size at least 2, then it follows from (1) that R is a hyperhole each bag of which has size at least 2, and hence it contains H_7, a contradiction. So we may assume without loss of generality that $|Y_1| = 1$. Thus each of Y_2, Y_3, Y_6 and Y_7 contain at least two vertices, and up to symmetry so does Y_4. Now by (1) we see that Y_2 is complete to Y_3, Y_3 is complete to Y_4, and Y_6 is complete to Y_7, and hence R contains H_7, a contradiction. This completes the proof of (2).

Towards a contradiction, suppose that R is not β-perfect. Since every proper induced subgraph of R contains a simplicial vertex or is a 7-ring, by Lemma 2.4 we may assume that R is minimally β-imperfect. In view of (2), we may assume without loss of generality that $|Y_1| = 1$, and $|Y_2| = 1$ or $|Y_3| = 1$.

First suppose that $|Y_2| = 1$. Let y_1 be the unique vertex of Y_1, and observe that $d_R(y_1) = |Y_7| + 1$. Thus, by Lemma 2.2, $|Y_7| \geq \beta(R) - 2$, and by symmetry $|Y_3| \geq \beta(R) - 2$. In particular, since no vertex of R is of degree 2 by Lemma 2.4, it follows that both Y_3 and Y_7 have size at least 2. Now by (1) we see that Y_3 is complete to Y_4 and Y_6 is complete to Y_7. Since $\omega(R) \leq \chi(R) < \beta(R)$, it follows that $|Y_4| = |Y_6| = 1$. But now R is a hyperhole, and it is β-perfect by Lemma 2.47, a contradiction.

So $|Y_3| = 1$. Observe that vertices from Y_2 have degree $|Y_2| + 1$, and hence $|Y_2| \geq \beta(R) - 2$ by Lemma 2.2. In particular, since no vertex of R is of degree 2 by Lemma 2.4, we have that $|Y_2| \geq 2$, and hence Y_5 is complete to Y_6 by (1). Fix vertices $y_4 \in Y_4$

and $y_7 \in Y_7$ such that $N_R[y_4] \subseteq N_R[y_4']$ for every $y_4' \in Y_4$ and $N_R[y_7] \subseteq N_R[y_7']$ for every $y_7' \in Y_7$. Let $Y_5' \subseteq Y_5$ and $Y_6' \subseteq Y_6$ be such that y_4 is complete to Y_5' and anticomplete to $Y_5 \setminus Y_5'$, and y_7 is complete to Y_6' and anticomplete to $Y_6 \setminus Y_6'$. Observe that $d_R(y_4) = |Y_4 \cup Y_5'|$ and $d_R(y_7) = |Y_6' \cup Y_7|$, and hence, by Lemma 2.2, $|Y_4 \cup Y_5'| \geq \beta(R) - 1$ and $|Y_6' \cup Y_7| \geq \beta(R) - 1$. It follows that the graph H induced by $(V(R) \setminus (Y_5 \cup Y_6)) \cup Y_5' \cup Y_6'$ is a hyperhole on at least $3\beta(R) - 2$ vertices. Therefore, by Theorem 2.19, $\chi(H) \geq \beta(R)$, and hence $\chi(R) \geq \beta(R)$, a contradiction. $\square$

Big rings

We now turn to odd rings of length at least 9. For the sake of brevity, we call such rings *big*. We use terminology (such as "sector" and "super-sector") that was defined in Section 2.2.2.

If $R = (Y_1, \ldots, Y_k)$ is a ring, then a *triad of* R is any triple (Y_{i-1}, Y_i, Y_{i+1}) such that $i \in \{1, \ldots, k\}$ and $|Y_{i-1}| = |Y_i| = |Y_{i+1}| = 1$. A *bad ring* is any big ring $R = (Y_1, \ldots, Y_k)$ that satisfies the following:

- for every $i \in \{1, \ldots, k\}$, $|Y_i| \leq 2$;
- for every $i \in \{1, \ldots, k\}$, if $|Y_i| = |Y_{i+1}| = 2$, then Y_i is not complete to Y_{i+1} and $|Y_{i-2}| = |Y_{i-1}| = |Y_{i+2}| = |Y_{i+3}| = 1$;
- R has no triad; and
- there exists at least one integer $i \in \{1, \ldots, k\}$ such that $|Y_i| = |Y_{i+1}| = 2$.

Lemma 2.50. *If R is a bad ring, then R is minimally β-imperfect.*

Proof. Let $R = (Y_1, \ldots, Y_k)$ be a bad ring. Clearly every proper induced subgraph of R contains a vertex of degree at most 2, and therefore, by Lemma 2.4, every proper induced subgraph of R is β-perfect. So it remains to prove that $\chi(R) < \beta(R)$. The minimum degree of R is 3, so $\beta(R) = 4$. Let $H = (X_1, \ldots, X_k)$ be a hyperhole contained in R such that $\chi(H) = \chi(R)$ (note that H exists by Theorem 2.43). Then H is a proper induced subgraph of R, and hence, by our earlier observation, H contains a vertex of degree 2. So without loss of generality X_1, X_2 and X_3 each consist of exactly one vertex, say x_1, x_2 and x_3 respectively. The graph $H \setminus \{x_2\}$ is a chordal graph with clique number at most 3, and hence there exists a 3-colouring of $H \setminus \{x_2\}$. Such a colouring can be extended to a 3-colouring of H by assigning to x_2 one of the three colours that has not been assigned to either of x_1 and x_3. Therefore $\chi(R) = \chi(H) = 3 < \beta(R) = 4$, so $\chi(R) < \beta(R)$, and this completes the proof that R is minimally β-imperfect. $\square$

For the remainder of the chapter, whenever we speak of a ring $R = (Y_1, \ldots, Y_k)$ that contains a hyperhole $H = (X_1, \ldots, X_k)$, we implicitly assume that $X_i \subseteq Y_i$ for each $i \in \{1, \ldots, k\}$. Moreover, if H is a base hyperhole, we assume in addition that $|X_3| = |X_4| = 1$.

We use the following notation: if $i \in \{1, \ldots, k\}$, then Y_i^1 denotes a set consisting of any one vertex from Y_i that is complete to $Y_{i-1} \cup Y_{i+1}$, and Y_i^2 denotes any set obtained from Y_i^1 by adding a single vertex from $Y_i \setminus Y_i^1$ (provided $Y_i \setminus Y_i^1 \neq \emptyset$).

Lemma 2.51. *Let $R = (Y_1, \ldots, Y_k)$ be a ring that contains a base hyperhole $H = (X_1, \ldots, X_k)$. If R contains no bad base hyperhole and no bad ring, then the following hold:*

- $|Y_i| = 1$ *for all even* $i \in \{8, \ldots, k\}$*;*
- $\min(|Y_1|, |Y_4|) = \min(|Y_3|, |Y_6|) = 1$*;*
- $Y_1 \cup Y_2$*,* $Y_3 \cup Y_4$ *and* $Y_5 \cup Y_6$ *are cliques.*

Proof. Since R contains a base hyperhole but no bad base hyperhole, R contains a good base hyperhole, and hence $|Y_i| \geq 2$ for $i = 2$ and for each odd $i \in \{5, \ldots, k\}$.

Suppose for some even $i \in \{8, \ldots, k\}$ that $|Y_i| \geq 2$. But now

$$R[(V(H) \setminus (X_{i-1} \cup X_i \cup X_{i+1})) \cup Y_{i-1}^1 \cup Y_i^2 \cup Y_{i+1}^1]$$

is a bad base hyperhole with 0-sectors (X_3, X_4), (X_{i-2}, Y_{i-1}^1) and (Y_{i+1}^1, X_{i+2}), a contradiction. This proves that the first bullet holds.

Suppose $|Y_1| \geq 2$ and $|Y_4| \geq 2$. But now

$$R[Y_k^1 \cup Y_1^2 \cup Y_2^1 \cup X_3 \cup Y_4^2 \cup Y_5^1 \cup X_6 \cup \cdots \cup X_{k-1}]$$

is a bad base hyperhole with 0-sectors (Y_2^1, X_3), (Y_5^1, X_6), and (X_{k-1}, Y_k^1), a contradiction. So $\min(|Y_1|, |Y_4|) = 1$, and by a symmetric argument we get that $\min(|Y_3|, |Y_6|) = 1$. This proves that the second bullet holds.

Suppose $Y_1 \cup Y_2$ is not a clique, and fix nonadjacent vertices $v_1 \in Y_1$ and $v_2 \in Y_2$. But now

$$R[Y_k^1 \cup (Y_1^1 \cup \{v_1\}) \cup (Y_2^1 \cup \{v_2\}) \cup X_3 \cup \cdots \cup X_{k-1}]$$

is a bad ring, a contradiction. So $Y_1 \cup Y_2$ is a clique, and by symmetry so is $Y_5 \cup Y_6$. If $Y_3 \cup Y_4$ is not a clique, then there exist nonadjacent vertices $v_3 \in Y_3$ and $v_4 \in Y_4$, and

$$R[X_1 \cup Y_2^1 \cup (Y_3^1 \cup \{v_3\}) \cup (Y_4^1 \cup \{v_4\}) \cup Y_5^1 \cup X_6 \cup \cdots \cup X_k]$$

is a bad ring, a contradiction. So $Y_3 \cup Y_4$ is a clique, and this completes the proof that the third bullet holds. □

Lemma 2.52. *Let $R = (Y_1, \dots, Y_k)$ be an odd ring. Suppose that R is minimally β-imperfect. Then R has no triad, and for all $i \in \{1, \dots, k\}$, if $|Y_i| = |Y_{i+1}| = 1$, then $|Y_{i+2}| = \beta(R) - 2$ and exactly one vertex of Y_{i+3} is complete to Y_{i+2}.*

Proof. If (Y_{i-1}, Y_i, Y_{i+1}) is a triad of R, then the unique vertex in Y_i is a simplicial extreme, contrary to Lemma 2.4. So R has no triad. Fix $i \in \{1, \dots, k\}$, and suppose $|Y_i| = |Y_{i+1}| = 1$. Let v be the unique vertex in Y_{i+1}, and observe that $d_R(v) = 1+|Y_{i+2}|$. It now follows from Lemma 2.2 that $|Y_{i+2}| \geq \beta(R) - 2$. Since $\omega(R) \leq \chi(R) < \beta(R)$, every clique of R has size at most $\beta(R) - 1$, and since $Y_{i+2} \cup Y_{i+3}^1$ is a clique, $|Y_{i+2}| \leq \beta(R) - 2$. Thus $|Y_{i+2}| = \beta(R) - 2$. If there are two vertices in Y_{i+3} that are complete to Y_{i+2}, then they together with the set Y_{i+2} form a clique of size $\beta(R)$, a contradiction; so exactly one vertex of Y_{i+3} is complete to Y_{i+2}. □

Lemma 2.53 (Lemma 2.7 from [41])**.** *Let $k \geq 4$ be an integer. Then every induced subgraph of a k-ring either contains a simplicial vertex or is a k-ring.*

Lemma 2.54. *Let $R = (Y_1, \dots, Y_k)$ be a ring that is not β-perfect. Let F be an induced subgraph of R that is minimally β-imperfect, and let $F_i = V(F) \cap Y_i$ for each $i \in \{1, \dots, k\}$. Then $F = (F_1, \dots, F_k)$ is a ring.*

Proof. If F is an even hole, then F is a ring. So we may assume that F is not an even hole, and therefore F contains no simplicial vertex by Lemma 2.4. □

Lemma 2.55. *Let $R = (Y_1, \dots, Y_k)$ be a big ring that contains a base hyperhole $H = (X_1, \dots, X_k)$. Then R is β-perfect if and only if R contains no bad base hyperhole and no bad ring.*

Proof. If R is β-perfect, then R contains no bad base hyperhole and no bad ring by Lemmas 2.28 and 2.50 respectively.

For the converse, suppose, towards a contradiction, that R contains no bad base hyperhole and no bad ring but R is not β-perfect. By Lemma 2.51, the following three claims hold.

(1) $|Y_i| = 1$ *for all even* $i \in \{8, \dots, k\}$.

(2) $\min(|Y_1|, |Y_4|) = \min(|Y_3|, |Y_6|) = 1$.

(3) $Y_1 \cup Y_2$, $Y_3 \cup Y_4$ *and* $Y_5 \cup Y_6$ *are cliques.*

Let F be an induced subgraph of R that is minimally β-imperfect, and let $F_i = V(F) \cap Y_i$ for each $i \in \{1, \dots, k\}$. By Lemma 2.54, $F = (F_1, \dots, F_k)$ is a ring.

In view of (2), there are, up to symmetry, three cases as follows:

- $|Y_1| = |Y_4| = |Y_3| = |Y_6| = 1$; or
- $|Y_1| \geq 2$ and $|Y_4| = 1$; or
- $|Y_1| = 1$ and $|Y_4| \geq 2$.

In the first case, it follows from (1), (2) and (3) that R is a hyperhole, and R clearly contains no even hole, bad base hyperhole, H_5 or H_7; but then, by Theorem 2.33, R is β-perfect, a contradiction. So the first case does not hold.

(4) F contains a base hyperhole.

Proof of (4): By (1), $|F_i| = 1$ for each even $i \in \{8, \dots, k\}$. Suppose $|F_1| = |F_6| = 1$. By Lemma 2.52, F has no triad, and hence $|F_i| \geq 2$ for each odd $i \in \{7, \dots, k\}$. Suppose $|F_2| = 1$. By Lemma 2.52, $|F_3| \geq 2$, and then by the same Lemma (together with (3)), $|F_4| = 1$, and therefore $|F_5| \geq 2$. Now clearly F contains a base hyperhole. So $|F_2| \geq 2$, and by symmetry $|F_5| \geq 2$, and again clearly F contains a base hyperhole. So we may assume it is not the case that $|F_1| = |F_6| = 1$.

Suppose $|F_1| \geq 2$. Then $|Y_1| \geq 2$, and hence by (2), $|Y_4| = 1$, and therefore $|F_4| = 1$. If $|F_3| = 1$, then (since (F_2, F_3, F_4) is not a triad by Lemma 2.52) $|F_2| \geq 2$; but, by (3), F_1 is complete to F_2, and the four bags F_1, F_2, F_3, F_4 contradict Lemma 2.52. So $|F_3| \geq 2$, and hence $|Y_6| = 1$ by (2). Since $|F_4| = |F_6| = 1$, it follows from Lemma 2.52 that $|F_5| \geq 2$. It is now easily seen that F contains a base hyperhole. Thus, if $|F_1| \geq 2$, then F contains a base hyperhole, and a symmetric argument shows that if $|F_6| \geq 2$, then F contains a base hyperhole. This proves (4).

By (4), F contains a base hyperhole, and since R contains no bad base hyperhole or bad ring, neither does F. So all our assumptions about R also hold for F, and therefore we may assume that $R = F$; i.e., we may assume R is minimally β-perfect. Recall from above that there are two cases: $|Y_1| \geq 2$ and $|Y_4| = 1$; or $|Y_1| = 1$ and $|Y_4| \geq 2$.

Suppose $|Y_1| \geq 2$ and $|Y_4| = 1$. If $|Y_3| = 1$, then by Lemma 2.52, Y_1 is not complete to Y_2, contradicting (3). So $|Y_3| \geq 2$, and therefore, by (2), $|Y_6| = 1$. So for any vertex $v \in Y_5$, $d_R(y_5) = |Y_5| + 1$, and hence $|Y_5| \geq \beta(R) - 2$ by Lemma 2.2. By a similar argument, and since $|Y_6| = 1$ and (by (1)) $|Y_i| = 1$ for each even $i \in \{8, \dots, k\}$, we have that $|Y_j| \geq \beta(R) - 2$ for each odd $j \in \{7, \dots, k-2\}$. So among $Y_4, \dots, Y_{k-1}$, there are $\lfloor \frac{k-4}{2} \rfloor$ bags of size at least $\beta(R) - 2$, and the remaining $\lceil \frac{k-4}{2} \rceil$ bags are of size 1.

Fix $y_3 \in Y_3$ with $|N_R(y_3) \cap Y_2|$ minimum and $y_k \in Y_k$ with $|N_R(y_k) \cap Y_1|$ minimum; and set $Z_2 = N_R(y_3) \cap Y_2$ and $Z_1 = N_R(y_k) \cap Y_1$. By assumption, $|Y_4| = 1$, and by (1), $|Y_{k-1}| = 1$, and therefore $d_R(y_3) = 1 + (|Y_3| - 1) + |Z_2| = |Y_3| + |Z_2|$, and similarly $d_R(y_k) = |Y_k| + |Z_1|$. By Lemma 2.2, $\delta(R) = \beta(R) - 1$, and hence $|Y_3| + |Z_2| \geq \beta(R) - 1$ and $|Y_k| + |Z_1| \geq \beta(R) - 1$. Now $Z = R[Z_1 \cup Z_2 \cup Y_3 \cup Y_4 \cup \dots \cup Y_k]$ is a hyperhole, and by adding together the above bounds, we get that

$$\begin{aligned} |V(Z)| &\geq 2\left(\beta(R) - 1\right) + \left\lfloor \frac{k-4}{2} \right\rfloor \left(\beta(R) - 2\right) + \left\lceil \frac{k-4}{2} \right\rceil \\ &= 2\left(\beta(R) - 1\right) + \frac{k-5}{2}\left(\beta(R) - 2\right) + \frac{k-3}{2} \\ &= \frac{k-1}{2}\beta(R) - \frac{k-1}{2} + 1. \end{aligned}$$

Thus, by Theorem 2.19,

$$\chi(Z) \geq \left\lceil \frac{(k-1)\beta(R) - (k-1) + 2}{k-1} \right\rceil = \left\lceil \beta(R) - 1 + \frac{2}{k-1} \right\rceil,$$

and since $\frac{2}{k-1} > 0$ for all $k > 1$, we get that $\chi(Z) \geq \beta(R)$, and hence $\chi(R) \geq \beta(R)$, a contradiction.

So $|Y_1| = 1$ and $|Y_4| \geq 2$. If $|Y_6| \geq 2$, then by (2), $|Y_3| = 1$, and hence (after relabeling the indices of $Y_1, \dots, Y_k$) we may apply the earlier argument which handles the case where $|Y_1| \geq 2$ and $|Y_4| = 1$. So we may assume $|Y_6| = 1$; and it follows from this assumption together with (1) that for each odd $i \in \{7, 9, \dots, k\}$, $|Y_{i-1}| = |Y_{i+1}| = 1$. By Lemma 2.2, $\delta(R) = \beta(R) - 1$, and therefore $|Y_i| \geq \beta(R) - 2$ for each odd $i \in \{7, 9, \dots, k\}$. So, among the bags $Y_6, \dots, Y_k, Y_1$, there are $\lfloor \frac{k-4}{2} \rfloor = \frac{k-5}{2}$ bags of size at least $\beta(R) - 2$, and the remaining $\lceil \frac{k-4}{2} \rceil = \frac{k-3}{2}$ bags are of size 1.

Fix $y_2 \in Y_2$ with $|N_R(y_2) \cap Y_3|$ minimum and $y_5 \in Y_5$ with $|N_R(y_5) \cap Y_4|$ minimum. Set $Z_3 = N_R(y_2) \cap Y_3$ and $Z_4 = N_R(y_5) \cap Y_4$. Since $d_R(y_2) = |Y_2| + |Z_3|$ and $d_R(y_5) = |Y_5| + |Z_4|$, it follows from Lemma 2.2 that $|Y_2| + |Z_3| \geq \beta(R) - 1$ and $|Y_5| + |Z_4| \geq \beta(R) - 1$. Since, by (3), Y_3 is complete to Y_4, it follows that $Z = R[Y_1 \cup Y_2 \cup Z_3 \cup Z_4 \cup Y_5 \cup \dots \cup Y_k]$ is a hyperhole, and by adding together the above bounds we get that

$$\begin{aligned} |V(Z)| &\geq 2(\beta(R) - 1) + \frac{k-5}{2}(\beta(R) - 2) + \frac{k-3}{2} \\ &= \frac{k-1}{2}\beta(R) - \frac{k-1}{2} + 1. \end{aligned}$$

Thus, by Theorem 2.19,

$$\chi(Z) \geq \left\lceil \frac{(k-1)\beta(R)-(k-1)+2}{k-1} \right\rceil = \left\lceil \beta(R) - 1 + \frac{2}{k-1} \right\rceil,$$

and since $\frac{2}{k-1} > 0$ for all $k > 1$, we get that $\chi(Z) \geq \beta(R)$, and hence $\chi(R) \geq \beta(R)$, a contradiction. □

A super-sector is a *2-super-sector* if it contains only 2-sectors.

Lemma 2.56. *Let R be an odd ring. If R has a triad or a 2-super-sector, then R is β-perfect.*

Proof. Towards a contradiction, suppose R has a triad or a 2-super-sector but is not β-perfect. Let F be an induced subgraph of R that is minimally β-imperfect, and let $F_i = V(F) \cap Y_i$ for each $i \in \{1, \dots, k\}$. By Lemma 2.11, F is not chordal, and hence $F_1, \dots, F_k$ are all nonempty. By Lemma 2.54, $F = (F_1, \dots, F_k)$ is a ring. It follows that if (Y_{i-1}, Y_i, Y_{i+1}) is a triad of R, then F has a vertex of degree 2, contrary to Lemma 2.4; thus, R has no triad. So R has a 2-super-sector, say $S = (Y_\ell, \dots, Y_r)$.

We now show for each 2-sector $(Y_s, Y_{s+1}, Y_{s+2}, Y_{s+3})$ contained in S that $|F_{s+1}| = \beta(F) - 2$ and $|F_{s+2}| = 1$. Since $Y_\ell, Y_{\ell+1}$ are of size 1, so are $F_\ell, F_{\ell+1}$, and hence it follows from Lemma 2.52 that $|F_{\ell+2}| = \beta(F) - 2$ and exactly one vertex in $F_{\ell+3}$ is complete to $F_{\ell+2}$. More specifically, since Y_i is complete to Y_{i+1} for each $i \in \{\ell, \ell+1, \dots, r-1\}$ (and hence every vertex in $F_{\ell+3}$ is complete to $F_{\ell+2}$), we get that $|F_{\ell+3}| = 1$. So our claim holds for the 2-sector $(Y_{\ell+1}, Y_{\ell+2}, Y_{\ell+3}, Y_{\ell+4})$. Since we now have that $|F_{\ell+3}| = |F_{\ell+4}| = 1$, we may repeat this argument for the 2-sector $(Y_{\ell+4}, Y_{\ell+5}, Y_{\ell+6}, Y_{\ell+7})$, if it exists, to get that $|F_{\ell+5}| = \beta(F) - 2$ and $|F_{\ell+6}| = 1$; and then for the 2-sector $(Y_{\ell+7}, Y_{\ell+8}, Y_{\ell+9}, Y_{\ell+10})$, if it exists, to get that $|F_{\ell+8}| = \beta(F) - 2$ and $|F_{\ell+9}| = 1$; and so on, until we get that $|F_{r-3}| = \beta(F) - 2$ and $|F_{r-2}| = 1$. But now the unique vertex of F_{r-1} is of degree 2 in F, contrary to Lemma 2.4. □

Lemma 2.57. *Let $R = (Y_1, \dots, Y_k)$ be an odd ring that has no triad, no 2-super-sector, and contains no bad ring. Then some induced subgraph H of R is a hyperhole that has no triad and no 2-super-sector.*

Proof. Towards a contradiction, suppose every hyperhole in R has a triad or a 2-super-sector. For any hyperhole $H = (X_1, \dots, X_k)$ in R, let $t(H)$ be the number of triads of H; i.e., $t(H)$ denotes the size of the set

$$\{i \in \{1, \dots, k\} : (X_{i-1}, X_i, X_{i+1}) \text{ is a triad of } H\};$$

and let $s(H)$ be the number of 2-super-sectors of H; i.e., $s(H)$ denotes the size of the set

$$\{(\ell, r) : \ell, r \in \{1, \dots, k\} \text{ and } (X_\ell, \dots, X_r) \text{ is a 2-super-sector of } H\}.$$

Fix a hyperhole $H = (X_1, \dots, X_k)$ in R that minimises $t(H)$, and subject to that, also minimises $s(H)$. For each $i \in \{1, \dots, k\}$, if $|X_i| = 1$, then we assume that the unique vertex in X_i is complete to $Y_{i-1} \cup Y_{i+1}$ (such a vertex exists by the definition of a ring).

(1) If (X_{i-1}, X_i, X_{i+1}) is a triad of H, then $X_i = Y_i$. Consequently, there exist no five bags $X_j, X_{j+1}, \dots, X_{j+4}$ of H all of size one.

Proof of (1): Suppose that (X_{i-1}, X_i, X_{i+1}) is a triad but $X_i \neq Y_i$. But now for any $y \in Y_i \setminus X_i$ the graph $H \cup \{y\}$ is a hyperhole with $t(H \cup \{y\}) < t(H)$, a contradiction. We now prove the second statement of the claim. Suppose, without loss of generality, that $X_1, X_2, \dots, X_5$ are all of size 1. It follows from the first statement of the claim applied to each of the triads (X_1, X_2, X_3), (X_2, X_3, X_4) and (X_3, X_4, X_5) that $|Y_2| = |Y_3| = |Y_4| = 1$. But then (Y_2, Y_3, Y_4) is a triad of R, a contradiction. This proves (1).

(2) Let (X_{i-1}, X_i, X_{i+1}) be a triad of H. If $Y_{i-1} \setminus X_{i-1} \neq \emptyset$, then:

- $|X_{i-2}| \geq 2$;
- $|X_{i-4}| = |X_{i-3}| = |Y_{i-3}| = 1$; *and*
- *no vertex in $Y_{i-1} \setminus X_{i-1}$ has at least two neighbours in Y_{i-2}.*

Similarly, if $Y_{i+1} \setminus X_{i+1} \neq \emptyset$, then:

- $|X_{i+2}| \geq 2$;
- $|Y_{i+3}| = |X_{i+3}| = |X_{i+4}| = 1$; *and*
- *no vertex in $Y_{i+1} \setminus X_{i+1}$ has at least two neighbours in Y_{i+2}.*

Proof of (2): If $|X_{i-4}| \geq 2$ or $|X_{i-3}| \geq 2$, then the graph $H' = (H \setminus (X_{i-2} \cup X_{i-1})) \cup Y^1_{i-2} \cup Y^2_{i-1}$ is a hyperhole with $t(H') < t(H)$, a contradiction. So $|X_{i-4}| = |X_{i-3}| = 1$. If $|X_{i-2}| = 1$, then (X_{i-2}, X_{i-1}, X_i) is a triad and $Y_{i-1} \setminus X_{i-1} \neq \emptyset$, contrary to (1); so $|X_{i-2}| \geq 2$. If $|Y_{i-3}| \geq 2$, then the graph $H' = (H \setminus (X_{i-3} \cup X_{i-2} \cup X_{i-1})) \cup Y^2_{i-3} \cup Y^1_{i-2} \cup Y^2_{i-1}$ is a hyperhole with $t(H') < t(H)$, a contradiction; so $|Y_{i-3}| = 1$. Suppose that some vertex $y \in Y_{i-1} \setminus X_{i-1}$ has two neighbours $y', y'' \in Y_{i-2}$. Then the graph

$H' = (H \setminus X_{i-2}) \cup \{y, y', y''\}$ is a hyperhole with $t(H') < t(H)$, a contradiction. The analogous statements when $Y_{i+1} \setminus X_{i+1} \neq \emptyset$ follow by symmetry. This proves (2).

(3) We may assume that no super-sector of H contains a sector of length at least 3.

Proof of (3): Suppose that some super-sector $S = (X_\ell, \ldots, X_r)$ of H contains a sector $T = (X_s, \ldots, X_t)$ of length at least 3. For each $i \in \{\ell, \ldots, r\}$ with $|X_i| = 1$, let $X'_i = X_i$. Let $Q = (X_a, \ldots, X_b)$ be any sector contained in the super-sector S. Suppose that Q is not a 2-sector. If the length of Q is odd, then for each $j \in \{a+1, \ldots, b-1\}$ let $X'_j = X^2_j$ if $j - a$ is odd, and let $X'_j = X^1_j$ if $j - a$ is even. If the length of Q is even, then let $X'_{b-1} = X^2_{b-1}$, let $X'_{b-2} = X^1_{b-2}$, and for each $j \in \{a+1, \ldots, b-3\}$ let $X'_j = X^2_j$ if $j - a$ is odd, and let $X'_j = X^1_j$ if $j - a$ is even. For each 2-sector $(X_s, X_{s+1}, X_{s+2}, X_{s+3})$ contained in S, let $X'_s = X_s$, $X'_{s+1} = X_{s+1}$, $X'_{s+2} = X_{s+2}$, and $X'_{s+3} = X_{s+3}$. Now the graph H' induced by $X'_\ell \cup \cdots \cup X'_r \cup (V(H) \setminus (X_\ell \cup \cdots \cup X_r))$ is a hyperhole with $t(H') = t(H)$, $s(H') = s(H)$, and with one fewer super-sector that contains a sector of length at least 3. By repeating this process, we obtain a hyperhole with the same number of triads as H and the same number of 2-super-sectors as H but with no super-sector that contains a sector of length at least 3. Thus we may assume that H has no super-sector that contains a sector of length at least 3. This proves (3).

By (3), we may assume that each sector contained in a super-sector of H is of length 1 or 2.

(4) Let $(X_\ell, \ldots, X_r)$ be a 2-super-sector of H. Then, for all $i \in \{\ell+1, \ldots, r-1\}$, if $|X_i| = 1$ then $|Y_i| = 1$.

Proof of (4): Observe that $|X_{\ell+1}| = |X_{r-1}| = 1$. Suppose that $|Y_{\ell+1}| \neq 1$. For each $i \in \{\ell, \ldots, r\} \setminus \{\ell+1\}$ such that $|X_i| = 1$, let $X'_i = X_i$. Let $X'_{\ell+1} = Y^2_{\ell+1}$. For each 2-sector $(X_s, X_{s+1}, X_{s+2}, X_{s+3})$ in $(X_\ell, \ldots, X_r)$, let $X'_{s+1} = X^1_{s+1}$, and let $X'_{s+2} = X^2_{s+2}$. Now the graph H' induced by $X'_\ell \cup \cdots \cup X'_r \cup (V(H) \setminus (X_\ell \cup \cdots \cup X_r))$ is a hyperhole with $t(H') = t(H)$ but with $s(H') < s(H)$, a contradiction. So $|Y_{\ell+1}| = 1$, and by a symmetric argument we get that $|Y_{r-1}| = 1$.

Now suppose that $|X_i| = 1$ but $|Y_i| \neq 1$ for some $i \in \{\ell+2, \ldots, r-2\}$. For each $j \in \{\ell, \ldots, r\} \setminus \{i\}$ such that $|X_j| = 1$, let $X'_j = X_j$. Let $X'_i = Y^2_i$. For each 2-sector $(X_s, X_{s+1}, X_{s+2}, X_{s+3})$ in the subsequence $(X_\ell, \ldots, X_i)$, let $X'_{s+1} = X^2_{s+1}$ and let $X'_{s+2} = X^1_{s+2}$; and for each 2-sector $(X_s, X_{s+1}, X_{s+2}, X_{s+3})$ in the subsequence $(X_i, \ldots, X_r)$, let $X'_{s+1} = X^1_{s+1}$ and let $X'_{s+2} = X^2_{s+2}$. Now the graph H' induced by

$X'_\ell \cup \cdots \cup X'_r \cup (V(H) \setminus (X_\ell \cup \cdots \cup X_r))$ is a hyperhole with $t(H') = t(H)$ but with $s(H') < s(H)$, a contradiction. Thus, for every $i \in \{\ell+2, \dots, r-2\}$, if $|X_i| = 1$ then $|Y_i| = 1$. This proves (4).

(5) Let $(X_\ell, \dots, X_r)$ be a 2-super-sector of H. If $|Y_\ell| \geq 2$, then $|X_{\ell-3}| = |X_{\ell-2}| = |Y_{\ell-2}| = 1$. Similarly, if $|Y_r| \geq 2$, then $|Y_{r+2}| = |X_{r+2}| = |X_{r+3}| = 1$.

Proof of (5): We prove the statement when $|Y_\ell| \geq 2$, and the case where $|Y_r| \geq 2$ follows by symmetry. Suppose that $|Y_\ell| \geq 2$. Let $X'_\ell = Y^2_\ell$ and $X'_r = X_r$. For each 2-sector $(X_s, X_{s+1}, X_{s+2}, X_{s+3})$ in $(X_\ell, \dots, X_r)$, let $X'_s = X_s$, $X'_{s+1} = X^1_{s+1}$, $X'_{s+2} = X^2_{s+2}$, and $X'_{s+3} = X_{s+3}$. Let $X'_{\ell-1} = X^1_{\ell-1}$. If $|X_{\ell-3}| \geq 2$, then let $X'_{\ell-2} = X^1_{\ell-2}$; now $X'_{\ell-2} \cup X'_{\ell-1} \cup X'_\ell \cup \cdots \cup X'_r \cup (V(H) \setminus (X_{\ell-2} \cup X_{\ell-1} \cup X_\ell \cup \cdots \cup X_r))$ induces a hyperhole H' with $t(H') = t(H)$ but with $s(H') < s(H)$, a contradiction. So $|X_{\ell-3}| = 1$. If $|X_{\ell-2}| \geq 2$, then $X'_{\ell-1} \cup X'_\ell \cup \cdots \cup X'_r \cup (V(H) \setminus (X_{\ell-1} \cup X_\ell \cup \cdots \cup X_r)$ again induces a hyperhole H' with $t(H') = t(H)$ but with $s(H') < s(H)$, a contradiction. A similar contradiction arises if $|Y_{\ell-2}| \geq 2$. Thus, if $|Y_\ell| \geq 2$, then $|X_{\ell-3}| = |X_{\ell-2}| = |Y_{\ell-2}| = 1$. This proves (5).

(6) Let $(X_\ell, \dots, X_r)$ be a 2-super-sector of H. Then we may assume that no vertex in $Y_\ell \setminus X_\ell$ has at least two neighbours in $Y_{\ell-1}$. Similarly, we may assume that no vertex in $Y_r \setminus X_r$ has at least two neighbours in Y_{r+1}.

Proof of (6): On the contrary, suppose that some vertex $y \in Y_\ell \setminus X_\ell$ has two neighbours $u, v \in Y_{\ell-1}$. It follows that $|Y_\ell| \geq 2$, and hence $|X_{\ell-3}| = |X_{\ell-2}| = 1$ by (5). Now consider the graph $H' = R[\{u, v, y\} \cup (V(H) \setminus X_{\ell-1})]$; it is a hyperhole, with $t(H') = t(H)$ and $s(H') = s(H)$, of which $(X_{\ell-3}, X_{\ell-2}, \{u, v\}, X_\ell \cup \{y\}, X_{\ell+1}, \dots, X_r)$ is a 2-super-sector. If some vertex in $Y_{\ell-3} \setminus X_{\ell-3}$ has at least two neighbours in $Y_{\ell-4}$ or if some vertex in $Y_r \setminus X_r$ has at least two neighbours in Y_{r+1}, then we may repeat this process. Since this process must terminate, we obtain in the end a hyperhole H^+ (say with bags $X'_1, \dots, X'_k$ satisfying $X'_i \subseteq Y_i$ for each $i \in \{1, \dots, k\}$) with $t(H^+) = t(H)$, $s(H^+) = s(H)$, and having a 2-super-sector $(X'_{\ell'}, \dots, X'_{r'})$ where $\{\ell, \dots, r\} \subseteq \{\ell', \dots, r'\}$, and no vertex in $Y_{\ell'} \setminus X'_{\ell'}$ has at least two neighbours in $Y_{\ell'-1}$, and no vertex in $Y_{r'} \setminus X'_{r'}$ has at least two neighbours in $Y_{r'+1}$. This proves (6).

(7) Let $S = (X_\ell, \dots, X_r)$ be a super-sector of H. If S contains a 2-sector, then we may assume that S is a 2-super-sector.

Proof of (7): Suppose that S contains at least one 2-sector. If S contains only 2-sectors, then we are done; thus, we may assume that S contains a sector of some other length, and by (3), each such sector is of length 1. Let (X_{t-1}, X_t, X_{t+1}) be a 1-sector contained in S. For each $i \in \{\ell, \dots, r\}$ with $|X_i| = 1$, let $X'_i = X_i$. For each 1-sector (X_{s-1}, X_s, X_{s+1}), let $X'_s = X_s$, and for each 2-sector $(X_s, X_{s+1}, X_{s+2}, X_{s+3})$, let $X'_{s+1} = X^2_{s+1}$ and $X'_{s+2} = X^1_{s+2}$ if $(X_s, X_{s+1}, X_{s+2}, X_{s+3})$ appears in the subsequence $(X_\ell, \dots, X_{t-1})$, and otherwise, let $X'_{s+1} = X^1_{s+1}$ and $X'_{s+2} = X^1_{s+2}$. Now the graph H' induced by $X'_\ell \cup \dots \cup X'_r \cup (V(H) \setminus (X_\ell \cup \dots \cup X_r))$ is a hyperhole with $t(H') = t(H)$, $s(H') = s(H)$, and $(X'_\ell, \dots, X'_r)$ is not a super-sector of H'. After repeating this procedure for each super-sector that contains a 2-sector, we obtain a hyperhole in which every super-sector containing a 2-sector is a 2-super-sector. This proves (7).

(8) Let $(X_\ell, \dots, X_r)$ be a 2-super-sector of H. If $|Y_\ell| \geq 2$, then $|X_{\ell-1}| \geq 2$. Similarly, if $|Y_r| \geq 2$, then $|X_{r+1}| \geq 2$.

Proof of (8): Suppose that $|Y_\ell| \geq 2$ but $|X_{\ell-1}| = 1$. Let $X'_\ell = Y^2_\ell$, for each $i \in \{\ell + 2, \ell + 5, \ell + 8, \dots, r - 3\}$, let $X'_i = X^1_i$, and for every other $i \in \{\ell, \dots, r\}$ let $X'_i = X_i$. Now the graph H' induced by $X'_\ell \cup \dots \cup X'_r \cup (V(H) \setminus (X_\ell \cup \dots \cup X_r))$ is a hyperhole with $t(H') = t(H)$ but with $s(H') < s(H)$, a contradiction. Thus, if $|Y_\ell| \geq 2$, then $|X_{\ell-1}| \geq 2$, and by symmetry, if $|Y_r| \geq 2$, then $|X_{r+1}| \geq 2$. This proves (8).

(9) We may assume that every sector of length at least 2 in H is contained in a super-sector of H.

Proof of (9): Let $S = (X_a, \dots, X_b)$ be a sector of length at least 2 in H, and say without loss of generality that $a = 1$. Suppose that H contains two 0-sectors (X_i, X_{i+1}) and (X_j, X_{j+1}) such that $i, i + 1, j, j + 1$ are all distinct. Assume i and j are chosen so that i is minimum (possibly $i = b$) and j is maximum (possibly $j = k$). By the definition of a sector we have that $\{i, i + 1, j, j + 1\} \cap \{2, \dots, b - 1\} = \emptyset$, and by our choice of i and j, there is no 0-sector in $(X_{j+1}, \dots, X_1, \dots, X_b, \dots, X_i)$. Now $(X_j, X_{j+1}, \dots, X_1, \dots, X_b, \dots, X_i, X_{i+1})$ is a super-sector of H that contains S, as required.

Thus, it remains to show that H has at least two 0-sectors that are formed by four distinct bags of H. If H has a super-sector $(X_\ell, \dots, X_r)$, then $(X_\ell, X_{\ell+1})$ and (X_{r-1}, X_r) are two such 0-sectors; thus, we may assume that H has no super-sector. If H has no triad, then H is as desired (i.e., H is an induced subgraph of R, and is a hyperhole, and has no triad and no 2-super-sector), and we are done; so suppose

(X_{i-1}, X_i, X_{i+1}) is a triad of H. Since R has no triad, it follows from (1) that $|Y_{i-1}| \geq 2$ or $|Y_{i+1}| \geq 2$. Suppose without loss of generality that $|Y_{i-1}| \geq 2$. Then, by (2), $|X_{i-4}| = |X_{i-3}| = 1$, and hence (X_{i-4}, X_{i-3}) and (X_{i-1}, X_i) are two 0-sectors of H, as required. This proves (9).

A 2-super-sector $(X_\ell, \dots, X_r)$ of H is of *type 1* if Y_{s+1} is complete to Y_{s+2} for every 2-sector $(X_s, X_{s+1}, X_{s+2}, X_{s+3})$ contained in $(X_\ell, \dots, X_r)$, and is of *type 2* otherwise. Thus, if a 2-super-sector $(X_\ell, \dots, X_r)$ of H is of type 2, then it contains at least one 2-sector $(X_s, X_{s+1}, X_{s+2}, X_{s+3})$ such that Y_{s+1} is not complete to Y_{s+2}.

Suppose $(X_\ell, \dots, X_r)$ is a 2-super-sector of type 1 of H. By (4), for each 2-sector $(X_s, X_{s+1}, X_{s+2}, X_{s+3})$ contained in $(X_\ell, \dots, X_r)$, we have that $|Y_s| = |Y_{s+3}| = 1$, and by the definition of type 1, Y_{s+1} is complete to Y_{s+2}. Since R has no 2-super-sector (so in particular $(Y_\ell, \dots, Y_r)$ is not a 2-super-sector of R), it follows that $|Y_\ell| \geq 2$ or $|Y_r| \geq 2$ (or both). We say a 2-super-sector $(X_\ell, \dots, X_r)$ of type 1 of H is *left* if $|Y_\ell| \geq 2$, and *right* otherwise; so $(X_\ell, \dots, X_r)$ is left or right, but not both, and if it is right, then $|Y_\ell| = 1$ and $|Y_r| \geq 2$.

(10) Let $S = (X_\ell, \dots, X_r)$ be a 2-super-sector of type 2 of H. Then we may assume that S contains only one 2-sector.

Proof of (10): Suppose $(X_\ell, \dots, X_r)$ contains at least two 2-sectors. We show that we may modify some of the bags among $X_\ell, \dots, X_r$ so that H contains one fewer 2-super-sector of type 2 that contains at least two 2-sectors but with $t(H)$ and $s(H)$ unchanged. We consider two cases; first, the case where there is some 2-sector $(X_s, X_{s+1}, X_{s+2}, X_{s+3})$ contained in S such that Y_{s+1} is complete to Y_{s+2}. Let $(X_a, \dots, X_b)$ be a maximal subsequence of $(X_\ell, \dots, X_r)$ such that:

- $(X_a, X_{a+1}, X_{a+2}, X_{a+3})$ and $(X_{b-3}, X_{b-2}, X_{b-1}, X_b)$ are 2-sectors (possibly not distinct); and
- every 2-sector $(X_t, X_{t+1}, X_{t+2}, X_{t+3})$ in $(X_a, \dots, X_b)$ is such that Y_{t+1} is complete to Y_{t+2}.

It follows from the existence of $(X_s, X_{s+1}, X_{s+2}, X_{s+3})$ that there exists such a subsequence. Note that, by maximality, if $(X_{a-3}, X_{a-2}, X_{a-1}, X_a)$ is a 2-sector contained in S, then Y_{a-2} is not complete to Y_{a-1}, and similarly, if $(X_b, X_{b+1}, X_{b+2}, X_{b+3})$ is a 2-sector contained in S, then Y_{b+1} is not complete to Y_{b+2}. Now, for each 2-sector $(X_t, X_{t+1}, X_{t+2}, X_{t+3})$ in the subsequence $(X_\ell, \dots, X_{a-1}, X_a)$, set $X_{t+2} = Y^1_{t+2}$, and for each 2-sector $(X_t, X_{t+1}, X_{t+2}, X_{t+3})$ in the subsequence $(X_b, X_{b+1}, \ldots, X_r)$, let $X_{t+1} = Y^1_{t+1}$. Observe that $(X_{a-1}, X_a, \ldots, X_b, X_{b+1})$ is now a 2-super-sector of type 1 of H, and that no other subsequence of $(X_\ell, \ldots, X_r)$ besides $(X_{a-1}, X_a, \ldots, X_b, X_{b+1})$ is a 2-super-sector of H. Thus, $s(H)$ remains unchanged. It is clear that $t(H)$ also remains unchanged, and H now has one fewer 2-super-sector of type 2 that contains at least two 2-sectors.

Now for the other case, i.e., for every 2-sector $(X_s, X_{s+1}, X_{s+2}, X_{s+3})$ contained in S, we have that Y_{s+1} is not complete to Y_{s+2}. For each 2-sector $(X_s, X_{s+1}, X_{s+2}, X_{s+3})$ contained in $(X_\ell, \ldots, X_r)$, besides $(X_{\ell+1}, X_{\ell+2}, X_{\ell+3}, X_{\ell+4})$, set $X_{s+1} = Y^1_{s+1}$. Observe that $(X_\ell, X_{\ell+1}, \ldots, X_{\ell+5})$ is now a 2-super-sector of type 2 of H, which contains only one 2-sector (namely, $(X_{\ell+1}, \ldots, X_{\ell+4})$), and that no other subsequence of $(X_\ell, \ldots, X_r)$ besides $(X_\ell, X_{\ell+1}, \ldots, X_{\ell+5})$ is a 2-super-sector of H. Thus, $s(H)$ remains unchanged. It is clear that $t(H)$ also remains unchanged, and H now has one fewer 2-super-sector of type 2 that contains at least two 2-sectors. This proves (10).

In view of (10), from here on we assume that every 2-super-sector of type 2 of H contains only one 2-sector.

Suppose (X_{i-1}, X_i, X_{i+1}) is a triad of H. Since R has no triad, it follows from (1) that $Y_{i-1} \setminus X_{i-1} \neq \emptyset$ or $Y_{i+1} \setminus X_{i+1} \neq \emptyset$ (or possibly both). We say (X_{i-1}, X_i, X_{i+1}) is *left* if $Y_{i-1} \setminus X_{i-1} \neq \emptyset$, and *right* otherwise. Thus each triad of H is left or right, but not both left and right.

We now construct an induced subgraph Z of R in the following way.

Step 1. Let $Z_i = X_i$ for each $i \in \{1, \ldots, k\}$.

Step 2. For each left triad (X_{i-1}, X_i, X_{i+1}) of H, set $Z_{i-1} = Y^2_{i-1}$, and for each right triad (X_{i-1}, X_i, X_{i+1}) of H, set $Z_{i+1} = Y^2_{i+1}$.

Step 3. For each left 2-super-sector $S = (X_\ell, \ldots, X_r)$ of type 1 of H, do the following:

- let $Z_\ell = Y^2_\ell$; and
- for each 2-sector $(X_s, X_{s+1}, X_{s+2}, X_{s+3})$ in S, let $Z_{s+1} = X^1_{s+1}$.

For each right 2-super-sector $S = (X_\ell, \ldots, X_r)$ of type 1 of H, do the following:

- let $Z_r = Y^2_r$; and
- for each 2-sector $(X_s, X_{s+1}, X_{s+2}, X_{s+3})$ in S, let $Z_{s+2} = X^1_{s+2}$.

For each 2-super-sector $S = (X_\ell, \ldots, X_r)$ of type 2 of H, do the following:

- let $y \in Y_{\ell+2}$ and $y' \in Y_{\ell+3}$ be nonadjacent vertices, and let $Z_{\ell+2} = Y^1_{\ell+2} \cup \{y\}$ and $Z_{\ell+3} = Y^1_{\ell+3} \cup \{y'\}$.

Step 4. For each $i \in \{1, \dots, k\}$ such that Z_i was not modified in Step 2 or 3, let $Z_i = X^1_i$ if $|X_i| = 1$, and otherwise let $Z_i = X^2_i$.

Let $Z_1, \dots, Z_k$ be as they are when the above algorithm terminates, and let $Z = R[Z_1 \cup \dots \cup Z_k]$.

Observation 1: $Z_i \subseteq Y_i$ for each $i \in \{1, \dots, k\}$.

Observation 2: $|Z_i| \leq 2$ for each $i \in \{1, \dots, k\}$.

Observation 3: If $|X_i| = 1$ and $|Z_i| = 2$, then the algorithm set Z_i to be of size 2 in Step 2 or Step 3, and therefore at least one of the following holds:

- (X_i, X_{i+1}, X_{i+2}) is a left triad of H;
- (X_{i-2}, X_{i-1}, X_i) is a right triad of H;
- $(X_i, X_{i+1}, \dots, X_r)$ is a left 2-super-sector of type 1 of H for some $r \in \{1, \dots, k\}$;
- $(X_\ell, \dots, X_{i-1}, X_i)$ is a right 2-super-sector of type 1 of H for some $\ell \in \{1, \dots, k\}$.

Observation 4: If $|X_i| \geq 2$ and $|Z_i| = 1$, then the algorithm set Z_i to be of size 1 in Step 3 as a result of:

- $(X_{i-1}, X_i, X_{i+1}, X_{i+2})$ being a 2-sector contained in a left 2-super-sector $(X_\ell, \dots, X_r)$ of type 1; or
- $(X_{i-2}, X_{i-1}, X_i, X_{i+1})$ being a 2-sector contained in a right 2-super-sector of type 1.

We use the above observations repeatedly throughout the remainder of the proof.

(11) For each left triad (X_{i-1}, X_i, X_{i+1}) of H, the following hold:

(a) $|Z_{i-4}| = |Z_{i-3}| = 1$;

(b) $|Z_{i-2}| = |Z_{i-1}| = 2$; and

(c) Z_{i-2} is not complete to Z_{i-1}.

Similarly, for each right triad (X_{i-1}, X_i, X_{i+1}) *of* H*, the following hold:*

(d) $|Z_{i+3}| = |Z_{i+4}| = 1$*;*

(e) $|Z_{i+1}| = |Z_{i+2}| = 2$*; and*

(f) Z_{i+1} *is not complete to* Z_{i+2}*.*

Proof of (11): We prove the three statements about left triads, and the analogous statements about right triads follow from a symmetric argument; so let (X_{i-1}, X_i, X_{i+1}) be a left triad of H. By (2), $|Y_{i-3}| = 1$, and therefore $|Z_{i-3}| = 1$ by Observation 1. Thus, in order to prove (a), it remains to show that $|Z_{i-4}| = 1$. To the contrary, suppose $|Z_{i-4}| \geq 2$. By (2), $|X_{i-4}| = 1$, and therefore it follows from Observation 3 that one of the following holds:

- $(X_{i-4}, X_{i-3}, X_{i-2})$ is a left triad of H;
- $(X_{i-6}, X_{i-5}, X_{i-4})$ is a right triad of H;
- $S = (X_{i-4}, X_{i-3}, \dots, X_r)$ is a left 2-super-sector of type 1 of H for some $r \in \{1, \dots, k\}$;
- $S = (X_\ell, \dots, X_{i-5}, X_{i-4})$ is a right 2-super-sector of type 1 of H for some $\ell \in \{1, \dots, k\}$.

Suppose the first bullet holds. Then the five bags $X_{i-4}, X_{i-3}, X_{i-2}, X_{i-1}, X_i$ are all of size 1, contrary to (1). Thus, the first bullet does not hold. Suppose the second bullet holds. Then, by (2) applied to $(X_{i-6}, X_{i-5}, X_{i-4})$, we get that $|X_{i-3}| \geq 2$, contrary to the already established fact that $|Y_{i-3}| = 1$. So the second bullet does not hold. Suppose the third bullet holds. Then $(X_{i-3}, X_{i-2}, X_{i-1}, X_i)$ is a 2-sector, and in particular $|X_{i-1}| \geq 2$. But this contradicts the fact that (X_{i-1}, X_i, X_{i+1}) is a triad. So the third bullet does not hold. Finally, if the fourth bullet holds, then $|Y_{i-4}| \geq 2$, and hence, by (8), $|X_{i-3}| \geq 2$, contrary to the fact that $|Y_{i-3}| = 1$. Thus, we conclude that $|Z_{i-4}| = 1$, and this completes the proof of (a).

We now prove (b). By Observation 2, it suffices to show that $|Z_{i-2}| \neq 1$ and $|Z_{i-1}| \neq 1$. Suppose $|Z_{i-2}| = 1$. Since, by (2), $|X_{i-2}| \geq 2$, it follows from Observation 4 that $(X_{i-3}, X_{i-2}, X_{i-1}, X_i)$ or $(X_{i-4}, X_{i-3}, X_{i-2}, X_{i-1})$ is a 2-sector in a 2-super-sector of H. If $(X_{i-3}, X_{i-2}, X_{i-1}, X_i)$ is a 2-sector, then $|X_{i-1}| \geq 2$, contrary to (X_{i-1}, X_i, X_{i+1}) being a triad; and if $(X_{i-4}, X_{i-3}, X_{i-2}, X_{i-1})$ is a 2-sector, then $|X_{i-3}| \geq 2$, and hence $|Y_{i-3}| \geq 2$, contrary to (2). Thus $|Z_{i-2}| \neq 1$. Suppose

$|Z_{i-1}| = 1$. Since (X_{i-1}, X_i, X_{i+1}) is a left triad of H, the set Z_{i-1} was set to be of size 2 in Step 2. Since on termination $|Z_{i-1}| = 1$, it follows that Z_{i-1} was modified in Step 3, and hence $(X_{i-2}, X_{i-1}, X_i, X_{i+1})$ or $(X_{i-3}, X_{i-2}, X_{i-1}, X_i)$ is a 2-sector in a 2-super-sector of H. But then, in both cases, $|X_{i-1}| \geq 2$, a contradiction. Therefore $|Z_{i-1}| \neq 1$, and this completes the proof of (b).

By (b), $|Z_{i-2}| = |Z_{i-1}| = 2$. Since $|X_{i-1}| = 1$, one vertex from Z_{i-1}, say y, belongs to $Y_{i-1} \setminus X_{i-1}$. By (2) applied to the triad (X_{i-1}, X_i, X_{i+1}), it follows that y has exactly one neighbour in Y_{i-2}, and hence y is not complete to Z_{i-2}. Therefore Z_{i-1} is not complete to Z_{i-2}, and this completes the proof of (c). This proves (11).

(12) For each left 2-super-sector $(X_\ell, \dots, X_r)$ of type 1 of H, the following hold:

- *(a)* $|Z_{\ell-3}| = |Z_{\ell-2}| = |Z_{r-1}| = |Z_r| = 1$*;*
- *(b)* $|Z_{\ell-1}| = |Z_\ell| = 2$*, and* $Z_{\ell-1}$ *is not complete to* Z_ℓ*; and*
- *(c) for each 2-sector* $(X_s, X_{s+1}, X_{s+2}, X_{s+3})$ *in* $(X_\ell, \dots, X_r)$*, we have that* $|Z_s| = |Z_{s+1}| = |Z_{s+3}| = 1$ *and* $|Z_{s+2}| = 2$.

Similarly, for each right 2-super-sector $(X_\ell, \dots, X_r)$ of type 1 of H, the following hold:

- *(d)* $|Z_\ell| = |Z_{\ell+1}| = |Z_{r+2}| = |Z_{r+3}| = 1$*;*
- *(e)* $|Z_r| = |Z_{r+1}| = 2$*, and* Z_r *is not complete to* Z_{r+1}*; and*
- *(f) for each 2-sector* $(X_s, X_{s+1}, X_{s+2}, X_{s+3})$ *in* $(X_\ell, \dots, X_r)$*, we have that* $|Z_s| = |Z_{s+2}| = |Z_{s+3}| = 1$ *and* $|Z_{s+1}| = 2$.

Proof of (12): We prove (a), (b) and (c) for a left 2-super-sector $(X_\ell, \dots, X_r)$ of type 1 of H, and the analogous statements (d), (e) and (f) for right 2-super-sectors of type 1 follow from a symmetric argument. We first prove (a). By Observation 2, each of $Z_{\ell-3}, Z_{\ell-2}, Z_{r-1}, Z_r$ has size at most 2. By (4), $|Y_{r-1}| = 1$, and by (5), $|Y_{\ell-2}| = 1$, and therefore $|Z_{\ell-2}| = |Z_{r-1}| = 1$ by Observation 1. By (5), we also get that $|X_{\ell-3}| = 1$. Suppose $|Z_{\ell-3}| = 2$. Then, by Observation 3, $(X_{\ell-3}, X_{\ell-2}, X_{\ell-1})$ is a left triad of H, or $(X_{\ell-5}, X_{\ell-4}, X_{\ell-3})$ is a right triad of H, or there exists $r' \in \{1, \dots, k\}$ such that $(X_{\ell-3}, X_{\ell-2}, \dots, X_{r'})$ is a left 2-super-sector of type 1 of H, or there exists $\ell' \in \{1, \dots, k\}$ such that $(X_{\ell'}, \dots, X_{\ell-4}, X_{\ell-3})$ is a right 2-super-sector of type 1 of H. In the first case, $|X_{\ell-1}| = 1$, contrary to the fact that, by (8), $|X_{\ell-1}| \geq 2$. In the second case, by (2) applied to the triad $(X_{\ell-5}, X_{\ell-4}, X_{\ell-3})$, we get that $|X_{\ell-2}| \geq 2$, contrary to the fact that $|Y_{\ell-2}| = 1$. In the third case, $(X_{\ell-2}, X_{\ell-1}, X_\ell, X_{\ell+1})$ is a

2-sector, and in particular, $|X_\ell| \geq 2$, contrary to $(X_\ell, \dots, X_r)$ being a 2-super-sector. In the fourth case, it follows from $(X_{\ell'}, \dots, X_{\ell-4}, X_{\ell-3})$ being a right 2-super-sector of type 1 that $|Y_{\ell-3}| \geq 2$, and hence, by (8), $|Y_{\ell-2}| \geq 2$, contrary to the already established fact that $|Y_{\ell-2}| = 1$. In each of the four cases we obtained a contradiction, and therefore we conclude that $|Z_{\ell-3}| = 1$. Suppose $|Z_r| \geq 2$. Then, by Observation 3, (X_r, X_{r+1}, X_{r+2}) is a left triad of H, or (X_{r-2}, X_{r-1}, X_r) is a right triad of H, or there exists $r' \in \{1, \dots, k\}$ such that $(X_r, X_{r+1}, \dots, X_{r'})$ is a left 2-super-sector of type 1 of H, or there exists $\ell' \in \{1, \dots, k\}$ such that $(X_{\ell'}, \dots, X_{r-1}, X_r)$ is a right 2-super-sector of type 1 of H. In the first case, by (2) applied to (X_r, X_{r+1}, X_{r+2}), we get that $|X_{r-1}| \geq 2$, contrary to the fact that $|X_{r-1}| = 1$. In the second case, $|X_{r-2}| = 1$, contrary to $(X_\ell, \dots, X_r)$ being a 2-super-sector. Suppose the third case holds, i.e., $(X_r, X_{r+1}, \dots, X_{r'})$ is a left 2-super-sector of type 1 of H. Then $|Y_r| \geq 2$, and hence by (8) applied to the 2-super-sector $(X_\ell, \dots, X_r)$, we get that $|X_{r+1}| \geq 2$, contrary to $(X_r, X_{r+1}, \dots, X_{r'})$ being a 2-super-sector. In the fourth case, clearly we must have that $\ell = \ell'$; but then $(X_\ell, \dots, X_r)$ is both left and right, a contradiction. So $|Z_r| = 1$, and this completes the proof of (a).

We now prove (b). By Observation 2, it suffices to show that $|Z_{\ell-1}| \neq 1$ and $|Z_\ell| \neq 1$. Suppose $|Z_{\ell-1}| = 1$. Since $|Y_\ell| \geq 2$, it follows from (8) that $|X_{\ell-1}| \geq 2$, and hence it follows from Observation 4 that $(X_{\ell-2}, X_{\ell-1}, X_\ell, X_{\ell+1})$ or $(X_{\ell-3}, X_{\ell-2}, X_{\ell-1}, X_\ell)$ is a 2-sector contained in a 2-super-sector of H. If $(X_{\ell-2}, X_{\ell-1}, X_\ell, X_{\ell+1})$ is a 2-sector, then $|X_\ell| \geq 2$, a contradiction; and if $(X_{\ell-3}, X_{\ell-2}, X_{\ell-1}, X_\ell)$ is a 2-sector, then $|Y_{\ell-2}| \geq 2$, contrary to (5). So $|Z_{\ell-1}| = 2$. By Step 3 of the algorithm, $|Z_\ell| = 2$. Finally, since $|X_\ell| = 1$, one vertex from Z_ℓ, say y, belongs to $Y_\ell \setminus X_\ell$. By (6) applied to the 2-super-sector $(X_\ell, \dots, X_r)$, y has exactly one neighbour in $Y_{\ell-1}$, and hence y is not complete to $Z_{\ell-1}$. Therefore $Z_{\ell-1}$ is not complete to Z_ℓ, and this completes the proof of (b).

Finally, we prove (c); let $(X_s, X_{s+1}, X_{s+2}, X_{s+3})$ be a 2-sector contained in $(X_\ell, \dots, X_r)$. By (4), $|Y_s| = |Y_{s+3}| = 1$, and hence $|Z_s| = |Z_{s+3}| = 1$ by Observation 1. By Step 3 of the algorithm, $|Z_{s+1}| = 1$, and $|Z_{s+2}| = 2$, and this completes the proof of (c). This proves (12).

Recall that, by (10), each 2-super-sector $(X_\ell, \dots, X_r)$ of type 2 of H contains only one 2-sector, and hence $(X_\ell, \dots, X_r) = (X_\ell, X_{\ell+1}, \dots, X_{\ell+5})$.

(13) For each 2-super-sector $(X_\ell, \dots, X_{\ell+5})$ of type 2 of H, the following hold:

(a) $|Z_\ell| = |Z_{\ell+1}| = |Z_{\ell+4}| = |Z_{\ell+5}| = 1$;

(b) $|Z_{\ell+2}| = |Z_{\ell+3}| = 2$, and $Z_{\ell+2}$ is not complete to $Z_{\ell+3}$.

Proof of (13): Let $(X_\ell, \ldots, X_r)$ be a 2-super-sector of type 2 of H. We first prove (a). By (4), $|Y_{\ell+1}| = |Y_{\ell+4}| = 1$, and therefore $|Z_{\ell+1}| = |Z_{\ell+4}| = 1$ by Observation 1; so it remains to prove that $|Z_\ell| = |Z_{\ell+5}| = 1$. Suppose $|Z_\ell| = 2$. By Observation 3, $(X_\ell, X_{\ell+1}, X_{\ell+2})$ is a left triad of H, or $(X_{\ell-2}, X_{\ell-1}, X_\ell)$ is a right triad of H, or $(X_\ell, X_{\ell+1}, \ldots, X_r)$ is a left 2-super-sector of type 1 of H for some $r \in \{1, \ldots, k\}$, or $(X_{\ell'}, \ldots, X_{\ell-1}, X_\ell)$ is a right 2-super-sector of type 1 of H for some $\ell' \in \{1, \ldots, k\}$. In the first case, $|X_{\ell+2}| = 1$, a contradiction. In the second case, it follows from (2) applied to the triad $(X_{\ell-2}, X_{\ell-1}, X_\ell)$ that $|X_{\ell+1}| \geq 2$, a contradiction. In the third case, clearly we must have that $(X_\ell, X_{\ell+1}, \ldots, X_{r'}) = (X_\ell, \ldots, X_{\ell+5})$, and hence this 2-super-sector is both of type 1 and type 2, a contradiction. In the fourth case, by (8) applied to $(X_{\ell'}, \ldots, X_{\ell-1}, X_\ell)$, we get that $|X_{\ell+1}| \geq 2$, a contradiction. We therefore conclude that $|Z_\ell| = 1$, and by a symmetric argument we get that $|Z_{\ell+5}| = 1$. This completes the proof of (a).

We now prove (b). It follows from Observation 2 and Step 3 that $|Z_{\ell+2}| = |Z_{\ell+3}| = 2$. That $Z_{\ell+2}$ is not complete to $Z_{\ell+3}$ follows from the choice of vertices y and y' in Step 3. Therefore (b) holds, and this completes the proof of (13).

It is clear from construction that Z is a ring of length k with bags $Z_1, \ldots, Z_k$. We now show that Z is a bad ring by checking that $Z = (Z_1, \ldots, Z_k)$ satisfies the following conditions from the definition of a bad ring.

- For every $i \in \{1, \ldots, k\}$, $|Z_i| \leq 2$.
- For every $i \in \{1, \ldots, k\}$, if $|Z_i| = |Z_{i+1}| = 2$, then Z_i is not complete to Z_{i+1} and $|Z_{i-2}| = |Z_{i-1}| = |Z_{i+2}| = |Z_{i+3}| = 1$.
- Z has no triad.
- There exists at least one integer $i \in \{1, \ldots, k\}$ such that $|Z_i| = |Z_{i+1}| = 2$.

By Observation 2, the first bullet holds.

We now prove that the second bullet holds; so assume $|Z_i| = |Z_{i+1}| = 2$ for some $i \in \{1, \ldots, k\}$. For our first step, suppose $|X_i| \geq 2$ and $|X_{i+1}| \geq 2$. Let $s, t \in \{1, \ldots, k\}$ be such that X_s is the only bag of size 1 in the sequence $(X_s, \ldots, X_i)$ and X_t is the only bag of size 1 in the sequence $(X_{i+1}, \ldots, X_t)$; that two such bags of size 1 exist follows from the fact that H has a triad or a 2-super-sector together with the observation that, since $|X_i| \geq 2$ and $|X_{i+1}| \geq 2$, $s \neq i$ and $t \neq i+1$. Then $(X_s, \ldots, X_i, X_{i+1}, \ldots, X_t)$ is a sector of H, and by (3), it is a 2-sector; thus, $s = i-1$ and $t = i+2$. By (7) and (9), $(X_{s-1}, X_s, X_{s+1}, X_{s+2})$ is contained in a 2-super-sector $(X_\ell, \ldots, X_r)$ of H.

Since $|Z_i| = |Z_{i+1}| = 2$, it follows from (12) that $(X_\ell, \dots, X_r)$ is of type 2, and then from (13) that $|Z_{i-2}| = |Z_{i-1}| = |Z_{i+2}| = |Z_{i+3}| = 1$ and Z_i is not complete to Z_{i+1}. Therefore, if $|X_i| \geq 2$ and $|X_{i+1}| \geq 2$, then the second bullet holds. So we may assume that $|X_i| = 1$ or $|X_{i+1}| = 1$.

Suppose $|X_i| = 1$. Then, since $|Z_i| = 2$, it follows from Observation 3 that (X_i, X_{i+1}, X_{i+2}) is a left triad of H, or (X_{i-2}, X_{i-1}, X_i) is a right triad of H, or $(X_i, X_{i+1}, \dots, X_r)$ is a left 2-super-sector of type 1 of H for some $r \in \{1, \dots, k\}$, or $(X_\ell, \dots, X_{i-1}, X_i)$ is a right 2-super-sector of type 1 of H. In the first case, by (1), $|Y_{i+1}| = 1$, and hence $|Z_{i+1}| = 1$, a contradiction. Suppose the second case holds, i.e., suppose that (X_{i-2}, X_{i-1}, X_i) is a right triad of H. It follows from (11) that $|Z_{i+2}| = |Z_{i+3}| = 1$ and Z_i is not complete to Z_{i+1}. By (1), $|Y_{i-1}| = 1$, and hence $|Z_{i-1}| = 1$ by Observation 1. If $|Z_{i-2}| = 1$, then we see from the facts just established that the second bullet holds; so we may assume $|Z_{i-2}| = 2$. Then, by Observation 3, (X_{i-2}, X_{i-1}, X_i) is a left triad of H (in which case the triad (X_{i-2}, X_{i-1}, X_i) is both left and right, a contradiction); or $(X_{i-4}, X_{i-3}, X_{i-2})$ is a right triad of H (in which case $X_{i-4}, X_{i-3}, X_{i-2}, X_{i-1}, X_i$ are five consecutive bags of size 1, contrary to (1)); or $(X_{i-2}, X_{i-1}, \dots, X_{r'})$ is a left 2-super-sector of type 1 of H (in which case Y_i is complete to Y_{i+1}, contrary to the already established fact that Z_i is not complete to Z_{i+1}); or $(X_\ell, \dots, X_{i-3}, X_{i-2})$ is a right 2-super-sector of type 1 of H (in which case, by (8), $|X_{i-1}| \geq 2$, contrary to the assumption that (X_{i-2}, X_{i-1}, X_i) is a triad). So $|Z_{i-2}| = 1$, and we now conclude that in the second case (i.e., when (X_{i-2}, X_{i-1}, X_i) is a right triad), the second bullet holds. So we may assume that the second case does not hold. In the third case, i.e., when $(X_i, X_{i+1}, \dots, X_r)$ is a left 2-super-sector of type 1 of H, we get from (4) that $|Y_{i+1}| = 1$, and hence $|Z_{i+1}| = 1$, a contradiction. In the fourth case, i.e., when $(X_\ell, \dots, X_{i-1}, X_i)$ is a right 2-super-sector of type 1, it follows from (12) that $|Z_{i-2}| = |Z_{i-1}| = |Z_{i+2}| = |Z_{i+3}| = 1$ and Z_i is not complete to Z_{i+1}, and hence the second bullet holds. Thus, we conclude that $|X_i| \neq 1$, and from a symmetric argument we get that $|X_{i+1}| \neq 1$, a contradiction. This completes the proof that the second bullet holds.

To prove the third bullet, suppose Z has a triad (Z_{i-1}, Z_i, Z_{i+1}). If (X_{i-1}, X_i, X_{i+1}) is a triad of H, then by part (b) or (e) of (11), $|Z_{i-1}| = 2$ or $|Z_{i+1}| = 2$, a contradiction. So at least one of X_{i-1}, X_i, X_{i+1} has size at least 2. Suppose $|X_{i-1}| \geq 2$. Then, by Observation 4, exactly one of $(X_{i-2}, X_{i-1}, X_i, X_{i+1})$ and $(X_{i-3}, X_{i-2}, X_{i-1}, X_i)$, call it S, is a 2-sector contained in a 2-super-sector $T = (X_\ell, \dots, X_r)$ of H. It follows from (13) that T is not of type 2, and hence T is of type 1. If $S = (X_{i-2}, X_{i-1}, X_i, X_{i+1})$, then by (12), T is a left 2-super-sector of type 1, in which case, by (12)(c), $|Z_i| = 2$, a contradiction. If $S = (X_{i-3}, X_{i-2}, X_{i-1}, X_i)$, then by (12), T is a right 2-super-sector of type 1, in which case, either: $(X_i, X_{i+1}, X_{i+2}, X_{i+3})$ is a 2-sector contained in T, and hence by (12)(f) we have that $|Z_{i+1}| = 2$, a contradiction; or $(X_i, X_{i+1}) = (X_{r-1}, X_r)$, and hence by (12)(e) we again have that $|Z_{i+1}| = 2$, a contradiction. We conclude that $|X_{i-1}| = 1$, and it follows from a symmetric argument that $|X_{i+1}| = 1$. So $|X_i| \geq 2$. Then, by Observation 4, either $(X_{i-1}, X_i, X_{i+1}, X_{i+2})$ is a 2-sector contained in a left 2-super-sector of type 1 of H (in which case, by (12)(c), $|Z_{i+1}| = 2$, a contradiction), or $(X_{i-2}, X_{i-1}, X_i, X_{i+1})$ is a 2-sector contained in a right 2-super-sector of type 1 of H (in which case, by (12)(f), $|Z_{i-1}| = 2$, a contradiction). This completes the proof that Z has no triad, and therefore the third bullet holds.

Suppose the fourth bullet does not hold. Then Z is a hyperhole, since each bag of Z is of size 1 or 2 (by the first bullet), no two of which are consecutive and of size 2 (by assumption); therefore clearly Z has no 2-super-sector; and by the third bullet, Z has no triad. But then Z is a hyperhole in R with no triad and no 2-super-sector, contrary to our initial assumption that R contains no such induced subgraph. Thus, the fourth bullet holds.

So Z is a bad ring, contrary to the fact that R contains no bad ring. □

Lemma 2.58. *Let $R = (Y_1, \ldots, Y_k)$ be a ring. Suppose R contains a hyperhole $H = (X_1, \ldots, X_k)$ that has exactly one 0-sector and all its other sectors are of length 2. If R contains no bad ring and no base hyperhole, then R is a hyperhole that contains exactly one 0-sector and all its other sectors are of length 2.*

Proof. Say (X_3, X_4) is the 0-sector of H, and suppose towards a contradiction that R contains no bad ring and no base hyperhole but R is not a hyperhole that has exactly one 0-sector and all its other sectors are of length 2. Then at least one of the following holds:

- (Y_3, Y_4) is not a 0-sector of R; or
- there is some 2-sector $(X_s, X_{s+1}, X_{s+2}, X_{s+3})$ of H such that $(Y_s, Y_{s+1}, Y_{s+2}, Y_{s+3})$ is not a 2-sector of R.

Note that the first bullet implies the second, for if $|Y_3| \geq 2$, then (Y_k, Y_1, Y_2, Y_3) is not a 2-sector of R, and if $|Y_4| \geq 2$, then (Y_4, Y_5, Y_6, Y_7) is not a 2-sector of R; so it suffices to consider only the second bullet. Let $(X_s, X_{s+1}, X_{s+2}, X_{s+3})$ be a 2-sector of H such that $(Y_s, Y_{s+1}, Y_{s+2}, Y_{s+3})$ is not a 2-sector of R. So $|Y_s| \geq 2$, or $|Y_{s+3}| \geq 2$, or Y_{s+1} is not complete to Y_{s+3}.

Suppose first that $|Y_s| \geq 2$. Let H' be the subgraph of R induced by $(V(H) \setminus (X_{s-1} \cup X_{s+1})) \cup Y^1_{s-1} \cup Y_s \cup Y^1_{s+1}$. Clearly H' is a hyperhole, and it can easily be checked that H' has no triad, no 2-super-sector, and that it is not the case that H' has exactly one 0-sector and all its other sectors are of length 2. Thus, by the definition of trivial, H' is a nontrivial hyperhole, and therefore it follows from Lemma 2.29 that R contains a base hyperhole, a contradiction. So $|Y_s| = 1$, and by a symmetric argument we get that $|Y_{s+3}| = 1$.

So Y_{s+1} is not complete to Y_{s+2}. Let $y \in Y_{s+1}$ and $y' \in Y_{s+2}$ be nonadjacent vertices of R, and let $Y'_{s+1} = Y^1_{s+1} \cup \{y\}$ and $Y'_{s+2} = Y^1_{s+2} \cup \{y'\}$. For each 2-sector $(X_t, X_{t+1}, X_{t+2}, X_{t+3})$ of H in the subsequence $(X_4, \dots, X_s)$, let $Y'_t = Y^1_t$, $Y'_{t+1} = Y_{t+1}$, $Y'_{t+2} = Y^1_{t+2}$, and $Y'_{t+3} = Y^1_{t+3}$. For each 2-sector of H in the subsequence $(X_{s+3}, \dots, X_k, X_1, X_2, X_3)$, let $Y'_t = Y^1_t$, $Y'_{t+1} = Y^1_{t+1}$, $Y'_{t+2} = Y_{t+2}$, and $Y'_{t+3} = Y^1_{t+3}$. Let $Y'_3 = Y^1_3$ and $Y'_4 = Y^1_4$ (this ensures that Y'_4 is defined in the case $(X_s, X_{s+1}, X_{s+2}, X_{s+3}) = (X_4, X_5, X_6, X_7)$ and that Y'_3 is defined in the case $(X_s, X_{s+1}, X_{s+2}, X_{s+3}) = (X_k, X_1, X_2, X_3)$). The graph induced by $Y'_1 \cup \cdots \cup Y'_k$ is a bad ring, a contradiction. □

Lemma 2.59. *Let $R = (Y_1, \dots, Y_k)$ be a big ring. If R is minimally β-imperfect, then R is a bad ring or R contains a base hyperhole.*

Proof. Suppose that R is minimally β-imperfect, is not a bad ring, and contains no base hyperhole. By minimality and by Lemma 2.50, R contains no bad ring. By Lemma 2.56, R has no triad and no 2-super-sector, and hence by Lemma 2.57, R contains a hyperhole $H = (X_1, \dots, X_k)$ that has no triad and no 2-super-sector. Since R contains no base hyperhole, it follows from Lemma 2.29 that H is trivial, and since H has no triad and no 2-super-sector, it follows from the definition of trivial that H contains exactly one 0-sector and all its other sectors are of length 2. Thus, by Lemma 2.58, R is a hyperhole that has exactly one 0-sector and all its other sectors are of length 2. That is, R is a trivial hyperhole, and hence, by Lemma 2.26, R is β-perfect, a contradiction. □

2.3.4 The characterisation

The following is our main result, a forbidden induced subgraph characterisation for the class of claw-free β-perfect graphs (an example of each of the forbidden induced subgraphs (besides an even hole) is given in Figure 2.9).

Theorem 2.60. *A claw-free graph is β-perfect if and only if it contains no even hole, bad base hyperhole, bad ring, H_5, R_5, H_7 or R_7.*

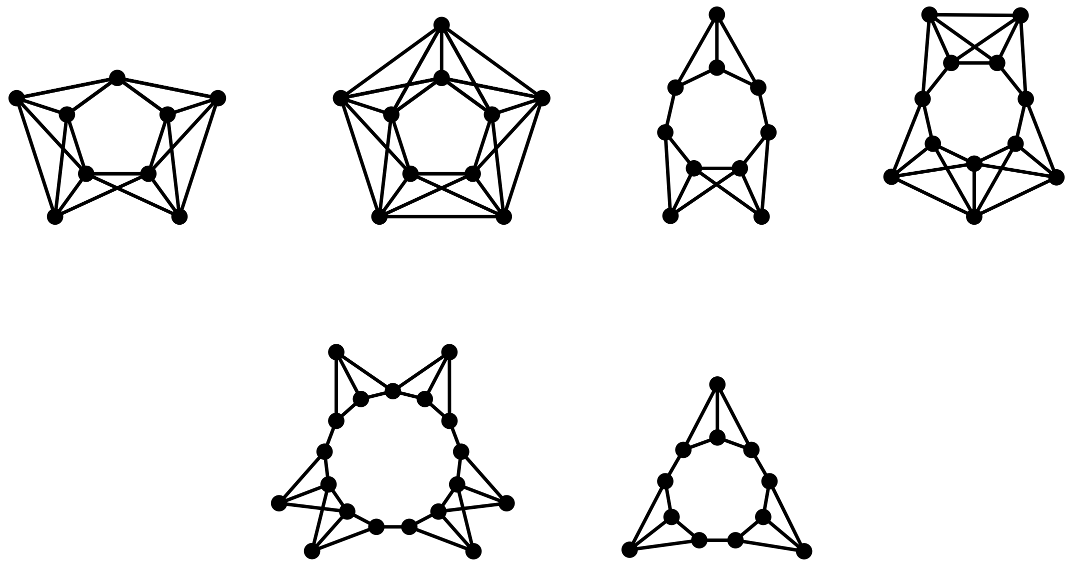


Figure 2.9: From left-to-right, top-to-bottom: the first four figures depict the graphs R_5, H_5, R_7 and H_7 respectively, and the other two depict an example of a bad ring and a bad base hyperhole respectively.

Proof. Let G be a claw-free graph. Suppose G is β-perfect. Then G contains no even hole, and by Lemmas 2.44, 2.45, 2.47, 2.48, 2.28 and 2.50 respectively, G contains no H_5, R_5, H_7, R_7, bad base hyperhole or bad ring.

We now prove the converse; suppose G contains no even hole, bad base hyperhole, bad ring, H_5, R_5, H_7 or R_7, and suppose towards a contradiction that G is not β-perfect. Since every induced subgraph of G is claw-free and contains none of the forbidden induced subgraphs mentioned in the statement of the present theorem, we may assume G is minimally β-imperfect. Then, by Lemma 2.38, G has no clique cutset, and hence by Lemma 2.42, G is a complete graph or an odd ring, or G contains a universal vertex. Clearly complete graphs are β-perfect, and it is easily seen that no minimal β-imperfect graph contains a universal vertex (adding to a β-perfect graph a universal vertex yields another β-perfect graph). Thus, G is an odd ring. By Lemma 2.46, G is not a 5-ring, and by Lemma 2.49, G is not a 7-ring. So G is a big ring. By Lemma 2.59, G contains a base hyperhole, and now it follows from Lemma 2.55 that G is β-perfect, a contradiction. Thus, if G is a claw-free graph that contains no even hole, bad base hyperhole, bad ring, H_5, R_5, H_7 or R_7, then G is β-perfect. $\square$

2.3.5 Recognition algorithm

In this section we present a polynomial-time algorithm that determines whether a given claw-free graph is β-perfect. We first need an algorithm that decides whether a ring is β-perfect.

Theorem 2.61. *There is an algorithm with the following specifications:*

> INPUT: A ring $R = (Y_1, \dots, Y_k)$.
> OUTPUT: Yes if R is β-perfect, and No otherwise.
> RUNNING TIME: $\mathcal{O}(n^4)$, where $n = |V(R)|$.

Proof. Consider the following algorithm.

Step 1. If k is even, then return No.

Step 2. For each $i \in \{1, \dots, k\}$, order Y_i as $Y_i = \{y_i^1, \dots, y_i^{|Y_i|}\}$ so that $N_R[y_i^{|Y_i|}] \subseteq \cdots \subseteq N_R[y_i^1] = Y_{i-1} \cup Y_i \cup Y_{i+1}$.

Step 3. If $k = 5$, then return No if both of the following hold:

- $|Y_i| \geq 2$ for each $i \in \{1, \dots, k\}$;
- y_i^2 is adjacent to y_{i+1}^2 for each $i \in \{1, \dots, 4\}$;

and also return No if, for some $i \in \{1, \dots, k\}$, both of the following hold:

- $|Y_i|, |Y_{i+1}|, |Y_{i+2}|, |Y_{i+3}| \geq 2$;
- there exist nonadjacent vertices $y_{i+1}^j \in Y_{i+1}$ and $y_{i+2}^{j'} \in Y_{i+2}$, and y_i^2 is adjacent to y_{i+1}^j and y_{i+3}^2 is adjacent to $y_{i+2}^{j'}$.

and otherwise, return Yes.

Step 4. If $k = 7$, then return No if, for some $i \in \{1, \dots, k\}$, both of the following hold:

- $|Y_i|, |Y_{i+1}|, |Y_{i+3}|, |Y_{i+4}|, |Y_{i+5}| \geq 2$;
- y_i^2 is adjacent to y_{i+1}^2, y_{i+3}^2 is adjacent to y_{i+4}^2, and y_{i+4}^2 is adjacent to y_{i+5}^2;

and also return No if, for some $i \in \{1, \dots, k\}$, both of the following hold:

- $|Y_i|, |Y_{i+3}|, |Y_{i+4}| \geq 2$;
- $y_{i+3}^{|Y_{i+3}|}$ is nonadjacent to $y_{i+4}^{|Y_{i+4}|}$;

and otherwise, return Yes.

Step 5. (At this point, $k \geq 9$.) If R contains a triad or a 2-super-sector, then return Yes. If R is a hyperhole, then apply the algorithm of Theorem 2.34; and if the output of that algorithm is Yes, then return Yes, and otherwise, return No. Now check whether, up to some cyclic permutation of the indices $1, \dots, k$,[1] the following three conditions hold:

- $|Y_i| = 1$ for each even $i \in \{8, \dots, k\}$;
- $\min(|Y_1|, |Y_4|) = \min(|Y_3|, |Y_6|) = 1$;
- $Y_1 \cup Y_2$, $Y_3 \cup Y_4$ and $Y_5 \cup Y_6$ are cliques.

If so, then return Yes, and otherwise, return No.

We now prove that this algorithm correctly decides whether a ring $R = (Y_1, \dots, Y_k)$ is β-perfect. First, suppose the algorithm returns Yes but R is not β-perfect. By Step 1, k is odd. Suppose $k = 5$. Then, by Lemma 2.46, R contains a ring, say $F = (X_1, \dots, X_k)$ where $X_i \subseteq Y_i$ for each $i \in \{1, \dots, k\}$, that is isomorphic to H_5 or R_5. If $F = H_5$, then clearly the first two bullets in Step 3 hold for any $i \in \{1, \dots, 5\}$, and hence the algorithm returns No, a contradiction. If $F = R_5$, say with X_1 being its unique bag of size 1, then the last two bullets in Step 3 hold for $i = 2$, and hence the algorithm returns No, a contradiction. So $k \neq 5$.

Suppose $k = 7$. Then, by Lemma 2.49, R contains a ring, say $F = (X_1, \dots, X_k)$ where $X_i \subseteq Y_i$ for each $i \in \{1, \dots, k\}$, that is isomorphic to H_7 or R_7. If $F = H_7$, say with X_1 and X_4 as its only two bags of size 1, then the first two bullets in Step 4 hold for $i = 2$, and hence the algorithm returns No, a contradiction. If $F = R_7$, say with X_1 not complete to X_2, then the last two bullets in Step 4 hold for $i = 5$, and hence the algorithm returns No, a contradiction. Therefore $k \neq 7$.

So $k \geq 9$. Since R is not β-perfect, it follows from Lemma 2.56 that R has no triad and no 2-super-sector. If R is a hyperhole, then the algorithm returned Yes as a result of the algorithm of Theorem 2.34 returning Yes, in which case R is β-perfect, a contradiction. Therefore the algorithm does not return Yes as a result of R being a hyperhole or having a triad or 2-super-sector. So, possibly after cyclically permuting indices $1, \dots, k$, the bags $Y_1, \dots, Y_k$ satisfy the three bullets in Step 5.

By Theorem 2.60, R contains a bad base hyperhole or bad ring, say $F = (F_1, \dots, F_k)$. Suppose F is a bad ring. By definition, there exists $i \in \{1, \dots, k\}$ such that $|F_i|, |F_{i+1}| =$

[1] $2, \dots, k, 1$, and $3, \dots, k, 1, 2$, and $k, 1, \dots, k-1$ are examples of cyclic permutations of $1, \dots, k$.

2, and F_i is not complete to F_{i+1}, and $|F_{i-2}| = |F_{i-1}| = |F_{i+2}| = |F_{i+3}| = 1$. Also by definition, F has no triad, and hence it follows that $|F_{i-3}| = 2$ and $|F_{i+4}| = 2$. If $\{i, i+1\} \subseteq \{7, \dots, k\}$, then one of $i, i+1$ is even and the bag of F indexed by it has size at least 2, contrary to the first bullet in Step 5. So $\{i, i+1\} \subseteq \{k, 1, \dots, 7\}$. If $(i, i+1) \in \{(k, 1), (4, 5)\}$, then $|Y_1| \geq 2$ and $|Y_4| \geq 2$, contrary to the second bullet in Step 5; and if $(i, i+1) \in \{(2, 3), (6, 7)\}$, then $|Y_3| \geq 2$ and $|Y_6| \geq 2$, again contradicting the second bullet in Step 5. Thus $(i, i+1) \in \{(1, 2), (3, 4), (5, 6)\}$. But then not all of $Y_1 \cup Y_2$, $Y_3 \cup Y_4$ and $Y_5 \cup Y_6$ are cliques, contrary to the third bullet in Step 5. Therefore F is not a bad ring.

So F is a bad base hyperhole. By definition, F contains (at least) two 0-sectors, say (F_i, F_{i+1}) and (F_j, F_{j+1}). It follows from F having no triad that all of Y_{i-1}, Y_{i+2}, Y_{j-1} and Y_{j+2} have size at least 2. If $(i, i+1) = (6, 7)$ or $(i, i+1) = (k, 1)$, then Y_{k-1} or Y_8 has size at least 2, contrary to the first bullet in Step 5; so $(i, i+1) \notin \{(6, 7), (k, 1)\}$, and by a symmetric argument, $(j, j+1) \notin \{(6, 7), (k, 1)\}$. Suppose $\{i, i+1\} \subseteq \{8, \dots, k-1\}$. (As a consequence of this assumption, and since k is odd, we have that $k \geq 11$.) Then one of $i-1, i+2$ is even, belongs to $\{10, \dots, k-1\}$, and the bag of F (and hence also the bag of R) indexed by it has size at least 2, contrary to the first bullet in Step 5. So $\{i, i+1\} \not\subseteq \{8, \dots, k-1\}$, and by a symmetric argument, $\{j, j+1\} \not\subseteq \{8, \dots, k-1\}$. Suppose $(i, i+1) = (7, 8)$. Then $|F_6| \geq 2$, and hence by the second bullet in Step 5, $|F_3| = 1$, and by the definition of a base hyperhole, $|F_5| = 1$. Since (F_3, F_4, F_5) is not a triad, $|F_4| \geq 2$, and thus by the second bullet in Step 5, $|F_1| = 1$. Similarly, since (F_1, F_2, F_3) is not a triad, $|F_2| \geq 2$. From these facts about the sizes of bags $F_1, \dots, F_6$, together with the fact established earlier that $\{(i, i+1), (j, j+1)\} \cap \{(6, 7), (k, 1)\} = \emptyset$, we deduce that there is no 0-sector in the subsequence $(F_k, F_1, \dots, F_7)$. Since $\{j, j+1\} \not\subseteq \{8, \dots, k-1\}$, it follows that $(j, j+1) = (k-1, k)$. But $|F_1| = 1$, contrary to the fact that $|F_{j+1}| \geq 2$. So $(i, i+1) \neq (7, 8)$, and by symmetry, $(i, i+1) \neq (k-1, k)$. By analogous argument, $(j, j+1) \neq (7, 8)$ and $(j, j+1) \neq (k-1, k)$. Putting all these things together, we see that $\{i, i+1, j, j+1\} \subseteq \{1, \dots, 6\}$, and hence up to symmetry (and since F has no triad and no two consecutive bags of size at least 2), there are two cases:

1. $(i, i+1) = (1, 2)$ and $(j, j+1) = (4, 5)$; or

2. $(i, i+1) = (2, 3)$ and $(j, j+1) = (5, 6)$.

In the first case, $|Y_3|, |Y_6| \geq 2$, and in the second, $|Y_1|, |Y_4| \geq 2$; in each case, the second bullet in Step 5 is contradicted. This completes the proof that if the algorithm returns Yes, then R is β-perfect.

We now prove the converse. Towards a contradiction, suppose R is β-perfect but the algorithm returns No. Since R is β-perfect, k is odd, and hence the algorithm does not return No in Step 1. Suppose the algorithm returns No in Step 3. If the algorithm returns No as a result of the first two bullets in Step 3 being satisfied, then clearly R contains H_5, and hence R is not β-perfect by Lemma 2.44. Similarly, if the algorithm returns No as a result of the last two bullets in Step 3 being satisfied, then it is clear that R contains R_5, and hence R is not β-perfect by Lemma 2.45. So the algorithm does not return No in Step 3. In a similar way (but using Lemmas 2.47 and 2.48 instead of Lemmas 2.44 and 2.44 respectively), we can show that the algorithm does not return No in Step 4. So the algorithm returns No in Step 5. By Step 5, R has no triad and no 2-super-sector, and (by the correctness of the algorithm of Theorem 2.34, which is called in Step 5) is not a hyperhole, and there is no cyclic permutation of $1, \dots, k$ for which the three bullets in Step 5 hold. Since R is β-perfect, it follows from Lemmas 2.28 and 2.50 that R contains no bad ring and no bad base hyperhole. If R contains a base hyperhole, then by Lemma 2.51 the three bullets in Step 5 hold, a contradiction. So R contains no base hyperhole. Now, By Lemma 2.57, R contains a hyperhole H that has no triad and no 2-super-sector. It follows from Lemma 2.29 that H is trivial, and since H has no triad and no 2-super-sector, it follows from the definition of trivial that H has exactly one 0-sector and all its other sectors are of length 2. By Lemma 2.58, R is a hyperhole that contains exactly one 0-sector and all its other sectors are of length 2. But this contradicts the fact that R is not a hyperhole, a contradiction. This completes the proof that if the algorithm returns No, then R is not β-perfect.

Finally, we show that this algorithm runs in time $\mathcal{O}(n^4)$, where $n = |V(R)|$. Step 1 takes $\mathcal{O}(1)$ time. Step 2 can be done in $\mathcal{O}(n^2)$ time, as observed in [41]. Step 3 takes $\mathcal{O}(n^2)$ time, and Step 4 takes $\mathcal{O}(n)$ time. For Step 5: checking whether R contains a triad can be done in $\mathcal{O}(n)$ time; checking whether R contains a 2-super-sector can be done in $\mathcal{O}(n^2)$ time; and checking whether R is a hyperhole can be done in $\mathcal{O}(n+m)$ time [4]. Then we check $\mathcal{O}(n)$ times whether the three bullets in Step 5 are satisfied, which can be done in $\mathcal{O}(n)$ time, $\mathcal{O}(1)$ time, and $\mathcal{O}(n^3)$ time respectively. Therefore Step 5 can be done in $\mathcal{O}(n^4)$ time. It follows that the algorithm has running time $\mathcal{O}(n^4)$. □

Our algorithm for deciding whether a claw-free graph is β-perfect involves a process of clique-cutset decomposition, and therefore we recall the following definition.

A *clique cutset decomposition tree* of a graph G is a tree T satisfying the following:

- the root of T is G;

- each non-leaf node H of T has a clique cutset C such that $V(H) \setminus C$ admits a partition (A, B) where A is anticomplete to B in H, and the children of H in T are the graphs $G[A \cup C]$ and $G[B \cup C]$, one of which has no clique cutset and is a leaf of T;

- the leaves of T are induced subgraphs of G that have no clique cutset.

Observe that such a clique cutset decomposition tree has $\mathcal{O}(|V(G)|)$ many leaves. One can compute in $\mathcal{O}(nm)$ time a clique cutset decomposition tree for an n-vertex m-edge graph [46].

Lemma 2.62. *Let G be a graph and let T be a clique cutset decomposition tree of G with leaves $H_1, \ldots, H_t$. Let F be an induced subgraph of G that has no clique cutset. Then F is an induced subgraph of one of $H_1, \ldots, H_t$.*

Proof. It suffices to show for any node B of T containing F that one of the children $B_1, \ldots, B_b$ of B in T also contains F. Suppose otherwise. Then there exist two nonadjacent vertices x, y of F such that, without loss of generality, $x \in V(B_1) \setminus V(B_2)$ and $y \in V(B_2) \setminus V(B_1)$. Now $V(B_1) \cap V(B_2)$ is a clique cutset of B that separates x and y, and therefore $V(F) \cap V(B_1) \cap V(B_2)$ is a clique cutset of F that separates x and y, contrary to the fact that F has no clique cutset. □

Let W_5^4 be the graph consisting of a hole of length five together with an additional vertex that has exactly four neighbours in this hole.

Lemma 2.63 (Boncompagni, Penev and Vušković [4]). *Let G be a graph and let T be a clique cutset decomposition tree of G with leaves $H_1, \ldots, H_t$. Then the following are equivalent.*

- *G is (3PC, proper wheel)-free.*

- *G is $(K_{2,3}, \overline{C_6}, W_5^4)$-free, and furthermore, for all $H_i \in \{H_1, \ldots, H_t\}$ and for all anticomponents H of H_i, either H is a long ring, or H contains no long holes, or $\alpha(H) \leq 2$.*

Lemma 2.64. *Let G be a claw-free graph and let T be a clique cutset decomposition tree of G with leaves $L_1, \ldots, L_t$. Then the following are equivalent.*

- *G is (C_4, 3PC, proper wheel)-free.*

- *For every $L_i \in \{L_1, \ldots, L_t\}$, either L_i is a chordal graph, or L_i contains a long ring R and every vertex in $V(L_i) \setminus V(R)$ is a universal vertex of L_i.*

Proof. Suppose that G is (C_4, 3PC, proper wheel)-free. By Lemma 2.63, G is $(K_{2,3}, \overline{C_6}, W_5^4)$-free, and furthermore, for all $L_i \in \{L_1, \dots, L_t\}$ and for all anticomponents H of L_i, either H is a long ring, or H contains no long holes, or $\alpha(H) \leq 2$.

Fix $L_i \in \{L_1, \dots, L_t\}$; we show that L_i is as described in the second bullet. If L_i is even-hole-free, then it follows from Lemma 2.42 that L_i is a complete graph, or is an odd (and therefore a long) ring, or contains a ring R and every vertex of $V(L_i) \setminus V(R)$ is a universal vertex of L_i. So we may assume that L_i contains an even hole, and in particular, since G is C_4-free, L_i contains a hole of length at least six; let H be an anticomponent of L_i containing such a hole. It follows that $\alpha(H) \geq 3$, and therefore, by Lemma 2.63, H is a long ring. Suppose F is an anticomponent of L_i different from H. If F contains at least two vertices, then F contains two nonadjacent vertices, which together with any two nonadjacent vertices from H induce a C_4, a contradiction. So every anticomponent of L_i different from H consists of a single vertex, and therefore every vertex of $V(L_i) \setminus V(H)$ is a universal vertex of L_i.

We now prove the converse. Suppose the second bullet holds. Then each of $L_1, \dots, L_t$ is chordal or consists of a long ring possibly together with some universal vertices and therefore contains no C_4, 3PC or proper wheel. If G contains a C_4, a 3PC or a proper wheel, then by Lemma 2.62 so does one of $L_1, \dots, L_t$, a contradiction. Therefore G is (C_4, 3PC, proper wheel)-free. □

Our main algorithmic result is the following.

Theorem 2.65. *There is an algorithm with the following specifications:*

> INPUT: A claw-free graph G.
> OUTPUT: Yes if G is β-perfect, and No otherwise.
> RUNNING TIME: $\mathcal{O}(n^5)$.

Proof. Consider the following algorithm.

Step 1. Compute a clique cutset decomposition tree T of G, and call its leaves $L_1, \dots, L_t$.

Step 2. For each $L_i \in \{L_1, \dots, L_t\}$, check whether L_i is chordal or the graph obtained from L_i by removing all universal vertices is a ring of odd length; if one of these checks fails, output No and terminate.

Step 3. For each graph $L_i \in \{L_1, \dots, L_t\}$ that is not a chordal graph, let L_i' be the graph obtained from L_i by removing all universal vertices (so, by Step 2, L_i' is a long ring); now compute a ring partition $(Y_1, \dots, Y_k)$ of L_i' and apply the algorithm of Theorem 2.61 to $L'_i = (Y_1, \dots, Y_k)$ to test whether L'_i is β-perfect. If for some L'_i the algorithm of Theorem 2.61 returns No, then output No and terminate; and otherwise, output Yes.

We now prove that this algorithm correctly decides whether a given claw-free graph G is β-perfect. Suppose G is β-perfect but the algorithm returns No. Suppose the algorithm returns No in Step 2. Then, by Lemma 2.64, G is not (C_4, 3PC, proper wheel)-free, and therefore it follows from Lemma 2.39 that G contains an even hole, in which case G is not β-perfect, a contradiction. So the algorithm does not return No in Step 2, and therefore the algorithm returns No in Step 3, in which case G contains some induced subgraph that was correctly determined by the algorithm of Theorem 2.61 to be β-imperfect, contrary to the β-perfection of G. Thus, if G is β-perfect, then the algorithm returns Yes.

Suppose G is not β-perfect but the algorithm returns Yes. If G contains an even hole, then by Lemma 2.62, one of $L_1, \dots, L_t$ contains an even hole; but by Step 2, each of $L_1, \dots, L_t$ is a chordal graph or consists of a ring of odd length together with a possibly empty set of universal vertices, and so in either case is even-hole-free. Therefore G is even-hole-free. It now follows from Theorem 2.60 that G contains an induced subgraph F isomorphic to H_5, R_5, H_7 or R_7 or to a bad base hyperhole or a bad ring; by Lemmas 2.44, 2.45, 2.47, 2.48, 2.28 and 2.50 respectively, F is not β-perfect. Clearly F has no clique cutset, and hence by Lemma 2.62, F is contained in one of $L_1, \dots, L_t$, say in L_1 without loss of generality. Since F is not chordal, neither is L_1, and therefore (as a result of Step 2) L_1 consists of a long ring together with a possibly empty set of universal vertices. Furthermore, since no vertex of F is universal in F, no vertex of F is universal in L_1, and hence F is contained in L'_1, where L'_1 is the graph obtained from L_1 by removing all universal vertices. Since F is not β-perfect, neither is L'_1, and therefore the algorithm of Theorem 2.61 returns No when given L'_1 as input. Thus the algorithm presented above returns No in Step 3, a contradiction.

Finally, we show that this algorithm runs in time $\mathcal{O}(n^5)$. Step 1 takes $\mathcal{O}(nm)$ time. Checking whether a graph is chordal can be done in $\mathcal{O}(n+m)$ time, and checking whether a graph is a ring of odd length can be done in $\mathcal{O}(n^2)$ time (see Lemma 8.14 from [4]), and therefore Step 2 takes $\mathcal{O}(n^2)$ time. In Step 3, for $\mathcal{O}(n)$ many graphs we compute a ring partition and run the algorithm of Theorem 2.61; and therefore Step 4 takes $\mathcal{O}(n^5)$ time. Therefore the algorithm has running time $\mathcal{O}(n^5)$. $\square$

Chapter 3

Graphs with all holes the same length

A graph is *ℓ-holed*, for an integer $\ell \geq 4$, if all its holes are of length ℓ. In this chapter we study the class of ℓ-holed graphs. Let us first relate the class of ℓ-holed graphs to other well-studied classes of graphs defined by forbidding as induced subgraphs holes of certain lengths. A graph is *Berge* if it and its complement contain no odd hole. The class of Berge graphs is arguably the most famous class of graphs defined by forbidding certain holes, due to its relation (and more specifically, by the strong perfect graph theorem, its equivalence) to the class of perfect graphs. The strong perfect graph theorem states that a graph is perfect if and only if it is Berge, the proof of which relies on the following decomposition theorem for Berge graphs (we leave terms undefined since we state the following theorem for illustrative purposes only):

Theorem 3.1 (Chudnovsky, Robertson, Seymour and Thomas [12]). *For every Berge graph G, either G is basic, or one of $G, \overline{G}$ admits a proper 2-join, or G admits a proper homogeneous pair, or G admits a balanced skew partition.*

This theorem tells us that Berge graphs are either basic or can be decomposed (by proper 2-joins, or proper 2-joins in the complement, or proper homogeneous pairs, or by balanced skew partitions) into basic graphs. However, this does not tell us how a given Berge graph can be constructed by piecing together simpler Berge graphs.

On the topic of Berge graphs, we note that for even $\ell \geq 6$, ℓ-holed graphs are in fact C_4-free Berge graphs, about which there are a number of known results. Chudnovsky, Lo, Maffray, Trotignon and Vušković [11] gave a purely graph-theoretical algorithm that colours C_4-free Berge graphs in polynomial time (in contrast, the only known polynomial time colouring algorithm for Berge graphs in general relies on the ellipsoid method). The maximum clique problem can also be solved (by a combinatorial algorithm) in polynomial time for C_4-free Berge graphs, since a C_4-free Berge graph on n vertices has $\mathcal{O}(n^2)$ maximal cliques [26], and one can list all K maximal cliques of a graph in $\mathcal{O}(n^3K)$ time [48].

Even-hole-free graphs are in some ways structurally similar to Berge (and therefore perfect) graphs, and many techniques developed in the world of even-hole-free graphs were used in the study of Berge graphs. Like for Berge graphs, we have decomposition theorems for even-hole-free graphs, but there is no known explicit construction that will generate all even-hole-free graphs.

Chordal graphs, on the other hand, are understood fully from a structural point of view, in the sense that we know how to decompose chordal graphs by clique cutsets into simpler graphs (namely, complete graphs), and we know that every chordal graph can be built by piecing together along clique cutsets simpler chordal graphs. The class of graphs G whose set of hole lengths is the empty set is one way to view the class of chordal graphs; and a natural generalisation is to consider the class of graphs G whose set of hole lengths is $\{\ell\}$ for some integer $\ell \geq 4$. We call these graphs ℓ-holed, and in this chapter we consider the question of whether the structure of ℓ-holed graphs can be fully understood in the way that we understand the structure of chordal graphs.

Two groups of researchers worked on this same problem; one group consisting of Linda Cook and Paul Seymour, and the other group consisting of the present author together with Myriam Preissmann, Cléophée Robin, Ni Luh Dewi Sintiari, Nicolas Trotignon and Kristina Vušković. Both groups independently obtained structure theorems for the class of ℓ-holed graphs where $\ell \geq 7$, but came together to submit a joint publication [19].

In Section 3.1 we state the structure theorem of Cook and Seymour, and in Section 3.2 we state the structure theorem of Preissmann, Robin, Sintiari, Trotignon, Vušković and the present author; the joint publication [19] contains the first structure theorem. Since only one of these structure theorems is to be peer reviewed, it is of interest to establish that the two structure theorems are equivalent; in Section 3.3, we prove that they are indeed equivalent.

Each of these structure theorems entail a clique cutset based decomposition theorem for ℓ-holed graphs: every ℓ-holed graph is "basic" or admits a clique cutset. Motivated by the problem of efficiently recognising ℓ-holed graphs, we introduce a 2-join-like cutset called a "special 2-join" in Section 3.4. A special 2-join is an edge cutset, and the removal of the edges of this edge cutset disconnect a graph into two or more components. So-called "blocks of decomposition" are produced from these components by the addition of certain vertices and edges. In Section 3.4, we show that special 2-joins preserve the property of being ℓ-holed, in the sense that for a graph G with a special 2-join and for $\ell \geq 7$, G is ℓ-holed if and only if its blocks of decomposition are ℓ-holed. We prove a clique cutset and special 2-join based decomposition theorem for ℓ-holed graphs (where $\ell \geq 7$ is odd), and together with an algorithm that detects whether a graph has a special 2-join, this leads to a polynomial time algorithm for deciding whether a graph is ℓ-holed for odd $\ell \geq 7$ (Section 3.5.2). In Section 3.5.3, we present a second recognition algorithm for the class of ℓ-holed graphs (where $\ell \geq 7$ is odd) that relies on a process of clique cutset decomposition and an algorithm that decides whether a graph is "basic" (where "basic" means an ℓ-holed graph with no clique cutset, no universal vertex, and that contains a theta, a pyramid or a prism; the structure of such graphs is described in Sections 3.1 and 3.2).

As an application of the structure theorem presented in Section 3.2, we give in Section 3.5.1 polynomial-time algorithms for solving the maximum weight clique and maximum weight stable set problems for ℓ-holed graphs where ℓ is odd and $\ell \geq 7$.

3.1 A structure theorem

In this section we present the structure theorem for ℓ-holed graphs that appears in [19]. This theorem describes exactly the structure of ℓ-holed graphs (where $\ell \geq 7$) that have no clique cutset or universal vertex; since being ℓ-holed is preserved by the addition of a universal vertex and by the operation that reverses decomposing by a clique cutset (this operation is sometimes called the "clique-sum"), this theorem describes how all ℓ-holed graphs ($\ell \geq 7$) may be generated.

Let G be a graph. If X and Y are disjoint subsets of $V(G)$, then we denote by $G[X, Y]$ the bipartite subgraph of G with vertex set $X \cup Y$ and edge set the set of edges of G between X and Y. A *half-graph* is a bipartite graph with no two edges uv and xy such that $\{u, v\}$ is anticomplete to $\{x, y\}$ (in other words, a bipartite graph with no induced two-edge matching). Therefore a graph with bipartition (X, Y) is a half-graph if and only if X can be ordered as $X = \{x_1, \dots, x_m\}$ and Y can be ordered as $Y = \{y_1, \dots, y_n\}$ such that for all i, i', j, j' with $1 \leq i \leq i' \leq m$ and $i \leq j \leq j' \leq n$, if $x_{i'}y_{j'}$ is an edge then x_iy_j is an edge. A half-graph together with such orderings of X and Y is called an *ordered half-graph*. If X, Y, Z are disjoint cliques of a graph G and $G[X, Y]$ and $G[X, Z]$ are half-graphs, then we say that $G[X, Y]$ and $G[X, Z]$ are *compatible* if $G[X, Y \cup Z]$ is a half-graph. When we say that two vertices are adjacent, we may specify that they are "G-adjacent" for a graph G if we mean that they are specifically adjacent in G.

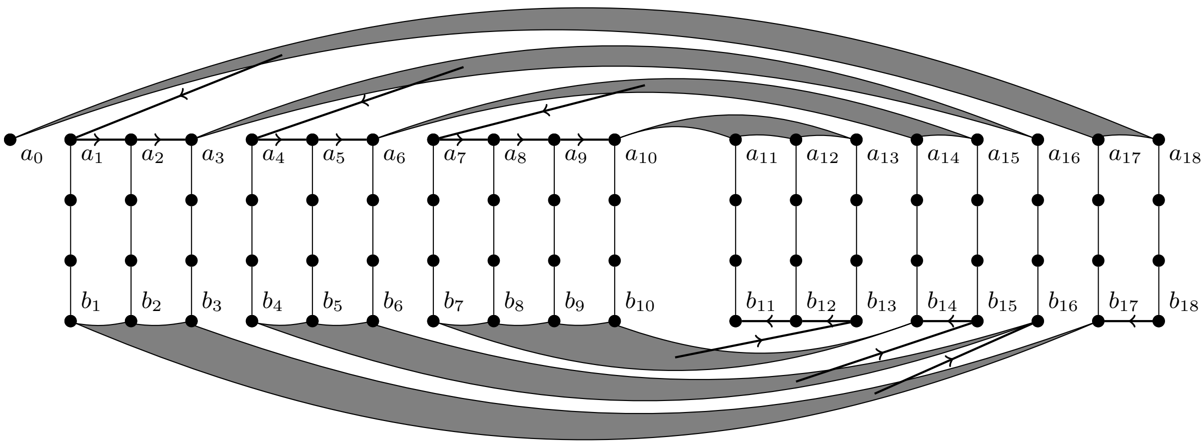


Figure 3.1: An 18-bar 9-framework.

We begin by defining an ℓ-framework for $\ell \geq 7$, which is best described with a figure (see Figure 3.1). Let us begin with the case where ℓ is odd. There are 19 vertices $a_0, \dots, a_{18}$ and 18 vertices $b_1, \dots, b_{18}$ (these could be any two numbers $k+1$ and k). For $1 \leq i \leq k$ there is a vertical path P_i of length $(\ell-3)/2$ between a_i and b_i. (In the case of Figure 3.1, $\ell = 9$.) The numbers $0, \dots, k$ break into two intervals $\{0, \dots, m\}$ and $\{m+1, \dots, k\}$ (in Figure 3.1, $m = 10$).

Let us call the grey shaded areas "tents". The tents are disjoint subsets of the plane, and each of the (four, in Figure 3.1) upper tents contains one vertex in $\{a_0, \dots, a_m\}$ called its "apex", and contains a nonempty interval of $\{a_{m+1}, \dots, a_k\}$ called its "base". Each of $a_{m+1}, \dots, a_k$ belongs to the base of an upper tent. The lower tents do the same with left and right switched. There can be any positive number of tents, but there must be a tent with apex a_0. (There is an odd number of tents in Figure 3.1, but there could be an even number.) Possibly $m = 0$, and if so there are no lower tents. The way the upper and lower tents interleave is important; for each upper tent (except the innermost when there is an odd number of tents), the leftmost vertex of its base is some a_i, and b_i is the apex of one of the lower tents; and for each lower tent (except the innermost when there is an even number of tents), the rightmost vertex of its base corresponds to the apex for one of the upper tents. This gives a sort of spiral running through all the apexes, in Figure 3.1 with vertices

$$a_0, a_{17}, b_{17}, b_3, a_3, a_{16}, b_{16}, b_6, a_6, a_{14}, b_{14}, b_{10}, a_{10}.$$

An *arborescence* is a tree T with its edges directed in such a way that no two edges have a common head; or equivalently, such that for some vertex $r(T)$ (called the *apex*), every edge is directed away from $r(T)$. A *leaf* is a vertex different from the apex, with outdegree zero, and $L(T)$ denotes the set of leaves of T. Each tent is meant to be an arborescence with the given apex and with set of leaves the base of the tent, and with its other vertices not drawn. (We call such an arborescence a *tent-arborescence.*) For each $i \in \{1, \dots, m\}$, if a_{i-1} is the apex of an upper tent-arborescence T_{i-1} say, there is a directed edge from some nonleaf vertex of T_{i-1} (possible from a_{i-1} to a_i; and if a_{i-1} is not the apex of a tent, there is a directed edge from a_{i-1} to a_i. So all these upper tent-arborescences and all the vertices $a_0, \dots, a_m$ are connected up in a sequence to form one big arborescence T with apex a_0, and with set of leaves either $\{a_{m+1}, \dots, a_k\}$ or $\{a_m, \dots, a_k\}$. There is a directed path of T that contains $a_0, a_1, \dots, a_m$ in order, possibly containing other vertices of T between them. Similarly for each $i \in \{m+1, \dots, k-1\}$, if b_{i+1} is an apex of a lower tent-arborescence S_{i+1}, there is a directed edge from some nonleaf vertex of S_{i+1} to b_i, and otherwise there is a directed edge from b_{i+1} to b_i. So similarly the lower tent-arborescences, and the vertices $b_{m+1}, \dots, b_k$ are joined up to make one arborescence S with apex b_k and with set of leaves either $\{b_1, \dots, b_m\}$ or $\{b_1, \dots, b_{m+1}\}$.

Thus the figure describes a graph in which some of the edges are directed: each directed edge belongs to one of two arborescences T, S and each undirected edge belongs to one of the paths P_i. We call such a graph an *ℓ-framework.*

Next we will describe a similar object for when ℓ is even, but we need another concept. Let T be an arborescence. For $v \in V(T)$, let D_v be the set of all vertices $w \in L(T)$ for which there is a directed path of T from v to w. Let S be a tree with $V(S) = L(T)$. We say that T *lives in* S if for each $v \in V(T)$, the set D_v is the vertex set of a subtree of S. Let T, T' be arborescences with $L(T) = L(T')$. We say they are *coarboreal* if there is a tree S with $V(S) = L(T) = L(T')$ such that T, T' both live in S. For instance, the first pair of arborescences in Figure 3.2 (with leaf set the four black vertices) are coarboreal, but the second pair are not. Finally, let T, T' be arborescences and let ϕ be a bijection from $L(T)$ onto $L(T')$. We say that T, T' are *coarboreal under* ϕ if identifying each vertex of $L(T)$ with its image under ϕ gives a coarboreal pair.

The structure we need when ℓ is even is shown in Figure 3.3. We have vertices

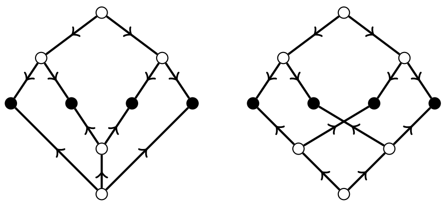


Figure 3.2: The first pair are coarboreal, the second pair are not.

$a_0, \ldots, a_k$ ($k = 18$ in the figure) and $b_1, \ldots, b_k$, but now there is an extra vertex b_0. There are paths P_i between a_i, b_i of length $\ell/2 - 1$ for $1 \leq i \leq m$, and length $\ell/2 - 2$ for $m + 1 \leq i \leq k$. ($\ell = 8$ and $m = 8$ in the figure.) There are upper and lower tents as before, but now all the tents have apex on the left. There must be an upper tent with apex a_0, and one with apex a_m, although $m = 0$ is permitted. The upper tents are paired with the lower tents; for each upper tent with base $\{a_i, \ldots, a_j\}$ there is also a lower tent with base $\{b_i, \ldots, b_j\}$, and vice versa. But the apexes shift by one; if an upper tent has apex a_i, the paired lower tent has apex b_{i+1} (or b_0 when $i = m$). An important condition, not shown in the figure, is:

- for each upper tent-arborescence T_i say, with apex a_i, the paired lower tent-arborescence S_{i+1} with apex b_{i+1} must be coarboreal with T_i under the bijection that maps a_j to b_j for each leaf a_j of T_i.

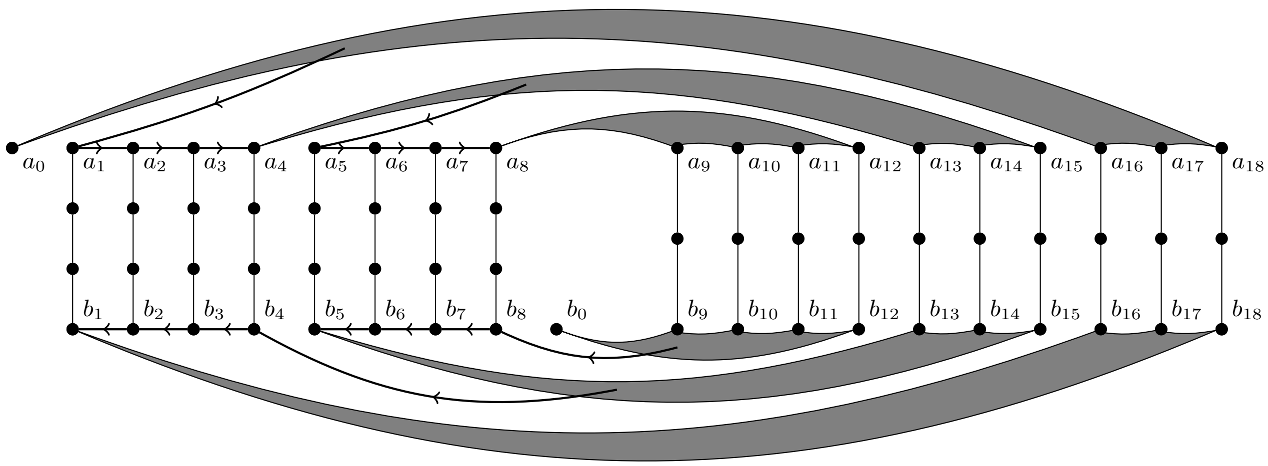


Figure 3.3: An 18-bar 8-framework.

As before, for each $i \in \{1, \ldots, m\}$, if a_{i-1} is the apex of an upper tent-arborescence

T_{i-1} say, there is a directed edge from some nonleaf vertex of T_{i-1} (possibly from a_{i-1}) to a_i; and if a_{i-1} is not the apex of a tent, there is a directed edge from a_{i-1} to a_i. So the upper tent-arborescences are connected up to form an arborescence T with apex a_0, and with set of leaves $\{a_{m+1}, \dots, a_k\}$. Also, for each $i \in \{1, \dots, m-1\}$, if b_{i+1} is the apex of a lower tent-arborescence S_{i+1} say, there is a directed edge from some nonleaf vertex of S_{i+1} (possibly from b_{i+1}) to b_i; and if b_{i+1} is not the apex of a tent, there is a directed edge from b_{i+1} to b_i. Finally, there is a directed edge from some nonleaf vertex of the tent-arborescence S_0 with apex b_0 (possibly from b_0 itself) to b_m. So the lower tent-arborescences are connected up to form an arborescence S with apex b_0, and with set of leaves $\{b_{m+1}, \dots, b_k\}$. We call this graph an *ℓ-framework*.

The *transitive closure* $\overrightarrow{T}$ of an arborescence T is the undirected graph with vertex set $V(T)$ in which vertices u, v are adjacent if and only if some directed path of T contains both u and v. Let F be an ℓ-framework (here, ℓ may be odd or even). Let $P_1, \dots, P_k, T, S$ and so on be as in the definition of an ℓ-framework. Let $D = \overrightarrow{T} \cup \overrightarrow{S} \cup P_1 \cup \dots \cup P_k$. Thus $V(D) = V(F)$, and distinct $u, v \in V(D)$ are D-adjacent if either they are adjacent in some P_i, or there is a directed path of one of S, T between u, v. We say a graph G is a *blow-up* of F if:

- D is an induced subgraph of G, and for each $t \in V(D)$ there is a clique W_t of G, all pairwise disjoint and with union $V(G)$; $W_t \cap V(D) = \{t\}$ for each $t \in V(D)$, and $W_t = \{t\}$ for each $t \in V(D) \setminus V(P_1 \cup \dots \cup P_k)$. (We will often have two graphs F, G, and a clique W_t of G for each $t \in V(F)$, pairwise vertex-disjoint. For an induced subgraph C of F, we denote $\bigcup_{t \in V(C)} W_t$ by $W(C)$. We use this notation in the final bullet of this definition, for instance.)

- For each $t \in V(D)$, there is a linear ordering of W_t with first term t, say $(x_1, \dots, x_n)$ where $x_1 = t$. It has the property that for every $v \in V(G) \setminus W_t$, and every $j \in \{1, \dots, n\}$, if v is G-adjacent to x_j, then v is G-adjacent to $x_1, \dots, x_j$. (It follows that for all distinct $t, t' \in V(D)$, if t, t' are not D-adjacent, then $W_t, W_{t'}$ are anticomplete, and if t, t' are D-adjacent, then $G[W_t, W_{t'}]$ is an ordered half-graph.)

- If $t, t' \in \{a_1, \dots, a_k\}$ or $t, t' \in \{b_1, \dots, b_k\}$, and t, t' are D-adjacent, then W_t is complete to $W_{t'}$.

- For each $t \in V(T)$, if $0 \le i \le m$ and a_i, t are D-adjacent, then W_t is complete to W_{a_i}. For each $t \in V(S)$, if either ℓ is odd and $i \in \{m+1, \dots, k\}$, or ℓ is even and $i \in \{0, \dots, m\}$, and b_i, t are D-adjacent, then W_t is complete to W_{b_i}.

- For each upper tent-arborescence T_j with apex a_j say, let $t \in L(T_j)$ and let the path Q of T from a_0 to t have vertices

$$a_0 = y_1, \ldots, y_p, a_j, z_1, \ldots, z_q = t$$

in order. Then W_t is complete to $\{y_1, \ldots, y_p, a_j\}$; W_t is anticomplete to $W(T \setminus V(Q))$; and $G[W_t, \{z_1, \ldots, z_{q-1}\}]$, with the given order of W_t and the order $z_1, \ldots, z_{q-1}$ of $\{z_1, \ldots, z_{q-1}\}$, is an ordered half-graph. The same holds for lower tent-arborescences with T, a_0 replaced by S, b_0.

Let G be a graph with vertex set partitioned into sets $W_1, \ldots, W_\ell$ with the following properties:

- $W_1, \ldots, W_\ell$ are all nonempty cliques;
- for $1 \leq i \leq \ell$, $G[W_{i-1}, W_i]$ is a half-graph (taking subscripts modulo ℓ);
- for all distinct $i, j \in \{1, \ldots, \ell\}$, if there is an edge between W_i and W_j then $j = i \pm 1$ (modulo ℓ);
- for $1 \leq i \leq \ell$, the graphs $G[W_i, W_{i+1}]$ and $G[W_i, W_{i-1}]$ are compatible.

Such a graph is called a *blow-up of an ℓ-cycle*. It is easily seen that blow-ups of ℓ-cycles (for $\ell \geq 4$) that have no clique cutset and rings are equivalent.

We are now ready to state the structure theorem for ℓ-holed graphs from [19].

Theorem 3.2 (Theorem 1.3 in [19])**.** *Let G be a graph with no clique cutset and no universal vertex, and let $\ell \geq 7$. Then G is ℓ-holed if and only if either G is a blow-up of a cycle of length ℓ, or G is a blow-up of a framework.*

3.2 Another structure theorem

In this section we present the structure theorem for ℓ-holed graphs that appears in [35]. This theorem describes exactly the structure of ℓ-holed graphs (where $\ell \geq 7$) that have no clique cutset or universal vertex; since being ℓ-holed is preserved by the addition of a universal vertex and by the operation that reverses decomposing by a clique cutset (this operation is sometimes called the "clique-sum"), this theorem describes how all ℓ-holed graphs ($\ell \geq 7$) may be generated. In Section 3.3, we show that the structure theorem presented in this section is equivalent to the one presented in the previous section.

To state the structure theorem, we need the definitions of so-called even templates and odd templates. These definitions appear in [35]. But first we must give a number of preliminary definitions.

A graph is a *threshold graph* if it is $(P_4, C_4, \overline{C_4})$-free. A *hypergraph* is a pair (V, E), where V is a finite set of vertices and E is a finite set of *hyperedges*, i.e., nonempty subsets of V. A hypergraph whose hyperedges are all of size 2 is a graph; in this way, hypergraphs are a generalisation of graphs. A hypergraph is *laminar* if for every pair X, Y of hyperedges we have that $X \subseteq Y$, or $Y \subseteq X$ or $X \cap Y = \emptyset$. For a graph G, a *module* of G is a set $X \subseteq V(G)$ such that every vertex in $V(G) \setminus X$ is either complete or anticomplete to X. Note that all subsets of $V(G)$ of cardinality 0, 1 or $|V(G)|$ are modules of G. If u and v are vertices of a graph G, then we write $u \leq_G v$ if $N(u) \setminus \{v\} \subseteq N(v) \setminus \{u\}$ and $u <_G v$ if $N(u) \setminus \{v\} \subsetneq N(v) \setminus \{u\}$. We define $\geq_G$ and $>_G$ accordingly, e.g., $u \geq_G v$ if and only if $v \leq_G u$, and we extend these relations to sets of vertices U and V by saying that $U \leq_G V$ if $u \leq_G v$ for every $u \in U$ and $v \in V$, and so on.

We are now ready to define odd and even templates. For an integer $\ell \geq 2$, an *odd ℓ-template* is any graph G that can be built according to the following process.

- Choose a threshold graph J on vertex set $\{1, \ldots, k\}$ for some $k \geq 3$.
- Choose a laminar hypergraph $\mathcal{H}$ on vertex set $\{1, \ldots, k\}$ such that:
  - every hyperedge X of $\mathcal{H}$ is a module of J of cardinality at least 2, and
  - at least one hyperedge W of $\mathcal{H}$ contains all vertices of $\mathcal{H}$.
- For each $i \in \{1, \ldots, k\}$, G contains two vertices v_i and v'_i that are linked by a path of G of length $\ell - 1$. The k paths built at this step are vertex disjoint and are called the *principal paths* of the odd template.
- The set of vertices of G is $V(G) = A \cup A' \cup B \cup B' \cup I$, where:
  - I is the set of all internal vertices of the principal paths,
  - $A = \{v_1, \ldots, v_k\}$,
  - $A' = \{v'_1, \ldots, v'_k\}$,
  - $B = \{v_X : X$ is a hyperedge of $\mathcal{H}$ such that $J[X]$ is anticonnected$\}$,
  - $B' = \{v'_X : X$ is a hyperedge of $\mathcal{H}$ such that $\overline{J}[X]$ is anticonnected$\}$.
- The set of edges of G is defined as follows.

- for every $v_i, v_j \in A$, $v_iv_j \in E(G)$ if and only if $ij \in E(J)$,
- for every $v'_i, v'_j \in A'$, $v'_iv'_j \in E(G)$ if and only if $ij \notin E(J)$,
- for every $v_X, v_Y \in B$, $v_Xv_Y \in E(G)$ if and only if $X \cap Y \neq \emptyset$,
- for every $v'_X, v'_Y \in B'$, $v'_Xv'_Y \in E(G)$ if and only if $X \cap Y \neq \emptyset$,
- for every $v_i \in A$, $v_X \in B$, $v_iv_X \in E(G)$ if and only if $i \in N_J[X]$,
- for every $v'_i \in A'$, $v'_X \in B'$, $v'_iv'_X \in E(G)$ if and only if $i \in N_{\overline{J}}[X]$,
- for every $v \in I$, v is incident to exactly two edges (those in its principal path).

We use the following notation: for every vertex $x \in B$ such that $x = v_X$ where X is a hyperedge of $\mathcal{H}$, we set $H_x = \{v_i : i \in X\}$. Similarly we define H'_x for $x \in B'$.

Lemma 3.3 (Lemma 4.4 in [35])**.** *Let G be an odd template with sets A, B, A', B', I as in the definition. Then $G[A \cup B]$ contains a universal vertex w and $G[A' \cup B']$ contains a universal vertex w' such that either $w \in A$ and $w' \in B'$, or $w \in B$ and $w' \in A'$.*

With G an odd ℓ-template and sets A, B, A', B', I as in the definition of an odd template, and with w and w' as in Lemma 3.3, we call the 7-tuple (A, B, A', B', I, w, w') an *ℓ-partition* of G.

Now we define even ℓ-templates. For an integer $\ell \geq 4$, an *even ℓ-template partition* of a graph G is a partition of the vertex-set of G into five sets A, B, A', B', I with the following properties.

- $A = A_K \cup A_S$ where $A_K = \{v_1, \dots, v_k\}$, $A_S = \{v_{k+1}, \dots, v_{k+s}\}$ and $k + s \geq 3$.
- $A' = A'_K \cup A'_S$ where $A'_K = \{v'_1, \dots, v'_k\}$ and $A'_S = \{v'_{k+1}, \dots, v'_{k+s}\}$.
- For each $i \in \{1, \dots, k\}$, v_i and v'_i are linked by a path of G of length $\ell - 1$ and for each $i \in \{1, \dots, s\}$, v_{k+i} and v'_{k+i} are linked by a path of G of length $\ell - 2$. These $k + s$ paths are vertex disjoint and they are called the *principal paths* of the partition.
- I is the set of all internal vertices of the principal paths.
- Both A_K and A'_K are cliques of G and both A_S and A'_S are stable sets of G. For $i \in \{1, \dots, k\}$ and $j \in \{1, \dots, s\}$, exactly one of v_iv_{k+j} and $v'_iv'_{k+j}$ is an edge. Furthermore, $G[A]$ and $G[A']$ are threshold graphs.
- There exists a laminar hypergraph $\mathcal{H}$ with vertex set $\{v_1, \dots, v_{k+s}\}$ such that:

  - every hyperedge X of $\mathcal{H}$ is an anticonnected module of $G[A]$ of cardinality at least 2, and
  - if $G[A]$ is not connected, then at least one hyperedge of $\mathcal{H}$ contains all vertices of A.

- There exists a laminar hypergraph $\mathcal{H}'$ with vertex set $\{v'_1, \dots, v'_{k+s}\}$ such that:
  - every hyperedge X' of $\mathcal{H}'$ is a module of $G[A']$ of cardinality at least 2, and
  - if $G[A']$ is not connected, then at least one hyperedge of $\mathcal{H}'$ contains all vertices of A'.

- $B = \{v_X : X \text{ is a hyperedge of } \mathcal{H}\}$ and $B' = \{v'_X : X \text{ is a hyperedge of } \mathcal{H}'\}$.

- The set of edges of G incident to vertices in $B \cup B'$ is defined as follows.
  - for every $v_X, v_Y \in B$, $v_X v_Y \in E(G)$ if and only if $X \cap Y \neq \emptyset$,
  - for every $v'_X, v'_Y \in B'$, $v'_X v'_Y \in E(G)$ if and only if $X \cap Y \neq \emptyset$,
  - for every $v_i \in A$, $v_X \in B$, $v_i v_X \in E(G)$ if and only if $v_i \in N_{G[A]}[X]$,
  - for every $v'_i \in A'$, $v'_X \in B'$, $v'_i v'_X \in E(G)$ if and only if $v'_i \in N_{G[A']}[X]$.

- There are no other edges of G than those mentioned above.

We use the following notation: for every vertex $x \in B$ such that $x = v_X$ where X is a hyperedge of $\mathcal{H}$, we set $H_x = X$. Similarly we define H'_x for $x \in B'$.

We extend $\mathcal{H}$ into a hypergraph $\mathcal{H}_A$ with vertex set A by adding to its hyperedge set the hyperedge $H_v = N_A[v] \cap \{u \in A : u \leq_{G[A]} v\}$ for every vertex $v \in A$. Similarly we extend $\mathcal{H}'$ into a hypergraph $\mathcal{H}'_{A'}$.

So far we have defined an even ℓ-template partition, but to define even templates we need the notion of a strong even ℓ-template partition.

Given an even ℓ-template partition of G, we define a hypergraph $\mathcal{H}_G$ whose vertex set is $\{k+1, \dots, k+s\}$ and whose hyperedges are sets of indices of the vertices of $A_S \cup A_{S'}$ in hyperedges of $\mathcal{H}_A \cup \mathcal{H}_{A'}$. More formally, $E(\mathcal{H}_G) = E_A \cup E_{A'}$, where:

- $E_A = \{\{i : v_i \in H \cap A_S\} : H \text{ is a hyperedge of } \mathcal{H}_A \text{ and } H \cap A_S \neq \emptyset\}$, and

- $E_{A'} = \{\{i : v_i \in H \cap A'_S\} : H \text{ is a hyperedge of } \mathcal{H}_{A'} \text{ and } H \cap A'_S \neq \emptyset\}$, and

A circular sequence $\mathcal{C} = (j_1, e_1, \dots, j_t, e_t, j_1)$, where each j_i is a distinct vertex of $\mathcal{H}_G$ and each e_i is a distinct hyperedge of $\mathcal{H}_G$, is said to be a *hyper cycle* of length t of $\mathcal{H}_G$ if:

- each j_i belongs to e_{i-1} and e_i (where $e_{t+1} = e_1$) and to no other hyperedge of $\mathcal{C}$, and
- any two distinct hyperedges of $\mathcal{C}$ that belong both to E_A or both to $E_{A'}$ are disjoint.

For an integer $\ell \geq 3$ and a graph G, a *strong even ℓ-template partition* of G is an even ℓ-template partition (A, A', B, B', I) of G, such that $\mathcal{H}_G$ contains no hyper cycle of length greater than 2. A graph G which has a strong even ℓ-template partition is called an *even ℓ-template*.

Lemma 3.4 (Lemma 8.1 in [35])**.** *Let G be an even template with sets A, B, A', B', I as in the definition. Then $G[A \cup B]$ contains a universal vertex w and $G[A' \cup B']$ contains a universal vertex w'.*

Now we define blowups of templates. Two vertices u, v in a graph G are *twins* if $N_G[u] = N_G[v]$, and G is twinless if no two distinct vertices of G are twins.

Let G be a twinless odd ℓ-template with an ℓ-partition (A, B, A', B', I, w, w'). An edge of G is *flat* if at least one of its ends belongs to I. An edge of G is *optional* if one end is a vertex $x \in B$ and the other end is a vertex $u \in H_x$ that is an isolated vertex of $G[H_x]$, or if one end is a vertex $x \in B'$ and the other end is a vertex $u \in H'_x$ that is an isolated vertex of $G[H'_x]$. An edge of G that is neither flat nor optional is *solid*. A *blowup* of G is any graph G^* that satisfies the following:

- For every vertex u of G there is a clique K_u in G^* on $k_u \geq 1$ vertices $u_1, \ldots, u_{k_u}$ such that: $u_{k_u} = u$; for distinct vertices u, v of G, $K_u \cap K_v = \emptyset$; and $V(G^*) = \bigcup_{u \in V(G)} K_u$.
- For all vertices $u \in V(G)$ and all integers $1 \leq i \leq j \leq k_u$, $N_{G^*}[u_i] \subseteq N_{G^*}[u_j]$.
- If u and v are nonadjacent vertices of G, then K_u is anticomplete to K_v.
- If uv is a solid edge of G, then K_u is complete to K_v.
- If ux is an optional edge of G with $u \in A$ and $x \in B$ (resp. $u \in A'$ and $x \in B'$), then u is complete to K_x.
- If ux and uy are optional edges of G with $u \in A$, $x, y \in B$ and $H_y \subsetneq H_x$ (resp. $u \in A'$, $x, y \in B'$ and $H'_y \subsetneq H'_x$), then every vertex of K_u with a neighbour in K_y is complete to K_x.

- w (resp. w') is a universal vertex of $G^*[\bigcup_{u\in A\cup B} K_u]$ (resp. $G^*[\bigcup_{u\in A'\cup B'} K_u]$).

An odd template is *proper* if one of $G[A], G[A']$ has at least two isolated vertices, and an even template is *proper* if all universal vertices of $G[A\cup B]$ belong to B and all universal vertices of $G[A'\cup B']$ belong to B'. The blowup G^* of G is *proper* if G is proper. We use the same definition as above to define proper blowups of twinless even templates.

We are now ready to state the structure theorems of [35], first for ℓ-holed graphs where ℓ is odd.

Theorem 3.5 (Theorem 7.1 in [35])**.** *Let $\ell \geq 7$ be an odd integer. If G is an ℓ-holed graph, then one of the following holds:*

- *G is a ring of length ℓ;*
- *G is a proper blowup of a twinless odd $(\ell-1)/2$-template;*
- *G has a universal vertex; or*
- *G has a clique cutset.*

The structural result of [35] for ℓ-holed graphs where ℓ is even is as follows.

Theorem 3.6 (Theorem 10.1 in [35])**.** *Let $\ell \geq 7$ be an even integer. If G is an ℓ-holed graph, then one of the following holds:*

- *G is a ring of length ℓ;*
- *G is a proper blowup of a twinless even $\ell/2$-template;*
- *G has a universal vertex; or*
- *G has a clique cutset.*

3.3 The equivalence of two structure theorems

In Sections 3.1 and 3.2 we presented two independently obtained structure theorems for ℓ-holed graphs (for $\ell \geq 7$) [19, 35]. In this section we prove that they are equivalent, in the following sense.

Theorem 3.7. *Let G be a graph with no clique cutset or universal vertex that contains a theta, a pyramid or a prism. Then the following are equivalent:*

- *G is ℓ-holed for some $\ell \geq 7$.*
- *G is a blow-up of an ℓ-framework.*
- *G is a proper blowup of a twinless odd $(\ell-1)/2$-template if ℓ is odd, and G is a proper blowup of a twinless even $\ell/2$-template if ℓ is even.*

This result is concerned with graphs that contain a theta, a pyramid or a prism. In the case that an ℓ-holed graph G (with no clique cutset and no universal vertex) contains neither a theta, nor a pyramid nor a prism, the result of [19] states that G is a "blow-up of an ℓ-cycle", while the result of [35] states that G is a "ring of length ℓ". It is evident from the definitions of blow-ups of cycles and rings (the definition of the former may be found in [19] and the definition of the latter may be found in [35] or Section 2.2) that they are equivalent (under the assumption that they have no clique cutset), so we do not prove that here.

3.3.1 A blow-up of a framework is a proper blowup of a twinless template

In this section we prove the following, namely that the second bullet of Theorem 3.7 implies the third.

Lemma 3.8. *Let $\ell \geq 7$ and let G be a blow-up of an ℓ-framework. Assume that G has no clique cutset. If ℓ is odd, then G is a proper blowup of a twinless odd $(\ell-1)/2$-template, and if ℓ is even, then G is a proper blowup of a twinless even $\ell/2$-template.*

We prove Lemma 3.8 in three steps:

1. First we prove that a blow-up of an ℓ-framework is a "bordered blow-up of an ℓ-frame" (Lemma 3.11);
2. then we prove that a bordered blow-up of an ℓ-frame (satisfying certain assumptions) is a "proper preblowup of a template" (Lemma 3.21);
3. and finally we apply results from [35] that say that a proper preblowup of a template is a proper blowup of a twinless template to establish Lemma 3.8.

Bordered blow-ups of frames

We now prove that a blow-up of an ℓ-framework is a so-called bordered blow-up of an ℓ-frame. First we need relevant definitions from [19].

We begin by defining an "ℓ-frame". Let $\ell \geq 5$ be odd. Let $k \geq 3$ be an integer, and take distinct vertices $a_1, \ldots, a_k, b_1, \ldots, b_k$. For $1 \leq i \leq k$ let P_i be a path of length $(\ell - 3)/2$ with ends a_i, b_i, pairwise vertex-disjoint. Let the subgraphs A and B induced on $\{a_1, \ldots, a_k\}$ and on $\{b_1, \ldots, b_k\}$ respectively be threshold graphs, and for $1 \leq i < j \leq k$, let b_i, b_j be adjacent if and only if a_i, a_j are nonadjacent. Moreover, let A either be disconnected or 2-connected (a graph is *k-connected* if it has more than k vertices and remains connected after the removal of fewer than k vertices), and the same for B. (See Figure 3.4.) For ℓ odd, a graph F constructible in this way is called an *ℓ-frame*; all its holes have length ℓ. We call $P_1, \ldots, P_k$ the *bars* of the frame, and A, B are its *sides*.

We remark that, since the subgraphs induced on $\{a_1, \ldots, a_k\}$ and $\{b_1, \ldots, b_k\}$ are complementary threshold graphs, one of them, such as a_6 in Figure 3.4, has a vertex of degree 0 (since every threshold graph has a universal vertex or an isolated vertex). Thus all ℓ-frames when ℓ is odd have one-vertex clique cutsets.

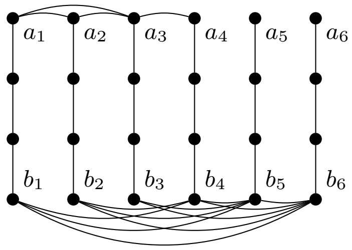


Figure 3.4: A 9-frame.

Now the case when ℓ is even. Let $m, n \geq 0$ be integers with $n \geq 2$ and $m + n \geq 3$; and let vertices

$$a_1, \ldots, a_n, c_1, \ldots, c_m$$

$$b_1, \ldots, b_n, d_1, \ldots, d_m$$

all be distinct. For $1 \leq i \leq n$ let P_i be a path with ends a_i, b_i of length $\ell/2 - 2$, and for $1 \leq i \leq m$ let Q_i be a path between c_i, d_i of length $\ell/2 - 1$, all pairwise vertex-disjoint. Let $\{c_1, \ldots, c_m\}$ and $\{d_1, \ldots, d_m\}$ be cliques; and let the bipartite subgraph with bipartition $(\{a_1, \ldots, a_n\}, \{c_1, \ldots, c_m\})$ be a half-graph. For $1 \leq i \leq n$ and $1 \leq j \leq m$, let b_i, d_j be adjacent if and only if a_i, c_j are nonadjacent. Let one of $a_1, \ldots, a_n$ and one of $b_1, \ldots, b_n$ have degree one in F. There are no other edges (thus $\{a_1, \ldots, a_n\}$ is a stable set, and so is $\{b_1, \ldots, b_n\}$). Let us call such a graph F an *ℓ-frame*. (See Figure 3.5.) Every hole in an ℓ-frame has length ℓ. Let A, B

be the subgraphs induced on $\{a_1, \ldots, a_n, c_1, \ldots, c_m\}$ and on $\{b_1, \ldots, b_n, d_1, \ldots, d_m\}$ respectively. (It follows that A, B are both disconnected threshold graphs.) We call $P_1, \ldots, P_n, Q_1, \ldots, Q_m$ the *bars* of the frame, and A, B its *sides*.

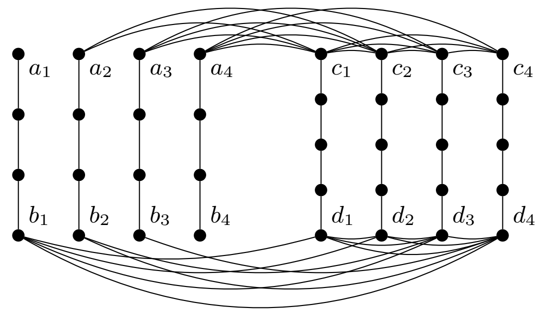


Figure 3.5: A 10-frame.

We now give the definition of a border. Let J be a graph, and let A be a threshold graph contained in J with $|A| \geq 3$. For each $t \in V(A)$ let W_t be a nonempty ordered clique of J, called a *bag*, all pairwise disjoint, such that if st is an edge of A then W_s is complete to W_t in J and otherwise they are anticomplete. Suppose that

- J is a connected chordal graph;
- each vertex in $V(J) \setminus W(A)$ has two nonadjacent neighbours in $V(A)$;
- for every induced path P of J with length at least three and with both ends in $W(A)$, some internal vertex of P belongs to the same bag as one of the ends of P;
- for each $t \in V(A)$ and each $v \in V(J) \setminus W_t$, let W_t be ordered as $\{x_1, \ldots, x_n\}$; then $t = x_1$, and for $1 \leq i < j \leq n$, if v, x_j are adjacent then v, x_i are adjacent (briefly, each vertex of $J \setminus W_t$ is adjacent to an *initial segment* of W_t).

In this case we call $(J, A, (W_t : t \in V(A)))$ a *border*.

With ℓ odd or even, let F be an ℓ-frame, in the usual notation. For each $t \in V(F)$, let W_t be an ordered clique where t is the first term of the ordering of W_t, all pairwise disjoint, and we will define a graph H with vertex set the union of these cliques. For every edge uv of $A \cup B$ we make W_u complete to W_v in H. For every other edge uv of F let $H[W_u, W_v]$ be an ordered half-graph. We call such a graph H a *blow-up of an ℓ-frame*.

Now take a graph H that is a blow-up of an ℓ-frame. Choose J, K such that $(J, A, (W_t : t \in V(A)))$ is a border, with $V(J) \cap V(H) = W(A)$, and $(K, B, (W_t : t \in V(B)))$ is a border, with $V(K) \cap V(H) = W(B)$, and $V(K) \cap V(J) = \emptyset$ and $V(J)$ is anticomplete to $V(K)$. We call the graph $H \cup J \cup K$ a *bordered blow-up of an ℓ-frame*, and that G is the *composition* of H, J, K.

We use the following result from [19]:

Lemma 3.9 (Lemma 8.1 in [19])**.** *Let $\ell \geq 7$, and let G be ℓ-holed, with no clique cutset or universal vertex. Then either G is a blow-up of an ℓ-cycle or a bordered blow-up of an ℓ-frame.*

We use the above in conjunction with:

Lemma 3.10 (Lemma 2.2 in [19])**.** *For $\ell \geq 5$, if G is a blow-up of an ℓ-cycle or G is a blow-up of an ℓ-framework, then G is ℓ-holed.*

We can now prove the following.

Lemma 3.11. *Let $\ell \geq 7$ be an integer. If G is a blow-up of an ℓ-framework with no clique cutset and no universal vertex, then G is a bordered blow-up of an ℓ-frame.*

Proof. Let G be a blow-up of an ℓ-framework with no clique cutset and no universal vertex. By Lemma 3.10, G is ℓ-holed. It then follows from Lemma 3.9 that G is a blow-up of an ℓ-cycle or G is a bordered blow-up of an ℓ-frame. Clearly G is not a blow-up of an ℓ-cycle, and therefore G is a bordered blow-up of an ℓ-frame. $\square$

Proper preblowups of templates

In this section we prove that a bordered blow-up of an ℓ-frame that has no clique cutset is a proper "preblowup" of a twinless template, and so we begin by defining proper preblowups of twinless templates (their definitions are from [35]).

A *preblowup* of an odd ℓ-template G with ℓ-partition (A, B, A', B', I, w, w') is any graph G^* obtained from G as follows. Every vertex u of $A \cup A' \cup I$ is replaced by a clique K_u on $k_u \geq 1$ vertices such that $u \in K_u$. We denote by A^* the set $\bigcup_{u \in A} K_u$ and use a similar notation for A'^* and I^*. The set B (resp. B') is replaced by a set B^* (resp. B'^*) of vertices such that $B \subseteq B^*$ (resp. $B' \subseteq B'^*$). So $V(G^*) = A^* \cup B^* \cup A'^* \cup B'^* \cup I^*$. The sets $A^*, B^*, A'^*, B'^*, I^*$ are disjoint. Vertices of G are adjacent in G^* if and only if they are adjacent in G. Finally, the following conditions must hold:

(a) For all $u \in A$, $N_{G^*}(K_u) \subseteq A^* \cup B^* \cup K_{u^+}$, where u^+ is the neighbour of u in I and:

1. For every $u^* \in K_u$, $N_A(u^*) = N_A[u]$.
2. Every vertex of K_u has a neighbour in K_{u^+}.

(b) $N(B^*) \subseteq A^*$, and:

1. If $w \in B$, then there exists $w^* \in B^*$ that is complete to A^*.
2. If $u^* \in B^*$, then there exist nonadjacent $a, b \in A$ such that u^* has neighbours in both K_a and K_b.

(i) For all $u \in I$, $N_{G^*}(K_u) \subseteq K_a \cup K_b$, where a, b are the neighbours of u in G, and:

1. Every vertex $u^* \in K_u$ has at least one neighbour in each of K_a and K_b.

Conditions (a') and (b') analogous to (a) and (b) hold for A' and B'.

If the underlying ℓ-partition of the template G is proper, then the preblowup G^* of G is also *proper*. The same definition as above is used to define (proper) preblowups of twinless even templates.

In order to prove that a graph G is a proper preblowup of a template, we must first identify some induced subgraph of G and prove that it is a template. Checking that all conditions from the definition of a template hold is tedious, and for that reason, in [35] the notion of a "pretemplate" is introduced. If a graph is a pretemplate and is ℓ-holed, then the graph is also a template (we state this more formally later on). So we now define pretemplates.

For an integer $\ell \geq 3$, an *odd ℓ-pretemplate* is a graph G whose vertex-set can be partitioned into five sets A, B, A', B', I with the following properties.

(a) $N(B) \subseteq A$ and $N(A \cup B) \subseteq I$.

(b) $N(B') \subseteq A'$ and $N(A' \cup B') \subseteq I$.

(c) $|A| = |A'| = k \geq 3$, $A = \{v_1, \dots, v_k\}$ and $A' = \{v'_1, \dots, v'_k\}$.

(d) For every $i \in \{1, \dots, k\}$, there exists a unique path P_i from v_i to v'_i whose interior is in I.

(e) Every vertex in I has degree 2 and lies on a path from v_i to v'_i for some $i \in \{1, \dots, k\}$.

(f) All paths $P_1, \dots, P_k$ have length $\ell - 1$.

(g) $G[A \cup B]$ and $G[A' \cup B']$ are both connected graphs.

(h) Every vertex of B is in the interior of a path of $G[A \cup B]$ with both ends in A.

(i) Every vertex of B' is in the interior of a path of $G[A' \cup B']$ with both ends in A'.

Under these circumstances, we say that (A, B, A', B', I) is an *ℓ-pretemplate partition* of G.

For every integer $\ell \geq 4$, an *even ℓ-pretemplate partition* of a graph G is a partition of the vertex-set of G into five sets $A = A_K \cup A_S$, B, $A' = A'_K \cup A'_S$, B' and I with the following properties.

(a) $N(B) \subseteq A$ and $N(A \cup B) \subseteq I$.

(b) $N(B') \subseteq A'$ and $N(A' \cup B') \subseteq I$.

(c) $|A_K| = |A'_K| = k$, $A_K = \{v_1, \dots, v_k\}$ and $A'_K = \{v'_1, \dots, v'_k\}$.

(d) $|A_S| = |A'_S| = s$, $A_S = \{v_{k+1}, \dots, v_{k+s}\}$ and $A'_S = \{v'_{k+1}, \dots, v'_{k+s}\}$ are stable sets of G where $k + s \geq 3$.

(e) For every $i \in \{1, \dots, k+s\}$, there exists a unique path P_i from v_i to v'_i whose interior is in I.

(f) Every vertex in I has degree 2 and lies on a path from v_i to v'_i for some $i \in \{1, \dots, k\}$.

(g) All paths $P_1, \dots, P_k$ have length $\ell - 1$ and all paths $P_{k+1}, \dots, P_{k+s}$ have length $\ell - 2$.

(h) $G[A \cup B]$ and $G[A' \cup B']$ are both connected graphs.

(i) Every vertex of B is in the interior of a path of $G[A \cup B]$ with both ends in A.

(j) Every vertex of B' is in the interior of a path of $G[A' \cup B']$ with both ends in A'.

We then say that (A, A', B, B', I) is an *even ℓ-pretemplate partition* of G.

The usefulness of pretemplates comes from the following two results.

Lemma 3.12 (Lemma 4.14 in [35]). *Let $\ell \geq 7$ be an odd integer. If G is an ℓ-holed odd $(\ell-1)/2$-pretemplate, then G is an odd $(\ell-1)/2$-template. Moreover, for every odd $(\ell-1)/2$-pretemplate partition (A, B, A', B', I) of G, there exist w and w' in $V(G)$ such that (A, B, A', B', I, w, w') is an $(\ell-1)/2$-partition of G.*

Lemma 3.13 (Lemma 8.8 in [35])**.** *Let $\ell \geq 7$ be an even integer. If G is an ℓ-holed even $\ell/2$-pretemplate, then G is an even $\ell/2$-template. Moreover, every even $\ell/2$-pretemplate partition (A, B, A', B', I) of G is a strong even template partition of G.*

Recall that our goal for the moment is to prove that bordered blow-ups of ℓ-frames are proper blowups of twinless templates. To that end, we now prove several lemmas about bordered blow-ups of ℓ-frames.

Lemma 3.14. *Let $\ell \geq 7$ be odd and let F be an ℓ-frame with sides A, B and bars $P_1, \dots, P_k$. Then one of A, B has at least two isolated vertices.*

Proof. Since A, B are complementary threshold graphs, one of them, say A without loss of generality, has a universal vertex, say a_1. It follows from the definition of an ℓ-frame that A is 2-connected. If A has another universal vertex, then B contains at least two isolated vertices and we are done. So we may assume that no other vertex of A is universal, and therefore $A \setminus \{a_1\}$ contains no universal vertex. Thus $A \setminus \{a_1\}$ contains an isolated vertex, and hence $A \setminus \{a_1\}$ is disconnected, contradicting the fact that A is 2-connected. $\square$

Lemma 3.15. *Let $\ell \geq 7$ and let G be an ℓ-holed graph with no clique cutset or universal vertex. Suppose that G is a bordered blow-up of an ℓ-frame, and G is the composition of H, J, K where:*

- *H is a blow-up of the ℓ-frame F, where F has sides A, B and bars $P_1, \dots, P_k$;*
- *$(J, A, (W_t : t \in V(A)))$ and $(K, B, (W_t : t \in V(B)))$ are borders; and*
- *$V(H \cap J) = W(A)$, and $V(H \cap K) = W(B)$, and $V(J), V(K)$ are disjoint and anticomplete.*

Then there is an ℓ-framework $T \cup S \cup P_1 \cup \dots \cup P_k$, of which G is a blow-up, such that $G[V(A) \cup (V(J) \setminus W(A))] = \overrightarrow{T}$ and $G[V(B) \cup (V(K) \setminus W(B))] = \overrightarrow{S}$.

Proof. This is shown in the proof of 10.2 in [19] for odd ℓ and in the proof of 12.1 in [19] for even ℓ. $\square$

Lemma 3.16. *Let $\ell \geq 7$ and let G be an ℓ-holed graph with no clique cutset and no universal vertex. Suppose that $G = H \cup J \cup K$ is a bordered blow-up of an ℓ-frame. Then some vertex c of J is complete to $V(A) \setminus \{c\}$ and some vertex c' of K is complete to $V(B) \setminus \{c'\}$.*

Proof. By Lemma 3.15, G is a blow-up of an ℓ-framework $S \cup T \cup P_1 \cup \cdots \cup P_k$ with the property that $G[V(A) \cup (V(J) \setminus W(A))] = \overrightarrow{T}$ and $G[V(B) \cup (V(K) \setminus W(B))] = \overrightarrow{S}$. Now the apex $r(T)$ of T is complete to $V(A) \setminus \{r(T)\}$ and the apex $r(S)$ of S is complete to $V(B) \setminus \{r(S)\}$. $\square$

Lemma 3.17. *Let $\ell \geq 7$ and let $G = H \cup J \cup K$ be a bordered blow-up of an ℓ-frame. Assume that G has no clique cutset and no universal vertex. Then both J and K contain a universal vertex.*

Proof. By Lemma 3.16, there is a vertex $c \in V(J)$ that is complete to $V(A) \setminus \{c\}$; among all such vertices, pick c so that $|N_G(c) \cap W(A)|$ is maximum. Suppose $c \in W(A)$. By the definition of a blow-up of an ℓ-frame, c is complete to $W(A) \setminus \{c\}$. Since J is chordal and every vertex of $V(J) \setminus W(A)$ has two nonadjacent neighbours in $V(A)$, it follows that c is complete to $V(J) \setminus W(A)$. Therefore c is a universal vertex of J. So we may assume that no vertex of $W(A)$ is universal, and hence $c \in V(J) \setminus W(A)$. We now need the following.

(1) If a is an isolated vertex of A, then every vertex in W_a has a neighbour in $V(J) \setminus W(A)$.

Proof of (1): Let a be an isolated vertex of A and fix $a' \in W_a$. Suppose that a' has no neighbour in $V(J) \setminus W(A)$. Let X be the set of all vertices in $W_a \setminus \{a'\}$ that belong to some path of J from a' to some vertex of $W(A) \setminus W_a$. Clearly every path in G from a' to some vertex in $W(A) \setminus W_a$ intersects $X \cup (N(a') \cap W_{a^+})$, where a^+ is the neighbour of a in the bar of F that contains a. We have that X is a clique, as is $N(a') \cap W_{a^+}$, and X is complete to $N(a') \cap W_{a^+}$ since $G[W_a, W_{a^+}]$ is a half-graph. It follows that $X \cup (N(a') \cap W_{a^+})$ is a clique cutset of G, a contradiction. This proves (1).

Since every vertex in $V(J) \setminus W(A)$ has two nonadjacent neighbours in $V(A)$, it follows from J being chordal that c is complete to $V(J) \setminus (W(A) \cup \{c\})$. So it remains to prove that c is complete to $W(A)$. Suppose there exists $a \in V(A)$ and $a' \in W_a$ such that c, a' are nonadjacent. Let I be the set of isolated vertices in A, and let us consider first the case where $a \in I$. By (1), a' has some neighbour c' in $V(J) \setminus W(A)$, clearly different from c. By what we proved in the first paragraph, c and c' are adjacent. Consequently, c' is adjacent to every $v \in N(c) \cap W_a$, for otherwise $\{a', v, c, c'\}$ induces a 4-hole, a contradiction. In particular, c' and a are adjacent. Furthermore, c' is adjacent to every $v \in (N(c) \cap W(A)) \setminus W_a$, for otherwise the path a', c', c, v violates the third bullet in the definition of a border. But now c' is complete to $V(A)$ and has more neighbours in $W(A)$ than c, contrary to our choice of c. So we now consider the case where $a \in V(A) \setminus I$. Since $|A \setminus I| \geq 2$ and $G[V(A) \setminus I]$ has a universal vertex (because A is a threshold graph and $A \setminus I$ has no isolated vertex), there is some $b \in V(A) \setminus (I \cup \{a\})$ adjacent to a. But now, for any $d \in I$, the path a', b, c, d violates the third bullet in the definition of a border. So c is complete to $W(A \setminus I)$, and hence c is complete to $W(A)$, as required. It follows by an analogous argument that K has a universal vertex. $\square$

Lemma 3.18. *Let $\ell \geq 7$ be even and let $G = H \cup J \cup K$ be a bordered blow-up of an ℓ-frame F. Then every universal vertex of J belongs to $V(J) \setminus W(A)$ and every universal vertex of K belongs to $V(K) \setminus W(B)$.*

Proof. By the definition of an ℓ-frame when ℓ is even, A and B each contain at least one isolated vertex, which together with the definition of a blow-up of an ℓ-frame implies that no universal vertex of J belongs to $W(A)$ and no universal vertex of K belongs to $W(B)$. $\square$

Lemma 3.19. *Let $\ell \geq 7$ and let $G = H \cup J \cup K$ be a bordered blow-up of an ℓ-frame F. Suppose that G has no clique cutset. Then we may assume that for every $u \in V(A)$, every vertex in W_u has a neighbour in W_v, where v is the unique neighbour of u in the bar of F that contains it.*

Proof. For each $t \in V(A)$, let $X_t = \{v \in W_t : N_G(v) \subseteq V(J)\}$. We claim that $(J, A, (W_t \setminus X_t : t \in V(A)))$ is a border. (So now the bags of J are the sets $W_t \setminus X_t$ for $t \in V(A)$, and therefore $W(A) = \bigcup_{t \in V(A)}(W_t \setminus X_t)$.) Let us check that the four bullets in the definition of a border are satisfied. Since $(J, A, (W_t : t \in V(A)))$ is a border, J is chordal by definition, so the first bullet holds. By the definition of a blow-up of an ℓ-frame, if $u \in V(A)$ has a neighbour in W_v for some $v \in V(A) \setminus \{u\}$, then u is complete to W_v; it follows that the fourth bullet holds.

We now verify that the second bullet holds. Let $t \in V(A)$ be such that $X_t \neq \emptyset$, and let v be the vertex that appears last in the ordering of W_t. We prove that v has two nonadjacent neighbours in $V(A)$, and from this we will deduce that every vertex in W_t has two nonadjacent neighbours in $V(A)$. By the half-graph condition in the definition of a blow-up of an ℓ-frame, we get that $v \in X_t$. Since G has no clique cutset, v has two nonadjacent neighbours u, x, and by the definition of X_t they both belong to $V(J)$. So we may assume up to symmetry that $x \in V(J) \setminus \bigcup_{s \in V(A)} W_s$ and either $u \in \bigcup_{s \in V(A)} W_s$ or $u \in V(J) \setminus \bigcup_{s \in V(A)} W_s$. Let y, z be two nonadjacent neighbours of x in $V(A)$.

Suppose first that $u \in \bigcup_{s\in V(A)} W_s$. If $u \in W_t$, then by the fourth bullet in the definition of a border we get that u appears after v in the ordering of W_t, contrary to our choice of v; so $u \notin W_t$. Suppose that $t \in \{y, z\}$, say $y = t$. Since W_t is complete to the bag that contains u, it follows that z does not belong to the same bag as u. But now the vertices u, v, x, z form either a path that contradicts the third bullet in the definition of a border, or a hole, which contradicts the first bullet. So $t \notin \{y, z\}$, and therefore we may assume that t is complete to $(N(x) \cap V(A)) \setminus \{t\}$. It follows immediately that t (and hence v) is adjacent to y and z, and we are done.

So both u and x belong to $V(J)\setminus\bigcup_{s\in V(A)} W_s$. If t is complete to $(N(x)\cap V(A))\setminus\{t\}$, then $t \notin \{y, z\}$, in which case y, z are two nonadjacent neighbours of t (and hence of v) in $V(A)$, as required. So we may assume that there exists $w \in (N(x) \cap V(A)) \setminus \{t\}$ nonadjacent to t, and by a similar argument we may fix some $w' \in (N(u)\cap V(A)) \setminus \{t\}$ that is nonadjacent to t. But now $\{u, x, w, w'\}$ induces a hole if $w = w'$, and otherwise the vertices w, x, v, u, w' form either a hole, or a path (whose vertices appear in that order) that contradicts the third bullet in the definition of a border.

Thus v has two nonadjacent neighbours in $V(A)$, and it follows from the definition of a blow-up of an ℓ-frame that every vertex in W_t has two nonadjacent neighbours in $V(A)$.

Finally we check that $(J, A, (W_t \setminus X_t : t \in V(A)))$ satisfies the third bullet in the definition of a border. On the contrary, suppose there exists a path P, with ends u, v say, that violates this condition. Let u_0 and v_0 be such that $u \in W_{u_0} \setminus X_{u_0}$ and $v \in W_{v_0} \setminus X_{v_0}$, and let x and y be the neighbours of u and v, respectively, in P. Clearly the interior of P does not intersect $W_{u_0} \cup W_{v_0}$. Since $(J, A, (W_t : t \in V(A)))$ is a border, it follows that x belongs to W_{u_0} or y belongs to W_{v_0}; suppose without loss of generality that $x \in W_{u_0}$. So $x \in X_{u_0}$. Call z the neighbour of x in P different from u. Since z is adjacent to x but not to u, and since $x \in W_{u_0}$, it follows that $z \notin \bigcup_{s\in V(A)} W_s$. Since $u \notin X_{u_0}$, it follows that u appears before x in the ordering of W_{u_0}. It then follows from the fourth bullet in the definition of a border that z is adjacent to u, a contradiction. This concludes the proof that all four bullets hold, and therefore $(J, A, (W_t \setminus X_t : t \in V(A)))$ is a border.

It follows by a symmetric argument that $(K, B, (W_t \setminus X_t : t \in V(B)))$ is a border, and therefore we conclude that G is a bordered blow-up of the ℓ-frame F where the borders of G are $(J, A, (W_t \setminus X_t : t \in V(A)))$ and $(K, B, (W_t \setminus X_t : t \in V(B)))$. □

Lemma 3.20. *Let $\ell \geq 7$ and let $G = H \cup J \cup K$ be a bordered blow-up of an ℓ-frame F. Suppose that G has no clique cutset. Fix $u \in V(F) \setminus V(J \cup K)$, and let a and b*

be the two neighbours of u in F. Then every vertex in W_u has a neighbour in both W_a and W_b.

Proof. Since $G[W_u, W_a]$, $G[W_u, W_b]$ are compatible half-graphs, there exists a vertex $v \in W_u$ such that $N_G(v) \subseteq N_G(v')$ for every $v' \in W_u$. With this in mind, it suffices to show that v has a neighbour in both W_a and W_b. Suppose otherwise; say v is anticomplete to W_b. Since G has no clique cutset, v (and therefore also t) has two nonadjacent neighbours x, y in G. Because W_u and W_a are cliques, up to symmetry we have that $x \in W_a$ and $y \in W_u$. It follows from $G[W_u, W_a]$, $G[W_u, W_b]$ being compatible half-graphs that y is also anticomplete to W_b. But now we have that $N_G(y) \subsetneq N_G(v)$, contrary to our choice of v. Therefore every vertex in W_u has a neighbour in both W_a and W_b. □

We are now ready to prove the main result of this section:

Lemma 3.21. *Let $\ell \geq 7$ and let $G = H \cup J \cup K$ be an ℓ-holed bordered blow-up of an ℓ-frame F. Assume that G has no clique cutset and no universal vertex. If ℓ is odd, then G is a proper preblowup of a twinless odd $(\ell - 1)/2$-template, and if ℓ is even, then G is a proper preblowup of a twinless even $\ell/2$-template.*

Proof. Set $D = F \cup ((J \cup K) \setminus (W(A) \cup W(B)))$ and let D^- be the graph obtained from D by removing twins. Define the following sets:

- $A_1 = V(A)$,
- $B_1 = (V(J) \setminus W(A)) \cap V(D^-)$,
- $A'_1 = V(B)$,
- $B'_1 = (V(K) \setminus W(B)) \cap V(D^-)$, and
- $I = V(D^-) \setminus (A_1 \cup B_1 \cup A'_1 \cup B'_1)$ (so I consists of all vertices of G that belong to the interior of some bar of F).

We point out that all twins of D belong to $(V(J) \setminus W(A)) \cup (V(K) \setminus W(B))$. It follows that $(A_1, B_1, A'_1, B'_1, I)$ is a partition of the vertex set of D^-.

(1) If ℓ is odd, then D^- is an odd $(\ell - 1)/2$-pretemplate.

Proof of (1): We claim that $(A_1, B_1, A'_1, B'_1, I)$ is an odd $(\ell-1)/2$-pretemplate partition of D^-. We check that this partition satisfies conditions (a) to (i) from the definition of an odd pretemplate. It follows easily from the relevant definitions that conditions (a) to (f) hold (for instance, conditions (a) to (c) follow from the definition of a border, and conditions (d) to (f) are satisfied by the bars of F). Conditions (h) and (i) hold by the second bullet in the definition of a border. By Lemma 3.17, $D^-[A_1 \cup B_1]$ contains a universal vertex, so $D^-[A_1 \cup B_1]$ is connected and by symmetry so is $D^-[A'_1 \cup B'_1]$. Therefore condition (g) holds, and this concludes the proof that D^- is an odd $(\ell-1)/2$-pretemplate with odd $(\ell-1)/2$-pretemplate partition $(A_1, B_1, A'_1, B'_1, I)$. This proves (1).

Now an analogous claim but for when ℓ is even:

(2) If ℓ is even, then D^- is an even $\ell/2$-pretemplate.

Proof of (2): Let:

- A_S be the set of isolated vertices of A,
- A'_S the set of isolated vertices of B,
- $A_K = V(A) \setminus A_S$, and
- $A'_K = V(B) \setminus A'_S$.

By the definition of an ℓ-frame when ℓ is even, $V(A)$ partitions into a clique and a stable set, and so does $V(B)$. With that in mind, A_S, A'_S are (nonempty) stable sets and A_K, A'_K are cliques.

We claim that $(A_1, B_1, A'_1, B'_1, I)$ is an even $\ell/2$-pretemplate partition of D^-. We check that this partition satisfies conditions (a) to (j) from the definition of an even pretemplate. It follows easily from the relevant definitions that conditions (a) to (g) hold (for instance, conditions (a) to (d) follow from the definition of a bordered blow-up of an ℓ-frame, and conditions (e) to (g) are satisfied by the bars of F). Conditions (i) and (j) hold by the second bullet in the definition of a border. By Lemma 3.17, $D^-[A_1 \cup B_1]$ contains a universal vertex, and therefore $D^-[A_1 \cup B_1]$ is connected. So condition (h) holds, and this concludes the proof that D^- is an even $\ell/2$-pretemplate with even $\ell/2$-pretemplate partition $(A_1, B_1, A'_1, B'_1, I)$. This proves (2).

If ℓ is odd, then by (1) and Lemma 3.12, D^- is an odd $(\ell-1)/2$-template, and if ℓ is even, then by (2) and Lemma 3.13, D^- is an even $\ell/2$-template. Furthermore, D^- is proper by Lemmas 3.14 and 3.18.

We now prove that G is a proper preblowup of D^-. We let:

- $K_u = W_u$ for each $u \in A_1 \cup A_1' \cup I$ (where W_u is as in the definition of a bordered blow-up of an ℓ-frame);
- $A^* = \bigcup_{u \in A_1} K_u$;
- $A'^* = \bigcup_{u \in A_1'} K_u$;
- $B^* = V(J) \setminus W(A)$;
- $B'^* = V(K) \setminus W(B)$; and
- $I^* = \bigcup_{u \in I} K_u$.

With respect to these sets, we check that conditions (a), (b) and (i) from the definition of a preblowup hold.

It is clear from the definition of a bordered blow-up of an ℓ-frame that:

- $N(K_u) \subseteq A^* \cup B^* \cup K_{u^+}$ for all $u \in A_1$, where u^+ is the neighbour of u in I;
- $N(B^*) \subseteq A^*$; and
- $N(K_u) \subseteq K_a \cup K_b$ for all $u \in I$, where a and b are the neighbours of u in G.

So the initial statements of (a), (b) and (i) hold. Part 1 of (a) holds by the definition of a blow-up of an ℓ-frame and part 2 holds by Lemma 3.19. Part 2 of (b) holds by the second bullet in the definition of a border. Part 1 of (i) holds by Lemma 3.20. It remains to prove part 1 of (b), i.e. that if w (the vertex from the ℓ-partition of D^-) belongs to B_1, then in G there is a vertex in B^* that is complete to A^*. So suppose $w \in B_1$. After possibly replacing w with another vertex from $V(J) \setminus W(A)$ that is complete to $V(A)$ and maximises $|N_G(w) \cap W(A)|$, the argument in the proof of Lemma 3.17 shows that w is complete to $W(A)$, and therefore part 1 of (b) holds. This completes the proof that G is a proper preblowup of D^-. $\square$

Proof of Lemma 3.8

Lemma 3.22 (Lemma 5.6 in [35]). *Let $\ell \geq 3$ and let G^* be a proper preblowup of an odd ℓ-template with $k \geq 3$ principal paths. If G^* is $(2\ell+1)$-holed, then G^* is a proper blowup of a twinless odd ℓ-template G with k principal paths (in particular, G is an induced subgraph of G^*).*

Lemma 3.23 (Lemma 8.15 in [35]). *Let $\ell \geq 3$ and let G^* be a proper preblowup of an even ℓ-template with $k \geq 3$ principal paths. If G^* is 2ℓ-holed, then G^* is a proper* *blowup of a twinless even ℓ-template G with k principal paths (in particular, G is an induced subgraph of G^*).*

We are now ready to prove Lemma 3.8. Let $\ell \geq 7$ be an integer and let G be a blow-up of an ℓ-framework. Assume that G has no clique cutset. By Lemma 3.21 together with Lemmas 3.22 and 3.23, G is a proper blowup of a twinless odd $(\ell-1)/2$-template if ℓ is odd, and G is a proper blowup of a twinless even $\ell/2$-template if ℓ is even. This completes the proof of Lemma 3.8.

3.3.2 Holes in proper blowups of twinless templates

We first state some lemmas from [35] about blowups of templates.

Lemma 3.24 (Lemma 5.5 in [35])**.** *Let $\ell \geq 3$ and let G^* be a blowup of a twinless odd ℓ-template G. Then every hole of G^* has length $2\ell+1$.*

Lemma 3.25 (Lemma 8.14 in [35])**.** *Let $\ell \geq 4$ and let G^* be a blowup of a twinless even ℓ-template G. Then every hole of G^* has length 2ℓ.*

We now obtain the main result of this section:

Lemma 3.26. *Let $\ell \geq 7$ and let G be a proper blowup of a twinless odd $(\ell-1)/2$-template if ℓ is odd, and a proper blowup of a twinless even $\ell/2$-template if ℓ is even. Then G is ℓ-holed.*

Proof. Follows immediately from Lemmas 3.24 and 3.25. □

3.3.3 Proof of Theorem 3.7

Theorem 3.7. *Let G be a graph with no clique cutset or universal vertex that contains a theta, a pyramid or a prism. Then the following are equivalent:*

- *G is ℓ-holed for some $\ell \geq 7$.*
- *G is a blow-up of an ℓ-framework.*
- *G is a proper blowup of a twinless odd $(\ell-1)/2$-template if ℓ is odd, and G is a proper blowup of a twinless even $\ell/2$-template if ℓ is even.*

Proof. By Theorem 3.2, an ℓ-holed graph with no clique cutset and no universal vertex that contains a theta, a pyramid or a prism is a blow-up of an ℓ-framework, and therefore the first bullet implies the second. By Lemma 3.8, the second bullet implies the third. By Lemma 3.26, the third bullet implies the first. It follows that the three bullets are equivalent. □

3.4 Special 2-joins

In this section we introduce special 2-joins, a variant of a type of edge cutset known as a 2-join. Let us begin by describing the motivation for special 2-joins. Standard 2-joins (defined in the next paragraph) appear in decomposition theorems for many hereditary graph classes, such as perfect graphs [12], even-hole-free graphs [22] and claw-free graphs [15]. The structure theorems for ℓ-holed graphs presented in Sections 3.1 and 3.2 describe how one may decompose ℓ-holed graphs by clique cutsets into "basic" graphs (specifically, into "blow-ups of frameworks" in the language of the theorem in Section 3.1, and into "proper blowups of twinless templates" in the language of the theorem in Section 3.2). In order to obtain decomposition-based recognition algorithms from these decomposition theorems, one must be able to decide whether a given graph is "basic". However, these basic graphs are in fact quite complicated, and it is not obvious from their definitions how one may determine algorithmically whether a graph is basic. But if one were able to further decompose, say by 2-joins, these basic graphs into "more-basic" graphs, and provided that decomposition by 2-joins preserves being ℓ-holed, the problem of recognition now reduces to deciding whether a graph is "more-basic"; and since "more-basic" graphs have a more restricted structure than "basic" graphs (that is, in addition to having no clique cutset they also have no 2-join), deciding whether a graph is "more-basic" is expected to be simpler than deciding whether a graph is "basic".

There are two problems with this idea, however. First, not all basic graphs admit 2-joins, and second, 2-joins fail to preserve being ℓ-holed. In order to illustrate the first problem, we need to define 2-joins (first defined by Cornuéjols and Cunningham in [21]). A partition (X_1, X_2) of the vertex set of a graph G is a *2-join* if there exist disjoint nonempty sets $A_1, B_1 \subseteq X_1$ and disjoint nonempty sets $A_2, B_2 \subseteq X_2$ such that the following hold:

- A_1 is complete to A_2, and B_1 is complete to B_2;
- there are no other edges between X_1 and X_2 besides those between A_1 and A_2 and between B_1 and B_2;
- $|X_1| \geq 3$ and $|X_2| \geq 3$;
- for each $i \in \{1, 2\}$, $G[X_i]$ is not a path of length 2 with an end in A_i, an end in B_i and its unique interior vertex in $X_i \setminus (A_i \cup B_i)$.

We now refer the reader to Figure 3.6, which depicts a blow-up of an ℓ-framework (or, equivalently, a proper blowup of a twinless template) that has no 2-join.

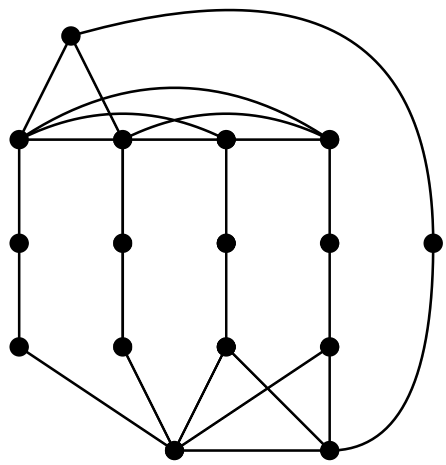


Figure 3.6: A blow-up of a 7-framework (or, equivalently, a proper blowup of a twinless template) that has no 2-join.

The second problem with the idea described above is that 2-joins do not preserve being ℓ-holed. That is, if G is a graph that has a 2-join and G_1 and G_2 are the blocks of decomposition of G with respect to this 2-join, then it is not necessarily the case that G is ℓ-holed if and only if both G_1 and G_2 are ℓ-holed.

3.4.1 Definitions

Let G be a graph. A partition (X_1, X_2) of $V(G)$ is a *frame* (*of* G) if for each $i \in \{1, 2\}$ there exist disjoint nonempty subsets A_i and B_i of X_i such that each of the following conditions hold.

- $|X_1| \geq 3$ and $|X_2| \geq 3$.
- The only edges between X_1 and X_2 are those between A_1 and A_2 and those between B_1 and B_2.
- For each $i \in \{1, 2\}$, $G[X_i]$ contains a path with one end in A_i, the other end in B_i, and interior in $X_i \setminus (A_i \cup B_i)$. Furthermore, if $|A_i| = |B_i| = 1$, then $G[X_i]$ is not a path.

Under these circumstances we call the tuple $(X_1, X_2, A_1, A_2, B_1, B_2)$ a *split* of the frame (X_1, X_2). Given a graph G and a frame (X_1, X_2) of G with split $S =$

$(X_1, X_2, A_1, A_2, B_1, B_2)$, we denote by $\mathcal{P}_i(G, S)$ (for $i \in \{1, 2\}$) the set of paths in $G[X_i]$ that have one end in A_i, the other end in B_i, and interior in $X_i \setminus (A_i \cup B_i)$. When the graph G and split S are clear from context, we may write $\mathcal{P}_i$ instead of $\mathcal{P}_i(G, S)$.

A frame (X_1, X_2) with split $(X_1, X_2, A_1, A_2, B_1, B_2)$ is a *special 2-join of type 1* if:

- A_1 is complete to A_2 and B_1 is complete to B_2,

and for some $i \in \{1, 2\}$:

- $X_i \setminus (A_i \cup B_i) \neq \emptyset$, and
- A_i and B_i are cliques, and at least one of $G[A_{3-i}]$ and $G[B_{3-i}]$ contains a universal vertex.

For a graph G, disjoint subsets A and B of $V(G)$ are *nested* if there are no four vertices, $a_1, a_2 \in A$ and $b_1, b_2 \in B$, such that a_1b_1 and a_2b_2 are edges of G but a_1b_2 and a_2b_1 are not. Consequently, for any $a, a' \in A$, one of $N_B(a), N_B(a')$ is contained in the other, and similarly for any $b, b' \in B$, one of $N_A(b), N_A(b')$ is contained in the other.

A frame (X_1, X_2) with split $(X_1, X_2, A_1, A_2, B_1, B_2)$ is a *special 2-join of type 2* if:

- A_1, A_2, B_1 and B_2 are cliques;
- A_1 and A_2 are nested, and B_1 is complete to B_2;
- some vertex of A_1 is complete to A_2 and some vertex of A_2 is complete to A_1;
- some vertex of $X_2 \setminus A_2$ is complete to A_2; and
- for every $i \in \{1, 2\}$ and every vertex $v \in N(A_i) \setminus A_{3-i}$, sets A_i and $A_{3-i} \cup \{v\}$ are nested.

A *special 2-join* is a frame that is a special 2-join of type 1 or 2. Note that a frame may be both a special 2-join of type 1 and a special 2-join of type 2.

Lemma 3.27. *Let G be a graph and let (X_1, X_2) be a special 2-join of type 2 of G with split $(X_1, X_2, A_1, A_2, B_1, B_2)$. For each $i \in \{1, 2\}$, there exists a path $Q_i = a_i, \dots, b_i$ in $\mathcal{P}_i$ with $a_i \in A_i$ and $b_i \in B_i$ such that a_i is complete to A_{3-i} and b_i is complete to B_{3-i}.*

Proof. Let $P = a_1, \dots, b_1$ be any path from $\mathcal{P}_1$, where $a_1 \in A_1$ and $b_1 \in B_1$. By definition, b_1 is complete to B_2, so we are done if a_1 is complete to A_2. Therefore we may assume that there exists a vertex $a_2 \in A_2 \setminus N(a_1)$. Let u be the unique neighbour of a_1 in P and let a be any vertex of A_1 that is complete to A_2. If a and u are nonadjacent, then the vertices a_1, u, a and a_2 contradict the fact that A_1 and $A_2 \cup \{u\}$ are nested. So a and u are adjacent, and now any path from a to b_1 in $G[V(P) \cup \{a\}]$ is as required. □

Let G be a graph and let (X_1, X_2) be a special 2-join of G with split $(X_1, X_2, A_1, A_2, B_1, B_2)$. Then *blocks of decomposition* (*of* G*, with respect to* (X_1, X_2)) are any two graphs G_1 and G_2 such that, for each $i \in \{1, 2\}$, $G_i = G[X_i \cup V(Q)]$, where $Q = a, \dots, b$ is a path from $\mathcal{P}_{3-i}$ with $a \in A_{3-i}$ and $b \in B_{3-i}$ such that a is complete to A_i and b is complete to B_i. The path Q is called the *marker path* of G_i. In the context of special 2-joins of type 1, the existence of a marker path follows immediately from the definition, and in the context of special 2-joins of type 2 it follows from Lemma 3.27.

Lemma 3.28. *Let G be a graph and let (X_1, X_2) be a frame of G with split $(X_1, X_2, A_1, A_2, B_1, B_2)$. Let $Q = a_2, \dots, b_2$ be a path in $\mathcal{P}_2$ where $a_2 \in A_2$ and $b_2 \in B_2$, and suppose that a_2 is complete to A_1 and b_2 is complete to B_1. If every hole of $G[X_1 \cup V(Q)]$ has the same length, then every path in $\mathcal{P}_1$ has the same length.*

Proof. For if there exist two paths $P, P' \in \mathcal{P}_1$ of different lengths, then $G[V(P) \cup V(Q)]$ and $G[V(P') \cup V(Q)]$ are holes of $G[X_1 \cup V(Q)]$ that have different lengths, a contradiction. □

A vertex is *simplicial* if its neighbourhood is a clique.

Lemma 3.29. *Let $\ell \geq 4$ be an integer. If v is a simplicial vertex or universal vertex of a graph G, then G is ℓ-holed if and only if $G \setminus v$ is ℓ-holed.*

Proof. The holes of G and $G \setminus v$ are the same. □

Lemma 3.30. *Let G be a graph, (X_1, X_2) a special 2-join of G, and G_1 and G_2 blocks of decomposition of G with respect to (X_1, X_2). Suppose that G has no clique cutset. Then G_1 (resp. G_2) has a clique cutset if and only if G_1 (resp. G_2) has a simplicial vertex.*

Proof. If one of the blocks G_1 or G_2 has a simplicial vertex, then clearly it has a clique cutset. To prove the converse, let $(X_1, X_2, A_1, A_2, B_1, B_2)$ be a split of (X_1, X_2), and suppose that G_1 has a clique cutset K but no simplicial vertex. Let D be a component of $G_1 \setminus K$ that contains a vertex of $(A_1 \cup B_1 \cup V(P)) \setminus K$, and let C be another component of $G_1 \setminus K$.

Case 1: $K \subseteq X_1$.
Then $(A_1 \cup B_1) \setminus K \subseteq V(D)$, and hence K is a clique cutset of G that separates $V(D) \cup X_2$ from $V(C)$, a contradiction.

Case 2: $K \subseteq V(P)$.
If A_1 and B_1 are both cliques, then since $\mathcal{P}_1 \neq \emptyset$, $A_1 \cup B_1 \cup (V(P) \setminus K) \subseteq V(D)$ and $V(C) \subseteq X_1 \setminus (A_1 \cup B_1)$, and hence G is disconnected, a contradiction. So A_1 and B_1 are not both cliques, A_2 and B_2 are both cliques, and without loss of generality $G[B_1]$ contains a universal vertex. Then we may assume that $B_1 \subseteq V(D)$ and hence A_2 is a clique cutset of G that separates $V(D) \cup X_2$ from $V(C)$, a contradiction.

Case 3: $K = A'_1 \cup \{a\}$ for some nonempty subset A'_1 of A_1.
Without loss of generality $B_1 \subseteq V(D)$. If A_2 is a clique, then $A_2 \cup A'_1$ is a clique cutset of G that separates $V(D) \cup X_2$ from $V(C)$, a contradiction. So A_2 is not a clique, and hence A_1 and B_1 are cliques. If $V(C) \not\subseteq A_1$, then A_1 is a clique cutset of G that separates $V(D) \cup X_2$ from $V(C) \setminus A_1$. So $V(C) \subseteq A_1 \setminus A'_1$ and hence any vertex of C is a simplicial vertex of G_1, a contradiction. □

3.4.2 Special 2-join decomposition is class-preserving

Lemma 3.31. *Let G be a graph, (X_1, X_2) a special 2-join of G, and G_1 and G_2 blocks of decomposition of G with respect to (X_1, X_2). Then for every integer $\ell \geq 5$, G is ℓ-holed if and only if both G_1 and G_2 are ℓ-holed.*

Proof. The "only if" direction follows from G_1 and G_2 being induced subgraphs of G. For the other direction, let us suppose for some $k \geq 5$ that G_1 and G_2 are ℓ-holed but G is not. Therefore G contains a hole H of length $\ell' \neq \ell$, and H contains vertices of both X_1 and X_2. Let $(X_1, X_2, A_1, A_2, B_1, B_2)$ be a split of (X_1, X_2).

First suppose that (X_1, X_2) is a special 2-join of type 2. We prove the following claims.

(1) H contains exactly one vertex from each of A_1, A_2, B_1 and B_2.

Proof of (1): Since A_1 and A_2 are cliques that are nested, clearly at most one edge of H is between A_1 and A_2. Similarly, at most one edge of H is between B_1 and B_2 since $B_1 \cup B_2$ is a clique. Since the number of edges of H between X_1 and X_2 must be even, and since H intersects both X_1 and X_2, it follows that exactly one edge of H is between A_1 and A_2 and exactly one edge of H is between B_1 and B_2. It follows immediately that $|V(H) \cap B_1| = |V(H) \cap B_2| = 1$.

Suppose H contains two vertices a_1 and a_1' from A_1. Fix $a_2 \in V(H) \cap A_2$. If a_1 is adjacent to a_2, then the neighbour x of a_1' in H different from a_1 belongs to $X_1 \setminus A_1$ (since A_1 is a clique), in which case A_1 and $A_2 \cup \{x\}$ are not nested. So $\{a_1, a_1'\}$ is anticomplete to a_2. But this contradicts the fact that there is an edge of H between A_1 and A_2. Therefore $|V(H) \cap A_1| = 1$, and by symmetry $|V(H) \cap A_2| = 1$. This proves (1).

It now follows from (1) that $H = G[V(P_1) \cup V(P_2)]$ for some $P_1 \in \mathcal{P}_1$ and $P_2 \in \mathcal{P}_2$. Since G_2 is ℓ-holed, by Lemma 3.28 every path in $\mathcal{P}_2$ has the same length, and in particular Q and P_2 have the same length. But then $G[(V(H) \setminus V(P_2)) \cup V(Q)]$ is a hole of length ℓ' in G_1, a contradiction.

Suppose now that (X_1, X_2) is a special 2-join of type 1. Without loss of generality we assume that A_1 and B_1 are cliques. We prove the following claims.

(2) H contains at least one vertex from each of A_1, A_2, B_1 and B_2.

Proof of (2): Suppose that $V(H) \cap A_1 = \emptyset$. Then there exist adjacent vertices $b_1 \in V(H) \cap B_1$ and $b_2 \in V(H) \cap B_2$. Since B_1 is a clique that is complete to B_2, we have that $V(H) \cap B_1 = \{b_1\}$. It follows that $V(H) \setminus \{b_1\} \subseteq X_2$. Let b_1' be the end of the marker path of G_2 that lies in B_1. But now $G[(V(H) \setminus \{b_1\}) \cup \{b_1'\}]$ is a hole of length ℓ' in G_2, a contradiction. So $V(H) \cap A_1 \neq \emptyset$, and by analogous argument $V(H) \cap B_1 \neq \emptyset$.

Suppose that $V(H) \cap A_2 = \emptyset$. Then there exist adjacent vertices $b_1 \in V(H) \cap B_1$ and $b_2 \in V(H) \cap B_2$. Since B_1 is a clique that is complete to B_2, we have that $V(H) \cap B_1 = \{b_1\}$. Fix $a_1 \in V(H) \cap A_1$, and let P be the subpath of H from a_1 to b_2 that does not contain b_1. Then $V(P) \cap (B_1 \setminus \{b_1\}) \neq \emptyset$, and hence $|V(H) \cap B_1| \geq 2$, a contradiction. So $V(H) \cap A_2 \neq \emptyset$, and by analogous argument $V(H) \cap B_2 \neq \emptyset$. This proves (2).

(3) $|V(H) \cap A_1| = |V(H) \cap B_1| = 1$.

Proof of (3): It follows immediately from (2) together with the fact that A_1 is a clique that is complete to A_2 and B_1 is a clique that is complete to B_2. This proves (3).

(4) Let $a_1 \in V(H) \cap A_1$ and $b_1 \in V(H) \cap B_1$. Then $V(H) \cap A_2 = N_H(a_1)$ and $V(H) \cap B_2 = N_H(b_1)$.

Proof of (4): Suppose otherwise. If $|V(H) \cap A_2| \geq 3$, then $d_H(a_1) \geq 3$, a contradiction. So $|V(H) \cap A_2| \leq 2$, and $|V(H) \cap B_2| \leq 2$ by symmetry. Suppose that $|V(H) \cap A_2| = |V(H) \cap B_2| = 1$. Then $H = G[V(P_1) \cup V(P_2)]$ for some $P_1 \in \mathcal{P}_1$ and $P_2 \in \mathcal{P}_2$. Let Q be the marker path of G_1. But now, by Lemma 3.28, $G[V(P_1) \cup V(Q)]$, is a hole of length ℓ' in G_1, a contradiction. So up to symmetry $N_H(a_1) \subseteq A_2$ and $N_H(b_1) \not\subseteq B_2$. Let b_2 be the unique vertex of $N_H(b_1) \cap B_2$ and let P be the subpath of H from a_1 to b_1 that does not contain b_2. Then $V(P) \subseteq X_1$, and hence $N_H(a_1) \cap X_1 \neq \emptyset$, a contradiction. This proves (4).

By (2), (3) and (4), there exist two paths $R_1 = u_1, \ldots, v_1$ and $R_2 = u_2, \ldots, v_2$ in $\mathcal{P}_2$ with $u_1, u_2 \in A_2$ and $v_1, v_2 \in B_2$, such that $H = G[V(R_1) \cup V(R_2) \cup \{a_1, b_1\}]$ for some $a_1 \in A_1$ and $b_1 \in B_1$. If the marker path Q of G_2 has length at least 2, then the ends of Q together with R_1 and R_2 form a hole of length ℓ' in G_2, a contradiction. So we may assume that Q has length 1, i.e. its ends $u \in A_1$ and $v \in B_1$ are adjacent. Assume without loss of generality that $G[A_2]$ contains a universal vertex s. Clearly $s \notin \{u_1, u_2\}$. For each $i \in \{1, 2\}$, fix $s_i \in V(R_i) \cap N(s)$ such that the length of the subpath $s_i R_i v_i$ is minimum. If $s_1 = u_1$ and $s_2 = u_2$, then $(V(H) \setminus \{a_1, b_1\}) \cup \{s, v\}$ induces a hole of length ℓ' in G_2, a contradiction. So without loss of generality $s_1 \neq u_1$. Suppose that $s_2 = u_2$. Then the three paths $ss_1R_1v_1v$, R_2v and Q form a pyramid in G_2. But Q is of length 1, contradicting the fact that the three paths of a pyramid in G_2 must have the same length (as G_2 is ℓ-holed). So $s_2 \neq u_2$. Now the three paths sQ, $ss_1R_1v_1v$ and $ss_2R_2v_2v$ form a theta in G_2. Since Q is of length 1 and all three paths of a theta in G_2 must have the same length, it follows that $s_1 = v_1$. But now $\{s, u, v, v_1\}$ induces a C_4 in G_2, a contradiction. □

3.4.3 Special 2-joins arising from templates

Let G be a twinless odd ℓ-template with ℓ-partition (A, B, A', B', I, w, w'). If a is a vertex of A, then a' is the end, different from a, of the principal path of G that contains a (and vice versa). We use this notation often in the proof of the following lemma. Also, if G^* is a blowup of a template G and X is a subset of $V(G)$, then we denote by $\mathcal{K}(X)$ the set $\bigcup_{x \in X} K_x$. (Recall from Section 3.2 that to obtain a blowup G^* of G each vertex x of G is replaced by a clique K_x.)

Lemma 3.32. *For every integer $\ell \geq 3$, every proper blowup of a twinless odd ℓ-template is a pyramid or admits a special 2-join.*

Proof. Fix an integer $\ell \geq 3$. Let (A, B, A', B', I, w, w') be a proper ℓ-partition of a twinless odd ℓ-template G, and let G^* be a proper blowup of G. Assume without loss of generality that $G[A]$ contains at least two isolated vertices. Let $S = \{v_1, \dots, v_{|S|}\}$ be the set of all isolated vertices of $G[A]$ and let $S' = \{v'_1, \dots, v'_{|S|}\}$. Let I_S be the set of all vertices from I that belong to a principal path of G whose ends are in $S \cup S'$. We divide the proof into two cases depending on whether $S = A$.

Case 1: $S \neq A$.
Thus $|A \setminus S| \geq 2$. Let $B_S = \{x \in B : H_x \cap S \neq \emptyset\}$ and $B_S^* = \{x \in B_S : H_x \cap (A \setminus S) \neq \emptyset\}$. Set $X_1 = \mathcal{K}(S \cup S' \cup B_S \cup I_S)$, $X_2 = V(G^*) \setminus X_1$, $A_1 = \mathcal{K}(B_S^*)$, $A_2 = \mathcal{K}((A \setminus S) \cup (B \setminus B_S))$, $B_1 = \mathcal{K}(S')$ and $B_2 = \mathcal{K}(B' \cup (A' \setminus S'))$. We claim that (X_1, X_2) is a special 2-join of type 1 of G^* with split $(X_1, X_2, A_1, A_2, B_1, B_2)$.

For each $i \in \{1, 2\}$, it is clear that A_i and B_i are nonempty disjoint subsets of X_i. It is also clear that (X_1, X_2) is a partition of $V(G^*)$ with $|X_1| \geq 3$, $|X_2| \geq 3$ and $X_1 \setminus (A_1 \cup B_1) \neq \emptyset$. We now prove the following claims.

(1) The only edges between X_1 and X_2 are those between A_1 and A_2 and those between B_1 and B_2.

Proof of (1): Observe that every edge between X_1 and X_2 is an edge of either $G^*[\mathcal{K}(A \cup B)]$ or $G^*[\mathcal{K}(A' \cup B')]$. Since $B_1 \cup B_2 = \mathcal{K}(A' \cup B')$, clearly every edge between X_1 and X_2 in $G^*[\mathcal{K}(A' \cup B')]$ is an edge between B_1 and B_2, so it remains to prove the analogous statement for $G^*[\mathcal{K}(A \cup B)]$. By condition (c) in the definition of a blowup, it suffices to prove that $S \cup (B_S \setminus B_S^*)$ is anticomplete to $(A \setminus S) \cup (B \setminus B_S)$. Since S consists of isolated vertices of $G[A]$, S is anticomplete to $A \setminus S$. Suppose there exist adjacent vertices $s \in S$ and $x \in B \setminus B_S$. By the definition of templates, $s \in N[H_x]$. Since $H_x \cap S = \emptyset$, we have that $s \notin H_x$ and hence $s \in N(H_x)$, contrary to s being an isolated vertex of $G[A]$. So S is anticomplete to $B \setminus B_S$. Suppose that there exist adjacent vertices $x \in B_S \setminus B_S^*$ and $a \in A \setminus S$. Then, by the definition of templates, $a \in N[H_x]$. Since $H_x \cap (A \setminus S) = \emptyset$, $a \in N(H_x)$ and hence $a \in N(S)$, contradicting the fact that S consists of isolated vertices of $G[A]$. So $B_S \setminus B_S^*$ is anticomplete to $A \setminus S$. Finally, suppose that there exist adjacent vertices $x \in B_S \setminus B_S^*$ and $y \in B \setminus B_S$. Then, by the definition of templates, $H_x \cap H_y \neq \emptyset$. But $H_x \subseteq S$ and $H_y \subseteq A \setminus S$, a contradiction. This proves (1).

(2) For each $i \in \{1,2\}$, $G^*[X_i]$ contains a path with one end in A_i, the other end in B_i, and interior in $X_i \setminus (A_i \cup B_i)$. Furthermore, if $|A_i| = |B_i| = 1$, then $G^*[X_i]$ is not a path.

Proof of (2): Since $|S| \geq 2$ and $|A \setminus S| \geq 2$, neither of $G^*[X_1]$ and $G^*[X_2]$ is a path. For every $x \in B_S^*$ and $s \in H_x \cap S$, the vertex x together with the principal path of G that contains s forms a path from A_1 to B_1 with interior in $X_1 \setminus (A_1 \cup B_1)$. For every $a \in A \setminus S$, the principal path of G that contains a is a path from A_2 to B_2 with interior in $X_2 \setminus (A_2 \cup B_2)$. This proves (2).

(3) A_1 is complete to A_2.

Proof of (3): Fix $x \in B_S^*$. By the definition of B_S^*, the set H_x contains a vertex u that is isolated in $G[A]$ and a vertex v that is not isolated in $G[A]$. Since H_x is a module of $G[A]$, the component C of $G[A]$ that contains v is contained in $G[H_x]$. As $G[A \setminus S]$ is a threshold graph that has no isolated vertex, $C = G[A \setminus S]$, and hence $A \setminus S \subseteq H_x$. It now follows immediately from the definition of a template that x is complete to $(A \setminus S) \cup (B \setminus B_S)$, and hence B_S^* is complete to $(A \setminus S) \cup (B \setminus B_S)$. Since $A \setminus S \subseteq H_x$, and in particular since H_x contains a universal vertex of $G[A \setminus S]$, it follows that no vertex of $H_x \setminus S$ is isolated in $G[H_x]$ and hence there exists no optional edge ux of G, where $u \in A \setminus S$ and $x \in B_S^*$. Therefore, by the definition of a blowup of a template, $\mathcal{K}(B_S^*) = A_1$ is complete to $\mathcal{K}((A \setminus S) \cup (B \setminus B_S)) = A_2$. This proves (3).

(4) B_1 is complete to B_2.

Proof of (4): Since S consists of isolated vertices of $G[A]$, S' consists of universal vertices of $G[A']$, and hence S' is complete to $A' \setminus S'$. Together with the fact that S' is a clique, this implies that $S' \subseteq N_{A'}[H'_x]$ for every $x \in B'$, and hence S' is complete to B'. Since S' consists of universal vertices of $G[A']$, no vertex of S' is isolated in any induced subgraph of $G[A']$ on at least 2 vertices, and hence G has no optional edge ux, where $u \in S'$ and $x \in B'$. Therefore, by the definition of a blowup of a template, $\mathcal{K}(S') = B_1$ is complete to $\mathcal{K}(B' \cup (A' \setminus S')) = B_2$. This proves (4).

(5) A_1 and B_1 are cliques, and $G^*[A_2]$ contains a universal vertex.

Proof of (5): In the proof of (3) it was shown that $A \setminus S \subseteq H_x$ for every $x \in B_S^*$. It thus follows from the definition of a template that B_S^* is a clique. Since S is a stable set, S' is a clique. It now follows from the definition of a blowup of a template that A_1

and B_1 are cliques. Let u be a universal vertex of $G[A \setminus S]$. Clearly $u \in N_A[H_x]$ for every $x \in B \setminus B_S$, and hence u is complete to $B \setminus B_S$. Since u is not an isolated vertex of any induced subgraph of $G[A \setminus S]$ on at least 2 vertices, no edge ux of G is optional, where $x \in B \setminus B_S$. It now follows from the definition of a blowup of a template that u is a universal vertex of $G^*[A_2]$. This proves (5).

It now follows from (1)–(5) that (X_1, X_2) is a special 2-join of type 1 of G^* with split $(X_1, X_2, A_1, A_2, B_1, B_2)$.

Case 2: $S = A$.
So A is a stable set and A' is a clique, and we may assume that G^* is not a pyramid. Note that $B' = \emptyset$ since A' is a clique.

Case 2.1: $\mathcal{K}(A \cup A' \cup I) = A \cup A' \cup I$.
Suppose that $|B| = 1$. Then $\mathcal{K}(B) \neq B$ or $|A| \geq 4$; let A^* be a subset of A of size 1 in the former case and of size 2 in the latter case (and making this choice arbitrarily if both cases hold). Let P^* be the set of all vertices belonging to a principal path of G that contains a vertex from A^*. Set $X_1 = \mathcal{K}(B) \cup P^*$, $X_2 = V(G^*) \setminus X_1$, $A_1 = \mathcal{K}(B)$, $A_2 = A \setminus A^*$, $B_1 = P^* \cap A'$ and $B_2 = A' \setminus B_1$. Clearly (X_1, X_2) is a frame of G^* with split $(X_1, X_2, A_1, A_2, B_1, B_2)$; in particular, the condition requiring that $G[X_1]$ and $G[X_2]$ are not paths (when $|A_1| = |B_1| = 1$ or $|A_2| = |B_2| = 1$ respectively) holds by our choice of A^*. Since $|B| = 1$ and $G[A]$ contains isolated vertices, the unique vertex of B is complete to A and hence, by the definition of a blowup of a template, A_1 is complete to A_2. Since A' is a clique, B_1 is complete to B_2, and any vertex of B_2 is a universal vertex of $G[B_2]$. Finally, clearly $X_1 \setminus (A_1 \cup B_1) \neq \emptyset$. It follows that (X_1, X_2) is a special 2-join of type 1 of G^* with split $(X_1, X_2, A_1, A_2, B_1, B_2)$.

So we may assume that $|B| \geq 2$. Fix $x \in B$ such that H_x is inclusion-wise minimal, i.e. $H_x \subset H_y$ for every $y \in B \setminus \{x\}$ with $H_x \cap H_y \neq \emptyset$ (note that the inclusion is strict since G is twinless). Let $B^* = \{y \in B : H_x \cap H_y \neq \emptyset\}$ and set $X_1 = \bigcup\{V(P) : P \text{ is a principal path of } G \text{ with } V(P) \cap H_x \neq \emptyset\}$, $X_2 = V(G^*) \setminus X_1$, $A_1 = H_x$, $A_2 = \mathcal{K}(B^*)$, $B_1 = \{a' \in A' : a \in H_x\}$ and $B_2 = A' \setminus B_1$. Clearly $X_1 \setminus (A_1 \cup B_1) \neq \emptyset$. By our choice of B^* and since A is a stable set, H_x is anticomplete to $\mathcal{K}(B \setminus B^*)$, and hence every edge between X_1 and X_2 is an edge between A_1 and A_2 or between B_1 and B_2. It is now easily seen that (X_1, X_2) is a frame of G^* with split $(X_1, X_2, A_1, A_2, B_1, B_2)$. Since A' is a clique, B_2 is a clique that is complete to B_1. By the minimality of H_x, we have that $H_x \subseteq H_y$ for every $y \in B^*$, and hence B^* is a clique that is complete to H_x. It follows from the definition of a blowup of a template that A_1 is complete to A_2. Finally, since A' is a clique, any vertex of B_1 is a universal vertex of $G[B_1]$. It now follows that (X_1, X_2) is a special 2-join of type 1 of G^* with split $(X_1, X_2, A_1, A_2, B_1, B_2)$.

Case 2.2: $\mathcal{K}(A \cup A' \cup I) \neq A \cup A' \cup I$.
Fix $a \in A$ such that the principal path $P = a, \ldots, a'$ of G that contains a satisfies $\mathcal{K}(V(P)) \neq V(P)$. Set $X_1 = V(G^*) \setminus \mathcal{K}(V(P))$, $X_2 = \mathcal{K}(V(P))$, $A_1 = \mathcal{K}(\{x \in B : a \in H_x\}$, $A_2 = K_a$, $B_1 = \mathcal{K}(A' \setminus \{a'\})$ and $B_2 = K_{a'}$. Since every vertex of A is isolated in $G[A]$, it follows that if $u \in A$ and $x \in B$ are adjacent, then $u \in H_x$. Thus by our choice of A_1 we have that A_2 is anticomplete to $\mathcal{K}(B) \setminus A_1$, and hence every edge between X_1 and X_2 is between A_1 and A_2 or between B_1 and B_2. It is now easily checked that (X_1, X_2) is a frame of G^* with split $(X_1, X_2, A_1, A_2, B_1, B_2)$. Since $a \in H_x$ for every $x \in A_1 \cap V(G)$, it follows that $A_1 \cap V(G)$ is a clique, and hence by the definition of a blowup of a template A_1 is a clique. Furthermore, clearly A_2, B_1 and B_2 are cliques. By the definition of a blowup of a template, a is complete to A_1, and hence some vertex of A_2 is complete to A_1. By condition (h) in the definition of a blowup of a template, some vertex of B is a universal vertex of $G^*[\mathcal{K}(A \cup B)]$, and hence some vertex of A_1 is complete to A_2. Let z be the unique neighbour of a in P. By the definition of a blowup of a template, z is complete to K_a, and hence some vertex of $X_2 \setminus A_2$ is complete to A_2.

That A_2 and $A_1 \cup \{v\}$ are nested for every vertex $v \in N(A_2) \setminus A_1$ follows from condition (b) in the definition of a blowup of a template. So to prove that (X_1, X_2) is a special 2-join of type 2 of G^*, it remains to prove that A_1 and $A_2 \cup \{v\}$ are nested for every vertex $v \in N(A_1) \setminus A_2$. Suppose otherwise. Then there exist four vertices, $x, y \in A_1$ and $u, v \in A_2 \cup N(A_1)$, such that ux and vy are edges but uy and vx are not. Since G^* is C_4-free, u and v are nonadjacent. Observe that $A_2 \cup N(A_1) \subseteq \mathcal{K}(A \cup B)$, and hence $\{u, v\} \subseteq \mathcal{K}(A \cup B)$. Moreover, at most one of u and v belongs to $\mathcal{K}(B)$. By symmetry, it suffices to consider the following two cases.

Case 2.2.1: $u, v \in \mathcal{K}(A)$.
Since u and v are nonadjacent and K_w is a clique for each $w \in A$, it follows that there exist two unique vertices u_0 and v_0 of G such that $u \in K_{u_0}$ and $v \in K_{v_0}$. Let P_u and P_v be the principal paths of G that contain u_0 and v_0 respectively. But now $\{u, v, x, y\} \cup (V(P_u) \cup V(P_v)) \setminus \{u_0, v_0\}$ induces a hole of length $2\ell + 2$, contradicting Lemma 3.24.

Case 2.2.2: $u \in \mathcal{K}(A)$ and $v \in \mathcal{K}(B)$.
Fix $w \in N_{G^*}(v) \cap A$. Since x and v are nonadjacent, it follows from the fact that S

is a stable set together with the definitions of a template and a blowup of a template that x and w are nonadjacent. By Case 1, w and y are nonadjacent. Let u_0 and w_0 be the unique vertices of G such that $u \in K_{u_0}$ and $w \in K_{w_0}$, and let P_u and P_w be the principal paths of G that contain u_0 and w_0 respectively. But now $\{u, v, w, x, y\} \cup (V(P_u) \cup V(P_w)) \setminus \{u_0, w_0\}$ induces a hole of length $2\ell + 3$, contradicting Lemma 5.5 in [19].

It follows that (X_1, X_2) is a special 2-join of type 2 of G^* with split $(X_1, X_2, A_1, A_2, B_1, B_2)$.

□

Theorem 3.33. *Let $\ell \geq 7$ be an odd integer. Then every ℓ-holed graph is a ring of length ℓ or a pyramid whose three paths are all of length $(\ell - 1)/2$, or has a universal vertex, a clique cutset or a special 2-join.*

Proof. Let $\ell \geq 7$ be odd and let G be an ℓ-holed graph. By Theorem 3.5, we may assume that G is a proper blowup of a twinless odd template. By Lemma 3.32, G is a pyramid or admits a special 2-join. If G is a pyramid, then clearly each of its three paths must be of length $(\ell - 1)/2$. □

3.4.4 Detecting special 2-joins

In this section we give algorithms for finding a special 2-join in a graph, if one exists. We use these algorithms in Section 3.5.3 to recognise ℓ-holed graphs (for odd $\ell \geq 7$) using a process of clique cutset and special 2-join decomposition.

Special 2-joins of type 1

A *1-configuration* for a graph G is a tuple (a_1, a_2, b_1, b_2, u) of five vertices such that:

- a_1a_2 and b_1b_2 are edges of G;
- a_1b_2 and a_2b_1 are not edges of G; and
- u is nonadjacent to both a_2 and b_2.

A special 2-join (X_1, X_2) of type 1 is *compatible* with a 1-configuration (a_1, a_2, b_1, b_2, u) if there exists a split $(X_1, X_2, A_1, A_2, B_1, B_2)$ of (X_1, X_2) such that $a_1 \in A_1$, $a_2 \in A_2$, $b_1 \in B_1$, $b_2 \in B_2$, $u \in X_1 \setminus (A_1 \cup B_1)$, a_2 is complete to $A_2 \setminus \{a_2\}$, and A_1 and B_1 are cliques.

The following algorithm is an adaptation of the one given in [18].

Lemma 3.34. *There exists an algorithm with the following specifications:*

INPUT: *A graph G that has no clique cutset, and a 1-configuration (a_1, a_2, b_1, b_2, u).*

OUTPUT: *"Yes", together with a special 2-join (X_1, X_2) of type 1 with split $(X_1, X_2, A_1, A_2, B_1, B_2)$ compatible with (a_1, a_2, b_1, b_2, u), if there is one, and "No" otherwise.*

RUNNING TIME: $\mathcal{O}(|V(G)|^4)$.

Proof. Consider the following algorithm.

Step 1. Initialise the following sets:

- $X_1 = \{a_1, b_1, u\}$,
- $X_2 = V(G) \setminus X_1$,
- $A_1 = N(a_2) \cap X_1$,
- $B_1 = N(b_2) \cap X_1$,
- $S_1 = X_1 \setminus (A_1 \cup B_1)$,
- $A_2 = \{x \in X_2 : N(x) \cap A_1 \neq \emptyset\}$,
- $B_2 = \{x \in X_2 : N(x) \cap B_1 \neq \emptyset\}$, and
- $S_2 = X_2 \setminus (A_2 \cup B_2)$.

Step 2. Repeatedly apply the following rules until no more rules can be applied. If at any point a vertex x that is adjacent to both a_2 and b_2 is moved from X_2 to X_1, then output "No" and terminate. After each application of a rule, update sets A_1, B_1 and S_1 by first setting $A_1 = N(a_2) \cap X_1$ and $B_1 = N(b_2) \cap X_1$, and then setting $S_1 = X_1 \setminus (A_1 \cup B_1)$. Similarly, update sets A_2, B_2 and S_2 by first setting $A_2 = \{x \in X_2 : N(x) \cap A_1 \neq \emptyset\}$ and $B_2 = \{x \in X_2 : N(x) \cap B_1 \neq \emptyset\}$, and then setting $S_2 = X_2 \setminus (A_2 \cup B_2)$. When it is no longer possible to apply any rule, proceed to Step 3.

Rule 1. If $x \in X_2$ has a neighbour in S_1, then move x from X_2 to X_1.

Rule 2. If $x \in A_2$ and $N(x) \cap (A_1 \cup B_1) \neq A_1$, then move x from X_2 to X_1.

Rule 3. If $x \in B_2$ and $N(x) \cap (A_1 \cup B_1) \neq B_1$, then move x from X_2 to X_1.

Rule 4. If $x \in A_2 \setminus \{a_2\}$ and x is nonadjacent to a_2, then move x from X_2 to X_1.

Step 3. Perform the following checks in order (thus if, say, Check 2 is peformed, then the algorithm did not terminate as a result of Check 1).

Check 1. If not both A_1 and B_1 are cliques, then output "No" and terminate.

Check 2. If $|X_2| = 2$, then output "No" and terminate.

Check 3. If $|A_2| = |B_2| = 1$ and $G[X_2]$ is a path, then output "No" and terminate.

Check 4. If $|A_1| \geq 2$ or $|B_1| \geq 2$, or if $|A_1| = |B_1| = 1$ but $G[X_1]$ is not a path, then output "Yes" together with (X_1, X_2) and $(X_1, X_2, A_1, A_2, B_1, B_2)$, and terminate.

Check 5. (Observe that, as a result of Check 4, $|A_1| = |B_1| = 1$ and $G[X_1]$ is a path.) For each $w \in X_2 \setminus \{a_2, b_2\}$, repeat Step 2 and Checks 1–4 of Step 3 but with:

- $X_1 = \{a_1, b_1, u, w\}$,
- $X_2 = V(G) \setminus X_1$,
- $A_1 = N(a_2) \cap X_1$,
- $B_1 = N(b_2) \cap X_1$,
- $S_1 = X_1 \setminus (A_1 \cup B_1)$,
- $A_2 = \{x \in X_2 : N(x) \cap A_1 \neq \emptyset\}$,
- $B_2 = \{x \in X_2 : N(x) \cap B_1 \neq \emptyset\}$, and
- $S_2 = X_2 \setminus (A_2 \cup B_2)$.

If the output is "No" for every $w \in X_2 \setminus \{a_2, b_2\}$, then output "No" and terminate.

We now prove that this algorithm is correct by way of the following two claims.

(1) If the algorithm outputs "Yes" together with (X_1, X_2) and $(X_1, X_2, A_1, A_2, B_1, B_2)$, then (X_1, X_2) is a special 2-join of type 1 of G that is compatible with (a_1, a_2, b_1, b_2, u).

Proof of (1): Suppose that the algorithm outputs "Yes" together with (X_1, X_2) and $(X_1, X_2, A_1, A_2, B_1, B_2)$. Clearly (X_1, X_2) is a partition of $V(G)$ such that $A_i \cup B_i \subseteq X_i$ for each $i \in \{1, 2\}$. By definition, $a_1 \in A_1$, $a_2 \in A_2$, $b_1 \in B_1$ and $b_2 \in B_2$. Therefore sets A_1, A_2, B_1 and B_2 are nonempty, and they are pairwise disjoint by Rules 2, 3 and the check performed during Step 2 regarding vertices that are adjacent to both a_2 and b_2. By Rules 1–3 of Step 2, every edge of G between X_1 and X_2 is either between A_1

and A_2 or between B_1 and B_2. By Rules 2 and 3, A_1 is complete to A_2 and B_1 is complete to B_2. By Rule 4, a_2 is a universal vertex of $G[A_2]$. Since u is nonadjacent to both a_2 and b_2, it follows by the definition of sets A_1 and B_1 that $u \in X_1 \setminus (A_1 \cup B_1)$, and therefore $X_1 \setminus (A_1 \cup B_1) \neq \emptyset$. By Check 1, A_1 and B_1 are cliques. Since $\{a_1, b_1, u\} \subseteq X_1$, we have that $|X_1| \geq 3$, and it follows from Check 2 that $|X_2| \geq 3$.

It remains to prove that for each $i \in \{1, 2\}$ there exists a path from A_i to B_i whose interior lies in $X_i \setminus (A_i \cup B_i)$. Observe that if there is a path from A_i to B_i in $G[X_i]$, then any shortest path from A_i to B_i in $G[X_i]$ has its interior in $X_i \setminus (A_i \cup B_i)$, so it suffices to prove that there is a path from A_i to B_i in $G[X_i]$. If there is no path from A_1 to B_1 in $G[X_1]$, then up to symmetry there is no path from u to A_1 in $G[X_1]$, and hence B_1 is a clique cutset of G that separates u from vertices of X_2, a contradiction. So there is a path from A_1 to B_1. If there is no path from A_2 to B_2 in $G[X_2]$, then A_1 is a clique cutset of G that separates vertices of A_2 from vertices of $X_1 \setminus A_1$, a contradiction. Therefore there is a path from A_2 to B_2 in $G[X_2]$. It follows that (X_1, X_2) is a special 2-join of type 1 of G that is compatible with (a_1, a_2, b_1, b_2, u). This proves (1).

(2) If the algorithm outputs "No", then there exists no special 2-join of type 1 of G that is compatible with (a_1, a_2, b_1, b_2, u).

Proof of (2): Towards a contradiction, suppose that the algorithm outputs "No" but there exists a special 2-join (X_1', X_2') of type 1 of G that is compatible with (a_1, a_2, b_1, b_2, u). Let $(X_1', X_2', A_1', A_2', B_1', B_2')$ be a split of (X_1', X_2') such that $a_1 \in A_1'$, $a_2 \in A_2'$, $b_1 \in B_1'$, $b_2 \in B_2'$ and $u \in X_1' \setminus (A_1' \cup B_1')$. Let sets X_1, X_2, A_1, A_2, B_1 and B_2 be as they were immediately before the algorithm terminated. Since all applications of Rules 1–4 were necessary, it follows that $X_1 \subseteq X_1'$, and in particular $A_1 \subseteq A_1'$ and $B_1 \subseteq B_1'$.

The algorithm outputted "No" and terminated either during Step 2, or as a result of Check 1, 2, 3 or 5. Let us consider each possibility in turn. Suppose that the algorithm terminated during Step 2. Then some vertex $x \in X_1$ is adjacent to both a_2 and b_2. But $x \in X_1'$ and therefore x belongs to both A_1' and B_1', contradicting the fact that A_1' and B_1' are disjoint. Suppose that the algorithm terminated as a result of Check 1. Then not both A_1 and B_1 are cliques, and since $A_1 \subseteq A_1'$ and $B_1 \subseteq B_1'$, it follows that not both A_1' and B_1' are cliques, a contradiction. Suppose that the algorithm terminated as a result of Check 2. Then $|X_2| = 2$, and since $X_1 \subseteq X_1'$ it follows that $|X_2'| = 2$, a contradiction. Suppose that the algorithm terminated as a result of Check 3, i.e. $|A_2| = |B_2| = 1$ and $G[X_2]$ is a path. Since $X_1 \subseteq X_1'$, it follows that $X_2' \subseteq X_2$. If

$X_2' = X_2$, then $|A_2'| = |B_2'| = 1$ and $G[X_2']$ is a path, and if $X_2' \neq X_2$ then $G[X_2']$ contains no path from A_2' to B_2'; in either case we obtain a contradiction. Finally, suppose that the algorithm terminates as a result of Check 5. So $|A_1| = |B_1| = 1$ and $G[X_1]$ is a path. Since it is not the case that $|A_1'| = |B_1'| = 1$ and $G[X_1']$ is a path, there exists some vertex $w \in X_1' \setminus X_1$. As part of Check 5, the algorithm checked whether there exists a special 2-join (X_1'', X_2'') of type 1 of G compatible with (a_1, a_2, b_1, b_2, u) such that $\{a_1, b_1, u, w\} \subseteq X_1''$ and $\{a_2, b_2\} \subseteq X_2''$. By what we have proved so far, it was correctly determined that no such special 2-join of type 1 exists, contradicting the existence of (X_1', X_2'). This proves (2).

Finally, we prove that the algorithm has time complexity $\mathcal{O}(|V(G)|^4)$. To perform Step 2, one must find a vertex for which some rule applies, and there are at most $\mathcal{O}(|V(G)|)$ such vertices. Given a vertex x, testing whether any one of Rules 1 to 4 applies and executing it takes $\mathcal{O}(|V(G)|)$ time. It follows that executing Step 2 takes $\mathcal{O}(|V(G)|^3)$ time. In Step 3, Check 1 takes $\mathcal{O}(|V(G)|^2)$ time, Check 2 takes $\mathcal{O}(1)$ time and Checks 3 and 4 each take $\mathcal{O}(|V(G)| + |E(G)|) = \mathcal{O}(|V(G)|^2)$ time. Check 5 involves executing $\mathcal{O}(|V(G)|)$ times Step 2 and Checks 1 to 4 of Step 3, and thus takes $\mathcal{O}(|V(G)|^4)$ time. It follows that the running time of this algorithm is $\mathcal{O}(|V(G)|^4)$. □

A *universal set of type 1* for a graph G is a set U of 1-configurations such that for every special 2-join (X_1, X_2) of type 1 of G, some 1-configuration from U is compatible with (X_1, X_2).

Lemma 3.35. *There exists an algorithm with the following specifications:*

INPUT: *A graph G that has no clique cutset, and a universal set U of type 1 for G.*

OUTPUT: *"Yes", together with a special 2-join (X_1, X_2) of type 1 with split $(X_1, X_2, A_1, A_2, B_1, B_2)$ compatible with (a_1, a_2, b_1, b_2, u), if there is one, and "No" otherwise.*

RUNNING TIME: *$\mathcal{O}(|V(G)|^4 \cdot |U|)$.*

Proof. For each 1-configuration $z \in U$, apply the algorithm of Lemma 3.34 with graph G and 1-configuration z as input. □

By enumerating all 5-tuples of a graph G, one can construct a universal set of size $\mathcal{O}(|V(G)|^5)$.

Lemma 3.36. *Let G be a graph. Then one can compute in $\mathcal{O}(n^6)$ time a universal set of type 1 of size $\mathcal{O}(n^5)$ for G.*

Proof. Let $\mathcal{U} = \emptyset$. Enumerate all 5-tuples (a_1, a_2, b_1, b_2, u) of vertices of G, and check whether (a_1, a_2, b_1, b_2, u) is a 1-configuration, adding it to $\mathcal{U}$ if it is. After this process completes, $\mathcal{U}$ is a universal set of type 1 of size $\mathcal{O}(n^5)$ for G. Since checking whether a 5-tuple is a 1-configuration consists of checking a constant number of times whether two vertices are adjacent or nonadjacent, the set $\mathcal{U}$ can be computed in $\mathcal{O}(n^6)$ time. □

Lemma 3.37. *There exists an algorithm with the following specifications:*

INPUT: *A graph G that has no clique cutset.*

OUTPUT: *"Yes", together with a special 2-join (X_1, X_2) of type 1 with split $(X_1, X_2, A_1, A_2, B_1, B_2)$, if there is one, and "No" otherwise.*

RUNNING TIME: $\mathcal{O}(|V(G)|^9)$.

Proof. Follows immediately from Lemmas 3.35 and 3.36. □

Special 2-joins of type 2

A *2-configuration* for a graph G is a tuple $(a_1, a_2, b_1, b_2, u, v)$ of six vertices such that:

- a_1a_2, b_1b_2 and va_2 are edges of G;
- a_1b_2, a_2b_1, va_1, vb_1 and uv are not edges of G; and
- u is adjacent to at most one of a_2 and b_2.

A special 2-join (X_1, X_2) of type 2 is *compatible* with a 2-configuration $(a_1, a_2, b_1, b_2, u, v)$ if there exists a split $(X_1, X_2, A_1, A_2, B_1, B_2)$ of (X_1, X_2) such that:

- $a_1 \in A_1$, $a_2 \in A_2$, $b_1 \in B_1$, $b_2 \in B_2$, $u \in X_1$ and $v \in X_2 \setminus A_2$; and
- a_1 is complete to A_2, a_2 is complete to A_1 and v is complete to A_2.

Lemma 3.38. *There exists an algorithm with the following specifications:*

INPUT: *A C_4-free graph G that has no clique cutset, and a 2-configuration $(a_1, a_2, b_1, b_2, u, v)$.*

Output: *"Yes", together with a special 2-join* (X_1, X_2) *of type 2 with split* $(X_1, X_2, A_1, A_2, B_1, B_2)$ *compatible with* $(a_1, a_2, b_1, b_2, u, v)$*, if there is one, and "No" otherwise.*

Running time: $\mathcal{O}(|V(G)|^6)$.

Proof. Consider the following algorithm.

Step 1. Initialise the following sets:

- $X_1 = \{a_1, b_1, u\}$,
- $X_2 = V(G) \setminus X_1$,
- $A_1 = N(a_2) \cap X_1$,
- $B_1 = N(b_2) \cap X_1$,
- $S_1 = X_1 \setminus (A_1 \cup B_1)$,
- $A_2 = \{x \in X_2 : N(x) \cap A_1 \neq \emptyset\}$,
- $B_2 = \{x \in X_2 : N(x) \cap B_1 \neq \emptyset\}$, and
- $S_2 = X_2 \setminus (A_2 \cup B_2)$.

Step 2. Repeatedly apply the following rules until no more rules can be applied. If at any point a vertex x that is adjacent to both a_2 and b_2 is moved from X_2 to X_1, then output "No" and terminate. After each application of a rule, update sets A_1, B_1 and S_1 by first setting $A_1 = N(a_2) \cap X_1$ and $B_1 = N(b_2) \cap X_1$, and then setting $S_1 = X_1 \setminus (A_1 \cup B_1)$. Similarly, update sets A_2, B_2 and S_2 by first setting $A_2 = \{x \in X_2 : N(x) \cap A_1 \neq \emptyset\}$ and $B_2 = \{x \in X_2 : N(x) \cap B_1 \neq \emptyset\}$, and then setting $S_2 = X_2 \setminus (A_2 \cup B_2)$. When it is no longer possible to apply any rule, proceed to Step 3.

Rule 1. If $x \in (A_2 \cap B_2) \setminus \{a_2, b_2\}$, then move x from X_2 to X_1.

Rule 2. If $x \in X_2 \setminus \{v\}$ has a neighbour in S_1, then move x from X_2 to X_1.

Rule 3. If $x \in B_2$ is not complete to B_1, then move x from X_2 to X_1.

Rule 4. If $x \in A_2 \setminus \{a_2\}$ is not adjacent to both a_1 and a_2, then move x from X_2 to X_1.

Rule 5. If $x \in A_2$ is not adjacent to v, then move x from X_2 to X_1.

Rule 6. If $x \in B_2 \setminus \{b_2\}$ is not adjacent to b_2, then move x from X_2 to X_1.

Rule 7. If there exist vertices $z \in A_1$, $x, y \in A_2 \setminus \{a_2\}$ and $w \in X_2 \setminus A_2$ such that $zx, wy, xa_2, ya_2 \in E(G)$ and $zy, xw \notin E(G)$, then move x from X_2 to X_1.

Step 3. Perform the following checks in order (thus if, say, Check 2 is peformed, then the algorithm did not terminate as a result of Check 1).

Check 1. If not all of A_1, A_2, B_1 and B_2 are cliques, then output "No" and terminate.

Check 2. If there exists some $w \in X_1 \setminus A_1$ such that A_1 and $A_2 \cup \{w\}$ are not nested, then output "No" and terminate.

Check 3. If $|X_2| = 2$, then output "No" and terminate.

Check 4. If $|A_2| = |B_2| = 1$ and $G[X_2]$ is a path, then output "No" and terminate.

Check 5. If $|A_1| \geq 2$ or $|B_1| \geq 2$, or if $|A_1| = |B_1| = 1$ but $G[X_1]$ is not a path, then output "Yes" together with (X_1, X_2) and $(X_1, X_2, A_1, A_2, B_1, B_2)$, and terminate.

Check 6. (Observe that, as a result of Check 5, $|A_1| = |B_1| = 1$ and $G[X_1]$ is a path.) For each $w \in X_2 \setminus \{a_2, b_2\}$, repeat Step 2 and Checks 1–5 of Step 3 but with:

- $X_1 = \{a_1, b_1, u, w\}$,
- $X_2 = V(G) \setminus X_1$,
- $A_1 = N(a_2) \cap X_1$,
- $B_1 = N(b_2) \cap X_1$,
- $S_1 = X_1 \setminus (A_1 \cup B_1)$,
- $A_2 = \{x \in X_2 : N(x) \cap A_1 \neq \emptyset\}$,
- $B_2 = \{x \in X_2 : N(x) \cap B_1 \neq \emptyset\}$, and
- $S_2 = X_2 \setminus (A_2 \cup B_2)$.

If the output is "No" for every $w \in X_2 \setminus \{a_2, b_2\}$, then output "No" and terminate.

We now prove that this algorithm is correct by way of the following two claims.

(1) If the algorithm outputs "Yes" together with (X_1, X_2) and $(X_1, X_2, A_1, A_2, B_1, B_2)$, then (X_1, X_2) is a special 2-join of type 2 of G that is compatible with $(a_1, a_2, b_1, b_2, u, v)$.

Proof of (1): Suppose that the algorithm outputs "Yes" together with (X_1, X_2) and $(X_1, X_2, A_1, A_2, B_1, B_2)$. Clearly (X_1, X_2) is a partition of $V(G)$ such that $A_i \cup B_i \subseteq X_i$ for each $i \in \{1, 2\}$. By definition, $a_1 \in A_1$, $a_2 \in A_2$, $b_1 \in B_1$ and $b_2 \in B_2$. Therefore sets A_1, A_2, B_1 and B_2 are nonempty, and they are pairwise disjoint by Rule 1 and the check performed during Step 2 regarding vertices that are adjacent to both a_2 and b_2. By Rules 1 and 2 of Step 2, every edge of G between X_1 and X_2 is either between A_1 and A_2 or between B_1 and B_2. By Check 1, all of A_1, A_2, B_1 and B_2 are cliques. It now follows from G being C_4-free that A_1 and A_2 are nested, and by Rule 3 that B_1 is complete to B_2. By Rule 4, some vertex of A_1, namely a_1, is complete to A_2, and some vertex of A_2, namely a_2, is complete to A_1. By Rule 5, some vertex of $X_2 \setminus A_2$, namely v, is complete to A_2. Since the only edges between A_2 and X_1 are those between A_2 and A_1, it follows that $N_G(A_2) \subseteq A_1 \cup (X_2 \setminus A_2)$. Similarly, $N_G(A_1) \subseteq A_2 \cup (X_1 \setminus A_1)$. Thus, by Rule 7, sets A_2 and $A_1 \cup \{w\}$ are nested for every $w \in N(A_2) \setminus A_1$. By Check 2, sets A_1 and $A_2 \cup \{w\}$ are nested for every $w \in N(A_1) \setminus A_2$. Since $\{a_1, b_1, u\} \subseteq X_1$, we have that $|X_1| \geq 3$, and it follows from Check 3 that $|X_2| \geq 3$. By Check 4, it is not the case that $|A_2| = |B_2| = 1$ and $G[X_2]$ is a path, and by Check 5 it is not the case that $|A_1| = |B_1| = 1$ and $G[X_1]$ is a path.

It remains to prove that for each $i \in \{1, 2\}$ there exists a path from A_i to B_i whose interior lies in $X_i \setminus (A_i \cup B_i)$. Observe that if there is a path from A_i to B_i in $G[X_i]$, then any shortest path from A_i to B_i in $G[X_i]$ has its interior in $X_i \setminus (A_i \cup B_i)$, so it suffices to prove that there is a path from A_i to B_i in $G[X_i]$. Suppose that there is no path from A_1 to B_1 in $G[X_1]$. Then every path from a vertex in A_1 to a vertex in $X_2 \setminus A_2$ intersects A_2, and therefore A_2 is a clique cutset of G that separates A_1 from $X_2 \setminus A_2$, a contradiction. It follows by symmetry that for each $i \in \{1, 2\}$ there exists a path from A_i to B_i whose interior lies in $X_i \setminus (A_i \cup B_i)$. This proves (1).

(2) If the algorithm outputs "No", then there exists no special 2-join of type 2 of G that is compatible with $(a_1, a_2, b_1, b_2, u, v)$.

Proof of (2): Towards a contradiction, suppose that the algorithm outputs "No" but there exists a special 2-join (X'_1, X'_2) of type 2 of G that is compatible with $(a_1, a_2, b_1, b_2, u, v)$. Let $(X'_1, X'_2, A'_1, A'_2, B'_1, B'_2)$ be a split of (X'_1, X'_2) such that $a_1 \in A'_1$, $a_2 \in A'_2$, $b_1 \in B'_1$, $b_2 \in B'_2$, $u \in X'_1$, $v \in X'_2 \setminus A'_2$, and a_1 is complete to A'_2, a_2 is complete to A'_1 and v is complete to A'_2. Let sets X_1, X_2, A_1, A_2, B_1 and B_2 be as they were immediately before the algorithm terminated.

It is clear that applications of Rules 1 to 6 are necessary. Let us prove that applications of Rule 7 are necessary. Suppose that at some point during the execution of the algorithm there exist vertices $z \in A_1$, $x, y \in A_2$ and $w \in X_2 \setminus A_2$ such that $zx, wy, xa_2, ya_2 \in E(G)$ and $zy, wx \notin E(G)$. Since we want that A_2 and $A_1 \cup \{w\}$ are nested, at least one of x, y and w must be moved from X_2 to X_1. Suppose that y is moved from X_2 to X_1. Then y would belong to A_1 since it is adjacent to a_2. But then A_1 would contain two nonadjacent vertices z and y, contradicting the fact that A_1 must be a clique. So y must not be moved from X_2 to X_1. Suppose that w is moved from X_2 to X_1. Since z and w are nonadjacent, w cannot belong to A_1 and hence w must belong to $X_1 \setminus A_1$. But now the edge yw contradicts the fact that an edge between X_1 and A_2 should be between A_1 and A_2. So neither of w and y can be moved from X_2 to X_1, and therefore x must be moved from X_2 to X_1, i.e. Rule 7 must be applied. Now, since all applications of Rules 1 to 7 are necessary, it follows that $X_1 \subseteq X_1'$, and in particular $A_1 \subseteq A_1'$ and $B_1 \subseteq B_1'$. Furthermore, $X_2' \subseteq X_2$.

The algorithm outputted "No" and terminated either during Step 2, or as a result of Check 1, 2, 3, 4 or 6. Suppose that the algorithm terminated during Step 2. Then some vertex $x \in X_1$ that is adjacent to both a_2 and b_2 is moved from X_2 to X_1. But then $x \in X_1'$, and therefore x belongs to both A_1' and B_1', contradicting the fact that A_1' and B_1' are disjoint. Suppose that the algorithm terminated as a result of Check 1. Then at least one of A_1, A_2, B_1 and B_2 is not a clique. If A_1 or B_1 is not a clique, then A_1' or B_1' is not a clique, a contradiction. Suppose that A_2 is not a clique. Then for any two nonadjacent vertices $x, y \in A_2$, the set $\{x, y, a_1, v\}$ induces a C_4, a contradiction (since v is complete to A_2 by Rule 5). So A_2 is a clique, and therefore B_2 is not a clique. Let b and b' be two nonadjacent vertices from B_2. By Rules 3 and 6, $b_2 \notin \{b, b'\}$ and $\{b, b'\}$ is complete to $\{b_1, b_2\}$. But then $\{b, b'\} \subseteq B_1' \cup B_2'$, contradicting the fact that $G[B_1' \cup B_2']$ is a complete graph.

Suppose that the algorithm terminated as a result of Check 2. Then there exist vertices $w \in X_1 \setminus A_1$, $u, y \in A_1$ and $x \in A_2$ such that $ux, wy \in E(G)$ and $uw, xy \notin E(G)$. Since x and y are nonadjacent, $x \neq a_2$, and therefore x and a_2 are adjacent by Rule 4. Since A_1' and $A_2' \cup \{w\}$ are nested, $x \notin A_2'$, and therefore $x \in X_1'$. Since x is adjacent to a_2, it follows that $x \in A_1'$. But now x and y are two vertices of A_1' that are nonadjacent, a contradiction. Suppose that the algorithm terminated as a result of Check 3, i.e. $|X_2| = 2$. But $X_2' \subseteq X_2$ and hence $|X_2'| \leq 2$, a contradiction. Suppose that the algorithm terminated as a result of Check 4, i.e. $|A_2| = |B_2| = 1$ and $G[X_2]$ is a path. If $X_2' = X_2$, then $|A_2'| = |B_2'| = 1$ and $G[X_2']$ is a path, and if $X_2' \neq X_2$ then $G[X_2']$ contains no path from A_2' to B_2'; in either case we obtain a contradiction. Finally, suppose that the algorithm terminates as a result of Check 6. So $|A_1| = |B_1| = 1$ and

$G[X_1]$ is a path. Since it is not the case that $|A'_1| = |B'_1| = 1$ and $G[X'_1]$ is a path, there exists some vertex $w \in X'_1 \setminus X_1$. As part of Check 6, the algorithm checked whether there exists a special 2-join (X''_1, X''_2) of type 2 of G compatible with $(a_1, a_2, b_1, b_2, u, v)$ such that $\{a_1, b_1, u, w\} \subseteq X''_1$ and $\{a_2, b_2, v\} \subseteq X''_2$. By what we have proved so far, it was correctly determined that no such special 2-join of type 2 exists, contradicting the existence of (X'_1, X'_2). This proves (2).

Finally, we prove that the algorithm has time complexity $\mathcal{O}(|V(G)|^6)$. To perform Step 2, consider each vertex x of X_2 and check whether any of Rules 1 to 6 apply, and if so, perform the move operation described by the rule. If no such rule applies to x, then for each $u \in A_1$ check whether Rule 7 applies. If so, then perform the move operation described by the rule. Checking whether the given vertex x satisfies the conditions of any of Rules 1 to 6 takes $\mathcal{O}(|V(G)|)$ time. To check whether $u \in A_1$ satisfies the conditions of Rule 7, we must check whether there exists $x, y \in A_2 \setminus \{a_2\}$ and $w \in X_2 \setminus A_2$ such that $ux, wy, xa_2, ya_2 \in E(G)$ and $uy, xw \notin E(G)$. Since A_1 and A_2 are cliques and G is C_4-free, it cannot be the case that $\{u, x, y, w\} \subseteq A_1 \cup A_2$. Therefore it suffices to test whether for each $w \in X_2 \setminus A_2$ the graph $G[(A_2 \setminus \{a_2\}) \cup \{u, w\}]$ is P_4-free. Testing P_4-freeness can be done in $\mathcal{O}(|V(G)| + |E(G)|) = \mathcal{O}(|V(G)|^2)$ time (see [20]). Therefore, given $u \in A_2$, it takes $\mathcal{O}(|V(G)|^3)$ time to check whether Rule 7 applies. It follows that executing Step 2 takes $\mathcal{O}(|V(G)|^5)$ time. In Step 3, Check 1 takes $\mathcal{O}(|V(G)|^2)$ time, Check 2 takes $\mathcal{O}(|V(G)|^3)$ time (this follows from a similar argument to the one given above regarding the complexity of checking whether Rule 7 applies), Check 3 takes $\mathcal{O}(1)$ time and Checks 4 and 5 each take $\mathcal{O}(|V(G)|^2)$ time. Check 6 involves executing $\mathcal{O}(|V(G)|)$ times Step 2 and Checks 1–5 of Step 3, and thus takes $\mathcal{O}(|V(G)|^6)$ time. It follows that the algorithm has time complexity $\mathcal{O}(|V(G)|^6)$. □

A *universal set of type 2* for a graph G is a set U of 2-configurations such that for every special 2-join (X_1, X_2) of type 2 of G, some 2-configuration from U is compatible with (X_1, X_2).

Lemma 3.39. *There exists an algorithm with the following specifications:*

INPUT: *A C_4-free graph G that has no clique cutset, and a universal set U of type 2 for G.*

OUTPUT: *"Yes", together with a special 2-join (X_1, X_2) of type 2 with split $(X_1, X_2, A_1, A_2, B_1, B_2)$, if there is one, and "No" otherwise.*

RUNNING TIME: $\mathcal{O}(|V(G)|^6 \cdot |U|)$.

Proof. For each 2-configuration $z \in U$, apply the algorithm of Lemma 3.38 with graph G and 2-configuration z as input. □

By enumerating all 6-tuples of a graph G, one can construct a universal set of size $\mathcal{O}(|V(G)|^6)$.

Lemma 3.40. *Let G be a graph. Then one can compute in $\mathcal{O}(n^7)$ time a universal set of type 2 of size $\mathcal{O}(n^6)$ for G.*

Proof. Let $\mathcal{U} = \emptyset$. Enumerate all 6-tuples $(a_1, a_2, b_1, b_2, u, v)$ of vertices of G, and check whether $(a_1, a_2, b_1, b_2, u, v)$ is a 2-configuration, adding it to $\mathcal{U}$ if it is. After this process completes, $\mathcal{U}$ is a universal set of type 2 of size $\mathcal{O}(n^6)$ for G. Since checking whether a 6-tuple is a 2-configuration consists of checking a constant number of times whether two vertices are adjacent or nonadjacent, the set $\mathcal{U}$ can be computed in $\mathcal{O}(n^7)$ time. □

Lemma 3.41. *There exists an algorithm with the following specifications:*

INPUT: *A C_4-free graph G that has no clique cutset.*

OUTPUT: *"Yes", together with a special 2-join (X_1, X_2) of type 2 with split $(X_1, X_2, A_1, A_2, B_1, B_2)$, if there is one, and "No" otherwise.*

RUNNING TIME: $\mathcal{O}(|V(G)|^{12})$.

Proof. Follows immediately from Lemmas 3.39 and 3.40. □

3.5 Algorithms

3.5.1 Maximum clique and maximum stable set

Let G be a graph. A *weight function* for G is a function $w : V(G) \to \mathbb{N}$ that assigns a natural number to each vertex of G. A *maximum weight clique* (resp. *maximum weight stable set*) of G is a clique (resp. stable set) S of G such that $\sum_{v \in S} w(v)$ is maximum.

In this section we present polynomial time algorithms for solving the maximum weight clique and maximum weight stable set problems for ℓ-holed graphs where ℓ is odd and $\ell \geq 7$. These algorithms rely on the structure theorem presented in Section 3.2.

A vertex v of a graph G is a *chordal-vertex* if $G[N[v]]$ is chordal.

Lemma 3.42. *For every odd integer $\ell \geq 7$, every ℓ-holed graph contains a chordal-vertex.*

Proof. We prove the following claims.

(1) Rings and proper blowups of twinless odd ℓ-templates contain at least two nonadjacent chordal-vertices.

Proof of (1): Every vertex of a ring is a chordal-vertex, so clearly a ring has at least two nonadjacent chordal-vertices. Suppose G^* is a proper blowup of a twinless odd ℓ-template G, and let (A, B, A', B', I, w, w') be a proper ℓ-partition of G. Let P and P' be any two principal paths of G. It is easily seen that any vertex from the interior of P together with any vertex from the interior of P' forms a pair of nonadjacent chordal-vertices of G^*. This proves (1).

(2) Let G be a C_4-free graph that is not complete, and let U be the set of all universal vertices of G. Then every chordal-vertex of $G \setminus U$ is a chordal-vertex of G, and if some vertex of U is a chordal-vertex of G, then every vertex of G is a chordal-vertex.

Proof of (2): Since G contains no C_4, U is a clique that is complete to $V(G) \setminus U$. It follows that any chordal-vertex of $G \setminus U$ is a chordal-vertex of G. If some vertex of U is a chordal-vertex of G, then G is a chordal graph, and hence every vertex of G is a chordal-vertex. This proves (2).

Fix an odd integer $\ell \geq 7$ and an ℓ-holed graph G. By Theorem 3.5, G is a ring or a proper blowup of a twinless odd $(\ell - 1)/2$-template, or G has a universal vertex or a clique cutset. In the first two cases, we are done by (1). Clearly we may assume that G is not complete, and therefore by (2), G has a chordal-vertex if and only if the graph obtained from G by removing all universal vertices has a chordal-vertex. So it remains to consider the case where G has a clique cutset. Then by Lemma 2.16, G has an extreme clique cutset C. Let (A, B) be a partition of $V(G) \setminus C$ such that A is anticomplete to B, and suppose without loss of generality that $G[A \cup C]$ has no clique cutset. By (1) and (2), some vertex in A is a chordal-vertex of $G[A \cup C]$, and hence it is a chordal-vertex of G. $\square$

Lemma 3.43. *There exists an algorithm with the following specifications:*

INPUT: *An ℓ-holed graph G for some odd $\ell \geq 7$, and a weight function $w : V(G) \to \mathbb{N}$.*

OUTPUT: *A maximum weight clique of G.*

RUNNING TIME: $\mathcal{O}(|V(G)|^2 \cdot |E(G)|)$.

Proof. Consider the following algorithm.

Step 1. Find a chordal-vertex x_1 of G, and set $N_1 = G[N[x_1]]$ and $G_1 = G \setminus \{x_1\}$.

Step 2. For each $i \in \{2, \dots, |V(G)|\}$, find a chordal-vertex x_i of G_{i-1}, and set $N_i = G[N_{G_{i-1}}[x_i]]$ and $G_i = G_{i-1} \setminus \{x_i\}$. (So $G_i = G \setminus \{x_1, \dots, x_i\}$.)

Step 3. Let $\mathcal{K}$ be the set that consists of all maximal cliques of each of the graphs $N_1, \dots, N_{|V(G)|}$.

Step 4. Output any clique $K \in \mathcal{K}$ that maximises $\sum_{x \in K} w(x)$ and terminate.

We first prove that this algorithm is correct. That the chordal-vertices x_1 and x_i in Steps 1 and 2 respectively exist follows from Lemma 3.42. Clearly any maximal clique of G is a maximal clique of exactly one of the graphs $N_1, \dots, N_{|V(G)|}$, and hence the set $\mathcal{K}$ that is constructed in Step 3 consists of all maximal cliques of G. In particular, $\mathcal{K}$ contains all maximum weight cliques of G, and hence the clique that is outputted in Step 4 is one of maximum weight. Therefore the algorithm is correct.

Let us analyse the running time of this algorithm. Checking whether a graph with n vertices and m edges is chordal can be done in $\mathcal{O}(n + m)$ time (see [45]), and therefore finding a vertex of G whose neighbourhood induces a chordal graph can be done in time $\mathcal{O}(\sum_{v \in V(G)}(d_G(v)+|E(G)|)) = \mathcal{O}(|V(G)| \cdot |E(G)|)$. Therefore constructing the graphs $N_1, \dots, N_{|V(G)|}$ can be done in $\mathcal{O}(|V(G)|^2 \cdot |E(G)|)$ time. Generating all maximal cliques of a chordal graph on n vertices and m edges (of which there are $\mathcal{O}(n)$ many; see [27]) can be done in time $\mathcal{O}(n + m)$ (see [33]), and therefore constructing the set $\mathcal{K}$ of $\mathcal{O}(|V(G)|^2)$ many maximal cliques of G takes time $\mathcal{O}(|V(G)|^2 + |V(G)| \cdot |E(G)|) = \mathcal{O}(|V(G)| \cdot |E(G)|)$. So the running time of this algorithm is dominated by building graphs $N_1, \dots, N_{|V(G)|}$, and hence the running time of this algorithm is $\mathcal{O}(|V(G)|^2 \cdot |E(G)|)$. □

A vertex v in a graph G is an *antichordal-vertex* if $G[V(G) \setminus N[v]]$ is chordal. If every graph in a hereditary class $\mathcal{C}$ contains an antichordal-vertex, then to solve the maximum weight stable set problem for this class one can use essentially the same method as above but by iteratively finding and deleting antichordal-vertices instead of chordal-vertices (and by computing maximum stable sets in chordal graphs intead of maximal cliques). However, it is not the case that all ℓ-holed graphs (for odd $\ell \geq 7$) contain an antichordal-vertex; see Figure 3.7 for an example. Therefore, to solve the

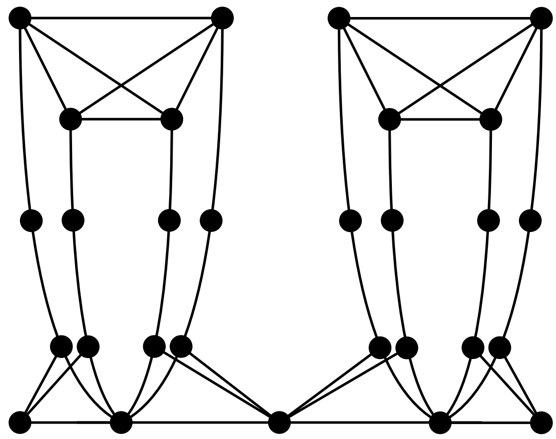


Figure 3.7: A 7-holed graph that has no antichordal-vertex.

maximum weight stable set problem for ℓ-holed graphs (for odd $\ell \geq 7$), we combine the method we just described with a process of clique cutset decomposition.

We first prove that rings and blowups of templates contain an antichordal-vertex. To do so, we need the following.

Lemma 3.44 (Lemma 4.11 in [35]). *Let $\ell \geq 3$ and let G be an odd ℓ-template with ℓ-partition (A, B, A', B', I, w, w'). Every hole of G is formed by two principal paths of G and a single vertex of $A \cup B \cup A' \cup B'$ that does not belong to these principal paths (it therefore has length $2\ell + 1$).*

Lemma 3.45 (Lemma 5.3 in [35]). *Let $\ell \geq 3$ and let G be an odd ℓ-template with ℓ-partition (A, B, A', B', I, w, w'). Let C be a cycle of G of length at least 4 with no solid chord. If C is not a hole, then there exist three consecutive vertices x, y, u in C such that:*

- *$u \in A$, $x, y \in B$, $\{u\} \subseteq H_y \subseteq H_x$ and u is an isolated vertex of H_x; or*
- *$u \in A'$, $x, y \in B'$, $\{u\} \subseteq H'_y \subseteq H'_x$ and u is an isolated vertex of H'_x.*

In particular, ux is an optional edge of G and a chord of C.

Lemma 3.46. *Let $\ell \geq 7$ be an odd integer and let G^* be a proper blowup of a twinless odd $(\ell - 1)/2$-template G. Let (A, B, A', B', I, w, w') be a proper $(\ell - 1)/2$-partition of G. Then $G^*[\bigcup_{u \in A \cup B} K_u]$ and $G^*[\bigcup_{u' \in A' \cup B'} K_{u'}]$ are chordal.*

Proof. Suppose that $G^*[\bigcup_{u \in A \cup B} K_u]$ contains a hole H, and let $H_0 = G[\{v \in V(G) : K_v \cap V(H) \neq \emptyset\}]$. By Lemma 3.44, H_0 is not a hole of G, and hence by conditions (c) and (d) in the definition of a blowup of a twinless odd template, H is isomorphic to some graph obtained from H_0 by removing optional edges. Therefore H_0 is a cycle of G with no solid chord. By Lemma 3.45, there exist three consecutive vertices x, y and u of H_0 such that $u \in A$, $x, y \in B$, $\{u\} \subsetneq H_y \subsetneq H_x$, and u is an isolated vertex of $G[H_x]$ (and hence also an isolated vertex of $G[H_y]$). So ux and uy are optional edges of G. Since a vertex $u' \in K_u \cap V(H)$ has a neighbour in K_y, it follows by condition (g) in the definition of a blowup of a twinless odd template that u' is complete to K_x. But then H has a chord, a contradiction. Therefore $G^*[\bigcup_{u \in A \cup B} K_u]$ is chordal, and a symmetric argument proves that $G^*[\bigcup_{u' \in A' \cup B'} K_{u'}]$ is chordal. □

Lemma 3.47. *Let $\ell \geq 7$ be an integer. If G is a ring or a proper blowup of a twinless odd $(\ell-1)/2$-template, then G contains an antichordal-vertex.*

Proof. Suppose that G is a ring of length k, and let $(X_1, \dots, X_k)$ be a ring partition of G. Observe that any hole of G intersects each of the cliques $X_1, \dots, X_k$. Fix $x \in X_1$. Since X_1 is a clique, $X_1 \subseteq N[x]$, and hence $G[V(G) \setminus N[x]]$ is chordal. It follows by symmetry that every vertex of a ring is an antichordal-vertex.

Suppose that G is a proper blowup of a twinless odd $(\ell-1)/2$-template G_0, and let (A, B, A', B', I, w, w') be a proper $(\ell-1)/2$-partition of G_0. By Lemma 3.46, every hole of G intersects $\bigcup_{u \in I} K_u$, and therefore it easily follows that every hole of G intersects both $\bigcup_{u \in A \cup B} K_u$ and $\bigcup_{u' \in A' \cup B'} K_{u'}$. By condition (h) in the definition of a blowup of a template, we have that $\bigcup_{u \in A \cup B} K_u \subseteq N_G[w]$, and therefore w is an antichordal-vertex. □

Lemma 3.48. *There exists an algorithm with the following specifications:*

INPUT: *A graph G that is a ring or a blowup of a twinless odd $(\ell-1)/2$-template (for odd $\ell \geq 7$), and a weight function $w : V(G) \to \mathbb{N}$.*

OUTPUT: *A maximum weight stable set of G.*

RUNNING TIME: $\mathcal{O}(|V(G)|^2 \cdot |E(G)|)$.

Proof. Consider the following algorithm.

Step 1. Find an antichordal-vertex x_1 of G, and set $N_1 = G[V(G) \setminus N(x_1)]$ and $G_1 = G \setminus \{x_1\}$.

Step 2. For each $i \in \{2, \dots, |V(G)|\}$, find an antichordal-vertex x_i of G_{i-1}, and set $N_i = G[V(G_{i-1}) \setminus N_{G_{i-1}}(x_i)]$ and $G_i = G_{i-1} \setminus \{x_i\}$. (So $G_i = G \setminus \{x_1, \dots, x_i\}$.)

Step 3. For each $i \in \{1, \ldots, |V(G)|\}$, compute a maximum weight stable set S_i of N_i.

Step 4. Output any set $S \in \{S_1, \ldots, S_{|V(G)|}\}$ such that $\sum_{s \in S} w(s)$ is maximum, and terminate.

Let us prove that this algorithm is correct. That the antichordal-vertices in Steps 1 and 2 exist follows from Lemma 3.47. To prove that the set S that is outputted in Step 4 is a maximum weight stable set of G, we show that every maximum weight stable set of G is a maximum weight stable set of one of the graphs $N_1, \ldots, N_{|V(G)|}$. Let S_0 be a maximum weight stable set of G, and let $i \in \{1, \ldots, |V(G)|\}$ be the smallest integer such that x_i belongs to S_0. If $i = 1$, then $S_0 = |V(G)|$ and hence S_0 is a maximum weight stable set of G_1. So we may assume that $i \geq 2$. Then, by our choice of i, $\{x_1, \ldots, x_{i-1}\} \cap S_0 = \emptyset$, and hence S_0 is a maximum weight stable set of G_{i-1}. Since $x_i \in S_0$, $N(x_i) \cap S_0 = \emptyset$, and hence S_0 is a maximum weight stable set of $G_{i-1} \setminus N(x_i) = N_i$. It follows that the set S outputted in Step 4 is a maximum weight stable set of G, and therefore the algorithm is correct.

The analysis of the running time of this algorithm is the same as the one for the algorithm given in the proof of Lemma 3.43, i.e. the algorithm has running time $\mathcal{O}(|V(G)|^2 \cdot |E(G)|)$. □

We define a class $\mathcal{B}$ of graphs inductively as follows:

- for every integer $\ell \geq 7$, every ring of length ℓ and every proper blowup of a twinless odd $(\ell-1)/2$-template belongs to $\mathcal{B}$;

- if G is a graph whose vertex set admits a partition (A, B) such that $G[A] \in \mathcal{B}$, and B is a clique that is complete to A, then $G \in \mathcal{B}$.

By Theorem 3.5, for every odd integer $\ell \geq 7$, every ℓ-holed graph either belongs to $\mathcal{B}$ or has a clique cutset.

Lemma 3.49. *There exists an algorithm with the following specifications:*

Input: *A graph $G \in \mathcal{B}$, and a weight function $w : V(G) \to \mathbb{N}$.*

Output: *A maximum weight stable set of G.*

Running time: $\mathcal{O}(|V(G)|^2 \cdot |E(G)|)$.

Proof. Consider the following algorithm. Let U be the (possibly empty) set of all universal vertices of G and let $G' = G \setminus U$. If $U = \emptyset$, then set $\alpha_U = 0$, and otherwise set $\alpha_U = \max_{u \in U} w(u)$. Apply the algorithm given in the proof of Lemma 3.48 with G' and the weight function w restricted to $V(G')$ as input, and let S be its output. If $\sum_{s \in S} w(s) > \alpha_U$, then output S and terminate, and otherwise output the set $\{u\}$, where u is any vertex of U satisfying $w(u) = \alpha_U$, and terminate. Clearly this algorithm is correct and has running time $\mathcal{O}(|V(G)|^2 \cdot |E(G)|)$. □

Lemma 3.50. *There exists an algorithm with the following specifications:*

INPUT: *An ℓ-holed graph G for some odd $\ell \geq 7$, and a weight function $w : V(G) \to \mathbb{N}$.*

OUTPUT: *A maximum weight stable set of G.*

RUNNING TIME: $\mathcal{O}(|V(G)|^3 \cdot |E(G)|)$.

Proof. By Lemma 3.49, and Theorem 3.5, together with Lemma 8.6 from [4]. □

3.5.2 Recognition via clique cutset and special 2-join decomposition

In Section 3.4.3, we showed that for odd $\ell \geq 7$, every ℓ-holed graph is a ring or a pyramid, or has a universal vertex, a clique cutset or a special 2-join (Theorem 3.33). In Section 3.4.4, we gave algorithms that detect whether a graph has a special 2-join. In this section, we put these things together to obtain a recognition algorithm for ℓ-holed graphs (where $\ell \geq 7$ is odd) based on a process of decomposition by clique cutsets and special 2-joins.

We make use of clique cutset decomposition trees, whose definition is given in Section 1.2. We make use of the following fact about computing clique cutset decomposition trees. Recall also that a clique cutset decomposition tree for a graph G has $\mathcal{O}(|V(G)|)$ leaves.

Theorem 3.51 (Tarjan [46])**.** *There exists an algorithm that computes an extreme clique cutset decomposition tree of any graph G in $\mathcal{O}(|V(G)| \cdot |E(G)|)$ time.*

Lemma 3.52. *There exists an algorithm with the following specifications:*

INPUT: *A graph G.*

OUTPUT: *A family $\mathcal{L}$ of induced subgraphs of G that satisfies the following properties:*

– *G is ℓ-holed for some odd $\ell \geq 7$ if and only if all the graphs in $\mathcal{L}$ are ℓ-holed.*

– *The graphs in $\mathcal{L}$ have no clique cutset.*

– *The number of graphs in $\mathcal{L}$ is $\mathcal{O}(|V(G)|)$.*

Running time: $\mathcal{O}(|V(G)|^3)$.

Proof. Execute the algorithm from Theorem 3.51 to obtain in $\mathcal{O}(|V(G)| \cdot |E(G)|) = \mathcal{O}(|V(G)|^3)$ time an extreme clique cutset decomposition tree T of G, and let $\mathcal{L}$ consist of all the leaves of T. Clearly no graph in $\mathcal{L}$ has a clique cutset, and $|\mathcal{L}| = \mathcal{O}(|V(G)|)$. If G is ℓ-holed for some odd $\ell \geq 7$, then all the graphs in $\mathcal{L}$ are also ℓ-holed because they are induced subgraphs of G. To see that the converse holds, it suffices to observe that if K is a clique cutset of G and G_1 and G_2 are two blocks of decomposition of G w.r.t. K, then any hole of G is a hole of G_1 or G_2. $\square$

Lemma 3.53. *There exists an algorithm with the following specifications:*

Input: *A C_4-free graph G that has no clique cutset.*

Output: *A family $\mathcal{L}$ of induced subgraphs of G that satisfies the following properties:*

– *G is ℓ-holed for some odd $\ell \geq 7$ if and only if all the graphs in $\mathcal{L}$ are ℓ-holed.*

– *The graphs in $\mathcal{L}$ have no universal vertex, clique cutset or special 2-join.*

– *The number of graphs in $\mathcal{L}$ is $\mathcal{O}(|V(G)|^2)$.*

Running time: $\mathcal{O}(|V(G)|^{14})$.

Proof. Consider the following algorithm. Initialise $\mathcal{L} = \emptyset$ and $\mathcal{L}' = \{G\}$. While $\mathcal{L}' \neq \emptyset$, repeatedly execute the following:

- Let H be any graph from $\mathcal{L}'$, and let H^* be the graph obtained from H by removing all simplicial vertices and universal vertices.
- Use the algorithms of Lemmas 3.37 and 3.41 to check whether H^* has a special 2-join.
- If H^* has no special 2-join, remove H from $\mathcal{L}'$ and add H^* to $\mathcal{L}$.
- If H^* has a special 2-join (X_1, X_2), then let H_1 and H_2 be the two blocks of decomposition of H^* w.r.t. (X_1, X_2). Remove H from $\mathcal{L}'$ and add graphs H_1 and H_2 to $\mathcal{L}'$.

When $\mathcal{L}' = \emptyset$, output $\mathcal{L}$ and terminate.

Let us prove that $\mathcal{L}$ has the desired properties. Since the blocks of decomposition of a graph w.r.t. a special 2-join are induced subgraphs, all graphs in $\mathcal{L}$ are induced subgraphs of G. Furthermore, it follows from this observation together with Lemmas 3.29 and 3.31 that G is ℓ-holed for some odd $\ell \geq 7$ if and only if all graphs in $\mathcal{L}$ are ℓ-holed. Clearly no graph in $\mathcal{L}$ has a universal vertex or a special 2-join. Since G has no clique cutset and we only add to $\mathcal{L}$ graphs that have no simplicial vertex, it follows from Lemma 3.30 that no graph in $\mathcal{L}$ has a clique cutset.

It remains to prove that $|\mathcal{L}| = \mathcal{O}(|V(G)|^2)$. For any graph F, let $\Phi(F) = |E(F)| - |V(F)| - 1$. Suppose that F is a graph with no clique cutset that has a special 2-join (X_1, X_2) with split $(X_1, X_2, A_1, A_2, B_1, B_2)$ and blocks of decomposition F_1 and F_2. Then

$$\begin{aligned}\Phi(F) \geq\ & |E(F[X_1])| + |E(F[X_2])| + (|A_1| + |A_2| - 1) + |B_1| \cdot |B_2| \\ & - |X_1| - |X_2| - 1.\end{aligned}$$

Since $|V(P)| = |E(P)| + 1$ for any path P, we have that for each $i \in \{1, 2\}$,

$$\Phi(F_i) = |E(F[X_i])| + |A_i| + |B_i| - |X_i| - 2.$$

Therefore

$$\begin{aligned}\Phi(F_1) + \Phi(F_2) =\ & |E(F[X_1])| + |E(F[X_2])| + |A_1| + |A_2| \\ & + |B_1| + |B_2| - |X_1| - |X_2| - 4,\end{aligned}$$

and since $p + q \leq pq + 1$ for all positive integers p and q it follows that

$$\Phi(F_1) + \Phi(F_2) < \Phi(F).$$

Since F has no clique cutset, F is 2-connected, and therefore so are F_1 and F_2. Thus $|E(F_i)| \geq |V(F_i)|$ for each $i \in \{1, 2\}$, and hence $\Phi(F_i) \geq -1$. If $\Phi(F_i) = -1$, then F_i is a hole, contradicting the fact that if $|A_{3-i}| = |B_{3-i}| = 1$, then $F[X_{3-i}]$ is not a path. Therefore $\Phi(F_i) \geq 0$, and $\Phi(F) > 0$ since $\Phi(F) > \Phi(F_1) + \Phi(F_2)$. It follows that the size of $\mathcal{L}$ is at most $2\Phi(G) = \mathcal{O}(|E(G)|) = \mathcal{O}(|V(G)|^2)$. The running time of this algorithm is therefore $\mathcal{O}(|V(G)|^{14})$. $\square$

Lemma 3.54. *There exists an algorithm with the following specifications:*

INPUT: *A graph G and an integer $\ell \geq 2$.*

OUTPUT: *Yes if G is a pyramid whose 3 paths are all of length ℓ, and No otherwise.*

RUNNING TIME: $\mathcal{O}(|V(G)|^2)$.

Proof. Consider the following algorithm.

Step 1. Let S be the set of vertices of G of degree 2. If $S = \emptyset$, then output No.

Step 2. Check if $G[S]$ has exactly three components P_1, P_2 and P_3, each a path of length $\ell-2$. If not, then output No. Say P_1 has endpoints u_1, v_1, P_2 has endpoints u_2, v_2 and P_3 has endpoints u_3, v_3.

Step 3. Check if $G \setminus S$ has exactly two components A and B, where A consists of a single vertex a of degree 3 in G, and B is a complete graph on 3 vertices x, y, z. If not, then output No. Check that the three neighbours of a belong to $\{u_1, u_2, u_3, v_1, v_2, v_3\}$; if not, then output No. From here on we assume that $N_G(a) = \{u_1, u_2, u_3\}$. Check that: each of x, y, z has exactly one neighbour in $G \setminus B$; that each such neighbour belongs to $\{v_1, v_2, v_3\}$; and that no two vertices of B are adjacent to the same vertex from $\{v_1, v_2, v_3\}$. If any of these checks fail, then output No.

Step 4. Output Yes.

It is clear that this algorithm correctly decides on input a graph G and an integer $\ell \geq 2$ whether G is a pyramid whose 3 paths are all of length ℓ.

The running time of this algorithm is $\mathcal{O}(n^2)$: Step 1 can be done in $\mathcal{O}(n^2)$ time and Steps 2 and 3 can be done in $\mathcal{O}(n+m)$ time. □

Theorem 3.55 (Boncompagni, Penev and Vušković [4])**.** *There exists an algorithm with the following specifications:*

INPUT: *A graph G.*

OUTPUT: *Either the true statement that G is a ring, together with the length and good partition of the ring, or the true statement that G is not a ring.*

RUNNING TIME: $\mathcal{O}(|V(G)|^2)$.

Theorem 3.56. *There exists an algorithm with the following specifications:*

INPUT: *A graph G and an odd integer $\ell \geq 7$.*

OUTPUT: *"Yes" if G is ℓ-holed, and "No" otherwise.*

RUNNING TIME: $\mathcal{O}(|V(G)|^{15})$.

Proof. Consider the following algorithm.

Step 1. Check whether G contains a C_4, and if so, output "No" and terminate.

Step 2. Execute the algorithm of Lemma 3.52 with input G, and let $\mathcal{L}_1$ be its output.

Step 3. For each graph $F \in \mathcal{L}_1$, execute the algorithm of Lemma 3.53 with input F, and let $\mathcal{L}_F$ be its output. Set $\mathcal{L}_2 = \bigcup_{F \in \mathcal{L}_1} \mathcal{L}_F$.

Step 4. Check that each graph in $\mathcal{L}_2$ is a complete graph, a ring of length k (using Lemma 3.55), or a pyramid whose three paths are of length $\frac{k-1}{2}$ (using Lemma 3.54); if not, then output No, and otherwise, output Yes.

By Lemmas 3.52 and 3.53, G is ℓ-holed if and only if every graph in $\mathcal{L}_2$ is ℓ-holed. If the algorithm outputs Yes, then every graph in $\mathcal{L}_2$ is a ring of length ℓ or a pyramid whose three paths are all of length $(\ell-1)/2$, and therefore is ℓ-holed. Thus, if the algorithm outputs Yes, then G is ℓ-holed. If the algorithm outputs No but G is ℓ-holed, then some $F \in \mathcal{L}_2$ is not a ring of length ℓ or a pyramid whose three paths are all of length $(\ell-1)/2$, contrary to Theorem 3.33. Thus, if the algorithm outputs No, then G is not ℓ-holed; this completes the proof that the above algorithm is correct.

Step 1 takes $\mathcal{O}(|V(G)|^4)$ time and Step 2 takes $\mathcal{O}(|V(G)|^3)$ time. Step 3 executes the algorithm of Lemma 3.53 for each of the $\mathcal{O}(|V(G)|)$ many graphs in $|\mathcal{L}_1|$, and hence this step takes $\mathcal{O}(|V(G)|^{15})$ time. Step 4 executes the algorithms of Lemma 3.54 and Theorem 3.55 for each of the $\mathcal{O}(|V(G)|^3)$ many graphs in $\mathcal{L}_2$, and hence this step takes $\mathcal{O}(|V(G)|^6)$ time. Therefore the overall running time of this algorithm is $\mathcal{O}(|V(G)|^{15})$. □

Theorem 3.57 (Nikolopoulos and Palios [44]). *There exists an algorithm with the following specifications:*

INPUT: *A graph G.*

OUTPUT: *A hole H of G, if one exists, and otherwise the true statement that G contains no hole.*

RUNNING TIME: $\mathcal{O}(|V(G)| + |E(G)|^2)$.

The following is the main result of this section.

Theorem 3.58. *There exists an algorithm with the following specifications:*

INPUT: *A graph G.*

OUTPUT: *"Yes" if G is ℓ-holed for some odd $\ell \geq 7$, and "No" otherwise.*

RUNNING TIME: $\mathcal{O}(|V(G)|^{15})$.

Proof. Run the algorithm of Theorem 3.57 and let H be a hole of G if it finds one. If it finds no hole, then output Yes, and if $|V(H)| < 7$ or if $|V(H)|$ is even, then output No. Now run the algorithm of Theorem 3.56 with the graph G and integer $|V(H)|$ as input. If we obtain the output "No", then output "No". Otherwise, output "Yes". $\square$

3.5.3 Recognition via clique cutset decomposition

In this section we give an algorithm that decides in polynomial time whether a given graph is a bordered blow-up of an ℓ-frame for some odd $\ell \geq 7$. Together with structural results from [19] (described in Sections 3.1 and 3.3), this entails a polynomial time recognition algorithm for ℓ-holed graphs where $\ell \geq 7$ and ℓ is odd.

We begin with the following. (We refer the reader to Section 3.3 for the definition of a bordered blow-up of an ℓ-frame.)

Lemma 3.59. *Let $\ell \geq 7$ be odd and let $G = H \cup J \cup K$ be a bordered blow-up of an ℓ-frame F. If G has no clique cutset, then some vertex of $V(J) \setminus W(A)$ is a universal vertex of J, or some vertex of $V(K) \setminus W(B)$ is a universal vertex of K.*

Proof. Suppose that G has no clique cutset. Since A and B are complementary threshold graphs, we may assume without loss of generality that A contains a vertex of degree 0; let I be the set of all such vertices of A. From the definition of a blow-up of an ℓ-frame, it follows that for each $a_i \in I$, the set $W(a_i)$ is anticomplete to $W(A \setminus \{a_i\})$. (Recall that vertices of A are labeled $a_1, \dots, a_k$, vertices of B are labeled $b_1, \dots, b_k$, and a_i, b_i are the ends of the bar P_i of F.) Assume $I = \{a_1, \dots, a_s\}$.

(1) If $I \neq V(A)$, then for every $a_i \in I$, each vertex of $W(a_i)$ has a neighbour in $V(J) \setminus W(A)$ that is adjacent to some vertex of $W(A \setminus I)$.

Proof of (1): Fix $a_i \in I$. If there is no path in J from some vertex of $W(a_i)$ to some vertex of $W(A \setminus I)$, then $W(\{b_1, \dots, b_s\})$ is a clique cutset of G that separates $W(a_i)$ from $W(A \setminus I)$, a contradiction. So there exists a path in J from some vertex of $W(a_i)$ to some vertex of $W(a_j)$ for some $a_j \in V(A) \setminus I$. We may assume a_j is chosen so that there is a path P in J with ends $a \in W(a_i)$ and $a' \in W(a_j)$ and with all other vertices belonging to $V(J) \setminus W(A)$. By the definition of a border, P has length 2. So a has a neighbour in $V(J) \setminus W(A)$ that is adjacent to some vertex of $W(A \setminus I)$; let X be the set of all vertices from $W(a_i)$ that have such a neighbour.

Suppose there exists $t \in W(a_i) \setminus X$, and let u be the neighbour of a_i in P_i. From what we have already proved, it follows that there is no path in J from t to some vertex of $W(A \setminus I)$ whose only vertex in $W(a_i)$ is t. Therefore every path in G from t to some vertex of $W(A \setminus I)$ intersects $X \cup (N(t) \cap W(u))$. Clearly both X and $N(t) \cap W(u)$ are cliques, and it follows from the fourth bullet in the definition of border that X is complete to $N(t) \cap W(u)$. But then $X \cup (N(t) \cap W(u))$ is a clique cutset of G, a contradiction. Therefore $X = W(a_i)$, and this proves (1).

If $I \neq V(A)$, then $V(J) \setminus W(A)$ is nonempty by (1), and if $I = V(A)$, then $V(J) \setminus W(A)$ is nonempty for otherwise G has a clique cutset (for instance the set $W(b)$ for any $b \in V(B)$ is a clique cutset). Among all vertices in $V(J) \setminus W(A)$, let c be one that maximises $|N(c) \cap V(A)|$, and subject to that, maximises $|N(c) \cap W(A)|$; and furthermore, if $I \neq V(A)$, then choose c from those vertices in $V(J) \setminus W(A)$ that have a neighbour in both $W(I)$ and $W(A \setminus I)$ (which is possible by (1)).

(2) If c is complete to $V(A)$, then c is a universal vertex of J.

Proof of (2): Suppose that c is complete to $V(A)$. By definition, every vertex in $V(J) \setminus W(A)$ has two nonadjacent neighbours in $V(A)$, and hence it follows from J being chordal that c is complete to $V(J) \setminus (W(A) \cup \{c\})$. So it remains to prove that c is complete to $W(A)$.

Suppose there exists some $a \in V(A)$ and $a' \in W(a)$ such that c, a' are nonadjacent. Suppose in addition that $a \in V(A) \setminus I$. Since $|A \setminus I| \geq 2$ and $G[V(A) \setminus I]$ has a universal vertex, there is some $b \in V(A) \setminus (I \cup \{a\})$ adjacent to a. But now, for any $d \in I$, the path a', b, c, d violates the third bullet in the definition of a border, a contradiction. So c is complete to $W(A \setminus I)$, and hence $a \in I$.

By (1), a' has some neighbour $c' \in V(J) \setminus W(A)$, and clearly it is different from c. By what we proved in the first paragraph, c and c' are adjacent. Consequently, c' is adjacent to every $v \in N(c) \cap W(a)$, for otherwise $\{a', v, c, c'\}$ induces a 4-hole, a contradiction; in particular, c' and a are adjacent. Furthermore, c' is adjacent to every $v \in (N(c) \cap W(A)) \setminus W(a)$, for otherwise the path a', c', c, v violates the third bullet in the definition of a border. But now c' is complete to $V(A)$ and has more neighbours in $W(A)$ than c, contrary to our choice of c. So c is complete to $W(A)$, and this proves (2).

We first handle the case where $I = V(A)$. By (2), we may assume that c is not complete to $V(A)$. Suppose there exists some $c' \in V(J) \setminus W(A)$ that is adjacent to some $a \in N_A(c)$ and $a' \in V(A) \setminus N(c)$. By maximality, there exists some $a'' \in N_A(c) \setminus N(c')$. But now one of a'', c, a, c', a' and a'', c, c', a' is a path that violates the third bullet in the definition of a border, a contradiction. So no vertex in $V(J) \setminus W(A)$ has neighbours in both $N_A(c)$ and $V(A) \setminus N_A(c)$. So for every $v \in V(J) \setminus W(A)$, either $N_A(v) \subseteq N_A(c)$ or $V(A) \setminus N_A(v)$; it follows from the third bullet in the definition of a border that if for $v, v' \in V(J) \setminus W(A)$ we have that $N_A(v) \subseteq N_A(c)$ and $N_A(v') \subseteq V(A) \setminus N_A(c)$, then v, v' are nonadjacent. But now, with $V(A) \setminus N(w) = \{a_1, \ldots, a_t\}$ say, the set $W(\{b_1, \ldots, b_t\})$ is a clique cutset of G, a contradiction. So $I \neq V(A)$, and since I contains all isolated vertices of A we have that $|A \setminus I| \geq 2$. Also, by our choice of c, it has a neighbour in both I and $V(A \setminus I)$, which we use in the proof of the following:

(3) c is complete to $W(A \setminus I)$.

Proof of (3): Fix $a \in N(c) \cap W(a_i)$ for some $a_i \in I$ and $a' \in N(c) \cap W(a_j)$ for some $a_j \in V(A \setminus I)$. Since $A \setminus I$ contains no vertex of degree 0, it contains a universal vertex, and therefore by the definition of a border so does $J[W(A \setminus I)]$; let $u \in V(A \setminus I)$ be a universal vertex of $J[W(A \setminus I)]$. Suppose that c is not adjacent to any universal vertex of $J[W(A \setminus I)]$. Then $a' \neq u$, and hence $u \notin W(a_j)$, but now the path a, c, a', u violates the third bullet in the definition of a border, a contradiction. So c is adjacent to some universal vertex of $J[W(A \setminus I)]$; thus, we may assume $a' = u$. It now follows that c is adjacent to every $v \in W(A \setminus (I \cup \{u\}))$, for otherwise the path a, c, u, v violates the third bullet in the definition of a border. It remains to prove that c is complete to $W(u)$. Suppose otherwise, and fix $u' \in W(u) \setminus N(c)$. Let v be any vertex in $V(A) \setminus (I \cup \{u\})$. But now the path a, c, v, u' contradicts the definition of a border, and therefore c is complete to $W(u)$. This proves (3).

(4) c is complete to I.

Proof of (4): Suppose not and fix $a \in I \setminus N(c)$. By (1) and (3), there is a vertex $c' \in V(J) \setminus W(A)$ adjacent to a and complete to $W(A \setminus I)$. By our choice of c, there is some $a' \in I$ adjacent to c and nonadjacent to c'. But now, for any $a'' \in V(A \setminus I)$, one of a, c', a'', c, a' and a, c', c, a' is a path that violates the third bullet in the definition of a border, a contradiction. This proves (4).

By (3) and (4), c is complete to $V(A)$, and thus it follows from (2) that c is a universal vertex of J. $\square$

We also need the following two lemmas:

Lemma 3.60. *Let $\ell \geq 7$ be odd and let $G = H \cup J \cup K$ be a bordered blow-up of an ℓ-frame F. If w is a universal vertex of J that belongs to $V(J) \setminus W(A)$, then:*

- $d_G(w, u) = 1$ *for all* $u \in V(J) \setminus \{w\}$;
- $1 < d_G(w, u) < (\ell - 3)/2 + 1$ *for all* $u \in V(H \setminus (J \cup K))$;
- $d_G(w, u) = (\ell - 3)/2 + 1$ *for all* $u \in W(B)$; *and*
- $d_G(w, u) = (\ell - 1)/2 + 1$ *for all* $u \in V(K) \setminus W(B)$.

Proof. The first bullet follows from w being a universal vertex of J.

Fix $u \in V(H \setminus (J \cup K))$. By definition, $V(J) \setminus W(A)$ is anticomplete to $V(H \setminus J)$, and therefore $d_G(w, u) > 1$. Let $P = p_1, \ldots, p_m$ be the bar of F and let $i \in \{1, \ldots, m\}$ be the integer such that $u \in W(p_i)$. Note that $p_1, p_m \in V(J \cup K)$, and hence $i \notin \{1, m\}$. Then $w, p_1, \ldots, p_{i-1}, u$ is a path in G from w to u, and its length is less than $(\ell-3)/2+1$ since the length of P is $(\ell - 3)/2$. Thus the second bullet holds.

Fix $u \in W(B)$. Clearly any path from w to u contains (possibly among other vertices) the vertex w together with exactly one vertex from each of the sets $W(p_1), \ldots, W(p_m)$ for some bar $p_1, \ldots, p_m$ of F. Such a path has length at least $(\ell - 3)/2 + 1$, and therefore $d_G(w, u) \geq (\ell - 3)/2 + 1$. That $d_G(w, u) \leq (\ell - 3)/2 + 1$ follows from the fact that there exists a bar $p_1, \ldots, p_m$ with $u \in W(p_m)$ (in which case $w, p_1, \ldots, p_{m-1}, u$ is the desired path). Thus the third bullet holds.

Finally, the fourth bullet follows from the third bullet together with the observation that $V(K) \setminus V(B)$ is anticomplete to $V(G) \setminus W(B)$ and every vertex of $V(K) \setminus W(B)$ has a neighbour in $W(B)$. $\square$

Recall that for a graph G and disjoint subsets X, Y of $V(G)$, we denote by $G[X, Y]$ the bipartite subgraph of G with vertex set $X \cup Y$ and edge set the set of edges of G between X and Y.

Lemma 3.61. *Let $\ell \geq 7$ be odd, let $G = H \cup J \cup K$ be a bordered blow-up of an ℓ-frame F and let $A_i \subseteq W(A)$ and $B_i \subseteq W(B)$ be nonempty sets. Then A_i is a bag of J if and only if there exists a component C_i of $G[W(A), V(G \setminus (J \cup K))]$ such that $V(C_i) \cap W(A) = A_i$. Similarly, B_i is a bag of K if and only if there exists a component C_i of $G[W(B), V(G \setminus (J \cup K))]$ such that $V(C_i) \cap W(B) = B_i$.*

Proof. We prove the statement about bags of J, and the analogous statement about bags of K follows by symmetry.

Let A_i be a bag of J, fix $a \in A_i \cap V(A)$, and let C_i be the component of $G[W(A), V(G \setminus (J \cup K))]$ that contains a. Let u be the unique neighbour of a in the bar of F that contains a. Since a, u are adjacent, $u \in V(C_i)$. By the definition of a bordered blow-up of an ℓ-frame, u is complete to $W(a) = A_i$, and therefore $A_i \subseteq V(C_i)$.

If C_i contains a vertex from a bag of J different from A_i, then there is a path of length 2 in C_i from a to some vertex belonging to a bag different from A_i and whose internal vertex belongs to $V(G \setminus (J \cup K))$, contradicting the fact that vertices from different bags of J share no neighbours outside of J.

To prove the converse, let C_i be a component of $G[W(A), V(G) \setminus (J \cup K)]$ and suppose that $V(C_i) \cap W(A)$ is not a bag of J. So either $V(C_i) \cap W(A)$ is a proper subset of some bag of J or $V(C_i) \cap W(A)$ intersects at least two bags of J. Since for each bag of J there is some vertex of $G \setminus (J \cup K)$ complete to it (namely the vertex with a neighbour in the bag and which belongs to a bar of F), it follows that if C_i contains some vertex of a bag, then it contains the entire bag. Thus $V(C_i) \cap W(A)$ is not a proper subset of some bag of J. Therefore C_i contains a vertex from some two bags A_s, A_t of J; assume A_s, A_t are chosen so that the length of a path from some vertex of A_s to some vertex of A_t in C_i is minimum. Applying the same argument as in the preceding paragraph, we get that some vertex of $G \setminus (J \cup K)$ has a neighbour in both A_s and A_t, contradicting the fact that vertices from different bags of J share no neighbours outside of J. $\square$

For a general graph G, a *blow-up of* G is any graph that can be obtained from G by substituting a nonempty clique for each vertex. (Blow-up means something more complicated for ℓ-frames and ℓ-frameworks.) If G is $(P_4, C_4, \overline{C_4})$-free, then clearly so is any blow-up of G. Thus:

Lemma 3.62. *If G is a blow-up of a threshold graph, then G is a threshold graph.*

Let T be a tree with root w. The *height* of T is defined to be $h(T) = \max_{v \in V(T)} d_T(w, v)$. For $i \in \{0, \ldots, h(T)\}$ we denote by $T(i)$ the set of vertices v

of T such that $d_T(w,v) = i$. A *breadth-first search tree* (or *BFS-tree* for short) is any tree obtained as output from the breadth-first search algorithm. It is well known that if T is a BFS-tree with root w for a graph G, then $d_T(w,v) = d_G(w,v)$ for all $v \in V(G)$.

Theorem 3.63. *There exists an algorithm with the following specifications:*

INPUT: *A graph G that has no clique cutset, and an odd integer $\ell \geq 7$.*

OUTPUT: *Yes if G is a bordered blow-up of an ℓ-frame, and No otherwise.*

RUNNING TIME: $\mathcal{O}(|V(G)|^7)$.

Proof. Consider the following algorithm. Let $Z = V(G)$.

Step 1. If $Z = \emptyset$, then output No; otherwise, pick a vertex $w \in Z$ and remove it from Z.

Step 2. Let T be a BFS-tree for G rooted at w. If the height $h = h(T)$ of T is neither $(\ell-1)/2+1$ nor $(\ell-3)/2+1$, then go to Step 1. Otherwise, let:

- $J' = G[T(0) \cup T(1)]$,
- $K' = G[T((\ell-3)/2+1) \cup T(h)]$,
- $M = V(G) \setminus V(J' \cup K')$,
- $J_A = \{v \in V(J') : N_G(v) \cap M \neq \emptyset\}$, and
- $K_B = \{v \in V(K') : N_G(v) \cap M \neq \emptyset\}$.

If it is not the case that J' and K' are chordal and $G[J_A]$ and $G[K_B]$ are threshold graphs, then go to Step 1.

Step 3. Let $C_1, \ldots, C_r$ be the components of the bipartite graph $G[J_A, M]$, and for each $i \in \{1, \ldots, r\}$ let $A_i = V(C_i) \cap J_A$. So $(A_1, \ldots, A_r)$ is a partition of J_A; similarly partition K_B into sets $B_1, \ldots, B_{r'}$ by considering the bipartite graph $G[K_B, M]$. Go to Step 1 if any of the following hold:

- $r \neq r'$ or $r < 3$;
- not all of $A_1, \ldots, A_r, B_1, \ldots, B_r$ are cliques;
- there exists distinct $i, j \in \{1, \ldots, r\}$ such that A_i is neither complete nor anticomplete to A_j, or that B_i and B_j are neither complete nor anticomplete to each other;

- some induced 4-vertex path of G has its two internal vertices in one of $A_1, \dots, A_r, B_1, \dots, B_r$;
- there exists distinct $i, j \in \{1, \dots, r\}$ such that some induced path from A_i to A_j with interior in $V(J') \setminus (A_i \cup A_j)$ has length greater than two, or some induced path from B_i to B_j with interior in $V(K') \setminus (B_i \cup B_j)$ has length greater than two;
- the neighbourhood of some vertex of $V(J' \cup K') \setminus (J_A \cup K_B)$ in $J_A \cup K_B$ is a clique.

Step 4. Go to Step 1 if the number of components of $G[M]$ is not exactly r; otherwise, let $M_1, \dots, M_r$ be these components. Check for each $i \in \{1, \dots, r\}$ that $N(M_i) = A_j \cup B_k$ for some unique pair of integers $j, k \in \{1, \dots, r\}$, and go to Step 1 if not. So from now on we assume that $N(M_i) = A_i \cup B_i$ for each $i \in \{1, \dots, r\}$. Now check for all distinct $i, j \in \{1, \dots, r\}$ that A_i is complete to A_j if and only if B_i is anticomplete to B_j, and go to Step 1 if not.

Step 5. For each $i \in \{1, \dots, r\}$ and $j \in \{1, \dots, (\ell-3)/2-1\}$, let $M_i^j = V(M_i) \cap T(j)$. (That this is well-defined follows from the fact that $B_1, \dots, B_r$ are subsets of $T((\ell-3)/2+1)$ and, by the check in Step 4, some vertex of M_i has a neighbour in one of $B_1, \dots, B_r$.) Let $M_i^0 = A_i$ and $M_i^{(\ell-3)/2} = B_i$. Check for all $i \in \{1, \dots, r\}$ and $j \in \{1, \dots, (\ell-3)/2-1\}$ that $G[M_i^j, M_i^{j-1}]$ and $G[M_i^j, M_i^{j+1}]$ are compatible half-graphs, and go to Step 1 if not.

Step 6. Output Yes.

We now prove that this algorithm is correct. Suppose that G is a bordered blow-up of an ℓ-frame F. Thus G is the composition of H, J, K, with notation as in the definition of bordered blow-ups of ℓ-frames. By Lemma 3.59, without loss of generality some vertex $w \in V(J) \setminus W(A)$ is a universal vertex of J. Let us consider the execution of Steps 2 to 6 when the vertex picked in Step 1 is this vertex w.

By Lemma 3.60, the height of T is either $(\ell-3)/2+1$ or $(\ell-1)/2+1$, and hence J', K', M, J_A and K_B are defined in Step 2. It also follows from Lemma 3.60 that $J = G[T(0) \cup T(1)] = J'$, and that $K = G[T((\ell-3)/2+1)]$ if $V(K) \setminus W(B) = \emptyset$ and $K = G[T((\ell-3)/2+1) \cup T((\ell-1)/2+1)]$ otherwise; therefore $K = K'$. So, by the definition of a border, J' and K' are chordal. The vertices of $W(A)$ are precisely those vertices of J that have a neighbour outside of J, and the vertices of $W(B)$ are precisely those vertices of K that have a neighbour outside of K; that is, $W(A) = J_A$

and $W(B) = K_B$. By the definition of an ℓ-frame, A and B are threshold graphs, and therefore by Lemma 3.62 so are $G[W(A)] = G[J_A]$ and $G[W(B)] = G[K_B]$. The facts established thus far show that the algorithm does not return to Step 1 as a result of the checks in Step 2.

By Lemma 3.61, the sets $A_1, \dots, A_r$ defined in Step 3 are the bags of J and the sets $B_1, \dots, B_{r'}$ are the bags of K. So $W(A) = A_1 \cup \dots \cup A_r$ and $W(B) = B_1 \cup \dots \cup B_{r'}$. It follows immediately from $(J, A, (W_t : t \in V(A)))$ being a border that:

- $r = r'$ and $r \geq 3$;
- all of $A_1, \dots, A_r, B_1, \dots, B_r$ are cliques any two of which are either complete or anticomplete to each other;
- if P is a path of J with length at least three and with both ends in $A_1 \cup \dots \cup A_r$, some internal vertex of P belongs to the same set among $A_1, \dots, A_r$ as one of the ends of P; and if P is a path of K with length at least three and with both ends in $B_1 \cup \dots \cup B_r$, some internal vertex of P belongs to the same set among $B_1, \dots, B_r$ as one of the ends of P; and
- for every vertex $v \in V(J' \cup K') \setminus (J_A \cup K_B)$, the neighbourhood of v in $J_A \cup K_B$ is not a clique.

That there is no induced 4-vertex path of G with its two internal vertices in one of $A_1, \dots, A_r, B_1, \dots, B_r$ follows from the fact that $G[W(u), W(v)]$ is a half-graph for all edges uv of G. Therefore the algorithm does not return to Step 1 as a result of the checks in Step 3.

Clearly each component of $G[M]$ is equal to $G[W(P) \setminus W(A \cup B)]$ for some bar P of F. It follows from the relevant definitions that the number of bags of J is equal to the number of bars of F, and F has r bars, so $G[M]$ has exactly r components. If P is a bar of F, say with ends $a \in V(A)$ and $b \in V(B)$, then every vertex of $W(a) \cup W(b)$ has a neighbour in $W(P) \setminus W(A \cup B)$, and for every $u \in V(A \cup B) \setminus \{a, b\}$ we have that $W(u)$ is anticomplete to $W(P) \setminus W(A \cup B)$. It follows that for each component M_i of $G[M]$ there exists a unique pair of integers $j, k \in \{1, \dots, r\}$ such that $N(M_i) = A_j \cup B_k$. With this in mind, and as we do in Step 4, assume that the indices of A_i, B_i, M_i are such that $N(M_i) = A_i \cup B_i$. It now follows from the construction of A and B in the definition of an ℓ-frame, together with the second bullet in the preceding paragraph, that A_i is complete to A_j if and only if B_i is anticomplete to B_j for all distinct $i, j \in \{1, \dots, r\}$. So the algorithm does not return to Step 1 as a result of Step 4.

Therefore the algorithm returns to Step 1 as a result of Step 5, i.e. there exists $i \in \{1, \dots, r\}$ and $j \in \{1, \dots, (\ell-3)/2+1\}$ such that $G[M_i^j, M_i^{j-1}]$ and $G[M_i^j, M_i^{j+1}]$ are not compatible half-graphs. By arguments in the previous paragraph, we have that: for each bar $p_1, \dots, p_m$ of F, where $p_1 \in V(A)$ and $p_m \in V(B)$, the sets $M_i^0, \dots, M_i^{(\ell-3)/2+2}$ are the sets $W(p_1), \dots, W(p_m)$, in that order. But by definition $G[W(p_s), W(p_{s-1})]$ and $G[W(p_s), W(p_{s+1})]$ are compatible half-graphs for every $s \in \{2, \dots, m-1\}$, a contradiction. So the algorithm does not return to Step 1 as a result of Step 5. It follows that the algorithm executes Step 6, and hence the algorithm outputs Yes, as required.

We now prove the converse: that if the algorithm outputs Yes, then the input graph G is a bordered blow-up of an ℓ-frame. To match the notation used in the definition of bordered blow-ups of ℓ-frames, we begin by defining the following sets (where J', K' and so on are as they were on execution of Step 6):

- $J = J'$ and $K = K'$;
- $W(A) = J_A$ and $W(B) = K_B$;
- $A = G[\{a_1, \dots, a_r\}]$ and $B = G[\{b_1, \dots, b_r\}]$, where for $i \in \{1, \dots, r\}$, a_i is a vertex in A_i of maximum degree in G among all vertices in A_i, and b_i is a vertex in B_i of maximum degree in G among all vertices in B_i;
- $H = A \cup B \cup G[M]$; and
- for each $t \in V(A \cup B)$, let W_t denote the unique set among $A_1, \dots, A_r, B_1, \dots, B_r$ that contains t (thus $(W_t : t \in V(A))$ is a partition of $W(A)$ and $(W_t : t \in V(B))$ is a partition of $W(B)$.)

We claim that G is the composition of H, J, K.

First we prove that $(J, A, (W_t : t \in V(A)))$ and $(K, B, (W_t : t \in V(B)))$ are borders. By Step 2, J is chordal and $G[J_A]$ is threshold, and since A is an induced subgraph of J_A it follows that A is threshold. By the first and second bullets of Step 3, $|A| \geq 3$, and W_t is a clique for each $t \in V(A)$. By the third bullet of Step 3 we have that for all distinct $s, t \in V(A)$, W_s is complete to W_t if st is an edge, and W_s is anticomplete to W_t otherwise. By the fourth bullet in Step 3, for every induced path P of J with length at least three and with both ends in $W(A)$, some internal vertex of P belongs to the same set W_t as one of the ends of P. By the sixth bullet in Step 3, every vertex in $V(J) \setminus W(A)$ has two nonadjacent neighbours in $W(A)$.

It remains to verify that the fourth bullet in the definition of a border holds. To do so, we first must order the vertices of W_t for $t \in V(A)$: order W_t as $\{x_1, \ldots, x_n\}$ so that $x_1 = t$ and $d_J(x_2) \geq \cdots \geq d_J(x_n)$. Now suppose, contrary to the fourth bullet in the definition of a border, that there exists $t \in V(A)$ and $v \in V(J) \setminus W_t$ such that, with $W_t = \{x_1, \ldots, x_n\}$ ordered as above, v is adjacent to x_j but not to x_i for some $j > i$. Since $d_J(x_i) \geq d_J(x_j)$, there exists some vertex $u \in V(J) \setminus (W_t \cup \{v\})$ adjacent to x_i but not to x_j. But now $\{u, v, x_i, x_j\}$ induces a 4-hole or a path that violates the fourth bullet of Step 3. So v is adjacent to x_i for all $1 \leq i < j$. Therefore the fourth bullet in the definition of a borders holds, and hence $(J, A, (W_t : t \in V(A)))$ is a border. An analogous argument shows that $(K, B, (W_t : t \in V(B)))$ is also a border.

Next we show that H is a blow-up of an ℓ-frame. For each $i \in \{1, \ldots, r\}$ and each $j \in \{1, \ldots, (\ell-3)/2 - 1\}$, fix $p_i^j \in M_i^j$ such that p_i^j is of maximum degree in G among all vertices of M_i^j, let p_i^0 be the unique neighbour of p_i^1 in $V(A)$, and let $p_i^{(\ell-3)/2}$ be the unique neighbour of $p_i^{(\ell-3)/2-1}$ in $V(B)$. For $i \in \{1, \ldots, r\}$ set $P_i = G[\{p_i^0, \ldots, p_i^{(\ell-3)/2}\}]$, and let $F = A \cup B \cup P_1 \cup \cdots \cup P_r$.

We claim that F is an ℓ-frame with sides A, B and bars $P_1, \ldots, P_r$. As argued in an earlier paragraph, A and B are threshold graphs with the same number of (at least 3) vertices. By the third bullet in Step 3, for distinct $i, j \in \{1, \ldots, r\}$, we have that a_i, a_j are adjacent if and only if b_i, b_j are nonadjacent. Clearly for all $i \in \{1, \ldots, r\}$ we have that: for each $j \in \{0, \ldots, (\ell-3)/2 - 1\}$, some vertex of M_i^j has a neighbour in M_i^{j+1}, and for each $j \in \{1, \ldots, (\ell-3)/2\}$, some vertex of M_i^j has a neighbour in M_i^{j-1}. It follows then by our choice of vertices p_i^j that for each $i \in \{1, \ldots, r\}$ the set $\{p_i^0, \ldots, p_i^{(\ell-3)/2}\}$ induces a path $p_i^0, \ldots, p_i^{(\ell-3)/2}$ of length $(\ell-3)/2 - 1$ with ends a_i, b_i. Clearly these paths are all pairwise vertex disjoint, and the only edges between them are those edges of A and B. Thus F is an ℓ-frame with sides A, B and bars $P_1, \ldots, P_r$.

Let s, t be vertices of F, let S, T be the unique sets among $A_1, \ldots, A_r, B_1, \ldots, B_r, M_i^j$ (where $1 \leq i \leq r$ and $1 \leq j \leq (\ell-3)/2$) such that $s \in S$ and $t \in T$, and suppose that $S \neq T$. So s and t are distinct. If $s, t \in V(A)$ or $s, t \in V(B)$, then S is complete to T by the third bullet in Step 3, and otherwise $G[S, T]$ is a half-graph by Step 5. It is easily seen that: $V(J)$ is anticomplete to $V(K)$; $V(J) \cap V(H) = W(A)$; $V(K) \cap V(H) = W(B)$; and $V(J) \cap V(K) = \emptyset$. Therefore G is a bordered blow-up of the ℓ-frame F, and in particular G is the composition of H, J, K. This completes the proof of correctness.

Finally, we analyse the running time. Let $n = |V(G)|$ and $m = |E(G)|$. Step 1 takes $\mathcal{O}(1)$ time. Computing a BFS-tree, checking whether a graph is chordal and checking whether a graph is a threshold graph can all be done in $\mathcal{O}(n + m)$ time, and therefore

Step 2 takes $\mathcal{O}(n+m)$ time. The complexity of Step 3 is dominated by the fourth bullet, which can be done in $\mathcal{O}(n^5)$ time, and therefore Step 3 takes $\mathcal{O}(n^5)$ time. Step 4 similarly takes $\mathcal{O}(n^5)$ time. The complexity of Step 5 is dominated by the complexity of checking whether $\mathcal{O}(n^2)$ pairs of half-graphs are compatible, which takes $\mathcal{O}(n^6)$ time. Steps 2 through 5 are executed $\mathcal{O}(n)$ times, and therefore the algorithm has running time $\mathcal{O}(n^7)$. $\square$

We now use the above algorithm that decides whether a graph is a bordered blow-up of an ℓ-frame to recognise ℓ-holed graphs, where ℓ is odd and $\ell \geq 7$. We restate the following Lemma from Section 3.5.2.

Lemma 3.52. *There exists an algorithm with the following specifications:*

INPUT: *A graph G.*

OUTPUT: *A family $\mathcal{L}$ of induced subgraphs of G that satisfies the following properties:*

- *G is ℓ-holed for some odd $\ell \geq 7$ if and only if all the graphs in $\mathcal{L}$ are ℓ-holed.*
- *The graphs in $\mathcal{L}$ have no clique cutset.*
- *The number of graphs in $\mathcal{L}$ is $\mathcal{O}(|V(G)|)$.*

RUNNING TIME: $\mathcal{O}(|V(G)|^3)$.

Theorem 3.64. *There exists an algorithm with the following specifications:*

INPUT: *A graph G and an odd integer $\ell \geq 7$.*

OUTPUT: *Yes if G is ℓ-holed, and No otherwise.*

RUNNING TIME: $\mathcal{O}(n^8)$.

Proof. Consider the following algorithm.

Step 1. Execute the algorithm of Lemma 3.52 with input G, and let $\mathcal{L}$ be its output.

Step 2. For each graph $G \in \mathcal{L}$, let G' be the graph obtained from G by removing all universal vertices, and then let $\mathcal{L}' = \{G' : G \in \mathcal{L}\}$.

Step 3. Check that each graph in $\mathcal{L}'$ is a complete graph, a ring of length ℓ (using Lemma 3.55) or a bordered blow-up of a ℓ-frame (using Lemma 3.63); if not, then output No, and otherwise, output Yes.

The correctness of this algorithm follows from Lemmas 3.9 and 3.10 and from the fact that (by Lemma 3.52) G is ℓ-holed if and only if all of the graphs in $\mathcal{L}'$ are ℓ-holed. □

Theorem 3.65. *There exists an algorithm with the following specifications:*

INPUT: *A graph G.*

OUTPUT: *Yes if G is ℓ-holed for some odd $\ell \geq 7$, and No otherwise.*

RUNNING TIME: $\mathcal{O}(n^8)$.

Proof. Run the algorithm of Theorem 3.57 and let H be a hole of G if it finds one. If it finds no hole, then output Yes, and if $|V(H)| < 7$ or if $|V(H)|$ is even, then output No. Now run the algorithm of Theorem 3.64 with the graph G and integer $|V(H)|$ as input. If from that algorithm we obtain the output No, then output No. Otherwise, output Yes. □

3.5.4 Recognition without decomposition

We conclude this chapter with a discussion on how to recognise ℓ-holed graphs without making use of decomposition.

Consider the following problem: given a graph G and two vertices s, t of G, does G contain an induced path between s and t whose length is greater than the length of a shortest path between s and t? Let us call this problem the *non-shortest induced path problem*. Suppose there exists an algorithm A that solves the non-shortest induced path problem in $\mathcal{O}(f(n))$ time. Such an algorithm may be used as a subroutine in deciding whether a graph is ℓ-holed, in the following way: for each three-vertex induced path a, b, c of G, let G' be the graph obtained from G by deleting b and all of the neighbours of b besides a and c; if there is no path in G' between a and c, then move on to the next three-vertex path; now check that the distance between a and c in G' is $\ell - 2$ (if not, then G is not ℓ-holed, and we stop); and then apply algorithm A to check that there is no induced path in G' between a and c whose length is greater than $\ell - 2$ (if such a path exists, then G is not ℓ-holed, and we stop). If after considering all three-vertex induced paths of G we have not determined that G is not ℓ-holed, then G is ℓ-holed. Thus, we may recognise ℓ-holed graphs in $\mathcal{O}(n^3 \cdot f(n))$ time.

Berger, Seymour and Spirkl [3] gave an algorithm that solves the non-shortest induced paths problem in $\mathcal{O}(n^{18})$ time, which leads to an $\mathcal{O}(n^{21})$-time algorithm for recognising ℓ-holed graphs. Chiu and Lu [9] significantly improved this running time by giving an algorithm that solves the non-shorted induced paths problem in $\mathcal{O}(n^{4.75})$-time, yielding an $\mathcal{O}(n^{7.75})$-time algorithm for recognising ℓ-holed graphs. Note that these algorithms recognise ℓ-holed graphs for any $\ell \geq 4$, while the algorithms presented in Sections 3.5.2 and 3.5.3 work only for odd $\ell \geq 7$.

Chapter 4

Even-hole-free circular-arc graphs

An *intersection graph* of a family $\mathcal{F} = \{F_1, \ldots, F_n\}$ of sets is a graph with vertices $v_1, \ldots, v_n$ such that, for distinct $i, j \in \{1, \ldots, n\}$, the vertices v_i and v_j are adjacent if and only if $F_i \cap F_j \neq \emptyset$. For example, the intersection graph of $\{\{1,2\}, \{2,3\}, \{1,3\}\}$ is the complete graph on three vertices.

An intersection graph of a family of intervals of the real line is called an *interval graph* (the family of intervals being an *interval model* for this graph; note that an interval model for a particular interval graph is not necessarily unique). Many real-world problems may be modelled as a problem to be solved on interval graphs. For instance, consider the following toy problem. Suppose there are n people, each of whom has booked a taxi; one person would like to be picked up at time s_1 and dropped off at time t_1, the next person picked up at time s_2 and dropped off at time t_2, and so on. The taxi service would like to minimise the number of taxi drivers needed to carry out these rides. Consider the intersection graph of the family of intervals $\{[s_1, t_1], [s_2, t_2], \ldots, [s_n, t_n]\}$, and assign colours to each of its vertices in such a way that no two adjacent vertices receive the same colour. By taking colours to be taxi drivers, one obtains an assignment of taxi drivers to rides with the property that no driver is assigned to two rides that overlap in time; thus, the chromatic number of this interval graph is the minimum number of drivers needed.

Interval graphs are chordal graphs, and therefore as is the case for chordal graphs, many problems that are NP-complete for the class of all graphs, such as colouring and finding the size of a maximum clique or a maximum stable set, are polynomial-time solvable when restricted to the class of interval graphs. Deciding whether a given graph G is an interval graph can be done in time $\mathcal{O}(|V(G)| + |E(G)|)$ [5].

The structure of interval graphs is well understood, and there exist several characterisations of interval graphs. An *asteroidal triple* (or *AT* for short) is a set of three pairwise nonadjacent vertices such that there is a path between each pair of these vertices that contains no neighbour of the third vertex. A graph is *AT-free* if no three of its vertices form an asteroidal triple.

Theorem 4.1 (Lekkerkerker and Boland [40]). *A graph is an interval graph if and only if it is chordal and AT-free.*

In the same paper, Lekkerkerker and Boland also provide a forbidden induced subgraph characterisation for interval graphs. A *net* (or *2-net*) is the graph on six vertices a, b, c, x, y, z with edge set $\{ab, bc, ac, ax, by, cz\}$. See Figure 4.3 for depictions of the bipartite claw, umbrella, k-nets ($k \geq 3$) and k-tents ($k \geq 3$).

Theorem 4.2 (Lekkerkerker and Boland [40]). *A graph G is an interval graph if and only if G contains no bipartite claw, umbrella, k-net (for $k \geq 2$), k-tent (for $k \geq 3$) or hole as an induced subgraph.*

We now turn to circular-arc graphs, a generalisation of interval graphs that form the focus of this chapter.

A *circular-arc model* $M = (C, \mathcal{A})$ is a circle C together with a collection $\mathcal{A}$ of arcs of C. A *circular-arc graph* G is the intersection graph of the arcs of a circular-arc model. Clearly every interval graph is a circular-arc graph. Circular-arc graphs and a number of subclasses have received much attention, both from a structural and algorithmic point of view. One such subclass is the class of proper circular-arc graphs. A circular-arc graph is *proper* if it is the intersection graph of a circular-arc model in which no arc properly contains another. Tucker gave a characterisation of proper circular-arc graphs in terms of forbidden induced subgraphs. See Figure 4.1 for depictions of the graphs armchair and stirrer.

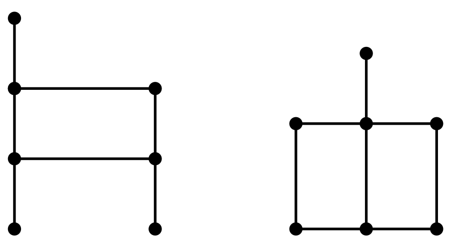


Figure 4.1: An armchair (left) and a stirrer (right).

Theorem 4.3 (Tucker [50]). *A graph G is a proper circular-arc graph if and only if it contains none of the following as induced subgraphs:*

- *$C_n \cup K_1$, $n \geq 4$;*
- *$\overline{C_{2j}}$, $j \geq 3$;*
- *$\overline{C_{2j+1} \cup K_1}$, $j \geq 1$ (this is a claw if $j = 1$);*
- *a net, or the complement of a net together with an isolated vertex;*
- *the complement of a stirrer, 3-tent, armchair, or bipartite claw.*

Tucker also gave the following characterisation of general circular-arc graphs in terms of circular orderings of vertices, but it remains open to characterise circular-arc graphs in terms of forbidden induced subgraphs. It also remains open to provide a forbidden induced subgraph characterisation for the class of chordal circular-arc graphs.

Theorem 4.4 (Tucker [49])**.** *A graph G is a circular-arc graph if and only if there is a circular ordering $v_1, \dots, v_n$ of its vertices such that, for $i < j$, if $v_i v_j$ is an edge of G then either $v_{i+1}, \dots, v_j \in N(v_i)$ or $v_{j+1}, \dots, v_i \in N(v_j)$.*

For a number of subclasses of circular-arc graphs, forbidden induced subgraph characterisations are known. A circular-arc graph G is *normal Helly* if there exists a circular-arc model for G no three arcs of which cover the circle. Cao, Grippo and Safe gave a forbidden induced subgraph characterisation for the class of normal Helly circular-arc graphs. See Figure 4.2 for depictions of G_1, G_2, G_3, G_4 and the domino. A *k-wheel* (for $k \geq 4$) is a hole of length k together with a vertex that is complete to the hole. By C_k^* we denote the graph consisting of a hole of length k together with a vertex that has no neighbour in the hole.

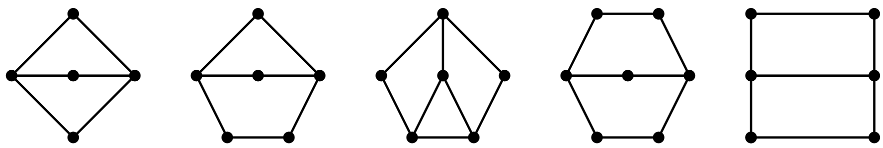


Figure 4.2: From left to right: the graphs G_1, G_2, G_3, G_4 and the domino.

Theorem 4.5 (Cao, Grippo and Safe [8])**.** *A graph G is a normal Helly circular-arc graph if and only if G contains no induced bipartite claw, umbrella, k-net for any $k \geq 2$, k-tent for any $k \geq 3$, k-wheel for any $k \geq 4$ and no G_1, G_2, G_3, G_4, domino, $\overline{C_6}$ or C_k^* for any $k \geq 4$.*

A number of problems that are NP-complete in general become polynomial-time solvable when restricted to circular-arc graphs. Gavril [29, 30] gave polynomial-time algorithms for the maximum stable set, maximum clique, and minimum clique cover problems on circular-arc graphs. However, the colouring problem remains NP-complete for circular-arc graphs [28], and even for Helly circular-arc graphs [31].

Circular-arc graphs appear in the study of a number of complex hereditary graph classes. For instance, proper circular-arc graphs constitute an important subclass in Chudnovsky and Seymour's structural study of claw-free graphs [14]. In [6], even-hole-free graphs that are also pan-free (a *pan* is a hole together with a vertex that has exactly one neighbour in the hole) are decomposed by clique cutsets into unit circular-arc graphs, which gives way to a linear-time recognition algorithm and a polynomial-time colouring algorithm for this class. Motivated by this result, in this chapter we work towards a characterisation of even-hole-free circular-arc graphs. As mentioned earlier, it remains open to characterise chordal circular-arc graphs in terms of forbidden induced subgraphs, and therefore our focus will be on even-hole-free circular-arc graphs that are not chordal.

The main result of this chapter is a partial characterisation of even-hole-free circular-arc graphs that are not chordal.

4.1 A partial characterisation of even-hole-free circular-arc graphs

Let (H, x) be a wheel, and let $x_1, \dots, x_r$ be the neighbours of x in H. A subpath of H with ends x_i and x_j is a *sector* if the interior of this path contains no neighbour of x. A *short sector* is a sector of length 1, and a *long sector* is a sector of length greater than 1. A *circular-arc wheel* (*CA-wheel* for short) is a wheel that has at most one long sector. In other words, a wheel (H, x) is a CA-wheel if $N_H(x)$ induces H or a subpath of H. A *non-circular-arc wheel* (*NCA-wheel* for short) is a wheel that is not a CA-wheel. A *0-wheel* is any graph that consists of a hole together with an additional vertex that has no neighbour in this hole.

Let $\mathcal{C}$ be the class of graphs that are (3PC, NCA-wheel, 0-wheel)-free and furthermore do not contain any of the graphs in Figure 4.3 as induced subgraphs. So that there is no ambiguity regarding k-nets and k-tents, we define them explicitly. For an integer $k \geq 3$, a *k-net* is any graph G consisting of a path $x_1, x_2, \dots, x_k$ on k vertices with four additional vertices a, b, c, d such that $N(a) = \{x_1\}$, $N(b) = \{c, x_1, x_2, \dots, x_k\}$, $N(c) = \{b\}$ and $N(d) = \{x_k\}$. For an integer $k \geq 3$, a *k-tent* is any graph G consisting of a path $x_1, x_2, \dots, x_k$ on k vertices with three additional vertices a, b, c such that $N(a) = \{b, c, x_1, x_2, \dots, x_{k-1}\}$, $N(b) = \{a, c, x_2, x_3, \dots, x_k\}$ and $N(c) = \{a, b\}$.

In this section we work towards resolving the following conjecture of Kathie Cameron, Kristina Vušković and the present author.

Conjecture. *Let G be an even-hole-free graph that is not chordal. Then G is a circular-arc graph if and only if G belongs to $\mathcal{C}$.*

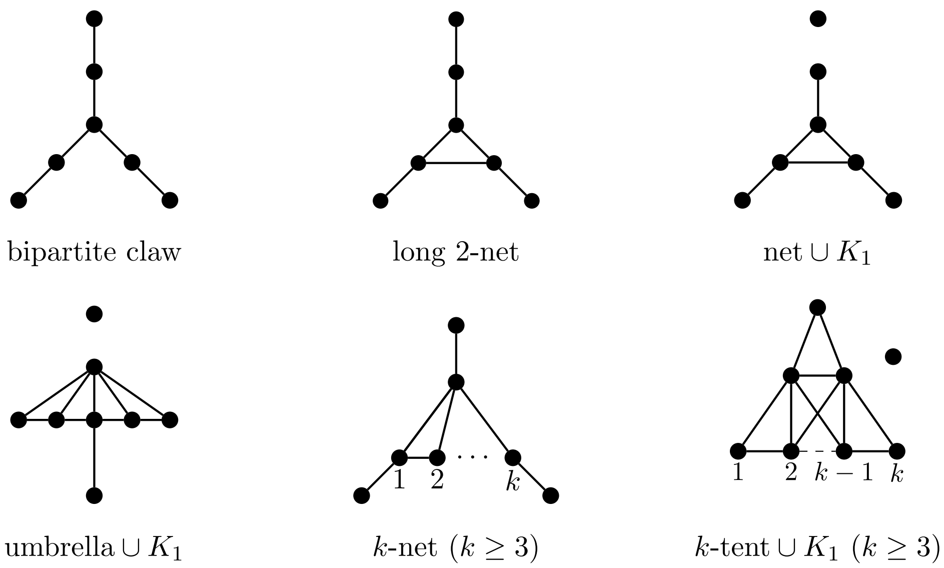
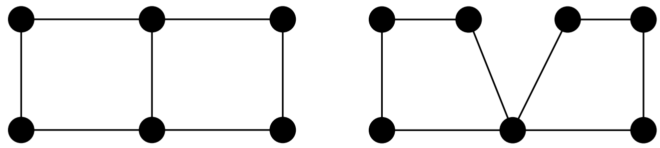


Figure 4.3: Some forbidden induced subgraphs for graphs in $\mathcal{C}$.

4.1.1 Circular-arc graphs belong to $\mathcal{C}$

In this section we show that every circular-arc graph is (3PC, NCA-wheel, 0-wheel)-free and also contains none of the graphs in Figure 4.3. It is easily seen that if H is a hole in a graph G, then the arcs of a circular-arc model for G that correspond to the vertices of H cover the circle, and therefore every vertex not in H has a neighbour in H; thus, 0-wheels are not circular-arc. In [47], it is shown that the graphs bipartite claw, net $\cup K_1$, k-net (for $k \geq 3$), umbrella $\cup K_1$ and k-tent $\cup K_1$ (for $k \geq 3$) are not circular-arc. It is easily checked that the long 2-net is not circular-arc (in attempting to build a circular-arc model for the long 2-net, one sees that the three arcs corresponding to the triangle of the long 2-net must cover the circle; but the long 2-net contains one vertex that has no neighbour in the triangle). Thus, it remains to show that 3PC's and NCA-wheels are not circular-arc.

In order to show that 3PC's and NCA-wheels are not circular-arc, we show that they are not "1-perfectly orientable". Before we can define 1-perfectly orientable graphs, we



F_1 F_2

Figure 4.4: Some non-1-p.o. graphs.

need some terminology.

An *orientation* D of a graph G assigns to each edge $uv \in E(G)$ an ordered pair; in particular, either (u, v) or (v, u). If $D(uv) = (u, v)$ then we write $u \to v$. A *tournament* is an orientation of a complete graph. We say that an orientation of a graph G is *1-perfect* if the out-neighbourhood of every vertex induces a tournament, and we say that G is *1-perfectly orientable* (or *1-p.o.*) if there exists a 1-perfect orientation of G. A graph is *non-1-p.o.* if it is not 1-p.o. A cycle C in an oriented graph G is *cyclically oriented* if every vertex of C has exactly one out-neighbour in C. A graph G is *hole-cyclically orientable* if there exists an orientation of G in which every hole of G is cyclically oriented. The following two results show the relations between circular-arc graphs, 1-perfectly orientable graphs, and hole-cyclically orientable graphs.

Theorem 4.6 (Urrutia and Gavril [51]). *Circular-arc graphs are 1-perfectly orientable.*

Theorem 4.7 (Hartinger and Milanič [34]). *In every 1-perfect orientation of a 1-perfectly orientable graph G, every chordless cycle of length at least 4 is oriented cyclically. In particular, 1-perfectly orientable graphs are hole-cyclically orientable.*

Let G be a graph and let xy be an edge of G. To *contract* the edge xy means to remove the vertices x and y from G and to add a new vertex whose neighbourhood is $(N(x) \cup N(y)) \setminus \{x, y\}$. A graph G is *contractible* to a graph F if F can be obtained from G by a sequence of edge contractions. A graph F is an *induced minor* of a graph G if F can be obtained from G by deleting vertices and contracting edges. The following result shows that the class of 1-p.o. graphs is closed under taking induced minors.

Theorem 4.8 (Hartinger and Milanič [34]). *The class of 1-perfectly orientable graphs is closed under vertex deletion and edge contraction.*

In [34], a number of non-1-p.o. graphs are given. Among them are the graphs $\overline{C_6}$ (i.e. the prism with the smallest number of vertices) and $\overline{K_2 + C_3}$ (i.e. the theta with the smallest number of vertices). They also show that graphs F_1 and F_2, depicted in Figure 4.4, are not 1-p.o.

Lemma 4.9. *If G is a circular-arc graph, then G is $3PC$-free.*

Proof. Since $\overline{C_6}$ (resp. $\overline{K_2 + C_3}$) can be obtained from a larger prism (resp. theta) by a series of edge contractions, it follows from Theorem 4.8 that all prisms and thetas are not 1-p.o. Then by Theorem 4.6, prisms and thetas are not circular-arc. Consider a pyramid Σ induced by paths $P_1 = x_1, \ldots, y$, $P_2 = x_2, \ldots, y$ and $P_3 = x_3, \ldots, y$. We show that Σ is not hole-cyclically orientable, and therefore not circular-arc (by Theorems 4.6 and 4.7). Suppose that Σ is hole-cyclically orientable with orientation D. Without loss of generality, say that P_1 is oriented as $x_1 \to \cdots \to y$. Since $P_1 \cup P_2$ is a hole, P_2 must be oriented as $y \to \cdots \to x_2$. Similarly since $P_2 \cup P_3$ is a hole, P_3 must be oriented as $x_3 \to \cdots \to y$. But then $P_1 \cup P_3$ is a hole that is not oriented cyclically, a contradiction. □

Lemma 4.10. *If G is a circular-arc graph, then G is NCA-wheel-free.*

Proof. Let G be a graph that contains a NCA-wheel (H, x) with rim $H = x_1, \ldots, x_n, x_1$. Suppose that G is circular-arc. Let $S_1 = x_i, \ldots, x_j$ and $S_2 = x_k, \ldots, x_l$ be distinct long sectors of (H, x) and w.l.o.g. assume that $i < j \leq k < l$. Let P_1 (resp. P_2) be the subpath of H with endpoints x_j and x_k (resp. x_l and x_i) and interior in $H \setminus (S_1 \cup S_2)$. By contracting every edge of P_1 and P_2 we obtain a theta, which is contractible to the non-1.p.o. graph $\overline{K_2 + C_3}$. It now follows from Theorem 4.8 that G is not 1-p.o., contradicting Theorem 4.6. □

Putting together everything so far, we have the following.

Theorem 4.11. *If G is a circular-arc graph, then G is (3PC, NCA-wheel, 0-wheel)-free and contains none of the graphs in Figure 4.3.*

4.1.2 Graphs from $\mathcal{C}$ with no crossing vertices are circular-arc

In this section we show that even-hole-free graphs from $\mathcal{C}$ that have no so-called crossing vertices are circular-arc.

Throughout this section we use the following notation. Let $H = x_1, \ldots, x_n, x_1$ be a hole in a graph G. A vertex $x \in V(G) \setminus V(H)$ is of *Type* i with respect to H if $|N_H(x)| = i$. We say that x is of *Type* $3.i$ w.r.t. H if x is of Type 3 w.r.t. H and $N_H(x) = \{x_{i-1}, x_i, x_{i+1}\}$, i.e. x is a twin of x_i in $G[V(H) \cup \{x\}]$. For $i \in 1, \ldots, n$, let

X_i^H be the set that consists of x_i together with all vertices of Type 3.i w.r.t. H. Let $X_H = \cup_{i=1}^n X_i^H$, U_H the set of all vertices of Type n w.r.t. H, and Y_H the set of all vertices of Type j w.r.t. H, for $3 < j < n$. When the underlying hole H is clear from context, we may write $X_1, \dots, X_n$, X, Y and U instead of $X_1^H, \dots, X_n^H$, X_H, Y_H and U_H respectively.

We now study how vertices outside of a hole attach to a hole in graphs from the class $\mathcal{C}$ that are also even-hole-free. We will make use of the following fact about even-hole-free graphs. A wheel (H, x) is *even* if x has an even number of neighbours in H.

Lemma 4.12. *If G an even-hole-free graph, then G contains no even wheel.*

Proof. By Theorem 1.2 together with the observation that even-hole-free graphs are odd-signable. □

Lemma 4.13. *Let G be an (even hole, 3PC, NCA-wheel)-free graph and let H be a hole of G. If $x \in V(G) \setminus V(H)$, then for some $i \in \{0, 1, 2\}$ or odd $i \in \{3, \dots, n\}$, x is of Type i w.r.t. H. Furthermore, if $i \notin \{0, n\}$ then the subgraph of H induced by the vertex set $N_H(x)$ is a path.*

Proof. Let $i = |N_H(x)|$. By definition, x is of Type i w.r.t. H. If $i \geq 3$, then i is odd, for otherwise (H, x) is an even wheel, contrary to Lemma 4.12. If $i \notin \{0, n\}$ and $N_H(x)$ does not induce a path in H, then $V(H) \cup \{x\}$ induces a theta or a NCA-wheel. □

From now on, when G is an (even hole, 3PC, NCA-wheel)-free graph, H is a hole of G, $x \in V(G) \setminus V(H)$ is of Type i w.r.t. H and $i \notin \{0, n\}$, then we denote by H_x the subgraph of H induced by $N_H(x)$. So by Lemma 4.13, H_x is a path. Furthermore, if $i \geq 3$ then we denote by H_x^* the subpath of H_x induced by the interior vertices of H_x. Going forward, we may use the fact that H_x is a path without justification.

Lemma 4.14. *Let G be an (even hole, 3PC, NCA-wheel)-free graph and let H be a hole of G. Let u and v be vertices of $G \setminus H$ such that u is of Type i w.r.t. H and v is of Type j w.r.t. H. Then the following hold:*

(i) if $|H_u \cap H_v| \geq 3$, then uv is an edge;
(ii) if $i \geq 3$, $j \geq 5$ and $|H_u \cap H_v| \geq 2$, then uv is an edge.

Proof. To prove (i), suppose that $|H_u \cap H_v| \geq 3$ but uv is not an edge, and let a and b be nonadjacent vertices in $H_u \cap H_v$. Then the vertex set $\{a, b, u, v\}$ induces a C_4 in G, a contradiction. This proves (i).

To prove (ii), assume $i \geq 3$, $j \geq 5$ and $|H_u \cap H_v| \geq 2$. By (i) we may in fact assume that $|H_u \cap H_v| = 2$. Suppose that uv is not an edge. Let a and b be the vertices in $H_u \cap H_v$. If a, b are nonadjacent, then the vertex set $\{a, b, u, v\}$ induces a C_4, a contradiction. So a and b are adjacent. But then $(H \setminus H_u^*) \cup \{u, v\}$ induces an even wheel with center v, contrary to Lemma 4.12. Therefore (ii) holds. □

Lemma 4.15. *Let G be an (even hole, 3PC, NCA-wheel)-free graph and let H be a hole of G. Let u and v be adjacent vertices of $G \setminus H$. Let u be of Type i w.r.t. H and v be of Type j w.r.t. H. Then the following hold:*

(i) if $i = 1$ and $j \geq 1$ then $H_u \subseteq H_v$;
(ii) if $i = j = 2$ then $H_u = H_v$;
(iii) if $i = 2$ and $j \geq 3$ then $H_u \cap H_v \neq \emptyset$;
(iv) if $i, j \geq 3$ then $|H_u \cap H_v| \geq 2$, and furthermore, exactly one of the following holds:

- *$|H_u \cap H_v|$ is odd; or*
- *$H_u \setminus H_v \neq \emptyset$, $H_v \setminus H_u \neq \emptyset$, $H_u \cap H_v$ induces a path, and $|H_u \cap H_v|$ is even.*

Proof. We may assume that $i, j \notin \{0, n\}$. Let $H_u = u_1, \ldots, u_i$ and $H_v = v_1, \ldots, v_j$. Suppose that $i = 1$, $j \geq 1$, but that $H_u \not\subseteq H_v$. Since $|H_u| = 1$, it follows that $H_u \cap H_v = \emptyset$. Observe that $\{v_1, v_j\} \cap N_H(u_1) = \emptyset$, for otherwise one of the vertex sets $\{u, u_1, v, v_1\}$ or $\{u, u_1, v, v_j\}$ would induce a C_4 in G, a contradiction. But then if $j = 1$ then $H \cup \{u, v\}$ induces a $3PC(u_1, v_1)$, if $j = 2$ then $H \cup \{u, v\}$ induces a $3PC(vv_1v_2, u_1)$, and if $j \geq 3$ then G contains a $3PC(u_1, v)$. This proves (i).

Suppose (ii) does not hold. If $H_u \cap H_v = \emptyset$, then the vertex set $V(H) \cup \{u, v\}$ induces a $3PC(uu_1u_2, vv_1v_2)$, so $H_u \cap H_v$ contains only one vertex. But then the vertex set $H \cup \{u, v\}$ induces an even wheel of G, contrary to Lemma 4.12. Thus (ii) holds. Part (iii) holds, for otherwise G contains a $3PC(uu_1u_2, v)$.

Finally, to prove (iv), assume $i, j \geq 3$. If $|H_u \cap H_v| \leq 1$ then $(H \setminus H_v^*) \cup \{u, v\}$ induces an even wheel with center u, a contradiction. Therefore $|H_u \cap H_v| \geq 2$.

It remains to prove that exactly one of the two bullets of part (iv) hold. If $|H_u \cap H_v|$ is odd then we are done, so suppose that $|H_u \cap H_v|$ is even. Further, suppose that $H_u \setminus H_v = \varnothing$, i.e. $H_u \subseteq H_v$. Thus $|H_u \cap H_v| = |H_u|$, and therefore $|H_u|$ is even. But then the vertex u contradicts Lemma 4.13. So $H_u \setminus H_v \neq \emptyset$, and by symmetry $H_v \setminus H_u \neq \emptyset$. Suppose that $H_u \cap H_v$ is not a path. Since both H_u and H_v are paths by Lemma 4.13, it follows that $H_u \cup H_v = H$. Therefore $|H_v| = |H| - |H_u| + |H_u \cap H_v|$. Since $|H|$ and $|H_u|$ are both odd and $|H_u \cap H_v|$ is even, it follows that $|H_v|$ is even. But then (H, v) is an even wheel, a contradiction. So $H_u \cap H_v$ is a path, and this completes the proof of (iv). □

We are now ready to prove the main result of this section. Let G be an even-hole-free graph that belongs to $\mathcal{C}$ and that is not chordal. Let $H = x_1, x_2, \dots, x_n, x_1$ be a hole of G of maximum length that furthermore maximises $|X|$. We say that two vertices $u, v \in Y$ are *crossing* (or that u and v *cross*) if the graph induced by $H_u \cap H_v$ is disconnected. Our result is that if G has no crossing vertices, then G is circular-arc. So assume that G has no crossing vertices.

For each $i \in \{1, \dots, n\}$, we define the following sets:

- W_i^- is the set of all vertices complete to $\{x_i, x_{i-1}\}$ and adjacent to a vertex of $V(G) \setminus \{x_{i-1}\}$ that is adjacent to x_i but not to x_{i-1}.
- W_i^+ is the set of all vertices complete to $\{x_i, x_{i+1}\}$ and adjacent to a vertex of $V(G) \setminus \{x_{i+1}\}$ that is adjacent to x_i but not to x_{i+1}.
- O_i^1 is the set of all Type 1 vertices w.r.t. H that are adjacent to x_i.
- O_i^2 is the set of all Type 2 vertices w.r.t. H that are adjacent to x_i and x_{i+1} and that do not belong to $W_i^+ \cup W_{i+1}^-$.

Lemma 4.16. *W_i^- and W_i^+ are cliques.*

Proof. By symmetry it suffices to prove that W_i^- is a clique. Suppose otherwise and let w_1 and w_2 be two nonadjacent vertices of W_i^-. By definition, for $j \in \{1, 2\}$, there exists a neighbour z_j of w_j such that z_j is adjacent to x_i but nonadjacent to x_{i-1}. If w_2 is adjacent to z_1, then $\{z_1, w_1, x_{i-1}, w_2\}$ induces a 4-hole, a contradiction. So w_2 is nonadjacent to z_1 and by symmetry w_1 is nonadjacent to z_2. In particular, $z_1 \neq z_2$, and since $\{z_1, z_2\} \cap H \subseteq \{x_{i+1}\}$, at most one of z_1, z_2 belongs to H. We establish the following facts.

(1) $\{w_1, w_2, z_1, z_2\}$ is anticomplete to x_{i-2}.

Proof of (1): Suppose not. First suppose that z_1 is adjacent to x_{i-2}. Since z_1 is adjacent to x_{i-2} and x_i but not to x_{i-1}, it follows that $\{x_{i-2}, x_{i-1}, x_i, z_1\}$ induces a 4-hole, a contradiction. So z_1 is nonadjacent to x_{i-2} and by symmetry it follows that $\{z_1, z_2\}$ is anticomplete to x_{i-2}. So without loss of generality w_1 is adjacent to x_{i-2}. By Lemma 4.14, w_2 is not adjacent to x_{i-2}.

Suppose first that w_1 is adjacent to x_{i+1}. Then without loss of generality $z_1 = x_{i+1}$, and hence w_2x_{i+1} is not an edge. In particular, by Lemma 4.14, $N_H(w_2) = \{x_{i-1}, x_i\}$. But then the hole in $H \cup \{w_1\}$ that contains w_1, together with w_2 induces a 0-wheel, a contradiction. Therefore w_1 is not adjacent to x_{i+1}, and in particular $z_1 \notin H$. Since $U = \emptyset$, by Lemma 4.13 $|H \setminus H_{w_1}| \geq 2$ and $H \setminus H_{w_1}$ is a path. Let $P = H_{w_1} \setminus x_i$. Let $t, t' \in H \setminus H_{w_1}$ be such that t is adjacent to the end of P different from x_{i-1}, and t' is adjacent to t. Since $|H| \geq 5$ and $|H \setminus H_{w_1}| \geq 2$, $P \cup \{t, t'\}$ induces a path. We claim that either $P \cup \{w_1, w_2, z_1, t\}$ induces a k-net for some $k \geq 3$, or $P \cup \{w_1, w_2, z_1, t, t'\}$ induces a long 2-net.

Suppose that $|H_{w_1}| = 3$, i.e. $N_H(w_1) = \{x_i, x_{i-1}, x_{i-2}\}$. If $\{z_1, w_2\}$ is anticomplete to $\{t, t'\}$, then $P \cup \{w_1, w_2, z_1, t, t'\}$ induces a long 2-net, a contradiction. If z_1 is adjacent to t, then $\{z_1, w_1, x_{i-2}, t\}$ induces a 4-hole, a contradiction. So z_1 is nonadjacent to t. If w_2 is adjacent to t, then $\{w_2, x_{i-1}, x_{i-2}, t\}$ induces a 4-hole, a contradiction. So w_2 is not adjacent to t. If both w_2 and z_1 are adjacent to t', then $\{w_1, x_{i-1}, x_{i-2}, t, t', z_1, w_2\}$ induces a $3PC(w_1x_{i-1}x_{i-2}, t')$, a contradiction. So not both w_2 and z_1 are adjacent to t'. Suppose that w_2 is adjacent to t'. But then the hole in $H \cup \{w_2\}$ that contains w_2, together with the vertex z_1, induces a 0-wheel, a contradiction. It follows that z_1t' is an edge, and this is the only edge between $\{z_1, w_2\}$ and $\{t, t'\}$. But now $\{x_{i-2}, w_1, z_1, t', t, w_2\}$ induces a 0-wheel, a contradiction.

So $|H_{w_1}| \geq 5$, and since $U = \emptyset$, this implies that $|H| \geq 7$. In particular, $|P| \geq 4$. If w_2 is adjacent to t or t', then $w_2 \in X \cup Y$. But then w_1 and w_2 contradict Lemma 4.14. So w_2 is anticomplete to $\{t, t'\}$. Suppose that z_1 is adjacent to t. If z_1 has no neighbour in P, then vertices t, z_1, w_1 together with the end of P different from x_{i-1} induces a 4-hole, a contradiction. So z_1 has a neighbour in P, and hence $|H_{z_1}| \geq 4$, so $z_1 \in Y$. But now z_1 and w_1 are two vertices of Y that cross, a contradiction. So z_1 is not adjacent to t. But now $P \cup \{w_1, w_2, z_1, t\}$ induces a k-net where $k = |H_{w_1}| - 1 \geq 3$, a contradiction. This completes the proof of (1).

(2) $\{w_1, w_2, z_1, z_2\}$ is anticomplete to x_{i-3}.

Proof of (2): Suppose not. Note first that since $|H| \geq 5$, $x_{i-3} \neq x_{i+1}$. If w_1 is adjacent to x_{i-3}, then since w_1 is nonadjacent to x_{i-2} by (1), $\{x_{i-3}, x_{i-2}, x_{i-1}, w_1\}$ induces a 4-hole, a contradiction. So w_1 is nonadjacent to x_{i-3} and it follows by symmetry that $\{w_1, w_2\}$ is anticomplete to x_{i-3}. So without loss of generality z_1 is adjacent to x_{i-3}. Observe that, by (1), $\{x_{i-3}, x_{i-2}, x_{i-1}, w_1, z_1\}$ induces a 5-hole. Since z_2 is anticomplete to $\{x_{i-2}, x_{i-1}, w_1\}$ and since G contains no 0-wheel, z_2 is adjacent to at least one of z_1, x_{i-3}. Suppose that z_2 is adjacent to both z_1 and x_{i-3}. Then the set $\{x_{i-1}, x_{i-2}, x_{i-3}, w_1, z_1, w_2, z_2\}$ induces a pyramid, a contradiction. So z_2 is not adjacent to both z_1 and x_{i-3}. If z_2 is adjacent to x_{i-3}, then $\{x_{i-3}, z_1, x_i, z_2\}$ induces a

4-hole, a contradiction. So z_2 is adjacent to z_1 and nonadjacent to x_{i-3}. But now the set $\{x_{i-1}, x_{i-2}, x_{i-3}, w_1, z_1, w_2, z_2\}$ induces a theta, a contradiction. This completes the proof of (2).

It now follows from (1) and (2) that $\{w_1, z_1, z_2, w_2, x_{i-1}, x_{i-3}\}$ induces a C_5^* if z_1z_2 is an edge, and $\{x_{i-3}, x_{i-2}, x_{i-1}, w_1, w_2, z_1, z_2\}$ induces a bipartite claw otherwise. In either case we obtain a contradiction, and this completes the proof. $\square$

A set $S \subseteq V(G)$ is *dominating (in G)* if every vertex of G outside of S has a neighbour in S. We extend this notion to induced subgraphs F of G by saying that F is dominating in G if $V(F)$ is dominating in G.

Lemma 4.17. *Let F be an induced subgraph of $G[W_i^- \cup O_i^1 \cup W_i^+]$, and set $V^- = W_i^- \cap F$ and $V^+ = W_i^+ \cap F$. If $P = v, \ldots, v'$ is an induced path in F where $v \in V^-$ and $v' \in V^+$, then P is dominating in F. If no vertex of F is universal in F, then the following hold:*

(i) $V^- \cap V^+ = \emptyset$.

(ii) If $v \in V^-$ and $v' \in V^+$, then $H_v \cap H_{v'} = \{x_i\}$.

(iii) If F is dominating in G, then both V^- and V^+ are nonempty.

Proof. Fix $v \in V^-$, $v' \in V^+$, and let P be an induced path from v to v' in F. Suppose that P is not dominating in F, i.e. there exists a vertex $u \in F$ such that $N[u] \cap P = \emptyset$. Let w be the vertex of P closest to v' that belongs to V^-, and let w' be the vertex of $P[w, v']$ closest to w that belongs to V^+. Then there is a hole H' in $H \cup P[w, w']$ that contains $P[w, w']$ but does not contain x_i. Since G contains no 0-wheel, u has a neighbour in H'. By our choice of u it follows that $N_{H'}(u) \subseteq H \backslash \{x_i\}$. So $u \in W_i^- \cup W_i^+$, but then u is adjacent to at least one of w, w' by Lemma 4.16, and hence $N[u] \cap P \neq \emptyset$, a contradiction. So for every induced path $P = v, \ldots, v'$ of F where $v \in V^-$ and $v' \in V^+$, P is dominating in F.

For the remainder of the proof we suppose that no vertex of F is universal in F. Contrary to (i), suppose there exists a vertex $v \in V^- \cap V^+$. But then the induced path $P = v$ is dominating in F and hence v is universal in F, a contradiction. This proves (i).

Contrary to (ii), suppose there exists $v \in V^-$ and $v' \in V^+$ such that $H_v \cap H_{v'} \neq \{x_i\}$. Clearly $x_i \in H_v \cap H_{v'}$, so there exists a vertex $x \in H \setminus \{x_i\}$ such that $x \in H_v \cap H_{v'}$. Since (by (i)) $x_{i-1} \in H_v \setminus H_{v'}$ and $x_{i+1} \in H_{v'} \setminus H_v$, it follows that if $v, v' \in Y$, then v and v' cross, a contradiction. So without loss of generality $v \notin Y$, and hence v is of

Type 2 or 3 w.r.t. H. By (i), v is not of Type 2 w.r.t. H. So v is of Type 3 w.r.t. H. But then $H_{v'} = H \setminus \{x_{i-1}\}$, contradicting Lemma 4.13. This proves (ii).

To prove (iii), suppose also that F is dominating in G. By (i), any neighbour of x_{i-1} in F must belong to V^- and any neighbour of x_{i+1} in F must belong to V^+. Therefore V^- and V^+ are nonempty. This proves (iii). □

We define the following notation. If v is a vertex outside of H such that H_v is a path of nonzero length, and x is an end of H_v, then for $i \in \{0,1,2\}$ we denote by $H^i_{v,x}$ the subpath of H of length i such that $H^i_{v,x} \cap H_v = \{x'\}$, where x' is the end of H_v different from x (for example, if $H_v = x_1, x_2, \ldots, x_5$, then $H^0_{v,x_1} = x_5$, $H^1_{v,x_1} = x_5, x_6$ and $H^2_{v,x_1} = x_5, x_6, x_7$). Such a subpath always exists by Lemma 4.13.

Lemma 4.18. *$G[W_i^- \cup O_i^1 \cup W_i^+]$ is chordal.*

Proof. Suppose not and let H' be a hole of $G[W_i^- \cup O_i^1 \cup W_i^+]$. First observe that x_i is complete to H', and since G contains no 0-wheel, every vertex of H has a neighbour in H'. It follows that every vertex of $H \setminus x_i$ has a neighbour in $H' \cap (W_i^- \cup W_i^+)$. So necessarily $H' \cap (W_i^- \cup W_i^+) \neq \emptyset$, and by Lemma 4.16, $|W_i^- \cap H'| \leq 2$ and $|W_i^+ \cap H'| \leq 2$. Furthermore, if $H' \cap (W_i^- \cup W_i^+)$ contains only one vertex, say w, then w is complete to H and hence $U \neq \emptyset$, a contradiction. So $|H' \cap (W_i^- \cup W_i^+)| \geq 2$.

Set $K^- = H' \cap W_i^-$ and $K^+ = H' \cap W_i^+$. By parts (i) and (iii) of Lemma 4.17, $K^- \cap K^+ = \emptyset$, and both K^- and K^+ are nonempty. Fix $w_1 \in K^-$ such that $|H_{w_1}|$ is maximum. Let u be the unique vertex of $H^0_{w_1,x_i}$ and v the unique vertex of $H^1_{w_1,x_i} \setminus H^0_{w_1,x_i}$. Note that since H is an odd hole, $x_i \neq v$. By the maximality of $|H_{w_1}|$ it follows that any neighbour of v in H' belongs to K^+. Let K_u be the set of vertices from K^- that are adjacent to u and let K_v be the set of vertices from K^+ that are adjacent to v. By (2), K_u is anticomplete to v and K_v is anticomplete to u. If $k_u \in K_u$ and $k_v \in K_v$ are adjacent, then $\{k_u, k_v, u, v\}$ induces a 4-hole, a contradiction. So K_u is anticomplete to K_v. But now $H' \cup \{u, v\}$ induces a theta if $|K_u| + |K_v| = 2$, a pyramid if $|K_u| + |K_v| = 3$ and a prism if $|K_u| + |K_v| = 4$. In any case we obtain a contradiction. Therefore $G[W_i^- \cup O_i^1 \cup W_i^+]$ is chordal. □

Lemma 4.19. *$G[W_i^- \cup O_i^1 \cup W_i^+]$ is an interval graph.*

Proof. In view of Lemma 4.18 and Theorem 4.2, it suffices to show that $G[W_i^- \cup O_i^1 \cup W_i^+]$ contains no umbrella, net, or k-tent (for every $k \geq 3$). Suppose otherwise and let F be an umbrella, net or k-tent (for some $k \geq 3$) in $G[W_i^- \cup O_i^1 \cup W_i^+]$. Since G is (umbrella $\cup K_1$, net $\cup K_1$, k-tent $\cup K_1$)-free for every $k \geq 3$, F is dominating in G, and it is easily checked that F contains no universal vertex. Therefore parts (i)–(iv) of Lemma 4.17 hold.

Set $V^- = W_i^- \cap F$ and $V^+ = W_i^+ \cap F$. By Lemma 4.17, V^- and V^+ are both nonempty. Fix $w \in V^-$ and $w' \in V^+$ such that $|H_w|$ and $|H_{w'}|$ are maximum (since G is NCA-wheel-free and $U = \emptyset$, both H_w and $H_{w'}$ are paths). By Lemma 4.17 (i), x_i is an end of both H_w and $H_{w'}$. Set $v = H^0_{w,x_i}$ and $v' = H^0_{w',x_i}$. By our choice of w and w' together with Lemma 4.17 (ii), $v \neq v'$, v is adjacent to v', and $N_H(V^-) \subseteq H_w$ and $N_H(V^+) \subseteq H_{w'}$. Since F is dominating in G, every vertex of H has a neighbour in F, so every vertex of $H \setminus \{x_i\}$ has a neighbour in $V^- \cup V^+$, and hence vv' must be an edge. It follows that w and w' are not adjacent, for otherwise $\{w, w', v, v'\}$ would induce a C_4. Note that since F is connnected, there is an induced path from w to w' in F.

Suppose first that F is a net. For all distinct $u, u' \in F$ and for any induced path P from u to u' in F, there is a vertex of F that neither belongs to P nor has a neighbour in P, so any induced path from w to w' in F contradicts Lemma 4.17. Hence $G[W_i^- \cup O_i^1 \cup W_i^+]$ does not contain a net.

Now suppose that F is an umbrella, i.e. F consists of an induced path $v_1, \dots, v_5$ on five vertices, and two additional nonadjacent vertices x and y such that x is complete to $\{v_1, \dots, v_5\}$, and y is adjacent to v_3 but anticomplete to $\{v_1, v_2, v_4, v_5\}$. By Lemma 4.17, up to symmetry we may assume that $w = x$ and $w' = y$ (since for any other choice of w, w', there exists an induced ww'-path in F that contradicts Lemma 4.17. Since y is anticomplete to $\{v_1, v_2, v_4, v_5\}$, by Lemma 4.16 we have that $\{v_1, v_2, v_4, v_5\} \cap W_i^+ = \emptyset$. So in particular no vertex of $\{v_1, v_2, v_4, v_5\}$ is adjacent to v'. Furthermore we see that v_3v' is not an edge, for otherwise $v_3 \in W_i^+$, so v_3v is not an edge and hence $\{x, v_3, v, v'\}$ induces a C_4. But now $\{v_1, v_2, v_3, v_4, v_5, y, v'\}$ induces a bipartite claw, a contradiction. So $G[W_i^- \cup O_i^1 \cup W_i^+]$ contains no umbrella.

We deduce that F is a k-tent for some $k \geq 3$. That is, F consists of an induced path $v_1, \dots, v_k$ and three additional vertices x, y, z such that $N_F(x) = \{v_1, \dots, v_{k-1}, y, z\}$, $N_F(y) = \{v_2, \dots, v_k, x, z\}$, and $N_F(z) = \{x, y\}$. By Lemma 4.17, up to symmetry we may assume that $w = x$ and $w' = v_k$ (for any other choice of w, w', either ww' is an edge or we can find an induced path from w to w' in F that contradicts Lemma 4.17). In particular, $F \cap (W_i^- \cup W_i^+) = \{w, w'\}$. But now if $k = 3$, then $\{v_1, \dots, v_k, y, z, x_{i-1}, x_{i+1}\}$ induces a net $\cup K_1$, and if $k \geq 4$, then $\{v_1, \dots, v_k, y, z, x_{i+1}\}$ induces a $(k-1)$-net. In either case we obtain a contradiction, and it follows that $G[W_i^- \cup O_i^1 \cup W_i^+]$ contains no k-tent for every $k \geq 3$. $\square$

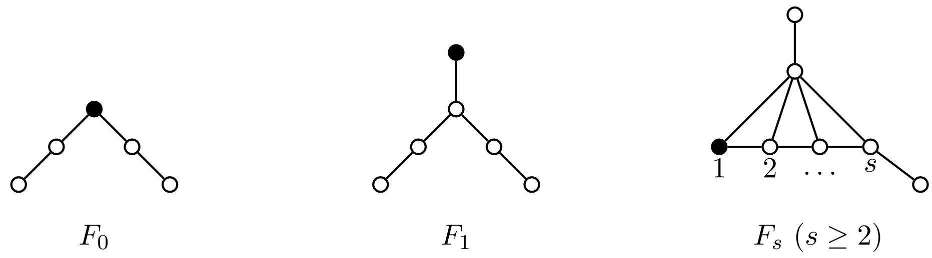


Figure 4.5: Obstructions for the filled vertex being an end vertex.

Let I be an interval graph and let $A, B \subseteq V(I)$. The pair (A, B) is a *left-right pair for I* if I admits an interval model in which all intervals corresponding to vertices of A have the same left endpoint and no other endpoints are further to the left, and all intervals corresponding to vertices of B have the same right endpoint and no other endpoints are further to the right. A vertex v of I is an *end vertex* if $(\{v\}, \emptyset)$ is a left-right pair for I.

Theorem 4.20 (Gimbel [32]). *Let I be an interval graph and let v be a vertex of I. Then v is an end vertex if and only if I contains none of the graphs in Figure 4.5 where the filled vertex represents v.*

Theorem 4.21 (de Figueiredo et al. [23]). *Let I be an interval graph and let $A, B \subseteq V(I)$. Then (A, B) is a left-right pair for I if and only if the following conditions hold:*

- *A and B are cliques.*
- *Each vertex of $A \cup B$ is an end vertex.*
- *For all distinct vertices u, v, both in A or both in B, there is no induced 4-vertex path P in I such that $P^* = \{u, v\}$.*
- *For each $a \in A$ and $b \in B$, every induced path from a to b in I is dominating in I.*

Lemma 4.22. *(W_i^-, W_i^+) is a left-right pair for $G[W_i^- \cup O_i^1 \cup W_i^+]$.*

Proof. We prove the following two claims.

(1) For all distinct vertices w, w', both in W_i^- or both in W_i^+, there is no induced 4-vertex path P in $G[W_i^- \cup O_i^1 \cup W_i^+]$ such that $P^* = \{w, w'\}$.

Proof of (1): Suppose otherwise and let $P = x, w, w', y$ be an induced 4-vertex path in

$G[W_i^- \cup O_i^1 \cup W_i^+]$ such that, without loss of generality, $w, w' \in W_i^-$. By Lemma 4.17 applied to subpaths wx and $w'y$ of P, and since W_i^- is a clique by Lemma 4.16 both x and y belong to O_i^1. By Lemma 4.17 (i), w and w' are nonadjacent to x_{i+1}, so x_i is an end of H_w and $H_{w'}$. If $H_w = H_{w'}$, then $P \cup H^2_{w,x_i}$ induces a long 2-net, a contradiction. So without loss of generality $H_{w'} \subsetneq H_w$. But now $P \cup H^1_{w,x_i} \cup H_w \setminus (H_{w'} \setminus H^0_{w',x_i})$ induces a k-net, where $k = |H_w| - |H_{w'}| + 2 \geq 3$, a contradiction. This completes the proof of (1).

(2) Each vertex of $W_i^- \cup W_i^+$ is an end vertex in $G[W_i^- \cup O_i^1 \cup W_i^+]$.

Proof of (2): By symmetry it suffices to prove that each vertex of W_i^- is an end vertex of $G[W_i^- \cup O_i^1 \cup W_i^+]$. Suppose otherwise. Then by Theorem 4.20 there exists some smallest integer $s \geq 0$ such that $G[W_i^- \cup O_i^1 \cup W_i^+]$ contains F_s (see Figure 4.5), where the filled vertex belongs to W_i^-.

Suppose first that $s = 0$, i.e. there exists an induced 5-vertex path $P = v_1, \dots, v_5$ in $G[W_i^- \cup O_i^1 \cup W_i^+]$ such that $v_3 \in W_i^-$. By (1) and Lemma 4.16, $\{v_1, v_2, v_4, v_5\} \cap W_i^- = \emptyset$. Furthermore, by Lemma 4.17 applied to P, no vertex of P belongs to W_i^+, therefore $\{v_1, v_2, v_4, v_5\} \subseteq O_i^1$. But now $P \cup H^1_{w,x_i}$ induces a bipartite claw, a contradiction, and hence $s \neq 0$.

Suppose now that $s = 1$, i.e. there exists an induced 5-vertex path $P = v_1, \dots, v_5$ together with an additional vertex x in $G[W_i^- \cup O_i^1 \cup W_i^+]$ such that $N_P(x) = \{v_3\}$ and $x \in W_i^-$. Then by Lemma 4.16 and since $s \neq 0$, $v_3 \notin W_i^-$ and hence $(P \cup \{x\}) \cap W_i^- = \{x\}$. By Lemma 4.17, no vertex of P belongs to W_i^+, so $P \subseteq O_i^1$. But now $P \cup \{x, x_{i-1}\}$ induces a bipartite claw, a contradiction.

So $s \geq 2$, i.e. $G[W_i^- \cup O_i^1 \cup W_i^+]$ contains an induced path $P = v_1, \dots, v_s$ such that $v_1 \in W_i^-$, together with additional vertices x, y, w such that x is complete to $P \cup \{y\}$ and nonadjacent to w, y is anticomplete to $P \cup \{w\}$, and w is adjacent to v_s and anticomplete to $P \setminus \{v_s\}$. By Lemma 4.16, $(P \cup \{x, y, w\}) \cap W_i^- \subseteq \{v_1, v_2, x\}$. If $\{v_2, x\} \cap W_i^- = \emptyset$, then by Lemma 4.17 $(P \setminus \{v_1\}) \cup \{x, y, w\} \subseteq O_i^1$. But then the graph induced by $P \cup \{x, y, w, x_{i-1}, x_{i+1}\}$ is a net $\cup K_1$ (if $s = 2$), or contains an s-net (if $s \geq 3$); in either case a contradiction. Therefore $\{v_2, x\} \cap W_i^- \neq \emptyset$.

If $v_2 \in W_i^+$, then by minimality $s = 2$ and by (1) $x \notin W_i^-$, in which case we may relabel v_2 and x so that $\{v_2, x\} \cap W_i^- = \{x\}$. So it suffices to consider the case $\{v_2, x\} \cap W_i^- = \{x\}$. By Lemma 4.17 (i), $\{v_1, x\}$ is anticomplete to x_{i+1}, so x_i is an end of H_{v_1} and H_x. If $H_{v_1} \subseteq H_x$, then $P \cup H^1_{x,x_i} \cup H_x \setminus (H_{v_1} \setminus H^0_{v_1,x_i}) \cup \{x, y, w\}$ induces an $(s+1)$-net for some $s \geq 2$, a contradiction. So there exists $z \in H_{v_1} \setminus H_x$. By Lemma 4.13

and since $U = \emptyset$, z and x_{i+1} are nonadjacent. But now $P \cup \{x, y, z, w, x_{i+1}\}$ induces a net $\cup K_1$, a contradiction. This completes the proof of (2).

It now follows from claims (1) and (2), Lemmas 4.16 and 4.17, and Theorems 4.20 and 4.21 that (W_i^-, W_i^+) is a left-right pair for $G[W_i^- \cup O_i^1 \cup W_i^+]$. $\square$

Lemma 4.23. *Let $w \in W_i^+$, $v \in O_i^2 \cup W_{i+1}^-$, and let A be the set of neighbours of w that are adjacent to x_i but not to x_{i+1}. If w and v are nonadjacent, then wx_{i+2} is not an edge, and A is anticomplete to v.*

Proof. Suppose that w and v are nonadjacent. If $a \in A$ is adjacent to v, then $\{w, a, v, x_{i+1}\}$ induces a C_4, a contradiction. So A is anticomplete to v.

Suppose that wx_{i+2} is an edge, and fix $a \in A$ (such a vertex exists by definition of W_i^+). It follows that $w \in W_{i+1}^-$, and hence $v \in O_i^2$ by Lemma 4.16. In particular, vx_{i+2} is not an edge. Furthermore, ax_{i+2} is not an edge, for otherwise $\{a, x_i, x_{i+1}, x_{i+2}\}$ induces a C_4, a contradiction.

Suppose that H is of length 5. By Lemma 4.13 and since $U = \emptyset$, w is nonadjacent to x_{i-1}. So $w \in X$. If ax_{i-1} is an edge, then by Lemma 4.14, a is of Type 2 w.r.t. H. But then $\{a, x_{i-1}, x_{i-2}, x_{i+2}, w, v\}$ induces a 0-wheel, a contradiction. So ax_{i-1} is not an edge. But now $\{a, w, v, x_{i+1}, x_{i+2}, x_{i+3}, x_{i-1}\}$ induces a long 2-net, a contradiction.

So H is of length at least 7. If wx_{i-1} is an edge, then there exists a hole H' in $H \cup \{w\}$ that contains w but not x_i or x_{i+1}. But then $H' \cup \{v\}$ induces a 0-wheel, a contradiction. So wx_{i-1} is not an edge, and by Lemma 4.13 and since $U = \emptyset$, wx_{i-2} is not an edge. But now $\{a, w, x_{i+2}, x_i, x_{i+1}, v, x_{i-2}\}$ induces a tent $\cup K_1$, a contradiction. Therefore wx_{i+2} is not an edge. $\square$

Lemma 4.24. *Let F be an induced subgraph of $G[W_i^+ \cup O_i^2 \cup W_{i+1}^-]$, and set $V_i = W_i^+ \cap F$ and $V_{i+1} = W_{i+1}^- \cap F$. If $P = v, \ldots, v'$ is an induced path in F where $v \in V_i$ and $v' \in V_{i+1}$, then P is dominating in F. If no vertex of F is universal in F, then the following hold:*

(i) $V_i \cap V_{i+1} = \emptyset$.

(ii) If $v \in V_i$ and $v' \in V_{i+1}$, then $H_v \cap H_{v'} = \{x_i, x_{i+1}\}$.

(iii) If F is dominating in G, then both V_i and V_{i+1} are nonempty.

Proof. We first prove that any induced path $P = v, \ldots, v'$ in F, where $v \in V_i$ and $v' \in V_{i+1}$, is dominating in F. Suppose otherwise. Let $P = v, \ldots, v'$ be a counterexample of minimum length and fix $u \in V(F)$ such that $N[u] \cap P = \emptyset$. By the minimality of P, we have that $P \cap V_i = \{v\}$, $P \cap V_{i+1} = \{v'\}$ and $P^* \subseteq O_i^2$. By Lemma 4.16, $u \in O_i^2$. By the definition of sets W_i^+ and W_{i+1}^- there exist vertices $z \in N(v) \setminus \{x_{i+1}\}$ and $z' \in N(v') \setminus \{x_i\}$ such that z is adjacent to x_i but not to x_{i+1} and z' is adjacent to x_{i+1} but not to x_i. If possible, let us choose z and z' so that they belong to H. By Lemma 4.23 applied to vertices v, u and to v', u, we have that none of $zu, z'u, vx_{i+2}, v'x_{i-1}$ are edges. Furthermore, z and z' are nonadjacent, for otherwise $\{z, x_i, x_{i+1}, z'\}$ induces a C_4. Let R be an induced path from z to z' in $P \cup \{z, z'\}$.

If $N_H(z) \setminus \{x_i\}$ and $N_H(z') \setminus \{x_{i+1}\}$ are both nonempty, then there exists an induced path from z to z' in $(H \setminus \{x_i, x_{i+1}\}) \cup \{z, z'\}$ which together with R and u forms a 0-wheel, a contradiction. So without loss of generality $N_H(z) = \{x_i\}$, and therefore $z \notin H$ and hence v is of Type 2 w.r.t. H. Observe that v' and x_{i-2} are nonadjacent by Lemma 4.13 and since H is of length at least 5. Suppose that $v \neq v'$. But now $R \cup \{x_{i-2}, x_i, x_{i+1}, u\}$ induces an $|R|$-tent$\,\cup K_1$ if $z'x_{i-2}$ is not an edge, and otherwise $R \cup \{x_{i+1}, x, u\}$ induces an $(|R|-1)$-net, where $x = H^0_{z', x_{i+1}}$ if $z' \notin H$ and $x = x_{i+3}$ otherwise (since $z' = x_{i+2}$ in this case). Since $|R| \geq 3$, in any case we obtain a contradiction. So $v = v'$. If $z' \in H$, then the graph induced by $(H_v \setminus \{x_i\}) \cup H^2_{v, x_i} \cup \{u, v, z\}$ is a long 2-net if $|H_v| = 3$ or contains a k-net for some $k \geq 3$ if $|H_v| \geq 5$. In either case we obtain a contradiction, so $z' \notin H$ and hence v is of Type 2 w.r.t. H. If z' is of Type 1 w.r.t. H, then $\{z, v, z', x_i, x_{i+1}, u\}$ together with any vertex of $H \setminus \{x_{i-1}, x_i, x_{i+1}, x_{i+2}\}$ induces a tent$\,\cup K_1$, a contradiction. So $|H_{z'}| > 1$, and hence by Lemma 4.15 applied to z' and $v = v'$, we have that $z' \in X \cup Y$. But now $H^1_{z', x_{i+1}} \cup \{z, v, z', x_{i+1}, u\}$ induces a long 2-net, a contradiction. Therefore any induced path in F with one end in V_i and the other in V_{i+1} is dominating in F.

For the remainder of the proof we assume that no vertex of F is universal in F. Contrary to (i), suppose there exists a vertex $v \in V_i \cap V_{i+1}$. Since no vertex of F is universal in F, the induced path $P = v$ is not dominating in F, a contradiction. So (i) holds.

Contrary to (ii), suppose that $(H_v \cap H_{v'}) \setminus \{x_i, x_{i+1}\} \neq \emptyset$ for some $v \in V_i$ and $v' \in V_{i+1}$. Therefore $v, v' \in X \cup Y$. By (i), v is nonadjacent to x_{i+2} and v' is nonadjacent to x_{i-1}, and therefore both v and v' must belong to Y. But then v and v' are two vertices of Y that cross, a contradiction. So (ii) holds.

Finally, if F is dominating in G, then by (i) any neighbour of x_{i-1} in F must belong to V_i and any neighbour of x_{i+2} in F must belong to V_{i+1}, so (iii) holds. □

Lemma 4.25. *If F is an induced subgraph of $G[W_i^+ \cup O_i^2 \cup W_{i+1}^-]$, then:*

- *F is not dominating in G, or*

- *some vertex of F is universal in F.*

Proof. On the contrary, suppose that there exists an induced subgraph F of $G[W_i^+ \cup O_i^2 \cup W_{i+1}^-]$ that is dominating in G but contains no universal vertex. Set $V^- = W_i^+ \cap F$ and $V^+ = W_{i+1}^- \cap F$. By Lemma 4.24 (iii), V^- and V^+ are nonempty. Fix $w \in V^-$ and $w' \in V^+$ so that $|H_w|$ and $|H_{w'}|$ are maximum. By Lemma 4.24 (i), $x_{i-1} \notin N(w')$ and $x_{i+2} \notin N(w)$, so x_i is an end of $H_{w'}$ and x_{i+1} is an end of H_w. Since F is dominating in G, it follows that $x_{i-1} \in N(w)$ and $x_{i+2} \in N(w')$. Therefore w and w' belong to $X \cup Y$ and hence $|H_w|$ and $|H_{w'}|$ are odd by Lemma 4.13. Since every vertex of H has a neighbour in F, it follows by our choice of w and w' that every vertex of H is adjacent to at least one of w and w', and hence $H_w \cup H_{w'} = V(H)$. Let v and v' be the (unique) vertices of $H^0_{w,x_{i+1}}$ and H^0_{w',x_i} respectively. By Lemma 4.24 (ii), vertices v and v' are distinct, and therefore they are adjacent (for otherwise some vertex of H would be nonadjacent to both w and w'). It follows that $|H| = |H_w| + |H_{w'}| - 2$, contradicting the fact that H is of odd length. $\square$

Lemma 4.26. *$G[W_i^+ \cup O_i^2 \cup W_{i+1}^-]$ is chordal.*

Proof. For suppose that $G[W_i^+ \cup O_i^2 \cup W_{i+1}^-]$ contains a hole H'. Since G contains no 0-wheel, H' is dominating in G. But clearly no vertex of H' is universal in H', contradicting Lemma 4.25. Therefore $G[W_i^+ \cup O_i^2 \cup W_{i+1}^-]$ is chordal. $\square$

Lemma 4.27. *$G[W_i^+ \cup O_i^2 \cup W_{i+1}^-]$ is an interval graph.*

Proof. By Theorem 4.2 and Lemma 4.26, it suffices to show that $G[W_i^+ \cup O_i^2 \cup W_{i+1}^-]$ contains no umbrella, net, or k-tent for every $k \geq 3$. Suppose otherwise and let F be an umbrella, net, or k-tent (for some $k \geq 3$) in $G[W_i^+ \cup O_i^2 \cup W_{i+1}^-]$. Since G is (umbrella $\cup K_1$, net $\cup K_1$, k-tent $\cup K_1$)-free for every $k \geq 3$, F is dominating in G, and in any case no vertex of F is universal in F. This contradicts Lemma 4.25. $\square$

Lemma 4.28. *(W_i^+, W_{i+1}^-) is a left-right pair for $G[W_i^+ \cup O_i^2 \cup W_{i+1}^-]$.*

Proof. We prove the following two claims.

(1) For all distinct vertices w, w', both in W_i^+ or both in W_{i+1}^-, there is no induced 4-vertex path P in $G[W_i^+ \cup O_i^2 \cup W_{i+1}^-]$ such that $P^* = \{w, w'\}$.

Proof of (1): Suppose otherwise and let $P = x, w, w', y$ be an induced 4-vertex path in $G[W_i^+ \cup O_i^2 \cup W_{i+1}^-]$ such that, without loss of generality, $w, w' \in W_i^+$. By Lemma 4.16, without loss of generality $x \in O_i^2$ and $y \in O_i^2 \cup W_{i+1}^-$. Since the path $w'y$ is not dominating in P, by Lemma 4.24 $y \notin W_{i+1}^-$ and hence $y \in O_i^2$. Let z (resp. z') be a neighbour of w (resp. w') in $V(G) \setminus \{x_{i+1}\}$ that is adjacent to x_i but not to x_{i+1}. Pick z and z' so that they belong to H, if possible. By Lemma 4.24 (i), $\{w, w'\}$ is anticomplete to $\{x_{i+2}\}$ and hence is also anticomplete to $\{x_{i+3}\}$ by Lemma 4.13. It follows that $H^2_{w,x_{i+1}} \setminus H^0_{w,x_{i+1}}$ is anticomplete to $\{w, x, y\}$ and $H^2_{w',x_{i+1}} \setminus H^0_{w',x_{i+1}}$ is anticomplete to $\{w', x, y\}$.

Suppose that $\{w, w'\} \cap (X \cup Y) \neq \emptyset$; without loss of generality $w \in X \cup Y$. If $H_w = H_{w'}$, then $H^2_{w,x_{i+1}} \cup \{w, w', x, y\}$ induces a long 2-net, a contradiction. So $H_w \neq H_{w'}$ and hence without loss of generality $H_{w'} \subset H_w$. But now the set $(H_w \setminus (H_{w'} \setminus H^0_{w',x_{i+1}})) \cup H^0_{w,x_{i+1}} \cup \{w, w', x, y\}$ induces a k-net where $k = |H_w| - (|H_{w'}| - 1) + 1 \geq 3$, a contradiction.

So both w and w' are of Type 2 w.r.t. H, and therefore $\{z, z'\} \cap H = \emptyset$. We first consider the case where one of z, z' is complete to $\{w, w'\}$. Suppose without loss of generality that z is complete to $\{w, w'\}$. By Lemma 4.23, z is anticomplete to $\{x, y\}$. If z is adjacent to x_{i+3}, then $H^0_{z,x_i} = x_{i+3}$, and hence $H^1_{z,x_i} \cup \{z, w, w', x, y\}$ induces a long 2-net, a contradiction. So z is nonadjacent to x_{i+3}. But then $\{w, w', x_{i+1}, x, y, z, x_{i+3}\}$ induces a tent $\cup K_1$, a contradiction.

So neither z nor z' is complete to $\{w, w'\}$, and hence $z \neq z'$. Furthermore, z and z' are nonadjacent, for otherwise $\{z, z', w, w'\}$ induces a C_4. If $\{z, z'\}$ is anticomplete to x_{i+3}, then $\{z, z', w, w', x_{i+1}, x_{i+2}, x_{i+3}\}$ induces a long 2-net, a contradiction. So without loss of generality z is adjacent to x_{i+3}, and z' is nonadjacent to x_{i+3} for otherwise $\{z, z', x_i, x_{i+3}\}$ induces a C_4. But then $\{z, w, x_{i+1}, x_{i+2}, x_{i+3}\}$ induces a hole which together with z' forms a 0-wheel, a contradiction. This completes the proof of (1).

(2) Each vertex of $W_i^+ \cup W_{i+1}^-$ is an end vertex in $G[W_i^+ \cup O_i^2 \cup W_{i+1}^-]$.

Proof of (2): By symmetry it suffices to prove that each vertex of W_i^+ is an end vertex of $G[W_i^+ \cup O_i^2 \cup W_{i+1}^-]$. Suppose otherwise. Then by Theorem 4.20 there exists some smallest integer $s \geq 0$ such that $G[W_i^+ \cup O_i^2 \cup W_{i+1}^-]$ contains F_s (see Figure 4.5).

Suppose first that $s = 0$, i.e. there exists an induced 5-vertex path $P = v_1, \ldots, v_5$ in $G[W_i^+ \cup O_i^2 \cup W_{i+1}^-]$, where $v_3 \in W_i^+$. By (1) and Lemma 4.16, $\{v_1, v_2, v_4, v_5\} \cap W_i^+ = \emptyset$. By Lemma 4.24 applied to P, no vertex of P belongs to W_{i+1}^-, and therefore $\{v_1, v_2, v_4, v_5\} \subseteq O_i^2$. By Lemma 4.23 applied to vertices v_1 and v_3, vertex v_3 is nonadjacent to x_{i+2} and therefore x_{i+1} is an end of H_{v_3}. If $v_3 \in X \cup Y$, then $P \cup H^1_{v_3,x_{i+1}}$

induces a bipartite claw, a contradiction. Therefore v_3 is of Type 2 w.r.t. H. By definition of W_i^+ there exists a neighbour z of v_3 in $V(G) \setminus \{x_{i+1}\}$ that is adjacent to x_i but not to x_{i+1}. Since $\{v_1, v_2, v_4, v_5\} \cap W_i^+ = \emptyset$, z is anticomplete to $\{v_1, v_2, v_4, v_5\}$. But then $P \cup \{z, x_{i+1}, x_{i+3}\}$ induces an umbrella $\cup K_1$ if zx_{i+3} is not an edge, and $P \cup \{z, x_{i+3}\}$ a bipartite claw if zx_{i+3} is an edge. In any case we obtain a contradiction, and therefore $s \neq 0$.

Suppose now that $s = 1$, i.e. there exists an induced 5-vertex path $P = v_1, \dots, v_5$ together with an additional vertex x in $G[W_i^+ \cup O_i^2 \cup W_{i+1}^-]$ such that $N_P(x) = \{v_3\}$ and $x \in W_i^+$. Then by Lemma 4.16 and since $s \neq 0$, $v_3 \notin W_i^+$ and hence $(P \cup \{x\}) \cap W_i^+ = \{x\}$. By Lemma 4.24, no vertex of P belongs to W_{i+1}^-, so $P \subseteq O_i^2 \setminus (W_i^+ \cup W_{i+1}^-)$. By definition of W_i^+, there exists a neighbour z of x in $V(G) \setminus \{x_{i+1}\}$ that is adjacent to x_i but not to x_{i+1}. Since $P \cap W_i^+ = \emptyset$, z is anticomplete to P. But then $P \cup \{x, z\}$ induces a bipartite claw, a contradiction. So $s \neq 1$.

Therefore $s \geq 2$. That is, $G[W_i^+ \cup O_i^2 \cup W_{i+1}^-]$ contains an induced path $P = v_1, \dots, v_s$ such that $v_1 \in W_i^+$, together with additional vertices x, y, w such that x is complete to $P \cup \{y\}$ and nonadjacent to w, y is adjacent to x and anticomplete to $P \cup \{w\}$, and w is adjacent to v_s and anticomplete to $(P \setminus \{v_s\}) \cup \{x, y\}$. By Lemma 4.16, $W_i^+ \cap (P \cup \{x, y, w\}) \subseteq \{v_1, v_2, x\}$. Fix $z \in N(v_1) \setminus \{x_{i+1}\}$ such that z is adjacent to x_i but not to x_{i+1}. By Lemma 4.23, no vertex of $P \cup \{x, y, z, w\}$ is adjacent to x_{i+2}.

If $\{v_2, x\} \cap W_i^+ = \emptyset$, then z is anticomplete to $P \cup \{x, y, w\}$ and hence $P \cup \{x, y, z, w, x_{i+2}\}$ induces an s-net $\cup K_1$ for some $s \geq 2$, a contradiction. So $\{v_2, x\} \cap W_i^+ \neq \emptyset$. If $v_2 \in W_i^+$, then by minimality $s = 2$ and by (1) $x \notin W_i^+$, in which case we may relabel v_2 and x so that $\{v_2, x\} \cap W_i^+ = \{x\}$. So it suffices to consider the case $\{v_2, x\} \cap W_i^+ = \{x\}$. Fix $z' \in N(x) \setminus \{x_{i+1}\}$ such that z' is adjacent to x_i but not to x_{i+1}. If possible, choose z' so that $z = z'$. If $z = z'$, then $P \cup \{x, y, z, w, x_{i+1}\}$ induces an $(s+2)$-tent (for some $s \geq 2$), which together with x_{i+3} forms an $(s+2)$-tent $\cup K_1$ if zx_{i+3} is not an edge. So zx_{i+3} is an edge, but then $P \cup \{x, y, z, w, x_{i+3}\}$ induces an $(s+1)$-net, a contradiction. So $z \neq z'$, and hence z and x are nonadjacent. But now $G[P \cup \{x, y, z, w, x_{i+2}\}]$ contains an s-net (for $s \geq 3$), a net $\cup K_1$ (if $s = 2$ and $zx_{i+2} \notin E(G)$), or a long 2-net (if $s = 2$ and $zx_{i+2} \in E(G)$). In any case we obtain a contradiction. This completes the proof of (2).

It now follows from claims (1) and (2), Lemmas 4.16 and 4.24, and Theorems 4.20 and 4.21 that (W_i^+, W_{i+1}^-) is a left-right pair for $G[W_i^+ \cup O_i^2 \cup W_{i+1}^-]$. $\square$

Lemma 4.29. *For each $i \in \{1, \dots, n\}$, the following hold.*

- *If $v \in O_i^1$, then $N_G(v) \subseteq \{x_i\} \cup W_i^- \cup O_i^1 \cup W_i^+$.*
- *If $v \in O_i^2$, then $N_G(v) \subseteq \{x_i, x_{i+1}\} \cup W_i^+ \cup O_i^2 \cup W_{i+1}^-$.*

Proof. Fix $i \in \{1,\dots,n\}$, $v \in O_i^1 \cup O_i^2$ and $w \in N_G(v)$. Since G contains no 0-wheel and $U = \emptyset$, by Lemma 4.13 there exists $j \in \{1,\dots,n-2\}$ such that w is of Type j w.r.t. H. We prove that $w \in \{x_i\} \cup W_i^- \cup O_i^1 \cup W_i^+$ if $v \in O_i^1$ and $w \in \{x_i, x_{i+1}\} \cup W_i^+ \cup O_i^2 \cup W_{i+1}^-$ if $v \in O_i^2$.

Suppose that $v \in O_i^1$. By definition of O_i^1, $N_H(v) = \{x_i\}$, so we may assume that $w \notin V(H)$. By Lemma 4.15, $w \in O_i^1$ if $j = 1$, and $w \in W_i^- \cup W_i^+$ if $j = 2$. So we may assume that $j \geq 3$, and hence up to symmetry $x_{i+1} \in H_w$ by Lemmas 4.13 and 4.15. It follows that $w \in W_i^+$, and this completes the proof of the first bullet.

Suppose now that $v \in O_i^2$. By definition of O_i^2, $N_H(v) = \{x_i, x_{i+1}\}$, so we may assume that $w \notin V(H)$. If $\{x_i, x_{i+1}\} \not\subseteq N(w)$, then by Lemma 4.15 $|N(w) \cap \{x_i, x_{i+1}\}| = 1$. But then v belongs to $W_i^+ \cup W_{i+1}^-$, a contradiction. So $\{x_i, x_{i+1}\} \subseteq N(w)$ and hence $w \in W_i^+ \cup O_i^2 \cup W_{i+1}^-$. This completes the proof of the second bullet. □

Lemma 4.30. *For each $i \in \{1,\dots,n\}$, we have that*

$$N_G(x_i) \setminus H = W_{i-1}^+ \cup O_{i-1}^2 \cup W_i^- \cup O_i^1 \cup W_i^+ \cup O_i^2 \cup W_{i+1}^-.$$

Proof. Fix $i \in \{1,\dots,n\}$. By definition of the sets in question we see that $W_{i-1}^+ \cup O_{i-1}^2 \cup W_i^- \cup O_i^1 \cup W_i^+ \cup O_i^2 \cup W_{i+1}^- \subseteq N_G(x_i) \setminus H$. For the reverse inclusion, let us fix $v \in N_G(x_i) \setminus H$. Since G contains no 0-wheel and $U = \emptyset$, v is of Type j w.r.t. H for some integer j satisfying $1 \leq j < n$. If $j = 1$, then $v \in O_i^1$, and if $j = 2$, then $N_H(v) = \{x_{i-1}, x_i\}$ or $N_H(v) = \{x_i, x_{i+1}\}$ and hence $v \in W_{i-1}^+ \cup O_{i-1}^2 \cup W_i^- \cup O_i^2 \cup W_i^+ \cup W_{i+1}^-$. So we may assume that $j \geq 3$, and therefore up to symmetry H_v contains $\{x_{i-1}, x_i, x_{i+1}\}$ or $\{x_i, x_{i+1}, x_{i+2}\}$ by Lemma 4.13. In the former case $v \in W_i^-$ and in the latter case $v \in W_{i+1}^-$. We conclude that $N_G(x_i) \setminus H = W_{i-1}^+ \cup O_{i-1}^2 \cup W_i^- \cup O_i^1 \cup W_i^+ \cup O_i^2 \cup W_{i+1}^-$. □

We define the following notation and terminology. Let $M = (C, \mathcal{A})$ be a circular-arc model, F its intersection graph, and fix a direction "clockwise" for C. For $A \in \mathcal{A}$ we denote by $\ell_M(A)$ and $r_M(A)$ respectively the left and right endpoints of A when traversing C clockwise, and for $v \in V(F)$ we denote by $A_M(v)$ the arc of M corresponding to v. For simplicity of notation we set $\ell_M(v) = \ell_M(A_M(v))$ and $r_M(v) = r_M(A_M(v))$ for each $v \in V(F)$. Say M is *open* (resp. *closed*) if all arcs in $\mathcal{A}$ are open (resp. closed). An arc A of M with endpoints a, b is *clockwise* if $\ell_M(A) = a$ and $r_M(A) = b$, and *anticlockwise* if $\ell_M(A) = b$ and $r_M(A) = a$.

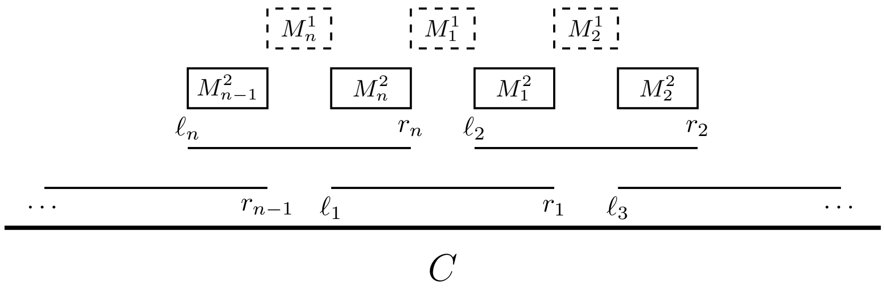


Figure 4.6: A depiction of a section of the circular-arc model $M' = (C, \mathcal{A}')$ that is constructed in the proof of Theorem 4.31. The thick line represents a segment of C and the five thinner lines are the arcs corresponding to vertices $x_{n-1}, x_n, x_1, x_2, x_3$ of H. A dashed rectangle represents a collection of open intervals while a solid rectangle represents a collection of closed intervals.

Theorem 4.31. *G is a circular-arc graph.*

Proof. Let $M = (C, \mathcal{A})$ be a closed circular-arc model for the hole H such that no two arcs share an endpoint. For brevity, let us set $\ell_i = \ell_M(x_i)$ and $r_i = r_M(x_i)$ for each $i \in \{1, \ldots, n\}$. Using Lemma 4.22, construct for each $i \in \{1, \ldots, n\}$ an open circular-arc model M_i^1 for $G[W_i^- \cup O_i^1 \cup W_i^+]$ such that:

- all arcs of M_i^1 are contained in the open clockwise arc (r_{i-1}, ℓ_{i+1}),
- $\ell_{M_i^1}(v) = r_{i-1}$ for all $v \in W_i^-$, and
- $r_{M_i^1}(v) = \ell_{i+1}$ for all $v \in W_i^+$.

Similarly, using Lemma 4.28, construct for each $i \in \{1, \ldots, n\}$ a closed circular-arc model M_i^2 for $G[W_i^+ \cup O_i^2 \cup W_{i+1}^-]$ such that:

- all arcs of M_i^2 are contained in the closed clockwise arc $[\ell_{i+1}, r_i]$,
- $\ell_{M_i^2}(v) = \ell_{i+1}$ for all $v \in W_i^+$, and
- $r_{M_i^2}(v) = r_i$ for all $v \in W_{i+1}^-$.

Now, for each $v \in G \setminus H$, let A_v be the union of all arcs corresponding to v among the models $M_1^1, M_1^2, \ldots, M_n^1, M_n^2$; for each $v \in H$, let $A_v = A_M(v)$; and let $\mathcal{A}' = \bigcup_{v \in V(G)} A_v$. See Figure 4.6 for a depiction of this construction. With the following two claims we verify that $M' = (C, \mathcal{A}')$ is a circular-arc model whose intersection graph is isomorphic to G.

(1) For each $v \in V(G)$, A_v is an arc on C.

Proof of (1): Fix $v \in V(G)$. If $v \in H$, then $A_v = A_M(v)$ is an arc on C by definition, so we may assume $v \in G \setminus H$. Furthermore, if exactly one model among $M_1^1, M_1^2, \ldots, M_n^1, M_n^2$ has an arc corresponding to v, then A_v is an arc on C by definition. So we may assume that at least two models among $M_1^1, M_1^2, \ldots, M_n^1, M_n^2$ have an arc corresponding to v. Since $O_1^1, O_1^2, \ldots, O_n^1, O_n^2$ are all pairwise disjoint, and each is also disjoint from $\bigcup_{i=1}^n (W_i^- \cup W_i^+)$, we may therefore assume that $v \in \bigcup_{i=1}^n (W_i^- \cup W_i^+)$. Up to symmetry we assume that $H_v = x_1, x_2, \ldots, x_k$ for some integer k satisfying $1 < k \leq n - 2$ (the latter inequality holds by Lemma 4.13). Define L_v, M_v, R_v as follows:

- L_v is the arc $A_{M_1^1}(v)$ if v has a neighbour different from x_2 that is adjacent to x_1 but not to x_2, and is the empty set otherwise.
- R_v is the arc $A_{M_k^1}(v)$ if v has a neighbour different from x_{k-1} that is adjacent to x_k but not to x_{k-1}, and is the empty set otherwise.
- $M_v = A_{M_1^2}(v) \cup (r_1, \ell_k) \cup A_{M_{k-1}^2}(v)$, where the arc (r_1, ℓ_k) is taken to be clockwise if $k \geq 3$ and anticlockwise otherwise. (Arcs $A_{M_1^2}(v)$ and $A_{M_{k-1}^2(v)}$ exist since $v \in W_2^- \cap W_{k-1}^+$ if $k \geq 3$ and $v \in W_1^+ \cup W_2^-$ if $k = 2$.)

We claim that $A_v = L_v \cup M_v \cup R_v$, and that $L_v \cup M_v \cup R_v$ is an arc on C. By definition, $L_v \cup R_v \subseteq A_v$. If $k \geq 3$, then $v \in \bigcap_{i=2}^{k-1} (W_i^- \cup W_i^+)$, so the models

$$M_2^1, M_2^2, \ldots, M_{k-2}^1, M_{k-2}^2, M_{k-1}^1$$

respectively contain clockwise arcs

$$(r_1, \ell_3), [\ell_3, r_2], \ldots, (r_{k-3}, \ell_{k-1}), [\ell_{k-1}, r_{k-2}], (r_{k-2}, \ell_k)$$

that correspond to v. The union of these arcs is the clockwise arc (r_1, ℓ_k), so $M_v \subseteq A_v$ if $k \geq 3$. Suppose that $k = 2$. Then $A_v \subseteq (r_n, \ell_3)$. Since at least two models among $M_1^1, M_1^2, \ldots, M_n^1, M_n^2$ contain arcs corresponding to v, it follows that $v \in W_1^+ \cap W_2^-$. Let z be a neighbour of v different from x_2 that is adjacent to x_1 but not to x_2, and let z' be a neighbour of v different from x_1 that is adjacent to x_2 but not to x_1. Since $k = 2$, $\{z, z'\} \cap H = \emptyset$, and since z is nonadjacent to x_2 and z' is nonadjacent to x_1 we see that the right endpoint (call it r_z) of A_z is contained in $A_{x_1} \setminus [\ell_2, r_1]$ and the left endpoint (call it $\ell_{z'}$) of $A_{z'}$ is contained in $A_{x_2} \setminus [\ell_2, r_1]$. Since the anticlockwise arc (r_1, ℓ_2) is contained in $[r_z, \ell_{z'}]$, it follows from z and z' being neighbours of v that $[r_z, \ell_{z'}] \subseteq A_v$. This concludes the proof that $L_v \cup M_v \cup R_v \subseteq A_v$. That $A_v \subseteq L_v \cup M_v \cup R_v$ follows from the fact that, by definition of sets W_i^- and W_i^+, $v \notin W_{k+1}^- \cup W_{k+2}^- \cup \cdots \cup W_n^- \cup W_1^-$ and $v \notin W_k^+ \cup W_{k+1}^+ \cup \cdots \cup W_n^+$. Finally, if L_v is nonempty then its right endpoint is ℓ_2, and if R_v is nonempty then its left endpoint is r_{k-1}, and hence $L_v \cup M_v \cup R_v$ is an arc on C. This completes the proof of (1).

In view of (1), let G' be the graph with vertex set $V(G)$ such that distinct vertices $u, v \in V(G')$ are adjacent in G' if and only if arcs A_u and A_v have nonempty intersection. We prove that:

(2) $G' = G$.

Proof of (2): Fix $v \in V(G')$. It suffices to prove that $N_{G'}(v) = N_G(v)$. By construction, if $v = x_i$ for some $i \in \{1, \ldots, n\}$, then A_v intersects $A_{x_{i-1}}$, $A_{x_{i+1}}$, and A_w for each $w \in W_{i-1}^+ \cup O_{i-1}^2 \cup W_i^- \cup O_i^1 \cup W_i^+ \cup O_i^2 \cup W_{i+1}^-$, and is disjoint from all other arcs, so $N_{G'}(v) = N_G(v)$ by Lemma 4.30. So we may assume that $v \notin V(H)$. Suppose that $v \in O_i^j$ for some $i \in \{1, \ldots, n\}$ and $j \in \{1, 2\}$. Since $O_1^1, O_1^2, \ldots, O_n^1, O_n^2$ are all pairwise disjoint, we have that $A_v = A_{M_i^j}(v)$ and hence $N_{G'}(v) = N_G(v)$ by Lemma 4.29. So we may assume that $v \in \bigcup_{i=1}^n (W_i^- \cup W_i^+)$. Fix $u \in \bigcup_{i=1}^n (W_i^- \cup W_i^+) \setminus \{v\}$. It remains to prove that uv is an edge of G' if and only if it is an edge of G.

Suppose that uv is an edge of G. Then by Lemma 4.15 $H_u \cap H_v$ induces a path P. If P has length at least 1, then $\{x_i, x_{i+1}\} \subseteq P$ for some $i \in \{1, \ldots, n\}$, and hence both u and v belong to $W_i^+ \cup O_i^2 \cup W_{i+1}^-$. It follows that uv is an edge of $G[W_i^+ \cup O_i^2 \cup W_{i+1}^-]$ and therefore $A_{M_i^2}(u) \cap A_{M_i^2}(v) \neq \emptyset$. So $A_u \cap A_v \neq \emptyset$ and therefore uv is an edge of G'. So we may assume that $P = x_i$ for some $i \in \{1, \ldots, n\}$, and therefore without loss of generality $H_u \cap \{x_{i-1}, x_i, x_{i+1}\} = \{x_{i-1}, x_i\}$ and $H_v \cap \{x_{i-1}, x_i, x_{i+1}\} = \{x_i, x_{i+1}\}$. It follows that $u \in W_i^-$ and $v \in W_i^+$, so uv is an edge of $G[W_i^- \cup O_i^1 \cup W_i^+]$. Therefore $A_{M_i^1}(u) \cap A_{M_i^1}(v) \neq \emptyset$ and hence $A_u \cap A_v \neq \emptyset$, so uv is an edge of G'.

Finally, it follows from our choice of models $M_1^1, M_1^2, \ldots, M_n^1, M_n^2$ that if A_u and A_v have nonempty intersection (i.e. if u and v are adjacent in G'), then there exist two arcs, one corresponding to u and the other to v, in one of the models $M_1^1, M_1^2, \ldots, M_n^1, M_n^2$, that have nonempty intersection and hence u and v are adjacent in G. This completes the proof of (2). $\square$

Putting together Theorems 4.11 and 4.31, we get the main result of this chapter.

Theorem 4.32. *Let G be an even-hole-free graph that is not chordal. Let H be a hole of G of maximum length that furthermore maximises $|X|$. Assume no two vertices of Y_H are crossing. Then G is circular-arc if and only if $G \in \mathcal{C}$.*

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