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Generalisations of Lévy Operators to Manifolds and Symmetric Spaces

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To the humans and cats that made this possible.

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Abstract

In this thesis, functional analytical methods are applied to the study of Lévy and Feller processes on manifolds. In the case of a compact Riemannian manifold, we prove that the Feller semigroup and generator of an isotropic Lévy process extend to L^p, and are self-adjoint in the case $p = 2$. When there is a non-trivial Brownian motion component to the process, we find that the generator has a discrete spectrum of non-positive eigenvalues, and that the semigroup is trace-class.

We also consider the case where the underlying manifold is a Riemannian symmetric space of noncompact type. Considering first the Lévy case, we use harmonic analysis to prove a sufficient condition for the associated convolution semigroup to possess an L^2 density, and calculate the spectrum of a self-adjoint Lévy generator. We then move on to consider Feller processes on a symmetric space of noncompact type. We develop a theory of pseudodifferential operators in this setting, prove that the semigroup and generator of a Feller process are both pseudodifferential operators in the sense we have defined, and calculate their symbols. Using the Hille–Yosida–Ray theorem, sufficient conditions are developed for a pseudodifferential operator to have a closed extension that generates a sub-Feller process. To demonstrate that these conditions are reasonable, we present a class of examples for which they are satisfied.

Contents

Introduction

The study of stochastic processes on manifolds is rich and diverse, and combines approaches from probability, geometry and analysis. On the one hand, one may study these probabilistic objects directly, using stochastic geometry [25, 26, 67]. On the other hand, one may apply techniques from functional analysis to study processes via their operator semigroups and infinitesimal generators. In this thesis, the latter approach is taken to the study of Lévy and Feller processes on Riemannian manifolds.

Roughly speaking, Lévy processes resemble Brownian motion interspersed with random jumps at random times. Viewed in this way, Lévy processes are comprised of a diffusion part and a jump part. The study of diffusion processes on manifolds has so far received more attention than the case in which processes are permitted to have jumps, although both are defined via the theory of stochastic differential equations [42, 51]. One important example of this is the construction of Brownian motion on a Riemannian manifold. First proposed by David Elworthy [24], this construction transfers sample paths of an $\mathbb{R}^d$-valued Brownian motion onto a manifold via stochastic development, or "rolling without slipping" ([42] pp. 44–51). In the more general Lévy case, Applebaum and Estrade [11] used rolling without slipping to develop the notion of a Lévy process on a Riemannian manifold. This topic is expanded further in Chapter 1, and some operator theoretic results are established concerning the infinitesimal generators of these processes (§1.3–§1.5). This work has now appeared as a publication, see [17].

When the manifold is a Lie group, the presence of a group law enables a more familiar definition of a Lévy process, as a stochastically continuous process with stationary and independent increments (see [12, 53] or §3.1.1). This definition can be extended to Riemannian symmetric spaces, by the identification of such a space with a homogeneous space G/K, where G is a connected real Lie group and K is a compact subgroup [54]. A process on G/K is then called a Lévy process if it is the projection of a Lévy process on G, under the projection map $\pi : G \to G/K$ (c.f. Definition 3.1.5). Results concerning Lévy processes on Lie groups have immediate implications for process on G/K.

A key advantage of the symmetric space setting is the presence of the Helgason–Harish-Chandra spherical transform in this setting [33, 34, 37]. Defined as an integral transform with respect to particular class of K-invariant function, the spherical transform may be used in place of the Fourier transform to apply classical Fourier analytic arguments to the study of Lévy and Feller processes on manifolds. One important example of its use in probability is

Gangolli's Lévy–Khinchine formula [29], a result that expresses the spherical transform of the law of a Lévy process in terms of a unique function, known as the Gangolli exponent of the process. This result is a direct analogue of the classical Lévy–Khinchine formula, and much of the theory of Lévy processes carries over. In particular, a Lévy process is uniquely determined by its Lévy characteristics, a triple (b, a, ν), where $b \in \mathbb{R}^d$, $a = (a_{ij})$ is a non-negative definite symmetric $d \times d$ matrix, and ν is a Lévy measure on G (to be defined — see Definition 3.1.9).

The Lévy characteristics of a Lévy process also determine its infinitesimal generator (see [43] Theorem 5.1 or Theorem 3.1.10 of this work). If the characteristics are permitted spatial dependence, the corresponding Lévy-type operator satisfies the positive maximum principle, and is an example of what will later be referred to as a Gangolli operator. The generators of a large class of Feller process take this form, and for this reason Feller processes are often considered to be spatially dependent generalisations of Lévy processes. When $X = \mathbb{R}^d$, this characterisation of Feller processes has been known for some time, and is a natural corollary of the much-celebrated Courrège theorem [21], which gives necessary and sufficient conditions for a densely defined linear operator to satisfy the positive maximum principle. More recently, Applebaum and Le Ngan [15, 16] have proven a generalised Courrège theorem that applies in symmetric spaces more generally, allowing Feller generators to be understood this way. This work built on that of Bony, Courrège and Priouret [20], who proved a generalisation of the Courrège theorem for manifolds, though their results are somewhat limited in that they are stated in terms of local coordinates. In contrast, the global form found in [15, 16] for operators satisfying the positive maximum principle is far better suited to a global harmonic analytical approach.

A natural question to ask is when does the reverse hold. That is, given a Gangolli operator, what conditions are sufficient for there to be a closed extension that generates a Feller process? When the manifold is $\mathbb{R}^d$, the answer to this question is known. In [45], Niels Jacob uses the theory of pseudodifferential operators to prove sufficient conditions for a given pseudodifferential operator to generate an $\mathbb{R}^d$-valued sub-Feller process. In Chapter 4 of this work, we apply similar methods to the symmetric space setting, developing a theory of pseudodifferential operators on symmetric spaces of noncompact type in the process. In particular, we prove that Gangolli operators are pseudodifferential operators in the sense defined, and use the Hille–Yosida–Ray theorem to find sufficient conditions on the symbol for a given pseudodifferential operator to extend to the generator of a sub-Feller process. This work has been submitted for publication — see [64].

Notation

For a locally compact Hausdorff topological space X, $\mathcal{B}(X)$ will denote the Borel σ-algebra associated with X, and $\mathcal{M}(X)$ the space of all Borel measures on X. Let $\mathcal{F}(X)$ denote the set of all functions from $X \to \mathbb{F}$, where $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. Given $f \in \mathcal{F}(X)$, the *support* of f is

$$\mathrm{Supp}(f) \coloneqq \overline{\{x \in X : f(x) \neq 0\}}.$$

Let $B(X)$ denote the space of Borel measurable functions, and $B_b(X)$ the subspace of bounded Borel functions. $B_b(X)$ is a Banach space with respect to the supremum norm $\|\cdot\|_\infty$, defined by

$$\|f\|_\infty \coloneqq \sup_{x \in X} |f(x)| \qquad \forall f \in B_b(X).$$

We write $C_b(X)$ for the closed subspace of $B_b(X)$ consisting of continuous bounded function, $C_0(X)$ for the closed subspace of $B_b(X)$ consisting of continuous functions vanishing at infinity, and $C_c(X)$ for the subspace consisting of compactly supported continuous functions. It is well-known that $C_c(X)$ is dense in $C_0(X)$.

If X is a smooth manifold and $k \in \mathbb{N} \cup \{\infty\}$, then we write $C^k(X)$ for the space of k-times continuously differentiable functions on X, and $C_c^k(X) = C_c(X) \cap C^k(X)$.

If (X, Σ, μ) is a measure space and $1 \leq p < \infty$, then the associated L^p space is denoted $L^p(X, \Sigma, \mu)$, and consists of equivalence classes of measurable functions whose p^{th} power is absolutely integrable, and where functions that agree almost everywhere are identified.

Each L^p space is a Banach space with respect to the p-norm $\|\cdot\|_p$, given by

$$\|f\|_p \coloneqq \left(\int_X |f(x)|^p \mu(dx) \right)^{\frac{1}{p}}, \qquad \forall f \in L^p(X, \Sigma, \mu).$$

In the case $p = 2$, $L^2(X, \Sigma, \mu)$ is also a Hilbert space, with inner product defined in the usual way.

In this work, all measures we consider will be Borel measures, in the sense that $\Sigma = \mathcal{B}(X)$. We therefore suppress Σ in notation, and just write $L^p(X, \mu)$. We may also suppress μ and just write $L^p(X)$, in cases where the measure is clear from the context.

For the majority of this thesis, we will take $\mathbb{F} = \mathbb{R}$. The main exception to this rule is §2.4, where we introduce some prerequisites from the harmonic analysis of Riemannian symmetric spaces.

If E is a complex Banach space, and L a linear operator on E, then $\mathrm{Dom}(L)$ will denote the domain of L, and $\mathrm{Ran}(L)$ its range. The *resolvent set* of L is the set

$$\rho(L) \coloneqq \{\lambda \in \mathbb{C} : \lambda I - L \text{ is invertible with bounded inverse}\}, \tag{0.0.1}$$

and the *spectrum* is

$$\mathrm{Spec}(L) \coloneqq \mathbb{C} \setminus \rho(L).$$

Note that $\rho(L)$ is an open subset of $\mathbb{C}$, and hence $\mathrm{Spec}(L)$ is a closed subset of $\mathbb{C}$.

Let $d \in \mathbb{N}$. By the *standard basis* of $\mathbb{R}^d$, we will mean the vectors $e_1, \ldots, e_d \in \mathbb{R}^d$, defined for each $1 \le i \le d$ by

$$e_i = (0, \ldots, 0, 1, 0, \ldots, 0),$$

where 1 is in the i^{th} position, and all other entries are zero.

Chapter 1

Lévy processes on compact Riemannian manifolds

We begin by investigating the case of a Lévy process on a general manifold. In this chapter, attention is restricted to the compact case, and we build on the work initially developed by Applebaum and Estrade [11]. This work has now been published — see [17].

§ 1.1 Preliminaries

1.1.1 Riemannian manifolds and the bundle of orthonormal frames

We summarise some notions from Riemannian geometry, needed in Section 1.2 when we consider Lévy processes on manifolds. For a more thorough exposition of the concepts discussed in this section, see Kobayashi and Nomizu [50] Chapters I–IV. Chapter II of David Elworthy's monograph [24] is also an excellent resource.

Let (M, g) be a connected Riemannian manifold of dimension $d \in \mathbb{N}$. For each $p \in M$, let OM_p denote the vector space of all linear isometries $r : \mathbb{R}^d \to T_pM$, where T_pM denotes the tangent space of M at p. Each element $r \in OM_p$ may be identified with a choice of orthonormal basis of T_pM, by taking the image under r of the standard basis vectors $e_1, \ldots, e_d$. The disjoint union

$$OM = \coprod_{p \in M} OM_p$$

is called the *bundle of orthonormal frames* for M. The orthogonal group $O(d)$ acts on OM by change of basis. This gives OM the structure of a principle fibre bundle over M — see Chapter I.5 of [50] for background.

For a smooth map $\psi : M \to N$ between smooth manifolds M and N, the *differential* of ψ will be denoted by $d\psi : TM \to TN$, and is a vector bundle homorphism from the tangent bundle of M to the tangent bundle of N. Explicitly, for each $p \in M$, $d\psi$ acts on T_pM via

$$d\psi_p : T_pM \to T_{\psi(p)}N; \;\; d\psi_p(X)(f) \coloneqq X(f \circ \psi),$$

for all $X \in T_pM$ and all $f \in C^\infty(N)$. Writing $\pi : OM \to M$ for the projection map, the *vertical subspace* at $r \in OM_p$ is defined by

$$V_rOM = \{X \in T_rOM : d\pi_r(X) = 0\},$$

and is a linear subspace of T_rOM. A connection of M is a specification of a complementary *horizontal subspace* H_rOM at every frame $r \in OM$, so that each tangent space may be decomposed as a direct sum

$$T_rOM \cong H_rOM \oplus V_rOM, \quad \forall r \in OM. \tag{1.1.1}$$

By Theorem 2.2 in Chapter IV of [50], the Riemannian structure of M gives rise to a unique torsion-free metric connection, called the *Levi-Civita connection* of M. We use this connection in what follows. For more details regarding connections more generally, see [50] Chapter II.

Observe that for $r \in OM_p$, the restriction of $d\pi_r$ to H_rOM defines a linear isomorphism $H_rOM \cong T_pM$. Given $X \in T_pM$, the *horizontal lift* of X is the unique $X^* \in H_rOM$ such that $d\pi_r(X^*) = X$. This construction extends naturally to vector fields.

The *canonical horizontal vector fields* $\{H_x : x \in \mathbb{R}^d\}$ of OM are defined by

$$H_x(r) = r(x)^*, \tag{1.1.2}$$

for each $r \in OM$ and $x \in \mathbb{R}^d$. Writing $H_i = H_{e_i}$ for each $1 \leq i \leq d$, where e_i denotes the i^{th} standard basis vector of $\mathbb{R}^d$, it is clear that $\{H_i : 1 \leq i \leq d\}$ is a basis for HOM.

It is a standard result that the Lie algebra $\mathfrak{o}(d)$ associated with the Lie group $O(d)$ consists of all real-valued skew-symmetric matrices. Given $A \in \mathfrak{o}(d)$, the *fundamental vector field of A* is the vector field A^* on OM associated with the action of the one-parameter subgroup $a_t = \exp(tA)$, so that for each $r \in OM$, $A^*(r)$ is the tangent vector to the curve ra_t at r. Fundamental vector fields are invariant under the action of $O(d)$, and are hence always vertical.

The *canonical form* of OM is the $\mathbb{R}^d$-valued 1-form θ given by

$$\theta(X) = r^{-1}(d\pi_r(X)),$$

for each $X \in T_rOM$. Observe that $\theta(H_x(r)) = x$ for all $r \in OM$ and $x \in \mathbb{R}^d$.

The *connection form* of OM is the $\mathfrak{o}(d)$-valued 1-form ω defined uniquely by the requirement that $\omega(X)^*$ is equal to the vertical part of X, for all $X \in TM$.

These 1-forms give rise to a trivialisation of the tangent bundle TOM, given fibre-wise by the following linear isomorphism

$$\psi_r : T_rOM \to \mathbb{R}^d \times \mathfrak{o}(d); \; X \mapsto (\theta(X), \omega(X)), \tag{1.1.3}$$

for each $r \in OM$ (see Elworthy [24] Chapter II Section 10 for details). Equipping $\mathfrak{o}(d)$ with the inner product

$$\langle A, B\rangle_{\mathfrak{o}(d)} = -\frac{1}{2}\operatorname{tr}(AB), \quad \forall A, B \in \mathfrak{o}(d),$$

we may define a metric $\tilde{g}$ on OM by requiring that (1.1.3) be an isometry, as in Elworthy [24] Chapter III Section 4. Specifically,

$$\tilde{g}(X,Y) = \theta(X)\cdot\theta(Y) - \frac{1}{2}\operatorname{tr}(\omega(X)\omega(Y)), \quad \forall r \in OM,\ \forall X, Y \in T_rOM.$$

Given $r \in OM$ and $X, Y \in H_rOM$, we have $\omega(X) = \omega(Y) = 0$, and so

$$\tilde{g}(X,Y) = r^{-1}(d\pi_r(X))\cdot r^{-1}(d\pi_r(Y)) = g(d\pi_r(X), d\pi_r(Y)),$$

using the fact that $r^{-1} : T_{\pi(r)}M \to \mathbb{R}^d$ is an isometry. It follows that for each $p \in M$ and $r \in OM_p$, $d\pi_r : H_rOM \to T_p(M)$ is an isometric isomorphism.

Since we have given OM the structure of a Riemannian manifold, we may consider the associated Riemannian measure $\tilde{\mu}$ on OM, which is called *Liouville measure* in the literature. By the above considerations, Liouville measure is projected down onto Riemannian measure μ via π, so that

$$\mu = \tilde{\mu}\circ\pi^{-1}. \tag{1.1.4}$$

Given a curve x_t on M and an initial frame $r_0 \in OM_{x_0}$, there is a unique horizontal curve r_t on OM for which $x_t = \pi(r_t)$, called the horizontal lift of x_t (see [50] Chapter II.3 for a proof). Parallel transport along x_t from a point $p = x_{t_0}$ to a point $q = x_{t_1}$ is given by

$$r_{t_1} r_{t_0}^{-1} : T_pM \to T_qM.$$

The *anti-development* of x_t is the curve y_t in $\mathbb{R}^d$ given by

$$y_t = \int_0^t r_s^{-1}\dot{x}_s ds.$$

Equivalently, horizontal lift and anti-development are characterised by the following ordinary differential equation on OM:

$$\dot{r}_t = H_i(r_t)\dot{y}_t^i. \tag{1.1.5}$$

In Section 1.2 we consider a stochastic version of this construction.

For each $p \in M$, let Exp_p denote the Riemannian exponential map at p, defined in a neighbourhood U of $0 \in T_pM$ as follows. Given $v \in T_pM$, there is a unique geodesic $\gamma_{X,v}$ for which $\gamma(0) = p$ and $\dot{\gamma}(0) = v$. If $\|v\|$ is sufficiently small, $\gamma_{X,v}$ may be defined up to time 1, and we may define

$$\operatorname{Exp}_p(v) = \gamma(1). \tag{1.1.6}$$

1.1.2 Some concepts from stochastic calculus

Given a probability space $(\Omega, \mathcal{F}, P)$, a family of sub-σ-algebras $\{\mathcal{F}_t, t \geq 0\}$ of $\mathcal{F}$ is called a a *filtration* if $\mathcal{F}_s \subseteq \mathcal{F}_t$ whenever $0 \leq s \leq t$. A stochastic process $X = \{X(t), t \geq 0\}$ taking values on a locally compact Hausdorff space E is *adapted* to this filtration if $X(t)$ is $\mathcal{F}_t$-measurable for all $t \geq 0$. Every process X is adapted to its *natural filtration*, $\mathcal{F}^X = (\mathcal{F}_t^X, t \geq 0\}$, where each $\mathcal{F}_t^X$ is taken to be the smallest sub-σ-algebra for which $X(s)$ is measurable for all $0 \leq s \leq t$.

Markov and Feller processes

An adapted process X taking values on a locally compact Hausdorff space E is called a *Markov process* if for all $f \in B_b(E)$ and $0 \leq s \leq t$,

$$\mathbb{E}(f(X(t))|\mathcal{F}_s) = \mathbb{E}\left(f(X(t))\middle|\, X(s)\right). \tag{1.1.7}$$

For our purposes, E will usually be a manifold. The transition semigroup of a Markov process X is a family $(T_t, t \geq 0)$ of bounded linear operators on $B_b(E)$, defined by

$$T_t f(x) = \mathbb{E}(f(X(t))|X(0) = x), \qquad \forall t \geq 0,\ f \in B_b(E),\ x \in E.$$

It may be shown using the Markov property (1.1.7) that $(T_t, t \geq 0)$ is an algebraic operator semigroup on $B_b(E)$, in the sense that

$$T_0 = I, \quad \text{and} \quad T_t T_s = T_{t+s}, \quad \forall t, s \geq 0 \tag{1.1.8}$$

— see [6] Chapter 3 for more details.

For each $x \in E$, $t \geq 0$ and $A \in \mathcal{B}(E)$, let

$$p_t(x, A) = \mathbb{P}(X(t) \in A|X(0) = x).$$

The mappings $p_t : E \times \mathcal{B}(E) \to [0, 1]$ are called the transition probabilities of X. Note that the transition semigroup may be expressed

$$T_t f(x) = \int_E f(y) p_t(x, dy), \qquad \forall t \geq 0,\ f \in B_b(E),\ x \in E.$$

We say that a Markov process X is a *Feller process* if the restriction of $(T_t, t \geq 0)$ to $C_0(E)$ forms a strongly continuous operator semigroup. In the presence of (1.1.8), it is sufficient that the transition semigroup satisfy

1. For all $t \geq 0$, $T_t\big[C_0(E)\big] \subseteq C_0(E)$,
2. For all $f \in C_0(E)$,
$$\lim_{t \to 0} \|T_t f - f\|_\infty = 0. \tag{1.1.9}$$

The *infinitesimal generator* of a Feller process X is a (typically unbounded) linear operator $\mathcal{L}$ with domain

$$\text{Dom}(\mathcal{L}) = \left\{ f \in C_0(E) : \exists g_f \in C_0(E) \text{ s.t. } \lim_{t \to 0} \left\| \frac{T_t f - f}{t} - g_f \right\|_\infty = 0 \right\},$$

defined by

$$\mathcal{L}f(x) = g_f(x) = \lim_{t \to 0} \frac{T_t f(x) - f(x)}{t}, \qquad \forall f \in \text{Dom}(\mathcal{L}),\ x \in E$$

The infinitesimal generator describes the local behaviour of a Feller process. For more details on Markov and Feller semigroups and their generators, see [10] Chapter 7.

Martingales and semimartingales

An adapted, integrable, $\mathbb{R}^d$-valued process X is a *martingale* if

$$\mathbb{E}(X(t)|\mathcal{F}_s) = X(s) \qquad \text{a.s.} \tag{1.1.10}$$

whenever $0 \leq s < t < \infty$.

A *stopping time* is a random variable $\tau : \Omega \to [0, \infty]$ such that $\{\tau \leq t\} \in \mathcal{F}_t$ for all $t \geq 0$. A *local martingale* is an adapted $\mathbb{R}^d$-valued process X for which there is an almost surely increasing sequence (τ_n) of stopping times tending to infinity and such that $\{X(t \wedge \tau_n), t \geq 0\}$ is a martingale. More generally still, an adapted $\mathbb{R}^d$-valued process Y is a *semimartingale* if there is a local martingale X and a right-continuous adapted process V of finite variation such that

$$Y(t) = Y(0) + X(t) + A(t) \tag{1.1.11}$$

for all $t \geq 0$. For more details, see Protter [58].

Semimartingales taking values on a d-dimensional Riemannian manifold M may also be considered, although one must then address the possibility that the process leaves the manifold in finite time. To resolve this issue, processes X are generally defined up to a stopping time η. An M-valued, adapted process $X = (X(t), 0 \leq t < \eta)$ is called a *semimartingale* if $f(X)$ is a real-valued semimartingale on $[0, \eta)$ for all $f \in C^\infty(M)$.

Lévy processes on $\mathbb{R}^d$

Given an $\mathbb{R}^d$-valued process $Y = (Y(t), t \geq 0)$, the random variables $Y(t) - Y(s)$, where $0 \leq s \leq t$, are called the *increments* of Y. Y is said to have *independent increments* if $Y(t) - Y(s)$ is independent of $\mathcal{F}_s^Y$, for all $t > s \geq 0$, and *stationary increments* if $Y(t) - Y(s) \sim Y(t-s) - Y(0)$, for all $t \geq s \geq 0$.

Definition 1.1.1. Y is called an *Lévy process* if the following are satisfied:

(i) $Y(0) = 0$ a.s.,

(ii) Y has stationary and independent increments.

(iii) *(Stochastic continuity)* For all $s \geq 0$ and all $\epsilon > 0$,

$$\lim_{t \to s} P(|Y(t) - Y(s)| > \epsilon) = 0.$$

One of the most important Lévy processes is Brownian motion. Suppose $a = (a_{ij})$ is a non-negative definite symmetric $d \times d$ matrix. An $\mathbb{R}^d$-valued Lévy process $B_a = (B_a(t), t \geq 0)$ is called a *Brownian motion with covariance matrix* a on $\mathbb{R}^d$ if it has continuous sample paths, and if $B_a(t) \sim N(0, ta)$ for all $t \geq 0$. In the case where a is the $d \times d$ identity matrix I, B_a is called a *standard Brownian motion* on $\mathbb{R}^d$, usually denoted by $B = (B(t), t \geq 0)$.

Suppose now that B is a standard Brownian motion on $\mathbb{R}^m$, where $1 \leq m \leq d$. Since a is non-negative definite and symmetric, we may choose a $d \times m$ matrix q such that $a = qq^T$. Then $(qB(t), t \geq 0)$ is a Brownian motion on $\mathbb{R}^d$ with covariance matrix a.

A Borel measure ν is called a *Lévy measure* if $\nu(\{0\}) = 0$ and $\int_{\mathbb{R}^d} 1 \wedge |x|^2 dx < \infty$.

Theorem 1.1.2 (Lévy–Khinchine formula)**.** *Let Y be a Lévy process on $\mathbb{R}^d$. Then, there exists $b \in \mathbb{R}^d$, a non-negative definite symmetric $d \times d$ matrix $a = (a_{ij})$, and a Lévy measure ν,*

$$\mathbb{E}\left(e^{iu \cdot Y(0)}\right) = e^{-\eta(u)}, \tag{1.1.12}$$

for all $u \in \mathbb{R}^d$, where

$$\eta(u) = -ib \cdot u + \frac{1}{2} u \cdot (au) + \int_{\mathbb{R}^d} \left(1 - e^{iu \cdot x} - \mathbf{1}_{|x|<1}\, iu \cdot y\right) \nu(dx). \tag{1.1.13}$$

Conversely, given a mapping η of this form, $u \mapsto e^{-\eta(u)}$ is the characteristic function of some Lévy process on $\mathbb{R}^d$.

For a proof of this theorem, see for example Sato [62] Chapter 2. We call the triple (b, a, ν) the *Lévy characteristics* of Y. In Chapter 3, we will see a generalisation of this theorem for Lévy processes on symmetric spaces, known as Gangolli's Lévy–Khinchine formula (see [29]).

Let Y be a Lévy process on $\mathbb{R}^d$. Every Lévy process has a càdlàg modification. If such a modification has been chosen, we may define the *jump process* ΔY of Y by $\Delta Y(t) = Y(t) - Y(t-)$. For each $t \geq 0$, and for $A \in \mathcal{B}(\mathbb{R}^d)$ bounded away from zero, there are at most finitely many $0 \leq s \leq t$ such that $\Delta Y(s) \in A$. We define

$$N(t, A) := |\{0 \leq s \leq t : \Delta Y \in A\}|, \qquad \text{and} \qquad N(t, \{0\}) = 0. \tag{1.1.14}$$

Then N is a Poisson random measure on $[0, \infty) \times \mathbb{R}^d$ with intensity $dt \otimes \nu$, where dt is Lebesgue measure on $[0, \infty)$ and ν is as in Theorem 1.1.2. The associated *compensated Poisson random measure* is defined

$$\tilde{N}(t, A) = N(t, A) - t\nu(A). \tag{1.1.15}$$

See Applebaum [6] §2.3, pp. 99–112, for more details.

Theorem 1.1.3 (Lévy–Itô representation)**.** *Let Y be a Lévy process with characteristics (b, a, ν). Then*

$$Y(t) = bt + B_a(t) + \int_{|x|<1} x\tilde{N}(t, dx) + \int_{|x|\geq 1} xN(t, dx), \qquad \forall t \geq 0, \tag{1.1.16}$$

where B_a is a Brownian motion with covariance matrix a, and N and $\tilde{N}$ are given by (1.1.14) and (1.1.15).

Lévy processes form an important class of Feller process on $\mathbb{R}^d$. Their Feller semigroups correspond precisely with the convolution semigroups of probability measures $(\mu_t, t \geq 0)$ on $\mathbb{R}^d$ (see Definition 2.4.3), via

$$\mu_t(A) = \mathbb{P}(X(t) \in A), \qquad t \geq 0,\ A \in \mathcal{B}(\mathbb{R}^d.$$

Lévy processes are also semimartingales, in the sense of (1.1.11). If Y is given by (1.1.16), then the martingale part is $B_a(t) + \int_{|x|<1} x\tilde{N}(t, dx)$, and $bt + \int_{|x|\geq 1} xN(t, dx)$ is a process of bounded variation. See [6] Proposition 2.7.1, pp. 137, for a proof.

§ 1.2 Lévy processes on manifolds

Suppose now that M is a compact Riemannian manifold. One can easily show that OM is also compact, using the fact that both its base manifold and structure group $O(d)$ are compact. In particular, both M and OM are complete, and μ and $\tilde{\mu}$ are finite measures. Note also that $C_c(M) = C(M) = C_0(M)$, and $C_c(OM) = C(OM) = C_0(OM)$.

Since OM is complete, the vector fields $\{H_x : x \in \mathbb{R}^d\}$ (see (1.1.2)) are complete; we write $\exp(tH_x)$ for the associated flows of diffeomorphisms. These flows are related to the Riemann exponential map (1.1.6) by

$$\pi\big(\exp(H_x)(r)\big) = \mathrm{Exp}_p\, r(x), \qquad \forall x \in \mathbb{R}^d,\ p \in M,\ r \in OM_p. \tag{1.2.1}$$

Let $Y = (Y(t), t \geq 0)$ be a $\mathbb{R}^d$-valued Lévy process, let $r \in OM$, and consider the stochastic differential equation (SDE)

$$dR(t) = \sum_{i=1}^{d} H_i(R(t-)) \diamond dY_i(t),\ t \geq 0; \quad R(0) = r \text{ (a.s.)} \tag{1.2.2}$$

where $\diamond$ denotes the *Marcus canonical integral*, defined in terms of the Itô integral by

$$\begin{aligned}\sum_{i=1}^{d} \int_0^t & H_i f(R(s-)) \diamond dY_i(s) \\ &= f(r) + \sum_{i=1}^{d} \int_0^t H_i f(R(s-)) dY_i(s) + \frac{1}{2} \sum_{i,j=1}^{d} \int_0^t a_{ij} H_i H_j f(R(s)) ds \\ &\quad + \sum_{s \leq t} \Big\{ f\Big(\exp\big(H_{\Delta Y(s)} R(s-)\big)\Big) - f\big(R(s-)\big) - H_{\Delta Y(s-)} f\big(R(s-)\big) \Big\},\end{aligned} \tag{1.2.3}$$

for each $f \in C^\infty(OM)$. For more information regarding the Itô integral, see for example [6] Chapter 4.

An OM-valued cádág semimartingale R defined up to a stopping time η is called a solution to (1.2.2) if

$$f(R(t)) = \sum_{i=1}^{d} \int_0^t H_i f(R(s-)) \diamond dY_i(s),$$

for all $f \in C^\infty(OM)$ and $0 \leq t < \eta$.

The Marcus integral is the best choice in this setting because it is invariant under changes in local coordinates (see [6] §6.10, pp. 417 or Protter [58] Theorem 36, pp. 82). For more information regarding SDEs on manifolds driven by processes with jumps, see Kunita [51] Chapter 7. When the driving process Y is continuous, the Marcus integral coincides with the Stratonovich integral.

It is proven in [3] that (1.2.2) has a unique càdlàg solution, defined up to an explosion time η. In [11], it is shown that $\eta = \infty$ almost surely, and that R is a Feller process on OM, with infinitesimal generator $\mathcal{L}$ given by

$$\begin{aligned}\mathcal{L}f(r) = \sum_{i=1}^{d} b^i H_i f(r) + \frac{1}{2} \sum_{i,j=1}^{d} a^{ij} H_i H_j f(r) \\ + \int_{\mathbb{R}^d} \left\{ f(\exp(H_x)(r)) - f(r) - \mathbf{1}_{|x|<1} H_x f(r) \right\} \nu(dx)\end{aligned}$$

for all $f \in C^\infty(OM)$ and $r \in OM$. We call R a *horizontal Lévy processes* with initial frame r.

As in [11], we impose the assumption that the driving process Y is isotropic, in that its law is $O(d)$-invariant. This assumption is sufficient for the process $X = \pi(R)$ obtained by projection onto the base manifold to be a Feller process on M ([11] Theorem 3.1). We call the projected process X an *isotropic Lévy process* on M.

Example 1.2.1. *(Brownian motion)* An important example of the above construction is when Y is a standard Brownian motion on $\mathbb{R}^d$. In this case, the solution to (1.2.2) is called a *horizontal Brownian motion on OM*. Its generator is $\frac{1}{2}\Delta_H$, where

$$\Delta_H \coloneqq \sum_{i=1}^{d} H_i^2$$

denotes Bochner's horizontal Laplacian. The projected process X is then precisely the Eels–Elworthy construction of Brownian motion on M (see [24] or [42]). Its infinitesimal generator is $\frac{1}{2}\Delta$, where

$$\Delta \coloneqq \operatorname{div} \circ \operatorname{grad} \tag{1.2.4}$$

denotes the Laplace–Beltrami operator on M (see [60] for more information regarding this operator).

Returning to the general Lévy case, since Y is isotropic, its Lévy characteristics take the form $(0, aI, \nu)$, where $a \geq 0$, and ν is $O(d)$-invariant (see [6] Corollary 2.4.22, pp. 128). The infinitesimal generators of R and X are then

$$\mathcal{L} = \frac{1}{2}a\Delta_H + \mathcal{L}_J \tag{1.2.5}$$

and

$$\mathcal{A} = \frac{1}{2}a\Delta + \mathcal{A}_J, \tag{1.2.6}$$

respectively, where the jump parts $\mathcal{L}_J$ and $\mathcal{A}_J$ are given by

$$\mathcal{L}_J g(r) = \int_{\mathbb{R}^d} \Big\{ g\big(\exp(H_x)(r)\big) - g(r) - \mathbf{1}_{|x|<1}\, H_x g(r) \Big\} \nu(dx), \tag{1.2.7}$$

for all $g \in C^\infty(OM)$ and $r \in OM$, and

$$\mathcal{A}_J f(p) = \int_{T_pM} \Big\{ f(\mathrm{Exp}_p\, y) - f(p) - \mathbf{1}_{|y|<1}\, y f(p) \Big\} \nu_p(dy), \qquad \forall f \in C^\infty(M),\ p \in M.$$

Here, the family of Lévy measures $\{\nu_p : p \in M\}$ act on each tangent space T_pM, and are defined by $\nu_p = \nu \circ r^{-1}$ for any frame $r \in OM_p$. The two generators also satisfy

$$\mathcal{A}f(p) = \mathcal{L}(f \circ \pi)(r), \qquad \forall f \in C^\infty(M),\ p \in M,\ r \in OM_p.$$

Observe that since ν is $O(d)$-invariant, the right hand side of (1.2.7) is invariant under the change of variable $x \mapsto -x$, and hence for all $f \in C^\infty(OM)$ and $r \in OM$,

$$\mathcal{L}_J g(r) = \int_{\mathbb{R}^d} \Big\{ g\big(\exp(H_{-x})(r)\big) - g(r) + \mathbf{1}_{|x|<1}\, H_x g(r) \Big\} \nu(dx), \tag{1.2.8}$$

where we have used the fact that $H_{-x}f = -H_x f$. Summing (1.2.7) and (1.2.8) and dividing by two,

$$\mathcal{L}_J g(r) = \frac{1}{2} \int_{\mathbb{R}^d} \Big\{ g\big(\exp(H_x)(r)\big) - 2g(r) + g\big(\exp(H_{-x})(r)\big) \Big\} \nu(dx) \tag{1.2.9}$$

for all $g \in C^\infty(OM)$ and $r \in OM$. It follows that

$$\mathcal{A}_J f(p) = \frac{1}{2} \int_{T_pM} \Big\{ f(\mathrm{Exp}_p\, y) - 2f(p) + f(\mathrm{Exp}_p(-y)) \Big\} \nu_p(dy) \tag{1.2.10}$$

for each $f \in C^\infty(M)$ and $p \in M$. Note the analogous expression (5.4.16) in [8] for symmetric Lévy motion on a Lie group.

§ 1.3 L^p properties of the semigroups

Let $(S_t, t \geq 0)$ and $(T_t, t \geq 0)$ denote, respectively, the transition semigroups of R and X, as defined in Section 1.1.2. Since $X = \pi(R)$, it is immediate that

$$T_t f(p) = S_t(f \circ \pi)(r), \tag{1.3.1}$$

for all $t \geq 0$, $f \in B_b(M)$ and $p = \pi(r) \in M$. Since R and X are Feller processes, the restriction of $(S_t, t \geq 0)$ and $(T_t, t \geq 0)$ to $C(OM)$ and $C(M)$, respectively, form strongly continuous contraction semigroups.

Any Riemannian manifold has a natural L^p structure arising from its Riemannian measure, and so we may consider the spaces $L^p(OM)$ and $L^p(M)$ for $1 \leq p \leq \infty$. For $p < \infty$, we prove that $(S_t, t \geq 0)$ and $(T_t, t \geq 0)$ extend to strongly continuous contraction semigroups on these spaces, and that they are self-adjoint when $p = 2$.

We begin by considering the semigroup $(S_t, t \geq 0)$ associated with the horizontal process R; analogous results for $(T_t, t \geq 0)$ are later obtained by projection down onto M.

Theorem 1.3.1. *For all $1 \leq p < \infty$, $(S_t, t \geq 0)$ extends to a strongly continuous semigroup of contractions on $L^p(OM)$.*

Proof. Let $q_t(\cdot, \cdot)$ denote the transition probabilities of R, so that

$$S_t f(r) = \int_{OM} f(u) q_t(r, du), \qquad \forall t \geq 0,\ f \in C(OM),\ r \in OM.$$

The horizontal fields H_x are divergence free for all $x \in \mathbb{R}^d$ (Proposition 4.1 of [56]), and so by Theorem 3.1 of [13], $\tilde{\mu}$ is invariant for S_t, in the sense that

$$\int_{OM} (S_t f)(r) \tilde{\mu}(dr) = \int_{OM} f(r) \tilde{\mu}(dr), \qquad \forall t \geq 0,\ f \in C(OM).$$

Therefore, by Jensen's inequality,

$$\begin{aligned} \|S_t f\|_p^p = \int_{OM} \left| \int_{OM} f(u) q_t(r, du) \right|^p \tilde{\mu}(dr) &\leq \int_{OM} \int_{OM} |f(u)|^p q_t(r, du) \tilde{\mu}(dr) \\ &= \int_{OM} (S_t |f|^p)(r) \tilde{\mu}(dr) = \int_{OM} |f(r)|^p \tilde{\mu}(dr) = \|f\|_p^p \end{aligned}$$

for all $t \geq 0$ and $f \in C(OM)$. Each S_t has domain $C(OM)$, which is a dense subspace of $L^p(OM)$ for all $1 \leq p < \infty$. It follows that each S_t extends to a unique contraction defined on the whole of $L^p(OM)$, which we also denote by S_t. By continuity, the semigroup property

$$S_t S_s = S_{t+s}, \qquad \forall s, t \geq 0$$

continues to hold in this larger domain. It remains to prove strong continuity, i.e. that

$$\lim_{t \to 0} \|S_t f - f\|_p = 0 \tag{1.3.2}$$

for all $f \in L^p(OM)$. By density of $C(OM)$ in $L^p(OM)$, it is sufficient to verify this for $f \in C(OM)$. The map $t \mapsto S_t$ is strongly continuous in $C(OM)$, and so $\lim_{t \to 0} \|S_t f - f\|_\infty = 0$ for all $f \in C(OM)$. Since $(OM, \mathcal{B}(OM), \tilde{\mu})$ is a finite measure space,

$$\|S_t f - f\|_p \leq \tilde{\mu}(OM)^{\frac{1}{p}} \|S_t f - f\|_\infty,$$

for all $f \in C(OM)$. Equation (1.3.2) now follows. □

Remark 1.3.2. The final part of the above proof applies more generally, in that if X is a compact space equipped with a finite measure m, and if $(P_t, t \geq 0)$ is a Feller semigroup on X, then $(P_t, t \geq 0)$ is strongly continuous in $L^p(X, m)$.

Projection down onto the base manifold yields the following.

Theorem 1.3.3. *For all $1 \leq p < \infty$, $(T_t, t \geq 0)$ extends to a strongly-continuous semigroup of contractions on $L^p(M)$.*

Proof. Let $1 \leq p < \infty$. By equations (1.1.4) and (1.3.1), many of the conditions we must check follow from their analogues on the frame bundle. Indeed, for all $f \in C^\infty(M)$ and $t \geq 0$, we have

$$\|T_t f\|^p_{L^p(M)} = \int_M |T_t f(p)|^p d\mu = \int_{OM} |S_t(f \circ \pi)(r)|^p d\tilde{\mu} = \|S_t(f \circ \pi)\|^p_{L^p(OM)},$$

and so, using the fact that S_t is a contraction of $L^p(OM)$,

$$\|T_t f\|_{L^p(M)} = \|S_t(f \circ \pi)\|_{L^p(OM)} \leq \|f \circ \pi\|_{L^p(OM)} = \|f\|_{L^p(M)}.$$

Hence T_t extends to a contraction of $L^p(M)$ for all $t \geq 0$. It is clear by continuity that the semigroup property continues to hold on this larger domain, as does equation (1.3.1). Strong continuity follows by Remark 1.3.2, or alternatively can be seen by the observation

$$\|T_t f - f\|_{L^p(M)} = \|S_t(f \circ \pi) - f \circ \pi\|_{L^p(OM)} \qquad \forall f \in L^p(M).$$

Thus $(T_t, t \geq 0)$ extends to a contraction semigroup on $L^p(M)$ for all $1 \leq p < \infty$. □

We continue to denote the generators of $(S_t, t \geq 0)$ and $(T_t, t \geq 0)$ by $\mathcal{L}$ and $\mathcal{A}$, respectively. Note that by Lemma 6.1.14 of [23], $\mathcal{L}$ and $\mathcal{A}$ are both closed operators on $L^p(OM)$.

§ 1.4 The Case $p = 2$

For the remainder of this chapter we focus on the case $p = 2$. Our aim in this section is to prove that the semigroups $(S_t, t \geq 0)$ and $(T_t, t \geq 0)$ are self-adjoint semigroups on $L^2(OM)$ and $L^2(M)$, respectively. By a standard result from semigroup theory ([22] Theorem 4.6, pp. 99), it will follow that $\mathcal{L}$ and $\mathcal{A}$ are self-adjoint linear operators.

Let us first impose the assumption that the Lévy measure ν is finite. In this case, $\mathcal{A}_J$ is the generator of a compound Poisson process on M (see [11]).

Lemma 1.4.1. *If ν is finite, then $\mathcal{L}$ is a self-adjoint operator on $L^2(OM)$.*

Proof. Since ν is finite, $\mathcal{L}_J$ is a bounded linear operator on $L^2(OM)$, and so equation (1.2.9) extends by continuity to the whole of $L^2(OM)$. It follows that $\mathcal{L}$ is a bounded perturbation of the horizontal Laplacian, and so its domain is $\mathrm{Dom}(\Delta_H)$. Clearly $\mathcal{L}$ is symmetric on this domain.

Since $\mathcal{L}$ is a closed, symmetric operator, by Theorem X.1 on page 136 of Reed and Simon [59], the spectrum $\mathrm{Spec}(\mathcal{L})$ of $\mathcal{L}$ is equal to one of the following:

1. The closed upper-half plane.
2. The closed lower-half plane.
3. The entire complex plane.
4. A subset of $\mathbb{R}$.

Moreover, $\mathcal{L}$ is self-adjoint if and only if Case 4 holds. By Theorem 8.2.1 of [23],

$$\sigma(\mathcal{L}) \subseteq (-\infty, 0], \tag{1.4.1}$$

from which we see that Case 4 is the only option. □

We now drop the assumption that ν is finite.

Theorem 1.4.2. *$(S_t, t \geq 0)$ and $(T_t, t \geq 0)$ are self-adjoint semigroups of operators on $L^2(OM)$ and $L^2(M)$, respectively.*

Proof. We will find it convenient to rewrite the process $R(t)$ (with initial condition $R(0) = r$ (a.s.)) as the action of a stochastic flow η_t on the point r, as in [11]. Then as shown in Section 4 of [11], there is a sequence $(\eta_t^{(n)})$ of stochastic flows on OM such that each $\eta_t^{(n)}$ is the flow of a horizontal Lévy process with finite Lévy measure, and

$$\lim_{n\to\infty} \eta_t^{(n)}(r) = \eta_t(r) \quad \text{(a.s.)},$$

for all $r \in OM$ and $t \geq 0$.

Let $(S_t^{(n)}, t \geq 0)$ be the transition semigroup corresponding to the flow $(\eta_t^{(n)}, t \geq 0)$, for each $n \in \mathbb{N}$. By Lemma 1.4.1, each $(S_t^{(n)}, t \geq 0)$ has a self-adjoint generator, and hence is a self-adjoint semigroup on $L^2(OM)$, by Theorem 4.6 on page 99 of [22].

By the dominated convergence theorem, for each $f \in C(OM)$ and $t \geq 0$, we have

$$\lim_{n\to\infty} \left\| S_t f - S_t^{(n)} f \right\|^2_{L^2(OM)} = \lim_{n\to\infty} \int_{OM} \left| \mathbb{E}\left(f\left(\eta_t(r)\right) - f\left(\eta_t^{(n)}(r)\right)\right) \right|^2 \tilde{\mu}(dr) = 0.$$

Then by the density of $C(OM)$ in $L^2(OM)$, and a standard $\epsilon/3$ argument (using the fact that $S_t^{(n)}$ is an L^2-contraction), we deduce that for all $f \in L^2(OM)$,

$$\lim_{n\to\infty} \left\| S_t f - S_t^{(n)} f \right\|_{L^2(OM)} = 0.$$

So S_t is the strong limit of a sequence of bounded self-adjoint operators, and hence is itself self-adjoint. To see that $(T_t, t \geq 0)$ is also self-adjoint, let $t \geq 0$ and $f, g \in L^2(M)$, and observe that by (1.1.4) and (1.3.1),

$$\langle T_t f, g \rangle_{L^2(M)} = \langle S_t(f \circ \pi), g \circ \pi \rangle_{L^2(OM)} = \langle f \circ \pi, S_t(g \circ \pi) \rangle_{L^2(OM)} = \langle f, T_t g \rangle_{L^2(M)}.$$

□

By Theorem 4.6 of Davies [22], $-\mathcal{L}$ and $-\mathcal{A}$ are positive self-adjoint operators on $L^2(OM)$ and $L^2(M)$, respectively.

§ 1.5 Spectral properties of the generator

For this final section of Chapter 1, we restrict attention to the case in which X has non-trivial Brownian part (that is, when $a > 0$), and prove some spectral results that are already well-established for the case of Brownian motion and the Laplace–Beltrami operator Δ (see (1.2.4)). For example, it is well known that Δ has a discrete spectrum of eigenvalues. Each eigenspace is finite-dimensional, and the eigenvectors may be normalised so as to form an orthonormal basis of $L^2(M)$ (see for example Lablée [52] Theorem 4.3.1). Moreover, such an eigenbasis (ψ_n) can be ordered so that the corresponding sequence of eigenvalues decreases to $-\infty$. For each $n \in \mathbb{N}$, write $-\mu_n$ for the eigenvalue associated with ψ_n, so that the real sequence (μ_n) satisfies

$$0 \leq \mu_1 \leq \mu_2 \leq \ldots \leq \mu_n \to \infty \text{ as } n \to \infty. \tag{1.5.1}$$

We prove an analogous result for $\mathcal{A}$, using a generalisation of the approach used by Lablée [52].

Theorem 1.5.1. *Let X be an isotropic Lévy process on M with non-trivial Brownian part. Then its generator $\mathcal{A}$ has a discrete spectrum*

$$\mathrm{Spec}(\mathcal{A}) = \{-\lambda_n : n \in \mathbb{N}\},$$

where (λ_n) is a sequence of real numbers satisfying

$$0 \leq \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n \to \infty \text{ as } n \to \infty. \tag{1.5.2}$$

Moreover, each of the associated eigenspaces is finite-dimensional, and there is a corresponding sequence (ϕ_n) of eigenvectors that forms an orthonormal basis of $L^2(M)$.

Remark 1.5.2. We will generally assume that (1.5.2) is listed with multiplicity, so that for all $n \in \mathbb{N}$, $-\lambda_n$ is the eigenvalue associated with ϕ_n.

Proof. Without loss of generality, assume that $a = 2$ so that

$$\mathcal{A} = \Delta + \mathcal{A}_J, \tag{1.5.3}$$

where $\mathcal{A}_J$ is given by (1.2.10). Both $\mathcal{A}$ and $\mathcal{A}_J$ are generators of self-adjoint contraction semigroups, and hence $-\mathcal{A}$ and $-\mathcal{A}_J$ are positive, self-adjoint operators ([22] Theorem 4.6). For $f, g \in \operatorname{Dom} \mathcal{A}$, define

$$\langle f, g\rangle_{\mathcal{A}} = \langle f, g\rangle_2 - \langle \mathcal{A} f, g\rangle_2. \tag{1.5.4}$$

The operator $I - \mathcal{A}$ is also positive and self-adjoint, and so by Theorem 11 of [19], there is a unique positive self-adjoint operator B such that $B^2 = I - \mathcal{A}$. By (1.4.1), $I - \mathcal{A}$ is invertible, and hence B is injective. Moreover,

$$\langle f, g\rangle_{\mathcal{A}} = \langle Bf, Bg\rangle_2, \quad \forall f, g \in \operatorname{Dom} \mathcal{A},$$

from which it is easy[1] to check that $\langle \cdot, \cdot\rangle_{\mathcal{A}}$ defines an inner product on $\operatorname{Dom} \mathcal{A}$.

Let V denote the completion of $C^\infty(M)$ with respect to $\langle \cdot, \cdot\rangle_{\mathcal{A}}$. By (1.5.3), we have

$$\langle f, g\rangle_{\mathcal{A}} = \langle f, g\rangle_{H^1} - \langle \mathcal{A}_J f, g\rangle_2 \quad \forall f, g \in C^\infty(M),$$

and since $-\mathcal{A}_J$ is a positive operator, it follows that $\|f\|_{\mathcal{A}} \geq \|f\|_{H^1}$ for all $f \in C^\infty(M)$. Similarly, (1.5.9) implies $\|f\|_{H^1} \geq \|f\|_2$ for all $f \in C^\infty(M)$. Hence

$$V \subseteq H^1(M) \subseteq L^2(M),$$

and

$$\|f\|_2 \leq \|f\|_{H^1} \leq \|f\|_{\mathcal{A}}, \quad \forall f \in V. \tag{1.5.5}$$

In particular, the inclusion $V \hookrightarrow H^1(M)$ is bounded. By Rellich's theorem (see Lablée [52] pp. 68), the inclusion $H^1(M) \hookrightarrow L^2(M)$ is compact, and hence so is the inclusion $i : V \hookrightarrow L^2(M)$ (it is the composition of a compact operator with a bounded operator).

Let $f \in L^2(M)$ and consider $l \in V^*$ given by

$$l(g) = \langle f, g\rangle_2 \quad \forall g \in V.$$

For all $g \in V$ we have by the Cauchy-Schwarz inequality

$$|l(g)| \leq \|f\|_2 \|g\|_2 \leq \|f\|_2 \|g\|_{\mathcal{A}}.$$

Hence

$$\|l\|_{V^*} \leq \|f\|_2, \tag{1.5.6}$$

[1] Bilinearity and symmetry are clear, and positive-definiteness is immediate by injectivity of B.

where $\|\cdot\|_{V^*}$ denotes the norm of V^*. By the Riesz representation theorem, there is a unique $v_f \in V$ for which

$$\langle v_f, g\rangle_{\mathcal{A}} = l(g), \qquad \forall g \in V.$$

Moreover,

$$\|v_f\|_{\mathcal{A}} = \sup_{g\in V\setminus\{0\}} \frac{|\langle v_f, g\rangle_{\mathcal{A}}|}{\|g\|_{\mathcal{A}}} = \|l\|_{V^*}.$$

Define $T : L^2(M) \to V$ by $Tf = v_f$ for all $f \in L^2(M)$. Then

$$\langle Tf, g\rangle_{\mathcal{A}} = \langle f, g\rangle_2 \qquad \forall f \in L^2(M), g \in V, \tag{1.5.7}$$

and T is bounded, since by (1.5.6),

$$\|Tf\|_{\mathcal{A}} = \|l\|_{V^*} \leq \|f\|_2,$$

for all $f \in L^2(M)$. By (1.5.5),

$$\|Tf\|_{\mathcal{A}} \leq \|f\|_{\mathcal{A}} \qquad \forall f \in V,$$

and so $T|_V$ is a bounded operator on V. We also have

$$T|_V = T \circ i,$$

and, since i is compact, so too is $T|_V$. By symmetry of inner products and equation (1.5.7), T is self-adjoint. Equation (1.5.7) also implies that $T|_V$ is a positive operator, and that 0 is not an eigenvalue of $T|_V$ (indeed, if $Tf = 0$, then $\|f\|_2^2 = \langle Tf, f\rangle_{\mathcal{A}} = 0$).

By the Hilbert–Schmidt theore (Simon [65] Section 3.2), the spectrum of $T|_V$ consists of a sequence (α_n) of positive eigenvalues that decreases to 0. Each eigenspace is finite-dimensional, and the corresponding eigenvectors can be normalised so as to form an orthonormal basis (v_n) of $(V, \langle\cdot,\rangle_{\mathcal{A}})$.

In fact, it is easy to see from the definition of $\langle\cdot,\cdot\rangle_{\mathcal{A}}$ that

$$T = (I - \mathcal{A})^{-1}, \tag{1.5.8}$$

and hence the spectrum of $\mathcal{A}$ is just $\{1 - \alpha_n^{-1} : n \in \mathbb{N}\}$, with corresponding eigenvectors still given by the v_n. Moreover, we may scale these eigenvectors so that they are L^2-orthonormal. Indeed, for each $n \in \mathbb{N}$, let

$$\phi_n = \frac{1}{\sqrt{\alpha_n}} v_n.$$

Then for all $m, n \in \mathbb{N}$,

$$\langle \phi_n, \phi_m\rangle_2 = \frac{1}{\sqrt{\alpha_n \alpha_m}} \langle Tv_n, v_m\rangle_{\mathcal{A}} = \sqrt{\frac{\alpha_n}{\alpha_m}} \langle v_n, v_m\rangle_{\mathcal{A}} = \delta_{m,n}.$$

By denseness of V in $L^2(M)$, the ϕ_n form an orthonormal basis of $L^2(M)$.

Finally, let $\lambda_n = \alpha_n^{-1} - 1$ for each $n \in \mathbb{N}$. Then (λ_n) satisfies equation (1.5.2), since $-\mathcal{A}$ is a positive operator, and (α_n) is a positive sequence that decreases to 0. □

Remark 1.5.3. The Hilbert space V introduced in the proof of Theorem 1.5.1 is a "Lévy analogue" of the Sobolev space $H^1(M)$ considered in Lablée [52] or Grigor'yan [32], where the completion is instead taken with respect to the Sobolev inner product

$$\langle f, g\rangle_{H^1} = \langle f, g\rangle_2 - \langle \Delta f, g\rangle_2, \qquad \forall f, g \in C^\infty(M). \tag{1.5.9}$$

In the case when $M = \mathbb{R}^d$, spaces of this type are discussed in Section 3.10 of Jacob [46], who refers to them as *anisotropic Sobolev spaces*. We will encounter generalisations of these spaces in Section 4.1.2, when we consider operators on Riemannian symmetric spaces of non-compact type.

It is well-known that the heat semigroup $(K_t, t \geq 0)$ associated with Brownian motion on a compact manifold is trace-class, and possesses an integral kernel. The final two results of this section extend this to the Lévy semigroup $(T_t, t \geq 0)$, subject to the assumption that $a > 0$.

Theorem 1.5.4. *Let X be an isotropic Lévy process on M with non-trivial Brownian part. Then the transition semigroup operator T_t is trace-class for all $t > 0$.*

Proof. We again assume $a = 2$, so that $\mathcal{A}$ has the form (1.5.3). The case for general $a > 0$ is very similar.

Let (λ_n) and (ϕ_n) be as in the statement of Theorem 1.5.1, and let (μ_n) and (ψ_n) be the analogous sequences for Δ, so that ψ_n is the n^{th} eigenvector of Δ, with associated eigenvalue $-\mu_n$.

Let $(K_t, t \geq 0)$ denote the heat semigroup associated with Brownian motion on M. This operator semigroup is known to possess many wonderful properties, including being trace-class. It follows that

$$\operatorname{tr} K_t = \sum_{n=1}^{\infty} \langle K_t \psi_n, \psi_n\rangle = \sum_{n=1}^{\infty} e^{-t\mu_n} < \infty \tag{1.5.10}$$

for all $t > 0$. As an element of $[0, \infty]$, the trace of each T_t is given by

$$\operatorname{tr} T_t = \sum_{n=1}^{\infty} \langle T_t \phi_n, \phi_n\rangle = \sum_{n=1}^{\infty} e^{-t\lambda_n}. \tag{1.5.11}$$

By the min-max principle for self-adjoint semibounded operators ([65] pp. 666), we have for all $n \in \mathbb{N}$,

$$\lambda_n = - \sup_{f_1, \ldots, f_{n-1} \in C^\infty(M)} \left[\inf_{f \in \{f_1, \ldots, f_{n-1}\}^\perp,\ \|f\|=1} \langle \mathcal{A} f, f\rangle \right],$$

and

$$\mu_n = - \sup_{f_1, \ldots, f_{n-1} \in C^\infty(M)} \left[\inf_{f \in \{f_1, \ldots, f_{n-1}\}^\perp,\ \|f\|=1} \langle \Delta f, f\rangle \right].$$

As noted in the proof of Theorem 1.5.1, for all $f \in C^\infty(M)$,

$$-\langle \mathcal{A} f, f\rangle \geq -\langle \Delta f, f\rangle \geq 0,$$

and hence $\lambda_n \geq \mu_n$ for all $n \in \mathbb{N}$. But then $e^{-t\lambda_n} \leq e^{-t\mu_n}$ for all $t > 0$ and $n \in \mathbb{N}$, and so, comparing (1.5.11) with (1.5.10),

$$\operatorname{tr} T_t < \operatorname{tr} K_t < \infty$$

for all $t > 0$. □

By Lemma 7.2.1 of Davies [23], we immediately obtain the following.

Corollary 1.5.5. *Let X be an isotropic Lévy process on M with non-trivial Brownian part. Then its semigroup $(T_t, t \geq 0)$ has a square-integrable kernel. That is, for all $t > 0$ there is a map $p_t \in L^2(M \times M)$ such that*

$$T_t f(x) = \int_M f(y) p_t(x, y) \mu(dy)$$

for all $f \in L^2(M)$ and $x \in M$. Moreover, we have the following L^2-convergent expansion:

$$p_t(x, y) = \sum_{n=1}^{\infty} e^{-\lambda_n t} \phi_n(x) \phi_n(y), \qquad \forall x, y \in M,\ t \geq 0.$$

It is natural to wonder whether similar results might hold in the pure jump case when $a = 0$ — perhaps under some further condition on ν. When M is a compact symmetric space, we can find an orthonormal basis of eigenfunctions that are common to the L^2–semigroups associated with all isotropic Lévy processes (see the results in Section 5 of [14]) . The key tool here, which enables a precise description of the spectrum of eigenvalues, is Gangolli's Lévy–Khinchine formula [29]. In the more general case considered in this chapter, such methods are not available, and we have been unable to make further progress at the present time.

Chapter 2

Analysis on Lie groups and Riemannian symmetric spaces

In the previous chapter, we considered Lévy processes on general manifolds, taking a functional analytic view and deriving properties of the generator and semigroup. However, progress was impeded, in part due to the lack of an analogue to the Lévy–Khinchine formula. To make progress, we refocus attention towards a different class of Riemannian manifold, known as Riemannian (globally) symmetric spaces.

In this chapter, we review key notions from the theory of Riemannian symmetric spaces, and establish necessary tools that will be useful in later chapters. For more information regarding these spaces, see Harish-Chandra [33], Helgason [39, 37, 36], Knapp [48] and Gangolli and Varadarajan [30].

§ 2.1 Lie groups and Lie algebras

We begin with a short section defining some notions from Lie group theory, which will soon be applied in the study of Riemannian symmetric spaces. A more detailed introduction to the subject may be found in Helgason [39] Chapter II.

A group G is called a *Lie group* if it is also a finite dimensional smooth manifold, and if $(g, h) \mapsto gh^{-1}$ is an infinitely differentiable mapping from $G \times G \to G$. In this work, all Lie groups will be real. A *Lie algebra* over a field $\mathbb{F}$, where $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, consist of a set $\mathfrak{l}$ together with an antisymmetric bilinear mapping $[\cdot, \cdot] : \mathfrak{l} \times \mathfrak{l} \to \mathfrak{l}$ for which the Jacobi identity holds, i.e.

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0 \qquad \forall X, Y, Z \in \mathfrak{l}. \tag{2.1.1}$$

The map $[\cdot, \cdot]$ is called the *Lie bracket* of $\mathfrak{l}$. All Lie algebras considered in this thesis will be over $\mathbb{R}$, unless stated otherwise.

If $[X, Y] = 0$ for all $X, Y \in \mathfrak{l}$, then $\mathfrak{l}$ is called *Abelian*. An *ideal* of $\mathfrak{l}$ is a linear subspace $\mathfrak{i} \subseteq \mathfrak{l}$ for which $[X, Y] \subseteq \mathfrak{i}$ whenever $X \in \mathfrak{i}$ and $Y \in \mathfrak{l}$. We say that $\mathfrak{l}$ is *simple* if it is non-Abelian, and its only ideals are itself and $\{0\}$. It is *semisimple* if it can be written as a finite direct sum of simple ideals.

Let G be a Lie group, and for each $g \in G$, let $l_g : G \to G$; $l_g(x) = gx$ denote left translation by g. A vector field X on G is *left invariant* if $dl_g X_h = X_{gh}$ for all $g \in G$. The set $\mathfrak{g}$ of all left invariant vector fields forms a Lie algebra, called the *Lie algebra of* G, with bracket operation given by the Lie derivative

$$[X, Y] = XY - YX, \qquad \forall X, Y \in \mathfrak{g}.$$

We identify $\mathfrak{g}$ with the tangent space at the identity $e \in G$ in the usual way.

The *Lie exponential map* is the smooth mapping $\exp : \mathfrak{g} \to G$, defined for each $X \in \mathfrak{g}$ by

$$\exp(X) \coloneqq \gamma_X(1),$$

where $\gamma_X : \mathbb{R} \to G$ is the unique solution to the initial valued problem

$$\dot{\gamma}(t) = X_{\gamma(t)}, \ \forall t \in \mathbb{R}; \qquad \gamma(0) = e.$$

The *Killing form* of $\mathfrak{g}$ is the symmetric bilinear form $B : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ given by

$$B(X, Y) \coloneqq \operatorname{tr}(\operatorname{ad} X \operatorname{ad} Y) \qquad \forall X, Y \in \mathfrak{g}.$$

Here, ad denotes the *adjoint representation of* $\mathfrak{g}$, defined by

$$\operatorname{ad}(X)Y \coloneqq [X, Y], \qquad \forall X, Y \in \mathfrak{g}.$$

A related concept is the *adjoint representation of* G, which is denoted Ad, and defined by

$$\operatorname{Ad}(g) \coloneqq (dc_g)_e \qquad \forall g \in G,$$

where for each $g \in G$, $c_g : G \to G$ is conjugation by g, given by

$$c_g(h) \coloneqq ghg^{-1} \qquad \forall h \in G.$$

One can show that $\operatorname{ad} = d \operatorname{Ad}_e$ (see Helgason [39] pp. 128).

The following property of the Lie bracket and Killing form will occasionally be useful:

$$B([X, Y], Z) = B([Y, Z], X), \qquad X, Y, Z \in \mathfrak{g}. \tag{2.1.2}$$

See for example equation (2) on page 131 of [39].

§ 2.2 Riemannian Symmetric Spaces

Let (M, g) be a Riemannian manifold. An isometry s of M is called an *involution* if $s^2 = Id$. M is called a *Riemannian (globally) symmetric space* if every point $p \in M$ is the unique fixed point of a non-trivial involution s_p of M. In fact, for such a space, each s_p is unique, and is *geodesic reversing* in the sense that it acts as minus the identity on T_pM.

A key advantage that Riemannian symmetric spaces have over more general manifolds is their Lie theoretic description. Let $I(M)$ denote the isometry group of M, considered with the compact–open topology. By Theorem II.2.5 and Lemma IV.3.2 of Helgason [39], this gives $I(M)$ the structure of a Lie transformation group, in the sense that it is a Lie group, and its action on M is smooth (c.f. [39] Ch. II §3). By Theorem IV.3.3(i) of Helgason [39], for any point $o \in M$, M is diffeomorphic to a homogeneous space G/K, where $G = I_0(M)$ is the identity component of $I(M)$, and K is the subgroup of G leaving o fixed, a compact subgroup of G. This isomorphism is given by the mapping

$$G/K \overset{\cong}{\to} M; \quad gK \mapsto g \cdot o.$$

The chosen point $o \in M$ is sometimes called the "origin" of M. Note that $o = \pi(K)$, where $\pi : G \to G/K$ denotes the natural surjection.

Let $\Theta : G \to G$; $g \mapsto s_o g s_o$. Then Θ is an involutive automorphism of G, and satisfies

$$G_0^\Theta \subseteq K \subseteq G^\Theta, \tag{2.2.1}$$

where G^Θ is the fixed point set of Θ, and G_0^Θ is the identity component of G^Θ. Alternatively, given any connected Lie group G with compact subgroup K, if there is an involutive automorphism Θ on G for which (2.2.1) is satisfied, we will call (G, K) a *Riemannian symmetric pair*. By Proposition IV.3.4 of [39], if (G, K) is a Riemannian symmetric pair, then subject to a choice of G-invariant Riemannian structure on G, G/K is a Riemannian symmetric space.

Let $\theta \coloneqq d\Theta_e$ denote the differential of Θ at e, an involution of $\mathfrak{g}$. By [39] Theorem IV.3.3(ii), the Lie algebra $\mathfrak{k}$ of K is the $+1$-eigenspace of θ,

$$\mathfrak{k} = \{X \in \mathfrak{g} : \theta(X) = X\}.$$

If $\mathfrak{p}$ denotes the -1-eigenspace of θ, then we have the direct sum

$$\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}. \tag{2.2.2}$$

One can check that

$$[\mathfrak{k}, \mathfrak{p}] \subseteq \mathfrak{p}, \quad \text{and} \quad [\mathfrak{p}, \mathfrak{p}] \subseteq \mathfrak{k}. \tag{2.2.3}$$

Indeed, if $X \in \mathfrak{p}$ and $Y \in \mathfrak{k}$, then

$$\theta([X, Y]) = [\theta(X), \theta(Y)] = [-X, Y] = -[X, Y].$$

The other assertion is proved similarly.

Consider the natural surjection $P : G \to M$; $g \mapsto g \cdot o$ associated with the G-action on M. By [39] Theorem IV.3.3(iii), its differential dP_e maps $\mathfrak{k}$ to $\{0\}$ and its restriction to $\mathfrak{p}$ yields an isomorphism $\mathfrak{p} \cong T_oM$. Moreover, for each $X \in \mathfrak{p}$, the curve $\gamma(t) = \exp(tX) \cdot o$ is the geodesic starting at o and with initial velocity $dP_e(X)$. Parallel translation along γ is given by $d[\exp(tX)]_o$.

To relate these notions to those of the previous chapter, the quotient map $\pi : G \to G/K$ gives G the structure of a principle K-bundle over G/K, so that (2.2.2) may be viewed as a symmetric space analogue of (1.1.1). Moreover, provided G acts *effectively* on $M = G/K$, in the sense that e is the only element of G fixing every point of M, then G viewed this way is a bundle reduction of OM. That is, there is an injective bundle homomorphism from $G \to OM$. Taking $G = I_0(M)$, as above, ensures that the G-action on M is effective.

We say that M is *irreducible* if the action of K on T_oM is irreducible, i.e. if T_oM has no non-trivial K-invariant subspaces. Equivalently, M is irreducible if the action of $\mathrm{Ad}(K)$ on $\mathfrak{p}$ is irreducible.

Integration on G and G/K

Every locally compact Hausdorff topological group has a nontrivial left (resp. right) invariant Radon measure that is unique up to multiplicative constants, called *left (resp. right) Haar measure*. The default measure for integration on G will be its left Haar measure, μ_G, normalised so that its restriction to K is a probability measure. This is possible, since K is compact, and Radon measures are finite on compact sets. In work to come, we will frequently find it useful to assume that G is semisimple. By [37] Proposition I.1.6, pp. 88, semisimple Lie groups are unimodular, in that every left Haar measure is also a right Haar measure. In this case, we just refer to it as Haar measure, denoted dg.

Integration on G/K will be with respect to the G-invariant measure induced by Haar measure on G,

$$\mu_{G/K} = \mu_G \circ \pi^{-1}. \tag{2.2.4}$$

G-invariant measures on G/K are unique up to a constant factor (c.f. [54] Proposition 1.10, pp. 12) and hence $\mu_{G/K}$ is proportional to the Riemannian measure of G/K — indeed, that Riemannian measure on G/K is G-invariant follows from the fact that Riemannian measure is invariant under all isometries (c.f. [37] Proposition 1.3, pp. 84).

For notational convenience, we will usually denote μ_G by $d\sigma$ (for a variable $\sigma \in G$), and $\mu_{G/K}$ by dx (for a variable $x \in G/K$).

§ 2.3 Symmetric spaces of noncompact type

We say that a symmetric space $M = G/K$ is of

1. *Euclidean type* if $\mathfrak{p}$ is an Abelian ideal of $\mathfrak{g}$,

2. *Compact type* if B is negative definite,

3. *Noncompact type* if B is negative definite on $\mathfrak{k}$ and positive definite on $\mathfrak{p}$.

Example 2.3.1. 1. $\mathbb{R}^d$ is a symmetric space of Euclidean type, for all $d \in \mathbb{N}$. In fact, every symmetric space of Euclidean type has sectional curvature zero ([39] Theorem V.3.1(iii)), and hence is isomorphic to $\mathbb{R}^d$.

2. The d-sphere S^d is a symmetric space of compact type, for all $d \in \mathbb{N}$. The identity component of its isotropy group is $I_0(S^d) = SO(d+1)$, and for any point $x \in S^d$, the subgroup of $SO(d+1)$ that fixes x is isomorphic to $SO(d)$. Hence $S^d \cong SO(d+1)/SO(d)$.

3. Hyperbolic space $\mathbb{H}^d$ is a symmetric space of noncompact type, for all $d \in \mathbb{N}$. Consider the hyperboloid model

$$\mathbb{H}^d = \left\{ x \in \mathbb{R}^{d+1} : x_1^2 - x_2^2 - \ldots - x_{d+1}^2 = 1, \text{ and } x_1 > 0 \right\},$$

which is a Riemannian manifold, with Riemannian metric given in geodesic polar coordinates by

$$dr^2 + (\sinh^2 r) d\alpha^2,$$

where $r > 0$, $\alpha \in S^d$, and $x = (\cosh r, (\sinh r)\alpha)$.

Let

$$J := \begin{pmatrix} 1 & 0 \\ 0 & -I_d \end{pmatrix},$$

where I_d denotes the $d \times d$ identity matrix. The *indefinite orthogonal group*, $O(d,1)$, consists of all $(d+1) \times (d+1)$ real matrices A that satisfy $A^T J A = J$. This group acts on $\mathbb{R}^{d+1}$ in the usual way, and the subgroup preserving the first coordinate of $x \in \mathbb{R}^{d+1}$ is called the *orthochronous Lorentz group*, denoted as $O^+(d,1)$. This group preserves $\mathbb{H}^d$, and in fact, $I(\mathbb{H}^d) = O^+(d,1)$. The connected component containing the identity is $I_0(\mathbb{H}^d) = SO^+(1,d)$, the subgroup of $O^+(1,d)$ consisting of all elements with determinant 1. The subgroup of $SO^+(1,d)$ that fixes the "origin" $o := (1, 0, \ldots, 0)$ is isomorphic to $SO(d)$, and hence $\mathbb{H}^d \cong SO^+(d,1)/SO(d)$.

Due to classification results (see [39] Chapter X), the study of probabilistic objects taking values on G/K can be simplified by assuming that G/K is one of these three types, without any reduction in generality. The Euclidean type is perhaps the least interesting, since they are all isomorphic to Euclidean space. Of the other two cases, the compact case has received more attention than the noncompact, from a probabilistic standpoint — see Trang Le Ngan's thesis [57], as well as papers by Applebaum and Le Ngan [14, 16], for more information. In work to come, we will focus on symmetric spaces of noncompact type.

Note that if G/K is of noncompact type, then G is simply connected, semisimple and noncompact. Moreover, B is nondegenerate, and so θ is a Cartan involution of $\mathfrak{g}$, in the sense that the bilinear form

$$B_\theta(X,Y) = -B(X, \theta Y) \qquad \forall X, Y \in \mathfrak{g}$$

is strictly positive definite. The associated Cartan decomposition is (2.2.2).

Fix an $\mathrm{Ad}(K)$-invariant inner product $\langle\cdot,\cdot\rangle$ on $\mathfrak{g}$, chosen so that restriction to $\mathfrak{p}$ induces the Riemannian structure of G/K. The space $\mathfrak{p}$ is the orthogonal complement of $\mathfrak{k}$, with respect to this inner product.

2.3.1 The Iwasawa decomposition of (G, K)

Suppose now that (G, K) is a Riemannian symmetric pair of noncompact type. We discuss an improvement to the decomposition (2.2.2), known as the Iwasawa decomposition, that will be of particular use in this setting. This decomposition expresses $\mathfrak{g}$ as a orthogonal direct sum of three subspaces, one nilpotent, one Abelian, and one compact. The compact Lie algebra $\mathfrak{k}$ we have already met. We now outline the construction of the nilpotent and Abelian subspaces, defined using the (restricted) root spaces of $\mathfrak{g}$.

Let $\mathfrak{a}$ be a maximal Abelian subspace of $\mathfrak{p}$. One can show that all such subspaces are isomorphic. By Lemma 1.2 on page 253 of Helgason [39], for each $X \in \mathfrak{p}$, $\mathrm{ad}\, X$ is symmetric, in the sense that

$$B_\theta(\mathrm{ad}(X)Y, Z) = B_\theta(Y, \mathrm{ad}(X)Z) \qquad \forall Y, Z \in \mathfrak{g}.$$

Therefore, the commutative family $\{\mathrm{ad}\, H : H \in \mathfrak{a}\}$ is simultaneously diagonalizable, by a choice of basis of $\mathfrak{a}$. In particular, these operators all have the same eigenvectors. For each $\lambda \in \mathfrak{a}^*$ define

$$\mathfrak{g}_\lambda = \{X \in \mathfrak{g} : \mathrm{ad}(H)X = \lambda(X)X \qquad \forall H \in \mathfrak{a}\}.$$

If $\lambda \neq 0$ and $\mathfrak{g}_\lambda \neq \{0\}$, then λ is called a *root* of $\mathfrak{g}$ with respect to $\mathfrak{a}$, and $\mathfrak{g}_\lambda$ is the corresponding *root space*. Let Σ denote the set of all roots.

Remark 2.3.2. In the literature, some authors (for example Helgason [39] Chapter VI §3, pp. 257–264) first consider root space decompositions of $\mathfrak{g}_{\mathbb{C}}$, and as a result, the associated roots belong to $\mathfrak{a}_{\mathbb{C}}^*$ rather than $\mathfrak{a}^*$. Restricting the domain of these roots to $\mathfrak{a}$ then gives the roots as defined above. Elements of Σ are then referred to as *restricted* roots, to distinguish them from roots defined on the whole $\mathfrak{a}_{\mathbb{C}}$. Since the complexification $\mathfrak{g}_{\mathbb{C}}$ will not feature explicitly in this work, we do not need to make this distinction.

Observe that for all $\lambda, \mu \in \mathfrak{a}^*$,

$$\theta(\mathfrak{g}_\lambda) = \mathfrak{g}_{-\lambda}, \quad \text{and} \quad [\mathfrak{g}_\lambda, \mathfrak{g}_\mu] \subset \mathfrak{g}_{\lambda+\mu}. \tag{2.3.1}$$

Indeed, the first statement follows by noting that for all $H \in \mathfrak{a}$ and $X \in \mathfrak{g}_\lambda$,

$$\theta(\mathrm{ad}(H)X) = \mathrm{ad}(\theta(H))\theta(X) = -\mathrm{ad}(H)\theta(X),$$

and the second statement follows from the Jacobi identity (2.1.1). Subject to an ordering on $\mathfrak{a}^*$, one may then identify a set $\Sigma^+ \subset \Sigma$ of *positive roots* for $\mathfrak{g}$. Let $\Sigma^- = \Sigma \setminus \Sigma^+$, the complementary set of *negative roots*.

The simultaneous diagonalisation discussed above then gives rise to the direct sum

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\lambda \in \Sigma} \mathfrak{g}_\lambda.$$

It is usedful to decompose $\mathfrak{g}$ further. By (2.2.2)

$$\mathfrak{g}_0 = (\mathfrak{k} \cap \mathfrak{g}_0) \oplus (\mathfrak{p} \cap \mathfrak{g}_0) = (\mathfrak{k} \cap \mathfrak{g}_0) \oplus \mathfrak{a}, \tag{2.3.2}$$

the second equality being due to the fact that $\mathfrak{a}$ is a maximal Abelian subalgebra of $\mathfrak{p}$.

A Lie algebra $\mathfrak{n}$ is called *nilpotent* if there exists $n \in \mathbb{N}$ such that $\mathrm{ad}_{X_1} \circ \ldots \circ \mathrm{ad}_{X_n} = 0$ for all $X_1, \ldots X_n \in \mathfrak{n}$. By Engel's theorem ([39] Theorem III.2.4, pp. 160), $\mathfrak{n}$ is nilpotent if and only if there is $n \in \mathbb{N}$ such that $(\mathrm{ad}\, X)^n = 0$ for all $X \in \mathfrak{n}$. By (2.3.1), the direct sum of positive root spaces

$$\mathfrak{n} = \bigoplus_{\lambda \in \Sigma^+} \mathfrak{g}_\lambda$$

forms a nilpotent Lie subalgebra of $\mathfrak{g}$.

If $X \in \sum_{\lambda \in \Sigma^-} \mathfrak{g}_\lambda$, then $\theta(X) \in \mathfrak{n}$ (by (2.3.1)), and $\theta(X) - X \in \mathfrak{k}$, since it is a -1-eigenvector of θ. Therefore

$$X = \theta(X) + (X - \theta(X)) \in \mathfrak{n} \oplus \mathfrak{k},$$

and combining (2.2.2) and (2.3.2), we have

$$\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{a} \oplus \mathfrak{k}. \tag{2.3.3}$$

A decomposition of Lie algebras of this kind is known as an Iwasawa decomposition.

On the level of Lie groups, we have the following global form. For a proof, see page 271 of Helgason [39]. Knapp [48] Chapter V, §2, (pp. 116–121) is another excellent reference for the ideas we have been discussing.

Theorem 2.3.3 (Iwasawa decomposition)**.** *Let (G, K) be a Riemannian symmetric pair of noncompact type, and let $\mathfrak{a}$ and $\mathfrak{n}$ be as above. Let A and N be connected Lie subgroups of G with Lie algebras $\mathfrak{a}$ and $\mathfrak{n}$, respectively. Then the mapping $(n, a, k) \mapsto nak$ is a diffeomorphism from $N \times A \times K$ onto G. The groups A and N are simply connected.*

Remark 2.3.4. It is common to write the decomposition of Theorem 2.3.3 as $G = NAK$. There is an equally valid statement in which G is instead decomposed as $G = KAN$.

For each $\sigma \in G$, let $A(\sigma)$ denote the unique element of $\mathfrak{a}$ such that

$$\sigma = n e^{A(\sigma)} k \tag{2.3.4}$$

for some $k \in K$ and $n \in N$. We will need this notation later, for example when discussing spherical functions.

2.3.2 The Cartan decomposition of (G, K)

The second decomposition of G that we would like to use is the Cartan decomposition[1] of G. In order to state this result, we introduce the Weyl group and associated Weyl chambers. For more details concerning these definitions, see [48] Chapter 5 §3, pp. 121–126.

We continue to consider a Riemannian symmetric pair of noncompact type, (G, K), with a fixed a choice of Iwasawa decomposition (2.3.3). The group K acts on $\mathfrak{a}$ via Ad; let the normaliser and centraliser of for this action be denoted

$$N_K(\mathfrak{a}) = \{k \in K : \mathrm{Ad}(k)\,\mathfrak{a} \subseteq \mathfrak{a}\} \quad \text{and} \quad Z_K(\mathfrak{a}) = \{k \in K : \mathrm{Ad}(k)H = H \quad \forall H \in \mathfrak{a}\}.$$

The *Weyl group* is then defined by

$$W \coloneqq N_K(\mathfrak{a})/Z_K(\mathfrak{a}). \tag{2.3.5}$$

W acts on $\mathfrak{a}$ via Ad, and on $\mathfrak{a}^*$ by duality.

For each root $\lambda \in \Sigma$, one can consider the hyperplane P_λ in $\mathfrak{a}$ of all $H \in \mathfrak{a}$ for which $\lambda(H) = 0$. The connected components of $\mathfrak{a} \setminus \left(\bigcup_{\lambda\in\Sigma} P_\lambda\right)$ are called the *Weyl chambers* of $\mathfrak{a}$. There is one Weyl chamber of particular importance, namely the *positive Weyl chamber*

$$\mathfrak{a}^+ = \{H \in \mathfrak{a} : \lambda(H) > 0,\ \forall\lambda \in \Sigma^+\}, \tag{2.3.6}$$

uniquely determined by the choice Σ^+ of positive roots.

If $\lambda \in \Sigma$ and $c \in \mathbb{R}$, then $c\lambda \in \Sigma$ implies $c \in \{\pm\frac{1}{2}, \pm1, \pm2\}$ (see Corollary 2.17 of Helgason [39] pp. 291). A root λ is called *indivisible* (or *short* — see Gangolli and Varadarajan [30] pp. 62) if neither of $\pm\frac{1}{2}\lambda$ are roots. Let Σ_0 denote the set of all indivisible roots, and let $\Sigma_0^+ = \Sigma_0 \cap \Sigma^+$. Then Σ_0^+ indexes the set of hyperplanes P_λ.

There is another characterisations of the Weyl group that will be of use. For each $\lambda \in \Sigma$, define $s_\lambda : \mathfrak{a}_{\mathbb{C}}^* \to \mathfrak{a}_{\mathbb{C}}^*$ by

$$s_\lambda(\mu) = \mu - \frac{2\langle\mu, \lambda\rangle}{\langle\lambda, \lambda\rangle} \qquad \forall\mu \in \mathfrak{a}_{\mathbb{C}}^*. \tag{2.3.7}$$

Then each s_λ is a reflection in the hyperplane P_λ, and together the s_λ generate a finite group of transformations of $\mathfrak{a}^*$, that acts on $\mathfrak{a}$ by permuting Weyl chambers. One can show (see Knapp [48], pp. 125) that this group is isomorphic to W as defined above, and is sometimes used as the definition of the Weyl group instead of (2.3.5).

By Theorem 2.12 on page 288 of Helgason [39], elements of W permute Weyl chambers, and W is simply transitive on the set of all Weyl chambers in $\mathfrak{a}$. In particular, $|W|$ is equal to the number of Weyl chambers of G/K For more details on how the Weyl group is constructed, see Helgason [39] pages 284 and 456.

Theorem 2.3.5 (Cartan decomposition)**.** *Let G, K and A be as above. Then every $\sigma \in G$ has a decomposition $\sigma = k_1 a k_2$, where $k_1, k_2 \in K$ and $a \in A$. Moreover, this decomposition is uniquely determined up to conjugation of a by elements of W.*

[1] This is not to be confused with (2.2.2), the Cartan decomposition of $\mathfrak{g}$.

For a proof, see Knapp [48] pp. 127. Because the A component is only determined up to W-conjugation, Theorem 2.3.5 is often instead expressed as

$$G = K \cdot \overline{\exp(\mathfrak{a}^+)} \cdot K \tag{2.3.8}$$

(see Theorem 6.7 on page 249 of Helgason [39]). This decomposition will be especially useful in calculations involving K-bi-invariant functions, which by (2.3.8) are determined entirely by their action on $\overline{\exp(\mathfrak{a}^+)}$.

The following notation will be useful. For each $\sigma \in G$, let $H(\sigma)$ denote the unique element of $\overline{\mathfrak{a}^+}$ such that

$$\sigma = ke^{H(\sigma)}k' \tag{2.3.9}$$

for some $k, k' \in K$.

2.3.3 Associated integral formulae

We finish this section with a theorem that captures how the Iwasawa and Cartan decompositions of G affect Haar measure. Given a root $\lambda \in \Sigma$, the *multiplicity* of λ is defined

$$m_\lambda := \dim g_\lambda.$$

We also introduce the half-sum of positive roots

$$\rho := \frac{1}{2} \sum_{\lambda \in \Sigma^+} m_\lambda \lambda, \tag{2.3.10}$$

an element of $\mathfrak{a}^*$ that will appear in many formulae to come.

Theorem 2.3.6. *1. For all $f \in C_c(G)$,*

$$\begin{aligned} \int_G f(\sigma) d\sigma &= \int_N \int_A \int_K f(nak) e^{-2\rho(\log a)} dk da dn \\ &= \int_K \int_A \int_N f(kan) e^{2\rho(\log a)} dn da dk, \end{aligned} \tag{2.3.11}$$

Here, $\log a$ denotes the unique element $H \in \mathfrak{a}$ for which $a = e^H$, for all $a \in A$.

2. For all $f \in C_c(G)$,

$$\int_G f(\sigma) d\sigma = \int_K \int_{\mathfrak{a}^+} \int_K f(k_1 e^H k_2) \delta(H) dk_2 dH dk_2, \tag{2.3.12}$$

where

$$\delta(H) = \prod_{\lambda \in \Sigma^+} \big(\sinh \lambda(H)\big)^{m_\lambda},$$

for all $H \in \mathfrak{a}^+$.

For a proof of both of these formulae, see Helgason [37] Chapter I, §5 (pp. 181, 186).

§ 2.4 Harmonic analysis on symmetric spaces of noncompact type

Riemannian symmetric spaces come equipped with an important integral transform, called the spherical transform. The spherical transform generalises the classical Fourier transform to symmetric spaces, though with the slightly more limited scope that functions must be K-invariant. We present some background in this theory next. In later chapters, we will apply it to the study of Lévy-type operators on symmetric spaces.

In this section only, we take $\mathbb{F} = \mathbb{C}$, so that functions may be complex-valued.

2.4.1 Functions, measures and convolution semigroups

For each $g \in G$, let $l_g, r_g : G \to G$ denote left and right multiplication by g, respectively. Let L_g and R_g denote the corresponding action on functions, so that for each $f \in \mathcal{F}(G)$,

$$L_g f(x) = f \circ l_g, \qquad \text{and} \qquad R_g f = f \circ r_g.$$

A function $f \in \mathcal{F}(G)$ is said to be *K-right-invariant* if $R_k f = f$ for all $k \in K$ and *K-left-invariant* if $L_k f = f$ for all $k \in K$. If f is K-right-invariant and K-left-invariant, we say it is K-bi-invariant. There is a one to one correspondence between the space of K-right-invariant functions on G and the space of all functions on G/K. We identify $\mathcal{F}(G/K)$ with the set of all K-right-invariant functions on G.

The group G acts on G/K by left multiplication. A function $f \in \mathcal{F}(G/K)$ is said to be K-invariant if $f(k \cdot x) = f(x)$ for all $k \in K$ and $x \in G/K$. The set $\mathcal{F}(K|G|K)$ of all real-valued functions on the double coset space

$$K|G|K = \{KgK : g \in G\},$$

can (and will) be identified with the set of all K-invariant elements of $\mathcal{F}(G/K)$. As such, elements of $\mathcal{F}(K|G|K)$ can be viewed as K-bi-invariant functions on G.

These conventions will be carried through to other important spaces of functions and measures, where applicable. For example, the space $C(G/K)$ of continuous functions on G/K will more often be viewed as the space of K-right-invariant functions on G.

We will also use some L^p spaces, where $1 \le p \le \infty$. Unless stated otherwise, these will be taken with respect to Haar measure on G and its projection onto G/K (see (2.2.4)).

The convolution product on G and G/K

Given $f, g, \in L^1(G)$, the *convolution* of f_1 and f_2 is denoted $f_1 * f_2$, and defined by

$$(f_1 * f_2)(\sigma) = \int_G f_1(\tau) f_2(\tau^{-1}\sigma) d\tau, \qquad \forall \sigma \in G. \tag{2.4.1}$$

The convolution of two functions $f_1, f_2 \in L^1(G/K)$ is defined by

$$(f_1 * f_2)(x) = \int_G f_1(\tau K) f_2(\tau^{-1}x) d\tau, \tag{2.4.2}$$

for all $x \in G/K$. Substituting $x = \sigma K$ into (2.4.2) and comparing with (2.4.1), it is clear that

$$(f_1 * f_2) \circ \pi = (f_1 \circ \pi) * (f_2 \circ \pi), \tag{2.4.3}$$

for all $f_1, f_2 \in L^1(G/K)$.

Equipped with convolution defined this way, one can prove using a standard argument that $L^1(G)$ and $L^1(G/K)$ are Banach algebras.

For the purposes of the next proposition only, let $L^1_{right}(G), L^1_{bi}(G)$ denote the closed subspaces of $L^1(G)$ consisting, respectively, of K-right- and K-bi-invariant elements, and let $L^1_K(G/K)$ denote the closed subspace of $L^1(G/K)$ consisting of K-invariant elements.

A measurable map $S : G/K \to G$ with the property that $\pi \circ S$ is the identity map on G/K is called a *section map.* Such a map always exists, and may even be chosen to be smooth in a chosen neighbourhood of G/K, if desired. For a proof of this, see [39] Lemma 4.1, pp. 123, combined with [54] Proposition A.2, pp. 352.

Proposition 2.4.1. *The Banach algebras $L^1(G/K)$ and $L^1_{right}(G)$ are isomorphic via the mapping*

$$\pi^* : f \mapsto f \circ \pi.$$

Moreover, restriction of π^* to $L^1_{bi}(G)$ gives an isomorphism of the Banach algebras $L^1_K(G/K)$ and $L^1_{bi}(G)$.

Proof. Clearly π^* is a linear isometry, and by (2.4.3), it is also multiplicative. We prove that π^* is bijective by showing that it has a well-defined inverse. Suppose $\tilde{f} \in L^1_{right}(G)$. Then for any section map S, we may take $f \in L^1(G/K)$ to be given by $f = \tilde{f} \circ S$, and obtain $\tilde{f} = \pi^* f$. Since $\tilde{f}$ is right-K-invariant, this choice of f is independent of our choice of section map S: indeed, if S' were some other section map, then for each $x \in G/K$ there would be some $k_x \in K$ such that $S'(x) = S(x)k_x$. But then $\tilde{f}(S'(x)) = \tilde{f}(S(x)k_x) = f(S(x))$ for almost all $x \in G/K$, and so $\tilde{f} \circ S = \tilde{f} \circ S'$. Therefore π^* has a well-defined inverse. □

Proposition 2.4.1 shows we may adopt notational conventions consistent with those described at the beginning of the section, and identify Banach algebras $L^1(G/K)$, $L^1_{right}(G)$, using $L^1(G/K)$ to denote both. Similarly, the space $L^1(K|G|K)$ will denote both $L^1_{bi}(G)$ and $L^1_K(G/K)$.

Since (G, K) is a Gelfand pair, the space $L^1(K|G|K)$ is a commutative Banach algebra — see for example Wolf [69] Chapter 8 for more details.

Recall that for a locally compact Hausdorff space X, $\mathcal{M}(X)$ denotes the set of all Borel measures on X. A measure $\mu \in \mathcal{M}(G)$ is said to be K-right-invariant if $\mu(Ak) = \mu(A)$ for all

$A \in \mathcal{B}(G)$ and all $k \in K$, and K-left-invariant if $\mu(kA) = \mu(A)$ for all $A \in \mathcal{B}(G)$ and $k \in K$. It is called K-bi-invariant if it is both K-right-invariant and K-left-invariant.

Given two measures $\mu_1, \mu_2 \in \mathcal{M}(G)$, we define their *convolution product* to be the measure $\mu_1 * \mu_2 \in \mathcal{M}(G)$ given by

$$(\mu_1 * \mu_2)(B) = \int_G \int_G \mathbf{1}_B(\sigma\tau)\mu_1(d\sigma)\mu_2(d\tau), \tag{2.4.4}$$

for all $B \in \mathcal{B}(G)$. Note that since G is unimodular, this operation is commutative. It is also clear from the definition that $\mu_1 * \mu_2$ is K-bi-invariant whenever μ_1 and μ_2 are.

One may also consider convolution on $\mathcal{M}(G/K)$. Let $S : G/K \to G$ be a section map. The convolution of two measures $\mu_1, \mu_2 \in \mathcal{M}(G/K)$ is

$$(\mu_1 * \mu_2)(B) = \int_{G/K} \int_{G/K} \int_K \mathbf{1}_B(S(x)ky)dk\mu_1(dx)\mu_2(dy) \tag{2.4.5}$$

for all $B \in \mathcal{B}(G/K)$. Note that this definition is independent of our choice of section map: if S' is another section map, then for each $x \in G/K$ there is $k_x \in K$ such that $S'(x) = S(x)k_x$. Replacing $S(x)$ with $S'(x)$ in (2.4.5) does not change the value of the right-hand side, as Haar measure dk on K is invariant under the change of variable $k \mapsto k_x^{-1}k$.

If μ_1 and μ_2 have densities f_1 and f_2 with respect to Riemannian measure on G/K, then by (2.2.4),

$$\begin{aligned}
(\mu_1 * \mu_2)(B) &= \int_{G/K} \int_{G/K} \mathbf{1}_B(S(x)ky)f_1(x)f_2(y)dkdxdy \\
&= \int_G \int_G \int_K \mathbf{1}_B(\tau k\sigma K)f_1(\tau K)f_2(\sigma K)dkd\tau d\sigma \\
&= \int_K \int_G \int_G \mathbf{1}_B(\tau k\sigma K)f_1(\tau K)f_2(\sigma K)d\tau d\sigma dk,
\end{aligned}$$

for all $B \in \mathcal{B}(G/K)$, where on the last line we have used Fubini's theorem to change the order of integration. Applying the change of variable $\tau \mapsto \tau k^{-1}$, it follows that

$$\begin{aligned}
(\mu_1 * \mu_2)(B) &= \int_G \int_G \mathbf{1}_B(\tau\sigma K)f_1(\tau K)f_2(\sigma K)d\tau d\sigma \\
&= \int_G \int_G \mathbf{1}_B(\sigma K)f_1(\tau K)f_2(\tau^{-1}\sigma K)d\tau d\sigma = \int_B (f_1 * f_2)(x)dx
\end{aligned}$$

for all $B \in \mathcal{B}(G/K)$. That is $\mu_1 * \mu_2$ has density $f_1 * f_2$ with respect to Riemannian measure on G/K.

The proof of the following proposition is very similar to that of Proposition 2.4.1, and in particular (2.4.6) can be verified in a similar manner to (2.4.3). See Proposition 1.9 of Liao [54], pp. 11 for details.

Proposition 2.4.2. *The map $\mu \mapsto \mu \circ \pi^{-1}$ defines a bijection from the set of K-right-invariant Borel measures on G onto $\mathcal{M}(G/K)$, and from the set of K-bi-invariant Borel measures on G onto the set of K-invariant Borel measures on G/K. It preserves convolution in the sense that*

$$(\mu_1 * \mu_2) \circ \pi^{-1} = \left(\mu_1 \circ \pi^{-1}\right) * \left(\mu_2 \circ \pi^{-1}\right), \tag{2.4.6}$$

for all K-right-invariant measures $\mu_1, \mu_2 \in \mathcal{M}(G)$.

In light of Proposition 2.4.2, we identify $\mathcal{M}(G/K)$ with the subspace of $\mathcal{M}(G)$ consisting of K-right-invariant measures. Similarly, we identify the subspace of $\mathcal{M}(G)$ consisting of all K-bi-invariant Borel measures on G with the subspace of $\mathcal{M}(G/K)$ consisting of K-invariant Borel measures on G/K, and denote both by $\mathcal{M}(K|G|K)$.

Convolution semigroups

A sequence (μ_n) in $\mathcal{M}(G)$ is said to converge *weakly* to $\mu \in \mathcal{M}(G)$ if for all $f \in C_b(G)$,

$$\lim_{n\to\infty} \int_G f d\mu_n = \int_G f d\mu. \tag{2.4.7}$$

If instead (2.4.7) holds for all $f \in C_0(G)$, we say $\mu_n \to \mu$ *vaguely*.

Definition 2.4.3. A family $(\mu_t, t \geq 0)$ of finite measures in $\mathcal{M}(G)$ will be called a *convolution semigroup (of probability measures)* if

1. $\mu_t(G) = 1$ for all $t \geq 0$,
2. $\mu_{s+t} = \mu_s * \mu_t$ for all $s, t \geq 0$, and
3. $\mu_t \to \mu_0$ vaguely as $t \to 0$.

Here, $*$ denotes convolution, as defined in (2.4.4). It is well-known that Definition 2.4.3 (2) and (3) together imply that $\mu_s \to \mu_t$ vaguely as $s \to t$, for any $t \geq 0$.

Some authors prefer to use weak convergence when defining convolution semigroups of probability measures. In fact, for probability measures in $\mathcal{M}(G)$, weak convergence is equivalent to vague convergence — see Heyer [40] Theorem 1.1.9, pp. 25, for a proof of this. Since vague convergence is defined with reference to $C_0(G)$, we find it to be the natural choice for working with Feller semigroups.

Note that by Definition 2.4.3 (2), μ_0 must be an idempotent measure, in the sense that $\mu_0 * \mu_0 = \mu_0$. By [40] Theorem 1.2.10, pp. 34, μ_0 must coincide with Haar measure on a compact subgroup of G. In the literature, it is very common to take $\mu_0 = \delta_e$, the delta mass at the identity. This is advantageous in the Lie group setting, since the corresponding family of convolution operators will form a strongly continuous operator semigroup on $C_0(G)$ (see Proposition 2.4.5). For similar reasons, we will see that in the symmetric space setting, a natural choice for μ_0 is normalised Haar measure on K, so that the the image of μ_0 after projecting onto G/K is δ_o, the delta mass at o. The corresponding family of convolution operators will form a strongly continuous operator semigroup on $C_0(G/K)$.

A family $(\mu_t, t \geq 0)$ of probability measures on G/K will be called a *convolution semigroup (of probability measures)* if it satisfies properties (1), (2) and (3) of Definition 2.4.3, but with convolution instead given by (2.4.5). By Proposition 2.4.2, convolution semigroups on G/K correspond precisely with the K-right-invariant convolution semigroups on G. Interestingly, a convolution semigroup on G is K-right-invariant if and only if it is K-left-invariant — see [54] Proposition 1.12, pp. 13 for more details as well as a proof of this. Therefore convolution semigroups on G/K correspond precisely with the K-bi-invariant convolution semigroups on G.

Given a probability measure $\mu \in \mathcal{M}(G)$, the associated *(left) convolution operator* $T_\mu : B_b(G) \to B_b(G)$ is defined by

$$T_\mu f(\sigma) \coloneqq (f * \mu)(\sigma) = \int_G f(\sigma\tau)\mu(d\tau), \qquad \forall f \in B_b(G),\ \sigma \in G. \tag{2.4.8}$$

By [8] Propositions 4.7.1, T_μ is a contraction, maps $C_0(G)$ into itself, is conservative (i.e. $T_\mu 1 = 1$), and positive (i.e. $f \geq 0$ implies $T_\mu f \geq 0$).

Definition 2.4.4. The family of convolution operators arising from a convolution semigroup are called *Hunt semigroups* in honour of Gilbert Hunt and his groundbreaking paper [43].

We state some of the above results more formally, for this specific class of convolution operator.

Proposition 2.4.5. *Let $(\mu_t, t \geq 0)$ be a convolution semigroup of probability measures on G, and let $(T_t, t \geq 0)$ be its Hunt semigroup. For all $t \geq 0$,*

1. $T_t 1 = 1$,
2. $f \geq 0 \implies T_t f \geq 0$, *for all* $f \in C_0(G)$,
3. $L_g T_t = T_t L_g$, *for all* $g \in G$.

Moreover, if $\mu_0 = \delta_e$, then $T_0 = I$, and the restriction of $(T_t, t \geq 0)$ to $C_0(G)$ defines a strongly continuous semigroup of contractions on $C_0(G)$.

Proof. Proposition 2.4.5 (3) is an easy check. The rest of the proposition is proved in exactly the same manner as [8] Propositions 4.7.1 and 5.3.1 pp. 107–8, 124. □

Remark 2.4.6. In Section 3.1.1, we explore the relationship between convolution semigroups, Hunt semigroups and Lévy processes on G and G/K.

2.4.2 Spherical functions and the spherical transform

Before defining spherical functions, we introduce some important spaces of invariant differential operators.

We define a *differential operator* on a smooth manifold M to be a linear mapping $D : C_c^\infty(M) \to C^\infty(M)$ that decreases supports, in the sense that

$$\mathrm{Supp}(Df) \subseteq \mathrm{Supp}(f) \qquad \forall f \in C_c^\infty(M).$$

For more information, see for example [39] pp. 239.

Let G again denote a Lie group. A differential operator D on G is said to be *left invariant* if $DL_g = L_g D$ for all $g \in G$, and *K-right-invariant* if $DR_k = R_k D$ for all $k \in K$. Let $\mathbf{D}(G)$ denote the space of all left-invariant differential operators on G, let $\mathbf{D}(G/K)$ denote the space of all G-invariant differential operators on G/K, and let $\mathbf{D}_K(G)$ denote the subspace of $\mathbf{D}(G)$ consisting of those operators that are also K-right-invariant.

The following result is essentially Theorem 4.6, from Helgason [37] Ch. II, pp. 285.

Theorem 2.4.7. *The mapping $\psi : \mathbf{D}_K(G) \to \mathbf{D}(G/K)$, where*

$$\psi(L)f \coloneqq L(f \circ \pi) \qquad \forall f \in C^\infty(G/K)$$

is a surjective homomorphism with kernel $\mathbf{D}_K(G) \cap \mathbf{D}(G)\mathfrak{k}$. As such, ψ induces an isomorphism

$$\mathbf{D}(G/K) \cong \mathbf{D}_K(G)/\mathbf{D}_K(G) \cap \mathbf{D}(G)\mathfrak{k}.$$

In light of our convention to identify functions on G/K with K-right-invariant functions on G, when restricted to $C^\infty(G/K)$, $\mathbf{D}_K(G)$ becomes indistinguishable from $\mathbf{D}(G/K)$. We will often choose to view elements of $\mathbf{D}(G/K)$ as elements of $\mathbf{D}_K(G)$ that have been restricted to $C^\infty(G/K)$.

We are now ready to state the definitions of the spherical functions on G/K and spherical functions on G.

Definition 2.4.8. A mapping $\phi : G \to \mathbb{C}$ is called *spherical* if it is K-bi-invariant, satisfies $\tilde{\phi}(e) = 1$, and if it is a simultaneous eigenfunction of every element of $\mathbf{D}_K(G)$.

A mapping $\tilde{\phi} : G/K \to \mathbb{C}$ is called spherical if it is K-invariant, if $\tilde{\phi}(o) = 1$, and if it is a simultaneous eigenfunction of every element of $\mathbf{D}(G/K)$.

By Theorem 2.4.7, as well as other previous considerations, one can check that $\phi : G/K \to \mathbb{C}$ is spherical if and only if $\phi \circ \pi$ is spherical on G.

There is a beautiful integral formula characterisation for spherical functions, which will prove useful in many calculations to come. A proof can be found in Helgason [37] pp. 400–402.

Proposition 2.4.9. *A function $\phi : G \to \mathbb{C}$ is spherical if and only if*

$$\phi(\sigma)\phi(\tau) = \int_K \phi(\sigma k \tau) dk \tag{2.4.9}$$

for all $\sigma, \tau \in G$.

Since we are assuming (G, K) is of noncompact type, we have the Iwasawa decomposition at our desposal. The following result is proved on page 418 of Helgason [37], and gives a famous formula relating spherical functions on G to a chosen Iwasawa decomposition.

Theorem 2.4.10 (Harish-Chandra Integral Formula)**.** *Let $G = NAK$ be an Iwasawa decomposition of G, and for each $\sigma \in G$ let $A(\sigma) \in \mathfrak{a}$ be as in (2.3.4). For $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$, define $\phi_\lambda : G \to \mathbb{C}$ by*

$$\phi_\lambda(\sigma) = \int_K e^{(i\lambda+\rho)(A(k\sigma))} dk, \qquad \forall \sigma \in G. \tag{2.4.10}$$

Then ϕ_λ is a spherical function, and all spherical functions on G arise this way. Moreover, given $\lambda, \nu \in \mathfrak{a}^*$, we have $\phi_\lambda = \phi_\nu$ if and only if $\lambda = s \cdot \nu$, for some $s \in W$.

By Corollary 11.5.11 on page 254 of Wolf [69], ϕ_λ is positive definite whenever $\lambda \in \mathfrak{a}^*$. By definition, this means that for all $n \in \mathbb{N}$, $x_1, \ldots, x_n \in G$ and $z_1, \ldots, z_n \in \mathbb{C}$, we have

$$\sum_{1 \le j,k \le n} \phi_\lambda(x_j^{-1} x_k) \overline{z_j} z_k \ge 0. \tag{2.4.11}$$

Proposition 2.4.11. *For all $\lambda \in \mathfrak{a}^*$ and $\sigma \in G$,*

1. $\overline{\phi_\lambda(\sigma)} = \phi_{-\lambda}(\sigma) = \phi_\lambda(\sigma^{-1})$
2. $|\phi_\lambda(\sigma)| \le 1$
3. $1 - \mathrm{Re}(\phi_\lambda(\sigma)) \ge 0$

Proof.

1. The first equality is immediate from (2.4.10). The second equality follows by noting that in (2.4.11), the $n \times n$ matrix with $(j,k)^{\text{th}}$ entry $\phi_\lambda(x_j^{-1} x_k)$ is Hermitian.
2. Substituting $n = 2$, $x_1 = e$, $x_2 = \sigma$, $z_1 = -\phi_\lambda(\sigma)$, $z_2 = 1$ into (2.4.11),

   $$-\phi_\lambda(\sigma)\overline{\phi_\lambda(\sigma)} - \phi_\lambda(\sigma^{-1})\phi_\lambda(\sigma) + \phi(e)\overline{\phi_\lambda(\sigma)}\phi_\lambda(\sigma) + \phi(e) \ge 0$$

   for all $\sigma \in G$. Since ϕ_λ is a spherical function, $\phi(e) = 1$. Using Proposition 2.4.11 (1), the above simplifies to give

   $$1 - |\phi_\lambda(\sigma)|^2 \ge 0.$$

   Proposition 2.4.11 (2) follows.
3. This follows from Proposition 2.4.11 (2), since $\mathrm{Re}(\phi_\lambda(\sigma)) \le |\phi_\lambda(\sigma)|$, for all $\sigma \in G$ and $\lambda \in \mathfrak{a}^*$.

∎

Remark 2.4.12. Note that Proposition 2.4.11 (2) and (3) hold for positive definite functions on G more generally — see [57] §2.4.

For each $D \in \mathbf{D}(G/K)$ and each $\lambda \in \mathfrak{a}^*$, let $\beta(D, \lambda)$ denote the eigenvalue of D associated with eigenfunction ϕ_λ. Below, we state a beautiful formula due to Harish-Chandra, that characterises these eigenvalues as polynomial functions on $\mathfrak{a}^*$.

Before stating the result, we introduce some notation. Let $\mathcal{T}(\mathfrak{a})$ denote the tensor algebra of $\mathfrak{a}$, and let $I \subset \mathcal{T}(\mathfrak{a})$ denote the two-sided ideal of $\mathcal{T}(\mathfrak{a})$ generated by $X \otimes Y - Y \otimes X$. The quotient $\mathrm{Sym}(\mathfrak{a}) := \mathcal{T}(\mathfrak{a})/I$ is called the *symmetric algebra* of $\mathfrak{a}$. Given any basis $H_1, \ldots, H_m$ of $\mathfrak{a}$, $\mathrm{Sym}(\mathfrak{a})$ may be identified with the algebra of polynomials in these basis vectors. As such, elements of $\mathrm{Sym}(\mathfrak{a})$ will usually be viewed as polynomial functions on $\mathfrak{a}$.

Recall that the Weyl group W acts on $\mathfrak{a}$ by permuting Weyl chambers. This action induces an action on $\mathrm{Sym}(\mathfrak{a})$; let $\mathrm{Sym}(\mathfrak{a})^W \subset \mathrm{Sym}(\mathfrak{a})$ denote the subalgebra of W-invariants. By duality, we may view elements of $\mathrm{Sym}(\mathfrak{a})^W$ as polynomial functions on $\mathfrak{a}^*$.

The following theorem is due to Harish-Chandra, see Section 4 of [33], pp. 260–263. A detailed account of this proof can also be found in Helgason [36] Chapter X, §6. Equation (2.4.12) corresponds to equation (12) on page 431 of [36].

Theorem 2.4.13. *There exists an algebra isomorphism* $\Gamma : \mathbf{D}(G/K) \to \mathrm{Sym}(\mathfrak{a})^W$*, such that for each* $D \in \mathbf{D}(G/K)$ *and each* $\lambda \in \mathfrak{a}^*$*,*

$$D\phi_\lambda = \Gamma(D)(i\lambda)\phi_\lambda, \tag{2.4.12}$$

In particular, each of the eigenvalues $\beta(D, \lambda)$ *is a* W*-invariant polynomial function on* $\mathfrak{a}^*$*, with complex coefficients, and with degree equal to that of* D*.*

Remark 2.4.14. In (2.4.12), the domain of the polynomial $\Gamma(D)$ has been extended to the complexification $\mathfrak{a}^*_{\mathbb{C}}$ in the obvious way.

Example 2.4.15. The Laplace–Beltrami operator Δ on G/K belongs to $\mathbf{D}(G/K)$. Viewed as an element of $\mathbf{D}_K(G)$, it satisfies the famous eigenrelation

$$\Delta\phi_\lambda = -(|\rho|^2 + |\lambda|^2)\phi_\lambda, \qquad \forall \lambda \in \mathfrak{a}^* \tag{2.4.13}$$

For more information, see [37] (7), pp. 427, as well as associated discussion.

The spherical transform

Definition 2.4.16. Let $f \in L^1(K|G|K)$. The *spherical transform of* f is the mapping $\hat{f} : \mathfrak{a}^* \to \mathbb{C}$ given by

$$\hat{f}(\lambda) = \int_G \phi_{-\lambda}(\sigma) f(\sigma) d\sigma, \qquad \forall \lambda \in \mathfrak{a}^* . \tag{2.4.14}$$

Let $\mu \in \mathcal{M}(K|G|K)$ and suppose μ is finite. The *spherical transform of* μ is the mapping $\hat{\mu} : \mathfrak{a}^* \to \mathbb{C}$ given by

$$\hat{\mu}(\lambda) = \int_G \phi_{-\lambda}(\sigma) \mu(d\sigma), \qquad \forall \lambda \in \mathfrak{a}^* . \tag{2.4.15}$$

Observe that if the measure μ has a density $f \in L^1(K|G|K)$ with respect to Haar measure, then $\hat{\mu} = \hat{f}$.

The next proposition demonstates that the spherical transform satisfies many of the properties we would expect for a generalisation of the Fourier transform.

Proposition 2.4.17. *1. For all $f, g \in L^1(K|G|K)$ and all $a, b \in \mathbb{R}$,*

$$(af + bg)^\wedge = a\hat{f} + b\hat{g}.$$

2. For all finite measures $\mu_1, \mu_2 \in \mathcal{M}(K|G|K)$,

$$(\mu_1 * \mu_2)^\wedge(\lambda) = \hat{\mu}_1(\lambda)\hat{\mu}_2(\lambda) \qquad \forall \lambda \in \mathfrak{a}^*.$$

3. For all $f_1, f_2 \in L^1(K|G|K)$,

$$(f_1 * f_2)^\wedge(\lambda) = \hat{f}_1(\lambda)\hat{g}_2(\lambda) \qquad \forall \lambda \in \mathfrak{a}^*.$$

Proof. The linearity property (1) is clear from the definition, and (3) will follow from (2) by taking $\mu_j(d\sigma) = f_j(\sigma)d\sigma$, for $j = 1, 2$. We verify (2). Given $\lambda \in \mathfrak{a}^*$,

$$\begin{aligned}(\mu_1 * \mu_2)^\wedge(\lambda) = \int_G \phi_{-\lambda}(\sigma)(\mu_1 * \mu_2)(d\sigma) &= \int_G \int_G \phi_{-\lambda}(\sigma\tau)\mu_1(d\sigma)\mu_2(d\tau) \\ &= \int_K \int_G \int_G \phi_{-\lambda}(\sigma k \tau)\mu_1(d\sigma)\mu_2(d\tau)dk,\end{aligned}$$

since μ_1 is invariant under the change of variable $\sigma \mapsto \sigma k^{-1}$, for all $k \in K$. All measures involved are finite, and $|\phi_{-\lambda}| \leq 1$, so we may apply the Fubini theorem to exchange the order of integration. Then by Proposition 2.4.9,

$$\begin{aligned}(\mu_1 * \mu_2)^\wedge(\lambda) &= \int_G \int_G \int_K \phi_{-\lambda}(\sigma k \tau)dk\mu_1(d\sigma)\mu_2(d\tau) \\ &= \int_G \int_G \phi_{-\lambda}(\sigma)\phi_{-\lambda}(\tau)\mu_1(d\sigma)\mu_2(d\tau) = \hat{\mu}_1(\lambda)\hat{\mu}_2(\lambda).\end{aligned}$$

□

Recall from the discussion surrounding (2.3.7) that the Weyl group W acts on $\mathfrak{a}^*$ via reflection in hyperplanes orthogonal to each root. This action induces an action on functions on $\mathfrak{a}^*$; we use the superscript W to indicate invariance under this action. For example, given a Borel measure ω on $\mathfrak{a}^*$, $L^2(\mathfrak{a}^*, \omega)^W$ will denote the space of all W-invariant, square-ω-integrable functions on $\mathfrak{a}^*$.

A proof of the next theorem may be found in Helgason [37], page 454. We first need a definition.

Definition 2.4.18. Let Σ_0^+ denote the set of all positive indivisible roots, as in Section 2.3.1. By results proven in Helgason [37] Ch. IV §6, the mapping

$$\mathbf{c}(\lambda) = c_0 \prod_{\alpha \in \Sigma_0^+} \frac{2^{-i\langle \lambda, \alpha_0 \rangle} \Gamma\big(\langle i\lambda, \alpha_0 \rangle\big)}{\Gamma\big(\frac{1}{2}(\frac{1}{2}m_\alpha + 1 + \langle i\lambda, \alpha_0\rangle)\big)\Gamma\big(\frac{1}{2}(\frac{1}{2}m_\alpha + m_{2\alpha} + \langle i\lambda, \alpha_0\rangle)\big)}, \tag{2.4.16}$$

where $\alpha_0 := \frac{\alpha}{|\alpha|^2}$ and c_0 is a constant chosen so that $\mathbf{c}(-i\rho) = 1$, defines a meromorphic function on $\mathfrak{a}^*_{\mathbb{C}}$, called *Harish-Chandra's* $\mathbf{c}$*-function.*

Theorem 2.4.19 (Spherical Inversion)**.** *There exists a Borel measure ω on $\mathfrak{a}^*$ such that for all $f \in C_c^\infty(K|G|K)$ and $\sigma \in G$,*

$$f(\sigma) = \int_{\mathfrak{a}^*} \phi_\lambda(\sigma)\hat{f}(\lambda)\omega(d\lambda). \tag{2.4.17}$$

This measure, known as Plancherel measure *is given by*

$$\omega(d\lambda) = c|\,\mathbf{c}(\lambda)|^{-2} d\lambda, \tag{2.4.18}$$

where $c > 0$ is a constant, and $\mathbf{c}$ is Harish-Chandra's $\mathbf{c}$-function. Furthermore, we have

$$\|f\|_{L^2(K|G|K)} = \|\hat{f}\|_{L^2(\mathfrak{a}^*,\omega)} \qquad \forall f \in C_c^\infty(K|G|K). \tag{2.4.19}$$

Let $L^2(\mathfrak{a}^*, \omega)^W$ denote the subspace of $L^2(\mathfrak{a}^*, \omega)$ consisting of W-invariants. Then the image of $C_c^\infty(K|G|K)$ under spherical transformation is a dense subspace of $L^2(\mathfrak{a}^*, \omega)^W$, and as such the spherical transform extends to an unitary isomorphism between the Hilbert spaces $L^2(K|G|K)$ and $L^2(\mathfrak{a}^*, \omega)^W$.

Equation (2.4.19) is known as the Plancherel formula, and generalises the famous formula of the same name associated with the Euclidean Fourier transform on $\mathbb{R}^d$. The unitarity of the spherical transform, i.e.

$$\int_G f(\sigma)\overline{g(\sigma)}d\sigma = \int_{\mathfrak{a}^*} \hat{f}(\lambda)\overline{\hat{g}(\lambda)}\omega(d\lambda) \qquad \forall f, g \in L^2(K|G|K), \tag{2.4.20}$$

will be referred to as Parseval's identity, and can be derived from (2.4.19) in the usual way using the polarisation identity for inner product spaces.

To simplify notation, from now on we will suppress subscripts for the inner product and norm of $L^2(K|G|K)$; all other norms and inner products will be identified using subscripts to minimise confusion.

Harish-Chandra's $\mathbf{c}$-function has many remarkable properties and connections, see for example the expository paper [38]. Its role in this work will primarily be through Plancherel measure.

The following estimate will be extremely useful in later work. See Helgason [37] page 450 for a proof.

Proposition 2.4.20. *There exist constants $C_1, C_2 > 0$ such that*

$$|\mathbf{c}(\lambda)|^{-1} \leq C_1 + C_2|\lambda|^{(\dim N)/2} \tag{2.4.21}$$

for all $\lambda \in \mathfrak{a}^*$.

There is a version of the Paley–Wiener theorem for the spherical transform:

Theorem 2.4.21 (Spherical Paley–Wiener theorem). *Given $f \in C_c^\infty(K|G|K)$ and $n \in \mathbb{N}$, there is a constant $k_n > 0$ such that*

$$|\hat{f}(\lambda)| \leq k_n(1 + |\lambda|)^{-n}$$

for all $\lambda \in \mathfrak{a}^*$.

For a proof, see Helgason [37] page 450.

2.4.3 Schwartz spaces on G and $\mathfrak{a}^*$

The main sources for this section are Gangolli and Varadarajan [30] Chapter 6, and Helgason [37] page 489, exercise 6*.

Definition 2.4.22. Given $\sigma \in G$, let $|\sigma|$ denote the geodesic distance between the cosets K and σK in G/K.

Geodesic distance on G/K is invariant under the left action of K, and hence $|\cdot|$ is K-bi-invariant. In particular,

$$|\sigma| := |H(\sigma)|, \qquad \forall \sigma \in G,$$

where $H(\sigma) \in \overline{\mathfrak{a}^+}$ is as defined in (2.3.9) — see also Theorem 2.3.5.

The following is contained in Proposition 4.6.11 of Gangolli and Varadarajan [30], pp. 167.

Proposition 2.4.23.

1. *The mapping $\sigma \to |\sigma|$ is continuous and K-bi-invariant,*
2. *For all $\sigma \in G$, $\left|\sigma^{-1}\right| = |\sigma|$,*
3. *For all $\sigma, \tau \in G$, $|\sigma\tau| \leq |\sigma| + |\tau|$,*
4. *Let $A : G \to \mathfrak{a}$ be as in (2.3.4). Then*

$$\|A(\sigma)\| \leq C|\sigma| \qquad \forall \sigma \in G,$$

for some constant $C > 0$.

Combining Theorems 4.6.4 and 4.6.5 of Gangolli and Varadarajan, pp. 161–2, we also have:

Theorem 2.4.24. *There exists $C > 0$ such that for all $\sigma \in G$*

$$e^{-\rho(H(\sigma))} \leq \phi_0(\sigma) \leq Ce^{-\rho(H(\sigma))}(1+|\sigma|)^{|\Sigma_0|}$$

where $H(\sigma)$ is as in (2.3.9), and Σ_0 denotes the set of all indivisible roots (defined in Section 2.3.2).

Definition 2.4.25. A function $f \in C^\infty(G)$ is called *rapidly decreasing* if

$$\sup_{\sigma \in G}(1+|\sigma|)^q \phi_0(\sigma)^{-1}(Df)(\sigma) < \infty, \qquad \forall D \in \mathbf{D}(G),\ q \in \mathbb{N} \cup \{0\}. \tag{2.4.22}$$

By Theorem 2.4.24, a function $f \in C^\infty(G)$ is rapidly decreasing if and only if

$$f(a) = O\left(e^{-\rho(\log a)}(1+|a|)^q\right) \tag{2.4.23}$$

for all $q \in \mathbb{N} \cup \{0\}$ and all $a \in \overline{\exp \mathfrak{a}^+}$.

The factor of the $e^{-\rho(\log a)}$ in (2.4.23) may seem at odds with the classical definition of a rapidly decreasing function. In fact, its presence is necessary if f is to be square-integrable. Recall the integral formulae (2.3.11) and (2.3.12). In both cases, integration over G is related to integration over A, and the associated Jacobian factors are each asymptotically equivalent to $e^{\rho(\log a)}$. See Gangolli and Varadarajan [30] pp. 250–253 for more details.

The space $\mathcal{S}(G)$ of all rapidly decreasing functions $f \in C^\infty(G)$ is called the *Schwartz space* of G. For $f \in \mathcal{S}(G)$, denote the left-hand side of (2.4.22) by $\|f\|_{D,q}$. One can check that $\|\cdot\|_{D,q}$ is a seminorm on $\mathcal{S}(G)$ for all $D \in \mathbf{D}(G)$ and $q \in \mathbb{N} \cup \{0\}$. The subspace

$$\mathcal{S}(K|G|K) := \mathcal{S}(G) \cap \mathcal{F}(K|G|K)$$

will be called the *Schwartz space of K-bi-invariant functions on G*.

By Proposition 6.1.3 on page 253 of [30], $\mathcal{S}(G)$ is a Frechét space with respect to the family of seminorms $\{\|\cdot\|_{D,q} : D \in \mathbf{D}(G), q \in \mathbb{N} \cup \{0\}\}$, and $\mathcal{S}(K|G|K)$ is a closed subspace of $\mathcal{S}(G)$.

Other useful facts proven in Section 6.1 of [30] include the following.

Proposition 2.4.26. *$C_c^\infty(G)$ is a dense subspace of $\mathcal{S}(G)$, and we have continuous inclusions*

$$C_c^\infty(G) \subseteq \mathcal{S}(G) \subseteq L^2(G).$$

Similarly, $C_c^\infty(K|G|K)$ is a dense subspace of $\mathcal{S}(K|G|K)$, and we have continuous inclusions

$$C_c^\infty(K|G|K) \subseteq \mathcal{S}(K|G|K) \subseteq L^2(K|G|K).$$

Remark 2.4.27. Some authors use a pair of elements a, b in the universal enveloping algebra $U(\mathfrak{g})$ to characterise $\mathcal{S}(G)$ instead of a differential operator $D \in \mathbf{D}(G)$, so that afb appears in the place of Df in (2.4.22). See [30] §2.6 pp. 84 for discussion of this.

Given $f, g \in \mathcal{S}(G)$, recall the convolution of f and g, as defined by (2.4.1).

The next result is stated as Theorem 6.1.10 in [30] page 255.

Theorem 2.4.28. *Convolution defines a continuous mapping from $\mathcal{S}(G) \times \mathcal{S}(G) \to \mathcal{S}(G)$. Moreover, $\mathcal{S}(G)$ is an algebra under convolution of functions, and $\mathcal{S}(K|G|K)$ is a commutative subalgebra.*

The space $\mathfrak{a}^*$ is a finite dimensional vector space, and so we may also consider the classical Schwartz space $\mathcal{S}(\mathfrak{a}^*)$, consisting of all rapidly decreasing maps f on $\mathfrak{a}^*$. Let $\mathcal{S}(\mathfrak{a}^*)^W$ denote the closed subspace of $\mathcal{S}(\mathfrak{a}^*)$ consisting of all functions that are invariant under the action of the Weyl group W. We regard $\mathcal{S}(\mathfrak{a}^*)^W$ as an algebra under pointwise multiplication.

Theorem 2.4.29. *The spherical transform defines a topological algebra isomorphism from $\mathcal{S}(K|G|K)$ onto $\mathcal{S}(\mathfrak{a}^*)^W$.*

For a proof, see [30] page 274.

It is also useful to consider the classical Schwartz space $\mathcal{S}(\mathfrak{a})$, as well as the subspace $\mathcal{S}(\mathfrak{a})^W$ comprising all maps in $\mathcal{S}(\mathfrak{a})$ that are invariant under the action of W. The Euclidean Fourier transform

$$\mathcal{F}(f)(\lambda) = \int_{\mathfrak{a}} e^{-i\lambda(H)} f(H)dH, \qquad \forall f \in \mathcal{S}(\mathfrak{a}), \lambda \in \mathfrak{a}^* \tag{2.4.24}$$

defines a topological isomorphism between the classical Schwartz spaces $\mathcal{S}(\mathfrak{a})$ and $\mathcal{S}(\mathfrak{a}^*)$, and the usual convolution/pointwise product relations hold (c.f. [70] Ch.VI.1, pp. 146).

There is an interesting connection between this isomorphism and the isomorphism of Theorem 2.4.29. Given $f \in \mathcal{S}(K|G|K)$ and $H \in \mathfrak{a}$, the *Abel transform* is defined by

$$\mathcal{A}f(H) = e^{\rho(H)} \int_N f((\exp H)n)dn.$$

The Abel transform is fascinating in its own right, and we refer to Sawyer [63] for more information. However, for our purposes we are mainly interested in its role in the following:

Theorem 2.4.30. *Writing $\mathcal{H}$ for the spherical transform, the diagram*

$$\begin{array}{ccc} & \mathcal{S}(\mathfrak{a}^*)^W & \\ \mathcal{H} \nearrow & & \nwarrow \mathcal{F} \\ \mathcal{S}(K|G|K) & \xrightarrow[\mathcal{A}]{} & \mathcal{S}(\mathfrak{a})^W \end{array}$$

commutes, up to normalizing constants. Each arrow describes an isomorphism of Fréchet algebras.

This result is Proposition 3 in Anker [1]; see also Gangolli and Varadarajan [30] page 265, and Helgason [37] 450.

L^2 characterisation and tempered distributions

One of the most useful properties of classical Schwartz spaces are their relationship to L^2 spaces. The following analogous result is Theorem 6.1.16 in [30] (pp. 257). For a proof, see [68] Section II.9.4, pp. 345–348.

Theorem 2.4.31. *Let $f \in C^\infty(G)$. Then $f \in \mathcal{S}(G)$ if and only if $(1+|\cdot|)^m Df \in L^2(G)$ for all $m \in \mathbb{N} \cup \{0\}$ and all $D \in \mathbf{D}(G)$. Moreover, the family of L^2-norms*

$$f \mapsto \|(1+|\cdot|)^m Df\|_{L^2(G)}, \qquad m \in \mathbb{N} \cup \{0\},\ D \in \mathbf{D}(G),$$

induce the topology of $\mathcal{S}(G)$.

The corresponding L^2 characterisation of $\mathcal{S}(K|G|K)$ may be arrived at by imposing K-bi-invariance in the usual way.

A distribution $\phi : C_c^\infty(G) \to \mathbb{C}$ on G is *tempered* if it is the restriction to $C_c^\infty(G)$ of a continuous linear functional on $\mathcal{S}(G)$. Linear functionals on $C_c^\infty(K|G|K)$ will be referred to as K-bi-invariant distributions on G. We call the topological dual of $\mathcal{S}(G)$, denoted $\mathcal{S}'(G)$, the *(Schwarz) space of tempered distributions on G* on G. The dual of $\mathcal{S}(K|G|K)$ is the *(Schwartz) space of K-bi-invariant tempered distributions*, denoted by $\mathcal{S}'(K|G|K)$.

Theorem 2.4.31 may be used to develop the theory of tempered distributions in much the same way as in the classical case — see [30] pp. 257–262 for more details. We will encounter $\mathcal{S}'(K|G|K)$ again in §4.1.2, where we use it to define spherical anisotropic Sobolev spaces.

Chapter 3

Lévy processes on Lie groups and symmetric spaces

Having developed the necessary background in symmetric space theory and its associated harmonic analysis, we are now ready to discuss Lévy processes on a Riemannian symmetric space M. We continue to view M as a homogeneous space G/K, where the Lie groups G and K are as described in §2.2. As with many objects defined on symmetric spaces, Lévy processes may be defined either directly as G/K-valued processes satisfying particular properties, or as the image under the projection map $\pi : G \to G/K$ of Lévy processes on G. We opt for the latter approach (c.f. Definition 3.1.5), and will often identify processes on G/K with K-right-invariant processes on G, in much the same way as we have for for functions, measures and operators in §2.4. Under this identification, Lévy processes on G/K coincide with K-bi-invariant Lévy processes on G. This is because, as we shall see, the laws of Lévy processes on G/K are automatically K-invariant (see also [54] Proposition 1.12).

§ 3.1 Background

We summarise key definitions and results from the theory of Lévy processes on Lie groups and Riemannian symmetric spaces.

3.1.1 Lévy processes on Lie groups

Let G be a Lie group, and let $Y = (Y(t), t \geq 0)$ a stochastic process taking values on G. The random variables

$$Y(s)^{-1}Y(t), \qquad 0 \leq s \leq t,$$

are called the *increments* of Y. Let $\mathcal{F}^Y$ denote the natural filtration of Y.

Definition 3.1.1. Y is called an *Lévy process* if the following are satisfied:

(i) *(Independent increments)* $Y(s)^{-1}Y(t)$ is independent of $\mathcal{F}_s^Y$, for all $t > s \geq 0$,

(ii) *(Stationary increments)* $Y(s)^{-1}Y(t) \sim Y(0)^{-1}Y(t-s)$ for all $t \geq s \geq 0$.

(iii) *(Stochastic continuity)* For all $s \geq 0$ and all $B \in \mathcal{B}(G)$ with $e \notin \overline{B}$,

$$\lim_{t \to s} P(Y(s)^{-1}Y(t) \in B) = 0.$$

Remark 3.1.2. Observe the similarity with the definition of a Lévy process given in §1.1.2.

Let Y be a Lévy process on G, and for each $t \geq 0$, let μ_t denote the law of $Y(0)^{-1}Y(t)$. By Theorem 1.7 on page 8 of [54], $(\mu_t, t \geq 0)$ is a convolution semigroup of probability measures on G (c.f. Definition 2.4.3).

Definition 3.1.3. We call $(\mu_t, t \geq 0)$ the *convolution semigroup associated with Y*.

Note that if $(\mu_t, t \geq 0)$ is the convolution semigroup associated with a Lévy process Y on G, then $\mu_0 = \delta_e$, irrespective of the initial distribution of Y. Moreover, given a convolution semigroup of probability measures $(\mu_t, t \geq 0)$ that satisfies $\mu_0 = \delta_e$, it may be shown using Kolmogorov's construction that there exists a Lévy process on G with $(\mu_t, t \geq 0)$ as its associated convolution semigroup — see [8] pp. 123 for further discussion of this.

A Feller process is called *left invariant* if its Feller semigroup $(T_t, t \geq 0)$ satisfies $L_g T_t = T_t L_g$ for all $g \in G$ and $t \geq 0$.

Proposition 3.1.4. *Let Y be a Lévy process on G with convolution semigroup $(\mu_t, t \geq 0)$. Then Y is a left invariant Feller process, with Feller semigroup $(T_t, t \geq 0)$ given by the Hunt semigroup of $(\mu_t, t \geq 0)$ (c.f. Definition 2.4.4).*

Conversely, every left invariant Feller process is equal in law to a Lévy process on G.

Proposition 3.1.4 generalises the well known classical result for $\mathbb{R}^d$-valued Lévy processes, and is proved in a very similar way. See [6] pp. 160–161 for the classical result, and [57] pp. 82–83 for a proof in the Lie group setting.

3.1.2 Lévy processes on symmetric spaces

Suppose now that $M = G/K$ is a Riemannian symmetric space, and continue to denote the associated canonical surjection map by $\pi : G \to G/K$.

Definition 3.1.5. A G/K-valued process Y is called a *Lévy process* if $Y = \pi(X)$ for some Lévy process X on G.

Remarks 3.1.6. 1. This definition is consistent with previous work, in the sense that every Lévy process on G/K may be realised as a solution to an SDE of the form (1.2.2). When G acts effectively on G/K, this may be made explicit using the bundle reduction of OM to G — for details, see Applebaum [4] §5.

2. Definition 3.1.5 is less restrictive than the assumptions made in Chapter 1, because the Lévy processes are not required to be isotropic.
3. By Proposition 3.1.4, Lévy processes on G/K correspond precisely to the G-invariant Feller processes on G/K.

Let Y be a Lévy process on G/K, and X a Lévy process on G for which $Y = \pi(X)$. Let $(p_t, t \geq 0)$ and $(q_t, t \geq 0)$ denote the transition probabilities of Y and X, respectively. Then for all $t \geq 0$, $\sigma \in G$ and $A \in \mathcal{B}(G/K)$,

$$p_t(\sigma K, A) = \mathbb{P}\left(\pi\big(X(t)\big) \in A \big| \, \pi\big(X\big) = \sigma K\right) = q_t\left(\sigma, \pi^{-1}(A)\right).$$

In particular, the prescription

$$\nu_t := p_t(o, \cdot), \qquad \forall t \geq 0$$

defines a convolution semigroup $(\nu_t, t \geq 0)$ on G/K. By [54] Proposition 1.12, pp. 13, $(\nu_t, t \geq 0)$ is K-invariant, and there is a K-bi-invariant convolution semigroup $(\mu_t, t \geq 0)$ on G for which

$$\nu_t = \mu_t \circ \pi^{-1}, \qquad \forall t \geq 0. \tag{3.1.1}$$

It may be tempting to think that $(\mu_t, t \geq 0)$ should be the convolution semigroup of X. In fact, this is not the case: if it were, then we would have $\mu_0 = \delta_0$, which is not a K-bi-invariant measure on G. However, if we denote the convolution semigroup of X by $(\mu_t^e, t \geq 0)$, and normalised Haar measure on K by ρ_K, then by [54] Theorem 3.14, pp. 88,

$$\mu_t := \rho_K * \mu_t^e, \qquad \forall t \geq 0$$

is a suitable choice for the K-bi-invariant convolution semigroup $(\mu_t, t \geq 0)$ on G, for which (3.1.1) is satisfied. In particular, $\mu_0 = \rho_K$.

In this way, Lévy processes on G/K may be understood through the study of K-bi-invariant convolution semigroups on G. The corresponding Lévy processes on G are called *K-bi-invariant Lévy processes.*

Let $(\mu_t, t \geq 0)$ is a K-bi-invariant convolution semigroup on G associated to a Lévy process Y on G/K. Let $(T_t, t \geq 0)$ denote the corresponding Hunt semigroup viewed as a family of operators acting on $C_0(G/K)$. Then since $\mu_0 = \rho_K$, we have $T_0 = I$. Moreover, in light of Proposition 2.4.5, $(T_t, t \geq 0)$ forms a strongly continuous operator semigroup on $C_0(G/K)$, and the restriction of each T_t to $C_0(K|G|K)$ yields a strongly continuous semigroup on $C_0(K|G|K)$.

Restricting to the K-bi-invariant functions in this way will be advantageous, as we have the spherical transform at our disposal. As an early application of this, we prove the following useful eigenvalue relation for the Hunt semigroup of a K-bi-invariant convolution semigroup.

Proposition 3.1.7. *Let $(\mu_t, t \geq 0)$ be a K-bi-invariant convolution semigroup on G, and let $(T_t, t \geq 0)$ denote the the restriction to $C_0(K|G|K)$ of the Hunt semigroup associated with $(\mu_t, t \geq 0)$. Then for all $t \geq 0$, $\lambda \in \mathfrak{a}^*$ and $\sigma \in G$,*

$$T_t \phi_\lambda(\sigma) = \hat{\mu}_t(\lambda) \phi_\lambda(\sigma).$$

Proof. Let $t \geq 0$, $\lambda \in \mathfrak{a}^*$ and $\sigma \in G$. Then

$$T_t\phi_\lambda(\sigma) = \int_G \phi_\lambda(\sigma\tau)\mu_t(d\tau)$$

Observe that since μ_t is invariant under all translations by K, for all $k \in K$ we have

$$T_t\phi_\lambda(\sigma) = \int_G \phi_\lambda(\sigma k\tau)\mu_t(d\tau).$$

Integrating over K, it follows that

$$T_t\phi_\lambda(\sigma) = \int_K \int_G \phi_\lambda(\sigma k\tau)\mu_t(d\tau)dk.$$

All the measures being considered are finite, and $|\phi_\lambda| \leq 1$. Therefore, by Fubini's theorem and Proposition 2.4.9,

$$T_t\phi_\lambda(\sigma) = \int_G \int_K \phi_\lambda(\sigma k\tau)dk\mu_t(d\tau) = \int_G \phi_\lambda(\sigma)\phi(\tau)\mu_t(d\tau) = \phi_\lambda(\sigma)\hat{\mu}_t(\lambda),$$

as desired. □

Hunt's formula for the generator

The infinitesimal generator of a Lévy process Y on G is given by the celebrated Hunt formula ([43] Theorem 5.1). We describe a version of this next, specialising to the K-bi-invariant case most relevant to our work on symmetric spaces. We first introduce a local coordinate system on G, defined in terms of the orthogonal decomposition (2.2.2).

Definition 3.1.8. Let $X_1, \ldots, X_l$ be an orthonormal basis of $\mathfrak{g}$, ordered according to (2.2.2), so that $X_1, \ldots, X_d$ is a basis of $\mathfrak{p}$ and $X_{d+1}, \ldots, X_l$ is a basis of $\mathfrak{k}$.

A collection $\{x_1, \ldots, x_l\}$ of smooth functions of compact support is called a *system of exponential coordinate functions* if there is a neighbourhood U of e for which

$$\sigma = \exp\left(\sum_{i=1}^{l} x_i(\sigma)X_i\right) \qquad \forall \sigma \in U. \tag{3.1.2}$$

The x_i may be chosen so as to be K-right-invariant for $i = 1, \ldots, d$, and such that

$$\sum_{i=1}^{d} x_i(k\sigma)X_i = \sum_{i=1}^{d} x_i(\sigma)\,\mathrm{Ad}(k)X_i \qquad \forall k \in K.$$

We assume this is the case in what follows. For more details, see Liao [54] pp.36–37, 83.

The choice of basis of $\mathfrak{p}$ enables us to view $\mathrm{Ad}(k)$ as a $d \times d$ matrix, for each $k \in K$. A vector $b \in \mathbb{R}^m$ is said to be $\mathrm{Ad}(K)$-*invariant* if

$$b = \mathrm{Ad}(k)^T b, \qquad \forall k \in K.$$

Similarly, a $d \times d$ real-valued matrix $a = (a_{ij})$ is $\mathrm{Ad}(K)$-*invariant* if

$$a = \mathrm{Ad}(k)^T a \, \mathrm{Ad}(k) \qquad \forall k \in K.$$

Definition 3.1.9. A Borel measure ν on G is called a *Lévy measure* if $\nu(\{e\}) = 0$, $\nu(U^c) < \infty$, and $\int_G \sum_{i=1}^l x_i(\sigma)^2 \nu(d\sigma) < \infty$.

Note that this definition is independent of the choice of basis $X_1, \ldots, X_l$ of $\}$ and coordinate functions $x_1, \ldots, x_l \in C_c^\infty(G)$ — see [54] pp. 38.

Theorem 3.1.10. *Let $\mathcal{A}$ be the infinitesimal generator associated with a Lévy process Y on G/K. Then $C_c^\infty(G/K) \subseteq \mathrm{Dom}\,\mathcal{A}$, and there is an $\mathrm{Ad}(K)$-invariant vector $b \in \mathbb{R}^d$, an $\mathrm{Ad}(K)$-invariant, non-negative definite, symmetric $d \times d$ matrix $a := (a_{ij})$, and a K-bi-invariant Lévy measure ν such that*

$$\begin{aligned} \mathcal{A}f(\sigma) = \sum_{i=1}^d b_i X_i f(\sigma) + \sum_{i,j=1}^d a_{ij} X_i X_j f(\sigma) \\ + \int_G \left(f(\sigma\tau) - f(\sigma) - \sum_{i=1}^d x_i(\tau) X_i f(\sigma) \right) \nu(d\sigma), \end{aligned} \tag{3.1.3}$$

for all $f \in C_c^\infty(G/K)$ and $\sigma \in G$. Moreover, the triple (b, a, ν) is completely determined by $\mathcal{A}$, and independent of the choice of exponential coordinate functions x_i, $i = 1, \ldots, d$.

Conversely, given a triple (b, a, ν) of this kind, there is a unique K-bi-invariant convolution semigroup of probability measures on G with infinitesimal generator given by $\mathcal{A}$.

Remarks 3.1.11.

1. If the assumption of K-bi-invariance is dropped in Theorem 3.1.10, the generator still takes the form (3.1.3), but without the $\mathrm{Ad}(K)$-invariance and K-bi-invariance restrictions. This more general result is known as the Hunt formula, first proven by Gilbert Hunt in [43]. For more details regarding the K-bi-invariant case, including a proof of Theorem 3.1.10, see [54] Section 3.2, pp. 78.

2. In fact, equation (3.1.3) is well-defined for any $f \in C_b^2(G)$. Indeed, by Lemma 2.3 on page 39 of [54], the integrand on the right-hand side of (3.1.4) is absolutely integrable with respect to ν, for all $f \in C_b(G)$. The diffusion term is well-defined for any twice continuously differentiable function. We will sometimes use this fact when discussing certain eigenvalue relations that hold on a larger domain — see for example equation (3.1.7).

We call (b, a, ν) from Theorem 3.1.10 the *Lévy characteristics* of Y.

Gangolli's Lévy–Khinchine formula

Let G/K be a symmetric space of noncompact type. Recall that in this case, G is semisimple. The following result is well-known in the literature. Its proof is identical to the argument made in the latter part of the proof of [55] Theorem 2, and is included for clarity.

Lemma 3.1.12. $\mathfrak{p}$ *has no non-zero* $\mathrm{Ad}(K)$*-invariant elements.*

Proof. Suppose $X \in \mathfrak{p}$ is $\mathrm{Ad}(K)$-invariant. Then $[Y, X] = 0$ for all $Y \in \mathfrak{k}$. By (2.2.3), $[X, Z] \in \mathfrak{k}$ for all $Z \in \mathfrak{p}$. Also, by (2.1.2),

$$B([X, Z], Y) = B(Z, [Y, X]) = 0 \qquad \forall Y \in \mathfrak{k},\ Z \in \mathfrak{p}.$$

That is, $[X, Z]$ is an element of $\mathfrak{k}$, and is orthogonal to every other element of $\mathfrak{k}$. It follows that $[X, Z] = 0$ for all $Z \in \mathfrak{p}$. But then $\mathrm{ad}(X) = 0$, and since G is semisimple, it follows that $X = 0$. □

An immediate consequence of Lemma 3.1.12 is that every K-bi-invariant Lévy process on G has a vanishing drift term. Specifically, the generator $\mathcal{A}$ takes the form

$$\mathcal{A}f(\sigma) = \sum_{i,j=1}^{d} a_{ij} X_i X_j f(\sigma) + \int_G \left(f(\sigma\tau) - f(\sigma) - \sum_{i=1}^{d} x_i(\tau) X_i f(\sigma) \right) \nu(d\sigma), \tag{3.1.4}$$

for all $f \in C_c^\infty(G)$, where $a = (a_{ij})$ is an $\mathrm{Ad}(K)$-invariant, non-negative definite symmetric matrix, and ν is a K-bi-invariant Lévy measure.

Proposition 3.1.13. $\sum_{i,j=1}^{d} a_{ij} X_i X_j \in \mathbf{D}_K(G)$.

Proof. For brevity, we write $\mathcal{A}_D \coloneqq \sum_{i,j=1}^{d} a_{ij} X_i X_j$ (the "diffusion part" of $\mathcal{A}$). Left invariance is clear, since the X_i belong to $\mathfrak{g}$. We show that $\mathcal{A}_D$ is K-right-invariant. Let $k \in K$, and let C_k denote the conjugation operator, so that $C_k f = f \circ c_k$, for all $f \in \mathcal{F}(G)$. By definition of Ad,

$$X C_k f = C_k \, \mathrm{Ad}(k) X f,$$

for all $X \in \mathfrak{g}$, $f \in C_c^\infty(G)$. Therefore, for $f \in C_c^\infty(G)$,

$$\begin{aligned} \mathcal{A}_D C_k f = \sum_{i,j=1}^{d} a_{ij} X_i X_j C_k f &= \sum_{i,j=1}^{d} a_{ij} C_k [\mathrm{Ad}(k) X_i][\mathrm{Ad}(k) X_j] f \\ &= \sum_{i,j=1}^{d} \left[\mathrm{Ad}(k)^T a \, \mathrm{Ad}(k) \right]_{ij} C_k X_i X_j f. \end{aligned}$$

Since a is $\mathrm{Ad}(K)$-invariant, it follows that $\mathcal{A}_D C_k = C_k \mathcal{A}_D$. But then, using $C_k = L_k^{-1} R_k$ and left invariance,

$$\mathcal{A}_D R_k = \mathcal{A}_D L_k C_k = L_k C_k \mathcal{A}_D = R_k \mathcal{A}_D.$$

That is, $\mathcal{A}_D$ is K-right-invariant. Hence $\mathcal{A}_D \in \mathbf{D}_K(G)$. □

It follows from the definition of a spherical function (Definition 2.4.8) that each ϕ_λ is an eigenfunction of $\sum_{i=1}^d a_{ij} X_i X_j$.

The following result was first proven by Ramesh Gangolli (see [29] Theorem 5.1), and is a direct generalisation of Theorem 1.1.2. For a proof of the specific statement below, see [54], pp. 139.

Theorem 3.1.14 (Gangolli's Lévy–Khinchine formula)**.** *Let* $(\mu_t, t \geq 0)$ *be a* K*-bi-invariant convolution semigroup of probability measures on* G *with infinitesimal generator* $\mathcal{A}$*, given by (3.1.4). Then*

$$\hat{\mu}_t = e^{-t\psi},$$

where

$$\psi(\lambda) = -\beta_a(\lambda) + \int_G (1 - \phi_\lambda(\sigma))\nu(d\sigma) \qquad \forall \lambda \in \mathfrak{a}^*, \tag{3.1.5}$$

and $\beta_a(\lambda)$ *denotes the* ϕ_λ*-eigenvalue of* $\sum_{i,j=1}^d a_{ij} X_i X_j$*.*

Remarks 3.1.15.

1. By Proposition 3.1.7, if $(T_t, t \geq 0)$ denotes the Hunt semigroup of $(\mu_t, t \geq 0)$, then
$$T_t \phi_\lambda = e^{-t\psi(\lambda)} \phi_\lambda \tag{3.1.6}$$
for all $t \geq 0$ and $\lambda \in \mathfrak{a}^*$.

2. By Theorem 2.4.13, β_a is a W-invariant quadratic polynomial function on $\mathfrak{a}^*$.

Definition 3.1.16. The function $\psi : \mathfrak{a}^* \to \mathbb{C}$ given by (3.1.5) will be called the *Gangolli exponent* of the process Y. It is uniquely determined by the Lévy characteristics of Y.

If the domain of a Lévy generator $\mathcal{A}$ is extended so as to include $C_b^2(K|G|K)$ (c.f. Remark 3.1.11 (2)), we will see that in this larger domain, $\mathcal{A}$ satisfies the eigenrelation

$$\mathcal{A}\phi_\lambda = -\psi(\lambda)\phi_\lambda \qquad \forall \lambda \in \mathfrak{a}^*\,. \tag{3.1.7}$$

Indeed, the proof of this relation is a special case of the proof of Theorem 4.2.7, to come. Moreover, (3.1.7) together with the spherical inversion formula imply that

$$(\mathcal{A}f)^\wedge(\lambda) = -\psi(\lambda)\hat{f}(\lambda), \qquad \forall f \in C_c^\infty(K|G|K),\ \lambda \in \mathfrak{a}^*\,. \tag{3.1.8}$$

This "Fourier multiplier" type relationship is something we will explore further in Chapter 4, when we consider pseudodifferential operators.

Note also that since $\phi_\lambda(e) = 1$, (3.1.7) implies

$$\psi(\lambda) = -\mathcal{A}\phi_\lambda(e), \qquad \forall \lambda \in \mathfrak{a}^*\,.$$

Example 3.1.17. We list some examples of well-known Lévy processes and their associated generators and Gangolli exponents.

1. *Brownian motion.* The Laplace–Beltrami operator on G/K may be viewed as a K-right-invariant operator on G in the usual way. It is then given by

$$\Delta = \sum_{i=1}^{d} X_i^2$$

(recall that the X_i have been ordered so that $X_1, \ldots, X_d$ is a basis of $\mathfrak{p}$).

Standard Brownian motion on G/K has generator $\frac{1}{2}\Delta$, and Lévy characteristics $(0, I, 0)$. By (2.4.13), the Gangolli exponent associated with standard Brownian motion is

$$\psi(\lambda) = \frac{1}{2}(|\rho|^2 + |\lambda|^2), \qquad \forall \lambda \in \mathfrak{a}^* .$$

The associated convolution semigroup $(\mu_t, t \geq 0)$ satisfies

$$\hat{\mu}_t(\lambda) = e^{-\frac{1}{2}t(|\rho|^2+|\lambda|^2)}, \tag{3.1.9}$$

for all $\lambda \in \mathfrak{a}^*$.

2. *Stable-like processes.* Let $\alpha \in (0, 2)$, and let $S = (S(t), t \geq 0)$ denote the $\frac{\alpha}{2}$-stable subordinator, so that S is a one-dimensional non-decreasing Lévy process on $[0, \infty)$ with symbol given by

$$\eta(u) = (-iu)^{\frac{\alpha}{2}} = \frac{\alpha/2}{\Gamma(1-\alpha/2)} \int_0^\infty (1 - e^{iut})t^{-1-\alpha} dt, \qquad \forall u \geq 0$$

(see [6] Section 1.7, page 80 for discussion of this fractional powers formula). Let $B = (B(t), t \geq 0)$ denote a Brownian motion on G/K with generator Δ (the factor of $\frac{1}{2}$ is now omitted for convenience). Let $Y = (Y(t), t \geq 0$ denote the process on G/K obtained by subordinating B by S, so that $Y(t) = B(S(t))$ almost surely, for all $t \geq 0$. Such a process is called an *α-stable-like process* on G/K. By standard results from the theory of subordination, Y is generated by $-(-\Delta)^{\frac{\alpha}{2}}$, and has Gangolli exponent

$$\psi_\alpha(\lambda) = (|\rho|^2 + |\lambda|^2)^{\frac{\alpha}{2}}, \qquad \forall \lambda \in \mathfrak{a}^* . \tag{3.1.10}$$

Observe that

$$\begin{aligned}\psi_\alpha(\lambda) &= \frac{\alpha/2}{\Gamma(1-\alpha/2)} \int_0^\infty t^{-1-\alpha/2}\left(1 - e^{-(|\lambda|^2+|\rho|^2)t}\right) dt \\ &= \frac{\alpha/2}{\Gamma(1-\alpha/2)} \int_0^\infty \int_G (1 - \phi_\lambda(\sigma))t^{-1-\alpha/2} h_t(\sigma) d\sigma dt\end{aligned}$$

for all $\lambda \in \mathfrak{a}^*$. Therefore the α-stable-like Lévy measure is given by

$$\nu_\alpha(A) = \frac{\alpha/2}{\Gamma(1-\alpha/2)} \int_0^\infty \int_A (1 - \phi_\lambda(\sigma))t^{-1-\alpha/2} h_t(\sigma) d\sigma dt, \tag{3.1.11}$$

for each $A \in \mathcal{B}(G \setminus \{e\})$, and $\nu_\alpha(\{e\}) = 0$. The convolution semigroup $(\mu_t, t \geq 0)$ is related to the convolution semigroup $(m_t, t \geq 0)$ of S by

$$\mu_t(A) = \int_0^\infty \mu_s(A) m_t(ds) \qquad \forall t \geq 0,\ A \in \mathcal{B}(G),$$

By Theorem 3.1.14,

$$\hat{\mu}_t(\lambda) = e^{-t(|\rho|^2+|\lambda|^2)^{\alpha/2}} = \exp\left\{-t \int_G (1 - \phi_\lambda(\sigma))\nu_\alpha(d\sigma)\right\}. \tag{3.1.12}$$

In particular, since there is no diffusion term in this representation, Y is a pure jump process.

3. *Compound Poisson process.* Let $Y_1, Y_2, \ldots$ be a sequence of i.i.d. random variable taking values in G, such that the common law μ of the Y_n is K-bi-invariant. Let $N = (N(t), t \geq 0)$ be a Poisson process with intensity $\alpha > 0$, which is independent of each of the Y_n. Let $\Pi = (\Pi(t), t \geq 0)$ be the process on G defined by

$$\Pi(t) = Y_0 \cdot Y_1 \cdot Y_2 \cdot \ldots Y_{N(t)}, \qquad \forall t > 0,$$

where Y_0 is uniformly distributed on K. Then Π is called a K-bi-invariant compound Poisson process on G. The projection of Π onto G/K is called a compound Poisson process on G/K. Note that since Y_0 is uniformly distributed on K, the projection of $\Pi(0)$ is $o \coloneqq eK$, almost surely. See Applebaum [4] for more details. Note that Π could instead have been defined directly on G/K, by choosing a sequence of i.i.d. random variables on G/K, and constructing their product via a section map.

The generator $\mathcal{A}$ of the semigroup associated with Π is given by

$$\mathcal{A}f(\sigma) = \int_G (f(\sigma\tau) - f(\tau))\alpha\mu(d\tau) \qquad \forall f \in C_c(K|G|K),\ \sigma \in G,$$

(see Theorem 1 of [4]), and hence the Gangolli exponent of Π is

$$\psi(\lambda) = \int_G (1 - \phi_\lambda(\sigma))\alpha\mu(d\sigma), \qquad \forall \lambda \in \mathfrak{a}^*.$$

In particular, Π is a pure jump process with Lévy characteristics $(0, 0, \alpha\mu)$.

Remark 3.1.18. Many of the results in this section have immediate analogues for the case where processes are allowed *killing* — that is, when they are defined up to a stopping time, so that their laws may have total mass less than 1. We follow convention and indicate that killing is in play by using the prefix "sub-"; for example, *sub*-diffusion processes, *sub*-Feller processes, and so on. In the case of killed K-bi-invariant Lévy processes, both Theorem 3.1.10 and Theorem 3.1.14 have direct analogues in which the generator (3.1.3) and symbol (3.1.5) gain a constant term. We investigate these more general formulae further in Chapter 4, where we consider the Feller case. For explicit discussion of the killed Lévy case, see [15] Theorem 4.5 and the preceding discussion.

§ 3.2 L^2 densities of convolution semigroups

Having introduced the topic of Lévy processes on symmetric spaces of noncompact type, we present a few new results on this topic.

Let G/K be a symmetric space of noncompact type. The following theorem is a generalisation of [8] Theorem 4.5.1 (see also [5]).

Theorem 3.2.1. *A K-bi-invariant probability measure μ on G has a square-integrable density if and only if $\hat{\mu} \in L^2(\mathfrak{a}^*, \omega)$. The density, when it exists, is given by*

$$f = \int_{\mathfrak{a}^*} \hat{\mu}(\lambda)\phi_\lambda \omega(d\lambda), \qquad a.e. \tag{3.2.1}$$

Remark 3.2.2. Of course if $\hat{\mu} \in L^2(\mathfrak{a}^*, \omega)$, then in fact $\hat{\mu} \in L^2(\mathfrak{a}^*, \omega)^W$: W-invariance is immediate from (2.4.15), since $\phi_{s(\lambda)} = \phi_\lambda$ for all $s \in W$.

Proof of Theorem 3.2.1. First note that if μ has density $f \in L^2(G)$, then comparing (2.4.15) with (2.4.14), we have $\hat{\mu} = \hat{f}$. By the Plancherel formula (2.4.19),

$$\int_{\mathfrak{a}^*} |\hat{\mu}(\lambda)|^2 \omega(d\lambda) = \int_G |f(\sigma)|^2 d\sigma < \infty,$$

and so $\hat{\mu} \in L^2(\mathfrak{a}^*, \omega)$.

Conversely, let μ be a K-bi-invariant probability measure on G for which $\hat{\mu} \in L^2(\mathfrak{a}^*, \omega)$, and let f be given by (3.2.1). K-bi-invariance of f follows immediately from that of ϕ_λ. By Theorem 2.4.19, we must have $\hat{f} = \hat{\mu}$ and

$$\int_G |f(\sigma)|^2 dg = \int_{\mathfrak{a}^*} |\hat{\mu}(\lambda)|^2 \omega(d\lambda) < \infty.$$

Thus $f \in L^2(K|G|K)$.

It remains to show $\mu(d\sigma) = f(\sigma)d\sigma$. For $g \in L^2(K|G|K)$ and $\sigma \in G$, we have by Parseval's identity (2.4.20),

$$\begin{aligned}\int_G g(\sigma)\overline{f(\sigma)}d\sigma = \int_{\mathfrak{a}^*} \hat{g}(\lambda)\overline{\hat{\mu}(\lambda)}\omega(d\lambda) &= \int_{\mathfrak{a}^*} \hat{g}(\lambda) \int_G \overline{\phi_{-\lambda}(\sigma)}\mu(d\sigma)\omega(d\lambda) \\ &= \int_{\mathfrak{a}^*} \int_G \hat{g}(\lambda)\phi_\lambda(\sigma)\mu(d\sigma)\omega(d\lambda),\end{aligned}$$

where we have used Proposition 2.4.11 (1) in the last line. By Proposition 2.4.11 (2), $|\phi_\lambda| \leq 1$ for all $\lambda \in \mathfrak{a}^*$, and so we may use Fubini's theorem to exchange the order of integration. Hence,

$$\int_G g(\sigma)\overline{f(\sigma)}d\sigma = \int_G \int_{\mathfrak{a}^*} \hat{g}(\lambda)\phi_\lambda(\sigma)\omega(d\lambda)\mu(d\sigma),$$

and by spherical inversion (2.4.17),

$$\int_G g(\sigma)\overline{f(\sigma)}d\sigma = \int_G g(\sigma)\mu(d\sigma)$$

for all $g \in L^2(K|G|K)$. To extend this to all $g \in L^2(G)$, write

$$g^\natural(\sigma) = \int_K \int_K g(k_1\sigma k_2)dk_1dk_2, \qquad \forall \sigma \in G.$$

A straightforward check (see [57] Corollary 2.2.3, pp. 19) shows that map $g \mapsto g^\natural$ is orthogonal projection of $L^2(G)$ onto $L^2(K|G|K)$. By K-bi-invariance of μ and Fubini's theorem,

$$\begin{aligned}\int_G g^\natural(\sigma)\mu(d\sigma) = \int_G \int_K \int_K g(k_1\sigma k_2)dk_1dk_2\mu(d\sigma) &= \int_K \int_K \int_G g(k_1\sigma k_2)\mu(d\sigma)dk_1dk_2 \\ &= \int_K \int_K \int_G g(\sigma)\mu(d\sigma)dk_1dk_2 \\ &= \int_G g(\sigma)\mu(d\sigma)\end{aligned}$$

for all $g \in L^2(G)$. Similarly, K-bi-invariance of f implies

$$\int_G h^\natural(\sigma)\overline{f(\sigma)}d\sigma = \int_G g(\sigma)\overline{f(\sigma)}d\sigma,$$

for all $g \in L^2(G)$. Hence by the previous calculation,

$$\int_G g(\sigma)\overline{f(\sigma)}d\sigma = \int_G g(\sigma)\mu(d\sigma),$$

for all $g \in L^2(G)$. In particular, this holds for all $g \in C_c(G)$. By the Riesz representation theorem, f is real-valued, and μ is absolutely continuous with respect to Haar measure, with density $f \in L^2(K|G|K)$. By the Jordan decomposition theorem for signed measures, f is non-negative almost everywhere. □

Having proven Theorem 3.2.1, we immediately obtain a result about K-bi-invariant Lévy processes and their convolution semigroups.

Corollary 3.2.3. *Let Y be a K-bi-invariant Lévy process on G, with convolution semigroup $(\mu_t, t \geq 0)$ and Gangolli exponent ψ. The random variable $Y(t)$ has a square-integrable probability density function for each $t > 0$ if and only if $\hat{\mu}_t \in L^2(\mathfrak{a}^*, \omega)$ for all $t > 0$. If so, for each $t > 0$, the density is given by*

$$f_t = \int_{\mathfrak{a}^*} e^{-t\overline{\psi(\lambda)}}\phi_\lambda \omega(d\lambda), \qquad a.e.$$

Note that this formula is consistent with Theorem 4 of [55].

Example 3.2.4. The following are some standard examples of convolution semigroups with L^2 densities.

1. *Brownian motion.* Let $(\mu_t, t \geq 0)$ be the convolution semigroup associated with a standard Brownian motion on G/K, as discussed in Example 3.1.17 (1). We show that for all $t > 0$, $\hat{\mu}_t \in L^2(\mathfrak{a}^*, \omega)^W$. Weyl group invariance is immediate from the fact that each μ_t is K-bi-invariant. Corollary 3.2.3 may be applied to show that μ_t has a square-integrable density for all $t > 0$, by proving that $\hat{\mu}_t \in L^2(\mathfrak{a}^*, \omega)^W$ for all $t > 0$. By (3.1.9), for all $t \geq 0$ and $\lambda \in \mathfrak{a}^*$,
$$|\hat{\mu}_t(\lambda)|^2 = \left| e^{-t(|\rho|^2 + |\lambda|^2)} \right|.$$
By (2.4.18) and (2.4.21), Plancherel measure ω has density function $c|\,\mathbf{c})\lambda)|^{-2}$ with respect to Lebesgue measure on $\mathfrak{a}^*$, and for some $C > 0$,
$$c|\,\mathbf{c}(\lambda)|^{-2} \leq C(1 + |\lambda|^{\dim N}), \qquad \forall \lambda \in \mathfrak{a}^*. \tag{3.2.2}$$
Claim. For all $s \geq 0$ and $t > 0$,
$$\int_{\mathfrak{a}^*} e^{-t|\lambda|^2} |\lambda|^s d\lambda = \frac{\mathrm{Vol}(S^{d-1})}{2} t^{-\frac{s+d}{2}} \Gamma\left(\frac{s+d}{2}\right). \tag{3.2.3}$$
Proof of Claim. This is a fairly straightforward exercise in integration by substitution. First note that, using the spherical polar substitution $\lambda = r\theta$, where $r > 0$ and $\theta \in S^{d-1}$,
$$\int_{\mathfrak{a}^*} e^{-t|\lambda|^2} |\lambda|^s d\lambda = \mathrm{Vol}(S^{d-1}) \int_0^\infty e^{-tr^2} r^{s+d-1} dr. \tag{3.2.4}$$
If we then substitute $u = tr^2$, we have
$$\int_0^\infty e^{-tr^2} r^{s+d-1} dr = \frac{1}{2t} \int_0^\infty e^{-u} \left(\frac{u}{t}\right)^{\frac{s+d-2}{2}} du = \frac{1}{2} t^{-\frac{s+d}{2}} \int_0^\infty e^{-u} u^{\frac{s+d}{2}-1} du.$$
Since $\mathrm{Re}\left(\frac{s+d}{2}\right) > 0$, the integral on the far right-hand side is finite and equal to $\Gamma\left(\frac{s+d}{2}\right)$. Combining with (3.2.4), the claim is proved.

Using the claim and (3.2.2), for all $t > 0$,
$$\begin{aligned}\int_{\mathfrak{a}^*} |\hat{\mu}_t(\lambda|^2 \omega(d\lambda) &\leq C \int_{\mathfrak{a}^*} e^{-t(|\rho|^2 + |\lambda|^2)} \left(1 + |\lambda|^{\dim N}\right) d\lambda \\ &= C \frac{\mathrm{Vol}(S^{d-1})}{2} e^{-t|\rho|^2} \left\{ t^{-\frac{d}{2}} \Gamma\left(\frac{d}{2}\right) + t^{-\frac{\dim N + d}{2}} \Gamma\left(\frac{\dim N + d}{2}\right) \right\} < \infty.\end{aligned}$$
That is, $\hat{\mu}_t \in L^2(\mathfrak{a}^*, \omega)^W$ for all $t > 0$. By Corollary 3.2.3, μ_t posesses a square-integrable density for each $t > 0$, given by
$$h_t = \int_{\mathfrak{a}^*} e^{-\frac{t}{2}(|\rho|^2 + |\lambda|^2)} \phi_\lambda \omega(d\lambda), \qquad \text{a.e.}$$

This density is usually known as the heat kernel. Closed form expressions for the heat kernel on a symmetric space are rarely available. However, one can study $(h_t, t \geq 0)$ using Gaussian estimates, such as the following, proven by Anker and Ostellari in [2]:

$$h_t(e^H) \asymp t^{-d/2} \prod_{\lambda \in \Sigma_0^+} (1+\lambda(H))(1+t+\lambda(H))^{\frac{m_\lambda + m_{2\lambda}}{2}-1} e^{-|\rho|^2 t - \rho(H) - \frac{|H|^2}{4t}}, \qquad (3.2.5)$$

for all $H \in \mathfrak{a}^+$ and $t \geq 0$, where $\Sigma_0^+ = \{\lambda \in \Sigma^+ : \frac{\lambda}{2} \notin \Sigma^+\}$ denotes the set of positive, indivisible roots of (G, K).

2. *Stable-like processes.* Let $\alpha \in (0,2)$, and recall the definition of an α-stable-like process on G/K, from Example 3.1.17 (2). Let $(\mu_t^\alpha, t \geq 0)$ be the convolution semigroup associated with such a process. We show that $\hat{\mu}_t^\alpha \in L^2(\mathfrak{a}^*, \omega)$, for all $t > 0$, and hence has an L^2 density. By (3.1.12), $|\hat{\mu}_t^\alpha|^2 = e^{-2t(|\rho|^2+|\lambda|^2)^{\alpha/2}}$, and so

$$|\hat{\mu}_t^\alpha|^2 \leq e^{-2t|\lambda|^\alpha} \qquad \forall \lambda \in \mathfrak{a}^* .$$

Using an almost identical proof to that of (3.2.3), one can check that

$$\int_{\mathfrak{a}^*} e^{-t|\lambda|^\alpha} |\lambda|^s d\lambda = \frac{\mathrm{Vol}(S^{d-1})}{\alpha} t^{-\frac{s+d}{\alpha}} \Gamma\left(\frac{s+d}{\alpha}\right), \qquad \forall s \geq 0,\ t > 0.$$

Therefore, for C as in (3.2.2),

$$\begin{aligned}\int_{\mathfrak{a}^*} |\hat{\mu}_t^\alpha|^2 \,\omega(d\lambda) &\leq C \int_{\mathfrak{a}^*} e^{-2t|\lambda|^\alpha} \left(1 + |\lambda|^{\dim N}\right) d\lambda \\ &= C \frac{\mathrm{Vol}(S^{d-1})}{\alpha} \left\{(2t)^{-\frac{d}{\alpha}} \Gamma\left(\frac{d}{\alpha}\right) + (2t)^{-\frac{\dim N + d}{\alpha}} \Gamma\left(\frac{\dim N + d}{\alpha}\right)\right\} < \infty.\end{aligned}$$

That is, $\hat{\mu}_t^\alpha \in L^2(\mathfrak{a}^*, \omega)^W$, for all $t > 0$. By Corollary 3.2.3, μ_t^α posesses a square-integrable density h_t^α for each $t > 0$, given by

$$h_t^\alpha = \int_{\mathfrak{a}^*} e^{-t(|\rho|^2+|\lambda|^2)^{\alpha/2}} \phi_\lambda \omega(d\lambda), \qquad \text{a.e.}$$

In fact, one can show using subordination methods that

$$h_t^\alpha(\sigma) = \int_0^\infty h_s(\sigma) \theta_t(ds),$$

where $(\theta_t, t \geq 0)$ denotes the convolution semigroup associated with the $\frac{\alpha}{2}$-stable subordinator — see [6] Theorem 3.3.15, pp. 145.

We now use Theorem 3.2.1 to prove another result concerning square-integrable densities of K-bi-invariant convolution semigroups.

The following is a generalisation Theorem 4.1 of [5].

Theorem 3.2.5. *Let $(\mu_t, t \geq 0)$ be a K-bi-invariant convolution semigroup of probability measures, and suppose that the Gangolli exponent ψ takes the form*

$$\psi(\lambda) = -c(|\lambda|^2 + |\rho|^2) + \int_G (1 - \phi_\lambda(\sigma))\nu(d\sigma), \tag{3.2.6}$$

where $c \geq 0$ and ν is the Lévy measure of $(\mu_t, t \geq 0)$. Let ν_α be as in (3.1.11), and suppose that for some for some $\alpha \in (0, 2)$, ν dominates ν_α, in the sense that

$$\nu(U) \geq \nu_\alpha(U) \tag{3.2.7}$$

whenever $U \in \mathcal{B}(G)$ is bounded away from e (that is, $e \notin \overline{U}$). Then μ_t has a square-integrable density for each $t > 0$.

Remark 3.2.6. In the special case where G/K is irreducible with $\dim(G/K) > 1$, the diffusion part of the generator must be a non-negative multiple of Δ (see Liao [54] Proposition 5.6, pp. 140), and so every Gangolli exponent takes the form (3.2.6).

Proof. Proceed as in [5], and consider $\mathrm{Re}(\psi)$, where ψ is the Gangolli exponent of $(\mu_t, t \geq 0)$. Then ψ takes the form (3.2.6), and $\hat{\mu}_t = e^{-t\psi}$. Recall from Proposition 2.4.11 that

$$1 - \mathrm{Re}\{\phi_\lambda(\sigma)\} \geq 0, \tag{3.2.8}$$

for all $\lambda \in \mathfrak{a}^*$ and $\sigma \in G$. Let (U_n) be an increasing sequence of Borel subsets of G, each bounded away from e, and such that $U_n \uparrow G$ as $n \to \infty$. Then by (3.2.8), for all $n \in \mathbb{N}$,

$$\begin{aligned}\mathrm{Re}\{\psi(\lambda)\} &= c(|\lambda|^2 + |\rho|^2) + \int_{U_n} (1 - \mathrm{Re}\{\phi_\lambda(\sigma)\})\nu(d\sigma) + \int_{G\setminus U_n} (1 - \mathrm{Re}\{\phi_\lambda(\sigma)\})\nu(d\sigma) \\ &\geq \int_{U_n} (1 - \mathrm{Re}\{\phi_\lambda(\sigma)\})\nu(d\sigma).\end{aligned}$$

Hence by (3.2.7),

$$\mathrm{Re}\{\psi(\lambda)\} \geq \int_{U_n} (1 - \mathrm{Re}\{\phi_\lambda(\sigma)\})\nu_\alpha(d\sigma).$$

Taking the limit as $n \to \infty$, for all $\lambda \in \mathfrak{a}^*$,

$$\mathrm{Re}\{\psi(\lambda)\} \geq \int_G (1 - \mathrm{Re}\{\phi_\lambda(\sigma)\})\nu_\alpha(d\sigma) = \psi_\alpha(\lambda)$$

where ψ_α denotes the α-stable-like symbol, as in (3.1.10). It follows that for all $t \geq 0$,

$$|\hat{\mu}_t| = e^{-t\,\mathrm{Re}\{\psi\}} \leq e^{-t\psi_\alpha} = \hat{\mu}_t^\alpha. \tag{3.2.9}$$

In Example 3.2.4 (2), we proved that $\hat{\mu}_t^\alpha \in L^2(\mathfrak{a}^*, \omega)^W$ for all $t > 0$. Hence by (3.2.9), $\mu_t \in L^2(\mathfrak{a}^*, \omega)^W$ for all $t > 0$, and the result follows by Theorem 3.2.1. ☐

To conclude this subsection, we make the following observation that convolution operators on a symmetric space of noncompact type cannot be Hilbert–Schmidt.

Proposition 3.2.7. *Let μ be a K-bi-invariant probability measure on G, and let T_μ be the associated convolution operator on $L^2(G)$, so that*

$$T_\mu f(\sigma) = \int_G f(\sigma\tau)\mu(d\tau), \qquad \forall f \in L^2(G).$$

Then T_μ is not Hilbert–Schmidt.

Proof. Suppose T_μ is Hilbert–Schmidt. Then T_μ has an L^2 integral kernel (see [65] Theorem 3.8.5, pp. 157). That is, there is $k_\mu \in L^2(G \times G)$ such that

$$T_\mu f(\sigma) = \int_G f(\tau)k_\mu(\sigma,\tau)d\tau, \qquad \forall f \in L^2(G), \sigma \in G. \tag{3.2.10}$$

Claim. μ has a square-integrable density, $m \in L^2(K|G|K)$, given by $m(\tau) = k_\mu(e,\cdot)$.

Proof of Claim. Let $m = k_\mu(e,\cdot)$. That m belongs to $L^2(G)$ is clear, since $k_\mu \in L^2(G \times G)$. Moreover, if μ' is the Borel measure with density m, then for all $U \in \mathcal{B}(G)$,

$$\mu(U) = T_\mu \mathbf{1}_U(e) = \int_G \mathbf{1}_U(\sigma)k_\mu(e,\sigma)d\sigma = \int_G \mathbf{1}_U(\sigma)m(\sigma)d\sigma = \mu'(U).$$

Hence $\mu(d\sigma) = m(\sigma)d\sigma$. It remains to show that m is K-bi-invariant. For all $k_1, k_2 \in K$ and all $U \in \mathcal{B}(G)$,

$$\mu(k_1 U k_2) = \int_G \mathbf{1}_U(k_1^{-1} g k_2^{-1})m(\sigma)d\sigma = \int_G \mathbf{1}_U(\sigma)m(k_1\sigma k_2)d\sigma,$$

by unimodularity of G. But also,

$$\mu(k_1 U k_2) = \mu(U) = \int_G \mathbf{1}_U(\sigma)m(\sigma)d\sigma.$$

Since U was arbitrary, m must be K-bi-invariant. Thus $m \in L^2(K|G|K)$, and the claim is proved.

Observe that by translation invariance of Haar measure, for all $f \in L^2(G)$,

$$T_\mu f(\sigma) = \int_G f(\sigma\tau)m(\tau)d\tau = \int_G f(\tau)m(\sigma^{-1}\tau)d\tau.$$

Comparing with (3.2.10), it follows by the Riesz representation theorem that $m(\sigma\tau) = k_\mu(\sigma,\tau)$ for almost all $\sigma, \tau \in G$. In particular, the mapping $(\sigma,\tau) \mapsto m(\sigma^{-1}\tau)$ belongs to $L^2(G \times G)$. That is,

$$\int_G \int_G m(\sigma^{-1}\tau)^2 d\tau d\sigma < \infty.$$

But using translation invariance once again, we have that for all $\sigma \in G$,

$$\int_G m(\sigma^{-1}\tau)^2 d\tau = \int_G m(\tau)^2 d\tau = \|m\|^2.$$

Therefore,

$$\int_G \int_G m(\sigma^{-1}\tau)^2 d\tau d\sigma = \|m\|^2 \int_G d\sigma$$

which cannot be finite, since G is noncompact. Hence we have reached a contradiction, and T_μ cannot be Hilbert–Schmidt. □

Remarks 3.2.8.

1. Since every trace-class operator is Hilbert–Schmidt, T_μ cannot be trace-class either.

2. It is interesting to compare Proposition 3.2.7 with [8] Theorem 4.7.1 on page 111, in which it is proven that a convolution operator T_μ on a compact Lie group is Hilbert–Schmidt if and only if μ has a square-integrable density. The proof of the claim in Proposition 3.2.7 follows a similar argument to the "only if" direction of [8] Theorem 4.7.1.

3. Replacing μ with μ_t, for some K-bi-invariant convolution semigroup $(\mu_t, t \geq 0)$, Proposition 3.2.7 says that the corresponding Hunt semigroup $(T_t, t \geq 0)$ cannot be Hilbert–Schmidt; indeed, T_t cannot be Hilbert–Schmidt for any $t \geq 0$.

§ 3.3 The spectrum of a self-adjoint Lévy generator

In this last section of Chapter 3, we present another new result, Theorem 3.3.1, connecting the spectrum of a self-adjoint Lévy generator to the range of its symbol. It is a symmetric space generalisation of a known result in $\mathbb{R}^d$ — see Applebaum [9] Theorem 3.2.

Let G/K be a symmetric space of noncompact type, and $(\mu_t, t \geq 0)$ a K-bi-invariant convolution semigroup of probability measures on G. Suppose that $(\mu_t, t \geq 0)$ is *symmetric*, in the sense that $\mu_t\left(B^{-1}\right) = \mu_t(B)$ for all $t \geq 0$. Then by [8] Theorem 5.4.1, pp. 140, the Lévy generator $\mathcal{A}$ associated with $(\mu_t, t \geq 0)$ is a self-adjoint linear operator on $L^2(K|G|K)$, and takes the form

$$\mathcal{A}f(\sigma) = \sum_{i,j=1}^{d} a_{ij} X_i X_j f(\sigma) + \frac{1}{2}\int_G (f(\sigma\tau) - 2f(\sigma) + f(\sigma\tau^{-1}))\nu(d\tau), \qquad \forall f \in C_c^2(K|G|K),$$

where $a = (a_{ij})$ is an $\mathrm{Ad}(K)$-invariant, non-negative definite, symmetric $d \times d$ matrix, and ν is a K-bi-invariant symmetric Lévy measure. Note also that the Gangolli exponent ψ associated with $(\mu_t, t \geq 0)$ must be real-valued — indeed, by (3.1.8) and Parseval's identity (2.4.20), for all $f, g \in L^2(K|G|K)$,

$$\langle \mathcal{A}f, g\rangle = \left\langle (\mathcal{A}f)^\wedge, \hat{g}\right\rangle_{L^2(\mathfrak{a}^*,\omega)^W} = -\int_{\mathfrak{a}^*} \psi(\lambda)\hat{f}(\lambda)\overline{\hat{g}(\lambda)}\omega(d\lambda),$$

while

$$\langle f, \mathcal{A}g\rangle = \left\langle \hat{f}, (\mathcal{A}g)^\wedge \right\rangle_{L^2(\mathfrak{a}^*,\omega)^W} = -\int_{\mathfrak{a}^*} \overline{\psi(\lambda)}\hat{f}(\lambda)\overline{\hat{g}(\lambda)}\omega(d\lambda).$$

By self-adjointness, $\psi = \overline{\psi}$.

Let $\beta_a(\lambda)$ again denote the ϕ_λ-eigenvalue of $\sum_{i,j=1}^d a_{ij} X_i X_j$, as in (3.1.5). Then

$$\psi(\lambda) = \beta_a(\lambda) + \int_G \big(\mathrm{Re}(1 - \phi_\lambda(\sigma))\big)\nu(d\sigma), \qquad \forall \lambda \in \mathfrak{a}^* .$$

By an argument analogous to that in Theorem 5.3.4 of [8], $C_c^\infty(K|G|K)$ is a core for $\mathcal{A}$. Using this, one can show that (3.1.8) holds on the entire domain of $\mathcal{A}$. This together with Theorem 2.4.19 implies that the $\mathrm{Dom}\,\mathcal{A}$ is the anisotropic Sobolev space $H^{\psi,2}$, to be introduced in Section 4.1.2.

The following result is a generalisation of [9] Theorem 3.2.

Theorem 3.3.1. $\mathrm{Spec}(\mathcal{A}) = \overline{\mathrm{Ran}(-\psi)}$.

Proof. First observe that by taking spherical transforms, an element $\alpha \in \mathbb{C}$ belongs to the resolvent set $\rho(\mathcal{A})$ (c.f. equation (0.0.1)) if and only if the following condition holds:

$$\forall f \in L^2(K|G|K), \exists u \in H^{\psi,2} \text{ s.t. } (\alpha + \psi)\hat{u} = \hat{f}. \tag{3.3.1}$$

To prove that $\overline{\mathrm{Ran}(-\psi)} \subseteq \mathrm{Spec}(\mathcal{A})$, let $\alpha \in \mathrm{Ran}(-\psi)$, and choose $\lambda \in \mathfrak{a}^*$ such that $-\psi(\lambda) = \alpha$. If $\alpha \notin \mathrm{Spec}(\mathcal{A})$, then (3.3.1) holds. Evaluating at λ, we have $\hat{f}(\lambda) = 0$ for all $f \in L^2(K|G|K)$. But since the spherical transform defines an isomorphism of $L^2(K|G|K)$ with $L^2(\mathfrak{a}^*, \omega)^W$, this implies that every element of $L^2(\mathfrak{a}^*, \omega)^W$ vanishes at λ, which is false. It follows that $\mathrm{Ran}(-\psi) \subseteq \mathrm{Spec}(\mathcal{A})$. Taking closures, we get that $\overline{\mathrm{Ran}(-\psi)} \subseteq \mathrm{Spec}(\mathcal{A})$.

We show that $\mathrm{Spec}(\mathcal{A}) \subseteq \overline{\mathrm{Ran}(-\psi)}$ by showing that $\mathbb{C} \setminus \overline{\mathrm{Ran}(-\psi)} \subseteq \rho(\mathcal{A})$. Since $-\mathcal{A}$ is positive and self-adjoint, $\mathrm{Spec}(-\mathcal{A}) \subseteq [0, \infty)$, and so $\mathrm{Spec}(\mathcal{A}) \subseteq (-\infty, 0]$. Therefore $\mathbb{C} \setminus (-\infty, 0] \subseteq \rho(\mathcal{A})$. Also, by the first part of the proof, $\overline{\mathrm{Ran}(-\psi)} \subseteq (-\infty, 0]$, and hence it is sufficient to prove that $(-\infty, 0] \setminus \overline{\mathrm{Ran}(-\psi)} \subseteq \rho(\mathcal{A})$. Let $\alpha \in (-\infty, 0] \setminus \overline{\mathrm{Ran}(-\psi)}$. There are two cases to consider:

- Case 1: $\overline{\mathrm{Ran}(-\psi)} = (-\infty, 0]$. Then $(-\infty, 0] \subseteq \mathrm{Spec}(\mathcal{A})$ by the $(\supseteq)$ direction. We have already noted that $\mathrm{Spec}(\mathcal{A}) \subseteq (-\infty, 0]$, and so we are done.
- Case 2: $\overline{\mathrm{Ran}(-\psi)} = (-K, 0]$ for some $K \geq 0$. Then $\alpha \in (-\infty, -K)$. Choose $\epsilon > 0$ such that $\alpha \in (-\infty, -K - \epsilon)$. Then for all $\lambda \in \mathfrak{a}^*$,

$$\frac{1}{|\alpha + \psi(\lambda)|} \leq \frac{1}{\epsilon},$$

and so $\frac{1}{\alpha + \psi}$ is a bounded map. Given $f \in L^2(K|G|K)$, we may define u to be the inverse spherical transform of $\frac{\hat{f}}{\alpha + \psi}$ and see that (3.3.1) is valid. Thus $\alpha \in \rho(\mathcal{A})$.

□

Chapter 4

Pseudodifferential operators that generate sub-Feller semigroups

A strongly continuous semigroup $(T_t, t \geq 0)$ defined on $C_0(E)$, for some locally compact Hausdorff space E, is called *sub-Feller* if for all $f \in C_0(E)$, and all $t \geq 0$,

$$0 \leq f \leq 1 \;\Rightarrow\; 0 \leq T_t f \leq 1.$$

Such semigroups naturally arise as transition semigroups of sub-Feller processes on E. The Hille–Yosida–Ray theorem (see Theorem 4.2.1) gives necessary and sufficient conditions for a densely defined linear operator $\mathcal{A}$ to be the generator of a sub-Feller semigroup. In Chapter 4 of his Habilitationsschrift [41], Walter Hoh uses this theorem to find a class of pseudodifferential operators on $\mathbb{R}^n$ that generate sub-Feller semigroups. This work built on a range of papers by Niels Jacob, see for example [44], [45]. In this chapter, we seek to generalise Hoh and Jacob's approach to the setting of symmetric spaces of noncompact type.

To do this, we first require a notion of a pseudodifferential operator on G/K; this will be the focus of Section 4.1. In Section 4.2, we introduce the Hille–Yosida–Ray theorem, and state a theorem due to Applebaum and Ngan [16] giving necessary and sufficient conditions for an operator on a symmetric space to satisfy the positive maximum principle. We prove that operators satisfying the criteria of this theorem are psudodifferential operators in the sense we have introduced, obtaining as a result a large class of examples of pseudodifferential operators on symmetric spaces of noncompact type. Section 4.3 is concerned with seeking sufficient conditions for a psudodifferential operator $q(\sigma, D)$ to extend to the generator of a sub-Feller semgrioup of operators on $C_0(K|G|K)$. Informed by the work of the previous section, as well as the Hille–Yosida–Ray theorem, we will see that this amounts to finding conditions that ensure

$$\overline{\mathrm{Ran}(\alpha I + q(\sigma, D))} = C_0(K|G|K)$$

for some $\alpha > 0$ (see Theorem 4.2.1 (3)).

§ 4.1 Operators and symbols

Let G/K be a symmetric space of noncompact type. In this section, we develop a theory of pseudodifferential operators (ΨDOs) on G/K. In keeping with previous conventions, we will view these operators as K-bi-invariant operators on G. In the compact setting, Ruzhansky and Turunen [61] have developed a theory of pseudodifferential operators on compact Lie groups. Applebaum and Ngan [15, 16] applied this theory to consider certain classes of pseudodifferential operators on compact Lie groups and symmetric spaces, as part of their study of the positive maximum principle. In the noncompact setting, we adopt a similar approach, and study densely defined operators $q(\sigma, D)$ on $L^2(K|G|K)$ of the form

$$q(\sigma, D)f(\sigma) = \int_{\mathfrak{a}^*} q(\sigma, \lambda)\phi_\lambda(\sigma)\hat{f}(\lambda)\omega(d\lambda), \qquad \forall f \in C_c^\infty(K|G|K),\ \sigma \in G,$$

where $q : G \times \mathfrak{a}^* \to \mathbb{C}$ belongs to some suitable class of symbols, to be defined rigorously in due course. Before we can do this, we require some results from the theory of positive and negative definite functions.

4.1.1 Positive and negative definite functions

By viewing $\mathfrak{a}^*$ as a finite-dimensional real vector space, we may consider positive and negative definite functions on $\mathfrak{a}^*$.

Definition 4.1.1. A mapping $\psi : \mathfrak{a}^* \to \mathbb{C}$ is *positive definite* if for all $n \in \mathbb{N}$, $\lambda_1 \ldots, \lambda_n \in \mathfrak{a}^*$, and $c_1, \ldots, c_n \in \mathbb{C}$,

$$\sum_{i,j=1}^{n} \psi(\lambda_i - \lambda_j)c_i\overline{c_j} \geq 0,$$

and *negative definite* if for all $n \in \mathbb{N}$, $\lambda_1 \ldots, \lambda_n \in \mathfrak{a}^*$, and $c_1, \ldots, c_n \in \mathbb{C}$,

$$\sum_{i,j=1}^{n} (\psi(\lambda_i) + \overline{\psi(\lambda_j)} - \psi(\lambda_i - \lambda_j))c_i\overline{c_j} \geq 0. \tag{4.1.1}$$

Remarks 4.1.2.

1. Some authors use the term conditionally negative definite where we use negative definite.
2. One can check that (4.1.1) is equivalent to the condition that

$$\sum_{i,j=1}^{n} \psi(\lambda_i - \lambda_j)c_i\overline{c_j} \leq 0 \quad \text{whenever} \quad \sum_{i=1}^{n} c_i = 0 \tag{4.1.2}$$

(hint: for $n \geq 2$, replace c_n with $-c_1 - \ldots - c_{n-1}$ in (4.1.2)).

We have already seen in Section 2.4.2 that ϕ_λ is positive definite as a function on G. In fact, $(\lambda, \sigma) \mapsto \phi_\lambda(\sigma)$ is positive definite in both of its arguments.

Proposition 4.1.3. *1. For all $\sigma \in G$, $\lambda \mapsto \phi_\lambda(\sigma)$ is positive definite.*

2. Let μ be a finite K-bi-invariant Borel measure. Then $\hat{\mu}$ is positive definite.

Proof. Let $\sigma \in G$, $n \in \mathbb{N}$, $\lambda_1, \ldots, \lambda_n \in \mathfrak{a}^*$, and $c_1, \ldots, c_n \in \mathbb{C}$, and note that

$$\sum_{\alpha,\beta=1}^{n} c_\alpha \overline{c_\beta} e^{(i(\lambda_\alpha - \lambda_\beta)+\rho)A(k\sigma)} = \left| \sum_{\alpha=1}^{n} c_\alpha e^{(i\lambda_\alpha + \frac{\rho}{2})A(k\sigma)} \right|^2 \geq 0.$$

Therefore, by the Harish-Chandra integral formula (2.4.10),

$$\sum_{\alpha,\beta=1}^{n} c_\alpha \overline{c_\beta} \phi_{\lambda_\alpha - \lambda_\beta}(\sigma) = \int_K \sum_{\alpha,\beta=1}^{n} c_\alpha \overline{c_\beta} e^{(i(\lambda_\alpha - \lambda_\beta)+\rho)A(k\sigma)} dk \geq 0. \tag{4.1.3}$$

Proposition 4.1.3 (1) follows.

For Proposition 4.1.3 (2), observe that since (4.1.3) holds for all $c_1, \ldots, c_n \in \mathbb{C}$, we can replace each c_j by its complex conjugate. Therefore, $\sum_{\alpha,\beta=1}^{n} \overline{c_\alpha} c_\beta \phi_{\lambda_\alpha - \lambda_\beta}(\sigma) \geq 0$ for all $\sigma \in G$, $n \in \mathbb{N}$, $\lambda_1, \ldots, \lambda_n \in \mathfrak{a}^*$, and $c_1, \ldots, c_n \in \mathbb{C}$. Taking complex conjugates,

$$\sum_{\alpha,\beta=1}^{n} c_\alpha \overline{c_\beta} \phi_{-(\lambda_\alpha - \lambda_\beta)}(\sigma) = \overline{\sum_{\alpha,\beta=1}^{n} \overline{c_\alpha} c_\beta \phi_{\lambda_\alpha - \lambda_\beta}} \geq 0,$$

for all $\sigma \in G$, $n \in \mathbb{N}$, $\lambda_1, \ldots, \lambda_n \in \mathfrak{a}^*$, and $c_1, \ldots, c_n \in \mathbb{C}$, and hence

$$\sum_{\alpha,\beta=1}^{n} c_\alpha \overline{c_\beta} \hat{\mu}(\lambda_\alpha - \lambda_\beta) = \int_{\mathfrak{a}^*} \sum_{\alpha,\beta=1}^{n} c_\alpha \overline{c_\beta} \phi_{-(\lambda_\alpha - \lambda_\beta)}(\sigma) \mu(d\sigma) \geq 0.$$

□

By choosing a basis of $\mathfrak{a}^*$, we may identify it with $\mathbb{R}^m$, and apply classical results about positive (resp. negative) definite functions on Euclidean space to functions on $\mathfrak{a}^*$, to obtain results about positive (resp. negative) definite functions in this new setting.

One useful application of this is to the Schoenberg correspondence: a map $\psi : \mathfrak{a}^* \to \mathbb{C}$ is negative definite if and only if $\psi(0) \geq 0$ and $e^{-t\psi}$ is positive definite for all $t > 0$. This is immediate by the Schoenberg correspondence on $\mathbb{R}^m$ — see [18] page 41 for a proof.

Proposition 4.1.4. *Let $\psi : \mathfrak{a}^* \to \mathbb{C}$ be the Gangolli exponent of a K-bi-invariant convolution semigroup on G. Then ψ is negative definite.*

Proof. Let $(\mu_t, t \geq 0)$ be a K-bi-invariant convolution semigroup on G with Gangolli exponent ψ. By Proposition 4.1.3, for each $t \geq 0$, $\hat{\mu}_t$ is positive definite. Therefore, by the Schoenberg correspondence, for each $t \geq 0$, there is a negative definite function $\psi_t : \mathfrak{a}^* \to \mathbb{C}$ such that $\psi_t(0) \geq 0$ and $\hat{\mu}_t = e^{-\psi_t}$. But since $(\mu_t, t \geq 0)$ is a convolution semigroup,

$$\hat{\mu}_t = e^{-t\psi_1}, \qquad \forall t \geq 0.$$

By uniqueness of Gangolli exponents, $\psi = \psi_1$, which is a negative definite function. □

We finish this subsection with a collection of results about negative definite functions, which will be useful in later sections.

Proposition 4.1.5. *Let $\psi : \mathfrak{a}^* \to \mathbb{C}$ be a continuous negative definite function. Then*

1. *For all $\lambda, \eta \in \mathfrak{a}^*$,*
$$\left| \sqrt{|\psi(\lambda)|} - \sqrt{|\psi(\eta)|} \right| \leq \sqrt{|\psi(\lambda - \eta)|}$$

2. *(Generalised Peetre inequality) For all $s \in \mathbb{R}$ and $\lambda, \eta \in \mathfrak{a}^*$,*
$$\left(\frac{1 + |\psi(\lambda)|}{1 + |\psi(\eta)|} \right)^s \leq 2^{|s|}(1 + |\psi(\lambda - \eta)|)^{|s|}.$$

3. *There is a constant $c_\psi > 0$ such that*
$$|\psi(\lambda)| \leq c_\psi(1 + |\lambda|^2) \qquad \forall \lambda \in \mathfrak{a}^* \, . \tag{4.1.4}$$

Proof. These results follow from their Euclidean analogues (see [41] pp. 16), by identifying $\mathfrak{a}^*$ with $\mathbb{R}^m$ through a choice of basis. □

4.1.2 Spherical anisotropic Sobolev spaces

First introduced by Niels Jacob (see for example [45]), and developed further by Hoh in [41], anisotropic Sobolev spaces are used in the Euclidean space setting to consider certain regularity properties for Lévy-type operators and generators of Feller processes (Lévy-type operators in this case are analogous to Gangolli operators on G/K). In this section, we define anisotropic Sobolev spaces on symmetric spaces of noncompact type, and prove results that will be useful in Section 4.3.

Suppose ψ is a real-valued continuous negative definite function, and let $s \in \mathbb{R}$. We define the *(spherical) anisotropic Sobolev space* associated with ψ and s to be

$$H^{\psi,s} := \left\{ u \in \mathcal{S}'(K|G|K) : \int_G (1 + \psi(\lambda))^s |\hat{u}(\lambda)|^2 \omega(d\lambda) < \infty \right\},$$

where $\mathcal{S}'(K|G|K)$ denotes the space of K-bi-invariant tempered distributions on G, defined in §2.4.3.

One can check that each $H^{\psi,s}$ is a Hilbert space with respect to the inner product

$$\langle u, v \rangle_{\psi,s} := \int_{\mathfrak{a}^*} (1 + \psi(\lambda))^s \hat{u}(\lambda) \overline{\hat{v}(\lambda)} \omega(d\lambda), \qquad \forall u, v \in H^{\psi,s}.$$

These spaces are a generalisation of the anisotropic Sobolev spaces first introduced by Niels Jacob, see [44], and developed further by Hoh, see [41]. For the special case $\psi(\lambda) = |\rho|^2 + |\lambda|^2$, we will write $H^{\psi,s} = H^s$. Note also that $H^{\psi,0} = L^2(K|G|K)$, by the Plancherel theorem. In this case, we will omit subscripts and just write $\langle \cdot, \cdot \rangle$ for the L^2 inner product.

Note that ψ is a non-negative function, since it is negative definite and real-valued. We impose an additional assumption, namely that there exist constants $r, c > 0$ such that

$$\psi(\lambda) \geq c|\lambda|^{2r} \qquad \forall \lambda \in \mathfrak{a}^*, \ |\lambda| \geq 1. \tag{4.1.5}$$

Analogous assumptions are made in [45] (1.5) and [41] (4.2), and the role of (4.1.5) will be similar to its role therein.

Theorem 4.1.6. *Let ψ be a real-valued, continuous negative definite symbol, satisfying (4.1.5). Then*

1. *$C_c^\infty(K|G|K)$ and $\mathcal{S}(K|G|K)$ are dense in each $H^{\psi,s}$, and we have continuous embeddings*

$$\mathcal{S}(K|G|K) \hookrightarrow H^{\psi,s} \hookrightarrow \mathcal{S}'(K|G|K)$$

2. *We have continuous embeddings*

$$H^{\psi,s_2} \hookrightarrow H^{\psi,s_1}$$

whenever $s_1, s_2 \in \mathbb{R}$ with $s_2 \geq s_1$. In particular, $H^{\psi,s} \hookrightarrow L^2(K|G|K)$ for all $s \geq 0$.

3. *Under the standard identification of $L^2(K|G|K)$ with its dual, the dual space of each $H^{\psi,s}$ is isomorphic to $H^{\psi,-s}$, with*

$$\|u\|_{\psi,-s} = \sup\left\{\frac{|\langle u, v\rangle|}{\|v\|_{\psi,s}} : v \in C_c^\infty(K|G|K), \ v \neq 0\right\}, \tag{4.1.6}$$

for all $s \in \mathbb{R}$.

4. *For $r > 0$ as in equation (4.1.5), we have continuous embeddings*

$$H^s \hookrightarrow H^{\psi,s} \hookrightarrow H^{rs},$$

for all $s \geq 0$.

5. *Let $s_3 > s_2 > s_1$. Then for all $\epsilon > 0$, there is $c(\epsilon) \geq 0$ such that*

$$\|u\|_{\psi,s_2} \leq \epsilon \|u\|_{\psi,s_3} + c(\epsilon)\|u\|_{\psi,s_1} \tag{4.1.7}$$

for all $u \in H^{\psi,s_3}$.

6. *There exist continuous embeddings*

$$H^{\psi,s} \hookrightarrow C_0(K|G|K)$$

for all $s > \frac{d}{r}$, where $d = \dim(G/K)$.

For brevity, let

$$\langle\lambda\rangle := \sqrt{1+|\lambda|^2}, \qquad \forall\lambda\in\mathfrak{a}^*, \tag{4.1.8}$$

and

$$\Psi(\lambda) := \sqrt{1+\psi(\lambda)}, \qquad \forall\lambda\in\mathfrak{a}^*\,. \tag{4.1.9}$$

The proof of Theorem 4.1.6 will be given after the next lemma.

Lemma 4.1.7. *Let $M > d := \dim(G/K)$. Then $\langle\cdot\rangle^{-M} \in L^1(\mathfrak{a}^*,\omega)$.*

Proof. First observe that $\int_{\mathbb{R}^d}\langle\xi\rangle^{-M}d\xi < \infty$, for all $M > d$. Indeed, for $R > 0$, then using spherical polar coordinates $(r,\theta)\in[0,\infty)\times S^{d-1}$,

$$\begin{aligned}
\int_{B_R(0)}\langle\xi\rangle^{-M}d\xi = V_{d-1}\int_0^R \frac{r^{d-1}}{(1+r^2)^{M/2}}dr &= V_{d-1}\int_0^R \frac{(r^2)^{(d-1)/2}}{(1+r^2)^{M/2}}dr \\
&\le V_{d-1}\int_0^R \frac{(1+r^2)^{(d-1)/2}}{(1+r^2)^{M/2}}dr \\
&= V_{d-1}\int_0^R \left(\frac{1}{\sqrt{1+r^2}}\right)^{M-d+1} dr,
\end{aligned}$$

where V^{d-1} denotes the volume of the unit $(d-1)$-ball in $\mathbb{R}^d$. Assuming $R > 1$ and $M > d$, we then get

$$\begin{aligned}
\int_{B_R(0)}\langle\xi\rangle^{-M}d\xi &\le V_{d-1}\left(\int_0^1\left(\frac{1}{\sqrt{1+r^2}}\right)^{M-d+1} dr + \int_1^R\left(\frac{1}{\sqrt{1+r^2}}\right)^{M-d+1} dr\right) \\
&\le V_{d-1}\left(\int_0^1 dr + \int_1^R\left(\frac{1}{\sqrt{1-2r+r^2}}\right)^{M-d+1} dr\right) \\
&= V_{d-1}\left(1+\int_1^R (r-1)^{-(M-d+1)}dr\right) \\
&= V_{d-1}\left(1-\frac{(R-1)^{-(M-d)}}{M-d}\right)
\end{aligned} \tag{4.1.10}$$

Since $M > d$, the expression on the last line of (4.1.10) converges as $R\to\infty$, and thus $\int_{\mathbb{R}^d}\langle\xi\rangle^{-M}d\xi < \infty$.

Next, writing $p = \frac{\dim N}{2}$, we have $d = \dim\mathfrak{a}^* + 2p$, and hence $\int_{\mathfrak{a}^*}\langle\lambda\rangle^{-M+2p}d\lambda < \infty$ whenever $M > d$. Noting (2.4.21), we may choose $C > 0$ such that

$$|\,\mathbf{c}(\lambda)|^{-1} \le C(1+|\lambda|^2)^p \qquad \forall\lambda\in\mathfrak{a}^*\,.$$

Then

$$\int_{\mathfrak{a}^*}\langle\lambda\rangle^{-M}\omega(d\lambda) = \int_{\mathfrak{a}^*}\langle\lambda\rangle^{-M}|\,\mathbf{c}(\lambda)|^{-2}d\lambda \le C\int_{\mathfrak{a}^*}\langle\lambda\rangle^{-M+2p}d\lambda < \infty,$$

whenever $M > d$. □

Proof of Theorem 4.1.6. Much of this theorem may be proved by adapting proofs from the $\mathbb{R}^d$ case.

1. Following the proof of Theorem 3.10.3 on page 208 of Jacob [46], let $\mathcal{L}^{\psi,s}$ denote the space of all measurable functions v on $\mathfrak{a}^*$ for which $\Psi^s v \in L^2(\mathfrak{a}^*,\omega)^W$. Each $\mathcal{L}^{\psi,s}$ is a Hilbert space, with inner product given by

$$\langle u,v\rangle = \int_{\mathfrak{a}^*} \Psi(\lambda)^{2s} u(\lambda)\overline{v(\lambda)}\omega(d\lambda), \qquad \forall u,v \in \mathcal{L}^{\psi,s}.$$

By Proposition 4.1.5 (3), there is a constant $c > 0$ such that

$$\Psi(\lambda)^s \leq c\langle\lambda\rangle^{|s|} \qquad \forall \lambda \in \mathfrak{a}^*\,. \tag{4.1.11}$$

It follows that $\mathcal{S}(\mathfrak{a}^*)^W \hookrightarrow \mathcal{L}^{\psi,s}$ in the sense of continuous embeddings. Indeed, that $\mathcal{S}(\mathfrak{a}^*)^W \subseteq \mathcal{L}^{\psi,s}$ is clear by (4.1.11), and if (v_n) is a sequence in $\mathcal{S}(\mathfrak{a}^*)^W$ converging to $v \in \mathcal{S}(\mathfrak{a}^*)^W$, then

$$\begin{aligned}\|v_n - v\|_{\mathcal{L}^{\psi,s}} = \|\Psi^s(v_n - v)\|_{L^2(\mathfrak{a}^*,\omega)} &\leq c\|\langle\cdot\rangle^{|s|}(v_n - v)\|_{L^2(\mathfrak{a}^*,\omega)} \\ &= c\int_{\mathfrak{a}^*}\langle\lambda\rangle^{2s}|v_n(\lambda) - v(\lambda)|\omega(d\lambda),\end{aligned}$$

for all $n \in \mathbb{N}$. But then by Proposition 2.4.20, writing $p = \frac{\dim N}{2}$,

$$\begin{aligned}\|v_n - v\|_{\mathcal{L}^{\psi,s}} &\leq cc_0\int_{\mathfrak{a}^*}\langle\lambda\rangle^{2s}(C_1 + C_2|\lambda|^p)|v_n(\lambda) - v(\lambda)|d\lambda \\ &\leq c'\int_{\mathfrak{a}^*}\langle\lambda\rangle^{2s+p}|v_n(\lambda) - v(\lambda)|d\lambda \\ &= c'\int_{\mathfrak{a}^*}\langle\lambda\rangle^{-(d+1)}\cdot\langle\lambda\rangle^{2s+p+d+1}|v_n(\lambda) - v(\lambda)|d\lambda,\end{aligned}$$

for some constant $c' > 0$. Now, $\langle\cdot\rangle^{-(d+1)} \in L^1(\mathfrak{a}^*,\omega)$ by Lemma 4.1.7, and

$$\sup_{\lambda\in\mathfrak{a}^*}\langle\lambda\rangle^{2s+p+d+1}|v_n(\lambda) - v(\lambda)|$$

is finite and tends to 0 as $n \to \infty$, since $v_n \to v$ in the Schwartz space topology. Thus

$$\|v_n - v\|_{\mathcal{L}^{\psi,s}} \leq c'\left\|\langle\cdot\rangle^{-(d+1)}\right\|_{L^1(\mathfrak{a}^*,\omega)}\sup_{\lambda\in\mathfrak{a}^*}\langle\lambda\rangle^{2s+p+d+1}|v_n(\lambda) - v(\lambda)|d\lambda \to 0$$

as $n \to \infty$. That is, $v_n \to v$ as a sequence in $\mathcal{L}^{\psi,s}$.

To prove $\mathcal{L}^{\psi,s} \hookrightarrow \mathcal{S}'(\mathfrak{a}^*)^W$, observe that by Cauchy–Schwarz, for all $\phi \in \mathcal{S}(\mathfrak{a}^*)^W$ and $v \in \mathcal{L}^{\psi,s}$,

$$\left|\int_{\mathfrak{a}^*}\phi(\lambda)v(\lambda)\omega(d\lambda)\right| \leq \left\|\Psi^{-s}\phi\right\|_{L^2(\mathfrak{a}^*,\omega)}\|\Psi^s v\|_{L^2(\mathfrak{a}^*,\omega)} \leq c\left\|\langle\cdot\rangle^{|s|}\phi\right\|_{L^2(\mathfrak{a}^*,\omega)}\|v\|_{\mathcal{L}^{\psi,s}}.$$

It follows that

$$\mathcal{S}(\mathfrak{a}^*)^W \hookrightarrow \mathcal{L}^{\psi,s} \hookrightarrow \mathcal{S}'(\mathfrak{a}^*)^W$$

in the sense of continuous embeddings. The spherical transform maps $\mathcal{S}(K|G|K)$ bijectively onto $\mathcal{S}(\mathfrak{a}^*)^W$, and $\mathcal{S}'(K|G|K)$ bijectively onto $\mathcal{S}'(\mathfrak{a}^*)^W$. Theorem 4.1.6 (1) follows.

2. Suppose $s_2 \geq s_1$. Then $\Psi^{s_2} \geq \Psi^{s_1}$, since $\psi \geq 0$. Therefore, for all $u \in C_c^\infty(K|G|K)$,

$$\|u\|_{\psi,s_1} = \|\Psi^{s_1}\hat{u}\| \leq \|\Psi^{s_2}\hat{u}\| = \|u\|_{\psi,s_2}. \tag{4.1.12}$$

By density of $C_c^\infty(K|G|K)$ in each of the spaces H^{ψ,s_1} and H^{ψ,s_2}, it follows that $H^{\psi,s_2} \subseteq H^{\psi,s_1}$, and (4.1.12) is valid for all $u \in H^{\psi,s_2}$.

3. By symmetry, it is sufficient to consider $s \geq 0$, so that $H^{\psi,s} \subseteq L^2(K|G|K)$. Under the standard identification of $L^2(K|G|K)$ with its dual, each $u \in H^{\psi,s}$ is identified with the functional F_u defined by $F_u(f) = \langle u, f\rangle$. Restriction of F_u to $H^{\psi,s}$ yields an element of $(H^{\psi,s})'$ if and only if $u \in H^{\psi,-s}$, since

$$F_u(f) = \langle u, f\rangle = \langle \Psi^{-s}u, \Psi^s f\rangle_{L^2(\mathfrak{a}^*,\omega)^W}.$$

Equation 4.1.6 is immediate from the definition of the operator norm.

4. Let $s \geq 0$, and observe that by (4.1.5) and Proposition 4.1.5 (3), there are constants $c_1, c_2 > 0$ such that

$$c_1\langle\lambda\rangle^{rs} \leq \Psi(\lambda)^s \leq c_2\langle\lambda\rangle^s,$$

for all $\lambda \in \mathfrak{a}^*$. Hence $H^s \subseteq H^{\psi,s} \subseteq H^{rs}$, with

$$c_1\|u\|_{rs} \leq \|u\|_{\psi,s} \leq c_2\|u\|_s$$

for all $u \in H^s$.

5. To prove (4.1.7), let $\epsilon > 0$ and $s_3 > s_2 > s_1$. Observe that

$$\Psi(\lambda)^{s_2-s_1} - \epsilon\Psi(\lambda)^{s_3-s_1} \to -\infty$$

as $|\lambda| \to \infty$. Indeed, for $|\lambda|$ large, $-\epsilon\psi(\lambda)^{(s_3-s_1)/2}$ dominates this expression, and by (4.1.5),

$$-\epsilon\psi(\lambda)^{(s_3-s_2)/2} \leq -\epsilon c\left(|\rho|^2 + |\lambda|^2\right)^{r(s_3-s_2)} \to -\infty$$

as $|\lambda| \to \infty$.

The mapping $\Psi^{s_2-s_1} - \epsilon\Psi^{s_3-s_1}$ is continuous, and hence is bounded above. Let $c(\epsilon) > 0$ such that

$$\Psi(\lambda)^{s_2-s_1} - \epsilon\Psi(\lambda)^{s_3-s_1} \leq c(\epsilon),$$

for all $\lambda \in \mathfrak{a}^*$. Then

$$\Psi^{s_2} \leq \epsilon\Psi^{s_3} + c(\epsilon)\Psi^{s_1},$$

and so for all $u \in H^{\psi,s_3}$,

$$\begin{aligned}\|u\|_{\psi,s_2} = \|\Psi^{s_2}\hat{u}\| &\leq \|(\epsilon\Psi^{s_3} + c(\epsilon)\Psi^{s_1})\,\hat{u}\| \\ &\leq \epsilon\|\Psi^{s_3}\hat{u}\| + c(\epsilon)\|\Psi^{s_1}\hat{u}\| = \epsilon\|u\|_{\psi,s_3} + c(\epsilon)\|u\|_{\psi,s_1}.\end{aligned}$$

6. By Theorem 4.1.6 (4) above, it suffices to prove that there is a continuous embedding

$$H^s \hookrightarrow C_0(K|G|K)$$

for all $s > d$, where $d = \dim(G/K)$. Suppose $s > d$, and observe that by Lemma 4.1.7, we have $\langle\cdot\rangle^{-s} \in L^2(\mathfrak{a}^*, \omega)$. Given $u \in \mathcal{S}(K|G|K)$, we have for all $\sigma \in G$,

$$\begin{aligned} |u(\sigma)| = \left| \int_{\mathfrak{a}^*} \phi_\lambda(\sigma)\hat{u}(\lambda)\omega(d\lambda) \right| &\leq \int_{\mathfrak{a}^*} |\hat{u}(\lambda)|\omega(d\lambda) \\ &= \int_{\mathfrak{a}^*} \langle\lambda\rangle^{-s}\langle\lambda\rangle^s |\hat{u}(\lambda)|\omega(d\lambda), \end{aligned}$$

by the spherical inversion formula. But then by the Cauchy–Schwarz inequality, for all $\sigma \in G$,

$$|u(\sigma)| \leq \|\langle\cdot\rangle^{-s}\|_{L^2(\mathfrak{a}^*,\omega)} \|\langle\cdot\rangle^s \hat{u}\|_{L^2(\mathfrak{a}^*,\omega)} = C\|u\|_s,$$

where $C = \|\langle\cdot\rangle^{-s}\|_{L^2(\mathfrak{a}^*,\omega)}$. It follows that

$$\|u\|_{C_0(K|G|K)} := \sup_{\sigma\in G} |u(\sigma)| \leq C\|u\|_s. \tag{4.1.13}$$

Let $u \in H^s$, and suppose (u_n) is a sequence in $\mathcal{S}(K|G|K)$ converging to u in H^s. Then (u_n) is a Cauchy sequence in H^s, and, by applying (4.1.13) to $u_j - u_k$ ($j, k \in \mathbb{N}$), it is immediate that (u_n) is also Cauchy as a sequence in $C_0(K|G|K)$. By completeness, (u_n) converges to some element $u_0 \in C_0(K|G|K)$. Recall that we have continuous embeddings $H^s \hookrightarrow L^2(K|G|K)$ and $C_0(K|G|K) \hookrightarrow L^2(K|G|K)$. Therefore, as a sequence in $L^2(K|G|K)$, we have $u_n \to u$ and $u_n \to u_0$. It follows that $u = u_0$ almost everywhere.

□

4.1.3 Pseudodifferential operators and their symbols

A measurable mapping $q : G \times \mathfrak{a}^* \to \mathbb{C}$ will be called a *negative definite symbol* if it is locally bounded, and if for each $\sigma \in G$, $q(\sigma, \cdot)$ is negative definite and continuous. If in addition q is continuous in its first argument, we will call q a *continuous negative definite symbol*.

Let $B(G)$ denote the set of all Borel measurable functions on G.

Theorem 4.1.8. *Let q be a negative definite symbol, and for each $f \in C_c^\infty(K|G|K)$ and $\sigma \in G$, define*

$$q(\sigma, D)f(\sigma) = \int_{\mathfrak{a}^*} \hat{f}(\lambda)\phi_\lambda(\sigma)q(\sigma, \lambda)\omega(d\lambda). \tag{4.1.14}$$

Then

1. *Equation (4.1.14) defines a linear operator $q(\sigma, D) : C_c^\infty(K|G|K) \to B(G)$.*
2. *If q is a continuous negative definite symbol, then $q(\sigma, D) : C_c^\infty(K|G|K) \to C(G)$.*

3. *If q is K-bi-invariant in its first argument, then $q(\sigma, D)f$ is K-bi-invariant for all $f \in C_c^\infty(K|G|K)$.*

Proof. 1. First observe that (4.1.14) is well-defined. By the spherical Paley–Wiener theorem (Theorem 2.4.21), given $f \in C_c^\infty(K|G|K)$, there exists for each $n \in \mathbb{N}$ a constant $k_n > 0$ such that

$$\left|\hat{f}(\lambda)\right| \leq k_n(1+|\lambda|)^{-n}, \qquad \forall \lambda \in \mathfrak{a}^*$$

Moreover, for each $\sigma \in G$, $q(\sigma, \cdot)$ is locally bounded and negative definite, and so by Proposition 4.1.5 (3), for all $\sigma \in G$ we may choose $\kappa_\sigma > 0$ such that

$$|q(\sigma, \lambda)| \leq \kappa_\sigma(1+|\lambda|^2), \qquad \forall \lambda \in \mathfrak{a}^* .$$

Recall that $\omega(d\lambda) = c_0|\,\mathbf{c}(\lambda)|^{-2}d\lambda$, where $\mathbf{c}$ is Harish-Chandra's $\mathbf{c}$-function, and $c_0 > 0$ is a constant. Let $C_1, C_2 > 0$ be the constants from Proposition 2.4.20, then

$$|\,\mathbf{c}(\lambda)|^{-1} \leq C_1 + C_2|\lambda|^p, \qquad \forall \lambda \in \mathfrak{a}^*,$$

where $p = \frac{\dim N}{2}$. By Proposition 2.4.11, $|\phi_\lambda(\sigma)| \leq 1$ for all $\lambda \in \mathfrak{a}^*$ and $\sigma \in G$. Therefore, for all $\sigma \in G$ and $n \in \mathbb{N}$, there is $\kappa_{n,\sigma} > 0$ such that

$$|\hat{f}(\lambda)\phi_\lambda(\sigma)q(\sigma,\lambda)||\,\mathbf{c}(\lambda)|^{-2} \leq \kappa_{n,\sigma}\frac{(1+|\lambda|^2)(C_1+C_2|\lambda|^p)^2}{(1+|\lambda|)^n}. \tag{4.1.15}$$

For n sufficiently large, the right-hand side of (4.1.15) is integrable over $\mathfrak{a}^*$, and hence

$$\int_{\mathfrak{a}^*} |\hat{f}(\lambda)\phi_\lambda(\sigma)q(\sigma,\lambda)|\omega(d\lambda) \leq \kappa_{n,\sigma}\int_{\mathfrak{a}^*}\frac{(1+|\lambda|^2)(C_1+C_2|\lambda|^p)^2}{(1+|\lambda|)^n}d\lambda < \infty.$$

To see that $q(\sigma, D)f$ is measurable, let $C \subseteq G$ be a compact set, and choose a constant $\kappa_C > 0$ such that

$$|q(\sigma,\lambda)| \leq \kappa_C(1+|\lambda|^2), \qquad \forall \lambda \in \mathfrak{a}^*,\ \sigma \in C. \tag{4.1.16}$$

The map $(\sigma, \lambda) \mapsto \mathbf{1}_C(\sigma)q(\sigma,\lambda)\hat{f}(\lambda)$ is measurable, and it is also $d\sigma \otimes \omega(d\lambda)$-integrable, since

$$|\,\mathbf{1}_C(\sigma)q(\sigma,\lambda)\hat{f}(\lambda)| \leq \kappa_C\,\mathbf{1}_C(\sigma)(1+|\lambda|^2)|\hat{f}(\lambda)|,$$

for all $\sigma \in G$ and $\lambda \in \mathfrak{a}^*$. Hence

$$\sigma \mapsto \int_{\mathfrak{a}^*} \mathbf{1}_C(\sigma)q(\sigma,\lambda)\hat{f}(\lambda)\omega(d\lambda) \tag{4.1.17}$$

is measurable for all compact sets $C \subseteq G$. Let (C_n) be an increasing sequence of compact sets with $C_n \uparrow G$ as $n \to \infty$. This is always possible, since G is a connected Lie group, and so is second countable. Alternatively, it may be shown directly, for example by taking $C_n := C^n$ for each $n \in \mathbb{N}$, where C is some compact neighbourhood of e. Apply (4.1.17) to each C_n. We have that $\sigma \mapsto \mathbf{1}_{C_n}(\sigma)q(\sigma,D)f(\sigma)$ is measurable for all $n \in \mathbb{N}$, and hence, letting $n \to \infty$, so is $\sigma \mapsto q(\sigma, D)f(\sigma)$.

2. Suppose q is a continuous negative definite symbol, and let $f \in C_c^\infty(K|G|K)$. Let $C \subseteq G$ be compact, and choose $\kappa_C > 0$ satisfying (4.1.16) once again. If $\sigma, \tau \in C$, then

$$|q(\sigma, D)f(\sigma) - q(\tau, D)f(\tau)| \leq \int_{\mathfrak{a}^*} \left|\hat{f}(\lambda)\right| \left|q(\sigma, \lambda)\phi_\lambda(\sigma) - q(\tau, \lambda)\phi_\lambda(\tau))\right|\omega(d\lambda).$$

The integrand on the right hand side is bounded above by the map $2\kappa_C(1 + |\cdot|^2)|\hat{f}|$, and since $\hat{f}$ is rapidly decreasing, this map belongs to $L^1(\mathfrak{a}^*, \omega)$. An application of the dominated convergence theorem then implies $\sigma \mapsto q(\sigma, D)f(\sigma)$ is continuous on C. Since C was arbitrary, Theorem 4.1.8 (2) follows.

3. This is immediate from the K-bi-invariance of each spherical function ϕ_λ.

□

Definition 4.1.9. Operators of the form (4.1.14), where q is a negative definite symbol, will be called *(spherical) pseudodifferential operators* on G.

An important subclass of these operators first appeared for irreducible symmetric spaces in [7], with the symbol arising as the Gangolli exponent of a K-bi-invariant Lévy process. Note that just as in the classical Euclidean case, the symbols arising from Lévy processes are spatially independent, in the sense that they are constant in their first argument. We explore some specific examples of this below. In Section 4.2, we introduce a large class of examples pseudodifferential operators with spatial dependence.

Example 4.1.10. 1. *Diffusion operators with constant coefficients.* Since G is semisimple, the generator of a K-bi-invariant diffusion-type Lévy process Y on G takes the form

$$\mathcal{A} := \sum_{i,j=1}^{d} a_{ij} X_i X_j,$$

where $a = (a_{ij})$ is an $\mathrm{Ad}(K)$-invariant, non-negative definite symmetric $d \times d$ matrix (c.f. (3.1.4)). By Proposition 3.1.13, $\mathcal{A} \in \mathbf{D}_K(G)$; let $\beta(\mathcal{A}, \lambda)$ denote the ϕ_λ-eigenvalue of $\mathcal{A}$, for each $\lambda \in \mathfrak{a}^*$. Note that $\lambda \mapsto -\beta(\mathcal{A}, \lambda)$ is the Gangolli exponent of Y.

We claim that $(\sigma, \lambda) \mapsto -\beta(\mathcal{A}, \lambda)$ is a continuous negative definite symbol, and that the associated pseudodifferential operator is $-\mathcal{A}$. To see this, let $(\mu_t, t \geq 0)$ denote the convolution semigroup generated by $\mathcal{A}$, and let $(T_t, t \geq 0)$ be the associated Hunt semigroup (c.f. Definition 2.4.4). Then, given $f \in C_c^\infty(K|G|K)$ and $\sigma \in G$,

$$\mathcal{A}f(\sigma) = \frac{d}{dt} T_t f(\sigma)\bigg|_{t=0}. \tag{4.1.18}$$

By the spherical inversion formula (2.4.17), for $t \geq 0$,

$$T_t f(\sigma) = \int_G f(\sigma\tau)\mu_t(d\tau) = \int_G \int_{\mathfrak{a}^*} \hat{f}(\lambda)\phi_\lambda(\sigma\tau)\omega(d\lambda)\mu_t(d\tau). \tag{4.1.19}$$

Since $\hat{f} \in \mathcal{S}(\mathfrak{a}^*)^W$ and $|\phi_\lambda| \leq 1$, a Fubini argument may be used to exchange the order of integration in (4.1.19) and conclude that

$$T_t f(\sigma) = \int_{\mathfrak{a}^*} \hat{f}(\lambda) T_t \phi_\lambda(\sigma) \omega(d\lambda).$$

By (3.1.6), $T_t \phi_\lambda = e^{t\beta(\mathcal{A},\lambda)} \phi_\lambda$, and thus

$$T_t f(\sigma) = \int_{\mathfrak{a}^*} \hat{f}(\lambda) e^{t\beta(\mathcal{A},\lambda)} \phi_\lambda(\sigma) \omega(d\lambda).$$

Now, by (4.1.18),

$$\mathcal{A} f(\sigma) = \lim_{t \to 0} \int_{\mathfrak{a}^*} \hat{f}(\lambda) \left(\frac{e^{t\beta(\mathcal{A},\lambda)} - 1}{t} \right) \phi_\lambda(\sigma) \omega(d\lambda). \tag{4.1.20}$$

Now, if $t > 0$ and $\lambda \in \mathfrak{a}^*$, then

$$\left| \hat{f}(\lambda) \left(\frac{e^{t\beta(\mathcal{A},\lambda)} - 1}{t} \right) \phi_\lambda(\sigma) \right| \leq \left| \hat{f}(\lambda) \right| \left| \frac{e^{t\beta(\mathcal{A},\lambda)} - 1}{t} \right| \leq \left| \hat{f}(\lambda) \right| |\beta(\mathcal{A}, \lambda)|.$$

Moreover, $|\hat{f}||\beta(\mathcal{A}, \cdot)| \in L^1(\mathfrak{a}^*, \omega)^W$, since $\hat{f} \in \mathcal{S}(\mathfrak{a}^*)^W$, and $\beta(\mathcal{A}, \cdot)$ is a W-invariant polynomial function. By the dominated convergence theorem, we may bring the limit through the integral sign in (4.1.20) to conclude that

$$\begin{aligned} \mathcal{A} f(\sigma) &= \int_{\mathfrak{a}^*} \hat{f}(\lambda) \lim_{t \to 0} \left(\frac{e^{t\beta(\mathcal{A},\lambda)} - 1}{t} \right) \phi_\lambda(\sigma) \omega(d\lambda) \\ &= \int_{\mathfrak{a}^*} \hat{f}(\lambda) \phi_\lambda(\sigma) \beta(\mathcal{A}, \lambda) \omega(d\lambda) \end{aligned} \tag{4.1.21}$$

for all $f \in C_c^\infty(K|G|K)$ and $\sigma \in G$.

2. *Brownian motion.* As a special case of the above, $-\Delta$ is a pseudodifferential operator with symbol $|\rho|^2 + |\lambda|^2$.

3. *Killed diffusions.* With minimal effort, the results of Example 4.1.10 (1) may be extended to include killing. To see this, note first that such operators are always of the form $\mathcal{A} - cI$, where $\mathcal{A}$ is a diffusion operator of the form considered above, and $c \geq 0$. The associated ϕ_λ-eigenvalues must satisfy

$$\beta(\mathcal{A} - c, \lambda) = \beta(\mathcal{A}, \lambda) - c,$$

and hence using (4.1.21) as well as the spherical inversion theorem,

$$\begin{aligned} (\mathcal{A} - c) f(\sigma) &= \int_{\mathfrak{a}^*} \hat{f}(\lambda) \phi_\lambda(\sigma) \beta(\mathcal{A}, \lambda) \omega(d\lambda) - c f(\sigma) \\ &= \int_{\mathfrak{a}^*} \hat{f}(\lambda) \phi_\lambda(\sigma) \beta(\mathcal{A} - c, \lambda) \omega(d\lambda), \end{aligned}$$

for all $f \in C_c^\infty(K|G|K)$ and $\sigma \in G$.

4. *Lévy generators.* More generally, if $\mathcal{A}$ is the infinitesimal generator of a K-bi-invariant Lévy process on G, and if ψ is the corresponding Gangolli exponent, then $(\sigma, \lambda) \mapsto \psi(\lambda)$ is a continuous negative definite symbol, and $-\mathcal{A}$ is the corresponding pseudodifferential operator. This is proven in [7] Theorem 5.2 in the case where G/K is irreducible, and later in this work as a special case of Theorem 4.2.7.

§ 4.2 Gangolli operators and the Hille–Yosida–Ray theorem

We will soon define the class of pseudodifferential operators that will be of primary interest, namely, the Gangolli operators. We motivate their definition with a short discussion of the Hille–Yosida–Ray theorem, and later prove that they are pseudodifferential operators in the sense of Definition 4.1.9. We finish the section with some examples.

Given a locally compact Hausdorff space E, a linear operator $\mathcal{A} : \mathrm{Dom}(\mathcal{A}) \to \mathcal{F}(E)$ is said to satisfy the *positive maximum principle*, if for all $f \in \mathrm{Dom}(\mathcal{A})$ and $x_0 \in E$ such that $f(x_0) = \sup_{x \in E} f(x) \geq 0$, we have $\mathcal{A}f(x_0) \leq 0$.

The following theorem is an extended version of the Hille–Yosida–Ray theorem, and fully characterises the operators that extend to generators of sub-Feller semigroups on $C_0(E)$. Similar versions in which $E = \mathbb{R}^d$ may found in [41], pp. 53, and [46], pp. 333. For a proof in the general case, see [27], pp. 165.

Theorem 4.2.1 (Hille–Yosida–Ray)**.** *A linear operator* $(\mathcal{A}, \mathrm{Dom}(\mathcal{A}))$ *on* $C_0(E)$ *is closable and its closure generates a sub-Feller semigroup on* $C_0(E)$ *if and only if the following is satisfied:*

1. $\mathrm{Dom}(\mathcal{A})$ *is dense in* $C_0(E)$*,*
2. $\mathcal{A}$ *satisfies the positive maximum principle, and*
3. *There exists* $\alpha > 0$ *such that* $\mathrm{Ran}(\alpha I - \mathcal{A})$ *is dense in* $C_0(E)$*.*

In their papers [15, 16], Applebaum and Ngan found necessary and sufficient conditions for an operator defined on $C_c^\infty(K|G|K)$ to satisfy Theorem 4.2.1 (2), for the cases $E = G$, G/K and $K|G|K$. We will focus primarily on the case $E = K|G|K$, since this is the realm in which the spherical transform is available.

A mapping $\nu : G \times \mathcal{B}(G) \to [0, \infty]$ will be called a *K-bi-invariant Lévy kernel* if it is K-bi-invariant in its first argument, and if for all $\sigma \in G$, $\nu(\sigma, \cdot)$ is a K-bi-invariant Lévy measure. Fix a system of exponential coordinate functions, as defined in Definition 3.1.8, and adopt all of the notation conventions from that definition.

Definition 4.2.2. An operator $\mathcal{A} : C_c^\infty(K|G|K) \to \mathcal{F}(G)$ will be called a *Gangolli operator* if there exist mappings $c, a_{i,j} \in \mathcal{F}(K|G|K)$ $(1 \leq i, j \leq d)$, as well as a K-bi-invariant Lévy kernel ν, such that for all $f \in C_c^\infty(K|G|K)$ and $\sigma \in G$,

$$\begin{aligned}\mathcal{A}f(\sigma) = -c(\sigma)f(\sigma) + \sum_{i,j=1}^{d} a_{i,j}(\sigma)X_iX_jf(\sigma) \\ + \int_G \left(f(\sigma\tau) - f(\sigma) - \sum_{i=1}^{d} x_i(\tau)X_if(\sigma) \right) \nu(\sigma, d\tau),\end{aligned} \tag{4.2.1}$$

and if for all $\sigma \in G$,

1. $c(\sigma) \geq 0$.
2. $a(\sigma) := (a_{i,j}(\sigma))$ is an $\mathrm{Ad}(K)$-invariant, non-negative definite, symmetric matrix.

Remarks 4.2.3. 1. Gangolli operators were first introduced in [16] for compact symmetric spaces and with a more restrictive form of (4.2.1). By Theorem 3.2 (3) of [16], Gangolli operators map into $\mathcal{F}(K|G|K)$, and satisfy the positive maximum principle.

2. Equation (4.2.1) may be viewed as a spatially dependent generalisation of (3.1.4), with an additional killing term c. As with previously, the absence of a drift term is due to the semisimplicity of G.

For a Gangolli operator $\mathcal{A}$ given by (4.2.1), and for each $\sigma \in G$, we will denote by $\mathcal{A}^\sigma$ the operator obtained by freezing the coefficients of $\mathcal{A}$ at σ. Explicitly, for all $f \in C_c^\infty(K|G|K)$ and $\sigma' \in G$,

$$\begin{aligned}\mathcal{A}^\sigma f(\sigma') = -c(\sigma)f(\sigma') + \sum_{i,j=1}^{d} a_{i,j}(\sigma)X_iX_jf(\sigma') \\ + \int_G \left(f(\sigma'\tau) - f(\sigma') - \sum_{i=1}^{d} x_i(\tau)X_if(\sigma') \right) \nu(\sigma, d\tau).\end{aligned}$$

For each $\sigma \in G$, $\mathcal{A}^\sigma$ is the generator of a killed K-bi-invariant Lévy process on G. We continue to adopt the notation $\mathcal{A}_D^\sigma$ for the diffusion part, and $\beta(\mathcal{A}_D^\sigma, \lambda)$ for the ϕ_λ-eigenvalue of $\mathcal{A}_D^\sigma$.

Consider the following continuity conditions on the coefficients (b, a, ν) of $\mathcal{A}$:

(c1) c, a_{ij} are continuous, for $1 \leq i, j \leq d$.

(c2) For each $f \in C_b(K|G|K)$, the mappings $\sigma \mapsto \int_U f(\tau) \sum_{i=1}^d x_i(\tau)^2 \nu(\sigma, d\tau)$ and $\sigma \mapsto \int_{U^c} f(\tau)\nu(\sigma, d\tau)$ are continuous from G to $[0, \infty)$.

Note that these conditions were first introduced in [15] Theorem 3.7.

Lemma 4.2.4. *Let $\mathcal{A}$ be a Gangolli operator, and define $q : G \times \mathfrak{a}^* \to \mathbb{C}$ by*

$$q(\sigma, \lambda) = -\beta(\mathcal{A}_D^\sigma, \lambda) + \int_G (1 - \phi_\lambda(\tau))\nu(\sigma, d\tau), \qquad \forall \sigma \in G, \lambda \in \mathfrak{a}^* . \tag{4.2.2}$$

Suppose (c1) and (c2) hold. Then q is a continuous negative definite symbol.

Proof. That q is continuous in its first argument is immediate from (c1) and (c2). Fix $\sigma \in G$ and consider $q(\sigma, \cdot) - c(\sigma)$. By Theorem 3.1.10, there is a convolution semigroup $(\mu_t^\sigma, t \geq 0)$ generated by $\mathcal{A}^\sigma + c(\sigma)$, and by Theorem 3.1.14, the corresponding Gangolli exponent is a continuous negative definite mapping on $\mathfrak{a}^*$, given by

$$\psi^\sigma(\lambda) = q(\sigma, \lambda) - c(\sigma) \qquad \forall \lambda \in \mathfrak{a}^* .$$

Therefore $q(\sigma, \cdot)$ is continuous, and negative definite since for fixed σ, $c(\sigma)$ is a non-negative constant. □

Definition 4.2.5. The symbols described by Lemma 4.2.4 will be referred to as *Gangolli symbols*, due to their connection with Gangolli's Lévy–Khinchine formula (Theorem 3.1.14).

Remarks 4.2.6.

1. Gangolli exponents are precisely those Gangolli symbols that are constant in their first argument.
2. The set of all Gangolli symbols forms a convex cone.

Theorem 4.2.7. *Let $\mathcal{A}$ and q be as in Lemma 4.2.4. Then $-\mathcal{A}$ is a pseudodifferential operator with symbol q.*

Proof. By Theorem 4.1.8 and Lemma 4.2.4, $f \mapsto -\int_{\mathfrak{a}^*} \hat{f}(\lambda)\phi_\lambda(\sigma)q(\sigma, \lambda)\omega(d\lambda)$ is a well-defined mapping from $C_c^\infty(K|G|K) \to C(G)$. We show that it is equal to $\mathcal{A}$.

Let $\mathcal{A}_J$ denote the non-local (i.e. jump) part of $\mathcal{A}$, so that

$$\mathcal{A}_J f(\sigma) = \int_G \left(f(\sigma\tau) - f(\sigma) - \sum_{i=1}^d x_i(\tau) X_i f(\sigma) \right) \nu(\sigma, d\tau) \tag{4.2.3}$$

for all $f \in C_c^\infty(K|G|K)$ and $\sigma \in G$. We then have

$$\mathcal{A}f(\sigma) = \mathcal{A}_D^\sigma f(\sigma) + \mathcal{A}_J f(\sigma), \qquad \forall f \in C_c^\infty(K|G|K),\ \sigma \in G. \tag{4.2.4}$$

For the diffusion part of $\mathcal{A}$, note that for each $\sigma \in G$, $\mathcal{A}_D^\sigma$ is an operator of the form considered in Example 4.1.10 (3), and in particular satisfies

$$\mathcal{A}_D^\sigma f(\sigma) = \int_{\mathfrak{a}^*} \hat{f}(\lambda)\beta(\mathcal{A}_D^\sigma, \lambda)\phi_\lambda(\sigma)\omega(d\lambda), \tag{4.2.5}$$

for all $f \in C_c^\infty(K|G|K)$.

Consider now the jump part $\mathcal{A}_J$. By Lemma 2.3 on page 39 of Liao [54], for each fixed $\sigma \in G$, and for all $f \in C_b^2(K|G|K)$, the integrand on the right-hand side of (4.2.3) is absolutely integrable with respect to $\nu(\sigma, \cdot)$. Therefore, (4.2.3) may be used to extend the domain of $\mathcal{A}_J$ so as to include $C_b^2(K|G|K)$. We do so now, and (without any loss of precision) denote the extension by $\mathcal{A}_J$.

Let us proceed similarly to Applebaum and Ngan [16] Section 5, and define for each $\sigma \in G$ a linear functional $\mathcal{A}_{J,\sigma} : C_b^2(K|G|K) \to \mathbb{C}$ by

$$\mathcal{A}_{J,\sigma} f := \mathcal{A}_J \left(L_\sigma^{-1} f \right)(\sigma), \qquad \forall \sigma \in G,\ f \in C_b^2(K|G|K).$$

Then $\mathcal{A}_J f(\sigma) = \mathcal{A}_{J,\sigma}(L_\sigma f)$, and hence

$$\mathcal{A}_{J,\sigma}\phi_\lambda = \int_G \left(L_\sigma^{-1}\phi_\lambda(\sigma\tau) - L_\sigma^{-1}\phi_\lambda(\sigma) - \sum_{i=1}^d x_i(\tau) X_i L_\sigma^{-1}\phi_\lambda(\sigma) \right) \nu(\sigma, d\tau),$$

for all $\sigma \in G$ and $f \in C_b^2(K|G|K)$. Moreover, the integrand on the right-hand side is absolutely $\nu(\sigma,\cdot)$-integrable, for all $\lambda \in \mathfrak{a}^*$ and $\sigma \in G$. Since $\phi_\lambda(e) = 1$, and $X\phi_\lambda(e) = 0$ for all $X \in \mathfrak{p}$ (Theorem 5.3 (b) of [54]),

$$L_\sigma^{-1}\phi_\lambda(\sigma\tau) - L_\sigma^{-1}\phi_\lambda(\sigma) - \sum_{i=1}^d x_i(\tau) X_i L_\sigma^{-1}\phi_\lambda(\sigma) = \phi_\lambda(\tau) - 1.$$

Thus, for all $\lambda \in \mathfrak{a}^*$ and $\sigma \in G$, $\phi_\lambda - 1$ is absolutely $\nu(\sigma,\cdot)$-integrable, and

$$\mathcal{A}_{J,\sigma}\phi_\lambda = \int_G (\phi_\lambda(\tau) - 1)\, \nu(\sigma, d\tau). \tag{4.2.6}$$

A standard argument involving the functional equation

$$\phi_\lambda(\sigma)\phi_\lambda(\tau) = \int_K \phi_\lambda(\sigma k \tau) dk, \qquad \forall \lambda \in \mathfrak{a}^*, \sigma, \tau \in G,$$

for spherical functions (see [37] Proposition 2.2, pp. 400) may now be applied in precisely the same way as in [16] (5.3)–(5.7), to infer that

$$\mathcal{A}_J\phi_\lambda(\sigma) = \int_G (\phi_\lambda(\sigma\tau) - \phi_\lambda(\sigma))\nu(\sigma, d\tau) = \int_G (\phi_\lambda(\tau) - 1)\phi_\lambda(\sigma)\nu(\sigma, d\tau) \tag{4.2.7}$$

for all $\sigma \in G$ and $\lambda \in \mathfrak{a}^*$.

Finally, let $f \in C_c^\infty(K|G|K)$, and observe that by the spherical inversion formula

$$\begin{aligned} \mathcal{A}_J f(\sigma) = \int_G \Bigg(&\int_{\mathfrak{a}^*} \phi_\lambda(\sigma\tau)\hat{f}(\lambda)\omega(d\lambda) - \int_{\mathfrak{a}^*} \phi_\lambda(\sigma)\hat{f}(\lambda)\omega(d\lambda) \\ &- \sum_{i=1}^d x_i(\tau) X_i \left[\int_{\mathfrak{a}^*} \phi_\lambda \hat{f}(\lambda)\omega(d\lambda) \right](\sigma) \Bigg) \nu(\sigma, d\tau) \end{aligned} \tag{4.2.8}$$

Claim. For all $X \in \mathfrak{p}$ and $f \in C_c^\infty(K|G|K)$,

$$X\left[\int_{\mathfrak{a}^*} \phi_\lambda \hat{f}(\lambda)\omega(d\lambda) \right](\sigma) = \int_{\mathfrak{a}^*} X\phi_\lambda(\sigma)\hat{f}(\lambda)\omega(d\lambda).$$

Proof of Claim. This is a fairly standard differentiation-through-integration-sign argument. First note that by translation invariance of X, it suffices to prove the claim for $\sigma = e$. Now,

$$X\left[\int_{\mathfrak{a}^*} \phi_\lambda \hat{f}(\lambda)\omega(d\lambda)\right](e) = \frac{d}{dt}\int_{\mathfrak{a}^*} \phi_\lambda(\exp tX)\hat{f}(\lambda)\omega(d\lambda)\bigg|_{t=0}$$
$$= \lim_{t\to 0}\int_{\mathfrak{a}^*} \frac{\phi_\lambda(\exp tX) - 1}{t}\hat{f}(\lambda)\omega(d\lambda).$$

The claim will follow if we can apply the dominated convergence theorem to bring the above limit through the integral sign. By the mean value theorem, for each $t > 0$ and $\lambda \in \mathfrak{a}^*$,

$$\frac{\phi_\lambda(\exp tX) - 1}{t} = X\phi_\lambda(\exp t'X),$$

for some $0 < t' < t$, and hence $\left|\frac{\phi_\lambda(\exp tX)-1}{t}\right| \leq \|X\phi_\lambda\|_\infty$ for all $t > 0$. By [35] Theorem 1.1 (iii), $\|X\phi_\lambda\|_\infty \leq C(1 + |\lambda|)$, for some some constant $C > 0$. Thus, for $f \in C_c^\infty(K|G|K)$, $\lambda \in \mathfrak{a}^*$ and $t > 0$,

$$\left|\frac{\phi_\lambda(\exp tX) - 1}{t}\hat{f}(\lambda)\right| \leq C(1 + |\lambda|)|\hat{f}(\lambda)|,$$

and clearly $C(1 + |\cdot|)\hat{f} \in L^1(\mathfrak{a}^*)^W$, since $\hat{f} \in \mathcal{S}(\mathfrak{a}^*)$. Hence we may apply dominated convergence as desired, and the claim follows.

Applying the claim to (4.2.8), for $f \in C_c^\infty(K|G|K)$ and $\sigma \in G$,

$$\mathcal{A}_J f(\sigma) = \int_G \Bigg(\int_{\mathfrak{a}^*} \phi_\lambda(\sigma\tau)\hat{f}(\lambda)\omega(d\lambda) - \int_{\mathfrak{a}^*} \phi_\lambda(\sigma)\hat{f}(\lambda)\omega(d\lambda)$$
$$- \sum_{i=1}^d x_i(\tau)\int_{\mathfrak{a}^*} X_i\phi_\lambda(\sigma)\hat{f}(\lambda)\omega(d\lambda)\Bigg)\nu(\sigma, d\tau)$$
$$= \int_G\int_{\mathfrak{a}^*} \hat{f}(\lambda)\left(\phi_\lambda(\sigma\tau) - \phi_\lambda(\sigma) - \sum_{i=1}^d x_i(\tau)X_i\phi_\lambda(\sigma)\right)\omega(d\lambda)\nu(\sigma, d\tau).$$

By the Fubini theorem,

$$\mathcal{A}_J f(\sigma) = \int_{\mathfrak{a}^*} \hat{f}(\lambda)\int_G \left(\phi_\lambda(\sigma\tau) - \phi_\lambda(\sigma) - \sum_{i=1}^d x_i(\tau)X_i\phi_\lambda(\sigma)\right)\nu(\sigma, d\tau)\omega(d\lambda)$$
$$= \int_{\mathfrak{a}^*} \hat{f}(\lambda)\mathcal{A}_J\phi_\lambda(\sigma)\omega(d\lambda)$$

for all $f \in C_c^\infty(K|G|K)$ and $\sigma \in G$. It follows by (4.2.7) that

$$\mathcal{A}_J f(\sigma) = \int_{\mathfrak{a}^*} \hat{f}(\lambda)\phi_\lambda(\sigma)\int_G (\phi_\lambda(\tau) - 1)\nu(\sigma, d\tau)\omega(d\lambda) \tag{4.2.9}$$

for all $f \in C_c^\infty(K|G|K)$ and $\sigma \in G$.

The result now follows by substituting (4.2.9) and (4.2.5) into (4.2.4). □

Example 4.2.8. 1. Let $u \in C(K|G|K)$ be non-negative, and let $v : \mathfrak{a}^* \to \mathbb{C}$ be a Gangolli exponent. Then $q : G \times \mathfrak{a}^* \to \mathbb{C}$ given by

$$q(\sigma, \lambda) = u(\sigma)v(\lambda) \qquad \forall \sigma \in G,\ \lambda \in \mathfrak{a}^*$$

is a continuous Gangolli symbol. Indeed, by Definition 3.1.16, there exists a sub-diffusion operator $\mathcal{L} \in \mathbf{D}_K(G)$ and a K-bi-invariant Lévy measure ν such that for all $\lambda \in \mathfrak{a}^*$,

$$v(\lambda) = -\beta(\mathcal{L}, \lambda) + \int_G (1 - \phi_\lambda(\sigma))\nu(d\tau).$$

Hence for all $\sigma \in G$ and $\lambda \in \mathfrak{a}^*$,

$$q(\sigma, \lambda) = -\beta(u(\sigma)\mathcal{L}, \lambda) + \int_G (1 - \phi_\lambda(\sigma))u(\sigma)\nu(d\tau).$$

If $\mathcal{L} = -c + \sum_{i,j=1}^{d} a_{ij} X_i X_j$, where $c \geq 0$ and $a = (a_{ij})$ is an $\mathrm{Ad}(K)$-invariant, non-negative definite symmetric matrix, then the characteristics are q are

$$c(\sigma) \coloneqq u(\sigma)c, \quad a(\sigma) = u(\sigma)a, \text{ and } \nu(\sigma, \cdot) = u(\sigma)\nu.$$

Since u is non-negative, continuous and K-bi-invariant, the conditions of Definition 4.2.2 are easily verified for these characteristics, as are (c1) and (c2).

2. *Hyperbolic plane.* As described in [37] (pp. 29–31), the Poincaré disc model D of the hyperbolic plane is isomorphic to $SU(1,1)/SO(2)$. Moreover, D is a symmetric space of noncompact type, with spherical functions given by the Legendre functions

$$\phi_\lambda(z) = P_{\frac{1}{2}+i\lambda}\big(\cosh d_{\mathbb{H}}(0, z)\big), \qquad \forall z \in D, \lambda \in \mathbb{R}$$

(see [39] Proposition 2.9, pp. 406). Since D is irreducible and $\dim D > 1$, by Theorem 3.3 of [16], diffusion operators on D must be multiples of the Laplace–Beltrami operator, and the symbols of Feller processes take the simplified form

$$q(z, \lambda) = c(z)\left(\frac{1}{4} + \lambda^2\right) + \int_0^\infty \big\{1 - P_{\frac{1}{2}+i\lambda}(\cosh r)\big\}\nu(z, dr),$$

for all $z \in D$ and $\lambda \in \mathbb{R}$. The constant coefficient (i.e. Lévy) case of this formula was discovered by Getoor — see [31] Theorem 7.4.

Remark 4.2.9. Example 4.2.8 (1) will be used later in Section 4.4, when we investigate a particular class of examples that satisfy Corollary 4.3.20.

§ 4.3 Construction of sub-Feller semigroups

In this section we tackle the third condition of Hille–Yosida–Ray (Theorem 4.2.1), when $E = K|G|K$. To this end, we seek conditions on a symbol q so that, for some $\alpha > 0$,

$$\overline{\mathrm{Ran}(\alpha I + q(\sigma, D))} = C_0(K|G|K). \tag{4.3.1}$$

Earlier work, especially that of Section 4.1, enables us to now apply the methods of [45] and [41] Section 4 to this problem.

For a mapping $q : G \times \mathfrak{a}^* \to \mathbb{R}$ and for each $\lambda, \eta \in \mathfrak{a}^*$, $\sigma \in G$, define

$$F_{\lambda,\eta}(\sigma) = \phi_{-\lambda}(\sigma) q(\sigma, \eta). \tag{4.3.2}$$

Observe that if $q(\cdot, \eta) \in L^2(K|G|K)$ for all $\eta \in \mathfrak{a}^*$, then $F_{\lambda,\eta} \in L^2(K|G|K)$, and we may consider the spherical transform $\hat{F}_{\lambda,\eta} \in L^2(\mathfrak{a}^*, \omega)$, given by

$$\hat{F}_{\lambda,\eta}(\mu) = \int_G \phi_{-\mu}(\sigma)\phi_{-\lambda}(\sigma) q(\sigma, \eta) d\sigma, \qquad \forall \mu \in \mathfrak{a}^* .$$

To motivate the introduction of $F_{\lambda,\eta}$, consider the case $G = \mathbb{R}^d$, $K = \{0\}$. In this case, the frequency shift property for the Fourier transform says that

$$\begin{aligned} \hat{F}_{\lambda,\eta}(\mu) &= \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} e^{-i\mu \cdot x} e^{-i\lambda \cdot x} q(x, \eta) dx \\ &= \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} e^{-i(\mu+\lambda)\cdot x} q(x, \eta) dx = \hat{q}(\lambda + \mu, \eta), \end{aligned} \tag{4.3.3}$$

where $\wedge$ denotes the Fourier transform taken in the first argument of q. [41] and [47] make use of bounds on $\hat{q}(\lambda - \mu, \eta)$, and $\hat{F}_{\lambda,\eta}(-\mu)$ will assume an analogous role in work to come.

As in previous work, let $\psi : \mathfrak{a}^* \to \mathbb{R}$ be a fixed real-valued, continuous negative definite function satisfying (4.1.5) for some fixed $r > 0$. The next lemma is an analogue of Lemma 2.1 of [45]. See also [41] Lemma 4.2, pp. 48. The primary difference in this work is the presence of integer powers of $\sqrt{-\Delta}$, which replace the multinomial powers of $\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_d}$ of the $\mathbb{R}^d$ setting.

One advantage of this approach is that $(-\Delta)^{\beta/2}$ ($\beta \in \mathbb{N}$) has a global definition that does not depend on our choice of local coordinates. Another advantage is that we know its symbol — see (4.3.8) below.

Lemma 4.3.1. *Let $M \in \mathbb{N}$, $q : G \times \mathfrak{a}^* \to \mathbb{R}$ and suppose $q(\cdot, \lambda) \in C_c^M(K|G|K)$ for all $\lambda \in \mathfrak{a}^*$. Suppose that for each $\beta \in \{0, 1, \ldots, M\}$, there is a non-negative function $\Phi_\beta \in L^1(K|G|K)$ such that*

$$\left|(-\Delta)^{\beta/2} F_{\lambda,\eta}(\sigma)\right| \le \Phi_\beta(\sigma) \langle\lambda\rangle^M (1 + \psi(\eta)), \tag{4.3.4}$$

for all $\lambda, \eta \in \mathfrak{a}^*$, $\sigma \in G$. Then there is a constant $C_M > 0$ such that

$$\left|\hat{F}_{\lambda,\eta}(\mu)\right| \le C_M \sum_{\beta=0}^{M} \|\Phi_\beta\|_1 \langle\lambda + \mu\rangle^{-M} (1 + \psi(\eta)), \tag{4.3.5}$$

for all $\lambda, \mu, \eta \in \mathfrak{a}^*$, where $\|\cdot\|_1$ denotes the usual norm on the Banach space $L^1(K|G|K)$.

Remarks 4.3.2. 1. As in (4.1.8), $\langle\lambda\rangle := \sqrt{1 + |\lambda|^2}$.

2. The condition (4.3.4) may seem quite obscure. The role of $\langle\lambda+\mu\rangle$ will hopefully become apparent in the proof of Theorem 4.3.6. For examples where it is satisfied, see §4.4.

3. Under the conditions of the lemma, and using the Fubini theorem, we have the following: for all $u \in C_c^\infty(K|G|K)$ and $\lambda \in \mathfrak{a}^*$,

$$\begin{aligned}(q(\sigma,D)u)^\wedge(\lambda) &= \int_G \int_{\mathfrak{a}^*} \phi_{-\lambda}(\sigma)\phi_\eta(\sigma)q(\sigma,\eta)\hat{u}(\eta)\omega(d\eta)d\sigma \\ &= \int_{\mathfrak{a}^*}\left(\int_G \phi_\eta(\sigma)F_{\lambda,\eta}(\sigma)d\sigma\right)\hat{u}(\eta)\omega(d\eta) \\ &= \int_{\mathfrak{a}^*}\hat{F}_{\lambda,\eta}(-\eta)\hat{u}(\eta)\omega(d\eta).\end{aligned} \tag{4.3.6}$$

Fubini's theorem does indeed apply here — a suitable bound for the integrand on the first line of (4.3.6) may be found by noting that, by (4.3.4),

$$|\phi_{-\lambda}(\sigma)\phi_\eta(\sigma)q(\sigma,\eta)\hat{u}(\eta)| \leq |q(\sigma,\eta)||\hat{u}(\eta)| \leq \Phi_0(\sigma)\big(1+\psi(\eta)\big)|\hat{u}(\eta)|, \tag{4.3.7}$$

for all $\lambda,\eta \in \mathfrak{a}^*$ and $\sigma \in G$. By Theorem 2.4.30, $\hat{u} \in \mathcal{S}(\mathfrak{a}^*)$, and the usual bound (2.4.21) on the density of Plancherel measure may be applied, similarly to (4.1.15), to conclude that the right-hand side of (4.3.7) is $\omega(d\eta)\times d\sigma$-integrable.

Proof of Lemma 4.3.1. Let $\beta \in \{0,1,\ldots,M\}$ and $\lambda,\eta \in \mathfrak{a}^*$ be fixed. Recall from Examples 3.1.17 (2) that the fractional Laplacian $-(-\Delta)^{\beta/2}$ generates a K-bi-invariant β-stable-like process on G, with symbol $\big(|\rho|^2+|\mu|^2\big)^{\beta/2}$. In particular,

$$\left((-\Delta)^{\beta/2}f\right)^\wedge(\mu) = \big(|\rho|^2+|\mu|^2\big)^{\beta/2}\hat{f}(\mu), \tag{4.3.8}$$

for all $f \in C_c^M(K|G|K)$ and $\mu \in \mathfrak{a}^*$. Then, using the definition of the spherical transform,

$$(|\rho|^2+|\mu|^2)^{\beta/2}\hat{f}(\mu) = \int_G \phi_{-\mu}(\sigma)(-\Delta)^{\beta/2}f(\sigma)d\sigma,$$

for all $f \in C_c^M(K|G|K)$ and all $\mu \in \mathfrak{a}^*$. Applying this to $f = F_{\lambda,\eta}$, we have for all $\mu \in \mathfrak{a}^*$,

$$\begin{aligned}\left|\big(|\rho|^2+|\mu|^2\big)^{\beta/2}\hat{F}_{\lambda,\eta}(\mu)\right| &\leq \int_G |\phi_{-\mu}(\sigma)|\left|(-\Delta)^{\beta/2}F_{\lambda,\eta}(\sigma)\right|d\sigma \\ &\leq \int_G \Phi_\beta(\sigma)\langle\lambda\rangle^M(1+\psi(\eta))d\sigma = \|\Phi_\beta\|_1\langle\lambda\rangle^M(1+\psi(\eta)),\end{aligned}$$

and summing over β,

$$\sum_{\beta=0}^M \big(|\rho|^2+|\mu|^2\big)^{\beta/2}\left|\hat{F}_{\lambda,\eta}(\mu)\right| \leq \sum_{\beta=0}^M \|\Phi_\beta\|_1\langle\lambda\rangle^M(1+\psi(\eta)), \tag{4.3.9}$$

for all $\lambda, \mu, \eta \in \mathfrak{a}^*$. Let $C'_M > 0$ be the smallest positive number such that

$$\langle \mu \rangle^M \leq C'_M \sum_{\beta=0}^{M} \left(|\rho|^2 + |\mu|^2\right)^{\beta/2} \qquad \forall \mu \in \mathfrak{a}^* .$$

Then, rearranging (4.3.9),

$$\left|\hat{F}_{\lambda,\eta}(\mu)\right| \leq C'_M \sum_{\beta=0}^{M} \|\Phi_\beta\|_1 \langle \mu \rangle^{-M} \langle \lambda \rangle^M (1 + \psi(\eta)), \tag{4.3.10}$$

for all $\lambda, \mu, \eta \in \mathfrak{a}^*$.

Finally, observe that by Peetre's inequality (see Proposition 4.1.5 (2)),

$$\langle \lambda \rangle^M \langle \lambda + \mu \rangle^{-M} = \left(\frac{1 + |\lambda|^2}{1 + |\lambda + \mu|^2}\right)^{M/2} \leq 2^{M/2}(1 + |\mu|^2)^{M/2} = 2^{M/2} \langle \mu \rangle^M$$

for all $\lambda, \mu \in \mathfrak{a}^*$. Therefore, for all $\lambda, \mu \in \mathfrak{a}^*$,

$$\langle \mu \rangle^{-M} \langle \lambda \rangle^M \leq 2^{M/2} \langle \lambda + \mu \rangle^{-M}$$

and by (4.3.10),

$$\left|\hat{F}_{\lambda,\eta}(\mu)\right| \leq 2^{M/2} C'_M \sum_{\beta=0}^{M} \|\Phi_\beta\|_1 \langle \lambda + \mu \rangle^{-M} (1 + \psi(\eta))$$

The result now follows by taking $C_M = 2^{M/2} C'_M$. □

Remark 4.3.3. The constant

$$C_M := 2^{M/2} \sup_{\lambda \in \mathfrak{a}^*} \frac{\langle \lambda \rangle^M}{\sum_{\beta=0}^{M} \left(|\rho|^2 + |\lambda|^2\right)^{\beta/2}} \tag{4.3.11}$$

appearing in the proof of Lemma 4.3.1 will remain relevant throughout this chapter.

Let now $q : G \times \mathfrak{a}^* \to \mathbb{R}$ be a continuous negative definite symbol, K-bi-invariant in its first argument, and W-invariant in its second (for example, q could be taken to be a Gangolli symbol, as in (4.2.2)). Similarly to [45] §4 and [41] (4.26), we write

$$q(\sigma, \lambda) = q_1(\lambda) + q_2(\sigma, \lambda), \qquad \forall \sigma \in G, \lambda \in \mathfrak{a}^*, \tag{4.3.12}$$

where $q_1(\lambda) = q(\sigma_0, \lambda)$ and $q_2(\sigma, \lambda) = q(\sigma, \lambda) - q(\sigma_0, \lambda)$, for some fixed $\sigma_0 \in G$. Observe that q_1 is necessarily a negative definite symbol. Though q_2 may not be, we may still define the operator $q_2(\sigma, D)$ in a meaningful way, by

$$q_2(\sigma, D) := q(\sigma, D) - q_1(D) = \int_{\mathfrak{a}^*} \phi_\lambda(\sigma) q_2(\sigma, \lambda) \hat{f}(\lambda) \omega(d\lambda), \qquad \forall \sigma \in G.$$

By decomposing q in this way, we view it as a perturbation of a negative definite function q_1 by q_2. The assumptions we place on q will control the size of this perturbation, as well as ensuring certain regularity properties of $q(\sigma, D)$ acting on the anisotropic Sobolev spaces introduced in Section 4.1.2.

Assumptions 4.3.4. *In the notation above, we impose the following:*

1. *There exist constants $c_0, c_1 > 0$ such that for all $\lambda \in \mathfrak{a}^*$ with $|\lambda| \geq 1$,*

$$c_0(1+\psi(\lambda)) \leq q_1(\lambda) \leq c_1(1+\psi(\lambda)). \tag{4.3.13}$$

2. *Let $M \in \mathbb{N}$, $M > \dim(G/K)$, and suppose that $q_2(\cdot,\lambda) \in C_c^M(K|G|K)$ for all $\lambda \in \mathfrak{a}^*$. Suppose further that for $\beta = 0, 1, \ldots, M$, there exists $\Phi_\beta \in L^1(K|G|K)$ such that*

$$\left|(-\Delta)^{\beta/2} F_{\lambda,\eta}(\sigma)\right| \leq \Phi_\beta(\sigma)\langle\lambda\rangle^M \big(1+\psi(\eta)\big), \tag{4.3.14}$$

for all $\lambda, \eta \in \mathfrak{a}^*$, $\sigma \in G$, where $F_{\lambda,\eta}(\sigma) = \phi_{-\lambda}(\sigma) q_2(\sigma,\eta)$ (c.f. (4.3.2)).

Remarks 4.3.5. 1. These assumptions are analogues to P.1, P.2.q of [45], pp. 156, or (A.1), (A.2.M) of [41], pp.54.

2. As noted in Remark 4.3.2 (3), the conditions in Assumption 4.3.4 (2) imply that

$$(q_2(\sigma,D)u)^{\wedge}(\lambda) = \int_{\mathfrak{a}^*} \hat{F}_{\lambda,\eta}(-\eta)\hat{u}(\eta)\omega(d\eta), \tag{4.3.15}$$

for all $\lambda \in \mathfrak{a}^*$ and $u \in C_c^\infty(K|G|K)$, a fact that will be useful several times more.

Theorem 4.3.6. *Subject to Assumptions 4.3.4, for all $s \in \mathbb{R}$, $q_1(D)$ extends to a continuous operator from $H^{\psi,s+2}$ to $H^{\psi,s}$, and $q(\sigma,D)$ extends to a continuous operator from $H^{\psi,2}$ to $L^2(K|G|K)$.*

Proof. Since q_1 is continuous, it is bounded on $B_1(0)$. Let $\kappa = 2\max\{\sup_{|\lambda|<1}|q_1(\lambda)|, c_1\}$. Then, by (4.3.13),

$$q_1(\lambda) \leq \kappa(1+\psi(\lambda)), \tag{4.3.16}$$

for all $\lambda \in \mathfrak{a}^*$. Therefore, by Plancherel,

$$\begin{aligned}\|q_1(D)u\|^2_{\psi,s} &= \int_{\mathfrak{a}^*}(1+\psi(\lambda))^s|q_1(\lambda)u(\lambda)|^2\omega(d\lambda)\\ &\leq \kappa^2\int_{\mathfrak{a}^*}(1+\psi(\lambda))^{(s+2)}|u(\lambda)|^2\omega(d\lambda) = \kappa^2\|u\|^2_{\psi,s+2}.\end{aligned}$$

Taking square roots, it follows that q_1 extends continuously to an operator $H^{\psi,s+2} \to H^{\psi,s}$. Thus the first part of the theorem is proved.

For the second part, observe that by Assumption 4.3.4 (2) we may set $\mu = -\eta$ in equation (4.3.5) to obtain

$$\left|\hat{F}_{\lambda,\eta}(-\eta)\right| \leq C_M \sum_{\beta=0}^{M} \|\Phi_\beta\|_{L^1(K|G|K)}\langle\lambda-\eta\rangle^{-M}(1+\psi(\eta)), \qquad \forall \lambda,\eta \in \mathfrak{a}^*.$$

Let Ψ be given by (4.1.9). By (4.3.6) and the Plancherel theorem, for $u, v \in C_c^\infty(K|G|K)$,

$$\begin{aligned}
|\langle q_2(\sigma, D)u, v\rangle| &= \left| \int_{\mathfrak{a}^*} (q_2(\sigma, D)u)^\wedge(\lambda)\overline{\hat{v}(\lambda)}\omega(d\lambda) \right| \\
&= \left| \int_{\mathfrak{a}^*} \int_{\mathfrak{a}^*} \hat{F}_{\lambda,\eta}(-\eta)\hat{u}(\eta)\overline{\hat{v}(\lambda)}\omega(d\eta)\omega(d\lambda) \right| \\
&\le \int_{\mathfrak{a}^*} \int_{\mathfrak{a}^*} \left| \hat{F}_{\lambda,\eta}(-\eta) \right| |\hat{u}(\eta)||\hat{v}(\lambda)|\omega(d\eta)\omega(d\lambda) \\
&\le C_M \sum_{\beta=0}^{M} \|\Phi_\beta\|_{L^1(K|G|K)} \int_{\mathfrak{a}^*} \int_{\mathfrak{a}^*} \langle \lambda - \eta \rangle^{-M} \Psi(\eta)^2 |\hat{u}(\eta)||\hat{v}(\lambda)|\omega(d\eta)\omega(d\lambda) \\
&= C_M \sum_{\beta=0}^{M} \|\Phi_\beta\|_{L^1(K|G|K)} \int_{\mathfrak{a}^*} \left[\langle \cdot \rangle^{-M} * \left(\Psi^2 |\hat{u}| \right) \right](\lambda) |\hat{v}(\lambda)| \omega(d\lambda) \\
&\le C_M \sum_{\beta=0}^{M} \|\Phi_\beta\|_{L^1(K|G|K)} \left\| \langle \cdot \rangle^{-M} * \left(\Psi^2 |\hat{u}| \right) \right\|_{L^2(\mathfrak{a}^*,\omega)} \|\hat{v}\|_{L^2(\mathfrak{a}^*,\omega)}
\end{aligned}$$

where on the last line we have used the Cauchy–Schwarz inequality. Noting that $M > \dim(G/K)$, we have by Lemma 4.1.7 that $\langle \cdot \rangle^{-M} \in L^1(\mathfrak{a}^*, \omega)$. By Young's (convolution) inequality (see Simon [66] Theorem 6.6.3, page 550),

$$\begin{aligned}
\left\| \langle \cdot \rangle^{-M} * \left(\Psi^2 |\hat{u}| \right) \right\|_{L^2(\mathfrak{a}^*,\omega)} \|\hat{v}\|_{L^2(\mathfrak{a}^*,\omega)} &\le \left\| \langle \cdot \rangle^{-M} \right\|_{L^1(\mathfrak{a}^*,\omega)} \|\Psi^2 \hat{u}\|_{L^2(\mathfrak{a}^*,\omega)} \|v\| \\
&= \|\langle \cdot \rangle^{-M}\|_{L^1(\mathfrak{a}^*,\omega)} \|u\|_{\psi,2} \|v\|.
\end{aligned}$$

It follows that

$$|\langle q_2(\sigma, D)u, v\rangle| \le C_M \sum_{\beta=0}^{M} \|\Phi_\beta\|_{L^1(K|G|K)} \|\langle \cdot \rangle^{-M}\|_{L^1(\mathfrak{a}^*,\omega)} \|u\|_{\psi,2} \|v\|,$$

for all $u, v \in C_c^\infty(K|G|K)$, and hence

$$\begin{aligned}
\|q_2(\sigma, D)u\| &= \sup\{ |\langle q_2(\sigma, D)u, v\rangle| : v \in C_c^\infty(K|G|K), \|v\| = 1 \} \\
&\le C_M \sum_{\beta=0}^{M} \|\Phi_\beta\|_{L^1(K|G|K)} \|\langle \cdot \rangle^{-M}\|_{L^1(\mathfrak{a}^*,\omega)} \|u\|_{\psi,2},
\end{aligned}$$

for all $u \in C_c^\infty(K|G|K)$. Thus $q_2(\sigma, D)$ extends to a bounded linear operator $H^{\psi,2} \to L^2(K|G|K)$. □

Under an additional assumption, we are able to obtain a more powerful result.

Theorem 4.3.7. *Suppose Assumptions 4.3.4 hold, and suppose further that $s \in \mathbb{R}$ satisfies $|s - 1| + 1 + \dim(G/K) < M$. Then $q(\sigma, D)$ extends to a continuous linear operator from $H^{\psi,s+2} \to H^{\psi,s}$.*

We first need a technical lemma.

Lemma 4.3.8. *Let $s \in \mathbb{R}$ and $M \in \mathbb{N}$ be such that $|s-1|+1+\dim(G/K) < M$. Then for all $\lambda, \eta \in \mathfrak{a}^*$,*

$$|\Psi(\lambda)^s - \Psi(\eta)^s| \le C_{s,\psi} \langle \lambda - \eta \rangle^{|s-1|+1} \Psi(\eta)^{s-1}, \tag{4.3.17}$$

where

$$C_{s,\psi} = 2^{(|s-1|+2)/2}(1+c_\psi)^{(|s-1|+1)/2}|s|, \tag{4.3.18}$$

and c_ψ is the constant from Proposition 4.1.5 (3).

Proof. Proceed as in Hoh [41] page 50. By the mean value theorem, for all $x, y > 0$ we have

$$|x^s - y^s| \le |s||x-y|(x^{s-1} + y^{s-1}).$$

Therefore, given $\lambda, \eta \in \mathfrak{a}^*$,

$$|\Psi(\lambda)^s - \Psi(\eta)^s| \le |s| \, |\Psi(\lambda) - \Psi(\eta)| \left(\Psi(\lambda)^{s-1} + \Psi(\eta)^{s-1}\right).$$

Applying Proposition 4.1.5 (1), with $1+\psi$ in place of ψ,

$$|\Psi(\lambda) - \Psi(\eta)| \le \Psi(\lambda - \eta),$$

and hence

$$|\Psi(\lambda)^s - \Psi(\eta)^s| \le |s|\Psi(\lambda-\eta)\left(\Psi(\lambda)^{s-1} + \Psi(\eta)^{s-1}\right).$$

By Proposition 4.1.5 (2), together with the fact that $\Psi = \sqrt{1+\psi} \ge 1$,

$$\begin{aligned}\Psi(\lambda)^{s-1} + \Psi(\eta)^{s-1} &\le 2^{|s-1|/2}\left[\Psi(\lambda-\eta)^{|s-1|} + 1\right]\Psi(\eta)^{s-1} \\ &\le 2 \cdot 2^{|s-1|/2}\Psi(\lambda-\eta)^{|s-1|}\Psi(\eta)^{s-1}.\end{aligned}$$

Therefore

$$|\Psi(\lambda)^s - \Psi(\eta)^s| \le 2^{(|s-1|+2)/2}|s|\Psi(\lambda-\eta)^{|s-1|+1}\Psi(\eta)^{s-1}.$$

Finally, by Proposition 4.1.5 (3),

$$|\Psi(\lambda)^s - \Psi(\eta)^s| \le 2^{(|s-1|+2)/2}(1+c_\psi)^{(|s-1|+1)/2}|s|\langle\lambda-\eta\rangle^{|s-1|+1}\Psi(\eta)^{s-1}.$$

□

Proof of Theorem 4.3.7. By Theorem 4.3.6, it suffices to prove that $q_2(\sigma, D)$ extends to a continuous operator from $H^{\psi,s+2} \to H^{\psi,s}$. Given $u \in C_c^\infty(K|G|K)$,

$$\begin{aligned}\|q_2(\sigma,D)u\|_{\psi,s} &= \|\Psi(D)^s q_2(\sigma,D)u\| \\ &\le \|q_2(\sigma,D)\Psi(D)^s u\| + \|[\Psi(D)^s, q_2(\sigma,D)]u\|.\end{aligned} \tag{4.3.19}$$

Also, by Theorem 4.3.6 and Theorem 4.1.6 (2),

$$\|q_2(\sigma,D)\Psi(D)^s u\| \le C\|\Psi(D)^s u\|_{\psi,2} = C\|u\|_{\psi,s+2}, \tag{4.3.20}$$

where

$$C = C_M \sum_{\beta=0}^{M} \|\Phi_\beta\|_{L^1(K|G|K)} \|\langle\cdot\rangle^{-M}\|_{L^1(\mathfrak{a}^*,\omega)}.$$

We will estimate

$$\|[\Psi(D)^s, q_2(\sigma, D)]u\|.$$

Write $F_{\lambda,\eta} = \phi_{-\lambda} q_2(\cdot, \eta)$ once again. Then by (4.3.6) we have for all $\lambda \in \mathfrak{a}^*$,

$$\begin{aligned}
([\Psi(D)^s, q_2(\sigma, D)]u)^\wedge(\lambda) &= (\Psi(D)^s q_2(\sigma, D)u)^\wedge(\lambda) - (q_2(\sigma, D)\Psi(D)^s u)^\wedge(\lambda) \\
&= \Psi(\lambda)^s (q_2(\sigma, D)u)^\wedge(\lambda) - \int_{\mathfrak{a}^*} \hat{F}_{\lambda,\eta}(-\eta) \left(\Psi(D)^s u\right)^\wedge(\eta)\omega(d\eta) \\
&= \int_{\mathfrak{a}^*} \hat{F}_{\lambda,\eta}(-\eta)\big\{\Psi(\lambda)^s - \Psi(\eta)^s\big\}\hat{u}(\eta)\omega(d\eta).
\end{aligned}$$

Hence for all $u, v \in C_c^\infty(K|G|K)$,

$$\begin{aligned}
\left|\langle[\Psi(D)^s, q_2(\sigma, D)]u, v\rangle\right| &= \left|\int_{\mathfrak{a}^*} ([\Psi(D)^s, q_2(\sigma, D)]u)^\wedge(\lambda)\overline{\hat{v}(\lambda)}\omega(d\lambda)\right| \\
&= \left|\int_{\mathfrak{a}^*}\int_{\mathfrak{a}^*} \hat{F}_{\lambda,\eta}(-\eta)\big\{\Psi(\lambda)^s - \Psi(\eta)^s\big\}\hat{u}(\eta)\overline{\hat{v}(\lambda)}\omega(d\eta)\omega(d\lambda)\right| \\
&\leq \int_{\mathfrak{a}^*}\int_{\mathfrak{a}^*} \left|\hat{F}_{\lambda,\eta}(-\eta)\right| |\Psi(\lambda)^s - \Psi(\eta)^s| \, |\hat{u}(\eta)||\hat{v}(\lambda)|\omega(d\eta)\omega(d\lambda)
\end{aligned}$$

By Lemmas 4.3.1 and 4.3.8,

$$\begin{aligned}
&\left|\langle[\Psi(D)^s, q_2(\sigma, D)]u, v\rangle\right| \\
&\quad\leq C_{s,\psi,M} \int_{\mathfrak{a}^*}\int_{\mathfrak{a}^*} \langle\lambda - \eta\rangle^{-M+|s-1|+1}\Psi(\eta)^{s+1}|\hat{u}(\eta)||\hat{v}(\lambda)|\omega(d\eta)\omega(d\lambda) \\
&\quad= C_{s,\psi,M} \int_{\mathfrak{a}^*} \left(\langle\cdot\rangle^{-M+|s-1|+1} * \left[\Psi^{s+1}|\hat{u}|\right]\right)(\lambda)|\hat{v}(\lambda)|\omega(d\lambda),
\end{aligned}$$

where

$$C_{s,\psi,M} = C_{s,\psi} C_M \sum_{\beta=0}^{M} \|\Phi_\beta\|_{L^1(K|G|K)}.$$

Now, we are assuming $M - |s-1| - 1 > \dim(G/K)$, and so by Lemma 4.1.7, $\langle\cdot\rangle^{-(M-|s-1|-2)} \in L^1(\mathfrak{a}^*, \omega)$. Thus by the Cauchy–Schwarz and Young inequalities,

$$\begin{aligned}
\left|\langle[\Psi(D)^s, q_2(\sigma, D)]u, v\rangle\right| &\leq C_{s,\psi,M} \left\|\langle\cdot\rangle^{-(M-|s-1|-1)} * \left[\Psi^{s+1}|\hat{u}|\right]\right\|_{L^2(\mathfrak{a}^*,\omega)} \|v\| \\
&\leq C_{s,\psi,M} \left\|\langle\cdot\rangle^{-(M-|s-1|-1)}\right\|_{L^1(\mathfrak{a}^*,\omega)} \left\|\Psi^{s+1}|\hat{u}|\right\|_{L^2(\mathfrak{a}^*,\omega)} \|v\| \\
&= C_{s,\psi,M} \left\|\langle\cdot\rangle^{-(M-|s-1|-1)}\right\|_{L^1(\mathfrak{a}^*,\omega)} \|u\|_{\psi,s+1}\|v\|.
\end{aligned}$$

Finally,

$$\begin{aligned}\|[\Psi(D)^s, q_2(\sigma,D)]u\| &= \sup\left\{\left|\left\langle[\Psi(D)^s, q_2(\sigma,D)]u, v\right\rangle\right| : v \in C_c^\infty(K|G|K), \|v\| = 1\right\}\\ &\leq C_{s,\psi,M}\left\|\langle\cdot\rangle^{-(M-|s-1|-1)}\right\|_{L^1(\mathfrak{a}^*,\omega)}\|u\|_{\psi,s+1},\end{aligned}$$

and so combining (4.3.19) and (4.3.20),

$$\|q_2(\sigma,D)u\|_{\psi,s} \leq C_M\sum_{\beta=0}^{M}\|\Phi_\beta\|_{L^1(\mathfrak{a}^*,\omega)}\Big(\|\langle\cdot\rangle^{-M}\|_{L^1(\mathfrak{a}^*,\omega)}\|u\|_{\psi,s+2} + C_{s,\psi}\|u\|_{\psi,s+1}\Big). \quad (4.3.21)$$

Inclusion $H^{\psi,s+2} \hookrightarrow H^{\psi,s+1}$ is continuous, and so it is clear from this that

$$\|q_2(\sigma,D)u\|_{\psi,s} \leq \kappa\|u\|_{\psi,s+2}$$

for some constant κ. □

To prove (4.3.1), we seek solutions u to the equation

$$(q(\sigma,D)+\alpha)u = f, \quad (4.3.22)$$

for a given function f and $\alpha > 0$. Consider the bilinear form B_α defined by

$$B_\alpha(u,v) = \langle(q(\sigma,D)+\alpha)u, v\rangle, \qquad \forall u,v \in C_c^\infty(K|G|K).$$

Theorem 4.3.9. *Suppose Assumptions 4.3.4 hold with $M > \dim(G/K)+1$. Then B_α extends continuously to $H^{\psi,1}\times H^{\psi,1}$.*

Proof. Let $u, v \in H^{\psi,1}$. By the same argument as that of equation (4.3.16), there is $\kappa_1 > 0$ such that $|q_1| \leq \kappa_1\Psi^2$. Therefore, by Parseval,

$$\begin{aligned}|\langle q_1(D)u, v\rangle| &\leq \int_{\mathfrak{a}^*}|q_1(\lambda)||\hat{u}(\lambda)||\hat{v}(\lambda)|\omega(d\lambda)\\ &\leq \kappa_1\int_{\mathfrak{a}^*}\Psi(\lambda)|\hat{u}(\lambda)|\Psi(\lambda)|\hat{v}(\lambda)|\omega(d\lambda) \leq \kappa_1\|u\|_{\psi,1}\|v\|_{\psi,1}.\end{aligned}$$

By (4.3.6),

$$\begin{aligned}|\langle q_2(\sigma,D)u, v\rangle| &\leq \int_{\mathfrak{a}^*}|(q_2(\sigma,D)u)^\wedge(\lambda)||\hat{v}(\lambda)|\omega(d\lambda)\\ &\leq \int_{\mathfrak{a}^*}\int_{\mathfrak{a}^*}\left|\hat{F}_{\lambda,\eta}(-\eta)\right||\hat{u}(\eta)||\hat{v}(\lambda)|\omega(d\eta)\omega(d\lambda)\\ &\leq C_M\sum_{\beta=0}^{M}\|\Phi_\beta\|_{L^1(\mathfrak{a}^*,\omega)}\int_{\mathfrak{a}^*}\int_{\mathfrak{a}^*}\langle\lambda-\eta\rangle^{-M}\Psi(\eta)^2|\hat{u}(\eta)\hat{v}(\lambda)|\omega(d\eta)\omega(d\lambda)\end{aligned}$$

By Proposition 4.1.5 (2) and (3),

$$\Psi(\eta) \leq \sqrt{2}\Psi(\lambda)\Psi(\lambda-\eta) \leq \sqrt{2(1+c_\psi)}\Psi(\lambda)\langle\lambda-\eta\rangle,$$

for all $\lambda, \eta \in \mathfrak{a}^*$. Hence, writing

$$\kappa_2 = C_M\sqrt{2(1+c_\psi)}\sum_{\beta=0}^{M}\|\Phi_\beta\|_{L^1(\mathfrak{a}^*,\omega)}, \tag{4.3.23}$$

we have

$$\begin{aligned}
|\langle q_2(\sigma,D)u,v\rangle| &\leq \kappa_2\int_{\mathfrak{a}^*}\langle\lambda-\eta\rangle^{-M+1}\Psi(\eta)|\hat{u}(\eta)|\Psi(\lambda)|\hat{v}(\lambda)|\omega(d\eta)\omega(d\lambda)\\
&= \kappa_2\int_{\mathfrak{a}^*}\left(\langle\cdot\rangle^{-M+1} * \Psi|\hat{u}|\right)(\lambda)\Psi(\lambda)|\hat{v}(\lambda)|\omega(d\lambda)\\
&\leq \kappa_2\left\|\langle\cdot\rangle^{-M+1} * \Psi|\hat{u}|\right\|_{L^2(\mathfrak{a}^*,\omega)}\|v\|_{\psi,1},
\end{aligned}$$

by the Cauchy–Schwarz inequality. By Lemma 4.1.7, $\langle\cdot\rangle^{-M+1} \in L^1(\mathfrak{a}^*,\omega)$, and so by Young's inequality,

$$|\langle q_2(\sigma,D)u,v\rangle| \leq \kappa_2\left\|\langle\cdot\rangle^{-M+1}\right\|_{L^1(\mathfrak{a}^*,\omega)}\|u\|_{\psi,1}\|v\|_{\psi,1}. \tag{4.3.24}$$

Inclusion $H^{\psi,1} \hookrightarrow L^2(K|G|K)$ is continuous, so we may choose $\kappa_3 > 0$ such that $\|u\| \leq \kappa_3\|u\|_{\psi,1}$. Then we have

$$\begin{aligned}
|B_\alpha(u,v)| &\leq |\langle q_1(D)u,v\rangle| + |\langle q_2(\sigma,D)u,v\rangle| + \alpha\,|\langle u,v\rangle|\\
&\leq \kappa_1\|u\|_{\psi,1}\|v\|_{\psi,2} + \kappa_2\|u\|_{\psi,1}\|v\|_{\psi,2} + \alpha\|u\|\|v\|\\
&\leq \left(\kappa_1+\kappa_2+\alpha\kappa_3^2\right)\|u\|_{\psi,1}\|v\|_{\psi,2},
\end{aligned}$$

which proves the theorem. □

Recall that a bilinear form B defined on a Hilbert space $(H,\langle\cdot,\cdot\rangle)$ is *coercive* if there is $c > 0$ such that

$$B(u,u) \geq c\langle u,u\rangle \qquad \forall u \in H.$$

Theorem 4.3.10 (Lax–Milgram)**.** *Let B be a bounded bilinear form, defined on a Hilbert space $(H,\langle\cdot,\cdot,\rangle)$, and suppose B is coercive with constant c. Then given $f \in H'$, there is a unique $u \in H$ such that*

$$B(u,v) = f(v) \qquad \forall v \in H.$$

Proof. See for example Yosida [70] page 92. □

The following assumption will ensure that for α sufficiently large, B_α is coercive on $H^{\psi,1}$. We will then use the Lax–Milgram theorem to obtain a weak solution to (4.3.22).

Assumption 4.3.11. *Let $M \in \mathbb{N}$, $M > \dim(G/K) + 1$, and write*

$$\gamma_M = \left(8C_M(2(1+c_\psi))^{1/2}\|\langle\cdot\rangle^{-M+1}\|_{L^1(\mathfrak{a}^*,\omega)}\right)^{-1},$$

where c_ψ and C_M are constants given by (4.1.4) and (4.3.11), respectively.

For c_0 is as in Assumption 4.3.4 (1), assume that

$$\sum_{\beta=0}^{M} \|\Phi_\beta\|_1 \leq \gamma_M c_0.$$

Remark 4.3.12. See [45] P.3 and P.4, pp. 161, or [41] (A.3.M), pp. 54, for comparison. Examples where Assumption 4.3.11 is satisfied are considered in Section 4.4.

The next theorem is an analogue of Theorem 3.1 of [45].

Theorem 4.3.13. *Suppose Assumptions 4.3.4 and 4.3.11 hold, with $M > \dim(G/K) + 1$. Then there is $\alpha_0 > 0$ such that*

$$B_\alpha(u,u) \geq \frac{c_0}{2}\|u\|^2_{1,\lambda},$$

for all $\alpha \geq \alpha_0$ and $u \in H^{\psi,1}$. In particular, B_α is coercive for all $\alpha \geq \alpha_0$.

Proof. Proceed exactly as in [41] page 57, lines 8–17. By Assumption 4.3.4 (1), there is $\alpha_0 > 0$ such that

$$q_1(\lambda) \geq c_0\Psi(\lambda)^2 - \alpha_0 \qquad \forall \lambda \in \mathfrak{a}^*. \tag{4.3.25}$$

Therefore, for all $u \in H^{\psi,1}$,

$$\begin{aligned}\langle q_1(D)u,u\rangle = \int_{\mathfrak{a}^*} q_1(\lambda)|\hat{u}(\lambda)|^2\omega(d\lambda) &\geq \int_{\mathfrak{a}^*}(c_0\Psi(\lambda)^2 - \alpha_0)|\hat{u}(\lambda)|^2\omega(d\lambda)\\ &= c_0\|u\|^2_{\psi,1} - \alpha_0\|u\|^2.\end{aligned}$$

Using equations (4.3.23) and (4.3.24), as well as Assumption 4.3.11, we also have

$$\begin{aligned}|\langle q_2(\sigma,D)u,u\rangle| &\leq C_M\sqrt{2(1+c_\psi)}\sum_{\beta=0}^{M}\|\Phi_\beta\|_{L^1(\mathfrak{a}^*,\omega)}\left\|\langle\cdot\rangle^{-M+1}\right\|_{L^1(\mathfrak{a}^*,\omega)}\|u\|^2_{\psi,1}\\ &= \frac{1}{8\gamma_M}\sum_{\beta=0}^{M}\|\Phi_\beta\|_{L^1(\mathfrak{a}^*,\omega)}\|u\|^2_{\psi,1} \leq \frac{c_0}{8}\|u\|_{\psi,1}\end{aligned}$$

Thus,

$$\begin{aligned}\langle q(\sigma,D)u,u\rangle &\geq \langle q_1(D)u,u\rangle - |\langle q_2(\sigma,D)u,u\rangle|\\ &\geq (c_0 - \frac{c_0}{8})\|u\|^2_{\psi,1} - \alpha_0\|u\|^2_{\psi,1}\\ &\geq \frac{c_0}{2}\|u\|^2_{\psi,1} - \alpha_0\|u\|^2_{\psi,1},\end{aligned}$$

and so for all $\alpha \geq \alpha_0$,

$$\begin{aligned} B_\alpha(u,u) &= \langle q(\sigma,D)u,u\rangle + \alpha\|u\| \\ &\geq \langle q(\sigma,D)u,u\rangle + \alpha_0\|u\| \geq \frac{c_0}{2}\|u\|_{\psi,1}^2. \end{aligned}$$

□

Theorem 4.3.14. *Let $\alpha \geq \alpha_0$. Then (4.3.22) has a weak solution in the following sense: for all $f \in L^2(K|G|K)$ there is a unique $u \in H^{\psi,1}$ such that for all $v \in H^{\psi,1}$,*

$$B_\alpha(u,v) = \langle f,v\rangle.$$

Proof. Apply the Lax–Milgram theorem (Theorem 1 of [28], pp. 297) to B_α, using the linear functional $v \mapsto \langle f,v\rangle$. □

Having found a weak solution to (4.3.22), the next task is to prove that this solution is also a strong solution, and belongs to $C_0(K|G|K)$. We use the Sobolev embedding of Theorem 4.1.6 (6).

Just as in [45] Theorem 3.1 and [41] Theorem 4.11, we have a useful lower bound for the pseudodifferential operator $q(\sigma,D)$ acting on $H^{\psi,s}$, when $s \geq 0$.

Theorem 4.3.15. *Let $s \geq 0$, and suppose the symbol q satisfies Assumptions 4.3.4 and 4.3.11, for some $M > |s-1| + 1 + \dim(G/K)$. Then there is $\kappa > 0$ such that for all $u \in H^{\psi,s+2}$,*

$$\|q(\sigma,D)u\|_{\psi,s} \geq \frac{c_0}{4}\|u\|_{\psi,s+2} - \kappa\|u\|.$$

Proof. The proof is formally no different to the sources mentioned. Let $u \in H^{\psi,s+2}$. By (4.3.25),

$$\begin{aligned} \|q_1(D)u\|_{\psi,s} = \|\Psi^s q_1 u\|_{L^2(\mathfrak{a}^*,\omega)} &\geq \|c_0\Psi^{s+2}u - \alpha_0\Psi^s u\|_{L^2(\mathfrak{a}^*,\omega)} \\ &\geq c_0\|u\|_{\psi,s+2} - \alpha_0\|u\|_{\psi,s}. \end{aligned}$$

By Theorem 4.1.6 (5), we may choose $\kappa_1 > 0$ such that

$$\alpha_0\|u\|_{\psi,s} \leq \frac{c_0}{2}\|u\|_{\psi,s+2} + \kappa_1\|u\|,$$

and thus

$$\|q_1(D)u\|_{\psi,s} \geq \frac{c_0}{2}\|u\|_{\psi,s+2} - \kappa_1\|u\|. \tag{4.3.26}$$

Apply the estimate (4.3.21) of $\|q_2(\sigma, D)u\|_{\psi,s}$ from the proof of Theorem 4.3.7. By Assumption 4.3.11,

$$\begin{aligned}
\|q_2(\sigma, D)u\|_{\psi,s} &\leq C_M \sum_{\beta=0}^{M} \|\Phi_\beta\|_{L^1(\mathfrak{a}^*,\omega)} \Big(\|\langle\cdot\rangle^{-M}\|_{L^1(\mathfrak{a}^*,\omega)} \|u\|_{\psi,s+2} + C_{s,\psi}\|u\|_{\psi,s+1} \Big) \\
&\leq C_M c_0 \gamma_M \Big(\|\langle\cdot\rangle^{-M}\|_{L^1(\mathfrak{a}^*,\omega)} \|u\|_{\psi,s+2} + C_{s,\psi}\|u\|_{\psi,s+1} \Big) \\
&= \frac{C_M c_0 \|\langle\cdot\rangle^{-M}\|_{L^1(\mathfrak{a}^*,\omega)} \|u\|_{\psi,s+2}}{8C_M\sqrt{2(1+c_\psi)}\|\langle\cdot\rangle^{-M+1}\|_{L^1(\mathfrak{a}^*,\omega)}} + c\|u\|_{\psi,s+1},
\end{aligned}$$

where $c > 0$ is a(nother) constant. Observe that, since $\langle\lambda\rangle \geq 1$ for all $\lambda \in \mathfrak{a}^*$, it is the case that $\|\langle\cdot\rangle^{-M}\|_{L^1(\mathfrak{a}^*,\omega)} \leq \|\langle\cdot\rangle^{-M+1}\|_{L^1(\mathfrak{a}^*,\omega)}$, and therefore

$$\|q_2(\sigma, D)u\|_{\psi,s} \leq \frac{c_0}{8}\|u\|_{\psi,s+2} + c\|u\|_{\psi,s+1}.$$

Using Theorem 4.1.6 (5) once more, we may then choose $\kappa_2 > 0$ such that

$$c\|u\|_{\psi,s+1} \leq \frac{c_0}{8}\|u\|_{\psi,s+2} + \kappa_2\|u\|.$$

Then, by the above,

$$\|q_2(\sigma, D)u\|_{\psi,s} \leq \frac{c_0}{4}\|u\|_{\psi,s+2} + \kappa_2\|u\|. \tag{4.3.27}$$

Combining (4.3.26) and (4.3.27), we get

$$\begin{aligned}
\|q(\sigma, D)u\|_{\psi,s} &\geq \|q_1(D)u\|_{\psi,s} - \|q_2(\sigma, D)u\|_{\psi,s} \\
&\geq \frac{c_0}{2}\|u\|_{\psi,s+2} - \kappa_1\|u\| - \Big(\frac{c_0}{4}\|u\|_{\psi,s+2} + \kappa_2\|u\|\Big) \\
&= \frac{c_0}{4}\|u\|_{\psi,s+2} - (\kappa_1+\kappa_2)\|u\|.
\end{aligned}$$

Putting $\kappa = \kappa_1 + \kappa_2$, the theorem follows. □

The proof of the next theorem makes use of a particular family $(J_\epsilon, 0 < \epsilon \leq 1)$ of bounded linear operators on $L^2(K|G|K)$, which will play the role of a Friedrich mollifier, but in the noncompact symmetric space setting.

First note that by identifying $\mathfrak{a}$ with $\mathbb{R}^m$ via our chosen basis, it makes sense to consider Friedrich mollifiers on $\mathfrak{a}$. For $0 < \epsilon \leq 1$ and $H \in \mathfrak{a}$, let

$$l(H) \coloneqq C_0 e^{\frac{1}{|H|^2-1}} \mathbf{1}_{B_1(0)}(H), \quad \text{and} \quad l_\epsilon(H) \coloneqq \epsilon^{-m} l(H/\epsilon),$$

where $C_0 > 0$ is a constant chosen so that $\int_{\mathfrak{a}} l(H)dH = 1$. This mollifier is used frequently in [28] (see Appendix C.4, pp. 629), and [45] and [41] use it to pass from a weak solution result to a strong solution result.

Observe that $l, l_\epsilon \in \mathcal{S}(\mathfrak{a})^W$ for all $0 < \epsilon \leq 1$. Using Theorem 2.4.30, let $j, j_\epsilon \in \mathcal{S}(K|G|K)$ be such that

$$\hat{j} = \mathcal{F}(l), \quad \text{and} \quad \hat{j}_\epsilon = \mathcal{F}(l_\epsilon), \qquad \forall 0 < \epsilon \leq 1,$$

where $\mathcal{F}$ denotes the Euclidean Fourier transform (see equation (2.4.24)). For $0 < \epsilon \leq 1$, let J_ϵ be the convolution operator defined on $L^2(K|G|K)$ by

$$J_\epsilon u = j_\epsilon * u \qquad \forall f \in L^2(K|G|K).$$

The most important properties of $(J_\epsilon, 0 < \epsilon \leq 1)$ needed for the proof of Theorem 4.3.18 are stated below, and proven in the appendix.

Proposition 4.3.16. *1. $\hat{j}_\epsilon(\lambda) = \hat{j}(\epsilon\lambda)$ for all $0 < \epsilon \leq 1$ and $\lambda \in \mathfrak{a}^*$.*

2. *For all $0 < \epsilon \leq 1$, J_ϵ is a self-adjoint contraction of $L^2(K|G|K)$.*

3. *$J_\epsilon u \in H^{\psi,s}$ for all $s \geq 0$, $u \in L^2(K|G|K)$ and $0 < \epsilon \leq 1$, and if $u \in H^{\psi,s}$, then*

$$\|J_\epsilon u\|_{\psi,s} \leq \|u\|_{\psi,s}.$$

4. *For all $s \geq 0$ and $u \in H^{\psi,s}$, $\|J_\epsilon u - u\|_{\psi,s} \to 0$ as $\epsilon \to 0$.*

The following commutator estimate will also be useful in the proof of Theorem 4.3.18.

Lemma 4.3.17. *Let $s \geq 0$, and suppose q is a continuous negative definite symbol satisfying Assumption 4.3.4 (2) for $M > |s-1| + 1 + \dim(G/K)$. Then there is $c > 0$ such that for all $0 < \epsilon \leq 1$ and all $u \in C_c^\infty(K|G|K)$,*

$$\|[J_\epsilon, q(\sigma, D)]u\|_{\psi,s} \leq c\|u\|_{\psi,s+1}.$$

Proof. Let $0 < \epsilon \leq 1$ and $u \in C_c^\infty(K|G|K)$, and observe that by Proposition 4.3.16 (2),

$$([J_\epsilon, q_1(D)]u)^\wedge(\lambda) = \hat{j}(\epsilon\lambda)q_1(\lambda)\hat{u}(\lambda) - q_1(\lambda)\hat{j}(\epsilon\lambda)\hat{u}(\lambda) = 0,$$

for all $\lambda \in \mathfrak{a}^*$, so $[J_\epsilon, q_1(D)]u = 0$. For $\lambda, \eta \in \mathfrak{a}^*$, let $F_{\lambda,\eta} = \phi_{-\lambda} q_2(\cdot, \eta)$, as previously (c.f. (4.3.2)). Then by (4.3.6) and Proposition 4.3.16 (1), for all $\lambda \in \mathfrak{a}^*$,

$$\begin{aligned} ([J_\epsilon, q(\sigma, D)]u)^\wedge(\lambda) &= (J_\epsilon q_2(\sigma, D)u)^\wedge(\lambda) - (q_2(\sigma, D)J_\epsilon u)^\wedge(\lambda) \\ &= \hat{j}(\epsilon\lambda)(q_2(\sigma, D)u)^\wedge(\lambda) - \int_{\mathfrak{a}^*} \hat{F}_{\lambda,\eta}(-\eta)(J_\epsilon u)^\wedge(\eta)\omega(d\eta) \\ &= \hat{j}(\epsilon\lambda)(q_2(\sigma, D)u)^\wedge(\lambda) - \int_{\mathfrak{a}^*} \hat{F}_{\lambda,\eta}(-\eta)\hat{j}(\epsilon\eta)\hat{u}(\eta)\omega(d\eta). \end{aligned}$$

Applying (4.3.6) once more,

$$([J_\epsilon, q(\sigma, D)]u)^\wedge(\lambda) = \int_{\mathfrak{a}^*} \hat{F}_{\lambda,\eta}(-\eta)\left(\hat{j}(\epsilon\lambda) - \hat{j}(\epsilon\eta)\right)\hat{u}(\eta)\omega(d\eta), \tag{4.3.28}$$

for all $\lambda \in \mathfrak{a}^*$.

Claim. There is a constant $c > 0$ such that for all $0 < \epsilon \leq 1$ and $\lambda, \eta \in \mathfrak{a}^*$,

$$\left|\hat{j}(\epsilon\lambda) - \hat{j}(\epsilon\eta)\right| \langle\lambda\rangle \leq c\langle\lambda - \eta\rangle. \tag{4.3.29}$$

Proof of Claim. Let $0 < \epsilon \leq 1$ and $\lambda, \eta \in \mathfrak{a}^*$. By the mean value theorem,

$$\left|\hat{j}(\epsilon\lambda) - \hat{j}(\epsilon\eta)\right| \leq |\epsilon\lambda - \epsilon\eta| \left|\nabla\hat{j}\big(\epsilon[(1-t)\lambda + t\eta]\big)\right|,$$

for some $0 < t < 1$. Since $\hat{j}$ is rapidly decreasing, we may choose $\kappa > 0$ such that $|\nabla\hat{j}(\lambda)| \leq \kappa(1 + |\lambda|^2)^{-1/2}$ for all $\lambda \in \mathfrak{a}^*$. Suppose first that $\lambda, \eta \in \mathfrak{a}^*$ satisfy $|\lambda - \eta| < \frac{|\lambda|}{2}$. Then

$$|(1-t)\lambda + t\eta)| \geq |\lambda| - t|\lambda - \eta| \geq |\lambda| - t\frac{|\lambda|}{2} \geq \frac{|\lambda|}{2},$$

and hence

$$\left|\hat{j}(\epsilon\lambda) - \hat{j}(\epsilon\eta)\right| \leq |\epsilon\lambda - \epsilon\eta|\kappa(1 + |\epsilon^2\lambda/2|^2)^{-1/2} \leq \kappa'\frac{\langle\lambda - \eta\rangle}{\langle\lambda\rangle},$$

for some constant $\kappa' > 0$. Note that for all $\lambda \in \mathfrak{a}^*$, $\left|\hat{j}(\lambda)\right| \leq C_0 \int_{\|H\|<1} e^{\frac{1}{|H|^2-1}} dH = 1$. Therefore, if $|\lambda - \eta| \geq \frac{|\lambda|}{2}$,

$$\left|\hat{j}(\epsilon\lambda) - \hat{j}(\epsilon\eta)\right| \langle\lambda\rangle \leq 2\langle\lambda\rangle = 2\sqrt{1 + |\lambda|^2} \leq 2\sqrt{1 + 4|\lambda - \eta|^2} \leq 4\langle\lambda - \eta\rangle.$$

Taking $c = \max\{4, \kappa'\}$, we have established (4.3.29).

We can now apply the claim to (4.3.28) to get

$$\begin{aligned}\left|\big([J_\epsilon, q(\sigma, D)]u\big)^\wedge(\lambda)\right| &\leq \int_{\mathfrak{a}^*} \left|\hat{F}_{\lambda,\eta}(-\eta)\right| \left|\hat{j}(\epsilon\lambda) - \hat{j}(\epsilon\eta)\right| |\hat{u}(\eta)|\omega(d\eta) \\ &\leq c\int_{\mathfrak{a}^*} \left|\hat{F}_{\lambda,\eta}(-\eta)\right| \frac{\langle\lambda - \eta\rangle}{\langle\lambda\rangle}|\hat{u}(\eta)|\omega(d\eta).\end{aligned}$$

for all $\lambda \in \mathfrak{a}^*$. By Lemma 4.3.1,

$$\left|\hat{F}_{\lambda,\eta}(-\eta)\right| \leq C_M \sum_{\beta=0}^{M} \|\Phi_\beta\|_{L^1(K|G|K)} \langle\lambda - \eta\rangle^{-M}\Psi(\eta)^2,$$

for all $\lambda, \eta \in \mathfrak{a}^*$. Therefore, writing $C = C_M \sum_{\beta=0}^{M} \|\Phi_\beta\|_{L^1(K|G|K)} c$,

$$\left|\big([J_\epsilon, q(\sigma, D)]u\big)^\wedge(\lambda)\right| \leq C\int_{\mathfrak{a}^*} \langle\lambda - \eta\rangle^{-M+1}\frac{\Psi(\eta)^2}{\langle\lambda\rangle}|\hat{u}(\eta)|\omega(d\eta),$$

for all $\lambda \in \mathfrak{a}^*$. Let $s \geq 0$ and $\lambda \in \mathfrak{a}^*$. Then,

$$\Psi(\lambda)^s \left|([J_\epsilon, q(\sigma, D)]u)^\wedge(\lambda)\right| \leq C\int_{\mathfrak{a}^*} \langle\lambda - \eta\rangle^{-M+1}\frac{\Psi(\eta)^2}{\langle\lambda\rangle}\Psi(\lambda)^s|\hat{u}(\eta)|\omega(d\eta).$$

By Proposition 4.1.5 (3), $\Psi(\lambda) \leq \sqrt{1+c_\psi}\langle\lambda\rangle$, and so

$$\Psi(\lambda)^s \left|([J_\epsilon, q(\sigma, D)]u)^\wedge(\lambda)\right| \leq C\sqrt{1+c_\psi}\int_{\mathfrak{a}^*}\langle\lambda-\eta\rangle^{-M+1}\Psi(\eta)^2\Psi(\lambda)^{s-1}|\hat{u}(\eta)|\omega(d\eta).$$

By Proposition 4.1.5 (2), $\Psi(\lambda)^{s-1} \leq 2^{|s-1|/2}\Psi(\eta)^{s-1}\Psi(\lambda-\eta)^{|s-1|}$, hence

$$\begin{aligned}\Psi(\lambda)^s &\left|([J_\epsilon, q(\sigma, D)]u)^\wedge(\lambda)\right| \\ &\leq 2^{|s-1|/2}C\sqrt{1+c_\psi}\int_{\mathfrak{a}^*}\langle\lambda-\eta\rangle^{-M+1}\Psi(\eta)^{s+1}\Psi(\lambda-\eta)^{|s-1|}|\hat{u}(\eta)|\omega(d\eta).\end{aligned}$$

Applying Proposition 4.1.5 (3) once more, and relabelling the constant C,

$$\begin{aligned}\Psi(\lambda)^s \left|([J_\epsilon, q(\sigma, D)]u)^\wedge(\lambda)\right| &\leq C\int_{\mathfrak{a}^*}\langle\lambda-\eta\rangle^{-M+1+|s-1|}\Psi(\eta)^{s+1}|\hat{u}(\eta)|\omega(d\eta) \\ &= C\left(\langle\cdot\rangle^{-M+1+|s-1|} * \Psi^{s+1}|\hat{u}|\right)(\lambda).\end{aligned}$$

By assumption, $M-1-|s-1| > \dim(G/K)$, and so $\langle\cdot\rangle^{-M+1+|s-1|} \in L^1(\mathfrak{a}, \omega)$ by Lemma 4.1.7. Hence by Parseval, Cauchy–Schwarz and Young's inequality,

$$\begin{aligned}\|[J_\epsilon, q(\sigma, D)]u\|_{\psi,s} &= \|\Psi^s\left([J_\epsilon, q(\sigma, D)]u\right)^\wedge\|_{L^2(\mathfrak{a}^*,\omega)} \\ &\leq C'\left\|\langle\cdot\rangle^{-M+1+|s-1|} * \Psi^{s+1}|\hat{u}|\right\|_{L^2(\mathfrak{a}^*,\omega)} \\ &\leq C'\|\langle\cdot\rangle^{-M+1+|s-1|}\|_{L^1(\mathfrak{a}^*,\omega)}\|u\|_{\psi,s+1}.\end{aligned}$$

□

We are now ready to state and prove that, subject to our conditions, a strong solution to (4.3.22) exists, and belongs to an anisotropic Sobolev space of suitably high order.

Theorem 4.3.18. *Let α_0 be as in Theorem 4.3.13, let $\alpha \geq \alpha_0$, and let $s \geq 0$. Suppose that the negative definite symbol q satisfies Assumptions 4.3.4 and 4.3.11, where $M > |s-1|+1+\dim(G/K)$. Then for all $f \in H^{\psi,s}$, there is a unique $u \in H^{\psi,s+2}$ such that*

$$(q(\sigma, D)+\alpha)u = f. \tag{4.3.30}$$

Proof. Let $f \in H^{\psi,s}$. By Theorem 4.1.6 we also have $f \in L^2(K|G|K)$, and so by Theorem 4.3.14 there is a unique $u \in H^{\psi,1}$ such that

$$B_\alpha(u,v) = \langle f, v\rangle \qquad \forall v \in C_c^\infty(K|G|K). \tag{4.3.31}$$

Let us first show that $u \in H^{\psi,s+2}$. The proof follows that of [45] Theorem 4.3, pp. 163 and [41] Theorem 4.12, pp. 59. Proceed by induction: we prove that $u \in H^{\psi,t}$ for $1 \leq t \leq s+2$. Note that since the $H^{\psi,t}$ are nested, it suffices to prove that if $u \in H^{\psi,t}$ for some $1 \leq t \leq s+1$, then $u \in H^{\psi,t+1}$ (the base case $t = 1$ is covered by Theorem 4.3.14). Suppose that $u \in H^{\psi,t}$ for some $1 \leq t \leq s+1$. We show that

$$\sup_{0<\epsilon\leq 1} \|J_\epsilon u\|_{\psi,t+1} < \infty, \tag{4.3.32}$$

from which $u \in H^{\psi,t+1}$ will follow. By Theorem 4.3.15,

$$\begin{aligned} \|J_\epsilon u\|_{\psi,t+1} &\leq \frac{4}{c_0}\big(\|q(\sigma,D)J_\epsilon u\|_{\psi,t-1} + \kappa\|J_\epsilon u\|\big) \\ &\leq \frac{4}{c_0}\big(\|(q(\sigma,D)+\alpha)J_\epsilon u\|_{\psi,t-1} + \alpha\|J_\epsilon u\|_{\psi,t-1} + \kappa\|J_\epsilon u\|\big) \\ &\leq \frac{4}{c_0}\big(\|(q(\sigma,D)+\alpha)J_\epsilon u\|_{\psi,t-1} + \alpha\|u\|_{\psi,t-1} + \kappa\|u\|\big) \end{aligned} \tag{4.3.33}$$

Claim. $\|(q(\sigma,D)+\alpha)J_\epsilon u\|_{\psi,t-1}$ is uniformly bounded in ϵ.

Proof of Claim. Let (u_n) be a sequence in $C_c^\infty(K|G|K)$ converging to u in $H^{\psi,t}$. Then for all $v \in C_c^\infty(K|G|K)$ and $0 < \epsilon \leq 1$,

$$\begin{aligned} \langle (q(\sigma,D)+\alpha)J_\epsilon u_n, v\rangle &= \langle J_\epsilon(q(\sigma,D)+\alpha)u_n, v\rangle - \langle [J_\epsilon, q(\sigma,D)]u_n, v\rangle \\ &= \langle (q(\sigma,D)+\alpha)u_n, J_\epsilon v\rangle - \langle [J_\epsilon, q(\sigma,D)]u_n, v\rangle, \end{aligned}$$

where we have used the fact that J_ϵ is a symmetric operator on $L^2(K|G|K)$. By Lemma 4.3.17, there is $\kappa_2 > 0$ such that

$$\|[J_\epsilon, q(\sigma,D)]u_i - [J_\epsilon, q(\sigma,D)]u_j\|_{\psi,t-1} \leq \kappa_2\|u_i - u_j\|_{\psi,t},$$

for all $i, j \in \mathbb{N}$, and this choice of κ_2 is independent of ϵ. It follows that $([J_\epsilon, q(\sigma,D)]u_n)$ is a Cauchy sequence in $H^{\psi,t-1}$, and hence converges to some $w_\epsilon \in H^{\psi,t-1}$, as $n \to \infty$. Moreover,

$$\|[J_\epsilon, q(\sigma,D)]u_n\|_{\psi,t-1} \leq \kappa_2\|u_n\|_{\psi,t},$$

and hence, taking limits as $n \to \infty$,

$$\|w_\epsilon\|_{\psi,t-1} \leq \kappa_2\|u\|_{\psi,t}, \qquad \forall 0 < \epsilon \leq 1.$$

In particular, $\|w_\epsilon\|_{\psi,t-1}$ is uniformly bounded in ϵ.

From the above, for all $0 < \epsilon \leq 1$ and $v \in C_c^\infty(K|G|K)$,

$$\begin{aligned} \langle (q(\sigma,D)+\alpha)J_\epsilon u, v\rangle &= \lim_{n\to\infty} \langle (q(\sigma,D)+\alpha)J_\epsilon u_n, v\rangle \\ &= \lim_{n\to\infty} \{\langle (q(\sigma,D)+\alpha)u_n, J_\epsilon v\rangle - \langle [J_\epsilon, q(\sigma,D)]u_n, v\rangle\} \\ &= \langle f, J_\epsilon v\rangle - \langle w_\epsilon, v\rangle = \langle J_\epsilon f - w_\epsilon, v\rangle, \end{aligned}$$

since J_ϵ is symmetric. Therefore, using duality of the spaces $H^{\psi,t-1}$ and $H^{\psi,1-t}$,

$$\begin{aligned} \|(q(\sigma,D)+\alpha)J_\epsilon u\|_{\psi,t-1} &= \sup\big\{\, |\langle (q(\sigma,D)+\alpha)J_\epsilon u, v\rangle| : v \in C_c^\infty(K|G|K), \|v\|_{1-t} = 1\big\} \\ &\leq \sup\big\{\, |\langle J_\epsilon f - w_\epsilon, v\rangle| : v \in C_c^\infty(K|G|K), \|v\|_{1-t} = 1\big\} \\ &= \|J_\epsilon f - w_\epsilon\|_{\psi,t-1} \\ &\leq \|J_\epsilon f\|_{\psi,t-1} + \|w_\epsilon\|_{\psi,t-1} \leq \|f\|_{\psi,t-1} + \kappa_2\|u\|_{\psi,t}, \end{aligned}$$

for all $0 < \epsilon \leq 1$. This completes the proof of the claim.

The claim together with (4.3.33) imply (4.3.32). Therefore, we may choose a sequence (ϵ_n) in $(0, 1]$, with $\epsilon_n \to 0$ as $n \to \infty$, such that $(J_{\epsilon_n} u)$ converges weakly to some $\tilde{u} \in H^{\psi,t+1}$. But we also know that $J_{\epsilon_n} u \to u$ in $H^{\psi,t}$, and since inclusion $H^{\psi,t+1} \hookrightarrow H^{\psi,t}$ is continuous, it follows that $u = \tilde{u} \in H^{\psi,t+1}$. This completes the inductive step, and we may conclude that $u \in H^{\psi,s+2}$ (for $s \geq 0$ as in the original statement of the theorem).

It remains to show that u satisfies the strong equation (4.3.30). Let (u_n) now denote a sequence in $C_c^\infty(K|G|K)$ converging in $H^{\psi,s+2}$ to u. Then for all $v \in C_c^\infty(K|G|K)$ and $n \in \mathbb{N}$,

$$B_\alpha(u_n, v) = \langle (q(\sigma, D) + \alpha) u_n, v \rangle.$$

Letting $n \to \infty$, we have by continuity that

$$B_\alpha(u, v) = \langle (q(\sigma, D) + \alpha) u, v \rangle, \qquad \forall v \in C_c^\infty(K|G|K),$$

and hence by (4.3.31),

$$\langle (q(\sigma, D) + \alpha) u, v \rangle = \langle f, v \rangle, \qquad \forall v \in C_c^\infty(K|G|K).$$

Equation (4.3.30) now follows by density of $C_c^\infty(K|G|K)$ in $L^2(K|G|K)$. □

Theorem 4.3.19. *Let q be a continuous negative definite symbol satisfying Assumptions 4.3.4 and 4.3.11 with $M > \max\left\{1, \frac{d}{r}\right\} + d$, where $d = \dim(G/K)$. Then for all $\alpha \geq \alpha_0$,*

$$\overline{\mathrm{Ran}(\alpha I + q(\sigma, D))} = C_0(K|G|K).$$

Proof. Let $\frac{d}{r} \wedge 1 < s < M - d$. By Theorem 4.1.6 (6), we have a continuous embedding $H^{\psi,t} \hookrightarrow C_0(K|G|K)$ for all $t \geq s$. Let $\mathcal{A}$ denote the linear operator on $C_0(K|G|K)$ with domain $H^{\psi,s+2}$, defined by $\mathcal{A}u = -q(\sigma, D)u$ for all $u \in \mathrm{Dom}(A)$. Note that $\mathcal{A}$ is well-defined, since by Theorem 4.3.7, its image is contained in $H^{\psi,s} \subseteq C_0(K|G|K)$.

Observe that $C_c^\infty(K|G|K)$ is a operator core for $\mathcal{A}$: indeed, given $f \in \mathrm{Dom}(\mathcal{A})$, let (f_n) be a sequence in $C_c^\infty(K|G|K)$ that converges to f in $H^{\psi,s+2}$. Then, since the embeddings $H^{\psi,s+2} \hookrightarrow C_0(K|G|K)$ and $H^{\psi,s} \hookrightarrow C_0(K|G|K)$ are continuous, there are constants $\kappa_1, \kappa_2 > 0$ such that

$$\|f_n - f\|_\infty + \|\mathcal{A}f_n - \mathcal{A}f\|_\infty \leq \kappa_1 \|f_n - f\|_{\psi,s+2} + \kappa_2 \|\mathcal{A}f_n - \mathcal{A}f\|_{\psi,s}$$

By Theorem 4.3.7, there is $\kappa > 0$ such that

$$\|f_n - f\|_\infty + \|\mathcal{A}f_n - \mathcal{A}f\|_\infty \leq \kappa \|f_n - f\|_{\psi,s+2},$$

and hence $f_n \to f$ in the graph norm for $\mathcal{A}$. Thus $C_c^\infty(K|G|K)$ is a core for $\mathcal{A}$. It follows that for all $\alpha \in \mathbb{R}$,

$$\overline{\mathrm{Ran}(\alpha I + q(\sigma, D))} = \overline{\mathrm{Ran}(\alpha I - \mathcal{A})}.$$

Let $f \in C_0(K|G|K)$, and choose a sequence (f_n) in $H^{\psi,s}$ such that $\|f_n - f\|_\infty \to 0$ as $n \to \infty$. Let α_0 be as in Theorem 4.3.18. Then $f_n \in \mathrm{Ran}(\alpha I - \mathcal{A})$ for all $\alpha \geq \alpha_0$, and thus $f \in \overline{\mathrm{Ran}(\alpha I - \mathcal{A})}$ for all $\alpha \geq \alpha_0$. □

Combining Theorem 4.3.19 with the work of Section 4.2 yields the following.

Corollary 4.3.20. *Let q be a Gangolli symbol that satisfies Assumptions 4.3.4 and 4.3.11 for some $M > \min\{1, d/r\} + d$. Then $-q(\sigma, D)$ extends to the infinitesimal generator of a strongly continuous sub-Feller semigroup on $C_0(K|G|K)$.*

Proof. By construction, $-q(\sigma, D)$ is a densely defined linear operator on $C_0(K|G|K)$. It is a Gangolli operator, and hence satisfies the positive maximum principle. By Theorems 4.2.1 and 4.3.19, $-q(\sigma, D)$ is closable, and its closure generates a strongly continuous sub-Feller semigroup. □

§ 4.4 A class of examples

We now present a class of Gangolli symbols that satisfy the conditions of Corollary 4.3.20. Let $M \in \mathbb{N}$ such that $M > \min\{1, d/r\} + d + 1$. We consider symbols $q : G \times \mathfrak{a}^* \to \mathbb{R}$ of the form

$$q(\sigma, \lambda) = \kappa\psi(\lambda) + u(\sigma)v(\lambda), \qquad \forall \sigma \in G, \lambda \in \mathfrak{a}^*, \tag{4.4.1}$$

where κ is a positive constant, $\psi : \mathfrak{a}^* \to \mathbb{R}$ is a Gangolli exponent satisfying (4.1.5), $u \in C_c^M(K|G|K)$ is non-negative, and $v : \mathfrak{a}^* \to \mathbb{R}$ is a Gangolli exponent satisfying, for some $c_v > 0$,

$$|v(\lambda)| \leq c_v(1 + \psi(\lambda)) \qquad \forall \lambda \in \mathfrak{a}^* . \tag{4.4.2}$$

By Remark 4.2.6 (2) and Example 4.2.8 (1), the mappings $(\sigma, \lambda) \mapsto c_0\psi(\lambda)$ and $(\sigma, \lambda) \mapsto u(\sigma)v(\lambda)$ are both Gangolli symbols, and hence so is q.

For each $\lambda \in \mathfrak{a}^*$ and $\sigma \in G$, let

$$q_1(\lambda) = \kappa\psi(\lambda), \quad \text{and} \quad q_2(\sigma, \lambda) = u(\sigma)v(\lambda). \tag{4.4.3}$$

Observe that q is of the form (4.3.12): since v has compact support, $\mathrm{Supp}(v) \neq G$, and if $\sigma_0 \in G \setminus \mathrm{Supp}(v)$, then $q_1 = q(\sigma_0, \cdot)$.

Proposition 4.4.1. *q_1 satisfies Assumption 4.3.4 (1).*

Proof. The upper bound of (4.3.13) may be easily verified by taking $c_1 = \kappa$. For the lower bound, suppose $|\lambda| \geq 1$. Then by (4.1.5),

$$q_1(\lambda) = \frac{\kappa}{2}(\psi(\lambda) + \psi(\lambda)) \geq \frac{\kappa}{2}(c|\lambda|^r + \psi(\lambda)) \geq \frac{\kappa}{2}\min\{1, c\}(1 + \psi(\lambda)),$$

and so taking $c_0 = \frac{\kappa}{2}\min\{1, c\}$, the result follows. □

For Assumption 4.3.4 (2), note that in the case we are considering,

$$F_{\lambda,\eta}(\sigma) = \phi_{-\lambda}(\sigma)u(\sigma)v(\eta), \qquad \forall \sigma \in G,\ \lambda, \eta \in \mathfrak{a}^*,$$

and so, for $\beta = 0, 1, \ldots, M$,

$$(-\Delta)^{\beta/2} F_{\lambda,\eta}(\sigma) = v(\eta)(-\Delta)^{\beta/2}(\phi_{-\lambda} u)(\sigma),$$

for all $\lambda, \eta \in \mathfrak{a}^*$ and $\sigma \in G$. By (4.4.2),

$$\left|(-\Delta)^{\beta/2} F_{\lambda,\eta}(\sigma)\right| = |v(\eta)| \left|(-\Delta)^{\beta/2}(\phi_{-\lambda} u)(\sigma)\right| \leq c_v \left|(-\Delta)^{\beta/2}(\phi_{-\lambda} u)(\sigma)\right| \big(1 + \psi(\eta)\big).$$

Let us first introduce some notation. For each $n \in \mathbb{N}$, a noncommutative version of the multinomial theorem tells us that

$$(-\Delta)^n(\phi_{-\lambda} u) = (-1)^n \left(\sum_{j=1}^{d} X_j^2\right)^n (\phi_{-\lambda} u) = \sum_{\substack{\alpha \in \mathbb{N}_0^d, \\ |\alpha| \leq r}} c_\alpha X^\alpha (\phi_{-\lambda} u) \tag{4.4.4}$$

for some coefficients c_α, where $|\alpha| = \alpha_1 + \ldots + \alpha_d$ and $X^\alpha \coloneqq X_1^{\alpha_1} \ldots X_d^{\alpha_d}$. This may also be seen by noting that $(-\Delta)^n$ belongs to the universal enveloping algebra $\mathcal{U}(\mathfrak{p})$, and applying the Poincaré-Birkoff–Witt theorem (see [49] Theorem 3.8, pp. 217) to write $(-\Delta)^n$ in the basis $\{X^\alpha : \alpha \in \mathbb{N}_0^d\}$ of $\mathcal{U}(\mathfrak{p})$.

Expanding the right-hand side of (4.4.4) using the fact that each X_j is a derivation will give a large sum of terms of the form

$$\kappa_{X,Y} X \phi_{-\lambda} Y u,$$

where the $\kappa_{X,Y}$ are constants, and $X, Y \in \mathbf{D}(G)$ are products of powers of $X_1, \ldots, X_d$, each with degree at most $2n$. Let $\mathscr{U}_n$ be the set of all the X's and $\mathscr{V}_n$ the set of all the Y's, so that

$$(-\Delta)^n(\phi_{-\lambda} u) = \sum_{\substack{X \in \mathscr{U}_n, \\ Y \in \mathscr{V}_n}} \kappa_{X,Y} X \phi_{-\lambda} Y u. \tag{4.4.5}$$

The following lemma is based on Theorem 1.1 (iii) of [35]. An earlier version in terms of the universal enveloping algebra can be found in [33] — see Lemma 46 on page 294.

Lemma 4.4.2. *For all $X \in \mathbf{D}(G)$, there is a constant $C_X > 0$ such that*

$$|X\phi_\lambda(\sigma)| \leq C_X \langle\lambda\rangle^{\deg X} \phi_0(\sigma), \tag{4.4.6}$$

for all $\lambda \in \mathfrak{a}^*$ and $\sigma \in G$.

Proof. By Theorem 1.1 (iii) of [35], there is a constant $C_1 > 0$ such that

$$|X\phi_\lambda(\sigma)| \leq C'_X (1 + |\lambda|)^{\deg X} \phi_{i\,\mathrm{Im}(\lambda)}(\sigma),$$

for all $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$ and $\sigma \in G$. Also,

$$1 + |\lambda| \leq \sqrt{2}\sqrt{1 + |\lambda|^2},$$

for all $\lambda \in \mathfrak{a}^*$. If $\sigma \in G$ and $\lambda \in \mathfrak{a}^*$, then $\mathrm{Im}(\lambda) = 0$, and hence

$$|X\phi_\lambda(\sigma)| \leq C'_X(1+|\lambda|)^{\deg X}\phi_0(\sigma) \leq C'_X\left(\sqrt{2}\sqrt{1+|\lambda|^2}\right)^{\deg X}\phi_0(\sigma) = C_X\langle\lambda\rangle^{\deg X}\phi_0(\sigma),$$

where $C_X = 2^{(\deg X)/2}C'_X$. □

Proposition 4.4.3. *The mapping q_2 in (4.4.3) satisfies Assumption 4.3.4 (2).*

Proof. It is clear by construction that $q_2(\cdot,\lambda) \in C_c^M(K|G|K)$ for all $\lambda \in \mathfrak{a}^*$.

To verify the rest of Assumption 4.3.4 (2), it will be useful to assume that M is even. Note that this is an acceptable assumption, since if M is odd, we may replace it with $M-1$ — the conditions of Corollary 4.3.20 will still be satisfied. Let $\beta \in \{0,1,\ldots,M\}$. We seek $\Phi_\beta \in L^1(K|G|K)$ for which

$$\left|(-\Delta)^{\beta/2}(\phi_{-\lambda}u)(\sigma)\right| \leq \Phi_\beta\langle\lambda\rangle^M, \qquad \forall\sigma\in G,\ \lambda\in\mathfrak{a}^*\,. \tag{4.4.7}$$

Let $n = \lfloor\beta\rfloor$. Assume first that β is even, so that $n = \beta/2$. By (4.4.5) and Lemma 4.4.2,

$$\begin{aligned}\left|(-\Delta)^{\beta/2}(\phi_{-\lambda}u)\right| &\leq \sum_{\substack{X\in\mathscr{U}_n,\\ Y\in\mathscr{V}_n}}|\kappa_{X,Y}||X\phi_{-\lambda}||Yu| \leq \sum_{\substack{X\in\mathscr{U}_n,\\ Y\in\mathscr{V}_n}}C_X|\kappa_{X,Y}||Yu|\langle\lambda\rangle^{\deg X}|\phi_0| \\ &\leq \sum_{\substack{X\in\mathscr{U}_n,\\ Y\in\mathscr{V}_n}}C_X|\kappa_{X,Y}||Yu|\langle\lambda\rangle^{\deg X},\end{aligned}$$

since $|\phi_0| \leq 1$. Now, $\deg X \leq 2n = \beta \leq M$ for all $X \in \mathscr{U}_n$, and therefore,

$$\left|(-\Delta)^{\beta/2}(\phi_{-\lambda}u)\right| \leq \kappa_\beta\sum_{Y\in\mathscr{V}_{\beta/2}}|Yu|\langle\lambda\rangle^M$$

where

$$\kappa_\beta = \sup\left\{C_X|\kappa_{X,Y}| : X\in\mathscr{U}_{\beta/2}, Y\in\mathscr{V}_{\beta/2}\right\}.$$

Let

$$\Phi_\beta \coloneqq \kappa_\beta\sum_{Y\in\mathscr{V}_{\beta/2}}|Yu|. \tag{4.4.8}$$

Then $\Phi_\beta \in L^1(K|G|K)$, since each Yu is a continuous function of compact support. Moreover,

$$\|\Phi_\beta\|_1 \leq \kappa_\beta\sum_{Y\in\mathscr{V}_{\beta/2}}\|Yu\|_1 \tag{4.4.9}$$

In particular, we have verified (4.4.7) when β is even.

Assume now that β is odd, so that $(-\Delta)^{\beta/2} = \sqrt{-\Delta}(-\Delta)^n$. Since M is even, note also that $1 \leq \beta \leq M-1$. Applying $\sqrt{-\Delta}$ to both sides of (4.4.5),

$$\left|(-\Delta)^{\beta/2}(\phi_\lambda u)\right| = \left|\sqrt{-\Delta}(-\Delta)^n(\phi_\lambda u)\right| \leq \sum_{\substack{X\in\mathscr{U}_n,\\ Y\in\mathscr{V}_n}}|\kappa_{X,Y}|\left|\sqrt{-\Delta}\big(X\phi_{-\lambda}Yu\big)\right|. \tag{4.4.10}$$

Note that the families $\mathscr{U}_n$ and $\mathscr{V}_n$ now each consist of differential operators of degree at most $2n = \beta - 1$.

We know that (c.f. Example 3.1.17 (2)) $-\sqrt{-\Delta}$ is the infinitesimal generator of the process obtained by subordinating Brownian motion on G/K by the standard $\frac{1}{2}$-stable subordinator on $\mathbb{R}$. By standard subordination theory (see [8] §5.7, pp. 154) $\sqrt{-\Delta}$ may be expressed as a Bochner integral

$$\sqrt{-\Delta} = \frac{1}{2\sqrt{\pi}} \int_{0+}^{\infty} t^{-3/2}(1 - T_t)dt, \tag{4.4.11}$$

where $(T_t, t \geq 0)$ denotes the heat semigroup generated by Δ.

Given $X \in \mathscr{U}_n$, $Y \in \mathscr{V}_n$ and $\sigma \in G$,

$$\begin{aligned} \left|\sqrt{-\Delta}(X\phi_{-\lambda}Yu)(\sigma)\right| &= \frac{1}{2\sqrt{\pi}} \left| \int_{0+}^{\infty} t^{-3/2}(1 - T_t)\big(X\phi_{-\lambda}Yu\big)dt \right| \\ &\leq \frac{1}{2\sqrt{\pi}} \left[\left| \int_{0+}^{1} t^{-3/2}(1 - T_t)\big(X\phi_{-\lambda}Yu\big)(\sigma)dt \right| \right. \\ &\qquad \left. + \left| \int_{1}^{\infty} t^{-3/2}(1 - T_t)\big(X\phi_{-\lambda}Yu\big)(\sigma)dt \right| \right]. \end{aligned} \tag{4.4.12}$$

Let $(h_t, t \geq 0)$ denote the heat kernel associated with $(T_t, t \geq 0)$. For the $\int_1^\infty$ term of (4.4.12), note that $\int_1^\infty t^{-3/2}dt = 2$, and so

$$\begin{aligned} &\left| \int_{1}^{\infty} t^{-3/2}(1 - T_t)\big(X\phi_{-\lambda}Yu\big)(\sigma)dt \right| \\ &\qquad = \left| \int_{1}^{\infty} t^{-3/2} X\phi_{-\lambda}(\sigma)Yu(\sigma)dt - \int_{1}^{\infty} t^{-3/2}T_t\big(X\phi_{-\lambda}Yu\big)(\sigma)dt \right| \\ &\qquad \leq 2|X\phi_{-\lambda}(\sigma)||Yu(\sigma)| + \left| \int_{1}^{\infty} t^{-3/2} \int_G X\phi_{-\lambda}(\sigma\tau)Yu(\sigma\tau)h_t(\tau)d\tau dt \right|. \end{aligned}$$

By Lemma 4.4.2 and the fact that $\deg X \leq \beta - 1$,

$$|X\phi_{-\lambda}| \leq C_X \langle\lambda\rangle^{\deg X} \leq C\langle\lambda\rangle^{\beta-1}, \qquad \forall \lambda \in \mathfrak{a}^*, \tag{4.4.13}$$

where C_X is as in (4.4.6), and $C = \max\{C_X : X \in \mathscr{U}_n\}$. Thus

$$\begin{aligned} &\left| \int_{1}^{\infty} t^{-3/2}(1 - T_t)\big(X\phi_{-\lambda}Yu\big)(\sigma)dt \right| \\ &\qquad \leq 2|X\phi_{-\lambda}(\sigma)||Yu(\sigma)| + \int_{1}^{\infty} t^{-3/2} \int_G |X\phi_{-\lambda}(\sigma\tau)||Yu(\sigma\tau)|h_t(\tau)d\tau dt \\ &\qquad \leq C \left(2|Yu(\sigma)| + \int_{1}^{\infty} t^{-3/2} \int_G |Yu(\sigma\tau)|h_t(\tau)d\tau dt \right) \langle\lambda\rangle^{\beta-1} \\ &\qquad = \Phi^{(1)}_{\beta,Y}(\sigma)\langle\lambda\rangle^{\beta-1}, \end{aligned}$$

where

$$\Phi^{(1)}_{\beta,Y} := C\left(2|Yu| + \int_1^\infty t^{-3/2} T_t\big(|Yu|\big)dt\right). \tag{4.4.14}$$

Since $\beta - 1 \leq M$ and $\langle\lambda\rangle \geq 1$ for all $\lambda \in \mathfrak{a}^*$, it follows that for all $\lambda \in \mathfrak{a}^*$,

$$\left|\int_1^\infty t^{-3/2}(1 - T_t)\big(X\phi_{-\lambda}Yu\big)dt\right| \leq \Phi^{(1)}_{\beta,Y}\langle\lambda\rangle^M. \tag{4.4.15}$$

We claim that $\Phi^{(1)}_{\beta,Y} \in L^1(K|G|K)$. Clearly $|Yu| \in L^1(K|G|K)$, since it is a continuous function of compact support. Each of the operators T_t is a positivity preserving contraction of $L^1(K|G|K)$, and so

$$\int_1^\infty t^{-3/2}\int_G T_t\big(|Yu|\big)(\sigma)d\sigma dt = \int_1^\infty t^{-3/2}\left\|T_t\big(|Yu|\big)\right\|_1 dt \leq \int_1^\infty t^{-3/2}\|Yu\|_1 dt = 2\|Yu\|_1.$$

By Fubini's theorem, $\int_1^\infty t^{-3/2}T_t\big(|Yu|\big)dt \in L^1(K|G|K)$, with

$$\left\|\int_1^\infty t^{-3/2}T_t\big(|Yu|\big)dt\right\|_{L_1(K|G|K)} \leq 2\|Yu\|_1.$$

It follows by (4.4.14) that $\Phi^{(1)}_{\beta,Y} \in L^1(K|G|K)$, and that

$$\|\Phi^{(1)}_{\beta,Y}\|_1 \leq 4C\|Yu\|_1. \tag{4.4.16}$$

For the $\int_{0+}^1$ term of (4.4.12), observe that by Lemma 6.1.12 of [23], pp. 169, as well as the Fubini theorem,

$$\begin{aligned}
\int_{0+}^1 t^{-3/2}(1 - T_t)\big(X\phi_{-\lambda}Yu\big)dt &= -\int_{0+}^1 t^{-3/2}\int_0^t T_s\Delta\big(X\phi_{-\lambda}Yu\big)dsdt \\
&= -\int_0^1\int_s^1 t^{-3/2}T_s\Delta\big(X\phi_{-\lambda}Yu\big)dtds \\
&= -\int_0^1 2(s^{-1/2} - 1)T_s\Delta\big(X\phi_{-\lambda}Yu\big)ds.
\end{aligned}$$

Hence, using the product formula for Δ,

$$\begin{aligned}
\int_{0+}^1 t^{-3/2}(1 - T_t)\big(X\phi_{-\lambda}Yu\big)dt = -2\int_0^1 (s^{-1/2} - 1)\Big\{T_s\big(X\phi_{-\lambda}\Delta Yu\big) \\
+ 2\sum_{j=1}^d T_s\big(X_jX\phi_{-\lambda}X_jYu\big) + T_s\big(\Delta X\phi_{-\lambda}Yu\big)\Big\}ds.
\end{aligned} \tag{4.4.17}$$

Let C and C_X be as in (4.4.13). Then for all $\sigma \in G$,

$$\begin{aligned}\left|T_s\big(X\phi_{-\lambda}\Delta Yu\big)(\sigma)\right| &= \left|\int_G X\phi_{-\lambda}(\sigma\tau)\Delta Yu(\sigma\tau)h_s(\tau)d\tau\right| \\ &\leq \int_G |X\phi_{-\lambda}(\sigma\tau)||\Delta Yu(\sigma\tau)|h_s(\tau)d\tau \\ &\leq C_X\langle\lambda\rangle^{\deg X}\int_G |\Delta Yu(\sigma\tau)|h_s(\tau)d\tau \leq C\langle\lambda\rangle^{\beta-1}T_s|\Delta Yu|(\sigma).\end{aligned}$$

In exactly the same way, for $j = 1, \ldots, d$,

$$\left|T_s\big(X_jX\phi_{-\lambda}X_jYu\big)\right| \leq C_X^{(j)}\langle\lambda\rangle^{\deg X+1}T_s|X_jYu| \leq C'\langle\lambda\rangle^{\beta}T_s|X_jYu|,$$

and also

$$\left|T_s\big(\Delta X\phi_{-\lambda}Yu\big)\right| \leq C_X^{(0)}\langle\lambda\rangle^{\deg X+2}T_s|Yu| \leq C'\langle\lambda\rangle^{\beta+1}T_s|Yu|,$$

where the constants $C_X^{(j)}, C^{(j)}$ are chosen so that for all $\lambda \in \mathfrak{a}^*$ and $j = 1, \ldots, d$,

$$|X\phi_{-\lambda}| \leq C_X^{(0)}\langle\lambda\rangle^{\deg X+2}, \qquad |X_jX\phi_{-\lambda}| \leq C_X^{(j)}\langle\lambda\rangle^{\deg X+1},$$

and $C' \coloneqq \max\{C_X^{(j)} : X \in \mathscr{U}_n, j = 0, 1, \ldots, d\}$. Such constants exist by Lemma 4.4.2. Now,

$$\langle\lambda\rangle^{\beta-1} \leq \langle\lambda\rangle^{\beta} \leq \langle\lambda\rangle^{\beta+1}$$

for all $\lambda \in \mathfrak{a}^*$, and hence by (4.4.17),

$$\begin{aligned}&\left|\int_{0+}^1 t^{-3/2}(1-T_t)\big(X\phi_{-\lambda}Yu\big)dt\right| \\ &\quad\leq 2\int_0^1 (s^{-1/2}-1)\Big\{\left|T_s\big(X\phi_{-\lambda}\Delta Yu\big)\right| + 2\sum_{j=1}^d \left|T_s\big(X_jX\phi_{-\lambda}X_jYu\big)\right| \\ &\qquad\qquad + \left|T_s\big(\Delta X\phi_{-\lambda}Yu\big)\right|\Big\}ds \\ &\quad\leq 2C'\langle\lambda\rangle^{\beta+1}\int_0^1 (s^{-1/2}-1)T_s\left(|\Delta Yu| + 2\sum_{j=1}^d |X_jYu| + |Yu|\right)ds.\end{aligned}$$

Since $\beta \leq M-1$, it follows that for all $X \in \mathscr{U}_n$ and $Y \in \mathscr{V}_n$,

$$\left|\int_{0+}^1 t^{-3/2}(1-T_t)\big(X\phi_{-\lambda}Yu\big)dt\right| \leq \Phi_{\beta,Y}^{(2)}\langle\lambda\rangle^M, \tag{4.4.18}$$

where

$$\Phi_{\beta,Y}^{(2)} = C'\int_0^1 (s^{-1/2}-1)T_s\left(|\Delta Yu| + 2\sum_{j=1}^d |X_jYu| + |Yu|\right)ds. \tag{4.4.19}$$

Observe that $\Phi^{(2)}_{\beta,Y} \in L^1(K|G|K)$ for all $Y \in \mathscr{V}_n$. Indeed, $u \in C_c^M(K|G|K)$, and $\deg Y \leq \beta - 1 \leq M - 2$, hence $|\Delta Y u|$, $\sum_{j=1}^d |X_j Y u|$ and $|Yu|$ are all continuous functions of compact support. Thus $T_s\big(|\Delta Y u| + 2\sum_{j=1}^d |X_j Y u| + |Yu|\big) \in L^1(K|G|K)$, and, since T_s is an $L^1(K|G|K)$-contraction,

$$\left\| T_s \left(|\Delta Y u| + 2\sum_{j=1}^{d} |X_j Y u| + |Yu| \right) \right\|_1 \leq \|\Delta Y u\|_1 + 2\sum_{j=1}^{d} \|X_j Y u\|_1 + \|Yu\|_1.$$

Noting that $\int_0^1 (s^{-1/2} - 1) ds = 1$, it follows by Fubini's theorem that $\Phi^{(2)}_{\beta,Y} \in L^1(K|G|K)$, with

$$\left\| \Phi^{(2)}_{\beta,Y} \right\|_1 \leq C'_X \left(\|\Delta Y u\|_1 + 2\sum_{j=1}^{d} \|X_j Y u\|_1 + \|Yu\|_1 \right). \tag{4.4.20}$$

Substituting (4.4.18) and (4.4.15) into (4.4.12), we obtain the pointwise estimate

$$\left| \sqrt{-\Delta}(X\phi_{-\lambda} Y u) \right| \leq \frac{1}{2\sqrt{\pi}} \left(\Phi^{(1)}_{\beta,Y} + \Phi^{(2)}_{\beta,Y} \right) \langle \lambda \rangle^M, \tag{4.4.21}$$

for all $X \in \mathscr{U}_n$, $Y \in \mathscr{V}_n$ and $\lambda \in \mathfrak{a}^*$, where the $\Phi^{(j)}_{\beta,Y}$ $(j = 1, 2)$ are given by (4.4.14) and (4.4.19). Hence by (4.4.10), for all $\lambda \in \mathfrak{a}^*$,

$$\left| (-\Delta)^{\beta/2} (\phi_\lambda u) \right| \leq \sum_{\substack{X \in \mathscr{U}_n, \\ Y \in \mathscr{V}_n}} |\kappa_{X,Y}| \left| \sqrt{-\Delta}\big(X\phi_{-\lambda} Y u\big) \right| \leq \Phi_\beta \langle \lambda \rangle^M,$$

where

$$\Phi_\beta := \frac{1}{2\sqrt{\pi}} \sum_{\substack{X \in \mathscr{U}_n, \\ Y \in \mathscr{V}_n}} |\kappa_{X,Y}| \left(\Phi^{(1)}_{\beta,Y} + \Phi^{(2)}_{\beta,Y} \right), \tag{4.4.22}$$

and β is still assumed to be odd. As already noted, $\Phi^{(1)}_{\beta,Y}, \Phi^{(2)}_{\beta,Y} \in L^1(K|G|K)$ for all $Y \in \mathscr{V}_n$, and hence $\Phi_\beta \in L^1(K|G|K)$. Moreover, by (4.4.16) and (4.4.20),

$$\|\Phi_\beta\|_1 \leq \kappa'_\beta \sum_{Y \in \mathscr{V}_n} \left(\|\Delta Y u\|_1 + \sum_{j=1}^{d} \|X_j Y u\|_1 + \|Yu\|_1 \right), \tag{4.4.23}$$

for some positive constant κ'_β. In particular, we have verified (4.4.7) when β is odd. □

Corollary 4.4.4. *Let $q : G \times \mathfrak{a}^* \to \mathbb{R}$ be of the form (4.4.1). Then for κ sufficiently large, the conditions of Corollary 4.3.20 are satisfied. In particular, $-q(\sigma, D)$ extends to the infinitesimal generator of a strongly continuous sub-Feller semigroup on $C_0(K|G|K)$.*

Appendix A

The Friedrich Mollifier J_ϵ

Recall the Friedrich mollifier on $\mathfrak{a}$ from Section 4.3, defined for each $0 < \epsilon \leq 1$ and $H \in \mathfrak{a}$ by

$$l_\epsilon(H) := \epsilon^{-m} l(H/\epsilon), \qquad \text{where} \qquad l(H) := C_0 e^{\frac{1}{|H|^2-1}} \mathbf{1}_{B_1(0)}(H),$$

where $C_0 > 0$ is a normalising constant, and the associated mappings $j, j_\epsilon \in \mathcal{S}(K|G|K)$ are again given by

$$\hat{j}_\epsilon = \mathcal{F}(l_\epsilon), \qquad \forall 0 < \epsilon \leq 1.$$

For $0 < \epsilon \leq 1$, the operators J_ϵ were defined

$$J_\epsilon u = j_\epsilon * u \qquad \forall f \in L^2(K|G|K).$$

This appendix will be devoted to proving Proposition 4.3.16, which we re-state below.

Proposition A.0.1. *1. $\hat{j}_\epsilon(\lambda) = \hat{j}(\epsilon\lambda)$ for all $0 < \epsilon \leq 1$ and $\lambda \in \mathfrak{a}^*$.*

2. For $0 < \epsilon \leq 1$, J_ϵ is a self-adjoint contraction of $L^2(K|G|K)$.

3. $J_\epsilon u \in H^{\psi,s}$ for all $s \geq 0$, $u \in L^2(K|G|K)$ and $0 < \epsilon \leq 1$, and if $u \in H^{\psi,s}$, then

$$\|J_\epsilon u\|_{\psi,s} \leq \|u\|_{\psi,s}.$$

4. For all $s \geq 0$ and $u \in H^{\psi,s}$, $\|J_\epsilon u - u\|_{\psi,s} \to 0$ as $\epsilon \to 0$.

Proof. 1. Let $0 < \epsilon \leq 1$ and $\lambda \in \mathfrak{a}^*$. Using a change of variable $H \mapsto \epsilon^{-1} H$,

$$\begin{aligned}\hat{j}_\epsilon(\lambda) = \mathscr{F}(l_\epsilon)(\lambda) &= \int_{\mathfrak{a}} e^{-i\lambda(H)} \epsilon^{-m} l(\epsilon^{-1} H) dH \\ &= \int_{\mathfrak{a}} e^{-i\epsilon\lambda(H)} l(H) dH = \mathscr{F}(l)(\epsilon\lambda) = \hat{j}(\epsilon\lambda).\end{aligned}$$

2. The map l is symmetric under $H \mapsto -H$, and hence $\mathscr{F}(l) = \hat{j}$ is real-valued. Therefore, given $u, v \in L^2(K|G|K)$ and $0 < \epsilon \leq 1$,

$$\langle J_\epsilon u, v\rangle = \int_{\mathfrak{a}^*} \hat{j}(\epsilon\lambda)\hat{u}(\lambda)\overline{\hat{v}(\lambda)}\omega(d\lambda) = \int_{\mathfrak{a}^*} \hat{u}(\lambda)\overline{\hat{j}(\epsilon\lambda)\hat{v}(\lambda)}\omega(d\lambda) = \langle u, J_\epsilon v\rangle.$$

To see that J_ϵ is a contraction, note that $|\hat{j}_\epsilon(\lambda)| = |\hat{j}(\epsilon\lambda)| \leq \hat{j}(0) = 1$ for all $\lambda \in \mathfrak{a}^*$, and so by Plancherel's identity

$$\|J_\epsilon u\| = \|\hat{j}_\epsilon \hat{u}\|_{L^2(\mathfrak{a}^*,\omega)} \leq \|\hat{u}\|_{L^2(K|G|K)} = \|u\|,$$

for all $u \in L^2(K|G|K)$ and all $0 < \epsilon \leq 1$.

3. Let $s \geq 0$ and $0 < \epsilon \leq 1$. By Theorem 2.4.30, $\hat{j} \in \mathcal{S}(\mathfrak{a}^*)^W$, and hence there is $\kappa > 0$ such that

$$\langle\lambda\rangle^s \left|\hat{j}(\epsilon\lambda)\right| \leq \kappa, \qquad \forall\lambda \in \mathfrak{a}^*.$$

Then, using Proposition 4.1.5 (3),

$$\Psi(\lambda)^s \left|\hat{j}(\epsilon\lambda)\right| \leq c_\psi^{s/2}\langle\lambda\rangle^s \left|\hat{j}(\epsilon\lambda)\right| \leq c_\psi^{s/2}\kappa,$$

for all $\lambda \in \mathfrak{a}^*$. Let $u \in L^2(K|G|K)$. By Plancherel's identity,

$$\int_{\mathfrak{a}^*} \Psi(\lambda)^{2s} \left|\hat{j}(\epsilon\lambda)\right|^2 |\hat{u}(\lambda)|^2\omega(d\lambda) \leq c_\psi^s \kappa^2 \|u\|^2 < \infty.$$

By Proposition 4.3.16 (1), $(J_\epsilon u)^\wedge(\lambda) = \hat{j}(\epsilon\lambda)\hat{u}(\lambda)$, for all $\lambda \in \mathfrak{a}^*$, and hence

$$\int_{\mathfrak{a}^*} \Psi(\lambda)^{2s}|(J_\epsilon u)^\wedge(\lambda)|^2\omega(d\lambda) < \infty.$$

That is, $J_\epsilon u \in H^{\psi,s}$.

Next, suppose $u \in H^{\psi,s}$. Then, since $|\hat{j}_\epsilon| \leq 1$,

$$\|J_\epsilon u\|_{\psi,s} = \|\Psi^s \hat{j}_\epsilon \hat{u}\|_{L^2(\mathfrak{a}^*,\omega)} \leq \|\Psi^s \hat{u}\|_{L^2(\mathfrak{a}^*,\omega)} = \|u\|_{\psi,s},$$

as desired.

4. By Theorem 1 on page 250 of [28], for all $v \in \mathcal{S}(\mathfrak{a})^W$, $l_\epsilon * v \to v$ as $\epsilon \to 0$, in the classical Sobolev space $W^s(\mathfrak{a}^*)$, for all $s \geq 0$. Therefore,

$$\lim_{\epsilon\to 0} \int_{\mathfrak{a}^*} (1 + |\lambda|^2)^s |\mathscr{F}(l_\epsilon * v - v)(\lambda)|^2 d\lambda = 0, \qquad \forall s \geq 0,\ v \in \mathcal{S}(\mathfrak{a})^W.$$

Let $u \in C_c^\infty(K|G|K)$ and $v = \mathscr{F}^{-1}(\hat{u})$. Then $v \in \mathcal{S}(\mathfrak{a})^W$, and

$$\mathscr{F}(l_\epsilon * v - v) = (\hat{j}_\epsilon - 1)\hat{u} = (J_\epsilon u - u)^\wedge.$$

Hence $\lim_{\epsilon\to 0}\int_{\mathfrak{a}^*}(1+|\lambda|^2)^s|(J_\epsilon u-u)^\wedge(\lambda)|^2 d\lambda = 0$, for all $s \geq 0$. By (2.4.21),

$$\begin{aligned}\int_{\mathfrak{a}^*}&(1+|\lambda|^2)^s|(J_\epsilon u-u)^\wedge(\lambda)|^2\omega(d\lambda)\\ &\leq \int_{\mathfrak{a}^*}(1+|\lambda|^2)^s|(J_\epsilon u-u)^\wedge(\lambda)|^2(C_1+C_2|\lambda|^p)^2 d\lambda\\ &\leq \kappa\int_{\mathfrak{a}^*}(1+|\lambda|^2)^{s+p}|(J_\epsilon u-u)^\wedge(\lambda)|^2 d\lambda,\end{aligned}$$

for some constant $\kappa > 0$, and where $p = \frac{\dim N}{2}$. Thus

$$\lim_{\epsilon\to 0}\int_{\mathfrak{a}^*}(1+|\lambda|^2)^s|(J_\epsilon u-u)^\wedge(\lambda)|^2\omega(d\lambda) = 0, \qquad \forall s \geq 0.$$

By Proposition 4.1.5 (3),

$$\begin{aligned}\|J_\epsilon u-u\|^2_{\psi,s} &= \int_{\mathfrak{a}^*}(1+\psi(\lambda))^s|(J_\epsilon u-u)^\wedge(\lambda)|^2\omega(d\lambda)\\ &\leq c_\psi\int_{\mathfrak{a}^*}(1+|\lambda|^2)^s|(J_\epsilon u-u)^\wedge(\lambda)|^2\omega(d\lambda)\to 0\end{aligned}$$

as $\epsilon \to 0$. Since $C_c^\infty(K|G|K)$ is dense in $H^{\psi,s}$, Proposition 4.3.16 (**??**) follows.

□

Index

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