Aspects of quantum charged scalar field theory in curved spacetime:

Black hole superradiance and Hadamard renormalisation

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Abstract

This thesis investigates the behaviour of charged scalar fields on curved background spacetimes. In Part I of this thesis, a brief review of quantum field theory in Schwarzschild spacetime is given.

In Part II, the Reissner-Nordström solution is introduced, before a massless, minimally-coupled charged scalar field is introduced and its dynamics on this spacetime are studied. The field is then quantised via canonical quantisation and various quantum states are defined in analogue with quantum states in Schwarzschild spacetime. This part concludes with analytical and numerical investigations of the expectation values of observables in various states.

In Part III, a charged scalar field of arbitrary mass and scalar coupling to the curvature is considered in a general background spacetime. The Hadamard renormalisation procedure is then developed for each of the three cases of two dimensions, an even number of dimensions and an odd number of dimensions. The renormalisation counterterms required for the evaluation of the RSET are derived explicitly. This Part concludes with a discussion of the renormalisation ambiguities associated with Hadamard renormalisation.

Acknowledgements

It is, in my mind, a far more achievable endeavour to discover the theory of quantum gravity underpinning the beautiful Universe we see around us than to properly thank all of those who have helped me arrive at this moment in time. With apologies to those who I do not mention by name, this acknowledgements section is my attempt to do so.

Firstly, I would like to convey my immense gratitude to my supervisor Prof. Elizabeth Winstanley. I could not have wished for a more encouraging, patient, understanding, modest and exceptionally intelligent supervisor in Elizabeth. I am thankful to her for having introduced me to the wonderful world of quantum field theory in curved spacetime and I feel incredibly privileged to have first explored this rich subject under her guidance.

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Preface

The material studied in Part I is entirely review material and no original work has been undertaken on the part of the author.

The material studied in Part II is the result of a collaboration between the author's supervisor, Prof Elizabeth Winstanley, Prof Luis Crispino, Dr Rafael Bernar and the author. While some of the material contained in Chapter 3 is well-established in the literature, almost all of the material in Chapters 4 and 5 represent new work. In particular, while the author contributed towards all of the material in Chapter 4 as well as the asymptotic calculations in Chapter 5, the author made very little contribution to the numerical plots contained in Chapter 5; this work was mainly conducted by Dr Rafael Bernar in close collaboration with Prof Elizabeth Winstanley and Prof Luis Crispino. In particular, Dr Rafael Bernar produced all of the plots that are included in Part II. Part of the work contained in §5.2 and §5.3.1 has been published in [1] and the remainder of the work in Chapters 4 and 5 will be published shortly.

The entirety of the material studied in Part III represents new work on the part of Prof Elizabeth Winstanley and the author, which has been published in [2].

The conventions used in this thesis are as follows. The metric has mostly minus signature, i.e. $(-,+,\ldots,+)$ and we are using units in which $8\pi G = c = \hbar = 1$.

Contents

Part I

Review

Chapter 1

Introduction

In §1.1, we introduce the general philosophy of quantum field theory in curved spacetime. We take the opportunity to establish our conventions for various geometric quantities in §1.2 and we also introduce the quantum observables that will be of interest throughout this thesis in §1.3. We then discuss key results obtained from considering the behaviour of quantum fields on various black hole spacetimes in §1.4; this sets the context for our investigation of quantum field theory in Reissner-Nordström spacetime in Part II. We then explain the importance of renormalisation as well as introducing Wald's axioms in §1.5. We conclude with a rapid overview of the Hadamard renormalisation scheme for neutral scalar fields in §1.6; we extend this procedure to charged scalar fields in Part III.

1.1 Quantum field theory in curved spacetime

This thesis studies the behaviour of quantum charged scalar fields on curved background spacetimes, which is an application of the rich subject known as quantum field theory in curved spacetime (QFTCS). The general philosophy of QFTCS is to consider the behaviour of a quantum field on a classical, fixed background spacetime. Furthermore, this thesis considers background spacetimes possessing a charge, which requires the presence of a background gauge field that is similarly left fixed and classical.

QFTCS has been responsible for some of the most profound advances in quantum gravity, such as the realisation by Hawking that black holes formed by gravitational collapse emit thermal radiation [3, 4], the discovery of Unruh that an accelerating observer experiences a thermal bath [5] and the insight of Parker that an expanding Universe leads to the creation of particles [6–8]. Furthermore, any successful theory of gravity must reduce to QFTCS in an appropriate limit meaning that a putative theory must reproduce the predictions of QFTCS. A selection of reviews on the subject can be found in [9–13].

In general, one can consider a variety of specific background spacetimes and gauge fields or we could leave these arbitrary. In this thesis, we will do both; in parts I and II, we consider the Schwarzschild and Reissner-Nordström black hole solutions as our background spacetimes and in part III, where we develop the Hadamard renormalisation procedure for a charged scalar field, we consider an arbitrary curved background spacetime with a general background gauge field. In order to study QFTCS further, we need to introduce a number of geometrical objects.

1.2 Geometrical preliminaries

Curved spacetimes are characterised by the existence of a $(0,2)$ symmetric tensor $g_{\mu\nu}$ called the metric, which describes the geometry of the spacetime. Unlike the Minkowski metric $\eta_{\mu\nu}$, it is not constant and its entries can, in general, depend on any coordinates we introduce on the spacetime which it describes. The determinant of the metric tensor is denoted by g. In differential geometry, vectors exist in tangent spaces at each spacetime point. We can relate vectors in nearby points using a connection; the unique, metric-compatible connection Γ on a Riemannian manifold is related to the metric $g_{\mu\nu}$ by [14]

$$\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2}\, g^{\lambda\rho}\left(\partial_\mu\, g_{\nu\rho} + \partial_\nu\, g_{\mu\rho} - \partial_\rho\, g_{\mu\nu}\right). \tag{1.1}$$

We refer to the $\Gamma^{\lambda}_{\mu\nu}$ in (1.1) as the Christoffel symbols. The partial derivative ∂_μ is not coordinate-independent. Instead, we define the covariant derivative of a (k,l) tensor as

$$\begin{aligned} \nabla_\rho\, T^{\mu_1\ldots\mu_k}{}_{\nu_1\ldots\nu_l} = {} & \partial_\rho\, T^{\mu_1\ldots\mu_k}{}_{\nu_1\ldots\nu_l} + \Gamma^{\mu_1}_{\rho\lambda}\, T^{\lambda\ldots\mu_k}{}_{\nu_1\ldots\nu_l} + \ldots + \Gamma^{\mu_k}_{\rho\lambda}\, T^{\mu_1\ldots\lambda}{}_{\nu_1\ldots\nu_l} \\ & - \Gamma^{\lambda}_{\rho\nu_1}\, T^{\mu_1\ldots\mu_k}{}_{\lambda\ldots\nu_l} - \ldots - \Gamma^{\lambda}_{\rho\nu_l}\, T^{\mu_1\ldots\mu_k}{}_{\nu_1\ldots\lambda}\,. \end{aligned} \tag{1.2}$$

We can use the $\Gamma^{\lambda}_{\mu\nu}$ to define a geodesic, which is a parameterised curve $x^\mu(\lambda)$ that satisfies

$$\frac{\mathrm{d}^2 x^\mu}{\mathrm{d}\lambda^2} + \Gamma^{\mu}_{\nu\rho}\frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\rho}{\mathrm{d}\lambda} = 0, \tag{1.3}$$

known as the geodesic equation and which describes the path of a freely-falling particle. The curvature of the spacetime is described by the Riemann tensor, which is given by

$$R^{\mu}{}_{\nu\rho\lambda} = \partial_\rho\, \Gamma^{\mu}_{\nu\lambda} - \partial_\lambda\, \Gamma^{\mu}_{\nu\rho} + \Gamma^{\mu}_{\rho\gamma}\, \Gamma^{\gamma}_{\nu\lambda} - \Gamma^{\mu}_{\lambda\gamma}\, \Gamma^{\gamma}_{\nu\rho}. \tag{1.4}$$

The Riemann tensor $R_{\mu\nu\rho\lambda}$ with lowered indices has several useful properties, which include

$$R_{\mu\nu\rho\lambda} = -R_{\nu\mu\rho\lambda} = -R_{\mu\nu\lambda\rho}, \tag{1.5a}$$

$$R_{\mu\nu\rho\lambda} = R_{\rho\lambda\mu\nu}, \tag{1.5b}$$

$$R_{\mu[\nu\rho\lambda]} = 0, \tag{1.5c}$$

$$\nabla_{[\gamma} R_{\mu\nu]\rho\lambda} = 0. \tag{1.5d}$$

In (1.5), indices contained within square brackets [.] are anti-symmetrised and indices contained within round brackets (.) are symmetrised. Equation (1.5d) is referred to as the Bianchi identity. Defining the Ricci tensor as the contraction of the Riemann tensor

$$R_{\mu\nu} = R^{\rho}{}_{\mu\rho\nu}, \tag{1.6}$$

and the Ricci scalar, or scalar curvature, as the contraction of the Ricci tensor

$$R = g^{\mu\nu} R_{\mu\nu}, \tag{1.7}$$

we derive, by contracting (1.5d) on two pairs of indices, the contracted Bianchi identity

$$\nabla^{\mu} R_{\mu\nu} = \frac{1}{2} \nabla_{\nu} R. \tag{1.8}$$

The Einstein tensor $G_{\mu\nu}$ is defined by

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R, \tag{1.9}$$

which leads to another expression of the contracted Bianchi identity (1.8) as $\nabla^{\mu} G_{\mu\nu} = 0$. The commutator of covariant derivatives $[\nabla_{\mu}, \nabla_{\nu}]$ acting on a general (k, l) tensor is

$$\begin{aligned} [\nabla_{\rho}, \nabla_{\lambda}]\, T^{\mu_1 \ldots \mu_k}{}_{\nu_1 \ldots \nu_l} &= R^{\mu_1}{}_{\gamma\rho\lambda}\, T^{\gamma \ldots \mu_k}{}_{\nu_1 \ldots \nu_l} + \ldots + R^{\mu_k}{}_{\gamma\rho\lambda}\, T^{\mu_1 \ldots \gamma}{}_{\nu_1 \ldots \nu_l} \\ &\quad - R^{\gamma}{}_{\nu_1\rho\lambda}\, T^{\mu_1 \ldots \mu_k}{}_{\gamma \ldots \nu_l} - \ldots - R^{\gamma}{}_{\nu_k\rho\lambda}\, T^{\mu_1 \ldots \mu_k}{}_{\nu_1 \ldots \gamma}\,. \end{aligned} \tag{1.10}$$

1.3 Quantum observables

Having introduced these geometrical quantities, an object of central importance in QFTCS is the expectation value of the stress-energy tensor (SET) $\langle \hat{T}_{\mu\nu} \rangle$, which gives information about the particle content, or flux of energy, associated to a quantum field. Since $\langle \hat{T}_{\mu\nu} \rangle$ acts as a source term in Einstein's semiclassical field equations, which are given by

$$G_{\mu\nu} = \langle \hat{T}_{\mu\nu} \rangle\,, \tag{1.11}$$

then evaluating the SET gives information about the quantum backreaction of the field on the spacetime geometry, which we can see from the appearance of the Einstein tensor $G_{\mu\nu}$ (1.9) on the l.h.s of (1.11). In the case of a charged scalar field, the expectation value of the current $\langle \hat{J}^{\mu} \rangle$ acts as a source for the semiclassical Maxwell equations

$$\nabla_{\mu} F^{\mu\nu} = 4\pi \langle \hat{J}^{\nu} \rangle\,, \tag{1.12}$$

which govern the quantum backreaction of the scalar field on the electromagnetic field. The final observable that we will consider is the renormalised expectation value of the square of the scalar field which, in this thesis, we will call the 'expectation value of the scalar condensate' $\langle \widehat{\mathcal{SC}} \rangle$. In order to evaluate the observables $\langle \widehat{\mathcal{SC}} \rangle$, $\langle \hat{J}^{\mu} \rangle$ and $\langle \hat{T}_{\mu\nu} \rangle$, we first need to specify an appropriate vacuum state for the field.

A major conceptual difference between QFT in Minkowski spacetime and QFTCS is the lack of a preferred vacuum state in the latter since curved spacetimes do not possess isometries in general. In parts I and II, when working in specific background spacetimes, we consider the expectation value of observables in specific quantum states. In part III, when the geometry of the background spacetime is left arbitrary, we develop the Hadamard renormalisation procedure for a charged scalar field for a general quantum state.

1.4 Quantum field theory in black hole spacetimes

In this thesis, we will be particularly interested in black holes which, despite all their elegance and rich structure, can be characterised by three parameters namely their mass M, electromagnetic charge Q and angular momentum J. While the most general solution is represented by the Kerr-Newman metric which has all three parameters non-vanishing, simplified solutions which set some parameters to zero are often easier to study.

1.4.1 The Schwarzschild solution

A solution with the latter two parameters set to zero is referred to as the Schwarzschild solution and represents the simplest possible black hole. The behaviour of quantum fields on this spacetime has been studied extensively and three different quantum states have been defined [15]. These are the Boulware state which is as empty as possible to a static observer far from the black hole [16], the Unruh state which is the natural state to use when modelling a black hole formed from gravitational collapse [5] and the Hartle-Hawking state which represents a black hole in an unstable equilibrium of thermal radiation at the Hawking temperature [17,18]. Calculations of renormalised expectation values of the scalar condensate $\langle\widehat{\mathcal{SC}}\rangle$ and the SET $\langle\hat{T}_{\mu\nu}\rangle$ in each of these quantum states have been performed widely; further details can be found in [19–34] and references therein.

1.4.2 The Reissner-Nordström solution

The Reissner-Nordström metric is the black hole solution that has only the angular momentum parameter set to zero and which represents a charged, non-rotating black hole. Such black holes are thought unlikely to be physically relevant since a charged black hole can preferentially attract particles of the opposite charge. Also, when quantum effects are taken into account, the black hole will preferentially emit Hawking radiation consisting of particles of the opposite charge [39]. Both of these processes will tend to reduce the charge of the black hole and ultimately neutralise it. Furthermore, most objects in the Universe are spinning and thus have nonzero angular momentum.

Nevertheless, Reissner-Nordström spacetime is mathematically interesting from the viewpoint of exploring how the presence of a black hole charge changes the behaviour of quantum fields from that of Schwarzschild spacetime. In Part II of this thesis, we construct a number of different analogues of the states defined in Schwarzschild spacetime in order to study the behaviour of quantum fields in Reissner-Nordström spacetime.

We will be particularly interested in the phenomenon of superradiance, which is a radiation enhancement effect involving dissipative systems [35]. In Reissner-Nordström spacetime, charge superradiance arises due to the interaction of the charges of the black hole and of the field [36–38]; this causes low-frequency modes of the classical field to undergo superradiant scattering by which incoming waves are reflected, from the black hole, back to infinity with a greater amplitude than they were incident with, thereby extracting charge from the Reissner-Nordström black hole. There is also a quantum analogue, whereby Reissner-Nordström black holes spontaneously emit a flux of particles in the classically superradiant modes [1,39] and which we study in detail in Part II.

The interaction of the charge of the black hole and the field charge also affects the Hawking radiation emitted by the black hole, thereby influencing the process of black hole evaporation [40–43]. The Unruh state has recently been constructed for a charged scalar field in Reissner-Nordström-de Sitter spacetime [44, 45]. Apart from the aforementioned work, there has been relatively little study of either the phenomenon of quantum superradiance or the definition of quantum states for a charged scalar field in Reissner-Nordström spacetime. Thus it is instructive to briefly review studies of quantum field theory in another black hole spacetime that also exhibits superradiance, namely the Kerr spacetime, and into which considerably more investigation has been conducted.

1.4.3 The Kerr solution

The Kerr metric is the black hole solution which has only the electromagnetic charge parameter set to zero. This is the most physically relevant solution since it is thought to be the category that is representative of astrophysical black holes. Superradiance arises in Kerr spacetimes due to the co-rotation of a field with the black hole. Interestingly, the only classical fields that exhibit superradiant scattering in Kerr spacetime are bosonic [46]. However, both bosonic and fermionic quantum fields cause Kerr black holes to emit Unruh-Starobinskii radiation, which is the quantum analogue of classical superradiance [47, 48].

The existence of quantum superradiance in the Kerr case diminishes our ability to define certain analogue quantum states with the same interpretations as their corresponding states in Schwarzschild spacetime. As we might expect from the absence of classical superradiance for fermionic fields, this is more apparent in the case of bosonic fields than fermionic ones [49–53]. For example, it is no longer possible to define a Boulware state (see §2.3.1) associated to a scalar field that is simultaneously empty at both past and future null infinity $\mathscr{I}^{\pm}$ [50, 52], and though it is possible to do so for a fermionic field, such a state is not everywhere regular, i.e. the expectation values of quantum observables in this state are divergent somewhere in the spacetime. Attempts to construct a Boulware state fail due to a superradiant flux of particles outgoing at $\mathscr{I}^{+}$, which further precludes defining a state with the time-reversal invariance of the Schwarzschild Boulware state [54].

Though it is possible to construct an analogue of the Schwarzschild Unruh state in Kerr spacetime with similar physical interpretations [55], this is not the case for the Schwarzschild Hartle-Hawking state which is a Hadamard state (these are defined in §1.6). One cannot define a state that is regular across both the past and future event horizons unlike the Schwarzschild Hartle-Hawking state by the Kay-Wald theorem [56, 57], which proves rigorously the nonexistence of stationary Hadamard states in Kerr spacetime. Indeed, attempts to do so either result in states which are divergent in some part of the spacetime exterior to the black hole [49–53] or that are not in a thermal equilibrium [15]. Instead Candelas, Chrzanowski and Howard were led to define the CCH state in [15], where the acronym "CCH" is derived from the authors' names. While the CCH state is indeed a thermal state, it is not regular across both the past and future event horizon [15].

As earlier stated, superradiance in Kerr spacetime arises due to the co-rotation of the field with the Kerr black hole. Thus, it is not entirely clear as to whether the difficulties in defining quantum states are primarily due to rotation or to superradiance. In Minkowski spacetime, in the absence of spacetime curvature, the construction of rotating states is considerably involved [58, 59] and rigidly-rotating states do not exist in the unbounded spacetime for bosonic fields [60]; on the other hand, for fermionic fields, such states can be constructed [61, 62] but they are not everywhere regular. The references above have studied the effects of rotation in flat spacetimes in the absence of superradiance.

1.4.4 Motivations for studying QFT in Reissner-Nordström spacetime

Thus, it is hoped that the work in Part II on quantum field theory in Reissner-Nordström spacetime, which studies black hole superradiance in the absence of rotation, can help to disentangle the physical effects of rotation and superradiance in the Kerr case. A further advantage of our study here is that the Kerr solution lacks spherical symmetry due to the black hole rotating about a given axis. Instead it is said to be axisymmetric, which is a significantly weaker symmetry constraint and also makes performing renormalisation more difficult [55]. In contrast, the irrotational nature of the Reissner-Nordström solution means that it retains spherical symmetry which considerably simplifies calculations in this spacetime and allows for potentially easier renormalisation.

We will not attempt direct calculations of renormalised expectation values of observables in Part II, instead relying on the geometric, state-independent divergent terms cancelling when considering the difference between expectation values of observables in two separate quantum states. We also consider components of observables which do not require renormalisation. However, the work in Part III of this thesis provides the general framework by which the Hadamard renormalisation procedure can be performed for a charged scalar field. We therefore introduce briefly the main concepts of renormalisation of observables associated to a neutral scalar field, particularly the Hadamard procedure.

1.5 Renormalisation and Wald's axioms

The expectation values $\langle\widehat{\mathcal{SC}}\rangle$, $\langle\hat{J}^{\mu}\rangle$ and $\langle\hat{T}_{\mu\nu}\rangle$ all contain products of field operators evaluated at the same spacetime point and are therefore formally divergent. For example, in the case of a neutral scalar field, the expectation value of the scalar condensate $\langle\widehat{\mathcal{SC}}\rangle$

$$\langle\widehat{\mathcal{SC}}\rangle = \langle\hat{\Phi}^2\rangle\,, \tag{1.13}$$

is infinite without subtracting off the divergent parts. This requires the introduction of a renormalisation scheme whereby we subtract off divergent terms in the expectation values of observables to leave a physically reasonable quantity that, in the case of $\langle\hat{T}_{\mu\nu}\rangle_{\text{ren}}$ and $\langle\hat{J}^{\mu}\rangle_{\text{ren}}$, can be used as source terms in Einstein's semiclassical field equations (1.11) and the semiclassical Maxwell equations (1.12) respectively.

There exist several such renormalisation schemes, each with its own advantages and disadvantages. The question then arises as to which renormalisation method is the most suitable for our study. The point-splitting approach developed in [63, 64] has proven to be an extremely powerful and general method. One of the advantages of this approach is that it was developed in conjunction with Wald's axioms, which are a set of four statements:

1. Conservation of the renormalised stress-energy tensor (RSET), i.e. $\nabla^\mu \langle \hat{T}_{\mu\nu} \rangle_{\rm ren} = 0$.
2. Natural causality requirements are satisfied.
3. Value of the matrix element $\langle A | \hat{T}_{\mu\nu} | B \rangle$ for orthogonal states $|A\rangle, |B\rangle$ is preserved.
4. $\langle \hat{T}_{\mu\nu} \rangle_{\rm ren}$ should reduce to that of a normal-ordered SET in Minkowski spacetime.

All renormalisation methods for the RSET that satisfy Wald's axioms result in a renormalised expectation value $\langle \hat{T}_{\mu\nu} \rangle_{\rm ren}$ which is unique up to the addition of a local conserved tensor; this ambiguity in the RSET corresponds to the freedom to add any local conserved to the r.h.s of Einstein's semiclassical field equations (1.11).

The general philosophy of the point-splitting approach is to consider the product of field operators evaluated at closely separated spacetime points [65–67]. In the case of the expectation value of the scalar condensate (1.13), we would instead consider the expression

$$\langle \widehat{\mathcal{SC}} \rangle = \langle \hat{\Phi}(x)\, \hat{\Phi}(x') \rangle\,, \tag{1.14}$$

where x is in a normal neighbourhood of x' such that there exists a unique geodesic connecting the two spacetime points. Through a suitable regularisation procedure, the divergences that arise in the coincidence limit $x' \to x$ can be identified and then subtracted off before we bring the two spacetime points together; such an approach is agnostic of the specific quantum state under consideration since the divergent terms are purely geometric and therefore state-independent [2, 68].

1.6 Hadamard renormalisation

In this section, we will give a rapid overview of the Hadamard renormalisation procedure for a neutral scalar field, which has already been developed in detail for a scalar field of arbitrary mass and coupling to the scalar curvature on a general background spacetime in any number of spacetime dimensions in [68].

It is useful to note that it has been rigorously demonstrated that the Hadamard renormalisation procedure described below results in an RSET that satisfies Wald's axioms and which is unique to the addition of a local conserved tensor [69–77]. Hadamard renormalisation has also been developed for the electromagnetic field [78], the Stuc̆kelberg massive electromagnetic field [79], one-loop quantum gravity [80], p-forms [81] and fermions [82–85]; in part III, we will extend Hadamard renormalisation to charged scalar fields.

Hadamard renormalisation uses the point-splitting approach; in this scheme we write the expectation value of the quantum observables in terms of the Feynman Green's function $G_{\rm F}(x, x')$ associated to the field, which is itself divergent in the coincidence limit $x' \to x$. We consider the Feynman Green's function, as opposed to the Wightman function, since in Part III we will generalise, to charged scalar fields, the treatment of Decanini and Folacci in [68] where the former is considered.

Consider a massive, neutral scalar field Φ satisfying the Klein-Gordon equation

$$\left(\Box - m^2 - \xi R\right)\Phi = 0, \tag{1.15}$$

where the d'Alembertian $\Box = \nabla^\mu \nabla_\mu$ and ξ is a coupling constant that describes the coupling of the field to the scalar curvature R. In parts I and II of this thesis, we will consider a minimally-coupled scalar field for which $\xi = 0$. In contrast, the field is said to be conformally coupled if the coupling constant ξ takes the value $\xi = \xi_{\mathrm{c}}$, where

$$\xi_{\mathrm{c}} = \frac{d-2}{4\,(d-1)}. \tag{1.16}$$

In part III, when developing Hadamard renormalisation for charged scalar fields, we will consider the coupling constant ξ to be arbitrary. In [86], it is shown to be a physically reasonable assumption that a quantised field is in a Hadamard state. Then, the Feynman Green's function $G_{\mathrm{F}}(x, x')$ associated to a Hadamard state can be written as

$$-\,\mathrm{i}\, G_{\mathrm{F}}^{(d)}\big(x, x'\big) = \langle T\,[\hat{\Phi}(x)\,\hat{\Phi}^{\dagger}\big(x'\big)]\rangle, \tag{1.17}$$

where T denotes normal-ordering and $G_{\mathrm{F}}^{(d)}(x, x')$ satisfies the inhomogeneous field equation

$$\big(\Box - m^2 - \xi R\big)\, G_{\mathrm{F}}^{(d)}\big(x, x'\big) = -\,[-g(x)]^{-\frac{1}{2}}\,\delta^{(d)}\big(x - x'\big)\,, \tag{1.18}$$

where $\delta^{(d)}(x - x')$ denotes the Dirac delta function in d spacetime dimensions. We see that $G_{\mathrm{F}}^{(d)}(x, x')$ (1.17) acts as a scalar in both of its arguments x and x'; it is thus known as a biscalar. Divergences in the various expectation values can then be identified from the divergences that arise in the Feynman Green's function as we take the limit $x' \to x$.

We therefore need to write the Feynman Green's function in a form where we can identify the divergent terms; we call this expansion the Hadamard parametrix. Given that the point-splitting approach involves taking one of the field operators to a nearby spacetime point x' distinct from x, it is intuitive that the Hadamard parametrix should depend on the geodetic distance between x and x', which can be written in terms of Synge's world function $\sigma(x, x')$. Assuming that the spacetime point x' is in a normal neighbourhood of x, then there is a unique geodesic connecting x' to x. We can parametrise this geodesic in terms of an affine parameter λ, where $\lambda_0 \le \lambda \le \lambda_1$, and the tensor $z^\mu(\lambda)$ with the values $z^\mu(\lambda_0) := x'$ and $z^\mu(\lambda_1) := x$. Defining the tangent vector t^μ to $z^\mu(\lambda)$ as t^μ to $\mathrm{d}z^\mu/\mathrm{d}\lambda$, we have the following definition for Synge's world function $\sigma(x, x')$:

$$\sigma\big(x, x'\big) = \frac{1}{2}\,(\lambda_1 - \lambda_0)\int_{\lambda_0}^{\lambda_1} g_{\mu\nu}(z)\, t^\mu t^\nu \mathrm{d}\lambda, \tag{1.19}$$

where the integral in (1.19) is evaluated on the unique geodesic between x and x'. Synge's world function $\sigma(x, x')$ is equal to half the geodetic distance between x and x' such that

$$2\sigma\big(x, x'\big) = g_{\mu\nu}\,\sigma^{;\mu}\sigma^{;\nu}. \tag{1.20}$$

In four spacetime dimensions, the Hadamard parametrix is given by

$$-\,\mathrm{i}\, G_{\mathrm{F}}^{(4)}\big(x, x'\big) = \frac{1}{4\pi}\left\{\frac{U^{(4)}(x, x')}{[\sigma(x, x') + \mathrm{i}\epsilon]} + V^{(4)}\big(x, x'\big)\ln\!\left[\frac{\sigma(x, x')}{\ell_{\mathrm{ren}}^2} + \mathrm{i}\epsilon\right] + W^{(4)}\big(x, x'\big)\right\}, \tag{1.21}$$

where $U^{(4)}(x,x')$, $V^{(4)}(x,x')$ and $W^{(4)}(x,x')$ are symmetric biscalars regular in the limit $x' \to x$, and which can be expanded in powers of the world function $\sigma(x,x')$ as

$$U^{(4)}(x,x') = U_0^{(4)}(x,x')\,, \tag{1.22a}$$
$$V^{(4)}(x,x') = \sum_{n=0}^{\infty} V_n^{(4)}(x,x')\,\sigma^n(x,x')\,, \tag{1.22b}$$
$$W^{(4)}(x,x') = \sum_{n=0}^{\infty} W_n^{(4)}(x,x')\,\sigma^n(x,x')\,. \tag{1.22c}$$

The renormalisation length scale ℓ_{ren} is required to make the argument of the logarithm in (1.21) dimensionless. The expansion of the Feynman Green's function in terms of the Hadamard parametrix in (1.21) allows us to define a Hadamard state in four dimensions as a state possessing a smooth $W^{(4)}(x,x')$ biscalar. More generally, a Hadamard state in any number of spacetime dimensions is one possessing a smooth $W^{(d)}(x,x')$ biscalar.

We can write $G_{\mathrm{F}}^{(4)}(x,x')$ (1.21) as the sum of a regular part $G_{\mathrm{R}}^{(4)}(x,x')$ and a part $G_{\mathrm{S}}^{(4)}(x,x')$ which is divergent in the coincidence limit according to

$$G_{\mathrm{F}}^{(4)}(x,x') = G_{\mathrm{S}}^{(4)}(x,x') + G_{\mathrm{R}}^{(4)}(x,x')\,, \tag{1.23}$$

where the quantity $G_{\mathrm{S}}^{(4)}(x,x')$ in (1.23) is defined by

$$-\,\mathrm{i}\,G_{\mathrm{S}}^{(4)}(x,x') = \frac{1}{4\pi}\left\{\frac{U^{(4)}(x,x')}{[\sigma(x,x')+\mathrm{i}\epsilon]} + V^{(4)}(x,x')\ln\left[\frac{\sigma(x,x')}{\ell_{\mathrm{ren}}^2}+\mathrm{i}\epsilon\right]\right\}, \tag{1.24}$$

and the quantity $G_{\mathrm{R}}^{(4)}(x,x')$ in (1.23) is defined by

$$-\,\mathrm{i}\,G_{\mathrm{R}}^{(4)}(x,x') = -\mathrm{i}\left[G_{\mathrm{F}}^{(4)}(x,x') - G_{\mathrm{S}}^{(4)}(x,x')\right] = \frac{1}{4\pi}\,W^{(4)}(x,x')\,. \tag{1.25}$$

We can then write the renormalised expectation values of observables in terms of the regularised Green's function $G_{\mathrm{R}}^{(4)}(x,x')$ (1.25). For example, in the case of the scalar condensate example given earlier in (1.14), we have simply

$$\langle\widehat{\mathcal{SC}}\rangle_{\mathrm{ren}} = \lim_{x'\to x}\Re\left\{-\mathrm{i}\,G_{\mathrm{R}}^{(d)}(x,x')\right\}. \tag{1.26}$$

In practice, we will need to evaluate explicitly the $U^{(4)}(x,x')$ and $V^{(4)}(x,x')$ biscalars up to a sufficient order for computation of the expectation value of the observables under consideration. In order to do this, we require the van Vleck-Morette determinant $\Delta(x,x')$

$$\Delta(x,x') = -\,[-g(x)]^{-\frac{1}{2}}\det\left[\sigma_{;\mu\nu'}(x,x')\right]\,\left[-g(x')\right]^{-\frac{1}{2}}\,, \tag{1.27}$$

where the subscript $;\mu'$ refers to the covariant derivative evaluated at the spacetime point x'. The van Vleck-Morette determinant $\Delta(x,x')$ (1.27) gives the rate at which geodesics converge or diverge away from each other [87]. It is related to Synge's world function by

$$\Box\,\sigma = d - 2\Delta^{-\frac{1}{2}}\Delta^{\frac{1}{2}}_{;\mu}\sigma^{;\mu}, \tag{1.28}$$

in d dimensions, and with the boundary condition

$$\lim_{x' \to x} \Delta\big(x, x'\big) = 1. \tag{1.29}$$

If we were to consider an particular quantum state, in a particular background spacetime with particular values of the scalar field mass m and coupling to the scalar curvature ξ, we could generate explicit expressions for the renormalised expectation values of observables.

However, in part III we will instead develop the general framework for Hadamard renormalisation of a charged scalar field in a general background spacetime of arbitrary dimension leaving the charge, mass and coupling to the scalar curvature of the field undetermined. We will derive explicitly the geometric renormalisation counterterms contained within the the biscalars $U^{(d)}(x, x')$ and $V^{(d)}(x, x')$ up to the required order for computation of the RSET in two, three and four spacetime dimensions. We will also derive expressions for the renormalised expectations values $\langle\widehat{\mathcal{SC}}\rangle_{\text{ren}}$, $\langle\hat{J}^{\mu}\rangle_{\text{ren}}$ and $\langle\hat{T}_{\mu\nu}\rangle_{\text{ren}}$ in terms of the biscalar $W^{(d)}(x, x')$. We will conclude with a discussion of the renormalisation ambiguities in each of the expectation values we derive.

Chapter 2

Quantum scalar field theory in Schwarzschild spacetime

In this chapter, we review quantum field theory in Schwarzschild spacetime. Our aim in this chapter is to provide context to our work in Part II and therefore we do not reproduce some of the more involved derivations; analogous but novel calculations pertaining to the Reissner-Nordström case can be found in Part II. In §2.1, we introduce the Schwarzschild solution. We introduce a scalar field on this spacetime in §2.2 and, lastly, we define the three main states associated to a scalar field in Schwarzschild spacetime in §2.3.

2.1 The Schwarzschild solution

The simplest possible black hole solution is Schwarzschild spacetime; it is convenient to use the Schwarzschild coordinate system (t, r, θ, φ) in order to study it. The first coordinate, t, is timelike and the rest are spacelike. The spacelike coordinates are the familiar spherical polar coordinates. Then, Schwarzschild spacetime is described by the line element

$$\mathrm{d}s^2 = -f_{\mathrm{s}}(r)\,\mathrm{d}t^2 + f_{\mathrm{s}}(r)^{-1}\,\mathrm{d}r^2 + r^2\mathrm{d}\theta^2 + r^2\sin^2\theta\,\mathrm{d}\varphi^2, \tag{2.1}$$

where the Schwarzschild metric function $f_{\mathrm{s}}(r)$ is given by

$$f_{\mathrm{s}}(r) = 1 - \frac{2M}{r}. \tag{2.2}$$

We have added a subscript "s" to the Schwarzschild metric function $f_{\mathrm{s}}(r)$, and more generally to any quantities defined in Schwarzschild spacetime that have analogues in Reissner-Nordström spacetime, in order to distinguish it from the metric function of Reissner-Nordström spacetime that we introduce in Chapter 3.

The Schwarzschild solution describes the exterior of a spherically symmetric ball of matter that is classically surrounded by an empty vacuum. However, when we consider a quantum field on this spacetime, the background gravitational fields give rise to radiation which can reach infinity and so the ball of matter is no longer surrounded by a vacuum.

Let us assume that the ball of matter is a star whose mass exceeds the Chandrasekhar limit; in this case we can reasonably assume that the star will eventually collapse under the force of gravity to form a black hole. We will consider this black hole system a long time after the collapse; in this case it is referred to as an eternal black hole. Then we can interpret the quantity M in (2.2) as the black hole mass. The Schwarzschild metric function $f_{\rm s}(r)$ (2.2) has a root $r_{\rm H}$ given by

$$r_{\rm H} = 2M. \tag{2.3}$$

This is the familiar location of the event horizon of the Schwarzschild black hole. The value of the metric function (2.3) on the Schwarzschild black hole horizon vanishes as

$$f_{\rm s}(r_{\rm H}) = 1 - \frac{2M}{2M} = 0, \tag{2.4}$$

and thus, from (2.1), the Schwarzschild metric (2.2) diverges on the horizon. This gives us a clue that the Schwarzschild coordinates do not give the full picture. In order to proceed, we define a new radial coordinate. Consider radial null geodesics; from (2.1), we have

$$\mathrm{d}t^2 = \left(1 - \frac{2M}{r}\right)^{-2} \mathrm{d}r^2. \tag{2.5}$$

If we define a new coordinate r_*, which we refer to as the tortoise coordinate, such that

$$\mathrm{d}r_*^2 = \left(1 - \frac{2M}{r}\right)^{-2} \mathrm{d}r^2, \tag{2.6}$$

then radial null geodesics take the simple form $\mathrm{d}t^2 = \mathrm{d}{r_*}^2$, and (2.1) becomes

$$\mathrm{d}s^2 = -f_{\rm s}(r)\,\mathrm{d}t^2 + f_{\rm s}(r)\,\mathrm{d}r_*^2 + r^2\,\mathrm{d}\theta^2 + r^2\sin^2\theta\,\mathrm{d}\varphi^2. \tag{2.7}$$

Demanding that r_* be real and monotonically increasing with r, we can solve (2.6) to give

$$r_* = r + 2M\ln\left(\frac{r-2M}{2M}\right). \tag{2.8}$$

The tortoise coordinate r_* has a number of useful properties. Firstly, the range $2M < r < \infty$ is mapped onto the range $-\infty < r_* < \infty$; we will see that this will be especially useful when examining the asymptotic behaviour of quantities near the Schwarzschild black hole horizon since $r_* \to -\infty$ as $r \to r_{\rm H}$. We can differentiate (2.8) to obtain the useful relation

$$\frac{\mathrm{d}r_*}{\mathrm{d}r} = f_{\rm s}(r)^{-1}. \tag{2.9}$$

Since the Schwarzschild metric function $f_{\rm s}(r)$ is analytic, we can invert (2.9) to obtain

$$\frac{\mathrm{d}r}{\mathrm{d}r_*} = f_{\rm s}(r)\,. \tag{2.10}$$

From (2.10), we see why r_* is referred to as the tortoise coordinate; since $\frac{\mathrm{d}r}{\mathrm{d}r_*} \to 0$ as $r \to 2M^+$, then r changes more and more slowly with r_* as we near the black hole event horizon. The second useful property of the tortoise coordinate is that we can use it to define new coordinates which will reveal more of the spacetime. We define a pair of new coordinates, which we refer to as lightcone coordinates, by the expressions

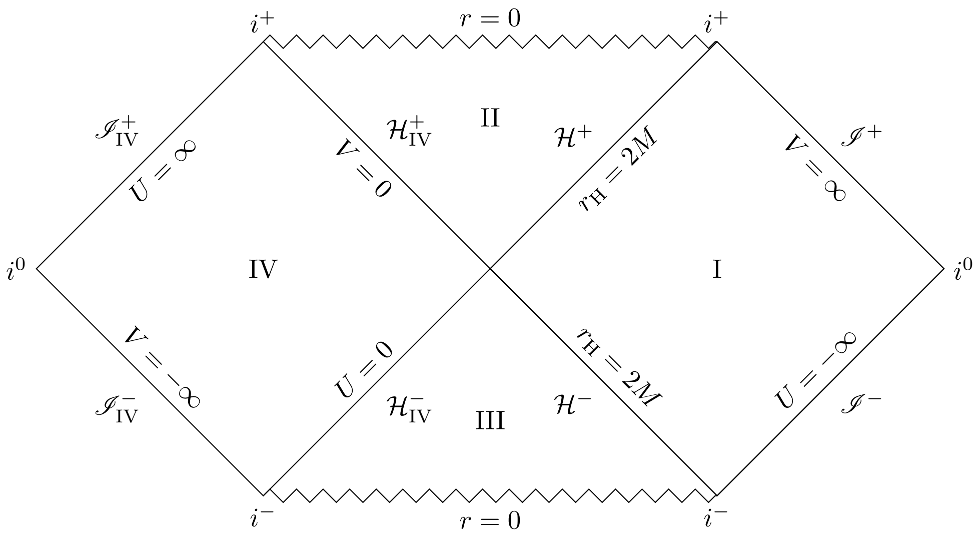

Figure 2.1. Penrose diagram for the maximally-extended Schwarzschild solution showing the spacetime regions I, II, III and IV. Lines of constant Schwarzschild-like radial coordinate r and Kruskal coordinate U, V corresponding to physically significant surfaces are shown. Surfaces of interest in region I include the past (future) event horizon $\mathcal{H}^-$ ($\mathcal{H}^+$) and past (future) null infinity $\mathscr{I}^-$ ($\mathscr{I}^+$); their corresponding surfaces in region IV are labelled with a subscript IV. Event horizons in region I correspond to a constant $r_{\rm H} = 2M$ and a spacetime singularity is located at $r = 0$. Past (future) timelike infinity is denoted by i^+ (i^-) and spacelike infinity is denoted by i^0.

$$u = t - r_*, \quad \text{and} \quad v = t + r_*, \tag{2.11}$$

such that lines of constant u correspond to outgoing null geodesics and lines of constant v correspond to ingoing null geodesics. Then the Schwarzschild line element (2.7) becomes

$$\mathrm{d}s^2 = -f_{\rm s}(r)\,\mathrm{d}u\,\mathrm{d}v + r^2\mathrm{d}\theta^2 + r^2\sin^2\theta\,\mathrm{d}\varphi^2. \tag{2.12}$$

A component of the inverse metric associated with the Schwarzschild metric (2.12) in terms of lightcone coordinates is still divergent on the event horizon. From (2.11), we define a new pair of coordinates, which we will refer to as Kruskal coordinates, given by

$$U = -\exp\left(-\frac{u}{4M}\right), \quad \text{and} \quad V = \exp\left(\frac{v}{4M}\right). \tag{2.13}$$

In terms of Kruskal coordinates, the Schwarzschild line element (2.12) becomes

$$\mathrm{d}s^2 = -\frac{32M^3}{r}\exp\left(-\frac{r}{2M}\right)\mathrm{d}U\mathrm{d}V + r^2\mathrm{d}\theta^2 + r^2\sin^2\theta\,\mathrm{d}\varphi^2. \tag{2.14}$$

From the expression for the Schwarzschild metric in terms of Kruskal coordinates (2.14), which is defined for nonzero values of the radial coordinate r, we see that the divergence in the line element in terms of Schwarzschild coordinates (2.1) is a coordinate singularity. However, the point $r = 0$ represents a physical singularity in the spacetime curvature. Having introduced Kruskal coordinates in (2.13), we can now access all regions of the maximally-extended Schwarzschild solution in Figure 2.1. We now proceed to introducing a scalar field Φ on a background Schwarzschild spacetime.

2.2 Scalar fields on a background Schwarzschild spacetime

In this section, we will restrict our attention to a massless neutral scalar field Φ which is minimally-coupled to the spacetime curvature. In general, the scalar field may possess a mass m as well as arbitrary coupling ξ to the scalar curvature. However, in Part II of this thesis, we will consider a massless, minimally-coupled charged scalar field on a background Reissner-Nordström spacetime; our purpose in Part I of this thesis is to study a similar set-up in Schwarzschild spacetime, which is a simpler black hole solution without the added complications of the charge possessed by a Reissner-Nordström black hole.

One might ask why we consider a neutral scalar field in Part I as opposed to a charged scalar field, which we consider later on. As we will see in Part II, most of the interesting physical phenomena that occur in charged scalar field theory in Reissner-Nordström spacetime are as a result of the interaction of the scalar field charge and the charge of the black hole. Since a Schwarzschild black hole does not possess a charge, it is simpler to introduce the general formalism of quantum field theory in curved spacetime in this chapter without the additional baggage of a scalar field charge.

A massless, minimally-coupled scalar field Φ is governed by the scalar field equation

$$\Box\,\Phi = \frac{1}{\sqrt{-g}}\,\partial_\mu\big(\sqrt{-g}\,\partial^\mu\Phi\big) = 0. \tag{2.15}$$

Our treatment of deriving solutions to the scalar field equation (2.15) is deliberately concise since we solve the scalar field equation associated to a charged scalar field in a background Reissner-Nordström spacetime in considerable detail in §3.2. We can expand (2.15) in terms of the Schwarzschild coordinates introduced in §2.1 to obtain

$$-\frac{1}{f(r)}\frac{\partial^2\Phi}{\partial t^2}+\frac{1}{r^2}\frac{\partial}{\partial r}\left[f(r)r^2\frac{\partial\Phi}{\partial r}\right]+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left[\sin\theta\,\frac{\partial\Phi}{\partial\theta}\right]+\frac{1}{r^2\sin^2\theta}\frac{\partial^2\Phi}{\partial\varphi^2}=0, \tag{2.16}$$

where we have used both the inverse and the determinant g of the metric (2.1). The scalar field equation (2.18) admits a separable solution of the form

$$\phi_{\omega\ell m} = \frac{\mathrm{e}^{-\mathrm{i}\omega t}}{r}\,\mathrm{N}_\omega\,X_{\omega\ell}(r)\,Y_{\ell m}(\theta,\varphi)\,. \tag{2.17}$$

The harmonic time-dependence $\mathrm{e}^{-\mathrm{i}\omega t}$ of the solution in (2.17) is a consequence of the fact that the Schwarzschild solution is stationary, as can be seen from the time-independence of the metric (2.1); a stationary spacetime is one which admits a time-translation Killing vector. Furthermore, a stationary spacetime can be considered static if the time-translation Killing vector is hypersurface-orthogonal. In terms of the spacetimes considered in this thesis, both Schwarzschild and Reissner-Nordström spacetimes are static, whereas Kerr spacetime is an example of a spacetime that is stationary but not static. The normalisation constant N_ω allows us to generate an orthonormal basis from the solutions to (2.17), which we require to quantise the field Φ, while $X_{\omega\ell}(r)$ and $Y_{\ell m}(\theta,\varphi)$ represent the radial and angular functions respectively. Given the spherical symmetry of Schwarzschild spacetime, which is apparent from the angular parts of the metric (2.1) being proportional to the metric on the 2-sphere $g_\Omega = \mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2$, we anticipate that the angular functions

$Y_{\ell m}(\theta,\varphi)$ will be the spherical harmonics. Substituting (2.17) into (2.16) and separating the radial and angular parts, we obtain

$$\frac{r^2\omega}{f(r)}+\frac{r}{X_{\omega\ell}(r)}\frac{\mathrm{d}}{\mathrm{d}r}\left[f(r)r^2\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{X_{\omega\ell}(r)}{r}\right)\right]$$
$$=-\frac{1}{Y_{\ell m}(\theta,\varphi)}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right]Y_{\ell m}(\theta,\varphi)=\lambda, \quad (2.18)$$

where the constant λ is a separation constant. The angular part of (2.18) is precisely the equation that is solved by the spherical harmonics; from their properties, the separation constant is written in terms of the total angular momentum quantum number ℓ as

$$\lambda=\ell(\ell+1)\,, \qquad \ell \text{ in } \mathbb{Z}_{\geq 0} \text{ and } \ell\geq|m|\,. \quad (2.19)$$

The spherical harmonics are explored in greater detail in §3.2.3 and further identities are derived in Appendix A. The explicit form of the separation constant (2.19) allows us to write the radial part of (2.18) in the form of the well-known Regge-Wheeler equation

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}r_*^2}-V_{\text{eff}}(r)\right]X_{\omega\ell}(r)=0, \quad (2.20)$$

where the scalar field effective potential $V_{\text{eff}}(r)$, which is the one-dimensional effective potential felt by a field mode $\phi_{\omega\ell m}$ in a background Schwarzschild spacetime, is given by

$$V_{\text{eff}}(r)=\frac{f(r)}{r^2}\left[\ell(\ell+1)+f'(r)r\right]-\omega^2. \quad (2.21)$$

Near the horizon and far from the black hole, the scalar field effective potential V_{eff} (2.21) takes the particularly simple asymptotic form

$$V_{\text{eff}}(r)\sim-\omega^2, \qquad r_*\to-\infty,\ r_*\to\infty. \quad (2.22)$$

Then, near the horizon and far from the back hole, the radial equation (2.20) becomes

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}r_*^2}+\omega^2\right]X_{\omega\ell}(r)=0, \quad (2.23)$$

which has asymptotic solutions of the form

$$X_{\omega\ell}(r)\sim \mathrm{e}^{\pm\mathrm{i}\omega r_*}, \qquad r_*\to-\infty,\ r_*\to\infty. \quad (2.24)$$

The asymptotic forms of the radial function $X_{\omega\ell}(r)$ allow us to generate various mode solutions $\phi_{\omega\ell m}$ to the scalar field equation (2.15). The mode solutions can be motivated by physical considerations of how the field should behave in the background spacetime, which is how we derive the modes in §3.3. For the sake of brevity, we will simply state the asymptotic forms of the radial functions associated to each mode here and we refer the reader to §3.3 for their detailed physical interpretation. The first mode solution we consider are the in-modes $\phi^{\text{in}}_{\omega\ell m}$, whose associated radial function $X^{\text{in}}_{\omega\ell}(r)$ has the asymptotic forms

$$X^{\text{in}}_{\omega\ell}(r) \sim \begin{cases} B^{\text{in}}_{\omega\ell}\, \mathrm{e}^{-\mathrm{i}\omega r_*} & r_* \to -\infty, \\ \mathrm{e}^{-\mathrm{i}\omega r_*} + A^{\text{in}}_{\omega\ell}\, \mathrm{e}^{\mathrm{i}\omega r_*} & r_* \to \infty. \end{cases} \tag{2.25}$$

Another solution is the up-modes $\phi^{\text{up}}_{\omega\ell m}$ with $X^{\text{up}}_{\omega\ell}(r)$ taking the asymptotic forms

$$X^{\text{up}}_{\omega\ell}(r) \sim \begin{cases} \mathrm{e}^{\mathrm{i}\omega r_*} + A^{\text{up}}_{\omega\ell}\, \mathrm{e}^{-\mathrm{i}\omega r_*} & r_* \to -\infty, \\ B^{\text{up}}_{\omega\ell}\, \mathrm{e}^{\mathrm{i}\omega r_*} & r_* \to \infty. \end{cases} \tag{2.26}$$

The in- and up-modes form an orthogonal basis of scalar field modes; we will refer to this basis as the 'past' basis. Another orthogonal basis of out-modes $\phi^{\text{out}}_{\omega\ell m}$ and down-modes $\phi^{\text{down}}_{\omega\ell m}$, which we will refer to as the 'future' basis, can be constructed from the complex conjugates of the radial functions $X^{\text{in}}_{\omega\ell}(r)$ and $X^{\text{up}}_{\omega\ell}(r)$ respectively. Then, the asymptotic forms of the out-mode radial function $X^{\text{out}}_{\omega\ell}(r)$ are given by

$$X^{\text{out}}_{\omega\ell}(r) \sim \begin{cases} B^{\text{in}*}_{\omega\ell}\, \mathrm{e}^{\mathrm{i}\omega r_*} & r_* \to -\infty, \\ \mathrm{e}^{\mathrm{i}\omega r_*} + A^{\text{in}*}_{\omega\ell}\, \mathrm{e}^{-\mathrm{i}\omega r_*} & r_* \to \infty, \end{cases} \tag{2.27}$$

and the asymptotic forms of the down-mode radial function $X^{\text{down}}_{\omega\ell}(r)$ are given by

$$X^{\text{down}}_{\omega\ell}(r) \sim \begin{cases} \mathrm{e}^{-\mathrm{i}\omega r_*} + A^{\text{up}*}_{\omega\ell}\, \mathrm{e}^{\mathrm{i}\omega r_*} & r_* \to -\infty, \\ B^{\text{up}*}_{\omega\ell}\, \mathrm{e}^{-\mathrm{i}\omega r_*} & r_* \to \infty. \end{cases} \tag{2.28}$$

Since $X^{\text{out}}_{\omega\ell}(r) = X^{\text{in}*}_{\omega\ell}(r)$, the in- and out-mode radial functions are linearly independent solutions of the radial equation (2.20); similar comments apply for the up- and down-mode radial functions since $X^{\text{down}}_{\omega\ell}(r) = X^{\text{up}*}_{\omega\ell}(r)$. The Wronskian $W(X_1, X_2)$, which is given by

$$W(X_1, X_2) = X_1 \frac{\mathrm{d}X_2}{\mathrm{d}r_*} - X_2 \frac{\mathrm{d}X_1}{\mathrm{d}r_*}, \tag{2.29}$$

of any two linearly independent solutions X_1, X_2 of (2.20) is independent of r_*. Then we can evaluate the Wronskian (2.29) for $X^{\text{in}}_{\omega\ell}(r)$ and $X^{\text{out}}_{\omega\ell}(r)$ near the horizon to obtain

$$W\left(X^{\text{in}}_{\omega\ell}, X^{\text{out}}_{\omega\ell}\right) = B^{\text{in}}_{\omega\ell}\, \mathrm{e}^{-\mathrm{i}\omega r_*}\left(\mathrm{i}\omega\, B^{\text{in}*}_{\omega\ell}\, \mathrm{e}^{\mathrm{i}\omega r_*}\right) - B^{\text{in}*}_{\omega\ell}\, \mathrm{e}^{\mathrm{i}\omega r_*}\left(-\mathrm{i}\omega B^{\text{in}}_{\omega\ell}\, \mathrm{e}^{-\mathrm{i}\omega r_*}\right) = 2\,\mathrm{i}\omega\left|B^{\text{in}}_{\omega\ell}\right|^2, \tag{2.30}$$

while evaluating the Wronskian (2.29) for $X^{\text{in}}_{\omega\ell}(r)$ and $X^{\text{out}}_{\omega\ell}(r)$ near infinity, we obtain

$$\begin{aligned} W\left(X^{\text{in}}_{\omega\ell}, X^{\text{out}}_{\omega\ell}\right) &= \left(\mathrm{e}^{-\mathrm{i}\omega r_*} + A^{\text{in}}_{\omega\ell}\, \mathrm{e}^{\mathrm{i}\omega r_*}\right)\left(\mathrm{i}\omega\, \mathrm{e}^{\mathrm{i}\omega r_*} - \mathrm{i}\omega A^{\text{in}*}_{\omega\ell}\, \mathrm{e}^{-\mathrm{i}\omega r_*}\right) \\ &\quad - \left(\mathrm{e}^{\mathrm{i}\omega r_*} + A^{\text{in}*}_{\omega\ell}\, \mathrm{e}^{-\mathrm{i}\omega r_*}\right)\left(-\mathrm{i}\omega \mathrm{e}^{-\mathrm{i}\omega r_*} + \mathrm{i}\omega A^{\text{in}}_{\omega\ell}\, \mathrm{e}^{\mathrm{i}\omega r_*}\right) = 2\,\mathrm{i}\omega\left[1 - \left|A^{\text{in}}_{\omega\ell}\right|^2\right]. \end{aligned} \tag{2.31}$$

Equating (2.30) and (2.31) for the Wronskian $W\left(X^{\text{in}}_{\omega\ell}, X^{\text{out}}_{\omega\ell}\right)$ leads to the relation

$$\left|B^{\text{in}}_{\omega\ell}\right|^2 = 1 - \left|A^{\text{in}}_{\omega\ell}\right|^2. \tag{2.32}$$

A similar set of calculations for the Wronskian $W\left(X^{\text{up}}_{\omega\ell}, X^{\text{down}}_{\omega\ell}\right)$ leads to the relation

$$\left|B^{\rm up}_{\omega\ell}\right|^2 = 1 - \left|A^{\rm up}_{\omega\ell}\right|^2. \tag{2.33}$$

Interpreting (2.32) and (2.33) as the standard scattering relations allows us to ascribe a natural interpretation to the complex constants $A_{\omega\ell}$ and $B_{\omega\ell}$ in (2.25 – 2.28) as reflection and transmission coefficients respectively. From (2.32) and (2.33), we have $|A_{\omega\ell}| \leq 1$ meaning that there is no superradiant scattering in Schwarzschild spacetime.

In order to quantise the field Φ, we need to normalise each of the mode solutions defined above by evaluating their normalisation constants N_ω. This is done by taking the Klein-Gordon inner product of any two similar mode solutions over a suitably chosen Cauchy surface Σ and demanding that the modes be orthonormal. The Klein-Gordon inner product for any two mode solutions ϕ_1, ϕ_2 of the scalar field equation (2.15) is [10]

$$\langle\phi_1,\phi_2\rangle = -\mathrm{i}\int_\Sigma \left[\,\phi_1^*\left(\nabla_\mu\,\phi_2\right) - \left(\nabla_\mu\,\phi_1^*\right)\phi_2\,\right]\sqrt{-g}\,\mathrm{d}\Sigma^\mu. \tag{2.34}$$

In §3.4, we evaluate both the norms and the normalisation constants of each of the in-, up-, out- and down-modes in Reissner-Nordström spacetime in considerable detail. Since the calculation of the corresponding quantities in Schwarzschild spacetime is similarly involved but somewhat less illuminating, given its comparative simplicity, we refer the reader to §3.4 for a detailed reading. Instead, using the notation $\mathrm{k} = \mathrm{in}, \mathrm{up}, \mathrm{out}, \mathrm{down}$ to label the specific mode solution, we write down the normalisation constants of each mode as

$$\mathrm{N}^{\rm k}_\omega = \frac{1}{\sqrt{4\pi|\omega|}}, \tag{2.35}$$

i.e. $\mathrm{N}^{\rm k}_\omega$ takes the same value for all modes. The normalisation constants (2.35) ensure

$$\langle\phi^{\rm k}_{\omega\ell m},\phi^{\rm k}_{\omega'\ell'm'}\rangle = \delta\big(\omega-\omega'\big)\,\delta_{\ell\ell'}\,\delta_{mm'}. \tag{2.36}$$

From (2.36), the norm of the complex conjugate of a general mode $\phi^{\rm k*}_{\omega\ell m}$ is given by

$$\langle\phi^{\rm k*}_{\omega\ell m},\phi^{\rm k*}_{\omega'\ell'm'}\rangle = -\delta\big(\omega-\omega'\big)\,\delta_{\ell\ell'}\,\delta_{mm'}. \tag{2.37}$$

Then the general form of a scalar field mode $\phi^{\rm k}_{\omega\ell m}$ is given by

$$\phi^{\rm k}_{\omega\ell m} = \frac{1}{\sqrt{4\pi|\omega|}}\,\frac{1}{r}\,\mathrm{e}^{-\mathrm{i}\omega t}X^{\rm k}_{\omega\ell}(r)\,Y_{\ell m}(\theta,\varphi)\,, \tag{2.38}$$

where the asymptotic forms of the radial functions $X^{\rm k}_{\omega\ell}(r)$ associated to each of the different mode solutions are those given in (2.25 – 2.28). Using relations derived by evaluating the Wronskian (2.29) for various combinations of $X^{\rm k}_{\omega\ell}(r)$ as well as the normalisation constants (2.35), we can write the out-modes and the down-modes in terms of in- and up-modes as

$$\phi^{\rm out}_{\omega\ell m} = A^{\rm in*}_{\omega\ell}\,\phi^{\rm in}_{\omega\ell m} + B^{\rm in*}_{\omega\ell}\,\phi^{\rm out}_{\omega\ell m}, \tag{2.39a}$$

$$\phi^{\rm down}_{\omega\ell m} = A^{\rm up*}_{\omega\ell}\,\phi^{\rm up}_{\omega\ell m} + B^{\rm up*}_{\omega\ell}\,\phi^{\rm in}_{\omega\ell m}. \tag{2.39b}$$

2.3 Quantum field theory in Schwarzschild spacetime

In order to canonically quantise the field, we need to decompose the scalar field modes introduced in §2.2 into sets of positive- and negative-frequency modes; upon quantisation positive-frequency modes will be multiplied by annihilation operators and negative-frequency modes will be multiplied by creation operators.

In Minkowski spacetime, there is a natural global vacuum. In QFTCS, however, there does not generally exist a unique way to decompose the field into positive- and negative-frequency modes; this, in turn, is intimately tied to the fact that there does not exist a unique vacuum state in curved spacetimes. In particular, we can consider the scalar field modes to be positive- and negative-frequency with respect to a variety of choices of time coordinate in order to define states with a certain physical interpretation.

There are three main states that have been defined in Schwarzschild spacetime, namely the Schwarzschild Boulware state $|\mathrm{B_s}\rangle$ [16], the Schwarzschild Unruh state $|\mathrm{U_s}\rangle$ [5] and the Schwarzschild Hartle-Hawking state $|\mathrm{H_s}\rangle$ [17]. We will now discuss each of these states in turn, taking care to explain the physical choice of time coordinate with respect to which we define positive- and negative-frequency modes in each case.

2.3.1 Schwarzschild Boulware state

The Schwarzschild Boulware state $|\mathrm{B_s}\rangle$ [16] has the physical interpretation of being as empty as possible to an observer far from the black hole, i.e. both past and future null infinity $\mathscr{I}^{\pm}$. In terms of the scalar field modes defined in §2.1, this corresponds to an absence of particles in both the in- and out-modes.

Schwarzschild spacetime is asymptotically flat and thus becomes indistinguishable from Minkowski spacetime as $r \to \infty$. Then, far from the black hole, the proper time experienced by a static observer is the Schwarzschild coordinate t and we can define positive- and negative-frequency modes in the way familiar from quantum field theory in Minkowski spacetime. A general scalar field mode $\phi^{\mathrm{k}}_{\omega\ell m}$ given by

$$\phi^{\mathrm{k}}_{\omega\ell m} = \frac{1}{\sqrt{4\pi|\omega|}}\frac{1}{r}\,\mathrm{e}^{-\mathrm{i}\omega t}X^{\mathrm{k}}_{\omega\ell}(r)\,Y_{\ell m}(\theta,\varphi)\,, \quad \omega > 0, \tag{2.40}$$

is considered to be positive-frequency with respect to t and its complex conjugate $\phi^{\mathrm{k}*}_{\omega\ell m}$

$$\phi^{\mathrm{k}*}_{\omega\ell m} = \frac{1}{\sqrt{4\pi|\omega|}}\frac{1}{r}\,\mathrm{e}^{\mathrm{i}\omega t}X^{\mathrm{k}*}_{\omega\ell}(r)\,Y^{*}_{\ell m}(\theta,\varphi)\,, \quad \omega > 0, \tag{2.41}$$

is considered to be negative-frequency with respect to t. Then, it is natural to expand the field $\Phi(x)$ in terms of in- and out-modes to obtain

$$\Phi(x) = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\int_0^{\infty}\mathrm{d}\omega\left\{\tilde{a}^{\mathrm{in}}_{\omega\ell m}\phi^{\mathrm{in}}_{\omega\ell m} + \tilde{a}^{\mathrm{in}\dagger}_{\omega\ell m}\phi^{\mathrm{in}*}_{\omega\ell m} + \tilde{a}^{\mathrm{out}}_{\omega\ell m}\phi^{\mathrm{out}}_{\omega\ell m} + \tilde{a}^{\mathrm{out}\dagger}_{\omega\ell m}\phi^{\mathrm{out}*}_{\omega\ell m}\right\}, \tag{2.42}$$

where the mode expansion coefficients $\tilde{a}^{\mathrm{in}}_{\omega\ell m}$ and $\tilde{a}^{\mathrm{out}}_{\omega\ell m}$ multiply positive-frequency modes and the mode expansion coefficients $\tilde{a}^{\dagger\mathrm{in}}_{\omega\ell m}$ and $\tilde{a}^{\dagger\mathrm{out}}_{\omega\ell m}$ multiply negative-frequency modes.

However, the in- and out-modes are not orthogonal to each other, and therefore do not form an orthonormal basis of scalar field modes which we require to quantise the field. From §2.2, the in- and up-modes do form an orthonormal basis; we would then like to re-express the out-modes in (2.42) in terms of in- and up-modes. Using (2.39a), we have

$$\Phi(x) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \int_0^{\infty} \mathrm{d}\omega \left\{ a^{\mathrm{in}}_{\omega\ell m} \phi^{\mathrm{in}}_{\omega\ell m} + a^{\mathrm{in}\dagger}_{\omega\ell m} \phi^{\mathrm{in}*}_{\omega\ell m} + a^{\mathrm{up}}_{\omega\ell m} \phi^{\mathrm{up}}_{\omega\ell m} + a^{\mathrm{up}\dagger}_{\omega\ell m} \phi^{\mathrm{up}*}_{\omega\ell m} \right\}, \tag{2.43}$$

where the mode coefficients in (2.43) are related to those in (2.42) by the expressions

$$\begin{aligned} a^{\mathrm{in}}_{\omega\ell m} &= \tilde{a}^{\mathrm{in}}_{\omega\ell m} + A^{\mathrm{in}*}_{\omega\ell}\, \tilde{a}^{\mathrm{out}}_{\omega\ell m}, \qquad && \omega > 0, \\ a^{\mathrm{up}}_{\omega\ell m} &= B^{\mathrm{in}*}_{\omega\ell}\, \tilde{a}^{\mathrm{out}}_{\omega\ell m}, && \omega > 0. \end{aligned} \tag{2.44}$$

We quantise the field by promoting the mode expansion coefficients in (2.43) to operators such that the field operator $\hat{\Phi}(x)$ is given by

$$\hat{\Phi}(x) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \int_0^{\infty} \mathrm{d}\omega \left\{ \hat{a}^{\mathrm{in}}_{\omega\ell m} \phi^{\mathrm{in}}_{\omega\ell m} + \hat{a}^{\mathrm{in}\dagger}_{\omega\ell m} \phi^{\mathrm{in}*}_{\omega\ell m} + \hat{a}^{\mathrm{up}}_{\omega\ell m} \phi^{\mathrm{up}}_{\omega\ell m} + \hat{a}^{\mathrm{up}\dagger}_{\omega\ell m} \phi^{\mathrm{up}*}_{\omega\ell m} \right\}, \tag{2.45}$$

where the operators in (2.45) obey the following commutation relations

$$\begin{aligned} \left[\hat{a}^{\mathrm{in}}_{\omega\ell m}, \hat{a}^{\mathrm{in}\dagger}_{\omega'\ell' m'} \right] &= \delta\big(\omega - \omega'\big)\, \delta_{\ell\ell'}\, \delta_{mm'}, \\ \left[\hat{a}^{\mathrm{up}}_{\omega\ell m}, \hat{a}^{\mathrm{up}\dagger}_{\omega'\ell' m'} \right] &= \delta\big(\omega - \omega'\big)\, \delta_{\ell\ell'}\, \delta_{mm'}, \end{aligned} \tag{2.46}$$

with any commutators not explicitly given in (2.46) vanishing. The Schwarzschild Boulware state $|\mathrm{B_s}\rangle$ is then defined as the state annihilated by the $\hat{a}^{\mathrm{in/up}}_{\omega\ell m}$ operators such that

$$\begin{aligned} \hat{a}^{\mathrm{in}}_{\omega\ell m} \, |\mathrm{B_s}\rangle &= 0, \\ \hat{a}^{\mathrm{up}}_{\omega\ell m} \, |\mathrm{B_s}\rangle &= 0. \end{aligned} \tag{2.47}$$

Using (2.47), we can act with the field operator $\hat{\Phi}$ (2.45) on $|\mathrm{B_s}\rangle$ to generate an expression for the scalar condensate $\langle \hat{\Phi}^2 \rangle_{|\mathrm{B_s}\rangle}$ in the Schwarzschild Boulware state. We have

$$\hat{\Phi}\, |\mathrm{B_s}\rangle = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \int_0^{\infty} \mathrm{d}\omega \left\{ \phi^{\mathrm{in}*}_{\omega\ell m} \hat{a}^{\mathrm{in}\dagger}_{\omega\ell m} \, |\mathrm{B_s}\rangle + \phi^{\mathrm{up}*}_{\omega\ell m} \hat{a}^{\mathrm{up}\dagger}_{\omega\ell m} \, |\mathrm{B_s}\rangle \right\}, \tag{2.48}$$

$$\langle \mathrm{B_s}|\, \hat{\Phi} = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \int_0^{\infty} \mathrm{d}\omega \left\{ \phi^{\mathrm{in}}_{\omega\ell m} \, \langle \mathrm{B_s}|\, \hat{a}^{\mathrm{in}}_{\omega\ell m} + \phi^{\mathrm{up}}_{\omega\ell m} \, \langle \mathrm{B_s}|\, \hat{a}^{\mathrm{up}}_{\omega\ell m} \right\}. \tag{2.49}$$

Putting (2.48) and (2.49) together, we obtain

$$\begin{aligned}\langle \mathrm{B_s}|\,\hat{\Phi}^2\,|\mathrm{B_s}\rangle \\ &= \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\sum_{\ell'=0}^{\infty}\sum_{m'=-\ell'}^{\ell'}\int_0^\infty \mathrm{d}\omega\,\mathrm{d}\omega'\,\Big\{\phi^{\mathrm{in}}_{\omega\ell m}\,\phi^{\mathrm{in}*}_{\omega'\ell'm'}\,\langle \mathrm{B_s}|\,\hat{a}^{\mathrm{in}}_{\omega\ell m}\,\hat{a}^{\mathrm{in}\dagger}_{\omega'\ell'm'}\,|\mathrm{B_s}\rangle \\ &\qquad + \phi^{\mathrm{up}}_{\omega\ell m}\,\phi^{\mathrm{up}*}_{\omega'\ell'm'}\,\langle \mathrm{B_s}|\,\hat{a}^{\mathrm{up}}_{\omega\ell m}\,\hat{a}^{\mathrm{up}\dagger}_{\omega'\ell'm'}\,|\mathrm{B_s}\rangle\Big\}. \end{aligned} \tag{2.50}$$

Using the commutation relations (2.46), (2.50) becomes

$$\langle \mathrm{B_s}|\hat{\Phi}^2\,|\mathrm{B_s}\rangle \tag{2.51}$$

$$= \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\sum_{\ell'=0}^{\infty}\sum_{m'=-\ell'}^{\ell'}\int_0^\infty \mathrm{d}\omega\,\mathrm{d}\omega'\,\left\{\phi^{\mathrm{in}}_{\omega\ell m}\,\phi^{\mathrm{in}*}_{\omega'\ell'm'} + \phi^{\mathrm{up}}_{\omega\ell m}\,\phi^{\mathrm{up}*}_{\omega'\ell'm'}\right\}\delta\big(\omega-\omega'\big)\,\delta_{\ell\ell'}\,\delta_{mm'}. \tag{2.52}$$

Then the scalar condensate $\langle\hat{\Phi}^2\rangle_{|\mathrm{B_s}\rangle}$ in the Schwarzschild Boulware state is given as

$$\langle \mathrm{B_s}|\,\hat{\Phi}^2\,|\mathrm{B_s}\rangle = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\int_0^\infty \mathrm{d}\omega\,\Big\{\big|\phi^{\mathrm{in}}_{\omega\ell m}\big|^2 + \big|\phi^{\mathrm{up}}_{\omega\ell m}\big|^2\Big\}. \tag{2.53}$$

The Schwarzschild Boulware state $|\mathrm{B_s}\rangle$ is as empty as possible to a static observer near past and future null infinity $\mathscr{I}^{\pm}$. It respects the underlying symmetries of the background Schwarzschild spacetime and it has the property of being time-reversal invariant [16, 88]. However, this state is divergent on the event horizon. It can be interpreted physically as the vacuum state in the exterior of a star which does not possess a horizon.

2.3.2 Schwarzschild Unruh state

The Schwarzschild Unruh state $|\mathrm{U_s}\rangle$ [5] has the physical interpretation of being as empty as possible at past null infinity $\mathscr{I}^-$ as well as exhibiting outgoing Hawking radiation at future null infinity $\mathscr{I}^+$. In terms of the scalar field modes defined in §2.2, this corresponds to an absence of particles in the in-modes and a thermalised flux of particles in the up-modes.

We have already defined in-modes that are positive- and negative-frequency w.r.t the Schwarzschild coordinate t, which is the proper time experienced by a static observer near past null infinity $\mathscr{I}^-$, in (2.40) and (2.41) respectively.

The outgoing Hawking radiation at future null infinity $\mathscr{I}^+$ in $|\mathrm{U_s}\rangle$ is emanating from the past horizon $\mathcal{H}^-$; the Kruskal coordinate U is the affine parameter along the null generators of this surface. Near $\mathcal{H}^-$, the natural choice of time coordinate is the Kruskal retarded time U. In defining positive-frequency up-modes w.r.t U we make use of the Lemma in Appendix H of [89] which states that, for positive real $\mathfrak{p}$ and arbitrary real $\mathfrak{q}$,

$$\int_{-\infty}^{\infty}\mathrm{d}\mathfrak{X}\,\mathrm{e}^{-\mathrm{i}\mathfrak{p}\mathfrak{X}}\,\Big\{\mathrm{e}^{-\mathrm{i}\mathfrak{q}\ln(\mathfrak{X})}\,\Theta(\mathfrak{X}) + \mathrm{e}^{-\pi\mathfrak{q}}\,\mathrm{e}^{-\mathrm{i}\mathfrak{q}\ln(-\mathfrak{X})}\,\Theta(-\mathfrak{X})\Big\} = 0. \tag{2.54}$$

The quantity in the curly brackets of (2.54) is positive-frequency w.r.t $\mathfrak{X}$ by the definition in (4.2). We begin by expressing the asymptotic form (2.26) of the up-modes near the past horizon $\mathcal{H}^-$ in Kruskal coordinates. Using (2.11) and (2.13), we have that near $\mathcal{H}^-$

$$\phi^{\rm up}_{\omega\ell m} = \frac{1}{\sqrt{4\pi|\omega|}}\frac{1}{r}\,{\rm e}^{{\rm i}\hat{\omega}\ln(-U)}\,Y_{\ell m}(\theta,\varphi)\,, \tag{2.55}$$

where we have defined $\hat{\omega} = 4M\omega$. The Kruskal coordinates are defined in all four regions of the Penrose diagram in Figure 2.1. The up-modes $\phi^{\rm up}_{\omega\ell m}$ are defined in regions I and III, where $U < 0$. We can trivially extend their definition to regions II and IV by using the Heaviside function (4.5) to demand that they vanish when $U > 0$. Then, (2.55) becomes

$$\phi^{\rm up}_{\omega\ell m} = \frac{1}{\sqrt{4\pi|\omega|}}\frac{1}{r}\,{\rm e}^{{\rm i}\hat{\omega}\ln(-U)}\,Y_{\ell m}(\theta,\varphi)\,\Theta(-U)\,. \tag{2.56}$$

The first term of (2.54) can be constructed from the asymptotic form of the up-modes (2.56) if we take $\mathfrak{X} = U$ and $\mathfrak{q} = -\hat{\omega}$. The second term of (2.54) can be constructed by taking the time-reverse complex conjugate to define a set of modes $\psi^{\rm down}_{\omega\ell m}$ near $\mathcal{H}^+_{\rm IV}$ as

$$\psi^{\rm down}_{\omega\ell m} = \frac{1}{\sqrt{4\pi|\omega|}}\frac{1}{r}\,{\rm e}^{-{\rm i}\hat{\omega}\ln(U)}\,Y^*_{\ell m}(\theta,\varphi)\,\Theta(U)\,, \tag{2.57}$$

where taking the time-reverse corresponds to making the transformations $U \to -U$, $V \to -V$, which means that the modes $\psi^{\rm down}_{\omega\ell m}$ (2.57) are defined in regions II and IV, vanishing in regions I and III. Since these modes are incident upon the future horizon $\mathcal{H}^+_{\rm IV}$, they can be understood as the region IV analogue of the down-modes defined in §2.1; we refer to them henceforth as the region IV down-modes and we will continue to use the notation $\psi_{\omega\ell m}$ to denote sets of modes that are defined in region IV, vanishing in region I.

We write the mathematical expression in (2.54), multiplied by an appropriate factor, as a linear combination of the asymptotic form of the up-modes near $\mathcal{H}^-$ (2.56) and the complex conjugate of the asymptotic form of the region IV down-modes near $\mathcal{H}^+_{\rm IV}$ (2.57):

$$\begin{aligned} 0 &= \frac{1}{\sqrt{4\pi|\omega|}}\frac{1}{r}\,Y_{\ell m}(\theta,\varphi)\int_{-\infty}^{\infty}{\rm d}U\,{\rm e}^{-{\rm i}\mathfrak{p}U}\left\{{\rm e}^{{\rm i}\hat{\omega}\ln(U)}\,\Theta(U) + {\rm e}^{\pi\hat{\omega}}\,{\rm e}^{{\rm i}\hat{\omega}\ln(-U)}\,\Theta(-U)\right\} \\ &= \int_{-\infty}^{\infty}{\rm d}U\,{\rm e}^{-{\rm i}\mathfrak{p}U}\left\{\psi^{\rm down *}_{\omega\ell m} + {\rm e}^{\pi\hat{\omega}}\phi^{\rm up}_{\omega\ell m}\right\}. \end{aligned} \tag{2.58}$$

By the lemma above, the quantity $\{\psi^{\rm down*}_{\omega\ell m} + {\rm e}^{\pi\hat{\omega}}\phi^{\rm up}_{\omega\ell m}\}$ is positive-frequency w.r.t U for all $\hat{\omega}$; multiplying by a normalisation factor $\mathfrak{N}^{\rm up+}_{\omega}{\rm e}^{-\frac{\pi\hat{\omega}}{2}}$, we define a set of modes $\chi^{\rm up+}_{\omega\ell m}$

$$\chi^{\rm up+}_{\omega\ell m} = \mathfrak{N}^{\rm up+}_{\omega}\left({\rm e}^{-\frac{\pi\hat{\omega}}{2}}\psi^{\rm down*}_{\omega\ell m} + {\rm e}^{\frac{\pi\hat{\omega}}{2}}\phi^{\rm up}_{\omega\ell m}\right) \tag{2.59}$$

that is positive-frequency w.r.t U for all $\hat{\omega}$. We define negative-frequency up-modes using the complex conjugate of the Lemma (2.54); for positive real $\mathfrak{p}$ and arbitrary real $\mathfrak{q}$,

$$\int_{-\infty}^{\infty}{\rm d}\mathfrak{X}\,{\rm e}^{{\rm i}\mathfrak{p}\mathfrak{X}}\left\{{\rm e}^{{\rm i}\mathfrak{q}\ln(\mathfrak{X})}\,\Theta(\mathfrak{X}) + {\rm e}^{-\pi\mathfrak{q}}\,{\rm e}^{{\rm i}\mathfrak{q}\ln(-\mathfrak{X})}\,\Theta(-\mathfrak{X})\right\} = 0. \tag{2.60}$$

The quantity in the curly brackets of (2.54) is negative-frequency with respect to the variable $\mathfrak{X}$ by the definition in (4.4). If we take $\mathfrak{X} = U$ and $\mathfrak{q} = \hat{\omega}$ then the mathematical expression in (2.60), multiplied by an appropriate factor, can be written as a linear combination of the asymptotic form of the up-modes near $\mathcal{H}^-$ (2.56) and the complex conjugate of the asymptotic form of the region IV down-modes near $\mathcal{H}^+_{\rm IV}$ (2.57); we obtain

$$0 = \frac{1}{\sqrt{4\pi|\omega|}} Y_{\ell m}(\theta,\varphi) \int_{-\infty}^{\infty} \mathrm{d}U \, \mathrm{e}^{\mathrm{i}\mathfrak{p}U} \left\{ \mathrm{e}^{\mathrm{i}\hat{\omega}\ln(U)} \, \Theta(U) + \mathrm{e}^{-\pi\hat{\omega}} \, \mathrm{e}^{\mathrm{i}\hat{\omega}\ln(-U)} \, \Theta(-U) \right\}$$
$$= \int_{-\infty}^{\infty} \mathrm{d}U \, \mathrm{e}^{\mathrm{i}\mathfrak{p}U} \left\{ \psi^{\mathrm{down}*}_{\omega\ell m} + \mathrm{e}^{-\pi\hat{\omega}} \phi^{\mathrm{up}}_{\omega\ell m} \right\}. \tag{2.61}$$

By the lemma above, the quantity $\{\psi^{\mathrm{down}*}_{\omega\ell m} + \mathrm{e}^{-\pi\hat{\omega}}\phi^{\mathrm{up}}_{\omega\ell m}\}$ is negative-frequency w.r.t U for all $\hat{\omega}$; multiplying by a normalisation factor $\mathfrak{N}^{\mathrm{up}-}_{\omega}\mathrm{e}^{\frac{\pi\hat{\omega}}{2}}$, we define a set of modes $\chi^{\mathrm{up}-}_{\omega\ell m}$

$$\chi^{\mathrm{up}-}_{\omega\ell m} = \mathfrak{N}^{\mathrm{up}-}_{\omega} \left(\mathrm{e}^{\frac{\pi\hat{\omega}}{2}} \psi^{\mathrm{down}*}_{\omega\ell m} + \mathrm{e}^{-\frac{\pi\hat{\omega}}{2}} \phi^{\mathrm{up}}_{\omega\ell m} \right) \tag{2.62}$$

that is negative-frequency w.r.t U for all $\hat{\omega}$. The norm of the region IV down-modes can be evaluated through a method similar to that used to evaluate the corresponding region IV down-modes in Reissner-Nordström spacetime in §4.4.1. Again, due to the calculation being considerably involved, we do not reproduce it here but instead refer the reader to §4.4.1 for a detailed reading of an analogous case. We give the norm of a set of general region IV modes $\psi^{\mathrm{k}}_{\omega\ell m}$ and its complex conjugate $\psi^{\mathrm{k}*}_{\omega\ell m}$ as

$$\langle \psi^{\mathrm{k}}_{\omega\ell m}, \psi^{\mathrm{k}}_{\omega'\ell' m'} \rangle = \delta\big(\omega - \omega'\big) \, \delta_{\ell\ell'} \, \delta_{mm'} \quad \Rightarrow \quad \langle \psi^{\mathrm{k}*}_{\omega\ell m}, \psi^{\mathrm{k}*}_{\omega'\ell' m'} \rangle = -\delta\big(\omega - \omega'\big) \, \delta_{\ell\ell'} \, \delta_{mm'}. \tag{2.63}$$

Then, we derive the form of the normalisation constants $\mathfrak{N}^{\mathrm{up}\pm}_{\omega}$ in (2.59) and (2.62) to be

$$\mathfrak{N}^{\mathrm{up}\pm}_{\omega} = \frac{1}{(2|\sinh(\pi\hat{\omega})|)^{\frac{1}{2}}}. \tag{2.64}$$

Since the in- and up-modes form an orthonormal basis of modes, we expand the field Φ as

$$\Phi(x) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \left\{ \int_{0}^{\infty} \mathrm{d}\omega \left(a^{\mathrm{in}}_{\omega\ell m} \phi^{\mathrm{in}}_{\omega\ell m} + a^{\mathrm{in}\dagger}_{\omega\ell m} \phi^{\mathrm{in}*}_{\omega\ell m} \right) + \int_{-\infty}^{\infty} \mathrm{d}\omega \left(a^{\mathrm{up}}_{\omega\ell m} \, \chi^{\mathrm{up}+}_{\omega\ell m} + a^{\mathrm{up}\dagger}_{\omega\ell m} \, \chi^{\mathrm{up}-}_{\omega\ell m} \right) \right\}, \tag{2.65}$$

where the mode expansion coefficients $a^{\mathrm{in}}_{\omega\ell m}$ and $a^{\mathrm{up}}_{\omega\ell m}$ multiply positive-frequency modes and the mode expansion coefficients $a^{\mathrm{in}\dagger}_{\omega\ell m}$ and $a^{\mathrm{up}\dagger}_{\omega\ell m}$ multiply negative-frequency modes. We quantise the field by promoting the mode expansion coefficients in (2.65) to operators such that the field operator $\hat{\Phi}(x)$ is given by

$$\Phi(x) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \left\{ \int_{0}^{\infty} \mathrm{d}\omega \left(\hat{a}^{\mathrm{in}}_{\omega\ell m} \phi^{\mathrm{in}}_{\omega\ell m} + \hat{a}^{\mathrm{in}\dagger}_{\omega\ell m} \phi^{\mathrm{in}*}_{\omega\ell m} \right) + \int_{-\infty}^{\infty} \mathrm{d}\omega \left(\hat{a}^{\mathrm{up}}_{\omega\ell m} \, \chi^{\mathrm{up}+}_{\omega\ell m} + \hat{a}^{\mathrm{up}\dagger}_{\omega\ell m} \, \chi^{\mathrm{up}-}_{\omega\ell m} \right) \right\}, \tag{2.66}$$

where the operators in (2.66) obey the following commutation relations

$$\left[\hat{a}^{\rm in}_{\omega\ell m}, \hat{a}^{\rm in\dagger}_{\omega'\ell' m'}\right] = \delta\left(\omega-\omega'\right)\delta_{\ell\ell'}\,\delta_{mm'},$$
$$\left[\hat{a}^{\rm up}_{\omega\ell m}, \hat{a}^{\rm up\dagger}_{\omega'\ell' m'}\right] = \delta\left(\omega-\omega'\right)\delta_{\ell\ell'}\,\delta_{mm'}, \tag{2.67}$$

with any commutators not explicitly given in (2.67) vanishing. The Schwarzschild Unruh state $|{\rm U_s}\rangle$ is then defined as the state annihilated by the $\hat{a}^{\rm in/up}_{\omega\ell m}$ operators such that

$$\hat{a}^{\rm in}_{\omega\ell m}\,|{\rm U_s}\rangle = 0,$$
$$\hat{a}^{\rm up}_{\omega\ell m}\,|{\rm U_s}\rangle = 0. \tag{2.68}$$

Using (2.68), we can act with the field operator $\hat{\Phi}$ (2.66) on $|{\rm U_s}\rangle$ to generate an expression for the scalar condensate $\langle\hat{\Phi}^2\rangle_{|{\rm U_s}\rangle}$ in the Schwarzschild Unruh state. We have

$$\hat{\Phi}\,|{\rm U_s}\rangle = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\left\{\int_0^{\infty}{\rm d}\omega\,\phi^{\rm in*}_{\omega\ell m}\hat{a}^{\rm in\dagger}_{\omega\ell m}\,|{\rm U_s}\rangle + \int_{-\infty}^{\infty}{\rm d}\omega\,\chi^{\rm up-}_{\omega\ell m}\hat{a}^{\rm up\dagger}_{\omega\ell m}\,|{\rm U_s}\rangle\right\}, \tag{2.69}$$

$$\langle{\rm U_s}|\,\hat{\Phi} = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\left\{\int_0^{\infty}{\rm d}\omega\,\phi^{\rm in}_{\omega\ell m}\,\langle{\rm U_s}|\,\hat{a}^{\rm in}_{\omega\ell m} + \int_{-\infty}^{\infty}{\rm d}\omega\left(\chi^{\rm up-}_{\omega\ell m}\right)^*\langle{\rm U_s}|\,\hat{a}^{\rm up}_{\omega\ell m}\right\}. \tag{2.70}$$

Putting (2.69) and (2.70) together, we obtain

$$\langle{\rm U_s}|\,\hat{\Phi}^2\,|{\rm U_s}\rangle = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\sum_{\ell'=0}^{\infty}\sum_{m'=-\ell'}^{\ell'}\left\{\int_0^{\infty}{\rm d}\omega\,{\rm d}\omega'\,\phi^{\rm in}_{\omega\ell m}\,\phi^{\rm in*}_{\omega'\ell' m'}\,\langle{\rm U_s}|\,\hat{a}^{\rm in}_{\omega\ell m}\,\hat{a}^{\rm in\dagger}_{\omega'\ell' m'}\,|{\rm U_s}\rangle\right.$$
$$\left. + \int_{-\infty}^{\infty}{\rm d}\omega\,{\rm d}\omega'\left(\chi^{\rm up-}_{\omega\ell m}\right)^*\chi^{\rm up-}_{\omega'\ell' m'}\,\langle{\rm U_s}|\,\hat{a}^{\rm up}_{\omega\ell m}\,\hat{a}^{\rm up\dagger}_{\omega'\ell' m'}\,|{\rm U_s}\rangle\right\}. \tag{2.71}$$

Using the commutation relations (2.67), (2.71) becomes

$$\langle{\rm U_s}|\,\hat{\Phi}^2\,|{\rm U_s}\rangle = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\sum_{\ell'=0}^{\infty}\sum_{m'=-\ell'}^{\ell'}\left\{\int_0^{\infty}{\rm d}\omega\,{\rm d}\omega'\,\phi^{\rm in}_{\omega\ell m}\,\phi^{\rm in*}_{\omega'\ell' m'}\right.$$
$$\left. + \int_{-\infty}^{\infty}{\rm d}\omega\,{\rm d}\omega'\left(\chi^{\rm up-}_{\omega\ell m}\right)^*\chi^{\rm up-}_{\omega'\ell' m'}\right\}\delta\left(\omega-\omega'\right)\delta_{\ell\ell'}\,\delta_{mm'}$$
$$= \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\left\{\int_0^{\infty}{\rm d}\omega\,\left|\phi^{\rm in}_{\omega\ell m}\right|^2 + \int_{-\infty}^{\infty}{\rm d}\omega\,\left|\chi^{\rm up-}_{\omega\ell m}\right|^2\right\}. \tag{2.72}$$

Restricting attention to region I, as well as using (2.62) and (2.64), (2.72) reduces to

$$\langle{\rm U_s}|\,\hat{\Phi}^2\,|{\rm U_s}\rangle = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\left\{\int_0^{\infty}{\rm d}\omega\,\left|\phi^{\rm in}_{\omega\ell m}\right|^2 + \int_{-\infty}^{\infty}{\rm d}\omega\,\frac{{\rm e}^{-\pi\hat{\omega}}}{(2|\sinh(\pi\hat{\omega})|)}\left|\phi^{\rm up}_{\omega\ell m}\right|^2\right\}, \tag{2.73}$$

where we have used the fact that the $\psi^{\rm down}_{\omega\ell m}$ (2.57) vanish in region I. We can simplify the integral over the square of the absolute value of the up-modes in (2.73) as

$$
\begin{aligned}
&\int_{-\infty}^{\infty} \mathrm{d}\omega \, \frac{\mathrm{e}^{-\pi\hat{\omega}}}{(2|\sinh(\pi\hat{\omega})|)} \left|\phi^{\mathrm{up}}_{\omega\ell m}\right|^2 \\
&\quad = \int_0^{\infty} \mathrm{d}\omega \, \frac{\mathrm{e}^{-\pi\hat{\omega}}}{(2\sinh(\pi\hat{\omega}))} \left|\phi^{\mathrm{up}}_{\omega\ell m}\right|^2 + \int_{-\infty}^{0} \mathrm{d}\omega \, \frac{\mathrm{e}^{-\pi\hat{\omega}}}{(-2\sinh(\pi\hat{\omega}))} \left|\phi^{\mathrm{up}}_{\omega\ell m}\right|^2 \\
&\quad = \int_0^{\infty} \mathrm{d}\omega \, \frac{\mathrm{e}^{-\pi\hat{\omega}}}{(2\sinh(\pi\hat{\omega}))} \left|\phi^{\mathrm{up}}_{\omega\ell m}\right|^2 + \int_{\infty}^{0} \mathrm{d}\,(-\omega) \, \frac{\mathrm{e}^{\pi\hat{\omega}}}{(-2\sinh(-\pi\hat{\omega}))} \left|\phi^{\mathrm{up}}_{-\omega\ell m}\right|^2 \\
&\quad = \int_0^{\infty} \mathrm{d}\omega \, \frac{\mathrm{e}^{-\pi\hat{\omega}}}{(2\sinh(\pi\hat{\omega}))} \left|\phi^{\mathrm{up}}_{\omega\ell m}\right|^2 + \int_0^{\infty} \mathrm{d}\omega \, \frac{\mathrm{e}^{\pi\hat{\omega}}}{(2\sinh(\pi\hat{\omega}))} \left|\phi^{\mathrm{up}}_{\omega\ell m}\right|^2 \\
&\quad = \int_0^{\infty} \mathrm{d}\omega \, \frac{\mathrm{e}^{-\pi\hat{\omega}} + \mathrm{e}^{\pi\hat{\omega}}}{(2\sinh(\pi\hat{\omega}))} \left|\phi^{\mathrm{up}}_{\omega\ell m}\right|^2 = \int_0^{\infty} \mathrm{d}\omega \, \coth(\pi\hat{\omega}) \left|\phi^{\mathrm{up}}_{\omega\ell m}\right|^2 ,
\end{aligned}
\tag{2.74}
$$

where we have performed the substitution $\omega \to -\omega$ in going from the first equality to the second and we have used the fact that $\left|\phi^{\mathrm{up}}_{-\omega\ell m}\right|^2 = \left|\phi^{\mathrm{up}}_{\omega\ell m}\right|^2$ in going from the second equality to the third, which follows from the general form of a mode (2.38) and the asymptotic form of the up-mode radial function $X^{\mathrm{up}}_{\omega\ell}(r)$ (2.26). Then the scalar condensate $\langle\hat{\Phi}^2\rangle_{|\mathrm{U_s}\rangle}$ in the Schwarzschild Unruh state $|\mathrm{U_s}\rangle$ is given as

$$
\langle \mathrm{U_s}| \, \hat{\Phi}^2 \, |\mathrm{U_s}\rangle = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \int_0^{\infty} \mathrm{d}\omega \left\{ \left|\phi^{\mathrm{in}}_{\omega\ell m}\right|^2 + \coth(4M\pi\omega) \left|\phi^{\mathrm{up}}_{\omega\ell m}\right|^2 \right\} . \tag{2.75}
$$

The Schwarzschild Unruh state $|\mathrm{U_s}\rangle$ is empty to a static observer near past null infinity $\mathscr{I}^-$ but contains an outgoing flux of Hawking radiation at future null infinity $\mathscr{I}^+$. While regular on the future horizon $\mathcal{H}^+$, it diverges on the past horizon $\mathcal{H}^-$; it is not time-reversal invariant [5, 88]. This state represents a black hole formed by gravitational collapse.

2.3.3 Schwarzschild Hartle-Hawking state

The Schwarzschild Hartle-Hawking state $|\mathrm{H_s}\rangle$ [17] has the physical interpretation of exhibiting both incoming Hawking radiation from past null infinity $\mathscr{I}^-$ as well as outgoing Hawking radiation at future null infinity $\mathscr{I}^+$. In terms of the scalar field modes defined in §2.2, this corresponds to a thermal flux of particles in both the up- and down-modes.

The outgoing Hawking radiation at future null infinity $\mathscr{I}^+$ in $|\mathrm{H_s}\rangle$ is emanating from the past horizon $\mathcal{H}^-$; the Kruskal coordinate U is the affine parameter along the null generators of this surface. We have already defined thermalised up-modes that are positive- and negative-frequency w.r.t U in (2.59) and (2.62) respectively.

The incoming Hawking radiation from past null infinity $\mathscr{I}^-$ in $|\mathrm{H_s}\rangle$ is incident upon the future horizon $\mathcal{H}^+$; the Kruskal coordinate V is the affine parameter along the null generators of this surface. Then, near $\mathcal{H}^+$, the natural choice of time coordinate is the Kruskal advanced time V. We define positive- and negative-frequency down-modes w.r.t V using the Lemma (2.54) and its complex conjugate (2.60) respectively.

The asymptotic form (2.28) of the down-modes near the future horizon $\mathcal{H}^+$ is given by

$$
\phi^{\mathrm{down}}_{\omega\ell m} = \frac{1}{\sqrt{4\pi|\omega|}} \frac{1}{r} \, \mathrm{e}^{-\mathrm{i}\hat{\omega}\ln(V)} \, Y_{\ell m}(\theta,\varphi) \, \Theta(V) \, . \tag{2.76}
$$

The first term of (2.54) can be constructed from the asymptotic form of the down-modes (2.76) if we take $\mathfrak{X} = V$ and $\mathfrak{q} = \hat{\omega}$; the second term can be constructed by taking the time-reverse complex conjugate to define the region IV up-modes $\psi^{\mathrm{up}}_{\omega\ell m}$ near $\mathcal{H}^-_{\mathrm{IV}}$ as

$$\psi^{\mathrm{up}}_{\omega\ell m} = \frac{1}{\sqrt{4\pi|\omega|}}\frac{1}{r}\,\mathrm{e}^{\mathrm{i}\hat{\omega}\ln(-V)}\,Y^*_{\ell m}(\theta,\varphi)\,\Theta(-V)\,. \tag{2.77}$$

Then, multiplying the Lemma by an appropriate factor, we can write (2.54) as a linear combination of the asymptotic form of the down-modes near $\mathcal{H}^+$ (2.76) and the complex conjugate of the asymptotic form of the region IV up-modes near $\mathcal{H}^-_{\mathrm{IV}}$ (2.77); we obtain

$$\begin{aligned} 0 &= \frac{1}{\sqrt{4\pi|\omega|}}\frac{1}{r}\,Y_{\ell m}(\theta,\varphi)\int_{-\infty}^{\infty}\mathrm{d}V\,\mathrm{e}^{-\mathrm{i}\mathfrak{p}V}\left\{\mathrm{e}^{-\mathrm{i}\hat{\omega}\ln(V)}\,\Theta(V)+\mathrm{e}^{-\pi\hat{\omega}}\,\mathrm{e}^{-\mathrm{i}\hat{\omega}\ln(-V)}\,\Theta(-V)\right\} \\ &= \int_{-\infty}^{\infty}\mathrm{d}V\,\mathrm{e}^{-\mathrm{i}\mathfrak{p}V}\left\{\phi^{\mathrm{down}}_{\omega\ell m}+\mathrm{e}^{-\pi\hat{\omega}}\psi^{\mathrm{up}*}_{\omega\ell m}\right\}. \end{aligned} \tag{2.78}$$

By the lemma (2.54), the quantity $\left\{\phi^{\mathrm{down}}_{\omega\ell m}+\mathrm{e}^{-\pi\hat{\omega}}\psi^{\mathrm{up}*}_{\omega\ell m}\right\}$ is positive-frequency w.r.t V for all $\hat{\omega}$; multiplying by a normalisation factor $\mathfrak{N}^{\mathrm{down}+}_{\omega}\mathrm{e}^{\frac{\pi\hat{\omega}}{2}}$, we define a set of modes $\chi^{\mathrm{down}+}_{\omega\ell m}$

$$\chi^{\mathrm{down}+}_{\omega\ell m} = \mathfrak{N}^{\mathrm{down}+}_{\omega}\left(\mathrm{e}^{\frac{\pi\hat{\omega}}{2}}\,\phi^{\mathrm{down}}_{\omega\ell m}+\mathrm{e}^{-\frac{\pi\hat{\omega}}{2}}\,\psi^{\mathrm{up}*}_{\omega\ell m}\right) \tag{2.79}$$

that is positive-frequency w.r.t V for all $\hat{\omega}$. We define negative-frequency down-modes using the complex conjugate (2.60) of the Lemma; if we take $\mathfrak{X} = V$ and $\mathfrak{q} = -\hat{\omega}$, a multiplication of (2.60) by an appropriate factor can be written as a linear combination of the asymptotic form of the down-modes near $\mathcal{H}^+$ (2.76) and the complex conjugate of the asymptotic form of the region IV up-modes near $\mathcal{H}^-_{\mathrm{IV}}$ (2.77); we obtain

$$\begin{aligned} 0 &= \frac{1}{\sqrt{4\pi|\omega|}}\,Y_{\ell m}(\theta,\varphi)\int_{-\infty}^{\infty}\mathrm{d}V\,\mathrm{e}^{\mathrm{i}\mathfrak{p}V}\left\{\mathrm{e}^{-\mathrm{i}\hat{\omega}\ln(V)}\,\Theta(V)+\mathrm{e}^{\pi\hat{\omega}}\,\mathrm{e}^{-\mathrm{i}\hat{\omega}\ln(-V)}\,\Theta(-V)\right\} \\ &= \int_{-\infty}^{\infty}\mathrm{d}V\,\mathrm{e}^{\mathrm{i}\mathfrak{p}V}\left\{\phi^{\mathrm{down}}_{\omega\ell m}+\mathrm{e}^{\pi\hat{\omega}}\psi^{\mathrm{up}*}_{\omega\ell m}\right\}. \end{aligned} \tag{2.80}$$

By the lemma (2.60), the quantity $\left\{\phi^{\mathrm{down}}_{\omega\ell m}+\mathrm{e}^{\pi\hat{\omega}}\psi^{\mathrm{up}*}_{\omega\ell m}\right\}$ is negative-frequency w.r.t V for all $\hat{\omega}$; multiplying by a normalisation factor $\mathfrak{N}^{\mathrm{down}-}_{\omega}\mathrm{e}^{-\frac{\pi\hat{\omega}}{2}}$, we define a set of modes $\chi^{\mathrm{down}-}_{\omega\ell m}$

$$\chi^{\mathrm{down}-}_{\omega\ell m} = \mathfrak{N}^{\mathrm{down}-}_{\omega}\left(\mathrm{e}^{-\frac{\pi\hat{\omega}}{2}}\,\phi^{\mathrm{down}}_{\omega\ell m}+\mathrm{e}^{\frac{\pi\hat{\omega}}{2}}\,\psi^{\mathrm{up}*}_{\omega\ell m}\right) \tag{2.81}$$

that is negative-frequency w.r.t V for all $\hat{\omega}$. Then, using (2.59), (2.62), (2.79) and (2.79), we expand the field Φ in a basis of up- and down-modes as

$$\Phi(x) = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\int_{-\infty}^{\infty}\mathrm{d}\omega\left\{\tilde{a}^{\mathrm{up}}_{\omega\ell m}\,\chi^{\mathrm{up}+}_{\omega\ell m}+\tilde{a}^{\mathrm{up}\dagger}_{\omega\ell m}\,\chi^{\mathrm{up}-}_{\omega\ell m}+\tilde{a}^{\mathrm{down}}_{\omega\ell m}\,\chi^{\mathrm{down}+}_{\omega\ell m}+\tilde{a}^{\mathrm{down}\dagger}_{\omega\ell m}\,\chi^{\mathrm{down}-}_{\omega\ell m}\right\}. \tag{2.82}$$

However, the up- and down-modes do not form an orthonormal basis, which we require to quantise the field; the in- and up-modes do however. We would instead like define a set of thermalised in-modes. Near $\mathcal{H}^+$, the asymptotic form (2.25) of the in-modes is given by

$$\phi^{\text{in}}_{\omega\ell m} = \frac{B^{\text{in}}_{\omega\ell}}{\sqrt{4\pi|\omega|}} \frac{1}{r} \, \mathrm{e}^{-\mathrm{i}\hat{\omega}\ln(V)} \, Y_{\ell m}(\theta,\varphi) \, \Theta(V) \, . \tag{2.83}$$

Comparing the near $\mathcal{H}^+$ asymptotic expressions of the in-modes (2.83) with that of the down-modes (2.76), we see that we can define a set of modes $\chi^{\text{in}+}_{\omega\ell m}$

$$\chi^{\text{in}+}_{\omega\ell m} = \mathfrak{N}^{\text{in}+}_{\omega} \left(\mathrm{e}^{\frac{\pi\hat{\omega}}{2}} \phi^{\text{in}}_{\omega\ell m} + \mathrm{e}^{-\frac{\pi\hat{\omega}}{2}} \psi^{\text{out}*}_{\omega\ell m} \right) \tag{2.84}$$

that is positive-frequency w.r.t V for all $\hat{\omega}$ and a set of modes $\chi^{\text{in}-}_{\omega\ell m}$

$$\chi^{\text{in}-}_{\omega\ell m} = \mathfrak{N}^{\text{in}-}_{\omega} \left(\mathrm{e}^{-\frac{\pi\hat{\omega}}{2}} \phi^{\text{in}}_{\omega\ell m} + \mathrm{e}^{\frac{\pi\hat{\omega}}{2}} \psi^{\text{out}*}_{\omega\ell m} \right) \tag{2.85}$$

that is negative-frequency w.r.t V for all $\hat{\omega}$, where the only difference in the derivation of (2.84) and (2.85) relative to (2.79) and (2.81) respectively is that the lemma and its complex conjugate have been multiplied by an extra factor of the transmission coefficient $B^{\text{in}}_{\omega\ell}$. Then, the normalisation constants $\mathfrak{N}^{\text{in}\pm}$ are given by

$$\mathfrak{N}^{\text{in}\pm}_{\omega} = \frac{1}{(2|\sinh(\pi\hat{\omega})|)^{\frac{1}{2}}} . \tag{2.86}$$

In order to rewrite the $\chi^{\text{down}\pm}_{\omega\ell m}$ in terms of $\chi^{\text{in}\pm}_{\omega\ell m}, \chi^{\text{up}\pm}_{\omega\ell m}$, we need to relate both their region I and IV parts. From the expression for $\phi^{\text{down}}_{\omega\ell m}$ in terms of $\phi^{\text{up}}_{\omega\ell m}, \phi^{\text{in}}_{\omega\ell m}$ (3.71b), we have

$$\psi^{\text{up}}_{\omega\ell m} = A^{\text{up}*}_{\omega\ell} \psi^{\text{down}}_{\omega\ell m} + B^{\text{up}*}_{\omega\ell} \psi^{\text{out}}_{\omega\ell m} . \tag{2.87}$$

Then, using (2.39b) and (2.87), we have

$$\chi^{\text{down}}_{\omega\ell m} = A^{\text{up}*}_{\omega\ell} \chi^{\text{up}}_{\omega\ell m} + B^{\text{up}*}_{\omega\ell} \chi^{\text{in}}_{\omega\ell m} . \tag{2.88}$$

Using (2.59), (2.62), (2.84) and (2.85), we can expand the field Φ in an orthonormal basis of in- and up-modes as

$$\Phi(x) \;=\; \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \int_{-\infty}^{\infty} \mathrm{d}\omega \left\{ a^{\text{up}}_{\omega\ell m} \chi^{\text{up}+}_{\omega\ell m} + a^{\text{up}\dagger}_{\omega\ell m} \chi^{\text{up}-}_{\omega\ell m} + a^{\text{in}}_{\omega\ell m} \chi^{\text{in}+}_{\omega\ell m} + a^{\text{in}\dagger}_{\omega\ell m} \chi^{\text{in}-}_{\omega\ell m} \right\} , \tag{2.89}$$

where the mode coefficients in (2.89) are related to those in (2.82) by the expressions

$$\begin{aligned} a^{\text{up}}_{\omega\ell m} &= \tilde{a}^{\text{up}}_{\omega\ell m} + A^{\text{up}*}_{\omega\ell} \, \tilde{a}^{\text{down}}_{\omega\ell m} , \\ a^{\text{in}}_{\omega\ell m} &= B^{\text{up}*}_{\omega\ell} \, \tilde{a}^{\text{down}}_{\omega\ell m} . \end{aligned} \tag{2.90}$$

We quantise the field by promoting the mode expansion coefficients in (2.89) to operators such that the field operator $\hat{\Phi}(x)$ is given by

$$\hat{\Phi}(x) \;=\; \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \int_{-\infty}^{\infty} \mathrm{d}\omega \left\{ \hat{a}^{\text{up}}_{\omega\ell m} \chi^{\text{up}+}_{\omega\ell m} + \hat{a}^{\text{up}\dagger}_{\omega\ell m} \chi^{\text{up}-}_{\omega\ell m} + \hat{a}^{\text{in}}_{\omega\ell m} \chi^{\text{in}+}_{\omega\ell m} + \hat{a}^{\text{in}\dagger}_{\omega\ell m} \chi^{\text{in}-}_{\omega\ell m} \right\} , \tag{2.91}$$

where the operators in (2.91) obey the following commutation relations

$$\left[\hat{a}^{\text{in}}_{\omega\ell m}, \hat{a}^{\text{in}\dagger}_{\omega'\ell'm'}\right] = \delta\left(\omega - \omega'\right) \delta_{\ell\ell'}\, \delta_{mm'},$$
$$\left[\hat{a}^{\text{up}}_{\omega\ell m}, \hat{a}^{\text{up}\dagger}_{\omega'\ell'm'}\right] = \delta\left(\omega - \omega'\right) \delta_{\ell\ell'}\, \delta_{mm'}, \tag{2.92}$$

with any commutators not explicitly given in (2.92) vanishing. The Schwarzschild Hartle-Hawking state is then defined as the state annihilated by the $\hat{a}^{\text{in/up}}_{\omega\ell m}$ operators such that

$$\hat{a}^{\text{in}}_{\omega\ell m} \left|\text{H}_\text{s}\right\rangle = 0,$$
$$\hat{a}^{\text{up}}_{\omega\ell m} \left|\text{H}_\text{s}\right\rangle = 0. \tag{2.93}$$

Using (2.93), we can act with the field operator $\hat{\Phi}$ (2.91) on $|\text{H}_\text{s}\rangle$ to generate an expression for the scalar condensate $\langle\hat{\Phi}^2\rangle_{|\text{H}_\text{s}\rangle}$ in the Schwarzschild Hartle-Hawking state. We have

$$\hat{\Phi}\left|\text{H}_\text{s}\right\rangle = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\int_{-\infty}^{\infty} \mathrm{d}\omega \left\{ \chi^{\text{in}-}_{\omega\ell m}\hat{a}^{\text{up}\dagger}_{\omega\ell m}\left|\text{H}_\text{s}\right\rangle + \chi^{\text{up}-}_{\omega\ell m}\hat{a}^{\text{up}\dagger}_{\omega\ell m}\left|\text{H}_\text{s}\right\rangle \right\}, \tag{2.94}$$

$$\left\langle\text{H}_\text{s}\right|\hat{\Phi} = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\int_{-\infty}^{\infty} \mathrm{d}\omega \left\{ \left(\chi^{\text{in}-}_{\omega\ell m}\right)^*\left\langle\text{H}_\text{s}\right|\hat{a}^{\text{in}}_{\omega\ell m} + \left(\chi^{\text{up}-}_{\omega\ell m}\right)^*\left\langle\text{H}_\text{s}\right|\hat{a}^{\text{up}}_{\omega\ell m} \right\}. \tag{2.95}$$

Putting (2.94) and (2.95) together, we obtain

$$\begin{aligned}\left\langle\text{H}_\text{s}\right|\hat{\Phi}^2\left|\text{H}_\text{s}\right\rangle = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\sum_{\ell'=0}^{\infty}\sum_{m'=-\ell'}^{\ell'}\int_{-\infty}^{\infty}\mathrm{d}\omega\,\mathrm{d}\omega' \Big\{ &\left(\chi^{\text{in}-}_{\omega\ell m}\right)^*\chi^{\text{in}-}_{\omega'\ell'm'}\left\langle\text{H}_\text{s}\right|\hat{a}^{\text{in}}_{\omega\ell m}\,\hat{a}^{\text{in}\dagger}_{\omega'\ell'm'}\left|\text{H}_\text{s}\right\rangle \\ &+ \left(\chi^{\text{up}-}_{\omega\ell m}\right)^*\chi^{\text{up}-}_{\omega'\ell'm'}\left\langle\text{H}_\text{s}\right|\hat{a}^{\text{up}}_{\omega\ell m}\,\hat{a}^{\text{up}\dagger}_{\omega'\ell'm'}\left|\text{H}_\text{s}\right\rangle \Big\}.\end{aligned} \tag{2.96}$$

Using the commutation relations (2.92), (2.96) becomes

$$\begin{aligned}\left\langle\text{H}_\text{s}\right|\hat{\Phi}^2\left|\text{H}_\text{s}\right\rangle &= \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\sum_{\ell'=0}^{\infty}\sum_{m'=-\ell'}^{\ell'}\int_{-\infty}^{\infty}\mathrm{d}\omega\,\mathrm{d}\omega' \Big\{ \left(\chi^{\text{in}-}_{\omega\ell m}\right)^*\chi^{\text{in}-}_{\omega'\ell'm'} \\ &\qquad + \left(\chi^{\text{up}-}_{\omega\ell m}\right)^*\chi^{\text{up}-}_{\omega'\ell'm'} \Big\}\,\delta\left(\omega-\omega'\right)\delta_{\ell\ell'}\,\delta_{mm'} \\ &= \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\int_{-\infty}^{\infty}\mathrm{d}\omega\left\{ \left|\chi^{\text{in}-}_{\omega\ell m}\right|^2 + \left|\chi^{\text{up}-}_{\omega\ell m}\right|^2 \right\}.\end{aligned} \tag{2.97}$$

Restricting attention to region I, (2.96) reduces to

$$\left\langle\text{H}_\text{s}\right|\hat{\Phi}^2\left|\text{H}_\text{s}\right\rangle = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\int_{-\infty}^{\infty}\mathrm{d}\omega\left\{ \frac{\mathrm{e}^{-\pi\hat{\omega}}}{(2|\sinh(\pi\hat{\omega})|)}\left|\phi^{\text{in}}_{\omega\ell m}\right|^2 + \frac{\mathrm{e}^{-\pi\hat{\omega}}}{(2|\sinh(\pi\hat{\omega})|)}\left|\phi^{\text{up}}_{\omega\ell m}\right|^2 \right\}, \tag{2.98}$$

where we have used the fact that the $\psi^{\text{out/down}}_{\omega\ell m}$ vanish in region I. Through a similar calculation to that in (2.74), we derive the expression for the scalar condensate $\langle\hat{\Phi}^2\rangle_{|\text{H}_\text{s}\rangle}$

$$\langle \mathrm{H_s} | \, \hat{\Phi}^2 \, | \mathrm{H_s} \rangle = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \int_0^\infty \mathrm{d}\omega \, \coth(4M\pi\omega) \left\{ \left| \phi^{\mathrm{in}}_{\omega\ell m} \right|^2 + \left| \phi^{\mathrm{up}}_{\omega\ell m} \right|^2 \right\}. \tag{2.99}$$

The Schwarzschild Hartle-Hawking state $|\mathrm{H_s}\rangle$ can be interpreted as a black hole in an unstable equilibrium of thermal radiation at the Hawking temperature [88]. It has attractive regularity properties in that it is regular on both the past and future horizons $\mathcal{H}^{\pm}$ as well as being time-reversal invariant [17, 18].

Part II

Quantum charged scalar fields in Reissner-Nordström spacetime

Chapter 3

Classical charged scalar fields in Reissner-Nordström spacetime

In §3.1, we introduce the Reissner-Nordström solution and discuss its geometry. We include a Penrose diagram. We introduce a scalar field on this spacetime in §3.2 and solve the associated scalar field equation. In §3.3, we motivate different mode solutions to the scalar field equation before demonstrating the phenomenon of classical superradiance. We conclude by calculating the inner products of the various mode solutions in §3.4 in order to evaluate their norms and associated normalisation constants.

3.1 The Reissner-Nordström solution

In this section, we introduce the Reissner-Nordström solution to Einstein-Maxwell theory and discuss three possible scenarios, before restricting our attention to the sub-extremal case that will be our focus for the rest of Part II of this thesis. We then introduce new coordinate systems that progressively reveal more regions of Reissner-Nordström spacetime, and which is summarised in the form of a Penrose diagram.

3.1.1 Einstein-Maxwell theory

The Einstein-Maxwell action, which describes the coupling of gravity to the electromagnetic field and also to a charged scalar field in four spacetime dimensions, is given by

$$S = \frac{1}{16\pi} \int \mathrm{d}^4 x \left(R - F_{\mu\nu} F^{\mu\nu} - D_\mu \Phi D^\mu \Phi \right) \sqrt{-g}, \tag{3.1}$$

where the electromagnetic field strength tensor $F_{\mu\nu}$, written in terms of the electromagnetic gauge potential A_μ, is defined by

$$F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu, \tag{3.2}$$

and the gauge covariant derivative operator D_μ, which depends upon the charge q possessed by the scalar field, is given by

$$D_\mu = \nabla_\mu - \mathrm{i} q A_\mu. \tag{3.3}$$

Variation of the action (3.1) with respect to the metric leads to the following equation of motion relating the curvature of the spacetime to the stress-energy tensors of the electromagnetic field and of the scalar field:

$$G_{\mu\nu} \coloneqq R_{\mu\nu} - \frac{1}{2}\, g_{\mu\nu} R = T^{\mathrm{F}}_{\mu\nu} + T^{\Phi}_{\mu\nu}, \tag{3.4}$$

where the Einstein tensor $G_{\mu\nu}$ is defined by the first equality in (3.4) and where $T^{\mathrm{F}}_{\mu\nu}$ is

$$T^{\mathrm{F}}_{\mu\nu} = F_{\mu\rho} F_{\nu}{}^{\rho} - \frac{1}{4}\, g_{\mu\nu} F_{\rho\tau} F^{\rho\tau}. \tag{3.5}$$

The homogeneous Maxwell's equation, which follows immediately from the definition of $F_{\mu\nu}$ in (3.2), is given by

$$\nabla_{[\mu} F_{\nu\lambda]} = 0. \tag{3.6}$$

The inhomogeneous Maxwell's equation can be derived by varying the action (3.1) with respect to the gauge field A_μ to give

$$\nabla_{\mu} F^{\mu\nu} = J^{\nu}. \tag{3.7}$$

Varying the action (3.1) with respect to the field Φ leads to the scalar field equation

$$D_{\mu} D^{\mu} \Phi = 0. \tag{3.8}$$

We will discuss the scalar field equation, as well as how to solve it, in detail in §3.2.2.

We introduce a new coordinate system (t, r, θ, φ), which is analogous to the Schwarzschild coordinate system introduced in §2.1; we will refer to this coordinate system as the Schwarzschild-like coordinates. The geometry of Reissner-Nordström spacetime is described, in terms of Schwarzschild-like coordinates, by the line element

$$\mathrm{d}s^2 = -f(r)\,\mathrm{d}t^2 + f(r)^{-1}\,\mathrm{d}r^2 + r^2 \mathrm{d}\theta^2 + r^2 \sin^2\theta\,\mathrm{d}\varphi^2, \tag{3.9}$$

where the (Reissner-Nordström) metric function $f(r)$ in (3.9) is given by

$$f(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2}. \tag{3.10}$$

We will drop the identifier "Reissner-Nordström" and refer to (3.10) as the metric function for the remainder of Part II of this thesis. The metric function (3.10) differs to that of Schwarzschild spacetime by the addition of a term proportional to Q, which we can interpret as the charge of the black hole. Its presence arises due to the electromagnetic gauge field potential A_μ, which is given explicitly by

$$A_{\mu} = \begin{pmatrix} -\frac{Q}{r} & 0 & 0 & 0 \end{pmatrix}. \tag{3.11}$$

The expression for A_μ in (3.11) corresponds to a choice of gauge; in this case, we have chosen a constant of integration such that the value of the gauge field potential A_μ vanishes far from the black hole as $r \to \infty$. It is shown in [44,45] that A_μ can be chosen to vanish at any fixed value of the radial coordinate r by means of a gauge transformation. In Part III of this thesis, which deals with Hadamard renormalisation of charged scalar fields, we will leave the value of the gauge potential A_μ arbitrary; however, where an explicit expression of the gauge field is specified, it will always be given by (3.11). One can easily see that the electromagnetic potential A_μ (3.11) satisfies the Lorenz gauge condition

$$\nabla^\mu A_\mu = \nabla^t A_t = 0, \tag{3.12}$$

since the expression in (3.11) is time-independent.

The metric function (3.10) has two roots r_+ and r_-, which are given by

$$r_\pm = M \pm \sqrt{M^2 - Q^2}. \tag{3.13}$$

From (3.13), there are three possible scenarios. The first, for which $M^2 > Q^2$, is called the sub-extremal case. In this case, r_+ is the location of the black hole event horizon and r_- is an inner Cauchy horizon; the latter does not exist in Schwarzschild spacetime. The second, for which $M^2 = Q^2$, is the extremal case. In this case, the two horizons coincide. The third, for which $M^2 < Q^2$, is called the super-extremal case and gives rise to a naked singularity. In this thesis, we will only consider the sub-extremal case, i.e. $M^2 > Q^2$. Therefore, the location of the black hole event horizon r_+ is given herein by

$$r_+ = M + \sqrt{M^2 - Q^2}, \tag{3.14}$$

and we note that the metric function (3.10) vanishes on the event horizon such that

$$f(r_+) = 0, \tag{3.15}$$

rendering the metric (3.9) of Reissner-Nordström spacetime singular on the black hole event horizon. As was the case in Schwarzschild spacetime, this is a coordinate singularity; in §3.1.2, we will introduce the analogues of the lightcone and Kruskal coordinate systems in Reissner-Nordström spacetime, which are not plagued by this singularity.

Before we do so, however, it is useful to consider some of the general properties of the Reissner-Nordström solution. The spacetime described by the Reissner-Nordström metric (3.9) is both static and spherically symmetric; the fact that it is static is evident from the metric (3.9) being independent of the time coordinate t and the fact that it is spherically symmetric is evident from the angular parts of the metric (3.9) being proportional to the metric on the 2-sphere $g_\Omega = \mathrm{d}\theta^2 + \sin^2\theta\, \mathrm{d}\varphi^2$. It turns out that the line element (3.9) together with the expression for the metric function (3.10) and the form of the gauge field potential (3.11) is the unique static, spherically symmetric solution to the Einstein-Maxwell equations, by a statement analogous to Birkhoff's theorem; the proof of this statement is outside the scope of this thesis. The fact that the Reissner-Nordström solution is static and spherically symmetric will continue to be important throughout Part II of this thesis and, in particular, will inform our choice of ansatz for the scalar field equation when we introduce a scalar field on this spacetime in §3.2.

Finally, for reference, we give the inverse metric $g^{\mu\nu}$ and the determinant of the metric g. From (3.9), the inverse metric is given, in terms of Schwarzschild-like coordinates, as

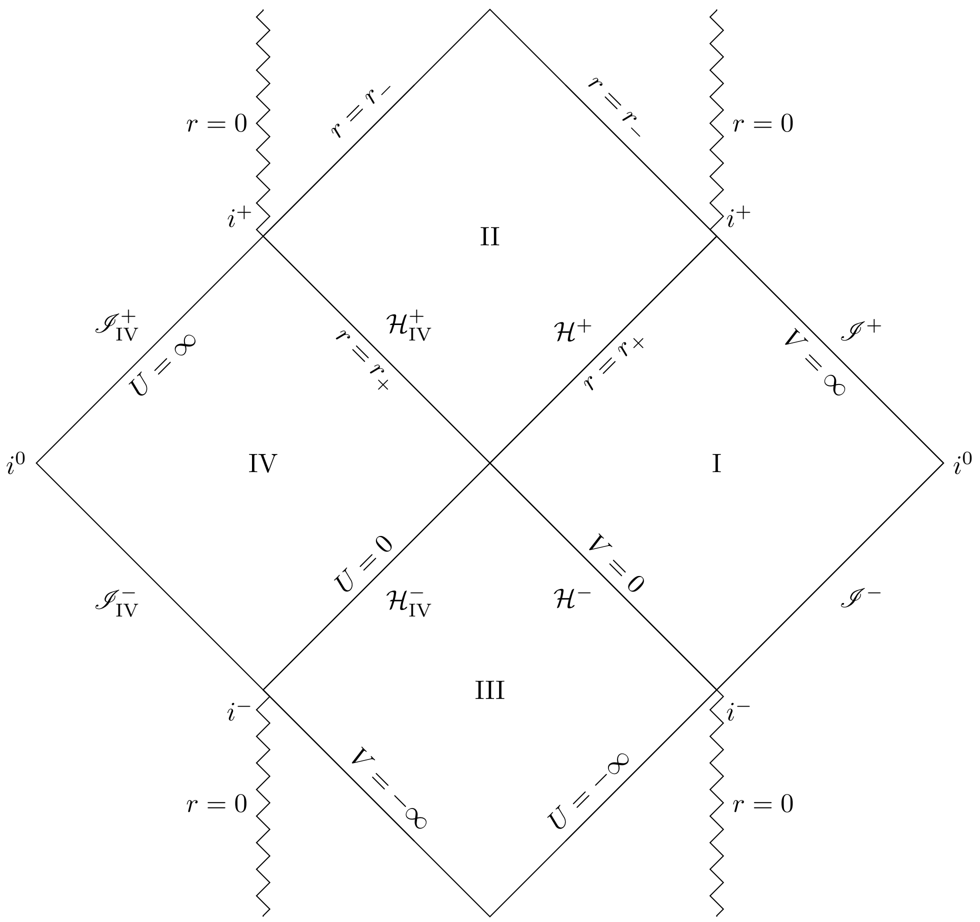

Figure 3.1. Penrose diagram for the maximally-extended sub-extremal Reissner-Nordström solution showing the spacetime regions I, II, III and IV. Lines of constant Schwarzschild-like radial coordinate r and Kruskal coordinate U, V corresponding to physically significant surfaces are shown. Surfaces of interest in region I include the past (future) event horizon $\mathcal{H}^-$ ($\mathcal{H}^+$) and past (future) null infinity $\mathscr{I}^-$ ($\mathscr{I}^+$); their corresponding surfaces in region IV are labelled with a subscript IV. Event horizons in both regions correspond to a constant $r = r_+$. The inner Cauchy horizon lies at $r = r_-$ and a spacetime singularity is located at $r = 0$. Past (future) timelike infinity is denoted by i^+ (i^-) and spacelike infinity is denoted by i^0.

$$g^{\mu\nu} = \mathrm{diag}\left(-f(r)^{-1}, f(r), r^{-2}, r^{-2}\,\mathrm{cosec}^2\theta\right), \tag{3.16}$$

and the metric determinant g is given by

$$g = -r^4 \sin^2\theta. \tag{3.17}$$

3.1.2 Geometry of Reissner-Nordström spacetime

The Schwarzschild-like coordinates used to describe the Reissner-Nordström solution (3.9) are only defined in region I of the Penrose diagram in Figure 3.1. We can reveal more regions of the spacetime by introducing lightcone coordinates. As was the case in §2.1, we first introduce the tortoise coordinate r_* which is defined only in region I, and which, in Reissner-Nordström spacetime, is related to the Schwarzschild-like radial coordinate r by

$$dr_* = f(r)^{-1}\, dr. \tag{3.18}$$

In region I, the range $r_+ < r < \infty$ of the Schwarzschild-like coordinate r is mapped on to the range $-\infty < r_* < \infty$ of the tortoise coordinate. This makes r_* particularly useful in describing the asymptotic behaviour of quantities near to the horizon and far from the black hole. Indeed, we will write down a basis of mode solutions to the scalar field equation in terms of r_* when we introduce a scalar field on Reissner-Nordström spacetime in §3.2. Rewriting the line element (3.9) in terms of the tortoise coordinate r_*, we have

$$ds^2 = -f(r)\, dt^2 + f(r)\, dr_*^2 + r^2 d\theta^2 + r^2 \sin^2\theta\, d\varphi^2. \tag{3.19}$$

Then, we can define the lightcone coordinates (u, v) according to

$$u = t - r_*, \quad \text{and} \quad v = t + r_*. \tag{3.20}$$

The equations in (3.20) can be inverted to give expressions for t and r_* as

$$t = \frac{1}{2}(u + v), \qquad r_* = \frac{1}{2}(v - u). \tag{3.21}$$

We further note, from (3.20), that

$$du = dt - dr_*, \quad \text{and} \quad dv = dt + dr_*. \tag{3.22}$$

Using (3.22), we rewrite the line element (3.19) in terms of the lightcone coordinates as

$$ds^2 = -f(r)\, du\, dv + r^2 d\theta^2 + r^2 \sin^2\theta\, d\varphi^2. \tag{3.23}$$

We can now define Kruskal coordinates in Reissner-Nordström spacetime, in relation to the lightcone coordinates, by the following expressions

$$U = -\frac{1}{\kappa}\, e^{-\kappa u}, \quad \text{and} \quad V = \frac{1}{\kappa}\, e^{\kappa v}, \tag{3.24}$$

where the surface gravity at the event horizon, κ, is given explicitly as

$$\kappa = \frac{1}{2} f'(r_+) = \frac{1}{r_+^2}(r_+ - M). \tag{3.25}$$

The equations in (3.24) can also be inverted to give expressions for the lightcone coordinates (u, v) in terms of the Kruskal coordinates (U, V) according to

$$u = -\frac{1}{\kappa}\ln(-\kappa U), \qquad v = \frac{1}{\kappa}\ln(\kappa V). \tag{3.26}$$

We further note, from (3.24), that

$$dU = e^{-\kappa u}\, du, \quad \text{and} \quad dV = e^{\kappa v}\, dv. \tag{3.27}$$

From (3.27), and using (3.20), we have

$$dU\, dV = e^{\kappa(v-u)} du\, dv = e^{2\kappa r_*} du\, dv, \tag{3.28}$$

Using (3.28), we can rewrite the line element in (3.23) as

$$\begin{aligned} \mathrm{d}s^2 &= -f(r)\,\mathrm{e}^{-2\kappa r_*}\,\mathrm{d}U\,\mathrm{d}V + r^2\mathrm{d}\theta^2 + r^2\sin^2\theta\,\mathrm{d}\varphi^2 \\ &= -2\,\zeta(r)\,\mathrm{d}U\,\mathrm{d}V + r^2\mathrm{d}\theta^2 + r^2\sin^2\theta\,\mathrm{d}\varphi^2, \end{aligned} \tag{3.29}$$

where we have defined

$$\zeta(r) = \frac{f(r)\,\mathrm{e}^{-2\kappa r_*}}{2}, \tag{3.30}$$

and the factor of $\frac{1}{2}$ in the definition of $\zeta(r)$ is such that we can simplify the expression of the metric in terms of Kruskal coordinates. In Kruskal coordinates, the inverse metric is

$$g^{\mu\nu} = \begin{pmatrix} 0 & -\dfrac{1}{\zeta(r)} & 0 & 0 \\ -\dfrac{1}{\zeta(r)} & 0 & 0 & 0 \\ 0 & 0 & r^{-2} & 0 \\ 0 & 0 & 0 & r^{-2}\mathrm{cosec}^2\theta \end{pmatrix}, \tag{3.31}$$

and the metric determinant g is given, in terms of Kruskal coordinates, by

$$g = -\zeta(r)^2\, r^4 \sin^2\theta. \tag{3.32}$$

Having defined Kruskal coordinates in (3.24), we can now access all regions of the maximally-extended Reissner-Nordström solution in Figure 3.1. In Part II, our primary concern is region I which is the only physically reasonable region in our opinion. One might ask why we took the time to introduce Kruskal coordinates at all then.

Of course, from the point of view of completeness in studying the Reissner-Nordström solution and its geometry, it is important to introduce Kruskal coordinates since these are the only coordinates which are defined throughout the spacetime. However, it will turn out that the Kruskal coordinate system will play a much larger role in Part II of this thesis. In introducing a scalar field in Reissner-Nordström spacetime, the various scalar field mode solutions that we will define will all take particularly simple asymptotic forms either near the horizon or far from the black hole. This, in turn, enables the comparatively simple evaluation of various quantities, be it the calculation of the norm of similar mode solutions, the definition of quantum states near the horizon or the expectation values of quantum observables in asymptotic regions, in terms of Kruskal coordinates. With this in mind, before concluding this section, it is useful to define the dimensionless quantities

$$\widetilde{U} = \kappa U, \qquad \widetilde{V} = \kappa V, \tag{3.33}$$

where κ is given by (3.25); the expressions for the lightcone coordinates (3.26) become

$$u = -\frac{1}{\kappa}\ln\left(-\widetilde{U}\right), \qquad v = \frac{1}{\kappa}\ln\left(\widetilde{V}\right). \tag{3.34}$$

We now have all of the necessary knowledge of Reissner-Nordström (RN) spacetime which, in the language of QFTCS, will be the background spacetime we are working on throughout

Part II of this thesis. We proceed to considering a classical charged scalar field in Reissner-Nordström spacetime with an electromagnetic gauge potential A_μ given by (3.11).

3.2 Solving the scalar field equation in RN spacetime

In this section, we study the behaviour of a classical massless, charged scalar field in RN spacetime. We introduce the scalar field equation, which admits a separable solution. We generate ODEs describing the behaviour of the radial function and the angular function separately. We introduce two orthogonal bases of scalar field modes, which allow us to demonstrate classical charge superradiance for a charged scalar field in RN spacetime. The section concludes with a calculation of the normalisation constants of these modes.

3.2.1 Introduction

In §3.1, we introduced the RN solution to the Einstein-Maxwell equations (3.1). In this section, we consider the propagation of a classical charged scalar field Φ in Reissner-Nordström spacetime with an electromagnetic gauge potential A_μ given by (3.11).

In general, the charged scalar field Φ can also possess a mass m and coupling to the scalar curvature R, where the strength of the coupling is described by the dimensionless constant ξ. In Part II, we will restrict ourselves to considering a massless, minimally coupled scalar field such that we have

$$m = 0, \qquad \text{and} \qquad \xi = 0. \tag{3.35}$$

We should add a caveat to say that (3.35) is only valid in Part II. In Part III, we will consider a charged scalar field of arbitrary mass m and coupling to the scalar curvature ξ.

3.2.2 Scalar field equation

The behaviour of a massless, minimally-coupled charged scalar field Φ is governed by

$$D_\mu D^\mu \Phi = 0. \tag{3.36}$$

We refer to (3.36) as the scalar field equation. Expanding out the gauge covariant derivatives in (3.36) and using the fact that the spacetime covariant derivative ∇_μ reduces to a partial derivative ∂_μ when acting on a scalar quantity, the scalar field equation becomes

$$\Box\,\Phi - \mathrm{i}q\left(\nabla_\mu A^\mu\right)\Phi - 2\,\mathrm{i}qA^\mu\partial_\mu\Phi - q^2A_\mu A^\mu\Phi = 0. \tag{3.37}$$

Rewriting the $\Box\,\Phi$ term in (3.37) in terms of the metric determinant g, we obtain

$$\frac{1}{\sqrt{-g}}\,\partial_\mu\big(\sqrt{-g}\,\partial^\mu\Phi\big) - 2\,\mathrm{i}qA^\mu\partial_\mu\Phi - q^2A_\mu A^\mu\Phi = 0, \tag{3.38}$$

where the term involving $(\nabla_\mu A^\mu)$ in (3.38) vanishes because the gauge field A_μ satisfies the Lorenz gauge condition (3.12). In Schwarzschild-like coordinates, (3.38) becomes

$$\partial_t \partial^t \Phi + \frac{1}{r^2}\,\partial_r\left(r^2 \partial^r \Phi\right) + \frac{1}{\sin\theta}\,\partial_\theta\left(\sin\theta\,\partial^\theta \Phi\right) + \partial_\varphi \partial^\varphi \Phi - 2\,\mathrm{i}qA^t\partial_t\Phi - q^2 A_t A^t \Phi = 0, \quad (3.39)$$

where we have used the metric determinant g (3.17). Then, (3.39) becomes

$$-\frac{1}{f(r)}\frac{\partial^2 \Phi}{\partial t^2} + \frac{1}{r^2}\frac{\partial}{\partial r}\left[f(r) r^2 \frac{\partial \Phi}{\partial r}\right] + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left[\sin\theta\,\frac{\partial \Phi}{\partial \theta}\right] + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 \Phi}{\partial \varphi^2} - \frac{2\,\mathrm{i}qQ}{f(r) r}\frac{\partial \Phi}{\partial t} + \frac{q^2 Q^2}{f(r) r^2}\,\Phi = 0. \quad (3.40)$$

The derivative operators in (3.40) only consist of partial derivatives acting with respect to one of the Schwarzschild-like coordinates t, r, θ or φ. Then, we see that (3.40) admits a separable solution in terms of scalar field modes, which we can represent by an ansatz consisting of functions of the Schwarzschild-like coordinates. We use the symmetries of the background Reissner-Nordström spacetime to inform us of the general form the ansatz should take. As described when introducing the Reissner-Nordström solution in §3.1, Reissner-Nordström spacetime is both static and spherically symmetric.

The staticity of the spacetime means that we would expect our ansatz to have a harmonic time-dependence; it is natural to expect that this time-dependence will be proportional to the frequency of the mode, which we will denote by ω, such that we can postulate a harmonic time dependence of the form $\mathrm{e}^{-\mathrm{i}\omega t}$.

We would expect the angular part of the ansatz to respect the spherical symmetry of RN spacetime. The functions which do this are the spherical harmonics $Y_{\ell m}(\theta, \varphi)$, where the quantum numbers ℓ and m correspond to the total angular momentum quantum number and the azimuthal quantum number respectively. While we expect the angular function to be given by the spherical harmonics, we will consider $Y_{\ell m}(\theta, \varphi)$ to be an arbitrary set of functions for now; we will derive the exact form of $Y_{\ell m}(\theta, \varphi)$ from the separated angular part of the solution to the scalar field equation (3.40).

There is no special symmetry of RN spacetime that might inform us of the possible form of the radial part of the ansatz and so we leave the radial function $X_{\omega\ell}(r)$ arbitrary.

If we were to consider a purely classical scalar field theory, then it would suffice to postulate an ansatz which is formed of the product of the functions $\mathrm{e}^{-\mathrm{i}\omega t}$, $Y_{\ell m}(\theta, \varphi)$ and $X_{\omega\ell}(r)$. However, we will eventually want to quantise the scalar field and we will be required to expand the field Φ in an orthonormal basis of mode solutions in order to do so. Thus, we will need to include in our ansatz a normalisation constant N_ω, which depends on the particular mode solution under consideration. Then, we can express our ansatz for the mode solutions of the scalar field equation (3.40) as

$$\phi_{\omega\ell m} = \frac{\mathrm{e}^{-\mathrm{i}\omega t}}{r}\,\mathrm{N}_\omega\, X_{\omega\ell}(r)\, Y_{\ell m}(\theta, \varphi)\,, \quad (3.41)$$

where we have included a factor of r^{-1} in (3.41) in anticipation of generating an equation for the radial function $X_{\omega\ell}(r)$ analogous to the Regge-Wheeler equation (2.20). We can substitute the mode solution ansatz (3.41) into the scalar field equation in (3.40) in order to obtain a set of separable differential equations; given the harmonic time-dependence of the ansatz (3.41), we expect two such differential equations, one containing the radial function $X_{\omega\ell}(r)$ and another containing the angular function $Y_{\ell m}(\theta,\varphi)$. We obtain

$$\begin{aligned}\frac{1}{f(r)}\,\omega^2 + \frac{1}{rX_{\omega\ell}(r)}\frac{\mathrm{d}}{\mathrm{d}r}\left[f(r)r^2\,\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{X_{\omega\ell}(r)}{r}\right)\right] \\ + \frac{1}{r^2\sin\theta}\frac{1}{Y_{\ell m}(\theta,\varphi)}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)Y_{\ell m}(\theta,\varphi) + \frac{1}{r^2\sin^2\theta}\frac{1}{Y_{\ell m}(\theta,\varphi)}\frac{\partial^2}{\partial\varphi^2}\,Y_{\ell m}(\theta,\varphi) \\ - \frac{1}{f(r)}\frac{2\omega qQ}{r} + \frac{1}{f(r)}\frac{q^2Q^2}{r^2} = 0. \quad (3.42)\end{aligned}$$

Rearranging and multiplying through by a factor of r^2 in order to separate the radial parts and angular parts of (3.42), we obtain

$$\begin{aligned}\frac{r^2}{f(r)}\left(\omega - \frac{qQ}{r}\right)^2 + \frac{r}{X_{\omega\ell}(r)}\frac{\mathrm{d}}{\mathrm{d}r}\left[f(r)r^2\,\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{X_{\omega\ell}(r)}{r}\right)\right] \\ = -\frac{1}{Y_{\ell m}(\theta,\varphi)}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right]Y_{\ell m}(\theta,\varphi) = \lambda, \quad (3.43)\end{aligned}$$

where λ is a separation constant. From (3.43), we can write down the PDE containing the angular function $Y_{\ell m}(\theta,\varphi)$, which we refer to as the angular equation, as

$$\frac{1}{Y_{\ell m}(\theta,\varphi)}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right]Y_{\ell m}(\theta,\varphi) = -\lambda, \quad (3.44)$$

and we can write down the ODE containing the radial function $X_{\omega\ell}(r)$, which we refer to as the radial equation, as

$$\frac{r^2}{f(r)}\left(\omega - \frac{qQ}{r}\right)^2 + \frac{r}{X_{\omega\ell}(r)}\frac{\mathrm{d}}{\mathrm{d}r}\left[f(r)r^2\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{X_{\omega\ell}(r)}{r}\right)\right] = \lambda. \quad (3.45)$$

We will examine each of the differential equations (3.44) and (3.45) in turn, beginning with the angular equation (3.44) since this will inform us as to the form of the constant λ.

3.2.3 Angular equation

Given the spherical symmetry of the background Reissner-Nordström spacetime, we reasoned that we would expect the angular functions in our mode solution ansatz (3.41) to be the spherical harmonics. Since the angular equation (3.44) is precisely the equation that is solved by the spherical harmonics, this is indeed the case; then, the separation constant is written in terms of the total angular momentum quantum number ℓ as

$$\lambda = \ell(\ell+1)\,, \qquad \ell \text{ in } \mathbb{Z}_{\geq 0} \text{ and } \ell \geq |m|\,. \quad (3.46)$$

The spherical harmonics are given by the expression

$$Y_{\ell m}(\theta,\varphi) = \mathrm{N}_{\ell m}\,P_\ell^m(\cos\theta)\,\mathrm{e}^{\mathrm{i}m\varphi}, \quad (3.47)$$

where $P_\ell^m(\cos\theta)$ is a real Legendre function and the form of the normalisation constant $\mathrm{N}_{\ell m}$ is determined by the normalisation convention we choose for the spherical harmonics $Y_{\ell m}(\theta,\varphi)$. We will choose a normalisation convention such that

$$\int Y_{\ell m}(\theta,\varphi)\, Y_{\ell' m'}(\theta,\varphi) \sin\theta\, \mathrm{d}\theta\, \mathrm{d}\varphi = \delta_{\ell\ell'}\, \delta_{mm'}. \tag{3.48}$$

Then, the normalisation constant $\mathrm{N}_{\omega\ell}$ associated to $Y_{\ell m}(\theta,\varphi)$ is given by [90, Eq.14.30.1]

$$\mathrm{N}_{\omega\ell} = \sqrt{\frac{(2\ell+1)}{4\pi}\frac{(\ell-m)!}{(\ell+m)!}}. \tag{3.49}$$

Thus, the spherical harmonics $Y_{\ell m}(\theta,\varphi)$ are given explicitly by the expression

$$Y_{\ell m}(\theta,\varphi) = \sqrt{\frac{(2\ell+1)}{4\pi}\frac{(\ell-m)!}{(\ell+m)!}}\, P_\ell^m(\cos\theta)\, \mathrm{e}^{\mathrm{i}m\varphi}. \tag{3.50}$$

The spherical harmonics $Y_{\ell m}(\theta,\varphi)$ (3.50) solve the angular part (3.44) of the scalar field equation (3.36), while respecting the spherical symmetry of the background RN spacetime. We derive further identities concerning the spherical harmonics in Appendix A, which will be useful when calculating the expectation values of quantum observables in Chapter 4.

Having derived the explicit form of the separation constant λ (3.46), we now proceed to solving the radial equation (3.45); this is the subject of the next section.

3.2.4 Radial equation and scalar field effective potential

Using the expression (3.46) for the separation constant λ in terms of the total angular momentum quantum number ℓ, the radial equation (3.45) becomes

$$\frac{r^2}{f(r)}\left(\omega - \frac{qQ}{r}\right)^2 + \frac{r}{X_{\omega\ell}(r)}\frac{\mathrm{d}}{\mathrm{d}r}\left[f(r) r^2 \frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{X_{\omega\ell}(r)}{r}\right)\right] = \ell\,(\ell+1)\,. \tag{3.51}$$

We can use the definition of the tortoise coordinate r_* (3.18) to simplify the second term on the l.h.s of (3.51) as

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}r}\left[f(r) r^2 \frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{X_{\omega\ell}(r)}{r}\right)\right] &= \frac{\mathrm{d}}{\mathrm{d}r}\left[f(r)\left\{r\,\frac{\mathrm{d}}{\mathrm{d}r}X_{\omega\ell}(r) - X_{\omega\ell}(r)\right\}\right] \\
&= \frac{\mathrm{d}}{\mathrm{d}r}\left[r\,\frac{\mathrm{d}r}{\mathrm{d}r_*}\frac{\mathrm{d}}{\mathrm{d}r}X_{\omega\ell}(r) - f(r)X_{\omega\ell}(r)\right] \\
&= \frac{\mathrm{d}}{\mathrm{d}r}\left[r\,\frac{\mathrm{d}}{\mathrm{d}r_*}X_{\omega\ell}(r) - f(r)X_{\omega\ell}(r)\right] \\
&= r\,\frac{\mathrm{d}}{\mathrm{d}r}\frac{\mathrm{d}}{\mathrm{d}r_*}X_{\omega\ell}(r) + \frac{\mathrm{d}}{\mathrm{d}r_*}X_{\omega\ell}(r) - f(r)\,\frac{\mathrm{d}}{\mathrm{d}r}X_{\omega\ell}(r) - f^{'}(r)X_{\omega\ell}(r) \\
&= \frac{r}{f(r)}\frac{\mathrm{d}^2}{\mathrm{d}r_*^2}X_{\omega\ell}(r) - f^{'}(r)X_{\omega\ell}(r)\,.
\end{aligned} \tag{3.52}$$

Substituting (3.52) into (3.51), the radial equation becomes

$$\frac{r}{X_{\omega\ell}(r)}\left[\frac{r}{f(r)}\frac{\mathrm{d}^2}{\mathrm{d}r_*^2}X_{\omega\ell}(r) - f^{'}(r)X_{\omega\ell}(r)\right] - \ell\,(\ell+1) + \frac{r^2}{f(r)}\left(\omega - \frac{qQ}{r}\right)^2 = 0. \tag{3.53}$$

Equation (3.53) simplifies to

$$\frac{\mathrm{d}^2}{\mathrm{d}r_*^2}X_{\omega\ell}(r) - \frac{f(r)}{r^2}\left[\ell(\ell+1) + f^{'}(r)r - \frac{r^2}{f(r)}\left(\omega - \frac{qQ}{r}\right)^2\right]X_{\omega\ell}(r) = 0. \tag{3.54}$$

We can write (3.54) as

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}r_*^2} - V_{\text{eff}}(r)\right]X_{\omega\ell}(r) = 0, \tag{3.55}$$

where the scalar field effective potential $V_{\text{eff}}(r)$, which is the one-dimensional effective potential felt by a charged scalar field mode $\phi_{\omega\ell m}$ in a background RN spacetime, is

$$V_{\text{eff}}(r) = \frac{f(r)}{r^2}\left[\ell(\ell+1) + f^{'}(r)r\right] - \left(\omega - \frac{qQ}{r}\right)^2. \tag{3.56}$$

From (3.56), we see that the scalar field effective potential $V_{\text{eff}}(r)$ takes particularly simple asymptotic forms; near the horizon, where $r \to r_+$ and $f(r) \to 0$ (3.15), we may neglect the first term in (3.56) while far away from the black hole, where $r \to \infty$, we need only consider leading order contributions in r. Then, we can summarise the asymptotic forms of the effective potential $V_{\text{eff}}(r)$ near the horizon and far away from the black hole as

$$V_{\text{eff}}(r) \sim \begin{cases} -\widetilde{\omega}^2 = -\left(\omega - \frac{qQ}{r_+}\right)^2, & r_* \to -\infty, \\ -\omega^2, & r_* \to \infty, \end{cases} \tag{3.57}$$

respectively, where we have defined the quantity

$$\widetilde{\omega} = \omega - \frac{qQ}{r_+}. \tag{3.58}$$

From the asymptotic forms of the scalar field effective potential $V_{\text{eff}}(r)$ (3.57), the radial equation in (3.55) must thus have asymptotic solutions of the form

$$X_{\omega\ell}(r) \sim \begin{cases} \mathrm{e}^{\pm \mathrm{i}\widetilde{\omega}r_*}, & r_* \to -\infty, \\ \mathrm{e}^{\pm \mathrm{i}\omega r_*}, & r_* \to \infty. \end{cases} \tag{3.59}$$

That the near-infinity expression for the scalar field effective potential $V_{\text{eff}}(r)$ depends neither on the charge of the black hole nor on the scalar field charge is to be expected since (3.11) corresponds to a choice of gauge such that the electromagnetic field potential A_μ vanishes far from the black hole. If we perform a gauge transformation of the form

$$A_\mu \to A_\mu + \partial_\mu \Upsilon, \qquad \Phi \to \mathrm{e}^{\mathrm{i}q\Upsilon}\Phi, \qquad \Upsilon = \frac{Qt}{r_0}, \tag{3.60}$$

for some fixed value r_0 of the radial coordinate, then A_μ becomes $(\underline{A}_0, 0, 0)$ with

$$\underline{A}_0 = -\frac{Q}{r} + \frac{Q}{r_0}. \tag{3.61}$$

In this sense, fixing the gauge in (3.11) corresponds to choosing $r_0 = \infty$. However, we could have equivalently chosen $r_0 = r_+$ such that the gauge potential A_μ instead vanishes at the black hole event horizon, rendering the near-horizon expression for the scalar field effective potential $V_{\text{eff}}(r)$ independent of both the charge of the black hole and the scalar field charge. Our choice of gauge is motivated by the desire to ascribe a natural physical interpretation to the frequency ω of a scalar field mode (3.41). In order to see this, consider how the frequency ω transforms under the gauge transformation given in (3.60); we have

$$\underline{\omega} = \omega - \frac{qQ}{r_0}. \tag{3.62}$$

Thus the frequency ω of a scalar field mode is not a gauge-invariant quantity and, moreover, a constant shift in the frequency corresponds to a gauge transformation of the form (3.60). Choosing $r_0 = \infty$, or equivalently making the same choice of gauge as in (3.11), enables us to interpret ω as the frequency of a mode as measured by a static observer near infinity.

From (3.59), the scalar field effective potential V_{eff} and, consequently, the radial function $X_{\omega\ell}(r)$ each have different forms near the horizon and far away from the black hole.

Far away from the black hole, $V_{\text{eff}}(r)$ and $X_{\omega\ell}(r)$ are independent of both the charge of the black hole Q and the scalar field charge q. This is intuitive since we chose a gauge (3.11) that made the electromagnetic potential A_μ vanish far from the black hole. Then the quantity ω has a natural physical interpretation; it is the frequency of a mode as measured by a static observer far from the black hole.

However, the charge of the black hole Q and the scalar field charge q are present in $V_{\text{eff}}(r)$ and $X_{\omega\ell}(r)$ near the horizon. This difference in effective potential near to and far from the black hole will be a key theme running throughout Part II of this thesis; it will give rise to the phenomenon of classical superradiance as well as having important consequences when defining quantum states and studying the expectation values of observables.

A further observation is that, unlike the case in Schwarzschild spacetime, the radial function $X_{\omega\ell}(r)$ is not invariant under the transformation $\omega \to -\omega$ due to the form the effective potential $V_{\text{eff}}(r)$ takes near the horizon in (3.57). This means that while $X^*_{\omega\ell}(r)$ satisfies the same radial equation (3.55), the function $X^*_{\omega\ell}(r)$ is not the same as $X_{-\omega\ell}(r)$. This will become important when defining vacuum states for the field.

3.3 Scalar field mode solutions

In this section, we use physical considerations of the behaviour of the scalar field in the background RN spacetime to introduce the in-, up-, out- and down-modes. We conclude by deriving relations between the reflection and transmission coefficients associated to each mode, which we use to demonstrate the phenomenon of classical superradiance.

3.3.1 Introduction

In §3.2.4, we derived the general form of the asymptotic solutions (3.59) of the radial function $X_{\omega\ell}(r)$ near the horizon as $r_* \to -\infty$ and far from the black hole as $r_* \to \infty$. Then, an intuitive way to develop an orthonormal basis of mode solutions is to impose suitable boundary conditions on the radial function $X_{\omega\ell}(r)$ in these asymptotic regions. The radial function $X_{\omega\ell}(r)$ and the harmonic time dependence of the scalar field modes, given by $e^{-i\omega t}$, together describe wave propagation in the background Reissner-Nordström spacetime. Thus we can motivate each mode solution by physical considerations of how the mode $\phi_{\omega\ell m}$ should behave in the asymptotic regions of the background spacetime.

Furthermore, since there are two surfaces in the asymptotic region $r_* \to \infty$, namely past null infinity $\mathscr{I}^-$ and future null infinity $\mathscr{I}^+$, we can generate two separate mode solutions by physical considerations near each surface separately. Similarly, we can generate another two mode solutions in the asymptotic region $r_* \to -\infty$ from physical considerations near the past horizon $\mathcal{H}^-$ and the future horizon $\mathcal{H}^+$ separately.

From the expression for the asymptotic forms of the radial function $X_{\omega\ell}(r)$ (3.59), the exponent, which contains $i\widetilde{\omega}r_*$ as $r_* \to -\infty$ and $i\omega r_*$ as $r_* \to \infty$, is given up to a sign in either case. This sign corresponds to the radial direction in which the wave is travelling; a positive sign in front of the aforementioned factors corresponds to waves emerging from the event horizon and travelling in the direction of increasing r_* towards infinity, while a negative sign corresponds to waves originating at infinity and travelling in the direction of decreasing r_* towards the event horizon.

Our final remark, before we begin explicitly deriving mode solutions, is that the exponential in the asymptotic forms of the radial function $X_{\omega\ell}(r)$ (3.59) is defined up to a multiplicative constant; taking the multiplicative constant to be unity describes a wave of unit flux. Thus it will be helpful to describe waves of unit flux near a certain surface. It stands to reason that incident waves will be either reflected at or transmitted through the potential barrier described by the scalar field effective potential $V_{\text{eff}}(r)$ (3.56); therefore, we will also specify the asymptotic form of the reflected and transmitted parts of the incident waves multiplied by the complex coefficients $A_{\omega\ell}$ and $B_{\omega\ell}$ respectively. We need not interpret the coefficients $A_{\omega\ell}$ and $B_{\omega\ell}$ as reflection and transmission coefficients for the purpose of deriving a basis of mode solutions; in §3.3.4, we will indeed interpret $A_{\omega\ell}$ and $B_{\omega\ell}$ as reflection and transmission coefficients, and doing so will lead us to the phenomenon of classical superradiance.

Now, we are ready to begin placing physical considerations on the behaviour of the field mode $\phi_{\omega\ell m}$ in the background Reissner-Nordström spacetime.

3.3.2 The mode solutions

In- and up-modes

The first such physical consideration is that there should be no outgoing radiation from the event horizon of the black hole, which means that no waves should emerge from the past horizon $\mathcal{H}^-$, while it allows the propagation of waves towards the future horizon $\mathcal{H}^+$. Consider a mode of unit flux coming up from past null infinity $\mathscr{I}^-$ with part of the mode being transmitted down the future horizon $\mathcal{H}^+$ and the other part being reflected towards future null infinity $\mathscr{I}^+$. Such a mode satisfies the requirement of an absence of outgoing radiation from the horizon. We refer to these field modes as the in-modes $\phi^{\text{in}}_{\omega\ell m}$, where

$$\phi^{\text{in}}_{\omega\ell m} = \mathrm{N}^{\text{in}}_{\omega}\, e^{-i\omega t}\, \frac{X^{\text{in}}_{\omega\ell}(r)}{r}\, Y_{\ell m}(\theta,\varphi)\,, \tag{3.63}$$

and the asymptotic forms of the associated radial function $X^{\text{in}}_{\omega\ell}(r)$ can be summarised as

$$X^{\rm in}_{\omega\ell}(r) \sim \begin{cases} B^{\rm in}_{\omega\ell}\, {\rm e}^{-{\rm i}\widetilde{\omega} r_*} & r_* \to -\infty, \\ {\rm e}^{-{\rm i}\omega r_*} + A^{\rm in}_{\omega\ell}\, {\rm e}^{{\rm i}\omega r_*} & r_* \to \infty. \end{cases} \tag{3.64}$$

The expressions (3.64) make explicit the requirements that the in-modes are incoming from past null infinity $\mathscr{I}^-$ with unit flux and vanish near the past horizon $\mathcal{H}^-$.

The second such physical consideration is that there should be no incoming radiation from infinity, which means that no waves should emerge from past null infinity $\mathscr{I}^-$, while it allows the propagation of waves towards future null infinity $\mathscr{I}^+$. Consider a mode of unit flux emerging from the past horizon $\mathcal{H}^-$ with part of the mode being transmitted towards future null infinity $\mathscr{I}^+$ and the other part being reflected towards the future horizon $\mathcal{H}^+$. Such a mode satisfies the requirement of an absence of incoming radiation from infinity. We will refer to these field modes as the up-modes $\phi^{\rm up}_{\omega\ell m}$, which are given by

$$\phi^{\rm up}_{\omega\ell m} = {\rm N}^{\rm up}_{\omega}\, {\rm e}^{-{\rm i}\omega t}\, \frac{X^{\rm up}_{\omega\ell}(r)}{r}\, Y_{\ell m}(\theta,\varphi)\,, \tag{3.65}$$

and the asymptotic forms of the associated radial function $X^{\rm up}_{\omega\ell}(r)$ can be summarised as

$$X^{\rm up}_{\omega\ell}(r) \sim \begin{cases} {\rm e}^{{\rm i}\widetilde{\omega} r_*} + A^{\rm up}_{\omega\ell}\, {\rm e}^{-{\rm i}\widetilde{\omega} r_*} & r_* \to -\infty, \\ B^{\rm up}_{\omega\ell}\, {\rm e}^{{\rm i}\omega r_*} & r_* \to \infty. \end{cases} \tag{3.66}$$

The expressions (3.66) make explicit the requirements that the up-modes are outgoing from the past horizon $\mathcal{H}^-$ with unit flux and vanish near past null infinity $\mathscr{I}^-$.

The in-modes (3.63) are defined near $\mathscr{I}^-$ where the up-modes vanish and the up-modes (3.65) are defined near $\mathcal{H}^-$ where the in-modes vanish; together, they therefore form an orthogonal basis of mode solutions in which we can expand the scalar field Φ. They do not yet, however, constitute an orthonormal basis which we require to quantise the field; in order for them to do so, we will need to evaluate the explicit forms of the normalisation constants ${\rm N}^{\rm in}_{\omega}$ and ${\rm N}^{\rm up}_{\omega}$ which we do in §3.4. When normalised, we will refer to a basis of in- and up-modes as the 'past' basis since we derived these mode solutions by placing physical considerations upon the past horizon $\mathcal{H}^-$ and past null infinity $\mathscr{I}^-$ respectively.

Out- and down-modes

Since the radial functions $X^{\rm in}_{\omega\ell}(r)$ and $X^{\rm up}_{\omega\ell}(r)$ each constitute separate solutions to the radial equation (3.55), then we may immediately find two further solutions by taking their complex conjugates; doing so will lead us to the out-modes $\phi^{\rm out}_{\omega\ell m}$ and the down-modes $\phi^{\rm down}_{\omega\ell m}$ respectively. The asymptotic forms of their associated radial functions $X^{\rm out}_{\omega\ell}(r)$ and $X^{\rm down}_{\omega\ell}(r)$ will then be given as the complex conjugates of $X^{\rm in}_{\omega\ell}(r)$ (3.64) and $X^{\rm up}_{\omega\ell}(r)$ (3.66) respectively. However, we can also derive these solutions by placing physical considerations on the behaviour of a mode $\phi_{\omega\ell m}$ in the background Reissner-Nordström spacetime as we did when deriving the in- and up-modes.

Then, another such physical consideration is that there should be no ingoing radiation incident upon the event horizon of the black hole, which means that no waves should be incident upon the future horizon $\mathcal{H}^+$, while it allows the propagation of waves outgoing from the past horizon $\mathscr{H}^-$. Consider a mode of unit flux outgoing at future null infinity $\mathscr{I}^+$; the parts of the mode that have been transmitted from the past horizon $\mathscr{H}^-$ and reflected from past null infinity $\mathscr{I}^-$, must interfere in such a way as to cancel exactly any radiation that might otherwise be scattered back down the horizon. Such a mode satisfies the requirement of an absence of incoming radiation incident upon the event horizon. We will refer to these field modes as the out-modes $\phi^{\text{out}}_{\omega\ell m}$, which are given by

$$\phi^{\text{out}}_{\omega\ell m} = \mathrm{N}^{\text{out}}_{\omega}\, \mathrm{e}^{-\mathrm{i}\omega t}\, \frac{X^{\text{out}}_{\omega\ell}(r)}{r}\, Y_{\ell m}(\theta,\varphi)\,, \tag{3.67}$$

and the asymptotic forms of the associated radial function $X^{\text{out}}_{\omega\ell}(r)$ can be summarised as

$$X^{\text{out}}_{\omega\ell}(r) \sim \begin{cases} B^{\text{in}*}_{\omega\ell}\, \mathrm{e}^{\mathrm{i}\widetilde{\omega} r_*} & r_* \to -\infty, \\ \mathrm{e}^{\mathrm{i}\omega r_*} + A^{\text{in}*}_{\omega\ell}\, \mathrm{e}^{-\mathrm{i}\omega r_*} & r_* \to \infty. \end{cases} \tag{3.68}$$

The expressions in (3.68) make explicit the requirements that the out-modes are outgoing at future null infinity $\mathscr{I}^+$ with unit flux and vanish near the future horizon $\mathscr{H}^+$. One can easily verify that the asymptotic forms of $X^{\text{out}}_{\omega\ell}(r)$ (3.68) are the complex conjugate of the corresponding asymptotic forms of $X^{\text{in}}_{\omega\ell}(r)$ (3.64).

The final physical consideration is that there should be no outgoing radiation at infinity, which means that no waves should escape to future null infinity $\mathscr{I}^+$, while it allows the propagation of waves incoming from past null infinity $\mathscr{I}^-$. Consider a mode of unit flux incident upon the future horizon $\mathscr{H}^+$; the parts of the mode that have been transmitted from past null infinity $\mathscr{I}^-$ and reflected from the past horizon $\mathscr{H}^-$, must interfere in such a way as to cancel exactly any radiation that might otherwise be scattered back to infinity [89]. Such a mode satisfies the requirement of an absence of radiation escaping to infinity. We will refer to these field modes as the down-modes $\phi^{\text{down}}_{\omega\ell m}$, where

$$\phi^{\text{down}}_{\omega\ell m} = \mathrm{N}^{\text{down}}_{\omega}\, \mathrm{e}^{-\mathrm{i}\omega t}\, \frac{X^{\text{down}}_{\omega\ell}(r)}{r}\, Y_{\ell m}(\theta,\varphi)\,, \tag{3.69}$$

and the asymptotic forms of the associated radial function $X^{\text{down}}_{\omega\ell}(r)$ are summarised as

$$X^{\text{down}}_{\omega\ell}(r) \sim \begin{cases} \mathrm{e}^{-\mathrm{i}\widetilde{\omega} r_*} + A^{\text{up}*}_{\omega\ell}\, \mathrm{e}^{\mathrm{i}\widetilde{\omega} r_*} & r_* \to -\infty, \\ B^{\text{up}*}_{\omega\ell}\, \mathrm{e}^{-\mathrm{i}\omega r_*} & r_* \to \infty. \end{cases} \tag{3.70}$$

The expressions in (3.70) make explicit the requirements that the down-modes are incoming at the future horizon $\mathscr{H}^+$ with unit flux and vanish near future null infinity $\mathscr{I}^+$. One can easily verify that the asymptotic forms of $X^{\text{down}}_{\omega\ell}(r)$ (3.70) are the complex conjugate of the corresponding asymptotic forms of $X^{\text{up}}_{\omega\ell}(r)$ (3.66).

The out-modes (3.67) are defined near $\mathscr{I}^+$ where the down-modes vanish and the down-modes (3.69) are defined near $\mathscr{H}^+$ where the out-modes vanish; together they form an orthogonal basis of mode solutions in which we can expand the scalar field Φ. Similar to the case of the in- and up-modes, they do not yet constitute an orthonormal basis since we are required to evaluate the explicit forms of the normalisation constants $\mathrm{N}^{\text{out}}_{\omega}$ and $\mathrm{N}^{\text{down}}_{\omega}$, which we do in §3.4. When normalised, we will refer to a basis of out- and down-modes as the 'future' basis since we derived these mode solutions by placing physical considerations upon the future horizon $\mathcal{H}^+$ and future null infinity $\mathscr{I}^+$ respectively.

We can write the radial functions of the out- (3.68) and down-modes (3.70) as linear combinations of the in- (3.64) and up-mode radial functions (3.66) according to

$$X_{\omega\ell}^{\mathrm{out}} = A_{\omega\ell}^{\mathrm{in}*} X_{\omega\ell}^{\mathrm{in}} + B_{\omega\ell}^{\mathrm{in}*} X_{\omega\ell}^{\mathrm{up}}, \tag{3.71a}$$

$$X_{\omega\ell}^{\mathrm{down}} = A_{\omega\ell}^{\mathrm{up}*} X_{\omega\ell}^{\mathrm{up}} + B_{\omega\ell}^{\mathrm{up}*} X_{\omega\ell}^{\mathrm{in}}, \tag{3.71b}$$

respectively. We can interpret (3.71a) and (3.71b) as expressing that up-modes transmitted from the past horizon $\mathcal{H}^-$ and the in-modes reflected from past null infinity $\mathscr{I}^-$ interfere in such a way as to generate the out-modes, which are outgoing at future null infinity $\mathscr{I}^+$ with unit flux and vanish near the future horizon $\mathcal{H}^+$, and the down-modes, which are incoming at the future horizon $\mathcal{H}^+$ with unit flux and vanish near future null infinity $\mathscr{I}^+$.

We can now derive the normalisation constants associated to each of the in-, up-, out- and down-modes in order to be able to expand the field Φ in an orthonormal basis before quantisation. Before we do so, however, it is instructive use the Wronskian function to derive expressions relating the complex coefficients $A_{\omega\ell}$ and $B_{\omega\ell}$ associated to each mode, which will enable us to demonstrate the phenomenon of classical superradiance.

3.3.3 Relations between reflection and transmission coefficients

Given any two linearly independent solutions X_1, X_2 of a second order, linear ODE with independent variable r_* of the form of the radial equation (3.55), the Wronskian $W(X_1, X_2)$

$$W(X_1, X_2) = X_1 \frac{\mathrm{d}X_2}{\mathrm{d}r_*} - X_2 \frac{\mathrm{d}X_1}{\mathrm{d}r_*}, \tag{3.72}$$

is independent of r_*. This enables us to derive a series of expressions relating the complex coefficients $A_{\omega\ell}$ and $B_{\omega\ell}$ of different field modes by evaluating the Wronskian (3.72) for the asymptotic forms of their corresponding radial functions $X_{\omega\ell}(r)$ near to the horizon and far from the black hole.

Since the in- and up-modes constitute an orthogonal basis of solutions to the radial equation (3.55), it is implied that they are linearly independent. Evaluating (3.72) for the radial functions $X_{\omega\ell}^{\mathrm{in}}(r)$ (3.64) and $X_{\omega\ell}^{\mathrm{up}}(r)$ (3.66) near the horizon, we obtain

$$\begin{aligned} W\left(X_{\omega\ell}^{\mathrm{in}}, X_{\omega\ell}^{\mathrm{up}}\right) = B_{\omega\ell}^{\mathrm{in}}\, \mathrm{e}^{-\mathrm{i}\widetilde{\omega} r_*} \left(\mathrm{i}\widetilde{\omega}\, \mathrm{e}^{\mathrm{i}\widetilde{\omega} r_*} - \mathrm{i}\widetilde{\omega} A_{\omega\ell}^{\mathrm{up}}\, \mathrm{e}^{-\mathrm{i}\widetilde{\omega} r_*}\right) \\ - \left(\mathrm{e}^{\mathrm{i}\widetilde{\omega} r_*} + A_{\omega\ell}^{\mathrm{up}}\, \mathrm{e}^{-\mathrm{i}\widetilde{\omega} r_*}\right)\left(-\mathrm{i}\widetilde{\omega} B_{\omega\ell}^{\mathrm{in}}\, \mathrm{e}^{-\mathrm{i}\widetilde{\omega} r_*}\right) = 2\,\mathrm{i}\widetilde{\omega} B_{\omega\ell}^{\mathrm{in}}, \end{aligned} \tag{3.73}$$

while evaluating (3.72) for the radial functions $X_{\omega\ell}^{\mathrm{in}}(r)$ and $X_{\omega\ell}^{\mathrm{up}}(r)$ near infinity, we obtain

$$\begin{aligned} W\left(X_{\omega\ell}^{\mathrm{in}}, X_{\omega\ell}^{\mathrm{up}}\right) = \left(\mathrm{e}^{-\mathrm{i}\omega r_*} + A_{\omega\ell}^{\mathrm{in}}\, \mathrm{e}^{\mathrm{i}\omega r_*}\right)\left(\mathrm{i}\omega B_{\omega\ell}^{\mathrm{up}}\, \mathrm{e}^{\mathrm{i}\omega r_*}\right) \\ - B_{\omega\ell}^{\mathrm{up}} \mathrm{e}^{\mathrm{i}\omega r_*}\left(-\mathrm{i}\omega \mathrm{e}^{-\mathrm{i}\omega r_*} + \mathrm{i}\omega A_{\omega\ell}^{\mathrm{in}}\, \mathrm{e}^{\mathrm{i}\omega r_*}\right) = 2\,\mathrm{i}\omega B_{\omega\ell}^{\mathrm{up}}. \end{aligned} \tag{3.74}$$

Since the Wronskian (3.72) is independent of r_*, equating the expressions for the Wronskian $W\left(X^{\text{in}}_{\omega\ell}, X^{\text{up}}_{\omega\ell}\right)$ near the horizon (3.73) and near infinity (3.74) leads to the relation

$$\widetilde{\omega} B^{\text{in}}_{\omega\ell} = \omega B^{\text{up}}_{\omega\ell}. \tag{3.75}$$

We may derive another relation by considering the Wronskian of the in- (3.64) and out-mode radial functions (3.68); that these are linearly independent follows from the fact that $X^{\text{out}}_{\omega\ell}(r) = X^{\text{in}*}_{\omega\ell}(r)$. Then, evaluating $W\left(X^{\text{in}}_{\omega\ell}, X^{\text{out}}_{\omega\ell}\right)$ near the horizon, we obtain

$$W\left(X^{\text{in}}_{\omega\ell}, X^{\text{out}}_{\omega\ell}\right) = B^{\text{in}}_{\omega\ell}\, e^{-i\widetilde{\omega} r_*}\left(i\widetilde{\omega}\, B^{\text{in}*}_{\omega\ell}\, e^{i\widetilde{\omega} r_*}\right) - B^{\text{in}*}_{\omega\ell}\, e^{i\widetilde{\omega} r_*}\left(-i\widetilde{\omega} B^{\text{in}}_{\omega\ell}\, e^{-i\widetilde{\omega} r_*}\right) = 2\, i\widetilde{\omega}\left|B^{\text{in}}_{\omega\ell}\right|^2, \tag{3.76}$$

while evaluating $W\left(X^{\text{in}}_{\omega\ell}, X^{\text{out}}_{\omega\ell}\right)$ near infinity, we obtain

$$\begin{aligned} W\left(X^{\text{in}}_{\omega\ell}, X^{\text{out}}_{\omega\ell}\right) &= \left(e^{-i\omega r_*} + A^{\text{in}}_{\omega\ell}\, e^{i\omega r_*}\right)\left(i\omega\, e^{i\omega r_*} - i\omega A^{\text{in}*}_{\omega\ell}\, e^{-i\omega r_*}\right) \\ &\quad - \left(e^{i\omega r_*} + A^{\text{in}*}_{\omega\ell}\, e^{-i\omega r_*}\right)\left(-i\omega e^{-i\omega r_*} + i\omega A^{\text{in}}_{\omega\ell}\, e^{i\omega r_*}\right) = 2\, i\omega\left[1 - \left|A^{\text{in}}_{\omega\ell}\right|^2\right]. \end{aligned} \tag{3.77}$$

Again, using the fact that the Wronskian (3.72) is independent of r_*, equating the expressions for $W\left(X^{\text{in}}_{\omega\ell}, X^{\text{out}}_{\omega\ell}\right)$ near the horizon (3.76) and near infinity (3.77) leads to

$$\widetilde{\omega}\left|B^{\text{in}}_{\omega\ell}\right|^2 = \omega\left[1 - \left|A^{\text{in}}_{\omega\ell}\right|^2\right]. \tag{3.78}$$

A similar calculation for the Wronskian $W\left(X^{\text{in}}_{\omega\ell}, X^{\text{out}}_{\omega\ell}\right)$ leads to the relation

$$\omega\left|B^{\text{up}}_{\omega\ell}\right|^2 = \widetilde{\omega}\left[1 - \left|A^{\text{up}}_{\omega\ell}\right|^2\right]. \tag{3.79}$$

The final relation we derive is obtained from the Wronskian of the in- and down-mode radial functions. That these are linearly independent follows from $X^{\text{down}}_{\omega\ell}(r)$ (3.70) being the complex conjugate of, and therefore linearly independent from, $X^{\text{up}}_{\omega\ell}(r)$; since $X^{\text{in}}_{\omega\ell}(r)$ (3.64) and $X^{\text{up}}_{\omega\ell}(r)$ are linearly independent of each other, it follows that $X^{\text{in}}_{\omega\ell}(r)$ and $X^{\text{down}}_{\omega\ell}(r)$ are also linearly independent. Then, near the horizon, we obtain

$$\begin{aligned} W\left(X^{\text{in}}_{\omega\ell}, X^{\text{down}}_{\omega\ell}\right) &= B^{\text{in}}_{\omega\ell}\, e^{-i\widetilde{\omega} r_*}\left(-i\widetilde{\omega}\, e^{-i\widetilde{\omega} r_*} + i\widetilde{\omega} A^{\text{up}*}_{\omega\ell}\, e^{i\widetilde{\omega} r_*}\right) \\ &\quad - \left(e^{-i\widetilde{\omega} r_*} + A^{\text{up}*}_{\omega\ell}\, e^{i\widetilde{\omega} r_*}\right)\left(-i\widetilde{\omega} B^{\text{in}}_{\omega\ell}\, e^{-i\widetilde{\omega} r_*}\right) = 2\, i\widetilde{\omega} A^{\text{up}*}_{\omega\ell} B^{\text{in}}_{\omega\ell}, \end{aligned} \tag{3.80}$$

while evaluating (3.72) for the radial functions $X^{\text{in}}_{\omega\ell}(r)$ and $X^{\text{down}}_{\omega\ell}(r)$ near infinity, we have

$$\begin{aligned} W\left(X^{\text{in}}_{\omega\ell}, X^{\text{down}}_{\omega\ell}\right) &= \left(e^{-i\omega r_*} + A^{\text{in}}_{\omega\ell}\, e^{i\omega r_*}\right)\left(-i\omega B^{\text{up}*}_{\omega\ell}\, e^{-i\omega r_*}\right) \\ &\quad - B^{\text{up}*}_{\omega\ell} e^{-i\omega r_*}\left(-i\omega e^{-i\omega r_*} + i\omega A^{\text{in}}_{\omega\ell}\, e^{i\omega r_*}\right) = -2\, i\omega A^{\text{in}}_{\omega\ell} B^{\text{up}*}_{\omega\ell}. \end{aligned} \tag{3.81}$$

Again, using the fact that the Wronskian (3.72) is independent of r_*, equating the expressions for $W\left(X^{\text{in}}_{\omega\ell}, X^{\text{down}}_{\omega\ell}\right)$ near the horizon (3.80) and near infinity (3.81) leads to

$$\widetilde{\omega} A^{\text{up}*}_{\omega\ell} B^{\text{in}}_{\omega\ell} = -\omega A^{\text{in}}_{\omega\ell} B^{\text{up}*}_{\omega\ell}. \tag{3.82}$$

We note that we could equivalently have obtained the complex conjugate of the expression in (3.82) had we considered the Wronskian $W\big(X^{\text{up}}_{\omega\ell}, X^{\text{out}}_{\omega\ell}\big)$ instead of $W\big(X^{\text{in}}_{\omega\ell}, X^{\text{down}}_{\omega\ell}\big)$.

3.3.4 Classical superradiance

Consider the expressions in (3.78) and (3.79), which relate the complex coefficients $A_{\omega\ell}$ and $B_{\omega\ell}$; if we now interpret the $A_{\omega\ell}$ as reflection coefficients and the $B_{\omega\ell}$ as transmission coefficients then, for $\omega\widetilde{\omega} < 0$, we have

$$\big|A^{\text{in}}_{\omega\ell}\big|^2 > 1, \quad \text{sgn}(\omega\widetilde{\omega}) = -1, \tag{3.83a}$$
$$\big|A^{\text{up}}_{\omega\ell}\big|^2 > 1, \quad \text{sgn}(\omega\widetilde{\omega}) = -1. \tag{3.83b}$$

This is the classical phenomenon of charge superradiance [36]. An in-mode incoming from past null infinity $\mathscr{I}^-$ with $\omega\widetilde{\omega} < 0$ will be reflected back to future null infinity $\mathscr{I}^+$ with an amplitude greater than it was incident with and, similarly, an up-mode outgoing from the past horizon $\mathcal{H}^-$ with $\omega\widetilde{\omega} < 0$ will be reflected back down the future horizon $\mathcal{H}^+$ with an amplitude greater than it was incident with. From the expression relating ω and $\widetilde{\omega}$ in (3.58), we see that there are two cases where $\omega\widetilde{\omega} < 0$ depending on the product of the scalar field charge q and the charge of the black hole Q. We can summarise these cases as

Case 1: If $qQ > 0$ then modes with $0 < \omega < \dfrac{qQ}{r_+}$ or $0 > \widetilde{\omega} > -\dfrac{qQ}{r_+}$ have $\omega\widetilde{\omega} < 0$,

Case 2: If $qQ < 0$ then modes with $0 > \omega > \dfrac{qQ}{r_+}$ or $0 < \widetilde{\omega} < -\dfrac{qQ}{r_+}$ have $\omega\widetilde{\omega} < 0$. (3.84)

From (3.84), we see that only low-frequency modes, i.e. those with $|\omega| < \left|\frac{qQ}{r_+}\right|$, are superradiantly scattered.

This is illustrated in Figure 3.2 in which we have plotted the in-mode reflection $\big|A^{\text{in}}_{\omega\ell}\big|^2$ and transmission coefficients $\omega^{-1}\widetilde{\omega}\,\big|B^{\text{in}}_{\omega\ell}\big|^2$ as a function of the frequency ω for particular choices of the total angular momentum number $\ell = 0$, the black hole charge $Q = M/2$ and scalar field charge $q = M/2$. Superradiance, which occurs when the reflection coefficient $\big|A^{\text{in}}_{\omega\ell}\big|^2$ is greater than unity, only takes place for small, positive values of ω; this is what we would expect since this situation corresponds to **Case 1** in (3.84). Furthermore, in this range $\widetilde{\omega} < 0$ and the transmission coefficient $\omega^{-1}\widetilde{\omega}\,\big|B^{\text{in}}_{\omega\ell}\big|^2$ is negative in accordance with the expression relating $A^{\text{in}}_{\omega\ell}$ and $B^{\text{in}}_{\omega\ell}$ in (3.78).

We find similar qualitative behaviour for other values of q and Q. Superradiant scattering of a charged scalar field arises in background Reissner-Nordström spacetimes as a result of the interaction between the charge of the black hole Q and the scalar field charge q. A similar effect, namely rotational superradiance, occurs in a background Kerr spacetime for a co-rotating field as a result of the interaction of the two angular momenta. Interestingly, the plot in Figure 3.2 suggests that the amplification of low-frequency modes is much greater than the corresponding case in Kerr spacetimes [35] (cf. Figure 16 in [91]).

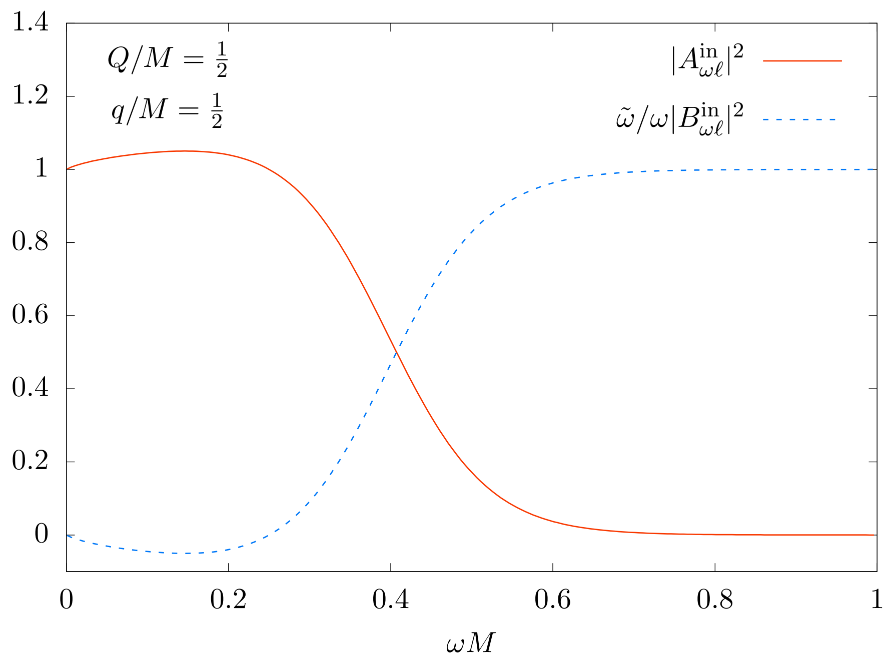

Figure 3.2. A plot of the in-mode reflection $\left|A^{\text{in}}_{\omega\ell}\right|^2$ and transmission coefficients $\omega^{-1}\widetilde{\omega}\left|B^{\text{in}}_{\omega\ell}\right|^2$ as a function of the frequency ω for $\ell = 0$ as well as black hole charge $Q = M/2$ and scalar field charge $q = M/2$. Superradiance occurs when the reflection coefficient is greater than unity. This plot, and all further plots in this thesis, were created by Dr. Rafael Bernar.

3.4 Normalisation constants and scalar field mode norms

3.4.1 Introduction

In §3.2.2, we wrote the expression for a generic mode solution (3.41) of the scalar field equation (3.36), which contains a normalisation constant N_ω that is yet to be determined. The exact form of N_ω depends on which of the modes introduced in §3.3 that we are considering. In this section, we derive the explicit expressions for the normalisation constants $\mathrm{N}^{\text{in}}_\omega$, $\mathrm{N}^{\text{up}}_\omega$, $\mathrm{N}^{\text{out}}_\omega$ and $\mathrm{N}^{\text{down}}_\omega$ associated to the in-, up-, out- and down-modes respectively.

We will do so by evaluating the Klein-Gordon inner product between any two similar mode solutions over a suitably chosen Cauchy surface Σ and demanding that the modes be orthonormal. In §2.1, we introduced the Klein-Gordon inner product of two real scalar field modes in a general curved spacetime; it differed from an inner product in the traditional sense since it is not positive-definite. In the case of the Klein-Gordon inner product of charged scalar field modes, the spacetime covariant derivatives will be generalised to gauge covariant derivatives; we do not expect this inner product to be positive-definite either.

Thus, evaluating the inner product between any two similar mode solutions will determine the conditions under which each of the in-, up-, out- and down-modes are of positive-norm or negative-norm. This, in turn, will have important consequences when we come to canonically quantising the field in §4.1, where the value of the norm of a particular mode solution affects the commutation relation between the creation and annihilation operators associated to that mode.

3.4.2 Klein-Gordon inner product

The Klein-Gordon inner product for any two mode solutions ϕ_1 and ϕ_2 of the scalar field equation (3.36) is given by

$$\langle \phi_1, \phi_2 \rangle = -\mathrm{i} \int_{\Sigma} \left[\phi_1^* \left(D_\mu \phi_2 \right) - \left(D_\mu \phi_1^* \right) \phi_2 \right] \sqrt{-g}\, \mathrm{d}\Sigma^\mu, \tag{3.85}$$

where $\mathrm{d}\Sigma^\mu$ is the surface element and the integral is performed over the 3D hypersurface Σ. Strictly speaking, our chosen Cauchy surface Σ must be spacelike by definition as well as spanning the entirety of the spacetime. However, it will suffice to consider a Cauchy surface that is arbitrarily close to a null surface, or the union of several such surfaces. Outside of these requirements, we are free to use any choice of Cauchy surface, particularly one that is conducive to simplifying the expressions obtained in (3.85); the resulting expression for the inner product of two scalar field modes will be independent of the choice of Σ since the integrand in (3.85) is a divergence-free vector which allows us to apply Gauss' theorem, the proof of which is outside the scope of this thesis.

The question then arises as to what is a suitable choice of Cauchy surface over which to evaluate the inner product of any two modes. The asymptotic forms of the in-modes (3.64) and up-modes (3.66) are both particularly simple near the past event horizon $\mathcal{H}^-$ as well as past null infinity $\mathscr{I}^-$. The union of the null surfaces $\mathcal{H}^-$ and $\mathscr{I}^-$ also spans the entirety of region I. We can extend the Cauchy surface throughout the Reissner-Nordström spacetime by including within it the union of the null surfaces $\mathcal{H}^+_{\mathrm{IV}}$ and $\mathscr{I}^+_{\mathrm{IV}}$ that span the entirety of region IV, in which the scalar field modes introduced in §3.3 vanish. Thus, a convenient choice of Cauchy surface on which to evaluate the in- and up-modes is the 'past' Cauchy surface Σ_{past}, which is given by

$$\Sigma_{\mathrm{past}} = \mathcal{H}^- \cup \mathscr{I}^- \cup \mathcal{H}^+_{\mathrm{IV}} \cup \mathscr{I}^+_{\mathrm{IV}}. \tag{3.86}$$

In contrast, the asymptotic forms of both the out-modes (3.68) and down-modes (3.70) are particularly simple near the future event horizon $\mathcal{H}^+$ as well as future null infinity $\mathscr{I}^+$. The union of the null surfaces $\mathcal{H}^+$ and $\mathscr{I}^+$ also spans the entirety of region I. We can extend the Cauchy surface throughout the entire Reissner-Nordström spacetime by including within it the union of the null surfaces $\mathcal{H}^-_{\mathrm{IV}}$ and $\mathscr{I}^-_{\mathrm{IV}}$ that span the entirety of region IV. Thus, a convenient choice of Cauchy surface on which to evaluate the out- and down-modes is the Cauchy surface Σ_{future}, which is given by

$$\Sigma_{\mathrm{future}} = \mathcal{H}^+ \cup \mathscr{I}^+ \cup \mathcal{H}^-_{\mathrm{IV}} \cup \mathscr{I}^-_{\mathrm{IV}}. \tag{3.87}$$

The past event horizon $\mathcal{H}^-$ and future null infinity $\mathscr{I}^+$ are both surfaces of constant Kruskal coordinate V. Similarly, the future event horizon $\mathcal{H}^+$ and past null infinity $\mathscr{I}^-$ are both surfaces of constant Kruskal coordinate U. Therefore, it will be convenient to evaluate the inner product (3.85) of each of the scalar field modes in terms of the Kruskal coordinates defined in (3.24); we will also require the lightcone coordinates (3.26) in order to re-express the asymptotic forms of each of the modes in terms of Kruskal coordinates.

The calculation differs slightly between the in- and out-modes, for which the derivation is simpler, and the up- and down-modes, for which the derivation is more subtle. This is due to the difference in the asymptotic forms of the scalar field effective potential (3.57) at the horizon and infinity; the exponents in the asymptotic forms of the radial functions $X^{\mathrm{up}}_{\omega\ell}$ and $X^{\mathrm{down}}_{\omega\ell}$ contain a factor of $\widetilde{\omega}$ that cannot be combined as easily with the factor of ω in the scalar field mode harmonic time dependence $\mathrm{e}^{-\mathrm{i}\omega t}$, as opposed to the case of the exponents in the asymptotic forms of $X^{\mathrm{in}}_{\omega\ell}$ and $X^{\mathrm{out}}_{\omega\ell}$ which also contain a factor of ω.

Therefore, we will begin by deriving an expression for the normalisation constant $\mathrm{N}^{\mathrm{in}}_{\omega}$ of the in-modes, before moving on to that of the normalisation constant $\mathrm{N}^{\mathrm{up}}_{\omega}$ of the up-modes. It turns out that the derivation of expressions for the normalisation constants $\mathrm{N}^{\mathrm{out}}_{\omega}$ of the out-modes and $\mathrm{N}^{\mathrm{down}}_{\omega}$ of the down-modes are very similar to that of $\mathrm{N}^{\mathrm{in}}_{\omega}$ and $\mathrm{N}^{\mathrm{up}}_{\omega}$ respectively. Therefore, we will treat the cases of $\mathrm{N}^{\mathrm{out}}_{\omega}$, in comparison with $\mathrm{N}^{\mathrm{in}}_{\omega}$, and $\mathrm{N}^{\mathrm{down}}_{\omega}$, in comparison with $\mathrm{N}^{\mathrm{up}}_{\omega}$, last.

In all cases, we shall derive the conditions upon which the particular mode under consideration is of positive-norm or negative-norm. As previously stated, this is of paramount importance when we come to canonically quantising the scalar field in §4.1. In all cases, we will use the following standard result from Fourier analysis:

$$\int_{x=-\infty}^{\infty} \mathrm{e}^{\mathrm{i}(\omega-\omega')x}\,\mathrm{d}x = 2\pi\,\delta\big(\omega-\omega'\big)\,. \tag{3.88}$$

3.4.3 Evaluating the inner product of the mode solutions

In-modes

In order to derive an expression for the normalisation constant $\mathrm{N}^{\mathrm{in}}_{\omega}$, as well as to determine the conditions under which the in-modes are of positive-norm or negative-norm, we can use the expression for the Klein-Gordon inner product (3.85) to evaluate the inner product of two in-modes (3.63) over the 'past' Cauchy surface Σ_{past} (3.86); we have

$$\begin{aligned}
&\langle\phi^{\mathrm{in}}_{\omega\ell m},\phi^{\mathrm{in}}_{\omega'\ell'm'}\rangle\\
&= \mathrm{i}\int_{\Sigma_{\mathrm{past}}}\big[\big(\partial_\mu\phi^{\mathrm{in}*}_{\omega\ell m}\big)\,\phi^{\mathrm{in}}_{\omega'\ell'm'} - \phi^{\mathrm{in}*}_{\omega\ell m}\,\partial_\mu\phi^{\mathrm{in}}_{\omega'\ell'm'} + 2\,\mathrm{i}qA_\mu\,\phi^{\mathrm{in}*}_{\omega\ell m}\phi^{\mathrm{in}}_{\omega'\ell'm'}\big]\sqrt{-g}\,\mathrm{d}\Sigma^{\mu}_{\mathrm{past}}\\
&= \mathrm{i}\int_{\mathscr{I}^-}\big[\big(\partial_\mu\phi^{\mathrm{in}*}_{\omega\ell m}\big)\,\phi^{\mathrm{in}}_{\omega'\ell'm'} - \phi^{\mathrm{in}*}_{\omega\ell m}\,\partial_\mu\phi^{\mathrm{in}}_{\omega'\ell'm'} + 2\,\mathrm{i}qA_\mu\,\phi^{\mathrm{in}*}_{\omega\ell m}\phi^{\mathrm{in}}_{\omega'\ell'm'}\big]\sqrt{-g}\,\mathrm{d}\Sigma^{\mu}_{\mathscr{I}^-}\,,
\end{aligned} \tag{3.89}$$

where we have used the fact that the flux of the in-modes through the $\mathcal{H}^-$ vanishes (see Appendix D), so the integral over the 'past' Cauchy surface Σ_{past} reduces to one over the past null infinity $\mathscr{I}^-$. Since $\mathscr{I}^-$ is a surface of constant Kruskal coordinate U, the normal vector n_μ to the surface which points in the direction of increasing U is given by

$$n_\mu = \mathrm{d}U = (1,0,0,0)\,. \tag{3.90}$$

Therefore, acting with the inverse metric in (3.31) on (3.90), we have

$$n^\mu = \big(0,-\zeta^{-1}(r)\,,0,0\big)\,, \tag{3.91}$$

where the function $\zeta(r)$ is defined in (3.30). Then, the volume element $\mathrm{d}\Sigma^{\mu}_{\mathscr{I}^-}$ in (3.89) is

$$\mathrm{d}\Sigma^{\mu}_{\mathscr{I}^-} = -\delta^{\mu}_{V}\, r^2 \sin\theta \,\mathrm{d}V \mathrm{d}\theta \,\mathrm{d}\varphi. \tag{3.92}$$

The factor of δ^{μ}_{V} in (3.92) means that the partial derivatives in (3.89) will only act on the in-mode $\phi^{\text{in}}_{\omega\ell m}$ and its complex conjugate with respect to the Kruskal coordinate V. Thus, it will useful to rewrite the asymptotic form of in-modes near $\mathscr{I}^-$ (3.64) in terms of the lightcone coordinates (3.20), which are readily differentiated w.r.t V from the expressions given in (3.24). Near $\mathscr{I}^-$, and using the fact that the flux of the second term in (3.64) through $\mathscr{I}^-$ vanishes through a calculation similar to that in Appendix D, we have

$$\begin{aligned}\phi^{\text{in}}_{\omega\ell m} &= \frac{1}{r}\,\mathrm{e}^{-\mathrm{i}\omega t}\,\mathrm{N}^{\text{in}}_{\omega}\,Y_{\ell m}(\theta,\varphi)\,\mathrm{e}^{\mathrm{i}\omega r_*}\\ &= \frac{1}{r}\,\mathrm{e}^{-\mathrm{i}\omega v}\,\mathrm{N}^{\text{in}}_{\omega}\,Y_{\ell m}(\theta,\varphi)\,. \end{aligned}\tag{3.93}$$

So the expression for the derivative of an in-mode near $\mathscr{I}^-$ is given by

$$\partial_V \phi^{\text{in}}_{\omega\ell m} = -\frac{1}{r^2}\,\mathrm{e}^{-\mathrm{i}\omega v}\,\mathrm{N}^{\text{in}}_{\omega}\,Y_{\ell m}(\theta,\varphi)\,\frac{\partial r}{\partial V} - \frac{\mathrm{i}\omega}{r}\,\mathrm{e}^{-\mathrm{i}\omega v}\,\mathrm{N}^{\text{in}}_{\omega}\,Y_{\ell m}(\theta,\varphi)\,\frac{\mathrm{d}v}{\mathrm{d}V}. \tag{3.94}$$

Since we are working near past null infinity $\mathscr{I}^-$, where $r \to \infty$, we can ignore sub-leading order contributions in the radial coordinate r in (3.94). Then, (3.94) reduces to

$$\partial_V \phi^{\text{in}}_{\omega\ell m} = -\frac{\mathrm{i}\omega}{r}\,\mathrm{e}^{-\mathrm{i}\omega v}\,\mathrm{N}^{\text{in}}_{\omega}\,Y_{\ell m}(\theta,\varphi)\,\frac{\mathrm{d}v}{\mathrm{d}V}. \tag{3.95}$$

In the expression for the inner product (3.89), we also require the V component of the gauge field A_μ. Using (3.11), and the usual procedure for changing coordinates, we have

$$A_V = \frac{\partial x^\mu}{\partial V} A_\mu = \frac{\partial t}{\partial V} A_t = \frac{\mathrm{d}v}{\mathrm{d}V}\frac{\partial t}{\partial v} A_t = -\frac{Q}{2r}\frac{\mathrm{d}v}{\mathrm{d}V}. \tag{3.96}$$

We also note that, since the partial derivative ∂_μ is real, we have $\partial_\mu \phi^{\text{in}*}_{\omega\ell m} = \left(\partial_\mu \phi^{\text{in}}_{\omega\ell m}\right)^*$. Then, substituting equations (3.92), (3.95) and (3.96) into (3.89), we obtain

$$\begin{aligned}\langle \phi^{\text{in}}_{\omega\ell m}, \phi^{\text{in}}_{\omega'\ell' m'}\rangle = \mathrm{i}\int_{V=0}^{\infty}\int_{\theta=0}^{\pi}\int_{\varphi=0}^{2\pi} \Bigg[&\frac{\mathrm{i}\omega}{r}\,\mathrm{e}^{\mathrm{i}\omega v}\,\mathrm{N}^{\text{in}*}_{\omega}\,Y^*_{\ell m}(\theta,\varphi)\,\frac{\mathrm{d}v}{\mathrm{d}V} \times \frac{1}{r}\,\mathrm{e}^{-\mathrm{i}\omega' v}\,\mathrm{N}^{\text{in}}_{\omega'}\,Y_{\ell' m'}(\theta,\varphi)\\ &-\frac{1}{r}\,\mathrm{e}^{\mathrm{i}\omega v}\,\mathrm{N}^{\text{in}*}_{\omega}\,Y^*_{\ell m}(\theta,\varphi) \times \frac{(-\mathrm{i}\omega')}{r}\,\mathrm{e}^{-\mathrm{i}\omega' v}\,\mathrm{N}^{\text{in}}_{\omega'}\,Y_{\ell' m'}(\theta,\varphi)\,\frac{\mathrm{d}v}{\mathrm{d}V}\\ &-\frac{\mathrm{i}qQ}{r}\,\mathrm{e}^{\mathrm{i}\omega v}\,\mathrm{N}^{\text{in}*}_{\omega}\,Y^*_{\ell m}(\theta,\varphi)\,\frac{\mathrm{d}v}{\mathrm{d}V} \times \frac{1}{r}\,\mathrm{e}^{-\mathrm{i}\omega' v}\,\mathrm{N}^{\text{in}}_{\omega'}\,Y_{\ell' m'}(\theta,\varphi)\Bigg]\\ &\times r^2 \sin\theta\,\mathrm{d}V\mathrm{d}\theta\,\mathrm{d}\varphi. \end{aligned}\tag{3.97}$$

Simplifying and rearranging (3.97), we have

$$\begin{aligned}\langle \phi^{\text{in}}_{\omega\ell m}, \phi^{\text{in}}_{\omega'\ell' m'}\rangle = \mathrm{N}^{\text{in}*}_{\omega}\,\mathrm{N}^{\text{in}}_{\omega'}\int_{V=0}^{\infty}\int_{\theta=0}^{\pi}\int_{\varphi=0}^{2\pi} &Y^*_{\ell m}(\theta,\varphi)\,Y_{\ell' m'}(\theta,\varphi)\,\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\varphi\\ &\times\left(\omega+\omega'-\frac{qQ}{r}\right)\mathrm{e}^{\mathrm{i}(\omega-\omega')v}\,\frac{\mathrm{d}v}{\mathrm{d}V}\,\mathrm{d}V. \end{aligned}\tag{3.98}$$

The integrals with respect to θ and φ are performed using (3.48), so (3.98) reduces to

$$\langle \phi^{\rm in}_{\omega\ell m}, \phi^{\rm in}_{\omega'\ell'm'} \rangle = \delta_{\ell\ell'}\,\delta_{mm'}\,{\rm N}^{\rm in*}_{\omega}\,{\rm N}^{\rm in}_{\omega'} \int_{V=0}^{\infty} \left(\omega + \omega' - \frac{qQ}{r}\right) {\rm e}^{{\rm i}(\omega-\omega')v}\,\frac{{\rm d}v}{{\rm d}V}\,{\rm d}V. \tag{3.99}$$

Since $r \to \infty$ near $\mathscr{I}^-$, we can ignore the sub-leading order $\frac{qQ}{r}$ term in (3.99); then

$$\langle \phi^{\rm in}_{\omega\ell m}, \phi^{\rm in}_{\omega'\ell'm'} \rangle = \left(\omega + \omega'\right)\delta_{\ell\ell'}\,\delta_{mm'}\,{\rm N}^{\rm in*}_{\omega}\,{\rm N}^{\rm in}_{\omega'} \int_{v=-\infty}^{\infty} {\rm e}^{{\rm i}(\omega-\omega')v}\,{\rm d}v, \tag{3.100}$$

where we have changed the limits of the integral in (3.100) to reflect that we are now integrating with respect to v instead of V. The integral with respect to the lightcone coordinate v can be performed using the identity (3.88) and (3.100) becomes

$$\begin{aligned}\langle \phi^{\rm in}_{\omega\ell m}, \phi^{\rm in}_{\omega'\ell'm'} \rangle &= 2\pi\left(\omega + \omega'\right)\delta\left(\omega - \omega'\right)\delta_{\ell\ell'}\,\delta_{mm'}\,{\rm N}^{\rm in*}_{\omega}\,{\rm N}^{\rm in}_{\omega'} \\ &= 4\pi\omega\,\delta\left(\omega - \omega'\right)\delta_{\ell\ell'}\,\delta_{mm'}\left|{\rm N}^{\rm in}_{\omega}\right|^2. \end{aligned} \tag{3.101}$$

Then, we obtain for the inner product of two in-modes with the same angular momentum ℓ and azimuthal m quantum numbers, the expression

$$\langle \phi^{\rm in}_{\omega\ell m}, \phi^{\rm in}_{\omega'\ell m} \rangle = 4\pi\omega\,\delta\left(\omega - \omega'\right)\left|{\rm N}^{\rm in}_{\omega}\right|^2. \tag{3.102}$$

Requiring the orthonormality of the inner product in (3.102) gives us the expression for the normalisation constant ${\rm N}^{\rm in}_{\omega}$, which is given by

$${\rm N}^{\rm in}_{\omega} = \frac{1}{\sqrt{4\pi|\omega|}}. \tag{3.103}$$

Then we can write the expression for the inner product of two generic in-modes as

$$\langle \phi^{\rm in}_{\omega\ell m}, \phi^{\rm in}_{\omega'\ell'm'} \rangle = {\rm sgn}\left(\omega\right)\delta\left(\omega - \omega'\right)\delta_{\ell\ell'}\,\delta_{mm'}. \tag{3.104}$$

The norm of two in-modes (3.104) is dependent upon the sign of ω, as is the case for the Schwarzschild in-modes. We summarise the value of the inner product of two in-modes as

$$\langle \phi^{\rm in}_{\omega\ell m}, \phi^{\rm in}_{\omega'\ell'm'} \rangle = \begin{cases} \delta(\omega - \omega')\,\delta_{\ell\ell'}\,\delta_{mm'}, & \text{for } \omega > 0, \\ -\delta(\omega - \omega')\,\delta_{\ell\ell'}\,\delta_{mm'}, & \text{for } \omega < 0, \end{cases} \tag{3.105}$$

meaning that in-modes are of positive-norm when $\omega > 0$ and of negative-norm when $\omega < 0$. In concluding, we can use the expression for ${\rm N}^{\rm in}_{\omega}$ (3.103) to write the in-modes as

$$\phi^{\rm in}_{\omega\ell m} = \frac{1}{\sqrt{4\pi|\omega|}}\,{\rm e}^{-{\rm i}\omega t}\,\frac{X^{\rm in}_{\omega\ell}(r)}{r}\,Y_{\ell m}(\theta,\varphi)\,, \tag{3.106}$$

where the asymptotic forms of the radial function $X^{\rm in}_{\omega\ell}(r)$ are given in (3.64).

Up-modes

In order to derive an expression for the normalisation constant $\mathrm{N}_{\omega}^{\mathrm{up}}$, as well as to determine the conditions under which the up-modes are of positive-norm or negative-norm, we can use the expression for the Klein-Gordon inner product (3.85) to evaluate the inner product of two up-modes (3.65) over the 'past' Cauchy surface Σ_{past} (3.86); we have

$$\begin{aligned}
&\langle \phi^{\mathrm{up}}_{\omega\ell m}, \phi^{\mathrm{up}}_{\omega'\ell'm'} \rangle \\
&= \mathrm{i} \int_{\Sigma_{\mathrm{past}}} \left[\left(\partial_\mu \phi^{\mathrm{up}*}_{\omega\ell m}\right) \phi^{\mathrm{up}}_{\omega'\ell'm'} - \phi^{\mathrm{up}*}_{\omega\ell m}\, \partial_\mu \phi^{\mathrm{up}}_{\omega'\ell'm'} + 2\,\mathrm{i}qA_\mu\, \phi^{\mathrm{up}*}_{\omega\ell m} \phi^{\mathrm{up}}_{\omega'\ell'm'} \right] \sqrt{-g}\, \mathrm{d}\Sigma^{\mu}_{\mathrm{past}} \\
&= \mathrm{i} \int_{\mathcal{H}^-} \left[\left(\partial_\mu \phi^{\mathrm{up}*}_{\omega\ell m}\right) \phi^{\mathrm{up}}_{\omega'\ell'm'} - \phi^{\mathrm{up}*}_{\omega\ell m}\, \partial_\mu \phi^{\mathrm{up}}_{\omega'\ell'm'} + 2\,\mathrm{i}qA_\mu\, \phi^{\mathrm{up}*}_{\omega\ell m} \phi^{\mathrm{up}}_{\omega'\ell'm'} \right] \sqrt{-g}\, \mathrm{d}\Sigma^{\mu}_{\mathcal{H}^-}, \qquad (3.107)
\end{aligned}$$

where we have used the fact that the flux of the up-modes through past null infinity $\mathscr{I}^-$ vanishes, so the integral over the 'past' Cauchy surface Σ_{past} reduces to one over the past horizon $\mathcal{H}^-$. Since $\mathcal{H}^-$ is a surface of constant Kruskal coordinate V, the normal vector n_μ to the surface which points in the direction of increasing V is given by

$$n_\mu = \mathrm{d}V = (0, 1, 0, 0)\,. \qquad (3.108)$$

Therefore, acting with the inverse metric (3.31) on (3.108), we have

$$n^\mu = \left(-\zeta^{-1}(r)\,, 0, 0, 0\right), \qquad (3.109)$$

where the function $\zeta(r)$ is defined in (3.30). Then, the volume element $\mathrm{d}\Sigma^{\mu}_{\mathcal{H}^-}$ in (3.107) is

$$\mathrm{d}\Sigma^{\mu}_{\mathcal{H}^-} = -\delta^{\mu}_{U}\, r^2 \sin\theta\, \mathrm{d}U \mathrm{d}\theta\, \mathrm{d}\varphi. \qquad (3.110)$$

The factor of δ^{μ}_{U} in (3.110) means that the partial derivatives in (3.107) will only act on the up-mode $\phi^{\mathrm{up}}_{\omega\ell m}$ and its complex conjugate with respect to the Kruskal coordinate U. Thus, it will useful to rewrite the asymptotic form of up-modes near $\mathcal{H}^-$ (3.66) in terms of the lightcone coordinates (3.20), which are readily differentiated with respect to U from the expressions given in (3.24). Near $\mathcal{H}^-$, and using the fact that the flux of the second term in (3.66) vanishes through a calculation similar to that in Appendix D, we have

$$\begin{aligned}
\phi^{\mathrm{up}}_{\omega\ell m} &= \frac{1}{r} \exp[-\mathrm{i}\omega t\,]\, \mathrm{N}^{\mathrm{up}}_{\omega}\, Y_{\ell m}(\theta,\varphi) \exp[\,\mathrm{i}\widetilde{\omega} r_*] \\
&= \frac{1}{r} \exp\left[-\mathrm{i}\omega\, \frac{(u+v)}{2}\right] \mathrm{N}^{\mathrm{up}}_{\omega}\, Y_{\ell m}(\theta,\varphi) \exp\left[\mathrm{i}\widetilde{\omega}\, \frac{(v-u)}{2}\right]. \qquad (3.111)
\end{aligned}$$

So the expression for the derivative of an up-mode near the past horizon $\mathcal{H}^-$ is given by

$$\begin{aligned}
\partial_U \phi^{\mathrm{up}}_{\omega\ell m} = &-\frac{1}{r^2} \exp\left[-\mathrm{i}\omega\, \frac{(u+v)}{2}\right] \mathrm{N}^{\mathrm{up}}_{\omega}\, Y_{\ell m}(\theta,\varphi) \exp\left[\mathrm{i}\widetilde{\omega}\, \frac{(v-u)}{2}\right] \frac{\partial r}{\partial U} \\
&- \frac{\mathrm{i}\,(\omega+\widetilde{\omega})}{2r} \exp\left[-\mathrm{i}\omega\, \frac{(u+v)}{2}\right] \mathrm{N}^{\mathrm{up}}_{\omega}\, Y_{\ell m}(\theta,\varphi) \exp\left[\mathrm{i}\widetilde{\omega}\, \frac{(v-u)}{2}\right] \frac{\mathrm{d}u}{\mathrm{d}U}. \qquad (3.112)
\end{aligned}$$

From the chain rule, and using (3.18) and (3.20), we have

$$\frac{\partial r}{\partial U} = \frac{\partial r}{\partial u}\frac{\mathrm{d}u}{\mathrm{d}U} = \frac{\mathrm{d}r}{\mathrm{d}r_*}\frac{\partial r_*}{\partial u}\frac{\mathrm{d}u}{\mathrm{d}U} = -\frac{f(r)}{2}\frac{\mathrm{d}u}{\mathrm{d}U}. \tag{3.113}$$

Then, substituting (3.113) into (3.112), we have

$$\partial_U \phi^{\mathrm{up}}_{\omega\ell m} = \left[\frac{f(r)}{2r^2} - \frac{\mathrm{i}\,(\omega+\widetilde{\omega})}{2r}\right]\exp\left[-\mathrm{i}\omega\,\frac{(u+v)}{2}\right]\mathrm{N}^{\mathrm{up}}_{\omega}\,Y_{\ell m}(\theta,\varphi)\exp\left[\mathrm{i}\widetilde{\omega}\,\frac{(v-u)}{2}\right]\frac{\mathrm{d}u}{\mathrm{d}U}. \tag{3.114}$$

Near the horizon, the metric function $f(r)$ vanishes from (3.15). Then, we can neglect the term proportional to $f(r)$ in (3.114) giving for the derivative of the up-modes near $\mathcal{H}^-$

$$\partial_U \phi^{\mathrm{up}}_{\omega\ell m} = -\frac{\mathrm{i}\,(\omega+\widetilde{\omega})}{2r}\exp\left[-\mathrm{i}\omega\,\frac{(u+v)}{2}\right]\mathrm{N}^{\mathrm{up}}_{\omega}\,Y_{\ell m}(\theta,\varphi)\exp\left[\mathrm{i}\widetilde{\omega}\,\frac{(v-u)}{2}\right]\frac{\mathrm{d}u}{\mathrm{d}U}. \tag{3.115}$$

In the expression for the inner product (3.107), we also require the U component of the gauge field A_μ. Using (3.11), and the usual procedure for changing coordinates, we have

$$A_U = \frac{\partial x^\mu}{\partial U}A_\mu = \frac{\partial t}{\partial U}A_t = \frac{\partial t}{\partial u}\frac{\mathrm{d}u}{\mathrm{d}U}A_t = -\frac{Q}{2r}\frac{\mathrm{d}u}{\mathrm{d}U}. \tag{3.116}$$

We note that, since the partial derivative ∂_μ is real, we have $\partial_\mu \phi^{\mathrm{up}*}_{\omega\ell m} = \left(\partial_\mu \phi^{\mathrm{up}}_{\omega\ell m}\right)^*$. Then, substituting equations (3.110), (3.115) and (3.116) into (3.107) and simplifying, we have

$$\begin{aligned}\langle \phi^{\mathrm{up}}_{\omega\ell m}, \phi^{\mathrm{up}}_{\omega'\ell' m'}\rangle \\ = -\mathrm{i}\int_{U=-\infty}^{0}\int_{\theta=0}^{\pi}\int_{\varphi=0}^{2\pi}\frac{\mathrm{i}}{r^2}\left[\frac{(\omega+\widetilde{\omega})}{2} + \frac{(\omega'+\widetilde{\omega}')}{2} - \frac{qQ}{r}\right]\exp\left[\mathrm{i}\,(\omega-\omega')\,\frac{(u+v)}{2}\right] \\ \times \mathrm{N}^{\mathrm{up}*}_{\omega}\,\mathrm{N}^{\mathrm{up}}_{\omega'}\,Y^*_{\ell m}(\theta,\varphi)\,Y_{\ell' m'}(\theta,\varphi)\exp\left[-\mathrm{i}\,(\widetilde{\omega}-\widetilde{\omega}')\,\frac{(v-u)}{2}\right]\frac{\mathrm{d}u}{\mathrm{d}U}\,r^2\sin\theta\,\mathrm{d}U\mathrm{d}\theta\,\mathrm{d}\varphi.\end{aligned} \tag{3.117}$$

Simplifying and rearranging (3.117), we have

$$\begin{aligned}\langle \phi^{\mathrm{up}}_{\omega\ell m}, \phi^{\mathrm{up}}_{\omega'\ell' m'}\rangle \\ = \mathrm{N}^{\mathrm{up}*}_{\omega}\,\mathrm{N}^{\mathrm{up}}_{\omega'}\int_{U=-\infty}^{0}\int_{\theta=0}^{\pi}\int_{\varphi=0}^{2\pi} Y^*_{\ell m}(\theta,\varphi)\,Y_{\ell' m'}(\theta,\varphi)\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\varphi \\ \times\left[\frac{(\omega+\widetilde{\omega}+\omega'+\widetilde{\omega}')}{2} - \frac{qQ}{r}\right]\exp\left[\mathrm{i}\,(\omega-\omega')\,\frac{(u+v)}{2}\right]\exp\left[-\mathrm{i}\,(\widetilde{\omega}-\widetilde{\omega}')\,\frac{(v-u)}{2}\right]\frac{\mathrm{d}u}{\mathrm{d}U}\,\mathrm{d}U.\end{aligned} \tag{3.118}$$

The integrals with respect to θ and φ are performed using (3.48), so (3.118) reduces to

$$\begin{aligned}\langle \phi^{\mathrm{up}}_{\omega\ell m}, \phi^{\mathrm{up}}_{\omega'\ell' m'}\rangle = \delta_{\ell\ell'}\,\delta_{mm'}\,\mathrm{N}^{\mathrm{up}*}_{\omega}\,\mathrm{N}^{\mathrm{up}}_{\omega'}\int_{U=-\infty}^{0}\frac{1}{2}\left(\omega+\widetilde{\omega}+\omega'+\widetilde{\omega}' - \frac{2qQ}{r}\right) \\ \times\exp\left[\mathrm{i}\,(\omega-\omega')\,\frac{(u+v)}{2}\right]\exp\left[-\mathrm{i}\,(\widetilde{\omega}-\widetilde{\omega}')\,\frac{(v-u)}{2}\right]\frac{\mathrm{d}u}{\mathrm{d}U}\,\mathrm{d}U.\end{aligned} \tag{3.119}$$

Since $r \to r_+$ near the past horizon $\mathcal{H}^-$, we can write the term $\frac{qQ}{r}$ in (3.119) as $\frac{qQ}{r_+}$. Then, having also simplified the exponential terms, (3.119) becomes

$$\begin{aligned}\langle \phi^{\mathrm{up}}_{\omega\ell m}, \phi^{\mathrm{up}}_{\omega'\ell' m'} \rangle = \delta_{\ell\ell'}\, \delta_{mm'}\, \mathrm{N}^{\mathrm{up}*}_{\omega}\, \mathrm{N}^{\mathrm{up}}_{\omega'} \int_{U=-\infty}^{0} \frac{1}{2}\left(\omega + \widetilde{\omega} + \omega' + \widetilde{\omega}' - \frac{2qQ}{r_+}\right) \\ \times \exp\left[\mathrm{i}\left(\omega - \omega' + \widetilde{\omega} - \widetilde{\omega}'\right)\frac{u}{2}\right] \exp\left[\mathrm{i}\left(\omega - \omega' - \widetilde{\omega} + \widetilde{\omega}'\right)\frac{v}{2}\right] \frac{\mathrm{d}u}{\mathrm{d}U}\, \mathrm{d}U. \quad (3.120)\end{aligned}$$

In order to simplify (3.120) further, we need to simplify the quantities in the brackets. We can use the relationship between ω and $\widetilde{\omega}$ (3.58) in order to do this. Beginning with the bracket in (3.120) that is not being exponentiated, we can simplify this as

$$\begin{aligned}\frac{1}{2}\left(\omega + \widetilde{\omega} + \omega' + \widetilde{\omega}' - \frac{2qQ}{r_+}\right) &= \frac{1}{2}\left(\omega + \omega - \frac{qQ}{r_+} + \omega' + \omega' - \frac{qQ}{r_+} - \frac{2qQ}{r_+}\right) \\ &= \omega + \omega' - \frac{2qQ}{r_+}. \quad (3.121)\end{aligned}$$

The exponentiated bracket in (3.120) that is being multiplied by u can be simplified as

$$\omega - \omega' + \widetilde{\omega} - \widetilde{\omega}' = \omega - \omega' + \omega - \frac{qQ}{r_+} - \omega' + \frac{qQ}{r_+} = 2\left(\omega - \omega'\right). \quad (3.122)$$

The exponentiated bracket in (3.120) that is being multiplied by v can be simplified as

$$\omega - \omega' - \widetilde{\omega} + \widetilde{\omega}' = \omega - \omega' - \omega + \frac{qQ}{r_+} + \omega' - \frac{qQ}{r_+} = 0. \quad (3.123)$$

Then, using (3.121), (3.122) and (3.123), equation (3.120) reduces considerably to

$$\langle \phi^{\mathrm{up}}_{\omega\ell m}, \phi^{\mathrm{up}}_{\omega'\ell' m'} \rangle = \delta_{\ell\ell'}\, \delta_{mm'}\, \mathrm{N}^{\mathrm{up}*}_{\omega}\, \mathrm{N}^{\mathrm{up}}_{\omega'} \int_{u=-\infty}^{\infty} \left(\omega + \omega' - \frac{2qQ}{r_+}\right) \exp\left[\mathrm{i}\left(\omega - \omega'\right) u\right] \mathrm{d}u, \quad (3.124)$$

where we have changed the limits of the integral in (3.124) to reflect that we are now integrating with respect to u instead of U. The integral with respect to the lightcone coordinate u can be performed using the identity in (3.88) and (3.124) becomes

$$\langle \phi^{\mathrm{up}}_{\omega\ell m}, \phi^{\mathrm{up}}_{\omega'\ell' m'} \rangle = 2\pi\left(\omega + \omega' - \frac{2qQ}{r_+}\right) \delta\left(\omega - \omega'\right) \delta_{\ell\ell'}\, \delta_{mm'}\, \mathrm{N}^{\mathrm{up}*}_{\omega}\, \mathrm{N}^{\mathrm{up}}_{\omega'}. \quad (3.125)$$

From (3.58), both $\widetilde{\omega}$ and $\widetilde{\omega}'$ in (3.125) are offset from ω and ω' by the same constant amount of $\frac{qQ}{r_+}$ respectively. We can use this to simplify (3.125) as

$$\begin{aligned}\langle \phi^{\mathrm{up}}_{\omega\ell m}, \phi^{\mathrm{up}}_{\omega'\ell' m'} \rangle &= 2\pi\left(\widetilde{\omega} + \widetilde{\omega}'\right) \delta\left(\omega - \omega'\right) \delta_{\ell\ell'}\, \delta_{mm'}\, \mathrm{N}^{\mathrm{up}*}_{\omega}\, \mathrm{N}^{\mathrm{up}}_{\omega'} \\ &= 4\pi\widetilde{\omega}\, \delta\left(\omega - \omega'\right) \delta_{\ell\ell'}\, \delta_{mm'}\, |\mathrm{N}^{\mathrm{up}}_{\omega}|^2. \quad (3.126)\end{aligned}$$

Then, we obtain for the inner product of two up-modes with the same angular momentum ℓ and azimuthal m quantum numbers, the expression

$$\langle \phi^{\mathrm{up}}_{\omega\ell m}, \phi^{\mathrm{up}}_{\omega'\ell m} \rangle = 4\pi\widetilde{\omega}\, \delta\left(\omega - \omega'\right) |\mathrm{N}^{\mathrm{up}}_{\omega}|^2. \quad (3.127)$$

Requiring the orthonormality of the inner product in (3.127) gives us the expression for the normalisation constant $\mathrm{N}_{\omega}^{\mathrm{up}}$, which is given by

$$\mathrm{N}_{\omega}^{\mathrm{up}} = \frac{1}{\sqrt{4\pi|\widetilde{\omega}|}}. \tag{3.128}$$

Then we can write the expression for the inner product of two generic up-modes as

$$\langle \phi^{\mathrm{up}}_{\omega\ell m}, \phi^{\mathrm{up}}_{\omega'\ell' m'} \rangle = \mathrm{sgn}\,(\widetilde{\omega})\,\delta\big(\omega-\omega'\big)\,\delta_{\ell\ell'}\,\delta_{mm'}, \tag{3.129}$$

where we see that the norm of two up-modes is dependent upon the sign of $\widetilde{\omega}$, in contrast to the case of the Schwarzschild up-modes. This is a crucial point that will lead to many subtleties when we come to canonically quantising the scalar field Φ and defining quantum states in Chapter 4. We can summarise the value of the norm of two up-modes as

$$\langle \phi^{\mathrm{up}}_{\omega\ell m}, \phi^{\mathrm{up}}_{\omega'\ell' m'} \rangle = \begin{cases} \delta(\omega-\omega')\,\delta_{\ell\ell'}\,\delta_{mm'}, & \text{for } \widetilde{\omega} > 0, \\ -\delta(\omega-\omega')\,\delta_{\ell\ell'}\,\delta_{mm'}, & \text{for } \widetilde{\omega} < 0, \end{cases} \tag{3.130}$$

meaning that up-modes are of positive-norm when $\widetilde{\omega} > 0$ and of negative-norm when $\widetilde{\omega} < 0$. In concluding, we can use the expression for $\mathrm{N}_{\omega}^{\mathrm{up}}$ (3.128) to write the up-modes as

$$\phi^{\mathrm{up}}_{\omega\ell m} = \frac{1}{\sqrt{4\pi|\widetilde{\omega}|}}\,\mathrm{e}^{-\mathrm{i}\omega t}\,\frac{X^{\mathrm{up}}_{\omega\ell}(r)}{r}\,Y_{\ell m}(\theta,\varphi)\,, \tag{3.131}$$

where the asymptotic forms of the radial function $X^{\mathrm{up}}_{\omega\ell}(r)$ are given in (3.66).

Out-modes

We will now evaluate the inner product of two out-modes. In doing so, we need not go through all of the details of the calculation since many of these will be analogous to the case of the calculation of the norm of two in-modes in §3.4.3. It will suffice to give the general outline and consider the differences between each calculation in order to derive an expression for the normalisation constant $\mathrm{N}_{\omega}^{\mathrm{out}}$ and determine the conditions under which the out-modes have positive-norm and negative-norm.

Using the expression for the Klein-Gordon inner product in (3.85), the inner product of two out-modes over the 'future' Cauchy surface Σ_{future} (3.87) is given by

$$\begin{aligned}&\langle \phi^{\mathrm{out}}_{\omega\ell m}, \phi^{\mathrm{out}}_{\omega'\ell' m'} \rangle \\ &= \mathrm{i}\int_{\mathscr{I}^+} \left[\left(\partial_\mu \phi^{\mathrm{out}*}_{\omega\ell m}\right)\phi^{\mathrm{out}}_{\omega'\ell' m'} - \phi^{\mathrm{out}*}_{\omega\ell m}\,\partial_\mu \phi^{\mathrm{out}}_{\omega'\ell' m'} + 2\,\mathrm{i}qA_\mu\,\phi^{\mathrm{out}*}_{\omega\ell m}\phi^{\mathrm{out}}_{\omega'\ell' m'}\right]\sqrt{-g}\,\mathrm{d}\Sigma^{\mu}_{\mathscr{I}^+},\end{aligned} \tag{3.132}$$

where we have used the fact that the flux of the out-modes through the future event horizon $\mathcal{H}^+$ vanishes, so the integral over the 'future' Cauchy surface Σ_{future} reduces to one over future null infinity $\mathscr{I}^+$. Since $\mathscr{I}^+$ is a surface of constant Kruskal coordinate V, the normal vector n_μ to the surface which points in the direction of decreasing V is

$$n_\mu = -\mathrm{d}V = (0,-1,0,0) \qquad \Rightarrow \qquad n^\mu = \left(\zeta^{-1}(r)\,,0,0,0\right), \tag{3.133}$$

where we have used the inverse metric (3.31). Then, the volume element $\mathrm{d}\Sigma^{\mu}_{\mathscr{I}^+}$ becomes

$$\mathrm{d}\Sigma^{\mu}_{\mathscr{I}^+} = \delta^{\mu}_{U}\, r^2 \sin\theta\, \mathrm{d}U \mathrm{d}\theta\, \mathrm{d}\varphi. \tag{3.134}$$

The factor of δ^{μ}_{U} in (3.134) means that the partial derivatives in (3.132) will only act on the out-mode $\phi^{\text{out}}_{\omega\ell m}$ and its complex conjugate with respect to the Kruskal coordinate U. Thus, it will useful to rewrite the asymptotic form of out-modes near $\mathscr{I}^+$ (3.68) in terms of the lightcone coordinates (3.20), which are readily differentiated with respect to U from the expressions given in (3.24). Near $\mathscr{I}^+$, as well as using (3.20) and (3.68), we have

$$\begin{aligned}\phi^{\text{out}}_{\omega\ell m} &= \frac{1}{r}\, \mathrm{e}^{-\mathrm{i}\omega t}\, \mathrm{N}^{\text{out}}_{\omega}\, Y_{\ell m}(\theta,\varphi)\, \mathrm{e}^{-\mathrm{i}\omega r_*} \\ &= \frac{1}{r}\, \mathrm{e}^{-\mathrm{i}\omega u}\, \mathrm{N}^{\text{out}}_{\omega}\, Y_{\ell m}(\theta,\varphi)\,. \end{aligned} \tag{3.135}$$

Comparing the expressions for the out-modes (3.135) and the volume element $\mathrm{d}\Sigma^{\mu}_{\mathscr{I}^+}$ (3.134) with that of the in-modes (3.93) and the volume element $\mathrm{d}\Sigma^{\mu}_{\mathscr{I}^-}$ (3.92), we see that the inner product of two out-modes (3.132) is very similar to the inner product of two in-modes (3.89); u and v are effectively dummy variables in (3.132) and (3.89) respectively, while $\phi^{\text{out}}_{\omega\ell m}$ has the same dependence on U, which it is being differentiated with respect to, as $\phi^{\text{in}}_{\omega\ell m}$ does on V. Furthermore, from (3.96) and (3.116), $A_U \frac{\mathrm{d}U}{\mathrm{d}u} = A_V \frac{\mathrm{d}V}{\mathrm{d}v}$ and we are similarly able to ignore sub-leading order contributions in r since $r \to \infty$ near $\mathscr{I}^+$.

However, there are two differences in the inner product of two out-modes as compared to that of two in-modes. Firstly, the volume element $\mathrm{d}\Sigma^{\mu}_{\mathscr{I}^+}$ (3.134) contains a minus sign relative to $\mathrm{d}\Sigma^{\mu}_{\mathscr{I}^-}$ (3.92), which induces a minus sign in $\langle \phi^{\text{out}}_{\omega\ell m}, \phi^{\text{out}}_{\omega'\ell'm'} \rangle$ relative to $\langle \phi^{\text{in}}_{\omega\ell m}, \phi^{\text{in}}_{\omega'\ell'm'} \rangle$. Secondly, where the in-modes are propagating away the surface of integration, $\mathscr{I}^-$, the out-modes are instead propagating towards the surface of integration $\mathscr{I}^+$; the corollary is that we integrate the out-modes from $u = \infty$ to $-\infty$ instead of $v = -\infty$ to ∞ as in the case of the in-modes (3.100), which induces a second minus sign in $\langle \phi^{\text{out}}_{\omega\ell m}, \phi^{\text{out}}_{\omega'\ell'm'} \rangle$ relative to $\langle \phi^{\text{in}}_{\omega\ell m}, \phi^{\text{in}}_{\omega'\ell'm'} \rangle$. These two minus signs cancel such that the two inner products are equal. Then the normalisation constant $\mathrm{N}^{\text{out}}_{\omega}$ is given by

$$\mathrm{N}^{\text{out}}_{\omega} = \frac{1}{\sqrt{4\pi|\omega|}}, \tag{3.136}$$

and the expression for the inner product of two generic out-modes is given by

$$\langle \phi^{\text{out}}_{\omega\ell m}, \phi^{\text{out}}_{\omega'\ell'm'} \rangle = \operatorname{sgn}(\omega)\, \delta\big(\omega - \omega'\big)\, \delta_{\ell\ell'}\, \delta_{mm'}. \tag{3.137}$$

The norm of two out-modes (3.137) is dependent upon the sign of ω, as is the case for the Schwarzschild in-modes. We summarise the value of the norm of two out-modes as

$$\langle \phi^{\text{out}}_{\omega\ell m}, \phi^{\text{out}}_{\omega'\ell'm'} \rangle = \begin{cases} \delta(\omega - \omega')\, \delta_{\ell\ell'}\, \delta_{mm'}, & \text{for } \omega > 0, \\ -\delta(\omega - \omega')\, \delta_{\ell\ell'}\, \delta_{mm'}, & \text{for } \omega < 0. \end{cases} \tag{3.138}$$

meaning that out-modes are of positive-norm when $\omega > 0$ and negative-norm when $\omega < 0$. In concluding, we can use the expression for $\mathrm{N}^{\text{out}}_{\omega}$ (3.136) to write the out-modes as

$$\phi^{\text{out}}_{\omega\ell m} = \frac{1}{\sqrt{4\pi|\omega|}}\, \mathrm{e}^{-\mathrm{i}\omega t}\, \frac{X^{\text{out}}_{\omega\ell}(r)}{r}\, Y_{\ell m}(\theta,\varphi)\,, \tag{3.139}$$

where the asymptotic forms of the radial function $X^{\text{out}}_{\omega\ell}(r)$ are given in (3.68).

Down-modes

We will now evaluate the inner product of two down-modes. As was the case for the out-modes in relation to the in-modes, the calculation of the norm of two down-modes is very similar to that of two up-modes in §3.4.3. Therefore, it will suffice to give the general outline and consider the differences between each calculation in order to derive an expression for the normalisation constant $\mathrm{N}^{\text{down}}_{\omega}$ and determine the conditions under which the down-modes have positive-norm and negative-norm.

Using the expression for the Klein-Gordon inner product in (3.85), the inner product of two down-modes over the 'future' Cauchy surface Σ_{future} (3.87) is given by

$$\begin{aligned}&\langle \phi^{\text{down}}_{\omega\ell m}, \phi^{\text{down}}_{\omega'\ell'm'}\rangle \\ &= \mathrm{i}\int_{\mathcal{H}^+}\left[\left(\partial_\mu \phi^{\text{down}*}_{\omega\ell m}\right)\phi^{\text{down}}_{\omega'\ell'm'} - \phi^{\text{down}*}_{\omega\ell m}\,\partial_\mu\phi^{\text{down}}_{\omega'\ell'm'} + 2\,\mathrm{i}qA_\mu\,\phi^{\text{down}*}_{\omega\ell m}\phi^{\text{down}}_{\omega'\ell'm'}\right]\sqrt{-g}\,\mathrm{d}\Sigma^{\mu}_{\mathcal{H}^+}.\end{aligned} \tag{3.140}$$

where we have used the fact that the flux of the down-modes through future null infinity $\mathscr{I}^+$ vanishes, so the integral over the 'future' Cauchy surface Σ_{future} reduces to one over the future event horizon $\mathcal{H}^+$. Since $\mathcal{H}^+$ is a surface of constant Kruskal coordinate U, the normal vector n_μ to the surface which points in the direction of decreasing U is given by

$$n_\mu = -\mathrm{d}U = (-1,0,0,0) \qquad \Rightarrow \qquad n^\mu = \left(0, \zeta^{-1}(r)\,, 0, 0\right), \tag{3.141}$$

where we have used the inverse metric (3.141). Then, the volume element $\mathrm{d}\Sigma^{\mu}_{\mathcal{H}^+}$ becomes

$$\mathrm{d}\Sigma^{\mu}_{\mathcal{H}^+} = \delta^{\mu}_{V}\, r^2 \sin\theta\, \mathrm{d}V \mathrm{d}\theta\, \mathrm{d}\varphi. \tag{3.142}$$

The factor of δ^{μ}_{V} in (3.142) means that the partial derivatives in (3.140) will only act on $\phi^{\text{down}}_{\omega\ell m}$ and its complex conjugate with respect to the Kruskal coordinate V. Thus, it will useful to rewrite the asymptotic form of down-modes near $\mathcal{H}^+$ (3.70) in terms of the lightcone coordinates (3.20), which are readily differentiated with respect to V from the expressions given in (3.24). Near $\mathcal{H}^+$, as well as using (3.20) and (3.70), we have

$$\begin{aligned}\phi^{\text{down}}_{\omega\ell m} &= \frac{1}{r}\exp[-\mathrm{i}\omega t\,]\,\mathrm{N}^{\text{down}}_{\omega}\, Y_{\ell m}(\theta,\varphi)\exp[-\mathrm{i}\widetilde{\omega} r_*] \\ &= \frac{1}{r}\exp\left[-\mathrm{i}\omega\,\frac{(u+v)}{2}\right]\mathrm{N}^{\text{down}}_{\omega}\, Y_{\ell m}(\theta,\varphi)\exp\left[\mathrm{i}\widetilde{\omega}\,\frac{(u-v)}{2}\right].\end{aligned} \tag{3.143}$$

Comparing the expressions for the down-modes (3.143) and the volume element $\mathrm{d}\Sigma^{\mu}_{\mathcal{H}^+}$ (3.134) with that of the up-modes (3.111) and the volume element $\mathrm{d}\Sigma^{\mu}_{\mathcal{H}^-}$ (3.110), we see that the inner product of two down-modes (3.140) is very similar to the inner product of two up-modes (3.107); both u and v are effectively dummy variables in (3.140) and (3.107), while $\phi^{\text{down}}_{\omega\ell m}$ has the same dependence on V, which it is being differentiated with respect to, as $\phi^{\text{up}}_{\omega\ell m}$ does on U. Furthermore, from (3.96) and (3.116), $A_U \frac{\mathrm{d}U}{\mathrm{d}u} = A_V \frac{\mathrm{d}V}{\mathrm{d}v}$ and we are similarly able to ignore contributions proportional to $f(r)$ since the metric function vanishes near the horizon from (3.15).

However, there are two differences in the inner product of two down-modes as compared to that of two up-modes. Firstly, the volume element $\mathrm{d}\Sigma^{\mu}_{\mathcal{H}^+}$ (3.142) contains a minus sign relative to $\mathrm{d}\Sigma^{\mu}_{\mathcal{H}^-}$ (3.110), which induces a minus sign in $\langle\phi^{\text{down}}_{\omega\ell m}, \phi^{\text{down}}_{\omega'\ell'm'}\rangle$ relative to $\langle\phi^{\text{up}}_{\omega\ell m}, \phi^{\text{up}}_{\omega'\ell'm'}\rangle$. Secondly, where the up-modes are propagating away the surface of integration, $\mathcal{H}^-$, the down-modes are instead propagating towards the surface of integration $\mathcal{H}^+$; the corollary is that we integrate the down-modes from $v = \infty$ to $-\infty$ instead of $u = -\infty$ to ∞ as in the case of the up-modes (3.120), which induces a second minus sign in $\langle\phi^{\text{down}}_{\omega\ell m}, \phi^{\text{down}}_{\omega'\ell'm'}\rangle$ relative to $\langle\phi^{\text{up}}_{\omega\ell m}, \phi^{\text{up}}_{\omega'\ell'm'}\rangle$. These two minus signs cancel such that the two inner products are equal. Then the normalisation constant $\mathrm{N}^{\text{down}}_{\omega}$ is given by

$$\mathrm{N}^{\text{down}}_{\omega} = \frac{1}{\sqrt{4\pi|\widetilde{\omega}|}}. \tag{3.144}$$

and the expression for the inner product of two generic down-modes is given by

$$\langle\phi^{\text{down}}_{\omega\ell m}, \phi^{\text{down}}_{\omega'\ell'm'}\rangle = \operatorname{sgn}(\widetilde{\omega})\, \delta\big(\omega - \omega'\big)\, \delta_{\ell\ell'}\, \delta_{mm'}. \tag{3.145}$$

The norm of two down-modes (3.145) is dependent upon the sign of $\widetilde{\omega}$, in contrast to the case of the Schwarzschild down-modes. This is a crucial point that will lead to many subtleties when we come to canonically quantising the scalar field Φ and when defining quantum states in Chapter 4. We summarise the value of the norm of two down-modes as

$$\langle\phi^{\text{down}}_{\omega\ell m}, \phi^{\text{down}}_{\omega'\ell'm'}\rangle = \begin{cases} \delta(\omega-\omega')\, \delta_{\ell\ell'}\, \delta_{mm'}, & \text{for } \widetilde{\omega} > 0, \\ -\delta(\omega-\omega')\, \delta_{\ell\ell'}\, \delta_{mm'}, & \text{for } \widetilde{\omega} < 0, \end{cases} \tag{3.146}$$

meaning that down-modes are of positive-norm when $\widetilde{\omega} > 0$ and of negative-norm when $\widetilde{\omega} < 0$. We can use the expression for $\mathrm{N}^{\text{down}}_{\omega}$ in (3.144) to write the down-modes as

$$\phi^{\text{down}}_{\omega\ell m} = \frac{1}{\sqrt{4\pi|\widetilde{\omega}|}}\, \mathrm{e}^{-\mathrm{i}\omega t}\, \frac{X^{\text{down}}_{\omega\ell}(r)}{r}\, Y_{\ell m}(\theta,\varphi)\,, \tag{3.147}$$

where the asymptotic forms of the radial function $X^{\text{down}}_{\omega\ell}(r)$ are given in (3.70).

3.4.4 Relations between scalar field modes revisited

Having derived the normalisation constants associated to each of the scalar field modes, we can use (3.71) to write the out-modes and the down-modes in terms of in- and up-modes:

$$\phi^{\text{out}}_{\omega\ell m} = A^{\text{in}*}_{\omega\ell}\, \phi^{\text{in}}_{\omega\ell m} + \left|\frac{\widetilde{\omega}}{\omega}\right|^{\frac{1}{2}} B^{\text{in}*}_{\omega\ell}\, \phi^{\text{up}}_{\omega\ell m}, \tag{3.148a}$$

$$\phi^{\text{down}}_{\omega\ell m} = A^{\text{up}*}_{\omega\ell}\, \phi^{\text{up}}_{\omega\ell m} + \left|\frac{\omega}{\widetilde{\omega}}\right|^{\frac{1}{2}} B^{\text{up}*}_{\omega\ell}\, \phi^{\text{in}}_{\omega\ell m}, \tag{3.148b}$$

Chapter 4

Canonical quantisation and definition of quantum states

In §4.1, we discuss the canonical quantisation procedure as well as the definition of positive- and negative-frequency modes. We discuss the problems that arise when using canonical quantisation to define states in RN spacetime in §4.2 and we outline two possible resolutions to these problems that we will employ throughout the rest of the chapter. In §4.3, we define analogues of the Schwarzschild Boulware state $|B_s\rangle$. In §4.4, we define analogues of the Schwarzschild Unruh state $|U_s\rangle$ and in §4.5, we define analogues of the Schwarzschild Hartle-Hawking state $|H_s\rangle$.

4.1 Canonical quantisation

4.1.1 Introduction

In Chapter 3, we described the properties of a classical massless, minimally-coupled charged scalar field propagating on a classical background Reissner-Nordström spacetime. In this section, we would like to quantise the field to define a variety of quantum states while leaving the background RN spacetime classical, in line with the philosophy of QFTCS. We choose to do so using the method of canonical quantisation.

The process of canonically quantising the scalar field relies upon being able to expand the field Φ into distinct sets of modes that are positive-frequency or negative-frequency with respect to a suitable time coordinate. This is because we will promote the coefficients multiplying these sets of modes, which we refer to as the mode expansion coefficients, to quantum operators with the understanding that those operators multiplying positive-frequency modes are annihilation operators and those operators multiplying negative-frequency modes are creation operators. The correct interpretation of the creation and annihilation operators, in turn, relies upon the positive-frequency modes being entirely of positive-norm and the negative-frequency modes being entirely of negative norm since only then will the canonical commutation relations associated to the operators be correct; we will make this point clearer by means of an example in §4.1.4.

Since the decomposition of a charged scalar field into positive-frequency modes, which are entirely of positive-norm, and negative-frequency modes, which are entirely of negative-norm, proceeds without issue in both Minkowski and Schwarzschild spacetimes, this requirement is not often emphasised in texts concerning those subjects. However, as we will see in this chapter, decomposing a charged scalar field in the aforementioned way in Reissner-Nordström spacetime is non-trivial due to the presence of superradiant up- and down-modes; again, this point will be made clearer by means of an example in §4.1.5.

We have already evaluated the norms of the in-, up-, out- and down-modes in §3.4; note that we did not make any statements about whether these modes were of positive- or negative-frequency. It is worth taking the time, now, to discuss how we define positive- and negative-frequency modes in the absence of considerations about their norms.

4.1.2 Defining positive- and negative-frequency modes

In §4.2, when we come to defining quantum states for a charged scalar field Φ in Reissner-Nordström spacetime, we will want to consider modes that are positive- and negative-frequency with respect to a variety of choices of time coordinate in order to define states with a certain physical interpretation. Thus, it is useful to consider how we can define positive- and negative-frequency modes with respect to a particular variable in general.

Consider the Fourier transform of an arbitrary function $f(\mathfrak{X})$ w.r.t a variable $\mathfrak{X}$

$$\int_{-\infty}^{\infty} d\mathfrak{X}\, e^{-i\mathfrak{p}\mathfrak{X}} f(\mathfrak{X}) = F(\mathfrak{p})\,, \tag{4.1}$$

where $F(\mathfrak{p})$ is the Fourier-transformed function. Then if

$$\int_{-\infty}^{\infty} d\mathfrak{X}\, e^{-i\mathfrak{p}\mathfrak{X}} f(\mathfrak{X}) = 0, \quad \mathfrak{p} > 0, \tag{4.2}$$

holds, the function $f(\mathfrak{X})$ is defined to be positive-frequency w.r.t the variable $\mathfrak{X}$. The statement in (4.2) can be understood as saying that $f(\mathfrak{X})$ is positive-frequency w.r.t $\mathfrak{X}$ if, when Fourier decomposed w.r.t $\mathfrak{X}$, it only contains positive-frequency elements; then the function $f(\mathfrak{X})$ is analytic in the lower-half of the complex plane. Similarly, if

$$\int_{-\infty}^{\infty} d\mathfrak{X}\, e^{i\mathfrak{p}\mathfrak{X}} f(\mathfrak{X}) = 0, \quad \mathfrak{p} > 0, \tag{4.3}$$

holds, the function $f(\mathfrak{X})$ is defined to be negative-frequency w.r.t the variable $\mathfrak{X}$. The condition for $f(\mathfrak{X})$ to be negative-frequency in (4.3) can be equivalently expressed as

$$\int_{-\infty}^{\infty} d\mathfrak{X}\, e^{-i\mathfrak{p}\mathfrak{X}} f(\mathfrak{X}) = 0, \quad \mathfrak{p} < 0. \tag{4.4}$$

The statement in (4.4) can be understood as saying that the function $f(\mathfrak{X})$ is said to be negative-frequency with respect to the variable $\mathfrak{X}$ if, when Fourier decomposed w.r.t $\mathfrak{X}$, it only contains negative-frequency elements; then the function $f(\mathfrak{X})$ is analytic in the upper-half of the complex plane. Given the form of the harmonic time-dependence $e^{-i\omega t}$ of the modes (3.41), the definitions in (4.2) and (4.4) suffice to define positive-frequency and negative-frequency modes with respect to the Schwarzschild-like coordinate t respectively.

However, when defining quantum states in which modes are thermalised, such as the analogues of the Schwarzschild Unruh (2.75) and Hartle-Hawking states (2.99), we will want to define positive- and negative-frequency quantities w.r.t the Kruskal coordinates U and V. Each of the scalar field modes we defined in §3.3 are non-zero in certain spacetime regions only; specifically the in- and down-modes are non-zero in regions I and II of the Penrose diagram in Figure 3.1, while the up- and out-modes are non-zero in regions I and III. However, the Kruskal coordinates (3.24) are well-defined throughout Reissner-Nordström spacetime. Thus, when defining positive- and negative-frequency modes w.r.t Kruskal coordinates we will need to enforce their vanishing in the regions in which they are defined to be zero; we do this by using the Heaviside function $\Theta(x)$

$$\Theta(x) = \begin{cases} 1, & x \geq 0, \\ 0, & x < 0. \end{cases} \tag{4.5}$$

Thus, when defining positive-frequency thermalised modes, we will make use of the Lemma in Appendix H of [89] which states that for positive real $\mathfrak{p}$ and arbitrary real $\mathfrak{q}$

$$\int_{-\infty}^{\infty} \mathrm{d}\mathfrak{X}\, \mathrm{e}^{-\mathrm{i}\mathfrak{p}\mathfrak{X}} \left\{ \mathrm{e}^{-\mathrm{i}\mathfrak{q}\ln(\mathfrak{X})}\, \Theta(\mathfrak{X}) + \mathrm{e}^{-\pi\mathfrak{q}}\, \mathrm{e}^{-\mathrm{i}\mathfrak{q}\ln(-\mathfrak{X})}\, \Theta(-\mathfrak{X}) \right\} = 0. \tag{4.6}$$

By (4.2), the quantity inside the curly brackets in (4.6) is positive-frequency w.r.t $\mathfrak{X}$. When defining negative-frequency thermalised modes, we will make use of the complex conjugate of the Lemma (4.6), which states that for positive real $\mathfrak{p}$ and arbitrary real $\mathfrak{q}$

$$\int_{-\infty}^{\infty} \mathrm{d}\mathfrak{X}\, \mathrm{e}^{\mathrm{i}\mathfrak{p}\mathfrak{X}} \left\{ \mathrm{e}^{\mathrm{i}\mathfrak{q}\ln(\mathfrak{X})}\, \Theta(\mathfrak{X}) + \mathrm{e}^{-\pi\mathfrak{q}}\, \mathrm{e}^{\mathrm{i}\mathfrak{q}\ln(-\mathfrak{X})}\, \Theta(-\mathfrak{X}) \right\} = 0. \tag{4.7}$$

By (4.3), the quantity inside the curly brackets in (4.7) is negative-frequency w.r.t $\mathfrak{X}$.

4.1.3 General outline of canonical quantisation

In this section, we will make our discussion of the canonical quantisation procedure concrete by means of an example; we consider a charged scalar field Φ on an arbitrary background spacetime. Consider the Klein-Gordon inner product (3.85); while this represents a natural choice of inner product for a free scalar field, it is not technically an inner product since it is not positive-definite. Let us assume that we are able to expand the classical field Φ in an orthonormal basis of mode solutions ϕ_j to the scalar field equation (3.36) where the label j indexes the solutions; note that we have not said anything about whether we can decompose the classical field Φ into distinct sets of positive- and frequency-modes yet. What we mean by expanding the field in an orthonormal basis of field modes ϕ_j is that the Klein-Gordon inner product (3.85) of any two normalised field modes ϕ_j, which solve the scalar field equation (3.36), is given by

$$\langle \phi_j, \phi_{j'} \rangle = \eta_j\, \delta_{jj'} \tag{4.8}$$

where $\delta_{jj'}$ is either the Kronecker delta function or the Dirac delta function depending on whether the spectrum of mode solutions is discrete or continuous respectively, and the factor of η_j in (4.8) is defined by

$$\eta_j = \begin{cases} 1 & \text{if } \phi_j \text{ has positive norm,} \\ -1 & \text{if } \phi_j \text{ has negative norm.} \end{cases} \tag{4.9}$$

The lack of positive-definiteness of the Klein-Gordon inner product is reflected in the factor of η_j (4.9) on the r.h.s of (4.8), which states that the inner product of two identical positive-norm modes is positive and the inner product of two identical negative-norm modes is negative. We have already seen this when calculating the norms of the in-, up-, out- and down-modes in §3.4; the inner product of the in- and out-modes depends on $\mathrm{sgn}(\omega)$ from (3.104) and (3.137) respectively, while the inner product of the up- and down-modes depends on $\mathrm{sgn}(\widetilde{\omega})$ from (3.130) and (3.145) respectively.

Now, we would like to decompose our orthonormal basis of field modes ϕ_j, which satisfy (4.8), into distinct sets of positive-frequency modes ϕ_j^+ and negative-frequency modes ϕ_j^-. We assume that all field modes ϕ_j have a harmonic time-dependence on a suitable timelike coordinate T such that

$$\phi_j \propto \exp\left[-\mathrm{i}\varpi T\right], \tag{4.10}$$

where ϖ is the frequency associated to the mode ϕ_j. From our discussion in §4.1.2, ϕ_j is considered to be a positive-frequency mode ϕ_j^+ if $\varpi > 0$ and a negative-frequency mode ϕ_j^- if $\varpi < 0$. We can then decompose our orthonormal basis of field modes ϕ_j into positive- and negative-frequency modes as

$$\Phi = \sum_j \left\{ a_j \phi_j^+ + b_j^\dagger \phi_j^- \right\}, \tag{4.11}$$

where the mode expansion coefficients a_j multiply positive-frequency modes ϕ_j^+ and the mode expansion coefficients b_j multiply negative-frequency modes ϕ_j^-. We will continue to use the labels $+$ and $-$ throughout this chapter to denote positive- and negative-frequency modes respectively and we will continue to use the notation a and b to denote mode expansion coefficients multiplying positive- and negative-frequency modes respectively. Canonical quantisation of the scalar field Φ proceeds by promoting the mode expansion coefficients a_j, b_j in (4.11) to operators such that the quantum field operator $\hat{\Phi}$ is given by

$$\hat{\Phi} = \sum_j \left\{ \hat{a}_j \phi_j^+ + \hat{b}_j^\dagger \phi_j^- \right\}, \tag{4.12}$$

where, in (4.12), the operators $\hat{a}_j$ multiply positive-frequency modes ϕ_j^+ and the operators $\hat{b}_j$ multiply negative-frequency modes ϕ_j^-. We can obtain the conjugate momentum operator $\hat{\Pi}$ associated to the field $\hat{\Phi}$ in (4.12) from the operation

$$\hat{\Pi} = \frac{\partial}{\partial T}\,\hat{\Phi}. \tag{4.13}$$

Then, the field operator $\hat{\Phi}$ (4.12) and its associated conjugate momentum operator $\hat{\Pi}$ (4.13) satisfy the canonical commutation relations

$$\left[\hat{\Phi}(T,\boldsymbol{x})\,,\hat{\Pi}\left(T,\boldsymbol{x}'\right)\right] = \mathrm{i}\,\delta^3\left(\boldsymbol{x},\boldsymbol{x}'\right), \tag{4.14a}$$

$$\left[\hat{\Phi}(T,\boldsymbol{x})\,,\hat{\Phi}\left(T,\boldsymbol{x}'\right)\right] = \left[\hat{\Pi}(T,\boldsymbol{x})\,,\hat{\Pi}\left(T,\boldsymbol{x}'\right)\right] = 0. \tag{4.14b}$$

Equations (4.14a, 4.14b), taken together, are often referred to as equal-time commutation relations since we have given these relations for a particular choice of the value of the time coordinate T. Using the inner product (4.8) of the field modes ϕ_j and the expansion of the field operator $\hat{\Phi}$ in terms of positive- and negative-frequency modes (4.12), we can derive the commutation relations for the $\hat{a}_j$ and $\hat{b}_j$ operators in (4.12) as

$$\left[\hat{a}_j,\hat{a}^\dagger_{j'}\right] = \eta_j\delta_{jj'}, \qquad \left[\hat{b}_j,\hat{b}^\dagger_{j'}\right] = -\eta_j\delta_{jj'}, \tag{4.15}$$

with any commutators not explicitly given in (4.15) vanishing. In particular, we note that the commutation relation between the operators $\hat{b}_j$ associated to negative-frequency modes ϕ^-_j includes a minus sign that is absent in the commutation relation between the operators $\hat{a}_j$ associated to positive-frequency modes ϕ^+_j.

Thus far, our discussion has been valid for a charged scalar field Φ on an arbitrary background spacetime. However, we now restrict our attention to charged scalar fields in two particular background spacetimes, namely Schwarzschild spacetime, which we treated in Chapter 2, and Reissner-Nordström spacetime, which is the subject of this chapter.

4.1.4 Canonical quantisation in Schwarzschild spacetime

In Chapter 2, when treating charged scalar fields in Schwarzschild spacetime, we saw that it is possible to choose a timelike coordinate such that positive-frequency modes ϕ^+_j are entirely of positive-norm and negative-frequency modes ϕ^-_j are entirely of negative-norm.

For example, when defining the Schwarzschild Boulware state, we chose to decompose the Schwarzschild in- and up-modes into distinct sets of positive- and negative-frequency modes w.r.t the Schwarzschild coordinate t. Since the harmonic time-dependence of all of the scalar field modes in Schwarzschild spacetime is given by $\mathrm{e}^{-\mathrm{i}\omega t}$ (2.17), then these modes are positive-frequency for $\omega > 0$ and negative-frequency for $\omega < 0$ by (4.2) and (4.4) respectively. Furthermore, we saw that all Schwarzschild modes are of positive-norm for $\omega > 0$ and negative-norm for $\omega < 0$, separately to whether they are considered positive-frequency or negative-frequency. From (4.9), η_j takes the value 1 in the case of positive-norm modes and -1 in the case of negative-norm modes. Then, the nonzero commutation relations (4.15) for the $\hat{a}_j$ and $\hat{b}_j$ operators associated to a charged scalar field Φ in a background Schwarzschild spacetime reduce to standard commutation relations

$$\left[\hat{a}_j,\hat{a}^\dagger_{j'}\right] = \delta_{jj'}, \qquad \left[\hat{b}_j,\hat{b}^\dagger_{j'}\right] = \delta_{jj'}, \tag{4.16}$$

allowing the $\hat{a}_j$, $\hat{b}_j$ and $\hat{a}^\dagger_j$, $\hat{b}^\dagger_j$ to retain their usual interpretation as annihilation and creation operators respectively.

The key point here is that in a background Schwarzschild spacetime, it is possible to decompose the scalar field Φ such that positive-frequency modes ϕ_j^+ are entirely of positive-norm and negative-frequency modes ϕ_j^- are entirely of negative-norm. This, in turn, allows the commutation relations for the creation and annihilation operators in (4.15), derived for a charged scalar field Φ in an arbitrary background spacetime, to reduce to standard commutation relations such as those in (4.16) in a background Schwarzschild spacetime. The commutation relations in (4.16) are required for the operators $\hat{a}_j$, $\hat{b}_j$ and $\hat{a}_j^\dagger$, $\hat{b}_j^\dagger$ to be interpreted correctly as annihilation and creation operators respectively.

4.1.5 Canonical quantisation in Reissner-Nordström spacetime

In Reissner-Nordström spacetime, however, the inability to define positive- and negative-frequency modes that are entirely of positive- and negative-norm respectively for all scalar field modes raises the prospect of operators that are labelled incorrectly. In §3.4, we saw that while the in- (3.104) and out-modes (3.137) are of positive- and negative-norm for $\omega > 0$ and $\omega < 0$ respectively, the up- (3.129) and down-modes (3.145) are of positive- and negative-norm for $\widetilde{\omega} > 0$ and $\widetilde{\omega} < 0$ respectively. All of the in-, up-, out- and down-modes share the same harmonic time-dependence on the Schwarzschild-like coordinate t given by $e^{-i\omega t}$ (3.41); thus, all of these mode solutions are positive-frequency for $\omega > 0$ and negative-frequency for $\omega < 0$ by (4.2) and (4.4) respectively.

Then, it follows that while for the in- and out-modes we are able to define positive-frequency modes that are entirely of positive-norm and negative-frequency modes that are entirely of negative-norm, we are not able to do this for the up- and down-modes because the norm of the up- and down-modes depends on $\text{sgn}(\widetilde{\omega})$ while the question as to whether the up- and down-modes are positive- or negative-frequency depends on $\text{sgn}(\omega)$.

From the expression for $\widetilde{\omega}$ (3.58) in terms of ω, we see there are two cases where the quantities $\text{sgn}(\omega)$ and $\text{sgn}(\widetilde{\omega})$ differ depending on the product of the scalar field charge q and the charge of the black hole Q. We can summarise these cases as

$$\begin{aligned}
&\textbf{Case 1:}\ \text{If } qQ > 0 \text{ then modes with } 0 < \omega < \frac{qQ}{r_+} \text{ have } \text{sgn}(\omega) = 1 \text{ and } \text{sgn}(\widetilde{\omega}) = -1,\\
&\textbf{Case 2:}\ \text{If } qQ < 0 \text{ then modes with } 0 > \omega > \frac{qQ}{r_+} \text{ have } \text{sgn}(\omega) = -1 \text{ and } \text{sgn}(\widetilde{\omega}) = 1.
\end{aligned} \tag{4.17}$$

If **Case 1** holds, then the up- and down-modes will contain some terms that are positive-frequency since $\text{sgn}(\omega) = 1$, but are of negative-norm since $\text{sgn}(\widetilde{\omega}) = -1$. Conversely, if **Case 2** holds, then the up- and down-modes will contain some terms that are negative-frequency since $\text{sgn}(\omega) = -1$, but are of positive-norm since $\text{sgn}(\widetilde{\omega}) = 1$. We note that (4.17) also holds for the in- and out-modes; however, as previously stated, there is no difficulty in decomposing these modes into positive-frequency modes that are entirely of positive-norm and negative-frequency modes that are entirely of negative-norm, since they are defined to be positive- and negative-frequency and of positive- and negative-norm with respect to the same variable ω. Furthermore, we note that the up- and down-modes specified in (4.17) are exactly those of the up- and down-modes in (3.84) that give rise to the phenomenon of classical superradiance.

We can use the fact that η_j takes the value 1 in the case of positive-norm modes and -1 in the case of negative-norm modes from (4.9) in order to see what the operator commutation relations (4.15) reduce to in the two cases stated in (4.17). Restricting our attention to the case of the superradiant up- and down-modes, if **Case 1** holds η_j takes the value -1 since the positive-frequency superradiant up- and down-modes are of negative-norm; then the nonzero commutation relations (4.15) for the $\hat{a}_j$ and $\hat{b}_j$ operators associated to these particular modes become

$$\left[\hat{a}_j, \hat{a}^\dagger_{j'}\right] = -\delta_{jj'} = \left[\hat{a}^\dagger_j, \hat{a}_{j'}\right], \qquad \left[\hat{b}_j, \hat{b}^\dagger_{j'}\right] = \delta_{jj'}, \tag{4.18}$$

leading to the mislabelling of the $\hat{a}_j$ and $\hat{a}^\dagger_j$ operators associated to superradiant up- and down-modes of positive frequency. In (4.18), for the operators $\hat{a}_j$ associated to positive-frequency modes, what we usually refer to as creation operators are in fact annihilation operators and what we usually refer to as annihilation operators are in fact creation operators. On the other hand, if **Case 2** holds η_j takes the value 1 since the negative-frequency superradiant up- and down-modes are of positive-norm; then the nonzero commutation relations (4.15) for the $\hat{a}_j$ and $\hat{b}_j$ operators associated to these particular modes become

$$\left[\hat{a}_j, \hat{a}^\dagger_{j'}\right] = \delta_{jj'}, \qquad \left[\hat{b}_j, \hat{b}^\dagger_{j'}\right] = -\delta_{jj'} = \left[\hat{b}^\dagger_j, \hat{b}_{j'}\right], \tag{4.19}$$

leading to the mislabelling of the $\hat{b}_j$ and $\hat{b}^\dagger_j$ operators associated to superradiant up- and down-modes of negative-frequency. In (4.19), for the negative-frequency mode operators $\hat{b}_j$, what we usually refer to as creation operators are in fact annihilation operators while what we usually refer to as annihilation operators are in fact creation operators.

Thus, we can see that superradiant up- and down-modes of positive- or negative-frequency that are of negative- or positive-norm respectively lead to the misinterpretation of the creation and annihilation operators associated to those particular modes. Note that if the creation and annihilation operators associated to all up- and down-modes were similarly mislabelled, we could simply interchange their interpretations and we would be able quantise the field consistently. However, operators multiplying sets of either positive- or negative-frequency modes that contain modes of different norm renders the ability to achieve a consistent interpretation of the creation and annihilation operators multiplying those modes challenging.

4.2 Consequences for defining quantum states in Reissner-Nordström spacetime

4.2.1 Possible resolutions

The question, then, is how do we proceed to define quantum states for a charged scalar field in Reissner-Nordström spacetime. We will explore two possible resolutions.

The first is to abandon defining both up- and down-modes as positive- and negative-frequency with respect to their frequency ω and, instead, to do so with respect to the quantity $\widetilde{\omega}$ (3.58). Then, since both the question as to whether the up- and down-modes will be positive- or negative-frequency as well as the question as to whether they are of positive- or negative-norm depends on $\mathrm{sgn}(\widetilde{\omega})$, we will not have sets of positive- or negative-frequency modes of varying norm in this case, which lead to operators with the wrong commutation relations such as those in (4.18) and (4.19). However, our ability to define quantum states becomes much more restrictive.

In particular, we can only define states in the 'past' and 'future' of the black hole in this case, i.e. with respect to the 'past' (3.86) or 'future' Cauchy surfaces (3.87). Thus, states that we define in this way in Reissner-Nordström spacetime, which are analogous to a particular state in Schwarzschild spacetime, will only share the properties of the corresponding Schwarzschild state on surfaces within either the 'past' or 'future' Cauchy surfaces. For example, we may define a 'past' Boulware state or 'future' Boulware state in Reissner-Nordström spacetime, and these states will only appear to be as empty as possible to a static observer at past null infinity $\mathscr{I}^-$ or future null infinity $\mathscr{I}^+$ respectively; this is in contrast to the Schwarzschild Boulware state which is as empty as possible to a static observer at both past and future null infinity $\mathscr{I}^\pm$. We will refer to states defined in this way as 'past' and 'future' states, and their definition will be made clearer in §4.2.3.

The second possible resolution is to continue to define all scalar field modes as positive-frequency for $\omega > 0$ and as negative-frequency for $\omega < 0$, but to include a factor in the commutation relations of the operators associated to the up- and down-modes that ensures the operators associated to these modes satisfy standard commutation relations such as those in (4.16). We will introduce this factor explicitly in §4.2.4.

In order to make concrete our discussion about the two possible resolutions to the problem of defining quantum states in Reissner-Nordström spacetime described in this section, we will now consider the example of defining an analogue of the Schwarzschild Boulware state in Reissner-Nordström spacetime; this is the subject of the next section.

4.2.2 Example: defining an analogue of Schwarzschild Boulware

Consider the process by which we defined the Schwarzschild Boulware state $|\mathrm{B_s}\rangle$ in §2.3.1; we expanded the field in terms of in- and out-modes since the absence of particles in these modes corresponds to the interpretation of the Schwarzschild Boulware state $|\mathrm{B_s}\rangle$ as being as empty as possible to a static observer at infinity. Due to the fact that the in- and out-modes did not form an orthogonal basis, we re-expressed the out-modes in terms of in- and up-modes, which do indeed form an orthogonal basis. Nevertheless, since the in-, up- and out-modes were all considered to be of positive- and negative-norm with respect to their frequency ω, then positive-norm out-modes were re-expressed in terms of positive-norm in- and up-modes and negative-norm out-modes were re-expressed in terms of negative-norm in- and up-modes. Thus, all of the positive-frequency in- and up-modes in our final expression for the Schwarzschild Boulware state were of positive-norm and, similarly, all of the negative-frequency in- and up-modes were of negative-norm. Therefore, we obtained standard commutation relations, such as those in (4.16), for the operators associated to all modes and, correspondingly, we were able to interpret the creation and annihilation operators associated to the Schwarzschild Boulware state $|\mathrm{B_s}\rangle$ in a consistent manner.

If we were to think about an analogous process in Reissner-Nordström spacetime where we expand the field Φ in terms of in- (3.106) and out-modes (3.139), before re-expressing the out-modes in terms of in-modes and up-modes (3.131), we would quickly find sets of positive- and negative-frequency modes of varying norm. Specifically, the positive- and negative-norm out-modes will be re-expressed in terms of positive- and negative-norm in-modes respectively, since the norm of both of these mode solutions depends on $\mathrm{sgn}(\omega)$. Furthermore, non-superradiant positive- and negative-norm out-modes will be re-expressed in terms of non-superradiant up-modes of positive- and negative-norm respectively since, though the norm of the former depend on $\mathrm{sgn}(\omega)$ while the norm of the latter depend on $\mathrm{sgn}(\widetilde{\omega})$, all non-superradiant modes have $\mathrm{sgn}(\omega\widetilde{\omega}) = 1$. However, the superradiant positive- and negative-norm out modes will be re-expressed in terms of negative- and positive-norm superradiant up-modes respectively, due to the fact that the values of $\mathrm{sgn}(\omega)$ and $\mathrm{sgn}(\widetilde{\omega})$ differ for these modes from (4.17). Thus, the set of positive-frequency up-modes in our final expression for this state will contain positive-norm non-superradiant modes and negative-norm superradiant modes. Similarly, the set of negative-frequency up-modes will contain negative-norm non-superradiant modes and positive-norm superradiant modes. From our discussion in §4.1.5, this will lead to commutation relations for the operators associated to the up-modes of the form (4.18) and (4.19), and the consequent inability to interpret creation and annihilation operators associated to the up-modes in a consistent manner.

4.2.3 'Past' and 'future' states

One resolution would be to instead expand the field Φ in terms of in- and up-modes directly, integrating the in-modes with respect to their frequency ω and the up-modes with respect to the quantity $\widetilde{\omega}$. The absence of particles in the in-modes corresponds to a state that is as empty as possible to an observer at past null infinity $\mathscr{I}^-$ only; we cannot tell if the state will appear as empty as possible to an observer at future null infinity $\mathscr{I}^+$ as we have not expanded the field Φ in terms of out-modes. However, the operators acting on the state will have well-defined interpretations since positive-frequency modes will be entirely of positive-norm and negative-frequency modes will be entirely of negative-norm. Expanding the field in precisely this manner will lead us to the 'past' Boulware state $|\mathrm{B}^-\rangle$, which we discuss in §4.3.1. We note that we could similarly expand the field Φ in terms of out- and down-modes instead, since these scalar field modes also form an orthonormal basis. Expanding the field in this manner will lead us to the 'future' Boulware state $|\mathrm{B}^+\rangle$, which appears as empty as possible to an observer at future null infinity $\mathscr{I}^-$ but is not necessarily empty to an observer at past null infinity $\mathscr{I}^+$; we discuss this state in §4.3.2.

The choice of Cauchy surface w.r.t which we define 'past' and 'future' quantum states underlines why they are referred to as such. The in- and up-modes, which we expand the field in terms of in order to define 'past' states, take particularly simple forms near the past horizon $\mathcal{H}^-$ and past null infinity $\mathscr{I}^-$. Thus, it makes sense to define 'past' quantum states with respect to the 'past' Cauchy surface Σ_{past} (3.86). Similarly, the out- and down-modes, which we expand the field in terms of in order to define 'future' states, take particularly simple forms near the future horizon $\mathcal{H}^+$ and future null infinity $\mathscr{I}^+$. Thus, it makes sense to define 'future' quantum states with respect to the 'future' Cauchy surface Σ_{future} (3.87). 'Future' states are considered in [50, 51].

Further examples of 'past' and future' states include the analogues of the Schwarzschild Unruh $|\mathrm{U_s}\rangle$ and Hartle-Hawking states $|\mathrm{H_s}\rangle$; we will define the 'past' Unruh state $|\mathrm{U}^-\rangle$ in §4.4.1, the 'future' Unruh state $|\mathrm{U}^+\rangle$ in §4.4.2, the 'past' CCH state $|\mathrm{CCH}^-\rangle$ in §4.5.1 and the 'future' CCH state $|\mathrm{CCH}^+\rangle$ in §4.5.2.

4.2.4 '-like' states

Returning to the example, in §4.2.2, of defining an analogue of the Schwarzschild Boulware state $|\mathrm{B_s}\rangle$ in Reissner Nordström spacetime, we can consider a different resolution to the issue of mislabelled operators associated to the up-modes in this putative state. We can continue to define the up-modes as being positive-frequency for $\omega > 0$ and negative-frequency for $\omega < 0$, and instead multiply the commutation relations of their corresponding operators by an appropriate factor so as to ensure that these operators obey standard commutation relations in the case of both superradiant and non-superradiant up-modes.

We can use our discussion in §4.1.5 to inform us of the specific form this factor should take. In (4.17) we saw that non-standard commutation relations between the operators associated to superradiant up- and down-modes arise due to the fact that these modes have $\text{sgn}(\widetilde{\omega}) \neq \text{sgn}(\omega)$; the result is an unwanted factor of -1 in either the commutation relations between the operators associated to positive-frequency modes if **Case 1** holds (4.18) or the commutation relations between the operators associated to negative-frequency modes if **Case 2** holds (4.19). The factor of -1, in turn, leads to the mislabelling of creation and annihilation operators in these two cases. Conversely, the operators associated to modes which have $\text{sgn}(\widetilde{\omega}) = \text{sgn}(\omega)$, i.e. in- and out-modes as well as non-superradiant up- and down-modes, satisfy standard commutation relations. This leads us to multiply the commutation relations of operators associated to the up- and down-modes by a factor $\eta_{\omega\widetilde{\omega}}$ that we will refer to as the eta-function, and which is defined by

$$\eta_{\omega\widetilde{\omega}} = \begin{cases} 1 & \text{if } \text{sgn}(\omega\widetilde{\omega}) = 1, \\ -1 & \text{if } \text{sgn}(\omega\widetilde{\omega}) = -1. \end{cases} \tag{4.20}$$

From (4.20), the eta-function $\eta_{\omega\widetilde{\omega}}$ takes the value 1 for modes that have $\text{sgn}(\widetilde{\omega}) = \text{sgn}(\omega)$, so as to retain standard commutation relations in this case, and -1 for modes that have $\text{sgn}(\widetilde{\omega}) \neq \text{sgn}(\omega)$, so as to cancel the unwanted factor of -1 in this case.

Furthermore, we will refer to states defined using the eta-function $\eta_{\omega\widetilde{\omega}}$ (4.20) as the '-like' states, since these states are those that are defined in Reissner-Nordström spacetime which remain as close in spirit as possible to their corresponding states in Schwarzschild spacetime. For example, the 'Boulware-like' state $|\mathrm{B}\rangle$ that we will define in §4.3.3 is a state in Reissner-Nordström spacetime that is defined with the intention of being as empty as possible to a static observer at both past and future null infinity $\mathscr{I}^\pm$, similar to the Schwarzschild Boulware state $|\mathrm{B_s}\rangle$ in §2.3.1. Whether the 'Boulware-like' state $|\mathrm{B}\rangle$ really does have this physical interpretation will be investigated in §5.5. Further examples of '-like' states include the analogues of the Schwarzschild Hartle-Hawking state $|\mathrm{H_s}\rangle$; we will define the 'Hartle-Hawking-like' state $|\mathrm{H}\rangle$ in §4.5.3 and the Frolov-Thorne state $|\mathrm{FT}\rangle$ in §4.5.4. Finally, we note that '-like' states were first developed in [49] to deal with an analogous situation that arises in the case of quantising a neutral scalar field in a background Kerr spacetime, where it was referred to as the "η-formalism".

4.2.5 Summary

In this section we have examined in detail the two resolutions to the problem, discussed in §4.1.5, that arises when defining quantum states in Reissner-Nordström spacetime due to the inability to define positive-frequency up- and down-modes that are entirely of positive-norm and negative-frequency up- and down-modes that are entirely of negative-norm. These are the 'past' and 'future' states, which are described in §4.2.3, and the '-like' states, which are described in §4.2.4.

We are now ready to define quantum states for a charged scalar field in Reissner-Nordström spacetime. We organise the remainder of this chapter as follows; we first define analogues of the Schwarzschild Boulware state $|\mathrm{B_s}\rangle$ in §4.3, before moving on to define analogues of the Schwarzschild Unruh state $|\mathrm{U_s}\rangle$ in §4.4 and, finally, we conclude this chapter by defining analogues of the Schwarzschild Hartle-Hawking state $|\mathrm{H_s}\rangle$ in §4.5.

4.3 Boulware states

In §2.3.1, we introduced the Schwarzschild Boulware state $|\mathrm{B_s}\rangle$, which has the physical interpretation of being as empty as possible to a static observer at both past and future null infinity $\mathscr{I}^{\pm}$ in Schwarzschild spacetime. We would like to define analogous states for a charged scalar field Φ in Reissner-Nordström spacetime.

In §3.3.4, we derived conditions for low-frequency modes of the classical scalar field Φ to undergo superradiant scattering in RN spacetime. This indicates that it may be impossible to define a Boulware state that is as empty as possible as seen by an observer at both past and future null infinity $\mathscr{I}^{\pm}$. While an observer at $\mathscr{I}^{-}$ may perceive the absence of radiation, this may not necessarily be true of an observer at $\mathscr{I}^{+}$ if the quantised field were to undergo superradiant scattering. Conversely, while a state may appear to be as empty as possible to an observer at $\mathscr{I}^{+}$, this may not necessarily be the case at $\mathscr{I}^{-}$.

As such, it will be prudent to define separate 'past' and 'future' Boulware states, namely the 'past' Boulware state $|\mathrm{B}^{-}\rangle$ in §4.3.1 that will have the physical interpretation of being as empty as possible at past null infinity $\mathscr{I}^{-}$, and the 'future' Boulware state $|\mathrm{B}^{+}\rangle$ in 4.3.2 that will have the physical interpretation of being as empty as possible at future null infinity $\mathscr{I}^{+}$. We will also define a '-like' state, namely the 'Boulware-like' state $|\mathrm{B}\rangle$ in §4.3.3, which is an attempt to remain as close in spirit as possible to the Schwarzschild Boulware state $|\mathrm{B_s}\rangle$. We will now define each of these states in turn.

4.3.1 'Past' Boulware state

We would like to construct a state that is as empty as possible to a static observer at past null infinity $\mathscr{I}^-$; this requirement corresponds to an absence of particles in the in-modes (3.106) of the field Φ. We require an orthonormal basis of scalar field modes in order to quantise the field; the in-modes together with the up-modes (3.131) constitute such a basis. Since the asymptotic forms of both the in-modes (3.64) and the up-modes (3.66) take particularly simple forms near the past horizon $\mathcal{H}^-$ and past null infinity $\mathscr{I}^-$, it is convenient to define the 'past' Boulware state $|\mathrm{B}^-\rangle$ with respect to the 'past' Cauchy surface Σ_{past} defined in (3.86). From our discussion in §4.1, we will first need to decompose the in- and up-modes into positive- and negative-frequency sets in order to canonically quantise the field. We note that this state was first constructed in [1] where it was referred to as the "in" vacuum.

Positive- and negative frequency in-modes

From the asymptotic forms of the in-mode radial function $X^{\mathrm{in}}_{\omega\ell}(r)$ (3.64), the flux of the in-modes through the past horizon $\mathcal{H}^-$ vanishes. Near past null infinity $\mathscr{I}^-$, the proper time experienced by a static observer is given by the Schwarzschild-like time coordinate t. The time-dependence of the in-modes (3.106) with respect to the coordinate t is given by

$$\phi^{\mathrm{in}}_{\omega\ell m} \propto \exp[-\mathrm{i}\omega t]\,. \tag{4.21}$$

Then, from (4.2), we define positive-frequency in-modes for $\omega > 0$ as

$$\phi^{\mathrm{in}+}_{\omega\ell m} = \frac{1}{\sqrt{4\pi|\omega|}}\,\mathrm{e}^{-\mathrm{i}\omega t}\,\frac{X^{\mathrm{in}}_{\omega\ell}(r)}{r}\,Y_{\ell m}(\theta,\varphi)\,, \quad \omega > 0, \tag{4.22}$$

where we note, from the expression for the norm of the in-modes (3.104), that the positive-frequency in-modes in (4.22) are entirely of positive-norm since they have $\mathrm{sgn}(\omega) = 1$. Similarly, from (4.3), we define negative-frequency in-modes for $\omega < 0$ as

$$\phi^{\mathrm{in}-}_{\omega\ell m} = \frac{1}{\sqrt{4\pi|\omega|}}\,\mathrm{e}^{-\mathrm{i}\omega t}\,\frac{X^{\mathrm{in}}_{\omega\ell}(r)}{r}\,Y_{\ell m}(\theta,\varphi)\,, \quad \omega < 0, \tag{4.23}$$

where we note, from the expression for the norm of the in-modes (3.104), that the negative-frequency in-modes in (4.23) are entirely of negative-norm since they have $\mathrm{sgn}(\omega) = -1$. Since the positive- and negative-frequency in-modes are entirely of positive- and negative-norm respectively then, from our discussion in §4.1, the operators associated to these modes will obey standard commutation relations such as those given in (4.16).

While we have defined positive- and negative-frequency in-modes in order to construct the 'past' Boulware state $|\mathrm{B}^-\rangle$, we will also use the definitions in (4.22) and (4.23) when defining the 'Boulware-like' state $|\mathrm{B}\rangle$ in §4.3.3 and the 'past' Unruh state $|\mathrm{U}^-\rangle$ in §4.4.1.

Positive- and negative frequency up-modes

From the asymptotic forms of the up-mode radial function $X^{\mathrm{up}}_{\omega\ell}(r)$ (3.66), the flux of the up-modes through past null infinity $\mathscr{I}^-$ vanishes. Near the past horizon $\mathcal{H}^-$, the proper time experienced by a static observer is still the Schwarzschild-like time coordinate t, although we note that a static observer near the event horizon of the black hole is necessarily accelerating. The time-dependence of the up-modes (3.131) w.r.t t is

$$\phi^{\mathrm{up}}_{\omega\ell m} \propto \exp[-\mathrm{i}\omega t]\,, \tag{4.24}$$

which suggests that we should similarly define up-modes to be positive- and negative-frequency for $\omega > 0$ and $\omega < 0$ respectively; however, from (3.129) the norms of the up-modes are proportional to $\mathrm{sgn}(\widetilde{\omega})$, not $\mathrm{sgn}(\omega)$. Thus, defining positive- and negative-frequency up-modes for $\omega > 0$ and $\omega < 0$ respectively lead, from our discussion in §4.1, to non-standard commutation relations for the creation and annihilation operators associated to superradiant up-modes, such as those in (4.18) and (4.19), and their consequent misinterpretation. Instead, we define positive-frequency up-modes for $\widetilde{\omega} > 0$ as

$$\phi^{\mathrm{up}+}_{\omega\ell m} = \frac{1}{\sqrt{4\pi|\widetilde{\omega}|}}\,\mathrm{e}^{-\mathrm{i}\omega t}\,\frac{X^{\mathrm{up}}_{\omega\ell}(r)}{r}\,Y_{\ell m}(\theta,\varphi)\,, \quad \widetilde{\omega} > 0, \tag{4.25}$$

where we note, from the expression for the norm of the up-modes (3.129), that the positive-frequency up-modes in (4.25) are entirely of positive-norm since they have $\mathrm{sgn}(\widetilde{\omega}) = 1$. Similarly, we define negative-frequency up-modes for $\widetilde{\omega} < 0$ as

$$\phi^{\mathrm{up}-}_{\omega\ell m} = \frac{1}{\sqrt{4\pi|\widetilde{\omega}|}}\,\mathrm{e}^{-\mathrm{i}\omega t}\,\frac{X^{\mathrm{up}}_{\omega\ell}(r)}{r}\,Y_{\ell m}(\theta,\varphi)\,, \quad \widetilde{\omega} < 0, \tag{4.26}$$

where we note, from the expression for the norm of the up-modes (3.129), that the negative-frequency up-modes in (4.26) are entirely of negative-norm since they have $\mathrm{sgn}(\widetilde{\omega}) = -1$. Since these positive- and negative-frequency up-modes are entirely of positive- and negative-norm respectively then, from our discussion in §4.1, the operators associated to these modes will obey standard commutation relations such as those given in (4.16).

Construction of the 'past' Boulware state

Together, the positive- (4.22) and negative-frequency in-modes (4.23) as well as positive- (4.25) and negative-frequency up-modes (4.26) constitute an orthonormal basis in which we can quantise the field Φ; doing precisely this will lead us to the 'past' Boulware state.

However, it is useful to pause for a moment to contrast our construction of the 'past' Boulware state $|\mathrm{B}^-\rangle$ in Reissner-Nordström spacetime with the definition of the Schwarzschild Boulware state $|\mathrm{B_s}\rangle$ in §2.3.1. When defining $|\mathrm{B_s}\rangle$, we initially expanded the field in terms of the corresponding in- and out-modes in Schwarzschild spacetime; the absence of radiation in these modes corresponds to the definition of the Schwarzschild Boulware state $|\mathrm{B_s}\rangle$ of being as empty as possible to an observer at both past and future null infinity $\mathscr{I}^\pm$. Since the in- and out-modes together do not constitute an orthonormal basis, we then re-expressed the classical field in terms of the Schwarzschild in- and up-modes; these do form an orthonormal basis of modes in which to quantise the field.

We would like to draw a distinction between the spirit of that discussion and our present one. In defining the 'past' Boulware state $|\mathrm{B}^-\rangle$ in Reissner-Nordström spacetime, we are seeking only a state which is as empty as possible to observers at past null infinity

$\mathscr{I}^-$ specifically, as opposed to both past and future null infinity $\mathscr{I}^\pm$. Thus, it does not make sense to include the out-modes in our expansion of the classical field at all, and instead we will expand the field Φ in terms of in- and up-modes directly.

Then, the scalar field can be expanded in an orthonormal basis of in-and up-modes as

$$\Phi(x) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \left\{ \int_0^\infty \mathrm{d}\omega\, a^{\mathrm{in}}_{\omega\ell m} \phi^{\mathrm{in}+}_{\omega\ell m} + \int_{-\infty}^0 \mathrm{d}\omega\, b^{\mathrm{in}\dagger}_{\omega\ell m} \phi^{\mathrm{in}-}_{\omega\ell m} + \int_0^\infty \mathrm{d}\widetilde{\omega}\, a^{\mathrm{up}}_{\omega\ell m} \phi^{\mathrm{up}+}_{\omega\ell m} + \int_{-\infty}^0 \mathrm{d}\widetilde{\omega}\, b^{\mathrm{up}\dagger}_{\omega\ell m} \phi^{\mathrm{up}-}_{\omega\ell m} \right\}. \tag{4.27}$$

We quantise the field by promoting the mode expansion coefficients in (4.27) to operators such that the field operator $\hat{\Phi}(x)$ is given by

$$\hat{\Phi}(x) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \left\{ \int_0^\infty \mathrm{d}\omega\, \hat{a}^{\mathrm{in}}_{\omega\ell m} \phi^{\mathrm{in}+}_{\omega\ell m} + \int_{-\infty}^0 \mathrm{d}\omega\, \hat{b}^{\mathrm{in}\dagger}_{\omega\ell m} \phi^{\mathrm{in}-}_{\omega\ell m} + \int_0^\infty \mathrm{d}\widetilde{\omega}\, \hat{a}^{\mathrm{up}}_{\omega\ell m} \phi^{\mathrm{up}+}_{\omega\ell m} + \int_{-\infty}^0 \mathrm{d}\widetilde{\omega}\, \hat{b}^{\mathrm{up}\dagger}_{\omega\ell m} \phi^{\mathrm{up}-}_{\omega\ell m} \right\}, \tag{4.28}$$

where the operators associated to the in-modes, $\hat{a}^{\mathrm{in}}_{\omega\ell m}$ and $\hat{b}^{\mathrm{in}}_{\omega\ell m}$, are defined for $\omega > 0$ and $\omega < 0$ respectively, and the operators associated to the up-modes, $\hat{a}^{\mathrm{up}}_{\omega\ell m}$ and $\hat{b}^{\mathrm{up}}_{\omega\ell m}$, are defined for $\widetilde{\omega} > 0$ and $\widetilde{\omega} < 0$ respectively. In (4.28), all of the positive-frequency modes $\phi^{\mathrm{in/up}+}_{\omega\ell m}$ are of positive-norm and all of the negative-frequency modes $\phi^{\mathrm{in/up}-}_{\omega\ell m}$ are of negative-norm; then, the operators $\hat{a}^{\mathrm{in/up}}_{\omega\ell m}$ and $\hat{b}^{\mathrm{in/up}}_{\omega\ell m}$ obey standard commutation relations:

$$\begin{aligned}
\left[\hat{a}^{\mathrm{in}}_{\omega\ell m}, \hat{a}^{\mathrm{in}\dagger}_{\omega'\ell' m'}\right] &= \delta\big(\omega-\omega'\big)\, \delta_{\ell\ell'}\, \delta_{mm'}, \quad \omega > 0,\\
\left[\hat{b}^{\mathrm{in}}_{\omega\ell m}, \hat{b}^{\mathrm{in}\dagger}_{\omega'\ell' m'}\right] &= \delta\big(\omega-\omega'\big)\, \delta_{\ell\ell'}\, \delta_{mm'}, \quad \omega < 0,\\
\left[\hat{a}^{\mathrm{up}}_{\omega\ell m}, \hat{a}^{\mathrm{up}\dagger}_{\omega'\ell' m'}\right] &= \delta\big(\omega-\omega'\big)\, \delta_{\ell\ell'}\, \delta_{mm'}, \quad \widetilde{\omega} > 0,\\
\left[\hat{b}^{\mathrm{up}}_{\omega\ell m}, \hat{b}^{\mathrm{up}\dagger}_{\omega'\ell' m'}\right] &= \delta\big(\omega-\omega'\big)\, \delta_{\ell\ell'}\, \delta_{mm'}, \quad \widetilde{\omega} < 0,
\end{aligned} \tag{4.29}$$

with any commutators not explicitly given in (4.29) vanishing. The 'past' Boulware state $|\mathrm{B}^-\rangle$ is then defined as the state annihilated by the $\hat{a}^{\mathrm{in/up}}_{\omega\ell m}$ and $\hat{b}^{\mathrm{in/up}}_{\omega\ell m}$ annihilation operators:

$$\begin{aligned}
\hat{a}^{\mathrm{in}}_{\omega\ell m}\, |\mathrm{B}^-\rangle &= 0, \quad \omega > 0,\\
\hat{b}^{\mathrm{in}}_{\omega\ell m}\, |\mathrm{B}^-\rangle &= 0, \quad \omega < 0,\\
\hat{a}^{\mathrm{up}}_{\omega\ell m}\, |\mathrm{B}^-\rangle &= 0, \quad \widetilde{\omega} > 0,\\
\hat{b}^{\mathrm{up}}_{\omega\ell m}\, |\mathrm{B}^-\rangle &= 0, \quad \widetilde{\omega} < 0.
\end{aligned} \tag{4.30}$$

The 'past' Boulware state $|\mathrm{B}^-\rangle$ contains no particles or antiparticles incoming from past null infinity $\mathscr{I}^-$ nor emanating from the past horizon $\mathcal{H}^-$. It is therefore the state which is as empty as possible as seen by a static observer at past null infinity $\mathscr{I}^-$. However, this state is not empty to an observer at future null infinity $\mathscr{I}^+$ where it contains an outgoing flux of particles in the superradiant modes [1]. We consider expectation values of observables in this state in §5.3.1.

4.3.2 'Future' Boulware state

We would like to construct a state that is as empty as possible to a static observer at future null infinity $\mathscr{I}^+$; this requirement corresponds to an absence of particles in the out-modes (3.139) of the field Φ. We require an orthonormal basis of scalar field modes in order to quantise the field; the out-modes together with the down-modes (3.147) constitute such a basis. Since the asymptotic forms of both the out-modes (3.68) and the down-modes (3.70) take particularly simple forms near the future horizon $\mathcal{H}^+$ and future null infinity $\mathscr{I}^+$, it is convenient to define the 'future' Boulware state $|\mathrm{B}^+\rangle$ with respect to the 'future' Cauchy surface Σ_{future} defined in (3.87). From our discussion in §4.1, we will first need to decompose the out- and down-modes into positive- and negative-frequency sets in order to canonically quantise the field Φ. We note that this state was first constructed in [1] where it was referred to as the "out" vacuum.

Positive- and negative-frequency out-modes

From the asymptotic forms of the out-mode radial function $X^{\mathrm{out}}_{\omega\ell}(r)$ (3.68), the flux of the out-modes through the future horizon $\mathcal{H}^+$ vanishes. Near future null infinity $\mathscr{I}^+$, the proper time experienced by a static observer is given by the Schwarzschild-like time coordinate t. The time-dependence of the out-modes (3.139) w.r.t t is given by

$$\phi^{\mathrm{out}}_{\omega\ell m} \propto \exp[-\mathrm{i}\omega t]\,. \tag{4.31}$$

Then, from (4.2), we define positive-frequency out-modes for $\omega > 0$ as

$$\phi^{\mathrm{out}+}_{\omega\ell m} = \frac{1}{\sqrt{4\pi|\omega|}}\,\mathrm{e}^{-\mathrm{i}\omega t}\,\frac{X^{\mathrm{out}}_{\omega\ell}(r)}{r}\,Y_{\ell m}(\theta,\varphi)\,, \quad \omega > 0, \tag{4.32}$$

where we note, from the expression for the norm of the out-modes (3.137), that the positive-frequency out-modes (4.32) are entirely of positive-norm since they have $\mathrm{sgn}(\omega) = 1$. Similarly, from (4.3), we define negative-frequency out-modes for $\omega < 0$ as

$$\phi^{\mathrm{out}-}_{\omega\ell m} = \frac{1}{\sqrt{4\pi|\omega|}}\,\mathrm{e}^{-\mathrm{i}\omega t}\,\frac{X^{\mathrm{out}}_{\omega\ell}(r)}{r}\,Y_{\ell m}(\theta,\varphi)\,, \quad \omega < 0, \tag{4.33}$$

where we note, from the expression for the norm of the out-modes (3.137), that the negative-frequency out-modes in (4.33) are entirely of negative-norm since they have $\mathrm{sgn}(\omega) = -1$. Since the positive- and negative-frequency out-modes are entirely of positive- and negative-norm respectively then, from our discussion in §4.1, the operators associated to these modes will obey standard commutation relations such as those given in (4.16).

While we have defined positive- and negative-frequency out-modes in order to construct the 'future' Boulware state $|\mathrm{B}^+\rangle$, we will also use the definitions in (4.32) and (4.33) when defining the 'future' Unruh state $|\mathrm{U}^+\rangle$ in §4.4.2.

Positive- and negative-frequency down-modes

From the asymptotic forms of the down-mode radial function $X^{\mathrm{down}}_{\omega\ell}(r)$ (3.70), the flux of the down-modes through future null infinity $\mathscr{I}^+$ vanishes. Near the future horizon

$\mathcal{H}^+$, the proper time experienced by a static observer is still the Schwarzschild-like time coordinate t, although we note that a static observer near the black hole event horizon is necessarily accelerating. The time-dependence of the down-modes (3.147) w.r.t t is

$$\phi^{\text{down}}_{\omega\ell m} \propto \exp[-\mathrm{i}\omega t]\,, \tag{4.34}$$

which suggests that we should similarly define down-modes to be positive- and negative-frequency for $\omega > 0$ and $\omega < 0$ respectively; however, from (3.145) the norms of the down-modes are proportional to $\text{sgn}(\widetilde{\omega})$, not $\text{sgn}(\omega)$. Thus, defining positive- and negative-frequency down-modes in this way would lead, from our discussion in §4.1, to non-standard commutation relations for the creation and annihilation operators associated to superradiant down-modes, such as those in (4.18) and (4.19), and their consequent misinterpretation. Instead, we define positive-frequency down-modes for $\widetilde{\omega} > 0$ as

$$\phi^{\text{down}+}_{\omega\ell m} = \frac{1}{\sqrt{4\pi|\widetilde{\omega}|}}\,\mathrm{e}^{-\mathrm{i}\omega t}\,\frac{X^{\text{down}}_{\omega\ell}(r)}{r}\,Y_{\ell m}(\theta,\varphi)\,, \quad \widetilde{\omega} > 0, \tag{4.35}$$

where we note, from the expression for the norm of the down-modes (3.145), that the positive-frequency down-modes in (4.35) are entirely of positive-norm since they have $\text{sgn}(\widetilde{\omega}) = 1$. Similarly, we define negative-frequency down-modes for $\widetilde{\omega} < 0$ as

$$\phi^{\text{down}-}_{\omega\ell m} = \frac{1}{\sqrt{4\pi|\widetilde{\omega}|}}\,\mathrm{e}^{-\mathrm{i}\omega t}\,\frac{X^{\text{down}}_{\omega\ell}(r)}{r}\,Y_{\ell m}(\theta,\varphi)\,, \quad \widetilde{\omega} < 0, \tag{4.36}$$

where, from the expression for the norm of the down-modes (3.145), the negative-frequency down-modes in (4.36) are entirely of negative-norm since they have $\text{sgn}(\widetilde{\omega}) = -1$. Since these positive- and negative-frequency down-modes are entirely of positive- and negative-norm respectively then, from our discussion in §4.1, the operators associated to these modes will obey standard commutation relations such as those given in (4.16).

Construction of the 'future' Boulware state

Taken together the positive- (4.32) and negative-frequency out-modes (4.33) and the positive- (4.35) and negative-frequency down-modes (4.36) constitute an orthonormal basis in which we can quantise the field Φ. Then, the scalar field can be expanded as

$$\Phi(x) = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\left\{\int_0^{\infty}\mathrm{d}\omega\, a^{\text{out}}_{\omega\ell m}\phi^{\text{out}+}_{\omega\ell m} + \int_{-\infty}^{0}\mathrm{d}\omega\, b^{\text{out}\dagger}_{\omega\ell m}\phi^{\text{out}-}_{\omega\ell m}\right.$$
$$\left. + \int_0^{\infty}\mathrm{d}\widetilde{\omega}\, a^{\text{down}}_{\omega\ell m}\phi^{\text{down}+}_{\omega\ell m} + \int_{-\infty}^{0}\mathrm{d}\widetilde{\omega}\, b^{\text{down}\dagger}_{\omega\ell m}\phi^{\text{down}-}_{\omega\ell m}\right\}. \tag{4.37}$$

We quantise the field by promoting the mode expansion coefficients in (4.37) to operators such that the field operator $\hat{\Phi}(x)$ is given by

$$\hat{\Phi}(x) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \left\{ \int_0^{\infty} \mathrm{d}\omega \, \hat{a}^{\mathrm{out}}_{\omega\ell m} \phi^{\mathrm{out}+}_{\omega\ell m} + \int_{-\infty}^{0} \mathrm{d}\omega \, \hat{b}^{\mathrm{out}\dagger}_{\omega\ell m} \phi^{\mathrm{out}-}_{\omega\ell m} \right.$$
$$\left. + \int_0^{\infty} \mathrm{d}\widetilde{\omega} \, \hat{a}^{\mathrm{down}}_{\omega\ell m} \phi^{\mathrm{down}+}_{\omega\ell m} + \int_{-\infty}^{0} \mathrm{d}\widetilde{\omega} \, \hat{b}^{\mathrm{down}\dagger}_{\omega\ell m} \phi^{\mathrm{down}-}_{\omega\ell m} \right\}, \quad (4.38)$$

where the operators associated to the out-modes, $\hat{a}^{\mathrm{out}}_{\omega\ell m}$ and $\hat{b}^{\mathrm{out}}_{\omega\ell m}$, are defined for $\omega > 0$ and $\omega < 0$ respectively, and the operators associated to the down-modes, $\hat{a}^{\mathrm{down}}_{\omega\ell m}$ and $\hat{b}^{\mathrm{down}}_{\omega\ell m}$, are defined for $\widetilde{\omega} > 0$ and $\widetilde{\omega} < 0$ respectively. In (4.38), all of the positive-frequency modes $\phi^{\mathrm{out/down}+}_{\omega\ell m}$ are of positive-norm and all of the negative-frequency modes $\phi^{\mathrm{out/down}-}_{\omega\ell m}$ are of negative-norm; then, the $\hat{a}^{\mathrm{out/down}}_{\omega\ell m}$ and $\hat{b}^{\mathrm{out/down}}_{\omega\ell m}$ obey standard commutation relations:

$$\begin{aligned}
\left[\hat{a}^{\mathrm{out}}_{\omega\ell m}, \hat{a}^{\mathrm{out}\dagger}_{\omega'\ell' m'}\right] &= \delta\big(\omega - \omega'\big)\, \delta_{\ell\ell'}\, \delta_{mm'}, \quad \omega > 0, \\
\left[\hat{b}^{\mathrm{out}}_{\omega\ell m}, \hat{b}^{\mathrm{out}\dagger}_{\omega'\ell' m'}\right] &= \delta\big(\omega - \omega'\big)\, \delta_{\ell\ell'}\, \delta_{mm'}, \quad \omega < 0, \\
\left[\hat{a}^{\mathrm{down}}_{\omega\ell m}, \hat{a}^{\mathrm{down}\dagger}_{\omega'\ell' m'}\right] &= \delta\big(\omega - \omega'\big)\, \delta_{\ell\ell'}\, \delta_{mm'}, \quad \widetilde{\omega} > 0, \\
\left[\hat{b}^{\mathrm{down}}_{\omega\ell m}, \hat{b}^{\mathrm{down}\dagger}_{\omega'\ell' m'}\right] &= \delta\big(\omega - \omega'\big)\, \delta_{\ell\ell'}\, \delta_{mm'}, \quad \widetilde{\omega} < 0,
\end{aligned} \quad (4.39)$$

with any commutators not explicitly given in (4.39) vanishing. The 'future' Boulware state is then defined as the state annihilated by the $\hat{a}^{\mathrm{out/down}}_{\omega\ell m}, \hat{b}^{\mathrm{out/down}}_{\omega\ell m}$ annihilation operators:

$$\begin{aligned}
\hat{a}^{\mathrm{out}}_{\omega\ell m} \left|\mathrm{B}^+\right\rangle &= 0, \quad \omega > 0, \\
\hat{b}^{\mathrm{out}}_{\omega\ell m} \left|\mathrm{B}^+\right\rangle &= 0, \quad \omega < 0, \\
\hat{a}^{\mathrm{down}}_{\omega\ell m} \left|\mathrm{B}^+\right\rangle &= 0, \quad \widetilde{\omega} > 0, \\
\hat{b}^{\mathrm{down}}_{\omega\ell m} \left|\mathrm{B}^+\right\rangle &= 0, \quad \widetilde{\omega} < 0.
\end{aligned} \quad (4.40)$$

The 'future' Boulware state $|\mathrm{B}^+\rangle$ contains no particles or antiparticles outgoing at future null infinity $\mathscr{I}^+$ or going down the future horizon $\mathcal{H}^+$. It is therefore the state which is as empty as possible as seen by a static observer at future null infinity $\mathscr{I}^+$ [1].

4.3.3 'Boulware-like' state

It is natural to ask as to whether it is possible to define a vacuum state in RN spacetime that is as empty as possible to a static observer at both past and future null infinity $\mathscr{I}^\pm$.

In §4.3.1, we defined the 'past' Boulware state $|\mathrm{B}^-\rangle$, which is as empty as possible to a static observer at past null infinity $\mathscr{I}^-$ but contains an outgoing flux of particles in the superradiant modes at future null infinity $\mathscr{I}^+$. Analogous comments apply for the 'future' Boulware state $|\mathrm{B}^+\rangle$, which we defined in §4.3.2. This is unlike the case of the Schwarzschild Boulware state $|\mathrm{B_s}\rangle$ in §2.3.1, where we are able to define a state that is as empty as possible as seen by an observer at both past and future null infinity $\mathscr{I}^\pm$.

The questions remains, then, as to whether it is possible to define an analogue of $|\mathrm{B_s}\rangle$ in Reissner-Nordström spacetime that is similarly as empty as possible at $\mathscr{I}^\pm$; this requirement corresponds to an absence of particles in both the in-modes (3.106) and the out-modes (3.139) of the field Φ. Using the definitions of positive- (4.22) and negative-frequency in-modes (4.23) as well as those of positive- (4.32) and negative-frequency out-modes (4.33), we can expand the field Φ as

$$\Phi(x) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \left\{ \int_0^{\infty} d\omega \left(\tilde{a}^{\text{in}}_{\omega\ell m} \phi^{\text{in}+}_{\omega\ell m} + \tilde{a}^{\text{out}}_{\omega\ell m} \phi^{\text{out}+}_{\omega\ell m} \right) \right. \\ \left. + \int_{-\infty}^{0} d\omega \left(\tilde{b}^{\text{in}\dagger}_{\omega\ell m} \phi^{\text{in}-}_{\omega\ell m} + \tilde{b}^{\text{out}\dagger}_{\omega\ell m} \phi^{\text{out}-}_{\omega\ell m} \right) \right\} . \quad (4.41)$$

In (4.41), the positive-frequency modes $\phi^{\text{in/out}+}_{\omega\ell m}$ are entirely of positive-norm since they have $\text{sgn}(\omega) = 1$ and the negative-frequency modes $\phi^{\text{in/out}-}_{\omega\ell m}$ are entirely of negative-norm since they have $\text{sgn}(\omega) = -1$.

However, the in-modes and the out-modes, together, do not form an orthonormal basis of scalar field modes, which we require to quantise the field Φ. The in-modes are orthogonal to the up-modes; we would then like to re-express the out-modes in (4.41) in terms of in- and up-modes in order to expand the field Φ in an orthonormal basis before quantisation. We note that, since the out-modes are defined to be positive-frequency for $\omega > 0$ (4.32) and negative-frequency for $\omega < 0$ (4.33) then the in- and up-modes, which we use to re-express the out-modes, should also be defined as being positive-frequency for $\omega > 0$ and negative-frequency for $\omega < 0$.

We can use the expression in (3.71a) for the out-mode radial function $X^{\text{out}}_{\omega\ell}$ in terms of the radial functions $X^{\text{in}}_{\omega\ell}(r)$ and $X^{\text{up}}_{\omega\ell}(r)$ of the in-modes and the up-modes respectively; then, multiplying (3.71a) by an appropriate factor, we have

$$\begin{aligned} &\frac{1}{\sqrt{4\pi|\omega|}}\, e^{-i\omega t}\, \frac{X^{\text{out}}_{\omega\ell}}{r}\, Y_{\ell m}(\theta,\varphi) \\ &\quad = \frac{1}{\sqrt{4\pi|\omega|}}\, e^{-i\omega t}\, \frac{1}{r} \left(A^{\text{in}*}_{\omega\ell} X^{\text{in}}_{\omega\ell} + B^{\text{in}*}_{\omega\ell} X^{\text{up}}_{\omega\ell} \right) Y_{\ell m}(\theta,\varphi) \\ &\quad = A^{\text{in}*}_{\omega\ell}\, \frac{1}{\sqrt{4\pi|\omega|}}\, e^{-i\omega t}\, \frac{X^{\text{in}}_{\omega\ell}}{r}\, Y_{\ell m}(\theta,\varphi) + B^{\text{in}*}_{\omega\ell} \left| \frac{\widetilde{\omega}}{\omega} \right|^{\frac{1}{2}} \frac{1}{\sqrt{4\pi|\widetilde{\omega}|}}\, e^{-i\omega t}\, \frac{X^{\text{up}}_{\omega\ell}}{r}\, Y_{\ell m}(\theta,\varphi) \,. \end{aligned} \quad (4.42)$$

Using the expression for the in-, up- and out-modes in (3.106), (3.131) and (3.139) respectively, (4.42) becomes

$$\phi^{\text{out}}_{\omega\ell m} = A^{\text{in}*}_{\omega\ell}\, \phi^{\text{in}}_{\omega\ell m} + \left| \frac{\widetilde{\omega}}{\omega} \right|^{\frac{1}{2}} B^{\text{in}*}_{\omega\ell}\, \phi^{\text{up}}_{\omega\ell m}, \quad (4.43)$$

and so the set of positive-frequency out-modes $\phi^{\text{out}+}_{\omega\ell m}$ in (4.41) is given, in terms of positive-frequency in-modes (4.22) and up-modes, by

$$\phi^{\text{out}+}_{\omega\ell m} = A^{\text{in}*}_{\omega\ell}\, \phi^{\text{in}+}_{\omega\ell m} + \left| \frac{\widetilde{\omega}}{\omega} \right|^{\frac{1}{2}} B^{\text{in}*}_{\omega\ell}\, \phi^{\text{up}}_{\omega\ell m}, \quad (4.44)$$

while the set of negative-frequency out-modes $\phi^{\text{out}-}_{\omega\ell m}$ in (4.41) is given, in terms of negative-frequency in-modes (4.23) and up-modes, by

$$\phi^{\text{out}-}_{\omega\ell m} = A^{\text{in}*}_{\omega\ell}\,\phi^{\text{in}-}_{\omega\ell m} + \left|\frac{\widetilde{\omega}}{\omega}\right|^{\frac{1}{2}} B^{\text{in}*}_{\omega\ell}\,\phi^{\text{up}}_{\omega\ell m}. \tag{4.45}$$

Note, we have dropped the notation $\phi^{\text{up}\pm}_{\omega\ell m}$ to denote positive- and negative-frequency up-modes in (4.44) and (4.45); while it is simple to rewrite the out-modes in terms of in-modes, there is a subtlety in re-writing the out-modes in terms of up-modes. Positive-frequency in- (4.22) and out-modes (4.32) are defined for $\omega > 0$ and negative-frequency in- (4.23) and out-modes (4.33) are defined for $\omega < 0$. In contrast, positive- and negative-frequency up-modes are defined, in (4.25) and (4.26), for $\widetilde{\omega} > 0$ and $\widetilde{\omega} < 0$ respectively.

In defining the 'Boulware-like' state $|\text{B}\rangle$, we are changing the interpretation of positive- and negative-frequency up-modes from the definitions in (4.25) and (4.26) respectively. Specifically, when defining the 'past' Boulware state $|\text{B}^-\rangle$ in §4.3.1, we defined up-modes to be positive-frequency for $\widetilde{\omega} > 0$ (4.25) and negative-frequency for $\widetilde{\omega} < 0$ (4.26). In defining the 'Boulware-like' state $|\text{B}\rangle$, we would like to define all field modes to be positive-frequency for $\omega > 0$ and negative-frequency for $\omega < 0$, meaning that to denote positive-frequency up-modes as $\phi^{\text{up}+}_{\omega\ell m}$ and negative-frequency up-modes as $\phi^{\text{up}-}_{\omega\ell m}$ in our expansion of the field Φ would be inconsistent with (4.25) and (4.26) respectively.

However, the labels $\phi^{\text{up}\pm}_{\omega\ell m}$ remain useful; to see why this is the case, consider the following. If $qQ > 0$, the up-modes in the set of positive-frequency out-modes $\phi^{\text{out}+}_{\omega\ell m}$ in (4.44) with $0 < \omega < \frac{qQ}{r_+}$, or equivalently $-\frac{qQ}{r_+} < \widetilde{\omega} < 0$, will be of negative-norm from (3.129). Similarly, if $qQ < 0$, the up-modes in the set of negative-frequency out-modes $\phi^{\text{out}-}_{\omega\ell m}$ in (4.45) with $0 > \omega > \frac{qQ}{r_+}$, or equivalently $-\frac{qQ}{r_+} > \widetilde{\omega} > 0$, will be of positive-norm from (3.129). Therefore, we can denote up-modes of positive-norm using the notation $\phi^{\text{up}+}_{\omega\ell m}$ and up-modes of negative-norm using the notation $\phi^{\text{up}-}_{\omega\ell m}$. We wish to remind the reader that denoting positive-norm up-modes as $\phi^{\text{up}+}_{\omega\ell m}$ and negative-norm up-modes as $\phi^{\text{up}-}_{\omega\ell m}$ is a change of notation from §4.3.1.

Now, we can return to the task of re-expressing the out-modes in (4.41) in terms of in- and up-modes. In line with our discussion in the preceding paragraph, we will separate the range of the integral of the positive-frequency up-modes, which is given by $0 < \omega < \infty$, and the range of the integral of the negative-frequency up-modes, which is given by $0 > \omega > -\infty$, into two separate ranges in both cases such that we can correctly denote up-modes that are of positive-norm as $\phi^{\text{up}+}_{\omega\ell m}$ and up-modes that are of negative-norm as $\phi^{\text{up}-}_{\omega\ell m}$. Then, using (4.44) and (4.45), the scalar field Φ can be expanded in an orthonormal basis of in- and up-modes as

$$\Phi(x) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \left\{ \int_0^\infty d\omega \, a_{\omega\ell m}^{\text{in}} \phi_{\omega\ell m}^{\text{in}+} + \int_{-\infty}^0 d\omega \, b_{\omega\ell m}^{\text{in}\dagger} \phi_{\omega\ell m}^{\text{in}-} \right.$$
$$+ \int_{\max\left\{\frac{qQ}{r_+},0\right\}}^\infty d\omega \, a_{\omega\ell m}^{\text{up}} \phi_{\omega\ell m}^{\text{up}+} + \int_0^{\max\left\{\frac{qQ}{r_+},0\right\}} d\omega \, a_{\omega\ell m}^{\text{up}} \phi_{\omega\ell m}^{\text{up}-}$$
$$\left. + \int_{\min\left\{\frac{qQ}{r_+},0\right\}}^0 d\omega \, b_{\omega\ell m}^{\text{up}\dagger} \phi_{\omega\ell m}^{\text{up}+} + \int_{-\infty}^{\min\left\{\frac{qQ}{r_+},0\right\}} d\omega \, b_{\omega\ell m}^{\text{up}\dagger} \phi_{\omega\ell m}^{\text{up}-} \right\}, \quad (4.46)$$

where the expansion coefficients in (4.46) are related to those in (4.41) by the expressions

$$\begin{aligned}
a_{\omega\ell m}^{\text{in}} &= \tilde{a}_{\omega\ell m}^{\text{in}} + A_{\omega\ell}^{\text{in}*} \, \tilde{a}_{\omega\ell m}^{\text{out}}, && \omega > 0, \\
b_{\omega\ell m}^{\text{in}\dagger} &= \tilde{b}_{\omega\ell m}^{\text{in}\dagger} + A_{\omega\ell}^{\text{in}*} \, \tilde{b}_{\omega\ell m}^{\text{out}\dagger}, && \omega < 0, \\
a_{\omega\ell m}^{\text{up}} &= \left| \frac{\widetilde{\omega}}{\omega} \right|^{\frac{1}{2}} B_{\omega\ell}^{\text{in}*} \, \tilde{a}_{\omega\ell m}^{\text{out}}, && \omega > 0, \\
b_{\omega\ell m}^{\text{up}\dagger} &= \left| \frac{\widetilde{\omega}}{\omega} \right|^{\frac{1}{2}} B_{\omega\ell}^{\text{in}*} \, \tilde{b}_{\omega\ell m}^{\text{out}\dagger}, && \omega < 0.
\end{aligned} \quad (4.47)$$

We quantise the field by promoting the mode expansion coefficients in (4.46) to operators such that the field operator $\hat{\Phi}(x)$ is given by

$$\hat{\Phi}(x) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \left\{ \int_0^\infty d\omega \, \hat{a}_{\omega\ell m}^{\text{in}} \phi_{\omega\ell m}^{\text{in}+} + \int_{-\infty}^0 d\omega \, \hat{b}_{\omega\ell m}^{\text{in}\dagger} \phi_{\omega\ell m}^{\text{in}-} \right.$$
$$+ \int_{\max\left\{\frac{qQ}{r_+},0\right\}}^\infty d\omega \, \hat{a}_{\omega\ell m}^{\text{up}} \phi_{\omega\ell m}^{\text{up}+} + \int_0^{\max\left\{\frac{qQ}{r_+},0\right\}} d\omega \, \hat{a}_{\omega\ell m}^{\text{up}} \phi_{\omega\ell m}^{\text{up}-}$$
$$\left. + \int_{\min\left\{\frac{qQ}{r_+},0\right\}}^0 d\omega \, \hat{b}_{\omega\ell m}^{\text{up}\dagger} \phi_{\omega\ell m}^{\text{up}+} + \int_{-\infty}^{\min\left\{\frac{qQ}{r_+},0\right\}} d\omega \, \hat{b}_{\omega\ell m}^{\text{up}\dagger} \phi_{\omega\ell m}^{\text{up}-} \right\}, \quad (4.48)$$

where the operators $\hat{a}_{\omega\ell m}^{\text{in/up}}$ and $\hat{b}_{\omega\ell m}^{\text{in/up}}$ are defined for $\omega > 0$, and the operators $\hat{a}_{\omega\ell m}^{\text{in/up}\dagger}$ and $\hat{b}_{\omega\ell m}^{\text{in/up}\dagger}$ are defined for $\omega < 0$.

In (4.48), the positive-frequency in-modes $\phi_{\omega\ell m}^{\text{in}+}$ are entirely of positive-norm and the negative-frequency modes $\phi_{\omega\ell m}^{\text{in}-}$ are entirely of negative-norm. Therefore, the operators $\hat{a}_{\omega\ell m}^{\text{in}}$ and $\hat{b}_{\omega\ell m}^{\text{in}}$ follow standard commutation relations.

Furthermore, the positive-norm up-modes $\phi_{\omega\ell m}^{\text{up}+}$ in the first term of the second line on the r.h.s of (4.48) are entirely of positive-frequency since they have $\text{sgn}(\omega) = 1$, meaning that we would indeed like to interpret the $\hat{a}_{\omega\ell m}^{\text{up}}$ as annihilation operators. From our discussion in §4.1, since these modes have $\text{sgn}(\omega\widetilde{\omega}) = 1$, i.e. they are both positive-frequency and of positive-norm, we require that the operators associated to these modes follow standard commutation relations when $\text{sgn}(\omega\widetilde{\omega}) = 1$.

However, the situation is complicated by the fact that the same operators $\hat{a}_{\omega\ell m}^{\text{up}}$ are also acting on the negative-norm up-modes $\phi_{\omega\ell m}^{\text{up}-}$ in the second term of the second line on the r.h.s of (4.48). If $qQ < 0$, the modes in this term also have $\text{sgn}(\omega) = -1$ from (3.58); however, in this case the integral in this term vanishes because $0 > \frac{qQ}{r_+}$ such that the limits of the integral coincide. If $qQ > 0$, on the other hand, the modes in this term have $\text{sgn}(\omega) = 1$, or equivalently $\omega > 0$, and are therefore positive-frequency modes. Therefore, we would still like to interpret the $\hat{a}^{\text{up}}_{\omega\ell m}$ as annihilation operators, despite them multiplying up-modes of negative-norm.

In summary, we would like to interpret the $\hat{a}^{\text{up}}_{\omega\ell m}$ as annihilation operators in both the cases where they are acting on positive-norm up-modes with $\text{sgn}(\omega\widetilde{\omega}) = 1$ and where they are acting on negative-norm up-modes with $\text{sgn}(\omega\widetilde{\omega}) = -1$. Analogous comments apply in terms of wanting to interpret the $\hat{b}^{\text{up}\dagger}_{\omega\ell m}$ as creation operators in both the cases where they are acting on negative-norm up-modes with $\text{sgn}(\omega\widetilde{\omega}) = 1$ and where they are acting on positive-norm up-modes with $\text{sgn}(\omega\widetilde{\omega}) = -1$.

Then, recalling our discussion in §4.2.4, we can multiply the commutation relations of the operators $\hat{a}^{\text{up}}_{\omega\ell m}$ and $\hat{b}^{\text{up}}_{\omega\ell m}$ by the eta-function $\eta_{\omega\widetilde{\omega}}$ (4.20). Therefore, the operators $\hat{a}^{\text{in/up}}_{\omega\ell m}$ and $\hat{b}^{\text{in/up}}_{\omega\ell m}$ obey the following non-standard commutation relations

$$\begin{aligned}
\left[\hat{a}^{\text{in}}_{\omega\ell m}, \hat{a}^{\text{in}\dagger}_{\omega'\ell'm'}\right] &= \delta\big(\omega-\omega'\big)\,\delta_{\ell\ell'}\,\delta_{mm'}, && \omega > 0,\\
\left[\hat{b}^{\text{in}}_{\omega\ell m}, \hat{b}^{\text{in}\dagger}_{\omega'\ell'm'}\right] &= \delta\big(\omega-\omega'\big)\,\delta_{\ell\ell'}\,\delta_{mm'}, && \omega < 0,\\
\left[\hat{a}^{\text{up}}_{\omega\ell m}, \hat{a}^{\text{up}\dagger}_{\omega'\ell'm'}\right] &= \eta_{\omega\widetilde{\omega}}\,\delta\big(\omega-\omega'\big)\,\delta_{\ell\ell'}\,\delta_{mm'}, && \omega > 0,\\
\left[\hat{b}^{\text{up}}_{\omega\ell m}, \hat{b}^{\text{up}\dagger}_{\omega'\ell'm'}\right] &= \eta_{\omega\widetilde{\omega}}\,\delta\big(\omega-\omega'\big)\,\delta_{\ell\ell'}\,\delta_{mm'}, && \omega < 0,
\end{aligned} \tag{4.49}$$

with any commutators not explicitly given in (4.49) vanishing. The 'Boulware-like' state $|\text{B}\rangle$ is then defined as the state annihilated by the $\hat{a}^{\text{in/up}}_{\omega\ell m}$ and $\hat{b}^{\text{in/up}}_{\omega\ell m}$ annihilation operators:

$$\begin{aligned}
\hat{a}^{\text{in}}_{\omega\ell m}\,|\text{B}\rangle &= 0, && \omega > 0,\\
\hat{b}^{\text{in}}_{\omega\ell m}\,|\text{B}\rangle &= 0, && \omega > 0,\\
\hat{a}^{\text{up}}_{\omega\ell m}\,|\text{B}\rangle &= 0, && \omega < 0,\\
\hat{b}^{\text{up}}_{\omega\ell m}\,|\text{B}\rangle &= 0 && \omega < 0.
\end{aligned} \tag{4.50}$$

The 'Boulware-like' state $|\text{B}\rangle$ contains no particles or antiparticles in the in-modes, as well as the non-superradiant up-modes, which is similar to the case of the 'past Boulware' state $|\text{B}^-\rangle$ that we defined in §4.3.1. However, the 'Boulware-like' state $|\text{B}\rangle$ differs from $|\text{B}^-\rangle$ in that it may contain particles and antiparticles in the superradiant 'up' modes. We explore this further in §5.5 when we study the expectation values of observables in this state.

4.4 Unruh states

In §2.3.2, we introduced the Schwarzschild Unruh state $|\text{U}_\text{s}\rangle$; this state exhibits an absence of incoming radiation from past null infinity $\mathscr{I}^-$ but predicts a thermalised flux of radiation, i.e. Hawking radiation, outgoing at future null infinity $\mathscr{I}^+$. We would like to define analogous states for a charged scalar field Φ in Reissner-Nordström spacetime.

In §4.2.3, we introduced the concept of 'past' and 'future' states; it turns out that the Schwarzschild Unruh state $|U_s\rangle$ that we defined in §2.3.2 is effectively a 'past' state. To see that this is the case, note that we defined $|U_s\rangle$ with respect to the corresponding 'past' Cauchy surface in Schwarzschild spacetime. Then $|U_s\rangle$ was defined using the Schwarzschild in- and up-modes, with the lack of incoming radiation at $\mathscr{I}^-$ corresponding to an absence of particles in the in-modes and the outgoing Hawking radiation at $\mathscr{I}^+$ corresponding to a thermalised flux of particles in the up-modes.

Then, the first analogue of the Schwarzschild Unruh state $|U_s\rangle$ that we will define in Reissner-Nordström spacetime is the 'past' Unruh state $|U^-\rangle$ in §4.4.1; this state is as empty as possible to a static observer at past null infinity $\mathscr{I}^-$ but predicts outgoing Hawking radiation at future null infinity $\mathscr{I}^+$. We note that the 'past' Unruh state $|U^-\rangle$ was first studied by Gibbons [39]. We need not define a '-like' state since the 'past' Unruh state will be defined with a similar physical interpretation to the Schwarzschild Unruh state. We will also define the 'future' Unruh state $|U^+\rangle$ in §4.4.2; this state is the time-reverse of the 'past' Unruh state $|U^-\rangle$.

4.4.1 'Past' Unruh state

We would like to construct a state that exhibits an absence of incoming radiation from past null infinity $\mathscr{I}^-$ but which predicts outgoing Hawking radiation at future null infinity $\mathscr{I}^+$; in terms of the scalar field modes defined in §3.3, this corresponds to as absence of particles in the in-modes (3.106) and a thermalised flux of particles in the up-modes (3.131).

Together, the in- and up-modes constitute an orthonormal basis which we require to quantise the field. From our discussion in §4.1, we first need to decompose the in- and up-modes into positive- and negative-frequency sets to canonically quantise the field.

The lack of incoming radiation in the 'past' Unruh state $|U^-\rangle$ corresponds to an absence of particles in the in-modes of the field Φ as seen by a static observer at past null infinity $\mathscr{I}^-$. Far from the black hole, the proper time experienced by a static observer is given by the Schwarzschild-like coordinate t. We have already defined in-modes that are positive- and negative-frequency w.r.t t when defining the 'past' Boulware state $|B^-\rangle$ in §4.3.1. Then, the definitions of positive- and negative-frequency in-modes that we require are those given in (4.22) and (4.23) respectively.

It should be noted that it is unsurprising that, in defining the 'past' Unruh state $|U^-\rangle$, we are able to use the same definitions of positive- and negative-frequency in-modes as when we defined the 'past' Boulware state $|B^-\rangle$; both of these states share the property of being as empty as possible to a static observer at past null infinity $\mathscr{I}^-$. We now need to define positive- and negative-frequency thermalised up-modes.

Positive- and negative-frequency thermalised up-modes

The construction of positive- and negative-frequency thermalised up-modes in this section differs in some aspects from the corresponding derivation for a neutral scalar field, such as that used in §2.3.2 or in [5, 89]. Therefore, this section will be necessarily detailed.

The 'past' Unruh state $|U^-\rangle$ has the additional interpretation of predicting Hawking radiation outgoing at future null infinity $\mathscr{I}^+$; this property corresponds to a thermalised flux of particles in the up-modes emanating from the past horizon $\mathcal{H}^-$. The Kruskal coordinate U (3.24) is the affine parameter along the null generators of the past horizon $\mathcal{H}^-$. Then, near $\mathcal{H}^-$, the natural choice of time-coordinate w.r.t which we can define positive- and negative-frequency up-modes is the Kruskal retarded time U.

From our discussion in §4.1.2 on defining positive- and negative-frequency modes w.r.t Kruskal coordinates, we will make use of the lemma (4.6) and its complex conjugate (4.7).

Up-modes in region I: We begin by expressing the asymptotic form of the up-modes (3.66) near $\mathcal{H}^-$ in terms of Kruskal coordinates; using (3.21) and (3.26), we have

$$\begin{aligned}\phi^{\text{up}}_{\omega\ell m} &= \frac{1}{\sqrt{4\pi|\widetilde{\omega}|}}\,\frac{1}{r}\,\mathrm{e}^{-\mathrm{i}\omega t}\,\mathrm{e}^{\mathrm{i}\widetilde{\omega}r_*}\,Y_{\ell m}(\theta,\varphi)\\ &= \frac{1}{\sqrt{4\pi|\widetilde{\omega}|}}\,\frac{1}{r}\exp\left[\mathrm{i}\widetilde{\omega}\,\frac{(v-u)}{2}\right]\exp\left[-\mathrm{i}\omega\,\frac{(u+v)}{2}\right]Y_{\ell m}(\theta,\varphi)\\ &= \frac{1}{\sqrt{4\pi|\widetilde{\omega}|}}\,\frac{1}{r}\exp\left[-\frac{\mathrm{i}\,(\omega+\widetilde{\omega})}{2}\,u\right]\exp\left[-\frac{\mathrm{i}\,(\omega-\widetilde{\omega})}{2}\,v\right]Y_{\ell m}(\theta,\varphi)\\ &= \frac{1}{\sqrt{4\pi|\widetilde{\omega}|}}\,\frac{1}{r}\exp\left[\frac{\mathrm{i}\,(\omega+\widetilde{\omega})}{2\kappa}\ln\left(-\widetilde{U}\right)\right]\exp\left[-\frac{\mathrm{i}\,(\omega-\widetilde{\omega})}{2\kappa}\ln\left(\widetilde{V}\right)\right]Y_{\ell m}(\theta,\varphi)\,, \end{aligned}\tag{4.51}$$

where we have used the dimensionless quantities $\widetilde{U}$ and $\widetilde{V}$ (3.33) to simplify the expression in (4.51). An important point to note is that the Kruskal coordinate U is defined throughout the entire Reissner-Nordström spacetime, i.e. on all four regions of the Penrose diagram in Figure 3.1. The up-modes are defined in regions I and III, where $\widetilde{U}<0$. We can trivially extend their definition to regions II and IV by using the Heaviside function (4.5) to demand that they vanish when $\widetilde{U}>0$. Then, (4.51) becomes

$$\phi^{\text{up}}_{\omega\ell m} = \frac{1}{\sqrt{4\pi|\widetilde{\omega}|}}\,\frac{1}{r}\exp\left[\frac{\mathrm{i}\,(\omega+\widetilde{\omega})}{2\kappa}\ln\left(-\widetilde{U}\right)\right]\exp\left[-\frac{\mathrm{i}\,(\omega-\widetilde{\omega})}{2\kappa}\ln\left(\widetilde{V}\right)\right]Y_{\ell m}(\theta,\varphi)\,\Theta\left(-\widetilde{U}\right).\tag{4.52}$$

Comparing the expression for the asymptotic form of the up-modes near the past horizon $\mathcal{H}^-$ in (4.52) with the lemma (4.6), we see that the second term in the lemma can be constructed from the expression in (4.52) if we take

$$\mathfrak{X}^{\text{up}+} = \widetilde{U} \quad\text{ and }\quad \mathfrak{q}^{\text{up}+} = -\frac{(\omega+\widetilde{\omega})}{2\kappa},\tag{4.53}$$

where the label up+ denotes that these values are chosen to define positive-frequency up-modes with respect to the Kruskal coordinate U. Then the lemma (4.6) becomes

$$\begin{aligned}\int_{-\infty}^{\infty}\mathrm{d}\widetilde{U}\,\mathrm{e}^{-\mathrm{i}\mathfrak{p}\widetilde{U}}\Bigg\{&\exp\left[\frac{\mathrm{i}\,(\omega+\widetilde{\omega})}{2\kappa}\ln\left(\widetilde{U}\right)\right]\Theta\left(\widetilde{U}\right)\\ &+\exp\left[\pi\frac{(\omega+\widetilde{\omega})}{2\kappa}\right]\exp\left[\frac{\mathrm{i}\,(\omega+\widetilde{\omega})}{2\kappa}\ln\left(-\widetilde{U}\right)\right]\Theta\left(-\widetilde{U}\right)\Bigg\} = 0.\end{aligned}\tag{4.54}$$

We now need to define a set of modes to construct the first term of the lemma in (4.54).

Region IV down-modes: We see that the first term of the lemma (4.54) can be constructed from a set of modes in regions II and IV, vanishing in regions I and III, with the same dependence on $\mathfrak{q}^{\mathrm{up}+}$ (4.53) as the asymptotic form of the up-modes in (4.52) but containing factors of $-\widetilde{U}$ as opposed to $\widetilde{U}$; this could be achieved by simply making the transformation $\widetilde{U} \to -\widetilde{U}$. However, the interpretation of such a set of modes is unclear.

It is attractive instead to define a set of modes by making both of the transformations $\widetilde{U} \to -\widetilde{U}$ and $\widetilde{V} \to -\widetilde{V}$ [89], which results in a set of modes that are nonzero in regions II and IV and vanishing elsewhere, as well as being orthogonal to the up-modes in (4.52) since the two sets of modes are defined in different regions of spacetime. Then, the asymptotic form of a set of modes $\psi^{\mathrm{down}}_{\omega\ell m}$ near the region IV future horizon $\mathcal{H}^+_{\mathrm{IV}}$ is given by

$$\psi^{\mathrm{down}}_{\omega\ell m} = \frac{1}{\sqrt{4\pi|\widetilde{\omega}|}}\,\frac{1}{r}\exp\left[\frac{\mathrm{i}\,(\omega+\widetilde{\omega})}{2\kappa}\ln\left(\widetilde{U}\right)\right]\exp\left[-\frac{\mathrm{i}\,(\omega-\widetilde{\omega})}{2\kappa}\ln\left(-\widetilde{V}\right)\right]Y_{\ell m}(\theta,\varphi)\,\Theta\left(\widetilde{U}\right). \tag{4.55}$$

In addition to having the desired dependence on the parameters $\mathfrak{q}^{\mathrm{up}+}$ and $\widetilde{U}$ (4.53), the modes in (4.55) have an intuitive interpretation in that they represent the region IV analogue of the down-modes (3.147) that are defined in region I. In order to see this is the case, recall that the up-modes in region I (3.131) are defined as emerging from the past horizon $\mathcal{H}^-$ with unit flux and are travelling towards future infinity $\mathscr{I}^+$, with modes being scattered down the future horizon $\mathcal{H}^+$; they are nonzero in regions I and III only.

Performing the transformations $\widetilde{U} \to -\widetilde{U}$ and $\widetilde{V} \to -\widetilde{V}$ then shifts the modes such that they are now nonzero in regions II and IV and vanish elsewhere, as we can see from the factor of $\Theta(\widetilde{U})$ in (4.55). Specifically, the modes $\psi^{\mathrm{down}}_{\omega\ell m}$ are now emerging from the future horizon $\mathcal{H}^+_{\mathrm{IV}}$ with unit flux, and are travelling to past null infinity $\mathscr{I}^-_{\mathrm{IV}}$ with some modes being scattered down the past horizon $\mathcal{H}^-_{\mathrm{IV}}$; the modes vanish near future null infinity $\mathscr{I}^+_{\mathrm{IV}}$. These conditions are similar to those used the define the down-modes (3.147) in region I, although the down-modes $\phi^{\mathrm{down}}_{\omega\ell m}$ are defined to be incident upon $\mathcal{H}^+$ with unit flux where the modes $\psi^{\mathrm{down}}_{\omega\ell m}$ are defined to be emanating from $\mathcal{H}^+_{\mathrm{IV}}$ with unit flux.

We have given the modes $\psi^{\mathrm{down}}_{\omega\ell m}$ in (4.55) the label "down" but the $\phi^{\mathrm{down}}_{\omega\ell m}$ and $\psi^{\mathrm{down}}_{\omega\ell m}$ remain distinct and defined in different regions of the spacetime diagram. We will henceforth refer to the modes $\psi^{\mathrm{down}}_{\omega\ell m}$ in (4.55) as the region IV down-modes; the notation $\psi^{\mathrm{down}}_{\omega\ell m}$ serves to distinguish these modes from the (region I) down-modes $\phi^{\mathrm{down}}_{\omega\ell m}$ in (3.147).

It is instructive to examine the norms of these modes before proceeding. In order to compare the calculation of the norms of the region IV down-modes with those of the scalar field modes in §3.4, it will be helpful to rewrite the expression for the asymptotic form of the region IV down-modes near $\mathcal{H}^+_{\mathrm{IV}}$ in (4.55) in terms of another set of Schwarzschild-like coordinates that are defined in region IV. This, in turn, will require us to first define another set of lightcone coordinates, which are similarly defined in region IV.

Coordinates in region IV: In defining the region IV down-modes, we made the transformations $\widetilde{U} \to -\widetilde{U}$ and $\widetilde{V} \to -\widetilde{V}$. This means that the Kruskal coordinate U, which ranged over $-\infty < U < 0$ in region I, now ranges over $0 < U < \infty$ in region IV. Similarly, the Kruskal coordinate V, which ranged over $0 < V < \infty$ in region I, now ranges over

$-\infty < V < 0$ in region IV. Thus, in region IV, the coordinate U ranges over the same interval that the coordinate V does in region I. Similarly, in region IV, the coordinate V ranges over the same interval that the coordinate U does in region I.

Then, we can define a set of region IV lightcone coordinates $(\bar{u}, \bar{v})$, where $\bar{v}$ shares the same relationship to the Kruskal coordinate V as the (region I) lightcone coordinate u does to U. Similarly, $\bar{u}$ should share the same relationship to the Kruskal coordinate U as the (region I) lightcone coordinate v does to V. Then, using the expressions for the (region I) lightcone coordinates in terms of the Kruskal coordinates (3.26), the region IV lightcone coordinates are related to Kruskal coordinates by

$$\bar{u} = \frac{1}{\kappa} \ln\left(\widetilde{U}\right) \quad \text{and} \quad \bar{v} = -\frac{1}{\kappa} \ln\left(-\widetilde{V}\right). \tag{4.56}$$

Furthermore, we can define a new set of Schwarzschild-like coordinates $\left(\bar{t}, \bar{r}, \bar{\theta}, \bar{\varphi}\right)$ in region IV. We need only consider $\bar{t}$ and $\bar{r}_*$ for our purposes and we recall that the natural logarithm is a monotonically increasing function. In order to derive relations between the region IV lightcone coordinates (4.56) and the region IV Schwarzschild-like coordinates, it is useful to consider, in region I, how the lightcone coordinates (3.20) and the (region I) Schwarzschild-like coordinates vary with respect to one another, where the relationship between the two is well-established. From the Penrose diagram in Figure 3.1, we see that

$$\text{As} \quad U \to -\infty, \quad \text{then} \quad u \to -\infty. \quad \text{We also have} \quad t \to -\infty \quad \text{and} \quad r_* \to \infty, \tag{4.57a}$$

$$\text{As} \quad V \to \infty, \quad \text{then} \quad v \to \infty. \quad \text{We also have} \quad t \to \infty \quad \text{and} \quad r_* \to \infty, \tag{4.57b}$$

$$\text{As} \quad U \to 0, \quad \text{then} \quad u \to \infty. \quad \text{We also have} \quad t \to \infty \quad \text{and} \quad r_* \to -\infty, \tag{4.57c}$$

$$\text{As} \quad V \to 0, \quad \text{then} \quad v \to -\infty. \quad \text{We also have} \quad t \to -\infty \quad \text{and} \quad r_* \to -\infty. \tag{4.57d}$$

Equations (4.57a – 4.57d) summarise the relationships between the Kruskal, lightcone and Schwarzschild-like coordinate systems in region I. We can now turn our attention to region IV in order to derive the corresponding relationships between the Kruskal coordinates, the region IV lightcones coordinates (4.56) and the region IV Schwarzschild-like coordinates.

We remind the reader that the value of the region IV Schwarzschild-like coordinate $\bar{t}$ increases from the past horizon $\mathcal{H}^-_{\text{IV}}$ to future null infinity $\mathscr{I}^+_{\text{IV}}$ and also increases from past null infinity $\mathscr{I}^-_{\text{IV}}$ to the future horizon $\mathcal{H}^+_{\text{IV}}$. Similarly, the value of the region IV Schwarzschild-like coordinate $\bar{r}_*$ increases from the past horizon $\mathcal{H}^-_{\text{IV}}$ to future null infinity $\mathscr{I}^+_{\text{IV}}$; however, it decreases from past null infinity $\mathscr{I}^-_{\text{IV}}$ to the future horizon $\mathcal{H}^+_{\text{IV}}$.

Then, examining region IV of the Penrose diagram in Figure 3.1, we see that the Kruskal coordinate U, which increases from $U = 0$ at the past horizon $\mathcal{H}^-_{\text{IV}}$ to $U = \infty$ at future null infinity $\mathscr{I}^+_{\text{IV}}$, is positively correlated to both $\bar{t}$ and $\bar{r}_*$; since $\bar{u}$ is positively correlated to U, we can conclude that $\bar{u}$ is also positively correlated with $\bar{t}$ and $\bar{r}_*$. Similarly, we see that the Kruskal coordinate V, which increases from $V = -\infty$ at past null infinity $\mathscr{I}^-_{\text{IV}}$ to $V = 0$ at the future horizon $\mathcal{H}^+_{\text{IV}}$, is positively correlated with $\bar{t}$ but negatively correlated with $\bar{r}_*$; since $\bar{v}$ is positively correlated to V, we can conclude that $\bar{v}$ is also positively correlated with $\bar{t}$ and negatively correlated with $\bar{r}_*$. More concisely, we have

$$
\begin{array}{llllll}
\text{As} \quad U \to \infty, & \text{then} \quad \bar{u} \to \infty. & \text{We also have} \quad \bar{t} \to -\infty & \text{and} \quad \bar{r}_* \to \infty, & (4.58\text{a}) \\
\text{As} \quad V \to -\infty, & \text{then} \quad \bar{v} \to -\infty. & \text{We also have} \quad \bar{t} \to \infty & \text{and} \quad \bar{r}_* \to \infty, & (4.58\text{b}) \\
\text{As} \quad U \to 0, & \text{then} \quad \bar{u} \to -\infty. & \text{We also have} \quad \bar{t} \to \infty & \text{and} \quad \bar{r}_* \to -\infty, & (4.58\text{c}) \\
\text{As} \quad V \to 0, & \text{then} \quad \bar{v} \to \infty. & \text{We also have} \quad \bar{t} \to -\infty & \text{and} \quad \bar{r}_* \to -\infty. & (4.58\text{d})
\end{array}
$$

Therefore, from (4.58a – 4.58d), we deduce that the region IV lightcone coordinates are related to the region IV Schwarzschild-like coordinates by the expressions

$$
\bar{u} = -\bar{t} + \bar{r}_* \quad \text{and} \quad \bar{v} = -\bar{t} - \bar{r}_*, \tag{4.59}
$$

which are different to the expressions (3.20) relating the lightcone coordinates and the Schwarzschild-like coordinates in region I. Restricting our attention to region IV such that we can ignore the factor of $\Theta\left(\widetilde{U}\right)$ in (4.55), we can express the asymptotic form of the modes $\psi^{\text{down}}_{\omega\ell m}$ near $\mathcal{H}^+_{\text{IV}}$ in terms of the region IV Schwarzschild-like coordinates as

$$
\begin{aligned}
\psi^{\text{down}}_{\omega\ell m} &= \frac{1}{\sqrt{4\pi|\widetilde{\omega}|}}\,\frac{1}{\bar{r}}\exp\left[\frac{\mathrm{i}\,(\omega+\widetilde{\omega})}{2\kappa}\ln\left(\widetilde{U}\right)\right]\exp\left[-\frac{\mathrm{i}\,(\omega-\widetilde{\omega})}{2\kappa}\ln\left(-\widetilde{V}\right)\right]Y_{\ell m}\left(\bar{\theta},\bar{\varphi}\right) \\
&= \frac{1}{\sqrt{4\pi|\widetilde{\omega}|}}\,\frac{1}{\bar{r}}\exp\left[\frac{\mathrm{i}\,(\omega+\widetilde{\omega})}{2}\,\bar{u}\right]\exp\left[\frac{\mathrm{i}\,(\omega-\widetilde{\omega})}{2}\,\bar{v}\right]Y_{\ell m}\left(\bar{\theta},\bar{\varphi}\right) \\
&= \frac{1}{\sqrt{4\pi|\widetilde{\omega}|}}\,\frac{1}{\bar{r}}\,\mathrm{e}^{-\mathrm{i}\omega\bar{t}}\,\mathrm{e}^{\mathrm{i}\widetilde{\omega}\bar{r}_*}Y_{\ell m}\left(\bar{\theta},\bar{\varphi}\right).
\end{aligned} \tag{4.60}
$$

Using (4.60), we are now ready to calculate the norms of the region IV down-modes.

Norms of the region IV down-modes: The expression for the region IV down-modes in (4.60) depends on all four of the region IV Schwarzschild-like coordinates $\left(\bar{t},\bar{r},\bar{\theta},\bar{\varphi}\right)$. Recalling the calculations in §3.4 of the norms of modes in region I however, when we evaluate the norm of the region IV down-modes each of the coordinates $\left(\bar{t},\bar{r},\bar{\theta},\bar{\varphi}\right)$ will appear as a dummy variable in the integrals within the Klein-Gordon inner product (3.85).

Then, in the understanding that all of the following coordinates are dummy variables, if we make the identifications $\bar{t} = t$, $\bar{r} = r$, $\bar{\theta} = \theta$ and $\bar{\varphi} = \varphi$ the asymptotic form of the region IV down-modes near $\mathcal{H}^+_{\text{IV}}$ (4.60) is identical to the asymptotic form of the up-modes near $\mathcal{H}^-$ in region I (3.66); we can use this, as well as the expression for the norm of two up-modes in (3.129), in order to determine the norms of the region IV down-modes.

We will need to choose a Cauchy surface over which to evaluate the Klein-Gordon inner-product (3.85) of two region IV down-modes. The modes in (4.55), take a particularly simple asymptotic form near the future horizon $\mathcal{H}^+_{\text{IV}}$ and vanish near future null infinity $\mathscr{I}^+_{\text{IV}}$ as well as in region I. Then, a convenient choice of Cauchy surface is the 'past' Cauchy surface Σ_{past} defined in (3.86). Then, the inner product of two region IV down-modes is

$$
\begin{aligned}
&\langle \psi^{\text{down}}_{\omega\ell m}, \psi^{\text{down}}_{\omega'\ell'm'} \rangle \\
&= \mathrm{i} \int_{\mathcal{H}^{+}_{\text{IV}}} \left[\left(\partial_{\mu} \psi^{\text{down}*}_{\omega\ell m} \right) \psi^{\text{down}}_{\omega'\ell'm'} - \psi^{\text{down}*}_{\omega\ell m} \partial_{\mu} \psi^{\text{down}}_{\omega'\ell'm'} + 2\,\mathrm{i}qA_{\mu} \psi^{\text{down}*}_{\omega\ell m} \psi^{\text{down}}_{\omega'\ell'm'} \right] \sqrt{-g}\, \mathrm{d}\Sigma^{\mu}_{\mathcal{H}^{+}_{\text{IV}}} ,
\end{aligned}
\tag{4.61}
$$

where the integral over the 'past' Cauchy surface Σ_{past} in (4.61) reduces to one over the future horizon $\mathcal{H}^{+}_{\text{IV}}$ since the region IV down-modes vanish near future null infinity $\mathscr{I}^{+}_{\text{IV}}$.

Contained within the integral over the future horizon in (4.61) is an integral w.r.t the Kruskal coordinate U whose limits are given by the range of values U takes in region IV, i.e. $0 < U < \infty$; using the relationship between the Kruskal coordinates and the region IV lightcone coordinates (4.56), this integral can be re-expressed w.r.t the region IV lightcone coordinate $\bar{u}$ with the limits $-\infty < \bar{u} < \infty$. In the calculation of the norm of the up-modes in §3.4.3, the limits of the integral w.r.t the (region I) lightcone coordinate u is given by $-\infty < u < \infty$ (3.124). Again, since the coordinates $\bar{u}$ and u are dummy variables in each of the aforementioned integrals respectively, then we see that integrating over the future horizon $\mathcal{H}^{+}_{\text{IV}}$ in region IV, as opposed to integrating over the past horizon $\mathcal{H}^{-}$ in region I, does not induce a minus sign in $\langle \psi^{\text{down}}_{\omega\ell m}, \psi^{\text{down}}_{\omega'\ell'm'} \rangle$ relative to $\langle \phi^{\text{up}}_{\omega\ell m}, \phi^{\text{up}}_{\omega'\ell'm'} \rangle$.

However, the normal to the region IV future horizon $\mathcal{H}^{+}_{\text{IV}}$ is given by $n_{\mu} = -\mathrm{d}V$. This is in contrast to the normal to the (region I) past horizon $\mathcal{H}^{-}$, which is given by $n_{\mu} = \mathrm{d}V$ (3.108). Therefore, integrating w.r.t the volume element $\Sigma^{\mu}_{\mathcal{H}^{+}_{\text{IV}}}$ as opposed to the volume element $\Sigma^{\mu}_{\mathcal{H}^{-}}$ induces a minus sign in $\langle \psi^{\text{down}}_{\omega\ell m}, \psi^{\text{down}}_{\omega'\ell'm'} \rangle$ relative to $\langle \phi^{\text{up}}_{\omega\ell m}, \phi^{\text{up}}_{\omega'\ell'm'} \rangle$.

Thus, the expression for the norm of the region IV down-modes in (4.61) becomes

$$
\begin{aligned}
\langle \psi^{\text{down}}_{\omega\ell m}, \psi^{\text{down}}_{\omega'\ell'm'} \rangle &= -\langle \phi^{\text{up}}_{\omega\ell m}, \phi^{\text{up}}_{\omega'\ell'm'} \rangle \\
&= -\text{sgn}(\widetilde{\omega})\, \delta\big(\omega - \omega'\big)\, \delta_{\ell\ell'}\, \delta_{mm'},
\end{aligned}
\tag{4.62}
$$

where we have used the expression for the norm of the up-modes (3.129). Then, we can write the expression for the inner product of two generic region IV down-modes as

$$
\langle \psi^{\text{down}}_{\omega\ell m}, \psi^{\text{down}}_{\omega'\ell'm'} \rangle = \begin{cases} -\delta(\omega - \omega')\, \delta_{\ell\ell'}\, \delta_{mm'}, & \text{for } \widetilde{\omega} > 0, \\ \delta(\omega - \omega')\, \delta_{\ell\ell'}\, \delta_{mm'}, & \text{for } \widetilde{\omega} < 0, \end{cases}
\tag{4.63}
$$

meaning that region IV down-modes are of negative-norm when $\widetilde{\omega} > 0$ and of positive-norm when $\widetilde{\omega} < 0$. This is different to the case of the up-modes (3.130) that have positive-norm when $\widetilde{\omega} > 0$ and negative-norm when $\widetilde{\omega} < 0$. This difference is crucial in the construction of positive- and negative-frequency thermalised up-modes and we will see below that it gives rise to the thermal factor of the modes.

Having evaluated the norm of the region IV down-modes, we now return to defining positive- and negative-frequency thermalised up-modes. The lemma in (4.54) involves integrating over a particular surface. We can use the properties of the asymptotic forms of the up-modes in (4.52) and of the region IV down-modes in (4.55) to physically reason what an appropriate surface of integration would be.

Choice of surface over which to integrate: We need to specify a surface over which to integrate the linear combination of up-modes and region IV down-modes given by the lemma in (4.54). Near the past horizon $\mathcal{H}^-$, the up-modes take the particularly simple asymptotic form given in (4.52) and these modes vanish in region IV. Near the future horizon $\mathcal{H}^+_{\text{IV}}$, the region IV down-modes take the particularly simple asymptotic form given in (4.55) and these modes vanish in region I. Both the past horizon $\mathcal{H}^-$ and the region IV future horizon $\mathcal{H}^+_{\text{IV}}$ are surfaces of constant $V = 0$.

Then it is convenient to choose our surface of integration to be a hypersurface of constant $V = \epsilon > 0$, where ϵ is a small, positive constant such that this surface lies inside region I, close to $\mathcal{H}^-$, and inside region II, close to $\mathcal{H}^+_{\text{IV}}$. Our chosen surface of integration will inform our choice of branch cut when simplifying the $\exp[\ln(-\widetilde{V})]$ term in the asymptotic form of the region IV down-modes in (4.55).

Choosing a branch of the logarithm for positive-frequency modes: Both the asymptotic form of the up-modes near $\mathcal{H}^-$ (4.52) and the asymptotic form of the region IV down-modes near $\mathcal{H}^+_{\text{IV}}$ (4.55) have the same dependence on $\mathfrak{q}^{\text{up}+}$ (4.53), as well as the correct dependence on $\widetilde{U}$ according to the lemma in (4.54); however, the former contain a factor of $\exp[\ln(\widetilde{V})]$, whereas the latter contain a factor of $\exp[\ln(-\widetilde{V})]$. We are attempting to define positive-frequency modes w.r.t U; these are analytic in the lower-half of the plane and so we need to use a branch of the logarithm that is also analytic in the lower-half plane. We can therefore choose to make a branch cut along the positive imaginary axis

$$\ln(-1) = -\mathrm{i}\pi. \tag{4.64}$$

Then, integrating over a hypersurface of constant $V = \epsilon > 0$, we have

$$\begin{aligned}
\exp\left[-\frac{\mathrm{i}\,(\omega-\widetilde{\omega})}{2\kappa}\ln\left(-\widetilde{V}\right)\right] &= \exp\left[-\frac{\mathrm{i}\,(\omega-\widetilde{\omega})}{2\kappa}\ln\left(\widetilde{V}\right)\right]\exp\left[-\frac{\mathrm{i}\,(\omega-\widetilde{\omega})}{2\kappa}\ln(-1)\right] \\
&= \exp\left[-\frac{\mathrm{i}\,(\omega-\widetilde{\omega})}{2\kappa}\ln\left(\widetilde{V}\right)\right]\exp\left[-\frac{\mathrm{i}\,(\omega-\widetilde{\omega})}{2\kappa}(-\mathrm{i}\pi)\right] \\
&= \exp\left[-\frac{\mathrm{i}\,(\omega-\widetilde{\omega})}{2\kappa}\ln\left(\widetilde{V}\right)\right]\exp\left[-\frac{\pi\,(\omega-\widetilde{\omega})}{2\kappa}\right].
\end{aligned} \tag{4.65}$$

Using (4.65), the asymptotic form of the region IV down-modes near $\mathcal{H}^+_{\text{IV}}$ (4.55) becomes

$$\begin{aligned}
\psi^{\text{down}}_{\omega\ell m} = \frac{1}{\sqrt{4\pi|\widetilde{\omega}|}}\,\frac{1}{r}\exp\left[\frac{\mathrm{i}\,(\omega+\widetilde{\omega})}{2\kappa}\ln\left(\widetilde{U}\right)\right]\exp\left[-\frac{\mathrm{i}\,(\omega-\widetilde{\omega})}{2\kappa}\ln\left(\widetilde{V}\right)\right] \\
\times\exp\left[-\frac{\pi\,(\omega-\widetilde{\omega})}{2\kappa}\right]Y_{\ell m}(\theta,\varphi)\,\Theta\left(\widetilde{U}\right).
\end{aligned} \tag{4.66}$$

We are now ready to construct a set of positive-frequency thermalised up-modes.

Defining positive-frequency thermalised up-modes: In order to define positive-frequency modes with respect to the Kruskal coordinate U, we will have to multiply the expression for the lemma in (4.54) by an appropriate factor such that we can write the lemma as a linear combination of the asymptotic forms of the up-modes in (4.52) and the region IV down-modes in (4.66); explicitly, we obtain

$$\begin{aligned}0 = \frac{1}{\sqrt{4\pi|\widetilde{\omega}|}}\,\frac{1}{r}\exp\left[-\frac{\mathrm{i}\,(\omega-\widetilde{\omega})}{2\kappa}\ln\!\left(\widetilde{V}\right)\right]\exp\left[-\frac{\pi\,(\omega-\widetilde{\omega})}{2\kappa}\right]Y_{\ell m}(\theta,\varphi)\\ \times\int_{-\infty}^{\infty}\mathrm{d}\widetilde{U}\,\mathrm{e}^{-\mathrm{i}\mathfrak{p}\widetilde{U}}\left\{\exp\left[\frac{\mathrm{i}\,(\omega+\widetilde{\omega})}{2\kappa}\ln\!\left(\widetilde{U}\right)\right]\Theta\!\left(\widetilde{U}\right)\right.\\ \left.+\exp\left[\frac{\pi\,(\omega+\widetilde{\omega})}{2\kappa}\right]\exp\left[\frac{\mathrm{i}\,(\omega+\widetilde{\omega})}{2\kappa}\ln\!\left(-\widetilde{U}\right)\right]\Theta\!\left(-\widetilde{U}\right)\right\}. \quad (4.67)\end{aligned}$$

The first term in (4.67) is exactly that of the asymptotic form of the region IV down-modes near $\mathcal{H}^{+}_{\mathrm{IV}}$ (4.66), while the second term in (4.67) is that of the asymptotic form of the up-modes near $\mathcal{H}^{-}$ (4.52) multiplied by a factor of

$$\exp\left[-\frac{\pi\,(\omega-\widetilde{\omega})}{2\kappa}\right]\exp\left[\frac{\pi\,(\omega+\widetilde{\omega})}{2\kappa}\right] = \exp\left[\frac{\pi\widetilde{\omega}}{\kappa}\right]. \quad (4.68)$$

Then we can write the linear combination of modes in (4.67) as

$$0 = \int_{-\infty}^{\infty}\mathrm{d}\widetilde{U}\,\mathrm{e}^{-\mathrm{i}\mathfrak{p}\widetilde{U}}\left\{\psi^{\mathrm{down}}_{\omega\ell m} + \mathrm{e}^{\frac{\pi\widetilde{\omega}}{\kappa}}\,\phi^{\mathrm{up}}_{\omega\ell m}\right\}, \quad \mathfrak{p} > 0. \quad (4.69)$$

By the statement (4.2), the quantity in the curly brackets in (4.69) is positive-frequency w.r.t $\widetilde{U}$ (and therefore U) for all values of $\widetilde{\omega}$. For reasons that will become apparent when we come to normalise, we can multiply this quantity by a factor of $\mathfrak{N}^{\mathrm{up}+}_{\omega}\mathrm{e}^{-\frac{\pi\widetilde{\omega}}{2\kappa}}$ where the normalisation constant $\mathfrak{N}^{\mathrm{up}+}_{\omega}$ is yet to be determined, to define a set of modes $\chi^{\mathrm{up}+}_{\omega\ell m}$

$$\chi^{\mathrm{up}+}_{\omega\ell m} = \mathfrak{N}^{\mathrm{up}+}_{\omega}\left(\mathrm{e}^{\frac{\pi\widetilde{\omega}}{2\kappa}}\,\phi^{\mathrm{up}}_{\omega\ell m} + \mathrm{e}^{-\frac{\pi\widetilde{\omega}}{2\kappa}}\,\psi^{\mathrm{down}}_{\omega\ell m}\right) \quad (4.70)$$

which is positive-frequency with respect to the Kruskal coordinate U for all values of $\widetilde{\omega}$. Here the label $+$ once again denotes that these are positive-frequency modes. The label "up", however, may be a little more mysterious. Eventually we will restrict our attention to region I of the spacetime diagram where (4.70) reduces to

$$\chi^{\mathrm{up}+}_{\omega\ell m} = \mathrm{e}^{\frac{\pi\widetilde{\omega}}{2\kappa}}\,\mathfrak{N}^{\mathrm{up}+}_{\omega}\,\phi^{\mathrm{up}}_{\omega\ell m}, \quad (4.71)$$

and the label "up" becomes more intuitive.

The modes (4.71) constitute a set of thermalised up-modes that are positive-frequency with respect to the Kruskal coordinate U. We will eventually normalise these modes such that they can be used to expand the field Φ in an orthonormal basis of scalar field modes before quantisation. Before we do so, however, we will define a set of thermalised up-modes that are negative-frequency with respect to the Kruskal coordinate U.

Defining negative-frequency thermalised up-modes: We would also like to define an analogous set of modes that are negative-frequency with respect to the Kruskal coordinate U. We can do this by considering the complex conjugate of the lemma in (4.7). Comparing the asymptotic expressions for the up-modes near $\mathcal{H}^{-}$ (4.52) and the region

IV down-modes near $\mathcal{H}^+_{\mathrm{IV}}$ (4.55) with the lemma in (4.7), we see that the terms in the lemma can be constructed from the expressions in (4.52) and (4.55) if we take

$$\mathfrak{x}^{\mathrm{up}-} = \widetilde{U}, \tag{4.72}$$

similar to the case when defining positive-frequency modes. However, in contrast to the case when defining positive-frequency modes, we need to take

$$\mathfrak{q}^{\mathrm{up}-} = \frac{(\omega + \widetilde{\omega})}{2\kappa}, \tag{4.73}$$

where the label up− denotes that these are the values we choose in order to define negative-frequency up-modes w.r.t the Kruskal coordinate U. Then the lemma (4.7) becomes

$$\int_{-\infty}^{\infty} \mathrm{d}\widetilde{U}\, \mathrm{e}^{\mathrm{i}\mathfrak{p}\widetilde{U}} \left\{ \exp\left[\frac{\mathrm{i}\,(\omega+\widetilde{\omega})}{2\kappa} \ln\left(\widetilde{U}\right)\right] \Theta\left(\widetilde{U}\right) \right. \\ \left. + \exp\left[-\frac{\pi\,(\omega+\widetilde{\omega})}{2\kappa}\right] \exp\left[\frac{\mathrm{i}\,(\omega+\widetilde{\omega})}{2\kappa} \ln\left(-\widetilde{U}\right)\right] \Theta\left(-\widetilde{U}\right) \right\} = 0. \tag{4.74}$$

We will again need to take a linear combination of the asymptotic forms of the up-modes near $\mathcal{H}^-$ (4.52) and of the region IV down-modes near $\mathcal{H}^+_{\mathrm{IV}}$ (4.55). The former contain a factor of $\exp[\ln(\widetilde{V})]$ while the latter contain a factor of $\exp[\ln(-\widetilde{V})]$. We are attempting to define negative-frequency modes w.r.t $\widetilde{U}$; these are analytic in the upper-half of the plane and so we need to use a branch of the logarithm that is also analytic in the upper-half of the plane. We therefore choose to make a branch cut along the negative imaginary axis

$$\ln(-1) = \mathrm{i}\pi. \tag{4.75}$$

Then, integrating over a hypersurface of constant $V = \epsilon > 0$, we have

$$\begin{aligned} \exp\left[-\frac{\mathrm{i}\,(\omega-\widetilde{\omega})}{2\kappa} \ln\left(-\widetilde{V}\right)\right] &= \exp\left[-\frac{\mathrm{i}\,(\omega-\widetilde{\omega})}{2\kappa} \ln\left(\widetilde{V}\right)\right] \exp\left[-\frac{\mathrm{i}\,(\omega-\widetilde{\omega})}{2\kappa} \ln(-1)\right] \\ &= \exp\left[-\frac{\mathrm{i}\,(\omega-\widetilde{\omega})}{2\kappa} \ln\left(\widetilde{V}\right)\right] \exp\left[-\frac{\mathrm{i}\,(\omega-\widetilde{\omega})}{2\kappa} (\mathrm{i}\pi)\right] \\ &= \exp\left[-\frac{\mathrm{i}\,(\omega-\widetilde{\omega})}{2\kappa} \ln\left(\widetilde{V}\right)\right] \exp\left[\frac{\pi\,(\omega-\widetilde{\omega})}{2\kappa}\right]. \end{aligned} \tag{4.76}$$

Using (4.76), the asymptotic form of the region IV down-modes near $\mathcal{H}^+_{\mathrm{IV}}$ (4.55) becomes

$$\psi^{\mathrm{down}}_{\omega\ell m} = \frac{1}{\sqrt{4\pi|\widetilde{\omega}|}} \frac{1}{r} \exp\left[\frac{\mathrm{i}\,(\omega+\widetilde{\omega})}{2\kappa} \ln\left(\widetilde{U}\right)\right] \exp\left[-\frac{\mathrm{i}\,(\omega-\widetilde{\omega})}{2\kappa} \ln\left(\widetilde{V}\right)\right] \\ \times \exp\left[\frac{\pi\,(\omega-\widetilde{\omega})}{2\kappa}\right] Y_{\ell m}(\theta,\varphi)\, \Theta\left(\widetilde{U}\right). \tag{4.77}$$

In order to define negative-frequency modes with respect to the Kruskal coordinate U, we will have to multiply the expression for the lemma in (4.74) by an appropriate factor such that we can write the lemma as a linear combination of the asymptotic forms of the up-modes near $\mathcal{H}^-$ (4.52) and the region IV down-modes near $\mathcal{H}^+_{\mathrm{IV}}$ (4.77); we have

$$0 = \frac{1}{\sqrt{4\pi|\widetilde{\omega}|}} \frac{1}{r} \exp\left[-\frac{\mathrm{i}\,(\omega-\widetilde{\omega})}{2\kappa}\ln\left(\widetilde{V}\right)\right] \exp\left[\frac{\pi\,(\omega-\widetilde{\omega})}{2\kappa}\right] Y_{\ell m}(\theta,\varphi) \\ \times \int_{-\infty}^{\infty} \mathrm{d}\widetilde{U}\, \mathrm{e}^{\mathrm{i}\mathfrak{p}\widetilde{U}} \left\{\exp\left[\frac{\mathrm{i}\,(\omega+\widetilde{\omega})}{2\kappa}\ln\left(\widetilde{U}\right)\right]\Theta\left(\widetilde{U}\right) \right. \\ \left. + \exp\left[-\frac{\pi\,(\omega+\widetilde{\omega})}{2\kappa}\right]\exp\left[\frac{\mathrm{i}\,(\omega+\widetilde{\omega})}{2\kappa}\ln\left(-\widetilde{U}\right)\right]\Theta\left(-\widetilde{U}\right)\right\}. \quad (4.78)$$

The first term in (4.78) is exactly that of the asymptotic form of the region IV down-modes near $\mathcal{H}^+_{\mathrm{IV}}$ (4.77), while the second term in (4.78) is that of the asymptotic form of the up-modes near $\mathcal{H}^-$ (4.52) multiplied by a factor of

$$\exp\left[\frac{\pi\,(\omega-\widetilde{\omega})}{2\kappa}\right]\exp\left[-\frac{\pi\,(\omega+\widetilde{\omega})}{2\kappa}\right] = \exp\left[-\frac{\pi\widetilde{\omega}}{\kappa}\right]. \quad (4.79)$$

Then we can write the linear combination of modes in (4.78) as

$$0 = \int_{-\infty}^{\infty} \mathrm{d}\widetilde{U}\, \mathrm{e}^{\mathrm{i}\mathfrak{p}\widetilde{U}} \left\{\psi^{\mathrm{down}}_{\omega\ell m} + \mathrm{e}^{-\frac{\pi\widetilde{\omega}}{\kappa}}\phi^{\mathrm{up}}_{\omega\ell m}\right\}, \quad \mathfrak{p} > 0. \quad (4.80)$$

By the statement (4.3), the quantity in the curly brackets in (4.80) is negative-frequency w.r.t $\widetilde{U}$ (and therefore U) for all values of $\widetilde{\omega}$. For reasons that will become apparent when we come to normalise, we can multiply this quantity by a factor of $\mathfrak{N}^{\mathrm{up}-}_{\omega}\mathrm{e}^{\frac{\pi\widetilde{\omega}}{2\kappa}}$, where the normalisation constant $\mathfrak{N}^{\mathrm{up}-}_{\omega}$ is yet to be determined, to define a set of modes $\chi^{\mathrm{up}-}_{\omega\ell m}$

$$\chi^{\mathrm{up}-}_{\omega\ell m} = \mathfrak{N}^{\mathrm{up}-}_{\omega}\left(\mathrm{e}^{-\frac{\pi\widetilde{\omega}}{2\kappa}}\phi^{\mathrm{up}}_{\omega\ell m} + \mathrm{e}^{\frac{\pi\widetilde{\omega}}{2\kappa}}\psi^{\mathrm{down}}_{\omega\ell m}\right) \quad (4.81)$$

which is negative-frequency with respect to the Kruskal coordinate U for all values of $\widetilde{\omega}$. Here the label $-$ once again denotes that these are negative-frequency modes. Restricting out attention to region I of the spacetime diagram, (4.81) reduces to

$$\chi^{\mathrm{up}-}_{\omega\ell m} = \mathrm{e}^{-\frac{\pi\widetilde{\omega}}{2\kappa}}\mathfrak{N}^{\mathrm{up}-}_{\omega}\phi^{\mathrm{up}}_{\omega\ell m}. \quad (4.82)$$

The modes in (4.82) constitute a set of thermalised up-modes that are negative-frequency w.r.t the Kruskal coordinate U. We have now defined positive- (4.70) and negative-frequency thermalised up-modes (4.81) w.r.t U for all $\widetilde{\omega}$. For these modes to form an orthonormal basis in which we can expand the field, we are required to normalise them.

Normalisation of the thermalised up-modes: We can normalise the positive- (4.70) and negative-frequency thermalised up modes (4.81) by using the Klein-Gordon inner product (3.85) to evaluate their norm; since both these sets of modes are linear combinations of the asymptotic forms of the up-modes near $\mathcal{H}^-$ and of the region IV down-modes near $\mathcal{H}^+_{\mathrm{IV}}$, we can use the expressions for the norm of the up-modes (3.129) and the norm of the region IV down-modes (4.63) to evaluate the inner product of the $\chi^{\mathrm{up}\pm}_{\omega\ell m}$ modes.

We also need to use the fact that the up-modes (3.131) and the region IV down-modes (4.55) are orthogonal since the up-modes are defined in region I while vanishing in region IV, and the region IV down-modes are defined in region I while vanishing in region IV. Therefore, the inner product $\langle\phi^{\mathrm{up}}_{\omega\ell m}, \psi^{\mathrm{down}}_{\omega'\ell' m'}\rangle$ vanishes.

Then, requiring orthonormality of the $\chi^{\text{up}\pm}_{\omega\ell m}$ modes, we have

$$\begin{aligned}\langle\chi^{\text{up}\pm}_{\omega\ell m},\chi^{\text{up}\pm}_{\omega'\ell'm'}\rangle\\ &= \mathfrak{N}^{\text{up}*\pm}_{\omega}\,\mathfrak{N}^{\text{up}\pm}_{\omega'}\langle(\mathrm{e}^{\pm\frac{\pi\widetilde{\omega}}{2\kappa}}\phi^{\text{up}}_{\omega\ell m}+\mathrm{e}^{\mp\frac{\pi\widetilde{\omega}}{2\kappa}}\psi^{\text{down}}_{\omega\ell m}),(\mathrm{e}^{\pm\frac{\pi\widetilde{\omega}}{2\kappa}}\phi^{\text{up}}_{\omega'\ell'm'}+\mathrm{e}^{\mp\frac{\pi\widetilde{\omega}}{2\kappa}}\psi^{\text{down}}_{\omega'\ell'm'})\rangle\\ &= \mathfrak{N}^{\text{up}*\pm}_{\omega}\,\mathfrak{N}^{\text{up}\pm}_{\omega'}\left(\mathrm{e}^{\pm\frac{\pi\widetilde{\omega}}{\kappa}}\langle\phi^{\text{up}}_{\omega\ell m},\phi^{\text{up}}_{\omega'\ell'm'}\rangle+2\,\langle\phi^{\text{up}}_{\omega\ell m},\psi^{\text{down}}_{\omega'\ell'm'}\rangle+\mathrm{e}^{\mp\frac{\pi\widetilde{\omega}}{\kappa}}\langle\psi^{\text{down}}_{\omega\ell m},\psi^{\text{down}}_{\omega'\ell'm'}\rangle\right)\\ &= \left|\mathfrak{N}^{\text{up}\pm}_{\omega}\right|^2\left(\mathrm{e}^{\pm\frac{\pi\widetilde{\omega}}{\kappa}}-\mathrm{e}^{\mp\frac{\pi\widetilde{\omega}}{\kappa}}\right)\operatorname{sgn}(\widetilde{\omega})\,\delta\big(\omega-\omega'\big)\,\delta_{\ell\ell'}\,\delta_{mm'},\end{aligned}\tag{4.83}$$

where we have used the expressions for the norms in (3.129) and (4.63) to go from the second equality to the third; (4.83) reduces to

$$\langle\chi^{\text{up}\pm}_{\omega\ell m},\chi^{\text{up}\pm}_{\omega'\ell'm'}\rangle = 2\left|\mathfrak{N}^{\text{up}\pm}_{\omega}\right|^2\sinh\!\left(\pm\frac{\pi\widetilde{\omega}}{\kappa}\right)\operatorname{sgn}(\widetilde{\omega})\,\delta\big(\omega-\omega'\big)\,\delta_{\ell\ell'}\,\delta_{mm'}.\tag{4.84}$$

Then, from (4.84), we have for the inner product of two generic $\chi^{\text{up}+}_{\omega\ell m}$ modes

$$\langle\chi^{\text{up}+}_{\omega\ell m},\chi^{\text{up}+}_{\omega'\ell'm'}\rangle = \begin{cases} 2\left|\mathfrak{N}^{\text{up}+}_{\omega}\right|^2\sinh\!\left(\dfrac{\pi\widetilde{\omega}}{\kappa}\right)\delta\big(\omega-\omega'\big)\,\delta_{\ell\ell'}\,\delta_{mm'}>0, & \widetilde{\omega}>0,\\ -2\left|\mathfrak{N}^{\text{up}+}_{\omega}\right|^2\sinh\!\left(-\dfrac{\pi\widetilde{\omega}}{\kappa}\right)\delta\big(\omega-\omega'\big)\,\delta_{\ell\ell'}\,\delta_{mm'}>0, & \widetilde{\omega}<0,\end{cases}\tag{4.85}$$

demonstrating that positive-frequency modes $\chi^{\text{up}+}_{\omega\ell m}$ have positive norm for all $\widetilde{\omega}\neq 0$. Similarly, from (4.84), we have for the inner product of two generic $\chi^{\text{up}-}_{\omega\ell m}$ modes

$$\langle\chi^{\text{up}-}_{\omega\ell m},\chi^{\text{up}-}_{\omega'\ell'm'}\rangle = \begin{cases} 2\left|\mathfrak{N}^{\text{up}-}_{\omega}\right|^2\sinh\!\left(\dfrac{\pi\widetilde{\omega}}{\kappa}\right)\delta\big(\omega-\omega'\big)\,\delta_{\ell\ell'}\,\delta_{mm'}<0, & \widetilde{\omega}>0,\\ -2\left|\mathfrak{N}^{\text{up}-}_{\omega}\right|^2\sinh\!\left(-\dfrac{\pi\widetilde{\omega}}{\kappa}\right)\delta\big(\omega-\omega'\big)\,\delta_{\ell\ell'}\,\delta_{mm'}<0, & \widetilde{\omega}<0,\end{cases}\tag{4.86}$$

demonstrating that negative-frequency modes $\chi^{\text{up}-}_{\omega\ell m}$ have negative norm for all $\widetilde{\omega}\neq 0$. Then, we can express the inner product of two generic $\chi^{\text{up}\pm}_{\omega\ell m}$ modes as

$$\langle\chi^{\text{up}\pm}_{\omega\ell m},\chi^{\text{up}\pm}_{\omega'\ell'm'}\rangle = 2\left|\mathfrak{N}^{\text{up}\pm}_{\omega}\right|^2\sinh\!\left(\frac{\pi\widetilde{\omega}}{\kappa}\right)\delta\big(\omega-\omega'\big)\,\delta_{\ell\ell'}\,\delta_{mm'},\quad \text{all }\widetilde{\omega},\tag{4.87}$$

where we have used $\sinh(-x) = -\sinh(x)$. Thus, for the inner products of two $\chi^{\text{up}\pm}_{\omega\ell m}$ modes with the same angular momentum ℓ and azimuthal m quantum numbers, we obtain

$$\langle\chi^{\text{up}\pm}_{\omega\ell m},\chi^{\text{up}\pm}_{\omega'\ell'm'}\rangle = 2\left|\mathfrak{N}^{\text{up}\pm}_{\omega}\right|^2\sinh\!\left(\frac{\pi\widetilde{\omega}}{\kappa}\right)\delta\big(\omega-\omega'\big)\,,\quad \text{all }\widetilde{\omega}.\tag{4.88}$$

Requiring the orthonormality of the inner products in (4.88) gives us the expression for the normalisation constants $\mathfrak{N}^{\text{up}\pm}_{\omega}$, which are given by

$$\mathfrak{N}^{\text{up}\pm}_{\omega} = \frac{1}{\sqrt{2\left|\sinh\!\left(\frac{\pi\widetilde{\omega}}{\kappa}\right)\right|}}.\tag{4.89}$$

Therefore, a set of normalised modes having positive-frequency w.r.t U is given by

$$\chi^{\mathrm{up}+}_{\omega\ell m} = \frac{1}{\sqrt{2\left|\sinh\left(\frac{\pi\widetilde{\omega}}{\kappa}\right)\right|}}\left(\mathrm{e}^{\frac{\pi\widetilde{\omega}}{2\kappa}}\phi^{\mathrm{up}}_{\omega\ell m} + \mathrm{e}^{-\frac{\pi\widetilde{\omega}}{2\kappa}}\psi^{\mathrm{down}}_{\omega\ell m}\right), \quad \text{all } \widetilde{\omega}, \tag{4.90}$$

and a set of normalised modes having negative-frequency w.r.t U is given by

$$\chi^{\mathrm{up}-}_{\omega\ell m} = \frac{1}{\sqrt{2\left|\sinh\left(\frac{\pi\widetilde{\omega}}{\kappa}\right)\right|}}\left(\mathrm{e}^{-\frac{\pi\widetilde{\omega}}{2\kappa}}\phi^{\mathrm{up}}_{\omega\ell m} + \mathrm{e}^{\frac{\pi\widetilde{\omega}}{2\kappa}}\psi^{\mathrm{down}}_{\omega\ell m}\right), \quad \text{all } \widetilde{\omega}. \tag{4.91}$$

Our final step is to restrict our attention to region I; in this case the $\psi^{\mathrm{down}}_{\omega\ell m}$, which are defined in regions II and IV, vanish such that (4.90) and (4.91) become

$$\chi^{\mathrm{up}+}_{\omega\ell m} = \frac{1}{\sqrt{2\left|\sinh\left(\frac{\pi\widetilde{\omega}}{\kappa}\right)\right|}}\,\mathrm{e}^{\frac{\pi\widetilde{\omega}}{2\kappa}}\phi^{\mathrm{up}}_{\omega\ell m}, \quad \text{all } \widetilde{\omega}, \tag{4.92a}$$

$$\chi^{\mathrm{up}-}_{\omega\ell m} = \frac{1}{\sqrt{2\left|\sinh\left(\frac{\pi\widetilde{\omega}}{\kappa}\right)\right|}}\,\mathrm{e}^{-\frac{\pi\widetilde{\omega}}{2\kappa}}\phi^{\mathrm{up}}_{\omega\ell m}, \quad \text{all } \widetilde{\omega}. \tag{4.92b}$$

Despite restricting our attention to region I, our earlier statements regarding (4.92a) and (4.92b) still hold; the modes in (4.92a) and (4.92b) constitute thermalised up-modes that are positive- and negative-frequency w.r.t the Kruskal coordinate U respectively.

Construction of the 'past' Unruh state

Recall that in defining the 'past' Unruh state $|\mathrm{U}^-\rangle$, we would like a state that is as empty as possible to a static observer at past null infinity $\mathscr{I}^-$ but which contains an outgoing flux of Hawking radiation at future null infinity $\mathscr{I}^+$; this corresponds to an absence of particles in the in-modes (3.106) as well as a thermalised flux of particles in the up-modes.

Then, we expand the scalar field Φ in terms of an orthonormal basis of in-modes and thermally populated up-modes, each divided into positive- and negative-frequency sets. We recall that the in-modes are defined to be positive- (4.22) and negative-frequency (4.23) with respect to the Schwarzschild-like coordinate t, which is the natural time coordinate to use near $\mathscr{I}^-$. Using (4.22), (4.23), (4.92a) and (4.92b), we have

$$\begin{aligned}\Phi(x) = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\Bigg\{&\int_0^{\infty}\mathrm{d}\omega\, a^{\mathrm{in}}_{\omega\ell m}\phi^{\mathrm{in}+}_{\omega\ell m} + \int_{-\infty}^{0}\mathrm{d}\omega\, b^{\mathrm{in}\dagger}_{\omega\ell m}\phi^{\mathrm{in}-}_{\omega\ell m}\\ &+ \int_{-\infty}^{\infty}\mathrm{d}\widetilde{\omega}\,\frac{1}{\sqrt{2\left|\sinh\left(\frac{\pi\widetilde{\omega}}{\kappa}\right)\right|}}\,\phi^{\mathrm{up}}_{\omega\ell m}\left[\mathrm{e}^{\frac{\pi\widetilde{\omega}}{2\kappa}}a^{\mathrm{up}}_{\omega\ell m} + \mathrm{e}^{-\frac{\pi\widetilde{\omega}}{2\kappa}}b^{\mathrm{up}\dagger}_{\omega\ell m}\right]\Bigg\}.\end{aligned} \tag{4.93}$$

We quantise the field by promoting the expansion coefficients to operators such that the field operator $\hat{\Phi}(x)$ is given by

$$\hat{\Phi}(x) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \left\{ \int_0^{\infty} \mathrm{d}\omega \, \hat{a}^{\mathrm{in}}_{\omega\ell m} \phi^{\mathrm{in}+}_{\omega\ell m} + \int_{-\infty}^{0} \mathrm{d}\omega \, \hat{b}^{\mathrm{in}\dagger}_{\omega\ell m} \phi^{\mathrm{in}-}_{\omega\ell m} \right.$$
$$\left. + \int_{-\infty}^{\infty} \mathrm{d}\widetilde{\omega} \, \frac{1}{\sqrt{2 \left| \sinh\left(\frac{\pi\widetilde{\omega}}{\kappa}\right) \right|}} \, \phi^{\mathrm{up}}_{\omega\ell m} \left[\mathrm{e}^{\frac{\pi\widetilde{\omega}}{2\kappa}} \hat{a}^{\mathrm{up}}_{\omega\ell m} + \mathrm{e}^{-\frac{\pi\widetilde{\omega}}{2\kappa}} \hat{b}^{\mathrm{up}\dagger}_{\omega\ell m} \right] \right\}, \quad (4.94)$$

where the operators associated to the in-modes, $\hat{a}^{\mathrm{in}}_{\omega\ell m}$ and $\hat{b}^{\mathrm{in}}_{\omega\ell m}$, are defined for $\omega > 0$ and $\omega < 0$ respectively, and the operators associated to the up-modes, $\hat{a}^{\mathrm{up}}_{\omega\ell m}$ and $\hat{b}^{\mathrm{up}}_{\omega\ell m}$, are each defined for all $\widetilde{\omega}$. In (4.94), all of the positive-frequency modes $\phi^{\mathrm{in/up}+}_{\omega\ell m}$ are of positive-norm and all of the negative-frequency modes $\phi^{\mathrm{in/up}-}_{\omega\ell m}$ are of negative-norm; then, the operators $\hat{a}^{\mathrm{in/up}}_{\omega\ell m}$ and $\hat{b}^{\mathrm{in/up}}_{\omega\ell m}$ obey the following, standard commutation relations

$$\begin{aligned}
\left[\hat{a}^{\mathrm{in}}_{\omega\ell m}, \hat{a}^{\mathrm{in}\dagger}_{\omega'\ell'm'}\right] &= \delta\big(\omega - \omega'\big)\, \delta_{\ell\ell'}\, \delta_{mm'}, \quad \omega > 0, \\
\left[\hat{b}^{\mathrm{in}}_{\omega\ell m}, \hat{b}^{\mathrm{in}\dagger}_{\omega'\ell'm'}\right] &= \delta\big(\omega - \omega'\big)\, \delta_{\ell\ell'}\, \delta_{mm'}, \quad \omega < 0, \\
\left[\hat{a}^{\mathrm{up}}_{\omega\ell m}, \hat{a}^{\mathrm{up}\dagger}_{\omega'\ell'm'}\right] &= \delta\big(\omega - \omega'\big)\, \delta_{\ell\ell'}\, \delta_{mm'}, \quad \text{all } \widetilde{\omega}, \\
\left[\hat{b}^{\mathrm{up}}_{\omega\ell m}, \hat{b}^{\mathrm{up}\dagger}_{\omega'\ell'm'}\right] &= \delta\big(\omega - \omega'\big)\, \delta_{\ell\ell'}\, \delta_{mm'}, \quad \text{all } \widetilde{\omega},
\end{aligned} \quad (4.95)$$

with any commutators not explicitly given in (4.95) vanishing. The 'past' Unruh state $|\mathrm{U}^-\rangle$ is then defined as the state annihilated by the $\hat{a}^{\mathrm{in/up}}_{\omega\ell m}$ and $\hat{b}^{\mathrm{in/up}}_{\omega\ell m}$ operators such that

$$\begin{aligned}
\hat{a}^{\mathrm{in}}_{\omega\ell m} \, |\mathrm{U}^-\rangle &= 0, \quad \omega > 0, \\
\hat{b}^{\mathrm{in}}_{\omega\ell m} \, |\mathrm{U}^-\rangle &= 0, \quad \omega < 0, \\
\hat{a}^{\mathrm{up}}_{\omega\ell m} \, |\mathrm{U}^-\rangle &= 0, \quad \text{all } \widetilde{\omega}, \\
\hat{b}^{\mathrm{up}}_{\omega\ell m} \, |\mathrm{U}^-\rangle &= 0, \quad \text{all } \widetilde{\omega}.
\end{aligned} \quad (4.96)$$

The 'past' Unruh state $|\mathrm{U}^-\rangle$ contains no particles or antiparticles incoming at past null infinity $\mathscr{I}^-$. However, it does contain a thermal flux of particles and antiparticles outgoing to future null infinity $\mathscr{I}^+$, which corresponds to Hawking radiation at all frequencies in agreement with [39]. We consider expectation values of observables in this state in §5.3.2.

4.4.2 'Future' Unruh state

The 'future' Unruh state $|\mathrm{U}^+\rangle$ can be understood as the time-reverse of the 'past' Unruh state $|\mathrm{U}^-\rangle$ that we defined in §4.4.1. Where the 'past' Unruh state $|\mathrm{U}^-\rangle$ was constructed using an orthonormal basis of in- and thermalised up-modes near surfaces contained within the 'past' Cauchy surface Σ_{past} (3.86) , the 'future' Unruh state $|\mathrm{U}^+\rangle$ is constructed using an orthonormal basis of out- and thermalised down-modes near surfaces contained within the 'future' Cauchy surface Σ_{future} (3.87).

The lack of outgoing radiation in the 'future' Unruh state $|\mathrm{U}^+\rangle$ corresponds to an absence of particles in the out-modes as seen by a static observer at future null infinity $\mathscr{I}^+$, where the proper time experienced by a static observer is given by the Schwarzschild-like coordinate t. We have already defined out-modes that are positive- and negative-frequency w.r.t t when defining the 'future' Boulware state $|\mathrm{B}^+\rangle$ in §4.3.2. Then, the definitions that we require are those given in (4.32) and (4.33) respectively.

We now need to define positive- and negative-frequency thermalised down-modes. Since their construction is analogous to the procedure used in §4.4.1 to construct positive- and negative-frequency thermalised up-modes, our treatment of which was necessarily detailed, our discussion of the construction of positive- and negative-frequency thermalised down-modes will be restricted to parts of the derivation that we deem essential.

Positive- and negative-frequency thermalised down-modes

The 'future' Unruh state $|\mathrm{U}^+\rangle$ has the additional interpretation of predicting Hawking radiation incoming from past null infinity $\mathscr{I}^-$; this property corresponds to a thermalised flux of particles in the down-modes incident upon the future horizon $\mathcal{H}^+$. The Kruskal coordinate V (3.24) is the affine parameter along the null generators of the future horizon $\mathcal{H}^+$. Then, near $\mathcal{H}^+$, the natural choice of time-coordinate w.r.t which we can define positive- and negative-frequency down-modes is the Kruskal advanced time V.

We can begin by expressing the asymptotic form of the down-modes (3.70) near the future horizon $\mathcal{H}^+$ in terms of Kruskal coordinates; using (3.21), (3.26) and (3.33) we have

$$\phi^{\text{down}}_{\omega\ell m} = \frac{1}{\sqrt{4\pi|\widetilde{\omega}|}}\,\frac{1}{r}\exp\left[\frac{\mathrm{i}\,(\omega-\widetilde{\omega})}{2\kappa}\ln\left(-\widetilde{U}\right)\right]\exp\left[-\frac{\mathrm{i}\,(\omega+\widetilde{\omega})}{2\kappa}\ln\left(\widetilde{V}\right)\right]Y_{\ell m}(\theta,\varphi)\,. \tag{4.97}$$

The down-modes are defined in regions I and II of the Penrose diagram in Figure 3.1, where $\widetilde{V} > 0$. We can trivially extend their definition to regions III and IV by using the Heaviside function (4.5) to demand that they vanish when $\widetilde{V} < 0$. Then, (4.97) becomes

$$\phi^{\text{down}}_{\omega\ell m} = \frac{1}{\sqrt{4\pi|\widetilde{\omega}|}}\,\frac{1}{r}\exp\left[\frac{\mathrm{i}\,(\omega-\widetilde{\omega})}{2\kappa}\ln\left(-\widetilde{U}\right)\right]\exp\left[-\frac{\mathrm{i}\,(\omega+\widetilde{\omega})}{2\kappa}\ln\left(\widetilde{V}\right)\right]Y_{\ell m}(\theta,\varphi)\,\Theta\left(\widetilde{V}\right). \tag{4.98}$$

Comparing the expression for the asymptotic form of the down-modes near the future horizon $\mathcal{H}^+$ in (4.98) with the lemma (4.6), we see that the first term in the lemma can be constructed from the expression in (4.98) if we take

$$\mathfrak{x}^{\text{down}+} = \widetilde{V} \quad \text{and} \quad \mathfrak{q}^{\text{down}+} = \frac{(\omega+\widetilde{\omega})}{2\kappa}. \tag{4.99}$$

Then the lemma (4.6) becomes

$$\int_{-\infty}^{\infty}\mathrm{d}\widetilde{V}\,\mathrm{e}^{-\mathrm{i}\mathfrak{p}\widetilde{V}}\left\{\exp\left[-\frac{\mathrm{i}\,(\omega+\widetilde{\omega})}{2\kappa}\ln\left(\widetilde{V}\right)\right]\Theta\left(\widetilde{V}\right)\right.$$
$$\left.+\exp\left[-\pi\frac{(\omega+\widetilde{\omega})}{2\kappa}\right]\exp\left[-\frac{\mathrm{i}\,(\omega+\widetilde{\omega})}{2\kappa}\ln\left(-\widetilde{V}\right)\right]\Theta\left(-\widetilde{V}\right)\right\} = 0. \tag{4.100}$$

We see that the second term of the lemma (4.100) can be constructed from a set of modes in regions III and IV, vanishing in regions I and II, with the same dependence on $\mathfrak{q}^{\text{down}+}$

(4.99) as the asymptotic form of the down-modes in (4.98) but containing factors of $-\widetilde{V}$ as opposed to $\widetilde{V}$. As was the case when defining the region IV down-modes in §4.4.1, we will make both of the transformations $\widetilde{U} \to -\widetilde{U}$ as well as $\widetilde{V} \to -\widetilde{V}$, which results in a set of modes that are nonzero in regions III and IV and vanishing elsewhere, as well as being orthogonal to the down-modes in (4.98) since the two sets of modes are defined in different regions of spacetime. Then, the asymptotic form of a set of modes $\psi^{\mathrm{up}}_{\omega\ell m}$ near the region IV past horizon $\mathcal{H}^-_{\mathrm{IV}}$ is given by

$$\psi^{\mathrm{up}}_{\omega\ell m} = \frac{1}{\sqrt{4\pi|\widetilde{\omega}|}}\,\frac{1}{r}\exp\left[\frac{\mathrm{i}\,(\omega-\widetilde{\omega})}{2\kappa}\ln\!\left(\widetilde{U}\right)\right]\exp\left[-\frac{\mathrm{i}\,(\omega+\widetilde{\omega})}{2\kappa}\ln\!\left(-\widetilde{V}\right)\right]Y_{\ell m}(\theta,\varphi)\,\Theta\!\left(-\widetilde{V}\right). \tag{4.101}$$

The modes in (4.101) have an intuitive interpretation in that they represent the region IV analogue of the up-modes (3.131) that are defined in region I. Through a similar process to that used to evaluate the norm of the region IV down-modes in §4.4.1, we find that the expression for the inner product of two generic region IV up-modes is given by

$$\langle \psi^{\mathrm{up}}_{\omega\ell m}, \psi^{\mathrm{up}}_{\omega'\ell' m'}\rangle = \begin{cases} -\delta(\omega-\omega')\,\delta_{\ell\ell'}\,\delta_{mm'}, & \text{for } \widetilde{\omega} > 0, \\ \delta(\omega-\omega')\,\delta_{\ell\ell'}\,\delta_{mm'}, & \text{for } \widetilde{\omega} < 0, \end{cases} \tag{4.102}$$

meaning that region IV up-modes are of negative-norm when $\widetilde{\omega} > 0$ and of positive-norm when $\widetilde{\omega} < 0$. This is different to the case of the down-modes (3.146) that have positive-norm when $\widetilde{\omega} > 0$ and negative-norm when $\widetilde{\omega} < 0$.

Near the future horizon $\mathcal{H}^+$, the down-modes take the particularly simple asymptotic form given in (4.98) and these modes vanish in region IV. Near the past horizon $\mathcal{H}^-_{\mathrm{IV}}$, the region IV up-modes take the particularly simple asymptotic form given in (4.101) and these modes vanish in region I. Both the future horizon $\mathcal{H}^+$ and the region IV past horizon $\mathcal{H}^-_{\mathrm{IV}}$ are surfaces of constant $U = 0$. Then it is convenient to choose our surface of integration of the modes in the lemma (4.100) to be a hypersurface of constant $U = \epsilon > 0$, where ϵ is a small, positive constant such that this surface lies inside region II, close to $\mathcal{H}^+$, and inside region IV, close to $\mathcal{H}^-_{\mathrm{IV}}$.

The asymptotic form of the region IV up-modes near $\mathcal{H}^-$ (4.101) contain a factor of $\exp[\ln(\widetilde{U})]$, whereas the asymptotic form of the down-modes near $\mathcal{H}^+$ (4.98) contain a factor of $\exp[\ln(-\widetilde{U})]$. We are attempting to define a set of positive-frequency modes with respect to the Kruskal coordinate V; these are analytic in the lower-half of the plane and so we need to use a branch of the logarithm that is also analytic in the lower-half plane. We can therefore choose to make a branch cut along the positive imaginary axis such that

$$\ln(-1) = -\mathrm{i}\pi. \tag{4.103}$$

Then, using the fact that we are integrating over a hypersurface of constant $U = \epsilon > 0$

$$\exp\left[\frac{\mathrm{i}\,(\omega-\widetilde{\omega})}{2\kappa}\ln\!\left(-\widetilde{U}\right)\right] = \exp\left[\frac{\mathrm{i}\,(\omega-\widetilde{\omega})}{2\kappa}\ln\!\left(\widetilde{U}\right)\right]\exp\left[\frac{\pi\,(\omega-\widetilde{\omega})}{2\kappa}\right]. \tag{4.104}$$

Using (4.104), the asymptotic form of the down-modes near $\mathcal{H}^+$ (4.98) becomes

$$\phi^{\text{down}}_{\omega\ell m} = \frac{1}{\sqrt{4\pi|\widetilde{\omega}|}}\,\frac{1}{r}\exp\left[\frac{\mathrm{i}\,(\omega-\widetilde{\omega})}{2\kappa}\ln\left(\widetilde{U}\right)\right]\exp\left[-\frac{\mathrm{i}\,(\omega+\widetilde{\omega})}{2\kappa}\ln\left(\widetilde{V}\right)\right] \times\exp\left[\frac{\pi\,(\omega-\widetilde{\omega})}{2\kappa}\right]Y_{\ell m}(\theta,\varphi)\,\Theta\left(\widetilde{V}\right). \quad (4.105)$$

Multiplying the expression for the lemma in (4.100) by an appropriate factor, we obtain

$$0 = \frac{1}{\sqrt{4\pi|\widetilde{\omega}|}}\,\frac{1}{r}\exp\left[\frac{\mathrm{i}\,(\omega-\widetilde{\omega})}{2\kappa}\ln\left(\widetilde{U}\right)\right]\exp\left[\frac{\pi\,(\omega-\widetilde{\omega})}{2\kappa}\right]Y_{\ell m}(\theta,\varphi) \times\int_{-\infty}^{\infty}\mathrm{d}\widetilde{V}\,\mathrm{e}^{-\mathrm{i}\mathfrak{p}\widetilde{V}}\left\{\exp\left[-\frac{\mathrm{i}\,(\omega+\widetilde{\omega})}{2\kappa}\ln\left(\widetilde{V}\right)\right]\Theta\left(\widetilde{V}\right) +\exp\left[-\frac{\pi\,(\omega+\widetilde{\omega})}{2\kappa}\right]\exp\left[-\frac{\mathrm{i}\,(\omega+\widetilde{\omega})}{2\kappa}\ln\left(-\widetilde{V}\right)\right]\Theta\left(-\widetilde{V}\right)\right\}. \quad (4.106)$$

The first term in (4.106) is exactly that of the asymptotic form of the down-modes near $\mathcal{H}^+$ (4.105), while the second term in (4.106) is that of the asymptotic form of the region IV up-modes near $\mathcal{H}^-_{\text{IV}}$ (4.101) multiplied by a factor of

$$\exp\left[\frac{\pi\,(\omega-\widetilde{\omega})}{2\kappa}\right]\exp\left[-\frac{\pi\,(\omega+\widetilde{\omega})}{2\kappa}\right] = \exp\left[-\frac{\pi\widetilde{\omega}}{\kappa}\right]. \quad (4.107)$$

Then we can write the linear combination of modes in (4.106) as

$$0 = \int_{-\infty}^{\infty}\mathrm{d}\widetilde{V}\,\mathrm{e}^{-\mathrm{i}\mathfrak{p}\widetilde{V}}\left\{\phi^{\text{down}}_{\omega\ell m} + \mathrm{e}^{-\frac{\pi\widetilde{\omega}}{\kappa}}\,\psi^{\text{up}}_{\omega\ell m}\right\}, \quad \mathfrak{p} > 0. \quad (4.108)$$

By the statement (4.2), the quantity in the curly brackets in (4.108) is positive-frequency w.r.t V for all values of $\widetilde{\omega}$. Multiplying this quantity by a factor of $\mathfrak{N}^{\text{down}+}_{\omega}\mathrm{e}^{\frac{\pi\widetilde{\omega}}{2\kappa}}$, where the normalisation constant $\mathfrak{N}^{\text{down}+}_{\omega}$ is yet to be determined, we define a set of modes $\chi^{\text{down}+}_{\omega\ell m}$

$$\chi^{\text{down}+}_{\omega\ell m} = \mathfrak{N}^{\text{down}+}_{\omega}\left(\mathrm{e}^{\frac{\pi\widetilde{\omega}}{2\kappa}}\,\phi^{\text{down}}_{\omega\ell m} + \mathrm{e}^{-\frac{\pi\widetilde{\omega}}{2\kappa}}\,\psi^{\text{up}}_{\omega\ell m}\right) \quad (4.109)$$

which is positive-frequency with respect to the Kruskal coordinate V for all values of $\widetilde{\omega}$.

We would also like to define a set of modes that are negative-frequency w.r.t the Kruskal coordinate V. We can do this by considering the complex conjugate of the lemma in (4.7). Comparing the asymptotic expressions for the down-modes near $\mathcal{H}^+$ (4.98) and the region IV up-modes near $\mathcal{H}^-_{\text{IV}}$ (4.101) with the lemma in (4.7), we see that the terms in the lemma can be constructed from the expressions in (4.98) and (4.101) if we take

$$\mathfrak{x}^{\text{down}-} = \widetilde{V} \quad \text{and} \quad \mathfrak{q}^{\text{down}-} = -\frac{(\omega+\widetilde{\omega})}{2\kappa}. \quad (4.110)$$

Then the lemma (4.7) becomes

$$\int_{-\infty}^{\infty}\mathrm{d}\widetilde{V}\,\mathrm{e}^{\mathrm{i}\mathfrak{p}\widetilde{V}}\left\{\exp\left[-\frac{\mathrm{i}\,(\omega+\widetilde{\omega})}{2\kappa}\ln\left(\widetilde{V}\right)\right]\Theta\left(\widetilde{V}\right) +\exp\left[\frac{\pi\,(\omega+\widetilde{\omega})}{2\kappa}\right]\exp\left[-\frac{\mathrm{i}\,(\omega+\widetilde{\omega})}{2\kappa}\ln\left(-\widetilde{V}\right)\right]\Theta\left(-\widetilde{V}\right)\right\} = 0. \quad (4.111)$$

We will again need to take a linear combination of the asymptotic forms of the down-modes near $\mathcal{H}^+$ (4.98) and of the region IV up-modes near $\mathcal{H}^-_{\rm IV}$ (4.101). The former contain a factor of $\exp[\ln(-\widetilde{U})]$ while the latter contain a factor of $\exp[\ln(\widetilde{U})]$. We are attempting to define negative-frequency modes w.r.t $\widetilde{V}$; these are analytic in the upper-half of the plane and so we need to use a branch of the logarithm that is also analytic in the upper-half of the plane. We therefore choose to make a branch cut along the negative imaginary axis

$$\ln(-1) = \mathrm{i}\pi. \tag{4.112}$$

Then, using the fact that we are integrating over a hypersurface of constant $U = \epsilon > 0$

$$\exp\left[\frac{\mathrm{i}(\omega-\widetilde{\omega})}{2\kappa}\ln\left(-\widetilde{U}\right)\right] = \exp\left[\frac{\mathrm{i}(\omega-\widetilde{\omega})}{2\kappa}\ln\left(\widetilde{U}\right)\right]\exp\left[-\frac{\pi(\omega-\widetilde{\omega})}{2\kappa}\right]. \tag{4.113}$$

Using (4.113), the asymptotic form of the down-modes near $\mathcal{H}^+$ (4.98) becomes

$$\begin{aligned}\phi^{\rm down}_{\omega\ell m} = \frac{1}{\sqrt{4\pi|\widetilde{\omega}|}}\frac{1}{r}\exp\left[\frac{\mathrm{i}(\omega-\widetilde{\omega})}{2\kappa}\ln\left(\widetilde{U}\right)\right]\exp\left[-\frac{\mathrm{i}(\omega+\widetilde{\omega})}{2\kappa}\ln\left(\widetilde{V}\right)\right]\\ \times\exp\left[-\frac{\pi(\omega-\widetilde{\omega})}{2\kappa}\right]Y_{\ell m}(\theta,\varphi)\,\Theta\left(\widetilde{V}\right).\end{aligned} \tag{4.114}$$

Multiplying the expression for the lemma in (4.111) by an appropriate factor, we obtain

$$\begin{aligned}0 = \frac{1}{\sqrt{4\pi|\widetilde{\omega}|}}\frac{1}{r}\exp\left[\frac{\mathrm{i}(\omega-\widetilde{\omega})}{2\kappa}\ln\left(\widetilde{U}\right)\right]\exp\left[-\frac{\pi(\omega-\widetilde{\omega})}{2\kappa}\right]Y_{\ell m}(\theta,\varphi)\\ \times\int_{-\infty}^{\infty}\mathrm{d}\widetilde{V}\,\mathrm{e}^{\mathrm{i}\mathfrak{p}\widetilde{V}}\left\{\exp\left[-\frac{\mathrm{i}(\omega+\widetilde{\omega})}{2\kappa}\ln\left(\widetilde{V}\right)\right]\Theta\left(\widetilde{V}\right)\right.\\ \left.+\exp\left[-\frac{\pi(\omega+\widetilde{\omega})}{2\kappa}\right]\exp\left[-\frac{\mathrm{i}(\omega+\widetilde{\omega})}{2\kappa}\ln\left(-\widetilde{V}\right)\right]\Theta\left(-\widetilde{V}\right)\right\}.\end{aligned} \tag{4.115}$$

The first term in (4.115) is exactly that of the asymptotic form of the down-modes near $\mathcal{H}^+$ (4.114), while the second term in (4.115) is that of the asymptotic form of the region IV up-modes near $\mathcal{H}^-_{\rm IV}$ (4.101) multiplied by a factor of

$$\exp\left[-\frac{\pi(\omega-\widetilde{\omega})}{2\kappa}\right]\exp\left[\frac{\pi(\omega+\widetilde{\omega})}{2\kappa}\right] = \exp\left[\frac{\pi\widetilde{\omega}}{\kappa}\right]. \tag{4.116}$$

Then we can write the linear combination of modes in (4.115) as

$$0 = \int_{-\infty}^{\infty}\mathrm{d}\widetilde{V}\,\mathrm{e}^{\mathrm{i}\mathfrak{p}\widetilde{V}}\left\{\phi^{\rm down}_{\omega\ell m} + \mathrm{e}^{\frac{\pi\widetilde{\omega}}{\kappa}}\psi^{\rm up}_{\omega\ell m}\right\}, \quad \mathfrak{p} > 0. \tag{4.117}$$

By the statement (4.3), the quantity in the curly brackets in (4.117) is negative-frequency w.r.t V for all values of $\widetilde{\omega}$. Multiplying this quantity by a factor of $\mathfrak{N}^{\rm down-}_{\omega}\mathrm{e}^{-\frac{\pi\widetilde{\omega}}{2\kappa}}$, where the normalisation constant $\mathfrak{N}^{\rm down-}_{\omega}$ is yet to be determined, we define a set of modes $\chi^{\rm down-}_{\omega\ell m}$

$$\chi^{\rm down-}_{\omega\ell m} = \mathfrak{N}^{\rm down-}_{\omega}\left(\mathrm{e}^{-\frac{\pi\widetilde{\omega}}{2\kappa}}\phi^{\rm down}_{\omega\ell m} + \mathrm{e}^{\frac{\pi\widetilde{\omega}}{2\kappa}}\psi^{\rm up}_{\omega\ell m}\right) \tag{4.118}$$

which is negative-frequency with respect to the Kruskal coordinate V for all values of $\widetilde{\omega}$.

We normalise the positive- (4.109) and negative-frequency thermalised down-modes (4.118) through a similar process to that used to normalise the thermalised up-modes in §4.4.1; using the fact that both sets of modes in (4.109) and (4.118) are a linear combination of the asymptotic forms of the down-modes $\phi^{\text{down}}_{\omega\ell m}$ and of the region IV up-modes $\psi^{\text{up}}_{\omega\ell m}$, we can use the norms of the $\phi^{\text{down}}_{\omega\ell m}$ and $\psi^{\text{up}}_{\omega\ell m}$ in (3.145) and (4.102) respectively to derive the normalisation constants $\mathfrak{N}^{\text{down}\pm}_{\omega}$

$$\mathfrak{N}^{\text{down}\pm}_{\omega} = \frac{1}{\sqrt{2\left|\sinh\left(\frac{\pi\widetilde{\omega}}{\kappa}\right)\right|}}. \tag{4.119}$$

Then, restricting our attention to region I where the $\psi^{\text{up}}_{\omega\ell m}$ vanish, a set of normalised down-modes positive-frequency with respect to the Kruskal coordinate V is given by

$$\chi^{\text{down}+}_{\omega\ell m} = \frac{1}{\sqrt{2\left|\sinh\left(\frac{\pi\widetilde{\omega}}{\kappa}\right)\right|}}\, e^{\frac{\pi\widetilde{\omega}}{2\kappa}}\, \phi^{\text{down}}_{\omega\ell m}, \quad \text{all } \widetilde{\omega}, \tag{4.120}$$

and a set of normalised down-modes having negative-frequency w.r.t V is given by

$$\chi^{\text{down}-}_{\omega\ell m} = \frac{1}{\sqrt{2\left|\sinh\left(\frac{\pi\widetilde{\omega}}{\kappa}\right)\right|}}\, e^{-\frac{\pi\widetilde{\omega}}{2\kappa}}\, \phi^{\text{down}}_{\omega\ell m}, \quad \text{all } \widetilde{\omega}. \tag{4.121}$$

Construction of the 'future' Unruh state

Recall that in defining the 'future' Unruh $|\mathrm{U}^+\rangle$ state, we would like a state that is as empty as possible to a static observer at future null infinity $\mathscr{I}^+$ but which contains an incoming flux of thermal radiation at past null infinity $\mathscr{I}^-$; this corresponds to an absence of particles in the out-modes (3.139) and a thermalised flux of particles in the down-modes.

Then, we may expand the scalar field Φ in terms of an orthonormal basis of out-modes and thermally populated down-modes, each divided into positive- and negative-frequency sets. We recall that the out-modes are defined to be positive- (4.32) and negative-frequency (4.33) with respect to the Schwarzschild-like coordinate t, which is the natural time coordinate to use near $\mathscr{I}^+$. Using (4.32), (4.33), (4.120) and (4.121), we have

$$\begin{aligned}\Phi(x) = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\Bigg\{ &\int_0^{\infty} d\omega\, a^{\text{out}}_{\omega\ell m}\phi^{\text{out}+}_{\omega\ell m} + \int_{-\infty}^{0} d\omega\, b^{\text{out}\dagger}_{\omega\ell m}\phi^{\text{out}-}_{\omega\ell m} \\ &+ \int_{-\infty}^{\infty} d\widetilde{\omega}\, \frac{1}{\sqrt{2\left|\sinh\left(\frac{\pi\widetilde{\omega}}{\kappa}\right)\right|}}\, \phi^{\text{down}}_{\omega\ell m}\left[e^{\frac{\pi\widetilde{\omega}}{2\kappa}} a^{\text{down}}_{\omega\ell m} + e^{-\frac{\pi\widetilde{\omega}}{2\kappa}} b^{\text{down}\dagger}_{\omega\ell m}\right]\Bigg\}. \end{aligned} \tag{4.122}$$

We quantise the field by promoting the expansion coefficients to operators such that the field operator $\hat{\Phi}(x)$ is given by

$$\hat{\Phi}(x) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \left\{ \int_0^{\infty} \mathrm{d}\omega\, \hat{a}^{\mathrm{out}}_{\omega\ell m} \phi^{\mathrm{out}+}_{\omega\ell m} + \int_{-\infty}^{0} \mathrm{d}\omega\, \hat{b}^{\mathrm{out}\dagger}_{\omega\ell m} \phi^{\mathrm{out}-}_{\omega\ell m} \right.$$
$$\left. + \int_{-\infty}^{\infty} \mathrm{d}\widetilde{\omega}\, \frac{1}{\sqrt{2\left|\sinh\left(\frac{\pi\widetilde{\omega}}{\kappa}\right)\right|}} \phi^{\mathrm{down}}_{\omega\ell m} \left[\mathrm{e}^{\frac{\pi\widetilde{\omega}}{2\kappa}} \hat{a}^{\mathrm{down}}_{\omega\ell m} + \mathrm{e}^{-\frac{\pi\widetilde{\omega}}{2\kappa}} \hat{b}^{\mathrm{down}\dagger}_{\omega\ell m} \right] \right\}, \quad (4.123)$$

where the operators associated to the out-modes, $\hat{a}^{\mathrm{out}}_{\omega\ell m}$ and $\hat{b}^{\mathrm{out}}_{\omega\ell m}$, are defined for $\omega > 0$ and $\omega < 0$ respectively, and the operators associated to the down-modes, $\hat{a}^{\mathrm{down}}_{\omega\ell m}$ and $\hat{b}^{\mathrm{down}}_{\omega\ell m}$, are defined for all $\widetilde{\omega}$. In (4.123), all of the positive-frequency modes $\phi^{\mathrm{out/down}+}_{\omega\ell m}$ are of positive-norm and all of the negative-frequency modes $\phi^{\mathrm{out/down}-}_{\omega\ell m}$ are of negative-norm; then, the operators $\hat{a}^{\mathrm{out/down}}_{\omega\ell m}$ and $\hat{b}^{\mathrm{out/down}}_{\omega\ell m}$ obey the following, standard commutation relations

$$\begin{aligned}
\left[\hat{a}^{\mathrm{out}}_{\omega\ell m}, \hat{a}^{\mathrm{out}\dagger}_{\omega'\ell'm'}\right] &= \delta\left(\omega-\omega'\right)\delta_{\ell\ell'}\,\delta_{mm'}, \quad && \omega > 0,\\
\left[\hat{b}^{\mathrm{out}}_{\omega\ell m}, \hat{b}^{\mathrm{out}\dagger}_{\omega'\ell'm'}\right] &= \delta\left(\omega-\omega'\right)\delta_{\ell\ell'}\,\delta_{mm'}, \quad && \omega < 0,\\
\left[\hat{a}^{\mathrm{down}}_{\omega\ell m}, \hat{a}^{\mathrm{down}\dagger}_{\omega'\ell'm'}\right] &= \delta\left(\omega-\omega'\right)\delta_{\ell\ell'}\,\delta_{mm'}, \quad && \text{all } \widetilde{\omega},\\
\left[\hat{b}^{\mathrm{down}}_{\omega\ell m}, \hat{b}^{\mathrm{down}\dagger}_{\omega'\ell'm'}\right] &= \delta\left(\omega-\omega'\right)\delta_{\ell\ell'}\,\delta_{mm'}, \quad && \text{all } \widetilde{\omega},
\end{aligned} \quad (4.124)$$

with any commutators not explicitly given in (4.124) vanishing. The 'future' Unruh state $|\mathrm{U}^{+}\rangle$ is defined as the state annihilated by the $\hat{a}^{\mathrm{out/down}}_{\omega\ell m}$ and $\hat{b}^{\mathrm{out/down}}_{\omega\ell m}$ operators such that

$$\begin{aligned}
\hat{a}^{\mathrm{out}}_{\omega\ell m}\,|\mathrm{U}^{+}\rangle &= 0, \quad && \omega > 0,\\
\hat{b}^{\mathrm{out}}_{\omega\ell m}\,|\mathrm{U}^{+}\rangle &= 0, \quad && \omega < 0,\\
\hat{a}^{\mathrm{down}}_{\omega\ell m}\,|\mathrm{U}^{+}\rangle &= 0, \quad && \text{all } \widetilde{\omega},\\
\hat{b}^{\mathrm{down}}_{\omega\ell m}\,|\mathrm{U}^{+}\rangle &= 0, \quad && \text{all } \widetilde{\omega}.
\end{aligned} \quad (4.125)$$

The 'future' Unruh state $|\mathrm{U}^{+}\rangle$ contains no particles or antiparticles outgoing at future null infinity $\mathscr{I}^{+}$. However, it does contain a thermal flux of particles and antiparticles incoming from past null infinity $\mathscr{I}^{-}$. We explore this state further when we study the expectation values of quantum observables in the 'future' Unruh state $|\mathrm{U}^{+}\rangle$ in §5.4.1.

4.5 Hartle-Hawking states

In §2.3.3, we introduced the Schwarzschild Hartle-Hawking state $|\mathrm{H_s}\rangle$; this state exhibits an unstable equilibrium of incoming thermal radiation from past null infinity $\mathscr{I}^{-}$ and outgoing thermal radiation at future null infinity $\mathscr{I}^{+}$. We would like to define analogous states for a charged scalar field in Reissner-Nordström spacetime.

In §3.3.4, we derived conditions for low-frequency modes of the classical scalar field to undergo superradiant scattering in Reissner-Nordström spacetime. This indicates that it may be impossible to define a Hartle-Hawking state which exhibits an equilibrium of incoming and outgoing thermal radiation.

As such, it will be prudent to define two separate 'past' and 'future' CCH states [15], namely the 'past' CCH state $|\text{CCH}^-\rangle$ in §4.5.1 that will have a thermal distribution of particles in the in- and up-modes, and the 'future' CCH state $|\text{CCH}^+\rangle$ in 4.5.2 that will have a thermal distribution of particles in the out- and down-modes; however, it remains to be seen whether either of these states represents a thermal equilibrium.

We will also define two '-like' states, namely the 'Hartle-Hawking-like' state $|\text{H}\rangle$ in §4.5.3, which is an attempt to remain as close in spirit as possible to the Schwarzschild Hartle-Hawking state $|\text{H}_\text{s}\rangle$, as well as the Frolov-Thorne state $|\text{FT}\rangle$ in §4.5.4, which will be defined in an analogous manner to the way in which the "Hartle-Hawking"-like state for a neutral scalar field in Kerr spacetime was defined in [49].

4.5.1 'Past' CCH state

We would like to construct a state that exhibits thermal radiation both incoming at past null infinity $\mathscr{I}^-$ as well as outgoing at future null infinity $\mathscr{I}^+$; in terms of the scalar field modes defined in §3.3, these requirements correspond to a thermalised flux of particles in both the in-modes (3.106) and the up-modes (3.131) of the field.

Together, the in- and up-modes constitute an orthonormal basis which we require in order to quantise the field. From our discussion in §4.1, we first need to decompose the in- and up-modes into positive- and negative-frequency setsto canonically quantise the field.

The thermal radiation outgoing to future null infinity $\mathscr{I}^+$ in the 'past' CCH state $|\text{CCH}^-\rangle$ corresponds to a thermalised flux of particles in the up-modes emanating from the past horizon $\mathcal{H}^-$. The Kruskal coordinate U (3.24) is the affine parameter along the null generators of the past horizon $\mathcal{H}^-$. Then, near $\mathcal{H}^-$, the natural choice of time-coordinate w.r.t which we can define positive- and negative-frequency up-modes is the Kruskal retarded time U. We have already defined thermalised up-modes that are positive- and negative-frequency w.r.t U when defining the 'past' Unruh state $|\text{U}^-\rangle$ in §4.4.1. Then, the definitions that we require are those given in (4.92a) and (4.92b) respectively.

It should be noted that it is unsurprising that, in defining the 'past' CCH state $|\text{CCH}^-\rangle$, we are able to use the same definitions of positive- and negative-frequency thermalised up-modes as when we defined the 'past' Unruh state $|\text{U}^-\rangle$; both of these states exhibit an outgoing flux of thermal radiation at future null infinity $\mathscr{I}^+$. We now need to define a set of positive- and negative-frequency thermalised in-modes.

Positive- and negative-frequency thermalised in-modes

The 'past' CCH state $|\text{CCH}^-\rangle$ has the additional interpretation of predicting thermal radiation incoming at past null infinity $\mathscr{I}^-$; this property corresponds to a thermalised flux of particles in the in-modes that is incident upon the future horizon $\mathcal{H}^+$. The Kruskal coordinate V (3.24) is the affine parameter along the null generators of the future horizon $\mathcal{H}^+$. Then, near $\mathcal{H}^+$, the natural choice of time-coordinate w.r.t which we can define positive- and negative-frequency in-modes is the Kruskal advanced time V.

We can begin by expressing the asymptotic form of the in-modes (3.64) near the future horizon $\mathcal{H}^+$ in terms of Kruskal coordinates; using (3.21), (3.26) and (3.33) we have

$$\phi^{\text{in}}_{\omega\ell m} = \frac{B^{\text{in}}_{\omega\ell}}{\sqrt{4\pi|\omega|}}\,\frac{1}{r}\,\exp\left[\frac{\mathrm{i}\,(\omega-\widetilde{\omega})}{2\kappa}\ln\!\left(-\widetilde{U}\right)\right]\exp\left[-\frac{\mathrm{i}\,(\omega+\widetilde{\omega})}{2\kappa}\ln\!\left(\widetilde{V}\right)\right]Y_{\ell m}(\theta,\varphi)\,. \tag{4.126}$$

The in-modes are defined in regions I and II of the Penrose diagram in Figure 3.1, where $\widetilde{V} > 0$. We can trivially extend their definition to regions III and IV by using the Heaviside function (4.5) to demand that they vanish when $\widetilde{V} < 0$. Then, (4.126) becomes

$$\phi^{\text{in}}_{\omega\ell m} = \frac{B^{\text{in}}_{\omega\ell}}{\sqrt{4\pi|\omega|}}\,\frac{1}{r}\,\exp\left[\frac{\mathrm{i}\,(\omega-\widetilde{\omega})}{2\kappa}\ln\!\left(-\widetilde{U}\right)\right]\exp\left[-\frac{\mathrm{i}\,(\omega+\widetilde{\omega})}{2\kappa}\ln\!\left(\widetilde{V}\right)\right]Y_{\ell m}(\theta,\varphi)\,\Theta\!\left(\widetilde{V}\right). \tag{4.127}$$

Comparing the expression for the asymptotic form of the in-modes near the future horizon $\mathcal{H}^+$ in (4.127) with the lemma (4.6), we see that the first term in the lemma can be constructed from the expression in (4.127) if we take

$$\mathfrak{X}^{\text{in}+} = \widetilde{V} \quad \text{and} \quad \mathfrak{q}^{\text{in}+} = \frac{(\omega+\widetilde{\omega})}{2\kappa}. \tag{4.128}$$

Then the lemma (4.6) becomes

$$\int_{-\infty}^{\infty}\mathrm{d}\widetilde{V}\,\mathrm{e}^{-\mathrm{i}\mathfrak{p}\widetilde{V}}\left\{\exp\left[-\frac{\mathrm{i}\,(\omega+\widetilde{\omega})}{2\kappa}\ln\!\left(\widetilde{V}\right)\right]\Theta\!\left(\widetilde{V}\right)\right.$$
$$\left.+\exp\left[-\frac{\pi\,(\omega+\widetilde{\omega})}{2\kappa}\right]\exp\left[-\frac{\mathrm{i}\,(\omega+\widetilde{\omega})}{2\kappa}\ln\!\left(-\widetilde{V}\right)\right]\Theta\!\left(-\widetilde{V}\right)\right\} = 0. \tag{4.129}$$

We see that the second term of the lemma (4.129) can be constructed from a set of modes in regions III and IV, vanishing in regions I and II, with the same dependence on $\mathfrak{q}^{\text{in}+}$ (4.128) as the asymptotic form of the in-modes in (4.127) but containing factors of $-\widetilde{V}$ as opposed to $\widetilde{V}$. As was the case when defining the region IV down-modes in §4.4.1, we will make both of the transformations $\widetilde{U} \to -\widetilde{U}$ as well as $\widetilde{V} \to -\widetilde{V}$, which results in a set of modes that are nonzero in regions III and IV and vanishing elsewhere, as well as being orthogonal to the in-modes in (4.127) since the two sets of modes are defined in different regions of spacetime. Then, the asymptotic form of a set of modes $\psi^{\text{out}}_{\omega\ell m}$ near the region IV past horizon $\mathcal{H}^-_{\text{IV}}$ is given by

$$\psi^{\text{out}}_{\omega\ell m} = \frac{B^{\text{in}}_{\omega\ell}}{\sqrt{4\pi|\omega|}}\,\frac{1}{r}\,\exp\left[\frac{\mathrm{i}\,(\omega-\widetilde{\omega})}{2\kappa}\ln\!\left(\widetilde{U}\right)\right]\exp\left[-\frac{\mathrm{i}\,(\omega+\widetilde{\omega})}{2\kappa}\ln\!\left(-\widetilde{V}\right)\right]Y_{\ell m}(\theta,\varphi)\,\Theta\!\left(-\widetilde{V}\right). \tag{4.130}$$

The modes in (4.130) have an intuitive interpretation in that they represent the region IV analogue of the out-modes (3.139) that are defined in region I. Through a similar process to that used to evaluate the norm of the region IV down-modes in §4.4.1, we find that the expression for the inner product of two generic region IV out-modes is given by

$$\langle \psi^{\text{out}}_{\omega\ell m}, \psi^{\text{out}}_{\omega'\ell' m'}\rangle = \begin{cases} -\delta(\omega-\omega')\,\delta_{\ell\ell'}\,\delta_{mm'}, & \text{for } \omega > 0, \\ \delta(\omega-\omega')\,\delta_{\ell\ell'}\,\delta_{mm'}, & \text{for } \omega < 0, \end{cases} \tag{4.131}$$

meaning that region IV out-modes are of negative-norm when $\omega > 0$ and of positive-norm when $\omega < 0$. This is different to the case of the in-modes (3.105) that have positive-norm when $\omega > 0$ and negative-norm when $\omega < 0$.

Near the future horizon $\mathcal{H}^+$, the in-modes take the particularly simple asymptotic form given in (4.127) and these modes vanish in region IV. Near the past horizon $\mathcal{H}^-_{\mathrm{IV}}$, the region IV out-modes take the particularly simple asymptotic form given in (4.130) and these modes vanish in region I. Both the future horizon $\mathcal{H}^+$ and the region IV past horizon $\mathcal{H}^-_{\mathrm{IV}}$ are surfaces of constant $U = 0$. Then it is convenient to choose our surface of integration of the modes in the lemma (4.129) to be a hypersurface of constant $U = -\epsilon < 0$, where ϵ is a small, positive constant such that this surface lies inside region I, close to $\mathcal{H}^+$, and inside region III, close to $\mathcal{H}^-_{\mathrm{IV}}$.

The asymptotic form of the in-modes near $\mathcal{H}^+$ (4.127) contain a factor of $\exp[\ln(-\widetilde{U})]$, whereas the asymptotic form of the region IV out-modes near $\mathcal{H}^-_{\mathrm{IV}}$ (4.130) contain a factor of $\exp[\ln(\widetilde{U})]$. We are attempting to define a set of positive-frequency modes with respect to the Kruskal coordinate V; these are analytic in the lower-half of the plane and so we need to use a branch of the logarithm that is also analytic in the lower-half plane. We can therefore choose to make a branch cut along the positive imaginary axis such that

$$\ln(-1) = -\mathrm{i}\pi. \tag{4.132}$$

Then, using the fact that we are integrating over a hypersurface of constant $U = -\epsilon < 0$

$$\exp\left[\frac{\mathrm{i}\,(\omega-\widetilde{\omega})}{2\kappa}\ln\!\left(\widetilde{U}\right)\right] = \exp\left[\frac{\mathrm{i}\,(\omega-\widetilde{\omega})}{2\kappa}\ln\!\left(-\widetilde{U}\right)\right]\exp\left[\frac{\pi\,(\omega-\widetilde{\omega})}{2\kappa}\right]. \tag{4.133}$$

Using (4.133), the asymptotic form of the in-modes near $\mathcal{H}^+$ (4.127) becomes

$$\begin{aligned}\phi^{\mathrm{in}}_{\omega\ell m} = \frac{B^{\mathrm{in}}_{\omega\ell}}{\sqrt{4\pi|\omega|}}\,\frac{1}{r}\exp\left[\frac{\mathrm{i}\,(\omega-\widetilde{\omega})}{2\kappa}\ln\!\left(\widetilde{U}\right)\right]\exp\left[-\frac{\mathrm{i}\,(\omega+\widetilde{\omega})}{2\kappa}\ln\!\left(\widetilde{V}\right)\right]&\\ \times\exp\left[-\frac{\pi\,(\omega-\widetilde{\omega})}{2\kappa}\right]Y_{\ell m}(\theta,\varphi)\,\Theta\!\left(\widetilde{V}\right).&\end{aligned} \tag{4.134}$$

Multiplying the expression for the lemma in (4.129) by an appropriate factor, we obtain

$$\begin{aligned}0 = \frac{B^{\mathrm{in}}_{\omega\ell}}{\sqrt{4\pi|\omega|}}\,\frac{1}{r}\exp\left[\frac{\mathrm{i}\,(\omega-\widetilde{\omega})}{2\kappa}\ln\!\left(\widetilde{U}\right)\right]\exp\left[-\frac{\pi\,(\omega-\widetilde{\omega})}{2\kappa}\right]Y_{\ell m}(\theta,\varphi)&\\ \times\int_{-\infty}^{\infty}\mathrm{d}\widetilde{V}\,\mathrm{e}^{-\mathrm{i}\mathfrak{p}\widetilde{V}}\left\{\exp\left[-\frac{\mathrm{i}\,(\omega+\widetilde{\omega})}{2\kappa}\ln\!\left(\widetilde{V}\right)\right]\Theta\!\left(\widetilde{V}\right)\right.&\\ \left.+\exp\left[-\frac{\pi\,(\omega+\widetilde{\omega})}{2\kappa}\right]\exp\left[-\frac{\mathrm{i}\,(\omega+\widetilde{\omega})}{2\kappa}\ln\!\left(-\widetilde{V}\right)\right]\Theta\!\left(-\widetilde{V}\right)\right\}.&\end{aligned} \tag{4.135}$$

The first term in (4.135) is exactly that of the asymptotic form of the in-modes near $\mathcal{H}^+$ (4.134), while the second term in (4.135) is that of the asymptotic form of the region IV out-modes near $\mathcal{H}^-_{\mathrm{IV}}$ (4.130) multiplied by a factor of

$$\exp\left[-\frac{\pi\left(\omega-\widetilde{\omega}\right)}{2\kappa}\right]\exp\left[-\frac{\pi\left(\omega+\widetilde{\omega}\right)}{2\kappa}\right]=\exp\left[-\frac{\pi\omega}{\kappa}\right]. \tag{4.136}$$

Then we can write the linear combination of modes in (4.135) as

$$0=\int_{-\infty}^{\infty}\mathrm{d}\widetilde{V}\,\mathrm{e}^{-\mathrm{i}\mathfrak{p}\widetilde{V}}\left\{\phi^{\mathrm{in}}_{\omega\ell m}+\mathrm{e}^{-\frac{\pi\omega}{\kappa}}\psi^{\mathrm{out}}_{\omega\ell m}\right\},\quad \mathfrak{p}>0. \tag{4.137}$$

By the statement (4.2), the quantity in the curly brackets in (4.137) is positive-frequency w.r.t V for all values of ω. Multiplying this quantity by a factor of $\mathfrak{N}^{\mathrm{in}+}_{\omega}\mathrm{e}^{\frac{\pi\omega}{2\kappa}}$, where the normalisation constant $\mathfrak{N}^{\mathrm{in}+}_{\omega}$ is yet to be determined, we define a set of modes $\chi^{\mathrm{in}+}_{\omega\ell m}$

$$\chi^{\mathrm{in}+}_{\omega\ell m}=\mathfrak{N}^{\mathrm{in}+}_{\omega}\left(\mathrm{e}^{\frac{\pi\omega}{2\kappa}}\phi^{\mathrm{in}}_{\omega\ell m}+\mathrm{e}^{-\frac{\pi\omega}{2\kappa}}\psi^{\mathrm{out}}_{\omega\ell m}\right) \tag{4.138}$$

which is positive-frequency with respect to the Kruskal coordinate V for all values of ω.

We would also like to define a set of modes that are negative-frequency w.r.t the Kruskal coordinate V. We can do this by considering the complex conjugate of the lemma in (4.7). Comparing the asymptotic expressions for the in-modes near $\mathcal{H}^+$ (4.127) and the region IV out-modes near $\mathcal{H}^-_{\mathrm{IV}}$ (4.130) with the lemma in (4.7), we see that the terms in the lemma can be constructed from the expressions in (4.127) and (4.130) if we take

$$\mathfrak{X}^{\mathrm{in}-}=\widetilde{V}\quad\text{and}\quad\mathfrak{q}^{\mathrm{in}-}=-\frac{\left(\omega+\widetilde{\omega}\right)}{2\kappa}. \tag{4.139}$$

Then the lemma (4.7) becomes

$$\begin{aligned}\int_{-\infty}^{\infty}\mathrm{d}\widetilde{V}\,\mathrm{e}^{\mathrm{i}\mathfrak{p}\widetilde{V}}&\left\{\exp\left[-\frac{\mathrm{i}\left(\omega+\widetilde{\omega}\right)}{2\kappa}\ln\left(\widetilde{V}\right)\right]\Theta\left(\widetilde{V}\right)\right.\\&\left.+\exp\left[\frac{\pi\left(\omega+\widetilde{\omega}\right)}{2\kappa}\right]\exp\left[-\frac{\mathrm{i}\left(\omega+\widetilde{\omega}\right)}{2\kappa}\ln\left(-\widetilde{V}\right)\right]\Theta\left(-\widetilde{V}\right)\right\}=0.\end{aligned} \tag{4.140}$$

We will again need to take a linear combination of the asymptotic forms of the in-modes near $\mathcal{H}^+$ (4.127) and of the region IV out-modes near $\mathcal{H}^-_{\mathrm{IV}}$ (4.130). The former contain a factor of $\exp[\ln(-\widetilde{U})]$ while the latter contain a factor of $\exp[\ln(\widetilde{U})]$. We are attempting to define negative-frequency modes w.r.t $\widetilde{V}$; these are analytic in the upper-half of the plane and so we need to use a branch of the logarithm that is also analytic in the upper-half of the plane. We therefore choose to make a branch cut along the negative imaginary axis

$$\ln\left(-1\right)=\mathrm{i}\pi. \tag{4.141}$$

Then, using the fact that we are integrating over a hypersurface of constant $U=-\epsilon<0$

$$\exp\left[\frac{\mathrm{i}\left(\omega-\widetilde{\omega}\right)}{2\kappa}\ln\left(\widetilde{U}\right)\right]=\exp\left[\frac{\mathrm{i}\left(\omega-\widetilde{\omega}\right)}{2\kappa}\ln\left(-\widetilde{U}\right)\right]\exp\left[-\frac{\pi\left(\omega-\widetilde{\omega}\right)}{2\kappa}\right]. \tag{4.142}$$

Using (4.142), the asymptotic form of the in-modes near $\mathcal{H}^+$ (4.127) becomes

$$\phi^{\rm in}_{\omega\ell m} = \frac{B^{\rm in}_{\omega\ell}}{\sqrt{4\pi|\widetilde{\omega}|}}\,\frac{1}{r}\,\exp\left[\frac{{\rm i}\,(\omega-\widetilde{\omega})}{2\kappa}\ln\Big(\widetilde{U}\Big)\right]\exp\left[-\frac{{\rm i}\,(\omega+\widetilde{\omega})}{2\kappa}\ln\Big(\widetilde{V}\Big)\right] \times\exp\left[\frac{\pi\,(\omega-\widetilde{\omega})}{2\kappa}\right]Y_{\ell m}(\theta,\varphi)\,\Theta\Big(\widetilde{V}\Big). \quad (4.143)$$

Multiplying the expression for the lemma in (4.140) by an appropriate factor, we obtain

$$0 = \frac{B^{\rm in}_{\omega\ell}}{\sqrt{4\pi|\widetilde{\omega}|}}\,\frac{1}{r}\,\exp\left[\frac{{\rm i}\,(\omega-\widetilde{\omega})}{2\kappa}\ln\Big(\widetilde{U}\Big)\right]\exp\left[\frac{\pi\,(\omega-\widetilde{\omega})}{2\kappa}\right]Y_{\ell m}(\theta,\varphi) \times\int_{-\infty}^{\infty}{\rm d}\widetilde{V}\,{\rm e}^{{\rm i}\mathfrak{p}\widetilde{V}}\left\{\exp\left[\frac{{\rm i}\,(\omega+\widetilde{\omega})}{2\kappa}\ln\Big(\widetilde{V}\Big)\right]\Theta\Big(\widetilde{V}\Big) +\exp\left[\frac{\pi\,(\omega+\widetilde{\omega})}{2\kappa}\right]\exp\left[-\frac{{\rm i}\,(\omega+\widetilde{\omega})}{2\kappa}\ln\Big(-\widetilde{V}\Big)\right]\Theta\Big(-\widetilde{V}\Big)\right\}. \quad (4.144)$$

The first term in (4.144) is exactly that of the asymptotic form of the in-modes near $\mathcal{H}^+$ (4.143), while the second term in (4.144) is that of the asymptotic form of the region IV out-modes near $\mathcal{H}^-_{\rm IV}$ (4.130) multiplied by a factor of

$$\exp\left[\frac{\pi\,(\omega-\widetilde{\omega})}{2\kappa}\right]\exp\left[\frac{\pi\,(\omega+\widetilde{\omega})}{2\kappa}\right] = \exp\left[\frac{\pi\omega}{\kappa}\right]. \quad (4.145)$$

Then we can write the linear combination of modes in (4.144) as

$$0 = \int_{-\infty}^{\infty}{\rm d}\widetilde{V}\,{\rm e}^{{\rm i}\mathfrak{p}\widetilde{V}}\left\{\phi^{\rm in}_{\omega\ell m}+{\rm e}^{\frac{\pi\omega}{\kappa}}\,\psi^{\rm out}_{\omega\ell m}\right\}, \quad \mathfrak{p}>0. \quad (4.146)$$

By the statement (4.3), the quantity in the curly brackets in (4.146) is negative-frequency w.r.t V for all values of ω. Multiplying this quantity by a factor of $\mathfrak{N}^{\rm in-}_{\omega}{\rm e}^{-\frac{\pi\omega}{2\kappa}}$, where the normalisation constant $\mathfrak{N}^{\rm in-}_{\omega}$ is yet to be determined, we define a set of modes $\chi^{\rm in-}_{\omega\ell m}$

$$\chi^{\rm in-}_{\omega\ell m} = \mathfrak{N}^{\rm in-}_{\omega}\left({\rm e}^{-\frac{\pi\omega}{2\kappa}}\,\phi^{\rm in}_{\omega\ell m}+{\rm e}^{\frac{\pi\omega}{2\kappa}}\,\psi^{\rm out}_{\omega\ell m}\right) \quad (4.147)$$

which is negative-frequency with respect to the Kruskal coordinate V for all values of ω.

We normalise the positive- (4.138) and negative-frequency thermal in-modes (4.147) through a similar process to that used to normalise the thermal up-modes in §4.4.1; using the fact that both sets of modes in (4.138) and (4.147) are a linear combination of the asymptotic forms of the in-modes $\phi^{\rm in}_{\omega\ell m}$ and of the region IV out-modes $\psi^{\rm out}_{\omega\ell m}$, we can use the norms of the $\phi^{\rm in}_{\omega\ell m}$ and $\psi^{\rm out}_{\omega\ell m}$ in (3.105) and (4.131) respectively to derive the normalisation constants $\mathfrak{N}^{\rm in\pm}_{\omega}$

$$\mathfrak{N}^{\rm in\pm}_{\omega} = \frac{1}{\sqrt{2\left|\sinh\left(\frac{\pi\omega}{\kappa}\right)\right|}}. \quad (4.148)$$

Then, restricting our attention to region I where the $\psi^{\rm out}_{\omega\ell m}$ vanish, a set of normalised in-modes having positive-frequency with respect to the Kruskal coordinate V is given by

$$\chi^{\rm in+}_{\omega\ell m} = \frac{1}{\sqrt{2\left|\sinh\left(\frac{\pi\omega}{\kappa}\right)\right|}}\,{\rm e}^{\frac{\pi\omega}{2\kappa}}\,\phi^{\rm in}_{\omega\ell m}, \quad \text{all } \omega, \quad (4.149)$$

and a set of normalised in-modes having negative-frequency w.r.t V is given by

$$\chi^{\text{in}-}_{\omega\ell m} = \frac{1}{\sqrt{2\left|\sinh\left(\frac{\pi\omega}{\kappa}\right)\right|}}\, \mathrm{e}^{-\frac{\pi\omega}{2\kappa}}\, \phi^{\text{in}}_{\omega\ell m}, \quad \text{all } \omega. \tag{4.150}$$

Construction of the 'past' CCH state

Recall that in defining the 'past' CCH state $|\text{CCH}^-\rangle$, we would like a state that exhibits thermal radiation both incoming from past null infinity $\mathscr{I}^-$ and outgoing to future null infinity $\mathscr{I}^+$; this corresponds to a thermalised flux of particles in both the in-modes (3.106) and the up-modes (3.131) of the field Φ.

Then, we may expand the scalar field Φ in terms of an orthonormal basis of thermally populated in- and up-modes, each divided into positive- and negative-frequency sets. We recall that the up-modes are defined to be positive- (4.92a) and negative-frequency (4.92b) with respect to the Kruskal coordinate U, which is the natural time coordinate to use near $\mathcal{H}^-$. Using (4.92a), (4.92b), (4.149) and (4.150), we have

$$\begin{aligned}\Phi(x) = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\Bigg\{ &\int_{-\infty}^{\infty} \mathrm{d}\omega \frac{1}{\sqrt{2\left|\sinh\left(\frac{\pi\omega}{\kappa}\right)\right|}}\, \phi^{\text{in}}_{\omega\ell m}\left[\mathrm{e}^{\frac{\pi\omega}{2\kappa}}\, a^{\text{in}}_{\omega\ell m} + \mathrm{e}^{-\frac{\pi\omega}{2\kappa}}\, b^{\text{in}\dagger}_{\omega\ell m}\right] \\ &+ \int_{-\infty}^{\infty} \mathrm{d}\widetilde{\omega} \frac{1}{\sqrt{2\left|\sinh\left(\frac{\pi\widetilde{\omega}}{\kappa}\right)\right|}}\, \phi^{\text{up}}_{\omega\ell m}\left[\mathrm{e}^{\frac{\pi\widetilde{\omega}}{2\kappa}}\, a^{\text{up}}_{\omega\ell m} + \mathrm{e}^{-\frac{\pi\widetilde{\omega}}{2\kappa}}\, b^{\text{up}\dagger}_{\omega\ell m}\right]\Bigg\}. \end{aligned}\tag{4.151}$$

We quantise the field by promoting the expansion coefficients to operators such that the field operator $\hat{\Phi}(x)$ is given by

$$\begin{aligned}\hat{\Phi}(x) = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\Bigg\{ &\int_{-\infty}^{\infty} \mathrm{d}\omega \frac{1}{\sqrt{2\left|\sinh\left(\frac{\pi\omega}{\kappa}\right)\right|}}\, \phi^{\text{in}}_{\omega\ell m}\left[\mathrm{e}^{\frac{\pi\omega}{2\kappa}}\, \hat{a}^{\text{in}}_{\omega\ell m} + \mathrm{e}^{-\frac{\pi\omega}{2\kappa}}\, \hat{b}^{\text{in}\dagger}_{\omega\ell m}\right] \\ &+ \int_{-\infty}^{\infty} \mathrm{d}\widetilde{\omega} \frac{1}{\sqrt{2\left|\sinh\left(\frac{\pi\widetilde{\omega}}{\kappa}\right)\right|}}\, \phi^{\text{up}}_{\omega\ell m}\left[\mathrm{e}^{\frac{\pi\widetilde{\omega}}{2\kappa}}\, \hat{a}^{\text{up}}_{\omega\ell m} + \mathrm{e}^{-\frac{\pi\widetilde{\omega}}{2\kappa}}\, \hat{b}^{\text{up}\dagger}_{\omega\ell m}\right]\Bigg\}. \end{aligned}\tag{4.152}$$

where the operators associated to the in-modes, $\hat{a}^{\text{in}}_{\omega\ell m}$ and $\hat{b}^{\text{in}}_{\omega\ell m}$, are each defined for all ω, and the operators associated to the up-modes, $\hat{a}^{\text{up}}_{\omega\ell m}$ and $\hat{b}^{\text{up}}_{\omega\ell m}$, are each defined for all $\widetilde{\omega}$. In (4.152), all of the positive-frequency modes $\phi^{\text{in/up}+}_{\omega\ell m}$ are of positive-norm and all of the negative-frequency modes $\phi^{\text{in/up}-}_{\omega\ell m}$ are of negative-norm; then, the operators $\hat{a}^{\text{in/up}}_{\omega\ell m}$ and $\hat{b}^{\text{in/up}}_{\omega\ell m}$ obey the following, standard commutation relations

$$\begin{aligned}\left[\hat{a}^{\text{in}}_{\omega\ell m}, \hat{a}^{\text{in}\dagger}_{\omega'\ell' m'}\right] &= \delta\big(\omega-\omega'\big)\,\delta_{\ell\ell'}\,\delta_{mm'}, \quad \text{all } \omega, \\ \left[\hat{b}^{\text{in}}_{\omega\ell m}, \hat{b}^{\text{in}\dagger}_{\omega'\ell' m'}\right] &= \delta\big(\omega-\omega'\big)\,\delta_{\ell\ell'}\,\delta_{mm'}, \quad \text{all } \omega, \\ \left[\hat{a}^{\text{up}}_{\omega\ell m}, \hat{a}^{\text{up}\dagger}_{\omega'\ell' m'}\right] &= \delta\big(\omega-\omega'\big)\,\delta_{\ell\ell'}\,\delta_{mm'}, \quad \text{all } \widetilde{\omega}, \\ \left[\hat{b}^{\text{up}}_{\omega\ell m}, \hat{b}^{\text{up}\dagger}_{\omega'\ell' m'}\right] &= \delta\big(\omega-\omega'\big)\,\delta_{\ell\ell'}\,\delta_{mm'}, \quad \text{all } \widetilde{\omega}, \end{aligned}\tag{4.153}$$

with any commutators not explicitly given in (4.153) vanishing. The 'past' CCH state is then defined as the state annihilated by the $\hat{a}^{\text{in/up}}_{\omega\ell m}$ and $\hat{b}^{\text{in/up}}_{\omega\ell m}$ operators such that

$$\begin{aligned} \hat{a}^{\text{in}}_{\omega\ell m}\,|\text{CCH}^-\rangle &= 0, \quad \text{all } \omega, \\ \hat{b}^{\text{in}}_{\omega\ell m}\,|\text{CCH}^-\rangle &= 0, \quad \text{all } \omega, \\ \hat{a}^{\text{up}}_{\omega\ell m}\,|\text{CCH}^-\rangle &= 0, \quad \text{all } \widetilde{\omega}, \\ \hat{b}^{\text{up}}_{\omega\ell m}\,|\text{CCH}^-\rangle &= 0, \quad \text{all } \widetilde{\omega}. \end{aligned} \tag{4.154}$$

The 'past' CCH state $|\text{CCH}^-\rangle$ exhibits incoming thermal radiation from past null infinity $\mathscr{I}^-$ and outgoing thermal radiation to future null infinity $\mathscr{I}^+$, which corresponds to a thermalised flux of particles in the in-modes and in the up-modes respectively. However, it is clear that there is no thermal equilibrium in this state, since the in- and up-modes each contain different thermal factors. We explore this state further when we study the expectation values of quantum observables in the 'past' CCH state $|\text{CCH}^-\rangle$ in §5.3.4.

4.5.2 'Future' CCH state

The 'future' CCH state $|\text{CCH}^+\rangle$ can be understood as the time-reverse of the 'past' CCH state $|\text{CCH}^-\rangle$ that we defined in §4.5.1. Where the 'past' CCH state $|\text{CCH}^-\rangle$ was constructed using an orthonormal basis of thermalised in- and up-modes near surfaces contained with the 'past' Cauchy surface Σ_{past} (3.86), the 'future' CCH state $|\text{CCH}^+\rangle$ is constructed using an orthonormal basis of thermalised out- and down-modes near surfaces contained with the 'future' Cauchy surface Σ_{future} (3.87).

The thermal radiation incoming from past null infinity $\mathscr{I}^+$ in the 'future' CCH state $|\text{CCH}^+\rangle$ corresponds to a thermalised flux of particles in the down-modes incident upon the future horizon $\mathcal{H}^+$. The Kruskal coordinate V (3.24) is the affine parameter along the null generators of the future horizon $\mathcal{H}^+$. Then, near $\mathcal{H}^+$, the natural choice of time-coordinate w.r.t which we can define positive- and negative-frequency up-modes is the Kruskal advanced time V. We have already defined thermalised down-modes that are positive- and negative-frequency w.r.t V when defining the 'future' Unruh state $|\text{U}^+\rangle$ in §4.4.2; the definitions that we require are those given in (4.120) and (4.121) respectively.

Positive- and negative-frequency thermalised out-modes

The 'future' CCH state $|\text{CCH}^+\rangle$ has the additional interpretation of predicting thermal radiation outgoing at future null infinity $\mathscr{I}^+$; this property corresponds to a thermalised flux of particles in the out-modes emanating from the past horizon $\mathcal{H}^-$. The Kruskal coordinate U (3.24) is the affine parameter along the null generators of the past horizon $\mathcal{H}^-$. Then, near $\mathcal{H}^-$, the natural choice of time-coordinate w.r.t which we can define positive- and negative-frequency in-modes is the Kruskal retarded time U.

We can begin by expressing the asymptotic form of the out-modes (3.68) near the past horizon $\mathcal{H}^-$ in terms of Kruskal coordinates; using (3.21), (3.26) and (3.33) we have

$$\phi^{\text{out}}_{\omega\ell m} = \frac{B^{\text{in}*}_{\omega\ell}}{\sqrt{4\pi|\omega|}}\,\frac{1}{r}\exp\left[\frac{\mathrm{i}\,(\omega+\widetilde{\omega})}{2\kappa}\ln\left(-\widetilde{U}\right)\right]\exp\left[-\frac{\mathrm{i}\,(\omega-\widetilde{\omega})}{2\kappa}\ln\left(\widetilde{V}\right)\right]Y_{\ell m}(\theta,\varphi)\,. \quad (4.155)$$

The out-modes are defined in regions I and III of the Penrose diagram in Figure 3.1, where $\widetilde{U} < 0$. We can trivially extend their definition to regions II and IV by using the Heaviside function (4.5) to demand that they vanish when $\widetilde{U} > 0$. Then, (4.155) becomes

$$\phi^{\text{out}}_{\omega\ell m} = \frac{B^{\text{in}*}_{\omega\ell}}{\sqrt{4\pi|\omega|}}\,\frac{1}{r}\exp\left[\frac{\mathrm{i}\,(\omega+\widetilde{\omega})}{2\kappa}\ln\left(-\widetilde{U}\right)\right]\exp\left[-\frac{\mathrm{i}\,(\omega-\widetilde{\omega})}{2\kappa}\ln\left(\widetilde{V}\right)\right]Y_{\ell m}(\theta,\varphi)\,\Theta\left(-\widetilde{U}\right). \quad (4.156)$$

Comparing the expression for the asymptotic form of the out-modes near the past horizon $\mathcal{H}^-$ in (4.156) with the lemma (4.6), we see that the second term in the lemma can be constructed from the expression in (4.156) if we take

$$\mathfrak{x}^{\text{out}+} = \widetilde{U} \quad \text{and} \quad \mathfrak{q}^{\text{out}+} = -\frac{(\omega+\widetilde{\omega})}{2\kappa}. \quad (4.157)$$

Then the lemma (4.6) becomes

$$\int_{-\infty}^{\infty} \mathrm{d}\widetilde{U}\,\mathrm{e}^{-\mathrm{i}\mathfrak{p}\widetilde{U}}\left\{\exp\left[\frac{\mathrm{i}\,(\omega+\widetilde{\omega})}{2\kappa}\ln\left(\widetilde{U}\right)\right]\Theta\left(\widetilde{U}\right)\right. \\ \left.+\exp\left[\frac{\pi\,(\omega+\widetilde{\omega})}{2\kappa}\right]\exp\left[\frac{\mathrm{i}\,(\omega+\widetilde{\omega})}{2\kappa}\ln\left(-\widetilde{U}\right)\right]\Theta\left(-\widetilde{U}\right)\right\} = 0. \quad (4.158)$$

We see that the first term of the lemma (4.158) can be constructed from a set of modes in regions II and IV, vanishing in regions I and III, with the same dependence on $\mathfrak{q}^{\text{out}+}$ as the asymptotic form of the out-modes in (4.156) but containing factors of $\widetilde{U}$ as opposed to $-\widetilde{U}$. As was the case when defining the region IV down-modes in §4.4.1, we will make both of the transformations $\widetilde{U} \to -\widetilde{U}$ as well as $\widetilde{V} \to -\widetilde{V}$, which results in a set of modes that are nonzero in regions II and IV and vanishing elsewhere, as well as being orthogonal to the out-modes in (4.156) since the two sets of modes are defined in different regions of spacetime. Then, the asymptotic form of a set of modes $\psi^{\text{in}}_{\omega\ell m}$ near the region IV future horizon $\mathcal{H}^+_{\text{IV}}$ is given by

$$\psi^{\text{in}}_{\omega\ell m} = \frac{B^{\text{in}*}_{\omega\ell}}{\sqrt{4\pi|\omega|}}\,\frac{1}{r}\exp\left[\frac{\mathrm{i}\,(\omega+\widetilde{\omega})}{2\kappa}\ln\left(\widetilde{U}\right)\right]\exp\left[-\frac{\mathrm{i}\,(\omega-\widetilde{\omega})}{2\kappa}\ln\left(-\widetilde{V}\right)\right]Y_{\ell m}(\theta,\varphi)\,\Theta\left(\widetilde{U}\right). \quad (4.159)$$

The modes in (4.159) have an intuitive interpretation in that they represent the region IV analogue of the in-modes (3.106) that are defined in region I. Through a similar process to that used to evaluate the norm of the region IV down-modes in §4.4.1, we find that the expression for the inner product of two generic region IV in-modes is given by

$$\langle\psi^{\text{in}}_{\omega\ell m},\psi^{\text{in}}_{\omega'\ell' m'}\rangle = \begin{cases} -\delta(\omega-\omega')\,\delta_{\ell\ell'}\,\delta_{mm'}, & \text{for } \omega > 0, \\ \delta(\omega-\omega')\,\delta_{\ell\ell'}\,\delta_{mm'}, & \text{for } \omega < 0, \end{cases} \quad (4.160)$$

meaning that region IV in-modes are of negative-norm when $\omega > 0$ and of positive-norm when $\omega < 0$. This is different to the case of the out-modes (3.138) that have positive-norm when $\omega > 0$ and negative-norm when $\omega < 0$.

Near the past horizon $\mathcal{H}^-$, the out-modes take the particularly simple asymptotic form given in (4.156) and these modes vanish in region IV. Near the future horizon $\mathcal{H}^+_{\mathrm{IV}}$, the region IV in-modes take the particularly simple asymptotic form given in (4.159) and these modes vanish in region I. Both the past horizon $\mathcal{H}^-$ and the region IV future horizon $\mathcal{H}^+_{\mathrm{IV}}$ are surfaces of constant $V = 0$. Then it is convenient to choose our surface of integration of the modes in the lemma (4.158) to be a hypersurface of constant $V = -\epsilon < 0$, where ϵ is a small, positive constant such that this surface lies inside region III, close to $\mathcal{H}^-$, and inside region IV, close to $\mathcal{H}^+_{\mathrm{IV}}$.

The asymptotic form of the out-modes near $\mathcal{H}^-$ (4.156) contain a factor of $\exp[\ln(\widetilde{V})]$, whereas the asymptotic form of the region IV in-modes near $\mathcal{H}^+_{\mathrm{IV}}$ (4.159) contain a factor of $\exp[\ln(-\widetilde{V})]$. We are attempting to define a set of positive-frequency modes with respect to the Kruskal coordinate U; these are analytic in the lower-half of the plane and so we need to use a branch of the logarithm that is also analytic in the lower-half plane. We can therefore choose to make a branch cut along the positive imaginary axis such that

$$\ln(-1) = -\mathrm{i}\pi. \tag{4.161}$$

Then, using the fact that we are integrating over a hypersurface of constant $V = -\epsilon < 0$

$$\exp\left[-\frac{\mathrm{i}\,(\omega-\widetilde{\omega})}{2\kappa}\ln\left(\widetilde{V}\right)\right] = \exp\left[-\frac{\mathrm{i}\,(\omega-\widetilde{\omega})}{2\kappa}\ln\left(-\widetilde{V}\right)\right]\exp\left[-\frac{\pi\,(\omega-\widetilde{\omega})}{2\kappa}\right]. \tag{4.162}$$

Using (4.162), the asymptotic form of the out-modes near $\mathcal{H}^+_{\mathrm{IV}}$ (4.127) becomes

$$\begin{aligned}\psi^{\mathrm{in}}_{\omega\ell m} = \frac{B^{\mathrm{in}*}_{\omega\ell}}{\sqrt{4\pi|\omega|}}\,\frac{1}{r}\exp\left[\frac{\mathrm{i}\,(\omega+\widetilde{\omega})}{2\kappa}\ln\left(\widetilde{U}\right)\right]\exp\left[-\frac{\mathrm{i}\,(\omega-\widetilde{\omega})}{2\kappa}\ln\left(\widetilde{V}\right)\right]\\ \times\exp\left[\frac{\pi\,(\omega-\widetilde{\omega})}{2\kappa}\right]Y_{\ell m}(\theta,\varphi)\,\Theta\left(\widetilde{U}\right).\end{aligned} \tag{4.163}$$

Multiplying the expression for the lemma in (4.158) by an appropriate factor, we obtain

$$\begin{aligned}0 = \frac{B^{\mathrm{in}*}_{\omega\ell}}{\sqrt{4\pi|\omega|}}\,\frac{1}{r}\exp\left[-\frac{\mathrm{i}\,(\omega-\widetilde{\omega})}{2\kappa}\ln\left(\widetilde{V}\right)\right]\exp\left[\frac{\pi\,(\omega-\widetilde{\omega})}{2\kappa}\right]Y_{\ell m}(\theta,\varphi)\\ \times\int_{-\infty}^{\infty}\mathrm{d}\widetilde{U}\,\mathrm{e}^{-\mathrm{i}\mathfrak{p}\widetilde{U}}\left\{\exp\left[\frac{\mathrm{i}\,(\omega+\widetilde{\omega})}{2\kappa}\ln\left(\widetilde{U}\right)\right]\Theta\left(\widetilde{U}\right)\right.\\ \left.+\exp\left[\frac{\pi\,(\omega+\widetilde{\omega})}{2\kappa}\right]\exp\left[\frac{\mathrm{i}\,(\omega+\widetilde{\omega})}{2\kappa}\ln\left(-\widetilde{U}\right)\right]\Theta\left(-\widetilde{U}\right)\right\}.\end{aligned} \tag{4.164}$$

The first term in (4.164) is exactly that of the asymptotic form of the region IV in-modes near $\mathcal{H}^+_{\mathrm{IV}}$ (4.163), while the second term in (4.164) is that of the asymptotic form of the out-modes near $\mathcal{H}^-$ (4.130) multiplied by a factor of

$$\exp\left[\frac{\pi\left(\omega-\widetilde{\omega}\right)}{2\kappa}\right]\exp\left[\frac{\pi\left(\omega+\widetilde{\omega}\right)}{2\kappa}\right]=\exp\left[\frac{\pi\omega}{\kappa}\right]. \tag{4.165}$$

Then we can write the linear combination of modes in (4.164) as

$$0=\int_{-\infty}^{\infty}\mathrm{d}\widetilde{U}\,\mathrm{e}^{-\mathrm{i}\mathfrak{p}\widetilde{U}}\left\{\psi^{\mathrm{in}}_{\omega\ell m}+\mathrm{e}^{\frac{\pi\omega}{\kappa}}\phi^{\mathrm{out}}_{\omega\ell m}\right\},\quad \mathfrak{p}>0. \tag{4.166}$$

By the statement (4.2), the quantity in the curly brackets in (4.166) is positive-frequency w.r.t U for all values of ω. Multiplying this quantity by a factor of $\mathfrak{N}^{\mathrm{out}+}_{\omega}\mathrm{e}^{-\frac{\pi\omega}{2\kappa}}$, where the normalisation constant $\mathfrak{N}^{\mathrm{out}+}_{\omega}$ is yet to be determined, we define a set of modes $\chi^{\mathrm{out}+}_{\omega\ell m}$

$$\chi^{\mathrm{out}+}_{\omega\ell m}=\mathfrak{N}^{\mathrm{out}+}_{\omega}\left(\mathrm{e}^{\frac{\pi\omega}{2\kappa}}\phi^{\mathrm{out}}_{\omega\ell m}+\mathrm{e}^{-\frac{\pi\omega}{2\kappa}}\psi^{\mathrm{in}}_{\omega\ell m}\right) \tag{4.167}$$

which is positive-frequency with respect to the Kruskal coordinate U for all values of ω.

We would also like to define a set of modes that are negative-frequency w.r.t the Kruskal coordinate U. We can do this by considering the complex conjugate of the lemma in (4.7). Comparing the asymptotic expressions for the out-modes near $\mathcal{H}^-$ (4.156) and the region IV in-modes near $\mathcal{H}^+_{\mathrm{IV}}$ (4.159) with the lemma in (4.7), we see that the terms in the lemma can be constructed from the expressions in (4.156) and (4.159) if we take

$$\mathfrak{x}^{\mathrm{in}-}=\widetilde{U}\quad\text{and}\quad\mathfrak{q}^{\mathrm{out}-}=\frac{\left(\omega+\widetilde{\omega}\right)}{2\kappa}. \tag{4.168}$$

Then the lemma (4.7) becomes

$$\begin{aligned}\int_{-\infty}^{\infty}\mathrm{d}\widetilde{U}\,\mathrm{e}^{\mathrm{i}\mathfrak{p}\widetilde{U}}\bigg\{&\exp\left[\frac{\mathrm{i}\left(\omega+\widetilde{\omega}\right)}{2\kappa}\ln\left(\widetilde{U}\right)\right]\Theta\left(\widetilde{U}\right)\\&+\exp\left[-\frac{\pi\left(\omega+\widetilde{\omega}\right)}{2\kappa}\right]\exp\left[\frac{\mathrm{i}\left(\omega+\widetilde{\omega}\right)}{2\kappa}\ln\left(-\widetilde{U}\right)\right]\Theta\left(-\widetilde{U}\right)\bigg\}=0.\end{aligned} \tag{4.169}$$

We will again need to take a linear combination of the asymptotic forms of the out-modes near $\mathcal{H}^-$ (4.156) and of the region IV in-modes near $\mathcal{H}^+_{\mathrm{IV}}$ (4.159). The former contain a factor of $\exp[\ln(\widetilde{V})]$ while the latter contain a factor of $\exp[\ln(-\widetilde{V})]$. We are attempting to define a set of negative-frequency modes w.r.t $\widetilde{U}$; these are analytic in the upper-half of the plane so we need to use a branch of the logarithm that is also analytic in the upper-half of the plane. We can therefore make a branch cut along the negative imaginary axis

$$\ln\left(-1\right)=\mathrm{i}\pi. \tag{4.170}$$

Then, using the fact that we are integrating over a hypersurface of constant $V=-\epsilon<0$

$$\exp\left[-\frac{\mathrm{i}\left(\omega-\widetilde{\omega}\right)}{2\kappa}\ln\left(\widetilde{V}\right)\right]=\exp\left[-\frac{\mathrm{i}\left(\omega-\widetilde{\omega}\right)}{2\kappa}\ln\left(-\widetilde{V}\right)\right]\exp\left[\frac{\pi\left(\omega-\widetilde{\omega}\right)}{2\kappa}\right]. \tag{4.171}$$

Using (4.171), the asymptotic form of the region IV in-modes near $\mathcal{H}^+_{\mathrm{IV}}$ (4.159) becomes

$$\psi^{\rm in}_{\omega\ell m} = \frac{B^{\rm in*}_{\omega\ell}}{\sqrt{4\pi|\widetilde{\omega}|}}\,\frac{1}{r}\exp\left[\frac{\mathrm{i}\,(\omega+\widetilde{\omega})}{2\kappa}\ln\left(\widetilde{U}\right)\right]\exp\left[-\frac{\mathrm{i}\,(\omega-\widetilde{\omega})}{2\kappa}\ln\left(\widetilde{V}\right)\right] \times \exp\left[-\frac{\pi\,(\omega-\widetilde{\omega})}{2\kappa}\right]Y_{\ell m}(\theta,\varphi)\,\Theta\left(\widetilde{U}\right). \quad (4.172)$$

Multiplying the expression for the lemma in (4.169) by an appropriate factor, we obtain

$$0 = \frac{B^{\rm in*}_{\omega\ell}}{\sqrt{4\pi|\widetilde{\omega}|}}\,\frac{1}{r}\exp\left[-\frac{\mathrm{i}\,(\omega-\widetilde{\omega})}{2\kappa}\ln\left(\widetilde{V}\right)\right]\exp\left[-\frac{\pi\,(\omega-\widetilde{\omega})}{2\kappa}\right]Y_{\ell m}(\theta,\varphi) \times\int_{-\infty}^{\infty}\mathrm{d}\widetilde{U}\,\mathrm{e}^{\mathrm{i}\mathfrak{p}\widetilde{U}}\left\{\exp\left[\frac{\mathrm{i}\,(\omega+\widetilde{\omega})}{2\kappa}\ln\left(\widetilde{U}\right)\right]\Theta\left(\widetilde{U}\right) +\exp\left[-\frac{\pi\,(\omega+\widetilde{\omega})}{2\kappa}\right]\exp\left[-\frac{\mathrm{i}\,(\omega+\widetilde{\omega})}{2\kappa}\ln\left(-\widetilde{U}\right)\right]\Theta\left(-\widetilde{U}\right)\right\}. \quad (4.173)$$

The first term in (4.173) is exactly that of the asymptotic form of the region IV in-modes near $\mathcal{H}^+_{\rm IV}$ (4.172), while the second term in (4.173) is that of the asymptotic form of the out-modes near $\mathcal{H}^-$ (4.156) multiplied by a factor of

$$\exp\left[-\frac{\pi\,(\omega-\widetilde{\omega})}{2\kappa}\right]\exp\left[-\frac{\pi\,(\omega+\widetilde{\omega})}{2\kappa}\right] = \exp\left[-\frac{\pi\omega}{\kappa}\right]. \quad (4.174)$$

Then we can write the linear combination of modes in (4.173) as

$$0 = \int_{-\infty}^{\infty}\mathrm{d}\widetilde{U}\,\mathrm{e}^{\mathrm{i}\mathfrak{p}\widetilde{U}}\left\{\psi^{\rm in}_{\omega\ell m} + \mathrm{e}^{-\frac{\pi\omega}{\kappa}}\phi^{\rm out}_{\omega\ell m}\right\}, \quad \mathfrak{p}>0. \quad (4.175)$$

By the statement (4.3), the quantity in the curly brackets in (4.175) is negative-frequency w.r.t U for all values of ω. Multiplying this quantity by a factor of $\mathfrak{N}^{\rm out-}_{\omega}\mathrm{e}^{\frac{\pi\omega}{2\kappa}}$, where the normalisation constant $\mathfrak{N}^{\rm out-}_{\omega}$ is yet to be determined, we define a set of modes $\chi^{\rm out-}_{\omega\ell m}$

$$\chi^{\rm out-}_{\omega\ell m} = \mathfrak{N}^{\rm out-}_{\omega}\left(\mathrm{e}^{-\frac{\pi\omega}{2\kappa}}\phi^{\rm out}_{\omega\ell m} + \mathrm{e}^{\frac{\pi\omega}{2\kappa}}\psi^{\rm in}_{\omega\ell m}\right) \quad (4.176)$$

which is negative-frequency with respect to the Kruskal coordinate U for all values of ω.

We can normalise the positive- (4.167) and negative-frequency thermalised out-modes (4.176) through a similar process to that used to normalise the thermalised up-modes in §4.4.1; using the fact that both sets of modes in (4.167) and (4.176) are a linear combination of the asymptotic forms of the out-modes $\phi^{\rm out}_{\omega\ell m}$ and of the region IV in-modes $\psi^{\rm in}_{\omega\ell m}$, we can use the norms of the $\phi^{\rm out}_{\omega\ell m}$ and $\psi^{\rm in}_{\omega\ell m}$ in (3.138) and (4.160) respectively to derive the normalisation constants $\mathfrak{N}^{\rm out\pm}_{\omega}$

$$\mathfrak{N}^{\rm out\pm}_{\omega} = \frac{1}{\sqrt{2\left|\sinh\left(\frac{\pi\omega}{\kappa}\right)\right|}}. \quad (4.177)$$

Then, restricting our attention to region I where the $\psi^{\rm in}_{\omega\ell m}$ vanish, a set of normalised out-modes having positive-frequency with respect to the Kruskal coordinate U is given by

$$\chi^{\rm out+}_{\omega\ell m} = \frac{1}{\sqrt{2\left|\sinh\left(\frac{\pi\omega}{\kappa}\right)\right|}}\,\mathrm{e}^{\frac{\pi\omega}{2\kappa}}\phi^{\rm out}_{\omega\ell m}, \quad \text{all } \omega, \quad (4.178)$$

and a set of normalised out-modes having negative-frequency w.r.t U is given by

$$\chi^{\text{out}-}_{\omega\ell m} = \frac{1}{\sqrt{2\left|\sinh\left(\frac{\pi\omega}{\kappa}\right)\right|}}\, \mathrm{e}^{-\frac{\pi\omega}{2\kappa}}\, \phi^{\text{out}}_{\omega\ell m}, \quad \text{all } \omega. \tag{4.179}$$

Construction of the 'future' CCH state

Recall that in defining the 'future' CCH state $|\text{CCH}^+\rangle$, we would like a state that exhibits thermal radiation at both past and future null infinity $\mathscr{I}^{\pm}$; this corresponds to a thermalised flux of particles in both the out-modes (3.139) and the down-modes (3.147).

Then, we may expand the scalar field Φ in terms of an orthonormal basis of thermally populated out- and down-modes, each divided into positive- and negative-frequency sets. We recall that the down-modes are defined to be positive- (4.120) and negative-frequency (4.121) with respect to the Kruskal coordinate V, which is the natural time coordinate to use near $\mathcal{H}^+$. Using (4.120), (4.121), (4.178) and (4.179), we have

$$\begin{aligned}\Phi(x) = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\Bigg\{ &\int_{-\infty}^{\infty} \mathrm{d}\omega \frac{1}{\sqrt{2\left|\sinh\left(\frac{\pi\omega}{\kappa}\right)\right|}}\, \phi^{\text{out}}_{\omega\ell m}\left[\mathrm{e}^{\frac{\pi\omega}{2\kappa}}\, a^{\text{out}}_{\omega\ell m} + \mathrm{e}^{-\frac{\pi\omega}{2\kappa}}\, b^{\text{out}\dagger}_{\omega\ell m}\right] \\ &+ \int_{-\infty}^{\infty} \mathrm{d}\widetilde{\omega} \frac{1}{\sqrt{2\left|\sinh\left(\frac{\pi\widetilde{\omega}}{\kappa}\right)\right|}}\, \phi^{\text{down}}_{\omega\ell m}\left[\mathrm{e}^{\frac{\pi\widetilde{\omega}}{2\kappa}}\, a^{\text{down}}_{\omega\ell m} + \mathrm{e}^{-\frac{\pi\widetilde{\omega}}{2\kappa}}\, b^{\text{down}\dagger}_{\omega\ell m}\right]\Bigg\}. \end{aligned} \tag{4.180}$$

We quantise the field by promoting the expansion coefficients to operators such that the field operator $\hat{\Phi}(x)$ is given by

$$\begin{aligned}\hat{\Phi}(x) = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\Bigg\{ &\int_{-\infty}^{\infty} \mathrm{d}\omega \frac{1}{\sqrt{2\left|\sinh\left(\frac{\pi\omega}{\kappa}\right)\right|}}\, \phi^{\text{out}}_{\omega\ell m}\left[\mathrm{e}^{\frac{\pi\omega}{2\kappa}}\, \hat{a}^{\text{out}}_{\omega\ell m} + \mathrm{e}^{-\frac{\pi\omega}{2\kappa}}\, \hat{b}^{\text{out}\dagger}_{\omega\ell m}\right] \\ &+ \int_{-\infty}^{\infty} \mathrm{d}\widetilde{\omega} \frac{1}{\sqrt{2\left|\sinh\left(\frac{\pi\widetilde{\omega}}{\kappa}\right)\right|}}\, \phi^{\text{down}}_{\omega\ell m}\left[\mathrm{e}^{\frac{\pi\widetilde{\omega}}{2\kappa}}\, \hat{a}^{\text{down}}_{\omega\ell m} + \mathrm{e}^{-\frac{\pi\widetilde{\omega}}{2\kappa}}\, \hat{b}^{\text{down}\dagger}_{\omega\ell m}\right]\Bigg\}, \end{aligned} \tag{4.181}$$

where the operators associated to the out-modes, $\hat{a}^{\text{out}}_{\omega\ell m}$ and $\hat{b}^{\text{out}}_{\omega\ell m}$, are each defined for all ω, and the operators associated to the down-modes, $\hat{a}^{\text{down}}_{\omega\ell m}$ and $\hat{b}^{\text{down}}_{\omega\ell m}$, are each defined for all $\widetilde{\omega}$. In (4.181), all of the positive-frequency modes $\phi^{\text{out/down}+}_{\omega\ell m}$ are of positive-norm and all of the negative-frequency modes $\phi^{\text{out/down}-}_{\omega\ell m}$ are of negative-norm; then, the operators $\hat{a}^{\text{out/down}}_{\omega\ell m}$ and $\hat{b}^{\text{out/down}}_{\omega\ell m}$ obey the following, standard commutation relations

$$\begin{aligned}\left[\hat{a}^{\text{out}}_{\omega\ell m}, \hat{a}^{\text{out}\dagger}_{\omega'\ell' m'}\right] &= \delta\left(\omega-\omega'\right)\delta_{\ell\ell'}\,\delta_{mm'}, \quad \text{all } \omega, \\ \left[\hat{b}^{\text{out}}_{\omega\ell m}, \hat{b}^{\text{out}\dagger}_{\omega'\ell' m'}\right] &= \delta\left(\omega-\omega'\right)\delta_{\ell\ell'}\,\delta_{mm'}, \quad \text{all } \omega, \\ \left[\hat{a}^{\text{down}}_{\omega\ell m}, \hat{a}^{\text{down}\dagger}_{\omega'\ell' m'}\right] &= \delta\left(\omega-\omega'\right)\delta_{\ell\ell'}\,\delta_{mm'}, \quad \text{all } \widetilde{\omega}, \\ \left[\hat{b}^{\text{down}}_{\omega\ell m}, \hat{b}^{\text{down}\dagger}_{\omega'\ell' m'}\right] &= \delta\left(\omega-\omega'\right)\delta_{\ell\ell'}\,\delta_{mm'}, \quad \text{all } \widetilde{\omega}, \end{aligned} \tag{4.182}$$

with any commutators not explicitly given in (4.182) vanishing. The 'future' CCH state is then defined as the state annihilated by the $\hat{a}^{\text{out/down}}_{\omega\ell m}$ and $\hat{b}^{\text{out/down}}_{\omega\ell m}$ operators such that

$$\begin{aligned} \hat{a}^{\text{out}}_{\omega\ell m}\left|\text{CCH}^+\right\rangle &= 0, \quad \text{all } \omega, \\ \hat{b}^{\text{out}}_{\omega\ell m}\left|\text{CCH}^+\right\rangle &= 0, \quad \text{all } \omega, \\ \hat{a}^{\text{down}}_{\omega\ell m}\left|\text{CCH}^+\right\rangle &= 0, \quad \text{all } \widetilde{\omega}, \\ \hat{b}^{\text{down}}_{\omega\ell m}\left|\text{CCH}^+\right\rangle &= 0, \quad \text{all } \widetilde{\omega}. \end{aligned} \tag{4.183}$$

We explore this state further when we study the expectation values of quantum observables in the 'future' CCH state $\left|\text{CCH}^+\right\rangle$ in §5.4.2.

4.5.3 'Hartle-Hawking-like' state

We would like to define a thermal state in RN spacetime that is as close in spirit as possible to the Schwarzschild Hartle-Hawking state $\left|\text{H}_\text{s}\right\rangle$, defined in §2.3.3, and which exhibits both incoming Hawking radiation from past null infinity $\mathscr{I}^-$ as well as outgoing Hawking radiation at future null infinity $\mathscr{I}^+$. In terms of the scalar field modes defined in §3.3, this corresponds to a thermalised flux of particles in both the up- and down-modes.

The outgoing Hawking radiation at future null infinity $\mathscr{I}^+$ in $\left|\text{H}\right\rangle$ is emanating from the past horizon $\mathcal{H}^-$; the Kruskal coordinate U is the affine parameter along the null generators of this surface. We have already defined thermalised up-modes that are positive- and negative-frequency w.r.t U in (4.92a) and (4.92b) respectively.

The incoming Hawking radiation from past null infinity $\mathscr{I}^-$ in $\left|\text{H}\right\rangle$ is incident upon the future horizon $\mathcal{H}^+$; the Kruskal coordinate V is the affine parameter along the null generators of this surface. We have already defined thermalised down-modes that are positive- and negative-frequency w.r.t V in (4.120) and (4.121) respectively.

Then, we may use (4.92a), (4.92b), (4.120) and (4.121) to expand the field Φ as

$$\Phi(x) = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\int_{-\infty}^{\infty} \mathrm{d}\widetilde{\omega}\left\{\tilde{a}^{\text{up}}_{\omega\ell m}\,\chi^{\text{up}+}_{\omega\ell m} + \tilde{b}^{\text{up}\dagger}_{\omega\ell m}\,\chi^{\text{up}-}_{\omega\ell m} + \tilde{a}^{\text{down}}_{\omega\ell m}\,\chi^{\text{down}+}_{\omega\ell m} + \tilde{b}^{\text{up}\dagger}_{\omega\ell m}\,\chi^{\text{down}-}_{\omega\ell m}\right\}. \tag{4.184}$$

In region I, equation (4.184) reduces to

$$\begin{aligned}\Phi(x) = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\int_{-\infty}^{\infty} \mathrm{d}\widetilde{\omega}\,\frac{1}{\sqrt{2\left|\sinh\left(\frac{\pi\widetilde{\omega}}{\kappa}\right)\right|}}&\left\{\left[\mathrm{e}^{\frac{\pi\widetilde{\omega}}{2\kappa}}\tilde{a}^{\text{up}}_{\omega\ell m} + \mathrm{e}^{-\frac{\pi\widetilde{\omega}}{2\kappa}}\tilde{b}^{\text{up}\dagger}_{\omega\ell m}\right]\phi^{\text{up}}_{\omega\ell m}\right. \\ &\left.+\left[\mathrm{e}^{\frac{\pi\widetilde{\omega}}{2\kappa}}\tilde{a}^{\text{down}}_{\omega\ell m} + \mathrm{e}^{-\frac{\pi\widetilde{\omega}}{2\kappa}}\tilde{b}^{\text{down}\dagger}_{\omega\ell m}\right]\phi^{\text{down}}_{\omega\ell m}\right\}, \end{aligned} \tag{4.185}$$

However, the up- and down-modes do not form an orthonormal basis of modes, which we require to quantise the field. The in-modes are orthogonal to the up-modes; we would then like to re-express the down-modes in (4.184) in terms of in- and up-modes. We note that since the thermal factor of the down-modes depends on $\widetilde{\omega}$, the in- and up-modes which we use to re-express the down-modes should also have thermal factors depending on $\widetilde{\omega}$.

While we have already defined thermalised in-modes that are positive- and negative-frequency w.r.t V in (4.149) and (4.150) respectively, these modes possess a thermal factor depending on ω. Thus, we need to define a new set of thermalised in-modes that are positive- and negative-frequency w.r.t V but whose thermal factor instead depends on $\widetilde{\omega}$.

Initially, their definition follows through in a similar way to the in-modes $\chi^{\text{in}}_{\omega\ell m}$ used to define the 'past' CCH state in §4.5.1. In particular, the asymptotic form of the in-modes near $\mathcal{H}^+$ is that given in (4.127), and we similarly make the transformations $U \to -U$, $V \to -V$ to define a set of region IV out-modes, whose asymptotic form near $\mathcal{H}^-_{\text{IV}}$ is that given in (4.130). Making the same choices $\mathfrak{X}^{\text{in}+} = \widetilde{V}$ and $\mathfrak{q}^{\text{in}+} = \frac{(\omega+\widetilde{\omega})}{2\kappa}$ to define positive-frequency modes as in (4.128), the lemma (4.6) takes the form given in (4.129).

However, when defining the 'past' CCH state $|\text{CCH}^-\rangle$, we chose our surface of integration to be a hypersurface of constant $U = -\epsilon < 0$, where ϵ is a small, positive constant such that this surface lies inside region I, close to $\mathcal{H}^+$, and inside region III, close to $\mathcal{H}^-_{\text{IV}}$. Now, we will instead choose our surface of integration of the modes in the lemma (4.129) to be a hypersurface of constant $U = \epsilon > 0$, where ϵ is a small, positive constant such that this surface lies inside region II, close to $\mathcal{H}^+$, and inside region IV, close to $\mathcal{H}^-_{\text{IV}}$.

The asymptotic form of the in-modes near $\mathcal{H}^+$ (4.127) contain a factor of $\exp[\ln(-\widetilde{U})]$, whereas the asymptotic form of the region IV out-modes near $\mathcal{H}^-_{\text{IV}}$ (4.130) contain a factor of $\exp[\ln(\widetilde{U})]$. Initially attempting to define a set of positive-frequency modes w.r.t V, which are analytic in the lower-half of the plane and therefore require us to use a branch of the logarithm that is also analytic in the lower-half plane, we choose to make the same branch cut along the positive imaginary axis such that $\ln(-1) = -\mathrm{i}\pi$, as in (4.132). Then, using the fact that we are integrating over a hypersurface of constant $U = \epsilon > 0$

$$\exp\left[\frac{\mathrm{i}\,(\omega-\widetilde{\omega})}{2\kappa}\ln\Big(-\widetilde{U}\Big)\right] = \exp\left[\frac{\mathrm{i}\,(\omega-\widetilde{\omega})}{2\kappa}\ln\Big(\widetilde{U}\Big)\right]\exp\left[\frac{\pi\,(\omega-\widetilde{\omega})}{2\kappa}\right]. \tag{4.186}$$

Using (4.186), the asymptotic form of the in-modes near $\mathcal{H}^+$ (4.127) becomes

$$\phi^{\text{in}}_{\omega\ell m} = \frac{B^{\text{in}}_{\omega\ell}}{\sqrt{4\pi|\omega|}}\,\frac{1}{r}\exp\left[\frac{\mathrm{i}\,(\omega-\widetilde{\omega})}{2\kappa}\ln\Big(\widetilde{U}\Big)\right]\exp\left[-\frac{\mathrm{i}\,(\omega+\widetilde{\omega})}{2\kappa}\ln\Big(\widetilde{V}\Big)\right] \\ \times\exp\left[\frac{\pi\,(\omega-\widetilde{\omega})}{2\kappa}\right]Y_{\ell m}(\theta,\varphi)\,\Theta\Big(\widetilde{V}\Big). \tag{4.187}$$

Multiplying the expression for the lemma in (4.129) by an appropriate factor, we obtain

$$0 = \frac{B^{\text{in}}_{\omega\ell}}{\sqrt{4\pi|\omega|}}\,\frac{1}{r}\exp\left[\frac{\mathrm{i}\,(\omega-\widetilde{\omega})}{2\kappa}\ln\Big(\widetilde{U}\Big)\right]\exp\left[\frac{\pi\,(\omega-\widetilde{\omega})}{2\kappa}\right]Y_{\ell m}(\theta,\varphi) \\ \times\int_{-\infty}^{\infty}\mathrm{d}\widetilde{V}\,\mathrm{e}^{-\mathrm{i}\mathfrak{p}\widetilde{V}}\left\{\exp\left[-\frac{\mathrm{i}\,(\omega+\widetilde{\omega})}{2\kappa}\ln\Big(\widetilde{V}\Big)\right]\Theta\Big(\widetilde{V}\Big)\right. \\ \left.+\exp\left[-\frac{\pi\,(\omega+\widetilde{\omega})}{2\kappa}\right]\exp\left[-\frac{\mathrm{i}\,(\omega+\widetilde{\omega})}{2\kappa}\ln\Big(-\widetilde{V}\Big)\right]\Theta\Big(-\widetilde{V}\Big)\right\}. \tag{4.188}$$

The first term in (4.188) is exactly that of the asymptotic form of the in-modes near $\mathcal{H}^+$ (4.187), while the second term in (4.188) is that of the asymptotic form of the region IV out-modes near $\mathcal{H}^-_{\text{IV}}$ (4.130) multiplied by a factor of

$$\exp\left[\frac{\pi\left(\omega-\widetilde{\omega}\right)}{2\kappa}\right]\exp\left[-\frac{\pi\left(\omega+\widetilde{\omega}\right)}{2\kappa}\right]=\exp\left[-\frac{\pi\widetilde{\omega}}{\kappa}\right]. \tag{4.189}$$

Then we can write the linear combination of modes in (4.188) as

$$0=\int_{-\infty}^{\infty}\mathrm{d}\widetilde{V}\,\mathrm{e}^{-\mathrm{i}\mathfrak{p}\widetilde{V}}\left\{\phi^{\mathrm{in}}_{\omega\ell m}+\mathrm{e}^{-\frac{\pi\widetilde{\omega}}{\kappa}}\psi^{\mathrm{out}}_{\omega\ell m}\right\},\quad \mathfrak{p}>0. \tag{4.190}$$

By the statement (4.2), the quantity in the curly brackets in (4.190) is positive-frequency w.r.t V for all values of $\widetilde{\omega}$. Multiplying this quantity by a factor of $\widetilde{\mathfrak{N}}^{\mathrm{in}+}_{\omega}\mathrm{e}^{\frac{\pi\omega}{2\kappa}}$, where the normalisation constant $\widetilde{\mathfrak{N}}^{\mathrm{in}+}_{\omega}$ is yet to be determined, we define a set of modes $\widetilde{\chi}^{\mathrm{in}+}_{\omega\ell m}$

$$\widetilde{\chi}^{\mathrm{in}+}_{\omega\ell m}=\widetilde{\mathfrak{N}}^{\mathrm{in}+}_{\omega}\left(\mathrm{e}^{\frac{\pi\widetilde{\omega}}{2\kappa}}\phi^{\mathrm{in}}_{\omega\ell m}+\mathrm{e}^{-\frac{\pi\widetilde{\omega}}{2\kappa}}\psi^{\mathrm{out}}_{\omega\ell m}\right) \tag{4.191}$$

which is positive-frequency w.r.t V for all values of $\widetilde{\omega}$ and where the notation $\widetilde{\chi}^{\mathrm{in}+}_{\omega\ell m}$ serves to distinguish these modes from the positive-frequency modes $\chi^{\mathrm{in}+}_{\omega\ell m}$ (4.138) in §4.5.1.

In order to define negative-frequency modes, we make the same choices $\mathfrak{X}^{\mathrm{in}-}=\widetilde{V}$ and $\mathfrak{q}^{\mathrm{in}-}=-\frac{(\omega+\widetilde{\omega})}{2\kappa}$ as in (4.139) and thus the complex conjugate (4.7) of the lemma takes the form given in (4.140). Since we are attempting to define a set of negative-frequency modes w.r.t V, which are analytic in the upper-half of the plane and therefore require us to use a branch of the logarithm that is also analytic in the upper-half plane, we choose to make the same branch cut along the negative imaginary axis such that $\ln(-1)=\mathrm{i}\pi$, as in (4.141). Using the fact that we are integrating over a hypersurface of constant $U=\epsilon>0$

$$\exp\left[\frac{\mathrm{i}\left(\omega-\widetilde{\omega}\right)}{2\kappa}\ln\left(-\widetilde{U}\right)\right]=\exp\left[\frac{\mathrm{i}\left(\omega-\widetilde{\omega}\right)}{2\kappa}\ln\left(\widetilde{U}\right)\right]\exp\left[-\frac{\pi\left(\omega-\widetilde{\omega}\right)}{2\kappa}\right]. \tag{4.192}$$

Using (4.192), the asymptotic form of the in-modes near $\mathcal{H}^+$ (4.127) becomes

$$\begin{aligned}\phi^{\mathrm{in}}_{\omega\ell m}=\frac{B^{\mathrm{in}}_{\omega\ell}}{\sqrt{4\pi|\widetilde{\omega}|}}\frac{1}{r}\exp\left[\frac{\mathrm{i}\left(\omega-\widetilde{\omega}\right)}{2\kappa}\ln\left(\widetilde{U}\right)\right]\exp\left[-\frac{\mathrm{i}\left(\omega+\widetilde{\omega}\right)}{2\kappa}\ln\left(\widetilde{V}\right)\right]\\ \times\exp\left[-\frac{\pi\left(\omega-\widetilde{\omega}\right)}{2\kappa}\right]Y_{\ell m}(\theta,\varphi)\,\Theta\left(\widetilde{V}\right).\end{aligned} \tag{4.193}$$

Multiplying the expression for the lemma in (4.140) by an appropriate factor, we obtain

$$\begin{aligned}0=\frac{B^{\mathrm{in}}_{\omega\ell}}{\sqrt{4\pi|\widetilde{\omega}|}}\frac{1}{r}\exp\left[\frac{\mathrm{i}\left(\omega-\widetilde{\omega}\right)}{2\kappa}\ln\left(\widetilde{U}\right)\right]\exp\left[-\frac{\pi\left(\omega-\widetilde{\omega}\right)}{2\kappa}\right]Y_{\ell m}(\theta,\varphi)\\ \times\int_{-\infty}^{\infty}\mathrm{d}\widetilde{V}\,\mathrm{e}^{\mathrm{i}\mathfrak{p}\widetilde{V}}\left\{\exp\left[\frac{\mathrm{i}\left(\omega+\widetilde{\omega}\right)}{2\kappa}\ln\left(\widetilde{V}\right)\right]\Theta\left(\widetilde{V}\right)\right.\\ \left.+\exp\left[\frac{\pi\left(\omega+\widetilde{\omega}\right)}{2\kappa}\right]\exp\left[-\frac{\mathrm{i}\left(\omega+\widetilde{\omega}\right)}{2\kappa}\ln\left(-\widetilde{V}\right)\right]\Theta\left(-\widetilde{V}\right)\right\}.\end{aligned} \tag{4.194}$$

The first term in (4.194) is exactly that of the asymptotic form of the in-modes near $\mathcal{H}^+$ (4.193), while the second term in (4.194) is that of the asymptotic form of the region IV out-modes near $\mathcal{H}^-_{\mathrm{IV}}$ (4.130) multiplied by a factor of

$$\exp\left[-\frac{\pi\,(\omega-\widetilde{\omega})}{2\kappa}\right]\exp\left[\frac{\pi\,(\omega+\widetilde{\omega})}{2\kappa}\right]=\exp\left[\frac{\pi\widetilde{\omega}}{\kappa}\right]. \tag{4.195}$$

Then we can write the linear combination of modes in (4.194) as

$$0=\int_{-\infty}^{\infty}\mathrm{d}\widetilde{V}\,\mathrm{e}^{\mathrm{i}\mathfrak{p}\widetilde{V}}\left\{\phi^{\text{in}}_{\omega\ell m}+\mathrm{e}^{\frac{\pi\widetilde{\omega}}{\kappa}}\,\psi^{\text{out}}_{\omega\ell m}\right\},\quad \mathfrak{p}>0. \tag{4.196}$$

By the statement (4.3), the quantity in the curly brackets in (4.196) is negative-frequency w.r.t V for all values of $\widetilde{\omega}$. Multiplying this quantity by a factor of $\widetilde{\mathfrak{N}}^{\text{in}-}_{\omega}\mathrm{e}^{-\frac{\pi\widetilde{\omega}}{2\kappa}}$, where the normalisation constant $\widetilde{\mathfrak{N}}^{\text{in}-}_{\omega}$ is yet to be determined, we define a set of modes $\widetilde{\chi}^{\text{in}-}_{\omega\ell m}$

$$\widetilde{\chi}^{\text{in}-}_{\omega\ell m}=\widetilde{\mathfrak{N}}^{\text{in}-}_{\omega}\left(\mathrm{e}^{-\frac{\pi\widetilde{\omega}}{2\kappa}}\,\phi^{\text{in}}_{\omega\ell m}+\mathrm{e}^{\frac{\pi\widetilde{\omega}}{2\kappa}}\,\psi^{\text{out}}_{\omega\ell m}\right) \tag{4.197}$$

which is negative-frequency w.r.t V for all values of $\widetilde{\omega}$ and where the notation $\widetilde{\chi}^{\text{in}-}_{\omega\ell m}$ serves to distinguish these modes from the negative-frequency modes $\chi^{\text{in}-}_{\omega\ell m}$ (4.147) in §4.5.1.

The derivation of the normalisation constants $\widetilde{\mathfrak{N}}^{\text{in}\pm}_{\omega}$ is somewhat subtle in that, while $\widetilde{\mathfrak{N}}^{\text{in}\pm}_{\omega}$ will contain a factor of $\widetilde{\omega}$, the norm of the modes $\phi^{\text{in}}_{\omega\ell m}$ and $\psi^{\text{out}}_{\omega\ell m}$ depend on ω from (3.104) and (4.131) respectively. We go through this calculation in detail now.

We need to use the fact that the in-modes and the region IV out-modes are orthogonal since the in-modes are defined in region I while vanishing in region IV, and the region IV out-modes are defined in region I while vanishing in region IV. Therefore, the inner product $\langle\phi^{\text{in}}_{\omega\ell m},\psi^{\text{out}}_{\omega'\ell'm'}\rangle$ vanishes. Requiring orthonormality of the $\widetilde{\chi}^{\text{in}\pm}_{\omega\ell m}$ modes, we have

$$\begin{aligned}
&\langle\widetilde{\chi}^{\text{in}\pm}_{\omega\ell m},\widetilde{\chi}^{\text{in}\pm}_{\omega'\ell'm'}\rangle\\
&\quad=\widetilde{\mathfrak{N}}^{\text{in}*\pm}_{\omega}\,\widetilde{\mathfrak{N}}^{\text{in}\pm}_{\omega'}\langle(\mathrm{e}^{\pm\frac{\pi\widetilde{\omega}}{2\kappa}}\,\phi^{\text{in}}_{\omega\ell m}+\mathrm{e}^{\mp\frac{\pi\widetilde{\omega}}{2\kappa}}\,\psi^{\text{out}}_{\omega\ell m}),(\mathrm{e}^{\pm\frac{\pi\widetilde{\omega}}{2\kappa}}\,\phi^{\text{in}}_{\omega'\ell'm'}+\mathrm{e}^{\mp\frac{\pi\widetilde{\omega}}{2\kappa}}\,\psi^{\text{out}}_{\omega'\ell'm'})\rangle\\
&\quad=\widetilde{\mathfrak{N}}^{\text{in}*\pm}_{\omega}\,\widetilde{\mathfrak{N}}^{\text{in}\pm}_{\omega'}\left(\mathrm{e}^{\pm\frac{\pi\widetilde{\omega}}{\kappa}}\langle\phi^{\text{in}}_{\omega\ell m},\phi^{\text{in}}_{\omega'\ell'm'}\rangle+2\,\langle\phi^{\text{in}}_{\omega\ell m},\psi^{\text{out}}_{\omega'\ell'm'}\rangle+\mathrm{e}^{\mp\frac{\pi\widetilde{\omega}}{\kappa}}\langle\psi^{\text{out}}_{\omega\ell m},\psi^{\text{out}}_{\omega'\ell'm'}\rangle\right)\\
&\quad=\left|\widetilde{\mathfrak{N}}^{\text{in}\pm}_{\omega}\right|^2\left(\mathrm{e}^{\pm\frac{\pi\widetilde{\omega}}{\kappa}}-\mathrm{e}^{\mp\frac{\pi\widetilde{\omega}}{\kappa}}\right)\operatorname{sgn}(\omega)\,\delta\big(\omega-\omega'\big)\,\delta_{\ell\ell'}\,\delta_{mm'},
\end{aligned} \tag{4.198}$$

where we have used the expressions for the norms in (3.104) and (4.131) to go from the second equality to the third; equation (4.198) reduces to

$$\langle\widetilde{\chi}^{\text{in}\pm}_{\omega\ell m},\widetilde{\chi}^{\text{in}\pm}_{\omega'\ell'm'}\rangle=2\left|\widetilde{\mathfrak{N}}^{\text{in}\pm}_{\omega}\right|^2\sinh\left(\pm\frac{\pi\widetilde{\omega}}{\kappa}\right)\operatorname{sgn}(\omega)\,\delta\big(\omega-\omega'\big)\,\delta_{\ell\ell'}\,\delta_{mm'}. \tag{4.199}$$

Then, from (4.199), we have for the inner product of two generic $\widetilde{\chi}^{\text{in}+}_{\omega\ell m}$ modes

$$\langle\widetilde{\chi}^{\text{in}+}_{\omega\ell m},\widetilde{\chi}^{\text{in}+}_{\omega'\ell'm'}\rangle=\begin{cases}2\left|\widetilde{\mathfrak{N}}^{\text{in}+}_{\omega}\right|^2\sinh\left(\dfrac{\pi\widetilde{\omega}}{\kappa}\right)\delta\big(\omega-\omega'\big)\,\delta_{\ell\ell'}\,\delta_{mm'}>0, & \omega>0,\\[2ex] -2\left|\widetilde{\mathfrak{N}}^{\text{in}+}_{\omega}\right|^2\sinh\left(\dfrac{\pi\widetilde{\omega}}{\kappa}\right)\delta\big(\omega-\omega'\big)\,\delta_{\ell\ell'}\,\delta_{mm'}>0, & \omega<0,\end{cases} \tag{4.200}$$

demonstrating that positive-frequency modes $\widetilde{\chi}^{\text{in}+}_{\omega\ell m}$ have positive norm if $\operatorname{sgn}(\omega\widetilde{\omega})=1$ and negative norm if $\operatorname{sgn}(\omega\widetilde{\omega})=-1$. Similarly, from (4.199), we have for $\langle\widetilde{\chi}^{\text{in}-}_{\omega\ell m},\widetilde{\chi}^{\text{in}-}_{\omega'\ell'm'}\rangle$

$$\langle \widetilde{\chi}^{\text{in}-}_{\omega\ell m}, \widetilde{\chi}^{\text{in}-}_{\omega'\ell'm'} \rangle = \begin{cases} 2\left|\widetilde{\mathfrak{N}}^{\text{in}-}_{\omega}\right|^2 \sinh\left(\dfrac{\pi\widetilde{\omega}}{\kappa}\right) \delta\left(\omega-\omega'\right) \delta_{\ell\ell'}\,\delta_{mm'} < 0, & \omega > 0, \\ -2\left|\widetilde{\mathfrak{N}}^{\text{in}-}_{\omega}\right|^2 \sinh\left(\dfrac{\pi\widetilde{\omega}}{\kappa}\right) \delta\left(\omega-\omega'\right) \delta_{\ell\ell'}\,\delta_{mm'} < 0, & \omega < 0, \end{cases} \tag{4.201}$$

demonstrating that negative-frequency modes $\widetilde{\chi}^{\text{in}-}_{\omega\ell m}$ have negative norm if $\text{sgn}(\omega\widetilde{\omega}) = 1$ and positive norm if $\text{sgn}(\omega\widetilde{\omega}) = -1$. The normalisation constants $\widetilde{\mathfrak{N}}^{\text{in}\pm}_{\omega}$ are given by

$$\widetilde{\mathfrak{N}}^{\text{in}\pm}_{\omega} = \frac{1}{\sqrt{2\left|\sinh\left(\frac{\pi\widetilde{\omega}}{\kappa}\right)\right|}}. \tag{4.202}$$

Therefore, a set of normalised modes having positive-frequency w.r.t V is given by

$$\widetilde{\chi}^{\text{in}+}_{\omega\ell m} = \frac{1}{\sqrt{2\left|\sinh\left(\frac{\pi\widetilde{\omega}}{\kappa}\right)\right|}} \left(\mathrm{e}^{\frac{\pi\widetilde{\omega}}{2\kappa}}\,\phi^{\text{in}}_{\omega\ell m} + \mathrm{e}^{-\frac{\pi\widetilde{\omega}}{2\kappa}}\,\psi^{\text{out}}_{\omega\ell m}\right), \quad \text{all } \widetilde{\omega}, \tag{4.203}$$

and a set of normalised modes having negative-frequency w.r.t V is given by

$$\widetilde{\chi}^{\text{in}-}_{\omega\ell m} = \frac{1}{\sqrt{2\left|\sinh\left(\frac{\pi\widetilde{\omega}}{\kappa}\right)\right|}} \left(\mathrm{e}^{-\frac{\pi\widetilde{\omega}}{2\kappa}}\,\phi^{\text{in}}_{\omega\ell m} + \mathrm{e}^{\frac{\pi\widetilde{\omega}}{2\kappa}}\,\psi^{\text{out}}_{\omega\ell m}\right), \quad \text{all } \widetilde{\omega}. \tag{4.204}$$

Having defined thermalised in-modes $\widetilde{\chi}^{\text{in}\pm}_{\omega\ell m}$ whose thermal factor depends on ω, we can return to rewriting the thermalised down-modes $\chi^{\text{down}\pm}_{\omega\ell m}$ in terms of the $\widetilde{\chi}^{\text{in}\pm}_{\omega\ell m}$ and the thermalised up-modes $\chi^{\text{up}\pm}_{\omega\ell m}$. In order to do so, we need to relate both their region I and IV parts. From the expression for $\phi^{\text{down}}_{\omega\ell m}$ in terms of $\phi^{\text{up}}_{\omega\ell m}$ and $\phi^{\text{in}}_{\omega\ell m}$ (3.148b), we have

$$\psi^{\text{up}}_{\omega\ell m} = A^{\text{up}*}_{\omega\ell}\,\psi^{\text{down}}_{\omega\ell m} + \left|\frac{\omega}{\widetilde{\omega}}\right|^{\frac{1}{2}} B^{\text{up}*}_{\omega\ell}\,\psi^{\text{out}}_{\omega\ell m}. \tag{4.205}$$

Thus, we have

$$\chi^{\text{down}\pm}_{\omega\ell m} = A^{\text{up}*}_{\omega\ell m}\,\chi^{\text{up}\pm}_{\omega\ell m} + \left|\frac{\omega}{\widetilde{\omega}}\right|^{\frac{1}{2}} B^{\text{up}*}_{\omega\ell m}\,\widetilde{\chi}^{\text{in}\pm}_{\omega\ell m}. \tag{4.206}$$

Using (4.92a), (4.92b), (4.203) and (4.204), we can expand the field Φ in an orthonormal basis of thermalised in- and up-modes as

$$\Phi(x) = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\left\{\int_{-\infty}^{\infty}\mathrm{d}\widetilde{\omega}\left(a^{\text{up}}_{\omega\ell m}\,\chi^{\text{up}+}_{\omega\ell m} + b^{\text{up}\dagger}_{\omega\ell m}\,\chi^{\text{up}-}_{\omega\ell m}\right)\right.$$
$$\left. + \int_{-\infty}^{\infty}\mathrm{d}\omega\left(a^{\text{in}}_{\omega\ell m}\,\widetilde{\chi}^{\text{in}+}_{\omega\ell m} + b^{\text{in}\dagger}_{\omega\ell m}\,\widetilde{\chi}^{\text{in}-}_{\omega\ell m}\right)\right\}, \tag{4.207}$$

where the mode coefficients in (4.207) are related to those in (4.184) by the expressions

$$\begin{aligned}
a^{\text{up}}_{\omega\ell m} &= \tilde{a}^{\text{up}}_{\omega\ell m} + A^{\text{up}*}_{\omega\ell}\,\tilde{a}^{\text{down}}_{\omega\ell m}\,, \\
b^{\text{up}\dagger}_{\omega\ell m} &= \tilde{b}^{\text{up}\dagger}_{\omega\ell m} + A^{\text{up}*}_{\omega\ell}\,\tilde{b}^{\text{down}\dagger}_{\omega\ell m}\,, \\
a^{\text{in}}_{\omega\ell m} &= \left|\frac{\omega}{\widetilde{\omega}}\right|^{\frac{1}{2}} B^{\text{up}*}_{\omega\ell}\,\tilde{a}^{\text{down}}_{\omega\ell m}\,, \\
b^{\text{in}\dagger}_{\omega\ell m} &= \left|\frac{\omega}{\widetilde{\omega}}\right|^{\frac{1}{2}} B^{\text{up}*}_{\omega\ell}\,\tilde{b}^{\text{down}\dagger}_{\omega\ell m}\,.
\end{aligned} \tag{4.208}$$

Restricting out attention to region I, (4.207) reduces to

$$\begin{aligned}
\Phi(x) = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\int_{-\infty}^{\infty} d\widetilde{\omega}\,\frac{1}{\sqrt{2\left|\sinh\left(\frac{\pi\widetilde{\omega}}{\kappa}\right)\right|}} &\left\{\left[e^{\frac{\pi\widetilde{\omega}}{2\kappa}}\,a^{\text{up}}_{\omega\ell m} + e^{-\frac{\pi\widetilde{\omega}}{2\kappa}}\,b^{\text{up}\dagger}_{\omega\ell m}\right]\phi^{\text{up}}_{\omega\ell m}\right. \\
&\left. + \left[e^{\frac{\pi\widetilde{\omega}}{2\kappa}}\,a^{\text{in}}_{\omega\ell m} + e^{-\frac{\pi\widetilde{\omega}}{2\kappa}}\,b^{\text{in}\dagger}_{\omega\ell m}\right]\phi^{\text{in}}_{\omega\ell m}\right\},
\end{aligned} \tag{4.209}$$

from which we can see that all modes will have thermal factors that depend on $\widetilde{\omega}$. We quantise the field by promoting the mode expansion coefficients in (4.209) to operators such that the field operator $\hat{\Phi}(x)$ is given by

$$\begin{aligned}
\hat{\Phi}(x) = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\int_{-\infty}^{\infty} d\widetilde{\omega}\,\frac{1}{\sqrt{2\left|\sinh\left(\frac{\pi\widetilde{\omega}}{\kappa}\right)\right|}} &\left\{\left[e^{\frac{\pi\widetilde{\omega}}{2\kappa}}\,\hat{a}^{\text{up}}_{\omega\ell m} + e^{-\frac{\pi\widetilde{\omega}}{2\kappa}}\,\hat{b}^{\text{up}\dagger}_{\omega\ell m}\right]\phi^{\text{up}}_{\omega\ell m}\right. \\
&\left. + \left[e^{\frac{\pi\widetilde{\omega}}{2\kappa}}\,\hat{a}^{\text{in}}_{\omega\ell m} + e^{-\frac{\pi\widetilde{\omega}}{2\kappa}}\,\hat{b}^{\text{in}\dagger}_{\omega\ell m}\right]\phi^{\text{in}}_{\omega\ell m}\right\}.
\end{aligned} \tag{4.210}$$

In (4.207), the positive-frequency modes $\chi^{\text{up}+}_{\omega\ell m}$ are entirely of positive-norm and the negative-frequency modes $\chi^{\text{up}-}_{\omega\ell m}$ are entirely of negative-norm. Therefore, the operators $\hat{a}^{\text{up}}_{\omega\ell m}$ and $\hat{b}^{\text{up}}_{\omega\ell m}$ follow standard commutation relations.

However, the positive-frequency modes $\widetilde{\chi}^{\text{in}+}_{\omega\ell m}$ are only of positive-norm when $\text{sgn}(\omega\widetilde{\omega}) = 1$ and are of negative-norm when $\text{sgn}(\omega\widetilde{\omega}) = -1$. Similarly, the negative-frequency modes $\widetilde{\chi}^{\text{in}-}_{\omega\ell m}$ are only of negative-norm when $\text{sgn}(\omega\widetilde{\omega}) = 1$ and are of positive-norm when $\text{sgn}(\omega\widetilde{\omega}) = -1$. Then, recalling our discussion in §4.2.4, we can multiply the commutation relations of the operators $\hat{a}^{\text{in}}_{\omega\ell m}$ and $\hat{b}^{\text{in}}_{\omega\ell m}$ by the eta-function $\eta_{\omega\widetilde{\omega}}$ (4.20). Therefore, the operators $\hat{a}^{\text{in/up}}_{\omega\ell m}$ and $\hat{b}^{\text{in/up}}_{\omega\ell m}$ obey the following non-standard commutation relations:

$$\begin{aligned}
\left[\hat{a}^{\text{up}}_{\omega\ell m}, \hat{a}^{\text{up}\dagger}_{\omega'\ell' m'}\right] &= \delta\big(\omega-\omega'\big)\,\delta_{\ell\ell'}\,\delta_{mm'}\,, && \text{all } \widetilde{\omega}, \\
\left[\hat{b}^{\text{up}}_{\omega\ell m}, \hat{b}^{\text{up}\dagger}_{\omega'\ell' m'}\right] &= \delta\big(\omega-\omega'\big)\,\delta_{\ell\ell'}\,\delta_{mm'}\,, && \text{all } \widetilde{\omega}, \\
\left[\hat{a}^{\text{in}}_{\omega\ell m}, \hat{a}^{\text{in}\dagger}_{\omega'\ell' m'}\right] &= \eta_{\omega\widetilde{\omega}}\,\delta\big(\omega-\omega'\big)\,\delta_{\ell\ell'}\,\delta_{mm'}\,, && \text{all } \widetilde{\omega}, \\
\left[\hat{b}^{\text{in}}_{\omega\ell m}, \hat{b}^{\text{in}\dagger}_{\omega'\ell' m'}\right] &= \eta_{\omega\widetilde{\omega}}\,\delta\big(\omega-\omega'\big)\,\delta_{\ell\ell'}\,\delta_{mm'}\,, && \text{all } \widetilde{\omega},
\end{aligned} \tag{4.211}$$

with any commutators not explicitly given in (4.211) vanishing. The 'Hartle-Hawking-like' state $|\text{H}\rangle$ is then defined as the state annihilated by the $\hat{a}^{\text{in/up}}_{\omega\ell m}$ and $\hat{b}^{\text{in/up}}_{\omega\ell m}$ operators:

$$\begin{aligned}
\hat{a}^{\mathrm{up}}_{\omega\ell m}\left|\mathrm{H}\right\rangle &= 0, \qquad \text{all } \widetilde{\omega}, \\
\hat{b}^{\mathrm{up}}_{\omega\ell m}\left|\mathrm{H}\right\rangle &= 0 \qquad \text{all } \widetilde{\omega}, \\
\hat{a}^{\mathrm{in}}_{\omega\ell m}\left|\mathrm{H}\right\rangle &= 0, \qquad \text{all } \widetilde{\omega}, \\
\hat{b}^{\mathrm{in}}_{\omega\ell m}\left|\mathrm{H}\right\rangle &= 0, \qquad \text{all } \widetilde{\omega}.
\end{aligned} \tag{4.212}$$

As well as the 'Hartle-Hawking-like' state $|\mathrm{H}\rangle$, the 'past' Unruh state $|\mathrm{U}^-\rangle$ and the 'past' CCH state $|\mathrm{CCH}^-\rangle$ are also defined using an orthonormal basis of in- and up-modes. While the particle distribution in the up-modes $\phi^{\mathrm{up}}_{\omega\ell m}$ in $|\mathrm{H}\rangle$ will be similar to that of $|\mathrm{U}^-\rangle$ and $|\mathrm{CCH}^-\rangle$, the particle distribution in the in-modes $\phi^{\mathrm{in}}_{\omega\ell m}$ in $|\mathrm{H}\rangle$ will be different to both $|\mathrm{U}^-\rangle$ and $|\mathrm{CCH}^-\rangle$.

4.5.4 Frolov-Thorne state

In [49], Frolov and Thorne defined a "Hartle-Hawking"-like state for a neutral scalar field in Kerr spacetime. In order to do so , they expanded the field in an orthonormal basis of up- and in-modes with thermal factors depending on $\widetilde{\omega}$, similar to the expansion (4.207) we used to define the 'Hartle-Hawking-like' state $|\mathrm{H}\rangle$ in RN spacetime in §4.5.3. However, Frolov and Thorne labelled the expansion coefficients multiplying each mode according to the norm of that mode. This is in contrast to the expansion (4.207), where the mode expansion coefficients were labelled according to the frequency of the mode they were multiplying; in (4.207), for example, positive-frequency modes $\chi^{\mathrm{in/up}+}_{\omega\ell m}$ are multiplied by the mode expansion coefficients $a^{\mathrm{in/up}}_{\omega\ell m}$ and negative-frequency modes $\chi^{\mathrm{in/up}-}_{\omega\ell m}$ are multiplied by the mode expansion coefficients $b^{\mathrm{in/up}\dagger}_{\omega\ell m}$.

We will expand the field in an analogous way to that used to define the "Hartle-Hawking"-like state in [49]; we will refer to the quantum state defined in this way as the Frolov-Thorne state $|\mathrm{FT}\rangle$. Using (4.92a), (4.92b), (4.203) and (4.204), we can expand the field Φ in an orthonormal basis of thermalised up- and in-modes as

$$\begin{aligned}
\Phi(x) = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\Bigg\{ &\int_{-\infty}^{\infty} \mathrm{d}\widetilde{\omega}\left(a^{\mathrm{up}}_{\omega\ell m}\,\chi^{\mathrm{up}+}_{\omega\ell m} + b^{\mathrm{up}\dagger}_{\omega\ell m}\,\chi^{\mathrm{up}-}_{\omega\ell m}\right) \\
&+ \int_{\max\left\{\frac{qQ}{r_+},0\right\}}^{\infty} \mathrm{d}\omega\left(a^{\mathrm{in}}_{\omega\ell m}\,\widetilde{\chi}^{\mathrm{in}+}_{\omega\ell m} + b^{\mathrm{in}\dagger}_{\omega\ell m}\,\widetilde{\chi}^{\mathrm{in}-}_{\omega\ell m}\right) \\
&+ \int_{-\infty}^{\min\left\{\frac{qQ}{r_+},0\right\}} \mathrm{d}\omega\left(a^{\mathrm{in}}_{\omega\ell m}\,\widetilde{\chi}^{\mathrm{in}+}_{\omega\ell m} + b^{\mathrm{in}\dagger}_{\omega\ell m}\,\widetilde{\chi}^{\mathrm{in}-}_{\omega\ell m}\right) \\
&+ \int_{\min\left\{\frac{qQ}{r_+},0\right\}}^{\max\left\{\frac{qQ}{r_+},0\right\}} \mathrm{d}\omega\left(a^{\mathrm{in}}_{\omega\ell m}\,\widetilde{\chi}^{\mathrm{in}-}_{\omega\ell m} + b^{\mathrm{in}\dagger}_{\omega\ell m}\,\widetilde{\chi}^{\mathrm{in}+}_{\omega\ell m}\right)\Bigg\}.
\end{aligned} \tag{4.213}$$

In (4.213), positive-frequency modes $\chi^{\mathrm{up}+}_{\omega\ell m}$ are entirely of positive-norm and negative-frequency modes $\chi^{\mathrm{up}-}_{\omega\ell m}$ are entirely of negative-norm. Thus the first term on the r.h.s of (4.213) is identical to the first term on the r.h.s of the expansion (4.207) of the field leading to the 'Hartle-Hawking-like' state $|\mathrm{H}\rangle$. However, the second term on the r.h.s of

(4.207), which involves the modes $\widetilde{\chi}^{\text{in}\pm}_{\omega\ell m}$, is split into three terms in (4.213) so that we may correctly label the expansion coefficients multiplying each mode according to the mode norm.

Specifically, positive-frequency modes $\widetilde{\chi}^{\text{in}+}_{\omega\ell m}$ with $\text{sgn}(\omega\widetilde{\omega}) = 1$ are of positive-norm and negative-frequency modes $\widetilde{\chi}^{\text{in}-}_{\omega\ell m}$ with $\text{sgn}(\omega\widetilde{\omega}) = 1$ are of negative-norm. In the second and third terms on the r.h.s of (4.213) therefore, positive-frequency modes $\chi^{\text{in}+}_{\omega\ell m}$ are multiplied by $a^{\text{in}}_{\omega\ell m}$ and negative-frequency modes $\chi^{\text{in}-}_{\omega\ell m}$ are multiplied by $b^{\text{in}\dagger}_{\omega\ell m}$.

However, positive-frequency modes $\widetilde{\chi}^{\text{in}+}_{\omega\ell m}$ with $\text{sgn}(\omega\widetilde{\omega}) = -1$ are of negative-norm and negative-frequency modes $\widetilde{\chi}^{\text{in}-}_{\omega\ell m}$ with $\text{sgn}(\omega\widetilde{\omega}) = -1$ are of positive-norm. Therefore, in the fourth term on the r.h.s of (4.213), positive-frequency modes $\chi^{\text{in}+}_{\omega\ell m}$ are multiplied by the expansion coefficients $b^{\text{in}\dagger}_{\omega\ell m}$ and negative-frequency modes $\chi^{\text{in}-}_{\omega\ell m}$ are multiplied by the expansion coefficients $a^{\text{in}}_{\omega\ell m}$. Recall from (4.17) that, if $qQ > 0$, modes with $0 < \omega < \frac{qQ}{r_+}$ have $\text{sgn}(\omega\widetilde{\omega}) = -1$ and, if $qQ < 0$, modes with $0 > \omega > \frac{qQ}{r_+}$ have $\text{sgn}(\omega\widetilde{\omega}) = -1$. Thus, the integral in the fourth term on the r.h.s of (4.213) reduces to the superradiant range $0 < \omega < \frac{qQ}{r_+}$ if $qQ > 0$ and to the superradiant range $0 > \omega > \frac{qQ}{r_+}$ if $qQ < 0$.

Restricting our attention to region I, (4.213) reduces to

$$\begin{aligned}
\Phi(x) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \Bigg\{ &\int_{-\infty}^{\infty} \mathrm{d}\widetilde{\omega} \, \frac{1}{\sqrt{2\left|\sinh\left(\frac{\pi\widetilde{\omega}}{\kappa}\right)\right|}} \left[\mathrm{e}^{\frac{\pi\widetilde{\omega}}{2\kappa}} a^{\text{up}}_{\omega\ell m} + \mathrm{e}^{-\frac{\pi\widetilde{\omega}}{2\kappa}} b^{\text{up}\dagger}_{\omega\ell m}\right] \phi^{\text{up}}_{\omega\ell m} \\
&+ \int_{\max\left\{\frac{qQ}{r_+},0\right\}}^{\infty} \mathrm{d}\omega \, \frac{1}{\sqrt{2\left|\sinh\left(\frac{\pi\widetilde{\omega}}{\kappa}\right)\right|}} \left[\mathrm{e}^{\frac{\pi\widetilde{\omega}}{2\kappa}} a^{\text{in}}_{\omega\ell m} + \mathrm{e}^{-\frac{\pi\widetilde{\omega}}{2\kappa}} b^{\text{in}\dagger}_{\omega\ell m}\right] \phi^{\text{in}}_{\omega\ell m} \\
&+ \int_{-\infty}^{\min\left\{\frac{qQ}{r_+},0\right\}} \mathrm{d}\omega \, \frac{1}{\sqrt{2\left|\sinh\left(\frac{\pi\widetilde{\omega}}{\kappa}\right)\right|}} \left[\mathrm{e}^{\frac{\pi\widetilde{\omega}}{2\kappa}} a^{\text{in}}_{\omega\ell m} + \mathrm{e}^{-\frac{\pi\widetilde{\omega}}{2\kappa}} b^{\text{in}\dagger}_{\omega\ell m}\right] \phi^{\text{in}}_{\omega\ell m} \\
&+ \int_{\min\left\{\frac{qQ}{r_+},0\right\}}^{\max\left\{\frac{qQ}{r_+},0\right\}} \mathrm{d}\omega \, \frac{1}{\sqrt{2\left|\sinh\left(\frac{\pi\widetilde{\omega}}{\kappa}\right)\right|}} \left[\mathrm{e}^{-\frac{\pi\widetilde{\omega}}{2\kappa}} a^{\text{in}}_{\omega\ell m} + \mathrm{e}^{\frac{\pi\widetilde{\omega}}{2\kappa}} b^{\text{in}\dagger}_{\omega\ell m}\right] \phi^{\text{in}}_{\omega\ell m} \Bigg\}.
\end{aligned} \tag{4.214}$$

We quantise the field by promoting the mode expansion coefficients in (4.214) to operators such that the field operator $\hat{\Phi}(x)$ is given by

$$\hat{\Phi}(x) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \left\{ \int_{-\infty}^{\infty} \mathrm{d}\widetilde{\omega} \, \frac{1}{\sqrt{2\left|\sinh\left(\frac{\pi\widetilde{\omega}}{\kappa}\right)\right|}} \left[\mathrm{e}^{\frac{\pi\widetilde{\omega}}{2\kappa}} \hat{a}^{\mathrm{up}}_{\omega\ell m} + \mathrm{e}^{-\frac{\pi\widetilde{\omega}}{2\kappa}} \hat{b}^{\mathrm{up}\dagger}_{\omega\ell m} \right] \phi^{\mathrm{up}}_{\omega\ell m} \right.$$
$$+ \int_{\max\left\{\frac{qQ}{r_+},0\right\}}^{\infty} \mathrm{d}\omega \, \frac{1}{\sqrt{2\left|\sinh\left(\frac{\pi\widetilde{\omega}}{\kappa}\right)\right|}} \left[\mathrm{e}^{\frac{\pi\widetilde{\omega}}{2\kappa}} \hat{a}^{\mathrm{in}}_{\omega\ell m} + \mathrm{e}^{-\frac{\pi\widetilde{\omega}}{2\kappa}} \hat{b}^{\mathrm{in}\dagger}_{\omega\ell m} \right] \phi^{\mathrm{in}}_{\omega\ell m}$$
$$+ \int_{-\infty}^{\min\left\{\frac{qQ}{r_+},0\right\}} \mathrm{d}\omega \, \frac{1}{\sqrt{2\left|\sinh\left(\frac{\pi\widetilde{\omega}}{\kappa}\right)\right|}} \left[\mathrm{e}^{\frac{\pi\widetilde{\omega}}{2\kappa}} \hat{a}^{\mathrm{in}}_{\omega\ell m} + \mathrm{e}^{-\frac{\pi\widetilde{\omega}}{2\kappa}} \hat{b}^{\mathrm{in}\dagger}_{\omega\ell m} \right] \phi^{\mathrm{in}}_{\omega\ell m}$$
$$\left. + \int_{\min\left\{\frac{qQ}{r_+},0\right\}}^{\max\left\{\frac{qQ}{r_+},0\right\}} \mathrm{d}\omega \, \frac{1}{\sqrt{2\left|\sinh\left(\frac{\pi\widetilde{\omega}}{\kappa}\right)\right|}} \left[\mathrm{e}^{-\frac{\pi\widetilde{\omega}}{2\kappa}} \hat{a}^{\mathrm{in}}_{\omega\ell m} + \mathrm{e}^{\frac{\pi\widetilde{\omega}}{2\kappa}} \hat{b}^{\mathrm{in}\dagger}_{\omega\ell m} \right] \phi^{\mathrm{in}}_{\omega\ell m} \right\}. \tag{4.215}$$

In (4.215), all modes with positive-norm are multiplied by annihilation operators $\hat{a}^{\mathrm{in/up}}_{\omega\ell m}$ and all modes with negative-norm are multiplied by creation operators $\hat{b}^{\mathrm{in/up}\dagger}_{\omega\ell m}$. Therefore, the operators $\hat{a}^{\mathrm{in/up}}_{\omega\ell m}$ and $\hat{b}^{\mathrm{in/up}}_{\omega\ell m}$ obey the following standard commutation relations:

$$\begin{aligned}
\left[\hat{a}^{\mathrm{in}}_{\omega\ell m}, \hat{a}^{\mathrm{in}\dagger}_{\omega'\ell'm'}\right] &= \delta\left(\omega-\omega'\right)\delta_{\ell\ell'}\,\delta_{mm'}, && \text{all } \omega, \\
\left[\hat{b}^{\mathrm{in}}_{\omega\ell m}, \hat{b}^{\mathrm{in}\dagger}_{\omega'\ell'm'}\right] &= \delta\left(\omega-\omega'\right)\delta_{\ell\ell'}\,\delta_{mm'}, && \text{all } \omega, \\
\left[\hat{a}^{\mathrm{up}}_{\omega\ell m}, \hat{a}^{\mathrm{up}\dagger}_{\omega'\ell'm'}\right] &= \delta\left(\omega-\omega'\right)\delta_{\ell\ell'}\,\delta_{mm'}, && \text{all } \widetilde{\omega}, \\
\left[\hat{b}^{\mathrm{up}}_{\omega\ell m}, \hat{b}^{\mathrm{up}\dagger}_{\omega'\ell'm'}\right] &= \delta\left(\omega-\omega'\right)\delta_{\ell\ell'}\,\delta_{mm'}, && \text{all } \widetilde{\omega},
\end{aligned} \tag{4.216}$$

with any commutators not explicitly given in (4.216) vanishing. The 'Frolov-Thorne' state $|\mathrm{FT}\rangle$ is then defined as the state annihilated by the $\hat{a}^{\mathrm{in/up}}_{\omega\ell m}$ and $\hat{b}^{\mathrm{in/up}}_{\omega\ell m}$ operators such that

$$\begin{aligned}
\hat{a}^{\mathrm{in}}_{\omega\ell m}\,|\mathrm{FT}\rangle &= 0, && \text{all } \omega, \\
\hat{b}^{\mathrm{in}}_{\omega\ell m}\,|\mathrm{FT}\rangle &= 0, && \text{all } \omega, \\
\hat{a}^{\mathrm{up}}_{\omega\ell m}\,|\mathrm{FT}\rangle &= 0, && \text{all } \widetilde{\omega}, \\
\hat{b}^{\mathrm{up}}_{\omega\ell m}\,|\mathrm{FT}\rangle &= 0 && \text{all } \widetilde{\omega}.
\end{aligned} \tag{4.217}$$

From the expansion (4.215), we anticipate that this state will contain a thermal flux of particles in both the in- and up-modes, with the thermal factor of each depending on $\widetilde{\omega}$. However, the superradiant in-modes will require careful treatment.

Chapter 5

Expectation values of quantum observables

In §5.1, we introduce each of the three main quantum observables considered in this thesis. We derive the asymptotic mode contributions to each of these observables in §5.2. In §5.3, we explore the expectation values of observables in the 'past' states. We evaluate the differences in expectation values between the 'future' states in §5.4. In §5.5, we investigate the 'Boulware-like' state. In §5.6, we investigate the Frolov-Thorne state. In §5.7, we investigate the 'Hartle-Hawking-like' state.

5.1 Quantum observables

There are three main observables of interest in a quantum charged scalar field theory, namely the scalar field condensate $\widehat{\mathcal{SC}}$, the charged scalar field current $\hat{J}^\mu$ and the stress-energy tensor $\hat{T}_{\mu\nu}$. In this section, we will introduce each of these observables in turn.

We will also use the classical expression for each of the observables, in terms of the field Φ, to give the explicit forms of the mode contributions to the scalar condensate $\mathcal{SC}$ and each component of the current J^μ and the stress-energy tensor $T_{\mu\nu}$. We use the notation O to denote the classical quantity corresponding to the quantum observable $\hat{O}$, and the scalar field mode contribution to the quantity O is denoted by $o_{\omega\ell m}$.

5.1.1 Scalar condensate

The simplest nontrivial observable associated to a quantum charged scalar field $\hat{\Phi}$ is the scalar condensate $\widehat{\mathcal{SC}}$, which is sometimes referred to as the vacuum polarisation; as the name implies, it is a scalar quantity. Classically, the scalar condensate $\mathcal{SC}$ corresponds to the square of the magnitude of the scalar field Φ; therefore, we have

$$\mathcal{SC} = |\Phi|^2. \tag{5.1}$$

Since the scalar condensate is a scalar quantity, it should not be able to distinguish between the 'past' states we defined in Chapter 4 and their corresponding 'future' states. In order to see this, consider that the out-mode radial function $X^{\text{out}}_{\omega\ell}$ (3.68) is the complex conjugate of the in-mode radial function $X^{\text{in}}_{\omega\ell}$ (3.64). Then, from the general form of a scalar field mode (3.41), we have $|\phi^{\text{out}}_{\omega\ell m}|^2 = |\phi^{\text{in}}_{\omega\ell m}|^2$. A similar line of reasoning shows that $|\phi^{\text{down}}_{\omega\ell m}|^2 = |\phi^{\text{up}}_{\omega\ell m}|^2$. The 'past' states were defined in terms of the in-modes (3.106) and the up-modes (3.131), while the 'future' states were defined in terms of the out-modes (3.139) and the down-modes (3.147). Therefore corresponding 'past' and 'future' states are indistinguishable to the scalar condensate.

In calculating the classical mode contribution $sc_{\omega\ell m}$ to the scalar condensate $\mathcal{SC}$, one would ordinarily need to consider the product of fields with different quantum numbers, i.e. different values of ω, ℓ and m. However, as we can see from the calculation of the expectation values of the quantum scalar condensate $\langle\hat{\Phi}^2\rangle$ in §2.3.1, only the square magnitudes of the individual modes contribute to the quantum expectation values since the creation and annihilation operators associated to modes of different quantum numbers commute. Thus, the classical mode contribution $sc_{\omega\ell m}$ to the scalar condensate $\mathcal{SC}$ can be given for general values of the quantum numbers ω, ℓ and m as

$$\begin{aligned} sc_{\omega\ell m} &= |\phi_{\omega\ell m}|^2 = \phi^*_{\omega\ell m}\,\phi_{\omega\ell m} \\ &= \mathrm{e}^{\mathrm{i}\omega t}\,\mathrm{N}^*_\omega\,\frac{X^*_{\omega\ell}(r)}{r}\,Y^*_{\ell m}(\theta,\varphi)\times\mathrm{e}^{-\mathrm{i}\omega t}\,\mathrm{N}_\omega\,\frac{X_{\omega\ell}(r)}{r}\,Y_{\ell m}(\theta,\varphi) \\ &= \frac{1}{r^2}\,|\mathrm{N}_\omega|^2\,|X_{\omega\ell}(r)|^2\,|Y_{\ell m}(\theta,\varphi)|^2. \end{aligned} \tag{5.2}$$

Performing the sum over the azimuthal number m, we obtain

$$sc_{\omega\ell} = \sum_{m=-\ell}^{\ell} sc_{\omega\ell m} = \frac{2\ell+1}{4\pi r^2}\,|\mathrm{N}_\omega|^2\,|X_{\omega\ell}(r)|^2. \tag{5.3}$$

where we have used the addition formula (A.4) for the spherical harmonics.

The scalar condensate of a quantum field

The scalar condensate $\widehat{\mathcal{SC}}$ associated to a quantum charged scalar field $\hat{\Phi}$ is given by

$$\widehat{\mathcal{SC}} = \frac{1}{2}\left[\hat{\Phi}\hat{\Phi}^\dagger + \hat{\Phi}^\dagger\hat{\Phi}\right]. \tag{5.4}$$

The expectation value $\langle\widehat{\mathcal{SC}}\rangle$ of the scalar condensate will be a function of the radial coordinate r only, irrespective of the quantum state under consideration. While this is straightforward to see from the form of the mode contribution $sc_{\omega\ell m}$ (5.3), it is illustrative to think about why this might be the case from physical considerations.

All of the quantum states defined in Chapter 4 are stationary, which means that all observables associated to these states are time-independent. Furthermore, Reissner-Nordström spacetime is both static and spherically symmetric, which corresponds to the field modes defined on this background spacetime having a harmonic time dependence $\mathrm{e}^{-\mathrm{i}\omega t}$ and their dependence on the angular coordinates being given by the spherical harmonics $Y_{\ell m}(\theta,\varphi)$ respectively, such as in the general form for a scalar field mode (3.41).

Since the scalar condensate $\widehat{\mathcal{SC}}$ (5.4) depends only on the product of the field operator $\hat{\Phi}$ and its Hermitian conjugate $\hat{\Phi}^\dagger$, then it is easy to see that the quantum scalar condensate $\widehat{\mathcal{SC}}$ is time-independent. Furthermore, from the addition formula of the spherical harmonics $Y_{\ell m}(\theta,\varphi)$ (A.4), we see that the scalar condensate is also independent of the angular coordinates θ and φ. Thus, the expectation values $\langle\widehat{\mathcal{SC}}\rangle$ of the scalar condensate will be functions of the radial coordinate r only.

5.1.2 Current

The next simplest observable associated to a quantum charged scalar field $\hat{\Phi}$ is the current $\hat{J}^\mu$. For a classical field Φ with scalar field charge q, the current J^μ is given by

$$J^\mu = -\frac{q}{4\pi}\,\Im\left[\Phi^* D^\mu \Phi\right], \tag{5.5}$$

The classical current is conserved

$$\nabla_\mu J^\mu = 0. \tag{5.6}$$

The mode contributions to each component of the current J^μ are derived in §B.1 of appendix B; the nonzero contributions are given by

$$j^t_{\omega\ell} = -\frac{q\,(2\ell+1)}{16\pi^2 r^2 f(r)}\left(\omega - \frac{qQ}{r}\right)|\mathrm{N}_\omega|^2\,|X_{\omega\ell}(r)\,|^2, \tag{5.7a}$$

$$j^r_{\omega\ell} = -\frac{qf(r)(2\ell+1)}{16\pi^2}\,|\mathrm{N}_\omega|^2\,\Im\left[\frac{X^*_{\omega\ell}(r)}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{X_{\omega\ell}(r)}{r}\right)\right]. \tag{5.7b}$$

The current of a quantum field

The current operator $\hat{J}^\mu$ associated to a quantum charged scalar field $\hat{\Phi}$ is given by

$$\hat{J}^\mu = \frac{\mathrm{i}q}{16\pi}\left[\hat{\Phi}^\dagger D^\mu\hat{\Phi} + (D^\mu\hat{\Phi})\hat{\Phi}^\dagger - \hat{\Phi}(D^\mu\hat{\Phi})^\dagger - (D^\mu\hat{\Phi})^\dagger\hat{\Phi}\right]. \tag{5.8}$$

The expectation values of the current operator $\hat{J}^\mu$ are also conserved

$$\nabla_\mu\langle\hat{J}^\mu\rangle = 0. \tag{5.9}$$

All of the quantum states defined in Chapter 4 are stationary and all observables associated to these states are therefore time-independent. Then, the derivative w.r.t t vanishes in the equation governing the conservation of the current operator $\langle\hat{J}^\mu\rangle$ and (5.9) reduces to

$$\begin{aligned} 0 = \nabla_\mu\langle\hat{J}^\mu\rangle &= \nabla_r\langle\hat{J}^r\rangle \\ &= \partial_r\langle\hat{J}^r\rangle + \Gamma^\mu_{r\mu}\langle\hat{J}^r\rangle \\ &= \partial_r\langle\hat{J}^r\rangle + \Gamma^t_{rt}\langle\hat{J}^r\rangle + \Gamma^r_{rr}\langle\hat{J}^r\rangle + \Gamma^\theta_{r\theta}\langle\hat{J}^r\rangle + \Gamma^\varphi_{r\varphi}\langle\hat{J}^r\rangle\,. \end{aligned} \tag{5.10}$$

Then, using the expressions for the Christoffel symbols in §3.1.2, (5.10) reduces to

$$0 = \partial_r\langle\hat{J}^r\rangle + \frac{2}{r}\,\langle\hat{J}^r\rangle\,. \tag{5.11}$$

We can integrate (5.11) to give, for any quantum state, the expression

$$\langle \hat{J}^r \rangle = -\frac{\mathcal{K}}{r^2}, \tag{5.12}$$

where the constant $\mathcal{K}$ is a state-dependent quantity that can be interpreted as the flux of charge emitted by the black hole in the particular quantum state under consideration. While the absolute value $|\mathcal{K}|$ gives the magnitude of the flux of charge, the direction of its flow depends on the product of the charge of the black hole Q and $\mathcal{K}$ as follows:

$$\mathcal{K}Q \begin{cases} > 0 & \Rightarrow \text{ the black hole is losing charge,} \\ = 0 & \Rightarrow \text{ there is no net flux of charge,} \\ < 0 & \Rightarrow \text{ the black hole is gaining charge.} \end{cases} \tag{5.13}$$

From the scalar field mode contributions to the classical current J^μ, we see that the only nonzero components of the expectation value $\langle \hat{J}^\mu \rangle$ of the current operator are $\langle \hat{J}^t \rangle$ and $\langle \hat{J}^r \rangle$. We prove in Appendix C that the $\langle \hat{J}^r \rangle$ component does not require renormalisation. Therefore, we may consider the expectation value $\langle \hat{J}^r \rangle$ of the radial component of the current in a particular state without needing to consider the differences in expectation values between two separate states; this allows us to investigate the flux of charge $\mathcal{K}$ associated with a particular quantum state. It is shown, however, in [45] that while the $\langle \hat{J}^t \rangle$ component does indeed require renormalisation, the renormalisation counterterms associated to this component are finite in the case of a RN-de Sitter background spacetime. We would expect such a result to extend to the case of a Reissner-Nordström spacetime.

5.1.3 Stress-energy tensor

The last observable we consider is the stress-energy tensor $T_{\mu\nu}$ associated to a massless, minimally coupled, charged scalar field on a Ricci-flat background spacetime, given by

$$T_{\mu\nu} = \Re\left[(D_\mu \Phi)^* \, D_\nu \Phi - \frac{1}{2}\, g_{\mu\nu}\, g^{\rho\lambda}\, (D_\rho \Phi)^* (D_\lambda \Phi) \right]. \tag{5.14}$$

The mode contributions to each component of the stress-energy tensor $T_{\mu\nu}$ are derived in

§B.2 of appendix B; the nonzero contributions are given by

$$
\begin{aligned}
t_{tt,\omega\ell} &= \frac{2\ell+1}{8\pi}\,|\mathrm{N}_\omega|^2 \left\{ \left[\frac{1}{r^2}\left(\omega - \frac{qQ}{r}\right)^2 + \frac{f(r)\,\ell(\ell+1)}{r^4} \right] |X_{\omega\ell}(r)|^2 \right. \\
&\quad \left. + f(r)^2 \left| \frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{X_{\omega\ell}(r)}{r}\right) \right|^2 \right\}, && \text{(5.15a)} \\
t_{tr,\omega\ell} &= -\frac{2\ell+1}{4\pi}\left(\omega - \frac{qQ}{r}\right) |\mathrm{N}_\omega|^2\, \Im\left[\frac{X^*_{\omega\ell}(r)}{r} \frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{X_{\omega\ell}(r)}{r}\right) \right], && \text{(5.15b)} \\
t_{rr,\omega\ell} &= \frac{2\ell+1}{8\pi}\,|\mathrm{N}_\omega|^2 \left\{ \left[\frac{1}{f(r)^2\,r^2}\left(\omega - \frac{qQ}{r}\right)^2 - \frac{\ell(\ell+1)}{f(r)\,r^4} \right] |X_{\omega\ell}(r)|^2 + \left| \frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{X_{\omega\ell}(r)}{r}\right) \right|^2 \right\}, && \text{(5.15c)} \\
t_{\theta\theta,\omega\ell} &= \frac{2\ell+1}{8\pi}\,|\mathrm{N}_\omega|^2 \left\{ \frac{1}{f(r)}\left(\omega - \frac{qQ}{r}\right)^2 |X_{\omega\ell}(r)|^2 - f(r) r^2 \left| \frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{X_{\omega\ell}(r)}{r}\right) \right|^2 \right\}, && \text{(5.15d)} \\
t_{\varphi\varphi,\omega\ell} &= t_{\theta\theta,\omega\ell}\,\sin^2\theta. && \text{(5.15e)}
\end{aligned}
$$

The stress-energy tensor associated to a quantum field

From (5.14), the corresponding quantum stress-energy tensor operator $\hat{T}_{\mu\nu}$ is given by

$$
\begin{aligned}
\hat{T}_{\mu\nu} = \frac{1}{4}\Big\{ & (D_\mu\hat{\Phi})^\dagger D_\nu\hat{\Phi} + D_\nu\hat{\Phi}\,(D_\mu\hat{\Phi})^\dagger + D_\mu\hat{\Phi}\,(D_\nu\hat{\Phi})^\dagger + (D_\nu\hat{\Phi})^\dagger D_\mu\hat{\Phi} \\
& - \frac{1}{2}\, g_{\mu\nu}\, g^{\rho\lambda}\left[(D_\rho\hat{\Phi})^\dagger D_\lambda\hat{\Phi} + D_\lambda\hat{\Phi}\,(D_\rho\hat{\Phi})^\dagger + D_\rho\hat{\Phi}\,(D_\lambda\hat{\Phi})^\dagger + (D_\lambda\hat{\Phi})^\dagger D_\rho\hat{\Phi} \right] \Big\}. \qquad \text{(5.16)}
\end{aligned}
$$

The expression for $\hat{T}_{\mu\nu}$ is derived in §8.3.3. The expectation values $\langle\hat{T}_{\mu\nu}\rangle$ of the stress-energy tensor operator (5.16) associated to the quantum charged scalar field $\hat{\Phi}$ are not conserved due to the coupling between the charged scalar field and the classical, background electromagnetic field A_μ. The expectation values $\langle\hat{T}_{\mu\nu}\rangle$ should instead satisfy

$$
\nabla^\mu\langle\hat{T}_{\mu\nu}\rangle = 4\pi F_{\mu\nu}\langle\hat{J}^\mu\rangle\,. \qquad \text{(5.17)}
$$

As we would expect, the total stress-energy tensor associated to both the scalar field $\hat{\Phi}$ and the background electromagnetic field A_μ is conserved; this is discussed in detail §8.3.3.

For static states on a spherically symmetric black hole, $\langle\hat{T}^\mu_\nu\rangle$ takes the form

$$
\langle\hat{T}^\mu_\nu\rangle = \begin{pmatrix} \mathcal{A}(r) & -\mathcal{P}(r)\,f^{-1}(r) & 0 & 0 \\ \mathcal{P}(r)\,f(r) & \mathsf{T}(r) - \mathcal{A}(r) - 2\mathcal{Q}(r) & 0 & 0 \\ 0 & 0 & \mathcal{Q}(r) & 0 \\ 0 & 0 & 0 & \mathcal{Q}(r) \end{pmatrix}, \qquad \text{(5.18)}
$$

where $\mathcal{A}(r)$, $\mathcal{P}(r)$ and $\mathcal{Q}(r)$ are functions of the radial coordinate r only, which are yet to be determined, and the trace $\mathsf{T}(r)$ of the stress-energy tensor is given by

$$
\mathsf{T}(r) = \langle\hat{T}^\mu_\nu\rangle\,\delta^\nu_\mu = \langle\hat{T}^\mu_\mu\rangle\,. \qquad \text{(5.19)}
$$

In the specific case of a minimally coupled, charged scalar field on a background spacetime with vanishing Ricci scalar curvature, i.e. $R = 0$, the trace $\mathsf{T}(r)$ has the form

$$\mathsf{T}(r) = \frac{1}{2880\pi^2} R^{\mu\nu\rho\lambda} R_{\mu\nu\rho\lambda} - \frac{1}{2880\pi^2} R^{\mu\nu} R_{\mu\nu} - \frac{q^2}{192\pi^2} F^{\mu\nu} F_{\mu\nu} - \frac{1}{2} \Box \langle \widehat{\mathcal{SC}} \rangle \,, \tag{5.20}$$

where we have derived (5.20) using equations (6.9), (7.274) and (8.130) from Part III of this thesis. The first two terms on the r.h.s of (5.20) depend only on the geometry of the classical background spacetime, while the third depends only on the field strength of the classical background electromagnetic field; these terms are therefore state-independent and will vanish when considering the differences of expectation values of observables in separate quantum states. The final term, however, depends on the quantum state under consideration; this term arises due to the minimal coupling of the scalar field to the background spacetime geometry and vanishes for a conformally coupled field.

Restricting ourselves to the case of a background Reissner-Nordström spacetime, the expression for the trace $\mathsf{T}(r)$ in (5.20) takes the explicit form

$$\mathsf{T}(r) = \frac{13Q^2 - 24MQ^2 r + 12M^2 r^2}{720\pi^2 r^8} - \frac{q^2 Q^2}{96\pi^2 r^4} - \frac{1}{2} \Box \langle \widehat{\mathcal{SC}} \rangle \,, \tag{5.21}$$

where we have used the expression for the electromagnetic field strength tensor $F_{\mu\nu}$ (3.2) and the geometry of Reissner-Nordström spacetime (3.9).

We can write the mode contribution $t^{\mu}_{\mu,\omega\ell}$ to the trace $\mathsf{T}(r)$ of the SET as

$$\begin{aligned} t^{\mu}_{\mu,\omega\ell} &= t^{t}_{t,\omega\ell} + t^{r}_{r,\omega\ell} + t^{\theta}_{\theta,\omega\ell} + t^{\varphi}_{\varphi,\omega\ell} \\ &= g^{tt}\, t_{tt,\omega\ell} + g^{rr}\, t_{rr,\omega\ell} + g^{\theta\theta}\, t_{\theta\theta,\omega\ell} + g^{\varphi\varphi}\, t_{\varphi\varphi,\omega\ell} \\ &= -f(r)^{-1}\, t_{tt,\omega\ell} + f(r)\, t_{rr,\omega\ell} + 2r^{-2}\, t_{\theta\theta,\omega\ell}, \end{aligned} \tag{5.22}$$

where we have used the expression (5.15e) for $t_{\varphi\varphi,\omega\ell}$ in terms of $t_{\theta\theta,\omega\ell}$ in the last line of (5.22). We can use the expressions for the mode contributions $t_{tt,\omega\ell}$ (5.15a), $t_{rr,\omega\ell}$ (5.15c) and $t_{\theta\theta,\omega\ell}$ (5.15d) to derive explicitly the mode contribution $t^{\mu}_{\mu,\omega\ell}$ to the trace $\mathsf{T}(r)$ as

$$\begin{aligned} t^{\mu}_{\mu,\omega\ell} &= \frac{2\ell+1}{4\pi} |\mathrm{N}_\omega|^2 \left\{ \left[\frac{1}{f(r) r^2} \left(\omega - \frac{qQ}{r} \right)^2 - \frac{\ell(\ell+1)}{r^4} \right] |X_{\omega\ell}(r)|^2 \right. \\ &\quad \left. - f(r) \left| \frac{\mathrm{d}}{\mathrm{d}r} \left(\frac{X_{\omega\ell}(r)}{r} \right) \right|^2 \right\} \\ &= -\frac{2\ell+1}{4\pi} |\mathrm{N}_\omega|^2 \, \Box \left(\frac{|X_{\omega\ell}(r)|^2}{r^2} \right), \end{aligned} \tag{5.23}$$

where we have used the radial equation (3.45) to simplify (5.23). Using the expression for the mode contribution $sc_{\omega\ell}$ to the scalar condensate (5.3), we can write the mode contribution $t^{\mu}_{\mu,\omega\ell}$ to the trace $\mathsf{T}(r)$ of the stress-energy tensor as

$$t^{\mu}_{\mu,\omega\ell} = -\frac{1}{2} \Box \, sc_{\omega\ell}. \tag{5.24}$$

Returning to the expression (5.20) for the trace $\mathsf{T}(r)$, the first three terms on the r.h.s arise as a result of the renormalisation process of the SET. Since we are considering differences, the only part of the trace $\mathsf{T}(r)$ that should receive a mode contribution is the $-\frac{1}{2}\Box\,\langle\widehat{\mathcal{SC}}\rangle$ term in (5.20), which our result for the mode contribution $t^{\mu}_{\mu,\omega\ell}$ in (5.24) reflects.

We can use the form of the expectation value of the RSET (5.18) to integrate the t-component of the equations (5.17) governing its conservation to give

$$\langle\hat{T}^r_t\rangle = -\frac{\mathcal{L}}{r^2} + \frac{4\pi\mathcal{QK}}{r^3}, \tag{5.25}$$

where the constant $\mathcal{L}$ is a state-independent quantity that can be interpreted as the flux of energy emitted by the black hole in the particular quantum state under consideration. The direction of the flux of energy depends on the value of $\mathcal{L}$ as follows:

$$\mathcal{L}\begin{cases} >0 & \Rightarrow \text{ the black hole is losing energy,} \\ =0 & \Rightarrow \text{ there is no net flux of energy,} \\ <0 & \Rightarrow \text{ the black hole is gaining energy.}\end{cases} \tag{5.26}$$

We prove in Appendix C that the $\langle\hat{T}^r_t\rangle$ component does not require renormalisation. Therefore, we may consider expectation values of the component $\langle\hat{T}^r_t\rangle$ in a particular state without needing to consider the differences in expectation values between two separate states; this allows us to investigate the flux of energy $\mathcal{L}$ in a particular quantum state.

5.1.4 Current and stress-energy tensor in Kruskal coordinates

The behaviour of the expectation value of the current $\langle\hat{J}^{\mu}\rangle$ and of the stress-energy tensor $\langle\hat{T}^{\mu}_{\nu}\rangle$ will aid us in investigating the regularity of the quantum states we defined in Chapter 4. Since the Schwarzschild-like coordinates are singular on the event horizon, we instead examine the regularity of states on the horizon in terms of Kruskal coordinates (3.24). We thus give the nonzero components of both observables in terms of Kruskal coordinates.

Recall that the J^{θ} and J^{φ} components of the current vanish from (B.3) and (B.4) respectively. Then the only nonzero components of the current J^{μ} in terms of Kruskal coordinates are J^U and J^V. Using the relations between the various sets of coordinates we have defined in (3.10), (3.20) and (3.26), we have

$$J^U = \frac{\partial U}{\partial u}J^u = -\kappa U\left[\frac{\partial u}{\partial t}J^t + \frac{\partial u}{\partial r_*}\frac{\partial r_*}{\partial r}J^r\right], \tag{5.27a}$$

$$J^V = \frac{\partial V}{\partial v}J^v = \kappa V\left[\frac{\partial v}{\partial t}J^t + \frac{\partial v}{\partial r_*}\frac{\partial r_*}{\partial r}J^r\right]. \tag{5.27b}$$

The nonzero components of the current are given in terms of Kruskal coordinates as

$$J^U = \kappa U\left[-J^t + f(r)^{-1}\,J^r\right], \tag{5.28a}$$

$$J^V = \kappa V\left[J^t + f(r)^{-1}\,J^r\right]. \tag{5.28b}$$

The only nonzero components of the stress-energy tensor $T_{\mu\nu}$ in terms of Schwarzschild-like coordinates are the T_{tt}, T_{tr}, T_{rr}, $T_{\theta\theta}$ and the $T_{\varphi\varphi}$ components. The $T_{\theta\theta}$ and the $T_{\varphi\varphi}$ components are invariant under the change to Kruskal coordinates. Then, the only components of the stress-energy tensor that we need to evaluate in terms of Kruskal coordinates are T_{UU}, T_{UV} and T_{VV}. Using the relations between the various sets of coordinates we have defined in (3.10), (3.20) and (3.26), we have

$$\begin{aligned} T_{UU} &= \frac{\partial u}{\partial U}\frac{\partial u}{\partial U}\,T_{uu} = \frac{1}{\kappa^2 U^2}\left[\frac{\partial t}{\partial u}\frac{\partial t}{\partial u}\,T_{tt} + 2\,\frac{\partial t}{\partial u}\frac{\partial r_*}{\partial u}\,T_{tr_*} + \frac{\partial r_*}{\partial u}\frac{\partial r_*}{\partial u}\,T_{r_*r_*}\right] \\ &= \frac{1}{\kappa^2 U^2}\left[\frac{1}{4}\,T_{tt} - \frac{1}{2}f(r)\,T_{tr} + \frac{1}{4}f(r)^2\,T_{rr}\right], \qquad (5.29a) \\ T_{UV} &= \frac{\partial u}{\partial U}\frac{\partial v}{\partial V}\,T_{uv} = -\frac{1}{\kappa^2 UV}\left[\frac{\partial t}{\partial u}\frac{\partial t}{\partial v}\,T_{tt} + \frac{\partial t}{\partial u}\frac{\partial r_*}{\partial v}\,T_{tr_*} + \frac{\partial r_*}{\partial u}\frac{\partial t}{\partial v}\,T_{r_*t} + \frac{\partial r_*}{\partial u}\frac{\partial r_*}{\partial v}\,T_{r_*r_*}\right] \\ &= -\frac{1}{\kappa^2 UV}\left[\frac{1}{4}\,T_{tt} + \frac{1}{4}f(r)\,T_{tr} - \frac{1}{4}f(r)\,T_{rt} - \frac{1}{4}f(r)^2\,T_{rr}\right] \\ &= -\frac{1}{\kappa^2 UV}\left[\frac{1}{4}\,T_{tt} - \frac{1}{4}f(r)^2\,T_{rr}\right], \qquad (5.29b) \\ T_{VV} &= \frac{\partial v}{\partial V}\frac{\partial v}{\partial V}\,T_{vv} = \frac{1}{\kappa^2 V^2}\left[\frac{\partial t}{\partial v}\frac{\partial t}{\partial v}\,T_{tt} + 2\frac{\partial t}{\partial v}\frac{\partial r_*}{\partial v}\,T_{tr_*} + \frac{\partial r_*}{\partial v}\frac{\partial r_*}{\partial v}\,T_{r_*r_*}\right] \\ &= \frac{1}{\kappa^2 V^2}\left[\frac{1}{4}\,T_{tt} + \frac{1}{2}f(r)\,T_{tr} + \frac{1}{4}f(r)^2\,T_{rr}\right], \qquad (5.29c) \end{aligned}$$

since $T_{tr} = T_{rt}$. The relevant components of the SET are given in Kruskal coordinates as

$$T_{UU} = \frac{1}{4}\kappa^{-2}\,U^{-2}\left[T_{tt} - 2f(r)\,T_{tr} + f(r)^2\,T_{rr}\right], \qquad (5.30a)$$

$$T_{UV} = -\frac{1}{4}\kappa^{-2}\,U^{-1}V^{-1}\left[T_{tt} - f(r)^2\,T_{rr}\right], \qquad (5.30b)$$

$$T_{VV} = \frac{1}{4}\kappa^{-2}\,V^{-2}\left[T_{tt} + 2f(r)\,T_{tr} + f(r)^2\,T_{rr}\right]. \qquad (5.30c)$$

5.1.5 A note on renormalisation

The quantum observables introduced in this section, namely the scalar condensate $\widehat{\mathcal{SC}}$, the current operator $\hat{J}^\mu$ and the stress-energy tensor operator $\hat{T}_{\mu\nu}$, all involve products of the field operator $\hat{\Phi}$ evaluated at the same spacetime point; therefore these observables are all formally divergent. The divergent terms only depend on the geometry of the background spacetime and the form of the background gauge field potential A_μ; in particular the divergent terms are state-independent and therefore identical regardless of the particular quantum state under consideration.

In Part III of this thesis, we develop the general framework for the Hadamard renormalisation procedure of the expectation values of observables associated to a charged scalar field. The actual implementation of this procedure in a specific background spacetime for a specific value of the background gauge potential A_μ is a subject that is still very much in its infancy [44, 45].

In this thesis, apart from the flux components $\langle \hat{J}^r \rangle$ and $\langle \hat{T}^r_t \rangle$ of the current and stress-energy tensor respectively which do not require renormalisation, we will restrict ourselves to considering the differences in expectation values between two separate quantum states. In this case the divergent terms in each of the expectation values cancel and we are left with a finite expression, thus alleviating the need to renormalise.

5.2 Asymptotic mode contributions to observables

5.2.1 Introduction

In §5.1, we derived expressions for the mode contributions to the scalar condensate as well as the components of the current and the stress-energy tensor. In general, these expressions contain factors of the radial coordinate r as well as the metric function $f(r)$. We would like to find the expectation values of quantum observables in the various quantum states that we defined in Chapter 4; evaluating these expectation values for arbitrary values of the radial coordinate r requires numerical computation.

In asymptotic regimes however, the mode contributions derived in §5.1 simplify considerably and we are able to derive comparatively simple expressions for the expectation values of observables. There are two asymptotic regimes, namely far from the black hole where $r \to \infty$, $f(r) \to 1$, and near the black hole horizon where $r \to r_+$, $f(r) \to 0$.

Later in this chapter, we will consider the differences in expectation values of observables in two separate quantum states. As previously stated, the expressions for the observables $\widehat{\mathcal{SC}}$ (5.4), $\hat{J}^\mu$ (5.8) and $\hat{T}_{\mu\nu}$ (5.16) involve products of field operators evaluated at the same spacetime point, and are therefore formally divergent. Such divergences are state-independent and cancel when considering differences in expectation values. Thus, any divergences remaining in the expressions we derive are likely to be indicative of pathological behaviour of (at least) one of the quantum states diverging in (at least) one of the asymptotic regimes. On the other hand, the absence of such divergences is a necessary, but not sufficient requirement for the quantum states under consideration to be regular.

In addition, our study of the expectation values of observables in asymptotic regimes will provide a useful consistency check on our numerical computations for general r.

5.2.2 The asymptotic regime far from the black hole

We would like to calculate the asymptotic mode contributions to observables far from the black hole. We will only consider leading order contributions and, since $r \to \infty$ far from the black hole, we need only consider the lowest-order contributions in the radial coordinate r. Furthermore, from the definition of the metric function $f(r)$ (3.10), we see that $f(r) \to 1$ far from the black hole.

Preliminaries

Several expressions for the mode contributions to the components of the current and the SET contain like terms; it will be useful to evaluate these terms now in order to simplify the process of evaluating the mode contributions later on. We can begin by calculating

$$\Im\left[\frac{X^*_{\omega\ell}(r)}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{X_{\omega\ell}(r)}{r}\right)\right],$$

for the asymptotic form of the up- and down-modes far from the black hole. Beginning with the asymptotic form of the up-modes (3.66) at infinity, we have

$$\begin{aligned}
\Im\left[\frac{X_{\omega\ell}^{\mathrm{up}*}(r)}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{X_{\omega\ell}^{\mathrm{up}}(r)}{r}\right)\right] &= \Im\left[\frac{X_{\omega\ell}^{\mathrm{up}*}(r)}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{B_{\omega\ell}^{\mathrm{up}}\,\mathrm{e}^{\mathrm{i}\omega r_*}}{r}\right)\right] \\
&= \Im\left[\frac{X_{\omega\ell}^{\mathrm{up}*}(r)}{r}\left(\frac{r\frac{\mathrm{d}r_*}{\mathrm{d}r}\frac{\mathrm{d}}{\mathrm{d}r_*}\left(B_{\omega\ell}^{\mathrm{up}}\,\mathrm{e}^{\mathrm{i}\omega r_*}\right)-B_{\omega\ell}^{\mathrm{up}}\,\mathrm{e}^{\mathrm{i}\omega r_*}}{r^2}\right)\right] \\
&= \Im\left[\frac{X_{\omega\ell}^{\mathrm{up}*}(r)}{r}\left(\frac{f(r)^{-1}\,r\left(\mathrm{i}\omega B_{\omega\ell}^{\mathrm{up}}\,\mathrm{e}^{\mathrm{i}\omega r_*}\right)-B_{\omega\ell}^{\mathrm{up}}\,\mathrm{e}^{\mathrm{i}\omega r_*}}{r^2}\right)\right] \\
&= \Im\left[\frac{B_{\omega\ell}^{\mathrm{up}*}\,\mathrm{e}^{-\mathrm{i}\omega r_*}}{r}\frac{B_{\omega\ell}^{\mathrm{up}}\,\mathrm{e}^{\mathrm{i}\omega r_*}}{r^2}\left(\frac{\mathrm{i}\omega r}{f(r)}-1\right)\right] \\
&= \Im\left[\frac{\left|B_{\omega\ell}^{\mathrm{up}}\right|^2}{r^3}\left(\frac{\mathrm{i}\omega r}{f(r)}-1\right)\right] \\
&\sim \frac{\omega\left|B_{\omega\ell}^{\mathrm{up}}\right|^2}{r^2}.
\end{aligned} \tag{5.31}$$

Since the down-mode radial function $X_{\omega\ell}^{\mathrm{down}}$ (3.70) is the complex conjugate of the up-mode radial function $X_{\omega\ell}^{\mathrm{up}}$ (3.66) then, from (5.31), we have immediately

$$\Im\left[\frac{X_{\omega\ell}^{\mathrm{down}*}(r)}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{X_{\omega\ell}^{\mathrm{down}}(r)}{r}\right)\right] \sim -\frac{\omega\left|B_{\omega\ell}^{\mathrm{up}}\right|^2}{r^2}. \tag{5.32}$$

We will also need to calculate the term

$$\left|\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{X_{\omega\ell}(r)}{r}\right)\right|^2,$$

for the asymptotic form of the up- and down-modes far from the black hole. Beginning with the asymptotic form of the up-modes (3.66) at infinity, we have

$$\begin{aligned}
\left|\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{X_{\omega\ell}^{\mathrm{up}}(r)}{r}\right)\right|^2 &= \left|\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{B_{\omega\ell}^{\mathrm{up}}\,\mathrm{e}^{\mathrm{i}\omega r_*}}{r}\right)\right|^2 \\
&= \left|\frac{r\frac{\mathrm{d}r_*}{\mathrm{d}r}\frac{\mathrm{d}}{\mathrm{d}r_*}\left(B_{\omega\ell}^{\mathrm{up}}\,\mathrm{e}^{\mathrm{i}\omega r_*}\right)-B_{\omega\ell}^{\mathrm{up}}\,\mathrm{e}^{\mathrm{i}\omega r_*}}{r^2}\right|^2 \\
&= \left|\frac{f(r)r\,\mathrm{i}\omega\left(B_{\omega\ell}^{\mathrm{up}}\,\mathrm{e}^{\mathrm{i}\omega r_*}\right)-B_{\omega\ell}^{\mathrm{up}}\,\mathrm{e}^{\mathrm{i}\omega r_*}}{r^2}\right|^2 \\
&= \left|\frac{B_{\omega\ell}^{\mathrm{up}}\,\mathrm{e}^{\mathrm{i}\omega r_*}}{r^2}\left(\frac{\mathrm{i}\omega r}{f(r)}-1\right)\right|^2 \\
&\sim \left|\frac{\mathrm{i}\omega B_{\omega\ell}^{\mathrm{up}}\,\mathrm{e}^{\mathrm{i}\omega r_*}}{r}\right|^2.
\end{aligned} \tag{5.33}$$

We therefore obtain

$$\left|\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{X_{\omega\ell}^{\mathrm{up}}(r)}{r}\right)\right|^2 \sim \frac{\omega^2\left|B_{\omega\ell}^{\mathrm{up}}\right|^2}{r^2}. \tag{5.34}$$

Since the down-mode radial function $X^{\text{down}}_{\omega\ell}$ (3.70) is the complex conjugate of the up-mode radial function $X^{\text{up}}_{\omega\ell}$ (3.66) and the l.h.s of (5.34) is an absolute value, we have immediately

$$\left|\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{X^{\text{down}}_{\omega\ell}(r)}{r}\right)\right|^2 \sim \frac{\omega^2 |B^{\text{up}}_{\omega\ell}|^2}{r^2}. \tag{5.35}$$

Near-infinity up-mode contributions to observables

We would like to evaluate the up-mode contributions to the scalar condensate $\mathcal{SC}$, the current J^μ and the SET $T_{\mu\nu}$ far from the black hole. Throughout this section, we will use the expression (3.128) for the normalisation constants $\mathrm{N}^{\text{up}}_\omega$ of the up-modes.

We denote the up-mode contributions to the scalar condensate SC (5.1) as $sc^{\text{up}}_{\omega\ell}$. Then, using the asymptotic form of the up-modes (3.66) at infinity, we have

$$\begin{aligned} sc^{\text{up}}_{\omega\ell} &\sim \frac{2\ell+1}{4\pi r^2}\frac{\omega}{4\pi|\widetilde{\omega}|}\,|B^{\text{up}}_{\omega\ell}|^2 \\ &= \frac{1}{16\pi^2 r^2}\frac{1}{|\widetilde{\omega}|}\,(2\ell+1)\,|B^{\text{up}}_{\omega\ell}|^2. \end{aligned} \tag{5.36}$$

We denote the up-mode contributions to the current J^μ (5.5) as $j^{\mu,\text{up}}_{\omega\ell}$. The only nonzero mode contributions to the components of the current are $j^t_{\omega\ell}$ (5.7a) and $j^r_{\omega\ell}$ (5.7b). Using the asymptotic form of the up-modes (3.66) at infinity, we have for $j^{t,\text{up}}_{\omega\ell}$

$$\begin{aligned} j^{t,\text{up}}_{\omega\ell} &\sim -\frac{q\,(2\ell+1)}{16\pi^2 r^2}\frac{\omega}{4\pi|\widetilde{\omega}|}\,|B^{\text{up}}_{\omega\ell}|^2 \\ &= -\frac{q}{64\pi^3 r^2}\frac{\omega}{|\widetilde{\omega}|}\,(2\ell+1)\,|B^{\text{up}}_{\omega\ell}|^2. \end{aligned} \tag{5.37}$$

Using (5.31), we have for $j^{r,\text{up}}_{\omega\ell}$

$$\begin{aligned} j^{r,\text{up}}_{\omega\ell} &\sim -\frac{q\,(2\ell+1)}{16\pi^2}\frac{1}{4\pi|\widetilde{\omega}|}\frac{\omega|B^{\text{up}}_{\omega\ell}|^2}{r^2} \\ &= -\frac{q}{64\pi^3 r^2}\frac{\omega}{|\widetilde{\omega}|}\,(2\ell+1)\,|B^{\text{up}}_{\omega\ell}|^2. \end{aligned} \tag{5.38}$$

Then, we can write the up-mode contribution $j^{\mu,\text{up}}_{\omega\ell}$ to the current J^μ as

$$j^{\mu,\text{up}}_{\omega\ell} \sim -\frac{q}{64\pi^3 r^2}\frac{\omega}{|\widetilde{\omega}|}\,(2\ell+1)\,|B^{\text{up}}_{\omega\ell}|^2 \begin{pmatrix}1 & 1 & 0 & 0\end{pmatrix}^{\intercal}. \tag{5.39}$$

We denote the up-mode contributions to the SET $T_{\mu\nu}$ (5.14) as $t^{\text{up}}_{\mu\nu,\omega\ell}$. The only nonzero mode contributions to the components of the stress-energy tensor are $t_{tt,\omega\ell}$ (5.15a), $t_{tr,\omega\ell}$ (5.15b), $t_{rr,\omega\ell}$ (5.15c), $t_{\theta\theta,\omega\ell}$ (5.15d) and $t_{\varphi\varphi,\omega\ell}$ (5.15e). Using the asymptotic form of the up-modes (3.66) at infinity and (5.34), we have for $t^{\text{up}}_{tt,\omega\ell}$

$$\begin{aligned} t^{\text{up}}_{tt,\omega\ell} &\sim \frac{2\ell+1}{8\pi}\frac{1}{4\pi|\widetilde{\omega}|}\left[\frac{1}{r^2}\,\omega^2|B^{\text{up}}_{\omega\ell}|^2 + \frac{1}{r^2}\,\omega^2|B^{\text{up}}_{\omega\ell}|^2\right] \\ &= \frac{1}{16\pi^2 r^2}\frac{\omega^2}{|\widetilde{\omega}|}\,(2\ell+1)\,|B^{\text{up}}_{\omega\ell}|^2. \end{aligned} \tag{5.40}$$

Using (5.31), we have for $t^{\text{up}}_{tr,\omega\ell}$

$$
\begin{aligned}
t^{\text{up}}_{tr,\omega\ell} &\sim -\frac{2\ell+1}{4\pi}\,\frac{\omega}{4\pi|\widetilde{\omega}|}\,\frac{\omega\big|B^{\text{up}}_{\omega\ell}\big|^2}{r^2} \\
&= -\frac{1}{16\pi^2 r^2}\,\frac{\omega^2}{|\widetilde{\omega}|}\,(2\ell+1)\,\big|B^{\text{up}}_{\omega\ell}\big|^2 .
\end{aligned} \tag{5.41}
$$

Using the asymptotic form of the up-modes (3.66) at infinity and (5.34), we have for $t^{\text{up}}_{rr,\omega\ell}$

$$
\begin{aligned}
t^{\text{up}}_{rr,\omega\ell} &\sim \frac{2\ell+1}{8\pi}\,\frac{1}{4\pi|\widetilde{\omega}|}\left[\frac{1}{r^2}\,\omega^2\big|B^{\text{up}}_{\omega\ell}\big|^2 + \frac{1}{r^2}\,\omega^2\big|B^{\text{up}}_{\omega\ell}\big|^2\right] \\
&= \frac{1}{16\pi^2 r^2}\,\frac{\omega^2}{|\widetilde{\omega}|}\,(2\ell+1)\,\big|B^{\text{up}}_{\omega\ell}\big|^2 .
\end{aligned} \tag{5.42}
$$

Using the asymptotic form of the up-modes (3.66) at infinity and (5.34), we have for $t^{\text{up}}_{\theta\theta,\omega\ell}$

$$
\begin{aligned}
t^{\text{up}}_{\theta\theta,\omega\ell} &\sim \frac{2\ell+1}{8\pi}\,\frac{1}{4\pi|\widetilde{\omega}|}\left[\frac{1}{r^2}\,\omega^2\big|B^{\text{up}}_{\omega\ell}\big|^2 + \mathcal{O}\big(r^{-4}\big) - \frac{1}{r^2}\,\omega^2\big|B^{\text{up}}_{\omega\ell}\big|^2\right] \\
&= \mathcal{O}\big(r^{-4}\big) ,
\end{aligned} \tag{5.43}
$$

where we have used that $\mathcal{O}\big(r^{-4}\big)$ is subleading order in r compared to the leading order $\mathcal{O}\big(r^{-2}\big)$ terms in the other nonzero up-mode contributions to the SET. We have for $t^{\text{up}}_{\varphi\varphi,\omega\ell}$

$$
\begin{aligned}
t^{\text{up}}_{\varphi\varphi,\omega\ell} &\sim t^{\text{up}}_{\theta\theta,\omega\ell}\sin^2\theta \\
&= \mathcal{O}\big(r^{-4}\big) ,
\end{aligned} \tag{5.44}
$$

which is similarly subleading order in r. Then, we can write the up-mode contribution $t^{\text{up}}_{\mu\nu,\omega\ell}$ to the stress-energy tensor $T_{\mu\nu}$ as

$$
t^{\text{up}}_{\mu\nu,\omega\ell} \sim \frac{1}{16\pi^2 r^2}\,\frac{\omega^2}{|\widetilde{\omega}|}\,(2\ell+1)\,\big|B^{\text{up}}_{\omega\ell}\big|^2
\begin{pmatrix}
1 & -1 & 0 & 0 \\
-1 & 1 & 0 & 0 \\
0 & 0 & \mathcal{O}\big(r^{-2}\big) & 0 \\
0 & 0 & 0 & \mathcal{O}\big(r^{-2}\big)
\end{pmatrix} . \tag{5.45}
$$

In §5.2.4, we will consider the expectation values $\langle \hat{T}^{\mu}_{\nu} \rangle$. Then, acting with the inverse metric (3.16) on (5.45), the up-mode contribution $t^{\mu,\text{up}}_{\nu,\omega\ell}$ to the SET T^{μ}_{ν} is given by

$$
t^{\mu,\text{up}}_{\nu,\omega\ell} \sim \frac{1}{16\pi^2 r^2}\,\frac{\omega^2}{|\widetilde{\omega}|}\,(2\ell+1)\,\big|B^{\text{up}}_{\omega\ell}\big|^2
\begin{pmatrix}
-1 & 1 & 0 & 0 \\
-1 & 1 & 0 & 0 \\
0 & 0 & \mathcal{O}\big(r^{-2}\big) & 0 \\
0 & 0 & 0 & \mathcal{O}\big(r^{-2}\big)
\end{pmatrix} . \tag{5.46}
$$

Near-infinity down-mode contributions to observables

We would like to evaluate the down-mode contributions to the scalar condensate $\mathcal{SC}$, the current J^μ and the SET $T_{\mu\nu}$ far from the black hole. Throughout this section, we will use the expression (3.144) for the normalisation constants $\mathrm{N}_\omega^{\mathrm{down}}$ of the down-modes.

Furthermore, since the down-mode radial function $X_{\omega\ell}^{\mathrm{down}}$ (3.70) is the complex conjugate of the up-mode radial function $X_{\omega\ell}^{\mathrm{up}}$ (3.66), where down-mode contributions only involve absolute values we may simply give the corresponding up-mode contribution that has already been calculated in §5.2.2.

We denote the down-mode contributions to the scalar condensate $\mathcal{SC}$ (5.1) as $sc_{\omega\ell}^{\mathrm{down}}$. Since the mode-contribution $sc_{\omega\ell}^{\mathrm{down}}$ only consists of absolute values, it is given immediately from the expression for $sc_{\omega\ell}^{\mathrm{up}}$ (5.36) as

$$sc_{\omega\ell}^{\mathrm{down}} \sim \frac{1}{16\pi^2 r^2}\frac{1}{|\widetilde{\omega}|}\left(2\ell+1\right)\left|B_{\omega\ell}^{\mathrm{up}}\right|^2. \tag{5.47}$$

This makes sense since we reasoned, in §5.1.1, that the scalar condensate should not be able to distinguish between 'past' and 'future' quantum states.

We denote the down-mode contributions to the current J^μ (5.5) as $j_{\omega\ell}^{\mu,\mathrm{down}}$. The only nonzero mode contributions to the components of the current are $j_{\omega\ell}^{t}$ (5.7a) and $j_{\omega\ell}^{r}$ (5.7b). Since the mode-contribution $j_{\omega\ell}^{t,\mathrm{down}}$ only consists of absolute values, it is given immediately from the expression for $j_{\omega\ell}^{t,\mathrm{up}}$ (5.37) as

$$j_{\omega\ell}^{t,\mathrm{down}} \sim -\frac{q}{64\pi^3 r^2}\frac{\omega}{|\widetilde{\omega}|}\left(2\ell+1\right)\left|B_{\omega\ell}^{\mathrm{up}}\right|^2. \tag{5.48}$$

Using (5.32), we have for $j_{\omega\ell}^{r,\mathrm{down}}$

$$\begin{aligned} j_{\omega\ell}^{r,\mathrm{down}} &\sim -\frac{q\left(2\ell+1\right)}{16\pi^2}\frac{1}{4\pi|\widetilde{\omega}|}\left(-\frac{\omega\left|B_{\omega\ell}^{\mathrm{up}}\right|^2}{r^2}\right) \\ &= \frac{q}{64\pi^3 r^2}\frac{\omega}{|\widetilde{\omega}|}\left(2\ell+1\right)\left|B_{\omega\ell}^{\mathrm{up}}\right|^2, \end{aligned} \tag{5.49}$$

which has the opposite sign to the expression for $j_{\omega\ell}^{r,\mathrm{up}}$ (5.38). Then, we can write the down-mode contribution $j_{\omega\ell}^{\mu,\mathrm{down}}$ to the current J^μ as

$$j_{\omega\ell}^{\mu,\mathrm{down}} \sim \frac{q}{64\pi^3 r^2}\frac{\omega}{|\widetilde{\omega}|}\left(2\ell+1\right)\left|B_{\omega\ell}^{\mathrm{up}}\right|^2\begin{pmatrix}-1 & 1 & 0 & 0\end{pmatrix}^{\intercal}. \tag{5.50}$$

We denote the down-mode contributions to the stress-energy tensor $T_{\mu\nu}$ (5.14) as $t_{\mu\nu,\omega\ell}^{\mathrm{down}}$. The only nonzero mode contributions to the components of the SET are $t_{tt,\omega\ell}^{\mathrm{down}}$ (5.15a), $t_{tr,\omega\ell}^{\mathrm{down}}$ (5.15b), $t_{rr,\omega\ell}^{\mathrm{down}}$ (5.15c), $t_{\theta\theta,\omega\ell}^{\mathrm{down}}$ (5.15d) and $t_{\varphi\varphi,\omega\ell}^{\mathrm{down}}$ (5.15e).

Since the mode-contributions $t_{tt,\omega\ell}^{\mathrm{down}}$, $t_{rr,\omega\ell}^{\mathrm{down}}$, $t_{\theta\theta,\omega\ell}^{\mathrm{down}}$ and $t_{\varphi\varphi,\omega\ell}^{\mathrm{down}}$ to the diagonal elements of the stress-energy tensor only consist of absolute values, they are given immediately from the corresponding expressions for the up-mode contributions in §5.2.2; then, we have

$$
\begin{aligned}
t^{\rm down}_{tt,\omega\ell} &\sim \frac{1}{16\pi^2 r^2}\frac{\omega^2}{|\widetilde{\omega}|}\,(2\ell+1)\left|B^{\rm up}_{\omega\ell}\right|^2, \\
t^{\rm down}_{rr,\omega\ell} &\sim \frac{1}{16\pi^2 r^2}\frac{\omega^2}{|\widetilde{\omega}|}\,(2\ell+1)\left|B^{\rm up}_{\omega\ell}\right|^2, \\
t^{\rm down}_{\theta\theta,\omega\ell} &\sim \mathcal{O}\left(r^{-4}\right), \\
t^{\rm down}_{\varphi\varphi,\omega\ell} &\sim \mathcal{O}\left(r^{-4}\right).
\end{aligned} \tag{5.51}
$$

Using (5.32), we have for $t^{\rm down}_{tr,\omega\ell}$

$$
\begin{aligned}
t^{\rm down}_{tr,\omega\ell} &\sim -\frac{2\ell+1}{4\pi}\frac{\omega}{4\pi|\widetilde{\omega}|}\left(-\frac{\omega\left|B^{\rm up}_{\omega\ell}\right|^2}{r^2}\right) \\
&= \frac{1}{16\pi^2 r^2}\frac{\omega^2}{|\widetilde{\omega}|}\,(2\ell+1)\left|B^{\rm up}_{\omega\ell}\right|^2,
\end{aligned} \tag{5.52}
$$

which has the opposite sign to the expression for $t^{\rm up}_{tr,\omega\ell}$ (5.41). Then, we can write the down-mode contribution $t^{\rm down}_{\mu\nu,\omega\ell}$ to the stress-energy tensor $T_{\mu\nu}$ as

$$
t^{\rm down}_{\mu\nu,\omega\ell} \sim \frac{1}{16\pi^2 r^2}\frac{\omega^2}{|\widetilde{\omega}|}\,(2\ell+1)\left|B^{\rm up}_{\omega\ell}\right|^2 \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & \mathcal{O}\left(r^{-2}\right) & 0 \\ 0 & 0 & 0 & \mathcal{O}\left(r^{-2}\right) \end{pmatrix}. \tag{5.53}
$$

In §5.2.4, we will consider the expectation values $\langle \hat{T}^{\mu}_{\nu}\rangle$. Then, acting with the inverse metric (3.16) on (5.53), the down-mode contribution $t^{\mu,\rm down}_{\nu,\omega\ell}$ to the SET T^{μ}_{ν} is given by

$$
t^{\mu,\rm down}_{\nu,\omega\ell} \sim \frac{1}{16\pi^2 r^2}\frac{\omega^2}{|\widetilde{\omega}|}\,(2\ell+1)\left|B^{\rm up}_{\omega\ell}\right|^2 \begin{pmatrix} -1 & -1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & \mathcal{O}\left(r^{-2}\right) & 0 \\ 0 & 0 & 0 & \mathcal{O}\left(r^{-2}\right) \end{pmatrix}. \tag{5.54}
$$

5.2.3 The asymptotic regime near the event horizon

We would like to calculate the asymptotic mode contributions to observables near the event horizon of the black hole. We will only consider leading order contributions which, since the metric function $f(r)$ diverges on the horizon, correspond to terms with the highest power of $f(r)$ in their denominator.

Preliminaries

Several expressions for the mode contributions to the components of the current and the SET contain like terms; it will be useful to evaluate these terms now in order to simplify the process of evaluating the mode contributions later on. We can begin by calculating

$$
\Im\left[\frac{X^*_{\omega\ell}(r)}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{X_{\omega\ell}(r)}{r}\right)\right],
$$

for the asymptotic form of the in- and out-modes near the event horizon of the black hole. Beginning with the asymptotic form of the in-modes (3.64) near the horizon, we have

$$\begin{aligned}
\Im\left[\frac{X^{\text{in}*}_{\omega\ell}(r)}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{X^{\text{in}}_{\omega\ell}(r)}{r}\right)\right] &= \Im\left[\frac{X^{\text{in}*}_{\omega\ell}(r)}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{B^{\text{in}}_{\omega\ell}\,\mathrm{e}^{-\mathrm{i}\widetilde{\omega}r_*}}{r}\right)\right] \\
&= \Im\left[\frac{X^{\text{in}*}_{\omega\ell}(r)}{r}\left(\frac{r\frac{\mathrm{d}r_*}{\mathrm{d}r}\frac{\mathrm{d}}{\mathrm{d}r_*}\left(B^{\text{in}}_{\omega\ell}\,\mathrm{e}^{-\mathrm{i}\widetilde{\omega}r_*}\right) - B^{\text{in}}_{\omega\ell}\,\mathrm{e}^{-\mathrm{i}\widetilde{\omega}r_*}}{r^2}\right)\right] \\
&= \Im\left[\frac{X^{\text{in}*}_{\omega\ell}(r)}{r}\left(\frac{f(r)^{-1}\,r\left(-\mathrm{i}\widetilde{\omega}B^{\text{in}}_{\omega\ell}\,\mathrm{e}^{-\mathrm{i}\widetilde{\omega}r_*}\right) - B^{\text{in}}_{\omega\ell}\,\mathrm{e}^{-\mathrm{i}\widetilde{\omega}r_*}}{r^2}\right)\right] \\
&= \Im\left[\frac{B^{\text{in}*}_{\omega\ell}\,\mathrm{e}^{\mathrm{i}\widetilde{\omega}r_*}}{r}\frac{B^{\text{in}}_{\omega\ell}\,\mathrm{e}^{-\mathrm{i}\widetilde{\omega}r_*}}{r^2}\left(-\frac{\mathrm{i}\widetilde{\omega}r}{f(r)}-1\right)\right] \\
&= -\Im\left[\frac{\left|B^{\text{in}}_{\omega\ell}\right|^2}{r^3}\left(\frac{\mathrm{i}\widetilde{\omega}r}{f(r)}+1\right)\right] \\
&\sim -\frac{\widetilde{\omega}\left|B^{\text{in}}_{\omega\ell}\right|^2}{f(r)\,r^2}.
\end{aligned} \tag{5.55}$$

Since the out-mode radial function $X^{\text{out}}_{\omega\ell}$ (3.68) is the complex conjugate of the in-mode radial function $X^{\text{in}}_{\omega\ell}$ (3.64) then, from (5.55), we have immediately

$$\Im\left[\frac{X^{\text{out}*}_{\omega\ell}(r)}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{X^{\text{out}}_{\omega\ell}(r)}{r}\right)\right] \sim \frac{\widetilde{\omega}\left|B^{\text{in}}_{\omega\ell}\right|^2}{f(r)\,r^2}. \tag{5.56}$$

We will also need to calculate the term

$$\left|\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{X_{\omega\ell}(r)}{r}\right)\right|^2,$$

for the asymptotic form of the in- and out-modes near the event horizon of the black hole. Beginning with the asymptotic form of the in-modes (3.64) near the horizon, we have

$$\begin{aligned}
\left|\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{X^{\text{in}}_{\omega\ell}(r)}{r}\right)\right|^2 &= \left|\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{B^{\text{in}}_{\omega\ell}\,\mathrm{e}^{-\mathrm{i}\widetilde{\omega}r_*}}{r}\right)\right|^2 \\
&= \left|\frac{r\frac{\mathrm{d}r_*}{\mathrm{d}r}\frac{\mathrm{d}}{\mathrm{d}r_*}\left(B^{\text{in}}_{\omega\ell}\,\mathrm{e}^{-\mathrm{i}\widetilde{\omega}r_*}\right) - B^{\text{in}}_{\omega\ell}\,\mathrm{e}^{-\mathrm{i}\widetilde{\omega}r_*}}{r^2}\right|^2 \\
&= \left|\frac{-f(r)^{-1}\,r\,\mathrm{i}\widetilde{\omega}\left(B^{\text{in}}_{\omega\ell}\,\mathrm{e}^{-\mathrm{i}\widetilde{\omega}r_*}\right) - B^{\text{in}}_{\omega\ell}\,\mathrm{e}^{-\mathrm{i}\widetilde{\omega}r_*}}{r^2}\right|^2 \\
&= \left|-\frac{B^{\text{in}}_{\omega\ell}\,\mathrm{e}^{-\mathrm{i}\widetilde{\omega}r_*}}{r^2}\left(\frac{\mathrm{i}\widetilde{\omega}r}{f(r)}+1\right)\right|^2 \\
&\sim \left|-\frac{\mathrm{i}\widetilde{\omega}B^{\text{in}}_{\omega\ell}\,\mathrm{e}^{-\mathrm{i}\widetilde{\omega}r_*}}{f(r)\,r}\right|^2.
\end{aligned} \tag{5.57}$$

We therefore obtain

$$\left| \frac{\mathrm{d}}{\mathrm{d}r} \left(\frac{X_{\omega\ell}^{\mathrm{in}}(r)}{r} \right) \right|^2 \sim \frac{\widetilde{\omega}^2 \left| B_{\omega\ell}^{\mathrm{in}} \right|^2}{f(r)^2 \, r^2}. \tag{5.58}$$

Since the down-mode radial function $X_{\omega\ell}^{\mathrm{down}}$ (3.70) is the complex conjugate of the up-mode radial function $X_{\omega\ell}^{\mathrm{up}}$ (3.66) and the l.h.s of (5.58) is an absolute value, we have immediately

$$\left| \frac{\mathrm{d}}{\mathrm{d}r} \left(\frac{X_{\omega\ell}^{\mathrm{out}}(r)}{r} \right) \right|^2 \sim \frac{\widetilde{\omega}^2 \left| B_{\omega\ell}^{\mathrm{in}} \right|^2}{f(r)^2 \, r^2}. \tag{5.59}$$

Near-horizon in-mode contributions to observables

We would like to evaluate the in-mode contributions to the scalar condensate $\mathcal{SC}$, the current J^μ and the stress-energy tensor $T_{\mu\nu}$ near the horizon. Throughout this section, we will use the expression (3.103) for the normalisation constants $\mathrm{N}_\omega^{\mathrm{in}}$ of the in-modes.

We denote the in-mode contributions to the scalar condensate $\mathcal{SC}$ (5.1) as $sc_{\omega\ell}^{\mathrm{in}}$. Then, using the asymptotic form of the in-modes (3.64) near the horizon, we have for $sc_{\omega\ell}^{\mathrm{in}}$

$$\begin{aligned} sc_{\omega\ell}^{\mathrm{in}} &\sim \frac{2\ell+1}{4\pi r^2} \frac{1}{4\pi |\omega|} \left| B_{\omega\ell}^{\mathrm{in}} \right|^2 \\ &= \frac{1}{16\pi^2 r^2} \frac{1}{|\omega|} \left(2\ell+1 \right) \left| B_{\omega\ell}^{\mathrm{in}} \right|^2. \end{aligned} \tag{5.60}$$

We denote the in-mode contributions to the current J^μ (5.5) as $j_{\omega\ell}^{\mu,\mathrm{in}}$. The only nonzero mode contributions to the components of the current are $j_{\omega\ell}^t$ (5.7a) and $j_{\omega\ell}^r$ (5.7b). Using the asymptotic form of the in-modes (3.64) near the horizon, we have for $j_{\omega\ell}^{t,\mathrm{in}}$

$$\begin{aligned} j_{\omega\ell}^{t,\mathrm{in}} &\sim -\frac{q\left(2\ell+1\right)}{16\pi^2 f(r) r^2} \left(\omega - \frac{qQ}{r_+} \right) \frac{1}{4\pi |\omega|} \left| B_{\omega\ell}^{\mathrm{in}} \right|^2 \\ &= -\frac{q}{64\pi^3 r^2} \frac{\widetilde{\omega}}{|\omega|} \left(2\ell+1 \right) \left| B_{\omega\ell}^{\mathrm{in}} \right|^2 \frac{1}{f(r)}. \end{aligned} \tag{5.61}$$

Using (5.55), we have for $j_{\omega\ell}^{r,\mathrm{in}}$

$$\begin{aligned} j_{\omega\ell}^{r,\mathrm{in}} &\sim -\frac{q f(r)(2\ell+1)}{16\pi^2} \frac{1}{4\pi |\omega|} \left(-\frac{\widetilde{\omega} \left| B_{\omega\ell}^{\mathrm{in}} \right|^2}{f(r) r^2} \right) \\ &= \frac{q}{64\pi^3 r^2} \frac{\widetilde{\omega}}{|\omega|} \left(2\ell+1 \right) \left| B_{\omega\ell}^{\mathrm{in}} \right|^2. \end{aligned} \tag{5.62}$$

Then, we can write the in-mode contribution $j_{\omega\ell}^{\mu,\mathrm{in}}$ to the current J^μ as

$$j_{\omega\ell}^{\mu,\mathrm{in}} \sim \frac{q}{64\pi^3 r^2} \frac{\widetilde{\omega}}{|\omega|} \left(2\ell+1 \right) \left| B_{\omega\ell}^{\mathrm{in}} \right|^2 \begin{pmatrix} -f(r)^{-1} & 1 & 0 & 0 \end{pmatrix}^{\intercal}. \tag{5.63}$$

We denote the in-mode contributions to the SET $T_{\mu\nu}$ as $t_{\mu\nu,\omega\ell}^{\mathrm{in}}$. The only nonzero mode contributions to the components of the stress-energy tensor are $t_{tt,\omega\ell}$ (5.15a), $t_{tr,\omega\ell}$ (5.15b), $t_{rr,\omega\ell}$ (5.15c), $t_{\theta\theta,\omega\ell}$ (5.15d) and $t_{\varphi\varphi,\omega\ell}$ (5.15e). Using the asymptotic form of the in-modes (3.64) near the horizon and (5.58), we have for $t_{tt,\omega\ell}^{\mathrm{in}}$

$$
\begin{aligned}
t^{\text{in}}_{tt,\omega\ell} &\sim \frac{2\ell+1}{8\pi}\frac{1}{4\pi|\omega|}\left[\frac{1}{r^2}\left(\omega-\frac{qQ}{r_+}\right)^2\left|B^{\text{in}}_{\omega\ell}\right|^2+f(r)^2\,\frac{\widetilde{\omega}^2\left|B^{\text{in}}_{\omega\ell}\right|^2}{f(r)^2r^2}\right] \\
&= \frac{1}{16\pi^2r^2}\frac{\widetilde{\omega}^2}{|\omega|}\left(2\ell+1\right)\left|B^{\text{in}}_{\omega\ell}\right|^2. \qquad (5.64)
\end{aligned}
$$

Using (5.55), we have for $t^{\text{in}}_{tr,\omega\ell}$

$$
\begin{aligned}
t^{\text{in}}_{tr,\omega\ell} &\sim -\frac{2\ell+1}{4\pi}\left(\omega-\frac{qQ}{r_+}\right)\frac{1}{4\pi|\omega|}\left(-\frac{\widetilde{\omega}\left|B^{\text{in}}_{\omega\ell}\right|^2}{f(r)r^2}\right) \\
&= \frac{1}{16\pi^2r^2}\frac{\widetilde{\omega}^2}{|\omega|}\left(2\ell+1\right)\left|B^{\text{in}}_{\omega\ell}\right|^2\frac{1}{f(r)}. \qquad (5.65)
\end{aligned}
$$

Using the asymptotic form of the in-modes (3.64) and (5.58), we have for $t^{\text{in}}_{rr,\omega\ell}$

$$
\begin{aligned}
t^{\text{in}}_{rr,\omega\ell} &\sim \frac{2\ell+1}{8\pi}\frac{1}{4\pi|\omega|}\left[\frac{1}{f(r)^2r^2}\left(\omega-\frac{qQ}{r_+}\right)^2\left|B^{\text{in}}_{\omega\ell}\right|^2+\mathcal{O}\Big(f(r)^{-1}\Big)+\frac{\widetilde{\omega}^2\left|B^{\text{in}}_{\omega\ell}\right|^2}{f(r)^2r^2}\right] \\
&= \frac{1}{16\pi^2r^2}\frac{\widetilde{\omega}^2}{|\omega|}\left(2\ell+1\right)\left|B^{\text{in}}_{\omega\ell}\right|^2\frac{1}{f(r)^2}. \qquad (5.66)
\end{aligned}
$$

Using the asymptotic form of the in-modes (3.64) and (5.58), we have for $t^{\text{in}}_{\theta\theta,\omega\ell}$

$$
\begin{aligned}
t^{\text{in}}_{\theta\theta,\omega\ell} &\sim \frac{2\ell+1}{8\pi}\frac{1}{4\pi|\omega|}\left[\frac{1}{f(r)}\left(\omega-\frac{qQ}{r_+}\right)^2\left|B^{\text{in}}_{\omega\ell}\right|^2-f(r)r^2\,\frac{\widetilde{\omega}^2\left|B^{\text{in}}_{\omega\ell}\right|^2}{f(r)^2r^2}+\mathcal{O}(1)\right] \\
&= \mathcal{O}(1)\,, \qquad (5.67)
\end{aligned}
$$

where we have used the fact that $\mathcal{O}(1)$ is subleading order in $f(r)$. We have, for the up-mode contribution $t^{\text{in}}_{\varphi\varphi,\omega\ell}$ to the component $T_{\varphi\varphi}$, the expression

$$
\begin{aligned}
t^{\text{in}}_{\varphi\varphi,\omega\ell} &\sim t^{\text{in}}_{\theta\theta,\omega\ell}\sin^2\theta \\
&= \mathcal{O}(1)\,, \qquad (5.68)
\end{aligned}
$$

which is similarly subleading order in $f(r)$. Then, we can write the in-mode contribution $t^{\text{in}}_{\mu\nu,\omega\ell}$ to the stress-energy tensor $T_{\mu\nu}$ as

$$
t^{\text{in}}_{\mu\nu,\omega\ell} \sim \frac{1}{16\pi^2r^2}\frac{\widetilde{\omega}^2}{|\omega|}\left(2\ell+1\right)\left|B^{\text{in}}_{\omega\ell}\right|^2\begin{pmatrix} 1 & f(r)^{-1} & 0 & 0 \\ f(r)^{-1} & f(r)^{-2} & 0 & 0 \\ 0 & 0 & \mathcal{O}(1) & 0 \\ 0 & 0 & 0 & \mathcal{O}(1)\end{pmatrix}. \qquad (5.69)
$$

In §5.2.4, we will consider the expectation values $\langle\hat{T}^{\mu}_{\nu}\rangle$. Then, acting with the inverse metric (3.16) on (5.69), the in-mode contribution $t^{\mu,\text{in}}_{\nu,\omega\ell}$ to the SET T^{μ}_{ν} is given by

$$t^{\mu,\text{in}}_{\nu,\omega\ell} \sim \frac{1}{16\pi^2 r^2}\frac{\widetilde{\omega}^2}{|\omega|}(2\ell+1)\left|B^{\text{in}}_{\omega\ell}\right|^2 \begin{pmatrix} -f(r)^{-1} & -f(r)^{-2} & 0 & 0 \\ 1 & f(r)^{-1} & 0 & 0 \\ 0 & 0 & \mathcal{O}(1) & 0 \\ 0 & 0 & 0 & \mathcal{O}(1) \end{pmatrix}. \tag{5.70}$$

Near-horizon out-mode contributions to observables

We would like to evaluate the out-mode contributions to the scalar condensate $\mathcal{SC}$, the current J^μ and the stress-energy tensor $T_{\mu\nu}$ near the horizon. Throughout this section, we will use the expression (3.136) for the normalisation constants $\text{N}^{\text{out}}_{\omega}$ of the out-modes.

Furthermore, since the out-mode radial function $X^{\text{out}}_{\omega\ell}(r)$ (3.139) is the complex conjugate of the in-mode radial function $X^{\text{in}}_{\omega\ell}(r)$ (3.106), where out-mode contributions only involve absolute values we may simply give the corresponding in-mode contribution that has already been calculated in §5.2.3.

We denote the out-mode contributions to the scalar condensate $\mathcal{SC}$ (5.1) as $sc^{\text{out}}_{\omega\ell}$. Since the mode-contribution $sc^{\text{out}}_{\omega\ell}$ only consists of absolute values, it is given immediately from the expression for $sc^{\text{in}}_{\omega\ell}$ (5.60) as

$$sc^{\text{out}}_{\omega\ell} \sim \frac{1}{16\pi^2 r^2}\frac{1}{|\omega|}(2\ell+1)\left|B^{\text{in}}_{\omega\ell}\right|^2. \tag{5.71}$$

This makes sense since we reasoned, in 5.1.1, that the scalar condensate should not be able to distinguish between 'past' and 'future' quantum states.

We denote the out-mode contributions to the current J^μ (5.5) as $j^{\mu,\text{out}}_{\omega\ell}$. The only nonzero mode contributions to the components of the current are $j^t_{\omega\ell}$ (5.7a) and $j^r_{\omega\ell}$ (5.7b). Since the mode-contribution $j^{t,\text{out}}_{\omega\ell}$ only consists of absolute values, it is given immediately from the expression for $j^{t,\text{in}}_{\omega\ell}$ (5.61) as

$$j^{t,\text{in}}_{\omega\ell} \sim -\frac{q}{64\pi^3 r^2}\frac{\widetilde{\omega}}{|\omega|}(2\ell+1)\left|B^{\text{in}}_{\omega\ell}\right|^2\frac{1}{f(r)}. \tag{5.72}$$

Using (5.56), we have for $j^{r,\text{out}}_{\omega\ell}$

$$\begin{aligned} j^{r,\text{out}}_{\omega\ell} &\sim -\frac{qf(r)(2\ell+1)}{16\pi^2}\frac{1}{4\pi|\omega|}\frac{\widetilde{\omega}\left|B^{\text{in}}_{\omega\ell}\right|^2}{f(r)r^2} \\ &= -\frac{q}{64\pi^3 r^2}\frac{\widetilde{\omega}}{|\omega|}(2\ell+1)\left|B^{\text{in}}_{\omega\ell}\right|^2. \end{aligned} \tag{5.73}$$

which has the opposite sign to the expression for $j^{r,\text{in}}_{\omega\ell}$ (5.62). Then, we can write the out-mode contribution $j^{\mu,\text{out}}_{\omega\ell}$ to the current J^μ as

$$j^{\mu,\text{out}}_{\omega\ell} \sim -\frac{q}{64\pi^3 r^2}\frac{\widetilde{\omega}}{|\omega|}(2\ell+1)\left|B^{\text{in}}_{\omega\ell}\right|^2\begin{pmatrix} f(r)^{-1} & 1 & 0 & 0\end{pmatrix}^{\mathsf{T}}. \tag{5.74}$$

We denote the out-mode contributions to the stress-energy tensor $T_{\mu\nu}$ (5.14) as $t^{\text{out}}_{\mu\nu,\omega\ell}$. The only nonzero mode contributions to the components of the SET are $t_{tt,\omega\ell}$ (5.15a), $t_{tr,\omega\ell}$ (5.15b), $t_{rr,\omega\ell}$ (5.15c), $t_{\theta\theta,\omega\ell}$ (5.15d) and $t_{\varphi\varphi,\omega\ell}$ (5.15e).

Since the mode-contributions $t_{tt,\omega\ell}$, $t_{rr,\omega\ell}$, $t_{\theta\theta,\omega\ell}$ and $t_{\varphi\varphi,\omega\ell}$ to the diagonal elements of the stress-energy tensor only consist of absolute values, they are given immediately from the corresponding expressions for the in-mode contributions in §5.2.3; then, we have

$$
\begin{aligned}
t^{\text{out}}_{tt,\omega\ell} &\sim \frac{1}{16\pi^2 r^2}\frac{\widetilde{\omega}^2}{|\omega|}\,(2\ell+1)\left|B^{\text{in}}_{\omega\ell}\right|^2, \\
t^{\text{out}}_{rr,\omega\ell} &\sim \frac{1}{16\pi^2 r^2}\frac{\widetilde{\omega}^2}{|\omega|}\,(2\ell+1)\left|B^{\text{in}}_{\omega\ell}\right|^2\frac{1}{f(r)^2}, \\
t^{\text{out}}_{\theta\theta,\omega\ell} &\sim \mathcal{O}(1)\,, \\
t^{\text{out}}_{\varphi\varphi,\omega\ell} &\sim \mathcal{O}(1)\,.
\end{aligned}
$$

Using (5.55), we have for $t^{\text{out}}_{tr,\omega\ell}$

$$
\begin{aligned}
t^{\text{out}}_{tr,\omega\ell} &\sim -\frac{2\ell+1}{4\pi}\left(\omega-\frac{qQ}{r_+}\right)\frac{1}{4\pi|\omega|}\frac{\widetilde{\omega}\left|B^{\text{in}}_{\omega\ell}\right|^2}{f(r)r^2} \\
&= -\frac{1}{16\pi^2 r^2}\frac{\widetilde{\omega}^2}{|\omega|}\,(2\ell+1)\left|B^{\text{in}}_{\omega\ell}\right|^2\frac{1}{f(r)},
\end{aligned}
\tag{5.75}
$$

which has the opposite sign to the expression for $t^{\text{in}}_{tr,\omega\ell}$ (5.65). Then, we can write the out-mode contribution $t^{\text{out}}_{\mu\nu,\omega\ell}$ to the stress-energy tensor $T_{\mu\nu}$ as

$$
t^{\text{out}}_{\mu\nu,\omega\ell} \sim \frac{1}{16\pi^2 r^2}\frac{\widetilde{\omega}^2}{|\omega|}\,(2\ell+1)\left|B^{\text{in}}_{\omega\ell}\right|^2
\begin{pmatrix}
1 & -f(r)^{-1} & 0 & 0 \\
-f(r)^{-1} & f(r)^{-2} & 0 & 0 \\
0 & 0 & \mathcal{O}(1) & 0 \\
0 & 0 & 0 & \mathcal{O}(1)
\end{pmatrix}.
\tag{5.76}
$$

In §5.2.4, we will consider the expectation values $\langle \hat{T}^{\mu}_{\nu}\rangle$. Then, acting with the inverse metric (3.16) on (5.76), the out-mode contribution $t^{\mu,\text{out}}_{\nu,\omega\ell}$ to the SET T^{μ}_{ν} is given by

$$
t^{\mu,\text{out}}_{\nu,\omega\ell} \sim \frac{1}{16\pi^2 r^2}\frac{\widetilde{\omega}^2}{|\omega|}\,(2\ell+1)\left|B^{\text{in}}_{\omega\ell}\right|^2
\begin{pmatrix}
-f(r)^{-1} & f(r)^{-2} & 0 & 0 \\
-1 & f(r)^{-1} & 0 & 0 \\
0 & 0 & \mathcal{O}(1) & 0 \\
0 & 0 & 0 & \mathcal{O}(1)
\end{pmatrix}.
\tag{5.77}
$$

5.2.4 Expressions for expectation values of observables

In Chapter 4, we defined several putative quantum states for a charged scalar field in Reissner-Nordström spacetime. We will now consider the differences in the expectation values of the observables $\widehat{\mathcal{SC}}$ (5.4), $\hat{J}^{\mu}$ (5.8) and $\hat{T}_{\mu\nu}$ (5.16) between separate quantum states as well the expectation values of the flux components $\langle \hat{J}^{r}\rangle$ and $\langle \hat{T}^{r}_{t}\rangle$ of the current and the SET respectively. Thus, it will be useful to write down the explicit form of the expectation value of a general quantum observable $\hat{O}$ in each of the states.

We can write down the expectation values of observables in the Boulware states, which are defined in §4.3, using the expression for the expectation value of the scalar condensate in the Schwarzschild Boulware state $|\mathrm{B_s}\rangle$ (2.53). Then, the expectation value of an observable $\hat{O}$ in the 'past' Boulware state $|\mathrm{B}^-\rangle$ (4.28) is

$$\langle \mathrm{B}^-|\,\hat{O}\,|\mathrm{B}^-\rangle = \frac{1}{2}\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\left\{\int_{-\infty}^{\infty} \mathrm{d}\omega\, o^{\mathrm{in}}_{\omega\ell m} + \int_{-\infty}^{\infty} \mathrm{d}\widetilde{\omega}\, o^{\mathrm{up}}_{\omega\ell m}\right\}. \tag{5.78}$$

The expectation value of an observable $\hat{O}$ in the 'future' Boulware state $|\mathrm{B}^+\rangle$ (4.38) is

$$\langle \mathrm{B}^+|\,\hat{O}\,|\mathrm{B}^+\rangle = \frac{1}{2}\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\left\{\int_{-\infty}^{\infty} \mathrm{d}\omega\, o^{\mathrm{out}}_{\omega\ell m} + \int_{-\infty}^{\infty} \mathrm{d}\widetilde{\omega}\, o^{\mathrm{down}}_{\omega\ell m}\right\}. \tag{5.79}$$

The expectation value of an observable $\hat{O}$ in the 'Boulware-like' state $|\mathrm{B}\rangle$ (4.48) is

$$\langle \mathrm{B}|\,\hat{O}\,|\mathrm{B}\rangle = \frac{1}{2}\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\left\{\int_{-\infty}^{\infty} \mathrm{d}\omega\, \left[o^{\mathrm{in}}_{\omega\ell m} + o^{\mathrm{up}}_{\omega\ell m}\right] - 2\int_{\min\left\{0,\frac{qQ}{r_+}\right\}}^{\max\left\{0,\frac{qQ}{r_+}\right\}} \mathrm{d}\omega\, o^{\mathrm{up}}_{\omega\ell m}\right\}. \tag{5.80}$$

We can write down the expectation values of observables in the Unruh states, which are defined in §4.4, using the expression for the expectation value of the scalar condensate in the Schwarzschild Unruh state $|\mathrm{U_s}\rangle$ (2.75). Then, the expectation value of an observable $\hat{O}$ in the 'past' Unruh state $|\mathrm{U}^-\rangle$ (4.94) is

$$\langle \mathrm{U}^-|\,\hat{O}\,|\mathrm{U}^-\rangle = \frac{1}{2}\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\left\{\int_{-\infty}^{\infty} \mathrm{d}\omega\, o^{\mathrm{in}}_{\omega\ell m} + \int_{-\infty}^{\infty} \mathrm{d}\widetilde{\omega}\, o^{\mathrm{up}}_{\omega\ell m}\coth\left|\frac{\pi\widetilde{\omega}}{\kappa}\right|\right\}. \tag{5.81}$$

The expectation value of an observable $\hat{O}$ in the 'future' Unruh state $|\mathrm{U}^+\rangle$ (4.123) is

$$\langle \mathrm{U}^+|\,\hat{O}\,|\mathrm{U}^+\rangle = \frac{1}{2}\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\left\{\int_{-\infty}^{\infty} \mathrm{d}\omega\, o^{\mathrm{out}}_{\omega\ell m} + \int_{-\infty}^{\infty} \mathrm{d}\widetilde{\omega}\, o^{\mathrm{down}}_{\omega\ell m}\coth\left|\frac{\pi\widetilde{\omega}}{\kappa}\right|\right\}. \tag{5.82}$$

We can write down the expectation values of observables in the Hartle-Hawking states, which are defined in §4.5, using the expression for the expectation value of the scalar condensate in the Schwarzschild Hartle-Hawking state $|\mathrm{H_s}\rangle$ (2.99). Then, the expectation value of an observable $\hat{O}$ in the 'past' CCH state $|\mathrm{CCH}^-\rangle$ (4.152) is

$$\langle \mathrm{CCH}^-|\,\hat{O}\,|\mathrm{CCH}^-\rangle = \frac{1}{2}\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\left\{\int_{-\infty}^{\infty} \mathrm{d}\omega\, o^{\mathrm{in}}_{\omega\ell m}\coth\left|\frac{\pi\omega}{\kappa}\right| + \int_{-\infty}^{\infty} \mathrm{d}\widetilde{\omega}\, o^{\mathrm{up}}_{\omega\ell m}\coth\left|\frac{\pi\widetilde{\omega}}{\kappa}\right|\right\}. \tag{5.83}$$

The expectation value of an observable $\hat{O}$ in the 'future' CCH state $|\mathrm{CCH}^+\rangle$ (4.181) is

$$\langle \mathrm{CCH}^+|\,\hat{O}\,|\mathrm{CCH}^+\rangle = \frac{1}{2}\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\left\{\int_{-\infty}^{\infty} \mathrm{d}\omega\, o^{\mathrm{out}}_{\omega\ell m}\coth\left|\frac{\pi\omega}{\kappa}\right| + \int_{-\infty}^{\infty} \mathrm{d}\widetilde{\omega}\, o^{\mathrm{down}}_{\omega\ell m}\coth\left|\frac{\pi\widetilde{\omega}}{\kappa}\right|\right\}. \tag{5.84}$$

The expectation value of an observable $\hat{O}$ in the 'Hartle-Hawking-like' state $|\mathrm{H}\rangle$ (4.212) is

$$\langle \mathrm{H}|\,\hat{O}\,|\mathrm{H}\rangle = \frac{1}{2}\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\left\{\int_{-\infty}^{\infty} \mathrm{d}\widetilde{\omega}\,\left[o^{\mathrm{in}}_{\omega\ell m} + o^{\mathrm{up}}_{\omega\ell m}\right]\coth\left|\frac{\pi\widetilde{\omega}}{\kappa}\right| - 2\int_{\min\left\{0,\frac{qQ}{r_+}\right\}}^{\max\left\{0,\frac{qQ}{r_+}\right\}} \mathrm{d}\omega\, o^{\mathrm{in}}_{\omega\ell m}\coth\left|\frac{\pi\widetilde{\omega}}{\kappa}\right|\right\}. \tag{5.85}$$

The expectation value of an observable $\hat{O}$ in the 'Frolov-Thorne' state $|\mathrm{FT}\rangle$ (4.217) is

$$\langle \mathrm{FT}|\,\hat{O}\,|\mathrm{FT}\rangle = \frac{1}{2}\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\left\{\int_{-\infty}^{\infty} \mathrm{d}\widetilde{\omega}\,\left[o^{\mathrm{in}}_{\omega\ell m} + o^{\mathrm{up}}_{\omega\ell m}\right]\coth\left|\frac{\pi\widetilde{\omega}}{\kappa}\right|\right\}. \tag{5.86}$$

Having defined the explicit expressions for the expectation values of observables in each quantum state, we will now explicitly construct the differences in expectation values between different quantum states by substituting into (5.78–5.86) the leading order classical mode contributions derived in §5.2. The expressions we generate will both inform and augment the numerical computations by demonstrating either the regularity or divergence of quantum states in asymptotic regimes.

Throughout the rest of this chapter, we will use the trigonometric identity:

$$\coth x - 1 = \frac{1}{\mathrm{e}^{2x}-1}. \tag{5.87}$$

We will also make use of the following identity:

$$-1 = \frac{1}{\exp(x)-1} + \frac{1}{\exp(-x)-1}. \tag{5.88}$$

5.3 The 'past' states

5.3.1 Fluxes of charge and energy in the 'past' Boulware state

In §4.3.1, we defined the 'past' Boulware state $|\mathrm{B}^-\rangle$ to be a state that is as empty as possible to a static observer as past null infinity $\mathscr{I}^-$, which corresponds to an absence of particles in the in-modes (3.106) of the field Φ. However, due to the phenomenon of classical superradiance that we studied in §3.3.4, we do not anticipate that this state will appear to be as empty as possible to a static observer at future null infinity $\mathscr{I}^+$.

Since the up-modes (3.131) are field modes emanating from the past horizon $\mathcal{H}^-$ and travelling towards future null infinity $\mathscr{I}^+$, then we may calculate the fluxes of charge $\mathcal{K}_{|\mathrm{B}^-\rangle}$ and energy $\mathcal{L}_{|\mathrm{B}^-\rangle}$ in terms of up-modes in order to investigate the properties of the 'past' Boulware state $|\mathrm{B}^-\rangle$ from the point of view of a static observer at future null infinity $\mathscr{I}^+$.

From Appendix C, the expectation values $\langle \hat{J}^r\rangle$ and $\langle \hat{T}^r_t\rangle$ do not require renormalisation; we can thus evaluate their expectation value in the 'past' Boulware state $|\mathrm{B}^-\rangle$ directly. Using the asymptotic near-infinity in- and up-mode contributions (5.38) to the current component J^r and (5.78), the expectation value $\langle \hat{J}^r\rangle_{|\mathrm{B}^-\rangle}$ at $\mathscr{I}^+$ becomes

$$\begin{aligned}
\langle \mathrm{B}^-| \hat{J}^r |\mathrm{B}^-\rangle &\sim \frac{q}{128\pi^3 r^2} \sum_{\ell=0}^{\infty} \left\{ \int_{-\infty}^{\infty} \mathrm{d}\omega \, \frac{\widetilde{\omega}}{|\omega|} \left| B^{\mathrm{in}}_{\omega\ell} \right|^2 - \int_{-\infty}^{\infty} \mathrm{d}\widetilde{\omega} \, \frac{\omega}{|\widetilde{\omega}|} \left| B^{\mathrm{up}}_{\omega\ell} \right|^2 \right\} (2\ell+1) \\
&= \frac{q}{128\pi^3 r^2} \sum_{\ell=0}^{\infty} \left\{ \int_{-\infty}^{\infty} \mathrm{d}\omega \, \frac{\widetilde{\omega}}{|\omega|} \frac{\omega^2}{\widetilde{\omega}^2} - \int_{-\infty}^{\infty} \mathrm{d}\widetilde{\omega} \, \frac{\omega}{|\widetilde{\omega}|} \right\} (2\ell+1) \left| B^{\mathrm{up}}_{\omega\ell} \right|^2 \\
&= \frac{q}{128\pi^3 r^2} \sum_{\ell=0}^{\infty} \int_{-\infty}^{\infty} \mathrm{d}\omega \left\{ \frac{|\omega|}{\widetilde{\omega}} - \frac{\omega}{|\widetilde{\omega}|} \right\} (2\ell+1) \left| B^{\mathrm{up}}_{\omega\ell} \right|^2 \\
&= \frac{q}{128\pi^3 r^2} \sum_{\ell=0}^{\infty} \int_{-\infty}^{\infty} \mathrm{d}\omega \, \frac{\omega}{|\widetilde{\omega}|} \left\{ \frac{|\omega|}{\omega} \frac{|\widetilde{\omega}|}{\widetilde{\omega}} - 1 \right\} (2\ell+1) \left| B^{\mathrm{up}}_{\omega\ell} \right|^2 \\
&= -\frac{q}{128\pi^3 r^2} \sum_{\ell=0}^{\infty} \int_{-\infty}^{\infty} \mathrm{d}\omega \, \frac{\omega}{|\widetilde{\omega}|} \left\{ 1 - \mathrm{sgn}(\omega)\, \mathrm{sgn}(\widetilde{\omega}) \right\} (2\ell+1) \left| B^{\mathrm{up}}_{\omega\ell} \right|^2 ,
\end{aligned} \tag{5.89}$$

where we have used the relationship between the in-mode transmission coefficient $B^{\mathrm{in}}_{\omega\ell}$ and the up-mode transmission coefficient $B^{\mathrm{up}}_{\omega\ell}$ in (3.75) to go from the first equality in (5.89) to the second. Furthermore, we have used the fact that the limits of the integral in (5.89) are from $-\infty$ to ∞; since $\widetilde{\omega}$ is offset from ω by a constant amount of $-\frac{qQ}{r_+}$ (3.58), we may simply replace the integration measure $\mathrm{d}\widetilde{\omega}$ by the measure $\mathrm{d}\omega$.

Equation (5.89) is nonzero only for $\mathrm{sgn}(\omega\widetilde{\omega}) = -1$, which corresponds to the case when $0 < \omega < \frac{qQ}{r_+}$ if $qQ > 0$, or to the case $0 > \omega > \frac{qQ}{r_+}$ if $qQ < 0$, i.e. the superradiant modes. Then, the expression in (5.89) reduces to

$$\langle \mathrm{B}^-| \hat{J}^r |\mathrm{B}^-\rangle \sim -\frac{q}{64\pi^3 r^2} \sum_{\ell=0}^{\infty} \int_{\min\left\{\frac{qQ}{r_+},0\right\}}^{\max\left\{\frac{qQ}{r_+},0\right\}} \mathrm{d}\omega \, \frac{\omega}{|\widetilde{\omega}|} (2\ell+1) \left| B^{\mathrm{up}}_{\omega\ell} \right|^2 . \tag{5.90}$$

Using (5.12) and (5.90), we derive the flux of charge $\mathcal{K}_{|\mathrm{B}^-\rangle}$ at $\mathscr{I}^+$ as

$$\mathcal{K}_{|\mathrm{B}^-\rangle} = \frac{q}{64\pi^3} \sum_{\ell=0}^{\infty} \int_{\min\left\{\frac{qQ}{r_+},0\right\}}^{\max\left\{\frac{qQ}{r_+},0\right\}} \mathrm{d}\omega \, \frac{\omega}{|\widetilde{\omega}|} (2\ell+1) \left| B^{\mathrm{up}}_{\omega\ell} \right|^2 . \tag{5.91}$$

From (5.91), the flux of charge $\mathcal{K}_{|\mathrm{B}^-\rangle}$ always has the same sign as the charge of the black hole Q. In order to see this, consider first the case that $qQ > 0$ in which case the integral in (5.91) is over the interval $0 < \omega < \frac{qQ}{r_+}$ such that ω is positive; if $Q > 0$ then we must have that q is positive such that $\mathcal{K}_{|\mathrm{B}^-\rangle}$ must also be positive, and if $Q < 0$ then we must have that q is negative such that $\mathcal{K}_{|\mathrm{B}^-\rangle}$ must also be negative. The reasoning follows through in an analogous way if $qQ < 0$. Therefore the product $\mathcal{K}_{|\mathrm{B}^-\rangle} Q$ must always be positive and, thus, the emission of particles in the superradiant modes in (5.90) corresponds to the Reissner-Nordström black hole discharging.

Using a similar process to that in (5.89), the expectation value $\langle \hat{T}^r_t \rangle_{|\mathrm{B}^-\rangle}$ at $\mathscr{I}^+$ is

$$\langle \mathrm{B}^-| \hat{T}^r_t |\mathrm{B}^-\rangle \sim -\frac{1}{16\pi^2 r^2} \sum_{\ell=0}^{\infty} \int_{\min\left\{\frac{qQ}{r_+},0\right\}}^{\max\left\{\frac{qQ}{r_+},0\right\}} \mathrm{d}\omega \, \frac{\omega^2}{|\widetilde{\omega}|} (2\ell+1) \left| B^{\mathrm{up}}_{\omega\ell} \right|^2 . \tag{5.92}$$

Using (5.25) and (5.92), we can derive the flux of energy $\mathcal{L}_{|\mathrm{B}^-\rangle}$ at $\mathscr{I}^+$ as

$$\mathcal{L}_{|\mathrm{B}^-\rangle} = \frac{1}{16\pi^2} \sum_{\ell=0}^{\infty} \int_{\min\left\{\frac{qQ}{r_+},0\right\}}^{\max\left\{\frac{qQ}{r_+},0\right\}} \mathrm{d}\omega \, \frac{\omega^2}{|\widetilde{\omega}|} \left(2\ell+1\right) \left|B^{\mathrm{up}}_{\omega\ell}\right|^2 . \tag{5.93}$$

The expression for the flux of energy $\mathcal{L}_{|\mathrm{B}^-\rangle}$ is always positive; thus the emission of particles in the superradiant modes in (5.92) corresponds to the RN black hole losing energy.

In conclusion, the nonzero flux of charge $\mathcal{K}_{|\mathrm{B}^-\rangle}$ (5.91) and nonzero flux of energy $\mathcal{L}_{|\mathrm{B}^-\rangle}$ (5.93) in the superradiant modes can be interpreted as the quantum analogue of the phenomenon of classical superradiance that we studied in §3.3.4. This suggests that we are correct in our intuition that the 'past' Boulware state $|\mathrm{B}^-\rangle$, while being as empty as possible to a static observer at past null infinity $\mathscr{I}^-$, is not empty to a static observer at future null infinity $\mathscr{I}^+$.

5.3.2 The 'past' Unruh state

In §4.4.1, we defined the 'past' Unruh state $|\mathrm{U}^-\rangle$ to be a state that is as empty as possible to a static observer at past null infinity $\mathscr{I}^-$ while containing an outgoing flux of thermal radiation at future null infinity $\mathscr{I}^+$. Since the 'past' Boulware state $|\mathrm{B}^-\rangle$ is defined to be as empty as possible to a static observer at past null infinity $\mathscr{I}^-$, then the difference in the expectation values of observables between the two states should correspond to an outgoing flux of thermal radiation at future null infinity $\mathscr{I}^+$; this is represented by a thermalised flux of particles in the up-modes (3.131) of the field Φ.

We can use the expectation value of a general observable $\hat{O}$ with classical mode contribution $o_{\omega\ell m}$ to construct an explicit expression for the difference $\langle \hat{O} \rangle_{|\mathrm{U}^-\rangle - |\mathrm{B}^-\rangle}$ in expectation values in the 'past' Boulware state (5.78) and the 'past' Unruh state (5.81):

$$\langle \mathrm{U}^- | \, \hat{O} \, | \mathrm{U}^- \rangle - \langle \mathrm{B}^- | \, \hat{O} \, | \mathrm{B}^- \rangle = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \int_{-\infty}^{\infty} \mathrm{d}\widetilde{\omega} \, \frac{1}{\exp\left|\frac{2\pi\widetilde{\omega}}{\kappa}\right| - 1} \, o^{\mathrm{up}}_{\omega\ell m}, \tag{5.94}$$

where we have used the identity in (5.87). In §5.2.2, we evaluated the asymptotic up-mode contributions to the classical quantities corresponding to the quantum observables $\widehat{\mathcal{SC}}$, $\hat{J}^\mu$ and $\hat{T}^\mu_\nu$ as $r \to \infty$. Using the near-infinity up-mode contribution (5.36) to the scalar condensate $\mathcal{SC}$, (5.94) becomes

$$\langle \mathrm{U}^- | \, \widehat{\mathcal{SC}} \, | \mathrm{U}^- \rangle - \langle \mathrm{B}^- | \, \widehat{\mathcal{SC}} \, | \mathrm{B}^- \rangle \sim \frac{1}{16\pi^2 r^2} \sum_{\ell=0}^{\infty} \int_{-\infty}^{\infty} \mathrm{d}\widetilde{\omega} \, \frac{2\ell+1}{|\widetilde{\omega}| \left(\exp\left|\frac{2\pi\widetilde{\omega}}{\kappa}\right| - 1 \right)} \left|B^{\mathrm{up}}_{\omega\ell}\right|^2 . \tag{5.95}$$

Using the near-infinity up-mode contribution (5.39) to the current J^μ, (5.94) becomes

$$\begin{aligned} &\langle \mathrm{U}^- | \, \hat{J}^\mu \, | \mathrm{U}^- \rangle - \langle \mathrm{B}^- | \, \hat{J}^\mu \, | \mathrm{B}^- \rangle \\ &\qquad \sim -\frac{q}{64\pi^3 r^2} \sum_{\ell=0}^{\infty} \int_{-\infty}^{\infty} \mathrm{d}\widetilde{\omega} \, \frac{\omega\left(2\ell+1\right)}{|\widetilde{\omega}| \left(\exp\left|\frac{2\pi\widetilde{\omega}}{\kappa}\right| - 1 \right)} \left|B^{\mathrm{up}}_{\omega\ell}\right|^2 \begin{pmatrix} 1 & 1 & 0 & 0 \end{pmatrix}^{\mathsf{T}} . \end{aligned} \tag{5.96}$$

Using the near-infinity up-mode contribution (5.46) to the SET T^{μ}_{ν}, (5.94) becomes

$$\langle \mathrm{U}^- | \, \hat{T}^{\mu}_{\nu} \, |\mathrm{U}^-\rangle - \langle \mathrm{B}^- | \, \hat{T}^{\mu}_{\nu} \, |\mathrm{B}^-\rangle$$
$$\sim \frac{1}{16\pi^2 r^2} \sum_{\ell=0}^{\infty} \int_{-\infty}^{\infty} \mathrm{d}\widetilde{\omega} \, \frac{\omega^2 \, (2\ell+1)}{|\widetilde{\omega}| \left(\exp\left| \frac{2\pi\widetilde{\omega}}{\kappa} \right| - 1 \right)} \left| B^{\mathrm{up}}_{\omega\ell} \right|^2 \begin{pmatrix} -1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & \mathcal{O}\left(r^{-2}\right) & 0 \\ 0 & 0 & 0 & \mathcal{O}\left(r^{-2}\right) \end{pmatrix}. \quad (5.97)$$

The differences (5.95 – 5.97) are regular when $\widetilde{\omega} = 0$ from the Wronskian relation in (3.75); the $\left|B^{\mathrm{up}}_{\omega\ell}\right|^2$ is of $\mathcal{O}\left(\widetilde{\omega}^2\right)$, which cancels the factor of $\mathcal{O}\left(\widetilde{\omega}^{-2}\right)$ in the denominators as $\widetilde{\omega} \to 0$.

Furthermore, these differences diverge near the horizon as $r \to r_+$. The Schwarzschild Boulware state $|\mathrm{B_s}\rangle$ is divergent everywhere on the horizon while the Schwarzschild Unruh state $|\mathrm{U_s}\rangle$ is divergent on the past horizon $\mathcal{H}^-$ but regular on the future horizon $\mathcal{H}^+$. Therefore, we expect that the divergence in the differences (5.95 – 5.97) arise due to the singular nature of the 'past' Boulware state $|\mathrm{B}^-\rangle$ on the horizon, although only a computation of the renormalised expectation values $\langle \widehat{\mathcal{SC}} \rangle_{|\mathrm{B}^-\rangle}$, $\langle \hat{J}^{\mu} \rangle_{|\mathrm{B}^-\rangle}$ and $\langle \hat{T}^{\mu}_{\nu} \rangle_{|\mathrm{B}^-\rangle}$, as opposed to calculating differences, would be able to confirm this.

Discussion of Figure 5.1: The plot of the difference $\langle \widehat{\mathcal{SC}} \rangle_{|\mathrm{U}^-\rangle - |\mathrm{B}^-\rangle}$ in the scalar condensate shows very little variation between different values of the scalar field charge q. Furthermore, this difference is positive for all of the positive values of q considered, indicating that the expectation value $\langle \widehat{\mathcal{SC}} \rangle_{|\mathrm{U}^-\rangle}$ is greater than $\langle \widehat{\mathcal{SC}} \rangle_{|\mathrm{B}^-\rangle}$, at least for $qQ > 0$.

The plot of the difference $\langle \hat{J}^t \rangle_{|\mathrm{U}^-\rangle - |\mathrm{B}^-\rangle}$ in the time component of the current shows that the magnitude of the difference vanishes in the uncharged limit $q \to 0$ and increases with increasing scalar field charge q. This behaviour is similar to that found in [45] for a massless, conformally coupled scalar field on a background RN-de Sitter spacetime.

The constant horizontal lines in the plot of $r^2 \langle \hat{J}^r \rangle_{|\mathrm{U}^-\rangle - |\mathrm{B}^-\rangle}$ illustrate that the difference in the radial component of the current is proportional to r^{-2}. Since $r^2 \langle \hat{J}^r \rangle_{|\mathrm{U}^-\rangle - |\mathrm{B}^-\rangle}$ is negative then, from (5.12), we have that the difference between the flux of charges $\mathcal{K}_{|\mathrm{U}^-\rangle - |\mathrm{B}^-\rangle}$ must be positive and so the flux of charge in the 'past' Unruh state $\mathcal{K}_{|\mathrm{U}^-\rangle}$ is greater than the flux of charge in the 'past' Boulware state $\mathcal{K}_{|\mathrm{B}^-\rangle}$ for a given value of the scalar field charge q. Recall, from §4.4.1, that $|\mathrm{U}^-\rangle$ is defined to exhibit outgoing Hawking radiation at $\mathscr{I}^+$. From our discussion in §5.3.1, $|\mathrm{B}^-\rangle$ exhibits an outgoing flux of charge $\mathcal{K}_{|\mathrm{B}^-\rangle}$ in the superradiant modes. Thus, it must be the case that the loss of charge due to Hawking radiation in $|\mathrm{U}^-\rangle$ is greater than the loss of charge due to quantum superradiance in $|\mathrm{B}^-\rangle$ for a fixed value of q.

Furthermore, since the difference $\langle \hat{J}^r \rangle_{|\mathrm{U}^-\rangle - |\mathrm{B}^-\rangle}$ increases with increasing scalar field charge q, then we reason that $\mathcal{K}_{|\mathrm{U}^-\rangle}$ must increase more rapidly with increasing q than $\mathcal{K}_{|\mathrm{B}^-\rangle}$ does, which also increases with increasing q from [1]. Then, it must also be the case that the loss of charge due to Hawking radiation in the 'past' Unruh state $|\mathrm{U}^-\rangle$ increases more rapidly with increasing q than the loss of charge due to quantum superradiance in the 'past' Boulware state $|\mathrm{B}^-\rangle$.

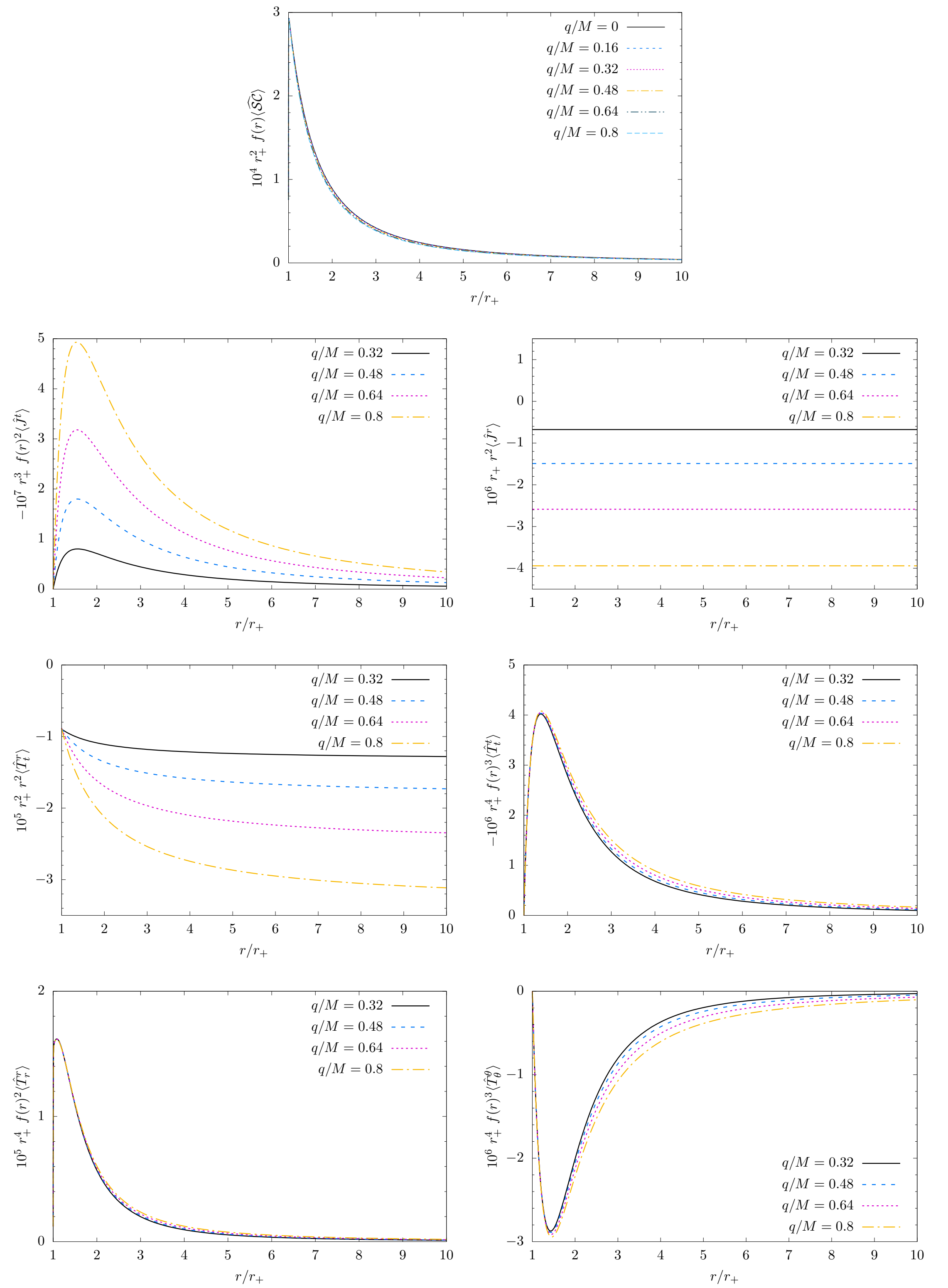

Figure 5.1. Difference in expectation values of the scalar condensate $\widehat{SC}$, components of the current $\hat{J}$ and components of the stress-energy tensor $\hat{T}^\mu_\nu$ between the 'past' Unruh state $|\mathrm{U}^-\rangle$ and the 'past' Boulware state $|\mathrm{B}^-\rangle$ in Reissner-Nordström spacetime for black hole charge $Q = 0.8M$ and positive values of the scalar field charge q. All expectation values are multiplied by powers of $f(r)$ so that the resulting quantities are regular at $r = r_+$.

The plots of the differences of the diagonal components of the stress-energy tensor $\langle \hat{T}^t_t \rangle_{|\mathrm{U}^-\rangle - |\mathrm{B}^-\rangle}$, $\langle \hat{T}^r_r \rangle_{|\mathrm{U}^-\rangle - |\mathrm{B}^-\rangle}$ and $\langle \hat{T}^\theta_\theta \rangle_{|\mathrm{U}^-\rangle - |\mathrm{B}^-\rangle}$ each show little variation between different values of the scalar field charge q.

The magnitude of the difference in the flux component $\langle \hat{T}^r_t \rangle_{|\mathrm{U}^-\rangle - |\mathrm{B}^-\rangle}$, however, increases rapidly for increasing scalar field charge q. Furthermore, $\langle \hat{T}^r_t \rangle_{|\mathrm{U}^-\rangle - |\mathrm{B}^-\rangle}$ is negative for all values of q which, from (5.25), indicates that the difference in the flux of energy $\mathcal{L}_{|\mathrm{U}^-\rangle - |\mathrm{B}^-\rangle}$ is greater than the difference in the flux of charge $\mathcal{K}_{|\mathrm{U}^-\rangle - |\mathrm{B}^-\rangle}$.

The differences in the scalar condensate, the time component of the current and the diagonal components of the stress-energy tensor, i.e. those differences that do not pertain to fluxes, all diverge near the horizon; as described earlier, we suspect that this divergence is due to the singular nature of $|\mathrm{B}^-\rangle$ on the horizon as opposed to $|\mathrm{U}^-\rangle$.

Lastly recall, from §5.3.1, that the 'past' Boulware state $|\mathrm{B}^-\rangle$ only contains a flux of particles in the superradiant modes at $\mathscr{I}^+$ and is otherwise defined to be as empty as possible to a static observer far from the black hole. All of the differences in expectation values plotted here decay as r^{-2} far from the black hole. Then, we reason that renormalised expectation values of observables $\langle \hat{O} \rangle_{|\mathrm{U}^-\rangle}$ in the 'past' Unruh state directly will also vanish far from the black hole.

5.3.3 Fluxes of charge and energy in the 'past' Unruh state

We would like to evaluate the flux of charge $\mathcal{K}_{|\mathrm{U}^-\rangle}$ and the flux of energy $\mathcal{L}_{|\mathrm{U}^-\rangle}$ in the 'past' Unruh state $|\mathrm{U}^-\rangle$ directly. We can do this by first calculating $\langle \hat{J}^r \rangle_{|\mathrm{U}^-\rangle}$ and $\langle \hat{T}^r_t \rangle_{|\mathrm{U}^-\rangle}$, from adding the expressions for $\langle \hat{J}^r \rangle_{|\mathrm{B}^-\rangle}$ (5.90) and $\langle \hat{T}^r_t \rangle_{|\mathrm{B}^-\rangle}$ (5.92) to the relevant components of the differences $\langle \hat{J}^\mu \rangle_{|\mathrm{U}^-\rangle - |\mathrm{B}^-\rangle}$ (5.96) and $\langle \hat{T}^\mu_\nu \rangle_{|\mathrm{U}^-\rangle - |\mathrm{B}^-\rangle}$ (5.97) respectively. For example, in the case of $\langle \hat{J}^r \rangle_{|\mathrm{U}^-\rangle}$, we can summarise the above as

$$\langle \hat{J}^r \rangle_{|\mathrm{U}^-\rangle} = \langle \mathrm{U}^- | \, \hat{J}^r \, | \mathrm{U}^- \rangle = \Big\{ \langle \mathrm{U}^- | \, \hat{J}^r \, | \mathrm{U}^- \rangle - \langle \mathrm{B}^- | \, \hat{J}^r \, | \mathrm{U}^- \rangle \Big\} + \langle \mathrm{B}^- | \, \hat{J}^r \, | \mathrm{U}^- \rangle \, . \tag{5.98}$$

The integral in the expression (5.90) for the difference $\langle \hat{J}^\mu \rangle_{|\mathrm{U}^-\rangle - |\mathrm{B}^-\rangle}$ has the measure $\mathrm{d}\widetilde{\omega}$. Since the limits of the integral are $-\infty < \omega < \infty$ and ω is offset from $\widetilde{\omega}$ by the constant amount of $\frac{qQ}{r_+}$, then we can replace the integration measure $\mathrm{d}\widetilde{\omega}$ in (5.90) by the measure $\mathrm{d}\omega$ in order to rewrite the expression for the difference $\langle \hat{J}^r \rangle_{|\mathrm{U}^-\rangle - |\mathrm{B}^-\rangle}$ as

$$\begin{aligned}
&\langle \mathrm{U}^- | \, \hat{J}^r \, | \mathrm{U}^- \rangle - \langle \mathrm{B}^- | \, \hat{J}^r \, | \mathrm{U}^- \rangle \\
&\quad \sim -\frac{q}{64\pi^3 r^2} \sum_{\ell=0}^{\infty} \int_{-\infty}^{\infty} \mathrm{d}\omega \, \frac{\omega}{|\widetilde{\omega}| \left(\exp\left| \frac{2\pi\widetilde{\omega}}{\kappa} \right| - 1 \right)} \, (2\ell+1) \left| B^{\mathrm{up}}_{\omega\ell} \right|^2 \\
&\quad = -\frac{q}{64\pi^3 r^2} \sum_{\ell=0}^{\infty} \left\{ \int_0^{\infty} \mathrm{d}\omega \, \frac{\omega\,(2\ell+1)}{|\widetilde{\omega}| \left(\exp\left| \frac{2\pi\widetilde{\omega}}{\kappa} \right| - 1 \right)} + \int_{-\infty}^{0} \mathrm{d}\omega \, \frac{\omega\,(2\ell+1)}{|\widetilde{\omega}| \left(\exp\left| \frac{2\pi\widetilde{\omega}}{\kappa} \right| - 1 \right)} \right\} \left| B^{\mathrm{up}}_{\omega\ell} \right|^2 \\
&\quad = \mathsf{I}_1 + \mathsf{I}_2 \, ,
\end{aligned} \tag{5.99}$$

where the integral I_1 is given by

$$\mathsf{I}_1 = -\frac{q}{64\pi^3 r^2}\sum_{\ell=0}^{\infty}\int_0^{\infty} \mathrm{d}\omega \, \frac{\omega\,(2\ell+1)}{|\widetilde{\omega}|\left(\exp\left|\frac{2\pi\widetilde{\omega}}{\kappa}\right| - 1\right)} \left|B^{\mathrm{up}}_{\omega\ell}\right|^2 , \tag{5.100}$$

and the integral I_2 is given by

$$\mathsf{I}_2 = -\frac{q}{64\pi^3 r^2}\sum_{\ell=0}^{\infty}\int_{-\infty}^{0} \mathrm{d}\omega \, \frac{\omega\,(2\ell+1)}{|\widetilde{\omega}|\left(\exp\left|\frac{2\pi\widetilde{\omega}}{\kappa}\right| - 1\right)} \left|B^{\mathrm{up}}_{\omega\ell}\right|^2 . \tag{5.101}$$

From our discussion in §3.3.4, one of the two integrals $\mathsf{I}_1, \mathsf{I}_2$ will contain superradiant modes. Whether the superradiant modes exist in either I_1 or I_2 depends on the sign of $\frac{qQ}{r_+}$; if $qQ > 0$, the superradiant modes have $0 < \omega < \frac{qQ}{r_+}$ such that they exist in I_1 and if $qQ < 0$, the superradiant modes have $0 > \omega > \frac{qQ}{r_+}$ such that they exist in I_2.

As discussed in §5.3.1, the integral in the expression (5.90) for $\langle \hat{J}^r\rangle_{|\mathrm{B}^-\rangle}$ is taken only over superradiant modes. Then we can combine the two expressions in (5.90) and (5.99) by considering in which of the two integrals $\mathsf{I}_1, \mathsf{I}_2$ the superradiant modes exist.

Consider firstly the case when $qQ > 0$; the superradiant modes occur as ω ranges through the interval $0 < \omega < \frac{qQ}{r_+}$, and are thus present in I_1. In this case, the expectation value $\langle \hat{J}^r\rangle_{|\mathrm{B}^-\rangle}$ (5.90) reduces to

$$\langle \hat{J}^r\rangle_{|\mathrm{B}^-\rangle} = \langle \mathrm{B}^-|\, \hat{J}^r \,|\mathrm{B}^-\rangle = -\frac{q}{64\pi^3 r^2}\sum_{\ell=0}^{\infty}\int_0^{\frac{qQ}{r_+}} \mathrm{d}\omega \, \frac{\omega}{|\widetilde{\omega}|}\,(2\ell+1)\left|B^{\mathrm{up}}_{\omega\ell}\right|^2 . \tag{5.102}$$

In the following calculation, it will be convenient to define

$$K = -\frac{q}{64\pi^3 r^2}\sum_{\ell=0}^{\infty}(2\ell+1) . \tag{5.103}$$

Then, splitting the integral I_1 into two separate intervals of $0 < \omega < \frac{qQ}{r_+}$ and $\frac{qQ}{r_+} < \omega < \infty$, we can add the expression for $\langle \hat{J}^r\rangle_{|\mathrm{B}^-\rangle}$ (5.102) to the integral I_1 (5.100) to give

$$\begin{aligned}
&\mathsf{I}_1 + \langle \mathrm{B}^-|\, \hat{J}^r \,|\mathrm{B}^-\rangle \\
&= K\left\{\int_0^{\frac{qQ}{r_+}} \mathrm{d}\omega \left(\frac{\omega}{|\widetilde{\omega}|\left[\exp\left(-\frac{2\pi\widetilde{\omega}}{\kappa}\right)-1\right]} + \frac{\omega}{|\widetilde{\omega}|}\right) + \int_{\frac{qQ}{r_+}}^{\infty} \mathrm{d}\omega \, \frac{\omega}{|\widetilde{\omega}|\left[\exp\left(\frac{2\pi\widetilde{\omega}}{\kappa}\right)-1\right]}\right\}\left|B^{\mathrm{up}}_{\omega\ell}\right|^2 \\
&= K\left\{\int_0^{\frac{qQ}{r_+}} \mathrm{d}\omega \, \frac{\omega}{|\widetilde{\omega}|}\left(\frac{1}{\left[\exp\left(-\frac{2\pi\widetilde{\omega}}{\kappa}\right)-1\right]} + 1\right) + \int_{\frac{qQ}{r_+}}^{\infty} \mathrm{d}\omega \, \frac{\omega}{|\widetilde{\omega}|\left[\exp\left(\frac{2\pi\widetilde{\omega}}{\kappa}\right)-1\right]}\right\}\left|B^{\mathrm{up}}_{\omega\ell}\right|^2 \\
&= K\left\{-\int_0^{\frac{qQ}{r_+}} \mathrm{d}\omega \, \frac{\omega}{|\widetilde{\omega}|}\frac{1}{\left[\exp\left(\frac{2\pi\widetilde{\omega}}{\kappa}\right)-1\right]} + \int_{\frac{qQ}{r_+}}^{\infty} \mathrm{d}\omega \, \frac{\omega}{|\widetilde{\omega}|\left[\exp\left(\frac{2\pi\widetilde{\omega}}{\kappa}\right)-1\right]}\right\}\left|B^{\mathrm{up}}_{\omega\ell}\right|^2 \\
&= K\left\{\int_0^{\frac{qQ}{r_+}} \mathrm{d}\omega \, \frac{\omega}{\widetilde{\omega}\left[\exp\left(\frac{2\pi\widetilde{\omega}}{\kappa}\right)-1\right]} + \int_{\frac{qQ}{r_+}}^{\infty} \mathrm{d}\omega \, \frac{\omega}{\widetilde{\omega}\left[\exp\left(\frac{2\pi\widetilde{\omega}}{\kappa}\right)-1\right]}\right\}\left|B^{\mathrm{up}}_{\omega\ell}\right|^2 \\
&= K\int_0^{\infty} \mathrm{d}\omega \, \frac{\omega}{\widetilde{\omega}\left[\exp\left(\frac{2\pi\widetilde{\omega}}{\kappa}\right)-1\right]}\left|B^{\mathrm{up}}_{\omega\ell}\right|^2 ,
\end{aligned} \tag{5.104}$$

where we have used the identity in (5.88) to go from the second equality in (5.104) to the third. Furthermore, we have used the fact that $\widetilde{\omega}$ is negative in the range $0 < \omega < \frac{qQ}{r_+}$ and that $\widetilde{\omega}$ is positive in the range $\frac{qQ}{r_+} < \omega < \infty$ in order to go from the third equality in (5.104) to the fourth. Using the fact that $\widetilde{\omega}$ is negative in the range $0 > \omega > -\infty$ when $qQ > 0$, we can rewrite the integral I_2 (5.101) as

$$
\begin{aligned}
\mathsf{I}_2 &= K \int_{-\infty}^{0} \mathrm{d}\omega \, \frac{\omega}{|\widetilde{\omega}| \left[\exp\left(-\frac{2\pi\widetilde{\omega}}{\kappa}\right) - 1\right]} \left|B^{\mathrm{up}}_{\omega\ell}\right|^2 \\
&= -K \int_{-\infty}^{0} \mathrm{d}\omega \, \frac{\omega}{\widetilde{\omega} \left[\exp\left[-\frac{2\pi}{\kappa}\left(\omega - \frac{qQ}{r_+}\right)\right] - 1\right]} \left|B^{\mathrm{up}}_{\omega\ell}\right|^2 \\
&= -K \int_{-\infty}^{0} \mathrm{d}\omega \, \frac{\omega}{\left(\omega - \frac{qQ}{r_+}\right)\left[\exp\left[\frac{2\pi}{\kappa}\left(-\omega + \frac{qQ}{r_+}\right)\right] - 1\right]} \left|B^{\mathrm{up}}_{\omega\ell}\right|^2 .
\end{aligned}
\tag{5.105}
$$

We can simplify (5.105) by making the substitution $u = -\omega$ to give

$$
\begin{aligned}
\mathsf{I}_2 &= -K \int_{\infty}^{0} \mathrm{d}\,(-u) \, \frac{-u}{\left(-u - \frac{qQ}{r_+}\right)\left[\exp\left[\frac{2\pi}{\kappa}\left(u + \frac{qQ}{r_+}\right)\right] - 1\right]} \left|B^{\mathrm{up}}_{-u\ell}\right|^2 \\
&= K \int_{\infty}^{0} \mathrm{d}u \, \frac{u}{\left(u + \frac{qQ}{r_+}\right)\left[\exp\left[\frac{2\pi}{\kappa}\left(u + \frac{qQ}{r_+}\right)\right] - 1\right]} \left|B^{\mathrm{up}}_{-u\ell}\right|^2 \\
&= -K \int_{0}^{\infty} \mathrm{d}u \, \frac{u}{\left(u + \frac{qQ}{r_+}\right)\left[\exp\left[\frac{2\pi}{\kappa}\left(u + \frac{qQ}{r_+}\right)\right] - 1\right]} \left|B^{\mathrm{up}}_{-u\ell}\right|^2 \\
&= -K \int_{0}^{\infty} \mathrm{d}\omega \, \frac{\omega}{\overline{\omega}\left[\exp\left(\frac{2\pi\overline{\omega}}{\kappa}\right) - 1\right]} \left|B^{\mathrm{up}}_{-\omega\ell}\right|^2 ,
\end{aligned}
\tag{5.106}
$$

where, in going from the penultimate line of (5.106) to the last one, we have used the fact that u is effectively a dummy variable enabling us to relabel u as ω in the last line and, furthermore, we have made the definition

$$
\overline{\omega} = \omega + \frac{qQ}{r_+}. \tag{5.107}
$$

Then, in the case that $qQ > 0$ such that the superradiant modes are contained in the integral I_1 (5.100), we obtain

$$
\begin{aligned}
&\left\{\mathsf{I}_1 + \langle \mathrm{B}^- | \, \hat{J}^r \, | \mathrm{B}^- \rangle \right\} + \mathsf{I}_2 \\
&= -\frac{q}{64\pi^3 r^2} \sum_{\ell=0}^{\infty} \int_0^{\infty} \mathrm{d}\omega \, (2\ell+1) \, \omega \left\{ \frac{\left|B^{\mathrm{up}}_{\omega\ell}\right|^2}{\widetilde{\omega}\left[\exp\left(\frac{2\pi\widetilde{\omega}}{\kappa}\right) - 1\right]} - \frac{\left|B^{\mathrm{up}}_{-\omega\ell}\right|^2}{\overline{\omega}\left[\exp\left[\frac{2\pi\overline{\omega}}{\kappa}\right] - 1\right]} \right\}.
\end{aligned}
\tag{5.108}
$$

A similar set of calculations show that in the case $qQ < 0$, when the superradiant modes are instead contained in the integral I_2 (5.101), we obtain

$$\begin{aligned}&\mathsf{I}_1 + \left\{ \mathsf{I}_2 + \langle \mathrm{B}^- | \, \hat{J}^r \, | \mathrm{B}^- \rangle \right\} \\ &\quad = -\frac{q}{64\pi^3 r^2} \sum_{\ell=0}^{\infty} \int_0^{\infty} \mathrm{d}\omega \, (2\ell+1) \, \omega \left\{ \frac{\left| B^{\mathrm{up}}_{\omega\ell} \right|^2}{\widetilde{\omega} \left[\exp\left(\frac{2\pi\widetilde{\omega}}{\kappa} \right) - 1 \right]} - \frac{\left| B^{\mathrm{up}}_{-\omega\ell} \right|^2}{\varpi \left[\exp\left[\frac{2\pi\varpi}{\kappa} \right] - 1 \right]} \right\}. \end{aligned} \tag{5.109}$$

Then, the expectation value $\langle \hat{J}^r \rangle_{|\mathrm{U}^-\rangle}$ is given by

$$\begin{aligned}&\langle \mathrm{U}^- | \, \hat{J}^r \, | \mathrm{U}^- \rangle \\ &\sim -\frac{q}{64\pi^3 r^2} \sum_{\ell=0}^{\infty} \int_0^{\infty} \mathrm{d}\omega \, (2\ell+1) \, \omega \left\{ \frac{\left| B^{\mathrm{up}}_{\omega\ell} \right|^2}{\widetilde{\omega} \left[\exp\left(\frac{2\pi\widetilde{\omega}}{\kappa} \right) - 1 \right]} - \frac{\left| B^{\mathrm{up}}_{-\omega\ell} \right|^2}{\varpi \left[\exp\left[\frac{2\pi\varpi}{\kappa} \right] - 1 \right]} \right\}. \end{aligned} \tag{5.110}$$

Using (5.12) and (5.110), we derive the flux of charge $\mathcal{K}_{|\mathrm{U}^-\rangle}$ in the 'past' Unruh state as

$$\mathcal{K}_{|\mathrm{U}^-\rangle} = \frac{q}{64\pi^3} \sum_{\ell=0}^{\infty} \int_0^{\infty} \mathrm{d}\omega \, (2\ell+1) \, \omega \left\{ \frac{\left| B^{\mathrm{up}}_{\omega\ell} \right|^2}{\widetilde{\omega} \left[\exp\left(\frac{2\pi\widetilde{\omega}}{\kappa} \right) - 1 \right]} - \frac{\left| B^{\mathrm{up}}_{-\omega\ell} \right|^2}{\varpi \left[\exp\left[\frac{2\pi\varpi}{\kappa} \right] - 1 \right]} \right\}. \tag{5.111}$$

A similar set of calculations shows that the expectation value $\langle \hat{T}^r_t \rangle_{|\mathrm{U}^-\rangle}$ is given by

$$\begin{aligned}&\langle \mathrm{U}^- | \, \hat{T}^r_t \, | \mathrm{U}^- \rangle \\ &\sim -\frac{1}{16\pi^2 r^2} \sum_{\ell=0}^{\infty} \int_0^{\infty} \mathrm{d}\omega \, (2\ell+1) \, \omega^2 \left\{ \frac{\left| B^{\mathrm{up}}_{\omega\ell} \right|^2}{\widetilde{\omega} \left[\exp\left(\frac{2\pi\widetilde{\omega}}{\kappa} \right) - 1 \right]} + \frac{\left| B^{\mathrm{up}}_{-\omega\ell} \right|^2}{\varpi \left[\exp\left[\frac{2\pi\varpi}{\kappa} \right] - 1 \right]} \right\}. \end{aligned} \tag{5.112}$$

Using (5.25) and (5.112), we derive the flux of energy $\mathcal{L}_{|\mathrm{U}^-\rangle}$ in the 'past' Unruh state as

$$\mathcal{L}_{|\mathrm{U}^-\rangle} = \frac{1}{16\pi^2} \sum_{\ell=0}^{\infty} \int_0^{\infty} \mathrm{d}\omega \, (2\ell+1) \, \omega^2 \left\{ \frac{\left| B^{\mathrm{up}}_{\omega\ell} \right|^2}{\widetilde{\omega} \left[\exp\left(\frac{2\pi\widetilde{\omega}}{\kappa} \right) - 1 \right]} + \frac{\left| B^{\mathrm{up}}_{-\omega\ell} \right|^2}{\varpi \left[\exp\left[\frac{2\pi\varpi}{\kappa} \right] - 1 \right]} \right\}. \tag{5.113}$$

The expressions for the flux of charge $\mathcal{K}_{|\mathrm{U}^-\rangle}$ (5.111) and the flux of energy $\mathcal{L}_{|\mathrm{U}^-\rangle}$ (5.113) each contain integrals over the entire spectrum of positive-frequency modes, i.e. $\omega > 0$. This begs the question as to what has happened to the negative-frequency modes; the second term in both the expressions (5.111) and (5.113) contain a factor of $\left| B^{\mathrm{up}}_{-\omega\ell} \right|^2$, which we can interpret as the square of the magnitude of the transmission coefficient $B^{\mathrm{up}}_{-\omega\ell}$ associated to negative-frequency modes. Then, as seen by a static observer near infinity, the first term in both (5.111) and (5.113) is the flux from the emission of positive-frequency modes with an effective chemical potential of $\frac{qQ}{r_+}$, while the second term is the flux from the emission of negative-frequency modes with an effective chemical potential of $-\frac{qQ}{r_+}$.

The contribution to the flux of charge $\mathcal{K}_{|\mathrm{U}^-\rangle}$ (5.111) from the emission of positive-frequency modes has the same sign as the scalar field charge q, while the contribution from the emission of negative-frequency modes has the opposite sign to q.

On the other hand, the emission of positive- and negative-frequency modes both give a positive contribution to the flux of energy $\mathcal{L}_{|\mathrm{U}^-\rangle}$ (5.113), as we would expect. Then, we deduce that the Reissner-Nordström black hole loses energy due to the emission of Hawking radiation in the 'past' Unruh state $|\mathrm{U}^-\rangle$.

Finally, using (5.105) and (5.106), we see that in the limit $\kappa \to 0$ the Hawking temperature $T_\mathrm{H} = \frac{\kappa}{2\pi}$ vanishes and both $\mathcal{K}_{|\mathrm{U}^-\rangle}$ (5.111) and $\mathcal{L}_{|\mathrm{U}^-\rangle}$ (5.113) reduce to $\mathcal{K}_{|\mathrm{B}^-\rangle}$ (5.91) and $\mathcal{L}_{|\mathrm{B}^-\rangle}$ (5.93), i.e. their corresponding fluxes in the 'past' Boulware state $|\mathrm{B}^-\rangle$, respectively. This suggests that the 'past' Unruh state $|\mathrm{U}^-\rangle$ reduces to the 'past' Boulware state $|\mathrm{B}^-\rangle$ in the limit $T_\mathrm{H} \to 0$.

5.3.4 The 'past' CCH state

In §4.5.1, we defined the 'past' CCH state $|\mathrm{CCH}^-\rangle$ to be a state that exhibits thermal radiation, both incoming at past null infinity $\mathscr{I}^-$ and outgoing at future null infinity $\mathscr{I}^+$. Since the 'past' Unruh state $|\mathrm{U}^-\rangle$ is defined to be as empty as possible to a static observer at past null infinity $\mathscr{I}^-$ while exhibiting thermal radiation outgoing at future null-infinity, then the difference in the expectation values of observables between the two states should correspond to an incoming flux of thermal radiation incident upon the future horizon $\mathcal{H}^+$; this is represented by a thermalised flux of particles in the in-modes (3.106) of the field.

We can use the expectation value of a general observable $\hat{O}$ with classical mode contribution $o_{\omega\ell m}$ to construct an expression for the difference $\langle\hat{O}\rangle_{|\mathrm{CCH}^-\rangle - |\mathrm{U}^-\rangle}$ in expectation values in the 'past' Unruh state (5.81) and the 'past' CCH state (5.83); we have

$$\langle \mathrm{CCH}^- |\, \hat{O}\, |\mathrm{CCH}^-\rangle - \langle \mathrm{U}^- |\, \hat{O}\, |\mathrm{U}^-\rangle = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\int_{-\infty}^{\infty} \mathrm{d}\omega\, \frac{1}{\exp\left|\frac{2\pi\omega}{\kappa}\right| - 1}\, o^{\mathrm{in}}_{\omega\ell m}, \tag{5.114}$$

where we have used the identity in (5.87). In §5.2.3, we evaluated the asymptotic in-mode contributions to the classical quantities corresponding to the quantum observables $\widehat{\mathcal{SC}}$, $\hat{J}^\mu$ and $\hat{T}^\mu_\nu$ as $r \to r_+$. Using the near-horizon in-mode contribution (5.60) to the scalar condensate $\mathcal{SC}$, (5.114) becomes

$$\langle \mathrm{CCH}^- |\, \widehat{\mathcal{SC}}\, |\mathrm{CCH}^-\rangle - \langle \mathrm{U}^- |\, \widehat{\mathcal{SC}}\, |\mathrm{U}^-\rangle \sim \frac{1}{16\pi^2 r^2}\sum_{\ell=0}^{\infty}\int_{-\infty}^{\infty} \mathrm{d}\omega\, \frac{2\ell+1}{|\omega|\left(\exp\left|\frac{2\pi\omega}{\kappa}\right| - 1\right)}\left|B^{\mathrm{in}}_{\omega\ell}\right|^2. \tag{5.115}$$

Using the near-horizon in-mode contribution (5.63) to the current J^μ, (5.114) becomes

$$\begin{aligned}&\langle \mathrm{CCH}^- |\, \hat{J}^\mu\, |\mathrm{CCH}^-\rangle - \langle \mathrm{U}^- |\, \hat{J}^\mu\, |\mathrm{U}^-\rangle \\ &\qquad \sim \frac{q}{64\pi^3 r^2}\sum_{\ell=0}^{\infty}\int_{-\infty}^{\infty} \mathrm{d}\omega\, \frac{\widetilde{\omega}\,(2\ell+1)}{|\omega|\left(\exp\left|\frac{2\pi\omega}{\kappa}\right| - 1\right)}\left|B^{\mathrm{in}}_{\omega\ell}\right|^2 \begin{pmatrix}-f(r)^{-1} & 1 & 0 & 0\end{pmatrix}^{\mathsf{T}}. \end{aligned} \tag{5.116}$$

Using the near-horizon in-mode contribution (5.70) to the SET T^μ_ν, (5.114) becomes

$$\langle \mathrm{CCH}^- | \hat{T}^\mu_\nu | \mathrm{CCH}^- \rangle - \langle \mathrm{U}^- | \hat{T}^\mu_\nu | \mathrm{U}^- \rangle$$
$$\sim \frac{1}{16\pi^2 r^2} \sum_{\ell=0}^{\infty} \int_{-\infty}^{\infty} \mathrm{d}\omega \, \frac{\widetilde{\omega}^2 (2\ell+1)}{|\omega| \left(\exp\left|\frac{2\pi\omega}{\kappa}\right| - 1\right)} \left|B^{\mathrm{in}}_{\omega\ell}\right|^2 \begin{pmatrix} -f(r)^{-1} & -f(r)^{-2} & 0 & 0 \\ 1 & f(r)^{-1} & 0 & 0 \\ 0 & 0 & \mathcal{O}(1) & 0 \\ 0 & 0 & 0 & \mathcal{O}(1) \end{pmatrix}. \tag{5.117}$$

The differences (5.115 – 5.117) are regular when $\omega = 0$ from the Wronskian relation (3.75); the $\left|B^{\mathrm{in}}_{\omega\ell}\right|^2$ is of $\mathcal{O}(\omega^2)$, which cancels the factor of $\mathcal{O}(\omega^{-2})$ in the denominators as $\omega \to 0$.

We can check whether the difference $\langle \hat{J}^\mu \rangle_{|\mathrm{CCH}^-\rangle - |\mathrm{U}^-\rangle}$ (5.116) of the current is regular on the horizon by changing to Kruskal coordinates; defining the quantity C as

$$C = \frac{q}{64\pi^3 r^2} \sum_{\ell=0}^{\infty} \int_{-\infty}^{\infty} \mathrm{d}\omega \, \frac{\widetilde{\omega}\, (2\ell+1)}{|\omega| \left(\exp\left|\frac{2\pi\omega}{\kappa}\right| - 1\right)} \left|B^{\mathrm{in}}_{\omega\ell}\right|^2, \tag{5.118}$$

and using (5.28), near the horizon we have

$$\langle \hat{J}^U \rangle_{|\mathrm{CCH}^-\rangle - |\mathrm{U}^-\rangle} \sim \kappa U \left[f(r)^{-1} C + f(r)^{-1} C \right] = 2\kappa C U f(r)^{-1}, \tag{5.119a}$$
$$\langle \hat{J}^V \rangle_{|\mathrm{CCH}^-\rangle - |\mathrm{U}^-\rangle} \sim \kappa V \left[-f(r)^{-1} C + f(r)^{-1} C \right] = \mathcal{O}(1). \tag{5.119b}$$

The difference $\langle \hat{J}^U \rangle_{|\mathrm{CCH}^-\rangle - |\mathrm{U}^-\rangle}$ (5.119a) contains a factor of $f(r)^{-1}$, which diverges as $r \to r_+$; the future horizon is a surface of constant $U = 0$, and so the factor of U cancels the divergence of the $f(r)^{-1}$ such that the difference is regular on $\mathcal{H}^+$ while, on the past horizon $\mathcal{H}^-$, the difference diverges. In contrast, the leading order divergences cancel in the difference $\langle \hat{J}^V \rangle_{|\mathrm{CCH}^-\rangle - |\mathrm{U}^-\rangle}$ (5.119b) such that it is regular everywhere. Therefore, the differences $\langle \hat{J}^\mu \rangle_{|\mathrm{CCH}^-\rangle - |\mathrm{U}^-\rangle}$ in the expectation values of the current operator (5.116) are regular on $\mathcal{H}^+$ but diverge on $\mathcal{H}^-$.

We can check whether the difference $\langle \hat{T}^\mu_\nu \rangle_{|\mathrm{CCH}^-\rangle - |\mathrm{U}^-\rangle}$ of the SET is regular on the horizon by changing to Kruskal coordinates; we define the quantity S as

$$S = \frac{1}{16\pi^2 r^2} \sum_{\ell=0}^{\infty} \int_{-\infty}^{\infty} \mathrm{d}\omega \, \frac{\widetilde{\omega}^2 (2\ell+1)}{|\omega| \left(\exp\left|\frac{2\pi\omega}{\kappa}\right| - 1\right)} \left|B^{\mathrm{in}}_{\omega\ell}\right|^2. \tag{5.120}$$

From (5.69), $T_{tt} \sim L$, $T_{tr} \sim f(r)^{-1} L$ and $T_{rr} \sim f(r)^{-2} L$. Using (5.30), near the horizon:

$$\langle \hat{T}_{UU} \rangle_{|\mathrm{CCH}^-\rangle - |\mathrm{U}^-\rangle} \sim \frac{1}{4} \kappa^{-2} U^{-2} \left[S - 2 f(r) f(r)^{-1} S + f(r)^2 f(r)^{-2} S \right], \tag{5.121a}$$
$$\langle \hat{T}_{UV} \rangle_{|\mathrm{CCH}^-\rangle - |\mathrm{U}^-\rangle} \sim -\frac{1}{4} \kappa^{-2} U^{-1} V^{-1} \left[S - f(r)^2 f(r)^{-2} S \right], \tag{5.121b}$$
$$\langle \hat{T}_{VV} \rangle_{|\mathrm{CCH}^-\rangle - |\mathrm{U}^-\rangle} \sim \frac{1}{4} \kappa^{-2} V^{-2} \left[S + 2 f(r) f(r)^{-1} S + f(r)^2 f(r)^{-2} S \right] = \kappa^{-2} V^{-2} S. \tag{5.121c}$$

The leading order divergences cancel in the differences $\langle \hat{T}_{UU} \rangle_{|\mathrm{CCH}^-\rangle - |\mathrm{U}^-\rangle}$ (5.121a) and $\langle \hat{T}_{UV} \rangle_{|\mathrm{CCH}^-\rangle - |\mathrm{U}^-\rangle}$ (5.121b). The difference (5.121c), which contains a factor of V^{-2}, diverges on the past horizon since $\mathcal{H}^-$ is a surface of constant $V = 0$. Therefore, the differences $\langle \hat{T}_{\mu\nu} \rangle_{|\mathrm{CCH}^-\rangle - |\mathrm{U}^-\rangle}$ in the expectation values of the SET (5.117) diverge on $\mathcal{H}^-$.

Thus the differences between the 'past' CCH state and the 'past' Unruh state in the expectation values of both the current $\langle \hat{J}^{\mu} \rangle_{|\mathrm{CCH}^-\rangle - |\mathrm{U}^-\rangle}$ and the SET $\langle \hat{T}_{\mu\nu} \rangle_{|\mathrm{CCH}^-\rangle - |\mathrm{U}^-\rangle}$ diverge on the past horizon $\mathcal{H}^-$, but may be regular on $\mathcal{H}^+$ if the 'past' Unruh state is regular as anticipated. We will return to this question in our discussion of Figure 5.2.

Discussion of Figure 5.2: All of the plots in Figure 5.2, exhibit a much greater variation between different values of the scalar field charge q compared to the corresponding plots in Figure 5.1 for the differences between the 'past' Unruh state $|\mathrm{U}^-\rangle$ and the 'past' Boulware state $|\mathrm{B}^-\rangle$. Recall, from our discussion of Figure 5.1, that we deduced that the expectation values of observables in $|\mathrm{U}^-\rangle$ must vanish far from the black hole because $|\mathrm{B}^-\rangle$ is defined to be as empty as possible to a static observer at infinity. Figure 5.2 illustrates that the differences between $|\mathrm{CCH}^-\rangle$ and $|\mathrm{U}^-\rangle$ do not vanish far from the black hole, leading us to infer that the 'past' CCH state is not empty at infinity, as we would expect of a thermal state.

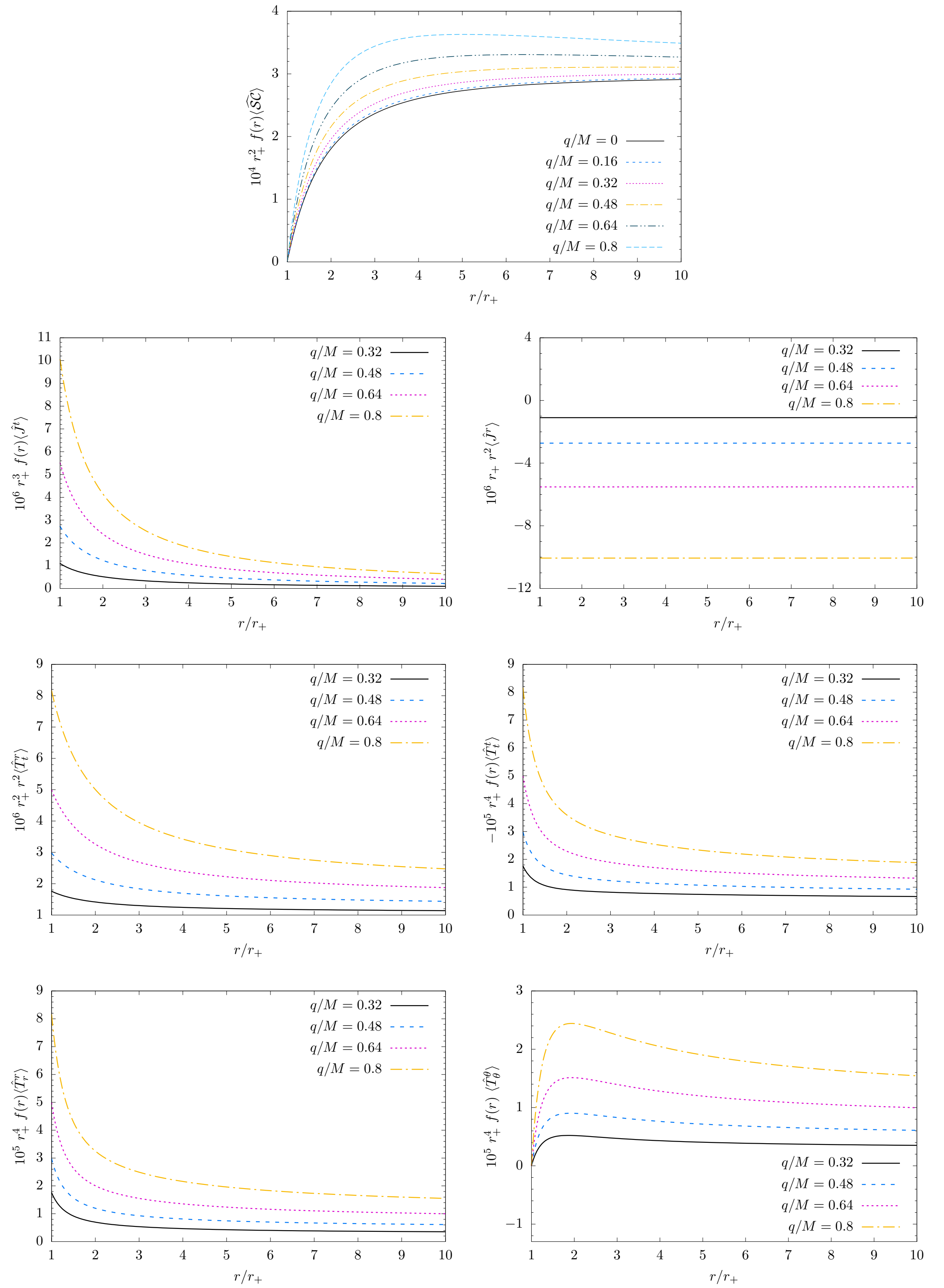

Figure 5.2. Difference in expectation values of the scalar condensate $\widehat{SC}$, components of the current $\hat{J}$ and components of the stress-energy tensor $\hat{T}^\mu_\nu$ between the 'past' CCH state $|\mathrm{CCH}^-\rangle$ and the 'past' Unruh state $|\mathrm{U}^-\rangle$ in Reissner-Nordström spacetime for black hole charge $Q = 0.8M$ and positive values of the scalar field charge q. All expectation values are multiplied by powers of $f(r)$ so that the resulting quantities are regular at $r = r_+$.

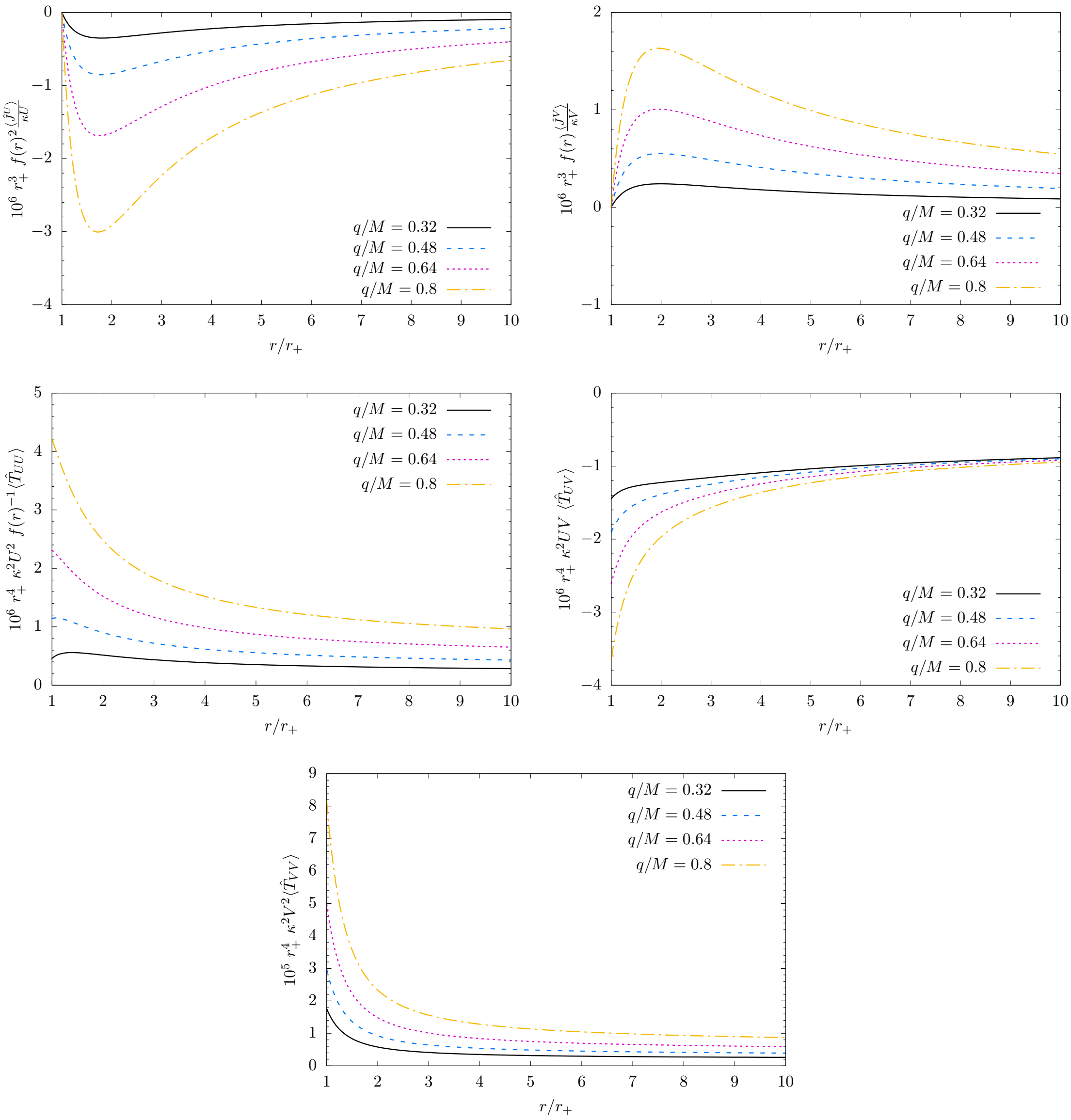

Figure 5.3. Difference in expectation values for the Kruskal components of the current $\hat{J}$ and stress-energy tensor $\hat{T}^{\mu}_{\nu}$ between the 'past' CCH state $|\mathrm{CCH}^-\rangle$ and the 'past' Unruh state $|\mathrm{U}^-\rangle$ in Reissner-Nordström spacetime for black hole charge $Q = 0.8M$ and positive values of the scalar field charge q. All expectation values are multiplied by powers of $f(r)$ so that the resulting quantities are regular at $r = r_+$.

The constant horizontal lines in the plot of $r^2\langle \hat{J}^r\rangle_{|\mathrm{CCH}^-\rangle - |\mathrm{U}^-\rangle}$ illustrate that the difference in the radial component of the current is proportional to r^{-2}. Since $r^2\langle \hat{J}^r\rangle_{|\mathrm{CCH}^-\rangle - |\mathrm{U}^-\rangle}$ is negative then, from (5.12), we have that the difference between the flux of charges $\mathcal{K}_{|\mathrm{CCH}^-\rangle} - \mathcal{K}_{|\mathrm{U}^-\rangle}$ must be positive and so the flux of charge $\mathcal{K}_{|\mathrm{CCH}^-\rangle}$ in the 'past' CCH state is greater than the flux of charge $\mathcal{K}_{|\mathrm{U}^-\rangle}$ in the 'past' Unruh state for a given value of the scalar field charge q.

Furthermore, comparing Figures 5.1 and 5.2, we see that the magnitude of the difference $\langle \hat{J}^r\rangle_{|\mathrm{CCH}^-\rangle - |\mathrm{U}^-\rangle}$ is several times larger than $\langle \hat{J}^r\rangle_{|\mathrm{U}^-\rangle - |\mathrm{B}^-\rangle}$. Recall that the differences in expectation values between $|\mathrm{CCH}^-\rangle$ and $|\mathrm{U}^-\rangle$ should correspond to an incoming flux of thermal radiation in the in-modes with a thermal factor proportional to ω, while the differences in expectation values between $|\mathrm{U}^-\rangle$ and $|\mathrm{B}^-\rangle$ should correspond to an outgoing flux of thermal radiation in the up-modes with a thermal factor proportional to $\widetilde{\omega}$. Then, we can infer that the outgoing flux of charge in the in-modes in $|\mathrm{CCH}^-\rangle$ is considerably greater than the outgoing flux of charge in the up-modes in $|\mathrm{U}^-\rangle$ due to the differing thermal factors.

Discussion of Figure 5.3: From the Penrose diagram of Reissner-Nordström spacetime in Figure 3.1, the past horizon $\mathcal{H}^-$ is a surface of constant $V = 0$ and the future horizon $\mathcal{H}^+$ is a surface of constant $U = 0$.

The difference $V^{-1}\langle \hat{J}^V \rangle_{|\mathrm{CCH}^-\rangle - |\mathrm{U}^-\rangle}$ is regular everywhere as $r \to r_+$. The difference $U^{-1}\langle \hat{J}^U \rangle_{|\mathrm{CCH}^-\rangle - |\mathrm{U}^-\rangle}$ diverges like $f(r)^{-1}$ as $r \to r_+$, which is in agreement with the expression in (5.119a), leading us to conclude that the difference $\langle \hat{J}^\mu \rangle_{|\mathrm{CCH}^-\rangle - |\mathrm{U}^-\rangle}$ is regular on $\mathcal{H}^+$ but diverges on $\mathcal{H}^-$.

The leading order terms in the difference $U^2\langle \hat{T}_{UU} \rangle_{|\mathrm{CCH}^-\rangle - |\mathrm{U}^-\rangle}$ as well as the difference $UV\langle \hat{T}_{UV} \rangle_{|\mathrm{CCH}^-\rangle - |\mathrm{U}^-\rangle}$ cancel as $r \to r_+$. The difference $V^2\langle \hat{T}_{VV} \rangle_{|\mathrm{CCH}^-\rangle - |\mathrm{U}^-\rangle}$ is finite as $r \to r_+$ but does not vanish, which is in agreement with the expression in (5.121c). This leads us to conclude that the difference $\langle \hat{T}^\mu_\nu \rangle_{|\mathrm{CCH}^-\rangle - |\mathrm{U}^-\rangle}$ is similarly regular on $\mathcal{H}^+$ and divergent on $\mathcal{H}^-$.

In §5.3.2, we explained that we expect the 'past' Unruh state $|\mathrm{U}^-\rangle$ to be regular on $\mathcal{H}^+$ but singular on $\mathcal{H}^-$. If we are correct then, from the expressions in (5.119) and (5.121) as well as the plots in Figure 5.2, we expect that the 'past' CCH state $|\mathrm{CCH}^-\rangle$ will be similarly singular on the past horizon $\mathcal{H}^-$ and may also be regular on the future horizon $\mathcal{H}^+$.

5.3.5 Fluxes of charge and energy in the 'past' CCH state

We would like to evaluate the flux of charge $\mathcal{K}_{|\mathrm{CCH}^-\rangle}$ and the flux of energy $\mathcal{L}_{|\mathrm{CCH}^-\rangle}$ with respect to the 'past' CCH state $|\mathrm{CCH}^-\rangle$ directly. We can do this by first simplifying the relevant components of the expressions for $\langle \hat{J}^\mu \rangle_{|\mathrm{CCH}^-\rangle - |\mathrm{U}^-\rangle}$ (5.116) and $\langle \hat{T}^\mu_\nu \rangle_{|\mathrm{CCH}^-\rangle - |\mathrm{U}^-\rangle}$ (5.117) before adding to them, respectively, the expressions for $\langle \hat{J}^r \rangle_{|\mathrm{U}^-\rangle}$ (5.110) and $\langle \hat{T}^r_t \rangle_{|\mathrm{U}^-\rangle}$ (5.112). Beginning with the radial component of the difference $\langle \hat{J}^\mu \rangle_{|\mathrm{CCH}^-\rangle - |\mathrm{U}^-\rangle}$ (5.116), we can simplify this as

$$\begin{aligned}
&\langle \mathrm{CCH}^-|\,\hat{J}^r\,|\mathrm{CCH}^-\rangle - \langle \mathrm{U}^-|\,\hat{J}^r\,|\mathrm{U}^-\rangle \\
&\sim \frac{q}{64\pi^3 r^2}\sum_{\ell=0}^{\infty}\left\{\int_0^{\infty} \mathrm{d}\omega\,\frac{\widetilde{\omega}\,|B^{\mathrm{in}}_{\omega\ell}|^2}{|\omega|\left(\exp\left|\frac{2\pi\omega}{\kappa}\right|-1\right)} + \int_{-\infty}^{0} \mathrm{d}\omega\,\frac{\widetilde{\omega}\,|B^{\mathrm{in}}_{\omega\ell}|^2}{|\omega|\left(\exp\left|\frac{2\pi\omega}{\kappa}\right|-1\right)}\right\}(2\ell+1) \\
&= \frac{q}{64\pi^3 r^2}\sum_{\ell=0}^{\infty}\left\{\int_0^{\infty} \mathrm{d}\omega\,\frac{\widetilde{\omega}\,|B^{\mathrm{in}}_{\omega\ell}|^2}{\omega\left[\exp\left(\frac{2\pi\omega}{\kappa}\right)-1\right]} - \int_{-\infty}^{0} \mathrm{d}\omega\,\frac{\widetilde{\omega}\,|B^{\mathrm{in}}_{\omega\ell}|^2}{\omega\left[\exp\left(-\frac{2\pi\omega}{\kappa}\right)-1\right]}\right\}(2\ell+1) \\
&= \frac{q}{64\pi^3 r^2}\sum_{\ell=0}^{\infty}\left\{\int_0^{\infty} \mathrm{d}\omega\,\frac{\widetilde{\omega}\,|B^{\mathrm{in}}_{\omega\ell}|^2}{\omega\left[\exp\left(\frac{2\pi\omega}{\kappa}\right)-1\right]} - \int_{\infty}^{0} \mathrm{d}(-u)\,\frac{\left(-u-\frac{qQ}{r_+}\right)|B^{\mathrm{in}}_{-u\ell}|^2}{(-u)\left[\exp\left(\frac{2\pi u}{\kappa}\right)-1\right]}\right\}(2\ell+1) \\
&= \frac{q}{64\pi^3 r^2}\sum_{\ell=0}^{\infty}\left\{\int_0^{\infty} \mathrm{d}\omega\,\frac{\widetilde{\omega}\,|B^{\mathrm{in}}_{\omega\ell}|^2}{\omega\left[\exp\left(\frac{2\pi\omega}{\kappa}\right)-1\right]} + \int_{\infty}^{0} \mathrm{d}u\,\frac{\left(u+\frac{qQ}{r_+}\right)|B^{\mathrm{in}}_{-u\ell}|^2}{u\left[\exp\left(\frac{2\pi u}{\kappa}\right)-1\right]}\right\}(2\ell+1) \\
&= \frac{q}{64\pi^3 r^2}\sum_{\ell=0}^{\infty}\left\{\int_0^{\infty} \mathrm{d}\omega\,\frac{\widetilde{\omega}\,|B^{\mathrm{in}}_{\omega\ell}|^2}{\omega\left[\exp\left(\frac{2\pi\omega}{\kappa}\right)-1\right]} - \int_{0}^{\infty} \mathrm{d}\omega\,\frac{\overline{\omega}\,|B^{\mathrm{in}}_{-\omega\ell}|^2}{\omega\left[\exp\left(\frac{2\pi\omega}{\kappa}\right)-1\right]}\right\}(2\ell+1) \\
&= \frac{q}{64\pi^3 r^2}\sum_{\ell=0}^{\infty}\int_0^{\infty}\mathrm{d}\omega\,(2\ell+1)\,\frac{1}{\omega\left[\exp\left(\frac{2\pi\omega}{\kappa}\right)-1\right]}\left(\widetilde{\omega}\,|B^{\mathrm{in}}_{\omega\ell}|^2 - \overline{\omega}\,|B^{\mathrm{in}}_{-\omega\ell}|^2\right), \qquad (5.122)
\end{aligned}$$

where $\overline{\omega}$ is given in (5.107). Using the Wronskian relation (3.75), (5.122) becomes

$$\begin{aligned}
&\langle \mathrm{CCH}^-|\hat{J}^r\,|\mathrm{CCH}^-\rangle - \langle \mathrm{U}^-|\,\hat{J}^r\,|\mathrm{U}^-\rangle \\
&\qquad \sim \frac{q}{64\pi^3 r^2}\sum_{\ell=0}^{\infty}\int_0^{\infty}\mathrm{d}\omega\,(2\ell+1)\,\frac{\omega}{\left[\exp\left(\frac{2\pi\omega}{\kappa}\right)-1\right]}\left[\frac{1}{\widetilde{\omega}}\,|B^{\mathrm{up}}_{\omega\ell}|^2 - \frac{1}{\overline{\omega}}\,|B^{\mathrm{up}}_{-\omega\ell}|^2\right]. \qquad (5.123)
\end{aligned}$$

Using (5.12) and (5.123), we can derive the flux of charge $\mathcal{K}_{|\mathrm{CCH}^-\rangle}$ as

$$\mathcal{K}_{|\mathrm{CCH}^-\rangle} = \mathcal{K}_{|\mathrm{U}^-\rangle} - \frac{q}{64\pi^3}\sum_{\ell=0}^{\infty}\int_0^{\infty}\mathrm{d}\omega\,(2\ell+1)\,\frac{\omega}{\left[\exp\left(\frac{2\pi\omega}{\kappa}\right)-1\right]}\left[\frac{1}{\widetilde{\omega}}\,|B^{\mathrm{up}}_{\omega\ell}|^2 - \frac{1}{\overline{\omega}}\,|B^{\mathrm{up}}_{-\omega\ell}|^2\right]. \qquad (5.124)$$

A similar set of calculations shows that the difference $\langle \hat{T}^r_t\rangle_{|\mathrm{CCH}^-\rangle-|\mathrm{U}^-\rangle}$ is given by

$$\begin{aligned}
&\langle \mathrm{CCH}^-|\hat{T}^r_t\,|\mathrm{CCH}^-\rangle - \langle \mathrm{U}^-|\,\hat{T}^r_t\,|\mathrm{U}^-\rangle \\
&\qquad \sim \frac{1}{16\pi^2 r^2}\sum_{\ell=0}^{\infty}\int_0^{\infty}\mathrm{d}\omega\,(2\ell+1)\,\frac{\omega^2}{\left[\exp\left(\frac{2\pi\omega}{\kappa}\right)-1\right]}\left[\frac{1}{\widetilde{\omega}}\,|B^{\mathrm{up}}_{\omega\ell}|^2 + \frac{1}{\overline{\omega}}\,|B^{\mathrm{up}}_{-\omega\ell}|^2\right]. \qquad (5.125)
\end{aligned}$$

Using (5.25) and (5.125), we can derive the flux of energy $\mathcal{L}_{|\mathrm{CCH}^-\rangle}$ in the 'past' CCH state $|\mathrm{CCH}^-\rangle$ as

$$\mathcal{L}_{|\mathrm{CCH}^-\rangle} = \mathcal{L}_{|\mathrm{U}^-\rangle} - \frac{1}{16\pi^2}\sum_{\ell=0}^{\infty}\int_0^{\infty}\mathrm{d}\omega\,(2\ell+1)\,\frac{\omega^2}{\left[\exp\left(\frac{2\pi\omega}{\kappa}\right)-1\right]}\left[\frac{1}{\widetilde{\omega}}\,|B^{\mathrm{up}}_{\omega\ell}|^2 + \frac{1}{\overline{\omega}}\,|B^{\mathrm{up}}_{-\omega\ell}|^2\right]. \qquad (5.126)$$

We immediately notice, from the argument of the exponential in the denominator of (5.126), that the difference in both fluxes between $|\mathrm{CCH}^-\rangle$ and $|\mathrm{U}^-\rangle$ corresponds to a thermalised flux of particles but without a chemical potential. When defining the 'past' CCH state in §4.5.1, we anticipated that the 'past' CCH state $|\mathrm{CCH}^-\rangle$ would not be an equilibrium state since the in-modes and up-modes in (4.152) had different thermal factors. From the expressions for $\mathcal{K}_{|\mathrm{U}^-\rangle}$ (5.111) and $\mathcal{L}_{|\mathrm{U}^-\rangle}$ (5.113), we see that the fluxes of charge $\mathcal{K}_{|\mathrm{CCH}^-\rangle}$ (5.124) and energy $\mathcal{L}_{|\mathrm{CCH}^-\rangle}$ (5.126) in the 'past' CCH state are indeed nonzero. Therefore, as expected, the 'past' CCH state is not an equilibrium state. However, it does have attractive regularity properties on the future horizon $\mathcal{H}^+$.

Considering first the flux of energy $\mathcal{L}_{|\mathrm{CCH}^-\rangle}$ (5.126) in the 'past' CCH state, it is apparent that nonsuperradiant modes, which have $\mathrm{sgn}(\omega\widetilde{\omega}) = 1$, act to reduce the flux of energy as compared to the flux of energy $\mathcal{L}_{|\mathrm{U}^-\rangle}$ in the 'past' Unruh state. In contrast, superradiant modes, which have $\mathrm{sgn}(\omega\widetilde{\omega}) = -1$, act to enhance $\mathcal{L}_{|\mathrm{CCH}^-\rangle}$ as compared to $\mathcal{L}_{|\mathrm{U}^-\rangle}$. Recall that in §5.3.4, we described that since the 'past' CCH state exhibits thermal radiation at both $\mathscr{I}^-$ and $\mathscr{I}^+$, whereas the 'past' Unruh state only exhibits thermal radiation at $\mathscr{I}^+$ while being as empty as possible to a static observer at $\mathscr{I}^-$, then the difference in observables between these two states should correspond to an incoming flux of thermal radiation from $\mathscr{I}^-$. Then we may tentatively interpret the reduction in $\mathcal{L}_{|\mathrm{CCH}^-\rangle}$ relative to $\mathcal{L}_{|\mathrm{U}^-\rangle}$ in nonsuperradiant modes as incoming thermal radiation from $\mathscr{I}^-$ that is incident upon $\mathcal{H}^+$ and the enhancement in $\mathcal{L}_{|\mathrm{CCH}^-\rangle}$ relative to $\mathcal{L}_{|\mathrm{U}^-\rangle}$ in superradiant modes as corresponding to the same incoming thermal radiation from $\mathscr{I}^-$ that has been reflected back towards $\mathscr{I}^+$ with a greater amplitude through a process of quantum superradiance.

The interpretation of the expression (5.124) for the flux of charge $\mathcal{K}_{|\mathrm{CCH}^-\rangle}$ in the 'past' CCH state is more subtle as a result of the relative sign difference between the contributions from positive- and negative-frequency modes. Of the superradiant modes, those of positive-frequency give a contribution to $\mathcal{K}_{|\mathrm{CCH}^-\rangle}$ of the opposite sign to q, while those of negative-frequency give a contribution of the same sign as q. Of the nonsuperradiant modes, however, those of positive-frequency give a contribution to $\mathcal{K}_{|\mathrm{CCH}^-\rangle}$ of the same sign as q, while those of negative-frequency give a contribution of the opposite sign to q.

Returning to the plot of the difference $\langle \hat{J}^r \rangle_{|\mathrm{CCH}^-\rangle - |\mathrm{U}^-\rangle}$ in Figure 5.2 we see that $\langle \hat{J}^r \rangle_{|\mathrm{CCH}^-\rangle - |\mathrm{U}^-\rangle}$ is negative and therefore that the difference between the fluxes $\mathcal{K}_{|\mathrm{CCH}^-\rangle} - \mathcal{K}_{|\mathrm{U}^-\rangle}$ must be positive for positive values of the scalar field charge q. Then, we deduce that there must be considerably more contribution to the flux of charge $\mathcal{K}_{|\mathrm{CCH}^-\rangle}$ in the 'past' CCH state from positive-frequency superradiant modes and negative-frequency nonsuperradiant modes, as opposed to negative-frequency superradiant modes and positive-frequency nonsuperradiant modes.

5.4 Differences in expectation values between 'future' states

5.4.1 The 'future' Unruh state

We can use the expectation value of a general observable $\hat{O}$ with classical mode contribution $o_{\omega\ell m}$ to construct an explicit expression for the difference $\langle \hat{O} \rangle_{|\mathrm{U}^+\rangle - |\mathrm{B}^+\rangle}$ in expectation values in the 'future' Boulware state (5.79) and the 'future' Unruh state (5.82); we have

$$\langle \mathrm{U}^+ |\, \hat{O}\, | \mathrm{U}^+ \rangle - \langle \mathrm{B}^+ |\, \hat{O}\, | \mathrm{B}^+ \rangle = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \int_{-\infty}^{\infty} \mathrm{d}\widetilde{\omega}\, \frac{1}{\exp\left|\frac{2\pi\widetilde{\omega}}{\kappa}\right| - 1}\, o^{\mathrm{down}}_{\omega\ell m}, \tag{5.127}$$

where we have used the identity in (5.87). In §5.2.2, we evaluated the asymptotic down-mode contributions to the classical quantities corresponding to the quantum observables $\widehat{\mathcal{SC}}$, $\hat{J}^\mu$ and $\hat{T}^\mu_\nu$ as $r \to \infty$. Using the near-infinity down-mode contribution (5.47) to the scalar condensate $\mathcal{SC}$, (5.127) becomes

$$\langle \mathrm{U}^+ |\, \widehat{\mathcal{SC}}\, | \mathrm{U}^+ \rangle - \langle \mathrm{B}^+ |\, \widehat{\mathcal{SC}}\, | \mathrm{B}^+ \rangle \sim \frac{1}{16\pi^2 r^2} \sum_{\ell=0}^{\infty} \int_{-\infty}^{\infty} \mathrm{d}\widetilde{\omega}\, \frac{2\ell+1}{|\widetilde{\omega}|\left(\exp\left|\frac{2\pi\widetilde{\omega}}{\kappa}\right| - 1\right)} \left|B^{\mathrm{up}}_{\omega\ell}\right|^2. \tag{5.128}$$

Using the near-infinity down-mode contribution (5.50) to the current J^μ, (5.127) becomes

$$\begin{aligned} &\langle \mathrm{U}^+ |\, \hat{J}^\mu\, | \mathrm{U}^+ \rangle - \langle \mathrm{B}^+ |\, \hat{J}^\mu\, | \mathrm{B}^+ \rangle \\ &\qquad \sim \frac{q}{64\pi^3 r^2} \sum_{\ell=0}^{\infty} \int_{-\infty}^{\infty} \mathrm{d}\widetilde{\omega}\, \frac{\omega\,(2\ell+1)}{|\widetilde{\omega}|\left(\exp\left|\frac{2\pi\widetilde{\omega}}{\kappa}\right| - 1\right)} \left|B^{\mathrm{up}}_{\omega\ell}\right|^2 \begin{pmatrix} -1 & 1 & 0 & 0 \end{pmatrix}^{\mathsf{T}}. \end{aligned} \tag{5.129}$$

Using the near-infinity down-mode contribution (5.54) to the SET T^μ_ν, (5.127) becomes

$$\begin{aligned} &\langle \mathrm{U}^+ |\, \hat{T}^\mu_\nu\, | \mathrm{U}^+ \rangle - \langle \mathrm{B}^+ |\, \hat{T}^\mu_\nu\, | \mathrm{B}^+ \rangle \\ &\sim \frac{1}{16\pi^2 r^2} \sum_{\ell=0}^{\infty} \int_{-\infty}^{\infty} \mathrm{d}\widetilde{\omega}\, \frac{\omega^2\,(2\ell+1)}{|\widetilde{\omega}|\left(\exp\left|\frac{2\pi\widetilde{\omega}}{\kappa}\right| - 1\right)} \left|B^{\mathrm{up}}_{\omega\ell}\right|^2 \begin{pmatrix} -1 & -1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & \mathcal{O}\left(r^{-2}\right) & 0 \\ 0 & 0 & 0 & \mathcal{O}\left(r^{-2}\right) \end{pmatrix}. \end{aligned} \tag{5.130}$$

The differences (5.128 – 5.130) are regular when $\widetilde{\omega} = 0$ from the Wronskian relation (3.75); the $\left|B^{\mathrm{up}}_{\omega\ell}\right|^2$ is of $\mathcal{O}\left(\widetilde{\omega}^{\,2}\right)$, which cancels the factor of $\mathcal{O}\left(\widetilde{\omega}^{-2}\right)$ in the denominators as $\widetilde{\omega} \to 0$.

5.4.2 The 'future' CCH state

We can use the expectation value of a general observable $\hat{O}$ with classical mode contribution $o_{\omega\ell m}$ to construct an explicit expression for the difference $\langle \hat{O} \rangle_{|\mathrm{CCH}^+\rangle - |\mathrm{U}^+\rangle}$ in expectation values in the 'future' Unruh state (5.82) and the 'future' CCH state (5.84); we have

$$\langle \mathrm{CCH}^+ |\, \hat{O}\, | \mathrm{CCH}^+ \rangle - \langle \mathrm{U}^+ |\, \hat{O}\, | \mathrm{U}^+ \rangle = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \int_{-\infty}^{\infty} \mathrm{d}\omega\, \frac{1}{\exp\left|\frac{2\pi\omega}{\kappa}\right| - 1}\, o^{\mathrm{out}}_{\omega\ell m}, \tag{5.131}$$

where we have used the identity in (5.87). In §5.2.3, we evaluated the asymptotic out-mode contributions to the classical quantities corresponding to the quantum observables

$\widehat{\mathcal{SC}}$, $\hat{J}^{\mu}$ and $\hat{T}^{\mu}_{\nu}$ as $r \to r_+$. Using the near-horizon out-mode contribution (5.71) to the scalar condensate $\mathcal{SC}$, (5.131) becomes

$$\langle \mathrm{CCH}^+ | \, \widehat{\mathcal{SC}} \, | \mathrm{CCH}^+ \rangle - \langle \mathrm{U}^+ | \, \widehat{\mathcal{SC}} \, | \mathrm{U}^+ \rangle \sim \frac{1}{16\pi^2 r^2} \sum_{\ell=0}^{\infty} \int_{-\infty}^{\infty} \mathrm{d}\omega \, \frac{2\ell+1}{|\omega| \left(\exp\left| \frac{2\pi\omega}{\kappa} \right| - 1 \right)} \left| B^{\mathrm{in}}_{\omega\ell} \right|^2 . \tag{5.132}$$

Using the near-horizon out-mode contribution (5.74) to the current J^{μ}, (5.131) becomes

$$\begin{aligned} &\langle \mathrm{CCH}^+ | \, \hat{J}^{\mu} \, | \mathrm{CCH}^+ \rangle - \langle \mathrm{U}^+ | \, \hat{J}^{\mu} \, | \mathrm{U}^+ \rangle \\ &\quad \sim -\frac{q}{64\pi^3 r^2} \sum_{\ell=0}^{\infty} \int_{-\infty}^{\infty} \mathrm{d}\omega \, \frac{\widetilde{\omega} \, (2\ell+1)}{|\omega| \left(\exp\left| \frac{2\pi\omega}{\kappa} \right| - 1 \right)} \left| B^{\mathrm{in}}_{\omega\ell} \right|^2 \begin{pmatrix} f(r)^{-1} & 1 & 0 & 0 \end{pmatrix}^{\mathsf{T}} . \end{aligned} \tag{5.133}$$

Using the near-horizon out-mode contribution (5.77) to the SET T^{μ}_{ν}, (5.131) becomes

$$\begin{aligned} &\langle \mathrm{CCH}^+ | \, \hat{T}^{\mu}_{\nu} \, | \mathrm{CCH}^+ \rangle - \langle \mathrm{U}^+ | \, \hat{T}^{\mu}_{\nu} \, | \mathrm{U}^+ \rangle \\ &\sim \frac{1}{16\pi^2 r^2} \sum_{\ell=0}^{\infty} \int_{-\infty}^{\infty} \mathrm{d}\omega \, \frac{\widetilde{\omega}^2 \, (2\ell+1)}{|\omega| \left(\exp\left| \frac{2\pi\omega}{\kappa} \right| - 1 \right)} \left| B^{\mathrm{in}}_{\omega\ell} \right|^2 \begin{pmatrix} -f(r)^{-1} & f(r)^{-2} & 0 & 0 \\ -1 & f(r)^{-1} & 0 & 0 \\ 0 & 0 & \mathcal{O}(1) & 0 \\ 0 & 0 & 0 & \mathcal{O}(1) \end{pmatrix} . \end{aligned} \tag{5.134}$$

The differences (5.132 – 5.134) are regular when $\omega = 0$ from the Wronskian relation (3.75); the $\left| B^{\mathrm{in}}_{\omega\ell} \right|^2$ is of $\mathcal{O}(\omega^2)$, which cancels the factor of $\mathcal{O}(\omega^{-2})$ in the denominators as $\widetilde{\omega} \to 0$.

5.4.3 Discussion of differences between 'future' states

In §5.1.1, we argued that the expectation value of the scalar condensate $\widehat{\mathcal{SC}}$ should not distinguish between 'past' and 'future' states. Comparing (5.95) with (5.128) and (5.115) with (5.132), we can see that this is indeed the case. Furthermore, we see that the differences in the current $\hat{J}^{\mu}$ (5.129, 5.133) as well as the SET $\hat{T}^{\mu}_{\nu}$ (5.130, 5.134) could have been obtained, by making the coordinate transformation $t \to -t$, from their corresponding expressions in the 'past' states in (5.96, 5.116) and (5.97, 5.117) respectively. In this light, we can consider the 'future' states as the time-reversal of their corresponding 'future' states. Since we have examined the properties of the 'past' states in considerable detail in §5.2.4, we will not consider the 'future' states any further.

5.5 The 'Boulware-like' state

In §4.3.3, we defined the 'Boulware-like' state $|\mathrm{B}\rangle$ which is an attempt to keep as close in spirit as possible to the Schwarzschild Boulware state $|\mathrm{B_s}\rangle$ in defining a state that is as empty as possible to a static observer at both past and future null infinity $\mathscr{I}^{\pm}$. Thus, the fluxes of charge $\mathcal{K}_{|\mathrm{B}\rangle}$ and energy $\mathcal{L}_{|\mathrm{B}\rangle}$ in the 'Boulware-like' state are of particular interest.

5.5.1 Differences in expectation values between Boulware states

Having studied the 'past' and 'future' Boulware states in considerable detail in §5.3.1 and §5.4.1 respectively, we can use the expectation value of a general observable $\hat{O}$ with classical mode contribution $o_{\omega\ell m}$ to construct an expression for the difference $\langle\hat{O}\rangle_{|\mathrm{B}\rangle-|\mathrm{B}^{\pm}\rangle}$ in expectation values in the 'Boulware-like' state (5.80) relative to both the 'past' (5.78) and the 'future' Boulware state (5.79); we have

$$\langle \mathrm{B}|\,\hat{O}\,|\mathrm{B}\rangle - \langle \mathrm{B}^-|\,\hat{O}\,|\mathrm{B}^-\rangle = -\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\int_{\min\left\{\frac{qQ}{r_+},0\right\}}^{\max\left\{\frac{qQ}{r_+},0\right\}} \mathrm{d}\omega\, o^{\mathrm{up}}_{\omega\ell m}, \tag{5.135}$$

and

$$\langle \mathrm{B}|\,\hat{O}\,|\mathrm{B}\rangle - \langle \mathrm{B}|\,\hat{O}\,|\mathrm{B}^+\rangle = -\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\int_{\min\left\{\frac{qQ}{r_+},0\right\}}^{\max\left\{\frac{qQ}{r_+},0\right\}} \mathrm{d}\omega\, o^{\mathrm{down}}_{\omega\ell m}, \tag{5.136}$$

where, in (5.136), we have used the fact that we could alternatively have defined the 'Boulware-like' state in an orthonormal basis of out- and down-modes. In §5.2.2, we evaluated the asymptotic up- and down-mode contributions to the classical quantities corresponding to the quantum observables $\widehat{\mathcal{SC}}$, $\hat{J}^{\mu}$ and $\hat{T}^{\mu}_{\nu}$ as $r\to\infty$. Using the near-infinity up-mode contribution (5.36) to the scalar condensate $\widehat{\mathcal{SC}}$, (5.135) becomes

$$\begin{aligned}\langle \mathrm{B}|\,\widehat{\mathcal{SC}}\,|\mathrm{B}\rangle - \langle \mathrm{B}^-|\,\widehat{\mathcal{SC}}\,|\mathrm{B}^-\rangle &= \langle \mathrm{B}|\,\widehat{\mathcal{SC}}\,|\mathrm{B}\rangle - \langle \mathrm{B}^+|\,\widehat{\mathcal{SC}}\,|\mathrm{B}^+\rangle \\ &\sim -\frac{1}{16\pi^2 r^2}\sum_{\ell=0}^{\infty}\int_{\min\left\{\frac{qQ}{r_+},0\right\}}^{\max\left\{\frac{qQ}{r_+},0\right\}} \mathrm{d}\omega\,\frac{1}{|\widetilde{\omega}|}\,(2\ell+1)\left|B^{\mathrm{up}}_{\omega\ell}\right|^2, \end{aligned}\tag{5.137}$$

where we have used the fact that the scalar condensate $\widehat{\mathcal{SC}}$ does not distinguish between 'past' and 'future' states to equate $\langle\widehat{\mathcal{SC}}\rangle_{|\mathrm{B}\rangle-|\mathrm{B}^-\rangle}$ with $\langle\widehat{\mathcal{SC}}\rangle_{|\mathrm{B}\rangle-|\mathrm{B}^+\rangle}$. Using the near-infinity up-mode contribution (5.39) to the current J^{μ}, (5.135) becomes

$$\begin{aligned}&\langle \mathrm{B}|\,\hat{J}^{\mu}\,|\mathrm{B}\rangle - \langle \mathrm{B}^-|\,\hat{J}^{\mu}\,|\mathrm{B}^-\rangle \\ &\qquad\sim \frac{q}{64\pi^3 r^2}\sum_{\ell=0}^{\infty}\int_{\min\left\{\frac{qQ}{r_+},0\right\}}^{\max\left\{\frac{qQ}{r_+},0\right\}} \mathrm{d}\omega\,\frac{\omega}{|\widetilde{\omega}|}\,(2\ell+1)\left|B^{\mathrm{up}}_{\omega\ell}\right|^2\begin{pmatrix}1 & 1 & 0 & 0\end{pmatrix}^{\mathsf{T}}. \end{aligned}\tag{5.138}$$

The difference in the flux of charge across past null infinity $\mathscr{I}^-$ between $|\mathrm{B}\rangle$ and $|\mathrm{B}^-\rangle$ is proportional to the difference in the Kruskal component $\langle\hat{J}^{U}\rangle_{|\mathrm{B}\rangle-|\mathrm{B}^-\rangle}$. Using the expression (5.28) for the current in terms of Kruskal coordinates, as $r\to\infty$ we have

$$\begin{aligned}&\langle \mathrm{B}|\,\hat{J}^{U}\,|\mathrm{B}\rangle - \langle \mathrm{B}^-|\,\hat{J}^{U}\,|\mathrm{B}^-\rangle \\ &\qquad\sim \frac{\kappa q U}{64\pi^3 r^2}\sum_{\ell=0}^{\infty}\int_{\min\left\{\frac{qQ}{r_+},0\right\}}^{\max\left\{\frac{qQ}{r_+},0\right\}} \mathrm{d}\omega\,\frac{\omega}{|\widetilde{\omega}|}\,(2\ell+1)\left|B^{\mathrm{up}}_{\omega\ell}\right|^2(-1+1)=0. \end{aligned}\tag{5.139}$$

Using the near-infinity up-mode contribution (5.46) to the SET T^{μ}_{ν}, (5.135) becomes

$$\begin{aligned}&\langle \mathrm{B}|\,\hat{T}^{\mu}_{\nu}\,|\mathrm{B}\rangle - \langle \mathrm{B}^{-}|\,\hat{T}^{\mu}_{\nu}\,|\mathrm{B}^{-}\rangle \\ &\sim \frac{1}{16\pi^2 r^2}\sum_{\ell=0}^{\infty}\int_{\min\left\{\frac{qQ}{r_+},0\right\}}^{\max\left\{\frac{qQ}{r_+},0\right\}} \mathrm{d}\omega\,\frac{\omega^2}{|\widetilde{\omega}|}\,(2\ell+1)\left|B^{\mathrm{up}}_{\omega\ell}\right|^2 \begin{pmatrix} 1 & -1 & 0 & 0\\ 1 & -1 & 0 & 0\\ 0 & 0 & \mathcal{O}\big(r^{-2}\big) & 0\\ 0 & 0 & 0 & \mathcal{O}\big(r^{-2}\big)\end{pmatrix}. \end{aligned} \tag{5.140}$$

The difference in the flux of energy across past null infinity $\mathscr{I}^-$, which is a surface of constant $U=-\infty$, between $|\mathrm{B}\rangle$ and $|\mathrm{B}^-\rangle$ is proportional to the difference $\langle\hat{T}_{UU}\rangle_{|\mathrm{B}\rangle-|\mathrm{B}^-\rangle}$. Using the expression (5.30) for the stress-energy tensor in terms of Kruskal coordinates, as $r\to\infty$ we have $\langle\hat{T}_{UU}\rangle_{|\mathrm{B}\rangle-|\mathrm{B}^-\rangle}\sim\mathcal{O}\big(r^{-2}U^{-2}\big)$ which vanishes at leading order.

Using the down-mode contribution (5.50) to the current J^{μ} as $r\to\infty$, (5.127) becomes

$$\begin{aligned}&\langle \mathrm{B}|\,\hat{J}^{\mu}\,|\mathrm{B}\rangle - \langle \mathrm{B}^{+}|\,\hat{J}^{\mu}\,|\mathrm{B}^{+}\rangle \\ &\sim \frac{q}{64\pi^3 r^2}\sum_{\ell=0}^{\infty}\int_{\min\left\{\frac{qQ}{r_+},0\right\}}^{\max\left\{\frac{qQ}{r_+},0\right\}} \mathrm{d}\omega\,\frac{\omega}{|\widetilde{\omega}|}\,(2\ell+1)\left|B^{\mathrm{up}}_{\omega\ell}\right|^2 \begin{pmatrix}1 & -1 & 0 & 0\end{pmatrix}^{\mathsf{T}}. \end{aligned} \tag{5.141}$$

The difference in the flux of charge across future null infinity $\mathscr{I}^+$ between $|\mathrm{B}\rangle$ and $|\mathrm{B}^+\rangle$ is proportional to the difference in the Kruskal component $\langle\hat{J}^{V}\rangle_{|\mathrm{B}\rangle-|\mathrm{B}^+\rangle}$. Using the expression (5.28) for the current in terms of Kruskal coordinates, as $r\to\infty$ we have

$$\begin{aligned}&\langle \mathrm{B}|\,\hat{J}^{V}\,|\mathrm{B}\rangle - \langle \mathrm{B}^{+}|\,\hat{J}^{V}\,|\mathrm{B}^{+}\rangle \\ &\sim \frac{\kappa q V}{64\pi^3 r^2}\sum_{\ell=0}^{\infty}\int_{\min\left\{\frac{qQ}{r_+},0\right\}}^{\max\left\{\frac{qQ}{r_+},0\right\}} \mathrm{d}\omega\,\frac{\omega}{|\widetilde{\omega}|}\,(2\ell+1)\left|B^{\mathrm{up}}_{\omega\ell}\right|^2(1-1)=0. \end{aligned} \tag{5.142}$$

Using the near-infinity down-mode contribution (5.54) to the SET T^{μ}_{ν}, (5.136) becomes

$$\begin{aligned}&\langle \mathrm{B}|\,\hat{T}^{\mu}_{\nu}\,|\mathrm{B}\rangle - \langle \mathrm{B}^{+}|\,\hat{T}^{\mu}_{\nu}\,|\mathrm{B}^{+}\rangle \\ &\sim \frac{1}{16\pi^2 r^2}\sum_{\ell=0}^{\infty}\int_{\min\left\{\frac{qQ}{r_+},0\right\}}^{\max\left\{\frac{qQ}{r_+},0\right\}} \mathrm{d}\omega\,\frac{\omega^2}{|\widetilde{\omega}|}\,(2\ell+1)\left|B^{\mathrm{up}}_{\omega\ell}\right|^2 \begin{pmatrix} 1 & 1 & 0 & 0\\ -1 & -1 & 0 & 0\\ 0 & 0 & \mathcal{O}\big(r^{-2}\big) & 0\\ 0 & 0 & 0 & \mathcal{O}\big(r^{-2}\big)\end{pmatrix}. \end{aligned} \tag{5.143}$$

The difference in the flux of energy across future null infinity $\mathscr{I}^+$, which is a surface of constant $V=\infty$, between $|\mathrm{B}\rangle$ and $|\mathrm{B}^+\rangle$ is proportional to the difference $\langle\hat{T}_{VV}\rangle_{|\mathrm{B}\rangle-|\mathrm{B}^-\rangle}$. Using the expression (5.30) for the stress-energy tensor in terms of Kruskal coordinates, as $r\to\infty$ we have $\langle\hat{T}_{VV}\rangle_{|\mathrm{B}\rangle-|\mathrm{B}^+\rangle}=\mathcal{O}\big(r^{-2}V^{-2}\big)$ which vanishes at leading order.

The differences (5.137 – 5.143) are regular when $\widetilde{\omega}=0$ from the relation (3.75); the $\left|B^{\mathrm{up}}_{\omega\ell}\right|^2$ is of $\mathcal{O}\big(\widetilde{\omega}^{\,2}\big)$, which cancels the factor of $\mathcal{O}\big(\widetilde{\omega}^{-1}\big)$ in the denominators as $\widetilde{\omega}\to0$.

5.5.2 Fluxes of charge and energy in the 'Boulware-like' state

We would like to evaluate the flux of charge $\mathcal{K}_{|\mathrm{B}\rangle}$ and the flux of energy $\mathcal{L}_{|\mathrm{B}\rangle}$ in the 'Boulware-like' state $|\mathrm{B}\rangle$ directly. Using the expression for the expectation value of the radial component of the current in the 'past' Boulware state $\langle \hat{J}^r\rangle_{|\mathrm{B}^-\rangle}$ (5.91) and the radial component of the difference $\langle \hat{J}^\mu\rangle_{|\mathrm{B}\rangle-|\mathrm{B}^-\rangle}$ (5.138), we have

$$\begin{aligned}\langle \hat{J}^r\rangle_{|\mathrm{B}\rangle} &= \Big(\langle \mathrm{B}|\, \hat{J}^r\, |\mathrm{B}\rangle - \langle \mathrm{B}^-|\, \hat{J}^r\, |\mathrm{B}^-\rangle\Big) + \langle \mathrm{B}^-|\, \hat{J}^r\, |\mathrm{B}^-\rangle \\ &\sim \frac{q}{64\pi^3 r^2}\sum_{\ell=0}^{\infty}\int_{\min\left\{\frac{qQ}{r_+},0\right\}}^{\max\left\{\frac{qQ}{r_+},0\right\}} \mathrm{d}\omega\, \frac{\omega}{|\widetilde{\omega}|}\,(2\ell+1)\left|B^{\mathrm{up}}_{\omega\ell}\right|^2(1-1) = 0. \end{aligned} \tag{5.144}$$

A similar calculation shows that the expectation value $\langle \hat{T}^r_t\rangle_{|\mathrm{B}\rangle}$ also vanishes. Therefore, we find that both the flux of charge $\mathcal{K}_{|\mathrm{B}\rangle}$ and the flux of energy $\mathcal{L}_{|\mathrm{B}\rangle}$ in the 'Boulware-like' state vanish, i.e.

$$\mathcal{K}_{|\mathrm{B}\rangle} = 0, \qquad \mathcal{L}_{|\mathrm{B}\rangle} = 0. \tag{5.145}$$

5.5.3 Interpretation of the 'Boulware-like' state

The fact that the fluxes of charge $\mathcal{K}_{|\mathrm{B}\rangle}$ and energy $\mathcal{L}_{|\mathrm{B}\rangle}$ in the 'Boulware-like' state $|\mathrm{B}\rangle$ vanish means that this state is an equilibrium state and it is invariant under time-reversal.

Recall that the 'past' Boulware state (4.28) is defined to be as empty as possible to a static observer at past null infinity $\mathscr{I}^-$ and that the 'future' Boulware state (4.38) is defined to be as empty as possible to a static observer at past null infinity $\mathscr{I}^+$. Since the differences $\langle \hat{J}^U\rangle_{|\mathrm{B}\rangle-|\mathrm{B}^-\rangle}$ and $\langle \hat{T}_{UU}\rangle_{|\mathrm{B}\rangle-|\mathrm{B}^-\rangle}$ vanish, then we can conclude that the 'Boulware-like' state has no incoming flux of particles at past null infinity $\mathscr{I}^-$. Furthermore, since the differences $\langle \hat{J}^V\rangle_{|\mathrm{B}\rangle-|\mathrm{B}^-\rangle}$ and $\langle \hat{T}_{VV}\rangle_{|\mathrm{B}\rangle-|\mathrm{B}^-\rangle}$ vanish, then we can conclude that the 'Boulware-like' state has no outgoing flux of particles at future null infinity $\mathscr{I}^+$.

Thus, it appears that, in defining the 'Boulware-like' state $|\mathrm{B}\rangle$, we have succeeded in defining an analogue of the Schwarzschild Boulware state for a charged scalar field in Reissner-Nordström spacetime, which is as empty as possible to a static observer at both past and future null infinity $\mathscr{I}^\pm$.

5.5.4 Discussion of Figures 5.4 and 5.5

All of the plots of the differences of observables between $|\mathrm{B}\rangle$ and $|\mathrm{B}^-\rangle$ in Figure 5.4 vanish in the uncharged limit $q \to 0$. Recall, from §5.3.1, that the 'past' Boulware state is defined to be as empty as possible to a static observer at $\mathscr{I}^-$ but exhibits an outgoing flux of radiation in the superradiant modes at $\mathscr{I}^+$. In the uncharged limit $q \to 0$, there is no superradiant scattering of field modes and so $|\mathrm{B}\rangle$ and $|\mathrm{B}^-\rangle$ coincide in this limit.

The difference in the scalar condensate vanishes far from the black hole but diverges near the horizon. Unlike the difference $\langle \widehat{\mathcal{SC}}\rangle_{|\mathrm{U}^-\rangle-|\mathrm{B}^-\rangle}$ (see Figure 5.1), the difference $\langle \widehat{\mathcal{SC}}\rangle_{|\mathrm{B}\rangle-|\mathrm{B}^-\rangle}$ is negative leading us to conclude that $\langle \widehat{\mathcal{SC}}\rangle_{|\mathrm{B}\rangle}$ is smaller than both $\langle \widehat{\mathcal{SC}}\rangle_{|\mathrm{B}^-\rangle}$ and $\langle \widehat{\mathcal{SC}}\rangle_{|\mathrm{U}^-\rangle}$.

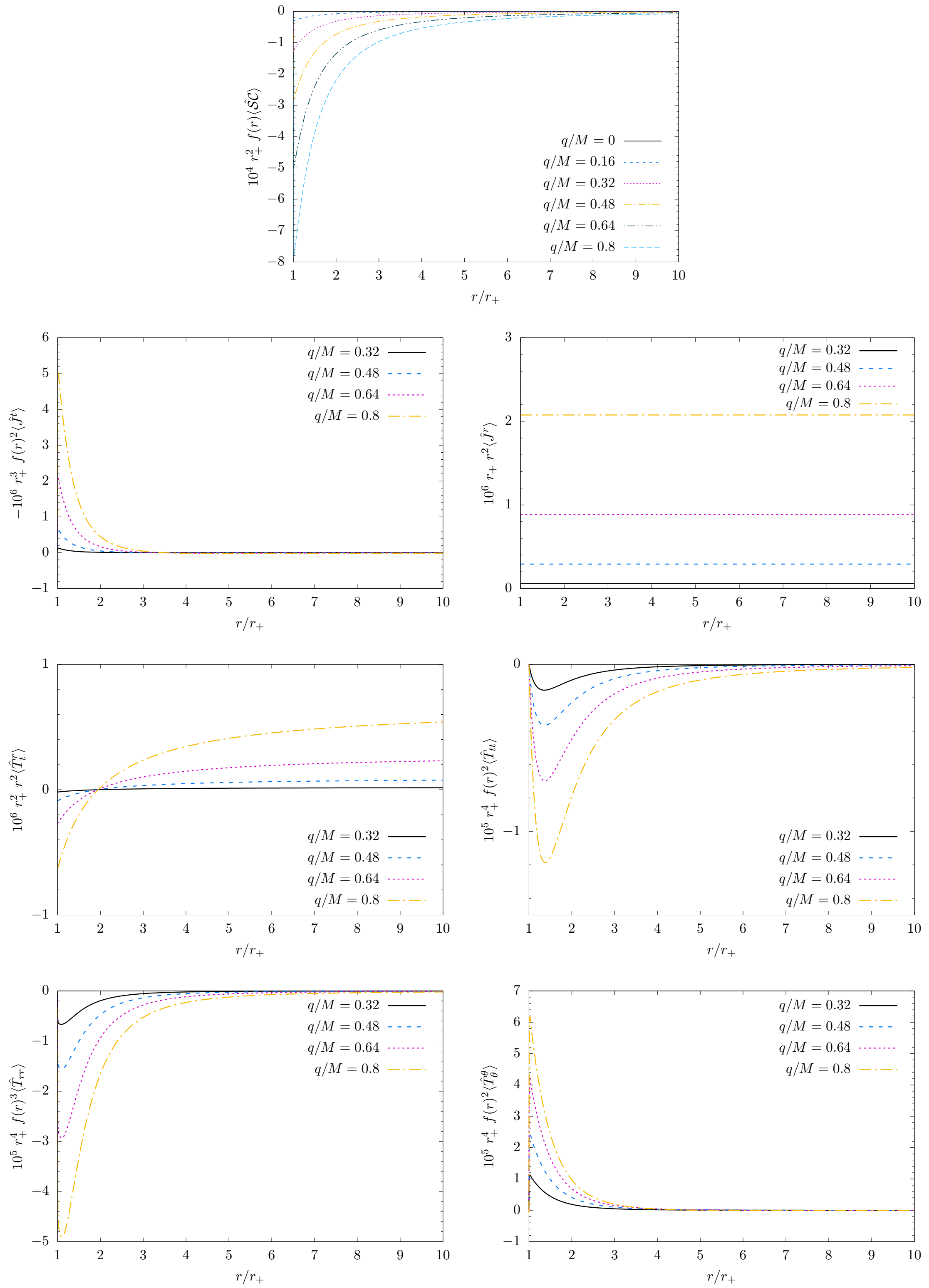

Figure 5.4. Difference in expectation values of the scalar condensate $\widehat{\mathcal{SC}}$, components of the current $\hat{J}$ and components of the stress-energy tensor $\hat{T}^\mu_\nu$ between the 'Boulware-like' state $|\mathrm{B}\rangle$ and the 'past' Boulware state $|\mathrm{B}^-\rangle$ in Reissner-Nordström spacetime for black hole charge $Q = 0.8M$ and positive values of the scalar field charge q. All expectation values are multiplied by powers of $f(r)$ so that the resulting quantities are regular at $r = r_+$.

The differences in $\langle \hat{J}^r \rangle_{|\mathrm{B}\rangle - |\mathrm{B}^-\rangle}$ and $\langle \hat{T}^r_t \rangle_{|\mathrm{B}\rangle - |\mathrm{B}^-\rangle}$ are exactly minus that of $\langle \hat{J}^r \rangle_{|\mathrm{B}^-\rangle}$ and $\langle \hat{T}^r_t \rangle_{|\mathrm{B}^-\rangle}$ in [1]. This is what we would expect; since the fluxes of charge $\mathcal{K}_{|\mathrm{B}\rangle}$ and energy $\mathcal{L}_{|\mathrm{B}\rangle}$ in the 'Boulware-like' state vanish, there is no contribution to the differences $\langle \hat{J}^r \rangle_{|\mathrm{B}\rangle - |\mathrm{B}^-\rangle}$ and $\langle \hat{T}^r_t \rangle_{|\mathrm{B}\rangle - |\mathrm{B}^-\rangle}$ from the 'Boulware-like' state and they reduce to $-\langle \hat{J}^r \rangle_{|\mathrm{B}^-\rangle}$ and $-\langle \hat{T}^r_t \rangle_{|\mathrm{B}^-\rangle}$ respectively.

The difference $\langle \hat{J}^t \rangle_{|\mathrm{B}\rangle - |\mathrm{B}^-\rangle}$ in the time component of the current as well as the differences in the diagonal components of the stress-energy tensor all decay rapidly as the radial coordinate r increases and vanish far from the black hole. Furthermore, the difference in the charge density $\langle \hat{J}^t \rangle_{|\mathrm{B}\rangle - |\mathrm{B}^-\rangle}$ and the energy density $\langle \hat{T}^t_t \rangle_{|\mathrm{B}\rangle - |\mathrm{B}^-\rangle}$ are negative near the horizon, but become positive further away from the black hole.

All of the differences, in terms of Kruskal coordinates, in Figure 5.5 diverge at the horizon. Since the Schwarzschild Boulware state $|\mathrm{B_s}\rangle$ is singular at the horizon, we expect that both the 'Boulware-like' state $|\mathrm{B}\rangle$ and the 'past' Boulware state $|\mathrm{B}^-\rangle$ will diverge at the horizon. Then, we can expect that either $|\mathrm{B}^-\rangle$ diverges more rapidly than $|\mathrm{B}\rangle$ as $r \to \infty$, or that both of these states diverge at the same rate but with different coefficients. We suspect that the latter is more likely, but only a computation of renormalised expectation values with respect to either state directly would be able to prove this.

The difference $U^{-1} \langle \hat{J}^U \rangle_{|\mathrm{B}\rangle - |\mathrm{B}^-\rangle}$ decays rapidly far from the black hole, as we would expect since past null infinity $\mathscr{I}^-$ is a surface of constant $U = \infty$. The difference $V^{-1} \langle \hat{J}^V \rangle_{|\mathrm{B}\rangle - |\mathrm{B}^-\rangle}$ also decays as $r \to \infty$, although not as rapidly.

In our interpretation of the 'Boulware-like' state, we reasoned that $|\mathrm{B}\rangle$ is time-reversal invariant. Then $U^{-1} \langle \hat{J}^U \rangle_{|\mathrm{B}\rangle}$ should equal $-V^{-1} \langle \hat{J}^V \rangle_{|\mathrm{B}\rangle}$. From the plots, however, we have that $U^{-1} \langle \hat{J}^U \rangle_{|\mathrm{B}\rangle - |\mathrm{B}^-\rangle} \neq -V^{-1} \langle \hat{J}^V \rangle_{|\mathrm{B}\rangle - |\mathrm{B}^-\rangle}$ constituting further evidence that the 'past' Boulware state $|\mathrm{B}^-\rangle$ is not time-reversal invariant.

5.5.5 Conclusions

The fact that the fluxes of charge $\mathcal{K}_{|\mathrm{B}\rangle}$ and energy $\mathcal{L}_{|\mathrm{B}\rangle}$ vanish (5.145), as well as the plots in Figures 5.4 and 5.5 lead us to the conclusion that our proposed 'Boulware-like' state is an equilibrium state which is regular everywhere outside the horizon and has vanishing fluxes of charge and energy.

However, when defining the 'Boulware-like' state in §4.3.3, we used non-standard commutation relations (4.49) that were multiplied by the eta-function $\eta_{\omega\widetilde{\omega}}$ (4.9). Therefore, $|\mathrm{B}\rangle$ is not a conventional vacuum state. In [50], it is shown that for a neutral scalar field on a background Kerr spacetime, there is no state which is as empty as possible at both past and future null infinity $\mathscr{I}^{\pm}$. In this light, our results for vanishing flux in the 'Boulware-like' state (5.145) are very interesting. However, further investigation is required in order to determine whether this state can be constructed rigorously or whether it suffers from any unforeseen pathologies.

5.6 The Frolov-Thorne state

In §4.5.4, we defined the Frolov-Thorne state $|\mathrm{FT}\rangle$ to be a state that exhibits thermal radiation at both past and future null infinity $\mathscr{I}^{\pm}$ with the thermal factors proportional

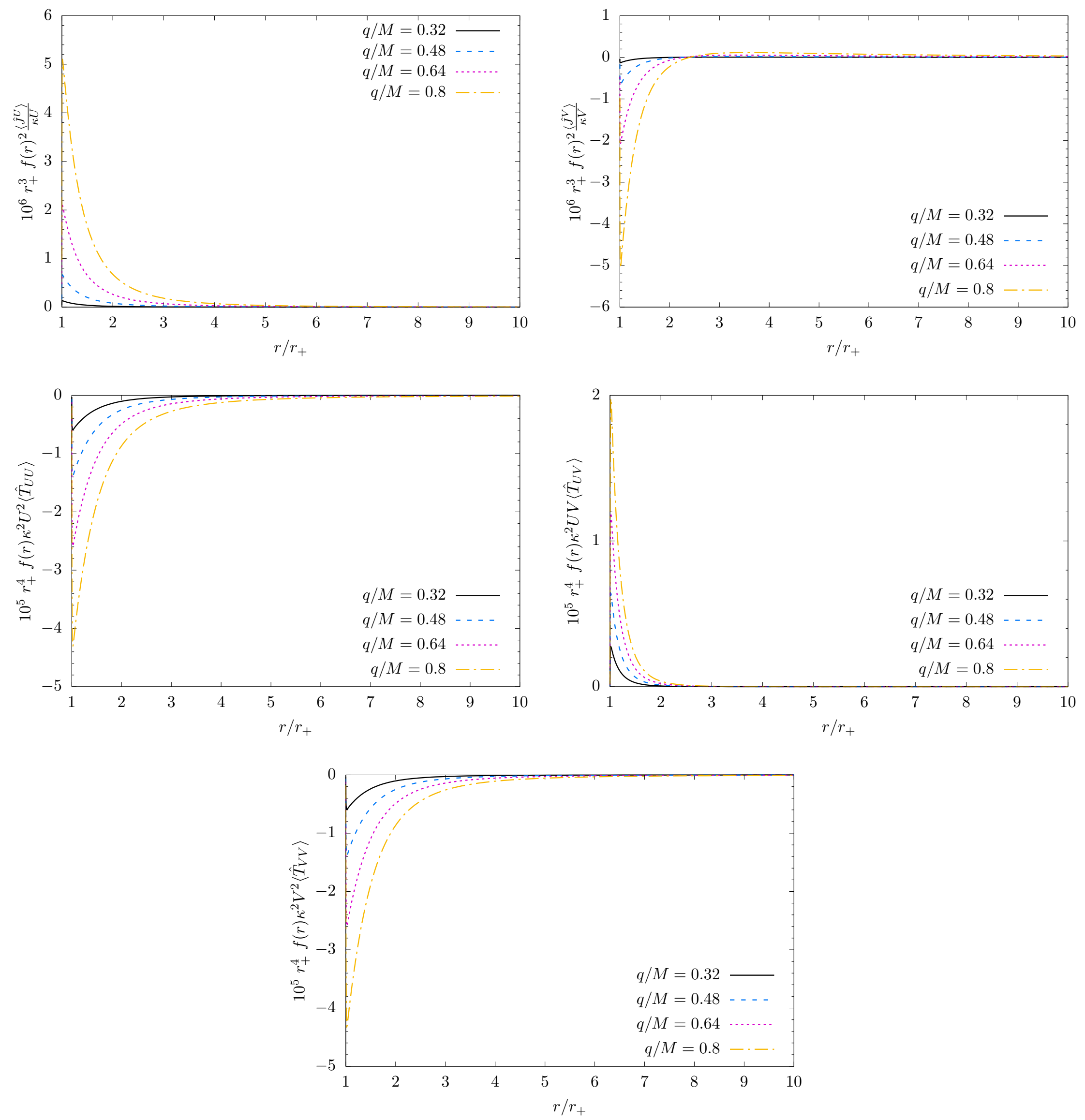

Figure 5.5. Difference in expectation values for the Kruskal components of the current $\hat{J}$ and stress-energy tensor $\hat{T}^{\mu}_{\nu}$ between the 'Boulware-like' state $|\mathrm{B}\rangle$ and the 'past' Boulware state $|\mathrm{B}^-\rangle$ in Reissner-Nordström spacetime for black hole charge $Q = 0.8M$ and positive values of the scalar field charge q. All expectation values are multiplied by powers of $f(r)$ so that the resulting quantities are regular at $r = r_+$.

to $\widetilde{\omega}$; this corresponds to a thermalised flux of particles in both the in-modes (3.106) and up-modes (3.131) of the field Φ, with the frequency $\widetilde{\omega}$ in all thermal factors.

5.6.1 Differences between Frolov-Thorne and 'past' Unruh

Recall that the 'past' Unruh state $|\mathrm{U}^-\rangle$ is defined, in §4.4.1, to be as empty as possible to a static observer at past null infinity $\mathcal{H}^-$ while exhibiting an outgoing thermal flux of radiation at $\mathscr{I}^+$. Since the Frolov-Thorne state $|\mathrm{FT}\rangle$ is defined to exhibit thermal radiation at both past and future null infinity $\mathscr{I}^\pm$, then the difference in the expectation values of observables between the two states should correspond to an incoming flux of thermal radiation at past null infinity $\mathscr{I}^-$; this is represented by a thermalised flux of particles in the in-modes (3.106) of the field $\hat{\Phi}$.

We can use the expectation value of a general observable $\hat{O}$ with classical mode contribution $o_{\omega\ell m}$ to construct an explicit expression for the difference $\langle\hat{O}\rangle_{|\mathrm{FT}\rangle-|\mathrm{U}^-\rangle}$ in expectation values in the 'Frolov-Thorne' state (5.86) and the 'past' Unruh state (5.81); we have

$$\langle\mathrm{FT}|\,\hat{O}\,|\mathrm{FT}\rangle - \langle\mathrm{U}^-|\,\hat{O}\,|\mathrm{U}^-\rangle = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\int_{-\infty}^{\infty}\mathrm{d}\omega\,\frac{1}{\exp\left|\frac{2\pi\widetilde{\omega}}{\kappa}\right|-1}\,o^{\mathrm{in}}_{\omega\ell m}, \tag{5.146}$$

where we have used the identity in (5.87). In §5.2.3, we evaluated the asymptotic in-mode contributions to the classical quantities corresponding to the quantum observables $\widehat{\mathcal{SC}}$, $\hat{J}^\mu$ and $\hat{T}^\mu_\nu$ as $r\to r_+$. Using the near-horizon in-mode contribution (5.63) to the current J^μ, (5.114) becomes

$$\begin{aligned}&\langle\mathrm{FT}|\,\hat{J}^\mu\,|\mathrm{FT}\rangle - \langle\mathrm{U}^-|\,\hat{J}^\mu\,|\mathrm{U}^-\rangle\\ &\qquad\sim\frac{q}{64\pi^3 r^2}\sum_{\ell=0}^{\infty}\int_{-\infty}^{\infty}\mathrm{d}\omega\,\frac{\widetilde{\omega}\,(2\ell+1)}{|\omega|\left(\exp\left|\frac{2\pi\widetilde{\omega}}{\kappa}\right|-1\right)}\left|B^{\mathrm{in}}_{\omega\ell}\right|^2\begin{pmatrix}-f(r)^{-1} & 1 & 0 & 0\end{pmatrix}^{\intercal}.\end{aligned} \tag{5.147}$$

We can check whether the difference $\langle\hat{J}^\mu\rangle_{|\mathrm{FT}\rangle-|\mathrm{U}^-\rangle}$ of the current operator is regular on the horizon by changing to Kruskal coordinates; defining the quantity D as

$$D=\frac{q}{64\pi^3 r^2}\sum_{\ell=0}^{\infty}\int_{-\infty}^{\infty}\mathrm{d}\omega\,\frac{\widetilde{\omega}\,(2\ell+1)}{|\omega|\left(\exp\left|\frac{2\pi\widetilde{\omega}}{\kappa}\right|-1\right)}\left|B^{\mathrm{in}}_{\omega\ell}\right|^2, \tag{5.148}$$

and using (5.28), near the horizon we have

$$\langle\hat{J}^U\rangle_{|\mathrm{FT}\rangle-|\mathrm{U}^-\rangle}\sim\kappa U\left[f(r)^{-1}D+f(r)^{-1}D\right]=2\kappa DUf(r)^{-1}, \tag{5.149a}$$

$$\langle\hat{J}^V\rangle_{|\mathrm{FT}\rangle-|\mathrm{U}^-\rangle}\sim\kappa V\left[-f(r)^{-1}D+f(r)^{-1}D\right]=\mathcal{O}(1). \tag{5.149b}$$

The difference $\langle\hat{J}^U\rangle_{|\mathrm{FT}\rangle-|\mathrm{U}^-\rangle}$ (5.149a) contains a factor of $f(r)^{-1}$, which diverges as $r\to r_+$; the future horizon is a surface of constant $U=0$, and so the factor of U cancels the divergence of the $f(r)^{-1}$ such that the difference is regular on $\mathcal{H}^+$ while, on the past horizon $\mathcal{H}^-$, the difference diverges. In contrast, the leading order divergences cancel in the difference $\langle \hat{J}^V \rangle_{|\mathrm{FT}\rangle - |\mathrm{U}^-\rangle}$ (5.149b) such that it is regular everywhere. Therefore, the differences $\langle \hat{J}^\mu \rangle_{|\mathrm{FT}\rangle - |\mathrm{U}^-\rangle}$ in the expectation values of the current operator (5.147) are regular on $\mathcal{H}^+$ but diverge on $\mathcal{H}^-$.

Using the near-horizon in-mode contribution (5.70) to the SET T^μ_ν, (5.146) becomes

$$\langle \mathrm{FT}| \hat{T}^\mu_\nu |\mathrm{FT}\rangle - \langle \mathrm{U}^-| \hat{T}^\mu_\nu |\mathrm{U}^-\rangle \sim \frac{1}{16\pi^2 r^2} \sum_{\ell=0}^{\infty} \int_{-\infty}^{\infty} \mathrm{d}\omega \, \frac{\widetilde{\omega}^2 \, (2\ell+1)}{|\omega| \left(\exp\left| \frac{2\pi \widetilde{\omega}}{\kappa} \right| - 1 \right)} \left| B^{\mathrm{in}}_{\omega\ell} \right|^2 \begin{pmatrix} -f(r)^{-1} & -f(r)^{-2} & 0 & 0 \\ 1 & f(r)^{-1} & 0 & 0 \\ 0 & 0 & \mathcal{O}(1) & 0 \\ 0 & 0 & 0 & \mathcal{O}(1) \end{pmatrix}. \tag{5.150}$$

We can check whether the difference $\langle \hat{T}^\mu_\nu \rangle_{|\mathrm{FT}\rangle - |\mathrm{U}^-\rangle}$ of the SET is regular on the horizon by changing to Kruskal coordinates; we define the quantity L as

$$L = \frac{1}{16\pi^2 r^2} \sum_{\ell=0}^{\infty} \int_{-\infty}^{\infty} \mathrm{d}\omega \, \frac{\widetilde{\omega}^2 \, (2\ell+1)}{|\omega| \left(\exp\left| \frac{2\pi \widetilde{\omega}}{\kappa} \right| - 1 \right)} \left| B^{\mathrm{in}}_{\omega\ell} \right|^2 . \tag{5.151}$$

From (5.69), $T_{tt} \sim L$, $T_{tr} \sim f(r)^{-1} L$ and $T_{rr} \sim f(r)^{-2} L$. Using (5.30), near the horizon:

$$\langle \hat{T}_{UU} \rangle_{|\mathrm{FT}\rangle - |\mathrm{U}^-\rangle} \sim \frac{1}{4} \kappa^{-2} U^{-2} \left[L - 2 f(r) f(r)^{-1} L + f(r)^2 f(r)^{-2} L \right] = \mathcal{O}(1) \,, \tag{5.152a}$$

$$\langle \hat{T}_{UV} \rangle_{|\mathrm{FT}\rangle - |\mathrm{U}^-\rangle} \sim -\frac{1}{4} \kappa^{-2} U^{-1} V^{-1} \left[L - f(r)^2 f(r)^{-2} L \right] = \mathcal{O}(1) \,, \tag{5.152b}$$

$$\langle \hat{T}_{VV} \rangle_{|\mathrm{FT}\rangle - |\mathrm{U}^-\rangle} \sim \frac{1}{4} \kappa^{-2} V^{-2} \left[L + 2 f(r) f(r)^{-1} L + f(r)^2 f(r)^{-2} L \right] = \kappa^{-2} V^{-2} L. \tag{5.152c}$$

The leading order divergences cancel in the differences $\langle \hat{T}_{UU} \rangle_{|\mathrm{FT}\rangle - |\mathrm{U}^-\rangle}$ (5.152a) and $\langle \hat{T}_{UV} \rangle_{|\mathrm{FT}\rangle - |\mathrm{U}^-\rangle}$ (5.152b). The difference (5.152c), which contains a factor of V^{-2}, diverges on the past horizon since $\mathcal{H}^-$ is a surface of constant $V = 0$, while it may be regular on the future horizon $\mathcal{H}^+$. Therefore, the differences $\langle \hat{T}_{\mu\nu} \rangle_{|\mathrm{FT}\rangle - |\mathrm{U}^-\rangle}$ in the expectation values of the stress-energy tensor operator (5.150) diverge on $\mathcal{H}^-$ but may be regular on $\mathcal{H}^+$.

Thus the differences, between the Frolov-Thorne state $|\mathrm{FT}\rangle$ and the 'past' Unruh state $|\mathrm{U}^-\rangle$, in the expectation values of both the current $\langle \hat{J}^\mu \rangle_{|\mathrm{FT}\rangle - |\mathrm{U}^-\rangle}$ and the stress-energy tensor $\langle \hat{T}_{\mu\nu} \rangle_{|\mathrm{FT}\rangle - |\mathrm{U}^-\rangle}$ diverge on the past horizon $\mathcal{H}^-$ but may be regular on the future horizon $\mathcal{H}^+$. Given that we anticipate that the 'past' Unruh state $|\mathrm{U}^-\rangle$ is regular on $\mathcal{H}^+$, then this suggests that the Frolov-Thorne state may also be regular on $\mathcal{H}^+$. We will return to this question in our discussion of Figure 5.7.

All of the expressions from (5.147 – 5.152) are regular when $\omega = 0$ and $\widetilde{\omega} = 0$. From the Wronskian relation in (3.75), the $\left| B^{\mathrm{in}}_{\omega\ell} \right|^2$ is of $\mathcal{O}\left(\omega^2\right)$, which cancels the factor of $\mathcal{O}\left(\omega^{-1}\right)$ in the denominators as $\omega \to 0$ and the positive powers of $\widetilde{\omega}$ cancel the factor of $\mathcal{O}\left(\widetilde{\omega}^{-1}\right)$ in the denominator as $\widetilde{\omega} \to 0$.

5.6.2 Fluxes of charge and energy in the Frolov-Thorne state

Since the up-modes (3.131) are field modes emanating from the past horizon $\mathcal{H}^-$, then we may calculate the fluxes of charge $\mathcal{K}_{|\mathrm{FT}\rangle}$ and energy $\mathcal{L}_{|\mathrm{FT}\rangle}$ in terms of up-modes in order to investigate the properties of the Frolov-Thorne state $|\mathrm{FT}\rangle$ near the horizon.

Since the expectation values $\langle \hat{J}^r \rangle$ and $\langle \hat{T}_t^r \rangle$ do not require renormalisation, we can evaluate their expectation value in the Frolov-Thorne state $|\mathrm{FT}\rangle$ directly. Using the in- and up-mode contributions (5.38) to the current component J^r, (5.86) becomes

$$\begin{aligned}&\langle \mathrm{FT}| \hat{J}^r |\mathrm{FT}\rangle \\ &\sim \frac{q}{128\pi^3 r^2} \sum_{\ell=0}^{\infty} \left\{ \int_{-\infty}^{\infty} \mathrm{d}\omega \, \frac{\widetilde{\omega}}{|\omega|} \left| B_{\omega\ell}^{\mathrm{in}} \right|^2 - \int_{-\infty}^{\infty} \mathrm{d}\widetilde{\omega} \, \frac{\omega}{|\widetilde{\omega}|} \left| B_{\omega\ell}^{\mathrm{up}} \right|^2 \right\} (2\ell+1) \coth\left| \frac{\pi\widetilde{\omega}}{\kappa} \right| . \qquad (5.153)\end{aligned}$$

The expression in (5.153) is exactly that of the first line on the r.h.s in the expression (5.89) for the expectation value of the flux component J^r in the 'past' Boulware state multiplied by a factor of $\coth\left|\frac{\pi\widetilde{\omega}}{\kappa}\right|$. Then, we may give the expression for the expectation value of the flux component J^r in the Frolov-Thorne state as

$$\langle \mathrm{FT}| \hat{J}^r |\mathrm{FT}\rangle \sim -\frac{q}{64\pi^3 r^2} \sum_{\ell=0}^{\infty} \int_{\min\left\{\frac{qQ}{r_+},0\right\}}^{\max\left\{\frac{qQ}{r_+},0\right\}} \mathrm{d}\omega \, \frac{\omega}{|\widetilde{\omega}|} (2\ell+1) \coth\left| \frac{\pi\widetilde{\omega}}{\kappa} \right| \left| B_{\omega\ell}^{\mathrm{up}} \right|^2 . \qquad (5.154)$$

Using (5.12) and (5.154), we can derive the flux of charge $\mathcal{K}_{|\mathrm{FT}\rangle}$ as

$$\mathcal{K}_{|\mathrm{FT}\rangle} = \frac{q}{64\pi^3} \sum_{\ell=0}^{\infty} \int_{\min\left\{\frac{qQ}{r_+},0\right\}}^{\max\left\{\frac{qQ}{r_+},0\right\}} \mathrm{d}\omega \, \frac{\omega}{|\widetilde{\omega}|} (2\ell+1) \coth\left| \frac{\pi\widetilde{\omega}}{\kappa} \right| \left| B_{\omega\ell}^{\mathrm{up}} \right|^2 . \qquad (5.155)$$

Through a similar reasoning to that given immediately after equation (5.91), the flux of charge $\mathcal{K}_{|\mathrm{FT}\rangle}$ (5.155) in the Frolov-Thorne state always has the same sign as the charge of the black hole Q; thus, the thermal radiation emitted in the Frolov-Thorne state corresponds to the Reissner-Nordström black hole discharging.

Using a similar process to that in (5.89), we can give the expectation value $\langle T_r^t \rangle_{|\mathrm{FT}\rangle}$:

$$\langle \mathrm{FT}| \hat{T}_t^r |\mathrm{FT}\rangle \sim \frac{1}{16\pi^2 r^2} \sum_{\ell=0}^{\infty} \int_{\min\left\{\frac{qQ}{r_+},0\right\}}^{\max\left\{\frac{qQ}{r_+},0\right\}} \mathrm{d}\omega \, \frac{\omega^2}{|\widetilde{\omega}|} (2\ell+1) \coth\left| \frac{\pi\widetilde{\omega}}{\kappa} \right| \left| B_{\omega\ell}^{\mathrm{up}} \right|^2 . \qquad (5.156)$$

Using (5.25) and (5.156), we can derive the flux of energy $\mathcal{L}_{|\mathrm{B}^-\rangle}$ as

$$\mathcal{L}_{|\mathrm{FT}\rangle} = \frac{1}{16\pi^2} \sum_{\ell=0}^{\infty} \int_{\min\left\{\frac{qQ}{r_+},0\right\}}^{\max\left\{\frac{qQ}{r_+},0\right\}} \mathrm{d}\omega \, \frac{\omega^2}{|\widetilde{\omega}|} (2\ell+1) \coth\left| \frac{\pi\widetilde{\omega}}{\kappa} \right| \left| B_{\omega\ell}^{\mathrm{up}} \right|^2 . \qquad (5.157)$$

The expression for the flux of energy $\mathcal{L}_{|\mathrm{FT}\rangle}$ in the Frolov-Thorne state is always positive; therefore the thermal radiation emitted in the Frolov-Thorne state corresponds to the Reissner-Nordström black hole losing energy.

In [49, 50], it is argued that, in Kerr spacetime, the Frolov-Thorne state is an equilibrium state. However, the expressions for the fluxes of charge (5.155) and energy (5.157) in the Frolov-Thorne state are both nonzero for $q \neq 0$. Therefore, the Frolov-Thorne state $|\mathrm{FT}\rangle$ is not an equilibrium state in Reissner-Nordström spacetime and neither is it time-reversal invariant.

5.6.3 Discussion of Figures 5.6 and 5.7

Since the Frolov-Thorne state is a state defined with the Schwarzschild Hartle-Hawking state $|\mathrm{H_s}\rangle$ in mind, we will make comparisons with the differences $\langle \hat{O} \rangle_{|\mathrm{CCH}^-\rangle - |\mathrm{U}^-\rangle}$ in expectation values of observables $\hat{O}$ between the 'past' CCH state and the 'past' Unruh state, which we studied in §5.3.4, throughout this discussion.

The difference $\langle \hat{J}^r \rangle_{|\mathrm{FT}\rangle - |\mathrm{U}^-\rangle}$ is positive, so the flux of charge $\mathcal{K}_{|\mathrm{U}^-\rangle}$ in the 'past' Unruh state must be greater than the flux of charge $\mathcal{K}_{|\mathrm{FT}\rangle}$ in the Frolov-Thorne state. This is similar to what we saw when calculating the difference $\langle \hat{J}^r \rangle_{|\mathrm{U}^-\rangle - |\mathrm{B}^-\rangle}$, but in that case the flux of charge $\mathcal{K}_{|\mathrm{U}^-\rangle}$ due to Hawking radiation was shown to be greater than the flux of charge $\mathcal{K}_{|\mathrm{B}^-\rangle}$ due to a non-thermal flux of particles that were superradiantly scattered. From (5.155), the flux of charge $\mathcal{K}_{|\mathrm{FT}\rangle}$ is due to a thermalised flux of particles in only the superradiant modes, which have exactly the same thermal factor as that of the Hawking radiation in the flux of charge in $|\mathrm{U}^-\rangle$. Then it is intuitive, despite the two fluxes of charge containing thermal radiation with the same thermal factor, that $\mathcal{K}_{|\mathrm{U}^-\rangle}$ is greater than $\mathcal{K}_{|\mathrm{CCH}^-\rangle}$ since the former contains thermal radiation from all field modes while the latter only contains thermal radiation in the superradiant modes.

From the above, we would also expect that the flux of energy $\mathcal{L}_{|\mathrm{U}^-\rangle}$ in the 'past' Unruh state is greater than the flux of energy $\mathcal{L}_{|\mathrm{CCH}^-\rangle}$ in the Frolov-Thorne state, which is indeed the case since $\langle \hat{T}_t^r \rangle_{|\mathrm{FT}\rangle - |\mathrm{U}^-\rangle}$ is positive.

The difference $\langle \hat{J}^t \rangle_{|\mathrm{FT}\rangle - |\mathrm{U}^-\rangle}$ as well as the differences in the diagonal elements of the SET all appear to tend to constant values far from the black hole; while this is clearly observable in the plots for small values of the scalar field charge q, we would need to consider very large values of the radial coordinate in order to verify this observation for large q. Of these differences, all but the difference in the energy density have the opposite sign as compared with their corresponding differences between $|\mathrm{CCH}^-\rangle$ and $|\mathrm{U}^-\rangle$. In contrast, the difference $\langle \hat{T}_t^t \rangle_{|\mathrm{FT}\rangle - |\mathrm{U}^-\rangle}$ in the energy densities has the same sign as $\langle \hat{T}_t^t \rangle_{|\mathrm{CCH}^-\rangle - |\mathrm{U}^-\rangle}$.

Noting the above, we can say that while the Frolov-Thorne state $|\mathrm{FT}\rangle$ and the 'past' CCH state $|\mathrm{CCH}^-\rangle$ share some properties, such as being non-empty far from the black hole, ultimately their physical behaviour and interpretation are very different. Recall that both states were defined to exhibit outgoing thermal radiation at $\mathscr{I}^+$ with the same thermal factor, which was proportional to $\widetilde{\omega}$. However, while each state also exhibits thermal radiation incoming at $\mathscr{I}^-$, the thermal factor of the in-modes in the $|\mathrm{FT}\rangle$ state is proportional to $\widetilde{\omega}$ while the thermal factor of the in-modes in the $|\mathrm{CCH}^-\rangle$ state is proportional to ω. From our analysis in this section, this clearly has a significant effect on the expectation values of observables in either state.

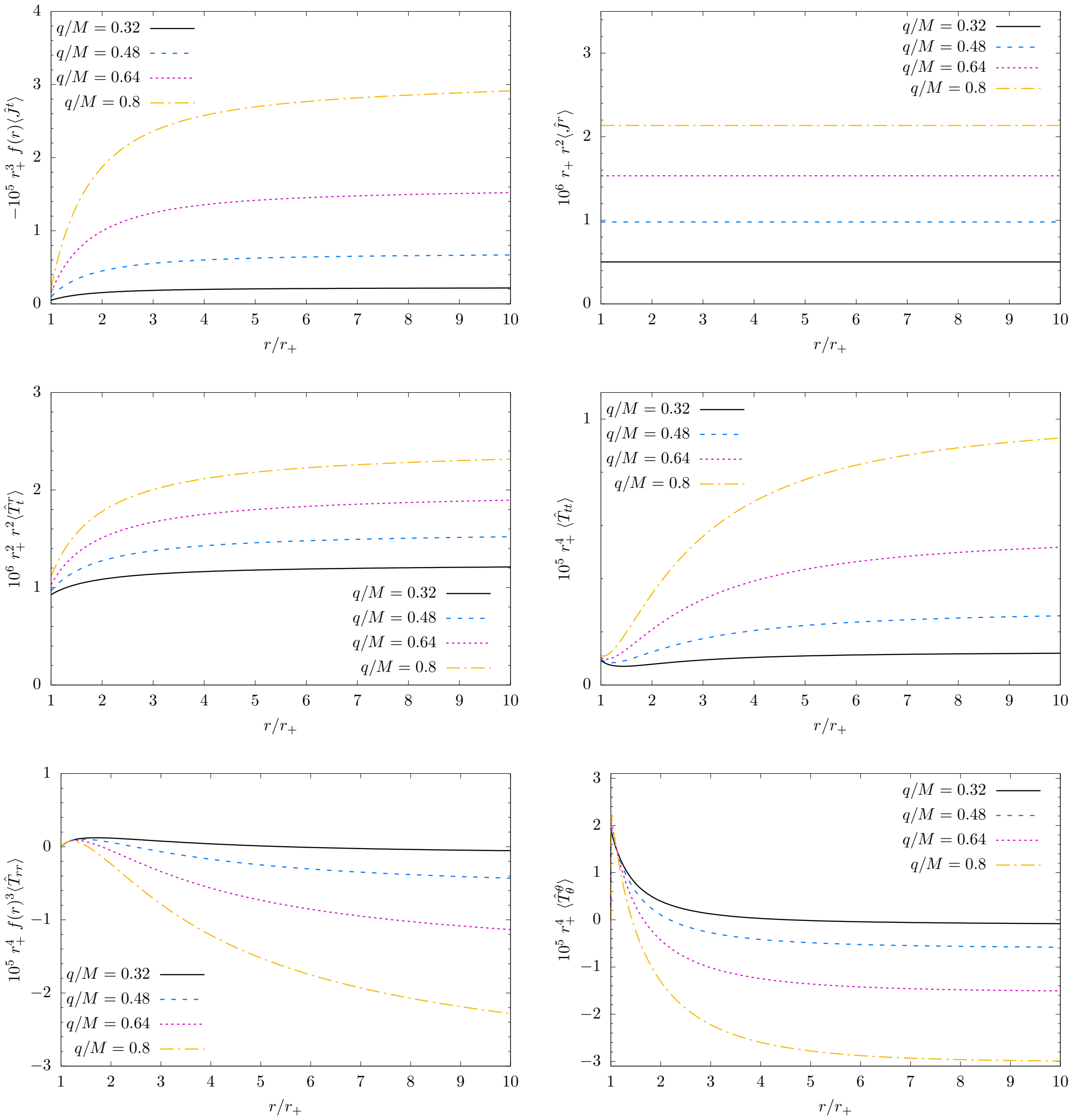

Figure 5.6. Difference in expectation values of nonzero components of the current $\hat{J}$ and various components of the stress-energy tensor $\hat{T}^{\mu}_{\nu}$ between the Frolov-Thorne state $|\mathrm{FT}\rangle$ and the 'past' Unruh state $|\mathrm{U}^-\rangle$ in Reissner-Nordström spacetime for black hole charge $Q = 0.8M$ and positive values of the scalar field charge q. All expectation values are multiplied by powers of $f(r)$ so that the resulting quantities are regular at $r = r_+$.

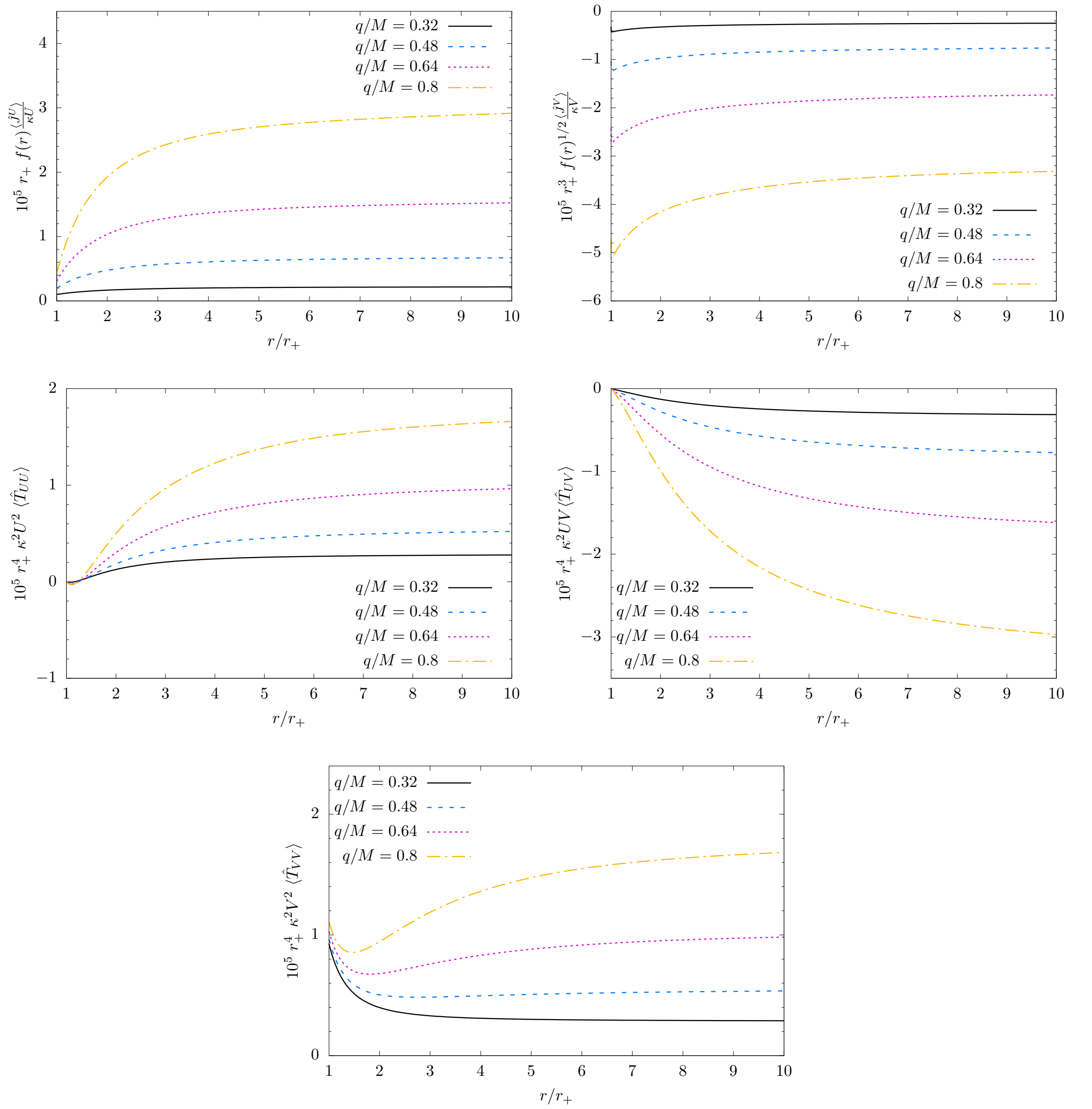

Figure 5.7. Difference in expectation values for the Kruskal components of the current $\hat{J}$ and stress-energy tensor $\hat{T}^{\mu}_{\nu}$ between the Frolov-Thorne state $|\mathrm{FT}\rangle$ and the 'past' Unruh state $|\mathrm{U}^-\rangle$ in Reissner-Nordström spacetime for black hole charge $Q = 0.8M$ and positive values of the scalar field charge q. All expectation values are multiplied by powers of $f(r)$ so that the resulting quantities are regular at $r = r_+$.

5.6.4 Conclusions

Before we conclude our analysis of the Frolov-Thorne state $|\mathrm{FT}\rangle$, we may evaluate the difference $\langle\widehat{\mathcal{SC}}\rangle_{|\mathrm{FT}\rangle-|\mathrm{U}^-\rangle}$ in the expectation values of the scalar condensate $\widehat{\mathcal{SC}}$ between $|\mathrm{FT}\rangle$ and $|\mathrm{U}^-\rangle$. Using the near-horizon in-mode contribution (5.60) to the scalar condensate $\mathcal{SC}$, (5.114) becomes

$$\langle\mathrm{FT}|\,\widehat{\mathcal{SC}}\,|\mathrm{FT}\rangle - \langle\mathrm{U}^-|\,\widehat{\mathcal{SC}}\,|\mathrm{U}^-\rangle \sim \frac{1}{16\pi^2 r^2}\sum_{\ell=0}^{\infty}\int_{-\infty}^{\infty}\mathrm{d}\omega\,\frac{2\ell+1}{|\omega|\left(\exp\left|\frac{2\pi\widetilde{\omega}}{\kappa}\right|-1\right)}\left|B^{\mathrm{in}}_{\omega\ell}\right|^2. \quad (5.158)$$

The expression for the difference $\langle\widehat{\mathcal{SC}}\rangle_{|\mathrm{FT}\rangle-|\mathrm{U}^-\rangle}$ is regular as $\omega\to 0$ since, from the Wronskian relation in (3.75), the $\left|B^{\mathrm{in}}_{\omega\ell}\right|^2$ is of $\mathcal{O}(\omega^2)$ as $\omega\to 0$. However, the integrand in (5.158) diverges when $\widetilde{\omega}=0$. Since we expect the 'past' Unruh state to be regular on the 'future' horizon $\mathcal{H}^+$ at least, then it appears that the Frolon-Thorne state is ill-defined on the horizon. We can investigate whether the Frolov-Thorne state is ill-defined in other regions of the spacetime by using the in-mode contribution to the scalar condensate $\mathcal{SC}$ for a general value of the radial coordinate r; then, (5.114) becomes

$$\langle\mathrm{FT}|\,\widehat{\mathcal{SC}}\,|\mathrm{FT}\rangle - \langle\mathrm{U}^-|\,\widehat{\mathcal{SC}}\,|\mathrm{U}^-\rangle = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\int_{-\infty}^{\infty}\mathrm{d}\omega\,\frac{1}{\exp\left|\frac{2\pi\widetilde{\omega}}{\kappa}\right|-1}\left|\phi^{\mathrm{in}}_{\omega\ell m}\right|^2. \quad (5.159)$$

The integrand in the expression for the difference $\langle\widehat{\mathcal{SC}}\rangle_{|\mathrm{FT}\rangle-|\mathrm{U}^-\rangle}$ outside the horizon (5.159) has a pole when $\widetilde{\omega}=0$ unless the magnitudes of the in-mode contributions $\left|\phi^{\mathrm{in}}_{\omega\ell m}\right|$ vanish at this frequency. However, numerical evaluations of (5.159) demonstrate that there is at least one in-mode (3.106) with non-vanishing magnitude when $\widetilde{\omega}=0$, rendering the difference $\langle\widehat{\mathcal{SC}}\rangle_{|\mathrm{FT}\rangle-|\mathrm{U}^-\rangle}$ divergent everywhere on the spacetime. This leads us to conclude that the Frolov-Thorne $|\mathrm{FT}\rangle$ state is ill-defined despite the differences $\langle\hat{J}^{\mu}\rangle_{|\mathrm{FT}\rangle-|\mathrm{U}^-\rangle}$ (5.147, 5.149) and $\langle\hat{T}^{\mu}_{\nu}\rangle_{|\mathrm{FT}\rangle-|\mathrm{U}^-\rangle}$ (5.150, 5.152) being well-behaved outside the horizon.

Calculations of the expectation value of the scalar condensate in the analogous Frolov-Thorne state in Kerr spacetime, where it was originally defined, are similarly divergent almost everywhere in the spacetime with the exception of the axis of symmetry [50]; here the contribution from modes undergoing rotational superradiance vanish and $|\mathrm{FT}\rangle$ reduces to the 'past' CCH state here. In RN spacetime, however, the event horizon receives a contribution from superradiant modes everywhere in the exterior of the black hole and so the Frolov-Thorne state defined in §4.5.4 is ill-defined throughout the spacetime.

5.7 The 'Hartle-Hawking-like' state

In §4.5.3, we defined the 'Hartle-Hawking-like' state $|\mathrm{H}\rangle$ to be a state that exhibits thermal radiation at both past and future null infinity $\mathscr{I}^{\pm}$ with the thermal factors proportional to $\widetilde{\omega}$; this corresponds to a thermalised flux of particles in both the in-modes (3.106) and up-modes (3.131) of the field Φ, with the frequency $\widetilde{\omega}$ in all thermal factors.

5.7.1 Differences between 'Hartle-Hawking-like' and Frolov-Thorne

In the previous section, §5.6, we investigated the Frolov-Thorne state and concluded that it is everywhere ill-defined on the spacetime. Despite this, in order to investigate the 'Hartle-Hawking-like' state $|\mathrm{H}\rangle$, it is convenient to consider the differences in the expectation values of observables between $|\mathrm{H}\rangle$ and $|\mathrm{FT}\rangle$.

We can use the expectation value of a general observable $\hat{O}$ with classical mode contribution $o_{\omega\ell m}$ to construct an explicit expression for the difference $\langle\hat{O}\rangle_{|\mathrm{H}\rangle-|\mathrm{FT}\rangle}$ in expectation values in the 'Hartle-Hawking-like' state (5.85) and the 'Frolov-Thorne' state (5.86) as

$$\langle\mathrm{H}|\,\hat{O}\,|\mathrm{H}\rangle - \langle\mathrm{FT}|\,\hat{O}\,|\mathrm{FT}\rangle = -\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\int_{\min\left\{\frac{qQ}{r_+},0\right\}}^{\max\left\{\frac{qQ}{r_+},0\right\}} \mathrm{d}\omega\, o^{\mathrm{in}}_{\omega\ell m}\coth\left|\frac{\pi\widetilde{\omega}}{\kappa}\right|. \tag{5.160}$$

Since the expectation value $\langle\widehat{\mathcal{SC}}\rangle_{|\mathrm{FT}\rangle}$ in the $|\mathrm{FT}\rangle$ state is divergent, we will only consider differences involving $\langle\hat{J}^\mu\rangle_{|\mathrm{FT}\rangle}$ and $\langle\hat{T}^\mu_\nu\rangle_{|\mathrm{FT}\rangle}$, which are well defined from §5.6. In §5.2.3, we evaluated the asymptotic in-mode contributions to the classical quantities corresponding to the quantum observables $\hat{J}^\mu$ and $\hat{T}^\mu_\nu$ as $r\to r_+$. Using the near-horizon in-mode contribution (5.63) to the current J^μ, (5.160) becomes

$$\begin{aligned}&\langle\mathrm{H}|\,\hat{J}^\mu\,|\mathrm{H}\rangle - \langle\mathrm{FT}|\,\hat{J}^\mu\,|\mathrm{FT}\rangle\\ &\quad\sim -\frac{q}{64\pi^3 r^2}\sum_{\ell=0}^{\infty}\int_{\min\left\{\frac{qQ}{r_+},0\right\}}^{\max\left\{\frac{qQ}{r_+},0\right\}} \mathrm{d}\omega\,\frac{\widetilde{\omega}}{|\omega|}\coth\left|\frac{\pi\widetilde{\omega}}{\kappa}\right|(2\ell+1)\left|B^{\mathrm{in}}_{\omega\ell}\right|^2\begin{pmatrix}-f(r)^{-1} & 1 & 0 & 0\end{pmatrix}^{\mathsf{T}}.\end{aligned} \tag{5.161}$$

We can check whether the difference $\langle\hat{J}^\mu\rangle_{|\mathrm{H}\rangle-|\mathrm{FT}\rangle}$ of the current operator is regular on the horizon by changing to Kruskal coordinates; defining the quantity K' as

$$K' = -\frac{q}{64\pi^3 r^2}\sum_{\ell=0}^{\infty}\int_{\min\left\{\frac{qQ}{r_+},0\right\}}^{\max\left\{\frac{qQ}{r_+},0\right\}} \mathrm{d}\omega\,\frac{\widetilde{\omega}}{|\omega|}\coth\left|\frac{\pi\widetilde{\omega}}{\kappa}\right|(2\ell+1)\left|B^{\mathrm{in}}_{\omega\ell}\right|^2, \tag{5.162}$$

and using (5.28), we have

$$\langle\hat{J}^U\rangle_{|\mathrm{H}\rangle-|\mathrm{FT}\rangle} \sim \kappa U\left[f(r)^{-1}K' + f(r)^{-1}K'\right] = 2\kappa K' U f(r)^{-1}, \tag{5.163a}$$

$$\langle\hat{J}^V\rangle_{|\mathrm{H}\rangle-|\mathrm{FT}\rangle} \sim \kappa V\left[-f(r)^{-1}K' + f(r)^{-1}K'\right] = \mathcal{O}(1). \tag{5.163b}$$

The difference $\langle\hat{J}^U\rangle_{|\mathrm{H}\rangle-|\mathrm{FT}\rangle}$ (5.163a) contains a factor of $f(r)^{-1}$, which diverges as $r\to r_+$; the future horizon is a surface of constant $U=0$, and so the factor of U cancels the divergence of the $f(r)^{-1}$ such that the difference is regular on $\mathcal{H}^+$ while, on the past horizon $\mathcal{H}^-$, the difference diverges. In contrast, the leading order divergences cancel in the difference $\langle\hat{J}^V\rangle_{|\mathrm{FT}\rangle-|\mathrm{U}^-\rangle}$ (5.163b) such that it is regular everywhere. Therefore, the differences $\langle\hat{J}^\mu\rangle_{|\mathrm{H}\rangle-|\mathrm{FT}\rangle}$ in the expectation values of the current operator (5.161) are regular on $\mathcal{H}^+$ but diverge on $\mathcal{H}^-$.

Using the near-horizon in-mode contribution (5.70) to the SET T^{μ}_{ν}, (5.160) becomes

$$\langle \mathrm{H}| \hat{T}^{\mu}_{\nu} |\mathrm{H}\rangle - \langle \mathrm{FT}| \hat{T}^{\mu}_{\nu} |\mathrm{FT}\rangle \sim -\frac{1}{16\pi^2 r^2} \sum_{\ell=0}^{\infty} \int_{\min\left\{\frac{qQ}{r_+},0\right\}}^{\max\left\{\frac{qQ}{r_+},0\right\}} \mathrm{d}\omega \, \frac{\widetilde{\omega}^2}{|\omega|} \coth\left|\frac{\pi\widetilde{\omega}}{\kappa}\right| (2\ell+1) \left|B^{\mathrm{in}}_{\omega\ell}\right|^2 \times \begin{pmatrix} -f(r)^{-1} & -f(r)^{-2} & 0 & 0 \\ 1 & f(r)^{-1} & 0 & 0 \\ 0 & 0 & \mathcal{O}(1) & 0 \\ 0 & 0 & 0 & \mathcal{O}(1) \end{pmatrix}. \quad (5.164)$$

The differences (5.161) and (5.164) are regular when $\omega = 0$ from the relation (3.75); the $\left|B^{\mathrm{in}}_{\omega\ell}\right|^2$ is of $\mathcal{O}(\omega^2)$, which cancels the factor of $\mathcal{O}(\omega^{-1})$ in the denominators as $\omega \to 0$.

We can check whether the difference $\langle \hat{T}^{\mu}_{\nu} \rangle_{|\mathrm{H}\rangle - |\mathrm{FT}\rangle}$ of the SET is regular on the horizon by changing to Kruskal coordinates; defining the quantity L' as

$$L' = -\frac{1}{16\pi^2 r^2} \sum_{\ell=0}^{\infty} \int_{\min\left\{\frac{qQ}{r_+},0\right\}}^{\max\left\{\frac{qQ}{r_+},0\right\}} \mathrm{d}\omega \, \frac{\widetilde{\omega}^2}{|\omega|} \coth\left|\frac{\pi\widetilde{\omega}}{\kappa}\right| (2\ell+1) \left|B^{\mathrm{in}}_{\omega\ell}\right|^2. \quad (5.165)$$

From (5.69), $T_{tt} \sim L'$, $T_{tr} \sim f(r)^{-1} L'$ and $T_{rr} \sim f(r)^{-2} L'$. Using (5.30), we have

$$\langle \hat{T}_{UU} \rangle_{|\mathrm{H}\rangle - |\mathrm{FT}\rangle} \sim \frac{1}{4} \kappa^{-2} U^{-2} \left[L' - 2f(r) f(r)^{-1} L' + f(r)^2 f(r)^{-2} L' \right] = \mathcal{O}(1), \quad (5.166a)$$

$$\langle \hat{T}_{UV} \rangle_{|\mathrm{H}\rangle - |\mathrm{FT}\rangle} \sim -\frac{1}{4} \kappa^{-2} U^{-1} V^{-1} \left[L' - f(r)^2 f(r)^{-2} L' \right] = \mathcal{O}(1), \quad (5.166b)$$

$$\langle \hat{T}_{VV} \rangle_{|\mathrm{H}\rangle - |\mathrm{FT}\rangle} \sim \frac{1}{4} \kappa^{-2} V^{-2} \left[L' + 2f(r) f(r)^{-1} L' + f(r)^2 f(r)^{-2} L' \right] = \kappa^{-2} V^{-2} L'. \quad (5.166c)$$

The leading order divergences cancel in the differences $\langle \hat{T}_{UU} \rangle_{|\mathrm{H}\rangle - |\mathrm{FT}\rangle}$ (5.166a) and $\langle \hat{T}_{UV} \rangle_{|\mathrm{H}\rangle - |\mathrm{FT}\rangle}$ (5.166b). The difference (5.166c), which contains a factor of V^{-2}, diverges on the past horizon since $\mathcal{H}^-$ is a surface of constant $V = 0$, while it may be regular on the future horizon $\mathcal{H}^+$. Therefore, the differences $\langle \hat{T}_{\mu\nu} \rangle_{|\mathrm{H}\rangle - |\mathrm{FT}\rangle}$ in the expectation values of the SET (5.164) diverge on $\mathcal{H}^-$ but may be regular on $\mathcal{H}^+$.

Thus the differences in the expectation values of both the current $\langle \hat{J}^{\mu} \rangle_{|\mathrm{H}\rangle - |\mathrm{FT}\rangle}$ and the SET $\langle \hat{T}_{\mu\nu} \rangle_{|\mathrm{H}\rangle - |\mathrm{FT}\rangle}$ diverge on the past horizon $\mathcal{H}^-$ but may be regular on the future horizon $\mathcal{H}^+$. This suggests that the 'Hartle-Hawking-like' state may be regular on $\mathcal{H}^+$.

5.7.2 Fluxes of charge and energy in the 'Hartle-Hawking-like' state

We would like to evaluate the flux of charge $\mathcal{K}_{|\mathrm{H}\rangle}$ and the flux of energy $\mathcal{L}_{|\mathrm{H}\rangle}$ in the 'Hartle-Hawking-like' state $|\mathrm{H}\rangle$ directly. Using the expression for the expectation value of the radial component of the current in the Frolov-Thorne state $\langle \hat{J}^r \rangle_{|\mathrm{FT}\rangle}$ (5.155) and the radial component of the difference $\langle \hat{J}^{\mu} \rangle_{|\mathrm{H}\rangle - |\mathrm{FT}\rangle}$ (5.161), we have

$$
\begin{aligned}
\langle \hat{J}^r \rangle_{|\mathrm{H}\rangle} &= \Big(\langle \mathrm{H} | \, \hat{J}^r \, | \mathrm{H} \rangle - \langle \mathrm{FT} | \, \hat{J}^r \, | \mathrm{FT} \rangle \Big) + \langle \mathrm{FT} | \, \hat{J}^r \, | \mathrm{FT} \rangle \\
&\sim -\frac{q}{64\pi^3 r^2} \sum_{\ell=0}^{\infty} \int_{\min\left\{\frac{qQ}{r_+},0\right\}}^{\max\left\{\frac{qQ}{r_+},0\right\}} \mathrm{d}\omega \, \frac{\widetilde{\omega}}{|\omega|} \coth\left|\frac{\pi\widetilde{\omega}}{\kappa}\right| (2\ell+1) \left|B^{\mathrm{in}}_{\omega\ell}\right|^2 (1-1) = 0. \qquad (5.167)
\end{aligned}
$$

A similar calculation shows that the expectation value $\langle \hat{T}^r_t \rangle_{|\mathrm{H}\rangle}$ also vanishes. Therefore, we find that both the flux of charge $\mathcal{K}_{|\mathrm{H}\rangle}$ and the flux of energy $\mathcal{L}_{|\mathrm{H}\rangle}$ vanish, i.e.

$$
\mathcal{K}_{|\mathrm{H}\rangle} = 0, \qquad \mathcal{L}_{|\mathrm{H}\rangle} = 0. \qquad (5.168)
$$

We therefore conclude, from (5.168), that the 'Hartle-Hawking-like' state $|\mathrm{H}\rangle$ is a time-reversal invariant, equilibrium state. Since, we have shown that $|\mathrm{H}\rangle$ is likely to be regular on the future horizon $\mathcal{H}^+$, then it being time-reversal invariant implies that indeed be regular everywhere on the horizon. However, only a study of the renormalised expectation values of observables with respect to $|\mathrm{H}\rangle$ directly could test this conjecture.

5.7.3 Discussion of Figures 5.8 and 5.9

The difference $\langle \hat{J}^r \rangle_{|\mathrm{H}\rangle - |\mathrm{FT}\rangle}$ is positive, so the difference in the flux of charge $\mathcal{K}_{|\mathrm{H}\rangle} - \mathcal{K}_{|\mathrm{FT}\rangle}$ is negative. However, from (5.168), $\mathcal{K}_{|\mathrm{H}\rangle}$ vanishes and so $\mathcal{K}_{|\mathrm{H}\rangle} - \mathcal{K}_{|\mathrm{FT}\rangle}$ reduces to $-\mathcal{K}_{|\mathrm{FT}\rangle} < 0$, meaning that the flux of charge $\mathcal{K}_{|\mathrm{FT}\rangle}$ is positive for a positive value of the black hole charge Q and positive values of the scalar field charge q. In fact, as we described in §5.6, $\mathcal{K}_{|\mathrm{FT}\rangle}$ always has the same sign as Q and so, from (5.13), the thermalised flux of charge in the superradiant modes in the Frolov-Thorne state $|\mathrm{FT}\rangle$ acts to discharge the RN black hole. Similarly, $\mathcal{L}_{|\mathrm{FT}\rangle}$ is always positive and so the thermalised flux of energy in the superradiant modes in the Frolov-Thorne state $|\mathrm{FT}\rangle$ causes the RN black hole to lose energy.

The difference $\langle \hat{J}^t \rangle_{|\mathrm{H}\rangle - |\mathrm{FT}\rangle}$ in the time component of the current as well as the differences in the diagonal elements of the stress-energy tensor approach constant values near infinity. The difference in the charge density $\langle \hat{J}^t \rangle_{|\mathrm{H}\rangle - |\mathrm{FT}\rangle}$ is negative and the difference in the energy density $\langle \hat{T}^t_t \rangle_{|\mathrm{H}\rangle - |\mathrm{FT}\rangle}$ is positive away from the horizon and considerably larger than the corresponding difference $\langle \hat{J}^t \rangle_{|\mathrm{FT}\rangle - |\mathrm{U}^-\rangle}$ between $|\mathrm{FT}\rangle$ and $|\mathrm{U}^-\rangle$.

In §5.6, we explained that we expect the Frolov-Thorne $|\mathrm{FT}\rangle$ to be regular on $\mathcal{H}^+$ but singular on $\mathcal{H}^-$. If we are correct then, from the expressions in (5.163) and (5.166) as well as the plots in Figure 5.9, we expect that the 'Hartle-Hawking-like' state $|\mathrm{H}\rangle$ will be similarly regular on the future horizon $\mathcal{H}^+$ with its behaviour on the past horizon $\mathcal{H}^-$ unclear. However, given the time-reversal invariance of $|\mathrm{H}\rangle$, we conclude that $|\mathrm{H}\rangle$ is regular everywhere on the horizon, i.e. both the past and future horizons $\mathcal{H}^\pm$.

In conclusion, while we have been able to define an equilibrium state that is also time-reversal invariant in the form of the 'Hartle-Hawking-like' state $|\mathrm{H}\rangle$, we cannot claim $|\mathrm{H}\rangle$ to be an analogue of the Schwarzschild Hartle-Hawking state $|\mathrm{H_s}\rangle$. The Kay-Wald theorem proves the nonexistence of a stationary Hadamard state in Kerr spacetime for a neutral scalar field [56, 57]. It is natural to anticipate that a generalised form of this theorem would apply to a charged scalar field. One of the assumptions in the Kay-Wald theorem is the positivity condition, which is discussed in detail in App. B of [49]. When defining

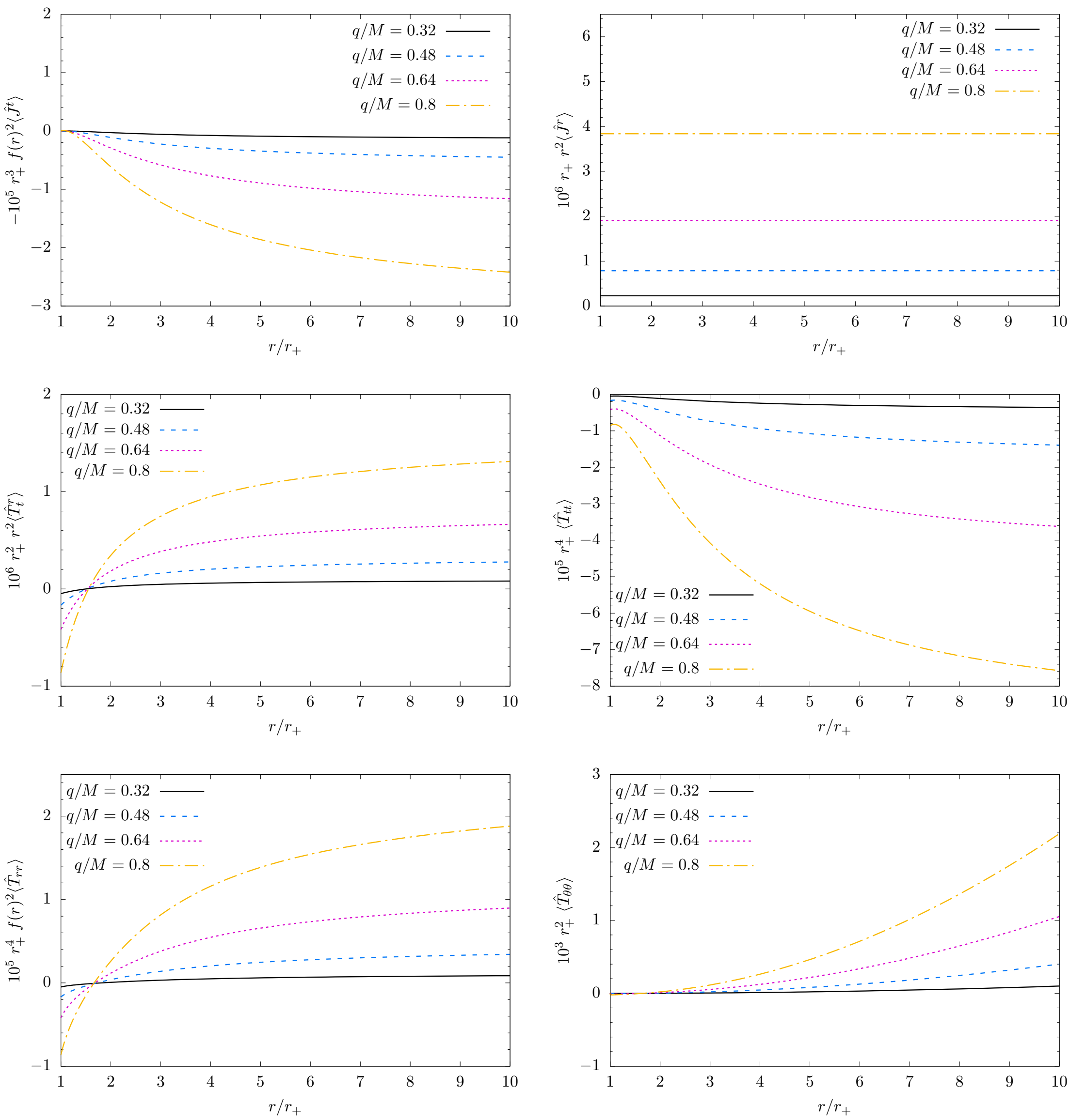

Figure 5.8. Difference in expectation values of nonzero components of the current $\hat{J}$ and various components of the stress-energy tensor $\hat{T}^{\mu}_{\nu}$ between the 'Hartle-Hawking-like' state $|\mathrm{H}\rangle$ and the Frolov-Thorne state $|\mathrm{FT}\rangle$ in Reissner-Nordström spacetime for black hole charge $Q = 0.8M$ and positive values of the scalar field charge q. All expectation values are multiplied by powers of $f(r)$ so that the resulting quantities are regular at $r = r_+$.

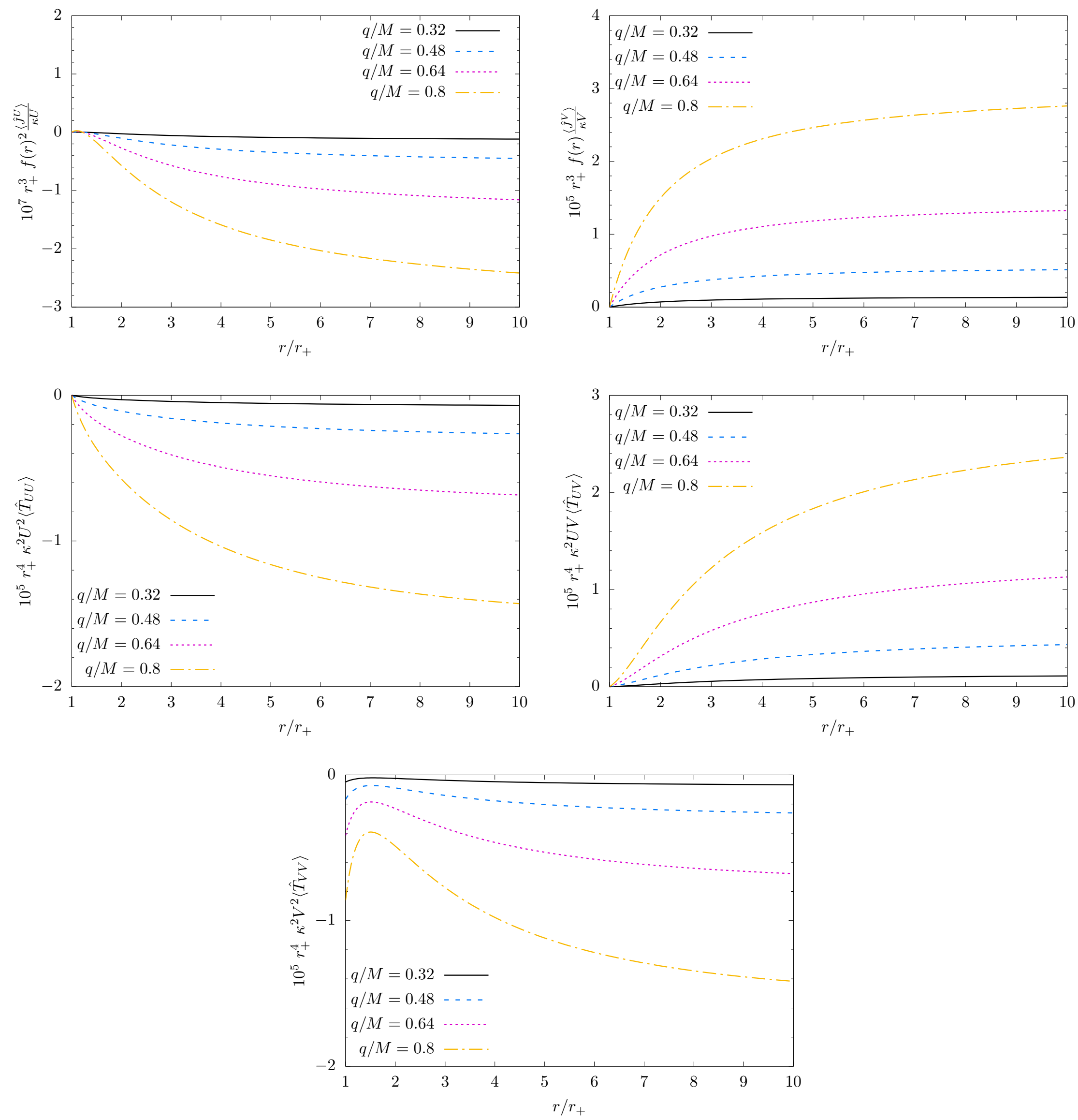

Figure 5.9. Difference in expectation values for the Kruskal components of the current $\hat{J}$ and stress-energy tensor $\hat{T}^{\mu}_{\nu}$ between the 'Hartle-Hawking-like' state $|\mathrm{H}\rangle$ and the Frolov-Thorne state $|\mathrm{FT}\rangle$ in Reissner-Nordström spacetime for black hole charge $Q = 0.8M$ and positive values of the scalar field charge q. All expectation values are multiplied by powers of $f(r)$ so that the resulting quantities are regular at $r = r_+$.

the $|\mathrm{H}\rangle$ in §4.5.3, we included the eta-function $\eta_{\omega\widetilde{\omega}}$ in the commutation relations of the operators associated to the in-modes. We anticipate that the nonstandard commutation relations will cause the state not to satisfy the usual positivity condition, thus not being in violation of a generalised form of the Kay-Wald theorem which we think would preclude the existence of a thermal equilibrium state for a charged scalar field.

Part III

Hadamard renormalisation of a charged scalar field

Chapter 6

The Hadamard parametrices

In §6.1, we outline the form of the Hadamard parametrices in all three cases, namely two spacetime dimensions, even numbers of dimensions and odd numbers of dimensions. In §6.2, we compare the properties of real symmetric biscalars, which arise in the Hadamard parametrices associated to neutral scalar fields, to complex sesquisymmetric biscalars, which arise in the Hadamard parametrices associated to complex scalar fields. We give an assortment of identities that will be useful in Part III in 6.3.

6.1 Introduction

Consider a massive, charged field Φ in a general background spacetime and with arbitrary coupling ξ to the scalar curvature. We henceforth assume that the field is in a Hadamard state; it is shown in [86] that this is a reasonable assumption for physical states. The equation satisfied by the scalar field is given by

$$\left(D_\mu D^\mu - m^2 - \xi R\right) \Phi = 0. \tag{6.1}$$

The form of the electromagnetic potential A_μ is also arbitrary. In line with the general philosophy of QFTCS, we consider the field Φ to have been quantised but the background spacetime and electromagnetic field A_μ remain classical. While the form of the scalar field equation (6.1) is changed by the inclusion of a scalar field charge q, this does not affect the principal part of the inhomogeneous scalar field equation in (1.18). Thus the Feynman Green's function $G_{\mathrm{F}}^{(d)}(x,x')$ satisfies

$$\left(D_\mu D^\mu - m^2 - \xi R\right) G_{\mathrm{F}}^{(d)}\left(x,x'\right) = -\left[-g(x)\right]^{-\frac{1}{2}} \delta^{(d)}\left(x-x'\right), \tag{6.2}$$

and the Hadamard parametrices in any number of spacetime dimensions are identical to that in [68]. However, in the charged case, the biscalar functions $U^{(d)}(x,x')$, $V^{(d)}(x,x')$ and $W^{(d)}(x,x')$ are sesquisymmetric in the interchange of x and x', as opposed to symmetric in the neutral case. Then we give the Hadamard parametrices in each case as follows.

In $d=2$, the Hadamard expansion of the Feynman propagator $G_{\mathrm{F}}^{(2)}(x,x')$ is given by

$$G_{\mathrm{F}}^{(2)}\left(x,x'\right) = \frac{\mathrm{i}\alpha_2}{2}\left\{V^{(2)}\left(x,x'\right)\ln\left[\frac{\sigma(x,x')}{\ell_{\mathrm{ren}}^2} + \mathrm{i}\epsilon\right] + W^{(2)}\left(x,x'\right)\right\}, \tag{6.3}$$

where $V^{(2)}(x,x')\,,W^{(2)}(x,x')$ are sesquisymmetric biscalars regular as $x' \to x$, given by

$$V^{(2)}\big(x,x'\big) = \sum_{n=0}^{\infty} V_n^{(2)}\big(x,x'\big)\,\sigma^n\big(x,x'\big)\,, \tag{6.4a}$$

$$W^{(2)}\big(x,x'\big) = \sum_{n=0}^{\infty} W_n^{(2)}\big(x,x'\big)\,\sigma^n\big(x,x'\big)\,. \tag{6.4b}$$

In even dimensions with $d = 2p$ and $p \neq 1$, the Hadamard expansion of the Feynman propagator $G_{\mathrm{F}}^{(2p)}(x,x')$ is given by

$$G_{\mathrm{F}}^{(2p)}\big(x,x'\big) = \frac{\mathrm{i}\alpha_d}{2}\left\{\frac{U^{(2p)}(x,x')}{[\sigma(x,x')+\mathrm{i}\epsilon]^{p-1}} + V^{(2p)}\big(x,x'\big)\ln\left[\frac{\sigma(x,x')}{\ell_{\mathrm{ren}}^2}+\mathrm{i}\epsilon\right] + W^{(2p)}\big(x,x'\big)\right\}, \tag{6.5}$$

where $U^{(2p)}(x,x')$, $V^{(2p)}(x,x')$ and $W^{(2p)}(x,x')$ are sesquisymmetric biscalars, regular in the limit $x' \to x$ and which can be expanded as

$$U^{(2p)}\big(x,x'\big) = \sum_{n=0}^{(p-2)} U_n^{(2p)}\big(x,x'\big)\,\sigma^n\big(x,x'\big)\,, \tag{6.6a}$$

$$V^{(2p)}\big(x,x'\big) = \sum_{n=0}^{\infty} V_n^{(2p)}\big(x,x'\big)\,\sigma^n\big(x,x'\big)\,, \tag{6.6b}$$

$$W^{(2p)}\big(x,x'\big) = \sum_{n=0}^{\infty} W_n^{(2p)}\big(x,x'\big)\,\sigma^n\big(x,x'\big)\,. \tag{6.6c}$$

In odd dimensions with $d = 2p+1$ and $p \geq 1$, the Hadamard expansion of the Feynman propagator $G_{\mathrm{F}}^{(2p+1)}(x,x')$ is given by

$$G_{\mathrm{F}}^{(2p+1)}\big(x,x'\big) = \frac{\mathrm{i}\alpha_d}{2}\left\{\frac{U^{(2p+1)}(x,x')}{[\sigma(x,x')+\mathrm{i}\epsilon]^{p-\frac{1}{2}}} + W^{(2p+1)}\big(x,x'\big)\right\}, \tag{6.7}$$

where $U^{(2p+1)}(x,x')$ and $W^{(2p+1)}(x,x')$ are symmetric biscalars, regular in the limit $x' \to x$ and which can be expanded as

$$U^{(2p+1)}\big(x,x'\big) = \sum_{n=0}^{\infty} U_n^{(2p+1)}\big(x,x'\big)\,\sigma^n\big(x,x'\big)\,, \tag{6.8a}$$

$$W^{(2p+1)}\big(x,x'\big) = \sum_{n=0}^{\infty} W_n^{(2p+1)}\big(x,x'\big)\,\sigma^n\big(x,x'\big)\,. \tag{6.8b}$$

We will adopt the convention of referring to the $U^{(d)}(x,x')$, $V^{(d)}(x,x')$ and $W^{(d)}(x,x')$ in the Hadamard expansion of $G_{\mathrm{F}}^{(d)}$ as the biscalar functions. Similarly, we will refer to the coefficients of their expansions, namely the $U_n^{(d)}(x,x')$, $V_n^{(d)}(x,x')$ and $W_n^{(d)}(x,x')$, as the Hadamard coefficients. In equations (6.3), (6.5) and (6.7), the coefficient α_d is given by

$$\alpha_d = \begin{cases} \frac{1}{2\pi} & \text{for } d = 2, \\ \Gamma\left[\frac{d-2}{2(2\pi)^{d/2}}\right] & \text{for } d \neq 2, \end{cases} \tag{6.9}$$

while the factor $i\epsilon$ with $\epsilon \to 0_+$ is introduced to give $G_{\rm F}^{(d)}(x,x')$ a singularity structure that is consistent with the definition of the Feynman propagator as a time-ordered product.

The biscalar functions $U^{(d)}(x,x')$ and $V^{(d)}(x,x')$ appearing in equations (6.3), (6.5) and (6.7) are geometric quantities and therefore uniquely determined. They contain the singular behaviour of Feynman Green's function $G_{\rm F}^{(d)}(x,x')$ entirely, which means we need to find $U^{(d)}(x,x')$ and $V^{(d)}(x,x')$ explicitly. Decanini and Folacci have given the general procedure for a neutral scalar field in [68].

We would like to extend this for a scalar field with an arbitrary charge q. We employ a covariant Taylor expansion method. Substituting the Hadamard expansion of $G_{\rm F}^{(d)}$ into (6.2) gives an expression in terms of $\sigma(x,x')$. Equating powers of $\sigma(x,x')$ will lead us to the equations that $U^{(d)}(x,x')$ and $V^{(d)}(x,x')$ must satisfy. Bearing in mind practical applications, it will be useful to expand the $U_n^{(d)}(x,x')$ and $V_n^{(d)}(x,x')$ as follows

$$U_n^{(d)}\big(x,x'\big) = \sum_{j=0}^{\infty} U^{(d)}_{nj\alpha_1\ldots\alpha_j}(x)\,\sigma^{\,;\alpha_1}\big(x,x'\big)\ldots\sigma^{\,;\alpha_j}\big(x,x'\big)\,, \tag{6.10a}$$

$$V_n^{(d)}\big(x,x'\big) = \sum_{j=0}^{\infty} V^{(d)}_{nj\alpha_1\ldots\alpha_j}(x)\,\sigma^{\,;\alpha_1}\big(x,x'\big)\ldots\sigma^{\,;\alpha_j}\big(x,x'\big)\,. \tag{6.10b}$$

6.2 Properties of sesquisymmetric biscalars

Before we proceed to derive explicit expressions for the biscalar functions, we can examine some of the properties that we expect the Feynman Green's function $G_{\rm F}^{(d)}(x,x')$ and the biscalar functions $U^{(d)}(x,x')$, $V^{(d)}(x,x')$ and $W^{(d)}(x,x')$, contained within (6.3), (6.5) and (6.7), to satisfy. In the case of a neutral scalar field, both $G_{\rm F}^{(d)}(x,x')$ as well as $U^{(d)}(x,x')$, $V^{(d)}(x,x')$ and $W^{(d)}(x,x')$ are real, symmetric biscalars; that is to say, they are symmetric in the interchange of the spacetime points x and x'. Given an arbitrary real, symmetric biscalar function $S(x,x')$, we can write this condition compactly as

$$S\big(x,x'\big) = S\big(x',x\big)\,. \tag{6.11}$$

The condition (6.11) has interesting corollaries; suppose the biscalar $S(x,x')$ admits a covariant Taylor series expansion of the form

$$S\big(x,x'\big) = s_0(x) + s_{1\mu}(x)\,\sigma^{\,;\mu} + s_{2(\mu\nu)}(x)\,\sigma^{\,;\mu}\sigma^{\,;\nu} + s_{3(\mu\nu\rho)}(x)\,\sigma^{\,;\mu}\sigma^{\,;\nu}\sigma^{\,;\rho} + \ldots\,, \tag{6.12}$$

where it is sufficient, for our purposes, to consider the lowest order terms in the covariant Taylor expansion. Then the symmetry condition (6.11) of the biscalar $S(x,x')$ constrains the expansion coefficients in (6.12) such that we can express odd coefficients in terms of even ones. The expressions relating the lowest order coefficients are given explicitly by [68]

$$s_{1\mu} = -\frac{1}{2}s_{0\,;\mu}, \tag{6.13a}$$

$$s_{3(\mu\nu\rho)} = -\frac{1}{2}s_{2(\mu\nu\,;\rho)} + \frac{1}{24}s_{0\,;(\mu\nu\rho)}. \tag{6.13b}$$

Equations (6.11) - (6.13) no longer hold in the case of a charged scalar field, where the aforementioned real, symmetric biscalar functions become complex, sequisymmetric biscalars. Given an arbitrary complex, sequisymmetric biscalar fucntion $K(x, x')$, we have

$$K\big(x, x'\big) = K^*\big(x', x\big)\,. \tag{6.14}$$

Now, suppose the biscalar $K(x, x')$ admits a covariant Taylor expansion of the form

$$K\big(x, x'\big) = k_0(x) + k_{1\mu}(x)\,\sigma^{\,;\mu} + k_{2(\mu\nu)}(x)\,\sigma^{\,;\mu}\sigma^{\,;\nu} + k_{3(\mu\nu\rho)}(x)\,\sigma^{\,;\mu}\sigma^{\,;\nu}\sigma^{\,;\rho} + \ldots, \tag{6.15}$$

where, again, it is sufficient for our purposes to consider the lowest order terms in the covariant Taylor expansion. We deduce an interesting property about the lowest order expansion coefficient $k_0(x)$ in (6.15) by working in the coincidence limit $x' \to x$. Since $\sigma^{\,;\mu}$ vanishes as $x' \to x$, we need only keep track of the lowest order term in (6.15). Taking the complex conjugate of (6.15) and interchanging the spacetime points x and x', we obtain

$$K^*\big(x', x\big) = k_0^*\big(x'\big) + \ldots, \tag{6.16}$$

Now taking the coincidence limit, (6.15) and (6.16) reduce to

$$K(x, x) = k_0(x)\,, \tag{6.17a}$$

$$K^*\big(x', x'\big) = K^*(x, x) = k_0^*(x)\,, \tag{6.17b}$$

respectively. By (6.14), equations (6.17a) and (6.17b) are equal, meaning

$$k_0(x) = k_0^*(x)\,. \tag{6.18}$$

Hence, the lowest order expansion coefficient $k_0(x)$ of a general complex, sesquisymmetric biscalar $K(x, x')$ is always real.

In general, the rest of the expansion coefficients in (6.15) are complex. However, since $K(x, x')$ should reduce to a real, symmetric biscalar when considering only its real part, then the real parts of the expansion coefficients in (6.15) satisfy analogous relations to (6.13). These are given explicitly by

$$\Re[k_{1\mu}] = -\frac{1}{2}\,k_{0\,;\mu}, \tag{6.19}$$

$$\Re\big[k_{3(\mu\nu\rho)}\big] = -\frac{1}{2}\,\Re\big[k_{2(\mu\nu\,;\rho)}\big] + \frac{1}{24}\,k_{0\,;(\mu\nu\rho)}, \tag{6.20}$$

where we have used the statement that $k_0(x)$ is real (6.18) in simplifying (6.19) and (6.20). Furthermore, we find an additional relation between the imaginary parts of the lowest order coefficients given by

$$\Im\big[k_{2(\mu\nu)}\big] = \frac{1}{4\mathrm{i}}\left[k_{0\,;\mu\nu} + 2\,k^*_{1(\mu\,;\nu)}\right]. \tag{6.21}$$

We can use (6.19) to simplify (6.21) by

$$\begin{aligned}\Im\big[k_{2(\mu\nu)}\big] &= \frac{1}{4\mathrm{i}}\Big[\nabla_\mu\, k_{0\,;\nu} + 2\,k^*_{1(\mu\,;\nu)}\Big] \\ &= \frac{1}{4\mathrm{i}}\left\{-2\,\nabla_{(\mu}\Re\big[k_{1\nu)}\big] + 2\,\nabla_{(\mu}\Re\big[k_{1\nu)}\big] - 2\,\mathrm{i}\,\nabla_{(\mu}\Im\big[k_{1\nu)}\big]\right\} \\ &= \frac{1}{4\mathrm{i}}\left\{-2\,\mathrm{i}\,\nabla_{(\mu}\Im\big[k_{1\nu)}\big]\right\}. \end{aligned} \tag{6.22}$$

Thus, for the imaginary parts of the lowest order expansion coefficients in (6.15), we obtain

$$\Im\big[k_{2(\mu\nu)}\big] = -\frac{1}{2}\,\Im\big[k_{1(\mu\,;\nu)}\big]\,. \tag{6.23}$$

6.3 Useful identities

In this section, we give a number of identities that will be useful throughout Part III.

When substituting the various Hadamard parametrices (6.3–6.8), the following identity will be useful; given an arbitrary biscalar $K(x, x')$ and suppressing arguments, we have

$$\begin{aligned}&\nabla^\mu\nabla_\mu K - \mathrm{i}q\,(\nabla^\mu A_\mu)K - 2\,\mathrm{i}qA_\mu[\nabla^\mu K] - q^2A^\mu A_\mu K - \big(m^2+\xi R\big)K \\ &\quad= \nabla^\mu\nabla_\mu K - \mathrm{i}q\,(\nabla^\mu A_\mu)K - \mathrm{i}qA_\mu[\nabla^\mu K] - \mathrm{i}q\,\nabla^\mu[A_\mu K] + \mathrm{i}q\,(\nabla^\mu A_\mu)K \\ &\quad\quad - q^2A^\mu A_\mu K - \big(m^2+\xi R\big)K \\ &\quad= \nabla^\mu\nabla_\mu K - \mathrm{i}qA_\mu[\nabla^\mu K] - \mathrm{i}q\,\nabla^\mu[A_\mu K] - q^2A^\mu A_\mu K - \big(m^2+\xi R\big)K \\ &\quad= \big(D_\mu D^\mu - m^2 - \xi R\big)K. \end{aligned} \tag{6.24}$$

In order to derive explicit expressions for the expansion coefficients (6.10) of the Hadamard coefficients $U_n^{(d)}(x,x')$, $V_n^{(d)}(x,x')$, we will require the covariant expansion of the second derivative $\sigma_{;\mu\nu}$ of Synge's world function (1.20) which is given by [92]

$$\begin{aligned}\sigma_{;\mu\nu} = g_{\mu\nu} - \frac{1}{3}R_{\mu(\theta|\nu|\phi)}\sigma^{;\theta}\sigma^{;\phi} + \frac{1}{12}R_{\mu(\theta|\nu|\phi\,;\psi)}\sigma^{;\theta}\sigma^{;\phi}\sigma^{;\psi} \\ - \left(\frac{1}{60}R_{\mu(\theta|\nu|\phi\,;\psi\gamma)} + \frac{1}{45}R_{\mu(\theta|\rho|\phi}R^{\rho}{}_{\psi|\nu|\gamma)}\right)\sigma^{;\theta}\sigma^{;\phi}\sigma^{;\psi}\sigma^{;\gamma} + \dots \end{aligned} \tag{6.25}$$

We also require the covariant expansion of the quantity $\Delta^{-\frac{1}{2}}\Delta^{\frac{1}{2}}{}_{;\mu}\sigma^{;\mu}$, where we remind the reader that $\Delta(x,x')$ is the van Vleck-Morette determinant (1.28); this is given by [92]

$$\begin{aligned}\Delta^{-\frac{1}{2}}\Delta^{\frac{1}{2}}{}_{;\mu}\sigma^{;\mu} = \frac{1}{6}R_{\mu\nu}\sigma^{;\mu}\sigma^{;\nu} - \frac{1}{24}R_{(\mu\nu;\rho)}\sigma^{;\mu}\sigma^{;\nu}\sigma^{;\rho} \\ + \left[\frac{1}{120}R_{(\mu\nu;\rho\tau)} + \frac{1}{90}R^{\lambda}{}_{(\mu|\beta|\nu}R^{\beta}{}_{\rho|\lambda|\tau)}\right]\sigma^{;\mu}\sigma^{;\nu}\sigma^{;\rho}\sigma^{;\tau} + \dots \end{aligned} \tag{6.26}$$

We can relate the gauge covariant derivatives to the EM field strength tensor by

$$\begin{aligned} D_\mu A_\nu - D_\nu A_\mu &= (\nabla_\mu - \mathrm{i}qA_\mu)A_\nu - (\nabla_\nu - \mathrm{i}qA_\nu)A_\mu \\ &= \nabla_\mu A_\nu - \mathrm{i}qA_\mu A_\nu - \nabla_\nu A_\mu + \mathrm{i}qA_\nu A_\mu \\ &= \nabla_\mu A_\nu - \nabla_\nu A_\mu \\ &= F_{\mu\nu}. \end{aligned} \tag{6.27}$$

The commutator of two gauge covariant derivatives acting on the gauge field is given by

$$\begin{aligned} \mathrm{i}q\,[D_\mu\,, D_\nu]A_\rho &= \mathrm{i}q\,[\nabla_\mu\,, \nabla_\nu]A_\rho + q^2\,[\nabla_\mu\,, A_\nu]A_\rho + q^2\,[A_\mu\,, \nabla_\nu]A_\rho - \mathrm{i}q^3\,[A_\mu\,, A_\nu]A_\rho \\ &= -\mathrm{i}qR^\lambda{}_{\rho\mu\nu}A_\lambda + q^2\,\nabla_\mu A_\nu A_\rho - q^2 A_\nu \nabla_\mu A_\rho + q^2 A_\mu \nabla_\nu A_\rho - q^2\,\nabla_\nu A_\mu A_\rho \\ &= -\mathrm{i}qR^\lambda{}_{\rho\mu\nu}A_\lambda + q^2(\nabla_\mu A_\nu)A_\rho - q^2(\nabla_\nu A_\mu)A_\rho \\ &= -\mathrm{i}qR^\lambda{}_{\rho\mu\nu}A_\lambda + q^2 A_\rho F_{\mu\nu}. \end{aligned} \tag{6.28}$$

Raising the index from the expression in (6.28), we also have

$$\mathrm{i}q\,[D_\mu\,, D_\nu]A^\rho = \mathrm{i}qR^\rho{}_{\lambda\mu\nu}A^\lambda + q^2 A^\rho F_{\mu\nu}. \tag{6.29}$$

Finally, the following quantity will be very useful:

$$\begin{aligned} \mathrm{i}q\,[D_\mu\,, D_\nu]\,D_\rho A_\tau &= \mathrm{i}q\,[\nabla_\mu - \mathrm{i}qA_\mu\,, \nabla_\nu - \mathrm{i}qA_\nu]\,D_\rho A_\tau \\ &= \mathrm{i}q\,[\nabla_\mu\,, \nabla_\nu]\,D_\rho A_\tau + q^2\,[A_\mu\,, \nabla_\nu]\,D_\rho A_\tau + q^2\,[\nabla_\mu\,, A_\nu]\,D_\rho A_\tau \\ &- \mathrm{i}q^3\,[A_\mu\,, A_\nu]\,D_\rho A_\tau \\ &= -\mathrm{i}q\left(R^\lambda{}_{\rho\mu\nu}D_\lambda A_\tau + R^\lambda{}_{\tau\mu\nu}D_\rho A_\lambda\right) + q^2 A_\mu \nabla_\nu D_\rho A_\tau - q^2\,\nabla_\nu A_\mu D_\rho A_\tau \\ &+ q^2\,\nabla_\mu A_\nu D_\rho A_\tau - q^2 A_\nu \nabla_\mu D_\rho A_\tau \\ &= -\mathrm{i}q\left(R^\lambda{}_{\rho\mu\nu}D_\lambda A_\tau + R^\lambda{}_{\tau\mu\nu}D_\rho A_\lambda\right) - q^2(\nabla_\nu A_\mu)D_\rho A_\tau + q^2(\nabla_\mu A_\nu)D_\rho A_\tau \\ &= -\mathrm{i}q\left(R^\lambda{}_{\rho\mu\nu}D_\lambda A_\tau + R^\lambda{}_{\tau\mu\nu}D_\rho A_\lambda\right) + q^2 F_{\mu\nu}D_\rho A_\tau. \end{aligned} \tag{6.30}$$

Using (6.30), we have

$$\mathrm{i}q\,[D_\mu\,, D_\nu]\,D_\rho A^\tau = \mathrm{i}q\left(-R^\lambda{}_{\rho\mu\nu}D_\lambda A^\tau + R^\tau{}_{\lambda\mu\nu}D_\rho A^\lambda\right) + q^2 F_{\mu\nu}D_\rho A^\tau. \tag{6.31}$$

Using (6.30), we also have

$$\mathrm{i}q\,[D_\mu\,, D_\nu]\,D^\rho A_\tau = \mathrm{i}q\left(R^\rho{}_{\lambda\mu\nu}D^\lambda A_\tau - R^\lambda{}_{\tau\mu\nu}D^\rho A_\lambda\right) + q^2 F_{\mu\nu}D^\rho A_\tau. \tag{6.32}$$

When simplifying products of Riemann tensor, we will make extensive use of the identity

$$R^{\mu\nu\rho}{}_\tau R_{\rho\nu\mu\lambda} = \frac{1}{2}\,R^{\mu\nu\rho}{}_\tau R_{\mu\nu\rho\lambda}\,. \tag{6.33}$$

We can prove this by noting

$$
\begin{aligned}
R^{\mu\nu\rho}{}_{\tau}R_{\mu\nu\rho\lambda} &= -R^{\mu\nu\rho}{}_{\tau}R_{\nu\rho\mu\lambda} - R^{\mu\nu\rho}{}_{\tau}R_{\rho\mu\nu\lambda} \\
&= R^{\mu\nu\rho}{}_{\tau}R_{\rho\nu\mu\lambda} + R^{\nu\mu\rho}{}_{\tau}R_{\rho\mu\nu\lambda} \\
&= R^{\mu\nu\rho}{}_{\tau}R_{\rho\nu\mu\lambda} + R^{\mu\nu\rho}{}_{\tau}R_{\rho\nu\mu\lambda} \\
&= 2R^{\mu\nu\rho}{}_{\tau}R_{\rho\nu\mu\lambda}.
\end{aligned}
\tag{6.34}
$$

Chapter 7

Renormalisation counterterms

In §7.1, we develop the Hadamard renormalisation procedure in two dimensions and derive the explicit renormalisation counterterms needed to evaluate the RSET. In 7.2, we develop the Hadamard renormalisation procedure in even dimensions and derive the explicit renormalisation counterterms in $d = 4$. In 7.3, we develop the Hadamard renormalisation procedure in odd dimensions and derive the explicit renormalisation counterterms in $d = 3$.

7.1 Two dimensions

In this section, we develop the general formalism for the Hadamard renormalisation procedure of charged scalar fields in a general background spacetime of two dimensions. We derive equations satisfied by the $V^{(2)}(x,x')$ and $W^{(2)}(x,x')$ biscalars in the Hadamard parametrix (6.3) and we derive explicit expressions for the renormalisation counterterms contained in the $V^{(2)}(x,x')$ biscalar up to the order required to evaluate the RSET.

We would like to evaluate the inhomogeneous scalar field equation (6.2) for the $d = 2$ Hadamard parametrix (6.3). From (6.9), we have $\alpha_2 = \frac{1}{2\pi}$. Then (6.2) becomes

$$\begin{aligned}
&\left(D_\mu D^\mu - m^2 - \xi R\right) G_{\mathrm{F}}^{(2)}\left(x,x'\right) \\
&\qquad = \frac{\mathrm{i}}{4\pi}\left(D_\mu D^\mu - m^2 - \xi R\right)\left[V^{(2)}\left(x,x'\right)\ln\sigma\left(x,x'\right) + W^{(2)}\left(x,x'\right)\right] \\
&\qquad = -\frac{1}{\sqrt{-g(x)}}\,\delta^{(2)}\left(x-x'\right).
\end{aligned} \tag{7.1}$$

It will be convenient instead to evaluate the equation

$$\begin{aligned}
&-4\pi\mathrm{i}\left(D_\mu D^\mu - m^2 - \xi R\right) G_{\mathrm{F}}^{(2)}\left(x,x'\right) \\
&\qquad = \left(D_\mu D^\mu - m^2 - \xi R\right)\left[V^{(2)}\left(x,x'\right)\ln\sigma\left(x,x'\right) + W^{(2)}\left(x,x'\right)\right] \\
&\qquad = \frac{4\pi\mathrm{i}}{\sqrt{-g(x)}}\,\delta^{(2)}\left(x-x'\right).
\end{aligned} \tag{7.2}$$

Then, suppressing arguments of the biscalar functions, we begin by calculating the quantity

$$
\begin{aligned}
-4\pi\mathrm{i}\, D_\mu G_{\mathrm{F}}^{(2)} &= (\nabla_\mu - \mathrm{i}qA_\mu)\left[V^{(2)}\ln\sigma + W^{(2)}\right] \\
&= \left[\nabla_\mu V^{(2)}\right]\ln\sigma + \sigma^{-1}\, V^{(2)}(\nabla_\mu\sigma) + \nabla_\mu W^{(2)} - \mathrm{i}qA_\mu V^{(2)}\ln\sigma - \mathrm{i}qA_\mu W^{(2)}.
\end{aligned} \tag{7.3}
$$

Acting on (7.3) with another gauge derivative, we obtain

$$
\begin{aligned}
-4\pi\mathrm{i}\, D_\mu D^\mu G_{\mathrm{F}}^{(2)} = {} & \left[\nabla^\mu\nabla_\mu V^{(2)}\right]\ln\sigma + 2\,\sigma^{-1}\left[\nabla_\mu V^{(2)}\right](\nabla^\mu\sigma) - \sigma^{-2}\, V^{(2)}(\nabla^\mu\sigma)(\nabla_\mu\sigma) \\
& + \sigma^{-1}\, V^{(2)}(\nabla^\mu\nabla_\mu\sigma) + \nabla^\mu\nabla_\mu W^{(2)} - \mathrm{i}q\,(\nabla^\mu A_\mu)V^{(2)}\ln\sigma \\
& - 2\,\mathrm{i}qA_\mu\left[\nabla^\mu V^{(2)}\right]\ln\sigma - 2\,\mathrm{i}q\,\sigma^{-1}A_\mu V^{(2)}(\nabla^\mu\sigma) - \mathrm{i}q\,(\nabla^\mu A_\mu)W^{(2)} \\
& - 2\,\mathrm{i}qA_\mu\left[\nabla^\mu W^{(2)}\right] - q^2A^\mu A_\mu V^{(2)}\ln\sigma - q^2A^\mu A_\mu W^{(2)}.
\end{aligned} \tag{7.4}
$$

We can simplify (7.4) using the identities involving Synge's world function (1.20) and the van Vleck-Morette determinant (1.28); we obtain

$$
\begin{aligned}
-4\pi\mathrm{i}\, D_\mu D^\mu G_{\mathrm{F}}^{(2)} = {} & \left[\nabla^\mu\nabla_\mu V^{(2)}\right]\ln\sigma + 2\,\sigma^{-1}\left[\nabla_\mu V^{(2)}\right](\nabla^\mu\sigma) - 2\,\sigma^{-1}\, V^{(2)}\Delta^{-\frac{1}{2}}\Delta^{\frac{1}{2}}{}_{;\mu}\,(\nabla^\mu\sigma) \\
& + \nabla^\mu\nabla_\mu W^{(2)} - \mathrm{i}q\,(\nabla^\mu A_\mu)V^{(2)}\ln\sigma - 2\,\mathrm{i}qA_\mu\left[\nabla^\mu V^{(2)}\right]\ln\sigma \\
& - 2\,\mathrm{i}q\,\sigma^{-1}A_\mu V^{(2)}(\nabla^\mu\sigma) - \mathrm{i}q\,(\nabla^\mu A_\mu)W^{(2)} - 2\,\mathrm{i}qA_\mu\left[\nabla^\mu W^{(2)}\right] \\
& - q^2A^\mu A_\mu V^{(2)}\ln\sigma - q^2A^\mu A_\mu W^{(2)}.
\end{aligned} \tag{7.5}
$$

This allows us to evaluate the l.h.s of the inhomogeneous Klein-Gordon equation (6.2) for $d = 2$ as follows

$$
\begin{aligned}
& -4\pi\mathrm{i}\,(D_\mu D^\mu - m^2 - \xi R)\, G_{\mathrm{F}}^{(2)} \\
& = \left[\nabla^\mu\nabla_\mu V^{(2)}\right]\ln\sigma + 2\,\sigma^{-1}\left[\nabla_\mu V^{(2)}\right](\nabla^\mu\sigma) - 2\,\sigma^{-1}\, V^{(2)}\Delta^{-\frac{1}{2}}\Delta^{\frac{1}{2}}{}_{;\mu}\,(\nabla^\mu\sigma) + \nabla^\mu\nabla_\mu W^{(2)} \\
& - \mathrm{i}q\,(\nabla^\mu A_\mu)\, V^{(2)}\ln\sigma - 2\,\mathrm{i}qA_\mu\left[\nabla^\mu V^{(2)}\right]\ln\sigma - 2\,\mathrm{i}q\,\sigma^{-1}A_\mu V^{(2)}(\nabla^\mu\sigma) - \mathrm{i}q\,(\nabla^\mu A_\mu)W^{(2)} \\
& - 2\,\mathrm{i}qA_\mu\left[\nabla^\mu W^{(2)}\right] - q^2A^\mu A_\mu V^{(2)}\ln\sigma - q^2A^\mu A_\mu W^{(2)} - \left(m^2 + \xi R\right)V^{(2)}\ln\sigma \\
& - \left(m^2 + \xi R\right)W^{(2)}.
\end{aligned} \tag{7.6}
$$

We can simplify (7.6) by use of the gauge covariant derivative, which then reduces to

$$
\begin{aligned}
& -4\pi\mathrm{i}\,\left(D_\mu D^\mu - m^2 + \xi R\right) G_{\mathrm{F}}^{(2)} \\
& \qquad = \left[\left(D_\mu D^\mu - m^2 - \xi R\right) V^{(2)}\right]\ln\sigma + \left(D_\mu D^\mu - m^2 - \xi R\right) W^{(2)} \\
& \qquad\qquad + 2\,\sigma^{-1}\left[\left(D_\mu - \Delta^{-\frac{1}{2}}\Delta^{\frac{1}{2}}{}_{;\mu}\right) V^{(2)}\right]\sigma^{;\mu}.
\end{aligned} \tag{7.7}
$$

From the r.h.s of the inhomogeneous scalar field equation (6.2), we know that (7.7) must be equal to zero when the two points x and x' are separated. Since Synge's world function is nonzero when $x' \neq x$, we deduce that (7.7) must vanish identically at each power of $\sigma(x,x')$. This allows us to generate two equations by considering terms proportional to $\ln\sigma$ and, separately, the remaining terms, i.e. those not proportional to $\ln\sigma$. In particular, the terms that are proportional to $\ln\sigma$ must vanish independently since no other terms can contain this factor; this allows us to write

$$\left(D_\mu D^\mu - m^2 - \xi R\right) V^{(2)}(x,x') = 0, \tag{7.8}$$

which means that the biscalar $V^{(2)}(x,x')$ solves the homogeneous scalar field equation. Equation (7.8) generalises (35) in [68] and it enables us to derive the recurrence relations for the Hadamard coefficients $V_n^{(2)}(x,x')$. Since the biscalar $V^{(2)}(x,x')$ admits a power series expansion in $\sigma(x,x')$ (6.4a), we can derive the recurrence relation for the $V_n^{(2)}(x,x')$ by expanding the terms in (7.8). Suppressing arguments, we first need to evaluate

$$D_\mu V^{(2)} = \sum_{n=0}^{\infty}\left\{\left[\nabla_\mu V_n^{(2)}\right]\sigma^n + nV_n^{(2)}\sigma^{n-1}\left(\nabla_\mu\sigma\right) - \mathrm{i}qA_\mu V_n^{(2)}\sigma^n\right\}, \tag{7.9}$$

and then

$$\begin{aligned}D_\mu D^\mu V^{(2)} = \sum_{n=0}^{\infty}\Big\{&\left[\nabla^\mu\nabla_\mu V_n^{(2)}\right]\sigma^n + \left[\nabla_\mu V_n^{(2)}\right]n\,\sigma^{n-1}(\nabla^\mu\sigma) + n\left[\nabla^\mu V_n^{(2)}\right]\sigma^{n-1}\left(\nabla_\mu\sigma\right)\\
&+ n\,(n-1)\,V_n^{(2)}\sigma^{n-2}\left(\nabla^\mu\sigma\right)(\nabla_\mu\sigma) + nV_n^{(2)}\sigma^{n-1}\left(\nabla^\mu\nabla_\mu\sigma\right)\\
&- \mathrm{i}q\left(\nabla^\mu A_\mu\right)V_n^{(2)}\sigma^n - \mathrm{i}qA_\mu\left[\nabla^\mu V_n^{(2)}\right]\sigma^n - \mathrm{i}qA_\mu\,n\,V_n^{(2)}\sigma^{n-1}\left(\nabla^\mu\sigma\right)\\
&- \mathrm{i}qA_\mu\left[\nabla_\mu V_n^{(2)}\right]\sigma^n - \mathrm{i}qA_\mu\,n\,V_n^{(2)}\sigma^{n-1}\left(\nabla_\mu\sigma\right) - q^2A^\mu A_\mu V_n^{(2)}\sigma^n\Big\}.\end{aligned} \tag{7.10}$$

Then, using (1.20), (1.28), (6.24) and the gauge covariant derivative, (7.10) becomes

$$\begin{aligned}D_\mu D^\mu V^{(2)} = \sum_{n=0}^{\infty}\Big\{&\left[D^\mu D_\mu V_n^{(2)}\right]\sigma^n + 2n\left[D_\mu V_n^{(2)}\right]\sigma^{n-1}(\nabla^\mu\sigma) + 2n^2\,V_n^{(2)}\sigma^{n-1}\\
&- 2n\,V_n^{(2)}\Delta^{-\frac{1}{2}}\left(\nabla_\mu\Delta^{\frac{1}{2}}\right)\sigma^{n-1}\left(\nabla^\mu\sigma\right)\Big\}.\end{aligned} \tag{7.11}$$

Substituting (7.11) into (7.8), we obtain

$$\begin{aligned}0 = \sum_{n=0}^{\infty}\Big\{&\left[\left(D_\mu D^\mu - m^2 - \xi R\right)V_n^{(2)}\right]\sigma^n + 2n\left[D_\mu V_n^{(2)}\right]\sigma^{n-1}(\nabla^\mu\sigma) + 2n^2\,V_n^{(2)}\sigma^{n-1}\\
&- 2n\,V_n^{(2)}\Delta^{-\frac{1}{2}}\left(\nabla_\mu\Delta^{\frac{1}{2}}\right)\sigma^{n-1}\left(\nabla^\mu\sigma\right)\Big\}.\end{aligned} \tag{7.12}$$

Performing the relabelling $n \to n+1$ in terms proportional to σ^{n-1} in (7.12), we obtain

$$\begin{aligned}0 = \sum_{n=0}^{\infty}\Big\{&\left[\left(D_\mu D^\mu - m^2 - \xi R\right)V_n^{(2)}\right]\sigma^n + 2(n+1)\left[D_\mu V_{n+1}^{(2)}\right]\sigma^n(\nabla^\mu\sigma)\\
&+ 2(n+1)^2\,V_{n+1}^{(2)}\,\sigma^n - 2(n+1)\,V_{n+1}^{(2)}\Delta^{-\frac{1}{2}}\left(\nabla_\mu\Delta^{\frac{1}{2}}\right)\sigma^n\left(\nabla^\mu\sigma\right)\Big\}.\end{aligned} \tag{7.13}$$

Since (7.13) must hold for each power of σ, this enables us to obtain the recurrence relation for the Hadamard coefficients $V_n^{(2)}(x,x')$ of the biscalar function $V^{(2)}(x,x')$. We have

$$2(n+1)^2\, V_{n+1}^{(2)} + 2(n+1)\,\sigma^{;\mu} D_\mu V_{n+1}^{(2)} - 2(n+1)\, V_{n+1}^{(2)}\, \Delta^{-\frac{1}{2}} \Delta^{\frac{1}{2}}_{;\mu}\, \sigma^{;\mu} + \big(D_\mu D^\mu - m^2 - \xi R\big)\, V_n^{(2)} = 0 \quad \text{for } n \text{ in } \mathbb{N}. \tag{7.14}$$

This generalises equation (33a) in [68]. Returning to (7.7), the remaining terms, i.e. those not proportional to $\ln\sigma$, give

$$\sigma\,\big(D_\mu D^\mu - m^2 - \xi R\big)\, W^{(2)} = -\,2\,\sigma^{;\mu} D_\mu V^{(2)} + 2\, V^{(2)} \Delta^{-\frac{1}{2}} \Delta^{\frac{1}{2}}{}_{;\mu}\, \sigma^{;\mu}, \tag{7.15}$$

which generalises (36) in [68]. It will be convenient to rewrite (7.15) as

$$\big(D_\mu D^\mu - m^2 - \xi R\big)\, W^{(2)} = -2\,\sigma^{-1}\sigma^{;\mu} D_\mu V^{(2)} + 2\,\sigma^{-1} V^{(2)} \Delta^{-\frac{1}{2}} \Delta^{\frac{1}{2}}{}_{;\mu}\, \sigma^{;\mu}, \tag{7.16}$$

which we refer to as the wave equation in $d = 2$, and from which we will derive identities concerning the expansion coefficients of the biscalar $W^{(2)}(x,x')$ in Chapter 8. We derive the recurrence relation for the Hadamard coefficients $W_n^{(2)}(x,x')$ by inserting the power series expansion for $V^{(2)}(x,x')$ and $W^{(2)}(x,x')$ into (7.16). Since the power series expansions for the biscalars $V^{(2)}(x,x')$ (6.4a) and the $W^{(2)}(x,x')$ (6.4b) are structurally similar expressions, we can use (7.11) to evaluate the $D_\mu D^\mu W^{(2)}$, remembering to interchange the $V_n^{(2)}$ for the $W_n^{(2)}$. Then the first term on the l.h.s of (7.16) is given by

$$\begin{aligned}
&\big(D_\mu D^\mu - m^2 - \xi R\big)\, W^{(2)} \\
&\quad = \sum_{n=0}^{\infty} \bigg\{ \Big[\big(D^\mu D_\mu - m^2 - \xi R\big)\, W_n^{(2)}\Big]\sigma^n + 2n\Big[D_\mu W_n^{(2)}\Big]\sigma^{n-1}(\nabla^\mu\sigma) \\
&\qquad + 2n^2\, W_n^{(2)}\sigma^{n-1} - 2n\, W_n^{(2)}\Delta^{-\frac{1}{2}}\left(\nabla_\mu \Delta^{\frac{1}{2}}\right)\sigma^{n-1}\,(\nabla^\mu\sigma)\bigg\}.
\end{aligned} \tag{7.17}$$

The first term on the r.h.s of (7.16) is given by

$$\begin{aligned}
2\,\sigma^{-1}\sigma^{;\mu} D_\mu V^{(2)} &= \sum_{n=0}^{\infty}\Big\{2\,\sigma^{-1}\sigma^{;\mu}\,\nabla_\mu\Big[V_n^{(2)}\sigma^n\Big] - 2\,\mathrm{i}q\,\sigma^{-1}\sigma^{;\mu} A_\mu V_n^{(2)}\sigma^n\Big\} \\
&= \sum_{n=0}^{\infty}\Big\{2\,\Big[\nabla_\mu V_n^{(2)}\Big]\sigma^{n-1}\sigma^{;\mu} + 2n\, V_n^{(2)}\sigma^{n-2}\sigma^{;\mu}\sigma_{;\mu} - 2\,\mathrm{i}q A_\mu V_n^{(2)}\sigma^{n-1}\sigma^{;\mu}\Big\} \\
&= \sum_{n=0}^{\infty}\Big\{2\,\Big[D_\mu V_n^{(2)}\Big]\sigma^{n-1}\sigma^{;\mu} + 4n\, V_n^{(2)}\sigma^{n-1}\Big\}.
\end{aligned} \tag{7.18}$$

The second term on the r.h.s of (7.16) is given by

$$-\,2\,\sigma^{-1} V^{(2)}\sigma^n \Delta^{-\frac{1}{2}} \Delta^{\frac{1}{2}}{}_{;\mu}\,\sigma^{;\mu} = -2\sum_{n=0}^{\infty} V_n^{(2)}\sigma^{n-1}\Delta^{-\frac{1}{2}}\Delta^{\frac{1}{2}}{}_{;\mu}\,\sigma^{;\mu}. \tag{7.19}$$

Inserting (7.17), (7.18) and (7.19) into (7.16), we obtain

$$
\begin{aligned}
0 = \sum_{n=0}^{\infty} \Big\{ & \Big[\left(D_\mu D^\mu - m^2 - \xi R\right) W_n^{(2)} \Big] \sigma^n + 2n \Big[D_\mu W_n^{(2)} \Big] \sigma^{n-1} (\nabla^\mu \sigma) + 2n^2\, W_n^{(2)} \sigma^{n-1} \\
& - 2n\, W_n^{(2)} \Delta^{-\frac{1}{2}} \left(\nabla_\mu \Delta^{\frac{1}{2}} \right) \sigma^{n-1} \left(\nabla^\mu \sigma \right) + 2 \left[D_\mu V_n^{(2)} \right] \sigma^{n-1} \sigma^{\,;\mu} + 4n\, V_n^{(2)} \sigma^{n-1} \\
& - 2\, V_n^{(2)} \sigma^{n-1} \Delta^{-\frac{1}{2}} {\Delta^{\frac{1}{2}}}_{;\mu}\, \sigma^{\,;\mu} \Big\} .
\end{aligned} \tag{7.20}
$$

Performing the relabelling $n \to n+1$ in terms proportional to σ^{n-1} in (7.20), we obtain

$$
\begin{aligned}
0 = \sum_{n=0}^{\infty} \Big\{ & \Big[\left(D_\mu D^\mu - m^2 - \xi R\right) W_n^{(2)} \Big] \sigma^n + 2(n+1) \Big[D_\mu W_{n+1}^{(2)} \Big] \sigma^n (\nabla^\mu \sigma) \\
& + 2(n+1)^2\, W_{n+1}^{(2)}\, \sigma^n - 2(n+1)\, W_{n+1}^{(2)}\, \Delta^{-\frac{1}{2}} \left(\nabla_\mu \Delta^{\frac{1}{2}} \right) \sigma^n (\nabla^\mu \sigma) + 2 \Big[D_\mu V_{n+1}^{(2)} \Big] \sigma^n \sigma^{\,;\mu} \\
& + 4(n+1)\, V_{n+1}^{(2)} \sigma^n - 2\, V_{n+1}^{(2)} \sigma^n \Delta^{-\frac{1}{2}} {\Delta^{\frac{1}{2}}}_{;\mu}\, \sigma^{\,;\mu} \Big\} + 2\, \sigma^{-1} \left(\sigma^{\,;\mu} D_\mu - \Delta^{-\frac{1}{2}} {\Delta^{\frac{1}{2}}}_{;\mu} \sigma^{\,;\mu} \right) V_0^{(2)} .
\end{aligned} \tag{7.21}
$$

Since (7.21) must hold for each power of σ, the terms proportional to σ^n enable us to obtain the recurrence relation for the Hadamard coefficients $W_n^{(2)}(x,x')$; we have

$$
\begin{aligned}
0 = 2(n+1)^2\, W_{n+1}^{(2)} + 2(n+1)\, \sigma^{\,;\mu} D_\mu W_{n+1}^{(2)} - 2(n+1)\, W_{n+1}^{(2)}\, \Delta^{-\frac{1}{2}} \Delta^{\frac{1}{2}}_{;\mu}\, \sigma^{\,;\mu} & \\
+ 4(n+1)\, V_{n+1}^{(2)} + 2\, \sigma^{\,;\mu} D_\mu V_{n+1}^{(2)} - 2\, V_{n+1}^{(2)}\, \Delta^{-\frac{1}{2}} {\Delta^{\frac{1}{2}}}_{;\mu}\, \sigma^{\,;\mu} & \\
+ \left[\left(D_\mu D^\mu - m^2 - \xi R\right) W_n^{(2)} \right] & \quad \text{for } n \text{ in } \mathbb{N}.
\end{aligned} \tag{7.22}
$$

This generalises equation (34) in [68]. The lowest order terms in $\sigma(x,x')$ in (7.21), i.e. those proportional to σ^{-1}, give the boundary condition for the $V_0^{(2)}(x,x')$ Hadamard coefficient:

$$
0 = \sigma^{\,;\mu} D_\mu V_0^{(2)} - V_0^{(2)} \Delta^{-\frac{1}{2}} {\Delta^{\frac{1}{2}}}_{;\mu} \sigma^{\,;\mu} . \tag{7.23}
$$

In the uncharged case, (7.23) reduces to

$$
0 = \sigma^{\,;\mu}\, V_{0\,;\mu}^{(2)} - \sigma^{\,;\mu}\, V_0^{(2)} \Delta^{-\frac{1}{2}} {\Delta^{\frac{1}{2}}}_{;\mu} , \tag{7.24}
$$

and we can see that (7.24) is solved by taking either $V_0^{(2)} = \Delta^{\frac{1}{2}}$ or $V_0^{(2)} = -\Delta^{\frac{1}{2}}$. Our guiding principle will be that the leading-order singularity in the Hadamard parametrix (6.3) matches that of Minkowski spacetime [68]; in the coincidence limit $x' \to x$, we have

$$
V_0^{(2)}(x,x) = -1 . \tag{7.25}
$$

Therefore, in the uncharged case, (7.24) is solved by

$$
V_0^{(2)}(x,x') = -\Delta^{\frac{1}{2}} . \tag{7.26}
$$

In the charged case (7.23) cannot be solved exactly and we expand $V_0^{(2)}(x,x')$ as a power series in σ up to the order required for the evaluation of the RSET. Rewriting (7.23) as

$$\left[D_\mu V_0^{(2)}\right]\sigma^{;\mu} - V_0^{(2)}\Delta^{-\frac{1}{2}}\Delta^{\frac{1}{2}}_{;\mu}\,\sigma^{;\mu} = 0, \tag{7.27}$$

we expand $V_0^{(2)}(x,x')$ as a covariant Taylor expansion according to (6.10b); in $d=2$, we are required to evaluate $V_0^{(2)}$ to $\mathcal{O}(\sigma)$ in order to evaluate the RSET. Then, we have

$$V_0^{(2)} = V_{00}^{(2)} + V_{01\mu}^{(2)}\,\sigma^{;\mu} + V_{02(\mu\nu)}^{(2)}\,\sigma^{;\mu}\sigma^{;\nu} + \mathcal{O}\left(\sigma^{\frac{3}{2}}\right) \tag{7.28}$$

We can evaluate $D_\mu V_0^{(2)}$ by first taking the covariant derivative of (7.28) to obtain

$$\begin{aligned}
V_{0\,;\mu}^{(2)} &= V_{00\,;\mu}^{(2)} + V_{01\nu\,;\mu}^{(2)}\sigma^{;\nu} + V_{01\nu}^{(2)}\sigma^{;\nu}{}_{\mu} + 2\,V_{02(\nu\rho)}^{(2)}\sigma^{;\nu}{}_{\mu}\sigma^{;\rho} + \mathcal{O}(\sigma)\\
&= V_{00\,;\mu}^{(2)} + V_{01\nu\,;\mu}^{(2)}\sigma^{;\nu} + V_{01\nu}^{(2)}\,g^{\nu\lambda}\left\{g_{\lambda\mu}\right\} + 2\,V_{02(\nu\rho)}^{(2)}\,g^{\nu\lambda}\left\{g_{\lambda\mu}\right\}\sigma^{;\rho} + \mathcal{O}(\sigma)\\
&= V_{00\,;\mu}^{(2)} + V_{01\mu}^{(2)} + V_{01\nu\,;\mu}^{(2)}\sigma^{;\nu} + 2\,V_{02(\mu\nu)}^{(2)}\sigma^{;\nu} + \mathcal{O}(\sigma)
\end{aligned} \tag{7.29}$$

where we have used the expansion for $\sigma_{;\mu\nu}$ (6.25) in going from the first equality in (7.29) to the second. Then, we have

$$D_\mu V_0^{(2)} = D_\mu V_{00}^{(2)} + V_{01\mu}^{(2)} + \left[D_\mu V_{01\nu}^{(2)} + 2\,V_{02(\mu\nu)}^{(2)}\right]\sigma^{;\nu} + \dots. \tag{7.30}$$

Then, the first term in the equation for $V_0^{(2)}(x,x')$ (7.27) is given by

$$\left[D_\mu V_0^{(2)}\right]\sigma^{;\mu} = \left[D_\mu V_{00}^{(2)} + V_{01\mu}^{(2)}\right]\sigma^{;\mu} + \left[D_\mu V_{01\nu}^{(2)} + 2\,V_{02(\mu\nu)}^{(2)}\right]\sigma^{;\mu}\sigma^{;\nu} + \dots. \tag{7.31}$$

The second term in the equation for $V_0^{(2)}(x,x')$ (7.27) is given by

$$-\,V_0^{(2)}\Delta^{-\frac{1}{2}}\Delta^{\frac{1}{2}}_{;\mu}\,\sigma^{;\mu} = -\frac{1}{6}\,V_{00}^{(2)}R_{\mu\nu}\sigma^{;\mu}\sigma^{;\nu} + \dots \tag{7.32}$$

We can find the explicit form of the expansion coefficients of the Hadamard coefficient $V_0^{(2)}(x,x')$ by equating (7.27) at each order of σ. We have, by definition, at $\mathcal{O}(1)$

$$V_{00}^{(2)} = -1. \tag{7.33}$$

The terms at $\mathcal{O}\left(\sigma^{1/2}\right)$ in the equation for $V_0^{(2)}(x,x')$ (7.27) give

$$0 = D_\mu V_{00}^{(2)} + V_{01\mu}^{(2)} = \mathrm{i}qA_\mu + V_{01\mu}^{(2)}, \tag{7.34}$$

where we have used (7.33). So we obtain for the $V_{01\mu}^{(2)}$ expansion coefficient

$$V_{01\mu}^{(2)} = -\mathrm{i}qA_\mu. \tag{7.35}$$

The terms at $\mathcal{O}(\sigma)$ in the equation for $V_0^{(2)}(x,x')$ (7.27) give

$$\begin{aligned}
0 &= D_{(\mu}V_{01\nu)}^{(2)} + 2\,V_{02(\mu\nu)}^{(2)} + \frac{1}{6}\,V_{00}^{(2)}R_{\mu\nu}\\
&= -\mathrm{i}q\,D_{(\mu}A_{\nu)} + 2\,V_{02(\mu\nu)}^{(2)} + \frac{1}{6}R_{\mu\nu},
\end{aligned} \tag{7.36}$$

where we have used (7.33) and (7.35), and we have symmetrised the $D_{(\mu}V^{(2)}_{01\nu)}$ term in the first line of (7.36) since it can be written in terms of the expansion coefficient $V^{(2)}_{02(\mu\nu)}$ and the Ricci tensor $R_{\mu\nu}$, both of which are symmetric tensors by definition. Then, we have

$$V^{(2)}_{02\mu\nu} = -\frac{1}{12}R_{\mu\nu} + \frac{1}{2}\,\mathrm{i}q\,D_{(\mu}A_{\nu)}, \tag{7.37}$$

In $d=2$, we can write $R_{\mu\nu}$ in terms of the Ricci scalar as $R_{\mu\nu} = \frac{1}{2}\,g_{\mu\nu}R$; then we have

$$V^{(2)}_{02\mu\nu} = -\frac{1}{24}\,g_{\mu\nu}R + \frac{1}{2}\,\mathrm{i}q\,D_{(\mu}A_{\nu)}. \tag{7.38}$$

As earlier stated, in $d=2$ we require terms up to $\mathcal{O}(\sigma)$ to evaluate the RSET. In the expansion of the $V^{(2)}(x,x')$ biscalar (6.4a), the $V^{(2)}_1(x,x')$ Hadamard coefficient is multiplied by σ; therefore, we are also required to evaluate the zeroth order term of the Taylor expansion of $V^{(2)}_1(x,x')$, i.e. $V^{(2)}_{10}$. We can obtain the equation satisfied by $V^{(2)}_1(x,x')$ by using (7.14) with $n=0$; doing so, we have

$$2\,V^{(2)}_1 + 2\,\sigma^{;\mu}D_\mu V^{(2)}_1 - 2\,V^{(2)}_1\Delta^{-\frac{1}{2}}\Delta^{\frac{1}{2}}_{;\mu}\,\sigma^{;\mu} + \left(D^\mu D_\mu - m^2 - \xi R\right)V^{(2)}_0 = 0. \tag{7.39}$$

We can expand $V^{(2)}_1(x,x')$ as a covariant Taylor expansion as follows

$$V^{(2)}_1 = V^{(2)}_{10} + \dots \tag{7.40}$$

Therefore, the first term in the equation for $V^{(2)}_1(x,x')$ (7.39) is given by

$$2\,V^{(2)}_1 = 2\,V^{(2)}_{10} + \dots \tag{7.41}$$

Neither the second term nor the third term in (7.39) contribute at $\mathcal{O}(1)$. We will evaluate the final term in (7.39) in steps, beginning with the gauge covariant derivative of (7.28). Since we only require the final expression up to $\mathcal{O}(\sigma)$, we require $D_\mu V^{(2)}_0$ to $\mathcal{O}(\sigma^{1/2})$. This is given by (7.30). We then need to act on (7.30) with another gauge covariant derivative:

$$\begin{aligned} D_\mu D^\mu V^{(2)}_0 &= g^{\mu\nu}D_\nu D_\mu V^{(2)}_{00} + g^{\mu\nu}D_\nu V^{(2)}_{01\mu} + g^{\mu\nu}\left[D_\mu V^{(2)}_{01\rho} + 2\,V^{(2)}_{02(\mu\rho)}\right]\sigma^{;\rho}{}_{\nu} + \dots \\ &= D_\mu D^\mu V^{(2)}_{00} + 2\,D^\mu V^{(2)}_{01\mu} + 2\,g^{\mu\nu}V^{(2)}_{02(\mu\nu)} + \dots \end{aligned} \tag{7.42}$$

Then, the fourth term in the equation for $V^{(2)}_1(x,x')$ (7.39) is given by

$$\left(D_\mu D^\mu - m^2 - \xi R\right)V^{(2)}_0 = \left(D_\mu D^\mu - m^2 - \xi R\right)V^{(2)}_{00} + 2\,D^\mu V^{(2)}_{01\mu} + 2\,g^{\mu\nu}V^{(2)}_{02(\mu\nu)} + \dots \tag{7.43}$$

Using (7.41) and (7.43), we can write the equation for $V^{(2)}_1(x,x')$ (7.39) as

$$0 = 2\,V^{(2)}_{10} - \left(D_\mu D^\mu - m^2 - \xi R\right)V^{(2)}_{00} - 2\,D^\mu V^{(2)}_{01\mu} + 2\,g^{\mu\nu}V^{(2)}_{02(\mu\nu)}. \tag{7.44}$$

We simplify (7.44) by using the relations between $V^{(2)}_{00}$, $V^{(2)}_{01\mu}$ and $V^{(2)}_{02(\mu\nu)}$ in (7.34, 7.36):

$$\begin{aligned}2\,V_{10}^{(2)} &= D_\mu D^\mu V_{00}^{(2)} - \left(m^2+\xi R\right)V_{00}^{(2)} - 2\,D_\mu D^\mu V_{00}^{(2)} - g^{\mu\nu}D_\mu V_{01\nu}^{(2)} + \frac{1}{12}\,g^{\mu\nu}g_{\mu\nu}R \\ &= D_\mu D^\mu V_{00}^{(2)} - \left(m^2+\xi R\right)V_{00}^{(2)} - 2\,D_\mu D^\mu V_{00}^{(2)} + g^{\mu\nu}D_\mu D_\nu V_{00}^{(2)} + \frac{1}{12}\,\delta^\mu_\mu R \\ &= -\left(m^2+\xi R\right)V_{00}^{(2)} + \frac{1}{6}R, \end{aligned} \tag{7.45}$$

where we have used the fact that $\delta^\mu_\mu = 2$ in $d=2$. Then, (7.45) simplifies to

$$V_{10}^{(2)} = -\frac{1}{2}\left(m^2+\xi R\right)V_{00}^{(2)} + \frac{1}{12}R. \tag{7.46}$$

Using the explicit expression for $V_{00}^{(2)}$ (7.33) , we obtain for the $V_{10}^{(2)}$ expansion coefficient

$$V_{10}^{(2)} = \frac{1}{2}\left[m^2 + \left(\xi - \frac{1}{6}\right)R\right]. \tag{7.47}$$

This is the same as in the uncharged case and so there are no gauge corrections to the zeroth order coefficient of the covariant Taylor expansion of $V_1^{(2)}$. This expression will be important when considering the trace anomaly of the RSET in $d=2$ in §8.131.

7.2 Hadamard form for even dimensions

In this section, we develop the general formalism for the Hadamard renormalisation procedure of charged scalar fields in a general background spacetime with an even number of dimensions (apart from the special case of $d=2$, which is treated in §7.1). We will then focus on the specific case of $d=4$ in order to derive explicit expressions for the renormalisation counterterms up to the order required to evaluate the RSET.

7.2.1 Hadamard renormalisation procedure in even dimensions

We would like to evaluate the inhomogeneous Klein-Gordon equation (6.2) for the even-dimensional Hadamard parametrix (6.5). We remind the reader that we write the number of spacetime dimensions as $d=2p$ with $p\neq 1$. Then (6.2) becomes

$$\begin{aligned}&\left(D_\mu D^\mu - m^2 - \xi R\right)G_{\rm F}^{(2p)} \\ &\quad= \frac{\mathrm{i}\,\alpha_{2p}}{2}\left(D_\mu D^\mu - m^2 - \xi R\right)\left\{\frac{U^{(2p)}(x,x')}{\left[\sigma(x,x')+\mathrm{i}\epsilon\right]^{p-1}} + V^{(2p)}(x,x')\ln\left[\sigma(x,x')+\mathrm{i}\epsilon\right] \right.\\ &\qquad\qquad \left. + W^{(2p)}(x,x')\right\} \\ &\qquad\qquad\qquad = -\frac{1}{\sqrt{-g\,(x)}}\,\delta^{(2p)}(x-x')\,. \end{aligned} \tag{7.48}$$

It will be convenient instead to evaluate the equation

$$
\begin{aligned}
&-\frac{2\,\mathrm{i}}{\alpha_{2p}}\left(D_\mu D^\mu - m^2 - \xi R\right) G_{\mathrm{F}}^{(2p)} \\
&\quad = \left(D_\mu D^\mu - m^2 - \xi R\right)\left\{\frac{U^{(2p)}(x,x')}{\left[\sigma(x,x') + \mathrm{i}\epsilon\right]^{p-1}} + V^{(2p)}\left(x,x'\right)\ln\left[\sigma\left(x,x'\right) + \mathrm{i}\epsilon\right] \right. \\
&\qquad\qquad \left. + W^{(2p)}\left(x,x'\right)\right\} \\
&\qquad\qquad = -\frac{2\,\mathrm{i}}{\alpha_{2p}}\frac{1}{\sqrt{-g\left(x\right)}}\,\delta^{(2p)}\left(x - x'\right). \qquad (7.49)
\end{aligned}
$$

Then, suppressing arguments of the biscalar functions, we begin by calculating the quantity

$$
\begin{aligned}
-\frac{2\,\mathrm{i}}{\alpha_{2p}}\,D_\mu G_{\mathrm{F}}^{(2p)} &= \left(\nabla_\mu - \mathrm{i}qA_\mu\right)\left[\sigma^{-p+1}\,U^{(2p)} + V^{(2p)}\ln\sigma + W^{(2p)}\right] \\
&= \sigma^{-p+1}\,\nabla_\mu U^{(2p)} - \left(p-1\right)\sigma^{-p}\,U^{(2p)}(\nabla_\mu\sigma) + \left[\nabla_\mu V^{(2p)}\right]\ln\sigma \\
&+ \sigma^{-1}\,V^{(2p)}(\nabla_\mu\sigma) + \nabla_\mu W^{(2p)} - \sigma^{-p+1}\,\mathrm{i}qA_\mu U^{(2p)} - \mathrm{i}qA_\mu V^{(2p)}\ln\sigma \\
&- \mathrm{i}qA_\mu W^{(2p)}. \qquad (7.50)
\end{aligned}
$$

Acting on (7.50) with another gauge derivative, we obtain

$$
\begin{aligned}
-\frac{2\,\mathrm{i}}{\alpha_{2p}}\,D_\mu D^\mu G_{\mathrm{F}}^{(2p)} &= \sigma^{-p+1}\,\nabla^\mu\nabla_\mu U^{(2p)} - 2\left(p-1\right)\sigma^{-p}\left[\nabla_\mu U^{(2p)}\right](\nabla^\mu\sigma) \\
&+ p\left(p-1\right)\sigma^{-p-1}\,U^{(2p)}(\nabla^\mu\sigma)(\nabla_\mu\sigma) - \left(p-1\right)\sigma^{-p}\,U^{(2p)}(\nabla^\mu\nabla_\mu\sigma) \\
&+ \left[\nabla_\mu\nabla^\mu V^{(2p)}\right]\ln\sigma + 2\,\sigma^{-1}\left[\nabla_\mu V^{(2p)}\right](\nabla^\mu\sigma) - \sigma^{-2}\,V^{(2p)}(\nabla^\mu\sigma)(\nabla_\mu\sigma) \\
&+ \sigma^{-1}\,V^{(2p)}(\nabla^\mu\nabla_\mu\sigma) + \nabla^\mu\nabla_\mu W^{(2p)} - \sigma^{-p+1}\,\mathrm{i}q\left(\nabla^\mu A_\mu\right)U^{(2p)} \\
&- 2\,\sigma^{-p+1}\,\mathrm{i}qA_\mu\left[\nabla^\mu U^{(2p)}\right] + 2\left(p-1\right)\sigma^{-p}\,\mathrm{i}qA_\mu U^{(2p)}(\nabla^\mu\sigma) \\
&- \mathrm{i}q\left(\nabla_\mu A^\mu\right)V^{(2p)}\ln\sigma - 2\,\mathrm{i}qA_\mu\left[\nabla^\mu V^{(2p)}\right]\ln\sigma - 2\,\mathrm{i}q\,\sigma^{-1}A_\mu V^{(2p)}(\nabla^\mu\sigma) \\
&- \mathrm{i}q\left(\nabla^\mu A_\mu\right)W^{(2p)} - 2\,\mathrm{i}qA_\mu\left[\nabla^\mu W^{(2p)}\right] - \sigma^{-p+1}\,q^2A_\mu A^\mu U^{(2p)} \\
&- q^2A_\mu A^\mu V^{(2p)}\ln\sigma - q^2A_\mu A^\mu W^{(2p)}. \qquad (7.51)
\end{aligned}
$$

We can simplify (7.51) using the identities involving Synge's world function (1.20) and the van Vleck-Morette determinant (1.28); we obtain

$$
\begin{aligned}
-\frac{2\,\mathrm{i}}{\alpha_{2p}}\,D_\mu D^\mu G_{\mathrm{F}}^{(2p)} &= \sigma^{-p+1}\,\nabla^\mu\nabla_\mu U^{(2p)} - 2\,(p-1)\,\sigma^{-p}\left[\nabla_\mu U^{(2p)}\right](\nabla^\mu\sigma) \\
&+ 2\,(p-1)\,\sigma^{-p}\,U^{(2p)}\Delta^{-\frac{1}{2}}\left(\nabla_\mu\Delta^{\frac{1}{2}}\right)(\nabla^\mu\sigma) + \left[\nabla_\mu\nabla^\mu V^{(2p)}\right]\ln\sigma \\
&+ 2\,\sigma^{-1}\left[\nabla_\mu V^{(2p)}\right](\nabla^\mu\sigma) + 2\,(p-1)\,\sigma^{-1}\,V^{(2p)} \\
&- 2\,\sigma^{-1}\,V^{(2p)}\Delta^{-\frac{1}{2}}\left(\nabla_\mu\Delta^{\frac{1}{2}}\right)(\nabla^\mu\sigma) + \nabla^\mu\nabla_\mu W^{(2p)} \\
&- \sigma^{-p+1}\,\mathrm{i}q\,(\nabla^\mu A_\mu)\,U^{(2p)} - 2\,\sigma^{-p+1}\,\mathrm{i}qA_\mu\left[\nabla^\mu U^{(2p)}\right] \\
&+ 2\,(p-1)\,\sigma^{-p}\,\mathrm{i}qA_\mu U^{(2p)}(\nabla^\mu\sigma) - \mathrm{i}q\,(\nabla_\mu A^\mu)V^{(2p)}\ln\sigma \\
&- 2\,\mathrm{i}qA_\mu\left[\nabla^\mu V^{(2p)}\right]\ln\sigma - 2\,\mathrm{i}q\,\sigma^{-1}A_\mu V^{(2p)}(\nabla^\mu\sigma) - \mathrm{i}q\,(\nabla^\mu A_\mu)W^{(2p)} \\
&- 2\,\mathrm{i}qA_\mu\left[\nabla^\mu W^{(2p)}\right] - \sigma^{-p+1}\,q^2A^\mu A_\mu U^{(2p)} - q^2A^\mu A_\mu V^{(2p)}\ln\sigma \\
&- q^2A^\mu A_\mu W^{(2p)}. \qquad (7.52)
\end{aligned}
$$

This allows us to evaluate the l.h.s of the inhomogeneous Klein-Gordon equation (6.2) for $d = 2p$ as follows

$$
\begin{aligned}
-\frac{2\,\mathrm{i}}{\alpha_{2p}}\,&(D^\mu D_\mu - m^2 - \xi R)\,G_{\mathrm{F}}^{(2p)} \\
&= \sigma^{-p+1}\,\nabla^\mu\nabla_\mu U^{(2p)} - 2\,(p-1)\,\sigma^{-p}\left[\nabla_\mu U^{(2p)}\right](\nabla^\mu\sigma) \\
&+ 2\,(p-1)\,\sigma^{-p}\,U^{(2p)}\Delta^{-\frac{1}{2}}\left(\nabla_\mu\Delta^{\frac{1}{2}}\right)(\nabla^\mu\sigma) + \left[\nabla_\mu\nabla^\mu V^{(2p)}\right]\ln\sigma \\
&+ 2\,\sigma^{-1}\left[\nabla_\mu V^{(2p)}\right](\nabla^\mu\sigma) + 2\,(p-1)\,\sigma^{-1}\,V^{(2p)} \\
&- 2\,\sigma^{-1}\,V^{(2p)}\Delta^{-\frac{1}{2}}\left(\nabla_\mu\Delta^{\frac{1}{2}}\right)(\nabla^\mu\sigma) + \nabla^\mu\nabla_\mu W^{(2p)} \\
&- \sigma^{-p+1}\,\mathrm{i}q\,(\nabla^\mu A_\mu)\,U^{(2p)} - 2\,\sigma^{-p+1}\,\mathrm{i}qA_\mu\left[\nabla^\mu U^{(2p)}\right] \\
&+ 2\,(p-1)\,\sigma^{-p}\,\mathrm{i}qA_\mu U^{(2p)}(\nabla^\mu\sigma) - \mathrm{i}q\,(\nabla_\mu A^\mu)V^{(2p)}\ln\sigma \\
&- 2\,\mathrm{i}qA_\mu\left[\nabla^\mu V^{(2p)}\right]\ln\sigma - 2\,\mathrm{i}q\,\sigma^{-1}A_\mu V^{(2p)}(\nabla^\mu\sigma) - \mathrm{i}q\,(\nabla^\mu A_\mu)W^{(2p)} \\
&- 2\,\mathrm{i}qA_\mu\left[\nabla^\mu W^{(2p)}\right] - \sigma^{-p+1}\,q^2A^\mu A_\mu U^{(2p)} - q^2A^\mu A_\mu V^{(2p)}\ln\sigma \\
&- q^2A^\mu A_\mu W^{(2p)} - \sigma^{-p+1}\big(m^2+\xi R\big)U^{(2p)} - \big(m^2+\xi R\big)V^{(2p)}\ln\sigma \\
&- \big(m^2+\xi R\big)W^{(2p)}. \qquad (7.53)
\end{aligned}
$$

We can simplify (7.53) by use of the gauge covariant derivative, which then reduces to

$$
\begin{aligned}
-\,&\frac{2\,\mathrm{i}}{\alpha_{2p}}\,\big(D^\mu D_\mu - m^2 - \xi R\big)\,G_{\mathrm{F}}^{(2p)} \\
&= \sigma^{-p+1}\big(D_\mu D^\mu - m^2 - \xi R\big)\,U^{(2p)} + \big(D_\mu D^\mu - m^2 - \xi R\big)\,V^{(2p)}\ln\sigma \\
&\qquad + \big(D_\mu D^\mu - m^2 - \xi R\big)\,W^{(2p)} \\
&\quad - 2\,(p-1)\,\sigma^{-p}\left[\left(D_\mu - \Delta^{-\frac{1}{2}}\Delta^{\frac{1}{2}}_{\ ;\mu}\right)U^{(2p)}\right]\sigma^{;\mu} \\
&\qquad + 2\,\sigma^{-1}\left[\left(\sigma^{;\mu}D_\mu + (p-1) - \Delta^{-\frac{1}{2}}\Delta^{\frac{1}{2}}_{\ ;\mu}\,\sigma^{;\mu}\right)V^{(2p)}\right]. \qquad (7.54)
\end{aligned}
$$

From the r.h.s of the inhomogeneous scalar field equation (6.2), we know that (7.54) must be equal to zero when the two points x and x' are separated. Since Synge's world function is nonzero when $x' \neq x$, we deduce that (7.54) must vanish identically at each power of $\sigma(x,x')$. This allows us to generate two equations by considering terms proportional to $\ln\sigma$ and. separately, the remaining terms, i.e. those not proportional to $\ln\sigma$. In particular, the terms that are proportional to $\ln\sigma$ must vanish independently since no other terms can contain this factor. This allows us to write

$$\left(D_\mu D^\mu - m^2 - \xi R\right) V^{(2p)}\left(x,x'\right) = 0, \tag{7.55}$$

which means that the biscalar $V^{(2p)}(x,x')$ solves the homogeneous scalar field equation. Equation (7.55) generalises (40) in [68] and it enables us to derive the recurrence relations for the Hadamard coefficients $V_n^{(2p)}(x,x')$. Since the biscalar $V^{(2p)}(x,x')$ admits a power series expansion in $\sigma(x,x')$ (6.5), we can derive the recurrence relation for the $V_n^{(2p)}(x,x')$ by expanding the terms in (7.55). Suppressing arguments, we first need to evaluate

$$D_\mu V^{(2p)} = \sum_{n=0}^{\infty}\left\{\left[\nabla_\mu V_n^{(2p)}\right]\sigma^n + nV_n^{(2p)}\sigma^{n-1}(\nabla_\mu\sigma) - \mathrm{i}qA_\mu V_n^{(2p)}\sigma^n\right\}, \tag{7.56}$$

and then

$$\begin{aligned} D_\mu D^\mu V^{(2p)} = \sum_{n=0}^{\infty}\Big\{&\left[\nabla^\mu\nabla_\mu V_n^{(2p)}\right]\sigma^n + \left[\nabla_\mu V_n^{(2p)}\right]n\,\sigma^{n-1}(\nabla^\mu\sigma) + n\left[\nabla^\mu V_n^{(2p)}\right]\sigma^{n-1}(\nabla_\mu\sigma) \\ &+ n\,(n-1)\,V_n^{(2p)}\sigma^{n-2}(\nabla^\mu\sigma)(\nabla_\mu\sigma) + nV_n^{(2p)}\sigma^{n-1}(\nabla^\mu\nabla_\mu\sigma) \\ &- \mathrm{i}q\,(\nabla^\mu A_\mu)\,V_n^{(2p)}\sigma^n - \mathrm{i}qA_\mu\left[\nabla^\mu V_n^{(2p)}\right]\sigma^n - \mathrm{i}qA_\mu\, n\,V_n^{(2p)}\sigma^{n-1}\,(\nabla^\mu\sigma) \\ &- \mathrm{i}qA^\mu\left[\nabla_\mu V_n^{(2p)}\right]\sigma^n - \mathrm{i}qA_\mu\, n\,V_n^{(2p)}\sigma^{n-1}(\nabla^\mu\sigma) - q^2A^\mu A_\mu V_n^{(2p)}\sigma^n\Big\}. \end{aligned} \tag{7.57}$$

Then using (1.20), (1.28), (6.24) and the gauge covariant derivative, (7.57) becomes

$$\begin{aligned} D_\mu D^\mu V^{(2p)} = \sum_{n=0}^{\infty}\Big\{&\left[D^\mu D_\mu V_n^{(2p)}\right]\sigma^n + 2n\left[D_\mu V_n^{(2p)}\right]\sigma^{n-1}(\nabla^\mu\sigma) \\ &+ 2n\,(n+p-1)\,V_n^{(2p)}\sigma^{n-1} - 2n\,V_n^{(2p)}\Delta^{-\frac{1}{2}}\left(\nabla_\mu\Delta^{\frac{1}{2}}\right)\sigma^{n-1}(\nabla^\mu\sigma)\Big\}. \end{aligned} \tag{7.58}$$

Substituting (7.58) into (7.55), we obtain

$$\begin{aligned} 0 = \sum_{n=0}^{\infty}\Big\{&\left[\left(D_\mu D^\mu - m^2 - \xi R\right)V_n^{(2p)}\right]\sigma^n + 2n\left[D_\mu V_n^{(2p)}\right]\sigma^{n-1}(\nabla^\mu\sigma) \\ &+ 2n\,(n+p-1)\,V_n^{(2p)}\sigma^{n-1} - 2n\,V_n^{(2p)}\Delta^{-\frac{1}{2}}\left(\nabla_\mu\Delta^{\frac{1}{2}}\right)\sigma^{n-1}(\nabla^\mu\sigma)\Big\}. \end{aligned} \tag{7.59}$$

Performing the relabelling $n \to n+1$ in term proportional to σ^{n-1} in (7.59), we obtain

$$0 = \sum_{n=0}^{\infty} \Big\{ \Big[\big(D_\mu D^\mu - m^2 - \xi R \big) V_n^{(2p)} \Big] \sigma^n + 2(n+1) \Big[D_\mu V_{n+1}^{(2p)} \Big] \sigma^n (\nabla^\mu \sigma) + 2\,(n+1)(n+p)\, V_{n+1}^{(2p)}\, \sigma^n - 2(n+1)\, V_{n+1}^{(2p)} \Delta^{-\frac{1}{2}} \Big(\nabla_\mu \Delta^{\frac{1}{2}} \Big) \sigma^n\, (\nabla^\mu \sigma) \Big\} . \qquad (7.60)$$

Since (7.60) must hold for each power of σ, this enables us to obtain the recurrence relation for the Hadamard coefficients $V_n^{(2p)}(x,x')$ of the biscalar function $V^{(2p)}(x,x')$. We have

$$2\,(n+1)(n+p)\, V_{n+1}^{(2p)} + 2(n+1)\, \sigma^{\,;\mu} D_\mu V_{n+1}^{(2p)} - 2(n+1)\, V_{n+1}^{(2p)}\, \Delta^{-\frac{1}{2}} \Delta^{\frac{1}{2}}_{\;;\mu}\, \sigma^{\,;\mu} + \big(D_\mu D^\mu - m^2 - \xi R \big)\, V_n^{(2p)} = 0 \quad \text{for } n \text{ in } \mathbb{N}. \qquad (7.61)$$

This generalises equation (38a) in [68]. Returning once again to (7.54), we can consider terms proportional to σ^{-1} to obtain the boundary condition for the $V_0^{(2p)}(x,x')$ Hadamard coefficient. Inserting the power series expansion for $U^{(2p)}(x,x')$ (6.6a), $V^{(2p)}(x,x')$ (6.6b) and $W^{(2p)}(x,x')$ (6.6c), we obtain

$$0 = \sigma^{-1} \Big[\big(D_\mu D^\mu - m^2 - \xi R \big)\, U_{p-2}^{(2p)} + 2 \Big(\sigma^{\,;\mu} D_\mu + (p-1) - \Delta^{-\frac{1}{2}} \Delta^{\frac{1}{2}}_{\;;\mu}\, \sigma^{\,;\mu} \Big)\, V_0^{(2p)} \Big] , \qquad (7.62)$$

where we have used that

$$\sigma^{-p+1} \sigma^{\,p-2} = \sigma^{-1}. \qquad (7.63)$$

Since (7.62) should hold for arbitrary $\sigma(x,x')$, we may write

$$(2p-2)\, V_0^{(2p)} + 2\, \sigma^{\,;\mu} D_\mu V_0^{(2p)} - 2\, V_0^{(2p)} \Delta^{-\frac{1}{2}} \Delta^{\frac{1}{2}}_{\;;\mu}\, \sigma^{\,;\mu} + \big(D_\mu D^\mu - m^2 - \xi R \big)\, U_{p-2}^{(2p)} = 0. \qquad (7.64)$$

This generalises equation (38b) in D+F. We can consider the terms in (7.54) that are proportional to σ^{-p+1} in order to derive the recurrence relations for the $U_n^{(2p)}(x,x')$. Remembering that $p > 1$, we see that the terms in (7.54) involving the biscalars $V^{(2p)}(x,x')$ and $W^{(2p)}(x,x')$ will not contribute at order $\mathcal{O}\big(\sigma^{-p+1}\big)$ as long as $p > 2$. This would be unsatisfactory as it means that the recurrence relations we will derive for the Hadamard coefficients $U_n^{(2p)}(x,x')$ below would not be valid when $p = 2$. However, looking at the form of the $U^{(2p)}(x,x')$ biscalar (6.6a), we can see that the power series expansion for $U^{(2p)}(x,x')$ truncates at zeroth order, i.e. $U^{(4)}(x,x') = U_0^{(4)}$, rendering the notion of recurrence relations meaningless in the case when $p = 2$. Thus, the recurrence relations we will derive below for the $U_n^{(2p)}(x,x')$ are valid for general p and we need only consider the terms in (7.54) involving the $U^{(2p)}(x,x')$ biscalar in order to derive them. Then, at $\mathcal{O}\big(\sigma^{-p+n+1}\big)$, we have

$$\sigma^{-p+1} \big(D_\mu D^\mu - m^2 - \xi R \big)\, U^{(2p)} - 2\,(p-1)\, \sigma^{-p} \Big[\Big(D_\mu - \Delta^{-\frac{1}{2}} \Delta^{\frac{1}{2}}_{\;;\mu} \Big)\, U^{(2p)} \Big] \sigma^{\,;\mu} = 0. \qquad (7.65)$$

Since the biscalar $U^{(2p)}(x,x')$ in (6.5) admits a power series expansion in $\sigma(x,x')$, we can derive the recursion relation for the Hadamard coefficients $U_n^{(2p)}(x,x')$ by expanding the terms in (7.65). We first need to evaluate $D_\mu D^\mu U^{(2p)}$; since the power series expansions for the biscalars $U^{(2p)}(x,x')$ and $V^{(2p)}(x,x')$ are similar, we may simply write down the quantity as

$$\begin{aligned} D_\mu D^\mu U^{(2p)} = \sum_{n=0}^{p-2} \Big\{ & \Big[D^\mu D_\mu U_n^{(2p)} \Big] \sigma^n + 2n \Big[D_\mu U_n^{(2p)} \Big] \sigma^{n-1} (\nabla^\mu \sigma) \\ & + 2n\,(n+p-1)\, U_n^{(2p)} \sigma^{n-1} - 2n\, U_n^{(2p)} \Delta^{-\frac{1}{2}} \Big(\nabla_\mu \Delta^{\frac{1}{2}} \Big) \sigma^{n-1} (\nabla^\mu \sigma) \Big\}. \end{aligned} \tag{7.66}$$

where we note that the upper limit of the summation in (7.65) is derived from the form of the power series expansion (6.6a) of the biscalar $U^{(2p)}(x,x')$. Thus the first term in (7.65) is given by

$$\begin{aligned} & \sigma^{-p+1} (D_\mu D^\mu - m^2 - \xi R)\, U^{(2p)} \\ & \quad = \sum_{n=0}^{p-2} \Big\{ \Big[(D_\mu D^\mu - m^2 - \xi R)\, U_n^{(2p)} \Big] \sigma^{-p+n+1} + 2n \Big[D_\mu U_n^{(2p)} \Big] \sigma^{-p+n} (\nabla^\mu \sigma) \\ & \quad + 2n\,(n+p-1)\, U_n^{(2p)} \sigma^{-p+n} - 2n\, U_n^{(2p)} \Delta^{-\frac{1}{2}} \Big(\nabla_\mu \Delta^{\frac{1}{2}} \Big) \sigma^{-p+n} (\nabla^\mu \sigma) \Big\} \end{aligned} \tag{7.67}$$

In order to evaluate the second term in (7.65), we will need to evaluate $D_\mu U^{(2p)}$; we can write this down by comparison with (7.56), giving us

$$\begin{aligned} D_\mu\, U^{(2p)} &= \sum_{n=0}^{p-2} \Big\{ \Big[\nabla_\mu U_n^{(2p)} \Big] \sigma^n + n U_n^{(2p)} \sigma^{n-1} (\nabla_\mu \sigma) - \mathrm{i} q A_\mu U_n^{(2p)} \sigma^n \Big\} \\ &= \sum_{n=0}^{p-2} \Big\{ \Big[D_\mu\, U_n^{(2p)} \Big] \sigma^n + n U_n^{(2p)} \sigma^{n-1} (\nabla_\mu \sigma) \Big\}. \end{aligned} \tag{7.68}$$

Then, the second term in (7.65) is given by

$$\begin{aligned} & -2\,(p-1)\,\sigma^{-p} \left[\left(D_\mu - \Delta^{-\frac{1}{2}} \Delta^{\frac{1}{2}}{}_{;\mu} \right) U^{(2p)} \right] \sigma^{\,;\mu} \\ & = -2\,(p-1)\,\sigma^{-p}\,\sigma^{\,;\mu} \left(\sum_{n=0}^{p-2} \Big\{ \Big[D_\mu\, U_n^{(2p)} \Big] \sigma^n + n U^{(2p)} \sigma^{n-1} \sigma_{\,;\mu} - U^{(2p)} \sigma^n \Delta^{-\frac{1}{2}} \Delta^{\frac{1}{2}}{}_{;\mu} \Big\} \right) \\ & = -2\,(p-1) \left(\sum_{n=0}^{p-2} \Big\{ \Big[\sigma^{\,;\mu} D_\mu\, U_n^{(2p)} \Big] \sigma^{-p+n} + 2n U^{(2p)} \sigma^{-p+n} - U^{(2p)} \sigma^{-p+n} \Delta^{-\frac{1}{2}} \Delta^{\frac{1}{2}}{}_{;\mu}\, \sigma^{\,;\mu} \Big\} \right). \end{aligned} \tag{7.69}$$

Substituting (7.67) and (7.69) into (7.65), we obtain

$$\begin{aligned} 0 = \sum_{n=0}^{p-2} \Big\{ & \Big[(D_\mu D^\mu - m^2 - \xi R)\, U_n^{(2p)} \Big] \sigma^{-p+n+1} + 2\,(n-p+1) \Big[\sigma^{\,;\mu} D_\mu\, U_n^{(2p)} \Big] \sigma^{-p+n} \\ & + 2n\,(n-p+1)\, U_n^{(2p)} \sigma^{-p+n} - 2\,(n-p+1)\, U_n^{(2p)} \sigma^{-p+n} \Delta^{-\frac{1}{2}} \Delta^{\frac{1}{2}}{}_{;\mu}\, \sigma^{\,;\mu} \Big\}. \end{aligned} \tag{7.70}$$

We may perform the relabelling $n \to n+1$ in the terms that are proportional to σ^{-p+n} in (7.70); then we obtain

$$0 = \sum_{n=0}^{p-2} \Big\{ \Big[\big(D_\mu D^\mu - m^2 - \xi R\big)\, U_n^{(2p)} \Big] \sigma^{-p+n+1} + 2\,(n-p+2) \Big[\sigma^{;\mu} D_\mu\, U_{n+1}^{(2p)} \Big] \sigma^{-p+n+1}$$
$$+ \, 2\,(n+1)(n-p+2)\, U_{n+1}^{(2p)} \sigma^{-p+n+1} - 2\,(n-p+2)\, U_{n+1}^{(2p)} \sigma^{-p+n+1} \Delta^{-\frac{1}{2}} \Delta^{\frac{1}{2}}_{;\mu}\, \sigma^{;\mu} \Big\} . \tag{7.71}$$

Since (7.71) must hold for each power of σ, this enables us to obtain the recurrence relation for the Hadamard coefficients $U_n^{(2p)}(x,x')$ of the biscalar function $U^{(2p)}(x,x')$. We have

$$0 = (n+1)(2n+4-2p)\, U_{n+1}^{(2p)} + (2n+4-2p)\, \sigma^{;\mu} D_\mu\, U_{n+1}^{(2p)}$$
$$- \, (2n+4-2p)\, U_{n+1}^{(2p)}\, \Delta^{-\frac{1}{2}} \Delta^{\frac{1}{2}}_{;\mu}\, \sigma^{;\mu}$$
$$+ \, \big(D_\mu D^\mu - m^2 - \xi R\big)\, U_n^{(2p)} \quad \text{for } n = 0, 1, \ldots, p-3. \tag{7.72}$$

This generalises equation (37a) in D+F. We can consider the terms in (7.54) that are proportional to positive powers of $\sigma(x,x')$ in order to derive the recurrence relations for the $W_n^{(2p)}(x,x')$. Since the power series expansion of the biscalar $U^{(2p)}(x,x')$ truncates at $\mathcal{O}\big(\sigma^{p-2}\big)$, we see that the terms in (7.54) involving the biscalar $U^{(2p)}(x,x')$ will not contribute at positive powers of $\sigma(x,x')$. Furthermore, the term containing a factor of $\ln \sigma$ will also not contribute. This allows us to write

$$\big(D_\mu D^\mu - m^2 - \xi R\big)\, W^{(2p)} + 2\,\sigma^{-1} \Big[\Big(\sigma^{;\mu} D_\mu + (p-1) - \Delta^{-\frac{1}{2}} \Delta^{\frac{1}{2}}_{;\mu}\, \sigma^{;\mu} \Big)\, V^{(2p)} \Big] = 0. \tag{7.73}$$

Since the biscalar $W^{(2p)}(x,x')$ in (6.6c) admits a power series expansion in $\sigma(x,x')$, we can derive the recursion relation for the Hadamard coefficients $W^{(2p)}(x,x')$ by expanding the terms in (7.73). We first need to evaluate $D_\mu D^\mu W^{(2p)}$; since the power series expansions for the biscalars $W^{(2p)}(x,x')$ and $W^{(2p)}(x,x')$ are similar, we may simply write down the quantity as

$$D_\mu D^\mu W^{(2p)} = \sum_{n=0}^{\infty} \Big\{ \Big[D^\mu D_\mu W_n^{(2p)} \Big] \sigma^n + 2n \Big[D_\mu W_n^{(2p)} \Big] \sigma^{n-1} (\nabla^\mu \sigma)$$
$$+ \, 2n\,(n+p-1)\, W_n^{(2p)} \sigma^{n-1} - 2n\, W_n^{(2p)} \Delta^{-\frac{1}{2}} \Big(\nabla_\mu \Delta^{\frac{1}{2}} \Big) \sigma^{n-1} (\nabla^\mu \sigma) \Big\} . \tag{7.74}$$

Thus the first term in (7.65) is given by

$$\big(D_\mu D^\mu - m^2 - \xi R\,\big)\, W^{(2p)}$$
$$= \sum_{n=0}^{\infty} \Big\{ \Big[\big(D_\mu D^\mu - m^2 - \xi R\big)\, W_n^{(2p)} \Big] \sigma^n + 2n \Big[D_\mu W_n^{(2p)} \Big] \sigma^{n-1} (\nabla^\mu \sigma)$$
$$+ \, 2n\,(n+p-1)\, W_n^{(2p)} \sigma^{n-1} - 2n\, W_n^{(2p)} \Delta^{-\frac{1}{2}} \Big(\nabla_\mu \Delta^{\frac{1}{2}} \Big) \sigma^{n-1} (\nabla^\mu \sigma) \Big\} \tag{7.75}$$

In order to evaluate the second term in (7.73), we will need to evaluate $D_\mu V^{(2p)}$; this is given by (7.56). Then, the second term in (7.73) is given by

$$2\,\sigma^{-1}\left[\left(\sigma^{\,;\mu}D_\mu + (p-1) - \Delta^{-\frac{1}{2}}\Delta^{\frac{1}{2}}_{\;;\mu}\,\sigma^{\,;\mu}\right)V^{(2p)}\right]$$
$$= 2\sum_{n=0}^{\infty}\left\{\left[\sigma^{\,;\mu}D_\mu V_n^{(2p)}\right]\sigma^{n-1} + (2n+p-1)\,V_n^{(2p)}\sigma^{n-1} - V_n^{(2p)}\sigma^{n-1}\Delta^{-\frac{1}{2}}\Delta^{\frac{1}{2}}_{\;;\mu}\,\sigma^{\,;\mu}\right\} \tag{7.76}$$

Substituting (7.75) and (7.76) into (7.73), we obtain

$$0 = \sum_{n=0}^{\infty}\left\{\left[\left(D_\mu D^\mu - m^2 - \xi R\right)W_n^{(2p)}\right]\sigma^n + 2n\left[\sigma^{\,;\mu}D_\mu W_n^{(2p)}\right]\sigma^{n-1}\right.$$
$$+\,2n\,(n+p-1)\,W_n^{(2p)}\sigma^{n-1} - 2n\,W_n^{(2p)}\sigma^{n-1}\Delta^{-\frac{1}{2}}\Delta^{\frac{1}{2}}_{\;;\mu}\,\sigma^{\,;\mu} + 2\left[\sigma^{\,;\mu}D_\mu V_n^{(2p)}\right]\sigma^{n-1}$$
$$\left.+\,2\,(2n+p-1)\,V_n^{(2p)}\sigma^{n-1} - 2\,V_n^{(2p)}\sigma^{n-1}\Delta^{-\frac{1}{2}}\Delta^{\frac{1}{2}}_{\;;\mu}\,\sigma^{\,;\mu}\right\}. \tag{7.77}$$

We may perform the relabelling $n \to n+1$ in the terms that are proportional to σ^{n-1} in (7.77); then we obtain

$$0 = \sum_{n=0}^{\infty}\left\{\left[\left(D_\mu D^\mu - m^2 - \xi R\right)W_n^{(2p)}\right]\sigma^n + 2\,(n+1)\left[\sigma^{\,;\mu}D_\mu W_{n+1}^{(2p)}\right]\sigma^{n}\right.$$
$$+\,2\,(n+1)(n+p)\,W_{n+1}^{(2p)}\sigma^{n} - 2\,(n+1)\,W_{n+1}^{(2p)}\sigma^{n}\Delta^{-\frac{1}{2}}\Delta^{\frac{1}{2}}_{\;;\mu}\,\sigma^{\,;\mu} + 2\left[\sigma^{\,;\mu}D_\mu V_{n+1}^{(2p)}\right]\sigma^{n}$$
$$\left.+\,2\,(2n+p+1)\,V_{n+1}^{(2p)}\sigma^{n} - 2\,V_{n+1}^{(2p)}\sigma^{n}\Delta^{-\frac{1}{2}}\Delta^{\frac{1}{2}}_{\;;\mu}\,\sigma^{\,;\mu}\right\}. \tag{7.78}$$

Since (7.73) must hold for each power of σ, this enables us to obtain the recurrence relation for the Hadamard coefficients $W_n^{(2p)}(x,x')$ of the biscalar function $W^{(2p)}(x,x')$. We have

$$\begin{aligned}0 = (n+1)(2n+2p)\,W_{n+1}^{(2p)} &+ 2\,(n+1)\,\sigma^{\,;\mu}D_\mu W_{n+1}^{(2p)}\\ &- 2\,(n+1)\,W_{n+1}^{(2p)}\Delta^{-\frac{1}{2}}\Delta^{\frac{1}{2}}_{\;;\mu}\,\sigma^{\,;\mu} + (4n+2+2p)\,V_{n+1}^{(2p)}\\ &+ 2\,\sigma^{\,;\mu}D_\mu V_{n+1}^{(2p)} - 2\,V_{n+1}^{(2p)}\Delta^{-\frac{1}{2}}\Delta^{\frac{1}{2}}_{\;;\mu}\,\sigma^{\,;\mu}\\ &+ \left(D_\mu D^\mu - m^2 - \xi R\right)W_n^{(2p)} \quad \text{for } n \text{in} \mathbb{N}.\end{aligned} \tag{7.79}$$

This generalises equation (39) in D+F. Returning to (7.54), we can consider terms proportional to σ^k where $k = -1, 0, 1, 2, \ldots$, in order to derive a relationship between the $V^{(2p)}(x,x')$ and $W^{(2p)}(x,x')$ biscalars (**check**). We obtain

$$\begin{aligned}0 = \left(D_\mu D^\mu - m^2 - \xi R\right)U_{p-2}^{(2p)} &+ \sigma\left(D_\mu D^\mu - m^2 - \xi R\right)W^{(2p)}\\ &+ 2\left[\left(\sigma^{\,;\mu}D_\mu + (p-1) - \Delta^{-\frac{1}{2}}\Delta^{\frac{1}{2}}_{\;;\mu}\,\sigma^{\,;\mu}\right)V^{(2p)}\right].\end{aligned} \tag{7.80}$$

It will be convenient to rewrite (7.80) as

$$\left(D_\mu D^\mu - m^2 - \xi R\right) W^{(2p)} = -\sigma^{-1}\left(D_\mu D^\mu - m^2 - \xi R\right) U^{(2p)}_{p-2} - 2\,\sigma^{-1}\left[\left(\sigma^{;\mu} D_\mu + (p-1) - \Delta^{-\frac{1}{2}}\Delta^{\frac{1}{2}}_{;\mu}\,\sigma^{;\mu}\right) V^{(2p)}\right]. \quad (7.81)$$

Returning to (7.54), we can consider the lowest order terms in $\sigma(x,x')$, i.e. those proportional to σ^{-p}, to obtain the boundary condition for the $U^{(2p)}_0(x,x')$ Hadamard coefficient. Inserting the power series expansion for $U^{(2p)}(x,x')$ (6.6a) into (7.54), we obtain

$$0 = \left(\sigma^{;\mu} D_\mu - \Delta^{-\frac{1}{2}}\Delta^{\frac{1}{2}}{}_{;\mu}\sigma^{;\mu}\right) U^{(2p)}_0, \quad (7.82)$$

where we have again used the fact that the r.h.s of (7.54) must be equal to zero at each power of $\sigma(x,x')$. In the uncharged case, (7.82) reduces to

$$0 = \sigma^{;\mu}\, U^{(2p)}_{0\,;\mu} - \sigma^{;\mu}\, U^{(2p)}_0 \Delta^{-\frac{1}{2}}\Delta^{\frac{1}{2}}{}_{;\mu}, \quad (7.83)$$

and we can see that equation (7.83) is solved by taking either $U^{(2p)}_0 = \Delta^{\frac{1}{2}}$ or $U^{(2p)}_0 = -\Delta^{\frac{1}{2}}$. Our guiding principle will be that the leading-order singularity in the Hadamard parametrix (6.5) matches that of Minkowski spacetime; in the coincidence limit we have

$$U^{(2p)}_0(x,x) = 1. \quad (7.84)$$

Therefore, in the uncharged case, (7.83) is solved by

$$U^{(2p)}_0(x,x') = \Delta^{\frac{1}{2}}. \quad (7.85)$$

In the charged case, (7.82) cannot be solved exactly and we need to expand $U^{(2p)}_0(x,x')$ as a power series in $\sigma(x,x')$ up to the required order for the evaluation of the RSET.

7.2.2 Explicit renormalisation counterterms in four dimensions

In order to derive explicit renormalisation counterterms, we focus on the specific case of $d=4$. In the $d=4$ Hadamard parametrix (6.5), there are terms contained within both the $U^{(4)}(x,x')$ and the $V^{(4)}(x,x')$ biscalars that we need to derive explicitly in order to be able to evaluate the RSET. We will begin with the terms contained within the $U^{(4)}(x,x')$ biscalar, since these are much simpler to derive.

Evaluating terms within $U^{(4)}(x,x')$ biscalar

From the expression (6.6a) for the series expansion of the $U^{(4)}(x,x')$ biscalar, we see that in $d=4$ we have $U^{(4)}(x,x') = U^{(4)}_0(x,x')$. This means that we only need to calculate the explicit expressions for the $U^{(4)}_0(x,x')$ Hadamard coefficient up to the order required for evaluating the RSET, i.e. up to $\mathcal{O}(\sigma^2)$, in order to have all of the necessary renormalisation counterterms contained within the $U^{(4)}(x,x')$ biscalar.

Then, rewriting (7.82) in anticipation of calculating $U^{(4)}_0(x,x')$ up to $\mathcal{O}(\sigma^2)$, we have

$$\sigma^{;\mu} D_\mu U_0^{(4)} - U_0^{(4)} \Delta^{-\frac{1}{2}} \Delta^{\frac{1}{2}}_{;\mu} \sigma^{;\mu} = 0. \tag{7.86}$$

We can expand $U_0^{(4)}(x,x')$ as a covariant Taylor expansion according to (6.10a):

$$U_0^{(4)} = U_{00}^{(4)} + U_{01\mu}^{(4)} \sigma^{;\mu} + U_{02(\mu\nu)}^{(4)} \sigma^{;\mu}\sigma^{;\nu} + U_{03(\mu\nu\rho)}^{(4)} \sigma^{;\mu}\sigma^{;\nu}\sigma^{;\rho} + U_{04(\mu\nu\rho\lambda)}^{(4)} \sigma^{;\mu}\sigma^{;\nu}\sigma^{;\rho}\sigma^{;\lambda} + \mathcal{O}\Big(\sigma^{5/2}\Big). \tag{7.87}$$

We note that the form of equation (7.86), which is satisfied by $U_0^{(4)}(x,x')$, is identical to that of (7.27), which is satisfied by $V_0^{(2)}(x,x')$. However, the zeroth order expansion coefficient of $V_0^{(2)}(x,x')$ was given as $V_{00}^{(2)} = -1$ (7.25), so that the leading-order singularity in the $d=2$ Hadamard parametrix (6.3) matches that of $d=2$ Minkowski spacetime. In the $d=4$ Hadamard parametrix (6.5) on the other hand, the mathcing of the leading-order singularity with $d=4$ Minkowski spacetime enforces the zeroth order expansion coefficient of $U_0^{(4)}(x,x')$ to be $U_{00}^{(4)} = 1$. Then, we may write down the first three expansion coefficients of $U_0^{(4)}(x,x')$ by multiplying the corresponding expansion coefficients of $V_0^{(2)}(x,x')$, given in (7.33), (7.35) and (7.38), by a factor of minus one; we obtain

$$U_{00}^{(4)} = 1, \tag{7.88}$$

$$U_{01\mu}^{(4)} = \mathrm{i}qA_\mu, \tag{7.89}$$

$$U_{02(\mu\nu)}^{(4)} = \frac{1}{12} R_{\mu\nu} - \frac{1}{2}\,\mathrm{i}q D_{(\mu}A_{\nu)}. \tag{7.90}$$

We now need to evaluate the $U_{03(\mu\nu\rho)}^{(4)}$ and $U_{04(\mu\nu\rho\lambda)}^{(4)}$ expansion coefficients of $U_0^{(4)}(x,x')$ since we did not require an expression for $V_0^{(2)}(x,x')$ beyond $\mathcal{O}(\sigma)$.

The first term in the equation (7.86) for $U^{(4)}(x,x')$ is $\sigma^{;\mu} D_\mu U_0^{(4)}$; since $\sigma^{;\mu}$ is $\mathcal{O}\big(\sigma^{1/2}\big)$ (1.20), we need only consider $D_\mu U_0^{(4)}$ up to $\mathcal{O}\big(\sigma^{3/2}\big)$. We begin by evaluating $\nabla_\mu U_0^{(4)}$ as

$$\begin{aligned} U_{0\,;\alpha}^{(4)} &= U_{00\,;\alpha}^{(4)} + U_{01\mu\,;\alpha}^{(4)} \sigma^{;\mu} + U_{01\mu}^{(4)} \sigma^{;\mu}{}_{\alpha} + U_{02(\mu\nu)\,;\alpha}^{(4)} \sigma^{;\mu}\sigma^{;\nu} + 2\,U_{02(\mu\nu)}^{(4)} \sigma^{;\mu}{}_{\alpha}\sigma^{;\nu} \\ &\quad + U_{03(\mu\nu\rho)\,;\alpha}^{(4)} \sigma^{;\mu}\sigma^{;\nu}\sigma^{;\rho} + 3\,U_{03(\mu\nu\rho)}^{(4)} \sigma^{;\mu}{}_{\alpha}\sigma^{;\nu}\sigma^{;\rho} + 4\,U_{04(\mu\nu\rho\tau)}^{(4)} \sigma^{;\mu}{}_{\alpha}\sigma^{;\nu}\sigma^{;\rho}\sigma^{;\tau} + \mathcal{O}\big(\sigma^2\big). \end{aligned} \tag{7.91}$$

Using the expansion for $\sigma_{;\mu\nu}$ (6.25), (7.91) becomes

$$\begin{aligned} U_{0\,;\alpha}^{(4)} &= U_{00\,;\alpha}^{(4)} + U_{01\mu\,;\alpha}^{(4)} \sigma^{;\mu} + U_{01\mu}^{(4)} g^{\mu\lambda} \Big\{ g_{\lambda\alpha} - \frac{1}{3} R_{\lambda(\theta|\alpha|\phi)} \sigma^{;\theta}\sigma^{;\phi} + \frac{1}{12} R_{\lambda(\theta|\alpha|\phi\,;\psi)} \sigma^{;\theta}\sigma^{;\phi}\sigma^{;\psi} \\ &\quad - \Big(\frac{1}{60} R_{\lambda(\theta|\alpha|\phi\,;\psi\gamma)} + \frac{1}{45} R_{\lambda(\theta|\rho|\phi} R^{\rho}{}_{\psi|\alpha|\gamma)} \Big) \sigma^{;\theta}\sigma^{;\phi}\sigma^{;\psi}\sigma^{;\gamma} \Big\} + U_{02(\mu\nu)\,;\alpha}^{(4)} \sigma^{;\mu}\sigma^{;\nu} \\ &\quad + 2\,U_{02(\mu\nu)}^{(4)} g^{\mu\lambda} \Big\{ g_{\lambda\alpha} - \frac{1}{3} R_{\lambda(\theta|\alpha|\phi)} \sigma^{;\theta}\sigma^{;\phi} + \frac{1}{12} R_{\lambda(\theta|\alpha|\phi\,;\psi)} \sigma^{;\theta}\sigma^{;\phi}\sigma^{;\psi} \Big\} \sigma^{;\nu} \\ &\quad + U_{03(\mu\nu\rho)\,;\alpha}^{(4)} \sigma^{;\mu}\sigma^{;\nu}\sigma^{;\rho} + 3\,U_{03(\mu\nu\rho)}^{(4)} g^{\mu\lambda} \Big\{ g_{\lambda\alpha} \Big\} \sigma^{;\nu}\sigma^{;\rho} \\ &\quad + 4\,U_{04(\mu\nu\rho\tau)}^{(4)} g^{\mu\lambda} \Big\{ g_{\lambda\alpha} \Big\} \sigma^{;\nu}\sigma^{;\rho}\sigma^{;\tau} + \mathcal{O}\big(\sigma^2\big), \end{aligned} \tag{7.92}$$

which simplifies to

$$\begin{aligned}
U^{(4)}_{0\,;\alpha} &= \Big[U^{(4)}_{00\,;\alpha} + U^{(4)}_{01\alpha}\Big] + \Big[U^{(4)}_{01\mu\,;\alpha} + 2\,U^{(4)}_{02(\alpha\mu)}\Big]\,\sigma^{\,;\mu} \\
&+ \bigg[U^{(4)}_{02(\mu\nu)\,;\alpha} + 3\,U^{(4)}_{03(\alpha\mu\nu)} - \frac{1}{3}U^{(4)}_{01\lambda}R^{\lambda}{}_{(\mu|\alpha|\nu)}\bigg]\,\sigma^{\,;\mu}\sigma^{\,;\nu} \\
&+ \bigg[\frac{1}{12}U^{(4)}_{01\lambda}R^{\lambda}{}_{(\mu|\alpha|\nu;\rho)} - \frac{1}{3}U^{(4)}_{02\lambda(\rho}R^{\lambda}{}_{\mu|\alpha|\nu)} - \frac{1}{3}U^{(4)}_{02(\rho|\lambda|}R^{\lambda}{}_{\mu|\alpha|\nu)} + U^{(4)}_{03(\mu\nu\rho)\,;\alpha} \\
&+ 4\,U^{(4)}_{04(\alpha\mu\nu\rho)}\bigg]\,\sigma^{\,;\mu}\sigma^{\,;\nu}\sigma^{\,;\rho} + \mathcal{O}\big(\sigma^2\big)\,. \qquad (7.93)
\end{aligned}$$

Thus, for the gauge covariant derivative of $U^{(4)}_0(x,x')$, we obtain

$$\begin{aligned}
D_\alpha U^{(4)}_0 &= \Big[U^{(4)}_{00\,;\alpha} - \mathrm{i}qA_\alpha U^{(4)}_{00} + U^{(4)}_{01\alpha}\Big] + \Big[U^{(4)}_{01\mu\,;\alpha} - \mathrm{i}qA_\alpha U^{(4)}_{01\mu} + 2\,U^{(4)}_{02(\alpha\mu)}\Big]\,\sigma^{\,;\mu} \\
&+ \bigg[U^{(4)}_{02(\mu\nu)\,;\alpha} - \mathrm{i}qA_\alpha U^{(4)}_{02(\mu\nu)} + 3\,U^{(4)}_{03(\alpha\mu\nu)} - \frac{1}{3}U^{(4)}_{01\lambda}R^{\lambda}{}_{(\mu|\alpha|\nu)}\bigg]\,\sigma^{\,;\mu}\sigma^{\,;\nu} \\
&+ \bigg[\frac{1}{12}U^{(4)}_{01\lambda}R^{\lambda}{}_{(\mu|\alpha|\nu;\rho)} - \frac{1}{3}U^{(4)}_{02\lambda(\rho}R^{\lambda}{}_{\mu|\alpha|\nu)} - \frac{1}{3}U^{(4)}_{02(\rho|\lambda|}R^{\lambda}{}_{\mu|\alpha|\nu)} + U^{(4)}_{03(\mu\nu\rho)\,;\alpha} \\
&- \mathrm{i}qA_\alpha U^{(4)}_{03(\mu\nu\rho)} + 4\,U^{(4)}_{04(\alpha\mu\nu\rho)}\bigg]\,\sigma^{\,;\mu}\sigma^{\,;\nu}\sigma^{\,;\rho} + \mathcal{O}\big(\sigma^2\big)\,. \qquad (7.94)
\end{aligned}$$

We can write (7.94) compactly using the gauge derivative as

$$\begin{aligned}
D_\alpha U^{(4)}_0 &= \Big[D_\alpha U^{(4)}_{00} + U^{(4)}_{01\alpha}\Big] + \Big[D_\alpha U^{(4)}_{01\mu} + 2\,U^{(4)}_{02(\alpha\mu)}\Big]\,\sigma^{\,;\mu} \\
&+ \bigg[D_\alpha U^{(4)}_{02(\mu\nu)} + 3\,U^{(4)}_{03(\alpha\mu\nu)} - \frac{1}{3}U^{(4)}_{01\lambda}R^{\lambda}{}_{(\mu|\alpha|\nu)}\bigg]\,\sigma^{\,;\mu}\sigma^{\,;\nu} \\
&+ \bigg[\frac{1}{12}U^{(4)}_{01\lambda}R^{\lambda}{}_{(\mu|\alpha|\nu;\rho)} - \frac{1}{3}U^{(4)}_{02\lambda(\rho}R^{\lambda}{}_{\mu|\alpha|\nu)} - \frac{1}{3}U^{(4)}_{02(\rho|\lambda|}R^{\lambda}{}_{\mu|\alpha|\nu)} + D_\alpha U^{(4)}_{03(\mu\nu\rho)} \\
&+ 4\,U^{(4)}_{04(\alpha\mu\nu\rho)}\bigg]\,\sigma^{\,;\mu}\sigma^{\,;\nu}\sigma^{\,;\rho} + \mathcal{O}\big(\sigma^2\big)\,. \qquad (7.95)
\end{aligned}$$

From (7.86), we need to contract (7.95) with a factor of $\sigma^{\,;\alpha}$; doing so, we obtain

$$\begin{aligned}
\sigma^{\,;\alpha}D_\alpha U^{(4)}_0 &= \Big[D_\alpha U^{(4)}_{00} + U^{(4)}_{01\alpha}\Big]\,\sigma^{\,;\alpha} + \Big[D_{(\alpha} U^{(4)}_{01\mu)} + 2\,U^{(4)}_{02(\alpha\mu)}\Big]\,\sigma^{\,;\alpha}\sigma^{\,;\mu} \\
&+ \bigg[D_{(\alpha} U^{(4)}_{02\mu\nu)} + 3\,U^{(4)}_{03(\alpha\mu\nu)} - \frac{1}{3}U^{(4)}_{01\lambda}R^{\lambda}{}_{(\mu\alpha\nu)}\bigg]\,\sigma^{\,;\alpha}\sigma^{\,;\mu}\sigma^{\,;\nu} \\
&+ \bigg[\frac{1}{12}U^{(4)}_{01\lambda}R^{\lambda}{}_{(\mu\alpha\nu;\rho)} - \frac{1}{3}U^{(4)}_{02\lambda(\rho}R^{\lambda}{}_{\mu\alpha\nu)} - \frac{1}{3}U^{(4)}_{02(\rho|\lambda|}R^{\lambda}{}_{\mu\alpha\nu)} + D_{(\alpha} U^{(4)}_{03\mu\nu\rho)} \\
&+ 4\,U^{(4)}_{04(\alpha\mu\nu\rho)}\bigg]\,\sigma^{\,;\alpha}\sigma^{\,;\mu}\sigma^{\,;\nu}\sigma^{\,;\rho} + \mathcal{O}\Big(\sigma^{5/2}\Big)\,. \qquad (7.96)
\end{aligned}$$

Given the antisymmetry of the lowered third and fourth indices of the Riemann tensor (1.5a), all terms containing the Riemann tensor vanish; then, the first term in (7.86) is

$$\begin{aligned}
\sigma^{\,;\alpha}D_\alpha\, U^{(4)}_0 &= \Big[D_\alpha U^{(4)}_{00} + U^{(4)}_{01\alpha}\Big]\,\sigma^{\,;\alpha} + \Big[D_{(\alpha} U^{(4)}_{01\mu)} + 2\,U^{(4)}_{02(\alpha\mu)}\Big]\,\sigma^{\,;\alpha}\sigma^{\,;\mu} \\
&+ \Big[D_{(\alpha} U^{(4)}_{02\mu\nu)} + 3\,U^{(4)}_{03(\alpha\mu\nu)}\Big]\,\sigma^{\,;\alpha}\sigma^{\,;\mu}\sigma^{\,;\nu} \\
&+ \Big[D_{(\alpha} U^{(4)}_{03\mu\nu\rho)} + 4\,U^{(4)}_{04(\alpha\mu\nu\rho)}\Big]\,\sigma^{\,;\alpha}\sigma^{\,;\mu}\sigma^{\,;\nu}\sigma^{\,;\rho} + \mathcal{O}\Big(\sigma^{5/2}\Big)\,. \qquad (7.97)
\end{aligned}$$

Using (6.26), the second term in the equation for $U^{(4)}(x,x')$ (7.86) is given by

$$\begin{aligned}
&-\left(\Delta^{-\frac{1}{2}}\,\Delta^{\frac{1}{2}}_{;\mu}\,\sigma^{\,;\mu}\right)U_0^{(4)} \\
&\quad= -\frac{1}{6}\,U_{00}^{(4)}R_{\mu\nu}\,\sigma^{\,;\mu}\sigma^{\,;\nu} + \frac{1}{24}\,U_{00}^{(4)}R_{(\mu\nu;\rho)}\,\sigma^{\,;\mu}\sigma^{\,;\nu}\sigma^{\,;\rho} + \\
&\quad -\left[\frac{1}{120}R_{(\mu\nu;\rho\tau)} + \frac{1}{90}R^{\lambda}{}_{(\mu|\beta|\nu}R^{\beta}{}_{\rho|\lambda|\tau)}\right]U_{00}^{(4)}\,\sigma^{\,;\mu}\sigma^{\,;\nu}\sigma^{\,;\rho}\sigma^{\,;\tau} - \frac{1}{6}\,U_{01(\mu}^{(4)}R_{\nu\rho)}\,\sigma^{\,;\mu}\sigma^{\,;\nu}\sigma^{\,;\rho} \\
&\quad + \frac{1}{24}\,U_{01(\mu}^{(4)}R_{\nu\rho;\tau)}\,\sigma^{\,;\mu}\sigma^{\,;\nu}\sigma^{\,;\rho}\sigma^{\,;\tau} - \frac{1}{6}\,U_{02(\mu\nu}^{(4)}R_{\rho\tau)}\,\sigma^{\,;\mu}\sigma^{\,;\nu}\sigma^{\,;\rho}\sigma^{\,;\tau} + \mathcal{O}\left(\sigma^{5/2}\right), \qquad (7.98)
\end{aligned}$$

which simplifies to

$$\begin{aligned}
&-\left(\Delta^{-\frac{1}{2}}\,\Delta^{\frac{1}{2}}_{;\mu}\,\sigma^{\,;\mu}\right)U_0^{(4)} \\
&\quad= -\frac{1}{6}\,U_{00}^{(4)}R_{\mu\nu}\sigma^{\,;\mu}\sigma^{\,;\nu} + \left[\frac{1}{24}\,U_{00}^{(4)}R_{(\mu\nu;\rho)} - \frac{1}{6}\,U_{01(\mu}^{(4)}R_{\nu\rho)}\right]\sigma^{\,;\mu}\sigma^{\,;\nu}\sigma^{\,;\rho} \\
&\quad -\left[\frac{1}{120}\,U_{00}^{(4)}R_{(\mu\nu;\rho\tau)} + \frac{1}{90}\,U_{00}^{(4)}R^{\lambda}{}_{(\mu|\beta|\nu}R^{\beta}{}_{\rho|\lambda|\tau)} - \frac{1}{24}\,U_{01(\mu}^{(4)}R_{\nu\rho;\tau)} + \frac{1}{6}\,U_{02(\mu\nu}^{(4)}R_{\rho\tau)}\right] \\
&\quad \times \sigma^{\,;\mu}\sigma^{\,;\nu}\sigma^{\,;\rho}\sigma^{\,;\tau} + \mathcal{O}\left(\sigma^{5/2}\right). \qquad (7.99)
\end{aligned}$$

The terms at $\mathcal{O}\left(\sigma^{3/2}\right)$ in the equation for $U_0^{(4)}(x,x')$ (7.86) give

$$D_{(\mu}U_{02\nu\rho)}^{(4)} + 3\,U_{03(\mu\nu\rho)}^{(4)} - \frac{1}{6}\,U_{01(\mu}^{(4)}R_{\nu\rho)} + \frac{1}{24}\,U_{00}^{(4)}R_{(\mu\nu;\rho)} = 0. \qquad (7.100)$$

Using (7.88), (7.89) and (7.90), then (7.100) becomes

$$\begin{aligned}
0 &= \frac{1}{12}\,D_{(\mu}R_{\nu\rho)} - \frac{1}{2}\,\mathrm{i}qD_{(\mu}D_{\nu}A_{\rho)} + 3\,U_{03(\mu\nu\rho)}^{(4)} - \frac{1}{6}\,\mathrm{i}qA_{(\mu}R_{\nu\rho)} + \frac{1}{24}R_{(\mu\nu;\rho)} \\
&= \frac{1}{12}\,D_{(\mu}R_{\nu\rho)} - \frac{1}{2}\,\mathrm{i}qD_{(\mu}D_{\nu}A_{\rho)} + 3\,U_{03(\mu\nu\rho)}^{(4)} - \frac{1}{8}\,\mathrm{i}qA_{(\mu}R_{\nu\rho)} + \frac{1}{24}D_{(\mu}R_{\nu\rho)} \\
&= \frac{1}{8}\,D_{(\mu}R_{\nu\rho)} - \frac{1}{2}\,\mathrm{i}qD_{(\mu}D_{\nu}A_{\rho)} + 3\,U_{03(\mu\nu\rho)}^{(4)} - \frac{1}{8}\,\mathrm{i}qA_{(\mu}R_{\nu\rho)}. \qquad (7.101)
\end{aligned}$$

Then, we obtain for the $U_{03(\mu\nu\rho)}^{(4)}$ expansion coefficient

$$U_{03(\mu\nu\rho)}^{(4)} = -\frac{1}{24}D_{(\mu}R_{\nu\rho)} + \frac{1}{24}\,\mathrm{i}qA_{(\mu}R_{\nu\rho)} + \frac{1}{6}\,\mathrm{i}qD_{(\mu}D_{\nu}A_{\rho)}. \qquad (7.102)$$

The terms at $\mathcal{O}\left(\sigma^{2}\right)$ in the equation for $U_0^{(4)}(x,x')$ (7.86) give

$$\begin{aligned}
&D_{(\mu}U_{03\nu\rho\tau)}^{(4)} + 4\,U_{04(\mu\nu\rho\tau)}^{(4)} - \frac{1}{120}\,U_{00}^{(4)}R_{(\mu\nu;\rho\tau)} - \frac{1}{90}\,U_{00}^{(4)}R^{\lambda}{}_{(\mu|\beta|\nu}R^{\beta}{}_{\rho|\lambda|\tau)} \\
&\qquad + \frac{1}{24}\,U_{01(\mu}^{(4)}R_{\nu\rho;\tau)} - \frac{1}{6}\,U_{02(\mu\nu}^{(4)}R_{\rho\tau)} = 0. \qquad (7.103)
\end{aligned}$$

Using (7.88), (7.89), (7.90) and (7.102), then (7.103) becomes

$$\begin{aligned}
0 &= -\frac{1}{24}D_{(\mu}D_{\nu}R_{\rho\tau)} + \frac{1}{24}\,\mathrm{i}qD_{(\mu}A_{\nu}R_{\rho\tau)} + \frac{1}{6}\,\mathrm{i}qD_{(\mu}D_{\nu}D_{\rho}A_{\tau)} + 4\,U_{04(\mu\nu\rho\tau)}^{(4)} - \frac{1}{120}R_{(\mu\nu;\rho\tau)} \\
&\quad - \frac{1}{90}R^{\lambda}{}_{(\mu|\beta|\nu}R^{\beta}{}_{\rho|\lambda|\tau)} + \frac{1}{24}\,\mathrm{i}qA_{(\mu}R_{\nu\rho;\tau)} - \frac{1}{72}R_{(\mu\nu}R_{\rho\tau)} + \frac{1}{12}\,\mathrm{i}q\left[D_{(\mu}A_{\nu}\right]R_{\rho\tau)}. \qquad (7.104)
\end{aligned}$$

We can expand out the terms containing gauge covariant derivatives acting on the Ricci tensor in (7.104) in order to simplify the expression; doing so, we obtain

$$
\begin{aligned}
0 = &-\frac{1}{24}\nabla_{(\mu}\nabla_{\nu}R_{\rho\tau)} + \frac{1}{24}\,\mathrm{i}q\,\nabla_{(\mu}A_{\nu}R_{\rho\tau)} + \frac{1}{24}\,\mathrm{i}qA_{(\mu}\nabla_{\nu}R_{\rho\tau)} + \frac{1}{24}\,q^2A_{(\mu}A_{\nu}R_{\rho\tau)}\\
&+ \frac{1}{24}\,\mathrm{i}q\,\nabla_{(\mu}A_{\nu}R_{\rho\tau)} + \frac{1}{24}\,q^2A_{(\mu}A_{\nu}R_{\rho\tau)} + \frac{1}{6}\,\mathrm{i}qD_{(\mu}D_{\nu}D_{\rho}A_{\tau)} + 4\,U^{(4)}_{04(\mu\nu\rho\tau)}\\
&- \frac{1}{120}\nabla_{(\mu}\nabla_{\nu}R_{\rho\tau)} - \frac{1}{90}R^{\lambda}{}_{(\mu|\beta|\nu}R^{\beta}{}_{\rho|\lambda|\tau)} + \frac{1}{24}\,\mathrm{i}qA_{(\mu}\nabla_{\nu}R_{\rho\tau)} - \frac{1}{72}R_{(\mu\nu}R_{\rho\tau)}\\
&+ \frac{1}{12}\,\mathrm{i}q\big[\nabla_{(\mu}A_{\nu}\big]R_{\rho\tau)} + \frac{1}{12}\,q^2A_{(\mu}A_{\nu}R_{\rho\tau)}\\
= &-\frac{1}{20}\nabla_{(\mu}\nabla_{\nu}R_{\rho\tau)} + \frac{1}{12}\,\mathrm{i}q\,\nabla_{(\mu}A_{\nu}R_{\rho\tau)} + \frac{1}{12}\,\mathrm{i}qA_{(\mu}\nabla_{\nu}R_{\rho\tau)} + \frac{1}{6}\,q^2A_{(\mu}A_{\nu}R_{\rho\tau)}\\
&+ \frac{1}{6}\,\mathrm{i}qD_{(\mu}D_{\nu}D_{\rho}A_{\tau)} + 4\,U^{(4)}_{04(\mu\nu\rho\tau)} - \frac{1}{90}R^{\lambda}{}_{(\mu|\beta|\nu}R^{\beta}{}_{\rho|\lambda|\tau)} - \frac{1}{72}R_{(\mu\nu}R_{\rho\tau)}\\
&+ \frac{1}{12}\,\mathrm{i}q\,\nabla_{(\mu}A_{\nu}R_{\rho\tau)} - \frac{1}{12}\,\mathrm{i}qA_{(\mu}\nabla_{\nu}R_{\rho\tau)}\\
= &-\frac{1}{20}\nabla_{(\mu}\nabla_{\nu}R_{\rho\tau)} + \frac{1}{6}\,\mathrm{i}q\,\nabla_{(\mu}A_{\nu}R_{\rho\tau)} + \frac{1}{6}\,q^2A_{(\mu}A_{\nu}R_{\rho\tau)} + \frac{1}{6}\,\mathrm{i}qD_{(\mu}D_{\nu}D_{\rho}A_{\tau)} + 4\,U^{(4)}_{04(\mu\nu\rho\tau)}\\
&- \frac{1}{90}R^{\lambda}{}_{(\mu|\beta|\nu}R^{\beta}{}_{\rho|\lambda|\tau)} - \frac{1}{72}R_{(\mu\nu}R_{\rho\tau)}\\
= &-\frac{1}{20}\nabla_{(\mu}\nabla_{\nu}R_{\rho\tau)} + \frac{1}{6}\,\mathrm{i}qD_{(\mu}A_{\nu}R_{\rho\tau)} + \frac{1}{6}\,\mathrm{i}qD_{(\mu}D_{\nu}D_{\rho}A_{\tau)} + 4\,U^{(4)}_{04(\mu\nu\rho\tau)}\\
&- \frac{1}{90}R^{\lambda}{}_{(\mu|\beta|\nu}R^{\beta}{}_{\rho|\lambda|\tau)} - \frac{1}{72}R_{(\mu\nu}R_{\rho\tau)}.
\end{aligned}
\tag{7.105}
$$

Then, we obtain for the $U^{(4)}_{04(\mu\nu\rho\tau)}$ expansion coefficient

$$
\begin{aligned}
U^{(4)}_{04(\mu\nu\rho\tau)} = &\,\frac{1}{80}\nabla_{(\mu}\nabla_{\nu}R_{\rho\tau)} + \frac{1}{288}R_{(\mu\nu}R_{\rho\tau)} + \frac{1}{360}R^{\lambda}{}_{(\mu|\beta|\nu}R^{\beta}{}_{\rho|\lambda|\tau)} - \frac{1}{24}\,\mathrm{i}qD_{(\mu}D_{\nu}D_{\rho}A_{\tau)}\\
&- \frac{1}{24}\,\mathrm{i}qD_{(\mu}A_{\nu}R_{\rho\tau)}.
\end{aligned}
\tag{7.106}
$$

From the expression (6.6a) for the series expansion of the $U^{(4)}(x,x')$ biscalar, we see that in $d=4$ we have $U^{(4)}(x,x') = U^{(4)}_0(x,x')$. This means that by calculating all of the explicit expressions for the $U^{(4)}_0(x,x')$ Hadamard coefficient up to the order required for evaluating the RSET, i.e. up to $\mathcal{O}(\sigma^2)$, we have calculated all of the necessary renormalisation counterterms contained within the $U^{(4)}(x,x')$ biscalar that we require to calculate of the RSET. However, there are still terms in the $d=4$ Hadamard parametrix (6.5) contained within the $V^{(4)}(x,x')$ biscalar that we need to evaluate explicitly in order to evaluate the RSET; this is the subject of the next section.

Evaluating terms within $V^{(4)}(x,x')$ biscalar

From the form of the $d=4$ Hadamard parametrix (6.5), we see that we need to evaluate renormalisation counterterms contained within the $V^{(4)}(x,x')$ biscalar up to $\mathcal{O}(\sigma)$. This means, from the expression (6.6b) for the series expansion of the $V^{(4)}(x,x')$ biscalar, that we need to evaluate the $V^{(4)}_0(x,x')$ Hadamard coefficient up to $\mathcal{O}(\sigma)$ and the $V^{(4)}_1(x,x')$ Hadamard coefficient to zeroth order in order to evaluate the RSET.

We can begin with the $V_0^{(4)}(x,x')$ Hadamard coefficient; the boundary condition for general $V_0^{(2p)}(x,x')$ is given by (7.64) and we therefore have in $d=4$:

$$2\,V_0^{(4)} + 2\,\sigma^{;\mu} D_\mu V_0^{(4)} - 2\,V_0^{(4)} \Delta^{-\frac{1}{2}} \Delta^{\frac{1}{2}}_{;\mu}\,\sigma^{;\mu} + \big(D_\mu D^\mu - m^2 - \xi R\big)\,U_0^{(4)} = 0. \tag{7.107}$$

We can expand $V_0^{(4)}(x,x')$ as a covariant Taylor expansion according to (6.10b):

$$V_0^{(4)} = V_{00}^{(4)} + V_{01\mu}^{(4)}\,\sigma^{;\mu} + V_{02(\mu\nu)}^{(4)}\,\sigma^{;\mu}\sigma^{;\nu} + \mathcal{O}\Big(\sigma^{3/2}\Big)\,. \tag{7.108}$$

Then the first term in the equation (7.107) for $V_0^{(4)}(x,x')$ is given by

$$2\,V_0^{(4)} = 2\,V_{00}^{(4)} + 2\,V_{01\mu}^{(4)}\,\sigma^{;\mu} + 2\,V_{02(\mu\nu)}^{(4)}\,\sigma^{;\mu}\sigma^{;\nu} + \mathcal{O}\Big(\sigma^{3/2}\Big)\,. \tag{7.109}$$

The second term in the equation for $V_0^{(4)}(x,x')$ (7.107) is $2\,\sigma^{;\mu}D_\mu V_0^{(4)}$; since $\sigma^{;\mu}$ is $\mathcal{O}\big(\sigma^{1/2}\big)$ (1.20), we need only consider $D_\mu V_0^{(4)}$ up to $\mathcal{O}\big(\sigma^{1/2}\big)$. We begin by evaluating $\nabla_\mu V_0^{(4)}$ as

$$\begin{aligned} V_{0\,;\mu}^{(4)} &= V_{00\,;\mu}^{(4)} + V_{01\nu}^{(4)}\,\sigma^{;\nu}{}_{\mu} + V_{01\nu\,;\mu}^{(4)}\,\sigma^{;\nu} + 2\,V_{02(\nu\rho)}^{(4)}\,\sigma^{;\nu}{}_{\mu}\sigma^{;\rho} + \mathcal{O}(\sigma) \\ &= V_{00\,;\mu}^{(4)} + V_{01\mu}^{(4)} + V_{01\nu\,;\mu}^{(4)}\,\sigma^{;\nu} + 2\,V_{02(\mu\nu)}^{(4)}\,\sigma^{;\nu} + \mathcal{O}(\sigma)\,, \end{aligned} \tag{7.110}$$

where we have used the expansion for $\sigma_{;\mu\nu}$ (6.25). Then, we obtain for $D_\mu V_0^{(4)}$

$$D_\mu V_0^{(4)} = D_\mu V_{00}^{(4)} + V_{01\mu}^{(4)} + \Big[D_\mu V_{01\nu}^{(4)} + 2\,V_{02(\mu\nu)}^{(4)}\Big]\sigma^{;\nu} + \mathcal{O}(\sigma)\,. \tag{7.111}$$

So the second term in the equation (7.107) for $V^{(4)}(x,x')$ is given by

$$2\,\sigma^{;\mu}D_\mu V_0^{(4)} = 2\,\Big[D_\mu V_{00}^{(4)} + V_{01\mu}^{(4)}\Big]\sigma^{;\mu} + 2\,\Big[D_{(\mu} V_{01\nu)}^{(4)} + 2\,V_{02(\mu\nu)}^{(4)}\Big]\sigma^{;\mu}\sigma^{;\nu} + \mathcal{O}\Big(\sigma^{3/2}\Big)\,. \tag{7.112}$$

Using (6.26), the third term in the equation (7.107) for $V^{(4)}(x,x')$ is given by

$$-\,2\,V_0^{(4)}\Delta^{-\frac{1}{2}}\Delta^{\frac{1}{2}}_{;\mu}\,\sigma^{;\mu} = -\frac{1}{3}\,V_{00}^{(4)} R_{\mu\nu}\,\sigma^{;\mu}\sigma^{;\nu} + \mathcal{O}\Big(\sigma^{3/2}\Big)\,. \tag{7.113}$$

We will evaluate the final term in (7.107) in steps, beginning with the calculation of $D_\mu U_0^{(4)}(x,x')$, which is given in (7.95). We can then act on (7.95) with another gauge covariant derivative to obtain an expression $D_\mu D^\mu U_0^{(4)}(x,x')$.

In (7.42), we calculated the quantity $D_\mu D^\mu V_0^{(2)}$ to $\mathcal{O}(1)$ in terms of the expansion coefficients of $V_0^{(2)}(x,x')$; from (6.10a) and (6.10b), the expansion of $U^{(4)}(x,x')$ in terms of its expansion coefficients is identical to that of $V^{(2)}(x,x')$; then we may write down $\big(D_\mu D^\mu - m^2 - \xi R\big)\,U_0^{(4)}$ to $\mathcal{O}(1)$ from (7.42) as

$$\big(D_\mu D^\mu - m^2 - \xi R\big)\,U_0^{(4)} = \big(D_\mu D^\mu - m^2 - \xi R\big)\,U_{00}^{(4)} + 2\,D^\mu U_{01\mu}^{(4)} + 2\,g^{\mu\nu} U_{02(\mu\nu)}^{(4)} + \mathcal{O}\Big(\sigma^{1/2}\Big)\,. \tag{7.114}$$

We now have all four terms in the equation for $V_0^{(4)}(x,x')$ up to $\mathcal{O}(1)$; considering only terms at $\mathcal{O}(1)$ in (7.109), (7.112), (7.113) and (7.114), the $V_{00}^{(4)}$ expansion coefficient is

$$0 = 2\,V^{(4)}_{00} + \big(D_\mu D^\mu - m^2 - \xi R\big)\,U^{(4)}_{00} + 2\,D^\mu U^{(4)}_{01\mu} + 2\,g^{\mu\nu} U^{(4)}_{02(\mu\nu)}. \tag{7.115}$$

Using the expressions for $U^{(4)}_{00}$ (7.88), $U^{(4)}_{01\mu}$ (7.89) and $U^{(4)}_{02(\mu\nu)}$ (7.90), (7.115) becomes

$$\begin{aligned} 0 &= 2\,V^{(4)}_{00} - \mathrm{i}q\,\nabla_\mu A^\mu - q^2 A_\mu A^\mu - \big(m^2 + \xi R\big) + 2\,\mathrm{i}q\,\nabla_\mu A^\mu + 2\,q^2 A_\mu A^\mu + \frac{1}{6}R - \mathrm{i}q\,\nabla_\mu A^\mu \\ &\quad - q^2 A_\mu A^\mu \\ &= 2\,V^{(4)}_{00} - \big(m^2 + \xi R\big) + \frac{1}{6}R. \end{aligned} \tag{7.116}$$

Examining (7.116), we see that all contributions containing the scalar field charge q cancel exactly. This is in line with our zeroth order results in $d = 2$ and for the $U^{(4)}_0(x, x')$ biscalar in $d = 4$ thus far. Then, we obtain for the $V^{(4)}_{00}$ expansion coefficient

$$V^{(4)}_{00} = \frac{1}{2}\left[m^2 + \left(\xi - \frac{1}{6}\right) R\right]. \tag{7.117}$$

This is the same as in the uncharged case [68] and so there are no gauge corrections to the lowest order expansion coefficient of the Hadamard coefficient $V^{(4)}_0(x, x')$.

Due to the computational complexity involved in evaluating $D_\mu D^\mu U^{(4)}_0$, we now take a slightly different approach in deriving the explicit expressions for the $V^{(4)}_{1\mu}$ and $V^{(4)}_{02(\mu\nu)}$ expansion coefficients. The complexity lies within the $\Box\, U^{(4)}_0$ term contained within $D_\mu D^\mu U^{(4)}_0$. Since $D_\mu D^\mu U^{(4)}_0$ reduces to $\Box\, U^{(4)}_0$ when we set $q = 0$, i.e. in the uncharged case, it makes sense to first re-derive the results due to [68], before generalising to the charged case. Our first task, then, is to calculate $\Box\, U^{(4)}_0$; in order to do so, we would like to consider $\nabla_\alpha U^{(4)}_0$ (7.93) up to $\mathcal{O}\big(\sigma^{3/2}\big)$, so we can get an expression for $\Box\, U^{(4)}_0$ up to second order. Since we have already calculated the zeroth order contribution to $V^{(4)}_0(x, x')$ in $V^{(4)}_{00}$ (7.117), we will ignore these terms in the following calculation and simply label them as $\mathcal{O}(1)$. Then, acting with another covariant derivative on (7.93), we obtain

$$\begin{aligned} \Box\, U^{(4)}_0 &= \mathcal{O}(1) + g^{\alpha\beta}\left[\nabla_\alpha U^{(4)}_{01\mu} + 2\,U^{(4)}_{02(\alpha\mu)}\right]\sigma^{;\mu}{}_{\beta} + \left[\Box\, U^{(4)}_{01\mu} + 2\,\nabla^\alpha U^{(4)}_{02(\alpha\mu)}\right]\sigma^{;\mu} \\ &\quad + 2\,g^{\alpha\beta}\left[\nabla_\alpha U^{(4)}_{02(\mu\nu)} + 3\,U^{(4)}_{03(\alpha\mu\nu)} - \frac{1}{3}\,U^{(4)}_{01\lambda} R^\lambda{}_{(\mu|\alpha|\nu)}\right]\sigma^{;\mu}{}_{\beta}\,\sigma^{;\nu} \\ &\quad + \left[\Box\, U^{(4)}_{02(\mu\nu)} + 3\,\nabla^\alpha U^{(4)}_{03(\alpha\mu\nu)} - \frac{1}{3}\left[\nabla^\alpha U^{(4)}_{01\lambda}\right] R^\lambda{}_{(\mu|\alpha|\nu)} - \frac{1}{3}\,U^{(4)}_{01\lambda}\,\nabla^\alpha R^\lambda{}_{(\mu|\alpha|\nu)}\right]\sigma^{;\mu}\sigma^{;\nu} \\ &\quad + 3\,g^{\alpha\beta}\left[\frac{1}{12}\,U^{(4)}_{01\lambda}\nabla_{(\mu} R^\lambda{}_{\nu|\alpha|\rho)} - \frac{1}{3}\,U^{(4)}_{02\lambda(\mu} R^\lambda{}_{\nu|\alpha|\rho)} - \frac{1}{3}\,U^{(4)}_{02(\mu|\lambda|} R^\lambda{}_{\nu|\alpha|\rho)} + \nabla_\alpha U^{(4)}_{03(\mu\nu\rho)} \right. \\ &\quad \left. + \,4\,U^{(4)}_{04(\alpha\mu\nu\rho)}\right]\sigma^{;\mu}{}_{\beta}\,\sigma^{;\nu}\sigma^{;\rho} + \mathcal{O}\Big(\sigma^{3/2}\Big). \end{aligned} \tag{7.118}$$

Using the expansion for $\sigma_{;\mu\nu}$ (6.25), (7.118) becomes

$$
\begin{aligned}
\Box\, U_0^{(4)} &= \mathcal{O}(1) + g^{\alpha\beta}\left[\nabla_\alpha U_{01\mu}^{(4)} + 2\, U_{02(\alpha\mu)}^{(4)}\right]\left\{-\frac{1}{3} R^\mu{}_{(\theta|\beta|\phi)}\sigma^{;\theta}\sigma^{;\phi}\right\} \\
&+ \left[\Box\, U_{01\mu}^{(4)} + 2\,\nabla^\alpha U_{02(\alpha\mu)}^{(4)}\right]\sigma^{;\mu} \\
&+ g^{\alpha\beta}\left[2\,\nabla_\alpha U_{02(\mu\nu)}^{(4)} + 6\, U_{03(\alpha\mu\nu)}^{(4)} - \frac{2}{3}\, U_{01\lambda}^{(4)} R^\lambda{}_{(\mu|\alpha|\nu)}\right]\left\{\delta^\mu_\beta\right\}\sigma^{;\nu} \\
&+ \left[\Box\, U_{02(\mu\nu)}^{(4)} + 3\,\nabla^\alpha U_{03(\alpha\mu\nu)}^{(4)} - \frac{1}{3}\left[\nabla^\alpha U_{01\lambda}^{(4)}\right] R^\lambda{}_{(\mu|\alpha|\nu)} - \frac{1}{3}\, U_{01\lambda}^{(4)}\,\nabla^\alpha R^\lambda{}_{(\mu|\alpha|\nu)}\right]\sigma^{;\mu}\sigma^{;\nu} \\
&+ g^{\alpha\beta}\left[\frac{1}{4}\, U_{01\lambda}^{(4)}\nabla_{(\mu} R^\lambda{}_{\nu|\alpha|\rho)} - U_{02\lambda(\mu}^{(4)} R^\lambda{}_{\nu|\alpha|\rho)} - U_{02(\mu|\lambda|}^{(4)} R^\lambda{}_{\nu|\alpha|\rho)} + 3\,\nabla_\alpha U_{03(\mu\nu\rho)}^{(4)}\right. \\
&\left. + 12\, U_{04(\alpha\mu\nu\rho)}^{(4)}\right]\left\{\delta^\mu_\beta\right\}\sigma^{;\nu}\sigma^{;\rho} + \mathcal{O}\left(\sigma^{3/2}\right).
\end{aligned}
\tag{7.119}
$$

Since we will be performing contractions over the μ and α indices in some of the terms in (7.119), it will be helpful to expand terms involving symmetrisations over μ and other indices into their constituent parts that do not involve symmetrisations over the μ index:

$$
\begin{aligned}
\Box\, U_0^{(4)} &= \mathcal{O}(1) - \frac{1}{3}\, g^{\alpha\beta}\left[\nabla_\alpha U_{01\mu}^{(4)} + U_{02\alpha\mu}^{(4)} + U_{02\mu\alpha}^{(4)}\right] R^\mu{}_{(\theta|\beta|\phi)}\sigma^{;\theta}\sigma^{;\phi} \\
&+ \left[\Box\, U_{01\mu}^{(4)} + 2\,\nabla^\alpha U_{02(\alpha\mu)}^{(4)}\right]\sigma^{;\mu} \\
&+ g^{\alpha\mu}\left[2\,\nabla_\alpha U_{02(\mu\nu)}^{(4)} + 6\, U_{03(\alpha\mu\nu)}^{(4)} - \frac{1}{3}\, U_{01\lambda}^{(4)} R^\lambda{}_{\mu\alpha\nu} - \frac{1}{3}\, U_{01\lambda}^{(4)} R^\lambda{}_{\nu\alpha\mu}\right]\sigma^{;\nu} \\
&+ \left[\Box\, U_{02(\mu\nu)}^{(4)} + 3\,\nabla^\alpha U_{03(\alpha\mu\nu)}^{(4)} - \frac{1}{3}\left[\nabla^\alpha U_{01\lambda}^{(4)}\right] R^\lambda{}_{(\mu|\alpha|\nu)} - \frac{1}{3}\, U_{01\lambda}^{(4)}\,\nabla^\alpha R^\lambda{}_{(\mu|\alpha|\nu)}\right]\sigma^{;\mu}\sigma^{;\nu} \\
&+ g^{\alpha\mu}\left[\frac{1}{12}\, U_{01\lambda}^{(4)}\nabla_\mu R^\lambda{}_{(\nu|\alpha|\rho)} + \frac{1}{12}\, U_{01\lambda}^{(4)}\nabla_{(\nu} R^\lambda{}_{|\mu\alpha|\rho)} + \frac{1}{12}\, U_{01\lambda}^{(4)}\nabla_{(\nu} R^\lambda{}_{\rho)\alpha\mu}\right. \\
&- \frac{1}{3}\, U_{02\lambda\mu}^{(4)} R^\lambda{}_{(\nu|\alpha|\rho)} - \frac{1}{3}\, U_{02\lambda(\nu}^{(4)} R^\lambda{}_{|\mu\alpha|\rho)} - \frac{1}{3}\, U_{02\lambda(\nu}^{(4)} R^\lambda{}_{\rho)\alpha\mu} - \frac{1}{3}\, U_{02\mu\lambda}^{(4)} R^\lambda{}_{(\nu|\alpha|\rho)} \\
&\left. - \frac{1}{3}\, U_{02(\nu|\lambda}^{(4)} R^\lambda{}_{\mu\alpha|\rho)} - \frac{1}{3}\, U_{02(\nu|\lambda|}^{(4)} R^\lambda{}_{\rho)\alpha\mu} + 3\,\nabla_\alpha U_{03(\mu\nu\rho)}^{(4)} + 12\, U_{04(\alpha\mu\nu\rho)}^{(4)}\right]\sigma^{;\nu}\sigma^{;\rho} \\
&+ \mathcal{O}\left(\sigma^{3/2}\right).
\end{aligned}
\tag{7.120}
$$

Now we may perform the contractions in (7.120); bearing in mind the symmetries of the Riemann tensor, some terms will vanish and some will become factors of the Ricci tensor:

$$\begin{aligned}
\Box U_0^{(4)} &= \mathcal{O}(1) \\
&- \left[\frac{1}{3}\left(\nabla^\alpha U^{(4)}_{01\lambda}\right) R^\lambda{}_{(\mu|\alpha|\nu)} + \frac{1}{3} g^{\alpha\beta} U^{(4)}_{02\alpha\lambda} R^\lambda{}_{(\mu|\beta|\nu)} + \frac{1}{3} g^{\alpha\beta} U^{(4)}_{02\lambda\alpha} R^\lambda{}_{(\mu|\beta|\nu)}\right] \sigma^{;\mu}\sigma^{;\nu} \\
&+ \left[\Box U^{(4)}_{01\mu} + 2\nabla^\alpha U^{(4)}_{02(\alpha\mu)}\right]\sigma^{;\mu} + \left[2\nabla^\alpha U^{(4)}_{02(\alpha\mu)} + 6 g^{\alpha\beta} U^{(4)}_{03(\alpha\beta\mu)} + \frac{1}{3} U^{(4)}_{01\lambda} R^\lambda{}_\mu\right]\sigma^{;\mu} \\
&+ \left[\Box U^{(4)}_{02(\mu\nu)} + 3\nabla^\alpha U^{(4)}_{03(\alpha\mu\nu)} - \frac{1}{3}\left[\nabla^\alpha U^{(4)}_{01\lambda}\right] R^\lambda{}_{(\mu|\alpha|\nu)} - \frac{1}{3} U^{(4)}_{01\lambda}\nabla^\alpha R^\lambda{}_{(\mu|\alpha|\nu)}\right]\sigma^{;\mu}\sigma^{;\nu} \\
&+ \left[\frac{1}{12} U^{(4)}_{01\lambda}\nabla^\alpha R^\lambda{}_{(\mu|\alpha|\nu)} - \frac{1}{12} U^{(4)}_{01\lambda}\nabla_{(\mu} R^\lambda{}_{\nu)} - \frac{1}{3} g^{\alpha\beta} U^{(4)}_{02\lambda\alpha} R^\lambda{}_{(\mu|\beta|\nu)} + \frac{1}{3} U^{(4)}_{02\lambda(\mu} R^\lambda{}_{\nu)}\right. \\
&- \left.\frac{1}{3} g^{\alpha\beta} U^{(4)}_{02\alpha\lambda} R^\lambda{}_{(\mu|\beta|\nu)} + \frac{1}{3} U^{(4)}_{02(\mu|\lambda} R^\lambda{}_{\nu)} + 3\nabla^\alpha U^{(4)}_{03(\alpha\mu\nu)} + 12 g^{\alpha\beta} U^{(4)}_{04(\alpha\beta\mu\nu)}\right]\sigma^{;\mu}\sigma^{;\nu} \\
&+ \mathcal{O}\left(\sigma^{3/2}\right). \qquad (7.121)
\end{aligned}$$

We can simplify like terms in (7.121) to obtain a final expression for $\Box U_0^{(4)}$, given by

$$\begin{aligned}
\Box U_0^{(4)} = \mathcal{O}(1) &+ \left[\Box U^{(4)}_{01\mu} + \frac{1}{3} U^{(4)}_{01\lambda} R^\lambda{}_\mu + 4\nabla^\alpha U^{(4)}_{02(\alpha\mu)} + 6 g^{\alpha\beta} U^{(4)}_{03(\alpha\beta\mu)}\right]\sigma^{;\mu} \\
&+ \left[-\frac{1}{12} U^{(4)}_{01\lambda}\nabla_{(\mu} R^\lambda{}_{\nu)} - \frac{1}{4} U^{(4)}_{01\lambda}\nabla^\alpha R^\lambda{}_{(\mu|\alpha|\nu)} - \frac{2}{3}\left(\nabla^\alpha U^{(4)}_{01\lambda}\right) R^\lambda{}_{(\mu|\alpha|\nu)} + \Box U^{(4)}_{02(\mu\nu)}\right. \\
&+ \frac{1}{3} U^{(4)}_{02\lambda(\mu} R^\lambda{}_{\nu)} + \frac{1}{3} U^{(4)}_{02(\mu|\lambda} R^\lambda{}_{\nu)} - \frac{2}{3} U^{(4)}_{02\lambda\alpha} R^\lambda{}_{(\mu}{}^\alpha{}_{\nu)} - \frac{2}{3} U^{(4)}_{02\alpha\lambda} R^\lambda{}_{(\mu}{}^\alpha{}_{\nu)} \\
&+ \left. 6\nabla^\alpha U^{(4)}_{03(\alpha\mu\nu)} + 12 g^{\alpha\beta} U^{(4)}_{04(\alpha\beta\mu\nu)}\right]\sigma^{;\mu}\sigma^{;\nu} + \mathcal{O}\left(\sigma^{3/2}\right). \qquad (7.122)
\end{aligned}$$

Having evaluated $\Box U_0^{(4)}$, we now calculate the other parts of the fourth term in equation (7.107) for $V_0^{(4)}(x,x')$. Using the expression for $\nabla_\alpha U_0^{(4)}$ (7.93) up to $\mathcal{O}(\sigma)$, we have

$$\begin{aligned}
-2\,\mathrm{i}qA^\alpha\nabla_\alpha U_0^{(4)} = \mathcal{O}(1) &- 2\,\mathrm{i}qA^\alpha\left[\nabla_\alpha U^{(4)}_{01\mu} + 2U^{(4)}_{02(\alpha\mu)}\right]\sigma^{;\mu} \\
&- 2\,\mathrm{i}qA^\alpha\left[\nabla_\alpha U^{(4)}_{02(\mu\nu)} + 3U^{(4)}_{03(\alpha\mu\nu)} - \frac{1}{3} U^{(4)}_{01\lambda} R^\lambda{}_{(\mu|\alpha|\nu)}\right]\sigma^{;\mu}\sigma^{;\nu} + \mathcal{O}\left(\sigma^{3/2}\right). \qquad (7.123)
\end{aligned}$$

Using the expression for $U_0^{(4)}$ (7.87) up to $\mathcal{O}(\sigma)$, we have

$$-q^2 A_\alpha A^\alpha U_0^{(4)} = \mathcal{O}(1) - q^2 A_\alpha A^\alpha U^{(4)}_{01\mu}\sigma^{;\mu} - q^2 A_\alpha A^\alpha U^{(4)}_{02(\mu\nu)}\sigma^{;\mu}\sigma^{;\nu} + \mathcal{O}\left(\sigma^{3/2}\right), \qquad (7.124)$$

$$-\mathrm{i}q\left(\nabla^\alpha A_\alpha\right) U_0^{(4)} = \mathcal{O}(1) - \mathrm{i}q\left(\nabla^\alpha A_\alpha\right) U^{(4)}_{01\mu}\sigma^{;\mu} - \mathrm{i}q\left(\nabla^\alpha A_\alpha\right) U^{(4)}_{02(\mu\nu)}\sigma^{;\mu}\sigma^{;\nu} + \mathcal{O}\left(\sigma^{3/2}\right), \qquad (7.125)$$

$$-\left(m^2+\xi R\right) U_0^{(4)} = \mathcal{O}(1) - \left(m^2+\xi R\right) U^{(4)}_{01\mu}\sigma^{;\mu} - \left(m^2+\xi R\right) U^{(4)}_{02(\mu\nu)}\sigma^{;\mu}\sigma^{;\nu} + \mathcal{O}\left(\sigma^{3/2}\right). \qquad (7.126)$$

Then the final term in (7.107) is given by

$$
\begin{aligned}
&\left(D_\mu D^\mu - m^2 - \xi R\right) U_0^{(4)} \\
&= \mathcal{O}(1) + \left[\left(D_\mu D^\mu - m^2 - \xi R\right) U_{01\mu}^{(4)} + \frac{1}{3}\, U_{01\lambda}^{(4)} R^\lambda{}_\mu + 4\, D^\alpha U_{02(\alpha\mu)}^{(4)} + 6\, g^{\alpha\beta}\, U_{03(\alpha\beta\mu)}^{(4)}\right] \sigma^{;\mu} \\
&+ \left[-\frac{1}{12}\, U_{01\lambda}^{(4)} \nabla_{(\mu} R^\lambda{}_{\nu)} - \frac{1}{4}\, U_{01\lambda}^{(4)} \nabla^\alpha R^\lambda{}_{(\mu|\alpha|\nu)} - \frac{2}{3} \left(D^\alpha U_{01\lambda}^{(4)}\right) R^\lambda{}_{(\mu|\alpha|\nu)} \right. \\
&+ \left(D_\mu D^\mu - m^2 - \xi R\right) U_{02(\mu\nu)}^{(4)} + \frac{1}{3}\, U_{02\lambda(\mu}^{(4)} R^\lambda{}_{\nu)} + \frac{1}{3}\, U_{02(\mu|\lambda}^{(4)} R^\lambda{}_{\nu)} - \frac{2}{3}\, U_{02\lambda\alpha}^{(4)} R^\lambda{}_{(\mu}{}^\alpha{}_{\nu)} \\
&\left. - \frac{2}{3}\, U_{02\alpha\lambda}^{(4)} R^\lambda{}_{(\mu}{}^\alpha{}_{\nu)} + 6\, D^\alpha U_{03(\alpha\mu\nu)}^{(4)} + 12\, g^{\alpha\beta}\, U_{04(\alpha\beta\mu\nu)}^{(4)}\right] \sigma^{;\mu} \sigma^{;\nu} + \mathcal{O}\left(\sigma^{3/2}\right).
\end{aligned}
\tag{7.127}
$$

One interesting observation to note in comparing (7.122) and (7.127) which, aside from the missing factor of $-\left(m^2 + \xi R\right) U_0^{(4)}$ in the former equation, is effectively the generalisation from the uncharged to the charged case, is that only the covariant derivatives acting on expansion coefficients of the $U_0^{(4)}$ Hadamard coefficient are generalised to gauge derivatives; the covariant derivatives that are acting on the Ricci and Riemann tensors in (7.122) remain as ordinary spacetime covariant derivatives in (7.127).

Now we will evaluate the $V_{01\mu}^{(4)}$ and $V_{02(\mu\nu)}^{(4)}$ expansion coefficients of the $V_0^{(4)}(x,x')$ Hadamard coefficient. Considering only those terms at $\mathcal{O}\left(\sigma^{1/2}\right)$ in (7.109), (7.112), (7.113) and (7.127), we can obtain the explicit expression for $V_{01\mu}^{(4)}$; we have

$$
\begin{aligned}
0 &= 2\, V_{01\mu}^{(4)} + 2\, D_\mu V_{00}^{(4)} + 2\, V_{01\mu}^{(4)} + \left(D_\mu D^\mu - m^2 - \xi R\right) U_{01\mu}^{(4)} + \frac{1}{3}\, U_{01\lambda}^{(4)} R^\lambda{}_\mu + 4\, D^\alpha U_{02(\alpha\mu)}^{(4)} \\
&\quad + 6\, g^{\alpha\beta}\, U_{03(\alpha\beta\mu)}^{(4)}.
\end{aligned}
\tag{7.128}
$$

Since our method will be to first recover the results due to [68], we will rewrite the gauge covariant derivatives in (7.128) in terms of ordinary spacetime derivatives and terms proportional to the gauge field A_μ. Then, (7.128) becomes

$$
\begin{aligned}
0 &= 4\, V_{01\mu}^{(4)} + 2\, V_{00\,;\mu}^{(4)} - 2\,\mathrm{i}q A_\mu V_{00}^{(4)} + \left(\Box - m^2 - \xi R\right) U_{01\mu}^{(4)} - 2\,\mathrm{i}q A^\alpha U_{01\mu\,;\alpha}^{(4)} - \mathrm{i}q \left(\nabla_\alpha A^\alpha\right) U_{01\mu}^{(4)} \\
&\quad - q^2 A_\alpha A^\alpha U_{01\mu}^{(4)} + \frac{1}{3}\, U_{01\lambda}^{(4)} R^\lambda{}_\mu + 4\, \nabla^\alpha U_{02(\alpha\mu)}^{(4)} - 4\,\mathrm{i}q A^\alpha U_{02(\alpha\mu)}^{(4)} + 6\, g^{\alpha\beta}\, U_{03(\alpha\beta\mu)}^{(4)}.
\end{aligned}
\tag{7.129}
$$

However, (7.129) also contains nontrivial contractions involving symmetrised indices; we will therefore expand these terms appropriately. Then, (7.129) becomes

$$
\begin{aligned}
0 &= 4\, V_{01\mu}^{(4)} + 2\, V_{00\,;\mu}^{(4)} - 2\,\mathrm{i}q A_\mu V_{00}^{(4)} + \left(\Box - m^2 - \xi R\right) U_{01\mu}^{(4)} - 2\,\mathrm{i}q A^\alpha U_{01\mu\,;\alpha}^{(4)} - \mathrm{i}q \left(\nabla_\alpha A^\alpha\right) U_{01\mu}^{(4)} \\
&\quad - q^2 A_\alpha A^\alpha U_{01\mu}^{(4)} + \frac{1}{3}\, U_{01\lambda}^{(4)} R^\lambda{}_\mu + 2\, \nabla^\alpha U_{02\alpha\mu}^{(4)} + 2\, \nabla^\alpha U_{02\mu\alpha}^{(4)} - 4\,\mathrm{i}q A^\alpha U_{02(\alpha\mu)}^{(4)} + 2\, g^{\alpha\beta}\, U_{03(\alpha\beta)\mu}^{(4)} \\
&\quad + 2\, g^{\alpha\beta}\, U_{03(\alpha|\mu|\beta)}^{(4)} + 2\, g^{\alpha\beta}\, U_{03\mu(\alpha\beta)}^{(4)}.
\end{aligned}
\tag{7.130}
$$

It will be instructive to evaluate each term in (7.130) individually. The first term on the r.h.s is trivial. Using (7.117), the second and third terms on the r.h.s are given by

$$2\,V^{(4)}_{00\,;\mu} = \nabla_\mu\left[m^2 + \left(\xi - \frac{1}{6}\right)R\right]$$
$$= \xi R_{;\mu} - \frac{1}{6}R_{;\mu}, \tag{7.131}$$
$$-2\,\mathrm{i}qA_\mu V^{(4)}_{00} = -\mathrm{i}qA_\mu\left[m^2 + \left(\xi - \frac{1}{6}\right)R\right]$$
$$= -\mathrm{i}qA_\mu\big(m^2 + \xi R\big) + \frac{1}{6}\,\mathrm{i}qA_\mu R. \tag{7.132}$$

Using (7.89), the terms proportional to $U^{(4)}_{01\mu}$ are given by

$$\big(\Box - m^2 - \xi R\big)\,U^{(4)}_{01\mu} = \mathrm{i}q\,\big(\Box - m^2 - \xi R\big)A_\mu, \tag{7.133}$$
$$-2\,\mathrm{i}qA^\alpha U^{(4)}_{01\mu\,;\alpha} = 2\,q^2A^\alpha(\nabla_\alpha A_\mu)\,, \tag{7.134}$$
$$-\mathrm{i}q\,(\nabla_\alpha A^\alpha)\,U^{(4)}_{01\mu} = q^2A_\mu\nabla_\alpha A^\alpha, \tag{7.135}$$
$$-q^2A_\alpha A^\alpha U^{(4)}_{01\mu} = -\mathrm{i}q^3A_\alpha A^\alpha A_\mu, \tag{7.136}$$
$$\frac{1}{3}U^{(4)}_{01\lambda}R^\lambda{}_\mu = \frac{1}{3}\,\mathrm{i}qA^\alpha R_{\alpha\mu}. \tag{7.137}$$

Using (7.90), the terms proportional to $U^{(4)}_{02(\mu\nu)}$ are given by

$$2\,\nabla^\alpha U^{(4)}_{02\alpha\mu} = \frac{1}{6}R_{\alpha\mu}{}^{;\alpha} - q^2\,\nabla_\alpha(A^\alpha A_\mu) - \mathrm{i}q\,\nabla^\alpha\nabla_\mu A_\alpha$$
$$= \frac{1}{6}R_{\alpha\mu}{}^{;\alpha} - q^2(\nabla_\alpha A^\alpha)A_\mu - q^2A^\alpha(\nabla_\alpha A_\mu) - \mathrm{i}q\,\nabla_\alpha\nabla_\mu A^\alpha, \tag{7.138}$$
$$2\,\nabla^\alpha U^{(4)}_{02\mu\alpha} = \frac{1}{6}R_{\mu\alpha}{}^{;\alpha} - q^2\,\nabla_\alpha(A_\mu A^\alpha) - \mathrm{i}q\,\nabla_\alpha\nabla^\alpha A_\mu$$
$$= \frac{1}{6}R_{\alpha\mu}{}^{;\alpha} - q^2(\nabla_\alpha A_\mu)A^\alpha - q^2A_\mu(\nabla_\alpha A^\alpha) - \mathrm{i}q\,\nabla_\alpha\nabla^\alpha A_\mu, \tag{7.139}$$
$$-4\,\mathrm{i}qA^\alpha U^{(4)}_{02(\alpha\mu)} = -4\,\mathrm{i}qA^\alpha\left[\frac{1}{12}R_{\alpha\mu} - \frac{1}{2}q^2A_\alpha A_\mu - \frac{1}{4}\mathrm{i}q\,(\nabla_\alpha A_\mu + \nabla_\mu A_\alpha)\right]$$
$$= -\frac{1}{3}\,\mathrm{i}qA^\alpha R_{\alpha\mu} + 2\,\mathrm{i}q^3A_\alpha A_\mu A^\alpha - q^2A^\alpha(\nabla_\alpha A_\mu) - q^2A^\alpha(\nabla_\mu A_\alpha)\,. \tag{7.140}$$

Using (7.102), the terms proportional to $U^{(4)}_{03(\mu\nu\rho)}$ are given by

$$
\begin{aligned}
2\,g^{\alpha\beta}\,U^{(4)}_{03(\alpha\beta)\mu} &= -\frac{1}{12}R_{\alpha}{}^{\alpha}{}_{;\mu} + \frac{1}{6}\,\mathrm{i}qA^{\alpha}R_{\alpha\mu} + q^{2}(\nabla_{\alpha}A^{\alpha})A_{\mu} - \frac{1}{3}\,\mathrm{i}q^{3}A_{\alpha}A^{\alpha}A_{\mu} \\
&\quad + \frac{1}{3}\,\mathrm{i}q\,\nabla_{\mu}\nabla_{\alpha}A^{\alpha} \\
&= -\frac{1}{12}R_{;\mu} + \frac{1}{6}\,\mathrm{i}qA^{\alpha}R_{\alpha\mu} + q^{2}(\nabla_{\alpha}A^{\alpha})\,A_{\mu} - \frac{1}{3}\,\mathrm{i}q^{3}A_{\alpha}A^{\alpha}A_{\mu} + \frac{1}{3}\,\mathrm{i}q\,\nabla_{\mu}\nabla_{\alpha}A^{\alpha},
\end{aligned}
\tag{7.141}
$$

$$
\begin{aligned}
2\,g^{\alpha\beta}\,U^{(4)}_{03(\alpha|\mu|\beta)} &= -\frac{1}{12}R_{\alpha\mu}{}^{;\alpha} + \frac{1}{6}\,\mathrm{i}qA^{\alpha}R_{\alpha\mu} + q^{2}(\nabla_{\mu}A_{\alpha})A^{\alpha} - \frac{1}{3}\,\mathrm{i}q^{3}A_{\alpha}A_{\mu}A^{\alpha} \\
&\quad + \frac{1}{3}\,\mathrm{i}q\,\nabla_{\alpha}\nabla_{\mu}A^{\alpha},
\end{aligned}
\tag{7.142}
$$

$$
\begin{aligned}
2\,g^{\alpha\beta}\,U^{(4)}_{03\mu(\alpha\beta)} &= -\frac{1}{12}R_{\mu\alpha}{}^{;\alpha} + \frac{1}{6}\,\mathrm{i}qA_{\mu}R_{\alpha}{}^{\alpha} + q^{2}(\nabla_{\alpha}A_{\mu})A^{\alpha} - \frac{1}{3}\,\mathrm{i}q^{3}A_{\mu}A_{\alpha}A^{\alpha} \\
&\quad + \frac{1}{3}\,\mathrm{i}q\,\nabla_{\alpha}\nabla^{\alpha}A_{\mu} \\
&= -\frac{1}{12}R_{\mu\alpha}{}^{;\alpha} + \frac{1}{6}\,\mathrm{i}qA_{\mu}R + q^{2}(\nabla_{\alpha}A_{\mu})A^{\alpha} - \frac{1}{3}\,\mathrm{i}q^{3}A_{\mu}A_{\alpha}A^{\alpha} + \frac{1}{3}\,\mathrm{i}q\,\nabla_{\alpha}\nabla^{\alpha}A_{\mu}.
\end{aligned}
\tag{7.143}
$$

Neutral scalar field: When the gauge field vanishes, we expect to recover the results for a neutral scalar field in [68]. In this sense, setting $A_{\mu} = 0$ is a useful sanity check. We can introduce the notation $\widehat{U}^{(4)}(x,x')$ and $\widehat{V}^{(4)}(x,x')$ to denote the biscalar functions that we obtain from the Hadamard parametrix in four dimensions (6.5) when setting the scalar field charge to zero. Then, the biscalars $\widehat{U}^{(4)}(x,x')$ and $\widehat{V}^{(4)}(x,x')$ admit a power series expansion analogous to those in (6.6a) and (6.6b), given explicitly by

$$
\widehat{U}^{(4)}(x,x') = \widehat{U}^{(4)}_{0}(x,x')\,, \tag{7.144}
$$

$$
\widehat{V}^{(4)}(x,x') = \sum_{n=0}^{\infty} \widehat{V}^{(4)}_{n}(x,x')\,\sigma^{n}(x,x')\,, \tag{7.145}
$$

respectively, where the expansion coefficients $\widehat{U}^{(4)}_{n}(x,x')$ and $\widehat{V}^{(4)}_{n}(x,x')$ can be expanded as power series analogous to (6.10a) and (6.10b), explicitly given by

$$
\widehat{U}^{(4)}_{n}(x,x') = \sum_{j=0}^{\infty} \widehat{U}^{(4)}_{nj\alpha_1\ldots\alpha_j}(x)\,\sigma^{;\alpha_1}(x,x')\ldots\sigma^{;\alpha_j}(x,x')\,, \tag{7.146}
$$

$$
\widehat{V}^{(4)}_{n}(x,x') = \sum_{j=0}^{\infty} \widehat{V}^{(4)}_{nj\alpha_1\ldots\alpha_j}(x)\,\sigma^{;\alpha_1}(x,x')\ldots\sigma^{;\alpha_j}(x,x')\,. \tag{7.147}
$$

Similarly, we can introduce the notation $\widetilde{U}^{(4)}(x,x')$ and $\widetilde{V}^{(4)}(x,x')$ to denote the correction to the biscalar function $\widehat{U}^{(4)}(x,x')$ and $\widehat{V}^{(4)}(x,x')$ respectively as a result of the presence of the gauge field. Then the biscalars $\widetilde{U}^{(4)}(x,x')$ and $\widetilde{V}^{(4)}(x,x')$ admit power series expansions analogous to those in (6.6a) and (6.6b), given explicitly by

$$
\widetilde{V}^{(4)}(x,x') = \sum_{n=0}^{\infty} \widetilde{V}^{(4)}_{n}(x,x')\,\sigma^{n}(x,x')\,, \tag{7.148}
$$

where the expansion coefficients $\widetilde{U}_n^{(4)}(x,x')$ and $\widetilde{V}_n^{(4)}(x,x')$ can be expanded as power series analogous to (6.10a) and (6.10b), given explicitly by

$$\widetilde{U}_n^{(4)}(x,x') = \sum_{j=0}^{\infty} \widetilde{U}^{(4)}_{nj\alpha_1\ldots\alpha_j}(x)\,\sigma^{;\alpha_1}(x,x')\ldots\sigma^{;\alpha_j}(x,x')\,, \tag{7.149}$$

$$\widetilde{V}_n^{(4)}(x,x') = \sum_{j=0}^{\infty} \widetilde{V}^{(4)}_{nj\alpha_1\ldots\alpha_j}(x)\,\sigma^{;\alpha_1}(x,x')\ldots\sigma^{;\alpha_j}(x,x')\,. \tag{7.150}$$

In this language, the biscalar functions $U^{(4)}(x,x')$ and $V^{(4)}(x,x')$ given in the Hadamard parametrix (6.5) in four spacetime dimensions are given by

$$U^{(4)}(x,x') = \widehat{U}^{(4)}(x,x') + \widetilde{U}^{(4)}(x,x')\,, \tag{7.151}$$

$$V^{(4)}(x,x') = \widehat{V}^{(4)}(x,x') + \widetilde{V}^{(4)}(x,x')\,, \tag{7.152}$$

i.e. they are the sum of the biscalar functions $\widehat{U}^{(4)}(x,x')$ and $\widehat{V}^{(4)}(x,x')$ when the gauge field vanishes and the corrections due to the gauge field $\widetilde{U}^{(4)}(x,x')$ and $\widetilde{V}^{(4)}(x,x')$ respectively. Equations (7.151) hold because all equations concerning the biscalars $U^{(4)}(x,x')$ and $V^{(4)}(x,x')$ are linear. Then, using (6.6a), (7.144) and (7.148), we have

$$U_0^{(4)}(x,x') = \widehat{U}_0^{(4)}(x,x') + \widetilde{U}_0^{(4)}(x,x')\,. \tag{7.153}$$

Similarly, using (6.6b), (7.144) and (7.148), we have

$$\begin{aligned}\sum_{n=0}^{\infty} V_n^{(4)}(x,x')\,\sigma^n(x,x') &= \sum_{n=0}^{\infty} \widehat{V}_n^{(4)}(x,x')\,\sigma^n(x,x') + \sum_{n=0}^{\infty} \widetilde{V}_n^{(4)}(x,x')\,\sigma^n(x,x') \\ &= \sum_{n=0}^{\infty}\Big[\widehat{V}_n^{(4)}(x,x') + \widetilde{V}_n^{(4)}(x,x')\Big]\sigma^n(x,x')\end{aligned} \tag{7.154}$$

Since (7.154) should hold for each power of $\sigma(x,x')$, then we may write

$$V_n^{(4)}(x,x') = \widehat{V}_n^{(4)}(x,x') + \widetilde{V}_n^{(4)}(x,x')\,. \tag{7.155}$$

A similar line of reasoning shows that

$$\begin{aligned}U^{(4)}_{nj\alpha_1\ldots\alpha_j}(x) &= \widehat{U}^{(4)}_{nj\alpha_1\ldots\alpha_j}(x) + \widetilde{U}^{(4)}_{nj\alpha_1\ldots\alpha_j}(x)\,,\\ V^{(4)}_{nj\alpha_1\ldots\alpha_j}(x) &= \widehat{V}^{(4)}_{nj\alpha_1\ldots\alpha_j}(x) + \widetilde{V}^{(4)}_{nj\alpha_1\ldots\alpha_j}(x)\,.\end{aligned} \tag{7.156}$$

Armed with this formalism, we now proceed to calculate the $\widehat{V}^{(4)}_{01\mu}$ expansion coefficient of the Hadamard coefficient $\widehat{V}^{(4)}(x,x')$. Setting $A_\mu = 0$, (7.129) immediately reduces to

$$\begin{aligned}0 &= 4\,\widehat{V}^{(4)}_{01\mu} + 2\,V^{(4)}_{00\,;\mu} + \left(\Box - m^2 - \xi R\right)\widehat{U}^{(4)}_{01\mu} + \frac{1}{3}\,\widehat{U}^{(4)}_{01\lambda}R^{\lambda}{}_{\mu} + 2\,\nabla^{\alpha}\widehat{U}^{(4)}_{02\alpha\mu} + 2\,\nabla^{\alpha}\widehat{U}^{(4)}_{02\mu\alpha}\\ &\quad + 2\,g^{\alpha\beta}\,\widehat{U}^{(4)}_{03(\alpha\beta)\mu} + 2\,g^{\alpha\beta}\,\widehat{U}^{(4)}_{03(\alpha|\mu|\beta)} + 2\,g^{\alpha\beta}\,\widehat{U}^{(4)}_{03\mu(\alpha\beta)},\end{aligned} \tag{7.157}$$

Substituting in the explicit expressions for each term in (7.157) using equations (7.131) - (7.143), we have

$$\begin{aligned}0 &= 4\,\widehat{V}^{(4)}_{01\mu} + \xi R_{;\mu} - \frac{1}{6}R_{;\mu} + \frac{1}{6}R_{\alpha\mu}{}^{;\alpha} + \frac{1}{6}R_{\alpha\mu}{}^{;\alpha} - \frac{1}{12}R_{;\mu} - \frac{1}{12}R_{\alpha\mu}{}^{;\alpha} - \frac{1}{12}R_{\alpha\mu}{}^{;\mu}\\ &= 4\,\widehat{V}^{(4)}_{01\mu} + \xi R_{;\mu} - \frac{1}{4}R_{;\mu} + \frac{1}{6}R_{\alpha\mu}{}^{;\alpha}\\ &= 4\,\widehat{V}^{(4)}_{01\mu} + \left(\xi R_{;\mu} - \frac{1}{6}R_{;\mu} + \frac{1}{6}R_{\mu\nu}{}^{;\nu} - \frac{1}{12}R_{;\mu}\right),\end{aligned} \tag{7.158}$$

where we have split the factor of $-\frac{1}{4}R_{;\mu}$ in going from the penultimate line of (7.158) to the last line in anticipation of using the Bianchi identity (1.8). Then, (7.158) reduces to

$$\begin{aligned}0 &= 4\,\widehat{V}^{(4)}_{01\mu} + \left(\xi R_{;\mu} - \frac{1}{6}R_{;\mu} + \frac{1}{12}R_{;\mu} - \frac{1}{12}R_{;\mu}\right)\\ &= 4\,\widehat{V}^{(4)}_{01\mu} + \left(\xi R_{;\mu} - \frac{1}{6}R_{;\mu}\right).\end{aligned} \tag{7.159}$$

So finally we obtain

$$\widehat{V}^{(4)}_{01\mu} = -\frac{1}{4}\left(\xi - \frac{1}{6}\right)R_{;\mu}, \tag{7.160}$$

which agrees with the results for a neutral scalar field in [68].

Charged scalar field: We want to calculate the $\widetilde{V}^{(4)}_{01\mu}$ expansion coefficient of the Hadamard coefficient $\widetilde{V}^{(4)}(x,x')$. Ignoring terms that do not involve A_μ, (7.130) becomes

$$\begin{aligned}0 &= 4\,\widetilde{V}^{(4)}_{01\mu} - 2\,\mathrm{i}qA_\mu V^{(4)}_{00} + \left(D_\mu D^\mu - m^2 - \xi R\right)\widetilde{U}^{(4)}_{01\mu} + \frac{1}{3}\,\widetilde{U}^{(4)}_{01\lambda}R^\lambda{}_\mu + 2\,D^\alpha \widetilde{U}^{(4)}_{02\alpha\mu}\\ &\quad + 2\,D^\alpha \widetilde{U}^{(4)}_{02\mu\alpha} + 2\,g^{\alpha\beta}\,\widetilde{U}^{(4)}_{03(\alpha\beta)\mu} + 2\,g^{\alpha\beta}\,\widetilde{U}^{(4)}_{03(\alpha|\mu|\beta)} + 2\,g^{\alpha\beta}\,\widetilde{U}^{(4)}_{03\mu(\alpha\beta)}.\end{aligned} \tag{7.161}$$

Using (7.117), the second term in (7.161) is given by

$$\begin{aligned}-2\,\mathrm{i}qA_\mu V^{(4)}_{00} &= -\mathrm{i}qA_\mu\left[m^2 + \left(\xi - \frac{1}{6}\right)R\right]\\ &= -\mathrm{i}qA_\mu\left(m^2 + \xi R\right) + \frac{1}{6}\,\mathrm{i}qA_\mu R,\end{aligned} \tag{7.162}$$

Using (7.89), the terms proportional to $U^{(4)}_{01\mu}$ in (7.161) are given by

$$\left(D_\alpha D^\alpha - m^2 - \xi R\right)\widetilde{U}^{(4)}_{01\mu} = \mathrm{i}q\,D_\alpha D^\alpha A_\mu - \mathrm{i}qA_\mu\left(m^2 + \xi R\right), \tag{7.163}$$

$$\frac{1}{3}\widetilde{U}^{(4)}_{01\lambda}R^\lambda{}_\mu = \frac{1}{3}\,\mathrm{i}qA^\alpha R_{\alpha\mu}. \tag{7.164}$$

Using (7.90), the terms proportional to $U^{(4)}_{02(\mu\nu)}$ in (7.161) are given by

$$2\,D^{\alpha}\widetilde{U}^{(4)}_{02\alpha\mu} = -\frac{1}{6}\,\mathrm{i}qA^{\alpha}R_{\alpha\mu} - \mathrm{i}q\,D^{\alpha}D_{\alpha}A_{\mu}, \tag{7.165}$$

$$2\,D^{\alpha}\widetilde{U}^{(4)}_{02\mu\alpha} = -\frac{1}{6}\,\mathrm{i}qA^{\alpha}R_{\alpha\mu} - \mathrm{i}q\,D^{\alpha}D_{\mu}A_{\alpha}. \tag{7.166}$$

Using (7.102), the terms proportional to $U^{(4)}_{03(\mu\nu\rho)}$ in (7.161) are given by

$$\begin{aligned}
2\,g^{\alpha\beta}\,\widetilde{U}^{(4)}_{03(\alpha\beta)\mu} &= \frac{1}{3}\,\mathrm{i}q\,g^{\alpha\beta}D_{(\alpha}D_{\beta)}A_{\mu} + \frac{1}{6}\,\mathrm{i}q\,g^{\alpha\beta}A_{(\alpha}R_{\beta)\mu} \\
&= \frac{1}{3}\,\mathrm{i}q\,D_{\alpha}D^{\alpha}A_{\mu} + \frac{1}{6}\,\mathrm{i}qA^{\alpha}R_{\alpha\mu}, \qquad (7.167)\\
2\,g^{\alpha\beta}\,\widetilde{U}^{(4)}_{03(\alpha|\mu|\beta)} &= \frac{1}{3}\,\mathrm{i}q\,g^{\alpha\beta}D_{(\alpha}D_{|\mu|}A_{\beta)} + \frac{1}{6}\,\mathrm{i}q\,g^{\alpha\beta}A_{(\alpha}R_{|\mu|\beta)} \\
&= \frac{1}{3}\,\mathrm{i}q\,D^{\alpha}D_{\mu}A_{\alpha} + \frac{1}{6}\,\mathrm{i}qA^{\alpha}R_{\alpha\mu}, \qquad (7.168)\\
2\,g^{\alpha\beta}\,\widetilde{U}^{(4)}_{03\mu(\alpha\beta)} &= \frac{1}{3}\,\mathrm{i}q\,g^{\alpha\beta}D_{\mu}D_{(\alpha}A_{\beta)} + \frac{1}{6}\,\mathrm{i}q\,g^{\alpha\beta}A_{\mu}R_{\alpha\beta} \\
&= \frac{1}{3}\,\mathrm{i}q\,D_{\mu}D_{\alpha}A^{\alpha} + \frac{1}{6}\,\mathrm{i}qA_{\mu}R. \qquad (7.169)
\end{aligned}$$

Substituting in the explicit expressions for each term in (7.161) using equations (7.162) - (7.169), we have

$$\begin{aligned}
0 &= 4\,\widetilde{V}^{(4)}_{01\mu} - \mathrm{i}qA_{\mu}\big(m^2+\xi R\big) + \frac{1}{6}\,\mathrm{i}qA_{\mu}R + \mathrm{i}q\,D_{\alpha}D^{\alpha}A_{\mu} - \mathrm{i}qA_{\mu}\big(m^2+\xi R\big) + \frac{1}{3}\,\mathrm{i}qA^{\alpha}R_{\alpha\mu} \\
&\quad - \frac{1}{6}\,\mathrm{i}qA^{\alpha}R_{\alpha\mu} - \mathrm{i}q\,D^{\alpha}D_{\alpha}A_{\mu} - \frac{1}{6}\,\mathrm{i}qA^{\alpha}R_{\alpha\mu} - \mathrm{i}q\,D^{\alpha}D_{\mu}A_{\alpha} + \frac{1}{3}\,\mathrm{i}q\,D_{\alpha}D^{\alpha}A_{\mu} + \frac{1}{6}\,\mathrm{i}qA^{\alpha}R_{\alpha\mu} \\
&\quad + \frac{1}{3}\,\mathrm{i}q\,D^{\alpha}D_{\mu}A_{\alpha} + \frac{1}{6}\,\mathrm{i}qA^{\alpha}R_{\alpha\mu} + \frac{1}{3}\,\mathrm{i}q\,D_{\mu}D_{\alpha}A^{\alpha} + \frac{1}{6}\,\mathrm{i}qA_{\mu}R\,. \qquad (7.170)
\end{aligned}$$

Simplifying like terms in (7.170), we obtain

$$\begin{aligned}
0 &= 4\,\widetilde{V}^{(4)}_{01\mu} - 2\,\mathrm{i}qA_{\mu}\big(m^2+\xi R\big) + \frac{1}{3}\,\mathrm{i}qA_{\mu}R + \frac{1}{3}\,\mathrm{i}qA^{\alpha}R_{\alpha\mu} + \frac{1}{3}\,\mathrm{i}q\,D^{\alpha}D_{\alpha}A_{\mu} - \frac{2}{3}\,\mathrm{i}q\,D^{\alpha}D_{\mu}A_{\alpha} \\
&\quad + \frac{1}{3}\,\mathrm{i}q\,D_{\mu}D_{\alpha}A^{\alpha}. \qquad (7.171)
\end{aligned}$$

In order to simplify (7.171) further, we can rewrite (7.171) as

$$\begin{aligned}
0 &= 4\,\widetilde{V}^{(4)}_{01\mu} - 2\,\mathrm{i}qA_{\mu}\big(m^2+\xi R\big) + \frac{1}{3}\,\mathrm{i}qA_{\mu}R + \frac{1}{3}\,\mathrm{i}qA^{\alpha}R_{\alpha\mu} + \frac{1}{3}\,\mathrm{i}q\,D^{\alpha}D_{\alpha}A_{\mu} - \frac{1}{3}\,\mathrm{i}q\,D^{\alpha}D_{\mu}A_{\alpha} \\
&\quad + \frac{1}{3}\,\mathrm{i}q\,D_{\mu}D_{\alpha}A^{\alpha} - \frac{1}{3}\,\mathrm{i}q\,D_{\alpha}D_{\mu}A^{\alpha} \\
&= 4\,\widetilde{V}^{(4)}_{01\mu} - 2\,\mathrm{i}qA_{\mu}\big(m^2+\xi R\big) + \frac{1}{3}\,\mathrm{i}qA_{\mu}R + \frac{1}{3}\,\mathrm{i}qA^{\alpha}R_{\alpha\mu} + \frac{1}{3}\,\mathrm{i}q\,D^{\alpha}(D_{\alpha}A_{\mu} - D_{\mu}A_{\alpha}) \\
&\quad + \frac{1}{3}\,\mathrm{i}q\,[D_{\mu}\,, D_{\alpha}]\,A^{\alpha}. \qquad (7.172)
\end{aligned}$$

Using the expression for the commutator of two gauge covariant derivatives (6.28), equation (7.172) becomes

$$\begin{aligned}
0 &= 4\,\widetilde{V}^{(4)}_{01\mu} - 2\,\mathrm{i}qA_\mu\big(m^2+\xi R\big) + \frac{1}{3}\,\mathrm{i}qA_\mu R + \frac{1}{3}\,\mathrm{i}qA^\alpha R_{\alpha\mu} + \frac{1}{3}\,\mathrm{i}q\,D^\alpha F_{\alpha\mu} - \frac{1}{3}\,\mathrm{i}qA^\alpha R_{\alpha\mu} \\
&\quad + \frac{1}{3}\,q^2 A^\alpha F_{\mu\alpha} \\
&= 4\,\widetilde{V}^{(4)}_{01\mu} - 2\,\mathrm{i}qA_\mu\big(m^2+\xi R\big) + \frac{1}{3}\,\mathrm{i}qA_\mu R + \frac{1}{3}\,\mathrm{i}q\,\nabla^\alpha F_{\alpha\mu} + \frac{1}{3}\,q^2 A^\alpha F_{\alpha\mu} + \frac{1}{3}\,q^2 A^\alpha F_{\mu\alpha} \\
&= 4\,\widetilde{V}^{(4)}_{01\mu} - 2\,\mathrm{i}qA_\mu\big(m^2+\xi R\big) + \frac{1}{3}\,\mathrm{i}qA_\mu R + \frac{1}{3}\,\mathrm{i}q\,\nabla^\alpha F_{\alpha\mu},
\end{aligned} \tag{7.173}$$

where, in the last equality, we have used the fact that the electromagnetic field strength tensor $F_{\mu\nu}$ is antisymmetric. So finally, the first order correction $\widetilde{V}^{(4)}_{01\mu}$ is given by

$$\widetilde{V}^{(4)}_{01\mu} = \frac{1}{2}\,\mathrm{i}q\left[m^2 + \left(\xi - \frac{1}{6}\right)R\right]A_\mu - \frac{1}{12}\,\mathrm{i}q\,\nabla^\alpha F_{\alpha\mu}. \tag{7.174}$$

From (7.156) the $V^{(4)}_{01\mu}$ expansion coefficient is given by

$$V^{(4)}_{01\mu} = \widehat{V}^{(4)}_{01\mu} + \widetilde{V}^{(4)}_{01\mu}. \tag{7.175}$$

Then, using the expressions for $\widehat{V}^{(4)}_{01\mu}$ (7.160) and $\widetilde{V}^{(4)}_{01\mu}$ (7.174), we have

$$V^{(4)}_{01\mu} = -\frac{1}{4}\left(\xi - \frac{1}{6}\right)R_{;\mu} + \frac{1}{2}\,\mathrm{i}q\left[m^2 + \left(\xi - \frac{1}{6}\right)R\right]A_\mu - \frac{1}{12}\,\mathrm{i}q\,\nabla^\alpha F_{\alpha\mu}. \tag{7.176}$$

Now we will evaluate the $V^{(4)}_{02(\mu\nu)}$ expansion coefficient of the $V^{(4)}_0(x,x')$ Hadamard coefficient. Considering only those terms at $\mathcal{O}(\sigma)$ in (7.109), (7.112), (7.113) and (7.127), we can obtain the explicit expression for $V^{(4)}_{02(\mu\nu)}$; we have

$$\begin{aligned}
0 &= 2\,V^{(4)}_{02(\mu\nu)} + 2\,D_{(\mu}V^{(4)}_{01\nu)} + 4\,V^{(4)}_{02(\mu\nu)} - \frac{1}{3}\,V^{(4)}_{00}R_{\mu\nu} - \frac{1}{12}\,U^{(4)}_{01\lambda}\nabla_{(\mu}R^\lambda{}_{\nu)} - \frac{1}{4}\,U^{(4)}_{01\lambda}\nabla^\alpha R^\lambda{}_{(\mu|\alpha|\nu)} \\
&\quad - \frac{2}{3}\left(D^\alpha U^{(4)}_{01\lambda}\right)R^\lambda{}_{(\mu|\alpha|\nu)} + \big(D_\mu D^\mu - m^2 - \xi R\big)\,U^{(4)}_{02(\mu\nu)} + \frac{1}{3}\,U^{(4)}_{02\lambda(\mu}R^\lambda{}_{\nu)} + \frac{1}{3}\,U^{(4)}_{02(\mu|\lambda}R^\lambda{}_{\nu)} \\
&\quad - \frac{2}{3}\,U^{(4)}_{02\lambda\alpha}R^\lambda{}_{(\mu}{}^\alpha{}_{\nu)} - \frac{2}{3}\,U^{(4)}_{02\alpha\lambda}R^\lambda{}_{(\mu}{}^\alpha{}_{\nu)} + 6\,D^\alpha U^{(4)}_{03(\alpha\mu\nu)} + 12\,g^{\alpha\beta}\,U^{(4)}_{04(\alpha\beta\mu\nu)}.
\end{aligned} \tag{7.177}$$

Again, since our method will be to first recover the results for a neutral scalar field in [68], we will rewrite the gauge covariant derivatives in (7.177) in terms of ordinary spacetime derivatives and terms proportional to the gauge field A_μ. However, (7.177) also contains nontrivial contractions involving symmetrised indices; therefore, we will expand these terms appropriately. Then, (7.177) becomes

$$\begin{aligned}
0 &= 6\,V^{(4)}_{02(\mu\nu)} + 2\,V^{(4)}_{01(\mu;\nu)} - 2\,\mathrm{i}qA_{(\mu}V^{(4)}_{01\nu)} - \frac{1}{3}\,V^{(4)}_{00}R_{\mu\nu} - \frac{1}{12}\,U^{(4)}_{01\lambda}\nabla_{(\mu}R^\lambda{}_{\nu)} \\
&\quad - \frac{1}{4}\,U^{(4)}_{01\lambda}\nabla^\alpha R^\lambda{}_{(\mu|\alpha|\nu)} - \frac{2}{3}\left(\nabla^\alpha U^{(4)}_{01\lambda}\right)R^\lambda{}_{(\mu|\alpha|\nu)} + \frac{2}{3}\,\mathrm{i}qA^\alpha U^{(4)}_{01\lambda}R^\lambda{}_{(\mu|\alpha|\nu)} \\
&\quad + \big(\Box - m^2 - \xi R\big)\,U^{(4)}_{02(\mu\nu)} - 2\,\mathrm{i}qA^\alpha U^{(4)}_{02(\mu\nu);\alpha} - \mathrm{i}q\,(\nabla_\alpha A^\alpha)\,U^{(4)}_{02(\mu\nu)} - q^2 A_\alpha A^\alpha U^{(4)}_{02(\mu\nu)} \\
&\quad + \frac{1}{3}\,U^{(4)}_{02\lambda(\mu}R^\lambda{}_{\nu)} + \frac{1}{3}\,U^{(4)}_{02(\mu|\lambda}R^\lambda{}_{\nu)} - \frac{2}{3}\,U^{(4)}_{02\lambda\alpha}R^\lambda{}_{(\mu}{}^\alpha{}_{\nu)} - \frac{2}{3}\,U^{(4)}_{02\alpha\lambda}R^\lambda{}_{(\mu}{}^\alpha{}_{\nu)} + 2\,\nabla^\alpha U^{(4)}_{03\alpha(\mu\nu)} \\
&\quad + 2\,\nabla^\alpha U^{(4)}_{03(\mu|\alpha|\nu)} + 2\,\nabla^\alpha U^{(4)}_{03(\mu\nu)\alpha} - 6\,\mathrm{i}qA^\alpha U^{(4)}_{03(\alpha\mu\nu)} + 2\,g^{\alpha\beta}\,U^{(4)}_{04\alpha\beta(\mu\nu)} + 2\,g^{\alpha\beta}\,U^{(4)}_{04\alpha(\mu|\beta|\nu)} \\
&\quad + 2\,g^{\alpha\beta}\,U^{(4)}_{04\alpha(\mu\nu)\beta} + 2\,g^{\alpha\beta}\,U^{(4)}_{04(\mu|\alpha\beta|\nu)} + 2\,g^{\alpha\beta}\,U^{(4)}_{04(\mu|\alpha|\nu)\beta} + 2\,g^{\alpha\beta}\,U^{(4)}_{04(\mu\nu)\alpha\beta}.
\end{aligned} \tag{7.178}$$

Neutral scalar field: From (7.156), the $V^{(4)}_{02(\mu\nu)}$ expansion coefficient is given by

$$V^{(4)}_{02(\mu\nu)} = \widehat{V}^{(4)}_{02(\mu\nu)} + \widetilde{V}^{(4)}_{02(\mu\nu)}. \tag{7.179}$$

We begin by calculating $\widehat{V}^{(4)}_{02(\mu\nu)}$, which is the expression we would obtain for $V^{(4)}_{02(\mu\nu)}$ in the absence of the gauge field. It will be useful to write down the explicit expressions for $\widehat{U}^{(4)}_{00}$, $\widehat{U}^{(4)}_{01\mu}$, $\widehat{U}^{(4)}_{02(\mu\nu)}$, $\widehat{U}^{(4)}_{03(\mu\nu\rho)}$ and $\widehat{U}^{(4)}_{04(\mu\nu\rho\tau)}$. From (7.88), (7.102) and (7.106), they are given by

$$\begin{aligned}
\widehat{U}^{(4)}_{00} &= 1,\\
\widehat{U}^{(4)}_{01\mu} &= 0,\\
\widehat{U}^{(4)}_{02(\mu\nu)} &= \frac{1}{12}R_{\mu\nu},\\
\widehat{U}^{(4)}_{03(\mu\nu\rho)} &= -\frac{1}{24}R_{(\mu\nu;\rho)},\\
\widehat{U}^{(4)}_{04(\mu\nu\rho\tau)} &= \frac{1}{80}R_{(\mu\nu;\rho\tau)} + \frac{1}{288}R_{(\mu\nu}R_{\rho\tau)} + \frac{1}{360}R^{\lambda}{}_{(\mu|\beta|\nu}R^{\beta}{}_{\rho|\lambda|\tau)}.
\end{aligned} \tag{7.180}$$

It will also be useful to write down the explicit expressions for $\widehat{V}^{(4)}_{00}$ and $\widehat{V}^{(4)}_{01\mu}$. We also note that $\widehat{V}^{(4)}_{00} = V^{(4)}_{00}$ given in (7.117) and that $\widehat{V}^{(4)}_{01\mu}$ is given in (7.160). Then, setting $A_\mu = 0$ and using (7.180), equation (7.178) reduces to

$$\begin{aligned}
0 &= 6\,\widehat{V}^{(4)}_{02(\mu\nu)} + 2\,\widehat{V}^{(4)}_{01(\mu\,;\nu)} - \frac{1}{3}\,V^{(4)}_{00}R_{\mu\nu} + \left(\Box - m^2 - \xi R\right)\widehat{U}^{(4)}_{02(\mu\nu)} + \frac{1}{3}\,\widehat{U}^{(4)}_{02\lambda(\mu}R^{\lambda}{}_{\nu)}\\
&\quad + \frac{1}{3}\,\widehat{U}^{(4)}_{02(\mu|\lambda}R^{\lambda}{}_{\nu)} - \frac{2}{3}\,\widehat{U}^{(4)}_{02\lambda\alpha}R^{\lambda}{}_{(\mu}{}^{\alpha}{}_{\nu)} - \frac{2}{3}\,\widehat{U}^{(4)}_{02\alpha\lambda}R^{\lambda}{}_{(\mu}{}^{\alpha}{}_{\nu)} + 2\,\nabla^{\alpha}\widehat{U}^{(4)}_{03\alpha(\mu\nu)} + 2\,\nabla^{\alpha}\widehat{U}^{(4)}_{03(\mu|\alpha|\nu)}\\
&\quad + 2\,\nabla^{\alpha}\widehat{U}^{(4)}_{03(\mu\nu)\alpha} + 2\,g^{\alpha\beta}\,\widehat{U}^{(4)}_{04\alpha\beta(\mu\nu)} + 2\,g^{\alpha\beta}\,\widehat{U}^{(4)}_{04\alpha(\mu|\beta|\nu)} + 2\,g^{\alpha\beta}\,\widehat{U}^{(4)}_{04\alpha(\mu\nu)\beta} + 2\,g^{\alpha\beta}\,\widehat{U}^{(4)}_{04(\mu|\alpha\beta|\nu)}\\
&\quad + 2\,g^{\alpha\beta}\,\widehat{U}^{(4)}_{04(\mu|\alpha|\nu)\beta} + 2\,g^{\alpha\beta}\,\widehat{U}^{(4)}_{04(\mu\nu)\alpha\beta},
\end{aligned} \tag{7.181}$$

where we have used the fact that (7.107) should vanish at each power of σ. It will be instructive to evaluate each term in (7.181) individually before substituting into the equation. The first term in (7.181) is trivial. Using (7.160), the second term in (7.181) is:

$$\begin{aligned}
2\,V^{(4)}_{01(\mu\,;\nu)} &= -\frac{1}{4}\left(\xi - \frac{1}{6}\right)\nabla_{(\mu}R_{;\nu)}\\
&= -\frac{1}{4}\left(\xi - \frac{1}{6}\right)R_{;\mu\nu}.
\end{aligned} \tag{7.182}$$

Using (7.117), the third term in (7.181) is given by

$$-\frac{1}{3}\,V^{(4)}_{00}R_{\mu\nu} = -\frac{1}{6}\left[m^2 + \left(\xi - \frac{1}{6}\right)R\right]R_{\mu\nu}. \tag{7.183}$$

Using (7.180), the terms proportional to $\widehat{U}^{(4)}_{02(\mu\nu)}$ in (7.181) are given by

$$\left(\Box - m^2 - \xi R\right) \widehat{U}^{(4)}_{02(\mu\nu)} = \frac{1}{12} \left(\Box - m^2 - \xi R\right) R_{\mu\nu}, \tag{7.184}$$

$$\begin{aligned}\frac{1}{3} \widehat{U}^{(4)}_{02\lambda(\mu} R^{\lambda}{}_{\nu)} + \frac{1}{3} \widehat{U}^{(4)}_{02(\mu|\lambda|} R^{\lambda}{}_{\nu)} &= \frac{1}{6} \widehat{U}^{(4)}_{02\lambda\mu} R^{\lambda}{}_{\nu} + \frac{1}{6} \widehat{U}^{(4)}_{02\lambda\nu} R^{\lambda}{}_{\mu} + \frac{1}{6} \widehat{U}^{(4)}_{02\mu\lambda} R^{\lambda}{}_{\nu} + \frac{1}{6} \widehat{U}^{(4)}_{02\nu\lambda} R^{\lambda}{}_{\mu} \\ &= \frac{1}{72} R_{\lambda\mu} R^{\lambda}{}_{\nu} + \frac{1}{72} R_{\lambda\nu} R^{\lambda}{}_{\mu} + \frac{1}{72} R_{\mu\lambda} R^{\lambda}{}_{\nu} + \frac{1}{72} R_{\nu\lambda} R^{\lambda}{}_{\mu} \\ &= \frac{1}{18} R_{\lambda\mu} R^{\lambda}{}_{\nu},\end{aligned} \tag{7.185}$$

as well as

$$\begin{aligned}&-\frac{2}{3} \widehat{U}^{(4)}_{02\lambda\alpha} R^{\lambda}{}_{(\mu}{}^{\alpha}{}_{\nu)} - \frac{2}{3} \widehat{U}^{(4)}_{02\alpha\lambda} R^{\lambda}{}_{(\mu}{}^{\alpha}{}_{\nu)} \\ &\quad = -\frac{1}{3} \widehat{U}^{(4)}_{02\lambda\alpha} \left(R^{\lambda}{}_{\mu}{}^{\alpha}{}_{\nu} + R^{\lambda}{}_{\nu}{}^{\alpha}{}_{\mu}\right) - \frac{1}{3} \widehat{U}^{(4)}_{02\alpha\lambda} \left(R^{\lambda}{}_{\mu}{}^{\alpha}{}_{\nu} + R^{\lambda}{}_{\nu}{}^{\alpha}{}_{\mu}\right) \\ &\quad = -\frac{1}{36} R^{\lambda\alpha} R_{\lambda\mu\alpha\nu} - \frac{1}{36} R^{\lambda\alpha} R_{\alpha\mu\lambda\nu} - \frac{1}{36} R^{\lambda\alpha} R_{\lambda\mu\alpha\nu} - \frac{1}{36} R^{\alpha\lambda} R_{\alpha\mu\lambda\nu} \\ &\quad = -\frac{1}{9} R^{\lambda\alpha} R_{\lambda\mu\alpha\nu}.\end{aligned} \tag{7.186}$$

Using (7.180), the terms proportional to $\widehat{U}^{(4)}_{03(\mu\nu\rho)}$ in (7.181) are given by

$$2\,\nabla^{\alpha} \widehat{U}^{(4)}_{03\alpha(\mu\nu)} = -\frac{1}{12} R_{\lambda(\mu\,;\nu)}{}^{\lambda} = -\frac{1}{24} R_{\lambda\mu\,;\nu}{}^{\lambda} - \frac{1}{24} R_{\lambda\nu\,;\mu}{}^{\lambda}, \tag{7.187}$$

$$2\,\nabla^{\alpha} \widehat{U}^{(4)}_{03(\mu|\alpha|\nu)} = -\frac{1}{12} R_{(\mu|\lambda|\,;\nu)}{}^{\lambda} = -\frac{1}{24} R_{\mu\lambda\,;\nu}{}^{\lambda} - \frac{1}{24} R_{\nu\lambda\,;\mu}{}^{\lambda}, \tag{7.188}$$

$$2\,\nabla^{\alpha} \widehat{U}^{(4)}_{03(\mu\nu)\alpha} = -\frac{1}{12} R_{\mu\nu\,;\lambda}{}^{\lambda} = -\frac{1}{12} \Box R_{\mu\nu}. \tag{7.189}$$

Using (7.180), the terms proportional to $\widehat{U}^{(4)}_{04(\mu\nu\rho\tau)}$ in (7.181) are given by

$$\begin{aligned}2\, g^{\alpha\beta}\, \widehat{U}^{(4)}_{04\alpha\beta(\mu\nu)} &= \frac{1}{80} \left(R^{\lambda}{}_{\lambda\,;\mu\nu} + R^{\lambda}{}_{\lambda\,;\nu\mu}\right) + \frac{1}{288} \left(R^{\lambda}{}_{\lambda} R_{\mu\nu} + R^{\lambda}{}_{\lambda} R_{\nu\mu}\right) \\ &\quad + \frac{1}{360} \left(R^{\alpha\lambda}{}_{\beta\lambda} R^{\beta}{}_{\mu\alpha\nu} + R^{\alpha\lambda}{}_{\beta\lambda} R^{\beta}{}_{\nu\alpha\mu}\right) \\ &= \frac{1}{40} R_{;\mu\nu} + \frac{1}{144} R R_{\mu\nu} + \frac{1}{360} \left(R^{\alpha}{}_{\beta} R^{\beta}{}_{\mu\alpha\nu} + R^{\alpha}{}_{\beta} R^{\beta}{}_{\nu\alpha\mu}\right) \\ &= \frac{1}{40} R_{;\mu\nu} + \frac{1}{144} R R_{\mu\nu} + \frac{1}{180} R^{\alpha\beta} R_{\alpha\mu\beta\nu},\end{aligned} \tag{7.190}$$

$$\begin{aligned}2\, g^{\alpha\beta}\, \widehat{U}^{(4)}_{04\alpha(\mu|\beta|\nu)} &= \frac{1}{80} \left(R^{\lambda}{}_{\mu\,;\lambda\nu} + R^{\lambda}{}_{\nu\,;\lambda\mu}\right) + \frac{1}{288} \left(R^{\lambda}{}_{\mu} R_{\lambda\nu} + R^{\lambda}{}_{\nu} R_{\lambda\mu}\right) \\ &\quad + \frac{1}{360} \left(R^{\alpha\lambda}{}_{\beta\mu} R^{\beta}{}_{\lambda\alpha\nu} + R^{\alpha\lambda}{}_{\beta\nu} R^{\beta}{}_{\lambda\alpha\mu}\right) \\ &= \frac{1}{80} \left(R^{\lambda}{}_{\mu\,;\lambda\nu} + R^{\lambda}{}_{\nu\,;\lambda\mu}\right) + \frac{1}{144} R^{\lambda}{}_{\mu} R_{\lambda\nu} \\ &\quad + \frac{1}{360} \left(R^{\alpha\lambda\beta}{}_{\mu} R_{\beta\lambda\alpha\nu} + R^{\beta\lambda\alpha}{}_{\mu} R_{\alpha\lambda\beta\nu}\right) \\ &= \frac{1}{80} \left(R^{\lambda}{}_{\mu\,;\lambda\nu} + R^{\lambda}{}_{\nu\,;\lambda\mu}\right) + \frac{1}{144} R^{\lambda}{}_{\mu} R_{\lambda\nu} + \frac{1}{360} R^{\alpha\lambda\beta}{}_{\mu} R_{\alpha\lambda\beta\nu},\end{aligned} \tag{7.191}$$

$$
\begin{aligned}
2\, g^{\alpha\beta}\, \widehat{U}^{(4)}_{04\alpha(\mu\nu)\beta} &= \frac{1}{80}\left(R^{\lambda}{}_{\mu\,;\nu\lambda} + R^{\lambda}{}_{\nu\,;\mu\lambda}\right) + \frac{1}{288}\left(R^{\lambda}{}_{\mu}R_{\nu\lambda} + R^{\lambda}{}_{\nu}R_{\mu\lambda}\right) \\
&+ \frac{1}{360}\left(R^{\alpha\lambda}{}_{\beta\mu}R^{\beta}{}_{\nu\alpha\lambda} + R^{\alpha\lambda}{}_{\beta\nu}R^{\beta}{}_{\mu\alpha\lambda}\right) \\
&= \frac{1}{80}\left(R^{\lambda}{}_{\mu\,;\nu\lambda} + R^{\lambda}{}_{\nu\,;\mu\lambda}\right) + \frac{1}{144}R^{\lambda}{}_{\mu}R_{\lambda\nu} \\
&+ \frac{1}{360}\left(R^{\alpha\lambda}{}_{\beta\mu}R^{\beta}{}_{\nu\alpha\lambda} + R^{\alpha\lambda}{}_{\beta\nu}R^{\beta}{}_{\mu\alpha\lambda}\right) \\
&= \frac{1}{80}\left(R^{\lambda}{}_{\mu\,;\nu\lambda} + R^{\lambda}{}_{\nu\,;\mu\lambda}\right) + \frac{1}{144}R^{\lambda}{}_{\mu}R_{\lambda\nu} + \frac{1}{180}R^{\alpha\lambda\beta}{}_{\mu}R_{\alpha\lambda\beta\nu}, \qquad (7.192)
\end{aligned}
$$

$$
\begin{aligned}
2\, g^{\alpha\beta}\, \widehat{U}^{(4)}_{04(\mu|\alpha\beta|\nu)} &= \frac{1}{80}\left(R_{\mu}{}^{\lambda}{}_{;\lambda\nu} + R_{\nu}{}^{\lambda}{}_{;\lambda\mu}\right) + \frac{1}{288}(R_{\mu}{}^{\lambda}R_{\lambda\nu} + R_{\nu}{}^{\lambda}R_{\lambda\mu}) \\
&+ \frac{1}{360}\left(R^{\alpha}{}_{\mu\beta}{}^{\lambda}R^{\beta}{}_{\lambda\alpha\nu} + R^{\alpha}{}_{\nu\beta}{}^{\lambda}R^{\beta}{}_{\lambda\alpha\mu}\right) \\
&= \frac{1}{80}\left(R_{\mu}{}^{\lambda}{}_{;\lambda\nu} + R_{\nu}{}^{\lambda}{}_{;\lambda\mu}\right) + \frac{1}{144}R^{\lambda}{}_{\mu}R_{\lambda\nu} \\
&+ \frac{1}{360}\left(R^{\alpha\lambda\beta}{}_{\mu}R_{\alpha\lambda\beta\nu} + R^{\alpha\lambda\beta}{}_{\mu}R_{\alpha\lambda\beta\nu}\right) \\
&= \frac{1}{80}\left(R_{\mu}{}^{\lambda}{}_{;\lambda\nu} + R_{\nu}{}^{\lambda}{}_{;\lambda\mu}\right) + \frac{1}{144}R^{\lambda}{}_{\mu}R_{\lambda\nu} + \frac{1}{180}R^{\alpha\lambda\beta}{}_{\mu}R_{\alpha\lambda\beta\nu}, \qquad (7.193)
\end{aligned}
$$

$$
\begin{aligned}
2\, g^{\alpha\beta}\, \widehat{U}^{(4)}_{04(\mu|\alpha|\nu)\beta} &= \frac{1}{80}\left(R_{\mu}{}^{\lambda}{}_{;\nu\lambda} + R_{\nu}{}^{\lambda}{}_{;\mu\lambda}\right) + \frac{1}{288}\left(R_{\mu}{}^{\lambda}R_{\nu\lambda} + R_{\nu}{}^{\lambda}R_{\mu\lambda}\right) \\
&+ \frac{1}{360}\left(R^{\alpha}{}_{\mu\beta}{}^{\lambda}R^{\beta}{}_{\nu\alpha\lambda} + R^{\alpha}{}_{\nu\beta}{}^{\lambda}R^{\beta}{}_{\mu\alpha\lambda}\right) \\
&= \frac{1}{80}\left(R_{\mu}{}^{\lambda}{}_{;\nu\lambda} + R_{\nu}{}^{\lambda}{}_{;\mu\lambda}\right) + \frac{1}{144}R^{\lambda}{}_{\mu}R_{\lambda\nu} \\
&+ \frac{1}{360}\left(\frac{1}{2}R^{\alpha\lambda\beta}{}_{\mu}R_{\alpha\lambda\beta\nu} + \frac{1}{2}R^{\alpha\lambda\beta}{}_{\mu}R_{\alpha\lambda\beta\nu}\right) \\
&= \frac{1}{80}\left(R_{\mu}{}^{\lambda}{}_{;\nu\lambda} + R_{\nu}{}^{\lambda}{}_{;\mu\lambda}\right) + \frac{1}{144}R^{\lambda}{}_{\mu}R_{\lambda\nu} + \frac{1}{360}R^{\alpha\lambda\beta}{}_{\mu}R_{\alpha\lambda\beta\nu}, \qquad (7.194)
\end{aligned}
$$

$$
\begin{aligned}
2\, g^{\alpha\beta}\, \widehat{U}^{(4)}_{04(\mu\nu)\alpha\beta} &= \frac{1}{80}\left(R_{\mu\nu\,;\lambda}{}^{\lambda} + R_{\nu\mu\,;\lambda}{}^{\lambda}\right) + \frac{1}{288}\left(R_{\mu\nu}R^{\lambda}{}_{\lambda} + R_{\nu\mu}R^{\lambda}{}_{\lambda}\right) \\
&+ \frac{1}{360}\left(R^{\alpha}{}_{\mu\beta\nu}R^{\beta\lambda}{}_{\alpha\lambda} + R^{\alpha}{}_{\nu\beta\mu}R^{\beta\lambda}{}_{\alpha\lambda}\right) \\
&= \frac{1}{40}R_{\mu\nu\,;\lambda}{}^{\lambda} + \frac{1}{144}RR_{\mu\nu} + \frac{1}{360}\left(R^{\alpha}{}_{\mu\beta\nu}R^{\beta}{}_{\alpha} + R^{\alpha}{}_{\nu\beta\mu}R^{\beta}{}_{\alpha}\right) \\
&= \frac{1}{40}R_{\mu\nu\,;\lambda}{}^{\lambda} + \frac{1}{144}RR_{\mu\nu} + \frac{1}{180}R^{\alpha\beta}R_{\alpha\mu\beta\nu}, \qquad (7.195)
\end{aligned}
$$

where we have used the symmetries of the Riemann tensor (1.5) to simplify (7.190 –7.195). Substituting in the explicit expressions for each term in (7.181) using equations (7.182) - (7.195) and simplifying like terms, we have

$$0 = 6\,\widehat{V}^{(4)}_{02(\mu\nu)} + \frac{13}{120}R_{;\mu\nu} - \frac{1}{2}\,\xi R_{;\mu\nu} - \frac{1}{4}\,m^2 R_{\mu\nu} - \frac{1}{4}\,\xi R R_{\mu\nu} + \frac{1}{24}RR_{\mu\nu} + \frac{1}{40}\Box R_{\mu\nu} - \frac{1}{10}R^{\alpha\beta}R_{\alpha\mu\beta\nu} + \frac{1}{12}R^{\alpha}{}_{\mu}R_{\alpha\nu} - \frac{7}{120}R^{\lambda}{}_{\mu\,;\nu\lambda} - \frac{7}{120}R^{\lambda}{}_{\nu\,;\mu\lambda} + \frac{1}{40}R^{\lambda}{}_{\mu\,;\lambda\nu} + \frac{1}{40}R^{\lambda}{}_{\nu\,;\lambda\mu} + \frac{1}{60}R^{\alpha\lambda\beta}{}_{\mu}R_{\alpha\lambda\beta\nu}. \tag{7.196}$$

We can simplify (7.196) by combining terms containing the Ricci tensor acted on by two covariant derivatives. Using the expression (1.10) for the commutator of two covariant derivative operators, we have for the aforementioned terms in (7.196) the expressions

$$\begin{aligned} R^{\lambda}{}_{\mu\,;\nu\lambda} = \nabla_\lambda\nabla_\nu R^{\lambda}{}_{\mu} &= R^{\alpha}{}_{\mu}R_{\alpha\nu} - R^{\alpha\beta}R_{\alpha\mu\beta\nu} + \nabla_\nu\nabla_\lambda R^{\lambda}{}_{\mu} \\ &= R^{\alpha}{}_{\mu}R_{\alpha\nu} - R^{\alpha\beta}R_{\alpha\mu\beta\nu} + R^{\lambda}{}_{\mu\,;\lambda\nu}, \end{aligned} \tag{7.197}$$

$$\begin{aligned} R^{\lambda}{}_{\nu\,;\mu\lambda} = \nabla_\lambda\nabla_\mu R^{\lambda}{}_{\nu} &= R^{\alpha}{}_{\nu}R_{\alpha\mu} - R^{\alpha\beta}R_{\alpha\nu\beta\mu} + \nabla_\mu\nabla_\lambda R^{\lambda}{}_{\nu} \\ &= R^{\alpha}{}_{\mu}R_{\alpha\nu} - R^{\alpha\beta}R_{\alpha\mu\beta\nu} + R^{\lambda}{}_{\nu\,;\lambda\mu}. \end{aligned} \tag{7.198}$$

Using (7.197) and (7.198), equation (7.196) becomes

$$0 = 6\,\widehat{V}^{(4)}_{02(\mu\nu)} + \frac{13}{120}R_{;\mu\nu} - \frac{1}{2}\,\xi R_{;\mu\nu} - \frac{1}{4}\,m^2 R_{\mu\nu} - \frac{1}{4}\,\xi R R_{\mu\nu} + \frac{1}{24}RR_{\mu\nu} + \frac{1}{40}\Box R_{\mu\nu} - \frac{1}{30}R^{\lambda}{}_{\mu\,;\lambda\nu} - \frac{1}{30}R^{\lambda}{}_{\nu\,;\lambda\mu} - \frac{1}{30}R^{\alpha}{}_{\mu}R_{\alpha\nu} + \frac{1}{60}R^{\alpha\beta}R_{\alpha\mu\beta\nu} + \frac{1}{60}R^{\alpha\lambda\beta}{}_{\mu}R_{\alpha\lambda\beta\nu}. \tag{7.199}$$

We can simplify (7.199) by using the covariant derivative of the Bianchi identity (1.8):

$$R^{\lambda}{}_{\nu\,;\lambda\mu} = \frac{1}{2}R_{;\mu\nu}. \tag{7.200}$$

Using the fact that covariant derivatives acting on a scalar commute, we also have

$$R^{\lambda}{}_{\mu\,;\lambda\nu} = \frac{1}{2}R_{;\mu\nu}. \tag{7.201}$$

Using (7.200) and (7.201), equation (7.199) simplifies to

$$\begin{aligned} 0 &= 6\,\widehat{V}^{(4)}_{02(\mu\nu)} + \frac{13}{120}R_{;\mu\nu} - \frac{1}{2}\,\xi R_{;\mu\nu} - \frac{1}{4}\,m^2 R_{\mu\nu} - \frac{1}{4}\,\xi R R_{\mu\nu} + \frac{1}{24}RR_{\mu\nu} + \frac{1}{40}\Box R_{\mu\nu} \\ &\quad - \frac{1}{60}R_{;\mu\nu} - \frac{1}{60}R_{;\mu\nu} - \frac{1}{30}R^{\alpha}{}_{\mu}R_{\alpha\nu} + \frac{1}{60}R^{\alpha\beta}R_{\alpha\mu\beta\nu} + \frac{1}{60}R^{\alpha\lambda\beta}{}_{\mu}R_{\alpha\lambda\beta\nu} \\ &= 6\,\widehat{V}^{(4)}_{02(\mu\nu)} + \frac{3}{40}R_{;\mu\nu} - \frac{1}{2}\,\xi R_{;\mu\nu} - \frac{1}{4}\,m^2 R_{\mu\nu} - \frac{1}{4}\,\xi R R_{\mu\nu} + \frac{1}{24}RR_{\mu\nu} + \frac{1}{40}\Box R_{\mu\nu} \\ &\quad - \frac{1}{30}R^{\alpha}{}_{\mu}R_{\alpha\nu} + \frac{1}{60}R^{\alpha\beta}R_{\alpha\mu\beta\nu} + \frac{1}{60}R^{\alpha\lambda\beta}{}_{\mu}R_{\alpha\lambda\beta\nu}. \end{aligned} \tag{7.202}$$

So finally, we obtain for the $\widehat{V}^{(4)}_{02(\mu\nu)}$ expansion coefficient

$$\widehat{V}^{(4)}_{02(\mu\nu)} = \frac{1}{24}\,m^2 R_{\mu\nu} + \frac{1}{12}\left(\xi - \frac{3}{20}\right)R_{;\mu\nu} - \frac{1}{240}\Box R_{\mu\nu} + \frac{1}{24}\left(\xi - \frac{1}{6}\right)RR_{\mu\nu} + \frac{1}{180}R^{\alpha}{}_{\mu}R_{\alpha\nu} - \frac{1}{360}R^{\alpha\beta}R_{\alpha\mu\beta\nu} - \frac{1}{360}R^{\alpha\lambda\beta}{}_{\mu}R_{\alpha\lambda\beta\nu}, \tag{7.203}$$

which agrees with the results for a neutral scalar field in [68].

Charged scalar field: Having already calculated the correct expression when the gauge field vanishes, we can ignore terms that do not involve A_μ and instead calculate the correction $\widetilde{V}^{(4)}_{02(\mu\nu)}$ due to the gauge field. It will be useful to write down the expressions for $\widetilde{U}^{(4)}_{00}$ (7.88), $\widetilde{U}^{(4)}_{01\mu}$ (7.89), $\widetilde{U}^{(4)}_{02(\mu\nu)}$ (7.90), $\widetilde{U}^{(4)}_{03(\mu\nu\rho)}$ (7.102) and $\widetilde{U}^{(4)}_{04(\mu\nu\rho\tau)}$ (7.106) as

$$
\begin{aligned}
\widetilde{U}^{(4)}_{00} &= 0, \\
\widetilde{U}^{(4)}_{01\mu} &= \mathrm{i}qA_\mu, \\
\widetilde{U}^{(4)}_{02(\mu\nu)} &= -\frac{1}{2}\,\mathrm{i}qD_{(\mu}A_{\nu)}, \\
\widetilde{U}^{(4)}_{03(\mu\nu\rho)} &= \frac{1}{6}\,\mathrm{i}qD_{(\mu}D_\nu A_{\rho)} + \frac{1}{12}\,\mathrm{i}qA_{(\mu}R_{\nu\rho)}, \\
\widetilde{U}^{(4)}_{04(\mu\nu\rho\tau)} &= -\frac{1}{24}\,\mathrm{i}qD_{(\mu}D_\nu D_\rho A_{\tau)} - \frac{1}{24}\,\mathrm{i}qD_{(\mu}\big[A_\nu R_{\rho\tau)}\big]\,.
\end{aligned}
\tag{7.204}
$$

It will also be useful to write down the expressions for $\widetilde{V}^{(4)}_{00}$ (7.117) and $\widetilde{V}^{(4)}_{01\mu}$ (7.174) as

$$
\begin{aligned}
\widetilde{V}^{(4)}_{00} &= 0, \\
\widetilde{V}^{(4)}_{01\mu} &= \frac{1}{2}\,\mathrm{i}q\left[m^2 + \left(\xi - \frac{1}{6}\right)R\right]A_\mu - \frac{1}{12}\,\mathrm{i}q\,\nabla^\alpha F_{\alpha\mu}.
\end{aligned}
\tag{7.205}
$$

Then, equation (7.178) becomes

$$
\begin{aligned}
0 = {} & 6\,\widetilde{V}^{(4)}_{02(\mu\nu)} + 2\,D_{(\mu}\widetilde{V}^{(4)}_{01\nu)} - 2\,\mathrm{i}qA_{(\mu}\widehat{V}^{(4)}_{01\nu)} - \frac{1}{12}\,\widetilde{U}^{(4)}_{01\lambda}\nabla_{(\mu}R^\lambda{}_{\nu)} - \frac{1}{4}\,\widetilde{U}^{(4)}_{01\lambda}\nabla^\alpha R^\lambda{}_{(\mu|\alpha|\nu)} \\
& - \frac{2}{3}\left[D^\alpha\widetilde{U}^{(4)}_{01\lambda}\right]R^\lambda{}_{(\mu|\alpha|\nu)} + \left(D_\alpha D^\alpha - m^2 - \xi R\right)\widetilde{U}^{(4)}_{02(\mu\nu)} - 2\,\mathrm{i}qA^\alpha\nabla_\alpha\widehat{U}^{(4)}_{02(\mu\nu)} \\
& - \mathrm{i}q\left(\nabla_\alpha A^\alpha\right)\widehat{U}^{(4)}_{02(\mu\nu)} - q^2A_\alpha A^\alpha\widehat{U}^{(4)}_{02(\mu\nu)} + \frac{1}{3}\,\widetilde{U}^{(4)}_{02\lambda(\mu}R^\lambda{}_{\nu)} + \frac{1}{3}\,\widetilde{U}^{(4)}_{02(\mu|\lambda|}R^\lambda{}_{\nu)} - \frac{2}{3}\,\widetilde{U}^{(4)}_{02\lambda\alpha}R^\lambda{}_{(\mu}{}^\alpha{}_{\nu)} \\
& - \frac{2}{3}\,\widetilde{U}^{(4)}_{02\alpha\lambda}R^\lambda{}_{(\mu}{}^\alpha{}_{\nu)} + 2\,D^\alpha\widetilde{U}^{(4)}_{03\alpha(\mu\nu)} + 2\,D^\alpha\widetilde{U}^{(4)}_{03(\mu|\alpha|\nu)} + 2\,D^\alpha\widetilde{U}^{(4)}_{03(\mu\nu)\alpha} - 6\,\mathrm{i}qA^\alpha\widehat{U}^{(4)}_{03(\alpha\mu\nu)} \\
& + 2\,g^{\alpha\beta}\,\widetilde{U}^{(4)}_{04\alpha\beta(\mu\nu)} + 2\,g^{\alpha\beta}\,\widetilde{U}^{(4)}_{04\alpha(\mu|\beta|\nu)} + 2\,g^{\alpha\beta}\,\widetilde{U}^{(4)}_{04\alpha(\mu\nu)\beta} + 2\,g^{\alpha\beta}\,\widetilde{U}^{(4)}_{04(\mu|\alpha\beta|\nu)} \\
& + 2\,g^{\alpha\beta}\,\widetilde{U}^{(4)}_{04(\mu|\alpha|\nu)\beta} + 2\,g^{\alpha\beta}\,\widetilde{U}^{(4)}_{04(\mu\nu)\alpha\beta}.
\end{aligned}
\tag{7.206}
$$

where we have used the fact that (7.107) should vanish at each power of σ. It is instructive to evaluate each term in (7.206) individually. The first term on the r.h.s is trivial. Using (7.174), the second term in (7.206) is given by

$$
\begin{aligned}
2\,D_{(\mu}\widetilde{V}^{(4)}_{01\nu)} &= \mathrm{i}q\left[m^2 + \left(\xi - \frac{1}{6}\right)R\right]A_{(\mu;\nu)} + \mathrm{i}q\left(\xi - \frac{1}{6}\right)R_{;(\mu}A_{\nu)} \\
&\quad + q^2\left[m^2 + \left(\xi - \frac{1}{6}\right)R\right]A_\mu A_\nu - \frac{1}{6}\,\mathrm{i}qD_{(\mu}\nabla^\alpha F_{|\alpha|\nu)} \\
&= \mathrm{i}q\left[m^2 + \left(\xi - \frac{1}{6}\right)R\right]D_{(\mu}A_{\nu)} + \mathrm{i}q\left(\xi - \frac{1}{6}\right)R_{;(\mu}A_{\nu)} - \frac{1}{6}\,\mathrm{i}qD_{(\mu}\nabla^\alpha F_{|\alpha|\nu)}
\end{aligned}
\tag{7.207}
$$

Using (7.160), the third term in (7.206) is given by

$$-2\,\mathrm{i}qA_{(\mu}\widehat{V}^{(4)}_{01\nu)} = \frac{1}{2}\,\mathrm{i}q\left(\xi-\frac{1}{6}\right)R_{;(\mu}A_{\nu)}. \tag{7.208}$$

Using (7.89), the terms proportional to $\widetilde{U}^{(4)}_{01\mu}$ in (7.206) are given by

$$-\frac{1}{12}\,\widetilde{U}^{(4)}_{01\lambda}\nabla_{(\mu}R^{\lambda}{}_{\nu)} = -\frac{1}{12}\,\mathrm{i}qA_{\lambda}\nabla_{(\mu}R^{\lambda}{}_{\nu)}, \tag{7.209}$$

$$-\frac{1}{4}\,\widetilde{U}^{(4)}_{01\lambda}\nabla^{\alpha}R^{\lambda}{}_{(\mu|\alpha|\nu)} = -\frac{1}{4}\,\mathrm{i}qA_{\lambda}\nabla^{\alpha}R^{\lambda}{}_{(\mu|\alpha|\nu)}, \tag{7.210}$$

$$-\frac{2}{3}\left[D^{\alpha}\widetilde{U}^{(4)}_{01\lambda}\right]R^{\lambda}{}_{(\mu|\alpha|\nu)} = -\frac{2}{3}\,\mathrm{i}qR^{\lambda}{}_{(\mu|\alpha|\nu)}\,D^{\alpha}A_{\lambda}. \tag{7.211}$$

Using (7.204), the term proportional to $\widetilde{U}^{(4)}_{02(\mu\nu)}$ in (7.206) is given by

$$\left(D_{\alpha}D^{\alpha}-m^2-\xi R\right)\widetilde{U}^{(4)}_{02(\mu\nu)} = -\frac{1}{2}\,\mathrm{i}qD_{\alpha}D^{\alpha}D_{(\mu}A_{\nu)} + \frac{1}{2}\,\mathrm{i}q\left(m^2+\xi R\right)D_{(\mu}A_{\nu)}. \tag{7.212}$$

Using (7.180), the terms proportional to $\widehat{U}^{(4)}_{02(\mu\nu)}$ in (7.206) are given by

$$-2\,\mathrm{i}qA^{\alpha}\nabla_{\alpha}\widehat{U}^{(4)}_{02(\mu\nu)} = -\frac{1}{6}\,\mathrm{i}qA^{\alpha}\nabla_{\alpha}R_{\mu\nu}, \tag{7.213}$$

$$-\mathrm{i}q\left(\nabla_{\alpha}A^{\alpha}\right)\widehat{U}^{(4)}_{02(\mu\nu)} = -\frac{1}{12}\,\mathrm{i}q\left(\nabla_{\alpha}A^{\alpha}\right)R_{\mu\nu}, \tag{7.214}$$

$$-q^2A_{\alpha}A^{\alpha}\widehat{U}^{(4)}_{02(\mu\nu)} = -\frac{1}{12}\,q^2A_{\alpha}A^{\alpha}R_{\mu\nu}. \tag{7.215}$$

For the following terms, we must be careful to use the unsymmetrised version of $\widetilde{U}^{(4)}_{02(\mu\nu)}$, i.e. $\widetilde{U}^{(4)}_{02\mu\nu}$, since we have already expanded out the symmetrisation in these terms. Then, using (7.204), the terms proportional to $\widetilde{U}^{(4)}_{02\mu\nu}$ in (7.206) are given by

$$\frac{1}{3}\,\widetilde{U}^{(4)}_{02\lambda(\mu}R^{\lambda}{}_{\nu)} = -\frac{1}{6}\,\mathrm{i}qR^{\lambda}{}_{(\mu}D_{|\lambda|}A_{\nu)}, \tag{7.216}$$

$$\frac{1}{3}\,\widetilde{U}^{(4)}_{02(\mu|\lambda|}R^{\lambda}{}_{\nu)} = -\frac{1}{6}\,\mathrm{i}qR^{\lambda}{}_{(\mu}D_{\nu)}A_{\lambda}, \tag{7.217}$$

$$\begin{aligned}
-\frac{2}{3}\,\widetilde{U}^{(4)}_{02\lambda\alpha}R^{\lambda}{}_{(\mu}{}^{\alpha}{}_{\nu)} - \frac{2}{3}\,\widetilde{U}^{(4)}_{02\alpha\lambda}R^{\lambda}{}_{(\mu}{}^{\alpha}{}_{\nu)} &= \frac{1}{3}\,\mathrm{i}qR_{\lambda(\mu|\alpha|\nu)}D^{\lambda}A^{\alpha} + \frac{1}{3}\,\mathrm{i}qR_{\lambda(\mu|\alpha|\nu)}D^{\alpha}A^{\lambda} \\
&= \frac{1}{3}\,\mathrm{i}qR_{\lambda(\mu|\alpha|\nu)}D^{\lambda}A^{\alpha} + \frac{1}{3}\,\mathrm{i}qR_{\alpha(\mu|\lambda|\nu)}D^{\lambda}A^{\alpha} \\
&= \frac{1}{3}\,\mathrm{i}qR_{\lambda(\mu|\alpha|\nu)}D^{\lambda}A^{\alpha} + \frac{1}{3}\,\mathrm{i}qR_{\lambda(\nu|\alpha|\mu)}D^{\lambda}A^{\alpha} \\
&= \frac{2}{3}\,\mathrm{i}qR_{\lambda(\mu|\alpha|\nu)}D^{\lambda}A^{\alpha},
\end{aligned} \tag{7.218}$$

where, in going from the first and second, the second and third, and the third and fourth equalities in (7.218), we have performed a simple relabelling of the summed over indices, used the symmetries of the Riemann tensor (1.5) and used the fact that we can arbitrarily swap symmetrised indices, respectively. Using (7.204), the terms proportional to $\widetilde{U}^{(4)}_{03(\mu\nu\rho)}$ in (7.206) are given by

$$2\, D^{\alpha}\widetilde{U}^{(4)}_{03\alpha(\mu\nu)} = \frac{1}{3}\,\mathrm{i}qD^{\alpha}D_{\alpha}D_{(\mu}A_{\nu)} + \frac{1}{6}\,\mathrm{i}qD^{\alpha}[A_{\alpha}R_{\mu\nu}]\,, \tag{7.219}$$

$$\begin{aligned} 2\, D^{\alpha}\widetilde{U}^{(4)}_{03(\mu|\alpha|\nu)} &= \frac{1}{3}\,\mathrm{i}qD^{\alpha}D_{(\mu}D_{|\alpha|}A_{\nu)} + \frac{1}{6}\,\mathrm{i}qD^{\alpha}\big[A_{(\mu}R_{|\alpha|\nu)}\big] \\ &= \frac{1}{3}\,\mathrm{i}qD^{\alpha}D_{(\mu}D_{|\alpha|}A_{\nu)} + \frac{1}{6}\,\mathrm{i}qD^{\alpha}\big[A_{(\mu}R_{\nu)\alpha}\big]\,, \end{aligned} \tag{7.220}$$

$$2\, D^{\alpha}\widetilde{U}^{(4)}_{03(\mu\nu)\alpha} = \frac{1}{3}\,\mathrm{i}qD^{\alpha}D_{(\mu}D_{\nu)}A_{\alpha} + \frac{1}{6}\,\mathrm{i}qD^{\alpha}\big[A_{(\mu}R_{\nu)\alpha}\big]\,. \tag{7.221}$$

Using (7.180), the term proportional to $\widehat{U}^{(4)}_{03(\mu\nu\rho)}$ in (7.206) is given by

$$\begin{aligned} -6\,\mathrm{i}qA^{\alpha}\widehat{U}^{(4)}_{03(\alpha\mu\nu)} &= \frac{1}{4}\,\mathrm{i}qA^{\alpha}\nabla_{(\alpha}R_{\mu\nu)} \\ &= \frac{1}{12}\,\mathrm{i}qA^{\alpha}\nabla_{\alpha}R_{\mu\nu} + \frac{1}{12}\,\mathrm{i}qA^{\alpha}\nabla_{(\mu}R_{|\alpha|\nu)} + \frac{1}{12}\,\mathrm{i}qA^{\alpha}\nabla_{(\mu}R_{\nu)\alpha} \\ &= \frac{1}{12}\,\mathrm{i}qA^{\alpha}\nabla_{\alpha}R_{\mu\nu} + \frac{1}{6}\,\mathrm{i}qA^{\alpha}\nabla_{(\mu}R_{\nu)\alpha}. \end{aligned} \tag{7.222}$$

Using (7.204), the terms proportional to $\widetilde{U}^{(4)}_{04(\mu\nu\rho\tau)}$ in (7.206) are given by

$$2\, g^{\alpha\beta}\,\widetilde{U}^{(4)}_{04\alpha\beta(\mu\nu)} = -\frac{1}{12}\,\mathrm{i}qD^{\alpha}D_{\alpha}D_{(\mu}A_{\nu)} - \frac{1}{12}\,\mathrm{i}qD^{\alpha}[A_{\alpha}R_{\mu\nu}]\,, \tag{7.223}$$

$$\begin{aligned} 2\, g^{\alpha\beta}\,\widetilde{U}^{(4)}_{04\alpha(\mu|\beta|\nu)} &= -\frac{1}{12}\,\mathrm{i}qD^{\alpha}D_{(\mu}D_{|\alpha|}A_{\nu)} - \frac{1}{12}\,\mathrm{i}qD^{\alpha}\big[A_{(\mu}R_{|\alpha|\nu)}\big] \\ &= -\frac{1}{12}\,\mathrm{i}qD^{\alpha}D_{(\mu}D_{|\alpha|}A_{\nu)} - \frac{1}{12}\,\mathrm{i}qD^{\alpha}\big[A_{(\mu}R_{\nu)\alpha}\big]\,, \end{aligned} \tag{7.224}$$

$$2\, g^{\alpha\beta}\,\widetilde{U}^{(4)}_{04\alpha(\mu\nu)\beta} = -\frac{1}{12}\,\mathrm{i}qD^{\alpha}D_{(\mu}D_{\nu)}A_{\alpha} - \frac{1}{12}\,\mathrm{i}qD^{\alpha}\big[A_{(\mu}R_{\nu)\alpha}\big]\,, \tag{7.225}$$

$$\begin{aligned} 2\, g^{\alpha\beta}\,\widetilde{U}^{(4)}_{04(\mu|\alpha\beta|\nu)} &= -\frac{1}{12}\,\mathrm{i}qD_{(\mu}D^{\alpha}D_{|\alpha|}A_{\nu)} - \frac{1}{12}\,\mathrm{i}qD_{(\mu}\big[A^{\alpha}R_{|\alpha|\nu)}\big] \\ &= -\frac{1}{12}\,\mathrm{i}qD_{(\mu}D^{\alpha}D_{|\alpha|}A_{\nu)} - \frac{1}{12}\,\mathrm{i}qD_{(\mu}\big[A^{\alpha}R_{\nu)\alpha}\big]\,, \end{aligned} \tag{7.226}$$

$$2\, g^{\alpha\beta}\,\widetilde{U}^{(4)}_{04(\mu|\alpha|\nu)\beta} = -\frac{1}{12}\,\mathrm{i}qD_{(\mu}D_{|\alpha|}D_{\nu)}A^{\alpha} - \frac{1}{12}\,\mathrm{i}qD_{(\mu}\big[A^{\alpha}R_{\nu)\alpha}\big]\,, \tag{7.227}$$

$$\begin{aligned} 2\, g^{\alpha\beta}\,\widetilde{U}^{(4)}_{04(\mu\nu)\alpha\beta} &= -\frac{1}{12}\,\mathrm{i}qD_{(\mu}D_{\nu)}D_{\alpha}A^{\alpha} - \frac{1}{12}\,\mathrm{i}qD_{(\mu}\big[A_{\nu)}R_{\alpha}{}^{\alpha}\big] \\ &= -\frac{1}{12}\,\mathrm{i}qD_{(\mu}D_{\nu)}D_{\alpha}A^{\alpha} - \frac{1}{12}\,\mathrm{i}qD_{(\mu}\big[A_{\nu)}R\big]\,, \end{aligned} \tag{7.228}$$

Substituting in the explicit expressions for each term in (7.206) using equations (7.207) - (7.228), we obtain

$$
\begin{aligned}
0 = 6\,\widetilde{V}^{(4)}_{02(\mu\nu)} &+ \mathrm{i}q\left[m^2 + \left(\xi - \frac{1}{6}\right)R\right]D_{(\mu}A_{\nu)} + \mathrm{i}q\left(\xi - \frac{1}{6}\right)R_{;(\mu}A_{\nu)} - \frac{1}{6}\,\mathrm{i}qD_{(\mu}\nabla^\alpha F_{|\alpha|\nu)} \\
&+ \frac{1}{2}\,\mathrm{i}q\left(\xi - \frac{1}{6}\right)R_{;(\mu}A_{\nu)} - \frac{1}{12}\,\mathrm{i}qA_\lambda\nabla_{(\mu}R^\lambda{}_{\nu)} - \frac{1}{4}\,\mathrm{i}qA_\lambda\nabla^\alpha R^\lambda{}_{(\mu|\alpha|\nu)} - \frac{2}{3}\,\mathrm{i}qR^\lambda{}_{(\mu|\alpha|\nu)}\,D^\alpha A_\lambda \\
&- \frac{1}{2}\,\mathrm{i}qD_\alpha D^\alpha D_{(\mu}A_{\nu)} + \frac{1}{2}\,\mathrm{i}q\left(m^2 + \xi R\right)D_{(\mu}A_{\nu)} - \frac{1}{6}\,\mathrm{i}qA^\alpha\nabla_\alpha R_{\mu\nu} - \frac{1}{12}\,\mathrm{i}q\,(\nabla_\alpha A^\alpha)R_{\mu\nu} \\
&- \frac{1}{12}\,q^2A_\alpha A^\alpha R_{\mu\nu} - \frac{1}{6}\,\mathrm{i}qR^\lambda{}_{(\mu}D_{|\lambda|}A_{\nu)} - \frac{1}{6}\,\mathrm{i}qR^\lambda{}_{(\mu}D_{\nu)}A_\lambda + \frac{2}{3}\,\mathrm{i}qR_{\lambda(\mu|\alpha|\nu)}D^\lambda A^\alpha \\
&+ \frac{1}{3}\,\mathrm{i}qD^\alpha D_\alpha D_{(\mu}A_{\nu)} + \frac{1}{6}\,\mathrm{i}qD^\alpha[A_\alpha R_{\mu\nu}] + \frac{1}{3}\,\mathrm{i}qD^\alpha D_{(\mu}D_{|\alpha|}A_{\nu)} + \frac{1}{6}\,\mathrm{i}qD^\alpha\left[A_{(\mu}R_{\nu)\alpha}\right] \\
&+ \frac{1}{3}\,\mathrm{i}qD^\alpha D_{(\mu}D_{\nu)}A_\alpha + \frac{1}{6}\,\mathrm{i}qD^\alpha\left[A_{(\mu}R_{\nu)\alpha}\right] + \frac{1}{12}\,\mathrm{i}qA^\alpha\nabla_\alpha R_{\mu\nu} + \frac{1}{6}\,\mathrm{i}qA^\alpha\nabla_{(\mu}R_{\nu)\alpha} \\
&- \frac{1}{12}\,\mathrm{i}qD^\alpha D_\alpha D_{(\mu}A_{\nu)} - \frac{1}{12}\,\mathrm{i}qD^\alpha[A_\alpha R_{\mu\nu}] - \frac{1}{12}\,\mathrm{i}qD^\alpha D_{(\mu}D_{|\alpha|}A_{\nu)} - \frac{1}{12}\,\mathrm{i}qD^\alpha\left[A_{(\mu}R_{\nu)\alpha}\right] \\
&- \frac{1}{12}\,\mathrm{i}qD^\alpha D_{(\mu}D_{\nu)}A_\alpha - \frac{1}{12}\,\mathrm{i}qD^\alpha\left[A_{(\mu}R_{\nu)\alpha}\right] - \frac{1}{12}\,\mathrm{i}qD_{(\mu}D^\alpha D_{|\alpha|}A_{\nu)} - \frac{1}{12}\,\mathrm{i}qD_{(\mu}\left[A^\alpha R_{\nu)\alpha}\right] \\
&- \frac{1}{12}\,\mathrm{i}qD_{(\mu}D_{|\alpha|}D_{\nu)}A^\alpha - \frac{1}{12}\,\mathrm{i}qD_{(\mu}\left[A^\alpha R_{\nu)\alpha}\right] - \frac{1}{12}\,\mathrm{i}qD_{(\mu}D_{\nu)}D_\alpha A^\alpha - \frac{1}{12}\,\mathrm{i}qD_{(\mu}\left[A_{\nu)}R\right].
\end{aligned}
\tag{7.229}
$$

Simplifying like terms in (7.229) and rearranging to group terms that are likely to combine, we have

$$
\begin{aligned}
0 = 6\,\widetilde{V}^{(4)}_{02(\mu\nu)} &+ \mathrm{i}q\left[m^2 + \left(\xi - \frac{1}{6}\right)R\right]D_{(\mu}A_{\nu)} + \frac{1}{2}\,\mathrm{i}q\left(m^2 + \xi R\right)D_{(\mu}A_{\nu)} \\
&+ \frac{3}{2}\,\mathrm{i}q\left(\xi - \frac{1}{6}\right)R_{;(\mu}A_{\nu)} - \frac{1}{6}\,\mathrm{i}qD_{(\mu}\nabla^\alpha F_{|\alpha|\nu)} - \frac{1}{4}\,\mathrm{i}qA_\lambda\nabla^\alpha R^\lambda{}_{(\mu|\alpha|\nu)} + \frac{2}{3}\,\mathrm{i}qR^\lambda{}_{(\mu|\alpha|\nu)}F^{\lambda\alpha} \\
&- \frac{1}{12}\,\mathrm{i}qA^\alpha\nabla_\alpha R_{\mu\nu} - \frac{1}{12}\,\mathrm{i}q\,(\nabla_\alpha A^\alpha)R_{\mu\nu} - \frac{1}{12}\,q^2A_\alpha A^\alpha R_{\mu\nu} + \frac{1}{12}\,\mathrm{i}qD^\alpha[A_\alpha R_{\mu\nu}] \\
&- \frac{1}{6}\,\mathrm{i}qR_{\alpha(\mu}D^\alpha A_{\nu)} + \frac{1}{6}\,\mathrm{i}qD^\alpha\left[A_{(\mu}R_{\nu)\alpha}\right] - \frac{1}{6}\,\mathrm{i}qR_{\alpha(\mu}D_{\nu)}A^\alpha - \frac{1}{6}\,\mathrm{i}qD_{(\mu}\left[A^\alpha R_{\nu)\alpha}\right] \\
&+ \frac{1}{12}\,\mathrm{i}qA^\alpha\nabla_{(\mu}R_{\nu)\alpha} - \frac{1}{12}\,\mathrm{i}qD_{(\mu}\left[A_{\nu)}R\right] - \frac{1}{4}\,\mathrm{i}qD^\alpha D_\alpha D_{(\mu}A_{\nu)} + \frac{1}{4}\,\mathrm{i}qD^\alpha D_{(\mu}D_{|\alpha|}A_{\nu)} \\
&- \frac{1}{12}\,\mathrm{i}qD_{(\mu}D^\alpha D_{|\alpha|}A_{\nu)} + \frac{1}{4}\,\mathrm{i}qD_\alpha D_{(\mu}D_{\nu)}A^\alpha - \frac{1}{12}\,\mathrm{i}qD_{(\mu}D_{|\alpha|}D_{\nu)}A^\alpha - \frac{1}{12}\,\mathrm{i}qD_{(\mu}D_{\nu)}D_\alpha A^\alpha,
\end{aligned}
\tag{7.230}
$$

where we have used

$$
\begin{aligned}
\frac{2}{3}\,\mathrm{i}qR_{\lambda(\mu|\alpha|\nu)}D^\lambda A^\alpha - \frac{2}{3}\,\mathrm{i}qR^\lambda{}_{(\mu|\alpha|\nu)}\,D^\alpha A_\lambda &= \frac{2}{3}\,\mathrm{i}qR_{\lambda(\mu|\alpha|\nu)}\left(D^\lambda A^\alpha - D^\alpha A^\lambda\right) \\
&= \frac{2}{3}\,\mathrm{i}qR_{\lambda(\mu|\alpha|\nu)}F^{\lambda\alpha},
\end{aligned}
\tag{7.231}
$$

to simplify (7.230). We also note that

$$
\begin{aligned}
-\frac{1}{12}\,\mathrm{i}qA^\alpha\nabla_\alpha R_{\mu\nu} - \frac{1}{12}\,\mathrm{i}q\,(\nabla_\alpha A^\alpha)\,R_{\mu\nu} - \frac{1}{12}\,q^2A_\alpha A^\alpha R_{\mu\nu}& \\
&= -\frac{1}{12}\,\mathrm{i}q\,\nabla_\alpha(A^\alpha R_{\mu\nu}) - \frac{1}{12}\,q^2A_\alpha A^\alpha R_{\mu\nu} \\
&= -\frac{1}{12}\,\mathrm{i}q\,D^\alpha[A_\alpha R_{\mu\nu}]\,,
\end{aligned}
\tag{7.232}
$$

which means that the third line of (7.230) cancels, leaving

$$
\begin{aligned}
0 &= 6\,\widetilde{V}^{(4)}_{02(\mu\nu)} + \mathrm{i}q\Big[m^2 + \Big(\xi - \frac{1}{6}\Big)R\Big]D_{(\mu}A_{\nu)} + \frac{1}{2}\,\mathrm{i}q\,\big(m^2 + \xi R\big)D_{(\mu}A_{\nu)} \\
&+ \frac{3}{2}\,\mathrm{i}q\,\Big(\xi - \frac{1}{6}\Big)R_{;(\mu}A_{\nu)} - \frac{1}{6}\,\mathrm{i}qD_{(\mu}\nabla^{\alpha}F_{|\alpha|\nu)} - \frac{1}{4}\,\mathrm{i}qA_{\lambda}\nabla^{\alpha}R^{\lambda}{}_{(\mu|\alpha|\nu)} + \frac{2}{3}\,\mathrm{i}qR^{\lambda}{}_{(\mu|\alpha|\nu)}F^{\lambda\alpha} \\
&- \frac{1}{6}\,\mathrm{i}qR_{\alpha(\mu}D^{\alpha}A_{\nu)} + \frac{1}{6}\,\mathrm{i}qD^{\alpha}\big[A_{(\mu}R_{\nu)\alpha}\big] - \frac{1}{6}\,\mathrm{i}qR_{\alpha(\mu}D_{\nu)}A^{\alpha} - \frac{1}{6}\,\mathrm{i}qD_{(\mu}\big[A^{\alpha}R_{\nu)\alpha}\big] \\
&+ \frac{1}{12}\,\mathrm{i}qA^{\alpha}\nabla_{(\mu}R_{\nu)\alpha} - \frac{1}{12}\,\mathrm{i}qD_{(\mu}\big[A_{\nu)}R\big] - \frac{1}{4}\,\mathrm{i}qD^{\alpha}D_{\alpha}D_{(\mu}A_{\nu)} + \frac{1}{4}\,\mathrm{i}qD^{\alpha}D_{(\mu}D_{|\alpha|}A_{\nu)} \\
&- \frac{1}{12}\,\mathrm{i}qD_{(\mu}D^{\alpha}D_{|\alpha|}A_{\nu)} + \frac{1}{4}\,\mathrm{i}qD_{\alpha}D_{(\mu}D_{\nu)}A^{\alpha} - \frac{1}{12}\,\mathrm{i}qD_{(\mu}D_{|\alpha|}D_{\nu)}A^{\alpha} - \frac{1}{12}\,\mathrm{i}qD_{(\mu}D_{\nu)}D_{\alpha}A^{\alpha}.
\end{aligned}
\tag{7.233}
$$

Equation (7.233) is obviously still very complicated. Rather than make further ad-hoc simplifications, we will attempt to be as systematic as possible in simplifying it. First let us address the terms that contain three gauge covariant derivatives acting on the gauge field. Our general strategy will be to commute the gauge covariant derivatives in the relevant terms such that we can simplify as much as possible. There are two types of these terms; the first are those where the gauge field itself has the index α and the second are those where it has an index of μ or ν. In the former type, one term has the outermost gauge covariant derivative with an index of α, one has the middle and one has the innermost. Thus, in the first type, it is simplest to commute gauge derivatives such that each term of this type has the middle gauge derivative with the index α. Using (6.29), we have

$$
-\frac{1}{12}\,\mathrm{i}qD_{(\mu}D_{\nu)}D_{\alpha}A^{\alpha} = -\frac{1}{12}\,\mathrm{i}qD_{(\mu}D_{|\alpha|}D_{\nu)}A^{\alpha} + \frac{1}{12}\,\mathrm{i}qD_{(\mu}R_{\nu)\alpha}A^{\alpha} - \frac{1}{12}\,q^2D_{(\mu}A^{\alpha}F_{\nu)\alpha}.
\tag{7.234}
$$

Using (6.31), we have

$$
\begin{aligned}
\frac{1}{4}\,\mathrm{i}qD_{\alpha}D_{(\mu}D_{\nu)}A^{\alpha} = \frac{1}{4}\,\mathrm{i}qD_{(\mu}D_{|\alpha|}D_{\nu)}A^{\alpha} + \frac{1}{4}\,\mathrm{i}qR_{\alpha(\mu}D_{\nu)}A^{\alpha} - \frac{1}{4}\,\mathrm{i}qR^{\lambda}{}_{(\mu|\alpha|\nu)}D_{\lambda}A^{\alpha} \\
+ \frac{1}{4}\,q^2F_{\alpha(\mu}D_{\nu)}A^{\alpha}.
\end{aligned}
\tag{7.235}
$$

Therefore, the terms proportional to three gauge derivatives acting on a gauge field with the index α in (7.233) are given by

$$
\begin{aligned}
&\frac{1}{4}\,\mathrm{i}qD_{\alpha}D_{(\mu}D_{\nu)}A^{\alpha} - \frac{1}{12}\,\mathrm{i}qD_{(\mu}D_{|\alpha|}D_{\nu)}A^{\alpha} - \frac{1}{12}\,\mathrm{i}qD_{(\mu}D_{\nu)}D_{\alpha}A^{\alpha} \\
&\quad= \frac{1}{4}\,\mathrm{i}qD_{(\mu}D_{|\alpha|}D_{\nu)}A^{\alpha} + \frac{1}{4}\,\mathrm{i}qR_{\alpha(\mu}D_{\nu)}A^{\alpha} - \frac{1}{4}\,\mathrm{i}qR^{\lambda}{}_{(\mu|\alpha|\nu)}D_{\lambda}A^{\alpha} \\
&\quad+ \frac{1}{4}\,q^2F_{\alpha(\mu}D_{\nu)}A^{\alpha} - \frac{1}{12}\,\mathrm{i}qD_{(\mu}D_{|\alpha|}D_{\nu)}A^{\alpha} - \frac{1}{12}\,\mathrm{i}qD_{(\mu}D_{|\alpha|}D_{\nu)}A^{\alpha} \\
&\quad+ \frac{1}{12}\,\mathrm{i}qD_{(\mu}R_{\nu)\alpha}A^{\alpha} - \frac{1}{12}\,q^2D_{(\mu}A^{\alpha}F_{\nu)\alpha} \\
&\quad= \frac{1}{12}\,\mathrm{i}qD_{(\mu}D_{|\alpha|}D_{\nu)}A^{\alpha} + \frac{1}{4}\,\mathrm{i}qR_{\alpha(\mu}D_{\nu)}A^{\alpha} - \frac{1}{4}\,\mathrm{i}qR^{\lambda}{}_{(\mu|\alpha|\nu)}D_{\lambda}A^{\alpha} \\
&\quad+ \frac{1}{4}\,q^2F_{\alpha(\mu}D_{\nu)}A^{\alpha} + \frac{1}{12}\,\mathrm{i}qD_{(\mu}R_{\nu)\alpha}A^{\alpha} - \frac{1}{12}\,q^2D_{(\mu}A^{\alpha}F_{\nu)\alpha}.
\end{aligned}
\tag{7.236}
$$

In the second type, one term has the innermost gauge covariant derivative with an index of μ or ν, one has the middle and one has the outermost. Thus, in the latter type, it is simplest to commute gauge derivatives such that each term of this type has the middle gauge derivative with the index μ or ν. Using (6.28), we have

$$-\frac{1}{4}\,\mathrm{i}qD^\alpha D_\alpha D_{(\mu}A_{\nu)} = -\frac{1}{4}\,\mathrm{i}qD^\alpha D_{(\mu}D_{|\alpha|}A_{\nu)} + \frac{1}{4}\,\mathrm{i}qD^\alpha R^\lambda{}_{(\mu|\alpha|\nu)}A_\lambda + \frac{1}{4}\,q^2 D^\alpha A_{(\mu}F_{\nu)\alpha}. \tag{7.237}$$

Using (6.32), we have

$$\begin{aligned}-\frac{1}{12}\,\mathrm{i}qD_{(\mu}D_{|\alpha|}D^\alpha A_{\nu)} = {}&-\frac{1}{12}\,\mathrm{i}qD_\alpha D_{(\mu}D^\alpha A_{\nu)} + \frac{1}{12}\,\mathrm{i}qR_{\alpha(\mu}D^\alpha A_{\nu)} - \frac{1}{12}\,\mathrm{i}qR^\lambda{}_{(\mu|\alpha|\nu)}D^\alpha A_\lambda \\ &+ \frac{1}{12}\,q^2 F_{\alpha(\mu}D^\alpha A_{\nu)}.\end{aligned} \tag{7.238}$$

Therefore, the terms proportional to three gauge derivatives acting on a gauge field with the index μ or ν are given by

$$\begin{aligned}-\frac{1}{4}\,&\mathrm{i}qD^\alpha D_\alpha D_{(\mu}A_{\nu)} + \frac{1}{4}\,\mathrm{i}qD^\alpha D_{(\mu}D_{|\alpha|}A_{\nu)} - \frac{1}{12}\,\mathrm{i}qD_{(\mu}D_{|\alpha|}D^\alpha A_{\nu)} \\ &= -\frac{1}{4}\,\mathrm{i}qD^\alpha D_{(\mu}D_{|\alpha|}A_{\nu)} + \frac{1}{4}\,\mathrm{i}qD^\alpha R^\lambda{}_{(\mu|\alpha|\nu)}A_\lambda + \frac{1}{4}\,q^2 D^\alpha A_{(\mu}F_{\nu)\alpha} \\ &+ \frac{1}{4}\,\mathrm{i}qD^\alpha D_{(\mu}D_{|\alpha|}A_{\nu)} - \frac{1}{12}\,\mathrm{i}qD_\alpha D_{(\mu}D^\alpha A_{\nu)} + \frac{1}{12}\,\mathrm{i}qR_{\alpha(\mu}D^\alpha A_{\nu)} \\ &- \frac{1}{12}\,\mathrm{i}qR^\lambda{}_{(\mu|\alpha|\nu)}D^\alpha A_\lambda + \frac{1}{12}\,q^2 F_{\alpha(\mu}D^\alpha A_{\nu)} \\ &= -\frac{1}{12}\,\mathrm{i}qD^\alpha D_{(\mu}D_{|\alpha|}A_{\nu)} + \frac{1}{4}\,\mathrm{i}qD^\alpha R^\lambda{}_{(\mu|\alpha|\nu)}A_\lambda + \frac{1}{4}\,q^2 D^\alpha A_{(\mu}F_{\nu)\alpha} \\ &+ \frac{1}{12}\,\mathrm{i}qR_{\alpha(\mu}D^\alpha A_{\nu)} - \frac{1}{12}\,\mathrm{i}qR^\lambda{}_{(\mu|\alpha|\nu)}D^\alpha A_\lambda + \frac{1}{12}\,q^2 F_{\alpha(\mu}D^\alpha A_{\nu)}.\end{aligned} \tag{7.239}$$

The sum of (7.236) and (7.239) gives us

$$\begin{aligned}&\frac{1}{4}\,\mathrm{i}qD_\alpha D_{(\mu}D_{\nu)}A^\alpha - \frac{1}{12}\,\mathrm{i}qD_{(\mu}D_{|\alpha|}D_{\nu)}A^\alpha - \frac{1}{12}\,\mathrm{i}qD_{(\mu}D_{\nu)}D_\alpha A^\alpha - \frac{1}{4}\,\mathrm{i}qD^\alpha D_\alpha D_{(\mu}A_{\nu)} \\ &+ \frac{1}{4}\,\mathrm{i}qD^\alpha D_{(\mu}D_{|\alpha|}A_{\nu)} - \frac{1}{12}\,\mathrm{i}qD_{(\mu}D_{|\alpha|}D^\alpha A_{\nu)} \\ &= \frac{1}{12}\,\mathrm{i}qD_{(\mu}D_{|\alpha|}D_{\nu)}A^\alpha + \frac{1}{4}\,\mathrm{i}qR_{\alpha(\mu}D_{\nu)}A^\alpha - \frac{1}{4}\,\mathrm{i}qR^\lambda{}_{(\mu|\alpha|\nu)}D_\lambda A^\alpha + \frac{1}{4}\,q^2F_{\alpha(\mu}D_{\nu)}A^\alpha \\ &+ \frac{1}{12}\,\mathrm{i}qD_{(\mu}R_{\nu)\alpha}A^\alpha - \frac{1}{12}\,q^2 D_{(\mu}A^\alpha F_{\nu)\alpha} - \frac{1}{12}\,\mathrm{i}qD^\alpha D_{(\mu}D_{|\alpha|}A_{\nu)} + \frac{1}{4}\,\mathrm{i}qD^\alpha R^\lambda{}_{(\mu|\alpha|\nu)}A_\lambda \\ &+ \frac{1}{4}\,q^2 D^\alpha A_{(\mu}F_{\nu)\alpha} + \frac{1}{12}\,\mathrm{i}qR_{\alpha(\mu}D^\alpha A_{\nu)} - \frac{1}{12}\,\mathrm{i}qR^\lambda{}_{(\mu|\alpha|\nu)}D^\alpha A_\lambda + \frac{1}{12}\,q^2F_{\alpha(\mu}D^\alpha A_{\nu)}.\end{aligned} \tag{7.240}$$

We can use the definition of the electromagnetic field strength (3.2) to combine the two terms proportional to three gauge derivatives acting on the gauge field. In order to do so, we need to commute the gauge derivatives in one of these terms. Using (6.32), we have

$$-\frac{1}{12}\,\mathrm{i}qD_\alpha D_{(\mu}D^\alpha A_{\nu)} = -\frac{1}{12}\,\mathrm{i}qD_{(\mu}D_{|\alpha|}D^\alpha A_{\nu)} - \frac{1}{12}\,\mathrm{i}qR_{\alpha(\mu}D^\alpha A_{\nu)} + \frac{1}{12}\,\mathrm{i}qR^\lambda{}_{(\mu|\alpha|\nu)}D^\alpha A_\lambda - \frac{1}{12}\,q^2F_{\alpha(\mu}D^\alpha A_{\nu)}. \tag{7.241}$$

Then, inserting (7.241) into (7.240) and simplifying, we have

$$\begin{aligned}
&\frac{1}{4}\,\mathrm{i}qD_\alpha D_{(\mu}D_{\nu)}A^\alpha - \frac{1}{12}\,\mathrm{i}qD_{(\mu}D_{|\alpha|}D_{\nu)}A^\alpha - \frac{1}{12}\,\mathrm{i}qD_{(\mu}D_{\nu)}D_\alpha A^\alpha - \frac{1}{4}\,\mathrm{i}qD^\alpha D_\alpha D_{(\mu}A_{\nu)}\\
&+\frac{1}{4}\,\mathrm{i}qD^\alpha D_{(\mu}D_{|\alpha|}A_{\nu)} - \frac{1}{12}\,\mathrm{i}qD_{(\mu}D_{|\alpha|}D^\alpha A_{\nu)}\\
&= \frac{1}{12}\,\mathrm{i}qD_{(\mu}D^\alpha F_{\nu)\alpha} + \frac{1}{4}\,\mathrm{i}qR_{\alpha(\mu}D_{\nu)}A^\alpha - \frac{1}{4}\,\mathrm{i}qR^\lambda{}_{(\mu|\alpha|\nu)}D_\lambda A^\alpha + \frac{1}{4}\,q^2F_{\alpha(\mu}D_{\nu)}A^\alpha\\
&+\frac{1}{12}\,\mathrm{i}qD_{(\mu}R_{\nu)\alpha}A^\alpha - \frac{1}{12}\,q^2D_{(\mu}A^\alpha F_{\nu)\alpha} + \frac{1}{4}\,\mathrm{i}qD^\alpha R^\lambda{}_{(\mu|\alpha|\nu)}A_\lambda + \frac{1}{4}\,q^2D^\alpha A_{(\mu}F_{\nu)\alpha}.
\end{aligned} \tag{7.242}$$

Substituting (7.242) back into (7.233), we have

$$\begin{aligned}
0 &= 6\,\widetilde{V}^{(4)}_{02(\mu\nu)} + \mathrm{i}q\left[m^2 + \left(\xi - \frac{1}{6}\right)R\right]D_{(\mu}A_{\nu)} + \frac{1}{2}\,\mathrm{i}q\left(m^2+\xi R\right)D_{(\mu}A_{\nu)}\\
&+\frac{3}{2}\,\mathrm{i}q\left(\xi - \frac{1}{6}\right)R_{;(\mu}A_{\nu)} - \frac{1}{6}\,\mathrm{i}qD_{(\mu}\nabla^\alpha F_{|\alpha|\nu)} - \frac{1}{4}\,\mathrm{i}qA_\lambda\nabla^\alpha R^\lambda{}_{(\mu|\alpha|\nu)} + \frac{2}{3}\,\mathrm{i}qR_{\lambda(\mu|\alpha|\nu)}F^{\lambda\alpha}\\
&-\frac{1}{6}\,\mathrm{i}qR_{\alpha(\mu}D^\alpha A_{\nu)} + \frac{1}{6}\,\mathrm{i}qD^\alpha\left[A_{(\mu}R_{\nu)\alpha}\right] - \frac{1}{6}\,\mathrm{i}qR_{\alpha(\mu}D_{\nu)}A^\alpha - \frac{1}{6}\,\mathrm{i}qD_{(\mu}\left[A^\alpha R_{\nu)\alpha}\right]\\
&+\frac{1}{12}\,\mathrm{i}qA^\alpha\nabla_{(\mu}R_{\nu)\alpha} - \frac{1}{12}\,\mathrm{i}qD_{(\mu}\left[A_{\nu)}R\right] + \frac{1}{12}\,\mathrm{i}qD_{(\mu}D^\alpha F_{\nu)\alpha} + \frac{1}{4}\,\mathrm{i}qR_{\alpha(\mu}D_{\nu)}A^\alpha\\
&-\frac{1}{4}\,\mathrm{i}qR^\lambda{}_{(\mu|\alpha|\nu)}D_\lambda A^\alpha + \frac{1}{4}\,q^2F_{\alpha(\mu}D_{\nu)}A^\alpha + \frac{1}{12}\,\mathrm{i}qD_{(\mu}R_{\nu)\alpha}A^\alpha - \frac{1}{12}\,q^2D_{(\mu}A^\alpha F_{\nu)\alpha}\\
&+\frac{1}{4}\,\mathrm{i}qD^\alpha R^\lambda{}_{(\mu|\alpha|\nu)}A_\lambda + \frac{1}{4}\,q^2D^\alpha A_{(\mu}F_{\nu)\alpha}.
\end{aligned} \tag{7.243}$$

Simplifying like terms in (7.243), we have

$$\begin{aligned}
0 &= 6\,\widetilde{V}^{(4)}_{02(\mu\nu)} + \mathrm{i}q\left[m^2 + \left(\xi - \frac{1}{6}\right)R\right]D_{(\mu}A_{\nu)} + \frac{1}{2}\,\mathrm{i}q\left(m^2+\xi R\right)D_{(\mu}A_{\nu)}\\
&+\frac{3}{2}\,\mathrm{i}q\left(\xi - \frac{1}{6}\right)R_{;(\mu}A_{\nu)} - \frac{1}{6}\,\mathrm{i}qD_{(\mu}\nabla^\alpha F_{|\alpha|\nu)} - \frac{1}{4}\,\mathrm{i}qA_\lambda\nabla^\alpha R^\lambda{}_{(\mu|\alpha|\nu)} + \frac{2}{3}\,\mathrm{i}qR_{\lambda(\mu|\alpha|\nu)}F^{\lambda\alpha}\\
&-\frac{1}{6}\,\mathrm{i}qR_{\alpha(\mu}D^\alpha A_{\nu)} + \frac{1}{6}\,\mathrm{i}qD^\alpha\left[A_{(\mu}R_{\nu)\alpha}\right] + \frac{1}{12}\,\mathrm{i}qR_{\alpha(\mu}D_{\nu)}A^\alpha - \frac{1}{12}\,\mathrm{i}qD_{(\mu}\left[A^\alpha R_{\nu)\alpha}\right]\\
&+\frac{1}{12}\,\mathrm{i}qA^\alpha\nabla_{(\mu}R_{\nu)\alpha} - \frac{1}{12}\,\mathrm{i}qD_{(\mu}\left[A_{\nu)}R\right] + \frac{1}{12}\,\mathrm{i}qD_{(\mu}D^\alpha F_{\nu)\alpha} - \frac{1}{4}\,\mathrm{i}qR^\lambda{}_{(\mu|\alpha|\nu)}D_\lambda A^\alpha\\
&+\frac{1}{4}\,q^2F_{\alpha(\mu}D_{\nu)}A^\alpha - \frac{1}{12}\,q^2D_{(\mu}A^\alpha F_{\nu)\alpha} + \frac{1}{4}\,\mathrm{i}qD^\alpha R^\lambda{}_{(\mu|\alpha|\nu)}A_\lambda + \frac{1}{4}\,q^2D^\alpha A_{(\mu}F_{\nu)\alpha}.
\end{aligned} \tag{7.244}$$

Examining (7.244), we see that it contains several terms with derivative operators acting on multiple quantities which can be simplified by using the product rule; then, we have

$$
\begin{aligned}
0 = {} & 6\,\widetilde{V}^{(4)}_{02(\mu\nu)} + \mathrm{i}q\left[m^2 + \left(\xi - \frac{1}{6}\right)R\right]D_{(\mu}A_{\nu)} + \frac{1}{2}\,\mathrm{i}q\left(m^2 + \xi R\right)D_{(\mu}A_{\nu)} \\
& + \frac{3}{2}\,\mathrm{i}q\left(\xi - \frac{1}{6}\right)A_{(\mu}\nabla_{\nu)}R - \frac{1}{6}\,\mathrm{i}qD_{(\mu}\nabla^{\alpha}F_{|\alpha|\nu)} - \frac{1}{4}\,\mathrm{i}qA_{\lambda}\nabla^{\alpha}R^{\lambda}{}_{(\mu|\alpha|\nu)} + \frac{2}{3}\,\mathrm{i}qR_{\lambda(\mu|\alpha|\nu)}F^{\lambda\alpha} \\
& - \frac{1}{6}\,\mathrm{i}qD^{\alpha}\left[A_{(\mu}R_{\nu)\alpha}\right] + \frac{1}{6}\,\mathrm{i}qA_{(\mu}\nabla^{\alpha}R_{\nu)\alpha} + \frac{1}{6}\,\mathrm{i}qD^{\alpha}\left[A_{(\mu}R_{\nu)\alpha}\right] + \frac{1}{12}\,\mathrm{i}qD_{(\mu}\left[A^{\alpha}R_{\nu)\alpha}\right] \\
& - \frac{1}{12}\,\mathrm{i}qA^{\alpha}\nabla_{(\mu}R_{\nu)\alpha} - \frac{1}{12}\,\mathrm{i}qD_{(\mu}\left[A^{\alpha}R_{\nu)\alpha}\right] + \frac{1}{12}\,\mathrm{i}qA^{\alpha}\nabla_{(\mu}R_{\nu)\alpha} - \frac{1}{12}\,\mathrm{i}qRD_{(\mu}A_{\nu)} \\
& - \frac{1}{12}\,\mathrm{i}qA_{(\mu}\nabla_{\nu)}R + \frac{1}{12}\,\mathrm{i}qD_{(\mu}D^{\alpha}F_{\nu)\alpha} - \frac{1}{4}\,\mathrm{i}qR^{\lambda}{}_{(\mu|\alpha|\nu)}D_{\lambda}A^{\alpha} + \frac{1}{4}\,q^2D_{(\mu}A^{\alpha}F_{|\alpha|\nu)} \\
& - \frac{1}{4}\,q^2A^{\alpha}\nabla_{(\mu}F_{|\alpha|\nu)} - \frac{1}{12}\,q^2D_{(\mu}A^{\alpha}F_{\nu)\alpha} + \frac{1}{4}\,\mathrm{i}qR^{\lambda}{}_{(\mu|\alpha|\nu)}D^{\alpha}A_{\lambda} + \frac{1}{4}\,\mathrm{i}qA_{\lambda}\nabla^{\alpha}R^{\lambda}{}_{(\mu|\alpha|\nu)} \\
& + \frac{1}{4}\,q^2D^{\alpha}A_{(\mu}F_{\nu)\alpha}. \qquad (7.245)
\end{aligned}
$$

Simplifying like terms, we have

$$
\begin{aligned}
0 = {} & 6\,\widetilde{V}^{(4)}_{02(\mu\nu)} + \frac{3}{2}\,\mathrm{i}q\left[m^2 + \left(\xi - \frac{1}{6}\right)R\right]D_{(\mu}A_{\nu)} + \frac{3}{2}\,\mathrm{i}q\left(\xi - \frac{1}{6}\right)A_{(\mu}\nabla_{\nu)}R - \frac{1}{6}\,\mathrm{i}qD_{(\mu}\nabla^{\alpha}F_{|\alpha|\nu)} \\
& + \frac{1}{12}\,\mathrm{i}qD_{(\mu}D^{\alpha}F_{\nu)\alpha} - \frac{1}{3}\,q^2D_{(\mu}A^{\alpha}F_{\nu)\alpha} + \frac{1}{4}\,q^2A^{\alpha}\nabla_{(\mu}F_{\nu)\alpha} + \frac{1}{4}\,q^2D^{\alpha}A_{(\mu}F_{\nu)\alpha}, \qquad (7.246)
\end{aligned}
$$

where we have used that

$$
\begin{aligned}
R_{\lambda(\mu|\alpha|\nu)}F^{\lambda\alpha} &= R_{\alpha(\nu|\lambda|\mu)}F^{\lambda\alpha} \\
&= R_{\alpha(\mu|\lambda|\nu)}F^{\lambda\alpha} \\
&= -R_{\alpha(\mu|\lambda|\nu)}F^{\alpha\lambda} \\
&= -R_{\lambda(\mu|\alpha|\nu)}F^{\lambda\alpha} = 0, \qquad (7.247)
\end{aligned}
$$

and that

$$
\begin{aligned}
\frac{1}{6}\,\mathrm{i}qA_{(\mu}\nabla^{\alpha}R_{\nu)\alpha} - \frac{1}{12}\,\mathrm{i}qA_{(\mu}\nabla_{\nu)}R &= \frac{1}{6}\,\mathrm{i}qA_{(\mu}\left[\nabla^{\alpha}R_{\nu)\alpha} - \frac{1}{2}\,\nabla_{\nu)}R\right] \\
&= \frac{1}{6}\,\mathrm{i}qA_{(\mu}\nabla^{\alpha}\left[R_{\nu)\alpha} - \frac{1}{2}\,g_{\nu)\alpha}R\right] \\
&= \frac{1}{6}\,\mathrm{i}qA_{(\mu}\nabla^{\alpha}G_{\nu)\alpha} \\
&= 0, \qquad (7.248)
\end{aligned}
$$

which follows from the Bianchi identity (1.8). We can further simplify by expanding the gauge covariant derivatives in (7.246) into spacetime covariant derivatives and the part which does not contain any derivative operators; then, we obtain

$$0 = 6\,\widetilde{V}^{(4)}_{02(\mu\nu)} + \frac{3}{2}\,\mathrm{i}q\left[m^2 + \left(\xi - \frac{1}{6}\right)R\right]D_{(\mu}A_{\nu)} + \frac{3}{2}\,\mathrm{i}q\left(\xi - \frac{1}{6}\right)A_{(\mu}\nabla_{\nu)}R + \frac{1}{6}\,\mathrm{i}q\,\nabla_{(\mu}\nabla^{\alpha}F_{\nu)\alpha}$$
$$+ \frac{1}{6}\,q^2A_{(\mu}\nabla^{\alpha}F_{\nu)\alpha} + \frac{1}{12}\,\mathrm{i}q\,\nabla_{(\mu}\nabla^{\alpha}F_{\nu)\alpha} + \frac{1}{12}\,q^2\,\nabla_{(\mu}A^{\alpha}F_{\nu)\alpha} + \frac{1}{12}\,q^2A_{(\mu}\nabla^{\alpha}F_{\nu)\alpha}$$
$$- \frac{1}{12}\,\mathrm{i}q^3A_{(\mu}A^{\alpha}F_{\nu)\alpha} - \frac{1}{3}\,q^2\,\nabla_{(\mu}A^{\alpha}F_{\nu)\alpha} + \frac{1}{3}\,\mathrm{i}q^3A_{(\mu}A^{\alpha}F_{\nu)\alpha} + \frac{1}{4}\,q^2A^{\alpha}\nabla_{(\mu}F_{\nu)\alpha}$$
$$+ \frac{1}{4}\,q^2\,\nabla^{\alpha}A_{(\mu}F_{\nu)\alpha} - \frac{1}{4}\,\mathrm{i}q^3A^{\alpha}A_{(\mu}F_{\nu)\alpha}. \tag{7.249}$$

Simplifying like terms and using the product rule in (7.249) gives us

$$0 = 6\,\widetilde{V}^{(4)}_{02(\mu\nu)} + \frac{3}{2}\,\mathrm{i}q\left[m^2 + \left(\xi - \frac{1}{6}\right)R\right]D_{(\mu}A_{\nu)} + \frac{3}{2}\,\mathrm{i}q\left(\xi - \frac{1}{6}\right)A_{(\mu}\nabla_{\nu)}R + \frac{1}{4}\,\mathrm{i}q\,\nabla_{(\mu}\nabla^{\alpha}F_{\nu)\alpha}$$
$$+ \frac{1}{6}\,q^2A_{(\mu}\nabla^{\alpha}F_{\nu)\alpha} + \frac{1}{12}\,q^2A^{\alpha}\nabla_{(\mu}F_{\nu)\alpha} + \frac{1}{12}\,q^2F_{(\mu|\alpha|}\nabla_{\nu)}A^{\alpha} + \frac{1}{12}\,q^2A_{(\mu}\nabla^{\alpha}F_{\nu)\alpha}$$
$$- \frac{1}{3}\,q^2A^{\alpha}\nabla_{(\mu}F_{\nu)\alpha} - \frac{1}{3}\,q^2F_{(\mu|\alpha|}\nabla_{\nu)}A^{\alpha} + \frac{1}{4}\,q^2A^{\alpha}\nabla_{(\mu}F_{\nu)\alpha} + \frac{1}{4}\,q^2A_{(\mu}\nabla^{\alpha}F_{\nu)\alpha}$$
$$+ \frac{1}{4}\,q^2F_{(\mu|\alpha|}\nabla^{\alpha}A_{\nu)}. \tag{7.250}$$

Simplifying like terms, we have

$$0 = 6\,\widetilde{V}^{(4)}_{02(\mu\nu)} + \frac{3}{2}\,\mathrm{i}q\left[m^2 + \left(\xi - \frac{1}{6}\right)R\right]D_{(\mu}A_{\nu)} + \frac{3}{2}\,\mathrm{i}q\left(\xi - \frac{1}{6}\right)A_{(\mu}\nabla_{\nu)}R + \frac{1}{4}\,\mathrm{i}q\,\nabla_{(\mu}\nabla^{\alpha}F_{\nu)\alpha}$$
$$+ \frac{1}{2}\,q^2A_{(\mu}\nabla^{\alpha}F_{\nu)\alpha} - \frac{1}{4}\,q^2F_{(\mu|\alpha|}\nabla_{\nu)}A^{\alpha} + \frac{1}{4}\,q^2F_{(\mu|\alpha|}\nabla^{\alpha}A_{\nu)}$$
$$= 6\,\widetilde{V}^{(4)}_{02(\mu\nu)} + \frac{3}{2}\,\mathrm{i}q\left[m^2 + \left(\xi - \frac{1}{6}\right)R\right]D_{(\mu}A_{\nu)} + \frac{3}{2}\,\mathrm{i}q\left(\xi - \frac{1}{6}\right)A_{(\mu}\nabla_{\nu)}R + \frac{1}{4}\,\mathrm{i}q\,\nabla_{(\mu}\nabla^{\alpha}F_{\nu)\alpha}$$
$$+ \frac{1}{2}\,q^2A_{(\mu}\nabla^{\alpha}F_{\nu)\alpha} + \frac{1}{4}\,q^2F^{\alpha}{}_{\mu}F_{\nu\alpha}. \tag{7.251}$$

So finally, the second order correction $\widetilde{V}^{(4)}_0(x,x')$ is given by

$$\widetilde{V}^{(4)}_{02(\mu\nu)} = -\frac{1}{4}\,\mathrm{i}q\left[m^2 + \left(\xi - \frac{1}{6}\right)R\right]D_{(\mu}A_{\nu)} - \frac{1}{4}\,\mathrm{i}q\left(\xi - \frac{1}{6}\right)A_{(\mu}\nabla_{\nu)}R - \frac{1}{24}\,\mathrm{i}q\,\nabla_{(\mu}\nabla^{\alpha}F_{\nu)\alpha}$$
$$- \frac{1}{12}\,q^2A_{(\mu}\nabla^{\alpha}F_{\nu)\alpha} - \frac{1}{24}\,q^2F^{\alpha}{}_{\mu}F_{\nu\alpha}. \tag{7.252}$$

From (7.156), the $V^{(4)}_{02(\mu\nu)}$ expansion coefficient is given by

$$V^{(4)}_{02(\mu\nu)} = \widehat{V}^{(4)}_{02(\mu\nu)} + \widetilde{V}^{(4)}_{02(\mu\nu)}. \tag{7.253}$$

Then, using (7.203) and (7.252), we have

$$V^{(4)}_{02(\mu\nu)} = \frac{1}{24}\,m^2R_{\mu\nu} + \frac{1}{12}\left(\xi - \frac{3}{20}\right)R_{;\mu\nu} - \frac{1}{240}\,\Box R_{\mu\nu} + \frac{1}{24}\left(\xi - \frac{1}{6}\right)RR_{\mu\nu} + \frac{1}{180}R^{\alpha}{}_{\mu}R_{\alpha\nu}$$
$$- \frac{1}{360}R^{\alpha\beta}R_{\alpha\mu\beta\nu} - \frac{1}{360}R^{\alpha\lambda\beta}{}_{\mu}R_{\alpha\lambda\beta\nu} - \frac{1}{4}\,\mathrm{i}q\left[m^2 + \left(\xi - \frac{1}{6}\right)R\right]D_{(\mu}A_{\nu)}$$
$$- \frac{1}{4}\,\mathrm{i}q\left(\xi - \frac{1}{6}\right)A_{(\mu}\nabla_{\nu)}R - \frac{1}{24}\,\mathrm{i}q\,\nabla_{(\mu}\nabla^{\alpha}F_{\nu)\alpha} - \frac{1}{12}\,q^2A_{(\mu}\nabla^{\alpha}F_{\nu)\alpha} - \frac{1}{24}\,q^2F^{\alpha}{}_{\mu}F_{\nu\alpha}. \tag{7.254}$$

Lastly, we are required to evaluate the Hadamard coefficient $V_1^{(4)}(x,x')$ to $\mathcal{O}(1)$, i.e. we are required to evaluate the expansion coefficient $V_{10}^{(4)}$, where we have used the expansion

$$V_1^{(4)}\big(x,x'\big) = V_{10}^{(4)} + \dots \tag{7.255}$$

The equation governing $V_1^{(4)}(x,x')$ is given by

$$4V_1^{(4)} + \big(D_\mu D^\mu - m^2 - \xi R\big)\, V_0^{(4)} = 0. \tag{7.256}$$

Expanding the Hadamard coefficient $V_0^{(4)}(x,x')$ as

$$V_0^{(4)} = V_{00}^{(4)} + V_{01\mu}^{(4)}\sigma^{;\mu} + V_{02(\mu\nu)}^{(4)}\sigma^{;\mu}\sigma^{;\nu} + \dots \tag{7.257}$$

Then, acting with the covariant derivative, we have

$$\nabla_\mu V_0^{(4)} = V_{00;\mu}^{(4)} + V_{01\nu;\mu}^{(4)}\sigma^{;\nu} + V_{01\nu}^{(4)}\sigma^{;\nu}{}_{\mu} + 2V_{02(\nu\rho)}^{(4)}\sigma^{;\nu}{}_{\mu}\sigma^{;\rho} + \dots \tag{7.258}$$

which gives

$$\begin{aligned} D_\mu V_0^{(4)} &= V_{00;\mu}^{(4)} + V_{01\nu;\mu}^{(4)}\sigma^{;\nu} + V_{01\mu}^{(4)} + 2V_{02(\mu\nu)}^{(4)}\sigma^{;\nu} - \mathrm{i}qA_\mu V_{00}^{(4)} - \mathrm{i}qA_\mu V_{01\nu}^{(4)}\sigma^{;\nu} + \dots \\ &= D_\mu V_{00}^{(4)} + V_{01\mu}^{(4)} + \sigma^{;\nu}D_\mu V_{01\nu}^{(4)} + 2V_{02(\mu\nu)}^{(4)}\sigma^{;\nu} + \dots \end{aligned} \tag{7.259}$$

Acting with another covariant derivative, we obtain

$$\nabla_\mu D^\mu V_0^{(4)} = g^{\mu\lambda}\nabla_\lambda D_\mu V_{00}^{(4)} + g^{\mu\lambda}\nabla_\lambda V_{01\mu}^{(4)} + \sigma^{;\nu}{}_{\lambda}g^{\mu\lambda}D_\mu V_{01\nu}^{(4)} + 2V_{02(\mu\nu)}^{(4)}g^{\mu\lambda}\sigma^{;\nu}{}_{\lambda} + \dots \tag{7.260}$$

Then, we have

$$\begin{aligned} D_\mu D^\mu V_0^{(4)} &= \nabla_\mu D^\mu V_{00}^{(4)} + \nabla^\mu V_{01\mu}^{(4)} + \delta^\nu_\lambda g^{\mu\lambda}D_\mu V_{01\nu}^{(4)} + 2V_{02(\mu\nu)}^{(4)}g^{\mu\lambda}\delta^\nu_\lambda - \mathrm{i}qA_\lambda g^{\mu\lambda}D_\mu V_{00}^{(4)} \\ &\quad - \mathrm{i}qA_\lambda g^{\mu\lambda}V_{01\mu}^{(4)} + \dots \\ &= D_\mu D^\mu V_{00}^{(4)} + 2D^\mu V_{01\mu}^{(4)} + 2V_{02(\mu\nu)}^{(4)}g^{\mu\nu} + \dots \end{aligned} \tag{7.261}$$

Neutral scalar field: In the uncharged case, (7.261) reduces to

$$4\widehat{V}_{10}^{(4)} + \left(\Box - m^2 - \xi R\right) V_{00}^{(4)} + 2\nabla^\mu \widehat{V}_{01\mu}^{(4)} + 2g^{\mu\nu}\widehat{V}_{02(\mu\nu)}^{(4)} = 0. \tag{7.262}$$

We have

$$\begin{aligned} \left(\Box - m^2 - \xi R\right) V_{00}^{(4)} &= \frac{1}{2}\left(\xi - \frac{1}{6}\right)\Box R - \frac{1}{2}m^2\left[m^2 + \left(\xi - \frac{1}{6}\right)R\right] \\ &\quad - \frac{1}{2}\xi R\left[m^2 + \left(\xi - \frac{1}{6}\right)R\right], \end{aligned} \tag{7.263}$$

$$2\nabla^{\mu}\widehat{V}^{(4)}_{01\mu} = -\frac{1}{2}\left(\xi-\frac{1}{6}\right)\Box R + \mathrm{i}q\left(\xi-\frac{1}{6}\right)A^{\mu}\nabla_{\mu}\Box R + \mathrm{i}q\left[m^2+\left(\xi-\frac{1}{6}\right)R\right]\nabla^{\mu}A_{\mu} - \frac{1}{6}\,\mathrm{i}q\,\nabla^{\mu}\nabla^{\alpha}F_{\alpha\mu}, \tag{7.264}$$

$$\begin{aligned}2g^{\mu\nu}\widehat{V}^{(4)}_{02(\mu\nu)} = {} & \frac{1}{12}\left[m^2+\left(\xi-\frac{1}{6}\right)R\right]R + \frac{1}{6}\left(\xi-\frac{3}{20}\right)\Box R - \frac{1}{2}m^2\left[m^2+\left(\xi-\frac{1}{6}\right)R\right] \\ & - \frac{1}{2}\,\xi R\left[m^2+\left(\xi-\frac{1}{6}\right)R\right] + \frac{1}{120}\Box R + \frac{1}{90}R^{\mu nu}R_{\mu\nu} - \frac{1}{180}R^{\mu\nu}R_{\mu\nu} \\ & - \frac{1}{180}R^{\mu\nu\rho\tau}R_{\mu\nu\rho\tau} - \frac{1}{2}\,\mathrm{i}q\left[m^2+\left(\xi-\frac{1}{6}\right)R\right]\nabla^{\mu}A_{\mu} - \frac{1}{2}\,\mathrm{i}q\left(\xi-\frac{1}{6}\right)A^{\mu}\nabla_{\mu}R \\ & + \frac{1}{12}q^2F^{\mu\nu}F_{\mu\nu} - \frac{1}{6}\,q^2A^{\mu}\nabla^{\alpha}F_{\mu\alpha} - \frac{1}{12}\,\mathrm{i}q\,\nabla^{\mu}\nabla^{\alpha}F_{\alpha\mu}.\end{aligned} \tag{7.265}$$

Then, we obtain for the $\widehat{V}^{(4)}_{10}$ expansion coefficient the expression

$$\widehat{V}^{(4)}_{10} = \frac{1}{8}\left[m^2+\left(\xi-\frac{1}{6}\right)R\right]^2 - \frac{1}{24}\left(\xi-\frac{1}{5}\right)\Box R - \frac{1}{720}R^{\mu\nu}R_{\mu\nu} + \frac{1}{720}R^{\mu\nu\rho\tau}R_{\mu\nu\rho\tau}, \tag{7.266}$$

which agrees with the results for a neutral scalar field in [68].

Charged scalar field: Having already calculated the correct expression when the gauge field vanishes, we can ignore terms that do not involve A_{μ} and instead calculate the correction $V^{(4)}_{10}$ due to the gauge field. Then, we have

$$0 = 4V^{(4)}_{10} - 2\,\mathrm{i}qA^{\mu}\nabla_{\mu}V^{(4)}_{00} - \mathrm{i}q\left(\nabla_{\mu}A^{\mu}\right)V^{(4)}_{00} - q^2A_{\mu}A^{\mu}V^{(4)}_{00} + 2D^{\mu}\widetilde{V}^{(4)}_{01\mu} - 2\,\mathrm{i}qA^{\mu}\widehat{V}^{(4)}_{01\mu} + 2\,g^{\mu\nu}\widetilde{V}^{(4)}_{02(\mu\nu)}. \tag{7.267}$$

We have

$$\begin{aligned}& -2\,\mathrm{i}qA^{\mu}\nabla_{\mu}V^{(4)}_{00} - \mathrm{i}q\left(\nabla_{\mu}A^{\mu}\right)V^{(4)}_{00} - q^2A_{\mu}A^{\mu}V^{(4)}_{00} \\ & = -\mathrm{i}q\left(\xi-\frac{1}{6}\right)A^{\mu}R_{;\mu} - \frac{1}{2}\,\mathrm{i}q\left[m^2+\left(\xi-\frac{1}{6}\right)R\right]D_{\mu}A^{\mu} - \frac{1}{2}q^2\left[m^2+\left(\xi-\frac{1}{6}\right)R\right]A_{\mu}A^{\mu},\end{aligned} \tag{7.268}$$

$$2D^{\mu}\widetilde{V}^{(4)}_{01\mu} = -\frac{1}{2}\left(\xi-\frac{1}{6}\right)\mathrm{i}q\left[m^2+\left(\xi-\frac{1}{6}\right)R\right]D_{\mu}A^{\mu} + \mathrm{i}q\left(\xi-\frac{1}{6}\right)A^{\mu}\nabla_{\mu}R - \frac{1}{6}\mathrm{i}q\,\nabla^{\mu}\nabla^{\alpha}F_{\alpha\mu} - \frac{1}{6}q^2A^{\mu}\nabla^{\alpha}F_{\alpha\mu}, \tag{7.269}$$

$$-2\,\mathrm{i}qA^{\mu}\widehat{V}^{(4)}_{01\mu} = \frac{1}{2}\,\mathrm{i}q\left(\xi-\frac{1}{6}\right)A^{\mu}R_{;\mu} + q^2\left[m^2+\left(\xi-\frac{1}{6}\right)R\right]A_{\mu}A^{\mu}, \tag{7.270}$$

$$2\, g^{\mu\nu}\widetilde{V}^{(4)}_{02(\mu\nu)} = -\frac{1}{2}\left[m^2 + \left(\xi - \frac{1}{6}\right)R\right]D_\mu A^\mu - \frac{1}{2}\,\mathrm{i}q\left(\xi - \frac{1}{6}\right)A^\mu\nabla_\mu R + \frac{1}{12}q^2F^{\mu\nu}F_{\mu\nu} - \frac{1}{6}q^2A^\mu\nabla^\nu F_{\mu\nu} - \frac{1}{12}\,\mathrm{i}q\,\nabla^\mu\nabla^\nu F_{\mu\nu}. \tag{7.271}$$

Then, we obtain for the $\widetilde{V}^{(4)}_{10}$ expansion coefficient the expression

$$\widetilde{V}^{(4)}_{10} = -\frac{1}{48}q^2F^{\mu\nu}F_{\mu\nu}. \tag{7.272}$$

From (7.156), the $V^{(4)}_{10}$ expansion coefficient is given by

$$V^{(4)}_{10} = \widehat{V}^{(4)}_{10} + \widetilde{V}^{(4)}_{10}. \tag{7.273}$$

Then, using (7.266) and (7.272), we have

$$V^{(4)}_{10} = \frac{1}{8}\left[m^2 + \left(\xi - \frac{1}{6}\right)R\right]^2 - \frac{1}{24}\left(\xi - \frac{1}{5}\right)\Box R - \frac{1}{720}R^{\mu\nu}R_{\mu\nu} + \frac{1}{720}R^{\mu\nu\rho\tau}R_{\mu\nu\rho\tau} - \frac{1}{48}q^2F^{\mu\nu}F_{\mu\nu}. \tag{7.274}$$

7.3 Hadamard form for odd dimensions

In this section, we develop the general formalism for the Hadamard renormalisation procedure of charged scalar fields in a general background spacetime with an odd number of dimensions. We will then focus on the specific case of $d = 3$ to derive explicit expressions for the renormalisation counterterms up to the order required to evaluate the RSET.

7.3.1 Hadamard renormalisation procedure in odd dimensions

We would like to evaluate the inhomogeneous Klein-Gordon equation (6.2) for the odd-dimensional Hadamard parametrix (6.7). We begin by noting, from (6.9), that $\alpha_3 = \frac{1}{4\pi\sqrt{2}}$. We remind the reader that we write the number of spacetime dimensions as $d = 2p+1$ with $p \geq 1$. Then (6.2) becomes

$$\begin{aligned}&\left(D_\mu D^\mu - m^2 - \xi R\right)G^{(2p+1)}_{\mathrm{F}}\\ &\quad= \frac{\mathrm{i}\,\alpha_{2p+1}}{2}\left(D_\mu D^\mu - m^2 - \xi R\right)\left\{\frac{U^{(2p+1)}(x,x')}{[\sigma(x,x') + \mathrm{i}\epsilon]^{p-\frac{1}{2}}} + W^{(2p+1)}(x,x')\right\}\\ &\quad= -\frac{1}{\sqrt{-g\,(x)}}\,\delta^{(2p+1)}(x-x'). \end{aligned} \tag{7.275}$$

It will be convenient instead to evaluate the equation

$$
\begin{aligned}
&-\frac{2\,\mathrm{i}}{\alpha_{2p+1}}\left(D_\mu D^\mu - m^2 - \xi R\right) G_{\mathrm{F}}^{(2p+1)} \\
&\qquad = \left(D_\mu D^\mu - m^2 - \xi R\right)\left\{\frac{U^{(2p+1)}(x,x')}{[\sigma(x,x')+\mathrm{i}\epsilon]^{p-\frac{1}{2}}} + W^{(2p+1)}(x,x')\right\} \\
&\qquad\qquad = -\frac{2\,\mathrm{i}}{\alpha_{2p+1}}\frac{1}{\sqrt{-g\,(x)}}\,\delta^{(2p+1)}(x-x')\,. \qquad (7.276)
\end{aligned}
$$

Then, suppressing arguments of the biscalar functions, we begin by calculating the quantity

$$
\begin{aligned}
-\frac{2\,\mathrm{i}}{\alpha_{2p+1}}\,D_\mu G_{\mathrm{F}}^{(2p+1)} &= \left(\nabla_\mu - \mathrm{i}qA_\mu\right)\left[\sigma^{-p+\frac{1}{2}}\,U^{(2p+1)} + W^{(2p+1)}\right] \\
&= \sigma^{-p+\frac{1}{2}}\,\nabla_\mu U^{(2p+1)} - \left(p-\frac{1}{2}\right)\sigma^{-p-\frac{1}{2}}\,U^{(2p+1)}\,\nabla_\mu\sigma + \nabla_\mu W^{(2p+1)} \\
&\quad - \sigma^{-p+\frac{1}{2}}\,\mathrm{i}qA_\mu U^{(2p+1)} - \mathrm{i}qA_\mu W^{(2p+1)}. \qquad (7.277)
\end{aligned}
$$

Acting on (7.277) with another gauge derivative, we obtain

$$
\begin{aligned}
-\frac{2\,\mathrm{i}}{\alpha_{2p+1}}\,D_\mu D^\mu G_{\mathrm{F}}^{(2p+1)} &= \sigma^{-p+\frac{1}{2}}\,\nabla^\mu\nabla_\mu U^{(2p+1)} - (2p-1)\,\sigma^{-p-\frac{1}{2}}\left[\nabla_\mu U^{(2p+1)}\right](\nabla^\mu\sigma) \\
&\quad + \left(p-\frac{1}{2}\right)\left(p+\frac{1}{2}\right)\sigma^{-p-\frac{3}{2}}\,U^{(2p+1)}(\nabla^\mu\sigma)(\nabla_\mu\sigma) \\
&\quad - \left(p-\frac{1}{2}\right)\sigma^{-p-\frac{1}{2}}\,U^{(2p+1)}(\nabla^\mu\nabla_\mu\sigma) + \nabla^\mu\nabla_\mu W^{(2p+1)} \\
&\quad - \sigma^{-p+\frac{1}{2}}\,\mathrm{i}q\,(\nabla^\mu A_\mu)\,U^{(2p+1)} - 2\,\sigma^{-p+\frac{1}{2}}\,\mathrm{i}qA_\mu\left[\nabla^\mu U^{(2p+1)}\right] \\
&\quad + (2p-1)\,\sigma^{-p-\frac{1}{2}}\,\mathrm{i}qA_\mu U^{(2p+1)}(\nabla^\mu\sigma) - \mathrm{i}q\,(\nabla^\mu A_\mu)\,W^{(2p+1)} \\
&\quad - 2\,\mathrm{i}qA_\mu\left[\nabla^\mu W^{(2p+1)}\right] - \sigma^{-p+\frac{1}{2}}\,q^2A^\mu A_\mu U^{(2p+1)} - q^2A^\mu A_\mu W^{(2p+1)}. \\
&\qquad (7.278)
\end{aligned}
$$

We can simplify (7.278) using the identities involving Synge's world function (1.20) and the van Vleck-Morette determinant (1.28); we obtain

$$
\begin{aligned}
-\frac{2\,\mathrm{i}}{\alpha_{2p+1}}\,D_\mu D^\mu G_{\mathrm{F}}^{(2p+1)} &= \sigma^{-p+\frac{1}{2}}\,\nabla^\mu\nabla_\mu U^{(2p+1)} - (2p-1)\,\sigma^{-p-\frac{1}{2}}\left[\nabla_\mu U^{(2p+1)}\right](\nabla_\mu\sigma) \\
&\quad + (2p-1)\,\sigma^{-p-\frac{1}{2}}\,U^{(2p+1)}\Delta^{-\frac{1}{2}}\left(\nabla_\mu\Delta^{\frac{1}{2}}\right)(\nabla^\mu\sigma) + \nabla^\mu\nabla_\mu W^{(2p+1)} \\
&\quad - \sigma^{-p+\frac{1}{2}}\,\mathrm{i}q\,(\nabla^\mu A_\mu)\,U^{(2p+1)} - 2\,\sigma^{-p+\frac{1}{2}}\,\mathrm{i}qA_\mu\left[\nabla^\mu U^{(2p+1)}\right] \\
&\quad + (2p-1)\,\sigma^{-p-\frac{1}{2}}\,\mathrm{i}qA_\mu U^{(2p+1)}(\nabla^\mu\sigma) - \mathrm{i}q\,(\nabla^\mu A_\mu)\,W^{(2p+1)} \\
&\quad - 2\,\mathrm{i}qA_\mu\left[\nabla^\mu W^{(2p+1)}\right] - \sigma^{-p+\frac{1}{2}}\,q^2A^\mu A_\mu U^{(2p+1)} - q^2A^\mu A_\mu W^{(2p+1)}. \\
&\qquad (7.279)
\end{aligned}
$$

This allows us to evaluate the l.h.s of the inhomogeneous Klein-Gordon equation (6.2) for $d = 2p+1$ as follows

$$
\begin{aligned}
-\frac{2\,\mathrm{i}}{\alpha_{2p+1}}\,&(D^\mu D_\mu - m^2 - \xi R)\, G_{\mathrm{F}}^{(2p+1)} \\
&= \sigma^{-p+\frac{1}{2}}\, \nabla^\mu \nabla_\mu U^{(2p+1)} - (2p-1)\, \sigma^{-p-\frac{1}{2}} \left[\nabla_\mu U^{(2p+1)}\right] (\nabla_\mu \sigma) \\
&+ (2p-1)\, \sigma^{-p-\frac{1}{2}}\, U^{(2p+1)} \Delta^{-\frac{1}{2}} \left(\nabla_\mu \Delta^{\frac{1}{2}}\right) (\nabla^\mu \sigma) + \nabla^\mu \nabla_\mu W^{(2p+1)} \\
&- \sigma^{-p+\frac{1}{2}}\, \mathrm{i}q\, (\nabla^\mu A_\mu)\, U^{(2p+1)} - 2\, \sigma^{-p+\frac{1}{2}}\, \mathrm{i}q A_\mu \left[\nabla^\mu U^{(2p+1)}\right] \\
&+ (2p-1)\, \sigma^{-p-\frac{1}{2}}\, \mathrm{i}q A_\mu U^{(2p+1)} (\nabla^\mu \sigma) - \mathrm{i}q\, (\nabla^\mu A_\mu)\, W^{(2p+1)} \\
&- 2\, \mathrm{i}q A_\mu \left[\nabla^\mu W^{(2p+1)}\right] - \sigma^{-p+\frac{1}{2}}\, q^2 A^\mu A_\mu U^{(2p+1)} - q^2 A^\mu A_\mu W^{(2p+1)} \\
&- \sigma^{-p+\frac{1}{2}} \left(m^2 + \xi R\right) U^{(2p+1)} - \left(m^2 + \xi R\right) W^{(2p+1)}. \qquad (7.280)
\end{aligned}
$$

We can simplify (7.280) by use of the gauge covariant derivative, which then reduces to

$$
\begin{aligned}
-\frac{2\,\mathrm{i}}{\alpha_{2p+1}}\, &\left(D^\mu D_\mu - m^2 - \xi R\right) G_{\mathrm{F}}^{(2p+1)} \\
&= \sigma^{-p+\frac{1}{2}} \left(D_\mu D^\mu - m^2 - \xi R\right) U^{(2p+1)} + \left(D_\mu D^\mu - m^2 - \xi R\right) W^{(2p+1)} \\
&\quad - (2p-1)\, \sigma^{-p-\frac{1}{2}} \left[\left(D_\mu - \Delta^{-\frac{1}{2}} \Delta^{\frac{1}{2}}_{;\mu}\right) U^{(2p+1)}\right] \sigma^{;\mu}. \qquad (7.281)
\end{aligned}
$$

From the r.h.s of the inhomogeneous scalar field equation (6.2), we know that (7.281) must be equal to zero when the two points x and x' are separated. Since Synge's world function is nonzero when $x' \neq x$, we deduce that (7.281) must vanish identically at each power of $\sigma(x, x')$. This allows us to generate two equations by considering terms proportional to integer powers of σ and, separately, terms proportional to fractional powers of σ. In particular, the terms that are proportional to integer powers of σ must vanish independently since no other terms can contain this factor due to the form of the expansion for the biscalar $U^{(2p+1)}(x, x')$ in (6.8a), which itself only contains terms proportional to integer powers of σ. This allows us to write

$$
\left(D_\mu D^\mu - m^2 - \xi R\right) W^{(2p+1)} \left(x, x'\right) = 0, \qquad (7.282)
$$

which means that the biscalar $W^{(2p+1)}(x, x')$ solves the homogeneous scalar field equation. Equation (7.282) generalises (44) in [68] and it enables us derive the recurrence relations for the Hadamard coefficients $W_n^{(2p+1)}(x, x')$. Since the biscalar $W^{(2p+1)}(x, x')$ in (6.8b) admits a power series expansion in $\sigma(x, x')$, we can derive the recurrence relation for the $W_n^{(2p+1)}(x, x')$ by expanding the terms in (7.282). Suppressing arguments, we first evaluate

$$
D_\mu W^{(2p+1)} = \sum_{n=0}^{\infty} \left\{ \left[\nabla_\mu W_n^{(2p+1)}\right] \sigma^n + n W_n^{(2p+1)} \sigma^{n-1} \left(\nabla_\mu \sigma\right) - \mathrm{i}q A_\mu W_n^{(2p+1)} \sigma^n \right\}, \qquad (7.283)
$$

and then

$$\begin{aligned} D_\mu D^\mu W^{(2p+1)} = \sum_{n=0}^{\infty} \Big\{ & \Big[\nabla^\mu \nabla_\mu W_n^{(2p+1)}\Big] \sigma^n + 2n \Big[\nabla_\mu W_n^{(2p+1)}\Big] \sigma^{n-1} (\nabla^\mu \sigma) \\ & + n(n-1)\, W_n^{(2p+1)} \sigma^{n-2} \, (\nabla^\mu \sigma)(\nabla_\mu \sigma) + n W_n^{(2p+1)} \sigma^{n-1} \, (\nabla^\mu \nabla_\mu \sigma) \\ & - \mathrm{i}q \, (\nabla^\mu A_\mu) \, W_n^{(2p+1)} \sigma^n - 2\, \mathrm{i}q A_\mu \Big[\nabla^\mu W_n^{(2p+1)}\Big] \sigma^n \\ & - 2\, \mathrm{i}q A_\mu \, n \, W_n^{(2p+1)} \sigma^{n-1} \, (\nabla^\mu \sigma) - q^2 A^\mu A_\mu W_n^{(2p+1)} \sigma^n \Big\}. \end{aligned} \tag{7.284}$$

Then, using (1.20), (1.28), (6.24) and the gauge covariant derivative, (7.284) becomes

$$\begin{aligned} D_\mu D^\mu W^{(2p+1)} = \sum_{n=0}^{\infty} \Big\{ & \Big[D^\mu D_\mu W_n^{(2p+1)}\Big] \sigma^n + 2n \Big[D_\mu W_n^{(2p+1)}\Big] \sigma^{n-1} (\nabla^\mu \sigma) \\ & + n \, [2(n-1) + (2p+1)] \, W_n^{(2p+1)} \sigma^{n-1} \\ & - 2n \, W_n^{(2p+1)} \Delta^{-\frac{1}{2}} \left(\nabla_\mu \Delta^{\frac{1}{2}}\right) \sigma^{n-1} \, (\nabla^\mu \sigma) \Big\}. \end{aligned} \tag{7.285}$$

Substituting (7.285) into (7.282), we obtain

$$\begin{aligned} 0 = \sum_{n=0}^{\infty} \Big\{ & \Big[\left(D_\mu D^\mu - m^2 - \xi R\right) W_n^{(2p+1)}\Big] \sigma^n + 2n \Big[D_\mu W_n^{(2p+1)}\Big] \sigma^{n-1} (\nabla^\mu \sigma) \\ & + n \, [2(n-1) + (2p+1)] \, W_n^{(2p+1)} \sigma^{n-1} \\ & - 2n \, W_n^{(2p+1)} \Delta^{-\frac{1}{2}} \left(\nabla_\mu \Delta^{\frac{1}{2}}\right) \sigma^{n-1} \, (\nabla^\mu \sigma) \Big\}. \end{aligned} \tag{7.286}$$

Performing the relabelling $n \to n+1$ in terms proportional to σ^{n-1} in (7.286), we obtain

$$\begin{aligned} 0 = \sum_{n=0}^{\infty} \Big\{ & \Big[\left(D_\mu D^\mu - m^2 - \xi R\right) W_n^{(2p+1)}\Big] \sigma^n + 2(n+1) \Big[D_\mu W_{n+1}^{(2p+1)}\Big] \sigma^{n} (\nabla^\mu \sigma) \\ & + (n+1)(2n+2p+1) \, W_{n+1}^{(2p+1)} \sigma^{n} \\ & - 2(n+1) \, W_{n+1}^{(2p+1)} \Delta^{-\frac{1}{2}} \left(\nabla_\mu \Delta^{\frac{1}{2}}\right) \sigma^{n} \, (\nabla^\mu \sigma) \Big\}. \end{aligned} \tag{7.287}$$

Since (7.287) must hold for each power of σ, this enables us to obtain the recurrence relation for the Hadamard coefficients $W_n^{(2p+1)}(x,x')$. We have

$$\begin{aligned} & (n+1)(2n+2p+1) \, W_{n+1}^{(2p+1)} + 2(n+1) \, \sigma^{;\mu} D_\mu W_{n+1}^{(2p+1)} \\ & \quad - 2(n+1) \, W_{n+1}^{(2p+1)} \, \Delta^{-\frac{1}{2}} \Delta^{\frac{1}{2}}{}_{;\mu} \, \sigma^{;\mu} + \left(D_\mu D^\mu - m^2 - \xi R\right) W_n^{(2p+1)} = 0 \\ & \qquad \text{for } n \text{ in } \mathbb{N}. \end{aligned} \tag{7.288}$$

This generalises equation (43) in [68]. Returning to (7.281), the terms that are proportional to fractional powers of σ, give

$$\sigma \left(D_\mu D^\mu - m^2 - \xi R\right) U^{(2p+1)} = (2p-1) \left[\left(D_\mu - \Delta^{-\frac{1}{2}} \Delta^{\frac{1}{2}}{}_{;\mu}\right) U^{(2p+1)}\right] \sigma^{;\mu}. \tag{7.289}$$

It will be convenient to rewrite (7.289) as

$$\left(D_\mu D^\mu - m^2 - \xi R\right) U^{(2p+1)} - (2p-1)\,\sigma^{-1}\sigma^{\,;\mu} D_\mu U^{(2p+1)} \\ + (2p-1)\,\sigma^{-1} U^{(2p+1)} \Delta^{-\frac{1}{2}} \Delta^{\frac{1}{2}}{}_{;\mu}\,\sigma^{\,;\mu} = 0. \quad (7.290)$$

We derive the recurrence relation for the Hadamard coefficients $U_n^{(2p+1)}(x,x')$ by inserting the power series expansion for $U^{(2p+1)}(x,x')$ into (7.290). Since the power series expansions for the biscalars $U^{(2p+1)}(x,x')$ (6.8a) and the $W^{(2p+1)}(x,x')$ (6.8b) are structurally similar expressions, we can use (7.285) to evaluate the $D_\mu D^\mu U^{(2p+1)}$, remembering to interchange the $W_n^{(2p+1)}$ for the $U_n^{(2p+1)}$. Then, the first term on the l.h.s of (7.290) is given by

$$\left(D_\mu D^\mu - m^2 - \xi R\right) U^{(2p+1)} = \sum_{n=0}^{\infty} \Big\{ \Big[\left(D^\mu D_\mu - m^2 - \xi R\right) U_n^{(2p+1)}\Big]\sigma^n \\ + 2n\Big[D_\mu U_n^{(2p+1)}\Big]\sigma^{n-1}(\nabla^\mu\sigma) + n\,[2(n-1) + (2p+1)]\, U_n^{(2p+1)}\sigma^{n-1} \\ - 2n\, U_n^{(2p+1)} \Delta^{-\frac{1}{2}} \Big(\nabla_\mu \Delta^{\frac{1}{2}}\Big)\sigma^{n-1}(\nabla^\mu\sigma)\Big\}. \quad (7.291)$$

The second term on the l.h.s of (7.290) is given by

$$-(2p-1)\,\sigma^{-1}\sigma^{\,;\mu} D_\mu U^{(2p+1)} \\ = -(2p-1)\,\sigma^{-1}\sigma^{\,;\mu}\,\nabla_\mu\Big[U_n^{(2p+1)}\sigma^n\Big] + (2p-1)\,\mathrm{i}q\,\sigma^{-1}\sigma^{\,;\mu} A_\mu U_n^{(2p+1)}\sigma^n \\ = -(2p-1)\Big[\nabla_\mu U_n^{(2p+1)}\Big]\sigma^{n-1}\sigma^{\,;\mu} - (2p-1)n\, U_n^{(2p+1)}\sigma^{n-2}\sigma^{\,;\mu}\sigma_{\,;\mu} \\ + (2p-1)\,\mathrm{i}qA_\mu\, U_n^{(2p+1)}\sigma^{n-1}\sigma^{\,;\mu} \\ = -(2p-1)\Big[D_\mu U_n^{(2p+1)}\Big]\sigma^{n-1}\sigma^{\,;\mu} - 2\,(2p-1)n\, U_n^{(2p+1)}\sigma^{n-1}. \quad (7.292)$$

The third term on the l.h.s of (7.290) is given by

$$(2p-1)\,\sigma^{-1}U_n^{(2p+1)}\sigma^n \Delta^{-\frac{1}{2}}\Delta^{\frac{1}{2}}{}_{;\mu}\,\sigma^{\,;\mu} = (2p-1)\,U_n^{(2p+1)}\sigma^{n-1}\Delta^{-\frac{1}{2}}\Delta^{\frac{1}{2}}{}_{;\mu}\,\sigma^{\,;\mu}. \quad (7.293)$$

Inserting (7.291), (7.292) and (7.293) into (7.290), we obtain

$$0 = \sum_{n=0}^{\infty}\Big\{\Big[\left(D_\mu D^\mu - m^2 - \xi R\right) U_n^{(2p+1)}\Big]\sigma^n + (2n-2p+1)\,\Big[D_\mu U_n^{(2p+1)}\Big]\sigma^{n-1}(\nabla^\mu\sigma) \\ + n\,(2n-2p+1)\,U_n^{(2p+1)}\sigma^{n-1} - (2n-2p+1)\,U_n^{(2p+1)}\Delta^{-\frac{1}{2}}\Big(\nabla_\mu\Delta^{\frac{1}{2}}\Big)\sigma^{n-1}(\nabla^\mu\sigma)\Big\}. \\ (7.294)$$

Performing the relabelling $n \to n+1$ in terms proportional to σ^{n-1} in (7.20), we obtain

$$0=\sum_{n=0}^{\infty}\Bigg\{\Big[\big(D_{\mu}D^{\mu}-m^2-\xi R\big)\,U_n^{(2p+1)}\Big]\sigma^n+[2n+4-(2p+1)]\Big[\sigma^{\,;\mu}D_{\mu}U_{n+1}^{(2p+1)}\Big]\sigma^n$$
$$+(n+1)\,[2n+4-(2p+1)]\,U_{n+1}^{(2p+1)}\sigma^n-[2n+4-(2p+1)]\,U_{n+1}^{(2p+1)}\sigma^n\Delta^{-\frac{1}{2}}\Delta^{\frac{1}{2}}_{;\mu}\,\sigma^{\,;\mu}\Bigg\}$$
$$-(2p-1)\,\sigma^{-1}\left\{\sigma^{\,;\mu}D_{\mu}U_0^{(2p+1)}-U_0^{(2p+1)}\Delta^{-\frac{1}{2}}\Delta^{\frac{1}{2}}_{;\mu}\,\sigma^{\,;\mu}\right\}. \tag{7.295}$$

Since (7.295) must hold for each power of σ, the terms proportional to σ^n enable us to obtain the recurrence relation for the Hadamard coefficients $U_n^{(2p+1)}(x,x')$; we have

$$0=(n+1)\,[2n+4-(2p+1)]\,U_{n+1}^{(2p+1)}+[2n+4-(2p+1)]\Big[D_{\mu}U_{n+1}^{(2p+1)}\Big]\sigma^{\,;\mu}$$
$$-[2n+4-(2p+1)]\,U_{n+1}^{(2p+1)}\Delta^{-\frac{1}{2}}\Delta^{\frac{1}{2}}_{;\mu}\,\sigma^{\,;\mu}$$
$$+\Big[\big(D_{\mu}D^{\mu}-m^2-\xi R\big)\,U_n^{(2p+1)}\Big]\quad\text{for } n \text{ in } \mathbb{N}. \tag{7.296}$$

This generalises equation (42a) in [68]. The lowest order terms in σ in (7.295), i.e. those proportional to σ^{-1}, give the boundary condition for the $U_0^{(2p+1)}$ Hadamard coefficient:

$$0=\left(\sigma^{\,;\mu}D_{\mu}-\Delta^{-\frac{1}{2}}\Delta^{\frac{1}{2}}{}_{;\mu}\sigma^{\,;\mu}\right)U_0^{(2p+1)}. \tag{7.297}$$

In the uncharged case, (7.297) reduces to

$$0=\sigma^{\,;\mu}\,U_{0\,;\mu}^{(2p+1)}-\sigma^{\,;\mu}\,U_0^{(2p+1)}\Delta^{-\frac{1}{2}}\Delta^{\frac{1}{2}}{}_{;\mu}, \tag{7.298}$$

and we can see that equation (7.298) is solved by taking either $U_0^{(2p+1)}=\Delta^{\frac{1}{2}}$ or $U_0^{(2p+1)}=-\Delta^{\frac{1}{2}}$. Our guiding principle will be that the leading-order singularity in the Hadamard parametrix (6.7) matches that of Minkowski spacetime [68]; in the limit $x'\to x$, we have

$$U_0^{(2p+1)}(x,x)=1. \tag{7.299}$$

Therefore, in the uncharged case, (7.298) is solved by

$$U_0^{(2p+1)}(x,x')=\Delta^{\frac{1}{2}}. \tag{7.300}$$

In the charged case, (7.297) cannot be solved exactly and we can expand $U_0^{(2p+1)}$ as a power series in $\sigma(x,x')$ up to the order required for the evaluation of the RSET.

7.3.2 Explicit renormalisation counterterms in three dimensions

In order to derive explicit renormalisation counterterms, we focus on the specific case of $d=3$, which is the simplest non-trivial number of odd spacetime dimensions that we can consider. We are required to evaluate $U_0^{(3)}(x,x')$ up to $\mathcal{O}\big(\sigma^{3/2}\big)$ in order to evaluate the RSET. Rewriting (7.297) as

$$\Big[D_{\mu}\,U_0^{(3)}\Big]\sigma^{\,;\mu}-U_0^{(3)}\Delta^{-\frac{1}{2}}\Delta^{\frac{1}{2}}_{;\mu}\,\sigma^{\,;\mu}=0, \tag{7.301}$$

we note that (7.301), which is satisfied by the $U_0^{(3)}(x,x')$ Hadamard coefficient, is structurally identical to (7.86), which is satisfied by the $U_0^{(4)}(x,x')$ Hadamard coefficient. Then, we may write down the required terms of the $U_0^{(3)}(x,x')$ Hadamard coefficient by simple comparison with (7.88), (7.89), (7.90) and (7.102). Then, we have

$$U_{00}^{(3)} = 1, \tag{7.302}$$

$$U_{01\mu}^{(3)} = \mathrm{i}qA_\mu, \tag{7.303}$$

$$U_{02(\mu\nu)}^{(3)} = \frac{1}{12}R_{\mu\nu} - \frac{1}{2}\,\mathrm{i}qD_{(\mu}A_{\nu)} \tag{7.304}$$

$$U_{03(\mu\nu\rho)}^{(3)} = -\frac{1}{24}R_{(\mu\nu;\rho)} + \frac{\mathrm{i}q}{6}D_{(\mu}D_\nu A_{\rho)} + \frac{\mathrm{i}q}{12}A_{(\mu}R_{\nu\rho)}. \tag{7.305}$$

As earlier stated, in $d=3$ we require terms up to $\mathcal{O}\big(\sigma^{3/2}\big)$ in order to evaluate the RSET. In the expansion of the $U^{(3)}(x,x')$ biscalar (6.8a), the $U_1^{(3)}(x,x')$ Hadamard coefficient is multiplied by σ; therefore, we are also required to evaluate the Taylor expansion of $U_1^{(3)}(x,x')$ up to $\mathcal{O}\big(\sigma^{1/2}\big)$. We can obtain the equation satisfied by $U_1^{(3)}(x,x')$ by using (7.296) with $n=0$ and $p=1$; doing so, we have

$$U_1^{(3)} + \sigma^{;\mu}D_\mu U_1^{(3)} - U_1^{(3)}\Delta^{-\frac{1}{2}}\Delta^{\frac{1}{2}}_{;\mu}\sigma^{;\mu} + \big(D^\mu D_\mu - m^2 - \xi R\big)\,U_0^{(3)} = 0. \tag{7.306}$$

At this point one may notice that the equation (7.306) satisfied by the $U_1^{(3)}(x,x')$ Hadamard coefficient in $d=3$ has a very similar form to the equation (7.107) satisfied by the $V_0^{(4)}(x,x')$ Hadamard coefficient in $d=4$. The difference lies in the fact that in (7.306), all terms share the same multiplicative constant, whereas in (7.107), the first three terms on the r.h.s are multiplied by a factor of 2, while the last term is multiplied by 1. Thus, adjusting for the minor difference in factor, we can write down the explicit expressions for $U_{10}^{(3)}$ and $U_{11\mu}^{(3)}$ from $V_{00}^{(4)}$ (7.117) and (7.176) respectively; we have

$$U_{10}^{(3)} = m^2 + \left(\xi - \frac{1}{6}\right)R, \tag{7.307}$$

$$U_{11\mu}^{(3)} = \frac{1}{2}\left(\xi - \frac{1}{6}\right)R_{;\mu}. \tag{7.308}$$

Chapter 8

Renormalised expectation values

In §8.1, we give a detailed introduction to the Hadamard renormalisation procedure. We derive several identities concerning the biscalars in §8.2 that will aid us in deriving expressions for the renormalised expectation values of observables in §8.3. In §8.4, we consider ambiguities in the renormalised expectation values of observables in and we examine the trace anomaly in §8.5.

8.1 Introduction

In the previous section, we derived the explicit expressions for the biscalar functions $U^{(d)}(x,x')$ and $V^{(d)}(x,x')$ that appear in the Hadamard parametrices (6.3), (6.5) and (6.7) associated to a charged scalar field, for a general spacetime geometry and gauge field, up to the required order in $\sigma(x,x')$ for the renormalisation of the stress-energy tensor. These expressions depend only on the background spacetime geometry and the fixed, classical, background gauge field.

The biscalar $U^{(d)}(x,x')$ is therefore uniquely determined in even dimensions with $d \neq 2$ by the boundary condition on $U_0^{(2p)}(x,x')$ (7.82) and the recurrence relations for the $U_n^{(2p)}(x,x')$ (7.72), and in odd dimensions by the boundary condition on $U_0^{(2p+1)}(x,x')$ (7.297) and the recurrence relations for the $U_n^{(2p+1)}(x,x')$ (7.296). Similarly, the biscalar $V^{(d)}(x,x')$ is uniquely determined in $d = 2$ by the boundary condition on $V_0^{(2)}(x,x')$ (7.25) and the recurrence relations for the $V_n^{(2)}(x,x')$ (7.14), and in even dimensions with $d \neq 2$ by the boundary condition on $V_0^{(2p)}(x,x')$ (7.64) and the recurrence relations for the $V_n^{(2p)}(x,x')$ (7.61). While the biscalar functions $U^{(d)}(x,x')$ and $V^{(d)}(x,x')$ are both regular in the coincidence limit $x' \to x$, they each multiply terms in the Hadamard parametrix that are divergent as $x' \to x$.

The biscalar $W^{(d)}(x,x')$, by contrast, neither depends purely on the spacetime geometry and the background gauge field, nor can it be uniquely determined. Indeed, this can be noted from the fact that one cannot derive a boundary condition for the lowest order Hadamard coefficient $W_0^{(d)}(x,x')$ of the $W^{(d)}(x,x')$ biscalar, as opposed to the case for the $U_0^{(d)}(x,x')$ Hadamard coefficient of the $U^{(d)}(x,x')$ biscalar or for the case of the $V_0^{(d)}(x,x')$ Hadamard coefficient of the $V^{(d)}(x,x')$ biscalar. As a consequence, this arbitrariness extends to all higher order Hadamard coefficients $W_n^{(d)}(x,x')$ of the $W^{(d)}(x,x')$

biscalar. The indeterminacy in the $W_0^{(d)}(x,x')$ Hadamard coefficient can be used to encode the details of the quantum state under consideration. Once $W_0^{(d)}(x,x')$ has been specified, however, the rest of the Hadmard coefficients of the $W^{(d)}(x,x')$ biscalar can be uniquely determined using the recurrence relations (7.22) in $d=2$, the recurrence relations (7.79) in even dimensions with $d\neq 2$ and the recurrence relations (7.288) in odd dimensions. The $W^{(d)}(x,x')$ biscalar is regular in the coincidence limit $x'\to x$.

We can summarise the above by writing the Feynman Green's function $G_{\mathrm{F}}^{(d)}(x,x')$ as the sum of a uniquely-determined, state-independent part $G_{\mathrm{S}}^{(d)}(x,x')$, which is singular in the coincidence limit, and a state-dependent part $G_{\mathrm{R}}^{(d)}(x,x')$, which is regular as $x'\to x$. Then, we have

$$G_{\mathrm{F}}^{(d)}(x,x') = G_{\mathrm{S}}^{(d)}(x,x') + G_{\mathrm{R}}^{(d)}(x,x')\,, \tag{8.1}$$

where the quantity $G_{\mathrm{S}}^{(d)}(x,x')$ in (8.1) is defined by

$$-\mathrm{i}\,G_{\mathrm{S}}^{(d)}(x,x') = \begin{cases} \alpha^{(2)}\,V^{(2)}(x,x')\ln\left[\dfrac{\sigma(x,x')}{\ell_{\mathrm{ren}}^2}+\mathrm{i}\epsilon\right] & d=2,\\[2ex] \alpha^{(2p)}\left\{\dfrac{U^{(2p)}(x,x')}{[\sigma(x,x')+\mathrm{i}\epsilon]^{p-1}}+V^{(2p)}(x,x')\ln\left[\dfrac{\sigma(x,x')}{\ell_{\mathrm{ren}}^2}+\mathrm{i}\epsilon\right]\right\} & d=2p,\\[2ex] \alpha^{(2p+1)}\,\dfrac{U^{(2p+1)}(x,x')}{[\sigma(x,x')+\mathrm{i}\epsilon]^{p-\frac{1}{2}}} & d=2p+1,\end{cases} \tag{8.2}$$

and the quantity $G_{\mathrm{R}}^{(d)}(x,x')$ in (8.1), which we will refer to as the regularised Green's function, is defined by

$$-\,\mathrm{i}\,G_{\mathrm{R}}^{(d)}(x,x') = -\mathrm{i}\left[G_{\mathrm{F}}^{(d)}(x,x') - G_{\mathrm{S}}^{(d)}(x,x')\right] = \alpha^{(d)}\,W^{(d)}(x,x')\,. \tag{8.3}$$

In this language, the first step of the Hadamard renormalisation procedure is to subtract, from the Feynman's Green function $G_{\mathrm{F}}^{(d)}(x,x')$, its singular part $G_{\mathrm{S}}^{(d)}(x,x')$ order by order in $\sigma(x,x')$ to give a Green's function $G_{\mathrm{R}}^{(d)}(x,x')$ that is regular in the coincidence limit $x'\to x$. The next step is to perform the relevant operation on the regularised Green's function $G_{\mathrm{R}}^{(d)}(x,x')$, where the form of the operation performed on $G_{\mathrm{R}}^{(d)}(x,x')$ depends on whether we are calculating the scalar field condensate, the renormalised current or the RSET, before taking the coincidence limit $x'\to x$.

Our aim is to provide the general framework for the extension of the Hadamard renormalisation procedure to charged scalar fields and, therefore, our discussion will be sufficiently general so as not to specify a quantum state under consideration. We will also consider an arbitrarily curved background spacetime as well as a general background gauge field. Thus, the biscalar $W^{(d)}(x,x')$ will be left undetermined in the following sections. If one were to use the framework developed below for a practical application, specifying the quantum state under consideration as well as the spacetime geometry and background gauge field, then explicit expressions for the biscalar $W^{(d)}(x,x')$, the vacuum polarisation and the renormalised expectation values of the current and stress-energy tensor could be computed.

It is useful, when using the Hadamard renormalisation procedure for practical applications, to replace the biscalar $W^{(d)}(x,x')$ in (8.3) by its covariant Taylor expansion in the geodetic distance $\sigma^{;\mu}(x,x')$ between x and x', which is given by

$$W^{(d)}(x,x') = w_0^{(d)}(x) + w_{1\mu}^{(d)}(x)\,\sigma^{;\mu} + w_{2(\mu\nu)}^{(d)}(x)\,\sigma^{;\mu}\sigma^{;\nu} + w_{3(\mu\nu\rho)}^{(d)}(x)\,\sigma^{;\mu}\sigma^{;\nu}\sigma^{;\rho} + \dots, \quad (8.4)$$

where we have not written down the expansion coefficients past $w_{3(\mu\nu\rho)}^{(d)}(x)$ in (8.4) since we will only require terms up to and including this expansion coefficient in our following analysis.

Having constructed a regularised Green's function $G_{\mathrm{R}}^{(d)}(x,x')$ in (8.3), we can now derive expressions for the scalar condensate $\langle\hat{\Phi}\hat{\Phi}^{\dagger}\rangle_{\mathrm{ren}}$, the renormalised expectation value of the current $\langle\hat{J}^{\mu}\rangle_{\mathrm{ren}}$, and the RSET $\langle\hat{T}_{\mu\nu}\rangle_{\mathrm{ren}}$, in terms of the expansion coefficients of $W^{(d)}(x,x')$ defined in (8.4). In order to simplify the resulting expressions, it will be useful to derive some identities concerning the expansion coefficients of the biscalar $W^{(d)}(x,x')$. This will be the topic of the next section, before returning to derive expressions for the quantities listed above in following sections.

8.2 Identities concerning the biscalar $W^{(d)}(x,x')$

In the previous subsection, we described the general Hadamard renormalisation procedure, which includes operating on the regularised Green's function $G_{\mathrm{R}}^{(d)}(x,x')$ in (8.3) before taking the coincidence limit $x' \to x$. Any terms produced by operating on $G_{\mathrm{R}}^{(d)}(x,x')$ and, from (8.3), thereby $W^{(d)}(x,x')$ that are $\mathcal{O}(\sigma^{1/2})$ or higher, vanish as $x' \to x$. Therefore, we need only keep track of the lowest order coefficients of the covariant Taylor expansion of $W^{(d)}(x,x')$ in (8.4), with the exact number depending on the form of the operation performed on $G_{\mathrm{R}}^{(d)}(x,x')$. It will turn out that the most complicated operation we will perform on $G_{\mathrm{R}}^{(d)}(x,x')$ is to apply a second-order differential operator to it when we calculate the RSET. Thus, we need only keep track of terms upto and including the $w_{02(\mu\nu)}^{(d)}(x)$ coefficient in (8.4) for the purpose of evaluating the RSET.

However, when we consider the conservation of the stress-energy tensor, we will obtain a factor of $\nabla^{\mu} w_{02(\mu\nu)}^{(d)}$ from taking the divergence of the RSET, i.e. $\nabla^{\mu}\langle\hat{T}_{\mu\nu}\rangle_{\mathrm{ren}}$. Therefore, we will derive identities, concerning the expansion coefficients of the biscalar $W^{(d)}(x,x')$, relating both $w_{02(\mu\nu)}^{(d)}$ and $\nabla^{\mu} w_{02(\mu\nu)}^{(d)}$ to lower order coefficients in (8.4) respectively.

This can be achieved by substituting the covariant Taylor expansion of $W^{(d)}(x,x')$ (8.4) into the scalar field equations satisfied by the $W^{(d)}(x,x')$ biscalar in $d=2$ (7.16), in even dimensions with $d \neq 2$ (7.81) and in odd dimensions (7.282) and considering the resulting terms up to $\mathcal{O}(\sigma^{1/2})$. Due to the presence of two derivative operators in the scalar field equation for $W^{(d)}(x,x')$ in all dimensions, we anticipate that we should be able to derive identities relating $w_{02(\mu\nu)}^{(d)}$ and $\nabla^{\mu} w_{02(\mu\nu)}^{(d)}$ to lower order coefficients using the resulting terms at $\mathcal{O}(1)$ and $\mathcal{O}(\sigma^{1/2})$ respectively.

We will treat the case with even dimensions (including $d = 2$) first. This case is more complex than that in odd dimensions since the wave equation satisfied by $W^{(2p+1)}(x, x')$ in odd dimensions is homogeneous, unlike that satisfied by $W^{(2)}(x, x')$ in $d = 2$, which contains contributions from $V^{(2)}(x, x')$, or that satisfied by $W^{(2p)}(x, x')$ in even dimensions with $d \neq 2$, which contains contributions from $V^{(2p)}(x, x')$ as well as $U^{(2p)}_{p-2}(x, x')$.

Since we will be substituting the covariant Taylor expansion of the biscalar $W^{(d)}(x, x')$ into the wave equation for $W^{(2)}(x, x')$ in $d = 2$ (7.16) and into the wave equation for $W^{(2p)}(x, x')$ in even dimensions with $d \neq 2$ (7.81), before equating terms at each power of $\sigma(x, x')$, we can also substitute the power series expansions of the $V^{(2)}(x, x')$ and the $V^{(2p)}(x, x')$ biscalars into (7.16) and (7.81) respectively. In either case, we consider terms upto $\mathcal{O}(\sigma^{1/2})$ to derive identities relating both $w^{(d)}_{02(\mu\nu)}$ and $\nabla^{\mu} w^{(d)}_{02(\mu\nu)}$ to lower order coefficients respectively.

Beginning with the wave equation in $d = 2$ (7.16) and substituting in the power series expansion for $V^{(2)}(x, x')$ (6.4a), the first term on the r.h.s is given by

$$\begin{aligned}
-2\,\sigma^{-1}\sigma^{;\mu}D_{\mu}V^{(2)} &= -2\,\sigma^{-1}\sigma^{;\mu}D_{\mu}V^{(2)}_0 - 2\,\sigma^{-1}\sigma^{;\mu}D_{\mu}\Big[V^{(2)}_1\sigma\Big] + \mathcal{O}(\sigma)\\
&= -2\,\sigma^{-1}\sigma^{;\mu}D_{\mu}V^{(2)}_0 - 2\,\sigma^{;\mu}D_{\mu}V^{(2)}_1 - 2\,\sigma^{-1}V^{(2)}_1\sigma^{;\mu}\sigma_{;\mu} + \mathcal{O}(\sigma)\\
&= -2\,\sigma^{-1}\sigma^{;\mu}D_{\mu}V^{(2)}_0 - 2\,\sigma^{;\mu}D_{\mu}V^{(2)}_1 - 4\,V^{(2)}_1 + \mathcal{O}(\sigma)\,,
\end{aligned}\tag{8.5}$$

where we have used the definition of Synge's world function (1.20) to go from the penultimate line to the last one. The second term on the r.h.s of (7.16) is given by

$$\begin{aligned}
2\,\sigma^{-1}V^{(2)}\Delta^{-\frac{1}{2}}\Delta^{\frac{1}{2}}_{;\mu}\,\sigma^{;\mu} &= 2\,\sigma^{-1}V^{(2)}_0\Delta^{-\frac{1}{2}}\Delta^{\frac{1}{2}}_{;\mu}\,\sigma^{;\mu} + 2\,V^{(2)}_1\Delta^{-\frac{1}{2}}\Delta^{\frac{1}{2}}_{;\mu}\,\sigma^{;\mu} + \mathcal{O}(\sigma)\\
&= 2\,\sigma^{-1}V^{(2)}_0\Delta^{-\frac{1}{2}}\Delta^{\frac{1}{2}}_{;\mu}\,\sigma^{;\mu} + \mathcal{O}(\sigma)\\
&= 2\,\sigma^{-1}\sigma^{;\mu}D_{\mu}V^{(2)}_0 + \mathcal{O}(\sigma)\,,
\end{aligned}\tag{8.6}$$

where we have used the expansion in (6.26) and the boundary condition for $V^{(2)}_0(x, x')$ (7.23) to go from the penultimate line to the last one. Then, using (8.5) and (8.6), the scalar field equation for $W^{(2)}(x, x')$ in $d = 2$ (7.16) becomes

$$\begin{aligned}
\big(D_{\mu}D^{\mu} - m^2 - \xi R\big)\,W^{(2)} &= -2\,\sigma^{-1}\sigma^{;\mu}D_{\mu}V^{(2)}_0 - 2\,\sigma^{;\mu}D_{\mu}V^{(2)}_1 - 4\,V^{(2)}_1 + 2\,\sigma^{;\mu}D_{\mu}V^{(2)}_0\\
&\quad + \mathcal{O}(\sigma)\,,
\end{aligned}\tag{8.7}$$

which simplifies to give

$$\big(D_{\mu}D^{\mu} - m^2 - \xi R\big)\,W^{(2)} = -2\,\sigma^{;\mu}D_{\mu}V^{(2)}_1 - 4\,V^{(2)}_1 + \mathcal{O}(\sigma)\,. \tag{8.8}$$

Treating the wave equation in even dimensions with $d \neq 2$ (7.81) next, we can first use (7.64) to eliminate $U^{(2p)}_{p-2}$ from (7.81); doing so, we obtain

$$\begin{aligned}
&\big(D_\mu D^\mu - m^2 - \xi R\big) W^{(2p)} \\
&\qquad = 2\,\sigma^{-1}\,(p-1)\,V_0^{(2p)} + 2\,\sigma^{-1}\sigma^{;\mu} D_\mu V_0^{(2p)} - 2\,\sigma^{-1} V_0^{(2p)} \Delta^{-\frac{1}{2}} \Delta^{\frac{1}{2}}_{;\mu}\,\sigma^{;\mu} \\
&\qquad - 2\,\sigma^{-1}\,(p-1)\,V^{(2p)} - 2\,\sigma^{-1}\sigma^{;\mu} D_\mu V^{(2p)} + 2\,\sigma^{-1} V^{(2p)} \Delta^{-\frac{1}{2}} \Delta^{\frac{1}{2}}_{;\mu}\,\sigma^{;\mu} \\
&\qquad = -2\,\sigma^{-1}\,(p-1)\left[V^{(2p)} - V_0^{(2p)}\right] - 2\,\sigma^{-1}\sigma^{;\mu} D_\mu \left[V^{(2p)} - V_0^{(2p)}\right] \\
&\qquad + 2\,\sigma^{-1}\left[V^{(2p)} - V_0^{(2p)}\right] \Delta^{-\frac{1}{2}} \Delta^{\frac{1}{2}}_{;\mu}\,\sigma^{;\mu}. \qquad (8.9)
\end{aligned}$$

From the expansion of the biscalar $V^{(2p)}(x,x')$ (6.6b), we have that

$$V^{(2p)}(x,x') - V_0^{(2p)} = \sum_{n=1}^{\infty} V_n^{(2p)}(x,x')\,\sigma^n(x,x')\,. \qquad (8.10)$$

Then, using (8.10), equation (8.9) becomes

$$\begin{aligned}
\big(D_\mu D^\mu - m^2 - \xi R\big)\,W^{(2p)} &= -2\,(p-1)\,V_1^{(2p)} - 2\,\sigma^{-1}\sigma^{;\mu} D_\mu\left[V_1^{(2p)}\sigma\right] + \mathcal{O}(\sigma) \\
&= -2\,(p-1)\,V_1^{(2p)} - 2\,\sigma^{;\mu} D_\mu V_1^{(2p)} - 2\,\sigma^{-1} V_1^{(2p)} \sigma^{;\mu}\sigma_{;\,\mu} + \mathcal{O}(\sigma) \\
&= -2\,(p-1)\,V_1^{(2p)} - 2\,\sigma^{;\mu} D_\mu V_1^{(2p)} - 4\,V_1^{(2p)} + \mathcal{O}(\sigma)\,, \qquad (8.11)
\end{aligned}$$

where we have used (1.20) to go from the second equality to the third, and which simplifies to

$$\big(D_\mu D^\mu - m^2 - \xi R\big)\,W^{(2p)} = -2\,(p+1)\,V_1^{(2p)} - 2\,\sigma^{;\mu} D_\mu V_1^{(2p)} + \mathcal{O}(\sigma)\,. \qquad (8.12)$$

We can see that (8.7) is equivalent to (8.12) for the special case $p = 1$; therefore we can treat (8.12) as being valid in arbitrary even dimensions including $d = 2$. Since (8.12) holds for arbitrary even dimensions including $d = 2$, we will derive the same identities relating the expansion coefficients of the biscalar $W^{(2)}(x,x')$ for the case of $d = 2$ as those relating the expansion coefficients of the biscalar $W^{(2p)}(x,x')$ for even dimensions with $d \neq 2$. Correspondingly, we will derive the same expressions for the scalar field condensate, the renormalised expectation values of the current and stress-energy tensor, as well as expressions for their divergence, and renormalisation ambiguities of each of the aforementioned quantities, in $d = 2$ as in even dimensions with $d \neq 2$. Therefore, we will drop the distinction between $d = 2$ and even dimensions with $d \neq 2$ for the rest of this chapter and the superscript $2p$ will hold for all $p \mathrm{in} \mathbb{N}$ from now on.

Interestingly, (8.12) contains the Hadamard coefficient $V_1^{(2p)}(x,x')$ but not the Hadamard coefficient $V_0^{(2p)}(x,x')$. It will be useful to substitute the covariant Taylor expansions for $W^{(2p)}(x,x')$ and $V_1^{(2p)}(x,x')$ into (8.12) in order to derive relations between the expansion coefficients of the $W^{(2p)}(x,x')$ biscalar and the expansion coefficients of the $V_1^{(2p)}(x,x')$ biscalar. We can begin by inserting the covariant Taylor expansion of $V_1^{(2p)}(x,x')$ (6.10b) into (8.12). Doing so, we obtain

$$\begin{aligned}\left(D_\mu D^\mu - m^2 - \xi R\right) W^{(2p)} &= -2\,(p+1)\,V_{10}^{(2p)} - 2\,(p+1)\,V_{11\mu}^{(2p)}\sigma^{\,;\mu} - 2\,\sigma^{\,;\mu} D_\mu V_{10}^{(2p)} \\ &\quad - 2\,\sigma^{\,;\mu} D_\mu \left[V_{11\nu}^{(2p)}\sigma^{\,;\nu}\right] + \mathcal{O}(\sigma) \\ &= -2\,(p+1)\,V_{10}^{(2p)} - 2\,(p+1)\,V_{11\mu}^{(2p)}\sigma^{\,;\mu} - 2\,\sigma^{\,;\mu} D_\mu V_{10}^{(2p)} \\ &\quad - 2\,\sigma^{\,;\mu}\,V_{11\nu}^{(2p)}\sigma^{\,;\nu}{}_{\mu} + \mathcal{O}(\sigma) \\ &= -2\,(p+1)\,V_{10}^{(2p)} - 2\,(p+1)\,V_{11\mu}^{(2p)}\sigma^{\,;\mu} - 2\,\sigma^{\,;\mu} D_\mu V_{10}^{(2p)} \\ &\quad - 2\,\sigma^{\,;\mu}\,V_{11\nu}^{(2p)}\,\delta^\nu_\mu + \mathcal{O}(\sigma)\,,\end{aligned} \tag{8.13}$$

where we have used the expression for $\sigma_{;\mu\nu}$ (6.25). So finally, we obtain

$$\left(D_\mu D^\mu - m^2 - \xi R\right) W^{(2p)} = -2\,(p+1)\,V_{10}^{(2p)} - \left[2\,D_\mu V_{10}^{(2p)} + 2\,(p+2)\,V_{11\mu}^{(2p)}\right]\sigma^{\,;\mu} + \mathcal{O}(\sigma)\,. \tag{8.14}$$

Now we can insert the covariant Taylor expansion of $W^{(2p)}(x,x')$ (8.4) into (8.14). Since we require (8.14) up to $\mathcal{O}\big(\sigma^{1/2}\big)$, we begin by evaluating $D_\mu W^{(2p)}$ up to $\mathcal{O}(\sigma)$:

$$\begin{aligned}D_\alpha W^{(2p)} &= w_{0\,;\alpha}^{(2p)} + w_{1\mu\,;\alpha}^{(2p)}\,\sigma^{\,;\mu} + w_{1\mu}^{(2p)}\sigma^{;\mu}{}_{\alpha} + w_{2(\mu\nu)\,;\alpha}^{(2p)}\,\sigma^{\,;\mu}\sigma^{\,;\nu} + 2\,w_{2(\mu\nu)}^{(2p)}\,\sigma^{;\mu}{}_{\alpha}\,\sigma^{\,;\nu} \\ &\quad + 3\,w_{3(\mu\nu\rho)}^{(2p)}\,\sigma^{;\mu}{}_{\alpha}\,\sigma^{\,;\nu}\sigma^{\,;\rho} - \mathrm{i}qA_\alpha w_0^{(2p)} - \mathrm{i}qA_\alpha w_{1\mu}^{(2p)}\sigma^{\,;\mu} - \mathrm{i}qA_\alpha w_{2(\mu\nu)}^{(2p)}\,\sigma^{\,;\mu}\sigma^{\,;\nu} \\ &\quad + \mathcal{O}\Big(\sigma^{3/2}\Big)\,.\end{aligned} \tag{8.15}$$

Acting on (8.15) with another gauge covariant derivative, we have

$$\begin{aligned}D^\alpha D_\alpha W^{(2p)} &= \Box\, w_0^{(2p)} + \Box\, w_{1\mu}^{(2p)}\sigma^{\,;\mu} + 2\,w_{1\mu\,;\alpha}^{(2p)}\,g^{\alpha\beta}\sigma^{;\mu}{}_{\beta} + w_{1\mu}^{(2p)}\Box\left(\sigma^{\,;\mu}\right) \\ &\quad + 4\,w_{2(\mu\nu)\,;\alpha}^{(2p)}\,g^{\alpha\beta}\sigma^{;\mu}{}_{\beta}\,\sigma^{\,;\nu} + 2\,w_{2(\mu\nu)}^{(2p)}\,\Box\left(\sigma^{\,;\mu}\right)\sigma^{\,;\nu} + 2\,w_{2(\mu\nu)}^{(2p)}\,\sigma^{;\mu}{}_{\alpha}\,g^{\alpha\beta}\sigma^{;\nu}{}_{\beta} \\ &\quad + 6\,w_{3(\mu\nu\rho)}^{(2p)}\,\sigma^{;\mu}{}_{\alpha}\,g^{\alpha\beta}\sigma^{;\nu}{}_{\beta}\,\sigma^{\,;\rho} - \mathrm{i}q\left(\nabla_\alpha A^\alpha\right) w_0^{(2p)} - \mathrm{i}qA^\alpha w_{0\,;\alpha}^{(2p)} \\ &\quad - \mathrm{i}q\left(\nabla_\alpha A^\alpha\right) w_{1\mu}^{(2p)}\sigma^{\,;\mu} - \mathrm{i}qA^\alpha w_{1\mu\,;\alpha}^{(2p)}\,\sigma^{\,;\mu} - \mathrm{i}qA^\alpha w_{1\mu}^{(2p)}\sigma^{;\mu}{}_{\alpha} \\ &\quad - 2\,\mathrm{i}qA^\alpha w_{2(\mu\nu)}^{(2p)}\,\sigma^{;\mu}{}_{\alpha}\,\sigma^{\,;\nu} - \mathrm{i}qA^\alpha w_{0\,;\alpha}^{(2p)} - \mathrm{i}qA^\alpha w_{1\mu\,;\alpha}^{(2p)}\,\sigma^{\,;\mu} - \mathrm{i}qA^\alpha w_{1\mu}^{(2p)}\sigma^{;\mu}{}_{\alpha} \\ &\quad - 2\,\mathrm{i}qA^\alpha w_{2(\mu\nu)}^{(2p)}\,\sigma^{;\mu}{}_{\alpha}\sigma^{\,;\nu} - q^2 A^\alpha A_\alpha w_0^{(2p)} - q^2 A^\alpha A_\alpha w_{1\mu}^{(2p)}\sigma^{\,;\mu} + \mathcal{O}(\sigma)\,.\end{aligned} \tag{8.16}$$

Using (6.25), equation (8.16) simplifies to

$$\begin{aligned}D^\alpha D_\alpha W^{(2p)} &= \Box\, w_0^{(2p)} + \Box\, w_{1\mu}^{(2p)}\sigma^{\,;\mu} + 2\,w_{1\mu\,;\alpha}^{(2p)}\,g^{\alpha\beta}\delta^\mu_\beta \\ &\quad + w_{1\mu}^{(2p)}\,\nabla^\alpha\left(\delta^\mu_\alpha - \frac{1}{3}R^\mu{}_{(\theta|\alpha|\phi)}\sigma^{\,;\theta}\sigma^{\,;\phi}\right) + 4\,w_{2(\mu\nu)\,;\alpha}^{(2p)}\,g^{\alpha\beta}\delta^\mu_\beta\,\sigma^{\,;\nu} \\ &\quad + 2\,w_{2(\mu\nu)}^{(2p)}\,\nabla^\alpha(\delta^\mu_\alpha)\,\sigma^{\,;\nu} + 2\,w_{2(\mu\nu)}^{(2p)}\,\delta^\mu_\alpha\,g^{\alpha\beta}\,\delta^\nu_\beta + 6\,w_{3(\mu\nu\rho)}^{(2p)}\,\delta^\mu_\alpha\,g^{\alpha\beta}\,\delta^\nu_\beta\,\sigma^{\,;\rho} \\ &\quad - \mathrm{i}q\left(\nabla_\alpha A^\alpha\right) w_0^{(2p)} - 2\,\mathrm{i}qA^\alpha w_{0\,;\alpha}^{(2p)} - \mathrm{i}q\left(\nabla_\alpha A^\alpha\right) w_{1\mu}^{(2p)}\sigma^{\,;\mu} - 2\,\mathrm{i}qA^\alpha w_{1\mu\,;\alpha}^{(2p)}\,\sigma^{\,;\mu} \\ &\quad - 2\,\mathrm{i}qA^\alpha w_{1\mu}^{(2p)}\,\delta^\mu_\alpha - 4\,\mathrm{i}qA^\alpha w_{2(\mu\nu)}^{(2p)}\,\delta^\mu_\alpha\,\sigma^{\,;\nu} - q^2 A^\alpha A_\alpha w_0^{(2p)} - q^2 A^\alpha A_\alpha w_{1\mu}^{(2p)}\sigma^{\,;\mu} \\ &\quad + \mathcal{O}(\sigma)\,.\end{aligned} \tag{8.17}$$

Then, (8.17) becomes

$$\begin{aligned}D^\alpha D_\alpha W^{(2p)} &= \left[\Box - \mathrm{i}q\left(\nabla_\alpha A^\alpha\right) - 2\,\mathrm{i}qA^\alpha\nabla_\alpha - q^2A^\alpha A_\alpha\right] w_0^{(2p)}\\ &+ \sigma^{;\mu}\left[\Box - \mathrm{i}q\left(\nabla_\alpha A^\alpha\right) - 2\,\mathrm{i}qA^\alpha\nabla_\alpha - q^2A^\alpha A_\alpha\right] w_{1\mu}^{(2p)} + 2\left[\nabla^\alpha - \mathrm{i}qA^\alpha\right] w_{1\alpha}^{(2p)}\\ &- \frac{2}{3}\, w_{1\mu}^{(2p)} R^\mu{}_{(\theta|\alpha|\phi)}\, \sigma^{;\theta}{}_\alpha\, \sigma^{;\phi} + 4\,\sigma^{;\nu}\left[\nabla^\alpha - \mathrm{i}qA^\alpha\right] w_{2(\alpha\nu)}^{(2p)} + 2\,g^{\mu\nu} w_{2\mu\nu}^{(2p)}\\ &+ 6\,g^{\mu\nu} w_{3(\mu\nu\rho)}^{(2p)}\,\sigma^{;\rho} + \mathcal{O}\left(\sigma\right). \end{aligned} \tag{8.18}$$

Equation (8.18) further simplifies to

$$\begin{aligned}D_\alpha D^\alpha W^{(2p)} &= D_\alpha D^\alpha w_0^{(2p)} + \sigma^{;\mu} D_\alpha D^\alpha w_{1\mu}^{(2p)} + 2\,D^\alpha w_{1\alpha}^{(2p)} - \frac{2}{3}\, w_{1\mu}^{(2p)} R^\mu{}_{(\theta|\alpha|\phi)}\, \delta^\theta_\alpha\, \sigma^{;\phi}\\ &+ 4\,\sigma^{;\nu} D^\alpha w_{2(\alpha\nu)}^{(2p)} + 2\,g^{\mu\nu} w_{2\mu\nu}^{(2p)} + 6\,g^{\mu\nu} w_{3(\mu\nu\rho)}^{(2p)}\,\sigma^{;\rho} + \mathcal{O}\left(\sigma\right). \end{aligned} \tag{8.19}$$

So finally, we obtain

$$\begin{aligned}\left(D_\alpha D^\alpha - m^2 - \xi R\right) W^{(2p)} &= \left(D_\alpha D^\alpha - m^2 - \xi R\right) w_0^{(2p)} + 2\,D^\alpha w_{1\alpha}^{(2p)} + 2\,g^{\mu\nu} w_{2\mu\nu}^{(2p)}\\ &+ \Bigg[\left(D_\alpha D^\alpha - m^2 - \xi R\right) w_{1\mu}^{(2p)} + \frac{1}{3}\, w_{1\alpha}^{(2p)} R_\mu{}^\alpha + 4\,D^\alpha w_{2(\alpha\mu)}^{(2p)}\\ &+ 6\,g^{\alpha\beta} w_{3(\alpha\beta\mu)}^{(2p)}\Bigg]\,\sigma^{;\mu} + \mathcal{O}\left(\sigma\right). \end{aligned} \tag{8.20}$$

Substituting (8.20) into (8.12) and rearranging, we obtain

$$\begin{aligned}0 &= \left(D_\alpha D^\alpha - m^2 - \xi R\right) w_0^{(2p)} + 2\,D^\alpha w_{1\alpha}^{(2p)} + 2\,g^{\mu\nu} w_{2\mu\nu}^{(2p)} + 2\left(p+1\right) V_{10}^{(2p)}\\ &+ \Bigg[\left(D_\alpha D^\alpha - m^2 - \xi R\right) w_{1\mu}^{(2p)} + \frac{1}{3}\, w_{1\alpha}^{(2p)} R_\mu{}^\alpha + 4\,D^\alpha w_{2(\alpha\mu)}^{(2p)} + 6\,g^{\alpha\beta} w_{3(\alpha\beta\mu)}^{(2p)} + 2D_\mu V_{10}^{(2p)}\\ &+ 2\left(p+2\right) V_{11\mu}^{(2p)}\Bigg]\,\sigma^{;\mu} + \mathcal{O}\left(\sigma\right). \end{aligned} \tag{8.21}$$

Since (8.21) should hold at each power of σ, the terms at $\mathcal{O}(1)$ in (8.21) give us

$$0 = \left(D_\alpha D^\alpha - m^2 - \xi R\right) w_0^{(2p)} + 2\,D^\alpha w_{1\alpha}^{(2p)} + 2\,g^{\mu\nu} w_{2\mu\nu}^{(2p)} + 2\left(p+1\right) V_{10}^{(2p)}. \tag{8.22}$$

Since the expansion coefficients in (8.22) are complex quantities, we can obtain two further relations by considering the real and imaginary parts of the equation respectively. Considering the real part of (8.22) first, we obtain

$$\begin{aligned}0 &= \left[\Box - q^2 A_\alpha A^\alpha - \left(m^2 + \xi R\right)\right] w_0^{(2p)} + 2\,\nabla^\alpha \Re\left[w_{1\alpha}^{(2p)}\right] + 2\,qA^\alpha \Im\left[w_{1\alpha}^{(2p)}\right] + 2\,g^{\mu\nu}\Re\left[w_{2\mu\nu}^{(2p)}\right]\\ &+ 2\left(p+1\right) V_{10}^{(2p)}, \end{aligned} \tag{8.23}$$

where we have used the fact that $\Im\left[w_0^{(2p)}\right] = 0$ by definition (6.18) and $\Im\left[V_{10}^{(2p)}\right] = 0$ from (7.47) and (7.274). (Note, we could also use the fact that the lowest order coefficient of the covariant expansion of a sequisymmetric biscalar is real by definition to deduce that $\Im\left[V_{10}^{(2p)}\right]=0$, as we did in the case of $w_0^{(2p)}$). Using (6.19), equation (8.23) reduces to

$$0=\left[\Box-q^2A_\alpha A^\alpha-\left(m^2+\xi R\right)\right]w_0^{(2p)}-2\,\nabla^\alpha\nabla_\alpha w_0^{(2p)}+2\,qA^\alpha\Im\left[w_{1\alpha}^{(2p)}\right]+2\,g^{\mu\nu}\Re\left[w_{2\mu\nu}^{(2p)}\right]$$
$$+2\,(p+1)\,V_{10}^{(2p)}, \tag{8.24}$$

giving us the identity

$$2\,g^{\mu\nu}\Re\left[w_{2\mu\nu}^{(2p)}\right]=-2qA^\alpha\Im\left[w_{1\alpha}^{(2p)}\right]+\left(q^2A_\alpha A^\alpha+m^2+\xi R\right)w_0^{(2p)}-2\,(p+1)\,V_{10}^{(2p)}. \tag{8.25}$$

Equation (8.25) generalises equation (56a) in [68] and its derivative will be useful in simplifying the expression obtained from taking the divergence of the RSET in even dimensions. We obtain, for the derivative of (8.25), the expression

$$\nabla_\nu\,g^{\lambda\tau}\Re\left[w_{2\lambda\tau}^{(2p)}\right]=-q\left(\nabla_\nu A^\alpha\right)\Im\left[w_{1\alpha}^{(2p)}\right]-qA^\alpha\nabla_\nu\Im\left[w_{1\alpha}^{(2p)}\right]+\left(q^2A_\alpha\nabla_\nu A^\alpha+\frac{1}{2}\,\xi R_{;\nu}\right)w_0^{(2p)}$$
$$+\frac{1}{2}\left(q^2A_\alpha A^\alpha+m^2+\xi R\right)\nabla_\nu\,w_0^{(2p)}-(p+1)\,\nabla_\nu V_{10}^{(2p)}. \tag{8.26}$$

Returning to (8.22) and this time considering the imaginary part, we have

$$0=-2\,qA^\alpha\nabla_\alpha w_0^{(2p)}-q\left(\nabla_\alpha A^\alpha\right)w_0^{(2p)}+2\,\nabla^\alpha\Im\left[w_{1\alpha}^{(2p)}\right]-2\,qA^\alpha\Re\left[w_{1\alpha}^{(2p)}\right]+2g^{\mu\nu}\Im\left[w_{2\mu\nu}^{(2p)}\right]. \tag{8.27}$$

Using (6.19) and (6.20), equation (8.27) becomes

$$0=-2\,qA^\alpha\nabla_\alpha w_0^{(2p)}-q\left(\nabla_\alpha A^\alpha\right)w_0^{(2p)}+2\,\nabla^\alpha\Im\left[w_{1\alpha}^{(2p)}\right]+qA^\alpha\nabla_\alpha w_0^{(2p)}-g^{\mu\nu}\Im\left[\nabla_{(\mu}w_{1\nu)}^{(2p)}\right], \tag{8.28}$$

giving us the identity

$$\nabla^\alpha\Im\left[w_{1\alpha}^{(2p)}\right]=qA^\alpha\nabla_\alpha w_0^{(2p)}+q\left(\nabla_\alpha A^\alpha\right)w_0^{(2p)}. \tag{8.29}$$

There is no analogue of (8.29) in the case of a neutral scalar field; this equation will be useful in simplifying the expressions obtained from taking the divergence of the renormalised expectation values of both the current and the stress-energy tensor in even dimensions.

Returning to (8.21), which must hold at each power of σ, the terms at $\mathcal{O}\left(\sigma^{1/2}\right)$ give

$$0=\left(D_\alpha D^\alpha-m^2-\xi R\right)w_{1\mu}^{(2p)}+\frac{1}{3}\,w_{1\alpha}^{(2p)}R_\mu{}^\alpha+4\,D^\alpha w_{2(\alpha\mu)}^{(2p)}+6\,g^{\alpha\beta}w_{3(\alpha\beta\mu)}^{(2p)}+2D_\mu V_{10}^{(2p)}$$
$$+2\,(p+2)\,V_{11\mu}^{(2p)}. \tag{8.30}$$

We would like to derive an identity relating the term $\nabla^\mu w_{2(\mu\nu)}^{(2p)}$ in (8.30) to the lower-order coefficients of the biscalar $W^{(2p)}(x,x')$, as this will help simplify the expression obtained from examining the conservation of the RSET; this term will not arise elsewhere in our analysis as only the expression for the RSET involves a second-order differential operator. The operator for the RSET also involves taking the real part of the expression obtained from operating on $G_{\rm R}^{(2p)}(x,x')$, and thereby $W^{(2p)}(x,x')$ (8.3). Therefore, we need only consider the real part of (8.30) to derive an identity relating $\nabla^{\mu}\Re\Big[w_{2(\mu\nu)}^{(2p)}\Big]$ to the real parts of the lower-order coefficients of $W^{(2p)}(x,x')$. Then, taking the real part of (8.30), we have

$$\begin{aligned}0 &= \left[\Box - q^2 A_\alpha A^\alpha - \left(m^2+\xi R\right)\right]\Re\Big[w_{1\mu}^{(2p)}\Big] + \left[2\,qA^\alpha\nabla_\alpha + q\left(\nabla_\alpha A^\alpha\right)\right]\Im\Big[w_{1\mu}^{(2p)}\Big]\\ &\quad + \frac{1}{3}\Re\Big[w_{1\alpha}^{(2p)}\Big]R_\mu{}^\alpha + 4\,\nabla^\alpha\Re\Big[w_{2(\mu\alpha)}^{(2p)}\Big] + 4\,qA^\alpha\Im\Big[w_{2(\mu\alpha)}^{(2p)}\Big] + 6\,g^{\alpha\beta}\,\Re\left[w_{3(\alpha\beta\mu)}^{(2p)}\right] + 2\,\nabla_\mu V_{10}^{(2p)}\\ &\quad + 2\,(p+2)\,\Re\Big[V_{11\mu}^{(2p)}\Big],\end{aligned} \tag{8.31}$$

where, again, we have used the fact that $V_{10}^{(2p)}$ is real. Using (6.19), (6.20) and (6.23), (8.31) becomes

$$\begin{aligned}0 &= -\frac{1}{2}\left[\Box - q^2A_\alpha A^\alpha - \left(m^2+\xi R\right)\right]w_{0\,;\mu}^{(2p)} + \left[2\,qA^\alpha\nabla_\alpha + q\left(\nabla_\alpha A^\alpha\right)\right]\Im\Big[w_{1\mu}^{(2p)}\Big]\\ &\quad - \frac{1}{6}\,w_{0\,;\alpha}^{(2p)}R_\mu{}^\alpha + 4\,\nabla^\alpha\Re\Big[w_{2(\mu\alpha)}^{(2p)}\Big] - 2\,qA^\alpha\Im\Big[w_{1(\mu\,;\alpha)}^{(2p)}\Big] - 3\,g^{\alpha\beta}\,\Re\left[w_{2(\alpha\beta\,;\mu)}^{(2p)}\right] + \frac{1}{4}\,g^{\alpha\beta}\,w_{0\,;(\alpha\beta\mu)}^{(2p)}\\ &\quad + 2\,\nabla_\mu V_{10}^{(2p)} - (p+2)\,\nabla_\mu V_{10}^{(2p)}.\end{aligned} \tag{8.32}$$

In order to simplify further, we expand the symmetric quantities in (8.32) to obtain

$$\begin{aligned}0 &= -\frac{1}{2}\,\nabla^\alpha\nabla_\alpha\nabla_\mu w_0^{(2p)} + \frac{1}{2}\left[q^2A_\alpha A^\alpha + \left(m^2+\xi R\right)\right]w_{0\,;\mu}^{(2p)} + 2\,qA^\alpha\nabla_\alpha\Im\Big[w_{1\mu}^{(2p)}\Big]\\ &\quad + q\left(\nabla_\alpha A^\alpha\right)\Im\Big[w_{1\mu}^{(2p)}\Big] - \frac{1}{6}\,w_{0\,;\alpha}^{(2p)}R_\mu{}^\alpha + 2\,\nabla^\alpha\Re\Big[w_{2\mu\alpha}^{(2p)}\Big] + 2\,\nabla^\alpha\Re\Big[w_{2\alpha\mu}^{(2p)}\Big] - qA^\alpha\nabla_\alpha\Im\Big[w_{1\mu}^{(2p)}\Big]\\ &\quad - qA^\alpha\nabla_\mu\Im\Big[w_{1\alpha}^{(2p)}\Big] - \nabla_\mu\,g^{\alpha\beta}\,\Re\Big[w_{2\alpha\beta}^{(2p)}\Big] - \nabla^\alpha\Re\Big[w_{2\alpha\mu}^{(2p)}\Big] - \nabla^\alpha\Re\Big[w_{2\mu\alpha}^{(2p)}\Big] + \frac{1}{12}\,\nabla_\mu\Big(\Box\,w_0^{(2p)}\Big)\\ &\quad + \frac{1}{12}\,\nabla^\alpha\nabla_\mu\nabla_\alpha w_0^{(2p)} + \frac{1}{12}\,\nabla^\alpha\nabla_\alpha\nabla_\mu w_0^{(2p)} - p\,\nabla_\mu V_{10}^{(2p)}.\end{aligned} \tag{8.33}$$

Simplifying like terms in (8.33) and using the fact that the commutator of covariant derivatives acting on a scalar vanishes, we have

$$\begin{aligned}0 &= -\frac{1}{3}\,\nabla_\alpha\nabla_\mu\nabla^\alpha w_0^{(2p)} + \frac{1}{2}\left[q^2A_\alpha A^\alpha + \left(m^2+\xi R\right)\right]w_{0\,;\mu}^{(2p)} + qA^\alpha\nabla_\alpha\Im\Big[w_{1\mu}^{(2p)}\Big]\\ &\quad + q\left(\nabla_\alpha A^\alpha\right)\Im\Big[w_{1\mu}^{(2p)}\Big] - \frac{1}{6}\,w_{0\,;\alpha}^{(2p)}R_\mu{}^\alpha + \nabla^\alpha\Re\Big[w_{2\mu\alpha}^{(2p)}\Big] + \nabla^\alpha\Re\Big[w_{2\alpha\mu}^{(2p)}\Big] - qA^\alpha\nabla_\mu\Im\Big[w_{1\alpha}^{(2p)}\Big]\\ &\quad - \nabla_\mu\,g^{\alpha\beta}\,\Re\Big[w_{2\alpha\beta}^{(2p)}\Big] + \frac{1}{12}\,\nabla_\mu\Big(\Box\,w_0^{(2p)}\Big) - p\,\nabla_\mu V_{10}^{(2p)}.\end{aligned} \tag{8.34}$$

Commuting the outermost covariant derivatives acting on $w_0^{(2p)}$ in the first term on the r.h.s of (8.34), we obtain

$$\begin{aligned}0 &= -\frac{1}{3}\,\nabla_\mu\Big(\Box\,w_0^{(2p)}\Big) - \frac{1}{3}\,w_{0\,;\alpha}^{(2p)}R_\mu{}^\alpha + \frac{1}{2}\left[q^2A_\alpha A^\alpha + \left(m^2+\xi R\right)\right]w_{0\,;\mu}^{(2p)} + qA^\alpha\nabla_\alpha\Im\Big[w_{1\mu}^{(2p)}\Big]\\ &\quad + q\left(\nabla_\alpha A^\alpha\right)\Im\Big[w_{1\mu}^{(2p)}\Big] - \frac{1}{6}\,w_{0\,;\alpha}^{(2p)}R_\mu{}^\alpha + 2\,\nabla^\alpha\Re\Big[w_{2(\alpha\mu)}^{(2p)}\Big] - q\,\nabla_\mu A^\alpha\Im\Big[w_{1\alpha}^{(2p)}\Big]\\ &\quad + q\left(\nabla_\mu A^\alpha\right)\Im\Big[w_{1\alpha}^{(2p)}\Big] - \nabla_\mu\,g^{\alpha\beta}\,\Re\Big[w_{2\alpha\beta}^{(2p)}\Big] + \frac{1}{12}\,\nabla_\mu\Big(\Box\,w_0^{(2p)}\Big) - p\,\nabla_\mu V_{10}^{(2p)},\end{aligned} \tag{8.35}$$

giving us the identity

$$\begin{aligned}2\,\nabla^{\alpha}\Re\Big[w^{(2p)}_{2(\alpha\mu)}\Big] &= \frac{1}{4}\,\nabla_{\mu}\Big(\Box\, w^{(2p)}_{0}\Big) + \nabla_{\mu}\, g^{\alpha\beta}\,\Re\Big[w^{(2p)}_{2\alpha\beta}\Big] + \frac{1}{2}R^{\alpha}{}_{\mu}\nabla_{\alpha}w^{(2p)}_{0} - qA^{\alpha}\nabla_{\alpha}\Im\Big[w^{(2p)}_{1\mu}\Big]\\ &\quad + q\,\nabla_{\mu}A^{\alpha}\Im\Big[w^{(2p)}_{1\alpha}\Big] - 2\,q\,\big(\nabla_{(\mu}A^{\alpha}\big)\,\Im\Big[w^{(2p)}_{1\alpha)}\Big]\\ &\quad - \frac{1}{2}\left[q^{2}A_{\alpha}A^{\alpha} + \big(m^{2}+\xi R\big)\right]\nabla_{\mu}w^{(2p)}_{0} + p\,\nabla_{\mu}V^{(2p)}_{10}.\end{aligned} \tag{8.36}$$

This generalises equation (56b) in [68]. We can combine the identities in (8.36) and (8.25) to establish another relation. Using (8.26) to replace the covariant derivative of the $g^{\alpha\beta}\Re\Big[w^{(2p)}_{2\alpha\beta}\Big]$ in (8.36), we obtain

$$\begin{aligned}2\,\nabla^{\alpha}\Re\Big[w^{(2p)}_{2(\alpha\mu)}\Big] &= \frac{1}{4}\,\nabla_{\mu}\Big(\Box\, w^{(2p)}_{0}\Big) - q\,\nabla_{\mu}A^{\alpha}\Im\Big[w^{(2p)}_{1\alpha}\Big] + \frac{1}{2}\left[q^{2}A_{\alpha}A^{\alpha} + \big(m^{2}+\xi R\big)\right]\nabla_{\mu}w^{(2p)}_{0}\\ &\quad + \left[q^{2}A^{\alpha}\nabla_{\mu}A_{\alpha} + \frac{1}{2}\,\xi R_{;\mu}\right]w^{(2p)}_{0} - (p+1)\,V^{(2p)}_{10} + \frac{1}{2}R^{\alpha}{}_{\mu}\nabla_{\alpha}w^{(2p)}_{0}\\ &\quad - qA^{\alpha}\nabla_{\alpha}\Im\Big[w^{(2p)}_{1\mu}\Big] + q\,\nabla_{\mu}A^{\alpha}\Im\Big[w^{(2p)}_{1\alpha}\Big] - 2\,q\,\big(\nabla_{(\mu}A^{\alpha}\big)\,\Im\Big[w^{(2p)}_{1\alpha)}\Big]\\ &\quad - \frac{1}{2}\left[q^{2}A_{\alpha}A^{\alpha} + \big(m^{2}+\xi R\big)\right]\nabla_{\mu}w^{(2p)}_{0} + p\,\nabla_{\mu}V^{(2p)}_{10},\end{aligned} \tag{8.37}$$

giving us the identity

$$\begin{aligned}2\,\nabla^{\alpha}\Re\Big[w^{(2p)}_{2(\alpha\mu)}\Big] &= \frac{1}{4}\,\nabla_{\mu}\Big(\Box\, w^{(2p)}_{0}\Big) + \frac{1}{2}R^{\alpha}{}_{\mu}\nabla_{\alpha}w^{(2p)}_{0} - qA^{\alpha}\nabla_{\alpha}\Im\Big[w^{(2p)}_{1\mu}\Big]\\ &\quad - 2\,q\,\big(\nabla_{(\mu}A^{\alpha}\big)\,\Im\Big[w^{(2p)}_{1\alpha)}\Big] + \left[q^{2}A^{\alpha}\nabla_{\mu}A_{\alpha} + \frac{1}{2}\,\xi R_{;\mu}\right]w^{(2p)}_{0} - \nabla_{\mu}V^{(2p)}_{10}.\end{aligned} \tag{8.38}$$

Equation (8.38) generalises (57) in [68] and will be useful in simplifying the expression obtained from taking the divergence of the RSET in even dimensions.

The identity in (8.25), its derivative in (8.26), and the identities in (8.36) and (8.38) are valid for even dimensions, including $d = 2$. It turns out that the identity in (8.29) is valid for all dimensions. We need not repeat the same process to derive the analogues of the aforementioned identities in odd dimensions; since the scalar field equation for $W^{(2p+1)}(x, x')$ in odd dimensions (7.282) is homogeneous then, when using the covariant expansion of $W^{(2p+1)}(x, x')$ (8.4) to expand (7.282), we obtain

$$0 = \big(D_{\alpha}D^{\alpha} - m^{2} - \xi R\big)\, w^{(2p+1)}_{0} + 2\, D^{\alpha}w^{(2p+1)}_{1\alpha} + 2\, g^{\mu\nu}\, w^{(2p+1)}_{2\mu\nu}, \tag{8.39}$$

for the terms in (7.282) at $\mathcal{O}(1)$, and

$$0 = \big(D_{\alpha}D^{\alpha} - m^{2} - \xi R\big)\, w^{(2p+1)}_{1\mu} + \frac{1}{3}\, w^{(2p+1)}_{1\alpha} R_{\mu}{}^{\alpha} + 4\, D^{\alpha}w^{(2p+1)}_{2(\alpha\mu)} + 6\, g^{\alpha\beta}w^{(2p+1)}_{3(\alpha\beta\mu)}, \tag{8.40}$$

for the terms in (7.282) at $\mathcal{O}\big(\sigma^{1/2}\big)$, where (8.39) and (8.40), which do not receive contributions from any another biscalar function due to the homogeneity of the scalar field equation for $W^{(2p+1)}(x, x')$ (7.282), correspond to the odd-dimensional analogues of (8.21) and (8.30) respectively. Then, it is easy to adapt (8.25), (8.26), (8.29), (8.36) and (8.38) to odd dimensions; we have

$$2\, g^{\mu\nu} \Re\Big[w^{(2p+1)}_{2\mu\nu}\Big] = -2qA^{\alpha} \Im\Big[w^{(2p+1)}_{1\alpha}\Big] + \left(q^2 A_{\alpha} A^{\alpha} + m^2 + \xi R\right) w^{(2p+1)}_{0}, \tag{8.41}$$

which generalises equation (58a) in [68]. The derivative of (8.41) is given by

$$\begin{aligned}\nabla_{\nu}\, g^{\lambda\tau} \Re\Big[w^{(2p+1)}_{2\lambda\tau}\Big] = {}& -q\left(\nabla_{\nu} A^{\alpha}\right) \Im\Big[w^{(2p+1)}_{1\alpha}\Big] - qA^{\alpha}\nabla_{\nu}\Im\Big[w^{(2p+1)}_{1\alpha}\Big] \\ & + \left(q^2 A_{\alpha}\nabla_{\nu}A^{\alpha} + \frac{1}{2}\,\xi R_{;\nu}\right) w^{(2p+1)}_{0} + \frac{1}{2}\left(q^2 A_{\alpha} A^{\alpha} + m^2 + \xi R\right)\nabla_{\nu}\, w^{(2p+1)}_{0}.\end{aligned} \tag{8.42}$$

Equation (8.42) will be useful in simplifying the expression obtained from taking the divergence of the RSET in odd dimensions. We also have

$$\nabla^{\alpha}\Im\Big[w^{(2p+1)}_{1\alpha}\Big] = qA^{\alpha}\nabla_{\alpha} w^{(2p+1)}_{0} + q\left(\nabla_{\alpha}A^{\alpha}\right) w^{(2p+1)}_{0}. \tag{8.43}$$

There is no analogue of (8.43) in the case of a neutral scalar field; this equation will be useful in simplifying the expressions obtained from taking the divergence of the renormalised expectation values of both the current and the stress-energy tensor in odd dimensions. In addition, we have

$$\begin{aligned}2\,\nabla^{\alpha}\Re\Big[w^{(2p+1)}_{2(\alpha\mu)}\Big] = {}& \frac{1}{4}\,\nabla_{\mu}\Big(\Box\, w^{(2p+1)}_{0}\Big) + \nabla_{\mu}\, g^{\alpha\beta}\,\Re\Big[w^{(2p+1)}_{2\alpha\beta}\Big] + \frac{1}{2} R^{\alpha}{}_{\mu}\nabla_{\alpha} w^{(2p+1)}_{0} \\ & - qA^{\alpha}\nabla_{\alpha}\Im\Big[w^{(2p+1)}_{1\mu}\Big] + q\,\nabla_{\mu}A^{\alpha}\Im\Big[w^{(2p+1)}_{1\alpha}\Big] - 2\,q\left(\nabla_{(\mu}A^{\alpha}\right)\Im\Big[w^{(2p+1)}_{1\alpha)}\Big] \\ & - \frac{1}{2}\left[q^2 A_{\alpha}A^{\alpha} + \left(m^2 + \xi R\right)\right]\nabla_{\mu} w^{(2p+1)}_{0},\end{aligned} \tag{8.44}$$

which generalises equation (58b) in [68], as well as

$$\begin{aligned}2\,\nabla^{\alpha}\Re\Big[w^{(2p+1)}_{2(\alpha\mu)}\Big] = {}& \frac{1}{4}\,\nabla_{\mu}\Big(\Box\, w^{(2p+1)}_{0}\Big) + \frac{1}{2} R^{\alpha}{}_{\mu}\nabla_{\alpha} w^{(2p+1)}_{0} - qA^{\alpha}\nabla_{\alpha}\Im\Big[w^{(2p+1)}_{1\mu}\Big] \\ & - 2\,q\left(\nabla_{(\mu}A^{\alpha}\right)\Im\Big[w^{(2p+1)}_{1\alpha)}\Big] + \left[q^2 A^{\alpha}\nabla_{\mu}A_{\alpha} + \frac{1}{2}\,\xi R_{;\mu}\right] w^{(2p+1)}_{0}.\end{aligned} \tag{8.45}$$

Equation (8.45) generalises (59) in [68] and will be useful in simplifying the expression obtained from taking the divergence of the RSET in odd dimensions.

We now have the machinery required to generate expressions for the scalar field condensate, the renormalised current and the RSET, as well the divergence of the latter two quantities. The identities derived in this section will also aid us in considering the renormalisation ambiguities in these quantities, as well as the trace anomaly of the RSET. We will now proceed to deriving these expressions in the next section, before considering renormalisation ambiguities in later sections.

8.3 Renormalised expectation values of observables

In this section, we will derive expressions for the scalar field condensate, the renormalised current and the RSET in terms of the expansion coefficients $w_0^{(d)}$, $w_{1\mu}^{(d)}$ and $w_{2(\mu\nu)}^{(d)}$ (8.4) of the biscalar $W^{(d)}(x,x')$ (8.3). As we have mentioned previously, if we were to consider a particular quantum state then we could derive explicit expressions for the aforementioned quantities in terms of the spacetime geometry and the background gauge field. However, our discussion is more general and our aim is to provide the general framework for the extension of the Hadamard renormalisation scheme to charged scalar fields.

The general philosophy of the Hadamard renormalisation procedure is to identify the divergent parts of the Hadamard parametrix contained in the $U^{(d)}(x,x')$ and $V^{(d)}(x,x')$ biscalars and subtract these parts from the Feynman Green's function $G_{\rm F}^{(d)}(x,x')$ in order to give a regularised Green's function $G_{\rm R}^{(d)}(x,x')$ (8.3). Having done so, we perform the relevant operation on $G_{\rm R}^{(d)}(x,x')$, the form of which will depend on whether we are calculating the scalar condensate, the renormalised current or the RSET, before taking the coincidence limit $x'\to x$. Since the covariant Taylor expansion (8.4) of the biscalar $W^{(d)}(x,x')$ is in terms of the derivative of Synge's world function $\sigma^{;\mu}$ and $\sigma^{;\mu}\to 0$ as $x'\to x$, then we will effectively be working to $\mathcal{O}(1)$ in $\sigma(x,x')$ in the remainder of the chapter.

We will begin by considering the simplest quantity, the scalar field condensate, before considering the renormalised expectation value of the current and finally the RSET.

8.3.1 Scalar field condensate

The scalar field condensate $\widehat{\mathcal{SC}}$ (5.1), which has no classical analogue, has a particularly simple form in terms of the regularised Green's function $G_{\rm R}^{(d)}(x,x')$ that is given by

$$\langle\widehat{\mathcal{SC}}\rangle_{\rm ren} = \lim_{x'\to x} \Re\left\{-{\rm i}\, G_{\rm R}^{(d)}(x,x')\right\}. \tag{8.46}$$

Then, using (8.1) and (8.4), equation (8.46) reduces to

$$\begin{aligned}\langle\hat{\Phi}\hat{\Phi}^{\dagger}\rangle_{\rm ren} &= \alpha^{(d)} \lim_{x'\to x}\Re\left\{W^{(d)}(x,x')\right\}\\ &= \alpha^{(d)} \lim_{x'\to x}\Re\left\{w_0^{(d)} + \mathcal{O}\left(\sigma^{1/2}\right)\right\}\\ &= \alpha^{(d)}\, w_0^{(d)},\end{aligned} \tag{8.47}$$

where we have used the fact that $w_0^{(d)}$ is real (6.18). From (8.47), we see that the scalar condensate depends only on the lowest order expansion coefficient of $W^{(d)}(x,x')$.

8.3.2 Renormalised expectation value of the current

The next quantity we will consider is the renormalised expectation value of the current. In classical field theory, the current J^{μ} associated to a charged, complex scalar field Φ is given by

$$J^\mu = \frac{\mathrm{i}q}{8\pi}\left[\Phi^* D^\mu \Phi - \Phi(D^\mu\Phi)^*\right]. \tag{8.48}$$

We can simplify (8.48) by expanding the field Φ in terms of its real and imaginary parts; then, we obtain

$$\begin{aligned}
J^\mu &= \frac{\mathrm{i}q}{8\pi}\Big\{\big(\Re[\Phi]-\mathrm{i}\,\Im[\Phi]\,\big)D^\mu\big(\Re[\Phi]+\mathrm{i}\,\Im[\Phi]\,\big) - \big(\Re[\Phi]+\mathrm{i}\,\Im[\Phi]\,\big)\big[D^\mu\big(\Re[\Phi]+\mathrm{i}\,\Im[\Phi]\,\big)\big]^*\Big\}\\
&= \frac{\mathrm{i}q}{8\pi}\Big\{\big(\Re[\Phi]-\mathrm{i}\,\Im[\Phi]\,\big)\big(D^\mu\Re[\Phi]+\mathrm{i}D^\mu\Im[\Phi]\,\big) - \big(\Re[\Phi]+\mathrm{i}\,\Im[\Phi]\,\big)\big(D^\mu\Re[\Phi]-\mathrm{i}D^\mu\Im[\Phi]\,\big)\Big\}\\
&= \frac{\mathrm{i}q}{8\pi}\Big\{\Big(\Re[\Phi]\,D^\mu\Re[\Phi] - \mathrm{i}\,(D^\mu\Re[\Phi])\,\Im[\Phi] + \mathrm{i}\,\Re[\Phi]\,D^\mu\Im[\Phi] + \Im[\Phi]\,D^\mu\Im[\Phi]\Big)\\
&- \Big(\Re[\Phi]\,D^\mu\Re[\Phi] + \mathrm{i}\,(D^\mu\Re[\Phi])\,\Im[\Phi] - \mathrm{i}\,\Re[\Phi]\,D^\mu\Im[\Phi] + \Im[\Phi]\,D^\mu\Im[\Phi]\Big)\Big\}\\
&= \frac{\mathrm{i}q}{8\pi}\Big\{-2\,\mathrm{i}\,(D^\mu\Re[\Phi])\,\Im[\Phi] + 2\,\mathrm{i}\,\Re[\Phi]\,D^\mu\Im[\Phi]\Big\}\\
&= -\frac{q}{4\pi}\,\Im\left[\Phi^* D^\mu\Phi\right],
\end{aligned} \tag{8.49}$$

where we have used the fact that

$$\Im\left[\Phi^* D^\mu\Phi\right] = -\,(D^\mu\Re[\Phi])\,\Im[\Phi] + \Re[\Phi]\,D^\mu\Im[\Phi], \tag{8.50}$$

to go from the penultimate line in (8.49) to the last. Therefore, the renormalised expectation value of the current is given by

$$\langle\hat{J}^\mu\rangle_{\mathrm{ren}} = -\frac{q}{4\pi}\lim_{x'\to x}\Im\left\{D^\mu\left[-\mathrm{i}\,G_{\mathrm{R}}^{(d)}(x,x')\right]\right\}. \tag{8.51}$$

Then, using (8.1) and (8.4), equation (8.51) becomes

$$\begin{aligned}
\langle\hat{J}_\mu\rangle_{\mathrm{ren}} &= -\frac{\alpha^{(d)}q}{4\pi}\lim_{x'\to x}\Im\left\{D_\mu\,W^{(d)}(x,x')\right\}\\
&= -\frac{\alpha^{(d)}q}{4\pi}\lim_{x'\to x}\Im\left\{D_\mu\left[w_0^{(d)} + w_{1\nu}^{(d)}\,\sigma^{;\nu} + \mathcal{O}\left(\sigma^{1/2}\right)\right]\right\}\\
&= -\frac{\alpha^{(d)}q}{4\pi}\lim_{x'\to x}\Im\left\{w_{0\,;\mu}^{(d)} - \mathrm{i}qA_\mu w_0^{(d)} + w_{1\nu}^{(d)}\,\sigma^{;\nu}{}_{\mu} + \mathcal{O}\left(\sigma^{1/2}\right)\right\}\\
&= -\frac{\alpha^{(d)}q}{4\pi}\Im\left\{-\mathrm{i}qA_\mu w_0^{(d)} + w_{1\nu}^{(d)}\,\delta^\nu_\mu\right\}\\
&= \frac{\alpha^{(d)}q}{4\pi}\left\{qA_\mu w_0^{(d)} - \Im\left[w_{1\mu}^{(d)}\right]\right\}.
\end{aligned} \tag{8.52}$$

where we have used the fact that $w_0^{(d)}$ is real (6.18). We require that the renormalised expectation value of the current be conserved, that is $\nabla^\mu\langle\hat{J}_\mu\rangle_{\mathrm{ren}} = 0$. This follows from

$$\begin{aligned}
\nabla^\mu\langle\hat{J}_\mu\rangle_{\mathrm{ren}} &= \frac{\alpha^{(d)}q}{4\pi}\left\{q\,(\nabla_\mu A^\mu)\,w_0^{(d)} + qA^\mu\nabla_\mu\,w_0^{(d)} - \nabla^\mu\Im\left[w_{1\mu}^{(d)}\right]\right\}\\
&= 0.
\end{aligned} \tag{8.53}$$

where we have gone from the first equality to the second by using (8.29) in the even-dimensional case and (8.43) in the odd-dimensional case.

8.3.3 Renormalised stress-energy tensor

The final quantity we will consider is the renormalised expectation value of the stress-energy tensor. In classical field theory, the SET $T_{\mu\nu}$ associated to a charged, complex scalar field Φ that is non-minimally coupled to the spacetime curvature is given by [2]

$$\begin{aligned} T_{\mu\nu} = & \frac{1}{2}(1-2\xi)\left[(D_\mu\Phi)^* D_\nu\Phi + D_\mu\Phi\,(D_\nu\Phi)^*\right] \\ & + \frac{1}{2}\left(2\,\xi - \frac{1}{2}\right) g_{\mu\nu}\, g^{\rho\sigma}\left[(D_\rho\Phi)^* D_\sigma\Phi + D_\rho\Phi\,(D_\sigma\Phi)^*\right] - \xi\left[\Phi^* D_\mu D_\nu\Phi + \Phi\,(D_\mu D_\nu\Phi)^*\right] \\ & + \xi\, g_{\mu\nu}\left[\Phi^* D_\rho D^\rho\,\Phi + \Phi\,(D_\rho D^\rho\,\Phi)^*\right] + \xi\left(R_{\mu\nu} - \frac{1}{2}\,g_{\mu\nu}R\right)\Phi^*\Phi - \frac{1}{2}\,m^2 g_{\mu\nu}\,\Phi^*\Phi. \end{aligned} \tag{8.54}$$

We can simplify (8.54) by expanding the field Φ in terms of its real and imaginary parts; examining the first term on the r.h.s in (8.54) first, we have

$$\begin{aligned} & (D_\mu\Phi)^* D_\nu\Phi + D_\mu\Phi\,(D_\nu\Phi)^* \\ & = \left[D_\mu\big(\Re[\Phi] + \mathrm{i}\,\Im[\Phi]\big)\right]^* D_\nu\big(\Re[\Phi] + \mathrm{i}\,\Im[\Phi]\big) + D_\mu\big(\Re[\Phi] + \mathrm{i}\,\Im[\Phi]\big)\left[D_\nu\big(\Re[\Phi] + \mathrm{i}\,\Im[\Phi]\big)^*\right] \\ & = \big(D_\mu\Re[\Phi] - \mathrm{i}D_\mu\Im[\Phi]\big)\big(D_\nu\Re[\Phi] + \mathrm{i}D_\nu\Im[\Phi]\big) \\ & + \big(D_\mu\Re[\Phi] + \mathrm{i}D_\mu\Im[\Phi]\big)\big(D_\nu\Re[\Phi] - \mathrm{i}D_\nu\Im[\Phi]\big) \\ & = \Big(D_\mu\Re[\Phi]\,D_\nu\Re[\Phi] + \mathrm{i}D_\mu\Re[\Phi]\,D_\nu\Im[\Phi] - \mathrm{i}D_\mu\Im[\Phi]\,D_\nu\Re[\Phi] + D_\mu\Im[\Phi]\,D_\nu\Im[\Phi]\Big) \\ & + \Big(D_\mu\Re[\Phi]\,D_\nu\Re[\Phi] - \mathrm{i}D_\mu\Re[\Phi]\,D_\nu\Im[\Phi] + \mathrm{i}D_\mu\Im[\Phi]\,D_\nu\Re[\Phi] + D_\mu\Im[\Phi]\,D_\nu\Im[\Phi]\Big) \\ & = 2\,D_\mu\Re[\Phi]\,D_\nu\Re[\Phi] + 2\,D_\mu\Im[\Phi]\,D_\nu\Im[\Phi] \\ & = 2\,\Re\left[(D_\mu\Phi)^* D_\nu\Phi\right], \end{aligned} \tag{8.55}$$

where we have used the fact that

$$\Re\left[(D_\mu\Phi)^* D_\nu\Phi\right] = D_\mu\Re[\Phi]\,D_\nu\Re[\Phi] + D_\mu\Im[\Phi]\,D_\nu\Im[\Phi] \tag{8.56}$$

to go from the penultimate line in (8.55) to the last. The relation (8.55) can also be used to simplify the second term on the r.h.s in (8.54). Examining the third term on the r.h.s in (8.54), we have

$$\begin{aligned} & \Phi^* D_\mu D_\nu\Phi + \Phi(D_\mu D_\nu\Phi)^* \\ & = \big(\Re[\Phi] - \mathrm{i}\,\Im[\Phi]\big) D_\mu D_\nu\big(\Re[\Phi] + \mathrm{i}\,\Im[\Phi]\big) + \big(\Re[\Phi] + \mathrm{i}\,\Im[\Phi]\big)\left[D_\mu D_\nu\big(\Re[\Phi] + \mathrm{i}\,\Im[\Phi]\big)\right]^* \\ & = \big(\Re[\Phi] - \mathrm{i}\,\Im[\Phi]\big)\big(D_\mu D_\nu\Re[\Phi] + \mathrm{i}D_\mu D_\nu\Im[\Phi]\big) \\ & + \big(\Re[\Phi] + \mathrm{i}\,\Im[\Phi]\big)\big(D_\mu D_\nu\Re[\Phi] - \mathrm{i}D_\mu D_\nu\Im[\Phi]\big) \\ & = \Big(\Re[\Phi]\,D_\mu D_\nu\Re[\Phi] + \mathrm{i}\,\Re[\Phi]\,D_\mu D_\nu\Im[\Phi] - \mathrm{i}\,\Im[\Phi]\,D_\mu D_\nu\Re[\Phi] + \Im[\Phi]\,D_\mu D_\nu\Im[\Phi]\Big) \\ & + \Big(\Re[\Phi]\,D_\mu D_\nu\Re[\Phi] - \mathrm{i}\,\Re[\Phi]\,D_\mu D_\nu\Im[\Phi] + \mathrm{i}\,\Im[\Phi]\,D_\mu D_\nu\Re[\Phi] + \Im[\Phi]\,D_\mu D_\nu\Im[\Phi]\Big) \\ & = 2\,\Re[\Phi]\,D_\mu D_\nu\Re[\Phi] + 2\,\Im[\Phi]\,D_\mu D_\nu\Im[\Phi]\Big) \\ & = 2\,\Re[\Phi^* D_\mu D_\nu\Phi], \end{aligned} \tag{8.57}$$

where we have used the fact that

$$\Re[\Phi^* D_\mu D_\nu \Phi] = \Re[\Phi]\, D_\mu D_\nu \Re[\Phi] + \Im[\Phi]\, D_\mu D_\nu \Im[\Phi]\,, \tag{8.58}$$

to go from the penultimate line in (8.57) to the last. The relation (8.57) can also be used to simplify the fourth term on the r.h.s in (8.54). The fifth and sixth terms on the r.h.s in (8.54) are easily simplified by using that

$$\Re[\Phi^*\Phi] = \Phi^*\Phi. \tag{8.59}$$

Then, using (8.55), (8.57) and (8.59), the classical SET (8.54) becomes

$$\begin{aligned} T_{\mu\nu} = \Re\Big\{ &(1-2\xi)\,(D_\mu\Phi)^* D_\nu\Phi + \left(2\,\xi - \frac{1}{2}\right) g_{\mu\nu}\, g^{\rho\sigma} (D_\rho\Phi)^* D_\sigma\Phi - 2\,\xi\,\Phi^* D_\mu D_\nu \Phi \\ &+ 2\,\xi\, g_{\mu\nu} \Phi^* D_\rho D^\rho\,\Phi + \xi\left(R_{\mu\nu} - \frac{1}{2}\, g_{\mu\nu} R\right)\Phi^*\Phi - \frac{1}{2}\, m^2 g_{\mu\nu}\,\Phi^*\Phi\Big\}. \end{aligned} \tag{8.60}$$

Then, we can write the expectation value $\langle \hat{T}_{\mu\nu}\rangle$ of the quantum SET operator as the limit

$$\langle \hat{T}_{\mu\nu}\rangle = \lim_{x'\to x} \Re\Big\{ \mathcal{T}_{\mu\nu}(x,x') \left[-\mathrm{i}\, G_{\mathrm{R}}^{(d)}(x,x')\right]\Big\}\,, \tag{8.61}$$

where the operator $\mathcal{T}_{\mu\nu}(x,x')$ is given by the expression [2]

$$\begin{aligned} \mathcal{T}_{\mu\nu} = &(1-2\,\xi)\, g_\nu{}^{\nu'} D_\mu D^*_{\nu'} + \left(2\,\xi - \frac{1}{2}\right) g_{\mu\nu}\, g^{\rho\sigma'} D_\rho D^*_{\sigma'} - 2\,\xi\, D_\mu D_\nu + 2\,\xi\, g_{\mu\nu}\, D_\rho D^\rho \\ &+ \xi \left(R_{\mu\nu} - \frac{1}{2}\, g_{\mu\nu} R\right) - \frac{1}{2}\, m^2 g_{\mu\nu}, \end{aligned} \tag{8.62}$$

and $g_\mu{}^{\mu'}$ is the bivector of parallel transport. From Wald's axioms, which are given in §1.5, the renormalised expectation value of the stress-energy tensor is unique only up to the addition of a local conserved tensor. Then, we can write the RSET as

$$\langle \hat{T}_{\mu\nu}\rangle_{\mathrm{ren}} = \alpha^{(d)} \lim_{x'\to x} \Re\Big[\mathcal{T}_{\mu\nu}(x,x')\, W^{(d)}(x,x')\Big] + \widetilde{\Theta}^{(d)}_{\mu\nu}, \tag{8.63}$$

where $\widetilde{\Theta}^{(d)}_{\mu\nu}$ is a local tensor whose exact form will be constrained by the conservation of the RSET. We would thus like to evaluate the quantity

$$\lim_{x'\to x} \Re\Big[\mathcal{T}_{\mu\nu}(x,x')\, W^{(d)}(x,x')\Big]\,, \tag{8.64}$$

We will need to use the result [68]

$$g_\nu{}^{\nu'} \sigma^\rho{}_{\nu'} = g_\nu{}^{\nu'} \sigma_{\mu\nu'} g^{\mu\rho} = (-g_{\mu\nu} + \ldots)\, g^{\mu\rho} = -\delta^\rho_\nu, \tag{8.65}$$

and that, in the coincidence limit $x'\to x$, we have [68]

$$\lim_{x'\to x} \sigma^{;\rho}{}_{\nu'\mu} = \lim_{x'\to x} \sigma^{;\rho}{}_{\nu\mu} = 0. \tag{8.66}$$

Since we will be taking the coincidence limit in (8.63), we need only consider (8.64) up to first order in sigma. Then the first term in (8.64) is given by

$$
\begin{aligned}
&(1-2\,\xi)\,g_{\nu}{}^{\nu'}D_{\mu}D^{*}_{\nu'}\left[w_0^{(d)}+w_{1\rho}^{(d)}\,\sigma^{\,;\rho}+w_{2(\rho\lambda)}^{(d)}\,\sigma^{\,;\rho}\sigma^{\,;\lambda}+\mathcal{O}\Big(\sigma^{\frac{3}{2}}\Big)\right]\\
&=(1-2\,\xi)\,g_{\nu}{}^{\nu'}D_{\mu}\Big[w_{1\rho}^{(d)}\,\sigma^{;\rho}{}_{\nu'}+2\,w_{2(\rho\lambda)}^{(d)}\,\sigma^{;\rho}{}_{\nu'}\,\sigma^{\,;\lambda}+\mathrm{i}qA_{\nu'}\,w_0^{(d)}+\mathrm{i}qA_{\nu'}\,w_{1\rho}^{(d)}\,\sigma^{\,;\rho}+\mathcal{O}(\sigma)\Big]\\
&=(1-2\,\xi)\,g_{\nu}{}^{\nu'}\left[w_{1\rho\,;\mu}^{(d)}\,\sigma^{;\rho}{}_{\nu'}+w_{1\rho}^{(d)}\,\sigma^{;\rho}{}_{\nu'\mu}+2\,w_{2(\rho\lambda)}^{(d)}\,\sigma^{;\rho}{}_{\nu'}\,\sigma^{;\lambda}{}_{\mu}+\mathrm{i}qA_{\nu'\,;\mu}\,w_0^{(d)}+\mathrm{i}qA_{\nu'}\,w_{0\,;\mu}^{(d)}\right.\\
&\left.+\,\mathrm{i}qA_{\nu'}\,w_{1\rho}^{(d)}\,\sigma^{;\rho}{}_{\mu}-\mathrm{i}qA_{\mu}\,w_{1\rho}^{(d)}\,\sigma^{;\rho}{}_{\nu'}+q^2A_{\mu}A_{\nu'}\,w_0^{(d)}+\mathcal{O}\Big(\sigma^{\frac{1}{2}}\Big)\right]\\
&=(1-2\,\xi)\left[-w_{1\rho\,;\mu}^{(d)}\,\delta^{\rho}_{\nu}-2\,w_{2(\rho\lambda)}^{(d)}\,\delta^{\rho}_{\nu}\,\delta^{\lambda}_{\mu}+\mathrm{i}qA_{\nu\,;\mu}\,w_0^{(d)}+\mathrm{i}qA_{\nu}\,w_{0\,;\mu}^{(d)}+\mathrm{i}qA_{\nu}\,w_{1\rho}^{(d)}\,\delta^{\rho}_{\mu}\right.\\
&\left.+\,\mathrm{i}qA_{\mu}\,w_{1\rho}^{(d)}\,\delta^{\rho}_{\nu}+q^2A_{\mu}A_{\nu}\,w_0^{(d)}+\mathcal{O}\Big(\sigma^{\frac{1}{2}}\Big)\right]\\
&=(1-2\,\xi)\left[-w_{1\nu\,;\mu}^{(d)}-2\,w_{2(\mu\nu)}^{(d)}+\mathrm{i}qA_{\nu\,;\mu}\,w_0^{(d)}+\mathrm{i}qA_{\nu}\,w_{0\,;\mu}^{(d)}+\mathrm{i}qA_{\nu}\,w_{1\mu}^{(d)}+\mathrm{i}qA_{\mu}\,w_{1\nu}^{(d)}\right.\\
&\left.+\,q^2A_{\mu}A_{\nu}\,w_0^{(d)}+\mathcal{O}\Big(\sigma^{\frac{1}{2}}\Big)\right]. \qquad (8.67)
\end{aligned}
$$

Taking the real part of (8.67), we have

$$
\begin{aligned}
&(1-2\,\xi)\,\Re\left\{g_{\nu}{}^{\nu'}D_{\mu}D_{\nu'}\left[w_0^{(d)}+w_{1\rho}^{(d)}\,\sigma^{\,;\rho}+w_{2(\rho\lambda)}^{(d)}\,\sigma^{\,;\rho}\sigma^{\,;\lambda}+\mathcal{O}\Big(\sigma^{\frac{3}{2}}\Big)\right]\right\}\\
&=(1-2\,\xi)\left\{-\nabla_{\mu}\Re\Big[w_{1\nu}^{(d)}\Big]-2\,\Re\Big[w_{2(\mu\nu)}^{(d)}\Big]-2\,qA_{(\mu}\Im\Big[w_{1\nu)}^{(d)}\Big]+q^2A_{\mu}A_{\nu}\,w_0^{(d)}+\mathcal{O}\Big(\sigma^{\frac{1}{2}}\Big)\right\}\\
&=(1-2\,\xi)\left\{\frac{1}{2}\,w_{0\,;\mu\nu}^{(d)}-2\,\Re\Big[w_{2(\mu\nu)}^{(d)}\Big]-2\,qA_{(\mu}\Im\Big[w_{1\nu)}^{(d)}\Big]+q^2A_{\mu}A_{\nu}\,w_0^{(d)}+\mathcal{O}\Big(\sigma^{\frac{1}{2}}\Big)\right\}. \qquad (8.68)
\end{aligned}
$$

The second term in (8.64) is given by

$$
\begin{aligned}
&\left(2\,\xi-\frac{1}{2}\right)g_{\mu\nu}\,g^{\rho\sigma'}D_{\rho}D^{*}_{\sigma'}\left[w_0^{(d)}+w_{1\lambda}^{(d)}\,\sigma^{\,;\lambda}+w_{2(\lambda\tau)}^{(d)}\,\sigma^{\,;\lambda}\sigma^{\,;\tau}+\mathcal{O}\Big(\sigma^{\frac{3}{2}}\Big)\right]\\
&=\left(2\,\xi-\frac{1}{2}\right)g_{\mu\nu}\,g^{\rho\sigma'}D_{\rho}\left[w_{1\lambda}^{(d)}\,\sigma^{;\lambda}{}_{\sigma'}+2\,w_{2(\lambda\tau)}^{(d)}\,\sigma^{;\lambda}{}_{\sigma'}\,\sigma^{\,;\tau}+\mathrm{i}qA_{\sigma'}\,w_0^{(d)}+\mathrm{i}qA_{\sigma'}\,w_{1\lambda}^{(d)}\,\sigma^{\,;\lambda}\right.\\
&\left.+\,\mathcal{O}(\sigma)\right]\\
&=\left(2\,\xi-\frac{1}{2}\right)g_{\mu\nu}\,g^{\rho\phi}g_{\phi}{}^{\sigma'}\left[w_{1\lambda\,;\rho}^{(d)}\,\sigma^{;\lambda}{}_{\sigma'}+w_{1\lambda}^{(d)}\,\sigma^{;\lambda}{}_{\sigma'\rho}+2\,w_{2(\lambda\tau)}^{(d)}\,\sigma^{;\lambda}{}_{\sigma'}\,\sigma^{;\tau}{}_{\rho}+\mathrm{i}qA_{\sigma'\,;\rho}\,w_0^{(d)}\right.\\
&\left.+\,\mathrm{i}qA_{\sigma'}\,w_{0\,;\rho}^{(d)}+\mathrm{i}qA_{\sigma'}\,w_{1\lambda}^{(d)}\,\sigma^{;\lambda}{}_{\rho}-\mathrm{i}qA_{\rho}\,w_{1\lambda}^{(d)}\,\sigma^{;\lambda}{}_{\sigma'}+q^2A_{\rho}A_{\sigma'}\,w_0^{(d)}+\mathcal{O}\Big(\sigma^{\frac{1}{2}}\Big)\right]\\
&=\left(2\,\xi-\frac{1}{2}\right)g_{\mu\nu}\,g^{\rho\phi}\left[-w_{1\lambda\,;\rho}^{(d)}\,\delta^{\lambda}_{\phi}-2\,w_{2(\lambda\tau)}^{(d)}\,\delta^{\lambda}_{\phi}\,\delta^{\tau}_{\rho}+\mathrm{i}qA_{\phi\,;\rho}\,w_0^{(d)}+\mathrm{i}qA_{\phi}\,w_{0\,;\rho}^{(d)}+\mathrm{i}qA_{\phi}\,w_{1\lambda}^{(d)}\,\delta^{\lambda}_{\rho}\right.\\
&\left.+\,\mathrm{i}qA_{\rho}\,w_{1\lambda}^{(d)}\,\delta^{\lambda}_{\phi}+q^2A_{\rho}A_{\phi}\,w_0^{(d)}+\mathcal{O}\Big(\sigma^{\frac{1}{2}}\Big)\right]\\
&=\left(2\,\xi-\frac{1}{2}\right)g_{\mu\nu}\,g^{\rho\phi}\left[-w_{1\phi\,;\rho}^{(d)}-2\,w_{2(\phi\rho)}^{(d)}+\mathrm{i}qA_{\phi\,;\rho}\,w_0^{(d)}+\mathrm{i}qA_{\phi}\,w_{0\,;\rho}^{(d)}+\mathrm{i}qA_{\phi}\,w_{1\rho}^{(d)}+\mathrm{i}qA_{\rho}\,w_{1\phi}^{(d)}\right.\\
&\left.+\,q^2A_{\rho}A_{\phi}\,w_0^{(d)}+\mathcal{O}\Big(\sigma^{\frac{1}{2}}\Big)\right]\\
&=\left(2\,\xi-\frac{1}{2}\right)g_{\mu\nu}\left[-\nabla^{\lambda}w_{1\lambda}^{(d)}-2\,g^{\lambda\tau}w_{2\lambda\tau}^{(d)}+\mathrm{i}q\left(\nabla_{\lambda}A^{\lambda}\right)w_0^{(d)}+\mathrm{i}qA^{\lambda}\,w_{0\,;\lambda}^{(d)}+2\,\mathrm{i}qA^{\lambda}\,w_{1\lambda}^{(d)}\right.\\
&\left.+\,q^2A^{\lambda}A_{\lambda}\,w_0^{(d)}+\mathcal{O}\Big(\sigma^{\frac{1}{2}}\Big)\right]. \qquad (8.69)
\end{aligned}
$$

Taking the real part of (8.69), we have

$$\begin{aligned}
&\left(2\,\xi-\frac{1}{2}\right)\Re\left\{g_{\mu\nu}\,g^{\rho\sigma'}D_{\rho}D^{*}_{\sigma'}\left[w_0^{(d)}+w_{1\lambda}^{(d)}\,\sigma^{\,;\lambda}+w_{2(\lambda\tau)}^{(d)}\,\sigma^{\,;\lambda}\sigma^{\,;\tau}+\mathcal{O}\left(\sigma^{\frac{3}{2}}\right)\right]\right\}\\
&=\left(2\,\xi-\frac{1}{2}\right)g_{\mu\nu}\left\{-\nabla^{\lambda}\Re\left[w_{1\lambda}^{(d)}\right]-2\,g^{\lambda\tau}\Re\left[w_{2\lambda\tau}^{(d)}\right]-2\,qA^{\lambda}\Im\left[w_{1\lambda}^{(d)}\right]+q^2A^{\lambda}A_{\lambda}\,w_0^{(d)}\right.\\
&\left.+\,\mathcal{O}\left(\sigma^{\frac{1}{2}}\right)\right\}\\
&=\left(2\,\xi-\frac{1}{2}\right)g_{\mu\nu}\left\{\frac{1}{2}\Box\,w_0^{(d)}-2\,g^{\lambda\tau}\Re\left[w_{2\lambda\tau}^{(d)}\right]-2\,qA^{\lambda}\Im\left[w_{1\lambda}^{(d)}\right]+q^2A^{\lambda}A_{\lambda}\,w_0^{(d)}+\mathcal{O}\left(\sigma^{\frac{1}{2}}\right)\right\}.
\end{aligned} \tag{8.70}$$

The third term in (8.64) is given by

$$\begin{aligned}
&-2\,\xi\,D_{\mu}D_{\nu}\left[w_0^{(d)}+w_{1\lambda}^{(d)}\,\sigma^{\,;\lambda}+w_{2(\lambda\tau)}^{(d)}\,\sigma^{\,;\lambda}\sigma^{\,;\tau}+\mathcal{O}\left(\sigma^{\frac{3}{2}}\right)\right]\\
&=-2\,\xi\,D_{\mu}\left[w_{0\,;\nu}^{(d)}+w_{1\lambda\,;\nu}^{(d)}\,\sigma^{\,;\lambda}+w_{1\lambda}^{(d)}\,\sigma^{\,;\lambda}{}_{\nu}+2\,w_{2(\lambda\tau)}^{(d)}\,\sigma^{\,;\lambda}{}_{\nu}\,\sigma^{\,;\tau}-\mathrm{i}qA_{\nu}\,w_0^{(d)}-\mathrm{i}qA_{\nu}\,w_{1\lambda}^{(d)}\,\sigma^{\,;\lambda}\right.\\
&\left.+\,\mathcal{O}(\sigma)\right]\\
&=-2\,\xi\left[w_{0\,;\mu\nu}^{(d)}+w_{1\lambda\,;\nu}^{(d)}\,\sigma^{\,;\lambda}{}_{\mu}+w_{1\lambda\,;\mu}^{(d)}\,\sigma^{\,;\lambda}{}_{\nu}+w_{1\lambda}^{(d)}\,\sigma^{\,;\lambda}{}_{\nu\mu}+2\,w_{2(\lambda\tau)}^{(d)}\,\sigma^{\,;\lambda}{}_{\nu}\,\sigma^{\,;\tau}{}_{\mu}-\mathrm{i}q\left(\nabla_{\mu}A_{\nu}\right)w_0^{(d)}\right.\\
&\left.-\,\mathrm{i}qA_{\nu}\,w_{0\,;\mu}^{(d)}-\mathrm{i}qA_{\nu}\,w_{1\lambda}^{(d)}\,\sigma^{\,;\lambda}{}_{\mu}-\mathrm{i}qA_{\mu}\,w_{0\,;\nu}^{(d)}-\mathrm{i}qA_{\mu}\,w_{1\lambda}^{(d)}\,\sigma^{\,;\lambda}{}_{\nu}-q^2A_{\mu}A_{\nu}\,w_0^{(d)}+\mathcal{O}(\sigma)\right]\\
&=-2\,\xi\left[w_{0\,;\mu\nu}^{(d)}+w_{1\lambda\,;\nu}^{(d)}\,\delta^{\lambda}_{\mu}+w_{1\lambda\,;\mu}^{(d)}\,\delta^{\lambda}_{\nu}+2\,w_{2(\lambda\tau)}^{(d)}\,\delta^{\lambda}_{\nu}\,\delta^{\tau}_{\mu}-\mathrm{i}q\left(\nabla_{\mu}A_{\nu}\right)w_0^{(d)}-2\,\mathrm{i}qA_{(\mu}\nabla_{\nu)}w_0^{(d)}\right.\\
&\left.-\,\mathrm{i}qA_{\nu}\,w_{1\lambda}^{(d)}\,\delta^{\lambda}_{\mu}-\mathrm{i}qA_{\mu}\,w_{1\lambda}^{(d)}\,\delta^{\lambda}_{\nu}-q^2A_{\mu}A_{\nu}\,w_0^{(d)}+\mathcal{O}(\sigma)\right]\\
&=-2\,\xi\left[w_{0\,;\mu\nu}^{(d)}+2\,w_{1(\mu\,;\nu)}^{(d)}+2\,w_{2(\mu\nu)}^{(d)}-\mathrm{i}q\left(\nabla_{\mu}A_{\nu}\right)w_0^{(d)}-2\,\mathrm{i}qA_{(\mu}\nabla_{\nu)}w_0^{(d)}-2\,\mathrm{i}qA_{(\mu}\,w_{1\nu)}^{(d)}\right.\\
&\left.-\,q^2A_{\mu}A_{\nu}\,w_0^{(d)}+\mathcal{O}(\sigma)\right].
\end{aligned} \tag{8.71}$$

Taking the real part of (8.71), we have

$$\begin{aligned}
&-2\,\xi\,\Re\left\{D_{\mu}D_{\nu}\left[w_0^{(d)}+w_{1\lambda}^{(d)}\,\sigma^{\,;\lambda}+w_{2(\lambda\tau)}^{(d)}\,\sigma^{\,;\lambda}\sigma^{\,;\tau}+\mathcal{O}\left(\sigma^{\frac{3}{2}}\right)\right]\right\}\\
&=-2\,\xi\left\{w_{0\,;\mu\nu}^{(d)}+2\,\nabla_{(\mu}\,\Re\left[w_{1\nu)}^{(d)}\right]+2\,\Re\left[w_{2(\mu\nu)}^{(d)}\right]+2\,qA_{(\mu}\,\Im\left[w_{1\nu)}^{(d)}\right]-q^2A_{\mu}A_{\nu}\,w_0^{(d)}+\mathcal{O}(\sigma)\right\}\\
&=-2\,\xi\left\{2\,\Re\left[w_{2(\mu\nu)}^{(d)}\right]+2\,qA_{(\mu}\,\Im\left[w_{1\nu)}^{(d)}\right]-q^2A_{\mu}A_{\nu}\,w_0^{(d)}+\mathcal{O}(\sigma)\right\}.
\end{aligned} \tag{8.72}$$

The fourth term in (8.64) is given by

$$
\begin{aligned}
&2\,\xi\, g_{\mu\nu}\, D_\rho D^\rho \left[w_0^{(d)} + w_{1\lambda}^{(d)}\,\sigma^{\,;\lambda} + w_{2(\lambda\alpha)}^{(d)}\,\sigma^{\,;\lambda}\sigma^{\,;\alpha} + \mathcal{O}\!\left(\sigma^{\frac{3}{2}}\right)\right] \\
&= 2\,\xi\, g_{\mu\nu}\, g^{\rho\tau} D_\rho D_\tau \left[w_0^{(d)} + w_{1\lambda}^{(d)}\,\sigma^{\,;\lambda} + w_{2(\lambda\alpha)}^{(d)}\,\sigma^{\,;\lambda}\sigma^{\,;\alpha} + \mathcal{O}\!\left(\sigma^{\frac{3}{2}}\right)\right] \\
&= 2\,\xi\, g_{\mu\nu}\, g^{\rho\tau} D_\rho \left[w_{0\,;\tau}^{(d)} + w_{1\lambda\,;\tau}^{(d)}\,\sigma^{\,;\lambda} + w_{1\lambda}^{(d)}\,\sigma^{;\lambda}{}_{\tau} + 2\,w_{2(\lambda\alpha)}^{(d)}\,\sigma^{;\lambda}{}_{\tau}\,\sigma^{\,;\alpha} - \mathrm{i}qA_\tau\, w_0^{(d)} \right.\\
&\left. -\, \mathrm{i}qA_\tau\, w_{1\lambda}^{(d)}\,\sigma^{\,;\lambda} + \mathcal{O}(\sigma)\right] \\
&= 2\,\xi\, g_{\mu\nu}\, g^{\rho\tau} \left[w_{0\,;\tau\rho}^{(d)} + w_{1\lambda\,;\tau}^{(d)}\,\sigma^{;\lambda}{}_{\rho} + w_{1\lambda\,;\rho}^{(d)}\,\sigma^{;\lambda}{}_{\tau} + w_{1\lambda}^{(d)}\,\sigma^{;\lambda}{}_{\tau\rho} + 2\,w_{2(\lambda\alpha)}^{(d)}\,\sigma^{;\lambda}{}_{\tau}\,\sigma^{;\alpha}{}_{\rho} - \mathrm{i}qA_{\tau\,;\rho}\, w_0^{(d)} \right.\\
&\left. -\, \mathrm{i}qA_\tau\, w_{0\,;\rho}^{(d)} - \mathrm{i}qA_\tau\, w_{1\lambda}^{(d)}\,\sigma^{;\lambda}{}_{\rho} - \mathrm{i}qA_\rho\, w_{0\,;\tau}^{(d)} - \mathrm{i}qA_\rho\, w_{1\lambda}^{(d)}\,\sigma^{;\lambda}{}_{\tau} - q^2 A_\rho A_\tau\, w_0^{(d)} + \mathcal{O}\!\left(\sigma^{\frac{1}{2}}\right)\right] \\
&= 2\,\xi\, g_{\mu\nu} \left[\Box\, w_0^{(d)} + 2\,\nabla^\lambda w_{1\lambda}^{(d)} + 2\, g^{\lambda\tau} w_{2\lambda\tau}^{(d)} - \mathrm{i}q\left(\nabla_\lambda A^\lambda\right) w_0^{(d)} - 2\,\mathrm{i}qA_\lambda \nabla^\lambda w_0^{(d)} - 2\,\mathrm{i}qA^\lambda\, w_{1\lambda}^{(d)} \right.\\
&\left. -\, q^2 A^\lambda A_\lambda\, w_0^{(d)} + \mathcal{O}\!\left(\sigma^{\frac{1}{2}}\right)\right]. \qquad (8.73)
\end{aligned}
$$

Taking the real part of (8.73), we have

$$
\begin{aligned}
&2\,\xi\, \Re\left\{g_{\mu\nu}\, D_\rho D^\rho \left[w_0^{(d)} + w_{1\lambda}^{(d)}\,\sigma^{\,;\lambda} + w_{2(\lambda\alpha)}^{(d)}\,\sigma^{\,;\lambda}\sigma^{\,;\alpha} + \mathcal{O}\!\left(\sigma^{\frac{3}{2}}\right)\right]\right\} \\
&= 2\,\xi\, g_{\mu\nu}\left\{\Box\, w_0^{(d)} + 2\,\nabla^\lambda \Re\!\left[w_{1\lambda}^{(d)}\right] + 2\, g^{\lambda\tau}\Re\!\left[w_{2\lambda\tau}^{(d)}\right] + 2\, qA^\lambda \Im\!\left[w_{1\lambda}^{(d)}\right] - q^2 A^\lambda A_\lambda\, w_0^{(d)} \right.\\
&\left. +\, \mathcal{O}\!\left(\sigma^{\frac{1}{2}}\right)\right\} \\
&= 2\,\xi\, g_{\mu\nu}\left\{2\, g^{\lambda\tau}\Re\!\left[w_{2\lambda\tau}^{(d)}\right] + 2\, qA^\lambda \Im\!\left[w_{1\lambda}^{(d)}\right] - q^2 A^\lambda A_\lambda\, w_0^{(d)} + \mathcal{O}\!\left(\sigma^{\frac{1}{2}}\right)\right\}. \qquad (8.74)
\end{aligned}
$$

The fifth term in (8.64) is given by

$$
\xi\left(R_{\mu\nu} - \frac{1}{2}\, g_{\mu\nu} R\right)\left[w_0^{(d)} + \mathcal{O}\!\left(\sigma^{\frac{1}{2}}\right)\right] = \xi\left(R_{\mu\nu} - \frac{1}{2}\, g_{\mu\nu} R\right) w_0^{(d)} + \mathcal{O}\!\left(\sigma^{\frac{1}{2}}\right). \qquad (8.75)
$$

The sixth term in (8.64) is given by

$$
-\,\frac{1}{2}\, g_{\mu\nu}\, m^2 \left[w_0^{(d)} + \mathcal{O}\!\left(\sigma^{\frac{1}{2}}\right)\right] = -\frac{1}{2}\, g_{\mu\nu}\, m^2\, w_0^{(d)} + \mathcal{O}\!\left(\sigma^{\frac{1}{2}}\right). \qquad (8.76)
$$

Equations (8.75) and (8.76) are obviously real. Inserting equations (8.68), (8.70), (8.72), (8.74), (8.75) and (8.76) into (8.64), we have

$$
\begin{aligned}
\langle \hat{T}_{\mu\nu}\rangle_{\mathrm{ren}} = \alpha^{(d)} \Bigg\{&(1-2\,\xi)\left(\frac{1}{2}\, w_{0\,;\mu\nu}^{(d)} - 2\,\Re\!\left[w_{2(\mu\nu)}^{(d)}\right] - 2\, qA_{(\mu}\Im\!\left[w_{1\nu)}^{(d)}\right] + q^2 A_\mu A_\nu\, w_0^{(d)}\right) \\
&+ \left(2\,\xi - \frac{1}{2}\right) g_{\mu\nu}\left(\frac{1}{2}\,\Box\, w_0^{(d)} - 2\, g^{\lambda\tau}\Re\!\left[w_{2\lambda\tau}^{(d)}\right] - 2\, qA^\lambda \Im\!\left[w_{1\lambda}^{(d)}\right] + q^2 A^\lambda A_\lambda\, w_0^{(d)}\right) \\
&- 2\,\xi\left(2\,\Re\!\left[w_{2(\mu\nu)}^{(d)}\right] + 2\, qA_{(\mu}\, \Im\!\left[w_{1\nu)}^{(d)}\right] - q^2 A_\mu A_\nu\, w_0^{(d)}\right) + 2\,\xi\, g_{\mu\nu}\left(2\, g^{\lambda\tau}\Re\!\left[w_{2\lambda\tau}^{(d)}\right]\right. \\
&\left. +\; 2\, qA^\lambda\Im\!\left[w_{1\lambda}^{(d)}\right] - q^2 A^\lambda A_\lambda\, w_0^{(d)}\right) + \xi\left(R_{\mu\nu} - \frac{1}{2}\, g_{\mu\nu} R\right) w_0^{(d)} - \frac{1}{2}\, g_{\mu\nu}\, m^2\, w_0^{(d)}\Bigg\} \\
&+ \widetilde{\Theta}_{\mu\nu}^{(d)}, \qquad (8.77)
\end{aligned}
$$

which simplifies to

$$\langle \hat{T}_{\mu\nu} \rangle_{\text{ren}} = \alpha^{(d)} \left\{ \frac{1}{2} w^{(d)}_{0\,;\mu\nu} + q^2 A_\mu A_\nu \, w^{(d)}_0 - 2\, q A_{(\mu} \Im\left[w^{(d)}_{1\nu)}\right] - 2\, \Re\left[w^{(d)}_{2(\mu\nu)}\right] - \frac{1}{4}\, g_{\mu\nu} \,\Box\, w^{(d)}_0 \right.$$
$$- \frac{1}{2}\, q^2 A_\lambda A^\lambda g_{\mu\nu}\, w^{(d)}_0 - \frac{1}{2}\, g_{\mu\nu}\, m^2\, w^{(d)}_0 + q A^\lambda g_{\mu\nu}\, \Im\left[w^{(d)}_{1\lambda}\right] + g_{\mu\nu}\, g^{\lambda\tau} \Re\left[w^{(d)}_{2\lambda\tau}\right]$$
$$\left. + \,\xi \left(- w^{(d)}_{0\,;\mu\nu} + R_{\mu\nu}\, w^{(d)}_0 + g_{\mu\nu} \,\Box\, w^{(d)}_0 - \frac{1}{2}\, g_{\mu\nu}\, w^{(d)}_0 R \right) \right\}$$
$$+ \widetilde{\Theta}^{(d)}_{\mu\nu}. \tag{8.78}$$

Then, finally we obtain for the RSET

$$\langle \widehat{T}_{\mu\nu} \rangle_{\text{ren}} = \alpha^{(d)} \left\{ -2\, \Re\left[w^{(d)}_{2(\mu\nu)}\right] - 2\, q A_{(\mu} \Im\left[w^{(d)}_{1\nu)}\right] - \left(\xi - \frac{1}{2}\right) w^{(d)}_{0\,;\mu\nu} \right.$$
$$+ \left(\xi R_{\mu\nu} + q^2 A_\mu A_\nu\right) w^{(d)}_0 + g_{\mu\nu} \left(g^{\lambda\tau} \Re\left[w^{(d)}_{2\lambda\tau}\right] + q A^\lambda \Im\left[w^{(d)}_{1\lambda}\right] \right.$$
$$\left.\left. + \left(\xi - \frac{1}{4}\right) \Box\, w^{(d)}_0 - \frac{1}{2} \left(m^2 + \xi R + q^2 A_\rho A^\rho\right) w^{(d)}_0 \right) \right\} + \widetilde{\Theta}^{(d)}_{\mu\nu}. \tag{8.79}$$

The expression for the RSET in (8.79) is manifestly symmetric in μ and ν, as we would expect, and reduces to (71) in [68] when we take the uncharged limit $q \to 0$.

From Wald's axioms in §1.5, we expect that the RSET is conserved, i.e. $\nabla^\mu \langle \widehat{T}_{\mu\nu} \rangle_{\text{ren}} = 0$. Taking the divergence of (8.79), we have

$$\nabla^\mu \langle \widehat{T}_{\mu\nu} \rangle_{\text{ren}} = \alpha^{(d)} \left\{ -2\, \nabla^\mu \Re\left[w^{(d)}_{2(\mu\nu)}\right] - 2\, q \left(\nabla^\mu A_{(\mu}\right) \Im\left[w^{(d)}_{1\nu)}\right] - 2\, q A_{(\mu} \nabla^\mu \Im\left[w^{(d)}_{1\nu)}\right] \right.$$
$$- \left(\xi - \frac{1}{2}\right) \nabla_\mu \nabla_\nu \nabla^\mu\, w^{(d)}_0 + \left[\xi\, \nabla^\mu R_{\mu\nu} + 2\, q^2 A_{(\mu} \nabla^\mu A_{\nu)}\right] w^{(d)}_0$$
$$+ \left(\xi R_{\mu\nu} + q^2 A_\mu A_\nu\right) \nabla^\mu w^{(d)}_0 + \nabla_\nu\, g^{\lambda\tau} \Re\left[w^{(d)}_{2\lambda\tau}\right] + q \left(\nabla_\nu A^\lambda\right) \Im\left[w^{(d)}_{1\lambda}\right]$$
$$+ q A^\lambda \nabla_\nu\, \Im\left[w^{(d)}_{1\lambda}\right] + \left(\xi - \frac{1}{4}\right) \nabla_\nu \left(\Box\, w^{(d)}_0\right) - \left(q^2 A_\rho \nabla_\nu A^\rho + \frac{1}{2}\, \xi R_{;\nu}\right) w^{(d)}_0$$
$$\left. - \frac{1}{2} \left(m^2 + \xi R + q^2 A_\rho A^\rho\right) \nabla_\nu\, w^{(d)}_0 \right\} + \nabla^\mu \widetilde{\Theta}^{(d)}_{\mu\nu}. \tag{8.80}$$

Thus far, our expressions for the RSET in terms of the expansion coefficients of the biscalar $W^{(d)}(x,x')$, as well as its divergence, have been valid in any number of dimensions (greater than one). In order to simplify (8.80), we will need to use the identities developed in §8.2, which differ depending on whether we are considering an even number of dimensions or an odd number. Given the presence of contributions, in the aforementioned identities, from the expansion coefficients of the $V^{(2p)}(x,x')$ biscalar in even dimensions, which do not appear in odd dimensions, we will consider an even number of dimensions in the following calculation with the generalisation to odd number of dimensions consisting of simply ignoring contributions from $V^{(2p)}(x,x')$.

In order to eliminate the $\nabla_\nu\, g^{\lambda\tau} \Re\left[w^{(2p)}_{2\lambda\tau}\right]$ term from (8.80), we can use the identity in (8.26); then (8.80) simplifies considerably and we obtain

$$\nabla^\mu \langle \widehat{T}_{\mu\nu}\rangle_{\rm ren} = \alpha^{(2p)} \Big\{ -2\,\nabla^\mu \Re\Big[w^{(2p)}_{2(\mu\nu)}\Big] - 2\,q\,(\nabla^\mu A_{(\mu})\,\Im\Big[w^{(2p)}_{1\nu)}\Big] - 2\,qA_{(\mu}\nabla^\mu \Im\Big[w^{(2p)}_{1\nu)}\Big]$$
$$- \left(\xi - \frac{1}{2}\right) \nabla_\mu\nabla_\nu\nabla^\mu\, w_0^{(2p)} + \left[\xi\,\nabla^\mu R_{\mu\nu} + 2\,q^2 A_{(\mu}\nabla^\mu A_{\nu)}\right] w_0^{(2p)}$$
$$+ \left(\xi R_{\mu\nu} + q^2 A_\mu A_\nu\right) \nabla^\mu w_0^{(2p)} + \left(\xi - \frac{1}{4}\right) \nabla_\nu\left(\Box\, w_0^{(2p)}\right) - (p+1)\,\nabla_\nu V_{10}^{(2p)} \Big\}$$
$$+ \nabla^\mu \widetilde{\Theta}^{(2p)}_{\mu\nu}. \tag{8.81}$$

In order to eliminate the $-2\,\nabla^\mu \Re\Big[w^{(2p)}_{2(\mu\nu)}\Big]$ term from (8.81), we can use the expression in (8.38); then (8.81) becomes

$$\nabla^\mu \langle \widehat{T}_{\mu\nu}\rangle_{\rm ren} = \alpha^{(2p)} \Big\{ -\frac{1}{2} R^\mu{}_\nu \nabla_\mu w_0^{(2p)} - q\,(\nabla^\mu A_\nu)\,\Im\Big[w^{(2p)}_{1\mu}\Big] + q\,(\nabla_\nu A^\mu)\,\Im\Big[w^{(2p)}_{1\mu}\Big]$$
$$- qA_\nu \nabla^\mu \Im\Big[w^{(2p)}_{1\mu}\Big] - \left[q^2 A^\mu \nabla_\nu A_\mu + \frac{1}{2}\,\xi R_{;\nu}\right] w_0^{(2p)} - \left(\xi - \frac{1}{2}\right) \nabla_\mu\nabla_\nu\nabla^\mu\, w_0^{(2p)}$$
$$+ \left[\xi\,\nabla^\mu R_{\mu\nu} + 2\,q^2 A_{(\mu}\nabla^\mu A_{\nu)}\right] w_0^{(2p)} + \left(\xi R_{\mu\nu} + q^2 A_\mu A_\nu\right) \nabla^\mu w_0^{(2p)}$$
$$+ \left(\xi - \frac{1}{2}\right) \nabla_\nu\left(\Box\, w_0^{(2p)}\right) - p\,\nabla_\nu V_{10}^{(2p)} \Big\} + \nabla^\mu \widetilde{\Theta}^{(2p)}_{\mu\nu}. \tag{8.82}$$

Commuting the spacetime derivatives (1.10) and using the definition of the EM field strength tensor (3.2), we have

$$\nabla^\mu \langle \widehat{T}_{\mu\nu}\rangle_{\rm ren} = \alpha^{(2p)} \Big\{ -\frac{1}{2} R^\mu{}_\nu \nabla_\mu w_0^{(2p)} - qF^\mu{}_\nu\,\Im\Big[w^{(2p)}_{1\mu}\Big] - qA_\nu \nabla^\mu \Im\Big[w^{(2p)}_{1\mu}\Big]$$
$$- \left[q^2 A^\mu \nabla_\nu A_\mu + \frac{1}{2}\,\xi R_{;\nu}\right] w_0^{(2p)} - \left(\xi - \frac{1}{2}\right) \nabla_\nu\left(\Box\, w_0^{(2p)}\right)$$
$$- \left(\xi - \frac{1}{2}\right) R^\mu{}_\nu \nabla_\mu w_0^{(2p)} + \left[\xi\,\nabla^\mu R_{\mu\nu} + 2\,q^2 A_{(\mu}\nabla^\mu A_{\nu)}\right] w_0^{(2p)}$$
$$+ \left(\xi R_{\mu\nu} + q^2 A_\mu A_\nu\right) \nabla^\mu w_0^{(2p)} + \left(\xi - \frac{1}{2}\right) \nabla_\nu\left(\Box\, w_0^{(2p)}\right) - p\,\nabla_\nu V_{10}^{(2p)} \Big\}$$
$$+ \nabla^\mu \widetilde{\Theta}^{(2p)}_{\mu\nu}. \tag{8.83}$$

Several terms in (8.83) cancel, with (8.83) simplifying considerably to give

$$\nabla^\mu \langle \widehat{T}_{\mu\nu}\rangle_{\rm ren} = \alpha^{(2p)} \Big\{ -qF^\mu{}_\nu\,\Im\Big[w^{(2p)}_{1\mu}\Big] - qA_\nu \nabla^\mu \Im\Big[w^{(2p)}_{1\mu}\Big] - q^2 A^\mu (\nabla_\nu A_\mu)\, w_0^{(2p)}$$
$$+ \xi\left(\nabla^\mu R_{\mu\nu} - \frac{1}{2} R_{;\nu}\right) w_0^{(2p)} + 2\,q^2 A_{(\mu}\left(\nabla^\mu A_{\nu)}\right) w_0^{(2p)} + q^2 A_\mu A_\nu \nabla^\mu w_0^{(2p)}$$
$$- p\,\nabla_\nu V_{10}^{(2p)} \Big\} + \nabla^\mu \widetilde{\Theta}^{(2p)}_{\mu\nu}. \tag{8.84}$$

Using the Bianchi identity (1.8) and expanding out the symmetrised quantity, we have

$$\nabla^\mu \langle \widehat{T}_{\mu\nu}\rangle_{\rm ren} = \alpha^{(2p)} \Big\{ -qF^\mu{}_\nu\,\Im\Big[w^{(2p)}_{1\mu}\Big] - qA_\nu \nabla^\mu \Im\Big[w^{(2p)}_{1\mu}\Big] - q^2 A^\mu (\nabla_\nu A_\mu)\, w_0^{(2p)}$$
$$+ q^2 A^\mu (\nabla_\mu A_\nu)\, w_0^{(2p)} + q^2 A_\nu (\nabla^\mu A_\mu)\, w_0^{(2p)} + q^2 A_\mu A_\nu \nabla^\mu w_0^{(2p)} - p\,\nabla_\nu V_{10}^{(2p)} \Big\}$$
$$+ \nabla^\mu \widetilde{\Theta}^{(2p)}_{\mu\nu}. \tag{8.85}$$

In order to eliminate the $-qA_\nu \nabla^\mu \Im\left[w_{1\mu}^{(2p)}\right]$ term from (8.85), we can use the expression in (8.29); then (8.85) becomes

$$\nabla^\mu \langle \widehat{T}_{\mu\nu}\rangle_{\rm ren} = \alpha^{(2p)} \left\{ q^2 A^\mu F_{\mu\nu}\, w_0^{(2p)} - qF^\mu{}_\nu\, \Im\left[w_{1\mu}^{(2p)}\right] - p\,\nabla_\nu V_{10}^{(2p)} \right\} + \nabla^\mu \widetilde{\Theta}_{\mu\nu}^{(2p)}, \quad (8.86)$$

where we have, again, used the definition of $F_{\mu\nu}$ (3.2). Noticing that the first two terms in (8.86) are multiples of the terms in the expression for the renormalised current (8.52), we can use (8.52) to write the divergence of the RSET as

$$\nabla^\mu \langle \widehat{T}_{\mu\nu}\rangle_{\rm ren} = -\alpha^{(2p)} p\,\nabla_\nu V_{10}^{(2p)} + 4\pi F_{\mu\nu} \langle \hat{J}^\mu \rangle_{\rm ren} + \nabla^\mu \widetilde{\Theta}_{\mu\nu}^{(2p)}. \quad (8.87)$$

which holds for even dimensions including $d = 2$. In odd dimensions, the corresponding expression is given by

$$\nabla^\mu \langle \widehat{T}_{\mu\nu}\rangle_{\rm ren} = 4\pi F_{\mu\nu} \langle \hat{J}^\mu \rangle_{\rm ren} + \nabla^\mu \widetilde{\Theta}_{\mu\nu}^{(2p+1)}. \quad (8.88)$$

Following the logic presented in [68] we can define another tensor $\Theta_{\mu\nu}^{(d)}$ which is locally conserved, in contrast to the quantity $\widetilde{\Theta}_{\mu\nu}^{(d)}$ introduced in (8.63), and which is defined as

$$\Theta_{\mu\nu}^{(d)} = \begin{cases} -\alpha^{(2p)} p\, V_{10}^{(2p)} g_{\mu\nu} + \widetilde{\Theta}_{\mu\nu}^{(2p)} & d = 2p, \\ \widetilde{\Theta}_{\mu\nu}^{(2p+1)} & d = 2p+1, \end{cases} \quad (8.89)$$

so that the divergence of the RSET is given, in all dimensions, by

$$\nabla^\mu \langle \hat{T}_{\mu\nu}\rangle_{\rm ren} = 4\pi F_{\mu\nu} \langle \hat{J}^\mu \rangle_{\rm ren}\,, \quad (8.90)$$

meaning that the RSET, associated to the quantised scalar field, is not conserved in any number of spacetime dimensions. There is an intuitive explanation for the result in (8.90). In the physical system we are considering, there are two matter fields present; the first is the quantised scalar field whose associated stress-energy tensor we are attempting to renormalise by extending the Hadamard renormalisation procedure. The second field present in the physical system under consideration is the classical, background gauge field that gives rise to the charge possessed by the quantum field. This field also has an associated classical, stress-energy tensor $T_{\mu\nu}^{\rm F}$ given by

$$T_{\mu\nu}^{\rm F} = F_{\mu\rho} F_\nu{}^\rho - \frac{1}{4}\, g_{\mu\nu} F_{\rho\tau} F^{\rho\tau}. \quad (8.91)$$

We then expect that the total stress-energy tensor, associated to both the quantised scalar field and to the background gauge field, is conserved, i.e. that

$$\nabla^\mu \left[\langle \hat{T}_{\mu\nu}\rangle_{\rm ren} + T_{\mu\nu}^{\rm F} \right] = \nabla^\mu \langle \hat{T}_{\mu\nu}\rangle_{\rm ren} + \nabla^\mu T_{\mu\nu}^{\rm F} = 0, \quad (8.92)$$

where we have used the fact that the covariant derivative ∇_μ is a linear operator in (8.92). Then, taking the divergence of (8.91), we have

$$
\begin{aligned}
\nabla^\mu T^{\rm F}_{\mu\nu} &= \nabla^\mu \left(F_{\mu\rho} F_\nu{}^\rho - \frac{1}{4}\, g_{\mu\nu} F_{\rho\tau} F^{\rho\tau} \right) \\
&= F_{\mu\rho} \nabla^\mu F_\nu{}^\rho + \left(\nabla^\mu F_{\mu\rho} \right) F_\nu{}^\rho - \frac{1}{2}\, g_{\mu\nu} F^{\rho\tau}\, \nabla^\mu F_{\rho\tau} \\
&= F^{\mu\rho}\, \nabla_\mu F_{\nu\rho} + \left(\nabla^\mu F_{\mu\rho} \right) F_\nu{}^\rho - \frac{1}{2} F^{\rho\tau}\, \nabla_\nu F_{\rho\tau} \\
&= F^{\rho\tau}\, \nabla_\rho F_{\nu\tau} + F_{\nu\rho} \nabla_\mu F^{\mu\rho} - \frac{1}{2} F^{\rho\tau}\, \nabla_\nu F_{\rho\tau} \\
&= F_{\nu\rho} \nabla_\mu F^{\mu\rho} + F^{\rho\tau} \left(\nabla_\rho F_{\nu\tau} - \frac{1}{2}\, \nabla_\nu F_{\rho\tau} \right) \\
&= F_{\nu\rho} \nabla_\mu F^{\mu\rho} + \frac{1}{2}\, F^{\rho\tau} \left(\nabla_\rho F_{\nu\tau} + \nabla_\rho F_{\nu\tau} - \nabla_\nu F_{\rho\tau} \right). \qquad (8.93)
\end{aligned}
$$

In order to simplify (8.93) further, we will need to use Maxwell's equations $\nabla_{[\rho} F_{\tau\nu]} = 0$ to cyclically permute the indices on the second term in the bracket on the r.h.s of the last line in (8.93). Then, we have

$$
\begin{aligned}
\nabla^\mu T^{\rm F}_{\mu\nu} &= F_{\nu\rho} \nabla_\mu F^{\mu\rho} + \frac{1}{2}\, F^{\rho\tau} \left(-\nabla_\rho F_{\tau\nu} + \nabla_\tau F_{\rho\nu} - \nabla_\nu F_{\rho\tau} \right) \\
&= F_{\nu\rho} \nabla_\mu F^{\mu\rho} + \frac{1}{2}\, F^{\rho\tau} \left(-\nabla_\rho F_{\tau\nu} - \nabla_\tau F_{\nu\rho} - \nabla_\nu F_{\rho\tau} \right) \\
&= F_{\nu\rho} \nabla_\mu F^{\mu\rho} - \frac{1}{2}\, F^{\rho\tau}\, \nabla_{[\rho} F_{\tau\nu]} \\
&= F_{\nu\rho} \nabla_\mu F^{\mu\rho}, \qquad (8.94)
\end{aligned}
$$

where we have used $\nabla_{[\rho} F_{\tau\nu]} = 0$ to go from the penultimate equality to the last one. Then, using the semiclassical Maxwell equation (1.12), we have

$$
\nabla^\mu T^{\rm F}_{\mu\nu} = 4\pi F_{\nu\rho} \langle \hat{J}^\rho \rangle_{\rm ren}\,. \qquad (8.95)
$$

Using (8.95), equation (8.92) becomes

$$
\nabla^\mu \left[\langle \hat{T}_{\mu\nu} \rangle_{\rm ren} + T^{\rm F}_{\mu\nu} \right] = 4\pi F_{\mu\nu} \langle \hat{J}^\mu \rangle_{\rm ren} + 4\pi F_{\nu\mu} \langle \hat{J}^\mu \rangle_{\rm ren} = 0, \qquad (8.96)
$$

from the antisymmetry of $F_{\mu\nu}$, meaning that the total stress-energy tensor associated to the entirety of the physical system under consideration is conserved, as required. The final act of this section, before proceeding to consider renormalisation ambiguities in the scalar field condensate as well as the renormalised expectation values of the current and stress-energy tensor, will be to use the definition of the locally conserved tensor $\Theta^{(d)}_{\mu\nu}$ in (8.89) as well as the identities (8.25), in even dimensions, and (8.41), in odd dimensions, to simplify the expression for the RSET in (8.79). In even dimensions, using (8.89) and (8.25), equation (8.79) reduces to

$$
\begin{aligned}
\langle \widehat{T}_{\mu\nu} \rangle_{\rm ren} = \alpha^{(2p)} \Bigg\{ &-2\, \Re\left[w^{(2p)}_{2(\mu\nu)} \right] - 2\, q A_{(\mu} \Im\left[w^{(2p)}_{1\nu)} \right] - \left(\xi - \frac{1}{2} \right) w^{(2p)}_{0\,;\mu\nu} \\
&+ \left(\xi R_{\mu\nu} + q^2 A_\mu A_\nu \right) w^{(2p)}_0 + g_{\mu\nu} \left[\left(\xi - \frac{1}{4} \right) \Box\, w^{(2p)}_0 - (p+1)\, V^{(2p)}_{10} \right] \Bigg\} \\
&+ \alpha^{(2p)} p\, V^{(2p)}_{10} g_{\mu\nu} + \Theta^{(2p)}_{\mu\nu}, \qquad (8.97)
\end{aligned}
$$

so that the expression for the RSET in even dimensions is given by

$$\langle\widehat{T}_{\mu\nu}\rangle_{\text{ren}} = \alpha^{(2p)}\left\{-2\,\Re\left[w^{(2p)}_{2(\mu\nu)}\right] - 2\,qA_{(\mu}\Im\left[w^{(2p)}_{1\nu)}\right] - \left(\xi-\frac{1}{2}\right)w^{(2p)}_{0\,;\mu\nu}\right.$$
$$\left. + \left(\xi R_{\mu\nu} + q^2 A_\mu A_\nu\right)w^{(2p)}_0 + g_{\mu\nu}\left[\left(\xi-\frac{1}{4}\right)\Box\,w^{(2p)}_0 - V^{(2p)}_{10}\right]\right\} + \Theta^{(2p)}_{\mu\nu}. \quad (8.98)$$

Using (8.89) and (8.25) to simplify (8.79), the expression for the RSET in odd dimensions is given immediately by

$$\langle\widehat{T}_{\mu\nu}\rangle_{\text{ren}} = \alpha^{(2p+1)}\left\{-2\,\Re\left[w^{(2p+1)}_{2(\mu\nu)}\right] - 2\,qA_{(\mu}\Im\left[w^{(2p+1)}_{1\nu)}\right] - \left(\xi-\frac{1}{2}\right)w^{(2p+1)}_{0\,;\mu\nu}\right.$$
$$\left. + \left(\xi R_{\mu\nu} + q^2 A_\mu A_\nu\right)w^{(2p+1)}_0 + g_{\mu\nu}\left(\xi-\frac{1}{4}\right)\Box\,w^{(2p+1)}_0\right\} + \Theta^{(2p+1)}_{\mu\nu}. \quad (8.99)$$

This concludes our study of the scalar field condensate, the renormalised current and the RSET. We will now proceed to consider renormalisation ambiguities in each of the aforementioned quantities.

8.4 Renormalisation ambiguities

In the previous section, we derived expressions for the scalar field condensate, the renormalised current and the RSET in terms of the expansion coefficients of the biscalar $W^{(d)}(x,x')$; the expansion coefficients of the biscalar $V^{(2p)}(x,x')$ also appeared in the expression for the RSET (8.98) in even dimensions. The expressions for the RSET in both even (8.98) and odd dimensions (8.99) contain a renormalisation ambiguity in the form of the locally conserved tensor $\Theta^{(d)}_{\mu\nu}$; this renormalisation ambiguity is to be expected and corresponds to the freedom to add any locally conserved tensor to the r.h.s of Einstein's semiclassical equations (1.11). Discussion about the possible forms of $\Theta^{(d)}_{\mu\nu}$ is beyond the scope of this thesis; see [68] and the references therein for a more detailed treatment.

It turns out that the form of the locally conserved tensor $\Theta^{(d)}_{\mu\nu}$ is the only renormalisation ambiguity in odd dimensions. In even dimensions however, there exists an ambiguity in the Hadamard parametrices (6.3) and (6.5) corresponding to the renormalisation length scale ℓ_{ren} in the denominator of the logarithm of the $V^{(2)}(x,x')$ and $V^{(2p)}(x,x')$ biscalars respectively. The corollary is that the arbitrariness of ℓ_{ren} leads to the freedom to make the replacement

$$W^{(2p)}(x,x') \to W^{(2p)}(x,x') + V^{(2p)}(x,x')\ln\ell^2_{\text{ren}}. \quad (8.100)$$

We can expand the biscalar $V^{(2p)}(x,x')$ in terms of the expansion coefficients of its Hadamard coefficients in order to derive expressions for the renormalisation ambiguities in the expansion coefficients of the covariant Taylor expansion of $W^{(2p)}(x,x')$. Considering terms up to $\mathcal{O}(\sigma)$, we have on the r.h.s of (8.100)

$$\begin{aligned}
V^{(2p)}\Big(x,x'\Big) &= V_0^{(2p)} + V_1^{(2p)}\sigma + \mathcal{O}\Big(\sigma^{3/2}\Big) \\
&= V_{00}^{(2p)} + V_{01\mu}^{(2p)}\,\sigma^{\,;\mu} + V_{02(\mu\nu)}^{(2p)}\,\sigma^{\,;\mu}\sigma^{\,;\nu} + V_{10}^{(2p)}\sigma + \mathcal{O}\Big(\sigma^{3/2}\Big) \\
&= V_{00}^{(2p)} + V_{01\mu}^{(2p)}\,\sigma^{\,;\mu} + V_{02(\mu\nu)}^{(2p)}\,\sigma^{\,;\mu}\sigma^{\,;\nu} + \frac{1}{2}\,V_{10}^{(2p)}\sigma^{\,;\mu}\sigma_{\,;\mu} + \mathcal{O}\Big(\sigma^{3/2}\Big) \\
&= V_{00}^{(2p)} + V_{01\mu}^{(2p)}\,\sigma^{\,;\mu} + \left\{V_{02(\mu\nu)}^{(2p)} + \frac{1}{2}\,g_{\mu\nu}V_{10}^{(2p)}\right\}\sigma^{\,;\mu}\sigma^{\,;\nu} + \mathcal{O}\Big(\sigma^{3/2}\Big)\,. \qquad (8.101)
\end{aligned}$$

Then, using (8.101) and equating terms order by order in $\sigma(x,x')$ on both sides of (8.100), we have

$$w_0^{(2p)} \to w_0^{(2p)} + V_{00}^{(2p)}\ln\ell_{\text{ren}}^2, \qquad (8.102a)$$

$$w_{1\mu}^{(2p)} \to w_{1\mu}^{(2p)} + V_{01\mu}^{(2p)}\ln\ell_{\text{ren}}^2, \qquad (8.102b)$$

$$w_{2(\mu\nu)}^{(2p)} \to w_{2(\mu\nu)}^{(2p)} + \left[V_{02(\mu\nu)}^{(2p)} + \frac{1}{2}\,g_{\mu\nu}V_{10}^{(2p)}\right]\ln\ell_{\text{ren}}^2, \qquad (8.102c)$$

We can first examine the renormalisation ambiguity in the simplest quantity of interest, namely the scalar condensate. Using (8.102), the renormalisation ambiguity in the scalar condensate in even dimensions is given by

$$\langle\hat{\Phi}\hat{\Phi}^\dagger\rangle_{\text{ren}} \to \alpha^{(2p)}\,w_0^{(2p)} + V_{00}^{(2p)}\ln\ell_{\text{ren}}^2. \qquad (8.103)$$

Using (8.103), the renormalisation ambiguity in the scalar condensate in $d=2$ is given by

$$\langle\hat{\Phi}\hat{\Phi}^\dagger\rangle_{\text{ren}} \to \alpha^{(2)}\,w_0^{(2)} - \ln\ell_{\text{ren}}^2, \qquad (8.104)$$

such that the renormalisation ambiguity in $d=2$ is a constant. Using (8.103), the renormalisation ambiguity in the scalar condensate in $d=4$ is given by

$$\langle\hat{\Phi}\hat{\Phi}^\dagger\rangle_{\text{ren}} \to \alpha^{(4)}\,w_0^{(4)} + \frac{1}{2}\left[m^2 + \left(\xi - \frac{1}{6}\right)R\right]\ln\ell_{\text{ren}}^2, \qquad (8.105)$$

such that the renormalisation ambiguity in $d=4$ depends upon both the mass of the scalar field and its coupling constant. From (8.105), we can see that the renormalisation ambiguity in the scalar condensate in $d=4$ vanishes for a massless, conformally coupled scalar field.

Next consider the renormalised current. Using (8.102), the renormalisation ambiguity in the renormalised expectation value of the current in even dimensions is given by

$$\langle\hat{J}_\mu\rangle_{\text{ren}} \to \frac{\alpha^{(2p)}q}{4\pi}\left\{qA_\mu\left[w_0^{(2p)} + V_{00}^{(2p)}\ln\ell_{\text{ren}}^2\right] - \Im\left[w_{1\mu}^{(2p)} + V_{01\mu}^{(2p)}\ln\ell_{\text{ren}}^2\right]\right\}, \qquad (8.106)$$

which simplifies to

$$\langle\hat{J}_\mu\rangle_{\text{ren}} \to \frac{\alpha^{(2p)}q}{4\pi}\left\{qA_\mu w_0^{(2p)} - \Im\left[w_{1\mu}^{(2p)}\right] + \left(qA_\mu V_{00}^{(2p)} - \Im\left[V_{01\mu}^{(2p)}\right]\right)\ln\ell_{\text{ren}}^2\right\}. \qquad (8.107)$$

Using (8.103), the renormalisation ambiguity in the current in $d = 2$ is given by

$$
\begin{aligned}
\langle \hat{J}_\mu \rangle_{\rm ren} &\to \frac{\alpha^{(2p)} q}{4\pi} \left\{ q A_\mu w_0^{(2p)} - \Im\left[w_{1\mu}^{(2p)} \right] + (-qA_\mu + qA_\mu) \ln \ell_{\rm ren}^2 \right\} \\
&= \frac{\alpha^{(2p)} q}{4\pi} \left\{ q A_\mu w_0^{(2p)} - \Im\left[w_{1\mu}^{(2p)} \right] \right\}. \qquad (8.108)
\end{aligned}
$$

Therefore, there is no renormalisation ambiguity in the renormalised expectation value of the current in $d = 2$. Using (8.103), the renormalisation ambiguity in the renormalised current in $d = 4$ is given by

$$
\begin{aligned}
\langle \hat{J}_\mu \rangle_{\rm ren} \to \frac{\alpha^{(2p)} q}{4\pi} \Bigg\{ & q A_\mu w_0^{(2p)} - \Im\left[w_{1\mu}^{(2p)} \right] + \left(\frac{1}{2}\, q \left[m^2 + \left(\xi - \frac{1}{6} \right) R \right] A_\mu \right. \\
& \left. - \frac{1}{2}\, q \left[m^2 + \left(\xi - \frac{1}{6} \right) R \right] A_\mu + \frac{1}{12}\, q\, \nabla^\alpha F_{\alpha\mu} \right) \ln \ell_{\rm ren}^2 \Bigg\}, \qquad (8.109)
\end{aligned}
$$

which simplifies to

$$
\langle \hat{J}_\mu \rangle_{\rm ren} \to \frac{\alpha^{(2p)} q}{4\pi} \left\{ q A_\mu w_0^{(2p)} - \Im\left[w_{1\mu}^{(2p)} \right] + \frac{1}{12}\, q \left(\nabla^\alpha F_{\alpha\mu} \right) \ln \ell_{\rm ren}^2 \right\}, \qquad (8.110)
$$

such that the renormalisation ambiguity in the renormalised current is proportional to the divergence of the electromagnetic field strength tensor $F_{\mu nu}$.

Lastly, we will consider the renormalisation ambiguity in the RSET. Given the complexity of the expressions involving the RSET, it will be helpful to write the renormalisation ambiguity in the RSET in the form

$$
\langle \hat{T}_{\mu\nu} \rangle_{\rm ren} \to \langle \hat{T}_{\mu\nu} \rangle_{\rm ren} + \Psi_{\mu\nu}^{(2p)} \ln \ell_{\rm ren}^2. \qquad (8.111)
$$

where $\Psi_{\mu\nu}^{(2p)}$ is a local tensor. Using the expression for the RSET (8.98) and (8.102), we have, in terms of the expansion coefficients of the biscalar $V^{(2p)}(x, x')$, the expression

$$
\begin{aligned}
\Psi_{\mu\nu}^{(2p)} = \alpha^{(2p)} \Bigg\{ & -2\, \Re\left[V_{02(\mu\nu)}^{(2p)} + \frac{1}{2}\, g_{\mu\nu} V_{10}^{(2p)} \right] - 2\, q A_{(\mu} \Im\left[V_{01\nu)}^{(2p)} \right] - \left(\xi - \frac{1}{2} \right) V_{00\,;\mu\nu}^{(2p)} \\
& + \left(\xi R_{\mu\nu} + q^2 A_\mu A_\nu \right) V_{00}^{(2p)} + \left(\xi - \frac{1}{4} \right) g_{\mu\nu} \Box V_{00}^{(2p)} \Bigg\}, \qquad (8.112)
\end{aligned}
$$

which simplifies to give for the tensor $\Psi_{\mu\nu}^{(2p)}$ the expression in even dimensions

$$
\begin{aligned}
\Psi_{\mu\nu}^{(2p)} = \alpha^{(2p)} \Bigg\{ & -2\, \Re\left[V_{02(\mu\nu)}^{(2p)} \right] - 2\, q A_{(\mu} \Im\left[V_{01\nu)}^{(2p)} \right] - \left(\xi - \frac{1}{2} \right) V_{00\,;\mu\nu}^{(2p)} \\
& + \left(\xi R_{\mu\nu} + q^2 A_\mu A_\nu \right) V_{00}^{(2p)} + \left(\xi - \frac{1}{4} \right) g_{\mu\nu} \Box V_{00}^{(2p)} - g_{\mu\nu} V_{10}^{(2p)} \Bigg\}. \qquad (8.113)
\end{aligned}
$$

Using (8.113), the renormalisation ambiguity in the RSET in $d = 2$ is given by

$$\begin{aligned}
\Psi^{(2p)}_{\mu\nu} &= \alpha^{(2)} \left\{ -2\,\Re\left[-\frac{1}{24}\, g_{\mu\nu} R + \frac{1}{2}\, \mathrm{i} q D_{(\mu} A_{\nu)} \right] + 2\, q^2 A_\mu A_\nu - \left(\xi R_{\mu\nu} + q^2 A_\mu A_\nu \right) \right. \\
&\quad \left. + \frac{1}{2} \left[m^2 + \left(\xi - \frac{1}{6} \right) R \right] g_{\mu\nu} \right\} \\
&= \alpha^{(2)} \left\{ \frac{1}{12}\, g_{\mu\nu} R - q^2 A_\mu A_\nu + q^2 A_\mu A_\nu - \xi R_{\mu\nu} + \frac{1}{2}\, m^2 g_{\mu\nu} + \frac{1}{2}\, \xi\, g_{\mu\nu} R - \frac{1}{12}\, g_{\mu\nu} R \right\} \\
&= \frac{\alpha^{(2)}}{2}\, m^2 g_{\mu\nu},
\end{aligned} \tag{8.114}$$

Therefore the renormalisation ambiguity in the RSET in $d = 2$ depends only on the mass of the field and vanishes for a massless scalar field. Using (8.113), the renormalisation ambiguity in the RSET in $d = 4$ is given by

$$\begin{aligned}
\Psi^{(2p)}_{\mu\nu} &= \alpha^{(2p)} \left\{ -\frac{1}{12} \left[m^2 + \left(\xi - \frac{1}{6} \right) R \right] R_{\mu\nu} - \frac{1}{6} \left(\xi - \frac{3}{20} \right) R_{;\mu\nu} + \frac{1}{120}\, \Box R_{\mu\nu} \right. \\
&\quad - \frac{1}{90} R^{\alpha}{}_{\mu} R_{\alpha\nu} + \frac{1}{180} R^{\alpha\beta} R_{\alpha\mu\beta\nu} + \frac{1}{180} R^{\alpha\lambda\beta}{}_{\mu} R_{\alpha\lambda\beta\nu} + \frac{1}{2}\, q^2 \left[m^2 + \left(\xi - \frac{1}{6} \right) R \right] A_\mu A_\nu \\
&\quad + \frac{1}{6}\, q^2 A_{(\mu} \nabla^\alpha F_{\nu)\alpha} + \frac{1}{12}\, q^2 F^{\alpha}{}_{\mu} F_{\nu\alpha} - q^2 \left[m^2 + \left(\xi - \frac{1}{6} \right) R \right] A_\mu A_\nu + \frac{1}{6}\, q^2 A_{(\mu} \nabla^\alpha F_{|\alpha|\nu)} \\
&\quad - \frac{1}{2} \left(\xi - \frac{1}{2} \right) \left(\xi - \frac{1}{6} \right) R_{;\mu\nu} + \frac{1}{2} \left(\xi R_{\mu\nu} + q^2 A_\mu A_\nu \right) \left[m^2 + \left(\xi - \frac{1}{6} \right) R \right] \\
&\quad + \frac{1}{2} \left(\xi - \frac{1}{4} \right) \left(\xi - \frac{1}{6} \right) g_{\mu\nu}\, \Box R - \frac{1}{8}\, g_{\mu\nu} \left[m^2 + \left(\xi - \frac{1}{6} \right) R \right]^2 + \frac{1}{24}\, g_{\mu\nu} \left(\xi - \frac{1}{5} \right) \Box R \\
&\quad \left. + \frac{1}{720}\, g_{\mu\nu} R^{\alpha\beta} R_{\alpha\beta} - \frac{1}{720}\, g_{\mu\nu} R^{\alpha\beta\lambda\tau} R_{\alpha\beta\lambda\tau} + \frac{1}{48}\, q^2 F^{\alpha\beta} F_{\alpha\beta}\, g_{\mu\nu} \right\},
\end{aligned} \tag{8.115}$$

which simplifies to

$$\begin{aligned}
\Psi^{(2p)}_{\mu\nu} &= \alpha^{(2p)} \left\{ \frac{1}{2} \left(\xi - \frac{1}{6} \right) \left[m^2 + \left(\xi - \frac{1}{6} \right) R \right] R_{\mu\nu} - \frac{1}{2} \left(\xi^2 - \frac{1}{3}\, \xi + \frac{1}{30} \right) R_{;\mu\nu} \right. \\
&\quad + \frac{1}{120}\, \Box R_{\mu\nu} - \frac{1}{90} R^{\alpha}{}_{\mu} R_{\alpha\nu} + \frac{1}{180} R^{\alpha\beta} R_{\alpha\mu\beta\nu} + \frac{1}{180} R^{\alpha\lambda\beta}{}_{\mu} R_{\alpha\lambda\beta\nu} + \frac{1}{12}\, q^2 F^{\alpha}{}_{\mu} F_{\nu\alpha} \\
&\quad + g_{\mu\nu} \left[\frac{1}{720} R^{\alpha\beta} R_{\alpha\beta} - \frac{1}{720} R^{\alpha\beta\lambda\tau} R_{\alpha\beta\lambda\tau} + \frac{1}{2} \left(\xi^2 - \frac{1}{3}\, \xi + \frac{1}{40} \right) \Box R \right. \\
&\quad \left.\left. - \frac{1}{8} \left[m^2 + \left(\xi - \frac{1}{6} \right) R \right]^2 + \frac{1}{48}\, q^2 F^{\alpha\beta} F_{\alpha\beta} \right] \right\}.
\end{aligned} \tag{8.116}$$

Thus, the renormalisation ambiguity in the RSET in $d = 4$ does not vanish for a massless field or for a conformally coupled field. Furthermore, the renormalisation ambiguity contains a term proportional to the square of the electromagnetic field strength tensor.

8.5 Trace anomaly

Now, let us make the definition

$$\Theta^{(d)}_{\mu\nu} = \begin{cases} \Psi^{(2p)}_{\mu\nu} \ln \ell^2_{\text{ren}} & d = 2p, \\ 0 & d = 2p+1, \end{cases} \tag{8.117}$$

so that we are only considering the renormalisation ambiguity arising from the choice of ℓ_{ren}. We would now like to consider the trace of the RSET. In order to do so, it will be helpful to establish a relationship between the expansion coefficients of the biscalar $V^{(2p)}(x,x')$. From (8.101), we have to $\mathcal{O}(1)$

$$\begin{aligned} D_\alpha V^{(2p)} = {} & V^{(2p)}_{00\,;\alpha} + V^{(2p)}_{01\mu\,;\alpha}\,\sigma^{\,;\mu} + V^{(2p)}_{01\mu}\sigma^{;\mu}{}_{\alpha} + 2\,V^{(2p)}_{02(\mu\nu)}\,\sigma^{;\mu}{}_{\alpha}\,\sigma^{\,;\nu} + g_{\mu\nu}\,V^{(2p)}_{10}\,\sigma^{;\mu}{}_{\alpha}\,\sigma^{\,;\nu} \\ & - \mathrm{i}qA_\alpha V^{(2p)}_{00} - \mathrm{i}qA_\alpha V^{(2p)}_{01\mu}\sigma^{\,;\mu} + \mathcal{O}(\sigma)\,. \end{aligned} \tag{8.118}$$

Acting on (8.118) with another gauge covariant derivative, we have

$$\begin{aligned} D^\alpha D_\alpha V^{(2p)} = {} & \Box\, V^{(2p)}_{00} + 2\,V^{(2p)}_{01\mu\,;\alpha}\,g^{\alpha\beta}\sigma^{;\mu}{}_{\beta} + V^{(2p)}_{01\mu}\Box\,(\sigma^{\,;\mu}) + 2\,V^{(2p)}_{02(\mu\nu)}\,\sigma^{;\mu}{}_{\alpha}\,g^{\alpha\beta}\sigma^{;\nu}{}_{\beta} \\ & + g_{\mu\nu}\,V^{(2p)}_{10}\,\sigma^{;\mu}{}_{\alpha}\,g^{\alpha\beta}\sigma^{;\nu}{}_{\beta} - \mathrm{i}q\,(\nabla_\alpha A^\alpha)\,V^{(2p)}_{00} - \mathrm{i}qA^\alpha V^{(2p)}_{00\,;\alpha} - \mathrm{i}qA^\alpha V^{(2p)}_{01\mu}\sigma^{;\mu}{}_{\alpha} \\ & - \mathrm{i}qA^\alpha V^{(2p)}_{00\,;\alpha} - \mathrm{i}qA^\alpha V^{(2p)}_{01\mu}\sigma^{;\mu}{}_{\alpha} - q^2A^\alpha A_\alpha V^{(2p)}_{00} + \mathcal{O}\Big(\sigma^{1/2}\Big)\,. \end{aligned} \tag{8.119}$$

Using (6.25), equation (8.119) simplifies to

$$\begin{aligned} D^\alpha D_\alpha V^{(2p)} = {} & \Box\, V^{(2p)}_{00} + 2\,V^{(2p)}_{01\mu\,;\alpha}\,g^{\alpha\beta}\delta^\mu_\beta + 2\,V^{(2p)}_{02(\mu\nu)}\,\delta^\mu_\alpha\,g^{\alpha\beta}\delta^\nu_\beta + g_{\mu\nu}\,V^{(2p)}_{10}\,\delta^\mu_\alpha\,g^{\alpha\beta}\delta^\nu_\beta \\ & - \mathrm{i}q\,(\nabla_\alpha A^\alpha)\,V^{(2p)}_{00} - \mathrm{i}qA^\alpha V^{(2p)}_{00\,;\alpha} - \mathrm{i}qA^\alpha V^{(2p)}_{01\mu}\delta^\mu_\alpha - \mathrm{i}qA^\alpha V^{(2p)}_{00\,;\alpha} - \mathrm{i}qA^\alpha V^{(2p)}_{01\mu}\delta^\mu_\alpha \\ & - q^2A^\alpha A_\alpha V^{(2p)}_{00} + \mathcal{O}\Big(\sigma^{1/2}\Big)\,. \end{aligned} \tag{8.120}$$

Then, (8.120) becomes

$$\begin{aligned} D^\alpha D_\alpha V^{(2p)} = {} & \left[\Box - \mathrm{i}q\,(\nabla_\alpha A^\alpha) - 2\,\mathrm{i}qA^\alpha\nabla_\alpha - q^2A^\alpha A_\alpha\right] V^{(2p)}_{00} + 2\,[\nabla^\alpha - \mathrm{i}qA^\alpha]\,V^{(2p)}_{11\alpha} \\ & + 2\,g^{\mu\nu}V^{(2p)}_{02(\mu\nu)} + g_{\mu\nu}\,g^{\mu\nu}V^{(2p)}_{10} + \mathcal{O}\left(\sigma^{1/2}\right). \end{aligned} \tag{8.121}$$

Since (8.121) should hold at each power of σ, the terms at $\mathcal{O}(1)$ in (8.121) give us

$$0 = \left(D_\alpha D^\alpha - m^2 - \xi R\right) V^{(2p)}_{00} + 2\,D^\alpha V^{(2p)}_{01\alpha} + 2\,g^{\mu\nu}V^{(2p)}_{02\mu\nu} + 2p\,V^{(2p)}_{10}. \tag{8.122}$$

Taking the real part, we have

$$\begin{aligned} 0 = {} & \Box\, V^{(2p)}_{00} - \left(q^2A_\alpha A^\alpha + m^2 + \xi R\right) V^{(2p)}_{00} + 2\,\nabla^\alpha V^{(2p)}_{01\alpha} + 2\,qA^\alpha\,\Im\Big[V^{(2p)}_{01\alpha}\Big] + 2\,g^{\mu\nu}\Re\Big[V^{(2p)}_{02\mu\nu}\Big] \\ & + 2p\,V^{(2p)}_{10}, \end{aligned} \tag{8.123}$$

which simplifies to give the identity

$$2\, g^{\mu\nu} \Re\Big[V^{(2p)}_{02\mu\nu}\Big] = \big(q^2 A_\alpha A^\alpha + m^2 + \xi R\big)\, V^{(2p)}_{00} - 2\, q A^\alpha \Im\Big[V^{(2p)}_{01\alpha}\Big] - 2p\, V^{(2p)}_{10}. \tag{8.124}$$

Then, taking the trace of (8.113), we have

$$\begin{aligned} g^{\mu\nu}\Psi^{(2p)}_{\mu\nu} &= \alpha^{(2p)} \bigg\{ -2\, g^{\mu\nu} \Re\Big[V^{(2p)}_{02(\mu\nu)}\Big] - 2\, q A^\alpha \Im\Big[V^{(2p)}_{01\alpha}\Big] - \left(\xi - \frac{1}{2}\right) \Box V^{(2p)}_{00} \\ &+ \big(\xi R + q^2 A^\alpha A_\alpha\big)\, V^{(2p)}_{00} + 2p \left(\xi - \frac{1}{4}\right) \Box V^{(2p)}_{00} - 2p\, V^{(2p)}_{10} \bigg\} \\ &= \alpha^{(2p)} \left\{ (2p-1)\, \xi \,\Box V^{(2p)}_{00} - \frac{1}{2}\, (p-1)\, \Box V^{(2p)}_{00} - m^2\, V^{(2p)}_{00} \right\} \\ &= \alpha^{(2p)} \left\{ (2p-1) \left[\xi - \frac{(p-1)}{(2p-1)}\right] \Box V^{(2p)}_{00} - m^2\, V^{(2p)}_{00} \right\}, \end{aligned} \tag{8.125}$$

which simplifies to give

$$g^{\mu\nu}\Psi^{(2p)}_{\mu\nu} = -\alpha^{(2p)} \left\{ m^2\, V^{(2p)}_{00} - (2p-1)\, (\xi - \xi_{\rm c})\, \Box V^{(2p)}_{00} \right\}. \tag{8.126}$$

As we can see, the trace of $\Psi^{(2p)}_{\mu\nu}$ vanishes for a massless, conformally coupled field. Now considering the trace of the RSET itself, we have, using (8.87)

$$\begin{aligned} \langle \hat{T}^\mu_\mu \rangle_{\rm ren} &= \alpha^{(2p)} \bigg\{ -2\, g^{\mu\nu} \Re\Big[w^{(2p)}_{2(\mu\nu)}\Big] - 2\, q A^\alpha \Im\Big[w^{(2p)}_{1\alpha}\Big] - \left(\xi - \frac{1}{2}\right) \Box w^{(2p)}_{0} \\ &+ \big(\xi R + q^2 A^\alpha A_\alpha\big)\, w^{(2p)}_{0} + 2p \left(\xi - \frac{1}{4}\right) \Box\, w^{(2p)}_{0} - 2p\, V^{(2p)}_{10} \bigg\} + g^{\mu\nu}\Theta^{(2p)}_{\mu\nu} \\ &= \alpha^{(2p)} \left\{ (2p-1)\, \xi \,\Box\, w^{(2p)}_{0} - \frac{1}{2}\, (p-1)\, \Box\, w^{(2p)}_{0} - m^2\, w^{(2p)}_{0} + 2\, V^{(2p)}_{10} \right\} + g^{\mu\nu}\Theta^{(2p)}_{\mu\nu}, \end{aligned} \tag{8.127}$$

which simplifies to give

$$\langle \hat{T}^\mu_\mu \rangle_{\rm ren} = -\alpha^{(2p)} \left\{ m^2\, w^{(2p)}_{0} - (2p-1)\, (\xi - \xi_{\rm c})\, \Box\, w^{(2p)}_{0} - 2\, V^{(2p)}_{10} \right\} + g^{\mu\nu}\Theta^{(2p)}_{\mu\nu}, \tag{8.128}$$

in even dimensions, and

$$\langle \hat{T}^\mu_\mu \rangle_{\rm ren} = -\alpha^{(2p+1)} \left\{ m^2\, w^{(2p+1)}_{0} - 2p\, (\xi - \xi_{\rm c})\, \Box\, w^{(2p)}_{0} \right\}, \tag{8.129}$$

in odd dimensions. Restricting our attention to massless, conformally coupled fields, we have

$$\langle \hat{T}^\mu_\mu \rangle_{\rm ren} = 2\alpha^{(2p)} V^{(2p)}_{10}. \tag{8.130}$$

Using (8.130), the trace of the RSET in $d = 2$ is given by

$$\langle \hat{T}^\mu_\mu \rangle_{\rm ren} = \frac{1}{24\pi} R, \tag{8.131}$$

that is to say that there is a constant, non-vanishing trace anomaly associated with a massless, conformally-coupled field in $d = 2$. Using (8.130), the trace of the RSET in $d = 4$ is given by

$$\begin{aligned}\langle \hat{T}^{\mu}_{\mu} \rangle_{\text{ren}} &= \frac{1}{24}\left(\frac{1}{6} - \frac{1}{5}\right)\Box R + \frac{1}{720} R^{\alpha\beta} R_{\alpha\beta} - \frac{1}{720} R^{\alpha\beta\lambda\tau} R_{\alpha\beta\lambda\tau} + \frac{1}{48}\, q^2 F^{\alpha\beta} F_{\alpha\beta} \\ &= \frac{1}{720}\Box R + \frac{1}{720} R^{\alpha\beta} R_{\alpha\beta} - \frac{1}{720} R^{\alpha\beta\lambda\tau} R_{\alpha\beta\lambda\tau} + \frac{1}{48}\, q^2 F^{\alpha\beta} F_{\alpha\beta}. \end{aligned} \tag{8.132}$$

In this case, the trace anomaly associated with a massless, conformally-coupled field in $d = 4$ depends on both the background spacetime geometry as well as the square of the electromagnetic field strength tensor $F_{\mu\nu}$.

Chapter 9

Conclusions and outlook

In Part II, we considered the behaviour of a massless, minimally-coupled charged scalar in Reissner-Nordström spacetime. We were particularly interested in this study due to the possibility of disentangling the effects of rotation and superradiance in Kerr spacetime, given that an RN black hole is an irrotational spacetime that gives rise to superradiant scattering. In Chapter 4, we defined a plethora of quantum states for the field, in analogue with the states in Schwarzschild spacetime that are already well-established.

In particular the 'past' states were defined here in a similar way to their corresponding states in Schwarzschild spacetime [50]. The 'past' Boulware state is as empty as possible to an observer at infinity apart from a flux of particles in the superradiant modes. The 'past' Unruh state contained an outgoing flux of thermalised particles with a nonzero chemical potential. The 'past' CCH state has in- and up-modes with different thermal factors. We also defined their time-reversals, which are the corresponding 'future' states namely the 'future' Boulware state, the 'future' Unruh state and the 'future' CCH state.

Furthermore, we also attempted to define states as close as possible in spirit to their corresponding states in Schwarzschild spacetime, namely the 'Boulware-like' state, the 'Hartle-Hawking-like' state and the 'Frolov-Thorne' state; we referred to these states as the '-like' states. All of these states relied on nonstandard commutation relations and thus the 'Boulware-like' state cannot be considered a vacuum states in the conventional sense. Nevertheless, it was as empty as possible to a static observer at infinity and was time-reversal invariant. The 'Frolov-Thorne' state is a thermal state, but it does not represent a thermal equilibrium and it was ill-defined everywhere in the spacetime. While we have managed to define an equilibrium state in the 'Hartle-Hawking-like' state, it is likely to evade a generalised version of the Kay-Wald theorem, as was explained in more detail above.

All of our investigations in this thesis have relied on considering the differences between two quantum states when the state-independent divergent terms in the Hadamard parametrix cancel, leaving finite quantities. Then the most obvious extension of our work would be to develop a concrete realisation of the extension of Hadamard renormalisation that we constructed in Part III. Such a realisation may result in generalising the extended coordinates method for neutral scalar fields developed in [93, 94]. Furthermore, in [68], explicit renormalisation counterterms were provided for Hadamard renormalisation of a neutral scalar field in two to six dimensions, whereas only two to four dimensions were considered in [95] with a partial extension to five dimensions in [95]. Another such extension of the work in Part III would be to develop Hadamard renormalisation in a systematic way, such as that given in [68], but for a fermionic field.

This would be an interesting extension since it has been shown in [50] that superradiance does not restrict the ability to define states for a fermionic field in spacetimes exhibiting superradiance, as much as it does for bosonic fields. Another interesting extension of the work in Part II then, is to study quantum charged fermionic fields in Reissner-Nordström spacetime. Finally, it would be interesting to rigorously extend the Kay-Wald theorem to a charged scalar field in order to see if our putative 'Hartle-Hawking-like' state would indeed evade this theorem.

Part IV

Appendices

Appendices

Appendix A

Identities concerning the spherical harmonics

The standard addition theorem for spherical harmonics is given by [90]

$$P_\ell(\cos\gamma) = \frac{4\pi}{2\ell+1} \sum_{m=-\ell}^{\ell} Y_{\ell m}(\theta,\varphi)\, Y^*_{\ell m}(\theta,\varphi)\,, \tag{A.1}$$

where the function $\cos\gamma$ in (A.1) is given by

$$\cos\gamma = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos\bigl(\varphi-\varphi'\bigr)\,. \tag{A.2}$$

The Legendre polynomials P_ℓ have the boundary condition

$$P_\ell(1) = 1. \tag{A.3}$$

Then, taking the coincidence limit $\theta'\to\theta$, $\varphi'\to\varphi$ in (A.1) as well as using the boundary condition (A.3), we obtain the well-known addition formula

$$\sum_{m=-\ell}^{\ell} |Y_{\ell m}(\theta,\varphi)|^2 = \frac{2\ell+1}{4\pi}. \tag{A.4}$$

Returning to the addition theorem of the spherical harmonics (A.1), we can differentiate both sides of the equation with respect to θ to obtain

$$\frac{4\pi}{2\ell+1} \sum_{m=-\ell}^{\ell} \frac{\partial Y_{\ell m}(\theta,\varphi)}{\partial\theta}\, Y^*_{\ell m}\bigl(\theta',\varphi'\bigr) = \frac{\partial\,(\cos\gamma)}{\partial\theta}\, P'_\ell(\cos\gamma)\,, \tag{A.5}$$

where the derivative of the function $\cos\gamma$ with respect to θ in (A.5) is given by

$$\frac{\partial(\cos\gamma)}{\partial\theta} = -\sin\theta\cos\theta' + \cos\theta\sin\theta'\cos\bigl(\varphi-\varphi'\bigr)\,. \tag{A.6}$$

Then, taking the coincidence limit $\theta'\to\theta$, $\varphi'\to\varphi$ in (A.5), we obtain the identity

$$\sum_{m=-\ell}^{\ell} \frac{\partial Y_{\ell m}(\theta,\varphi)}{\partial\theta}\, Y^*_{\ell m}(\theta,\varphi) = 0. \tag{A.7}$$

Returning to the expression in (A.5), we now differentiate (A.5) with respect to θ' to obtain

$$\frac{4\pi}{2\ell+1}\sum_{m=-\ell}^{\ell}\frac{\partial Y_{\ell m}(\theta,\varphi)}{\partial\theta}\frac{\partial Y^*_{\ell m}(\theta',\varphi')}{\partial\theta'} = \frac{\partial^2(\cos\gamma)}{\partial\theta'\partial\theta}P'_\ell(\cos\gamma)+\frac{\partial(\cos\gamma)}{\partial\theta}\frac{\partial(\cos\gamma)}{\partial\theta'}P''_\ell(\theta,\varphi)\,, \quad \text{(A.8)}$$

where the derivatives of the function $\cos\gamma$ in (A.8) are given by

$$\frac{\partial(\cos\gamma)}{\partial\theta'} = -\sin\theta'\cos\theta+\cos\theta'\sin\theta\cos(\varphi-\varphi')\,, \quad \text{(A.9a)}$$

$$\frac{\partial^2(\cos\gamma)}{\partial\theta'\partial\theta} = \sin\theta\sin\theta'+\cos\theta\cos\theta'\cos(\varphi-\varphi')\,. \quad \text{(A.9b)}$$

The derivative of the Legendre polynomials P'_ℓ has the boundary condition

$$P'_\ell(1) = \frac{\ell(\ell+1)}{2}. \quad \text{(A.10)}$$

Then, taking the coincidence limit $\theta'\to\theta$, $\varphi'\to\varphi$ in (A.8) as well as using the boundary condition (A.10), we obtain the identity

$$\sum_{m=-\ell}^{\ell}\left|\frac{\partial Y_{\ell m}(\theta,\varphi)}{\partial\theta}\right|^2 = \frac{2\ell+1}{4\pi}P'_\ell(1) = \frac{1}{8\pi}\ell(\ell+1)(2\ell+1)\,. \quad \text{(A.11)}$$

Returning to the addition theorem of the spherical harmonics (A.1), we can differentiate (A.1) first with respect to φ and then with respect to φ' to obtain

$$\frac{4\pi}{2\ell+1}\sum_{m=-\ell}^{\ell}\frac{\partial Y_{\ell m}(\theta,\varphi)}{\partial\varphi}\frac{\partial Y^*_{\ell m}(\theta',\varphi')}{\partial\varphi'} = \frac{\partial^2(\cos\gamma)}{\partial\varphi'\partial\varphi}P'_\ell(\cos\gamma)+\frac{\partial(\cos\gamma)}{\partial\varphi}\frac{\partial(\cos\gamma)}{\partial\varphi'}P''_\ell(\cos\gamma)\,, \quad \text{(A.12)}$$

where the derivatives of the function $\cos\gamma$ in (A.11) are given by

$$\frac{\partial(\cos\gamma)}{\partial\varphi} = -\sin\theta\sin\theta'\sin(\varphi-\varphi') = -\frac{\partial(\cos\gamma)}{\partial\varphi'}, \quad \text{(A.13a)}$$

$$\frac{\partial^2(\cos\gamma)}{\partial\varphi'\partial\varphi} = \sin\theta\sin\theta'\cos(\varphi-\varphi')\,. \quad \text{(A.13b)}$$

Then, taking the coincidence limit $\theta'\to\theta$, $\varphi'\to\varphi$ in (A.11) as well as using the explicit expression for the spherical harmonics $Y_{\ell m}(\theta,\varphi)$ (3.50) and the boundary condition (A.10), we obtain the identity

$$\sum_{m=-\ell}^{\ell}\left|\frac{\partial Y_{\ell m}(\theta,\varphi)}{\partial\varphi}\right|^2 = \sum_{m=-\ell}^{\ell}m^2\,|Y_{\ell m}(\theta,\varphi)|^2 = \frac{1}{8\pi}\ell(\ell+1)(2\ell+1)\sin^2\theta. \quad \text{(A.14)}$$

Appendix B

Mode contributions to the current and stress-energy tensor

In this section we will calculate the classical mode contributions to the components of quantum observables, several of which contain products of the square magnitude of field operators. In general, these classical mode contributions will contain products of modes with different quantum numbers. However, for reasons similar to those outlined when evaluating the classical mode contribution $sc_{\omega\ell m}$ to the scalar condensate $\mathcal{SC}$ in §5.1.1, we need only calculate the square magnitudes of the individual modes in this case.

B.1 Mode contributions to components of the current

In this section, we derive the explicit expressions for the classical mode contributions $j^{\mu}_{\omega\ell m}$ to each component of the current J^{μ} using the general form of a scalar field mode (3.41) as well as the expression for the current (5.5). The mode contribution $j^{t}_{\omega\ell m}$ is given by

$$\begin{aligned}
j^{t}_{\omega\ell m} &= -\frac{q}{4\pi}\, g^{tt}\, \Im\left[\, \phi^{*}_{\omega\ell m}\left(\nabla_t - \mathrm{i}qA_t\right)\phi_{\omega\ell m}\,\right] \\
&= \frac{q}{4\pi}\,\frac{1}{f(r)}\, \Im\left[\phi^{*}_{\omega\ell m}\left(\partial_t - \mathrm{i}qA_t\right)\times \frac{\mathrm{e}^{-\mathrm{i}\omega t}}{r}\,\mathrm{N}_{\omega}\, X_{\omega\ell}(r)\, Y_{\ell m}(\theta,\varphi)\right] \\
&= \frac{q}{4\pi f(r)}\, \Im\left[\phi^{*}_{\omega\ell m}\left(-\mathrm{i}\omega + \frac{\mathrm{i}qQ}{r}\right)\phi_{\omega\ell m}\right] \\
&= -\frac{q}{4\pi r^2 f(r)}\left(\omega - \frac{qQ}{r}\right)|\mathrm{N}_{\omega}|^2\, |X_{\omega\ell}(r)\,|^2\, |Y_{\ell m}(\theta,\varphi)|^2. \qquad \text{(B.1)}
\end{aligned}$$

The mode contribution $j^{r}_{\omega\ell m}$ is given by

$$\begin{aligned}
j^r_{\omega\ell m} &= -\frac{q}{4\pi}\, g^{rr}\, \Im\left[\, \phi^*_{\omega\ell m}\left(\nabla_r - \mathrm{i}qA_r\right)\phi_{\omega\ell m}\,\right] \\
&= -\frac{q}{4\pi}\, f(r)\, \Im\left[\phi^*_{\omega\ell m}\,\partial_r\left(\frac{\mathrm{e}^{-\mathrm{i}\omega t}}{r}\,\mathrm{N}_\omega\, X_{\omega\ell}(r)\, Y_{\ell m}(\theta,\varphi)\right)\right] \\
&= -\frac{qf(r)}{4\pi}\,\Im\left[\frac{\mathrm{e}^{\mathrm{i}\omega t}}{r}\,\mathrm{N}^*_\omega\, X^*_{\omega\ell}(r)\, Y^*_{\ell m}(\theta,\varphi)\times \mathrm{e}^{-\mathrm{i}\omega t}\,\mathrm{N}_\omega\left[\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{X_{\omega\ell}(r)}{r}\right)\right]Y_{\ell m}(\theta,\varphi)\right] \\
&= -\frac{qf(r)}{4\pi}\,|\mathrm{N}_\omega|^2\,\Im\left[\frac{X^*_{\omega\ell}(r)}{r}\,\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{X_{\omega\ell}(r)}{r}\right)\right]|Y_{\ell m}(\theta,\varphi)|^2. \qquad \text{(B.2)}
\end{aligned}$$

The mode contribution $j^\theta_{\omega\ell m}$ is given by

$$\begin{aligned}
j^\theta_{\omega\ell m} &= -\frac{q}{4\pi}\, g^{\theta\theta}\, \Im\left[\, \phi^*_{\omega\ell m}\left(\nabla_\theta - \mathrm{i}qA_\theta\right)\phi_{\omega\ell m}\,\right] \\
&= -\frac{q}{4\pi}\, r^{-2}\, \Im\left[\phi^*_{\omega\ell m}\,\partial_\theta\left(\frac{\mathrm{e}^{-\mathrm{i}\omega t}}{r}\,\mathrm{N}_\omega\, X_{\omega\ell}(r)\, Y_{\ell m}(\theta,\varphi)\right)\right] \\
&= -\frac{q}{4\pi}\, r^{-2}\, \Im\left[\frac{\mathrm{e}^{\mathrm{i}\omega t}}{r}\,\mathrm{N}^*_\omega\, X^*_{\omega\ell}(r)\, Y^*_{\ell m}(\theta,\varphi)\times\frac{\mathrm{e}^{-\mathrm{i}\omega t}}{r}\,\mathrm{N}_\omega\, X_{\omega\ell}(r)\,\partial_\theta\, Y_{\ell m}(\theta,\varphi)\right] \\
&= -\frac{q}{4\pi r^4}\,|\mathrm{N}_\omega|^2\,|X_{\omega\ell}(r)\,|^2\,\Im\left[Y^*_{\ell m}(\theta,\varphi)\,\frac{\partial}{\partial\theta}\,Y_{\ell m}(\theta,\varphi)\right] = 0, \qquad \text{(B.3)}
\end{aligned}$$

where we have used the identity (A.7) for the spherical harmonics to show that the expression in the last line of (B.3) vanishes when the summation over m is taken. The mode contribution $j^\varphi_{\omega\ell m}$ is given by

$$\begin{aligned}
j^\varphi_{\omega\ell m} &= -\frac{q}{4\pi}\, g^{\varphi\varphi}\, \Im\left[\, \phi^*_{\omega\ell m}\left(\nabla_\varphi - \mathrm{i}qA_\varphi\right)\phi_{\omega\ell m}\,\right] \\
&= -\frac{q}{4\pi}\, r^{-2}\,\mathrm{cosec}^2\theta\, \Im\left[\, \phi^*_{\omega\ell m}\partial_\varphi\,\phi_{\omega\ell m}\right] \\
&= -\frac{mq}{4\pi r^2\sin^2\theta}\,|\phi_{\omega\ell m}|^2. \qquad \text{(B.4)}
\end{aligned}$$

While the mode contribution $j^\varphi_{\omega\ell m}$ (B.4) does not vanish, the expectation values with respect to all quantum states involve sums over the azimuthal number $m = -\ell, \ldots, \ell$. However, the expression in (B.4) contains a factor of $|\phi_{\omega\ell m}|^2$. From the properties of the spherical harmonics (3.50), we have $|\phi_{\omega\ell m}|^2 = \left|\phi_{\omega\ell(-m)}\right|^2$. Therefore, the expression in (B.4) is an odd function of m, which vanishes when the summation with respect to m is performed over $-\ell \geq m \leq \ell$.

Then, using the addition formula (A.4) for the spherical harmonics $Y_{\ell m}(\theta,\varphi)$, we can perform the sum over the azimuthal number m in the nonzero components of the current J^μ. The mode contribution $j^t_{\omega\ell}$ to the component J^t is given by

$$j^t_{\omega\ell} = \sum_{m=-\ell}^{\ell} j^t_{\omega\ell m} = -\frac{q\,(2\ell+1)}{16\pi^2 r^2 f(r)}\left(\omega - \frac{qQ}{r}\right)|\mathrm{N}_\omega|^2\,|X_{\omega\ell}(r)\,|^2. \qquad \text{(B.5)}$$

The mode contribution $j^r_{\omega\ell}$ to the component J^r is given by

$$j^r_{\omega\ell} = \sum_{m=-\ell}^{\ell} j^r_{\omega\ell m} = -\frac{qf(r)(2\ell+1)}{16\pi^2}\,|\mathrm{N}_\omega|^2\,\Im\left[\frac{X^*_{\omega\ell}(r)}{r}\,\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{X_{\omega\ell}(r)}{r}\right)\right]. \qquad \text{(B.6)}$$

B.2 Mode contributions to components of the SET

In this section, we derive the explicit expressions for the classical mode contributions $t_{\mu\nu,\omega\ell m}$ to each component of the stress-energy tensor $T_{\mu\nu}$ using the general form of a scalar field mode (3.41) as well as the expression for the SET (5.14).

In order to do so, it will be convenient to first calculate the quantity $g^{\rho\sigma}(D_\rho\Phi)^* D_\sigma\Phi$ that will appear in many of the mode contributions to the stress-energy tensor components, and which is given by

$$\begin{aligned}
&g^{\rho\sigma}(D_\rho\Phi)^* D_\sigma\Phi \\
&= g^{tt}(D_t\Phi)^* D_t\Phi + g^{rr}(D_r\Phi)^* D_r\Phi + g^{\theta\theta}(D_\theta\Phi)^* D_\theta\Phi + g^{\varphi\varphi}(D_\varphi\Phi)^* D_\varphi\Phi \\
&= -\frac{1}{f(r)}\left[(\partial_t - \mathrm{i}qA_t)\,\phi_{\omega\ell m}\right]^* \times (\partial_t - \mathrm{i}qA_t)\,\phi_{\omega\ell m} + f(r)\,(\partial_r\phi_{\omega\ell m})^*\partial_r\phi_{\omega\ell m} \\
&+ \frac{1}{r^2}\,(\partial_\theta\phi_{\omega\ell m})^*\partial_\theta\phi_{\omega\ell m} + \frac{1}{r^2\sin^2\theta}\,(\partial_\varphi\phi_{\omega\ell m})^*\partial_\varphi\phi_{\omega\ell m} \\
&= -\frac{1}{f(r)}\left[\left(-\mathrm{i}\omega + \frac{\mathrm{i}qQ}{r}\right)\phi_{\omega\ell m}\right]^* \times \left(-\mathrm{i}\omega + \frac{\mathrm{i}qQ}{r}\right)\phi_{\omega\ell m} \\
&+ f(r)\,|\mathrm{N}_\omega|^2\left[\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{X_{\omega\ell}(r)}{r}\right)\right]^*\left[\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{X_{\omega\ell}(r)}{r}\right)\right]|Y_{\ell m}(\theta,\varphi)|^2 \\
&+ \frac{1}{r^2}\frac{1}{r^2}\,|\mathrm{N}_\omega|^2\,|X_{\omega\ell}(r)|^2\left[\frac{\partial}{\partial\theta}\,Y_{\ell m}(\theta,\varphi)\right]^*\left[\frac{\partial}{\partial\theta}\,Y_{\ell m}(\theta,\varphi)\right] + \frac{1}{r^2\sin^2\theta}\,(\mathrm{i}m\,\phi_{\omega\ell m})^*\,\mathrm{i}m\,\phi_{\omega\ell m} \\
&= -\frac{1}{f(r)}\left(\omega - \frac{qQ}{r}\right)^2|\phi_{\omega\ell m}|^2 + f(r)\,|\mathrm{N}_\omega|^2\left|\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{X_{\omega\ell}(r)}{r}\right)\right|^2|Y_{\ell m}(\theta,\varphi)|^2 \\
&+ \frac{1}{r^4}\,|\mathrm{N}_\omega|^2\,|X_{\omega\ell}(r)|^2\left|\frac{\partial}{\partial\theta}\,Y_{\ell m}(\theta,\varphi)\right|^2 + \frac{m^2}{r^2\sin^2\theta}\,|\phi_{\omega\ell m}|^2 \\
&= \left[-\frac{1}{f(r)}\left(\omega - \frac{qQ}{r}\right)^2 + \frac{m^2}{r^2\sin^2\theta}\right]|\phi_{\omega\ell m}|^2 + f(r)\,|\mathrm{N}_\omega|^2\left|\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{X_{\omega\ell}(r)}{r}\right)\right|^2|Y_{\ell m}(\theta,\varphi)|^2 \\
&+ \frac{1}{r^4}\,|\mathrm{N}_\omega|^2\,|X_{\omega\ell}(r)|^2\left|\frac{\partial}{\partial\theta}\,Y_{\ell m}(\theta,\varphi)\right|^2 .
\end{aligned}\tag{B.7}$$

Then, using (B.7), the mode contribution $t_{tt,\omega\ell m}$ is given by

$$\begin{aligned}
t_{tt,\omega\ell m} &= \Re\left\{(D_t\Phi)^* D_t\Phi - \frac{1}{2}\,g_{tt}\,g^{\rho\sigma}(D_\rho\Phi)^* D_\sigma\Phi\right\} \\
&= \Re\left\{\left[(\partial_t - \mathrm{i}qA_t)\,\phi_{\omega\ell m}\right]^* \times (\partial_t - \mathrm{i}qA_t)\,\phi_{\omega\ell m} - \frac{1}{2}\,g_{tt}\,g^{\rho\sigma}(D_\rho\Phi)^* D_\sigma\Phi\right\} \\
&= \Re\left\{\left[\left(-\mathrm{i}\omega + \frac{\mathrm{i}qQ}{r}\right)\phi_{\omega\ell m}\right]^* \times \left(-\mathrm{i}\omega + \frac{\mathrm{i}qQ}{r}\right)\phi_{\omega\ell m} + \frac{1}{2}\,f(r)\,g^{\rho\sigma}(D_\rho\Phi)^* D_\sigma\Phi\right\} \\
&= \Re\left\{\left(\omega - \frac{qQ}{r}\right)^2|\phi_{\omega\ell m}|^2 + \frac{1}{2}\,f(r)\,g^{\rho\sigma}(D_\rho\Phi)^* D_\sigma\Phi\right\} \\
&= \frac{1}{2}\left[\left(\omega - \frac{qQ}{r}\right)^2 + \frac{m^2 f(r)}{r^2\sin^2\theta}\right]|\phi_{\omega\ell m}|^2 + \frac{f(r)^2}{2}\,|\mathrm{N}_\omega|^2\left|\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{X_{\omega\ell}(r)}{r}\right)\right|^2|Y_{\ell m}(\theta,\varphi)|^2 \\
&+ \frac{f(r)}{2r^4}\,|\mathrm{N}_\omega|^2\,|X_{\omega\ell}(r)|^2\left|\frac{\partial}{\partial\theta}\,Y_{\ell m}(\theta,\varphi)\right|^2 .
\end{aligned}\tag{B.8}$$

The mode contribution $t_{tr,\omega\ell m}$ is given by

$$\begin{aligned}
t_{tr,\omega\ell m} &= \Re\left\{ (D_t\Phi)^* D_r\Phi - \frac{1}{2}\, g_{tr}\, g^{\rho\sigma} (D_\rho\Phi)^* D_\sigma\Phi \right\} \\
&= \Re\left\{ [\,(\partial_t - \mathrm{i}qA_t)\,\phi_{\omega\ell m}\,]^* \partial_r\phi_{\omega\ell m} \right\} \\
&= \Re\left\{ \left[\left(-\mathrm{i}\omega + \frac{\mathrm{i}qQ}{r} \right) \phi_{\omega\ell m} \right]^* \times \mathrm{e}^{-\mathrm{i}\omega t}\, \mathrm{N}_\omega \left[\frac{\mathrm{d}}{\mathrm{d}r} \left(\frac{X_{\omega\ell}(r)}{r} \right) \right] Y_{\ell m}(\theta,\varphi) \right\} \\
&= \Re\left\{ \mathrm{i} \left(\omega - \frac{qQ}{r} \right) |\mathrm{N}_\omega|^2 \left[\frac{X^*_{\omega\ell}(r)}{r} \frac{\mathrm{d}}{\mathrm{d}r} \left(\frac{X_{\omega\ell}(r)}{r} \right) \right] |Y_{\ell m}(\theta,\varphi)|^2 \right\} \\
&= \left(\omega - \frac{qQ}{r} \right) |\mathrm{N}_\omega|^2 |Y_{\ell m}(\theta,\varphi)|^2\, \Re\left\{ \mathrm{i} \left[\frac{X^*_{\omega\ell}(r)}{r} \frac{\mathrm{d}}{\mathrm{d}r} \left(\frac{X_{\omega\ell}(r)}{r} \right) \right] \right\} \\
&= -\left(\omega - \frac{qQ}{r} \right) |\mathrm{N}_\omega|^2\, |Y_{\ell m}(\theta,\varphi)|^2\, \Im\left[\frac{X^*_{\omega\ell}(r)}{r} \frac{\mathrm{d}}{\mathrm{d}r} \left(\frac{X_{\omega\ell}(r)}{r} \right) \right].
\end{aligned} \tag{B.9}$$

The mode contribution $t_{t\theta,\omega\ell m}$ is given by

$$\begin{aligned}
t_{t\theta,\omega\ell m} &= \Re\left\{ (D_t\Phi)^* D_\theta\Phi - \frac{1}{2}\, g_{t\theta}\, g^{\rho\sigma} (D_\rho\Phi)^* D_\sigma\Phi \right\} \\
&= \Re\left\{ [\,(\partial_t - \mathrm{i}qA_t)\,\phi_{\omega\ell m}\,]^* \partial_\theta\phi_{\omega\ell m} \right\} \\
&= \Re\left\{ \left[\left(-\mathrm{i}\omega + \frac{\mathrm{i}qQ}{r} \right) \phi_{\omega\ell m} \right]^* \times \frac{\mathrm{e}^{-\mathrm{i}\omega t}}{r}\, \mathrm{N}_\omega\, X_{\omega\ell}(r)\, \frac{\partial}{\partial\theta}\, Y_{\ell m}(\theta,\varphi) \right\} \\
&= \Re\left\{ \frac{\mathrm{i}}{r^2} \left(\omega - \frac{qQ}{r} \right) |\mathrm{N}_\omega|^2\, |X_{\omega\ell}(r)|^2 \left[Y^*_{\ell m}(\theta,\varphi)\, \frac{\partial}{\partial\theta}\, Y_{\ell m}(\theta,\varphi) \right] \right\} \\
&= \frac{1}{r^2} \left(\omega - \frac{qQ}{r} \right) |\mathrm{N}_\omega|^2\, |X_{\omega\ell}(r)|^2\, \Re\left\{ \mathrm{i} \left[Y^*_{\ell m}(\theta,\varphi)\, \frac{\partial}{\partial\theta}\, Y_{\ell m}(\theta,\varphi) \right] \right\} \\
&= -\frac{1}{r^2} \left(\omega - \frac{qQ}{r} \right) |\mathrm{N}_\omega|^2\, |X_{\omega\ell}(r)|^2\, \Im\left[Y^*_{\ell m}(\theta,\varphi)\, \frac{\partial}{\partial\theta}\, Y_{\ell m}(\theta,\varphi) \right] \\
&= 0,
\end{aligned} \tag{B.10}$$

where we have used the identity (A.7) for the spherical harmonics to show that the expression in the last line of (B.10) vanishes when the summation over m is taken. The mode contribution $t_{t\varphi,\omega\ell m}$ is given by

$$\begin{aligned}
t_{t\varphi,\omega\ell m} &= \Re\left\{ (D_t\Phi)^* D_\varphi\Phi - \frac{1}{2}\, g_{t\varphi}\, g^{\rho\sigma} (D_\rho\Phi)^* D_\sigma\Phi \right\} \\
&= \Re\left\{ [\,(\partial_t - \mathrm{i}qA_t)\,\phi_{\omega\ell m}\,]^* \partial_\varphi\phi_{\omega\ell m} \right\} \\
&= \Re\left\{ \left[\left(-\mathrm{i}\omega + \frac{\mathrm{i}qQ}{r} \right) \phi_{\omega\ell m} \right]^* (\mathrm{i}m)\, \phi_{\omega\ell m} \right\} \\
&= -m \left(\omega - \frac{qQ}{r} \right) |\phi_{\omega\ell m}|^2.
\end{aligned} \tag{B.11}$$

Using (B.7), the mode contribution $t_{rr,\omega\ell m}$ is given by

$$
\begin{aligned}
&t_{rr,\omega\ell m}\\
&= \Re\biggl\{ (D_r\Phi)^* D_r\Phi - \frac{1}{2}\, g_{rr}\, g^{\rho\sigma} (D_\rho\Phi)^* D_\sigma\Phi \biggr\}\\
&= \Re\biggl\{ (\partial_r\phi_{\omega\ell m})^* \partial_r\phi_{\omega\ell m} - \frac{1}{2}\, g_{rr}\, g^{\rho\sigma} (D_\rho\Phi)^* D_\sigma\Phi \biggr\}\\
&= \Re\biggl\{ \left(\mathrm{e}^{-\mathrm{i}\omega t}\, \mathrm{N}_\omega \left[\frac{\mathrm{d}}{\mathrm{d}r} \left(\frac{X_{\omega\ell}(r)}{r} \right) \right] Y_{\ell m}(\theta,\varphi) \right)^* \times \mathrm{e}^{-\mathrm{i}\omega t}\, \mathrm{N}_\omega \left[\frac{\mathrm{d}}{\mathrm{d}r} \left(\frac{X_{\omega\ell}(r)}{r} \right) \right] Y_{\ell m}(\theta,\varphi)\\
&- \frac{1}{2} \frac{1}{f(r)}\, g^{\rho\sigma} (D_\rho\Phi)^* D_\sigma\Phi \biggr\}\\
&= \Re\biggl\{ |\mathrm{N}_\omega|^2 \left| \frac{\mathrm{d}}{\mathrm{d}r} \left(\frac{X_{\omega\ell}(r)}{r} \right) \right|^2 |Y_{\ell m}(\theta,\varphi)|^2 + \frac{1}{2} \frac{1}{f(r)} \left[\frac{1}{f(r)} \left(\omega - \frac{qQ}{r} \right)^2 - \frac{m^2}{r^2 \sin^2\theta} \right] |\phi_{\omega\ell m}|^2\\
&- \frac{1}{2}\, |\mathrm{N}_\omega|^2 \left| \frac{\mathrm{d}}{\mathrm{d}r} \left(\frac{X_{\omega\ell}(r)}{r} \right) \right|^2 |Y_{\ell m}(\theta,\varphi)|^2 - \frac{1}{2r^4 f(r)} |\mathrm{N}_\omega|^2\, |X_{\omega\ell}(r)|^2 \left| \frac{\partial}{\partial\theta}\, Y_{\ell m}(\theta,\varphi) \right|^2 \biggr\}\\
&= \frac{1}{2}\, |\mathrm{N}_\omega|^2 \left| \frac{\mathrm{d}}{\mathrm{d}r} \left(\frac{X_{\omega\ell}(r)}{r} \right) \right|^2 |Y_{\ell m}(\theta,\varphi)|^2 + \frac{1}{2f(r)^2} \left[\left(\omega - \frac{qQ}{r} \right)^2 - \frac{m^2 f(r)}{r^2 \sin^2\theta} \right] |\phi_{\omega\ell m}|^2\\
&- \frac{1}{2r^4 f(r)}\, |\mathrm{N}_\omega|^2\, |X_{\omega\ell}(r)|^2 \left| \frac{\partial}{\partial\theta}\, Y_{\ell m}(\theta,\varphi) \right|^2 .
\end{aligned}
\tag{B.12}
$$

The mode contribution $t_{r\theta,\omega\ell m}$ is given by

$$
\begin{aligned}
t_{r\theta,\omega\ell m} &= \Re\biggl\{ (D_r\Phi)^* D_\theta\Phi - \frac{1}{2}\, g_{r\theta}\, g^{\rho\sigma} (D_\rho\Phi)^* D_\sigma\Phi \biggr\}\\
&= \Re\biggl\{ (\partial_r\phi_{\omega\ell m})^* \partial_\theta\phi_{\omega\ell m} \biggr\}\\
&= \Re\biggl\{ \left(\mathrm{e}^{-\mathrm{i}\omega t}\, \mathrm{N}_\omega \left[\frac{\mathrm{d}}{\mathrm{d}r} \left(\frac{X_{\omega\ell}(r)}{r} \right) \right] Y_{\ell m}(\theta,\varphi) \right)^* \times \frac{\mathrm{e}^{-\mathrm{i}\omega t}}{r}\, \mathrm{N}_\omega\, X_{\omega\ell}(r)\, \frac{\partial}{\partial\theta}\, Y_{\ell m}(\theta,\varphi) \biggr\}\\
&= |\mathrm{N}_\omega|^2\, \Re\biggl\{ \frac{X_{\omega\ell}(r)}{r} \frac{\mathrm{d}}{\mathrm{d}r} \left(\frac{X^*_{\omega\ell}(r)}{r} \right) \times Y^*_{\ell m}(\theta,\varphi)\, \frac{\partial}{\partial\theta}\, Y_{\ell m}(\theta,\varphi) \biggr\} .
\end{aligned}
\tag{B.13}
$$

The mode contribution $t_{r\varphi,\omega\ell m}$ is given by

$$
\begin{aligned}
t_{r\varphi,\omega\ell m} &= \Re\biggl\{ (D_r\Phi)^* D_\varphi\Phi - \frac{1}{2}\, g_{r\varphi}\, g^{\rho\sigma} (D_\rho\Phi)^* D_\sigma\Phi \biggr\}\\
&= \Re\biggl\{ (\partial_r\phi_{\omega\ell m})^* \partial_\varphi\phi_{\omega\ell m} \biggr\}\\
&= \Re\biggl\{ \left[\mathrm{e}^{-\mathrm{i}\omega t}\, \mathrm{N}_\omega\, \frac{\mathrm{d}}{\mathrm{d}r} \left(\frac{X_{\omega\ell}(r)}{r} \right) Y_{\ell m}(\theta,\varphi) \right]^* \times (\mathrm{i}m)\, \frac{\mathrm{e}^{-\mathrm{i}\omega t}}{r}\, \mathrm{N}_\omega\, X_{\omega\ell}(r)\; Y_{\ell m}(\theta,\varphi) \biggr\}\\
&= \Re\biggl\{ \mathrm{i}m\, |\mathrm{N}_\omega|^2 \left[\frac{X_{\omega\ell}(r)}{r} \frac{\mathrm{d}}{\mathrm{d}r} \left(\frac{X^*_{\omega\ell}(r)}{r} \right) \right] |Y_{\ell m}(\theta,\varphi)|^2 \biggr\}\\
&= \frac{m}{r}\, |\mathrm{N}_\omega|^2\, \Re\biggl\{ \mathrm{i}X_{\omega\ell}(r)\, \frac{\mathrm{d}}{\mathrm{d}r} \left(\frac{X^*_{\omega\ell}(r)}{r} \right) \biggr\} |Y_{\ell m}(\theta,\varphi)|^2\\
&= -\frac{m}{r}\, |\mathrm{N}_\omega|^2\, \Im\biggl[X_{\omega\ell}(r)\, \frac{\mathrm{d}}{\mathrm{d}r} \left(\frac{X^*_{\omega\ell}(r)}{r} \right) \biggr] |Y_{\ell m}(\theta,\varphi)|^2 .
\end{aligned}
\tag{B.14}
$$

Using (B.7), the mode contribution $t_{\theta\theta,\omega\ell m}$ is given by

$$
\begin{aligned}
&t_{\theta\theta,\omega\ell m}\\
&= \Re\Big\{ (D_\theta\Phi)^* D_\theta\Phi - \frac{1}{2}\, g_{\theta\theta}\, g^{\rho\sigma} (D_\rho\Phi)^* D_\sigma\Phi \Big\}\\
&= \Re\Big\{ (\partial_\theta\phi_{\omega\ell m})^*\partial_\theta\phi_{\omega\ell m} - \frac{1}{2}\, g_{\theta\theta}\, g^{\rho\sigma} (D_\rho\Phi)^* D_\sigma\Phi \Big\}\\
&= \Re\Big\{ \Big(\frac{\mathrm{e}^{-\mathrm{i}\omega t}}{r}\,\mathrm{N}_\omega\, X_{\omega\ell}(r)\,\frac{\partial}{\partial\theta}\,Y_{\ell m}(\theta,\varphi)\Big)^* \times \frac{\mathrm{e}^{-\mathrm{i}\omega t}}{r}\,\mathrm{N}_\omega\, X_{\omega\ell}(r)\,\frac{\partial}{\partial\theta}\,Y_{\ell m}(\theta,\varphi)\\
&- \frac{1}{2}\, r^2 g^{\rho\sigma} (D_\rho\Phi)^* D_\sigma\Phi \Big\}\\
&= \Re\left\{ \frac{1}{r^2}\,|\mathrm{N}_\omega|^2\,|X_{\omega\ell}(r)|^2\left|\frac{\partial}{\partial\theta}\,Y_{\ell m}(\theta,\varphi)\right|^2 + \frac{1}{2}\left[\frac{r^2}{f(r)}\Big(\omega-\frac{qQ}{r}\Big)^2 - \frac{m^2}{\sin^2\theta}\right]|\phi_{\omega\ell m}|^2\right.\\
&\left. - \frac{f(r)r^2}{2}\,|\mathrm{N}_\omega|^2\left|\frac{\mathrm{d}}{\mathrm{d}r}\Big(\frac{X_{\omega\ell}(r)}{r}\Big)\right|^2 |Y_{\ell m}(\theta,\varphi)|^2 - \frac{1}{2r^2}\,|\mathrm{N}_\omega|^2\,|X_{\omega\ell}(r)|^2\left|\frac{\partial}{\partial\theta}\,Y_{\ell m}(\theta,\varphi)\right|^2\right\}\\
&= \frac{1}{2r^2}\,|\mathrm{N}_\omega|^2\,|X_{\omega\ell}(r)|^2\left|\frac{\partial}{\partial\theta}\,Y_{\ell m}(\theta,\varphi)\right|^2 + \frac{1}{2}\left[\frac{r^2}{f(r)}\Big(\omega-\frac{qQ}{r}\Big)^2 - \frac{m^2}{\sin^2\theta}\right]|\phi_{\omega\ell m}|^2\\
&- \frac{f(r)r^2}{2}\,|\mathrm{N}_\omega|^2\left|\frac{\mathrm{d}}{\mathrm{d}r}\Big(\frac{X_{\omega\ell}(r)}{r}\Big)\right|^2 |Y_{\ell m}(\theta,\varphi)|^2.
\end{aligned}
\tag{B.15}
$$

The mode contribution $t_{\theta\varphi,\omega\ell m}$ is given by

$$
t_{\theta\varphi,\omega\ell m} = \Re\Big\{ (D_\theta\Phi)^* D_\varphi\Phi - \frac{1}{2}\, g_{\theta\varphi}\, g^{\rho\sigma} (D_\rho\Phi)^* D_\sigma\Phi \Big\} \tag{B.16}
$$

$$
\begin{aligned}
&= \Re\Big\{ (\partial_\theta\phi_{\omega\ell m})^*\partial_\varphi\phi_{\omega\ell m} \Big\}\\
&= \Re\Big\{ \Big(\frac{\mathrm{e}^{-\mathrm{i}\omega t}}{r}\,\mathrm{N}_\omega\, X_{\omega\ell}(r)\,\frac{\partial}{\partial\theta}\,Y_{\ell m}(\theta,\varphi)\Big)^* \times (\mathrm{i}m)\,\frac{\mathrm{e}^{-\mathrm{i}\omega t}}{r}\,\mathrm{N}_\omega\, X_{\omega\ell}(r)\,\frac{\partial}{\partial\theta}\,Y_{\ell m}(\theta,\varphi)\Big\}\\
&= \Re\Big\{ \frac{\mathrm{i}m}{r^2}\,|\mathrm{N}_\omega|^2\,|X_{\omega\ell}(r)|^2\,Y_{\ell m}(\theta,\varphi)\,\frac{\partial}{\partial\theta}\,Y^*_{\ell m}(\theta,\varphi)\Big\}\\
&= \frac{m}{r^2}\,|\mathrm{N}_\omega|^2\,|X_{\omega\ell}(r)|^2\,\Re\Big\{ \mathrm{i}\,Y_{\ell m}(\theta,\varphi)\,\frac{\partial}{\partial\theta}\,Y^*_{\ell m}(\theta,\varphi)\Big\}\\
&= -\frac{m}{r^2}\,|\mathrm{N}_\omega|^2\,|X_{\omega\ell}(r)|^2\,\Im\Big[Y_{\ell m}(\theta,\varphi)\,\frac{\partial}{\partial\theta}\,Y^*_{\ell m}(\theta,\varphi)\Big]\\
&= 0,
\end{aligned}
\tag{B.17}
$$

where we have used the identity (A.7) for the spherical harmonics to show that the expression in the last line of (B.16) vanishes when the summation over m is taken. Using (B.7), the mode contribution $t_{\varphi\varphi,\omega\ell m}$ is given by

$$
\begin{aligned}
&t_{\varphi\varphi,\omega\ell m}\\
&= \Re\left\{ (D_\varphi \Phi)^* D_\varphi \Phi - \frac{1}{2} g_{\varphi\varphi}\, g^{\rho\sigma} (D_\rho \Phi)^* D_\sigma \Phi \right\}\\
&= \Re\left\{ (\partial_\varphi \phi_{\omega\ell m})^* \partial_\varphi \phi_{\omega\ell m} - \frac{1}{2} g_{\varphi\varphi}\, g^{\rho\sigma} (D_\rho \Phi)^* D_\sigma \Phi \right\}\\
&= \Re\left\{ (\mathrm{i}m\, \phi_{\omega\ell m})^* \,(\mathrm{i}m\, \phi_{\omega\ell m}) - \frac{1}{2} r^2 \sin^2\!\theta\, g^{\rho\sigma} (D_\rho \Phi)^* D_\sigma \Phi \right\}\\
&= \Re\Bigg\{ m^2 \left|\phi_{\omega\ell m}\right|^2 - \frac{r^2 \sin^2\!\theta}{2} \left[-\frac{1}{f(r)} \left(\omega - \frac{qQ}{r} \right)^2 + \frac{m^2}{r^2 \sin^2\!\theta} \right] \left|\phi_{\omega\ell m}\right|^2\\
&- \frac{f(r) r^2 \sin^2\!\theta}{2} \left|\mathrm{N}_\omega\right|^2 \left| \frac{\mathrm{d}}{\mathrm{d}r} \left(\frac{X_{\omega\ell}(r)}{r} \right) \right|^2 \left|Y_{\ell m}(\theta,\varphi)\right|^2\\
&- \frac{\sin^2\!\theta}{2r^2} \left|\mathrm{N}_\omega\right|^2 \left|X_{\omega\ell}(r)\right|^2 \left| \frac{\partial}{\partial\theta}\, Y_{\ell m}(\theta,\varphi) \right|^2 \Bigg\}\\
&= \frac{1}{2} \left[m^2 + \frac{r^2 \sin^2\!\theta}{f(r)} \left(\omega - \frac{qQ}{r} \right)^2 \right] \left|\phi_{\omega\ell m}\right|^2\\
&- \frac{f(r) r^2 \sin^2\!\theta}{2} \left|\mathrm{N}_\omega\right|^2 \left| \frac{\mathrm{d}}{\mathrm{d}r} \left(\frac{X_{\omega\ell}(r)}{r} \right) \right|^2 \left|Y_{\ell m}(\theta,\varphi)\right|^2 - \frac{\sin^2\!\theta}{2r^2} \left|\mathrm{N}_\omega\right|^2 \left|X_{\omega\ell}(r)\right|^2 \left| \frac{\partial}{\partial\theta}\, Y_{\ell m}(\theta,\varphi) \right|^2 .
\end{aligned}
\tag{B.18}
$$

While the mode contributions $t_{t\varphi,\omega\ell m}$ (B.11) and $t_{r\varphi,\omega\ell m}$ (B.14) do not vanish, the expectation values with respect to all quantum states involve sums over the azimuthal number $m = -\ell, \ldots, \ell$. The expressions in (B.11) and (B.14) are odd functions of m, which vanish when the summation with respect to m is performed over $-\ell \geq m \leq \ell$. Furthermore, the mode contribution $t_{r\theta,\omega\ell m}$ (B.13) also vanishes when the summation with respect to m is performed from the spherical harmonic identity in (A.7).

Then, using the identities involving the spherical harmonics $Y_{\ell m}(\theta, \varphi)$ in §A, we can perform the sum over the azimuthal number m in the nonzero components of the stress-energy tensor $T_{\mu\nu}$.

Performing the sum over m in the expression for the mode contribution $t_{tt,\omega\ell m}$, we obtain

$$
\begin{aligned}
t_{tt,\omega\ell} &= \frac{1}{2r^2} \left|\mathrm{N}_\omega\right|^2 \left[\frac{(2\ell+1)}{4\pi} \left(\omega - \frac{qQ}{r} \right)^2 + \frac{f(r)\, \ell(\ell+1)(2\ell+1) \sin^2\!\theta}{8\pi r^2 \sin^2\!\theta} \right] \left|X_{\omega\ell}(r)\right|^2\\
&+ \frac{f(r)^2 (2\ell+1)}{8\pi} \left|\mathrm{N}_\omega\right|^2 \left| \frac{\mathrm{d}}{\mathrm{d}r} \left(\frac{X_{\omega\ell}(r)}{r} \right) \right|^2 + \frac{f(r)\, \ell(\ell+1)(2\ell+1)}{16\pi r^4} \left|\mathrm{N}_\omega\right|^2 \left|X_{\omega\ell}(r)\right|^2\\
&= \frac{2\ell+1}{8\pi} \left|\mathrm{N}_\omega\right|^2 \left[\frac{1}{r^2} \left(\omega - \frac{qQ}{r} \right)^2 + \frac{f(r)\, \ell(\ell+1)}{2r^4} + \frac{f(r)\, \ell(\ell+1)}{2r^4} \right] \left|X_{\omega\ell}(r)\right|^2\\
&+ \frac{2\ell+1}{8\pi} \left|\mathrm{N}_\omega\right|^2 f(r) \left| \frac{\mathrm{d}}{\mathrm{d}r} \left(\frac{X_{\omega\ell}(r)}{r} \right) \right|^2 ,
\end{aligned}
\tag{B.19}
$$

where we have used the spherical harmonic identities (A.4), (A.11) and (A.14). Then, the mode contribution $t_{tt,\omega\ell}$ to the component T_{tt} is given by

$$t_{tt,\omega\ell} = \frac{2\ell+1}{8\pi}\,|\mathrm{N}_\omega|^2 \left\{ \left[\frac{1}{r^2}\left(\omega - \frac{qQ}{r}\right)^2 + \frac{f(r)\,\ell(\ell+1)}{r^4} \right] |X_{\omega\ell}(r)|^2 + f(r)^2 \left| \frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{X_{\omega\ell}(r)}{r}\right) \right|^2 \right\}. \tag{B.20}$$

Using the spherical harmonic identity in (A.4), the mode contribution $t_{tr,\omega\ell}$ to the component T_{tr} is given by

$$t_{tr,\omega\ell} = -\frac{2\ell+1}{4\pi}\left(\omega - \frac{qQ}{r}\right)|\mathrm{N}_\omega|^2\,\Im\left[\frac{X^*_{\omega\ell}(r)}{r}\,\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{X_{\omega\ell}(r)}{r}\right)\right]. \tag{B.21}$$

Performing the sum over m in the expression for the mode contribution $t_{rr,\omega\ell m}$, we obtain

$$\begin{aligned} t_{rr,\omega\ell} &= \frac{2\ell+1}{8\pi}\,|\mathrm{N}_\omega|^2 \left| \frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{X_{\omega\ell}(r)}{r}\right) \right|^2 \\ &\quad + \frac{1}{2f(r)^2 r^2}\,|\mathrm{N}_\omega|^2 \left[\frac{2\ell+1}{4\pi}\left(\omega - \frac{qQ}{r}\right)^2 - \frac{f(r)\,\ell(\ell+1)\,(2\ell+1)\sin^2\theta}{8\pi r^2 \sin^2\theta} \right] |X_{\omega\ell}(r)|^2 \\ &\quad - \frac{\ell(\ell+1)(2\ell+1)}{16\pi f(r) r^4}\,|\mathrm{N}_\omega|^2\,|X_{\omega\ell}(r)|^2 \\ &= \frac{2\ell+1}{8\pi}\,|\mathrm{N}_\omega|^2 \left| \frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{X_{\omega\ell}(r)}{r}\right) \right|^2 \\ &\quad + \frac{2\ell+1}{8\pi}\,|\mathrm{N}_\omega|^2 \left[\frac{1}{f(r)^2 r^2}\left(\omega - \frac{qQ}{r}\right)^2 - \frac{\ell(\ell+1)}{2f(r)\,r^4} - \frac{\ell(\ell+1)}{2f(r)\,r^4} \right] |X_{\omega\ell}(r)|^2, \end{aligned} \tag{B.22}$$

where we have used the spherical harmonic identities (A.4), (A.11) and (A.14). Then, the mode contribution $t_{rr,\omega\ell m}$ to the component T_{rr} is given by

$$t_{rr,\omega\ell} = \frac{2\ell+1}{8\pi}\,|\mathrm{N}_\omega|^2 \left\{ \left[\frac{1}{f(r)^2\,r^2}\left(\omega - \frac{qQ}{r}\right)^2 - \frac{\ell(\ell+1)}{f(r)\,r^4} \right] |X_{\omega\ell}(r)|^2 + \left| \frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{X_{\omega\ell}(r)}{r}\right) \right|^2 \right\}. \tag{B.23}$$

Performing the sum over m in the expression for the mode contribution $t_{\theta\theta,\omega\ell m}$, we obtain

$$\begin{aligned} t_{\theta\theta,\omega\ell} &= \frac{\ell(\ell+1)(2\ell+1)}{16\pi r^2}\,|\mathrm{N}_\omega|^2\,|X_{\omega\ell}(r)|^2 \\ &\quad + \frac{1}{2r^2}\,|\mathrm{N}_\omega|^2 \left[\frac{(2\ell+1)\,r^2}{4\pi f(r)}\left(\omega - \frac{qQ}{r}\right)^2 - \frac{\ell(\ell+1)\,(2\ell+1)\sin^2\theta}{8\pi\sin^2\theta} \right] |X_{\omega\ell}(r)|^2 \\ &\quad - \frac{f(r)(2\ell+1)r^2}{8\pi}\,|\mathrm{N}_\omega|^2 \left| \frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{X_{\omega\ell}(r)}{r}\right) \right|^2 \\ &= \frac{2\ell+1}{8\pi}\,|\mathrm{N}_\omega|^2 \left[\frac{\ell(\ell+1)}{2r^2} + \frac{1}{f(r)}\left(\omega - \frac{qQ}{r}\right)^2 - \frac{\ell(\ell+1)}{2r^2} \right] |X_{\omega\ell}(r)|^2 \\ &\quad - \frac{2\ell+1}{8\pi}\,|\mathrm{N}_\omega|^2\,f(r) r^2 \left| \frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{X_{\omega\ell}(r)}{r}\right) \right|^2, \end{aligned} \tag{B.24}$$

where we have used the spherical harmonic identities (A.4), (A.11) and (A.14). Then, the mode contribution $t_{\theta\theta,\omega\ell m}$ to the component $T_{\theta\theta}$ is given by

$$t_{\theta\theta,\omega\ell} = \frac{2\ell+1}{8\pi}\,|\mathrm{N}_\omega|^2 \left\{ \frac{1}{f(r)}\left(\omega - \frac{qQ}{r}\right)^2 |X_{\omega\ell}(r)|^2 - f(r)r^2 \left| \frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{X_{\omega\ell}(r)}{r}\right)\right|^2 \right\}. \tag{B.25}$$

Performing the sum over m in the expression for the mode contribution $t_{\varphi\varphi,\omega\ell m}$, we obtain

$$\begin{aligned} t_{\varphi\varphi,\omega\ell} &= \frac{1}{2r^2}\,|\mathrm{N}_\omega|^2 \left[\frac{\ell(\ell+1)(2\ell+1)\sin^2\theta}{8\pi} + \frac{(2\ell+1)\,r^2\sin^2\theta}{4\pi f(r)}\left(\omega - \frac{qQ}{r}\right)^2 \right] |X_{\omega\ell}(r)|^2 \\ &\quad - \frac{(2\ell+1)\,f(r)r^2\sin^2\theta}{8\pi}\,|\mathrm{N}_\omega|^2 \left| \frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{X_{\omega\ell}(r)}{r}\right)\right|^2 \\ &\quad - \frac{\ell(\ell+1)(2\ell+1)\sin^2\theta}{16\pi r^2}\,|\mathrm{N}_\omega|^2\,|X_{\omega\ell}(r)|^2 \\ &= \frac{(2\ell+1)\sin^2\theta}{8\pi}\,|\mathrm{N}_\omega|^2 \left[\frac{\ell(\ell+1)}{2r^2} - \frac{\ell(\ell+1)}{2r^2} + \frac{1}{f(r)}\left(\omega - \frac{qQ}{r}\right)^2 \right] |X_{\omega\ell}(r)|^2 \\ &\quad - \frac{(2\ell+1)\sin^2\theta}{8\pi}\,|\mathrm{N}_\omega|^2\, f(r)r^2 \left| \frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{X_{\omega\ell}(r)}{r}\right)\right|^2 \end{aligned} \tag{B.26}$$

where we have used the spherical harmonic identities (A.4), (A.11) and (A.14). Then, the mode contribution $t_{\varphi\varphi,\omega\ell m}$ to the component $T_{\varphi\varphi}$ is given by

$$\begin{aligned} t_{\varphi\varphi,\omega\ell} &= \frac{(2\ell+1)\sin^2\theta}{8\pi}\,|\mathrm{N}_\omega|^2 \left\{ \frac{1}{f(r)}\left(\omega - \frac{qQ}{r}\right)^2 |X_{\omega\ell}(r)|^2 + f(r)r^2 \left| \frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{X_{\omega\ell}(r)}{r}\right)\right|^2 \right\} \\ &= t_{\theta\theta,\omega\ell}\,\sin^2\theta. \end{aligned} \tag{B.27}$$

Appendix C

Non-renormalisation of flux components

We would like to show that the expectation values of the flux components of the current $\langle \hat{J}^r \rangle$ and the SET $\langle \hat{T}_{tr} \rangle$ do not require renormalisation. In order to do so, we follow the procedure used in [49] to prove the corresponding results for a neutral scalar field in Kerr spacetime. We work in $d = 4$.

Let $G_{\mathrm{F}}(x, x')$ be the Feynman Green's function associated to a charged scalar field in an arbitrary quantum state. The renormalised expressions for the current and SET in this state are given in (8.51) and (8.61) respectively, where $\mathcal{T}_{\mu\nu}$ takes the form given in (8.62).

Restricting our attention to the specific case of a massless, minimally-coupled charged scalar fieldin a background Reissner-Nordström metric, the expansion coefficients of the biscalars $U^{(4)}(x,x')$ and $V^{(4)}(x,x')$ take the form

$$U^{(4)}_{00} = 1, \tag{C.1a}$$
$$U^{(4)}_{01\mu} = \mathrm{i}qA_\mu, \tag{C.1b}$$
$$U^{(4)}_{02(\mu\nu)} = \frac{1}{12}R_{\mu\nu} - \frac{\mathrm{i}q}{2}\nabla_{(\mu}A_{\nu)} - \frac{q^2}{2}A_\mu A_\nu, \tag{C.1c}$$
$$U^{(4)}_{03(\mu\nu\lambda)} = -\frac{1}{24}R_{(\mu\nu;\lambda)} + \frac{\mathrm{i}q}{6}\nabla_{(\mu}\nabla_\nu A_{\lambda)} + \frac{q^2}{2}A_{(\mu}\nabla_\nu A_{\lambda)} - \frac{\mathrm{i}q^3}{6}A_\mu A_\nu A_\lambda + \frac{\mathrm{i}q}{12}R_{(\mu\nu}A_{\lambda)}, \tag{C.1d}$$
$$\begin{aligned} U^{(4)}_{04\mu\nu\lambda\tau} = {} & \frac{1}{80}R_{(\mu\nu;\lambda\tau)} + \frac{1}{288}R_{(\mu\nu}R_{\lambda\tau)} + \frac{1}{360}R^\rho{}_{(\mu|\psi|\nu}R^\psi{}_{\lambda|\rho|\tau)} - \frac{\mathrm{i}q}{24}\nabla_{(\mu}\nabla_\nu\nabla_\lambda A_{\tau)} \\ & - \frac{q^2}{6}A_{(\mu}\nabla_\nu\nabla_\lambda A_{\tau)} - \frac{q^2}{8}\left[\nabla_{(\mu}A_\nu\right]\left[\nabla_\lambda A_{\tau)}\right] + \frac{\mathrm{i}q^3}{4}A_{(\mu}A_\nu\nabla_\lambda A_{\tau)} + \frac{q^4}{24}A_\mu A_\nu A_\lambda A_\tau \\ & - \frac{\mathrm{i}q}{24}A_{(\mu}\nabla_\nu R_{\lambda\tau)} - \frac{\mathrm{i}q}{24}R_{(\mu\nu}\nabla_\lambda A_{\tau)} - \frac{q^2}{24}R_{(\mu\nu}A_\lambda A_{\tau)}, \end{aligned} \tag{C.1e}$$
$$V^{(4)}_{00} = 0, \tag{C.1f}$$
$$V^{(4)}_{01\mu} = -\frac{\mathrm{i}q}{12}\nabla^\alpha F_{\alpha\mu}, \tag{C.1g}$$
$$\begin{aligned} V^{(4)}_{02(\mu\nu)} = {} & -\frac{1}{240}\Box R_{\mu\nu} + \frac{1}{180}R^\alpha{}_\mu R_{\alpha\nu} - \frac{1}{360}R^{\alpha\beta}R_{\alpha\mu\beta\nu} - \frac{1}{360}R^{\alpha\beta\gamma}{}_\mu R_{\alpha\beta\gamma\nu} - \frac{q^2}{24}F^\alpha{}_\mu F_{\nu\alpha} \\ & - \frac{q^2}{12}A_{(\mu}\nabla^\alpha F_{\nu)\alpha} - \frac{iq}{24}\nabla_{(\mu}\nabla^\alpha F_{\nu)\alpha}, \end{aligned} \tag{C.1h}$$
$$V^{(4)}_{10} = \frac{1}{720}R^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta} - \frac{1}{720}R^{\alpha\beta}R_{\alpha\beta} - \frac{q^2}{48}F^{\alpha\beta}F_{\alpha\beta}. \tag{C.1i}$$

To show that $\langle \hat{J}^r\rangle$ and $\langle \hat{T}_{tr}\rangle$ do not require renormalization, we seek to prove that

$$\mathfrak{F}_1 \equiv \Im\big\{D^r\left[-\mathrm{i}\,G_{\mathrm{S}}(x,x')\right]\big\} = 0, \tag{C.2a}$$
$$\mathfrak{F}_2 \equiv \Re\big\{\mathcal{T}_{tr}\left[-\mathrm{i}\,G_{\mathrm{S}}(x,x')\right]\big\} = 0. \tag{C.2b}$$

Since the Reissner-Nordström metric (3.9) is static and spherically symmetric, without loss of generality we may consider two space-time points x and x' as follows:

$$x = (0, r, \theta, 0), \qquad x' = (0, r', \theta', 0). \tag{C.3}$$

Then the unique geodesic connecting the points x and x' lies in the surface $\Sigma = \{t = 0, \varphi = 0\}$. Using the letter $\mathcal{X}$ to denote the indices t, φ, and $\mathcal{A}$ to denote r, θ, we have [49]

$$\sigma^{;\mu} = \delta^\mu_{\mathcal{A}}\sigma^{;\mathcal{A}}, \tag{C.4a}$$
$$g_\nu{}^{\nu'} = \delta^{\nu'}_{\mathcal{A}'}\delta^{\mathcal{A}}_\nu g_{\mathcal{A}}{}^{\mathcal{A}'} + \delta^{\nu'}_{\mathcal{X}'}\delta^{\mathcal{X}}_\nu g_{\mathcal{X}}{}^{\mathcal{X}'}. \tag{C.4b}$$

We can write the gauge potential (3.11) as

$$A_\mu = \delta^{\mathcal{X}}_\mu A_{\mathcal{X}}, \tag{C.4c}$$

where $A_{\mathcal{X}}$ depends only on $\mathcal{A}$ coordinates. Therefore the quantities (C.2) take the form

$$\mathfrak{F}_1 = \Im\left\{\nabla^r\left[-\mathrm{i}\,G_{\mathrm{S}}(x,x')\right]\right\}, \tag{C.5a}$$
$$\mathfrak{F}_2 = \Re\left\{-i\left[g_r{}^{\mathcal{A}'}D_t\nabla_{\mathcal{A}'}\right]G_{\mathrm{S}}(x,x')\right\}. \tag{C.5b}$$

The biscalar $\sigma(x,x')$ and its derivatives are real, as are the gauge field potential A_μ and field strength $F_{\mu\nu}$, as well as all curvature tensors and their derivatives. From (C.4), we have $A_\mu \sigma^{;\mu} = 0$, which immediately simplifies the form of $G_{\rm S}(x,x')$.

The symmetries of the metric mean that Christoffel symbols $\Gamma^{\mu}_{\nu\lambda}$ having an odd number of $\mathcal{X}$ indices vanish, while those with an even number of $\mathcal{X}$ indices are nonzero. Therefore the nonzero components of all covariant derivatives of the gauge potential A_μ contain at least one $\mathcal{X}$ index and hence all terms in (C.2) containing covariant derivatives of A_μ do not contribute to $U^{(4)}(x,x')$ or $V^{(4)}(x,x')$ when contracted with $\sigma^{;\mu}$. As a result, $U^{(4)}(x,x')$ is real and depends only on curvature tensors; the gauge potential does not contribute.

The gauge field strength has the form

$$F_{\mu\nu} = \left[\delta^{\mathcal{A}}_{\mu}\delta^{\mathcal{X}}_{\nu} - \delta^{\mathcal{X}}_{\mu}\delta^{\mathcal{A}}_{\nu}\right] F_{\mathcal{A}\mathcal{X}}, \tag{C.6}$$

where $F_{\mathcal{A}\mathcal{X}}$ depends only on the $\mathcal{A}$ coordinates. Hence we have

$$\nabla^{\alpha} F_{\alpha\mu} = \delta^{\mathcal{X}}_{\mu} \nabla^{\mathcal{A}} F_{\mathcal{A}\mathcal{X}}. \tag{C.7}$$

Therefore $V^{(4)}(x,x')$ is also real. We deduce that $-\mathrm{i}\, G_{\rm S}(x,x')$ is real and hence $\mathfrak{F}_1$ (C.5a) is trivally zero, while $\mathfrak{F}_2$ (C.5b) simplifies to

$$\mathfrak{F}_2 = {g_r}^{\mathcal{A}'} \nabla_t \nabla_{\mathcal{A}'} \left[-\mathrm{i}\, G_{\rm S}\big(x,x'\big)\right]. \tag{C.8}$$

The derivatives in the above expression commute since they are evaluated at different space-time points and $G_{\rm S}(x,x')$ is a biscalar. Furthermore, $G_{\rm S}(x,x')$ depends only on the space-time geometry and the background electromagnetic field. Therefore $G_{\rm S}(x,x')$ does not depend on t and thus $\nabla_t(-iG_S)$ must be zero. We then have $\mathfrak{F}_2 = 0$, as required.

In conclusion, the components $\langle \hat{J}^r \rangle$ and $\langle \hat{T}_{tr} \rangle$ do not require renormalization.

Appendix D

Vanishing of mode fluxes

Throughout this thesis we have referred to a set of modes vanishing near a certain hypersurface. For example, in §3.4.3, we have referred to the in-modes vanishing near $\mathcal{H}^-$. In fact, what we mean specifically by this statement is that the flux of the in-modes through the past horizon $\mathcal{H}^-$ vanishes. We can demonstrate this by the following calculation. Using the expression for the volume element $\mathrm{d}\Sigma^\mu_{\mathcal{H}^-}$ in (3.110), we have for the flux of the in-modes through the past horizon

$$
\begin{aligned}
&\mathrm{d}\Sigma^\mu D_\mu \phi^{\mathrm{in}}_{\omega\ell m}|_{\mathcal{H}^-} \\
&\quad= \mathrm{d}\Sigma^\mu_{\mathcal{H}^-} \nabla_\mu \phi^{\mathrm{in}}_{\omega\ell m} - \mathrm{i}q\,\mathrm{d}\Sigma^\mu_{\mathcal{H}^-} A_\mu\,\phi^{\mathrm{in}}_{\omega\ell m} \\
&\quad= \mathrm{d}\Sigma^\mu_{\mathcal{H}^-} \partial_\mu \phi^{\mathrm{in}}_{\omega\ell m} - \mathrm{i}q\,\mathrm{d}\Sigma^\mu_{\mathcal{H}^-} A_\mu\,\phi^{\mathrm{in}}_{\omega\ell m} \\
&\quad= -\delta^\mu_U r^2 \sin\theta\,\mathrm{d}U \mathrm{d}\theta\,\mathrm{d}\varphi\,\partial_\mu \left\{ \frac{1}{\sqrt{4\pi\omega}} \frac{\mathrm{e}^{-\mathrm{i}\omega t}}{r} B^{\mathrm{in}}_{\omega\ell}\,\mathrm{e}^{-\mathrm{i}\widetilde{\omega} r_*} Y_{\ell m}(\theta,\varphi) \right\} \\
&\quad+ \mathrm{i}q\,\delta^\mu_U r^2 \sin\theta\,\mathrm{d}U \mathrm{d}\theta\,\mathrm{d}\varphi A_\mu\,\phi^{\mathrm{in}}_{\omega\ell m} \\
&\quad= -r^2 \sin\theta\,\mathrm{d}U \mathrm{d}\theta\,\mathrm{d}\varphi\,\frac{\mathrm{d}u}{\mathrm{d}U}\frac{\partial}{\partial u}\left\{ \frac{1}{\sqrt{4\pi\omega}}\frac{1}{r}\exp\left[-\mathrm{i}\omega\frac{(u+v)}{2}\right] B^{\mathrm{in}}_{\omega\ell}\exp\left[-\mathrm{i}\widetilde{\omega}\frac{(v-u)}{2}\right] Y_{\ell m}(\theta,\varphi)\right\} \\
&\quad+ \mathrm{i}q\,r^2 \sin\theta\,\mathrm{d}U \mathrm{d}\theta\,\mathrm{d}\varphi A_U \phi^{\mathrm{in}}_{\omega\ell m} \\
&\quad= -r^2 \sin\theta\,\mathrm{d}U \mathrm{d}\theta\,\mathrm{d}\varphi\,\frac{\mathrm{d}u}{\mathrm{d}U}\,\frac{\mathrm{i}(\widetilde{\omega}-\omega)}{2}\phi^{\mathrm{in}}_{\omega\ell m} \\
&\quad= \frac{\mathrm{i}qQ}{2r_+}\,r^2 \sin\theta\,\mathrm{d}U \mathrm{d}\theta\,\mathrm{d}\varphi\,\frac{\mathrm{d}u}{\mathrm{d}U}\,\phi^{\mathrm{in}}_{\omega\ell m} - \frac{\mathrm{i}qQ}{2r_+}\,r^2 \sin\theta\,\mathrm{d}U \mathrm{d}\theta\,\mathrm{d}\varphi\,\frac{\mathrm{d}u}{\mathrm{d}U}\,\phi^{\mathrm{in}}_{\omega\ell m} \\
&\quad= 0,
\end{aligned} \tag{D.1}
$$

which vanishes identically. Wherever in this thesis we have referred to a set of modes vanishing near a certain hypersurface, it is meant that the flux of the modes under consideration through the hypersurface vanishes through a similar calculation to (D.1).

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