A Cohomological Bundle Theory for Sheaf Homology

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Abstract. The construction of the Khovanov homology of links motivates an interest in decorated Boolean lattices. Placing this work in the context of a bundle theory of presheaves on small categories, we produce, for a certain set of naturally occurring cases, a Leray-Serre type spectral sequence relating the bundle to the cohomology of the total sheaf. This gives a reduction property for the cohomology of sheaves on certain posets.

In the 19th century, Lord Kelvin came to the idea that atoms are knots of swirling vortices in the æther. A 'table of elements' then, was a table of distinct knots – the Scottish physicist and Kelvin's collaborator Peter Guthrie Tait prepared meticulous lists of unique knots, believing he was describing something fundamental to the material world. And while the vortex theory became obsolete, the mathematics of knots has found modern applications in biology and chemistry, from protein folding to determining the chirality of molecules. Among the various applications of knot theory to physics is the topological quantum computer [Kit03], which uses braids for its logical gates.

The most relevant point on the chain of inspiration for this work is Mikhail Khovanov's breakthrough discovery of a link invariant [Kho00] that generalises the Jones polynomial. As with other fruitful advancements, 'Khovanov homology' intersects two relatively disjoint fields – knot theory, with its chiefly combinatorial approach, and homological algebra. This connection was realised early, but it is the work of Brent Everitt and Paul Turner ([ET09, ET12, ET15]) that makes it explicit in our context. Indeed, Everitt and Turner realise Khovanov's construction as a presheaf of modules on a poset.

The main aim of this thesis is to generalise and expand the results of [ET12]. That paper devises a homology theory, called 'coloured poset homology', and shows how a Leray-Serre style spectral sequence converges to the coloured poset homology of the total sheaf of a bundle. Here, we move to the broader context of sheaves on small categories and employ the usual definition of *co*homology for such objects, namely the values of the derived limits. Our main theorem, proved in Section 7.5, is

Main Theorem. *Let* $\xi : \mathbf{B} \to \mathbf{Sh}$ *be a poset bundle of sheaves with* $\mathbf{B}$ *a recursively admissible finite poset, and* $(\mathbf{E}_\xi; F_\xi)$ *the associated total sheaf. Then there is a spectral sequence that converges to the cohomology of the total sheaf:*

$$E_2^{p,q} = H^p(\mathbf{B}; \mathcal{H}^q_{fib}(\xi)) \Rightarrow H^\bullet(\mathbf{E}_\xi; F_\xi).$$

We proceed as follows. Chapter 1 traces the main beats in the development of knot theory, culminating in the detailed definitions of the Jones polynomial and of Khovanov homology. After that, in Chapter 2, we lay out the categorical notation and the homological apparatus that the following arguments are based on. Particularly useful in certain cases is the formulation of adjointness in terms of universal arrows.

Sheaves on small categories are introduced in Chapter 3 and the construction of a simplicial complex from the nerve of a category $\mathbf{C}$ is given in Chapter 4. Section 3.2 sets up the argument in Section 4.2 that the higher limits of a sheaf F on a small category $\mathbf{C}$ are isomorphic to the simplicial cohomology of that sheaf.

Chapter 5 brings in the theory of spectral sequences, with constructions for filtrations of complexes and for bicomplexes. The bundle of sheaves, defined in Chapter 6, naturally defines a bicomplex $\mathcal{K}^{\bullet,\bullet}$ and thus gives rise to a spectral sequence. Section 6.3 considers this construction for a constant sheaf and finds an explicit quasi-isomorphism, employed in the proof of the main theorem.

For the main theoretical result in Chapter 7 we impose the general assumption that the base $\mathbf{B}$ of our bundle $\xi : \mathbf{B} \to \mathbf{Sh}$ is a poset category and that for each

$x \in \mathbf{B}$, the small category of $\xi(x)$ is also a poset category; we call such a bundle a *poset bundle of sheaves*. We further explain the technical requirement of 'recursive admissibility' on the base poset category of our bundles and then give an explicit chain map $\omega^\bullet$ between the simplicial complex of the total sheaf and the total complex of the bicomplex $\mathcal{K}^{\bullet,\bullet}$. Two long exact sequences in terms of the above two objects are discussed in Sections 7.3 and 7.4. The main theorem stitches the two long exact sequences via the chain map $\omega^\bullet$ and completes the result by induction on the size of the base poset category. Section 7.6 gives an example of a bundle over a non-poset base and shows that the result of the main theorem remains true, suggesting it might apply more broadly than what we prove here.

The final chapter explores some consequences of Theorem 7.14. In the context of sheaf cohomology, the spectral sequence for a poset bundle of sheaves converges to the cohomology of the fiber at the maximum of the base. Thus, while the main theorem of [ET12] is able to model Khovanov homology, out key application is as follows.

Main Application. *Let* $\mathbf{E}$ *and* $\mathbf{B}$ *be posets, with* $\mathbf{B}$ *recursively admissible. Suppose that* $\pi : \mathbf{E} \to \mathbf{B}$ *is an onto poset map such that for all* $x < y$ *in* $\mathbf{B}$*, the subposet* $\pi^{-1}(x) \cup \pi^{-1}(y)$ *of* $\mathbf{E}$ *is admissible for* $\pi^{-1}(x), \pi^{-1}(y)$*. Then*

$$H^\bullet(\mathbf{E}; F) \cong H^\bullet(\pi^{-1}(1); F),$$

for all $F \in \mathbf{Sh}(\mathbf{E})$*, where* 1 *is the unique maximum of* $\mathbf{B}$.

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Author's declaration

I declare that this thesis is original work and I am the sole author. A truncated version of the results presented in Chapter 7 has appeared as a pre-print on the arXiv [Hur20]. This work has not previously been presented for an award at this, or any other, University. All sources are acknowledged as references.

Variable naming conventions

Some effort has been made to keep notation between chapters consistent. The following table gives commonly used objects and possible variable names associated to them.

Object	Variable names
Knot/link or knot/link diagram	L
Boolean sequence	μ, ν
Category	$\mathbf{C}, \mathbf{D}$
Object of category (Chapter 2)	A, B, C
Object of category (Chapters 3-8)	x, y, z
Morphism in a category	f, g, h
Functor	F, G
Natural transformation	α, β
(Co)chain complex	$C^\bullet, D_\bullet, M^\bullet, N_\bullet$
(Co)chain map	$\varphi_\bullet, \psi^\bullet, \theta_\bullet$
Ring	R
R-module	A, B
R-module homomorphism	f, g, h
Object of the category **Sh**	$(\mathbf{C}; F), (\mathbf{D}; G)$
Morphism in the category **Sh**	$\gamma = (\gamma_1, \gamma_2)$
Simplex in the nerve of a category	σ, τ
Spectral sequence	E
Filtration	$\mathcal{F}, \mathcal{J}$
Bundle of sheaves	ξ
Total sheaf of a bundle ξ	$(\mathbf{E}_\xi; F_\xi)$
Bicomplex	$\mathcal{K}^{\bullet,\bullet}, \mathcal{L}^{\bullet,\bullet}$
Total complex of a bicomplex $\mathcal{K}$	$T^\bullet_{\mathcal{K}}$

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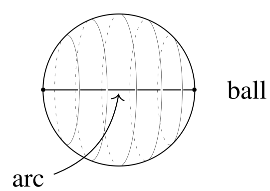
Motivation: knot invariants

1.1 The basics

A *link* L of n components is a subset of $\mathbb{R}^3 \subseteq \mathbb{R}^3 \cup \{\infty\} = S^3$, consisting of n disjoint piecewise linear simple closed curves, where each curve has finitely many pieces. A *knot* is a link with one component. Occasionally, the components of L are also oriented, in which case we have an *oriented link*. The 3-sphere S^3 is always oriented.

Some authors (eg. [Kau87b]) use a more topological definition of a link. A link L of n components is a subset of $\mathbb{R}^3 \subseteq S^3$ consisting of n disjoint embeddings of S^1. This version allows for *wild knots* – knots where there is a point p in S^3, such that each neighbourhood of p contains infinitely many crossings. We will be interested only in *tame links* – a link L is tame if any point on a component of the link has a neighbourhood in S^3 that intersects only a neighbourhood of that component. In other words, a neighbourhood of any point of the link is homeomorphic to the 'ball and arc' pair below.



Two links L_1 and L_2 are *equivalent*, or *ambient isotopic*, if there exists an orientation-preserving homeomorphism $h : S^3 \to S^3$ such that $h(L_1) = L_2$. If L_1, L_2 are oriented, then $h(L_1)$ must be oriented the same way as L_2.

We can also focus on the combinatorial nature of knots and links. To that effect, we consider *link diagrams* – projections of links onto $\mathbb{R}^2 \subseteq S^2 \subseteq S^3$ that correspond to four-regular plane multigraphs, where each vertex is decorated with a crossing indicator ╳ in some orientation (see [Liv93, §2.4]). When the link is oriented, the link diagram inherits the appropriate orientation for the plane graph. For convenience the same label is used for a link and (any of) its diagram(s).

Reidemeister [Rei27] proved in the 1920s that two links are ambient isotopic if and only if their diagrams can be transformed into each other by planar isotopy (continuous deformations in $\mathbb{R}^2$) and the three *Reidemeister moves*, shown in Figure 1.1.

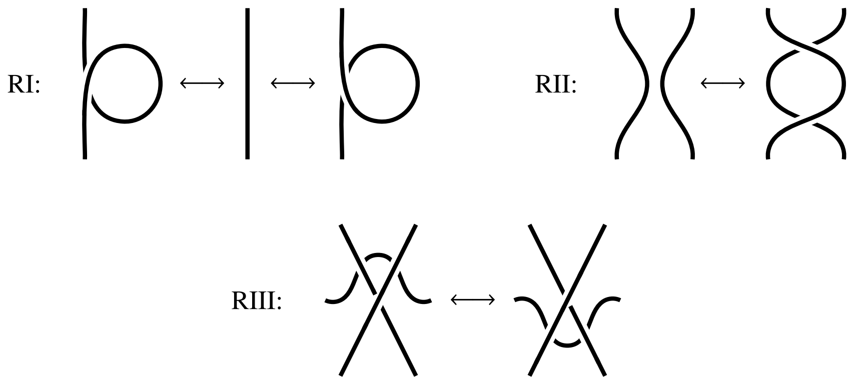


Fig. 1.1: The three Reidemeister moves.

One can also define a finer notion of equivalence – two links are *regularly isotopic* if they can be transformed into each other by planar isotopy and Reidemeister moves II and III only.

Reidemeister's theorem also applies to oriented links – the oriented Reidemeister moves are simply the moves in Figure 1.1 with any orientation assigned to the appearing strands.

1.2 Some topological invariants of knots and links

Knot theorists have historically been interested in determining when two knot diagrams represent the same knot up to isotopy. Using Reidemeister moves directly becomes hopelessly difficult in practice – for a diagram of the unknot with n crossings, [Lac15] shows that up to $(236n)^{11}$ moves are needed to remove all crossings. The way forward is to use *invariants*: an invariant of a link L is a mathematical object (a space, a polynomial, etc.) that does not depend on the particular diagram or geometric realisation of L. Thus, if the invariants of two links differ, we know that those links are not equivalent.

One invariant suggests itself directly from the definition of equivalence. If two links L_1, L_2 are equivalent, then the orientation-preserving homeomorphism that shows their equivalence also shows that the two complements $S^3 \backslash L_1, S^3 \backslash L_2$ are homeomorphic as oriented spaces. The converse is not true in general ([Rol76, §3.A]), but it is true for knots:

Theorem 1.1 ([Gor89])**.** *If two knots have complements that are homeomorphic by an orientation-preserving homeomorphism, then they are [ambient] isotopic.*

A simpler invariant is to consider just the fundamental group of the complement of a knot – this is the *knot group*. While clearly not nearly as discerning as considering the entirety of the complement, the knot group is easily computable via its Wirtinger presentation [Tie08] and distinguishes prime knots ([Gor89, Corollary 2.1]). An example of non-equivalent knots with isomorphic knot groups is given in Figure 1.2.

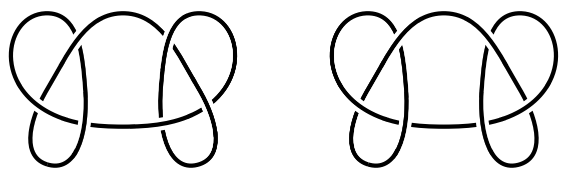


Fig. 1.2: The Granny knot (left) and the Square knot (right) both have $\langle x, y, z \mid xyx = yxy, xzx = zxz \rangle$ as their knot group

A link L is called *hyperbolic* if its complement admits a complete metric of constant curvature -1. Equivalently, L is hyperbolic if $S^3 \backslash L = \mathbb{H}^3/\Gamma$, where $\mathbb{H}^3$ is hyperbolic 3-space and Γ is a discrete, torsion-free group of isometries, isomorphic to $\pi_1(S^3 \backslash L)$ (the knot group, if L is a knot). By Mostow-Prasad rigidity ([Mos73]), the hyperbolic structure is unique up to isometry and thus the volume of $S^3 \backslash L$ as a hyperbolic manifold is an invariant of the link. A theorem by Jørgensen and Thurston [Thu97] implies that there are only finitely many links with a given hyperbolic volume. If we include the *canonical decomposition* [Sak21] of $S^3 \backslash L$ into ideal polyhedra, we have a complete invariant of hyperbolic knots (due to Theorem 1.1).

1.3 Combinatorial invariants

It turns out that even just a link diagram of a link gives a lot of ways to distinguish it from other links. When defining invariants via diagrams, we need to first consider whether the different diagrams of the same link give the same result. Conveniently, we only need to check the Reidemeister moves given above – since any two diagrams of the same link are related by a finite number of moves, if the invariant does not change when applying a move, then it is indeed an invariant of the link.

We start with a simple construction (see [Liv93, §3.2]). In a knot diagram D, an *arc* is a path in the plane multigraph, passing through over-crossings and ending at under-crossings; visually, an arc is a connected curve in the diagram, disconnected by under-crossings. A knot is called *tricolourable* if for a diagram D all arcs can be coloured with three colours, such that

- at least two colours appear, and
- at every crossing the three incident arcs are either all the same colour or three different colours.

Figure 1.3 shows the trefoil with its three arcs in different colours. The proof that tricolourability is a knot invariant consists of just colouring the Reidemeister moves – see Figure 1.4. Since the unknot cannot be tricoloured (we cannot get more than one colour to appear), this invariant gives a straightforward proof that the trefoil is knotted. For if the trefoil and the unknot were equivalent, then they would be connected by a finite number of Reidemeister moves; but those moves preserve whether a knot is tricolourable or not.

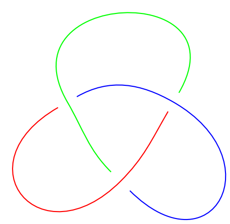


Fig. 1.3: The trefoil knot with a valid tricolouring.

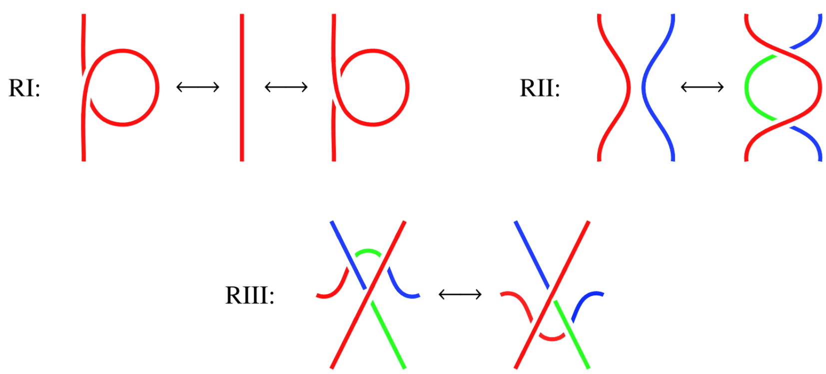


Fig. 1.4: Tricolouring the Reidemeister moves.

The first knot polynomial invariant was described by Alexander in the 1920s [Ale28]. It is originally constructed by considering the infinite cyclic cover of the knot complement. More than 40 years later, John Conway [Con70] rediscovered it in a different form. The Alexander-Conway polynomial $\nabla(z)$ has integer coefficients and is entirely determined for any knot by the following requirements:

- $\nabla(z)$ is an ambient isotopy invariant,
- $\nabla(\bigcirc) = 1$, and
- $\nabla(\overcrossing) - \nabla(\undercrossing) = z\nabla(\smoothing)$.

The culmination of this line of inquiry came after Vaughan Jones constructed another polynomial invariant $V_L(t)$ ([Jon85]). Jones' original definition stems from his work on von Neumann algebras, but his paper also gives a combinatorial description via the skein relation

$$\frac{1}{t}V_{\overcrossing} - tV_{\undercrossing} = \left(\sqrt{t} - \frac{1}{\sqrt{t}}\right)V_{\smoothing}.$$

The following theorem (appearing in [FYH+85] and independently in [PT88]) gives a three-variable link polynomial P_L that specialises to both $\nabla(z)$ and $V_L(t)$:

Theorem 1.2. *There is a unique function P from the set of isotopy classes of tame oriented links to the set of homogeneous Laurent polynomials of degree 0 in x, y, z such that*

- $xP(\overcrossing) + yP(\undercrossing) + zP(\smoothing) = 0$,
- $P_L(x, y, z) = 1$ *if L consists of a single unknotted component.*

We then have the specialisations

$$\nabla_L(z) = P_L(1, -1, z), V_L(t) = P_L(t, -t^{-1}, t^{\frac{1}{2}} - t^{-\frac{1}{2}}).$$

1.4 The Jones polynomial and Khovanov homology

The last stop on the way to the main motivation for this thesis is the bracket polynomial. Kauffman [Kau87a] realised that while $V_L(t)$ is an invariant of oriented links, 'most' of what it measures is recoverable from only the unoriented diagram of the link. The rest of the Jones polynomial comes from the 'twistedness', or *writhe*, of the diagram:

Definition 1.3. The *writhe* of an oriented link diagram L is

$$w(L) = \#\{\overcrossing \text{ crossings in } L\} - \#\{\undercrossing \text{ crossings in } L\}.$$

The two types of crossings are called *positive* and *negative*, respectively.

If L is an unoriented diagram, a *state* S of L is a full resolution, i.e. a diagram where each crossing of L is replaced by either $\smoothing$ or $\hsmoothing$. For a crossing $\crossing$, we call $\hsmoothing$ its *0-smoothing* and $\smoothing$ its *1-smoothing*.

Definition 1.4. The Kaufmann bracket $\langle L\rangle$ of a link diagram L is a Laurent polynomial in A defined by:

- $\langle \bigcirc \rangle = 1$,
- if S is a state of L, then define $\langle L \mid S \rangle$ to be:

$$\langle L \mid S \rangle := A^{i-j},$$

  where i and j are the number of 0-smoothings and 1-smoothings in S, respectively,
- if $|S|$ denotes the number of disjoint components in the diagram S, then

$$\langle L \rangle = \sum_S \langle L \mid S \rangle \cdot (-A^2 - A^{-2})^{|S|-1}.$$

The bracket polynomial is not a link invariant – it is not invariant under Reidemeister move I. It is invariant under moves II and III however, making it an invariant of regular isotopy (recall end of Section 1.1). Combining the bracket and the writhe for an oriented link diagram gives a link invariant that is a change-of-variable away from the Jones polynomial.

Theorem 1.5 ([Kau87a]). *Suppose L is an oriented link diagram and L' is the same diagram with the orientation removed. Then*

$$f[L](A) = (-A)^{-3w(L)} \langle L' \rangle$$

is a link invariant. Moreover,

$$V_L(t) = f[L](t^{-1/4}).$$

It is useful to visualise the state sum for a given link. We do that with the help of a certain poset that will be widely used in what is to follow.

Definition 1.6. A *Boolean lattice* $\mathbb{B}_n$ of rank n is a poset with elements the n-long Boolean sequences $\{0,1\}^n$ and a relation $\leq$ defined by $\mu \leq \nu$ if and only if $\mu_i \leq \nu_i$ for all $i \in \{1, \cdots, n\}$.

This is isomorphic to the poset of all subsets of a set of size n ordered by inclusion, but it will be useful for our purposes to consider it as defined above.

If $d = (A^2 + A^{-2})$, then the calculation of the bracket of the trefoil T can be seen as constructing the Boolean lattice of rank 3, where each vertex is a state of T associated to a Boolean sequence μ. The picture in Figure 1.5 has a satisfyingly similar analogue for constructing Khovanov homology – both consider the same states of a given diagram, foreshadowing the description of the (unnormalised) Jones polynomial as the Euler characteristic of Khovanov homology (Theorem 1.8).

Finally, we come to the description of Khovanov's celebrated construction in [Kho00] of a double-graded R-module link invariant. The following is based on Bar-Natan's survey on the topic in [BN02]. The basic ingredient here is the graded R-module V with $V_1 = V_{-1} = R$:

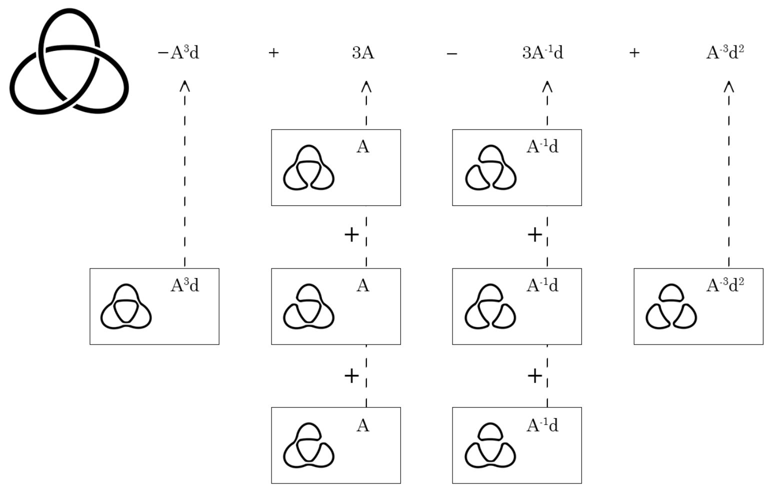


Fig. 1.5: Calculating the bracket polynomial of the trefoil T from the states of T.

$V =$

1	R
0	
−1	R

In this thesis we will have $R = \mathbb{Z}$, but part of the literature also looks at $R = \mathbb{Q}$. Our preference for integral Khovanov homology (as opposed to rational) comes from the richer (and more complex) values of the invariant, including non-obvious appearances of finite quotients of $\mathbb{Z}$ in the final results.

For readability, whenever we have a copy of V, we will write the generator of the module in graded (or q-)degree 1 as a 1 and the generator in q-degree −1 as u. Displaying the generators of V, then, would look like the following (note the gray fill – tables with gray fill always give generators of modules as opposed to the modules themselves).

$V =$

1	1
0	
−1	u

An operation on graded modules we will frequently be using is the *q-degree shift*:

$$V\{k\}_i = V_{i-k}.$$

For example,

$$V\{3\} = \begin{array}{r|c|} \cline{2-2} 4 & 1 \\ \cline{2-2} 3 & \\ \cline{2-2} 2 & u \\ \cline{2-2} \end{array}$$

Finally, we often take tensor products of copies of V:

$$(V \otimes W)_i = \bigoplus_{k+\ell=i} V_k \otimes W_\ell.$$

For example,

$$V^{\otimes 2} = \begin{array}{r|c|} \cline{2-2} 1 & 1 \\ \cline{2-2} 0 & \\ \cline{2-2} -1 & u \\ \cline{2-2} \end{array} \otimes \begin{array}{r|c|} \cline{2-2} 1 & 1 \\ \cline{2-2} 0 & \\ \cline{2-2} -1 & u \\ \cline{2-2} \end{array} = \begin{array}{r|c|} \cline{2-2} 2 & 1 \otimes 1 \\ \cline{2-2} 1 & \\ \cline{2-2} 0 & 1 \otimes u, u \otimes 1 \\ \cline{2-2} -1 & \\ \cline{2-2} -2 & u \otimes u \\ \cline{2-2} \end{array}$$

Now suppose L is a link diagram. We can construct the lattice of full resolutions of L as before, but we now assign a graded module to each state S_μ associated to a Boolean sequence μ. Define

$$\overline{Kh}(L \mid S_\mu) := V^{\otimes |S_\mu|}\{\textstyle\sum \mu\},$$

where $|S_\mu|$ is the number of disjoint components in the diagram S_μ and $\sum \mu$ is the number of 1-smoothings in S_μ. For the trefoil again, we get the picture in Figure 1.6.

The arrows in the figure indicate a single switch of a 0-smoothing to a 1-smoothing. There are two options for the effect of that switch on a diagram – it can either merge two disjoint components, or split one component in two. We assign one of two maps m and Δ, depending on whether the effect is 'merge' or 'split', respectively. In both cases, the copies of V that are involved in the morphism are the ones associated to the concerned components.

$$m : (V \otimes V)\{k\} \to V\{k+1\} : \begin{array}{r|c|} \cline{2-2} k+2 & 1 \otimes 1 \\ \cline{2-2} k+1 & \\ \cline{2-2} k & 1 \otimes u, u \otimes 1 \\ \cline{2-2} k-1 & \\ \cline{2-2} k-2 & u \otimes u \\ \cline{2-2} \end{array} \begin{array}{c} \longrightarrow \\ \\ \longrightarrow \\ \\ \\ \end{array} \begin{array}{r|c|} \cline{2-2} k+2 & 1 \\ \cline{2-2} k+1 & \\ \cline{2-2} k & u \\ \cline{2-2} \\ \\ \end{array}$$

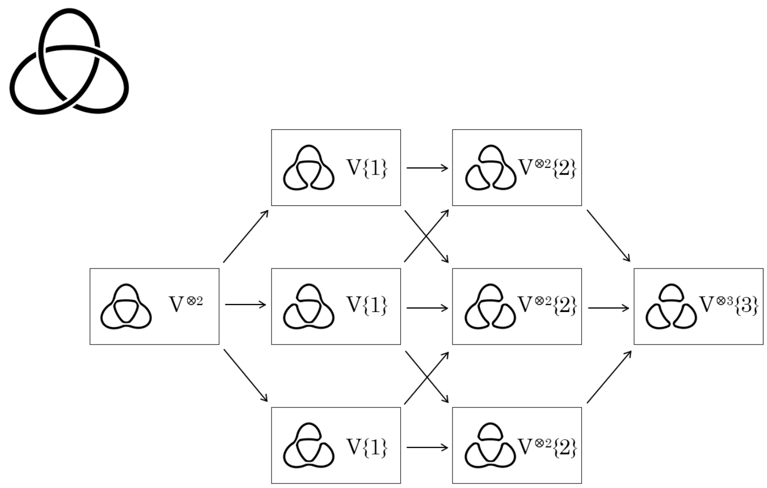
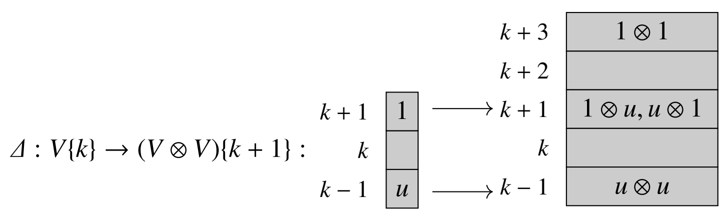


Fig. 1.6: The graded modules assigned to each state of the trefoil.



On the generators, m and Δ act as follows.

$$\begin{aligned}
&m(1 \otimes 1) = 1\\
&m(1 \otimes u) = m(u \otimes 1) = u\\
&m(u \otimes u) = 0\\
&\Delta(1) = 1 \otimes u + u \otimes 1\\
&\Delta(u) = u \otimes u.
\end{aligned}$$

We refer to a pair (μ, ν) of n-long Boolean sequences as *j-adjacent* if they differ only in their j-th place, with $\mu_j = 0$ and $\nu_j = 1$. We call a pair (μ, ν) just *adjacent*, if it is j-adjacent for some j. Thus, for adjacent (μ, ν), we have defined a morphism of graded modules $d_\mu^\nu : \overline{Kh}(L \mid S_\mu) \to \overline{Kh}(L \mid S_\nu)$.

The final piece of the machinery that needs setting up arises from the fact that as currently defined, the morphisms assigned to arrows make the squares in the Boolean lattice commute. We want to build a chain complex, so we would have to flip the sign of some of them, so they anti-commute. If (μ, ν) are j-adjacent, then define

$$\epsilon_\mu^\nu = \mu_1 + \mu_2 + \cdots + \mu_{j-1}.$$

All we have left is to construct the chain complex $C^\bullet(L)$. Let

$$C^i(L) = \bigoplus_{\sum \mu = i} \overline{Kh}(L \mid S_\mu)$$

and define the differential $d^i : C^i(L) \to C^{i+1}(L)$ as

$$d^i = \bigoplus_{(\mu,\nu)} (-1)^{\epsilon_\mu^\nu} d_\mu^\nu,$$

where the sum ranges over adjacent pairs (μ, ν) with $\sum \mu = i$. The 'Jedi sign trick' of adding the $(-1)^{\epsilon_\mu^\nu}$ factor on the morphisms ensures that all squares anti-commute. Therefore $d^2 = 0$ and this is indeed a chain complex.

Definition 1.7. The *unnormalised Khovanov homology* $\overline{Kh}^\bullet(L)$ of a link diagram L is the homology of the above complex:

$$\overline{Kh}^\bullet(L) = HC^\bullet(L).$$

If L is oriented and N_+ and N_- are the number of positive and negative crossings, respectively, then the *normalised Khovanov homology* of L is

$$Kh^\bullet(L) = \overline{Kh}^{\bullet - N_-}(L)\{N_+ - 2N_-\},$$

where curly brackets denote q-degree shift.

Theorem 1.8. *The double-graded $\mathbb{Z}$-module $Kh^\bullet(L)$ is an invariant of L. Moreover, its graded Euler characteristic is the (renormalised) Jones polynomial.*

Remark 1.9. A link L with n crossings produces a chain complex $C^\bullet(L)$ with $n + 1$ non-zero degrees coming from 2^n states (or full resolutions) of L. To make explicit calculations more tractable, several techniques have been developed.

- For any chosen crossing, there is a skein exact sequence ([BN02, Vir02, Kho00])

$$0 \to C^\bullet(\rangle\langle) \to C^\bullet(\times) \to C^\bullet(\asymp) \to 0.$$

- Unnormalised Khovanov homology can be interpreted as the derived limits of a slightly modified Boolean lattice $\mathbb{B}_n^+$ (see [ET14] and Section 4.2)

$$\overline{Kh}^\bullet(L) \cong \varprojlim\nolimits^\bullet_{\mathbb{B}_n^+} F_{Kh}.$$

- Spectral sequences can also be constructed, effectively extending the short exact skein sequence ([Tur08, ET12]).

2

Category-theoretical preliminaries

In this chapter we lay out the (standard) category-theoretical setup used for the rest of the thesis. This author's understanding of Category Theory and its applications has been permanently shaped by Paolo Aluffi's 'Algebra: Chapter 0' [Alu09]. Thus, most of the exposition here is based on that book. One exception to this is the material on universal arrows; this can be found in Mac Lane's 'Categories for the Working Mathematician' [ML98, III].

2.1 Categories

Definition 2.1. A category $\mathbf{C}$ consists of the following data

- a class $\mathrm{Obj}(\mathbf{C})$ of *objects*,
- for all $A, B \in \mathrm{Obj}(\mathbf{C})$, a class $\mathbf{C}(A, B)$ of *arrows* (or *morphisms*) from A to B,
- for all $A, B, C \in \mathrm{Obj}(\mathbf{C})$, a (class) function

$$\begin{aligned} \mathbf{C}(A, B) \times \mathbf{C}(B, C) &\to \mathbf{C}(A, C) \\ (f, g) &\mapsto g \circ f \end{aligned}$$

  called *composition*,
- for each $A \in \mathrm{Obj}(\mathbf{C})$, an arrow $1_A \in \mathbf{C}(A, A)$ (or id_A), called the *identity*

subject to the following axioms:

- *associativity*: for all $f : A \to B$, $g : B \to C$, $h : C \to D$, we have

$$h \circ (g \circ f) = (h \circ g) \circ f,$$

- *unit*: for all $f : A \to B$, we have $1_B \circ f = f$ and $f \circ 1_A = f$.

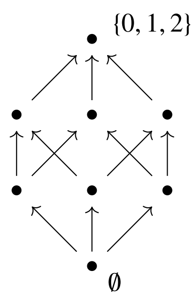
We usually omit the circle in $g \circ f$ and simply write gf for the composition of f and g.

Definition 2.2. Let $\mathbf{C}$ be a category.

- The category $\mathbf{C}$ is said to be *small* if its class of objects $\mathrm{Obj}(\mathbf{C})$ is a set and the class $\mathbf{C}(A, B)$ is a set for all $A, B \in \mathbf{C}$. If $\mathrm{Obj}(\mathbf{C})$ is a proper class, but $\mathbf{C}(A, B)$ is a set for all $A, B \in \mathbf{C}$, then $\mathbf{C}$ is called *locally small*. If $\mathrm{Obj}(\mathbf{C})$ is a proper class and there are $A, B \in \mathbf{C}$ such that $\mathbf{C}(A, B)$ is a proper class, then $\mathbf{C}$ is *large* (for example, the category $\mathbf{Cat}$ of all small categories).
- The category $\mathbf{C}$ is said to be a *poset* if $\mathbf{C}$ is small, the set $\mathbf{C}(A, B)$ consists of at most one element for all $A, B \in \mathbf{C}$, and the set of all arrows in $\mathbf{C}$ forms a partial order (denoted $\leq$) on $\mathrm{Obj}\,\mathbf{C}$.
- A category $\mathbf{D}$ is said to be a *subcategory* of $\mathbf{C}$ if
  - $\mathrm{Obj}\,\mathbf{D} \subseteq \mathrm{Obj}\,\mathbf{C}$;
  - $\mathbf{D}(A, B) \subseteq \mathbf{C}(A, B)$ for all $A, B \in \mathrm{Obj}\,\mathbf{D}$;
  - $\mathrm{id}_A \in \mathbf{D}(A, A)$ for all $A \in \mathbf{D}$;
  - $gf \in \mathbf{D}(A, C)$ for all $f \in \mathbf{D}(A, B), g \in \mathbf{D}(B, C)$, where $A, B, C \in \mathrm{Obj}\,\mathbf{D}$.

  A subcategory $\mathbf{D}$ of $\mathbf{C}$ is a *full subcategory* if $\mathbf{D}(A, B) = \mathbf{C}(A, B)$ for all objects $A, B \in \mathbf{D}$.

Example 2.3. If S is a set, then we can construct the poset $\mathbf{P}_S$ of subsets of S: the set of objects of $\mathbf{P}_S$ is the powerset $\mathcal{P}(S)$ of S and for $A, B \in \mathcal{P}(S)$ there is a unique arrow $A \to B$ if and only if $A \subseteq B$. For finite sets S with $|S| = n$, the poset $\mathbf{P}_S$ is the Boolean lattice $\mathbb{B}_n$ of rank n. For example, if $S = \{0, 1, 2\}$, we get the Boolean lattice of rank 3.



Definition 2.4. An arrow $f \in \mathbf{C}(A, B)$ is a *monomorphism* (or just *monic*) if for all objects C and all $\alpha', \alpha'' \in \mathbf{C}(C, A)$, we have that $f\alpha' = f\alpha''$ implies $\alpha' = \alpha''$.

An arrow $g \in \mathbf{C}(A, B)$ is an *epimorphism* (or *epic*) if for all objects C and all $\beta', \beta'' \in \mathbf{C}(B, C)$, we have that $\beta' g = \beta'' g$ implies $\beta' = \beta''$.

Definition 2.5. An arrow $f \in \mathbf{C}(A, B)$ is an *isomorphism* if and only if there exists $g : B \to A$ such that

$$fg = 1_B \quad \text{and} \quad gf = 1_A.$$

Example 2.6.
- The isomorphisms in the category $\mathbf{Set}$ of sets are the bijections.
- The isomorphisms in the category $\mathbf{Grp}$ of groups are the group isomorphisms.
- The isomorphisms in the category $\mathbf{Top}$ of topological spaces are the homeomorphisms.

- The familiar statement 'isomorphism if and only if monic and epic' holds in any *abelian* category (Definition 2.19), but not in general. For example, in the category defined by $\leq$ on $\mathbb{Z}$, *every* morphism is both monic and epic, while the only isomorphisms are the identities.

Definition 2.7. If $\mathbf{C}$ is a category, the *opposite category* or *dual category* $\mathbf{C}^{op}$ is obtained by formally reversing the direction of the arrows. More precisely:

- the set of objects $\mathrm{Obj}(\mathbf{C}^{op})$ is the same as $\mathrm{Obj}(\mathbf{C})$,
- the sets of arrows of $\mathbf{C}^{op}$ are

$$\mathbf{C}^{op}(B, A) = \{f^{op} \mid f \in \mathbf{C}(A, B)\},$$

- the composition of arrows in $\mathbf{C}^{op}$ agrees with the composition in $\mathbf{C}$:

$$f^{op} g^{op} = (gf)^{op},$$

- the identity arrows are preserved, i.e. 1_A^{op} is the identity arrow of $A \in \mathrm{Obj}(\mathbf{C}^{op})$.

Definition 2.8. Given categories $\mathbf{C}$ and $\mathbf{D}$, a *(covariant) functor*

$$F : \mathbf{C} \to \mathbf{D}$$

consists of

- for each $A \in \mathbf{C}$, an object $FA \in \mathbf{D}$,
- for each $f \in \mathbf{C}(A, B)$, an arrow $Ff \in \mathbf{D}(FA, FB)$,

such that

- for all $f \in \mathbf{C}(A, B)$, $g \in \mathbf{C}(B, C)$, we have

$$F(gf) = (Fg)(Ff),$$

- for all $A \in \mathbf{C}$, we have $F(1_A) = 1_{FA}$.

A *contravariant functor* $F : \mathbf{C} \to \mathbf{D}$ is a covariant functor $\mathbf{C}^{op} \to \mathbf{D}$.

Definition 2.9. Let $F, G : \mathbf{C} \to \mathbf{D}$ be functors. A *natural transformation*

$$\alpha : F \to G$$

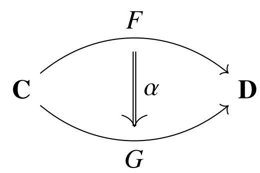
consists of an arrow

$$\alpha_A : FA \to GA \text{ in } \mathbf{D}$$

for each $A \in \mathrm{Obj}(\mathbf{C})$, such that the square

$$\begin{array}{ccc} FA & \xrightarrow{\alpha_A} & GA \\ {\scriptstyle Ff}\downarrow & & \downarrow{\scriptstyle Gf} \\ FB & \xrightarrow[\alpha_B]{} & GB \end{array}$$

commutes for every $f \in \mathbf{C}(A, B)$. This can also be denoted by the diagram



If α_A is an isomorphism for every $A \in \mathrm{Obj}(\mathbf{C})$, then α is called a *natural isomorphism*.

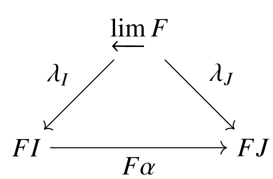
Definition 2.10. Let $\mathbf{C}$ be a category.

- We say that $I \in \mathrm{Obj}(\mathbf{C})$ is *initial* in $\mathbf{C}$ if for every $A \in \mathrm{Obj}(\mathbf{C})$ there exists exactly one morphism $I \to A$ in $\mathbf{C}$.
- An object $J \in \mathbf{C}$ if *final* in $\mathbf{C}$ if for every $A \in \mathrm{Obj}(\mathbf{C})$ there exists exactly one morphism $A \to J$ in $\mathbf{C}$.
- An object is called a *zero object* if it is both initial and final.

Example 2.11.
- If S is a set, then the empty set $\emptyset \in \mathrm{Obj}(\mathbf{P}_S)$ is initial in $\mathbf{P}_S$ (recall Example 2.3).
- The trivial group $\{e\}$ is a zero object of **Grp**.
- The space $\{\bullet\}$ consisting of a single point is a final object of **Top**.

Definition 2.12. Let $\mathbf{I}, \mathbf{C}$ be categories and $F : \mathbf{I} \to \mathbf{C}$ be a covariant functor. A *limit* of F is an object $\varprojlim F \in \mathbf{C}$, endowed with morphisms $\lambda_I : \varprojlim F \to FI$ for all objects $I \in \mathbf{I}$, satisfying the following properties.

- If $\alpha \in \mathbf{I}(I, J)$, then $\lambda_J = F(\alpha)\lambda_I$:



- $\varprojlim F$ is final with respect to this property: that is, if A is another object, endowed with morphisms μ_I, also satisfying the above requirement, then there exists a unique morphism $A \to \varprojlim F$ making all the relevant diagrams commute.

The 'dual notion' to the limit is the *colimit* of a functor $F : \mathbf{I} \to \mathbf{C}$. The colimit is an object $\varinjlim F \in \mathbf{C}$, endowed with morphisms $\chi_I : F(I) \to \varinjlim F$ for all objects $I \in \mathbf{I}$ such that $\chi_I = \chi_J F\alpha$ for all $\alpha \in \mathbf{C}(I, J)$ and that $\varinjlim F$ is initial with respect to this requirement.

Remark 2.13. If the functor F in the above definition is instead *contra*variant, we have that $F\alpha : FJ \to FI$. The required commutativity is then $\lambda_I = F(\alpha)\lambda_J$.

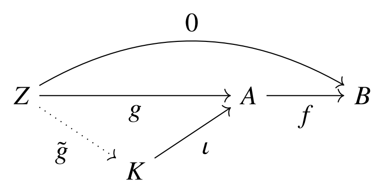
Example 2.14. If $\mathbf{I}$ is the 'discrete category' consisting of $n \in \mathbb{N}$ objects with only identity morphisms, then if $F : \mathbf{I} \to \mathbf{C}$ is a functor, the limit $\varinjlim F$ (if it exists) is called the *coproduct* of the objects $\{FI\}_{I \in \mathbf{I}}$. The colimit $\varprojlim F$ is called the *product* of those objects.

Definition 2.15. A category $\mathbf{C}$ is *additive* if it has a zero-object, both finite products and finite coproducts exist, and each set of morphisms $\mathbf{C}(A, B)$ is endowed with an abelian group structure, in such a way that the composition maps are bilinear. A functor between two additive categories is *additive* if it preserves the abelian group structures of the sets of morphisms.

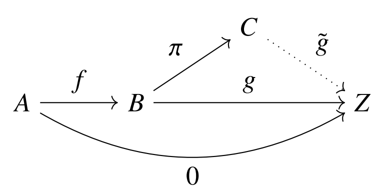
2.2 Homological algebra

If a category is additive, it makes sense to talk about 'zero morphisms' (denoted by 0) and to use addition and subtraction for the group operation in the sets of morphisms.

Definition 2.16. Let $\mathbf{C}$ be an additive category and let $f \in \mathbf{C}(A, B)$. An arrow $\iota \in \mathbf{C}(K, A)$ is a *kernel* of f if $f\iota = 0$ and for all morphisms $g \in \mathbf{C}(Z, A)$ such that $fg = 0$, there exists a unique $\tilde{g} \in \mathbf{C}(Z, K)$ making the diagram



commute. A morphism $\pi \in \mathbf{C}(B, C)$ is a *cokernel* of f if $\pi f = 0$ and for all morphisms $g \in \mathbf{C}(B, Z)$ such that $gf = 0$, there exists a unique $\tilde{g} \in \mathbf{C}(C, Z)$ making the diagram



commute.

Lemma 2.17. *In any additive category, kernels are monomorphisms and cokernels are epimorphisms.*

Remark 2.18. It is convenient to think of monomorphisms $A \to B$ as defining A as a 'subobject' of B. Similarly, it is convenient to think of epimorphisms as 'quotients': if $\varphi : A \to B$ is a monomorphism, we can use B/A to denote (the target of) $\operatorname{coker} \varphi$.

Definition 2.19. An additive category $\mathbf{C}$ is *abelian* if kernels and cokernels exist in $\mathbf{C}$, every monomorphism is the kernel of some morphism, and every epimorphism is the cokernel of some morphism.

Definition 2.20. A (possibly infinite) sequence of objects and morphisms in an abelian category

$$\cdots \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow \cdots$$

is *exact at B* if

- $gf = 0$ and
- $\operatorname{coker} f \ker g = 0$.

If this is true for every object in the sequence, then we say that the sequence is exact.

Definition 2.21. Let $f \in \mathbf{C}(A, B)$ be a morphism in an abelian category. The *image* of f is defined as $\operatorname{im} f := \ker(\operatorname{coker} f)$. The *coimage* of f is $\operatorname{coim} f := \operatorname{coker}(\ker f)$.

This means that the condition defining exactness can be summarised simply as $\operatorname{im} f = \ker g$.

Definition 2.22. Let $\mathbf{C}$ and $\mathbf{D}$ be categories and $F : \mathbf{C} \to \mathbf{D}$ be a functor. The *functor F is exact* if for any exact sequence in $\mathbf{C}$

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

the image sequence in $\mathbf{D}$

$$0 \longrightarrow FA \xrightarrow{Ff} FB \xrightarrow{Fg} FC \longrightarrow 0$$

is exact.

We say that F is *left-exact* if whenever

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C$$

is exact in $\mathbf{C}$, then so is

$$0 \longrightarrow FA \xrightarrow{Ff} FB \xrightarrow{Fg} FC \quad .$$

Similarly for F being *right-exact*.

Definition 2.23. A *chain complex* $(M_\bullet, d_\bullet)$ in an abelian category $\mathbf{C}$ is a sequence of objects and morphisms,

$$\cdots \xrightarrow{d_{i+2}} M_{i+1} \xrightarrow{d_{i+1}} M_i \xrightarrow{d_i} M_{i-1} \xrightarrow{d_{i-1}} \cdots$$

such that $d_i d_{i+1} = 0$ for all i. We can just as well use ascending indices (which are then traditionally written as superscripts),

$$\cdots \xrightarrow{d^{i-2}} M^{i-1} \xrightarrow{d^{i-1}} M^i \xrightarrow{d^i} M^{i+1} \xrightarrow{d^{i+1}} \cdots$$

and impose $d^i d^{i-1} = 0$. This is a *cochain complex* $(M^\bullet, d^\bullet)$. The morphisms d_i (or d^i) are the *differentials* of the complex.

Definition 2.24. The *homology* of a chain complex $(M_\bullet, d_\bullet)$ in an abelian category is a collection of objects $\{H_i(M_\bullet)\}_{i\in\mathbb{Z}}$, where

$$H_i(M_\bullet) := \frac{\ker d_i}{\operatorname{im} d_{i+1}}.$$

The *cohomology* of a cochain complex $(M^\bullet, d^\bullet)$ is a collection of objects $\{H^i(M^\bullet)\}_{i\in\mathbb{Z}}$, where

$$H^i(M^\bullet) := \frac{\ker d^i}{\operatorname{im} d^{i-1}}.$$

Definition 2.25. Let $\mathbf{C}$ be an abelian category. The *category* $\mathbf{Ch}(\mathbf{C})$ *of cochain complexes in* $\mathbf{C}$ is defined by

- $\mathrm{Obj}(\mathbf{Ch}(\mathbf{C})) = \{$cochain complexes in $\mathbf{C}\}$;
- for cochain complexes $M^\bullet = (M^\bullet, d_M^\bullet)$ and $N^\bullet = (N^\bullet, d_N^\bullet)$, the morphism set $\mathbf{Ch}(\mathbf{C})(M^\bullet, N^\bullet)$ consists of *cochain maps*, that is commutative diagrams

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & M^{i-1} & \xrightarrow{d_M^{i-1}} & M^i & \xrightarrow{d_M^i} & M^{i+1} & \longrightarrow & \cdots \\ & & \downarrow{\scriptstyle\varphi^{i-1}} & & \downarrow{\scriptstyle\varphi^i} & & \downarrow{\scriptstyle\varphi^{i+1}} & & \\ \cdots & \longrightarrow & N^{i-1} & \xrightarrow{d_N^{i-1}} & N^i & \xrightarrow{d_N^i} & N^{i+1} & \longrightarrow & \cdots \end{array}$$

  in $\mathbf{C}$. We denote by $\varphi^\bullet$ the cochain map determined by the collection $\{\varphi^i\}_{i\in\mathbb{N}}$.

Remark 2.26. We occasionally use a superscript to indicate where the non-zero objects of a complex are. For example, $M^\bullet \in \mathbf{Ch}^{\leq 0}$ has $M^i = 0$ for all $i > 0$.

Let $\mathbf{C}$ be an abelian category and $C \in \mathbf{C}$ be an object. A trivial (but convenient) example of a cochain complex is the one with C in degree 0, with all other objects and morphisms 0. We denote this complex by $\iota^\bullet(C)$.

$$\iota^\bullet(C): \quad \cdots \longrightarrow 0 \longrightarrow C \longrightarrow 0 \longrightarrow \cdots$$

Lemma 2.27. *If* **C** *is an abelian category, then so is* **Ch**(**C**).

Lemma 2.28. *For every integer i, the assignment*

$$H^i : M^\bullet \mapsto H^i(M^\bullet)$$

defines an additive covariant functor **Ch**(**C**) → **C**.

Definition 2.29. A (co)chain map $\varphi^\bullet$ of cochain complexes is a *quasi-isomorphism* if it induces an isomorphism in cohomology.

Definition 2.30. A *homotopy* between two morphisms of cochain complexes

$$\varphi^\bullet, \psi^\bullet : L^\bullet \to M^\bullet$$

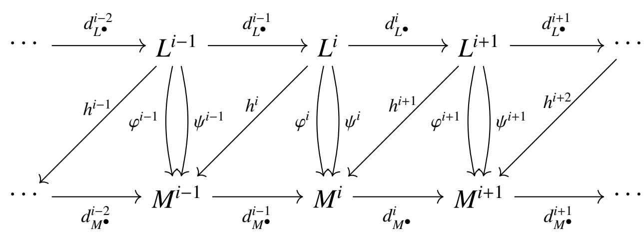
is a collection of morphisms

$$h^i : L^i \to M^{i-1}$$

such that for each i we have

$$\varphi^i - \psi^i = d^{i-1}_{M^\bullet} h^i + h^{i+1} d^i_{L^\bullet}.$$

We say that $\varphi^\bullet$ is *homotopic* to $\psi^\bullet$ and write $\varphi^\bullet \sim \psi^\bullet$ if there is a homotopy between $\varphi^\bullet$ and $\psi^\bullet$. The following diagram of the setup is *not* assumed to be commutative:



Definition 2.31. A morphism $\varphi^\bullet : L^\bullet \to M^\bullet$ is a *homotopy equivalence* if there is a morphism $\psi^\bullet : M^\bullet \to L^\bullet$ such that $\varphi^\bullet\psi^\bullet \sim \mathrm{id}_{M^\bullet}$ and $\psi^\bullet\varphi^\bullet \sim \mathrm{id}_{L^\bullet}$. In this case, the complexes $M^\bullet, L^\bullet$ are called *homotopy equivalent*. In particular, a homotopy equivalence $\varphi^\bullet$ is a quasi-isomorphism, but a quasi-isomorphism is not necessarily a homotopy equivalence.

Proposition 2.32. *Homotopy equivalent complexes have isomorphic cohomology.*

Remark 2.33. We can, of course, define homotopy and homotopy equivalence for *chain* complexes in an analogous way. The last proposition also extends to this case – homotopy equivalent *chain* complexes have isomorphic *homology*.

Lemma 2.34. *A (short) exact sequence in* **Ch**(**C**)

$$0 \to L^\bullet \to M^\bullet \to N^\bullet \to 0$$

determines a (long) exact sequence in **C**

$$\cdots \to H^{i-1}(N^\bullet) \to H^i(L^\bullet) \to H^i(M^\bullet) \to H^i(N^\bullet) \to H^{i+1}(L^\bullet) \to \cdots.$$

Lemma 2.35 (5-Lemma). *Let the following be a commutative diagram in* $\mathbf{C}$

$$\begin{array}{ccccccccc} A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & D & \longrightarrow & E \\ \downarrow f & & \downarrow g & & \downarrow h & & \downarrow i & & \downarrow j \\ A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & D' & \longrightarrow & E' \end{array}$$

where the rows are exact, the maps g and i are isomorphisms and the maps f and j are epic and monic, respectively. Then h is an isomorphism.

Definition 2.36. Let $\mathbf{C}$ be an abelian category. An object $P \in \mathbf{C}$ is *projective* if and only if for any epimorphism $f \in \mathbf{C}(M, N)$ and any morphism $g \in \mathbf{C}(P, N)$, there exists a morphism $\tilde{g} \in \mathbf{C}(P, M)$ such that the diagram

$$\begin{array}{ccccc} & \overset{\tilde{g}}{\swarrow} & P & & \\ & & \downarrow g & & \\ M & \xrightarrow[f]{} & N & \longrightarrow & 0 \end{array}$$

commutes. An object $Q \in \mathbf{C}$ is *injective* if and only if for any monomorphism $k \in \mathbf{C}(L, M)$ and any morphism $\ell \in \mathbf{C}(L, Q)$, there exists a morphism $\tilde{\ell} \in \mathbf{C}(M, Q)$ such that the diagram

$$\begin{array}{ccccc} & & Q & \overset{\tilde{\ell}}{\nwarrow} & \\ & & \uparrow \ell & & \\ 0 & \longrightarrow & L & \xrightarrow[k]{} & M \end{array}$$

commutes.

Definition 2.37. An abelian category $\mathbf{C}$ has *enough projectives* if for every object $C \in \mathbf{C}$ there exists a projective object $P \in \mathbf{C}$ and an epimorphism $P \to C$. The category has *enough injectives* if for every object $C \in \mathbf{C}$ there is an injective object $Q \in \mathbf{C}$ and a monomorphism $C \to Q$.

Definition 2.38. Let C be an object of an abelian category $\mathbf{C}$. A *projective resolution* of C is an exact sequence

$$P_0 \leftarrow P_1 \leftarrow P_2 \leftarrow \cdots$$

and a quasi-isomorphism $P_\bullet \to \iota_\bullet(C)$, where $P_\bullet \in \mathbf{Ch}(\mathbf{C})$ with each of its objects projective. An *injective resolution* of C is an exact sequence

$$Q_0 \to Q_1 \to Q_2 \to \cdots$$

and a quasi-isomorphism $\iota^\bullet(C) \to Q^\bullet$, where $Q^\bullet \in \mathbf{Ch}(\mathbf{C})$ with each of its objects injective. We usually refer to the resolutions as just $P_\bullet$ or $Q^\bullet$, leaving the quasi-isomorphism understood.

Definition 2.39. Let $\mathbf{C}$ and $\mathbf{D}$ be abelian categories and assume $\mathbf{C}$ has enough projectives. Let $F : \mathbf{C} \to \mathbf{D}$ be an additive, covariant, right exact functor and C an object of $\mathbf{C}$. The *left-derived functor* $L_i F(C)$ of F at C is obtained by finding any projective resolution $P^C_\bullet$ of C, applying the functor F to the complex $P^C_\bullet$ to obtain a complex in $\mathbf{Ch}(\mathbf{D})$, and taking the i-th homology of this complex. Similarly, the *right-derived functor* $R^i G(C)$ of a left exact functor G at C is obtained by finding any injective resolution $Q^\bullet_C$ of C, applying the functor G to the complex $Q^\bullet_C$ to obtain a complex in $\mathbf{Ch}(\mathbf{D})$, and taking the i-th cohomology of this complex.

Remark 2.40. Since contravariant functors $F : \mathbf{C} \to \mathbf{D}$ are just covariant functors $\mathbf{C}^{op} \to \mathbf{D}$, if the functor F in the definition above is contravariant, then the roles of injectives and projectives should be swapped – the right-derived functors of an additive contravariant functor $\mathbf{C} \to \mathbf{D}$ will be defined if $\mathbf{C}$ has enough projectives (i.e. $\mathbf{C}^{op}$ will have enough injectives, as needed).

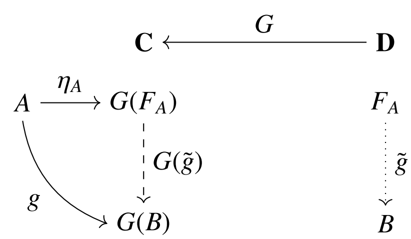
Proposition 2.41. *The above description results in well-defined functors, i.e. any two resolutions $P^C_\bullet, P'^C_\bullet$ (or $Q^\bullet_C, Q'^\bullet_C$) of an object C of $\mathbf{C}$ give quasi-isomorphic complexes $F(P^C_\bullet), F(P'^C_\bullet)$ (or $F(Q^\bullet_C), F(Q'^\bullet_C)$).*

Definition 2.42. Let $G : \mathbf{D} \to \mathbf{C}$ be a functor and $A \in \mathbf{C}$. A *universal arrow from A to G* is a pair

$$(F_A, \eta_A : A \to G(F_A))$$

such that $F_A \in \mathbf{D}$ and for every $B \in \mathbf{D}$ and $g \in \mathbf{C}(A, GB)$ there is a unique arrow $\tilde{g} \in \mathbf{D}(F_A, B)$ such that

$$G(\tilde{g})\eta_A = g.$$



Definition 2.43. Let $G : \mathbf{D} \to \mathbf{C}$ and $F : \mathbf{C} \to \mathbf{D}$ be functors. An *adjunction* between F and G is a family of bijections

$$\theta_{A,B} : \mathbf{D}(FA, B) \to \mathbf{C}(A, GB)$$

that are natural in $A \in \mathbf{C}$ and $B \in \mathbf{D}$. We say that (F, G) is an adjoint pair (and we say that F is *left-adjoint* to G; G is *right-adjoint* to F).

Theorem 2.44. *Let $G : \mathbf{D} \to \mathbf{C}$ be a functor. The following are equivalent:*

- *To give a functor $F : \mathbf{C} \to \mathbf{D}$ and a family of bijections*

$$\theta_{A,B} : \mathbf{D}(FA, B) \to \mathbf{C}(A, GB)$$

that are natural in $A \in \mathbf{C}$ and $B \in \mathbf{D}$.

- *To give, for every $A \in \mathbf{C}$, a universal arrow from A to G*

$$(F_A, \eta_A : A \to G(F_A)).$$

Proposition 2.45. *Every right-(left-)adjoint functor between two abelian categories is left-(right-)exact.*

Proposition 2.46. *If F is left-adjoint to an exact functor, then FA is projective whenever A is projective.*

3

Sheaves and the category Sh

3.1 Sheaves

In Section 1.4, we assigned graded modules to each element of a Boolean lattice and described morphisms between them. This construction is an example of a *presheaf* of modules on a small category. In this section, we lay out the particular definitions we will be working with. Our treatment follows, but also aims to generalise, the treatment of 'coloured posets' in [ET09].

One further note. It occasionally happens that the same concept is given different names in order to reflect a specific perspective or attitude. In this way, the term 'presheaf' is a *concept with an attitude* – it is called a *pre*sheaf, because it is not yet 'sheafified' into a sheaf, or because it indicates interest in the 'presheaf topos'. We will, in fact, not be engaging explicitly with either and thus will contend ourselves with using *sheaf* for the relevant functor, as opposed to *presheaf*.

From now on, let $\mathbf{C}$ be a small category and R be a commutative ring with 1. For the rest of the thesis we will switch to using x, y, z for objects of categories (as opposed to A, B, C) and will reserve A, B, C, etc. for R-modules.

Definition 3.1. A *sheaf* F on $\mathbf{C}$ is a contravariant functor $F : \mathbf{C} \to {}_R Mod$.

$$
\begin{array}{llcl}
F: & Fx & \xleftarrow{\quad Fg \quad} & Fy \\
 & \uparrow & & \uparrow \\
\mathbf{C}: & x & \xrightarrow{\quad g \quad} & y
\end{array}
$$

We write F_x^y for $F(x \to y) : F(y) \to F(x)$. These are the *structure maps* of F.

Example 3.2. If $\mathbf{P}$ is a poset with a unique maximal element, then the coloured poset $(\mathbf{P}, F)$ ([ET09, Definition 1]) is a sheaf F on $\mathbf{P}^{op}$.

Definition 3.3. A *map of sheaves* is a morphism $\alpha : F \to G$, where F and G are sheaves on $\mathbf{C}$, such that α is a natural transformation of functors (recall Definition 2.9).

$$
\begin{array}{llcl}
G : & Gx & \xleftarrow{\quad Gg \quad} & Gy \\
 & \uparrow{\scriptstyle \alpha_x} & & \uparrow{\scriptstyle \alpha_y} \\
F : & Fx & \xleftarrow[\quad Fg \quad]{} & Fy \\
 & & & \\
\mathbf{C} : & x & \xrightarrow{\quad g \quad} & y
\end{array}
$$

The category of sheaves on $\mathbf{C}$ is denoted $\mathbf{Sh}(\mathbf{C})$.

Example 3.4. A basic example is the *constant sheaf* on $\mathbf{C}$. Let $A \in {}_R Mod$. Then define $\Delta A : \mathbf{C} \to {}_R Mod$ by $\Delta A(x) = A$ and for any $x \to y \in \mathbf{C}(x, y)$

$$
\Delta A(x \to y) : \quad \underset{\underset{A}{\parallel}}{\Delta A(x)} \xleftarrow{\text{id}} \underset{\underset{A}{\parallel}}{\Delta A(y)}.
$$

If A and B are R-modules and $f : A \to B$ is an R-module homomorphism, then we have an induced map of sheaves $\alpha : \Delta A \to \Delta B$.

$$
\begin{array}{ccccc}
B & \xleftarrow{\text{id}} & B & \quad & \Delta B \\
\uparrow{\scriptstyle f} & & \uparrow{\scriptstyle f} & & \uparrow{\scriptstyle \alpha} \\
A & \xleftarrow{\text{id}} & A & & \Delta A \\
\vdots & & \vdots & & \\
x & \xrightarrow[g]{} & y & & \mathbf{C}
\end{array}
$$

This makes Δ a covariant functor ${}_R Mod \to \mathbf{Sh}(\mathbf{C})$.

We can also functorially get an R-module from a sheaf. If F is a sheaf on $\mathbf{C}$, then we have a functor $F : \mathbf{C} \to {}_R Mod$ and we can explicitly construct the limit $\varprojlim_{\mathbf{C}} F$ (recall Definition 2.12) as a submodule of the product $\prod_{x \in \mathbf{C}} F(x)$. The product consists of arbitrary sequences $(a_x)_{x \in \mathbf{C}}$ of elements $a_x \in F(x)$. Say that a sequence $(a_x)_{x \in \mathbf{C}}$ is *coherent* if for every $x_1 \to x_2 \in \mathbf{C}$ we have $a_{x_1} = F_{x_1}^{x_2}(a_{x_2})$. Define

$$
\varprojlim_{\mathbf{C}} F = \left\{ (a_x)_{x \in \mathbf{C}} \in \prod_{x \in \mathbf{C}} F(x) \mid (a_x)_{x \in \mathbf{C}} \text{ is coherent} \right\}.
$$

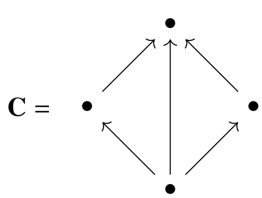
The canonical projections $\pi_x : \prod_y F(y) \to F(x)$ restrict to $\varprojlim_{\mathbf{C}} F$ and so give the required limit morphisms that commute with the module morphisms in the sheaf.

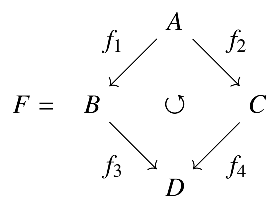
We can also explicitly construct the colimit $\varinjlim_{\mathbf{C}} F$ as a quotient of the sum:

$$\varinjlim_{\mathbf{C}} F = \bigoplus_{x \in \mathbf{C}} F(x)/I,$$

where I is generated by all the $a_y - F_x^y(a_y)$ for $x \to y$ in $\mathbf{C}$ and $a_y \in F(y)$. The quotient maps of the canonical inclusions $F(y) \to \bigoplus_x F(x)$ provide the colimit morphisms.

Example 3.5. Let $\mathbf{C}$ be a poset category (Definition 2.2) and F be a sheaf on $\mathbf{C}$ as represented below.





If $(a, b, c, d) \in \varprojlim_{\mathbf{C}} F$, then

$$(a, b, c, d) = (a, f_1 a, f_2 a, f_3 f_1 a),$$

so $\varprojlim_{\mathbf{C}} F \cong A$.

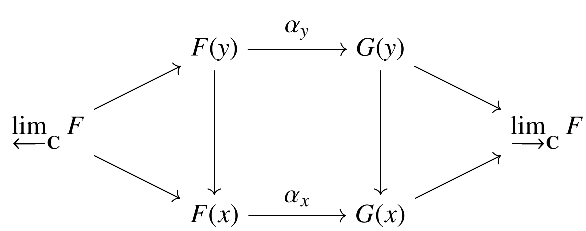
If $(a, b, c, d) \in \varinjlim_{\mathbf{C}} F$, then

$$(a, b, c, d) = (0, 0, 0, d - f_3 f_1 a - f_3 b - f_4 c),$$

so $\varinjlim_{\mathbf{C}} F \cong D$.

More generally ([Yuz91]), whenever we have a poset category $\mathbf{C}$ with a unique maximum (or a unique minimum) x, the same argument gives $\varprojlim_{\mathbf{C}} F \cong F(x)$ (or $\varinjlim_{\mathbf{C}} F \cong F(x)$).

We have constructed two R-modules $\varprojlim_{\mathbf{C}} F$ and $\varinjlim_{\mathbf{C}} F$ from a sheaf F. Now suppose we have two sheaves F, G on $\mathbf{C}$ and a sheaf morphism $\alpha : F \to G$. For an arrow $x \to y \in \mathbf{C}$ consider the following diagram



The left and right triangles commute by the definition of the limit and colimit, respectively. The square commutes since α is a natural transformation. We can then compose the limit maps with the α morphisms and the α morphisms with the colimit maps. The universal properties of the limit and the colimit imply that there are unique morphisms

$$\varprojlim_{\mathbf{C}} F \to \varprojlim_{\mathbf{C}} G \quad \text{and} \quad \varinjlim_{\mathbf{C}} F \to \varinjlim_{\mathbf{C}} G$$

and so $\varprojlim_{\mathbf{C}}$ and $\varinjlim_{\mathbf{C}}$ are covariant functors $\mathbf{Sh}(\mathbf{C}) \to {}_R Mod$.

Proposition 3.6. *Let* $\mathbf{C}$ *be a small abelian category. The functors* $\Delta : {}_R Mod \to \mathbf{Sh}(\mathbf{C})$ *and* $\varprojlim_{\mathbf{C}} : \mathbf{Sh}(\mathbf{C}) \to {}_R Mod$ *form an adjoint pair* $(\Delta, \varprojlim_{\mathbf{C}})$.

Proof. Theorem 2.44 means that we only need to define for each $A \in {}_R Mod$ a universal arrow $(\Delta A, \eta_A : A \to \varprojlim_{\mathbf{C}} \Delta A)$. Let $B \in \mathbf{Sh}(\mathbf{C})$ and $g : A \to \varprojlim_{\mathbf{C}} B$. Finally, let $x \to y$ be an arrow in $\mathbf{C}$ and consider the diagram in Figure 3.1.

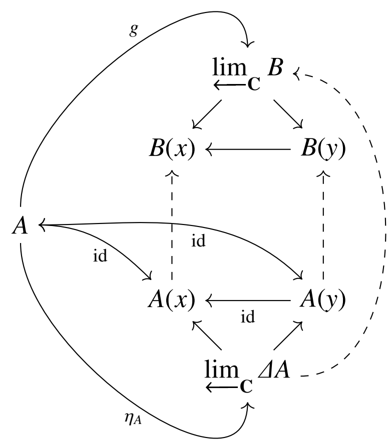


Fig. 3.1: Construction of the universal arrow $A \to \varprojlim_{\mathbf{C}} \Delta A$.

If $A(x) := \Delta A(x) \cong A$ and $A(y) = \Delta A \cong A$, then the identity maps $A \to A(z)$ for every $z \in \mathbf{C}$ give a unique morphism $\eta_A : A \to \varprojlim_{\mathbf{C}} \Delta A$ from the universal property of the limit. This means that the portion of the diagram with solid arrows now commutes. We want to find the unique morphism $\tilde{g} : \Delta A \to B$ such that $\varprojlim_{\mathbf{C}}(\tilde{g})\eta_A = g$.

In order to maintain commutativity, for each $z \in \mathbf{C}$, we can only construct the arrow $A(z) \to B(z)$ as the composition $A(z) \to A \to \varprojlim_{\mathbf{C}} B \to B(z)$. This gives the sheaf morphism $\tilde{g}$. The curved dashed arrow represents $\varprojlim_{\mathbf{C}}(\tilde{g})$ and, since everything in sight commutes, we have verified that $\varprojlim_{\mathbf{C}}(\tilde{g})\eta_A = g$. □

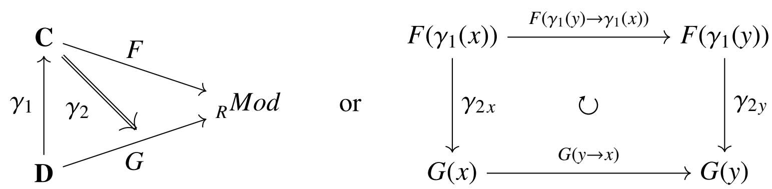
Similarly, $(\varinjlim_{\mathbf{C}}, \Delta)$ also form an adjoint pair. Therefore $\varprojlim_{\mathbf{C}}$ is a left exact functor and $\varinjlim_{\mathbf{C}}$ is a right exact functor.

Definition 3.7. The *higher limits* $\varprojlim^i_{\mathbf{C}}$ are defined as the derived functors $R^i \varprojlim_{\mathbf{C}}$. The *higher colimits* $\varinjlim^i_{\mathbf{C}}$ are defined as the derived functors $L^i \varinjlim_{\mathbf{C}}$. We also define the cohomology and homology of $\mathbf{C}$ with coefficients in F by

$$H^i(\mathbf{C}; F) := \varprojlim^i_{\mathbf{C}} F \quad \text{and} \quad H_i(\mathbf{C}; F) := \varinjlim^i_{\mathbf{C}} F.$$

We can also vary the small category $\mathbf{C}$ that encodes the shape of the sheaf.

Definition 3.8 (Category Sh). An object $(\mathbf{C}, F)$ of **Sh** consists of a small category $\mathbf{C}$ and a sheaf F on $\mathbf{C}$. A **Sh**-morphism $\gamma : (\mathbf{C}, F) \to (\mathbf{D}, G)$ is a pair of maps (γ_1, γ_2), where $\gamma_1 : \mathbf{D} \to \mathbf{C}$ is a covariant functor and $\gamma_2 : F\gamma_1 \to G$ is a natural transformation:



The composition of two morphisms $\gamma : (\mathbf{C}, F) \to (\mathbf{D}, G)$ and $\delta : (\mathbf{D}, G) \to (\mathbf{E}, H)$ is then $(\gamma_1\delta_1, \delta_2\gamma_2) : (\mathbf{C}, F) \to (\mathbf{E}, H)$.

3.2 Computing the cohomology of a sheaf

The next chapter will give a 'simplicial' homology theory for computing the higher colimits of a sheaf. This section gives an alternative way to compute $H^i(\mathbf{C}; F)$. We start by collecting some facts into the following proposition.

Proposition 3.9. *Let* $\mathbf{C}$ *be a small category. Then* $\mathbf{Sh}(\mathbf{C})$ *is abelian, has enough projectives and injectives; kernels, cokernels, and exactness in* $\mathbf{Sh}(\mathbf{C})$ *can be determined locally, or 'pointwise'.*

Most of Proposition 3.9 can be found in Chapters 5 and 6 of [Rot09], for example Corollary 5.94, Propositions 6.2 and 6.5, etc.

A special case of the adjointness of $(\varDelta, \varprojlim_{\mathbf{C}})$ (Proposition 3.6) when $A = R$ gives

$$\mathrm{Hom}_{\mathbf{Sh}(\mathbf{C})}(\varDelta R, \_) \cong \mathrm{Hom}_R(R, \varprojlim_{\mathbf{C}}) \cong \varprojlim_{\mathbf{C}}.$$

Proposition 3.10. *For any* $F, G \in \mathbf{Sh}(\mathbf{C})$*, we have*

$$R^i \,\mathrm{Hom}_{\mathbf{Sh}(\mathbf{C})}(F, \_)(G) \cong R^i \,\mathrm{Hom}_{\mathbf{Sh}(\mathbf{C})}(\_, G)(F).$$

Proof. In view of Proposition 3.9, the proof goes through analogously to any standard proof in $_R Mod$ (for example, [Wei94, §2.7]). □

Therefore we have

$$\varprojlim^i_{\mathbf{C}} F \cong R^i \operatorname{Hom}_{\mathbf{Sh}(\mathbf{C})}(\varDelta R, \_)(F) \cong R^i \operatorname{Hom}_{\mathbf{Sh}(\mathbf{C})}(\_, F)(\varDelta R).$$

Suppose then that we have a projective resolution $P_\bullet$ of $\varDelta R$ in $\mathbf{Sh}(\mathbf{C})$, i.e. an exact sequence

$$\cdots \to P_2 \to P_1 \to P_0 \to \varDelta R \to 0,$$

where the P_i's are projective sheaves. Then the above isomorphism means that $\varprojlim^i_{\mathbf{C}} F$ is isomorphic to the degree-i cohomology of

$$\cdots \leftarrow \operatorname{Hom}_{\mathbf{Sh}(\mathbf{C})}(P_1, F) \leftarrow \operatorname{Hom}_{\mathbf{Sh}(\mathbf{C})}(P_0, F) \leftarrow 0.$$

4

Combinatorial Cohomology

In the previous chapter we gave the theoretical procedure for finding the cohomology of a sheaf via the higher limits. The goal now is to define a cochain complex that computes the limit of a sheaf, but that is not obscured behind the abstract veil of derived functors. In the first section we define such a complex, with the proof that it indeed computes the higher limits left to the second. A version of the exposition in this chapter can be found in [Moe95].

4.1 The complex $\mathcal{S}^\bullet$

From a small category $\mathbf{C}$ we will define a simplicial complex $N\mathbf{C}$ called the *nerve* of $\mathbf{C}$:

- An *i-simplex* σ is a chain $x_0 \to x_1 \to \cdots \to x_i$, where each x_j is an object of $\mathbf{C}$ and each $x_j \to x_{j+1}$ is an arrow in $\mathbf{C}$.
- A *subsimplex* $\tau \subseteq \sigma$ is a choice of a set $\{i_0, \ldots, i_k\} \subseteq \{0, \ldots, i\}$ with $i_j < i_{j+1}$. This gives a k-simplex $x_{i_0} \to x_{i_1} \to \cdots \to x_{i_k}$, where the arrows are the appropriate compositions of arrows from σ.

A sheaf F on $\mathbf{C}$ gives a covariant functor F_h from the poset of simplices of $N\mathbf{C}$ to ${}_R\mathit{Mod}$:

$$F_h(x_0 \to x_1 \to \cdots \to x_i) = F(x_0),$$

$$F_h(x_{i_0} \to \cdots \to x_{i_k} \subseteq x_0 \to \cdots \to x_{i_0}) = F(x_0 \to x_i) = F(x_{i_0}) \to F(x_0),$$

where the arrow $x_0 \to x_{i_0}$ is given by the appropriate composition of arrows in σ.

We form the module for the k-cochains ($k \geq 0$):

$$\mathcal{S}^k(N\mathbf{C}; F) = \prod_{\sigma} F_h(\sigma),$$

where the product ranges over all k-simplices σ. For $k < 0$, $\mathcal{S}^k(N\mathbf{C}; F) = 0$.

The differential $d^k : \mathcal{S}^{k-1}(N\mathbf{C}; F) \to \mathcal{S}^k(N\mathbf{C}; F)$ is defined for $k > 0$ by

$$d^k u|_\sigma = \sum_{j=0}^{k} (-1)^j F_h(\sigma_j \subseteq \sigma)(u|_{\sigma_j}),$$

where $\sigma = x_0 \to x_1 \to \cdots \to x_k$, $u \in \mathcal{S}^k(N\mathbf{C}; F)$, $s|_\sigma$ is the component of u indexed by σ, and

$$\begin{aligned}\sigma_j &= x_0 \to \cdots \to \hat{x}_j \to \cdots \to x_k \\ &= x_0 \to \cdots \to x_{j-1} \to x_{j+1} \to \cdots \to x_k.\end{aligned}$$

For $k \leq 0$, $d^k = 0$.

Lemma 4.1. *$\mathcal{S}^\bullet(N\mathbf{C}; F)$ is a cochain complex.*

Proof. We need to show that $d^2 u$ is zero at each coordinate. If σ is as above, then it is not hard to see that $d^k d^{k-1} u|_\sigma$ is a sum of terms of u in coordinates corresponding to $x_0 \to \cdots \to \hat{x}_i \to \cdots \to \hat{x}_j \to \cdots \to x_k$, where each of these terms appears exactly twice. One of these terms arises when x_i is deleted first, x_j second. Its sign is $(-1)^i(-1)^{j-1} = (-1)^{i+j-1}$. If x_j is deleted first, x_i second, the sign of the term is $(-1)^j(-1)^i = (-1)^{i+j}$. So the two terms cancel.

In the case where $i = 0, j = 1$, the two terms are not coordinates of u, but $F(x_1 \to x_2) \circ F(x_0 \to x_1)$ and $F(x_0 \to x_2)$ applied to a coordinate of u. The two are the same by the functoriality of F. □

Given a **Sh**-morphism $\gamma : (\mathbf{C}, F) \to (\mathbf{D}, G)$, there is an induced map

$$\gamma^\bullet : \mathcal{S}^\bullet(N\mathbf{C}; F) \to \mathcal{S}^\bullet(N\mathbf{D}; G)$$

defined by

$$\gamma^\bullet u|_\sigma = \gamma_{2x_0}(u|_{\gamma_1(\sigma)}).$$

Lemma 4.2. *The induced map $\gamma^\bullet$ is a well-defined chain map.*

Proof. We want to show that $d\gamma^\bullet = \gamma^\bullet d$. If $u \in \mathcal{S}^{k-1}(N\mathbf{C}; F)$ and $\sigma = x_0 \to \cdots \to x_k$ is a k-simplex in $N\mathbf{D}$, then

$$\begin{aligned}d\gamma^\bullet u|_\sigma &= G_h(\sigma_0 \subseteq \sigma)(\gamma^\bullet u|_{\sigma_0}) + \sum_{j=1}^{k} (-1)^j G_h(\sigma_j \subseteq \sigma)(\gamma^\bullet u|_{\sigma_j}) \\ &= G(x_0 \to x_1)(\gamma_{2x_1}(u|_{\gamma_1(\sigma_0)})) + \sum_{j=1}^{k} (-1)^j \gamma_{2x_0}(u|_{\gamma_1(\sigma_j)}) \\ &= \gamma_{2x_0}(F_h(\gamma_1(\sigma)_0 \subseteq \gamma_1(\sigma))(u|_{\gamma_1(\sigma_0)})) + \\ &\qquad + \gamma_{2x_0}\left(\sum_{j=1}^{k} (-1)^j F_h(\gamma_1(\sigma)_j \subseteq \gamma_1(\sigma))(u|_{\gamma_1(\sigma)_j})\right) \\ &= \gamma_{2x_0}(du|_{\gamma_1(\sigma)}) \\ &= \gamma^\bullet du|_\sigma,\end{aligned}$$

where the middle equals holds because of the naturality of γ_2. □

Remark 4.3. We can also define the *chain* modules $\mathcal{S}_\bullet(N\mathbf{C}; F)$ in a similar way:

$$\mathcal{S}_k(N\mathbf{C}; F) = \bigoplus_{\sigma} F_h(\sigma),$$

with the differential $d^k : \mathcal{S}_k(N\mathbf{C}; F) \to \mathcal{S}_{k-1}(N\mathbf{C}; F)$ given by

$$u|_\sigma \mapsto \sum_{j=0}^{k} (-1)^j F_h(\sigma_j \subseteq \sigma)(u|_{\sigma_j}).$$

Analogously to the above two lemmas, we have that $\mathcal{S}_\bullet(N\mathbf{C}; F)$ is a chain complex and a **Sh**-morphism γ induces a chain map $\gamma_\bullet$.

We have thus defined a covariant functor

$$\mathcal{S}^\bullet : \mathbf{Sh} \to \mathbf{Ch}_R,$$

from pairs of small categories and sheaves to chain complexes over R. In particular, for each $q \in \mathbb{Z}$ we have a covariant functor

$$\mathcal{S}^q : \mathbf{Sh} \to {}_R Mod.$$

Since homology is a functor from chain complexes to graded R-modules, we also have a covariant functor

$$H^\bullet \mathcal{S}^\bullet : \mathbf{Sh} \to \mathbf{Gr}\, {}_R Mod.$$

In particular, for each $q \in \mathbb{Z}$ we have a covariant functor

$$H^q \mathcal{S}^\bullet : \mathbf{Sh} \to {}_R Mod.$$

In the next section, we will make use of one particular fact about the chain complex $\mathcal{S}_\bullet(N\mathbf{C}; \Delta R)$ when $\mathbf{C}$ has an initial object.

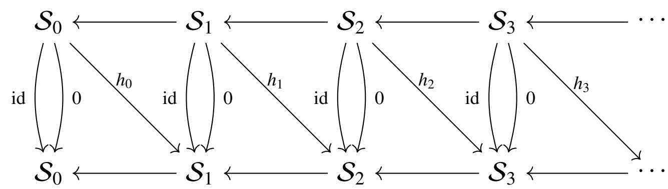
Proposition 4.4. *Suppose* $\mathbf{C}$ *is a small category with an initial object* x *and let* ΔR *be the constant sheaf on* $\mathbf{C}$*. Then*

$$H_n \mathcal{S}_\bullet(N\mathbf{C}; \Delta R) \cong \begin{cases} R, & n = 0, \\ 0, & \textit{otherwise.} \end{cases}$$

Proof. We construct a homotopy $h^i : \mathcal{S}_i(N\mathbf{C}; \Delta R) \to \mathcal{S}_{i+1}(N\mathbf{C}; \Delta R)$ between the chain maps $\mathrm{id}_\bullet$ and $0_\bullet$. If σ is an n-simplex $x_0 \to \cdots \to x_n$ of $N\mathbf{C}$, then

$$h^n : u|_\sigma \mapsto u|_{x \to \sigma},$$

where $x \to \sigma = x \to x_0 \to \cdots \to x_n$ and the arrow $x \to x_0$ is the unique arrow from the initial object x. We have the following (non-commutative) diagram



We need to show that id $= dh + hd$. Indeed, if $\sigma = x_0 \to \cdots \to x_n$, then

$$\begin{aligned}(dh + hd)(u|_\sigma) &= d(u|_{x\to\sigma}) + h(\sum_{j=0}^{n}(-1)^j u|_{\sigma_j}) \\ &= \sum_{j=1}^{n+1}(-1)^j u|_{x\to\sigma_{j-1}} + u|_\sigma + \sum_{j=0}^{n}(-1)^j u|_{x\to\sigma_j} \\ &= u|_\sigma.\end{aligned}$$

This means that for all $n > 0$ we have $H_n\mathcal{S}_\bullet(N\mathbf{C}; \Delta R) = 0$. The 0-th homology we can find directly. The differential d_1 sends elements of the form $u|_{x_0\to x_1}$ to $u|_{x_1} - u|_{x_0}$. Since there is a unique arrow $x \to x_0$ for each object x_0, the module $H_0\mathcal{S}_\bullet(N\mathbf{C}; \Delta R)$ is generated by one copy of R associated to $\sigma = x$. □

4.2 $\mathcal{S}^\bullet$ computes the higher limits of the sheaf

Definition 4.5. Let $\mathbf{C}$ be a small category and let $A \in {}_R\mathit{Mod}$. The *Yoneda embedding* $\mathrm{Yon}_A : \mathbf{C} \to \mathbf{Sh}(\mathbf{C})$ is defined as follows:

- if $x, y \in \mathrm{Obj}(\mathbf{C})$, then
$$(\mathrm{Yon}_A(x))(y) = \bigoplus_{\mathbf{C}(y,x)} A;$$
- if $x, y, z \in \mathrm{Obj}(\mathbf{C})$, $f \in \mathbf{C}(y, z)$, then
$$(\mathrm{Yon}_A(x))(f) = \bigoplus_{\mathbf{C}(z,x)} A \to \bigoplus_{\mathbf{C}(y,x)} A,$$
where the last map is defined by identity mapping an A-summand associated with $g \in \mathbf{C}(z, x)$ to an A-summand associated with $gf \in \mathbf{C}(y, x)$.

We want to show that higher limits and $\mathcal{S}^\bullet$ compute the same objects. The next few results will be used to prove the following proposition by the end of the section.

Proposition 4.6. *Let* $(\mathbf{C}, F) \in \mathbf{Sh}$. *Then*

$$\varprojlim\nolimits^i_{\mathbf{C}} F \cong H^i\mathcal{S}^\bullet(N\mathbf{C}, F).$$

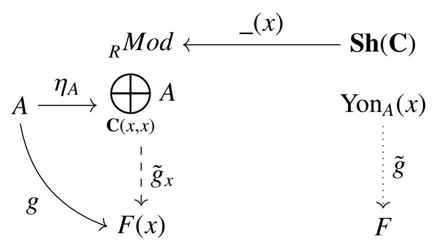
Following Section 3.2, we construct a projective resolution $P_\bullet$ of ΔR. Let $\mathbf{C}$ be a small category. Define

$$P_n := \bigoplus_{x_0 \to \cdots \to x_n \in N\mathbf{C}} \mathrm{Yon}_R(x_0)$$

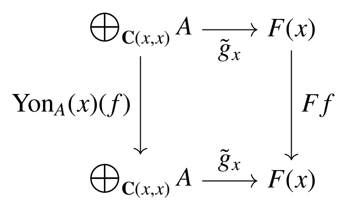
for $n \geq 0$. Since the coproduct (direct sum in this case) of projective objects is projective, it is enough to show that $\mathrm{Yon}_R(x)$ is projective for any $x \in \mathbf{C}$. In the following, $\mathrm{Yon}\_(x)$ is the functor from R-modules to sheaves on $\mathbf{C}$ that takes a module A to the sheaf $\mathrm{Yon}_A(x)$; the functor $\_(x) : \mathbf{Sh}(\mathbf{C}) \to {}_R Mod$ is the 'evaluation at x' functor that sends a sheaf F to the module $F(x)$.

Lemma 4.7. *Let x be a fixed object of $\mathbf{C}$. Then $(\mathrm{Yon}\_(x), \_(x))$ is an adjoint pair.* [1]

Proof. We again use Theorem 2.44. Assume $A \in {}_R Mod$ and set $F_A = \mathrm{Yon}_A(x)$. Then $\_(x)(\mathrm{Yon}_A(x)) = \bigoplus_{\mathbf{C}(x,x)} A$, so define $\eta_A : A \to \bigoplus_{\mathbf{C}(x,x)} A$ as the identity homomorphism onto the summand associated to $\mathrm{id} \in \mathbf{C}(x,x)$. Now let F be a sheaf on $\mathbf{C}$ and $g : A \to F(x)$ be a homomorphism. We have the following diagram from the definition of the universal arrow.



For the left triangle to commute, we need $\tilde{g}_x$ to send the summand associated to $\mathrm{id} \in \mathbf{C}(x,x)$ to $F(x)$ by g. But if $\tilde{g}$ is to be a sheaf morphism, then $\tilde{g}_x$ needs to map a summand associated to $f \in \mathbf{C}(x,x)$ via $F(f)g$. To see this, suppose $\tilde{g}$ is a sheaf morphism, i.e. a natural transformation, so the following diagram commutes.



The summand associated to the identity is sent to $F(f)g$ on the left. Since it is mapped onto the summand associated to f by $\mathrm{Yon}_A(x)(f)$, in order for the diagram to commute, we need that summand sent to $F(f)g$. We have thus only one choice for $\tilde{g}_x$.

[1] This is a version of the so-called 'Yoneda Lemma'. The story goes that Saunders Mac Lane met with a young Nobuo Yoneda in Paris while interviewing category theorists for a book. The contents of their conversation later appeared in Mac Lane's writings as a lemma dedicated to Yoneda.

Considering any $f \in \mathbf{C}(x, y)$ and a diagram similar to that above, there is always a unique choice for building $\tilde{g}$. $\square$

Now, Proposition 3.9 means that $\_(x)$ is an exact functor (since exactness in $\mathbf{Sh}(\mathbf{C})$ is checked 'pointwise'). Then Proposition 2.46, together with the fact that R is projective in ${}_R Mod$, ensures that $\mathrm{Yon}_R(x)$ is a projective object of $\mathbf{Sh}(\mathbf{C})$.

Next, we define the maps $\delta_n : P_n \to P_{n-1}$. If $f \in \mathbf{C}(x, y)$, then there is an induced map $\mathrm{Yon}_R^f : \mathrm{Yon}_R(x) \to \mathrm{Yon}_R(y)$ defined at $z \in \mathbf{C}$ by

$$\bigoplus_{\mathbf{C}(z,x)} R \to \bigoplus_{\mathbf{C}(z,y)} R,$$

where an R-summand associated to $g \in \mathbf{C}(z, x)$ is identically mapped to an R-summand associated to $fg \in \mathbf{C}(z, y)$. Then for $\sigma = x_0 \to \cdots \to x_n \in N\mathbf{C}$ we have

$$\delta_n : \mathrm{Yon}_R(x_0)|_\sigma \mapsto \sum_{j=1}^{n} (-1)^j \,\mathrm{Yon}_R(x_0)|_{\sigma_j} + \mathrm{Yon}_R^{x_0 \to x_1}(\mathrm{Yon}_R(x_0))|_{\sigma_0}.$$

Lemma 4.8. *The object $P_\bullet$ is a chain complex, i.e. $P_\bullet \in \mathbf{Ch}(\mathbf{Sh}(\mathbf{C}))$. Moreover,*

$$H_n P_\bullet \cong \begin{cases} \Delta R, & n = 0, \\ 0, & \textit{otherwise.} \end{cases}$$

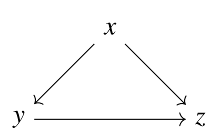
Proof. The δ_n maps define the usual simplicial differential, so it is clear that $P_\bullet$ is a chain complex.

Now fix $x \in \mathbf{C}$ and consider

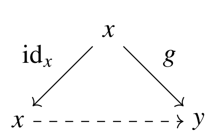
$$P_n(x) = \left(\bigoplus_{x_0 \to \cdots \to x_n} \mathrm{Yon}_R(x_0) \right)(x) = \bigoplus_{x_0 \to \cdots \to x_n} \bigoplus_{\mathbf{C}(x, x_0)} R = \bigoplus_{x \to x_0 \to \cdots \to x_n} R.$$

We define the *under category* $x/\mathbf{C}$ as follows

- the objects of $x/\mathbf{C}$ consist of morphisms $x \to y$ in $\mathbf{C}$,
- the morphisms of $x/\mathbf{C}$ consist of commuting triangles in $\mathbf{C}$:



Note that this category has an initial object $\mathrm{id}_x \in \mathbf{C}(x, x)$. Explicitly, if $g : x \to y$ is an object of $x/\mathbf{C}$, then the dashed arrow in the diagram



can only be $g \in \mathbf{C}(x, y)$ if the triangle is to commute. Therefore, there is a unique morphism in $x/\mathbf{C}$ from id_x to any object of $x/\mathbf{C}$.

Looking back at the expression for $P_n(x)$ above, we can rephrase the direct sum as

$$P_n(x) = \bigoplus_{z_0 \to \cdots \to z_n} R,$$

where $z_i \in \mathrm{Obj}(x/\mathbf{C})$. Thus, we have

$$P_\bullet(x) = \mathcal{S}_\bullet(Nx/\mathbf{C}; \Delta R).$$

Now Proposition 4.4 implies that

$$H_n \mathcal{S}_\bullet(Nx/\mathbf{C}; F) \cong \begin{cases} R, & n = 0, \\ 0, & \text{otherwise}, \end{cases}$$

and so 'gluing up' $P_\bullet$ (and using Proposition 3.9 again) gives the required result. □

We now have our projective resolution $P_\bullet$ of ΔR and thus $\mathrm{Hom}_{\mathbf{Sh(C)}}(P_\bullet, F)$ computes the higher limits $\varprojlim^i_{\mathbf{C}} F$ of a sheaf F over $\mathbf{C}$.

Proof of Proposition 4.6. Let P_n be the sheaves on $\mathbf{C}$ constructed above. Since R is projective in ${}_R Mod$, Lemma 4.7 and Proposition 2.46 imply that P_n is projective (as the direct sum of projective objects). The chain complex $P_\bullet$ (with differential δ_n defined earlier in this section) forms a projective resolution of ΔR (due to Lemma 4.8), so by Proposition 3.10

$$\varprojlim\nolimits^i_{\mathbf{C}} F \cong H^i \,\mathrm{Hom}_{\mathbf{Sh(C)}}(P_\bullet, F).$$

But (again by Lemma 4.7) we have a natural isomorphism

$$\mathrm{Hom}_{\mathbf{Sh(C)}}(\mathrm{Yon}_R(x), F) \cong \mathrm{Hom}_R(R, F(x)) \cong F(x)$$

and so

$$\mathrm{Hom}_{\mathbf{Sh(C)}}(P_n, F) = \mathrm{Hom}_{\mathbf{Sh(C)}}\left(\bigoplus_{x_0 \to \cdots \to x_n} \mathrm{Yon}_R(x_0), F \right) \cong \prod_{x_0 \to \cdots \to x_n} F(x_0) = \mathcal{S}^n(N\mathbf{C}; F).$$

Therefore

$$H^i(\mathbf{C}; F) = \varprojlim\nolimits^i_{\mathbf{C}} F \cong H^i \mathcal{S}^\bullet(N\mathbf{C}; F).$$ □

4.3 Computing Khovanov homology with $\mathcal{S}^\bullet$

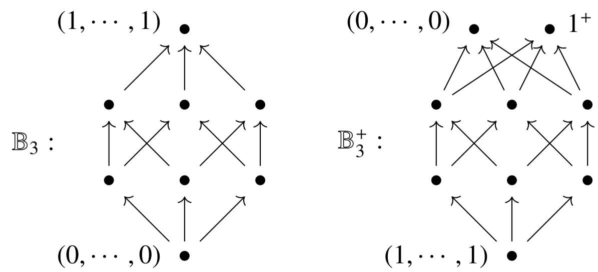
The method described in Section 3.2 and employed in the previous section can also connect the higher limits of a sheaf to other homology theories. Most relevant to our discussion is the reinterpretation of unnormalised Khovanov homology of a link as the derived limit over a modified Boolean lattice. The following exposition is based on [ET15, §1].

We'll need to construct a contravariant Khovanov sheaf on a modified poset. Since the differentials given in Section 1.4 increase $\sum \mu$, we take $\mathbb{B}_n^{op}$ as the starting point of our modification.

Definition 4.9. A *Boolean$^+$ lattice* $\mathbb{B}_n^+$ of rank n is the poset with objects

$$\mathrm{Obj}\,\mathbb{B}_n^+ = \mathrm{Obj}\,\mathbb{B}_n \cup \{1^+\},$$

such that if $\mu_1, \mu_2 \in \mathbb{B}_n$, then $\mu_1 \leq \mu_2$ in $\mathbb{B}_n^+$ if and only if $\mu_1 \geq \mu_2$ in $\mathbb{B}_n$; and $\mu \leq 1^+$ in $\mathbb{B}_n^+$ for all $\mu \in \mathbb{B}_n \backslash \{(0, \cdots, 0)\}$, where $(0, \cdots, 0)$ is the unique object of $\mathbb{B}_n$ with sum 0. For ease of reference, we adopt the convention $\sum 1^+ = 0$.



The Boolean$^+$ posets are also the cell posets of certain CW complexes. To see this, take the $(n-1)$-simplex Δ^{n-1}. Let X be the suspension $S\Delta^{n-1}$, which is homeomorphic to the closed n-dimensional ball B^n. Let the two suspension points (1 and 1^+) in X be the two 0-cells. Each $(k-1)$-cell of Δ^{n-1} determines a k-cell suspension of that cell in X. The simplex Δ^{n-1} has $\binom{n}{k}$-many $(k-1)$-cells, so for $1 \leq k \leq n$, X has $\binom{n}{k}$-many k-cells. We can define a partial order on the cells of X by $x \leq y$ if and only if $\overline{x} \supseteq y$, where $\overline{x}$ is the (CW-)closure of the cell x. This is the cell poset of the CW complex X. It is clear from the description above that this poset is $\mathbb{B}_n^+$; Figure 4.1 illustrates the construction for $n = 3$.

Recall that in the context of Khovanov homology we made the choice to have $R = \mathbb{Z}$. To match the definitions given there, for the rest of this section we will consider our sheaves as functors to ${}_{\mathbb{Z}}Mod$, i.e. the category **Ab** of abelian groups.

We now construct another projective resolution, this time of $\Delta\mathbb{Z}$ over $\mathbb{B}_n^+$. For $m \in \mathbb{N}$, define

$$P_m := \bigoplus_{\sum \mu = m} \mathrm{Yon}_{\mathbb{Z}}(\mu).$$

The P_m's are projective for the same reason the P_n's in the previous section were: sums of projectives are projective and $\mathrm{Yon}\_(\mu)$ is left-adjoint to the exact functor $\_(\mu)$.

Using Lemma 4.7 again, if $F \in \mathbf{Sh}(\mathbb{B}_n^+)$ we have

$$\mathrm{Hom}_{\mathbf{Sh}(\mathbb{B}_n^+)}(P_m, F) = \mathrm{Hom}_{\mathbf{Sh}(\mathbb{B}_n^+)}\left(\bigoplus_{\sum \mu = m} \mathrm{Yon}_{\mathbb{Z}}(\mu), F\right) \cong \bigoplus_{\sum \mu = m} F(\mu).$$

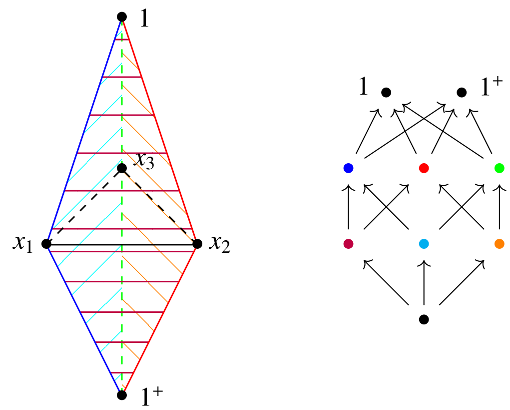


Fig. 4.1: The suspension $S\Delta^2$ with each cell coloured and indicated in the cell poset.

Now recall the signage given by the ϵ_μ^ν symbols, defined towards the end of Section 1.4. We can extend that signage to the whole of $\mathbb{B}_n^+$ by setting $\epsilon_{1^+}^\mu = 1$ for all $\mu < 1^+$. Assembling the resolution, define $\delta_{m,\mu} : P_m(\mu) \to P_{m-1}(\mu)$ by

$$\delta_{m,\mu}(\nu) = \sum_{\lambda > \nu} (-1)^{\epsilon_\lambda^\nu} \lambda,$$

where $\sum \nu = m$.

The key property of ϵ_μ^ν makes the squares in $\mathbb{B}_n^+$ anti-commute, so the above map δ gives a chain complex of sheaves

$$\cdots \to P_m \xrightarrow{\delta_m} P_{m-1} \xrightarrow{\delta_{m-1}} P_{m-2} \to \cdots .$$

By Proposition 3.9, the complex $P_\bullet$ is exact if and only if $P_\bullet(\mu)$ is exact for each $\mu \in \mathbb{B}_n^+$. If we look at the cell poset interpretation of $\mathbb{B}_n^+$ again, $\text{Yon}_\mathbb{Z}(\mu)$ has one copy of $\mathbb{Z}$ at μ and at each $\nu < \mu$. This means that if $P_m(\mu) = \mathbb{Z}^k$, then there are k many objects with sum m that are $\geq \mu$; equivalently, there are k many m-cells in the closure of the cell μ. But then $P_\bullet(\mu)$ is just the cell decomposition of a single cell, connected with boundary maps. Therefore

$$H_m P_\bullet(\mu) = \begin{cases} \mathbb{Z}, & m = 0, \\ 0, & \text{otherwise}, \end{cases}$$

and so $P_\bullet$ is a projective resolution of $\Delta\mathbb{Z}$.

For a given link diagram L with n crossings, we define the Khovanov sheaf $F_{Kh} : \mathbb{B}_n^+ \to \mathbf{Ab}$ as follows (recall the terminology from Section 1.4).

- $F_{Kh}(1^+) = 0$ and $F_{Kh}(\mu \leq 1^+) = 0$ for all μ;
- $F_{Kh}(\mu) = \overline{Kh}(L \mid S_\mu)$ for $\mu \neq 1^+$;

- $F_{Kh}(\mu \prec \nu) = d_\nu^\mu : F_{Kh}(\nu) \to F_{Kh}(\mu)$ for (ν, μ) adjacent and $\nu \neq 1^+$.

It remains to show that

$$\mathrm{Hom}_{\mathbf{Sh}(\mathbb{B}_n^+)}(P_\bullet, F_{Kh}) \cong C^\bullet(L).$$

We have already established that there is an isomorphism

$$f : \mathrm{Hom}_{\mathbf{Sh}(\mathbb{B}_n^+)}(P_m, F_{Kh}) \to \bigoplus_{\sum \mu = m} F_{Kh}(\mu),$$

so we only need to show that the following diagram commutes.

$$\begin{array}{ccc}
\mathrm{Hom}_{\mathbf{Sh}(\mathbb{B}_n^+)}(P_{m+1}, F_{Kh}) & \xleftarrow{\ \delta\ } & \mathrm{Hom}_{\mathbf{Sh}(\mathbb{B}_n^+)}(P_m, F_{Kh}) \\
\Big\downarrow f & & \Big\downarrow f \\
C^{m+1}(L) & \xleftarrow{\ d\ } & C^m(L)
\end{array}$$

Let $\alpha : P_m \to F_{Kh}$. This natural transformation is determined by what it does to the modules associated to objects with sum m. In particular,

$$f\alpha = \sum_{\sum \mu = m} \alpha_\mu(\mu).$$

We thus have

$$d(f\alpha) = \sum_{\sum \mu = m} \sum_{\nu \prec \mu} (-1)^{\epsilon_\mu^\nu} F_{Kh}(\nu \prec \mu)(\alpha_\mu(\mu)).$$

Similarly, $\delta(\alpha) = \alpha\delta$ is determined by what it does to the modules associated to objects with sum $m+1$. Using the naturality of α, we have that, for ν with $\sum \nu = m+1$,

$$\alpha\delta(\nu) = \alpha_\nu\left(\sum_{\nu \prec \mu} (-1)^{\epsilon_\mu^\nu} \mu\right) = \sum_{\nu \prec \mu} (-1)^{\epsilon_\mu^\nu} F_{Kh}(\nu \prec \mu)(\alpha_\mu(\mu)).$$

Therefore,

$$f(\delta(\alpha)) = \sum_{\sum \nu = m+1} \sum_{\nu \prec \mu} (-1)^{\epsilon_\mu^\nu} F_{Kh}(\nu \prec \mu)(\alpha_\mu(\mu)).$$

Note that the addition of 1^+ does not affect the map f. It does, however, affect the sheaf cohomology; as we have seen, a poset with a unique maximum has zero cohomology in all degrees $\neq 0$. The result of the above discussion is the following theorem.

Theorem 4.10. *Let L be a link diagram with n crossings and let $\mathbb{B}_n^+$ and F_{Kh} be as above. Then*

$$\overline{Kh}^\bullet(L) \cong \varprojlim\nolimits^\bullet_{\mathbb{B}_n^+} F_{Kh}.$$

5

Spectral sequences

5.1 Definition of spectral sequence

In the next chapters we will be using a spectral sequence to recover the cohomology of a sheaf. Here we set out the standard definitions and basic facts about cohomological spectral sequences. This exposition can be found in [Wei94] and [ML95].

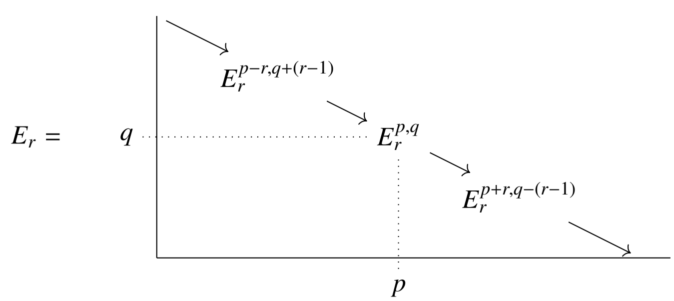
Definition 5.1. A (cohomological) *spectral sequence* consists of R-modules $E_r^{p,q}$, collected in pages $E_a, E_{a+1}, \ldots$ (usually $a = 0, 1$, or 2), and maps

$$E_r^{p,q} \to E_r^{p+r,q-(r-1)},$$

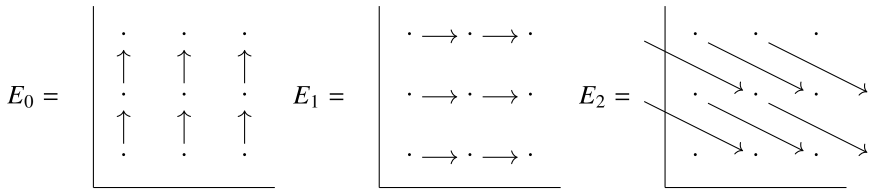
such that

$$\cdots \to E_r^{p-r,q+(r-1)} \to E_r^{p,q} \to E_r^{p+r,q-(r-1)} \to \cdots$$

is a chain complex, for each $p, q \in \mathbb{Z}$, $r \in \{a, a+1, \ldots\}$. Furthermore, $E_{r+1}^{p,q}$ is the homology of the above complex at the p, q position.



In particular,



What all of these maps have in common is they increase the *total degree* by 1. If $n = p + q$ is the total degree, then on the E_r page we have a differential

$$E_r^{p,q} \to E_r^{p+r,q-(r-1)}$$

where the total degree of $E_r^{p+r,q-(r-1)}$ is

$$p + r + q - (r - 1) = p + q + 1 = n + 1.$$

Definition 5.2. If for all $p, q \in \mathbb{Z}$ there exists $r_0 = r_0(p, q)$ such that

$$E_r^{p,q} = E_{r_0}^{p,q},$$

for all $r \geq r_0$, then we say *the E_∞ page exists* and we set

$$E_\infty^{p,q} = E_{r_0(p,q)}^{p,q}.$$

Definition 5.3. A page in a spectral sequence is *bounded* if for all $n \in \mathbb{Z}$ there are only finitely many non-zero entries with total degree n. A spectral sequence is *bounded* if it has a bounded page.

Proposition 5.4. *Bounded spectral sequences have E_∞ pages.*

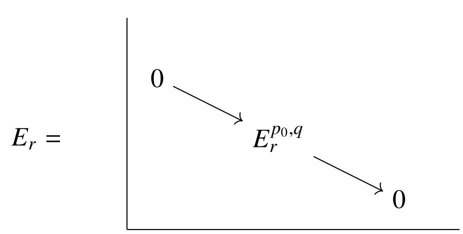
Proof. As $E_{r+1}^{p,q}$ is a subquotient of $E_r^{p,q}$ for all p, q, r, if $E_r^{p,q} = 0$, we have $E_{r+1}^{p,q} = 0$. Therefore if the E_r page is bounded, then so are all pages E_s for $s \geq r$.

Now pick $p, q \in \mathbb{Z}$ and suppose E_r is a bounded page. Since E_r is bounded, there is an $r_0 = r_0(p, q)$ such that for all $s \geq r_0$ both $E_r^{p-s,q+(s-1)}$ and $E_r^{p+s,q-(s-1)}$ are zero. But they are also zero on the E_s page and on that page there are maps

$$\cdots \to \underset{\underset{0}{\parallel}}{E_s^{p-s,q+(s-1)}} \to E_s^{p,q} \to \underset{\underset{0}{\parallel}}{E_s^{p+s,q-(s-1)}} \to \cdots,$$

therefore $E_{s+1}^{p,q} = E_s^{p,q}$ and so $E_{r_0}^{p,q} = E_\infty^{p,q}$. □

Example 5.5 (Collapsing). Suppose the E_r page ($r > 1$) has only one non-zero row, say the p_0-th row:



Then each $E_{r+1}^{p_0,q}$ is the homology of the complex

$$\cdots \to 0 \to E_r^{p_0,q} \to 0 \to \cdots,$$

so $E_{r+1}^{p_0,q} \cong E_r^{p_0,q}$ for each q. But then the $(r+1)$-th page also has only one non-zero row, hence the E_r page coincides with the E_∞ page.

5.2 Convergence and mapping

Definition 5.6. A *filtration* $(\mathcal{F}, C^\bullet)$ (or just $\mathcal{F}$) of a chain complex $C^\bullet$ is a collection $\{\mathcal{F}^p C^\bullet\}_{p\in\mathbb{Z}}$ of complexes with

$$\cdots \subseteq \mathcal{F}^{p+1}C^\bullet \subseteq \mathcal{F}^p C^\bullet \subseteq \cdots \subseteq C^\bullet,$$

i.e. each $\mathcal{F}^p C^\bullet$ is a subcomplex of $C^\bullet$ with the given inclusions.

Definition 5.7. Let $(\mathcal{F}_1, C_1^\bullet)$ and $(\mathcal{F}_2, C_2^\bullet)$ be filtrations. A *morphism of filtrations* $\varphi : (\mathcal{F}_1, C_1^\bullet) \to (\mathcal{F}_2, C_2^\bullet)$ is a chain map $\varphi^\bullet : C_1^\bullet \to C_2^\bullet$ with $\varphi^\bullet(\mathcal{F}_1^p C_1^\bullet) \subseteq \mathcal{F}_2^p C_2^\bullet$.

Definition 5.8. Let $\mathcal{F}$ be a filtration of a chain complex $C^\bullet$.

- We say $\mathcal{F}$ is *bounded above* if for any $n \in \mathbb{Z}$ there are integers t_n such that $\mathcal{F}^{t_n}C^n = C^n$.
- We say $\mathcal{F}$ is *bounded below* if for any $n \in \mathbb{Z}$ there are integers s_n such that $\mathcal{F}^{s_n}C^n = 0$.
- We say $\mathcal{F}$ is *bounded* if it is both bounded above and below, i.e.

$$0 = \mathcal{F}^{s_n}C^n \subseteq \mathcal{F}^{s_n-1}C^n \subseteq \cdots \subseteq \mathcal{F}^{t_n}C^n = C^n.$$

- We say $\mathcal{F}$ is *convergent above* if

$$\bigcup_p \mathcal{F}^p C^\bullet = C^\bullet.$$

Now suppose $H^\bullet = \{H^i\}_{i\in\mathbb{Z}}$ are R-modules, usually the cohomology of some object. We say that $\mathcal{F}$ is a filtration of $H^\bullet$ if there are R-modules $\{\mathcal{F}^p H^n\}_{p\in\mathbb{Z}}$ for each n such that

$$\cdots \subseteq \mathcal{F}^{p+1}H^n \subseteq \mathcal{F}^p H^n \subseteq \mathcal{F}^{p-1}H^n \subseteq \cdots \subseteq H^n.$$

Equivalently, we can extend the definition of a filtration to $H^\bullet$ by (artificially) defining 'differentials' $d^i_{H^\bullet} = 0$.

Definition 5.9. A spectral sequence E *converges* to $H^\bullet$, written $E \Rightarrow H^\bullet$, if and only if

a) the spectral sequence has an E_∞ page, and
b) there is a bounded filtration $\mathcal{F}$ of $H^\bullet$ with

$$E_\infty^{p,q} = \frac{\mathcal{F}^p H^{p+q}}{\mathcal{F}^{p+1}H^{p+q}}.$$

Definition 5.10. A *morphism* $f : E \to E'$ *of spectral sequences* is a collection of maps $f_r^{p,q} : E_r^{p,q} \to E_r'^{p,q}$ for $r \in \{r_0, r_0 + 1, \cdots\}$ with $r_0 \geq a$, such that $d_r f_r = f_r d_r'$, and where $f_{r+1}^{p,q} : E_{r+1}^{p,q} \to E_{r+1}'^{p,q}$ is the map induced by $f_r^{p,q} : E_r^{p,q} \to E_r'^{p,q}$ on the homologies of the concerned modules.

Definition 5.11. A spectral sequence E is *bounded below* if for each degree n there is an integer $s = s(n)$ such that $E_0^{p,q} = 0$ when $p < s$ and $p + q = n$.

Lemma 5.12 (Mapping Lemma). *Let $f : E \to E'$ be a morphism of spectral sequences and suppose for some r that $f_r^{p,q} : E_r^{p,q} \to E_r'^{p,q}$ is an isomorphism for each p and q. Then $f_s^{p,q} : E_s^{p,q} \to E_s'^{p,q}$ is also an isomorphism for each p and q when $s \geq r$ (by the Five Lemma 2.35). If E and E' are bounded below, then $f_\infty^{p,q} : E_\infty^{p,q} \to E_\infty'^{p,q}$ is also an isomorphism.*

5.3 Construction of spectral sequences

Abstractly defined, the differentials at each page of a spectral sequence do not necessarily have anything to do with each other. In practice, however, all differentials of a spectral sequence are induced by morphisms between other objects. We will be making use of two such constructions – the spectral sequence of a filtration and the spectral sequence of a bicomplex. Descriptions of how the first few pages are constructed will follow the relevant theorems; we will fall short of giving a detailed exposition of the (opaque) general definition of all differentials.

Theorem 5.13. *A filtration $\mathcal{F}$ of a complex $C^\bullet$ naturally determines a spectral sequence starting with $E_0^{p,q} = \mathcal{F}^p C^{p+q} / \mathcal{F}^{p+1} C^{p+q}$. If $\mathcal{F}$ is a bounded filtration, then*

$$E \Rightarrow H^\bullet(C^\bullet).$$

By construction of the filtration, the differentials on the E_0 page are

$$d_0^{p,q} : \frac{\mathcal{F}^p C^{p+q}}{\mathcal{F}^{p+1} C^{p+q}} \to \frac{\mathcal{F}^p C^{p+q+1}}{\mathcal{F}^{p+1} C^{p+q+1}},$$

induced by the differential of the complex $C^\bullet$. The E_1 page thus has modules

$$E_1^{p,q} = H^q\left(\frac{\mathcal{F}^p C^{p+\bullet}}{\mathcal{F}^{p+1} C^{p+\bullet}}\right).$$

Lemma 5.14 (Mapping Lemma for filtrations). *Let $\mathcal{F}_1, \mathcal{F}_2$ be convergent below and bounded above filtrations of $C_1^\bullet, C_2^\bullet$, respectively, and let E, E' be the spectral sequences determined by $\mathcal{F}_1$ and $\mathcal{F}_2$, respectively. If a morphism of filtrations $\varphi : (\mathcal{F}_1, C_1^\bullet) \to (\mathcal{F}_2, C_2^\bullet)$ is such that for some r the induced map*

$$\varphi_r^{p,q} : E_r^{p,q} \to E'^{p,q}_r$$

is an isomorphism for each $p, q \in \mathbb{Z}$, then φ induces an isomorphism

$$\varphi^\bullet : H^\bullet(C_1^\bullet) \to H^\bullet(C_2^\bullet).$$

We can also start with a bicomplex, giving us explicitly two full pages of sensible differentials.

Definition 5.15. A *bicomplex* $\mathcal{K}^{\bullet,\bullet}$ is a family $\{\mathcal{K}^{p,q}\}_{p,q\in\mathbb{Z}}$ of R-modules, together with maps

$$d^h : \mathcal{K}^{p,q} \to \mathcal{K}^{p+1,q} \qquad d^v : \mathcal{K}^{p,q} \to \mathcal{K}^{p,q+1}$$

such that $d^h d^h = d^v d^v = d^h d^v + d^v d^h = 0$. The *total complex* $T^\bullet_{\mathcal{K}}$ of a bicomplex $\mathcal{K}^{\bullet,\bullet}$ is

$$T^n_{\mathcal{K}} = \prod_{p+q=n} \mathcal{K}^{p,q}.$$

Note that this is a (co)chain complex due to the restrictions on d^h and d^v.

Note that we can 'transpose' a bicomplex $\mathcal{K}^{\bullet,\bullet}$ to get another bicomplex $\mathcal{K}_t^{\bullet,\bullet}$ with $\mathcal{K}_t^{p,q} = \mathcal{K}^{q,p}$ and $d^h_{\mathcal{K}_t} = d^v_{\mathcal{K}}, d^v_{\mathcal{K}_t} = d^h_{\mathcal{K}}$.

Definition 5.16. Let $\mathcal{K}^{\bullet,\bullet}$ and $\mathcal{L}^{\bullet,\bullet}$ be bicomplexes. A *morphism of bicomplexes* $\psi : \mathcal{K}^{\bullet,\bullet} \to \mathcal{L}^{\bullet,\bullet}$ consists of homomorphisms $\psi^{p,q} : \mathcal{K}^{p,q} \to \mathcal{L}^{p,q}$ for each $p, q \in \mathbb{Z}$, such that $\psi^{p,\bullet}$ and $\psi^{\bullet,q}$ are chain maps.

Theorem 5.17. *A bicomplex $\mathcal{K}^{\bullet,\bullet}$ naturally determines a spectral sequence starting with $E_0^{p,q} = \mathcal{K}^{p,q}$. If $\mathcal{K}^{p,q} = 0$ when $p < 0$ or $q < 0$, then*

$$E \Rightarrow H^\bullet(T^\bullet_{\mathcal{K}}).$$

The standard approach for constructing the spectral sequence of a bicomplex is to filter the total complex in one of two ways (by rows or by columns) and then follow Theorem 5.13. One can also be slightly more explicit. Setting $E_0^{p,q} = \mathcal{K}^{p,q}$, the E_0 page differentials are the vertical differentials d^v of $\mathcal{K}^{\bullet,\bullet}$. The E_1 page differentials are induced by d^h on the modules $H^q\mathcal{K}^{p,\bullet}$. Now if we assume we have defined differentials $d_r^{p,q}$ and $d_{r+1}^{p,q}$ on the E_r and E_{r+1} pages, respectively, then

$$d_{r+2}^{p,q} : E_{r+2}^{p,q} \to E_{r+2}^{p+r+2,q-r-1}$$

can be defined by chasing the following diagram:

$$\begin{array}{ccccc}
E_r^{p,q} & \xrightarrow{d_{r+1}} & E_r^{p+r+1,q-r} & & \\
\uparrow{\scriptstyle d_r} & & \uparrow{\scriptstyle d_r} & & \\
E_r^{p-r,q+r-1} & \xrightarrow{d_{r+1}} & E_r^{p+1,q-1} & \xrightarrow{d_{r+1}} & E_r^{p+r+2,q-r+-1}
\end{array}$$

Lemma 5.18 (Mapping Lemma for bicomplexes). *Let $\psi : \mathcal{K}^{\bullet,\bullet} \to \mathcal{L}^{\bullet,\bullet}$ be a morphism of bicomplexes and let E, E' be the spectral sequences determined by $\mathcal{K}^{\bullet,\bullet}$ and $\mathcal{L}^{\bullet,\bullet}$, respectively. If for some r the induced map*

$$\psi_r^{p,q} : E_r^{p,q} \to {E'}_r^{p,q}$$

is an isomorphism for each $p, q \in \mathbb{Z}$, then ψ induces an isomorphism

$$\psi^\bullet : H^\bullet(T^\bullet_{\mathcal{K}}) \to H^\bullet(T^\bullet_{\mathcal{L}}).$$

6

Bundles of sheaves

One of the key results of [ET12] is that we can break down the calculation of $H^\bullet(\mathbf{C}; F)$ for some large finite $\mathbf{C}$ into more manageable chunks by splitting $\mathbf{C}$. The way to do this is via a *bundle of sheaves* – some of the morphisms 'stay in the fiber' and some become parts of the maps of sheaves connecting the bundle. We can 'glue-up' these fibers to recover the large $\mathbf{C}$ we started with. It turns out that we can calculate the cohomology of each fiber of the bundle separately and then, via a spectral sequence, recover the cohomology of $(\mathbf{C}; F)$.

This chapter lays out the final prerequisites for completing the above procedure.

6.1 Bundle of sheaves

Definition 6.1. Let $\mathbf{B}$ be a small category. A *bundle of sheaves* with base $\mathbf{B}$ is a contravariant functor $\xi : \mathbf{B} \to \mathbf{Sh}$.

Example 6.2. a) A *constant bundle* $\xi = \mathbf{B} \times (\mathbf{C}; F)$ is a bundle of sheaves with $\xi(x) = (\mathbf{C}; F)$ for all $x \in \mathbf{B}$ and $\xi(x \to y) = \mathrm{id}_{(\mathbf{C};F)}$ for all arrows $x \to y$.

b) A bundle of coloured posets with base $\mathbf{B}$ in the language of [ET12] is a covariant functor $\zeta : \mathbf{B} \to \mathbf{CP}_R$ where $\mathbf{B}$ is a poset with a unique maximum and $\mathbf{CP}_R$ is the category of coloured posets. Such a bundle of coloured posets gives rise to a bundle of sheaves $\xi : \mathbf{B}^{op} \to \mathbf{Sh}$, where if $\zeta(x) = (\mathbf{P}, F)$, then $\xi(x) = (\mathbf{P}^{op}; F)$.

c) If $\mathbf{P}$ and $\mathbf{Q}$ are posets, then an object $F \in \mathbf{Sh}(\mathbf{P} \times \mathbf{Q})$ determines a bundle of sheaves $\xi : \mathbf{P} \to \mathbf{Sh}$. For any $x \in \mathbf{P}$, denote by F_x the functor from the full subcategory $\{x\} \times \mathbf{Q}$ of $\mathbf{P} \times \mathbf{Q}$ that agrees with F. Then $\xi(x) = (\mathbf{Q}, F_x)$ for all $x \in \mathbf{P}$ and $\xi(x \to y) = (\mathrm{id}_{\mathbf{Q}}, F_{x\to y})$, where

$$F_{x\to y}|_z : F_y(y, z) \to F_x(x, z)$$

agrees with F.

d) We can also model a group action on a sheaf $(\mathbf{C}; F)$. Let the category $\mathbf{C}_G$ have one object $\bullet$ and let the morphisms of $\mathbf{C}_G$ be given by G, with composition given by the group operation. Then a bundle of sheaves $\xi : \mathbf{C}_G \to \mathbf{Sh}$ with $\xi(\bullet) = (\mathbf{C}; F)$ describes the action of G on $(\mathbf{C}; F)$.

For clarity, if ξ is a bundle of sheaves with base $\mathbf{B}$ and $x \in \mathbf{B}$, then we will use the notation $\mathbf{E}_x$ for the small category that is the first coordinate of $\xi(x)$ and F_x for the second coordinate of $\xi(x)$. Also, if $y \in \mathbf{E}_x$, then $\pi(y) = x$, i.e. π indicates which fiber y comes from. Finally, we write $\xi_1(x_1 \to x_2) : \mathbf{E}_{x_1} \to \mathbf{E}_{x_2}$ for the first coordinate of the $\mathbf{Sh}$-morphism $\xi(x_1 \to x_2) : (\mathbf{E}_{x_2}; F_{x_2}) \to (\mathbf{E}_{x_1}; F_{x_1})$ instead of $\xi(x_1 \to x_2)_1$, similarly $\xi_2(x_1 \to x_2) : F_{x_2}\xi_1(x_1 \to x_2) \to F_{x_1}$ instead of $\xi(x_1 \to x_2)_2$.

Definition 6.3. Let $\mathbf{B}$ be a small category and ξ a bundle of sheaves with base $\mathbf{B}$. The associated *total sheaf* $(\mathbf{E}_\xi; F_\xi)$ consists of a small category $\mathbf{E}_\xi$ and a sheaf $F_\xi : \mathbf{E}_\xi \to {}_R Mod$, defined as follows (also see Figure 6.1):

- As a small category,
$$\mathrm{Obj}(\mathbf{E}_\xi) = \bigsqcup_{x \in \mathbf{B}} \mathrm{Obj}(\mathbf{E}_x).$$
The simple arrows of $\mathbf{E}_\xi$ are of two types. There is an arrow $y_1 \to y_2$ in $\mathbf{E}_\xi$ if
  a) $y_1, y_2 \in \mathbf{E}_x$ for some $x \in \mathbf{B}$ and $y_1 \to y_2$ is an arrow in $\mathbf{E}_x$;
  b) $x_1 \to x_2$ is a non-identity arrow in $\mathbf{B}$, y_1 and y_2 are objects of $\mathbf{E}_{x_1}$ and $\mathbf{E}_{x_2}$, respectively, and we have $\xi_1(x_1 \to x_2)(y_1) = y_2$.

  The set of all arrows of $\mathbf{E}_\xi$ is the smallest set containing the simple arrows that is closed under composition, where
  – for any $x \in \mathbf{B}$, composition of arrows of type a) from $\mathbf{E}_x$ is given by the composition in $\mathbf{E}_x$,
  – composition of arrows of type b) (and identity arrows) is given by composition in $\mathbf{B}$.

  Additionally, we impose the commutativity of squares: if $x_1 \to x_2$ is an arrow in $\mathbf{B}$ and $y_1 \to y_2$ is an arrow in $\mathbf{E}_{x_1}$, then the square below commutes in $\mathbf{E}_\xi$:
$$\begin{array}{ccc} y_2 & \longrightarrow & \xi_1(x_1 \to x_2)(y_2) \\ \uparrow & & \uparrow \\ y_1 & \longrightarrow & \xi_1(x_1 \to x_2)(y_1) \\ & & \\ x_1 & \longrightarrow & x_2 \end{array}$$

- As a sheaf, F_ξ sends an object $y \in \mathbf{E}_\xi$ with $\pi(y) = x$ to $F_x(y)$. Arrows $y_1 \to y_2$ of type a) from some $\mathbf{E}_x$ are sent to the map $F_x(y_1 \to y_2)$; arrows $y_1 \to y_2$ of type b) with $\pi(y_1) = x_1, \pi(y_2) = x_2$ are sent to $\xi_2(x_1 \to x_2)_{y_1}$. Composition arrows are sent to the appropriate composition of the above maps.

Remark 6.4. The commutativity of squares imposed on $\mathbf{E}_\xi$ above enables us to prove Proposition 6.5 at the category level. Indeed, a similar proposition necessarily holds at the level of the sheaf, since the module homomorphisms at type b) arrows come

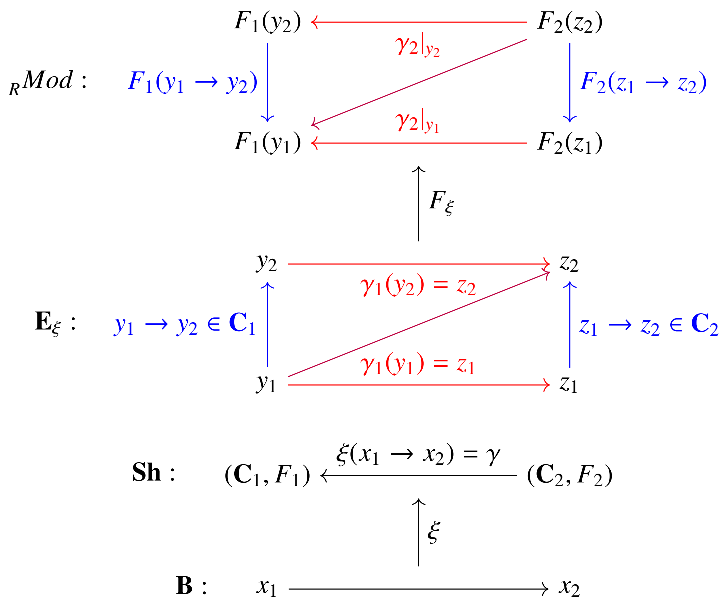
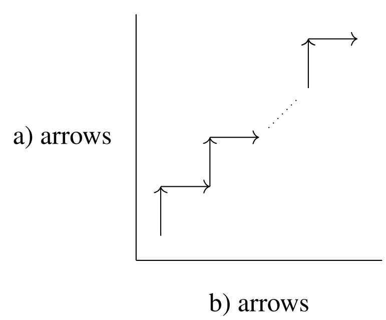


Fig. 6.1: Constructing the total sheaf $(\mathbf{E}_\xi; F_\xi)$. Arrows of type a) are in blue, arrows of type b) are in red, and composition arrows are in purple.

from the natural transformations $\xi_2(x_1 \to x_2)$ and so the relevant squares commute. We prefer pushing the commutativity to the category $\mathbf{E}_\xi$, because of certain later arguments (e.g. Lemma 7.5).

Proposition 6.5. *Any composition arrow f in $\mathbf{E}_\xi$ is equal to gh, for some type a) arrow g and some type b) arrow h.*

Proof. Since compositions of arrows of type a) are still arrows of the same type (similarly for type b)), a composition arrow in $\mathbf{E}_\xi$ is an alternating sequence of arrows of type a) and b). Suppose f starts with a type a) arrow and ends with a type b). We have the following picture in $\mathbf{E}_\xi$:



Consider any sequence of an arrow of type a) followed by an arrow of type b):

$$\begin{array}{ccc} y_1' & \longrightarrow & y_2' \\ \uparrow & & \\ y_0' & & \end{array}$$

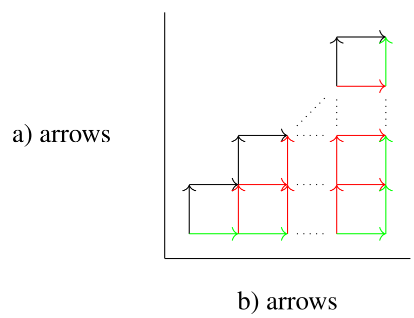
Now, the horizontal map is of type b), so it comes from an arrow $\pi(y_1') \to \pi(y_2') \in \mathbf{B}$, whereas the vertical arrow comes from $\mathbf{E}_{\pi(y_0')} = \mathbf{E}_{\pi(y_1')}$. Therefore

$$y_2' = \xi_1(\pi(y_1') \to \pi(y_2'))(y_1')$$

and the square below commutes in $\mathbf{E}_\xi$:

$$\begin{array}{ccc} y_1' & \longrightarrow & \xi_1(\pi(y_1') \to \pi(y_2'))(y_1') \\ \uparrow & & \uparrow \\ y_0' & \longrightarrow & \xi_1(\pi(y_1') \to \pi(y_2'))(y_0') \\ \pi(y_1') & \longrightarrow & \pi(y_2') \end{array}$$

Applying this to the picture of f above, we get a commutative grid



Therefore f is equal to the composition of the green arrows. But all vertical arrows are of type a), so the composition of all the green vertical arrows is some type a) arrow g. Analogously, the composition of the green horizontal arrows is some type b) arrow h.

Finally, if f starts with a type b) or ends with a type a), the last paragraph subsumes those with the other horizontal or vertical green arrows, again giving a resulting composition of gh for some h of type b) and some g of type a). □

Proposition 6.6. *The pair* $(\mathbf{E}_\xi; F_\xi)$ *above is an object of* **Sh**.

Proof. The identity arrow at an object $y \in \mathbf{E}_\xi$ is given by the identity arrow at $y \in \mathbf{E}_{\pi(y)}$. The requirement that composition of arrows of type a) and type b) is given by composition in $\mathbf{E}_x$ and $\mathbf{B}$, respectively, ensures that the property of the identity arrow is satisfied. Associativity follows from the previous proposition.

Finally, since the action of F_ξ on composition arrows is defined as the composition of actions on simple arrows, functoriality of F_ξ follows from the functoriality of ξ and F_x for all $x \in \mathbf{B}$. □

Example 6.7. Let P and Q be posets and $F \in \mathbf{Sh}(\mathbf{P} \times \mathbf{Q})$. We can define a bundle of sheaves $\xi : \mathbf{P} \to \mathbf{Sh}(\mathbf{Q})$ (recall Example 6.2 c)). We claim that $(\mathbf{E}_\xi, F_\xi) = (\mathbf{P} \times \mathbf{Q}, F)$: arrows of type a) in $\mathbf{E}_\xi$ connect elements of the form $(y, z_1) \leq (y, z_2)$ with $z_1 \leq z_2 \in \mathbf{Q}$, while arrows of type b) connect $(y_1, z) \leq (y_2, z)$ with $y_1 \leq y_2 \in \mathbf{P}$. Thus, if we have $(y_1, z_1) \leq (y_2, z_2)$ in $\mathbf{E}_\xi$, then $(y_1, z_1) \leq (y_2, z_2)$ in $\mathbf{P} \times \mathbf{Q}$.

Conversely, if $(y_1, z_1) \leq (y_2, z_2)$ in $\mathbf{P} \times \mathbf{Q}$, then $(y_1, z_1) \leq (y_1, z_2) \leq (y_2, z_2)$, but the first inequality is given by an arrow of type a) and the second by an arrow of type b). Therefore, $\mathbf{E}_\xi$ and $\mathbf{P} \times \mathbf{Q}$ are the same category. And since F_ξ and F agree on simple arrows (type a) and type b)), by construction this means that $(\mathbf{E}_\xi, F_\xi) = (\mathbf{P} \times \mathbf{Q}, F)$.

6.2 The bicomplex $\mathcal{S}^p(\mathbf{B}, \mathcal{S}^q)$

Definition 6.8. Given a bundle $\xi : \mathbf{B} \to \mathbf{Sh}$, for any $q \in \mathbb{Z}$ the *q-cochain sheaf* on $\mathbf{B}$ is the sheaf $\mathcal{S}^q : \mathbf{B} \to {}_R Mod$, i.e. the composition

$$\mathbf{B} \xrightarrow{\xi} \mathbf{Sh} \xrightarrow{\mathcal{S}^q} {}_R Mod.$$

Similarly, the *q-homology of the fibers* sheaf on $\mathbf{B}$ is the sheaf $\mathcal{H}^q_{fib} : \mathbf{B} \to {}_R Mod$, i.e. the composition

$$\mathbf{B} \xrightarrow{\xi} \mathbf{Sh} \xrightarrow{\mathcal{S}^\bullet} \mathbf{Ch}_R \xrightarrow{\mathcal{H}^q} {}_R Mod.$$

Explicitly, if $x \in \mathbf{B}$, then $\mathcal{H}^q_{fib}(x) = H^q(\mathbf{E}_x; F_x)$.

Let $\xi : \mathbf{B} \to \mathbf{Sh}$ be a bundle of sheaves and suppose $x \to y$ is an arrow in $\mathbf{B}$. We have the commutative square

$$\begin{array}{ccc} \mathcal{S}^{q-1}(\mathbf{E}_x; F_x) & \xrightarrow{\quad d \quad} & \mathcal{S}^q(\mathbf{E}_x; F_x) \\ \uparrow & & \uparrow \\ \mathcal{S}^{q-1}(\mathbf{E}_y; F_y) & \xrightarrow{\quad d \quad} & \mathcal{S}^q(\mathbf{E}_y; F_y) \end{array}$$

where the vertical maps are the chain map from Lemma 4.2 induced by $\xi(x \to y)$. In particular, the differential d induces a $\mathbf{Sh}$-morphism $\gamma : (\mathbf{B}; \mathcal{S}^{q-1}) \to (\mathbf{B}; \mathcal{S}^q)$, where γ_1 is the identity functor and γ_2 are the differentials at each object of $\mathbf{B}$. This gives us the induced map

$$\gamma^\bullet : \mathcal{S}^\bullet(\mathbf{B}; \mathcal{S}^{q-1}) \to \mathcal{S}^\bullet(\mathbf{B}; \mathcal{S}^q).$$

Applying this for all $q \in \mathbb{Z}$ we get the commutative grid

$$\begin{array}{ccccccc} & & \vdots & & \vdots & & \\ & & \uparrow & & \uparrow & & \\ \cdots & \longrightarrow & \mathcal{S}^{p-1}(\mathbf{B}; \mathcal{S}^q) & \longrightarrow & \mathcal{S}^p(\mathbf{B}; \mathcal{S}^q) & \longrightarrow & \cdots \\ & & \uparrow & & \uparrow & & \\ \cdots & \longrightarrow & \mathcal{S}^{p-1}(\mathbf{B}; \mathcal{S}^{q-1}) & \longrightarrow & \mathcal{S}^p(\mathbf{B}; \mathcal{S}^{q-1}) & \longrightarrow & \cdots \\ & & \uparrow & & \uparrow & & \\ & & \vdots & & \vdots & & \end{array}$$

To make the squares anti-commute instead, we apply the usual 'Jedi sign trick', i.e. we include a factor of -1 in every other horizontal map. We will be concerned with this bicomplex in particular in later chapters, so we will sometimes refer to it as just $\mathcal{K}_\xi^{p,q}$. Explicitly, we have

$$\mathcal{K}_\xi^{p,q} = \mathcal{S}^p(\mathbf{B}; \mathcal{S}^q);$$

if we denote

$$\sigma = x_0 \to \ldots \to x_p \in N\mathbf{B} \text{ and } \tau = y_0 \to \ldots \to y_q \in N\mathbf{E}_{x_0},$$

then the vertical differential $d^v : \mathcal{S}^p(\mathbf{B}; \mathcal{S}^{q-1}) \to \mathcal{S}^p(\mathbf{B}; \mathcal{S}^q)$ is defined by

$$d^v(u)|_{\sigma,\tau} = F_{x_0}(y_0 \to y_1)(u|_{\sigma,\tau_0}) + \sum_{j=1}^{q} (-1)^j (u|_{\sigma,\tau_j})$$

and the horizontal differential $d^h : \mathcal{S}^{p-1}(\mathbf{B}; \mathcal{S}^q) \to \mathcal{S}^p(\mathbf{B}; \mathcal{S}^q)$ is defined by

$$d^h(u)|_{\sigma,\tau} = (-1)^{p+q} \left(\gamma_{y_0}(u|_{\sigma_0,\gamma_1(\tau)}) + \sum_{i=1}^{p} (-1)^i (u|_{\sigma_j,\tau}) \right),$$

where $\xi_2(x_0 \to x_1) = \gamma$.

We can place the modules $\mathcal{K}_\xi^{p,q}$ on the E_0 page of a spectral sequence and use the vertical maps as the differentials on that page. We can further use the quotients of the horizontal maps for the differentials on the E_1 page.

Proposition 6.9. *The E_2 page of the spectral sequence defined above has*

$$E_2^{p,q} = H^p(\mathbf{B}; \mathcal{H}^q_{fib}(\xi)).$$

Proof. Note that the differentials on the E_2 page are of degree $(2, -1)$. Consider the following diagram

$$\begin{array}{lllll} \xi \xrightarrow{\mathcal{S}^\bullet} (\mathbf{B}; \mathcal{S}^\bullet) & \xrightarrow{\mathcal{S}^p} & \mathcal{S}^p(\mathbf{B}; \mathcal{S}^\bullet) & \xrightarrow{H^\bullet} & H^\bullet(\mathcal{S}^p(\mathbf{B}; \mathcal{S}^\bullet)) \\ & \searrow^{H^\bullet} & & & \\ & & (\mathbf{B}; \mathcal{H}^\bullet_{fib}(\xi)) & \xrightarrow{\mathcal{S}^p} & \mathcal{S}^p(\mathbf{B}; \mathcal{H}^\bullet_{fib}(\xi)) \end{array}$$

The top path is how we get the modules in a given column on the E_1 page – we take vertical homology of a column in E_0. On the other hand, taking horizontal homology of rows formed by $\mathcal{S}^p(\mathbf{B}; \mathcal{H}^q_{fib}(\xi))$ clearly gives the required modules $H^p(\mathbf{B}; \mathcal{H}^q_{fib}(\xi))$. It is then enough to show that the two graded modules at the ends of the two paths are equal for each $p \in \mathbb{Z}$. This follows directly from cohomology commuting with the direct product. □

Now, recall that there is a total complex associated to $\mathcal{K}^{p,q}_\xi$. To reduce notational clutter, instead of naming this total complex $T^\bullet_{\mathcal{K}_\xi}$, we will denote it as $T^\bullet_\xi$. Explicitly,

$$T^n_\xi := \prod_{p+q=n} \mathcal{K}^{p,q}_\xi,$$

with $d = d^h + d^v$. Then, from the general construction of a spectral sequence from a bicomplex (Theorem 5.17) and from the above proposition, we have the sheaf cohomological version of [ET12, Proposition 2.2]:

Proposition 6.10. *If $\xi : \mathbf{B} \to \mathbf{Sh}$ is a bundle of sheaves, then there is a spectral sequence*

$$E^{p,q}_2 = H^p(B; \mathcal{H}^q_{fib}) \Longrightarrow H^\bullet(T^\bullet_\xi).$$

Calculating $H^\bullet(T^\bullet_\xi)$ on its own is usually unfeasible, but in certain situations we can relate it to the cohomology of the total sheaf $(\mathbf{E}_\xi; F_\xi)$. Before doing that in the next chapter, it will be useful to examine a case where it is possible to easily identify what $H^\bullet(T^\bullet_\xi)$ is.

6.3 Constant bundles over posets with a unique minimum

Proposition 6.11. *Suppose $\mathbf{B}$ is a poset (recall Definition 2.2), $x \in \mathbf{B}$ is a unique minimum, and $(\mathbf{C}; F)$ is an object of $\mathbf{Sh}$. If $\xi = \mathbf{B} \times (\mathbf{C}; F)$ is a constant bundle (recall Example 6.2), then there is a chain map*

$$\varphi^\bullet : \mathcal{S}^\bullet(\mathbf{C}; F) \to T^\bullet_\xi$$

such that the induced map

$$\varphi^\bullet : H^\bullet(\mathbf{C}; F) \to H^\bullet T^\bullet_\xi$$

is an isomorphism.

Proof. It is straightforward to see why $\mathcal{S}^\bullet(\mathbf{C}; F)$ is quasi-isomorphic to $T^\bullet_\xi$. The E_2 page of the spectral sequence for ξ has

$$E_2^{p,q} = H^p(\mathbf{B}, \varDelta H^q(\mathbf{C}; F)).$$

Since the right-hand side is the cohomology of a constant sheaf, the only non-zero positions on the E_2 page are in the column $p = 0$; so the sequence collapses and we can read off $H^\bullet T^\bullet_\xi$. Explicitly,

$$H^p(\mathbf{B}; \varDelta H^q(\mathbf{C}; F)) = \begin{cases} H^q(\mathbf{C}; F), & \text{if } p = 0, \\ 0, & \text{otherwise.} \end{cases}$$

So $H^\bullet(\mathbf{C}; F) \cong H^\bullet T^\bullet_\xi$. It is, still, useful to describe the explicit quasi-isomorphism; we will use a version of this explicit chain map in the proof of Proposition 7.11.

First consider the constant sheaf $(\mathbf{P}; \varDelta A)$, where $\mathbf{P}$ is a poset with a unique minimum. Recall that

$$H^n(\mathbf{P}; \varDelta A) \cong \begin{cases} A, & \text{if } n = 0, \\ 0, & \text{otherwise.} \end{cases}$$

Our first goal is to find an explicit map for the isomorphism above. So let $u \in \mathcal{S}^0(\mathbf{P}; \varDelta A)$ be such that $du = 0$. Since we have a unique minimum 0, for any $x \in \mathbf{P}$, there is an arrow $0 \le x$ in $\mathbf{P}$. Then

$$0 = du|_{0\le x} = u|_x - u|_0,$$

so $u|_x = u|_0$ for all $x \in \mathbf{P}$. Denote such a constant element of $\mathcal{S}^0(\mathbf{P}; \varDelta A)$ by u_a if $u_a|_x = a \in A$ for all $x \in \mathbf{P}$. So the isomorphism we are looking for is $\theta : A \to H^0(P, \varDelta A) : a \mapsto u_a$.

Now consider the (trivial) chain complex $\iota^\bullet(A)$ defined by

$$\iota^n(A) = \begin{cases} A, & \text{if } n = 0, \\ 0, & \text{otherwise,} \end{cases}$$

and $d^n_{\iota^\bullet(A)} = 0$ for all n. Define the map $\psi^\bullet : \iota^\bullet(A) \to \mathcal{S}^\bullet(\mathbf{P}; \varDelta A)$ as

$$\psi^n = \begin{cases} \theta, & \text{if } n = 0, \\ 0, & \text{otherwise.} \end{cases}$$

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & 0 & \longrightarrow & A & \longrightarrow & 0 & \longrightarrow & \cdots \\ & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle \theta} & & \downarrow{\scriptstyle 0} & & \\ \cdots & \longrightarrow & 0 & \longrightarrow & \mathcal{S}^0(\mathbf{P}; \varDelta A) & \longrightarrow & \mathcal{S}^1(\mathbf{P}; \varDelta A) & \longrightarrow & \cdots \end{array}$$

To see this is a chain map, note that $\theta(a) \in \ker(d^0)$, so $d\theta = 0$. All other squares commute since all compositions are the 0 map.

Crucially, $\psi^\bullet$ is a quasi-isomorphism. This is because $H^0\iota^\bullet(A) = A$ and by construction θ induces the isomorphism $H^0\iota^\bullet(A) \to H^0\mathcal{S}^\bullet(\mathbf{P}; \varDelta A)$. Note that the map $-\psi^\bullet$ is also a quasi-isomorphism, since $-\theta$ induces $-\operatorname{id} : A \to A$ in homology.

Returning to the case of the constant bundle $\xi = \mathbf{B} \times (\mathbf{P}; F)$, we can now define $\varphi^n : \mathcal{S}^n(\mathbf{C}; F) \to T^n_\xi$ by

$$\varphi u|_{\sigma,\tau} = \begin{cases} u|_\tau, & \text{if } \operatorname{length}(\sigma) = 0, \\ 0, & \text{otherwise.} \end{cases}$$

To see that this is a chain map, note that if $\operatorname{length}(\sigma) \geq 2$, both φdu and $d\varphi u$ are 0. If $\operatorname{length}(\sigma) = 1$, then

$$\begin{aligned} d\varphi u|_{x_0 \leq x_1, y_0 \leq \cdots \leq y_n} &= (-1)^{n+1}\varphi u|_{x_1, y_0 \leq \cdots \leq y_n} - (-1)^{n+1}\varphi u|_{x_0, y_0 \leq \cdots \leq y_n} + \\ &\quad + (-1)^{n+1}\sum_{i=0}^{n}(-1)^i \varphi u|_{x_0 \leq x_1, y_0 \leq \cdots \leq \hat{y}_i \leq \cdots \leq y_n} \\ &= 0 \\ &= \varphi du|_{x_0 \leq x_1, y_0 \leq \cdots \leq y_n}. \end{aligned}$$

Finally, if $\operatorname{length}(\sigma) = 0$, then

$$\begin{aligned} d\varphi u|_{x_0, y_0 \leq \cdots \leq y_n} &= \sum_{i=0}^{n}(-1)^i \varphi u|_{x_0, y_0 \leq \cdots \leq \hat{y}_i \leq \cdots \leq y_n} \\ &= \sum_{i=0}^{n}(-1)^i u|_{y_0 \leq \cdots \leq \hat{y}_i \leq \cdots \leq y_n} \\ &= du|_{y_0 \leq \cdots \leq y_n} \\ &= \varphi du|_{x_0, y_0 \leq \cdots \leq y_n}. \end{aligned}$$

We define a bicomplex $\mathcal{L}^{\bullet,\bullet}$ by

$$\mathcal{L}^{p,q} = \begin{cases} \mathcal{S}^q(\mathbf{C}; F) & \text{if } p = 0, \\ \quad 0, & \text{otherwise} \end{cases}$$

and we let $d^h_{\mathcal{L}} = 0$, $d^v_{\mathcal{L}} = 0$ on the non-zero columns, and $d^v_{\mathcal{L}} = d_{\mathcal{S}^\bullet(\mathbf{C};F)}$ on the 0-th column.

$$\mathcal{L}^{\bullet,\bullet} : \quad \begin{array}{ccccccc} \uparrow & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & \mathcal{S}^1(\mathbf{C}; F) & \longrightarrow & 0 & \longrightarrow & \cdots \\ \uparrow & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & \mathcal{S}^0(\mathbf{C}; F) & \longrightarrow & 0 & \longrightarrow & \cdots \\ \uparrow & & \uparrow & & \uparrow & & \end{array}$$

Now take the bicomplex defined in Section 6.2.

$$\mathcal{K}_\xi^{p,q} = \mathcal{S}^p(\mathbf{B};\mathcal{S}^q)$$

$$\mathcal{K}_\xi^{\bullet,\bullet}: \quad \begin{array}{ccccccc} & & \uparrow & & \uparrow & & \\ \cdots & \longrightarrow & \mathcal{S}^p(\mathbf{B};\mathcal{S}^{q+1}) & \longrightarrow & \mathcal{S}^{p+1}(\mathbf{B};\mathcal{S}^{q+1}) & \longrightarrow & \cdots \\ & & \uparrow & & \uparrow & & \\ \cdots & \longrightarrow & \mathcal{S}^p(\mathbf{B};\mathcal{S}^{q}) & \longrightarrow & \mathcal{S}^{p+1}(\mathbf{B};\mathcal{S}^{q}) & \longrightarrow & \cdots \\ & & \uparrow & & \uparrow & & \end{array}$$

We want to show that φ induces a morphism of these two bicomplexes. To that effect, we need three facts:

a) First, it is clear that $\varphi(\mathcal{S}^q(\mathbf{C};F)) \subseteq \mathcal{S}^0(\mathbf{B};\mathcal{S}^q)$.
b) Second, we need φ to induce a chain map on the vertical complexes. This is the zero map for $p \neq 0$. Consider the diagram

$$\begin{array}{ccc} \uparrow & & \uparrow \\ \mathcal{S}^{q+1}(\mathbf{C};F) & \xrightarrow{\ \varphi\ } & \mathcal{S}^0(\mathbf{B};\mathcal{S}^{q+1}) \\ d\uparrow & & d^v\uparrow \\ \mathcal{S}^{q}(\mathbf{C};F) & \xrightarrow{\ \varphi\ } & \mathcal{S}^0(\mathbf{B};\mathcal{S}^{q}) \\ \uparrow & & \uparrow \end{array}$$

We want to show that $d^v\varphi = \varphi d$. Let $u \in \mathcal{S}^0(\mathbf{B};\mathcal{S}^{q+1})$, $x \in \mathbf{B}$, $y_0 \leq \cdots \leq y_{q+1} \in \mathbf{C}$.

$$\begin{aligned} d^v\varphi u|_{x,y_0\leq\cdots\leq y_{q+1}} &= \sum_{i=0}^{q+1} \varphi u|_{x,y_0\leq\cdots\leq\hat{y}_i\leq\cdots\leq y_{q+1}} \\ &= \sum_{i=0}^{q+1} u|_{x,y_0\leq\cdots\leq\hat{y}_i\leq\cdots\leq y_{q+1}} \\ &= du|_{y_0\leq\cdots\leq y_{q+1}} \\ &= \varphi du|_{x,y_0\leq\cdots\leq y_{q+1}}. \end{aligned}$$

Therefore φ induces a chain map on vertical complexes.
c) Finally, we need φ to induce chain maps on horizontal complexes. Consider the diagram

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & 0 & \longrightarrow & \mathcal{S}^q(\mathbf{C};F) & \longrightarrow & 0 & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle 0} & & \\
\cdots & \longrightarrow & 0 & \longrightarrow & \mathcal{S}^0(\mathbf{B};\mathcal{S}^q) & \longrightarrow & \mathcal{S}^1(\mathbf{B};\mathcal{S}^q) & \longrightarrow & \cdots
\end{array}$$

If we denote $\mathcal{S}^q(\mathbf{C};F) = A$, this is just an instance of the map ψ.

Now consider the two spectral sequences E and E' associated to the bicomplexes $\mathcal{L}^{\bullet,\bullet}$ and $\mathcal{K}_\xi^{\bullet,\bullet}$, respectively. The morphism of bicomplexes φ induces a morphism $E \to E'$ of spectral sequences. Note also that both E and E' are bounded below. The first pages of E and E' are as follows.

$$E_1 : \quad \begin{array}{ccccccc}
\cdots & \longrightarrow & H^1(\mathbf{C};F) & \longrightarrow & 0 & \longrightarrow & \cdots \\
\cdots & \longrightarrow & H^0(\mathbf{C};F) & \longrightarrow & 0 & \longrightarrow & \cdots
\end{array}$$

$$E'_1 : \quad \begin{array}{ccccccc}
\cdots & \longrightarrow & \mathcal{S}^0(\mathbf{B};\mathcal{H}^1_{fib}) & \longrightarrow & \mathcal{S}^1(\mathbf{B};\mathcal{H}^1_{fib}) & \longrightarrow & \cdots \\
\cdots & \longrightarrow & \mathcal{S}^0(\mathbf{B};\mathcal{H}^0_{fib}) & \longrightarrow & \mathcal{S}^1(\mathbf{B};\mathcal{H}^0_{fib}) & \longrightarrow & \cdots
\end{array}$$

As in the case of a constant bundle, the induced maps φ are quasi-isomorphisms on the horizontal complexes. This means that φ induces isomorphisms on the second pages of E and E'. By the Mapping Lemma (Lemma 5.18), we have an induced isomorphism

$$\varphi : E_\infty^{p,q} \to E'^{p,q}_\infty.$$

By the above, the construction of the total complex of a bicomplex, and Proposition 6.10, we can conclude that φ gives an isomorphism

$$\varphi : H^\bullet(\mathbf{C};F) \to H^\bullet T_\xi^\bullet. \qquad \square$$

7

The spectral sequence and the total sheaf

Up to this point, for a bundle of sheaves $\xi : \mathbf{B} \to \mathbf{Sh}$, we have constructed the total sheaf $(\mathbf{E}_\xi, F_\xi)$ and its simplicial complex $\mathcal{S}^\bullet(\mathbf{E}_\xi, F_\xi)$, as well as the bicomplex $\mathcal{K}_\xi^{\bullet,\bullet}$ and its total complex $T_\xi^\bullet$. We know that the spectral sequence of the bicomplex converges to $H^\bullet T_\xi^\bullet$, but we would like to identify cases where it converges to the cohomology of the total sheaf. In this chapter, we will define a chain map ω between the two complexes and show that, under certain (fairly strong) assumptions, it is a quasi-isomorphism. This puts the results of [ET12] into the sheaf cohomology setting.

7.1 Assumptions

For most of this chapter, we will have to (substantially) restrict the categories we consider.

Definition 7.1. A bundle of sheaves $\xi : \mathbf{B} \to \mathbf{Sh}$ is a *poset bundle of sheaves* if both $\mathbf{B}$ and $\mathbf{E}_x$ for all $x \in \mathbf{B}$ are finite posets (recall Definition 2.2).

Unless otherwise stated, all small categories in sight are assumed to be finite posets. If $x, y \in \mathbf{B}$, we say that *y covers x* (denoted $x \prec y$) if, whenever $z \in \mathbf{B}$ is such that $x \leq z \leq y$, we have $z = x$ or $z = y$. We also say that $\mathbf{B}$ *has a* 0 (or is a poset with 0) if $\mathbf{B}$ has a unique minimal element $0 \in \mathbf{B}$.

Now, for an element $x \in \mathbf{B}$, define $\mathbf{B}_{\geq x}$ and $\mathbf{B}_{\not\geq x}$ to be the full subcategories of $\mathbf{B}$ with

$$\operatorname{Obj} \mathbf{B}_{\geq x} := \{z \in \operatorname{Obj} \mathbf{B} \mid x \leq z\} \text{ and } \operatorname{Obj} \mathbf{B}_{\not\geq x} := \operatorname{Obj} \mathbf{B} \setminus \operatorname{Obj} \mathbf{B}_{\geq x}.$$

Note that both $\mathbf{B}_{\geq x}$ and $\mathbf{B}_{\not\geq x}$ inherit the poset structure of $\mathbf{B}$. We will occasionally omit Obj when we refer to the objects of a poset category if the meaning is clear from context.

Lemma 7.2. *If $\xi : \mathbf{B} \to \mathbf{Sh}$ is a poset bundle of sheaves, then the small category $\mathbf{E}_\xi$ associated to ξ is also a poset.*

Proof. Any arrow f in $\mathbf{E}_\xi$ is either of type a), of type b), or a composition arrow (recall Definition 6.3). If f is of type a), then it comes from one of the posets $\mathbf{E}_x$. If it is of type b), then it comes from the poset $\mathbf{B}$. And if f is a composition arrow, then it is equal to the composition gh, for an arrow g of type a) and an arrow h of type b) (by Proposition 6.5). But both of those come from posets, so the composition is unique. □

The key property we will exploit in this chapter is the following.

Definition 7.3. Assume $\mathbf{B}$ is a poset.

a) Let $\mathbf{B}_1$ and $\mathbf{B}_2$ be full subposets (full subcategories) of $\mathbf{B}$. We call $\mathbf{B}$ *admissible for* $\mathbf{B}_1, \mathbf{B}_2$ if
   - $\mathbf{B}_1 \cap \mathbf{B}_2 = \emptyset$,
   - $\mathbf{B}_1 \cup \mathbf{B}_2 = \mathbf{B}$,
   - there are no $x \in \mathbf{B}_2$ and $y \in \mathbf{B}_1$ with $x \leq y$, and
   - for all $x \in \mathbf{B}_1$, the full subposet $\{y \in \mathbf{B}_2 \mid x \leq y\} \subseteq \mathbf{B}_2$ is non-empty and has a unique minimum.

b) We call $\mathbf{B}$ *admissible for* $x \in \mathbf{B}$ if $\mathbf{B}$ is admissible for $\mathbf{B}_{\ngeq x}, \mathbf{B}_{\geq x}$. Note that the first three requirements of admissibility are automatically satisfied for $\mathbf{B}_{\ngeq x}, \mathbf{B}_{\geq x}$ (see Figure 7.1). We also denote the poset in the last requirement by

$$\mathbf{B}_{\geq x}^{\geq y} := \{z \in \mathbf{B}_{\geq x} \mid y \leq z\} = \mathbf{B}_{\geq x} \cap \mathbf{B}_{\geq y}.$$

c) We call $\mathbf{B}$ *recursively admissible* if $\mathbf{B}$ has a 0 and either
   - $\mathbf{B}$ is Boolean of rank 1, or
   - $\mathbf{B}$ is admissible for some $x > 0$ and both $\mathbf{B}_{\geq x}$ and $\mathbf{B}_{\ngeq x}$ are recursively admissible.

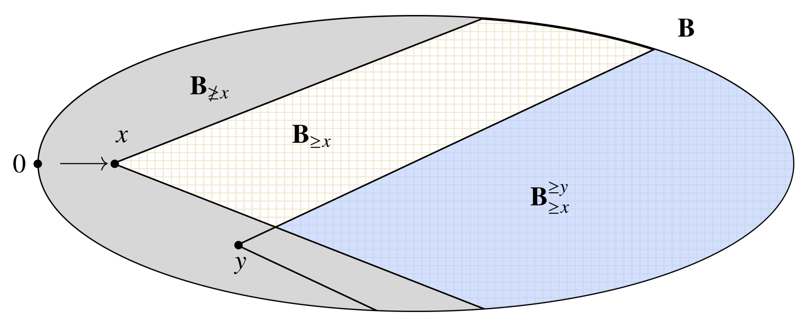


Fig. 7.1: A poset $\mathbf{B}$ with $x > 0$ and $y \in \mathbf{B}_{\ngeq x}$.

Example 7.4. • The Boolean lattices $\mathbb{B}_n$ (recall Definition 1.6) are recursively admissible (Figure 7.2). Indeed, if $n > 1$ and $\mu = (1, 0, \cdots, 0)$, then $\mu \succ 0$ and

$$\mathbb{B}_{n,\geq\mu} \cong \mathbb{B}_{n,\ngeq\mu} \cong \mathbb{B}_{n-1}.$$

Moreover, if $\nu \in \mathbb{B}_{n,\ngeq\mu}$, then $\min \mathbb{B}^{\geq\nu}_{n,\geq\mu} = \mu + \nu$.

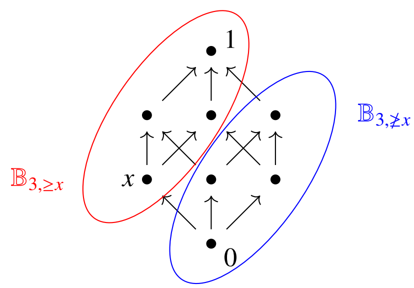


Fig. 7.2: The poset $\mathbb{B}_3$ is admissible for x.

• In the homological setup of [ET12], the Bruhat posets of type $I_2(m)$ are specially admissible (see [ET12, Example 3.7]). In the language of this thesis they are just admissible (Figure 7.3): let $x \prec 1$, $I_2(m) = \mathbf{B}$, and consider

$$\mathbf{B}_2 := \{x \to 1\} \quad \text{and} \quad \mathbf{B}_1 := \mathbf{B} \backslash \mathbf{B}_2.$$

If $y \in \mathbf{B}_1$ with $y \prec 1$, then $\min\{z \in \mathbf{B}_2 \mid y \leq z\} = 1$; if y is any other object of $\mathbf{B}_1$, then $\min\{z \in \mathbf{B}_2 \mid y \leq z\} = x$. We will see in the next chapter that we do indeed retain a property similar to 'special admissibility' for $I_2(m)$.

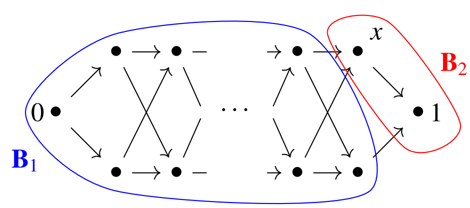


Fig. 7.3: The poset $I_2(m)$ is admissible for $\mathbf{B}_1, \mathbf{B}_2$.

• Let $\mathbb{B}^+_n$ be the poset with objects

$$\mathrm{Obj}\, \mathbb{B}^+_n = \mathrm{Obj}\, \mathbb{B}_n \cup \{1^+\},$$

such that if $x_1, x_2 \in \mathbb{B}_n$, then $x_1 \leq x_2$ in $\mathbb{B}^+_n$ if and only if $\mathbb{B}_n$; and $x \leq 1^+$ for all $x \in \mathbb{B}_n \backslash \{1\}$ (where 1 is the maximum of $\mathbb{B}_n$). By inspection, both $\mathbb{B}^+_1$ and $\mathbb{B}^+_2$ are

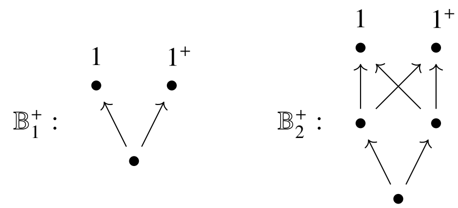


Fig. 7.4: The posets $\mathbb{B}_1^+$ and $\mathbb{B}_2^+$ are non-admissible.

non-admissible (Figure 7.4). In fact, $\mathbb{B}_n^+$ is not recursively admissible for any n (Proposition 8.1).

If we have a poset bundle of sheaves $\xi : \mathbf{B} \to \mathbf{Sh}$ and a subcategory $\mathbf{C}$ of $\mathbf{B}$, we can restrict the bundle ξ to $\mathbf{C}$ to obtain another bundle $\xi_{\mathbf{C}} : \mathbf{C} \to \mathbf{Sh}$ with total sheaf $(\mathbf{E}_{\xi_{\mathbf{C}}}; F_{\xi_{\mathbf{C}}})$. When the bundle ξ is clear from context, we will just use $(\mathbf{E}_{\mathbf{C}}; F_{\mathbf{C}})$. Note that we use $(\mathbf{E}_x; F_x)$ for the sheaf $\xi(x)$ when x is an object of $\mathbf{B}$, which (almost) coincides with $(\mathbf{E}_{\mathbf{C}}; F_{\mathbf{C}})$ when $\mathbf{C}$ is the subcategory of $\mathbf{B}$ consisting only of x and its identity arrow.

The next lemma shows how admissibility of $\mathbf{B}$ extends to $\mathbf{E}_\xi$.

Lemma 7.5. *Let* $\mathbf{B}$ *be admissible for some* $x \in \mathbf{B}$ *and* $\xi : \mathbf{B} \to \mathbf{Sh}$ *be a poset bundle of sheaves with total sheaf* $(\mathbf{E}_\xi; F_\xi)$*. Then* $\mathbf{E}_\xi$ *is admissible for* $\mathbf{E}_{\mathbf{B}_{\not\geq x}}, \mathbf{E}_{\mathbf{B}_{\geq x}}$.

Proof. It is immediate that $\mathbf{E}_{\mathbf{B}_{\not\geq x}}$ and $\mathbf{E}_{\mathbf{B}_{\geq x}}$ are disjoint, that $\mathbf{E}_{\mathbf{B}_{\not\geq x}} \cup \mathbf{E}_{\mathbf{B}_{\geq x}} = \mathbf{E}_\xi$, and that there is no arrow from an object of $\mathbf{E}_{\mathbf{B}_{\not\geq x}}$ to an object of $\mathbf{E}_{\mathbf{B}_{\geq x}}$. It remains to show that for all $w \in \mathbf{E}_{\mathbf{B}_{\not\geq x}}$, the subposet

$$\{z \in \mathbf{E}_{\mathbf{B}_{\geq x}} \mid w \leq z\}$$

has a unique minimal element.

Since $w \in \mathbf{E}_{\mathbf{B}_{\not\geq x}}$, w is an element of a particular $\mathbf{E}_y$ for some $y \in \mathbf{B}_{\not\geq x}$. By the admissibility of $\mathbf{B}$, that means that the poset $\mathbf{B}_{\geq x}^{\geq y}$ has a unique minimum, say v. Then $y \leq v$ and thus there is an arrow $y \to v$ in $\mathbf{B}$. Denote the sheaf morphism given by this arrow as γ. By the construction of the total sheaf, we have that $w \leq \gamma_1(w)$.

Suppose $w \leq z$ for some $z \in \mathbf{E}_{\mathbf{B}_{\geq x}}$ and suppose $z \in \mathbf{E}_u$, $u \in \mathbf{B}_{\geq x}$. Then by our argument in Proposition 6.5 we have a $z_0 \in \mathbf{E}_u$ with $w \leq z_0 \leq z$ and an arrow $y \to u$ giving rise to a sheaf morphism γ'. Thus u is in $\mathbf{B}_{\geq x}^{\geq y}$, not just $\mathbf{B}_{\geq x}$. Since v is the minimal element of $\mathbf{B}_{\geq x}^{\geq y}$, we have that $v \leq u$. But there is a unique arrow $y \to u$, so γ'_1 factors through $\mathbf{E}_v$ and the sheaf morphism given by $v \to u$ maps $\gamma_1(w)$ to z_0. This means that $\gamma_1(w) \leq z_0 \leq z$, therefore $\gamma_1(w)$ is the minimum of the set $\{z \in \mathbf{E}_{\mathbf{B}_{\geq x}} \mid w \leq z\}$. Refer to Figure 7.5 for the relevant objects. □

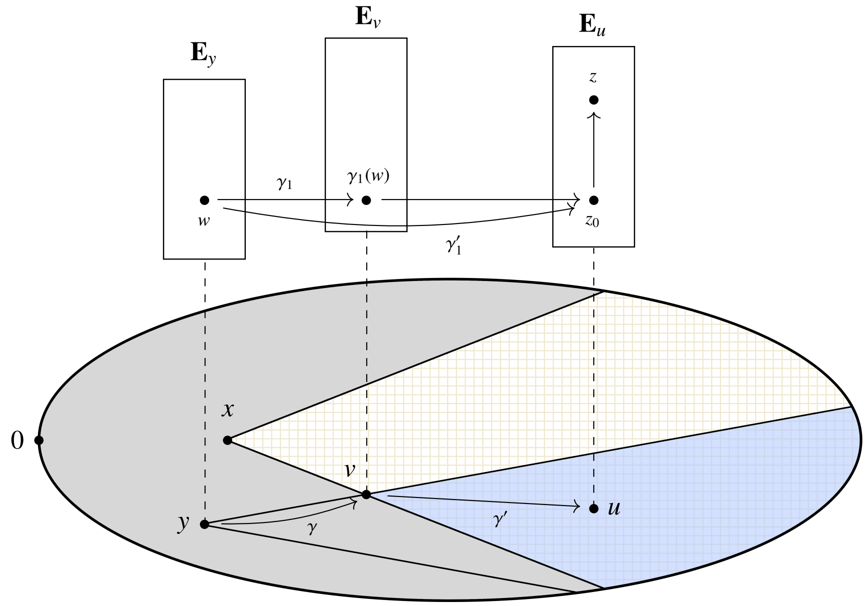


Fig. 7.5: The poset **B** and the fibers over y, v and u.

7.2 Grid Traversals

For a given poset bundle of sheaves ξ, we define a chain map $\omega : \mathcal{S}^\bullet(\mathbf{E}_\xi; F_\xi) \to T_\xi^\bullet$, where $\mathcal{S}^\bullet(\mathbf{E}_\xi; F_\xi)$ is the chain complex constructed in Section 4.1 on the total sheaf of ξ (recall Definition 6.3), and $T_\xi^\bullet$ is the total complex associated to the bicomplex $\mathcal{K}_\xi^{\bullet,\bullet}$ constructed in Section 6.2. To do that, if $\sigma = x_0 \to \cdots \to x_p \in N\mathbf{B}$ and $\tau \in N\mathbf{E}_{x_0}$, then to each pair (σ, τ) we will associate a (signed) combination of all traversals of a particular grid in $\mathbf{E}_\xi$.

To form this grid, we lay out σ and τ and complete the grid using the morphisms $\xi(x_i \to x_{i+1})$:

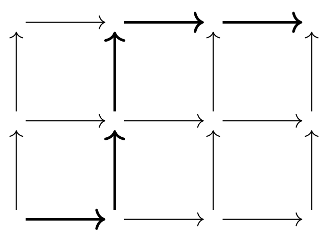
$$
\begin{array}{ccccccccc}
 & & \tau & & & & & & \\
 & & \shortparallel & & & & & & \\
 & & y_{0,q} & \longrightarrow & y_{1,q} & \longrightarrow & \cdots & \longrightarrow & y_{p,q} \\
 & & \uparrow & & \uparrow & & & & \uparrow \\
 & & \vdots & & \vdots & & & & \vdots \\
 & & \uparrow & & \uparrow & & & & \uparrow \\
 & & y_{0,0} & \longrightarrow & y_{1,0} & \longrightarrow & \cdots & \longrightarrow & y_{p,0} \\
 & & & & & & & & \\
\sigma & = & x_0 & \longrightarrow & x_1 & \longrightarrow & \cdots & \longrightarrow & x_p
\end{array}
$$

where $y_{0,j} = y_j$ and $y_{i+1,j} = \xi_1(x_i \to x_{i+1})(y_{i,j})$.

Definition 7.6. If $\sigma = x_0 \to \cdots \to x_p \in N\mathbf{B}$ and $\tau = y_0 \to \cdots \to y_q \in N\mathbf{E}_{x_0}$, then a *grid traversal* $z \in N\mathbf{E}_\xi$ of the grid of (σ, τ) is a chain of length $(p+q)$ of arrows in the grid. In particular, each arrow in z is either

$$\xi_1(x_0 \to x_i)(y_j \to y_{j+1}) \text{ or } y_{i,j} \to \xi_1(x_i \to x_{i+1})(y_{i,j}).$$

Note that these correspond to type a) and type b) in Definition 6.3.



Definition 7.7. For each grid traversal z of the grid of (σ, τ), define

$$m(z) = \#\{\text{squares in the grid below and to the right of } z\}.$$

Furthermore, define $\varsigma(q) = \left\lceil \frac{q}{2} \right\rceil = \min\left\{n \in \mathbb{Z} \mid n \geq \frac{q}{2}\right\}$.

We can now define the chain map we are interested in.

Definition 7.8. The map $\omega : \mathcal{S}^\bullet(\mathbf{E}_\xi; F_\xi) \to T^\bullet_\xi$ is defined, for any $u \in \mathcal{S}^\bullet(\mathbf{E}_\xi; F_\xi)$, by

$$\omega(u)|_{\sigma,\tau} = (-1)^{\varsigma(q)} \sum_z (-1)^{m(z)} u|_z,$$

where the sum is taken over all traversals z of the grid of (σ, τ).

Proposition 7.9. *The map ω defined above is a chain map.*

Proof. We need to show that

$$\omega du|_{\sigma,\tau} = d\omega u|_{\sigma,\tau}$$

for all appropriate d, σ, and τ.

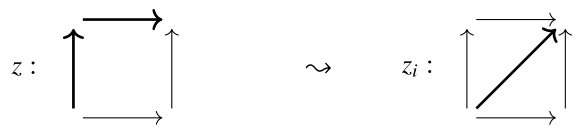
For the rest of this proof we allow a slight abuse of notation – in cases where the head of a chain of arrows is deleted, we will write $u|_{\sigma_0}$ instead of $F(x_0 \to x_1)(u|_{\sigma_0})$.

If $\sigma = x_0 \to \cdots \to x_p$ and $\tau = y_0 \to \cdots \to y_q$, writing out the various formulae gives

$$\begin{aligned}\omega du|_{\sigma,\tau} &= (-1)^{\varsigma(q)} \sum_z (-1)^{m(z)} \sum_{i=0}^{p+q} (-1)^i u_{z_i},\\ d\omega u|_{\sigma,\tau} &= \sum_{r=0}^{p} (-1)^r \omega u|_{\sigma_r,\tau} + (-1)^{p+q} \sum_t \omega u|_{\sigma,\tau_t}\\ &= \sum_{r=0}^{p} \sum_{\tilde{z}} (-1)^{r+m(\tilde{z})+\varsigma(q)} u|_{\tilde{z}} + \sum_{t=0}^{q} \sum_{\hat{z}} (-1)^{p+q+t+m(\hat{z})+\varsigma(q-1)} u|_{\hat{z}},\end{aligned}$$

where z traverses (σ, τ), $\tilde{z}$ traverses (σ_r, τ), and $\hat{z}$ traverses (σ, τ_t).

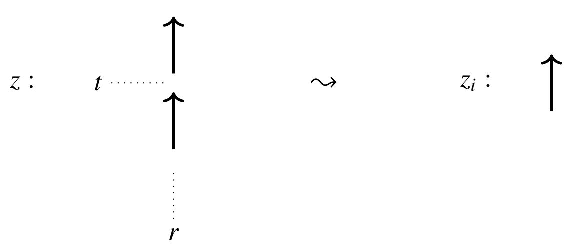
Now, *a priori* there are more summands in $\omega du|_{\sigma,\tau}$. The extra summands arise from deleting the corners of traversals:



But all of these corners come in pairs – a lower-right and an upper-left. The difference in squares below and to the right in the grid for these paired traversals is exactly one, and so $m(z)$ is of the opposite parity. Thus the summands corresponding to paired corner-cuts cancel out in $\varphi du|_{\sigma,\tau}$.

We are left with two cases – when z_i is shortened along a vertical stretch and when it is shortened along a horizontal stretch.

(Case 1). Suppose z_i is shortened along a vertical stretch of z:



The traversal $\hat{z}$ matching z_i in $d\varphi u|_{\sigma,\tau}$ appears when τ is shortened at t. The coefficient of z_i is $(-1)^{\varsigma(q)+m(z)+i}$ and the coefficient of the matching traversal is $(-1)^{p+q+t+m(\hat{z})+\varsigma(q-1)}$. There are $p - r$ squares in the grid to the right of any of the arrows pictured. This means that

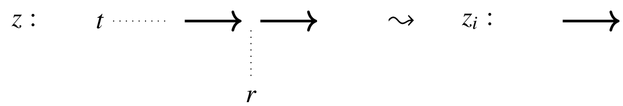
$$m(z) = m(\hat{z}) + p - r.$$

Also note that $i = t + r$. We have

$$\begin{aligned}&\varsigma(q) + m(z) + i + p + q + t + m(\hat{z}) + \varsigma(q-1) =\\ &\quad = \varsigma(q) + \varsigma(q-1) + 2m(\hat{z}) + 2p + i + r + t \equiv\\ &\quad \equiv \varsigma(q) + \varsigma(q-1) + q + 2i\\ &\quad \equiv 0, \qquad \text{mod } 2\end{aligned}$$

thus the two coefficients are the same.

(Case 2). Suppose z_i is shortened along a horizontal stretch of z:



The traversal $\tilde{z}$ matching z_i in $d\omega u|_{\sigma,\tau}$ appears when σ is shortened at r. The coefficient of z_i is $(-1)^{\varsigma(q)+m(z)+i}$ and the coefficient of the matching traversal is $(-1)^{r+m(\tilde{z})+\varsigma(q)}$. There are t squares in the grid below any of the arrows pictured. This means that

$$m(z) = m(\hat{z}) + t.$$

Again note that $i = t + r$. We have

$$\begin{aligned}\varsigma(q)+&m(z) + i + r + m(\tilde{z}) + \varsigma(q) = \\ &= 2\varsigma(q) + 2m(\tilde{z}) + i + r + t \equiv \\ &\equiv 2i \\ &\equiv 0, \qquad \text{mod } 2\end{aligned}$$

thus the two coefficients are the same.

□

7.3 Long exact sequence in the cohomology of the total complex

If we have a poset bundle $\xi : \mathbf{B} \to \mathbf{Sh}$ and a subcategory $\mathbf{C}$ of $\mathbf{B}$, then we will denote the chain complex $T^\bullet_{\xi_\mathbf{C}}$ (recall Section 6.2) by just $T^\bullet_\mathbf{C}$. Below we headline the main result of this section and leave the proof until we have built up the required machinery.

Theorem 7.10. *Let $\xi : \mathbf{B} \to \mathbf{Sh}$ be a poset bundle of sheaves with $\mathbf{B}$ an admissible poset for $x > 0$. Then there is a long exact sequence*

$$\cdots \to H^{n-1}T^\bullet_{\mathbf{B}_{\not\geq x}} \to H^nT^\bullet_\xi \to H^nT^\bullet_{\mathbf{B}_{\geq x}} \oplus H^nT^\bullet_{\mathbf{B}_{\not\geq x}} \to H^nT^\bullet_{\mathbf{B}_{\not\geq x}} \to H^{n+1}T^\bullet_\xi \to \cdots$$

We will need to leverage the admissibility condition in the theorem to establish the connection between the total complex of the whole sheaf and those of the two smaller parts $\mathbf{B}_{\geq x}$ and $\mathbf{B}_{\not\geq x}$, determined by the element $x > 0$. Recall that we assume all the $\mathbf{E}_y$ are posets.

Where possible, we will use x's to refer to objects in $\mathbf{B}_{\not\geq x}$ and z's to refer to objects of $\mathbf{B}_{\geq x}$. We can write down explicitly what T^n_ξ, $T^n_{\mathbf{B}_{\geq x}}$, and $T^n_{\mathbf{B}_{\not\geq x}}$ are:

$$T^n_\xi = \bigoplus_{p+q=n} \prod_{x_0 \le \cdots \le x_p \in \mathbf{B}} \prod_{y_0 \le \cdots \le y_q \in \mathbf{E}_{x_0}} F_{x_0}(y_0).$$

$$T^n_{\mathbf{B}_{\ge x}} = \bigoplus_{p+q=n} \prod_{z_0 \le \cdots \le z_p \in \mathbf{B}_{\ge x}} \prod_{y_0 \le \cdots \le y_q \in \mathbf{E}_{z_0}} F_{z_0}(y_0).$$

$$T^n_{\mathbf{B}_{\ngeq x}} = \bigoplus_{p+q=n} \prod_{x_0 \le \cdots \le x_p \in \mathbf{B}_{\ngeq x}} \prod_{y_0 \le \cdots \le y_q \in \mathbf{E}_{x_0}} F_{x_0}(y_0).$$

Define the quotient map

$$\rho : T^n_\xi \to T^n_{\mathbf{B}_{\ge x}} \oplus T^n_{\mathbf{B}_{\ngeq x}}$$

by setting to 0 any coordinate corresponding to a sequence $x_0 \le \cdots \le x_p \in \mathbf{B}$ that has objects in both $\mathbf{B}_{\ge x}$ and $\mathbf{B}_{\ngeq x}$. Explicitly, if $u \in T^{p+q}_\xi$, $\sigma = x_0 \le \cdots \le x_p \in \mathbf{B}_{\ge x}$ or $\mathbf{B}_{\ngeq x}$, and $\tau = y_0 \le \cdots \le y_q \in \mathbf{E}_{x_0}$, then

$$\rho u|_{\sigma,\tau} = u|_{\sigma,\tau}.$$

To see that ρ is a chain map, let $x_i \in \mathbf{B}_{\ge x}$ for all i. We have

$$\begin{aligned}
\rho du|_{\sigma,\tau} &= du|_{\sigma,\tau} \\
&= \sum_{i=0}^{p} (-1)^i u|_{\sigma_i,\tau} + (-1)^{p+q} \sum_{j=0}^{q} (-1)^j u|_{\sigma,\tau_j} \\
&= \sum_{i=0}^{p} (-1)^i \rho u|_{\sigma_i,\tau} + (-1)^{p+q} \sum_{j=0}^{q} (-1)^j \rho u|_{\sigma,\tau_j} \\
&= d\rho u|_{\sigma,\tau}.
\end{aligned}$$

The calculation is analogous if $x_i \in \mathbf{B}_{\ngeq x}$ for all i. Therefore ρ is a chain map.

It is also clearly surjective, so we have a short exact sequence

$$0 \to M^\bullet \to T^\bullet_\xi \to T^\bullet_{\mathbf{B}_{\ge x}} \oplus T^\bullet_{\mathbf{B}_{\ngeq x}} \to 0$$

for a particular chain complex $M^\bullet$.

We describe $M^\bullet$ explicitly:

$$M^n = \bigoplus_{p+q=n} \prod_{x_0 \le \cdots \le x_p} \prod_{y_0 \le \cdots \le y_q \in \mathbf{E}_{x_0}} F_{x_0}(y_0),$$

where $x_0 \in \mathbf{B}_{\ngeq x}$, $x_p \in \mathbf{B}_{\ge x}$.

We can rewrite $M^\bullet$ to pay attention to how many of the x_i's are in $\mathbf{B}_{\ngeq x}$ and how many are in $\mathbf{B}_{\ge x}$:

$$M^n = \bigoplus_{s+t+q=n} \prod_{x_0 \le \cdots \le x_s \le z_0 \le \cdots \le z_{t-1}} \prod_{y_0 \le \cdots \le y_q \in \mathbf{E}_{x_0}} F_{x_0}(y_0),$$

where $x_i \in \mathbf{B}_{\ngeq x}$, $z_i \in \mathbf{B}_{\ge x}$, $s \ge 0, t \ge 1$.

Proposition 7.11. *Let* $\xi : \mathbf{B} \to \mathbf{Sh}$ *be a poset bundle of sheaves with* $\mathbf{B}$ *an admissible poset for* $x > 0$*. If* $M^\bullet$ *is as above, there is a chain map*

$$\varphi_1 : T^{n-1}_{\mathbf{B}_{\not\geq x}} \to M^n$$

that induces an isomorphism in cohomology.

Proof. In an attempt to keep the notation less cluttered, denote

$$K^n = T^{n-1}_{\mathbf{B}_{\not\geq x}}.$$

We define the chain map $\varphi_1 : K^n \to M^n$, which will extend to a morphism of filtered complexes. By showing that φ_1 induces isomorphisms on the first pages of the two spectral sequences associated to the two filtrations, the Mapping Lemma 5.14 implies that it is a quasi-isomorphism.

Let $\sigma = x_0 \leq \cdots \leq x_s \leq z_0 \leq \cdots \leq z_{t-1}$ be a sequence in $\mathbf{B}$ with $x_i \in \mathbf{B}_{\not\geq x}$, $z_i \in \mathbf{B}_{\geq x}$, $s \geq 0$, $t \geq 1$. Denote $\sigma' = x_0 \leq \cdots \leq x_s$. Also let $\tau = y_0 \leq \cdots \leq y_q$ be a sequence in $\mathbf{E}_{x_0}$. Now if $s + t + q = n$, we define $\varphi_1 : K^n \to M^n$ by

$$\varphi_1 u|_{\sigma,\tau} = \begin{cases} (-1)^q u|_{\sigma',\tau} & \text{if } t = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Intuitively, φ_1 acts like the map φ in Proposition 6.11 on the portion of $M^\bullet$ that matches $T^\bullet_{\mathbf{B}_{\not\geq x}}$. To see that φ_1 is a chain map, note that if $t \geq 3$, both $\varphi_1 du$ and $d\varphi_1 u$ are 0. If $t = 2$, then

$$\begin{aligned} d\varphi_1 u|_{x_0\leq\cdots\leq x_s\leq z_0\leq z_1, y_0\leq\cdots\leq y_q} &= \sum_{i=0}^{s} (-1)^i \varphi_1 u|_{x_0\leq\cdots\leq\hat{x}_i\leq\cdots\leq x_s\leq z_0\leq z_1, y_0\leq\cdots\leq y_q} + \\ &\quad (-1)^{s+1}\varphi_1 u|_{x_0\leq\cdots\leq x_s\leq z_1, y_0\leq\cdots\leq y_q} + \\ &\quad (-1)^{s+2}\varphi_1 u|_{x_0\leq\cdots\leq x_s\leq z_0, y_0\leq\cdots\leq y_q} + \\ &\quad (-1)^n \sum_{j=0}^{q} \varphi_1 u|_{x_0\leq\cdots\leq x_s\leq z_0\leq z_1, y_0\leq\cdots\leq\hat{y}_i\leq\cdots\leq y_q} \\ &= 0 \\ &= \varphi_1 du|_{x_0\leq\cdots\leq x_s\leq z_0\leq z_1, y_0\leq\cdots\leq y_q}. \end{aligned}$$

Finally, if $t = 1$, then

$$
\begin{aligned}
d\varphi_1 u|_{x_0\le\cdots\le x_s\le z_0, y_0\le\cdots\le y_q} &= \sum_{i=0}^{s} \varphi_1 u|_{x_0\le\cdots\le\hat{x}_i\le\cdots\le x_s\le z_0, y_0\le\cdots\le y_q} + \\
&\quad (-1)^{s+q+1}\sum_{j=0}^{q} \varphi_1 u|_{x_0\le\cdots\le x_s\le z_0, y_0\le\cdots\le\hat{y}_i\le\cdots\le y_q} \\
&= (-1)^q \sum_{i=0}^{s} u|_{x_0\le\cdots\le\hat{x}_i\le\cdots\le x_s, y_0\le\cdots\le y_q} + \\
&\quad (-1)^{s+2q}\sum_{j=0}^{q} u|_{x_0\le\cdots\le x_s\le z_0, y_0\le\cdots\le\hat{y}_i\le\cdots\le y_q} \\
&= (-1)^q du|_{x_0\le\cdots\le x_s, y_0\le\cdots\le y_q} \\
&= \varphi_1 du|_{x_0\le\cdots\le x_s\le z_0, y_0\le\cdots\le y_q}.
\end{aligned}
$$

Now we define filtrations of $M^\bullet$ and $K^\bullet$:

$$\mathcal{F}^p M^n = \{u \in M^n : u|_{\sigma,\tau} \neq 0 \Rightarrow \sigma = x_0 \le \cdots \le x_s \le z_0 \le \cdots \le z_{t-1} \text{ with } s \ge p\},$$

$$\mathcal{J}^p K^n = \{u \in K^n : u|_{\sigma,\tau} \neq 0 \Rightarrow \sigma = x_0 \le \cdots \le x_s \text{ with } s \ge p\}.$$

We want to use the Mapping Lemma 5.14 for these two filtrations, so the next step is establishing all the assumptions of the lemma. We prove them for $\mathcal{F}$ with the arguments for $\mathcal{J}$ being analogous.

($\mathcal{F}$ is a filtration). It is clear from the definition of $\mathcal{F}$ that $\mathcal{F}^{p+1}M^n \subseteq \mathcal{F}^p M^n$ for each p and n. Remains to show that $\mathcal{F}^p M^\bullet$ is a complex for each p. Let $u \in \mathcal{F}^p M^n$ and $\sigma = x_0 \le \cdots \le x_s \le z_0 \le \cdots \le z_{t-1}$ with $s < p$. Then for any sequence $\tau \in \mathbf{E}_{x_0}$ (of appropriate length q) we have

$$du|_{\sigma,\tau} = \sum_{i=0}^{s}(-1)^i u|_{\sigma_i,\tau} + (-1)^{s+1}\sum_{k=0}^{t-1}(-1)^k u|_{\sigma_{s+k},\tau} + +(-1)^{s+t+q}\sum_{\ell=0}^{q}(-1)^\ell u|_{\sigma,\tau_\ell}.$$

The summands in the first sum correspond to x-sequences of length $s-1 < p$, while the summands in the other two sums correspond to x-sequences of length $s < p$. All those coordinates are 0 in $u \in \mathcal{F}^p M^n$, so d induces a differential on $\mathcal{F}^p M^\bullet$.

($\mathcal{F}$ is convergent below). Observe that $\mathcal{F}^0 M^n = M^n$, since M^n does not have any coordinates corresponding to sequences in $\mathbf{B}$ not containing elements of $\mathbf{B}_{\not\ge x}$.

($\mathcal{F}$ is bounded above). Observe that $\mathcal{F}^n M^n = 0$, since we need $s + t + q = n$ and $t \ge 1$.

(φ_1 is a morphism of filtrations). Let $u \in \mathcal{J}^p K^n$, set $\sigma = x_0 \le \cdots \le x_s \le z$ and $\tau = y_0 \le \cdots \le y_q$. First suppose $s + q + 1 \neq n$. The potentially non-zero coordinates of $\varphi_1 u|_{\sigma,\tau}$ correspond to sequences of combined length satisfying $s + q \neq n - 1$, so they are also 0. Now suppose $s < p$. Again, the potentially non-zero coordinates of $\varphi_1 u|_{\sigma,\tau}$ correspond to x-sequences of length $s < p$, so are also 0. Thus $\varphi_1(\mathcal{J}^p K^n) \subseteq \mathcal{F}^p M^n$.

To see that φ_1 induces chain maps $\mathcal{J}^p K^\bullet \to \mathcal{F}^p M^\bullet$ for every p, note that we already know that $d\varphi_1 = \varphi_1 d$ and that $\varphi_1(\mathcal{J}^p K^n) \subseteq \mathcal{F}^p M^n$.

Let E, E' be the spectral sequences associated to the filtrations $\mathcal{F}, \mathcal{J}$, respectively. We have

$$E_0^{p,q} = \frac{\mathcal{F}^p M^{p+q}}{\mathcal{F}^{p+1} M^{p+q}} = \{u \in M^{p+q} : u|_{\sigma,\tau} \neq 0 \Rightarrow \sigma = x_0 \leq \cdots \leq x_p \leq z_0 \leq \cdots \leq z_{t-1}\},$$
$$E_0'^{p,q} = \frac{\mathcal{J}^p K^{p+q}}{\mathcal{J}^{p+1} K^{p+q}} = \{u \in K^{p+q} : u|_{\sigma,\tau} \neq 0 \Rightarrow \sigma = x_0 \leq \cdots \leq x_p\}.$$

The vertical differentials in E_0 are given by

$$du|_{x_0\leq\cdots\leq x_p\leq z_0\leq\cdots\leq z_{t-1},y_0\leq\cdots\leq y_{q-t}} = (-1)^{p+1}\sum_{i=0}^{t-1}(-1)^i u|_{x_0\leq\cdots\leq x_p\leq z_0\leq\cdots\leq\hat{z}_i\leq\cdots\leq z_{t-1},y_0\leq\cdots\leq y_{q-t}} + \\ + (-1)^{p+q}\sum_{\ell=0}^{q-t}(-1)^\ell u|_{x_0\leq\cdots\leq x_p\leq z_0\leq\cdots\leq z_{t-1},y_0\leq\cdots\leq\hat{y}_\ell\leq\cdots\leq y_{q-t}}$$

and the vertical differentials in E_0' are given by

$$du|_{x_0\leq\cdots\leq x_p,y_0\leq\cdots\leq y_q} = (-1)^{p+q}\sum_{\ell=0}^{q}(-1)^\ell u|_{x_0\leq\cdots\leq x_p,y_0\leq\cdots\leq\hat{y}_\ell\leq\cdots\leq y_q}.$$

Using the notation from Definition 7.3 we can thus rewrite

$$E_0^{p,\bullet} = \prod_{x_0\leq\cdots\leq x_p}(-1)^{p+1}T^{\bullet-1}_{\mathbf{B}_{\geq x}^{\geq x_p}\times(\mathbf{E}_{x_0};F_{x_0})}$$

and

$$E_0'^{p,\bullet} = \prod_{x_0\leq\cdots\leq x_p}(-1)^{p+q}\mathcal{S}^{\bullet-1}(\mathbf{E}_{x_0};F_{x_0}).$$

Now note that φ_1 acts as the product over all p-long x-sequences in $\mathbf{B}_{\not\geq x}$ of the maps in Proposition 6.11, since $\mathbf{B}$ is an admissible poset and thus the subposet $\mathbf{B}_{\geq x}^{\geq x_p}$ has a unique minimum. This means that $\varphi_1 : E_0'^{p,\bullet} \to E_0^{p,\bullet}$ is a quasi-isomorphism and thus

$$E_1'^{p,q} = H^p(E_0'^{p,\bullet}) \overset{\varphi_1^\bullet}{\cong} H^p(E_0^{p,\bullet}) = E_1^{p,q}.$$

The Mapping Lemma 5.14 then implies that

$$\varphi_1^\bullet : H^{n-1}T^\bullet_{\mathbf{B}_{\not\geq x}} \cong H^n(M^\bullet).$$ □

We can now easily complete the proof of the theorem, headlined in this section.

Proof of Theorem 7.10. We have the short exact sequence from before

$$0 \to M^\bullet \to T_\xi^\bullet \to T^n_{\mathbf{B}_{\geq x}} \oplus T^n_{\mathbf{B}_{\not\geq x}} \to 0,$$

from which we get a long exact sequence in homology

$$\cdots \to H^{n-1}T^\bullet_{\mathbf{B}_{\geq x}} \oplus H^{n-1}T^\bullet_{\mathbf{B}_{\not\geq x}} \to H^n M^\bullet \to H^n T^\bullet_\xi \to H^n T^\bullet_{\mathbf{B}_{\geq x}} \oplus H^n T^\bullet_{\mathbf{B}_{\not\geq x}} \to \\ \to H^{n+1}M^\bullet \to \cdots$$

Replacing the occurrences of $H^n M^\bullet$ with $H^{n-1}T^\bullet_{\mathbf{B}_{\not\geq x}}$ and the maps around those occurrences with the appropriate compositions with $\varphi_1^\bullet$ and $\varphi_1^{\bullet-1}$ gives the required long exact sequence. □

7.4 Long exact sequence in sheaf cohomology

We now repeat this procedure for the cochain complex of the total sheaf $(\mathbf{E}_\xi; F_\xi)$. The story is fairly similar to that of the previous section, so we are a little briefer. Again, we headline the main result, with the proof delayed until the end of the section.

Theorem 7.12. *Let $\xi : \mathbf{B} \to \mathbf{Sh}$ be a poset bundle of sheaves with $\mathbf{B}$ an admissible poset. Then there is a long exact sequence*

$$\cdots \to H^{n-1}(\mathbf{E}_{\mathbf{B}_{\not\geq x}}; F_{\mathbf{B}_{\not\geq x}}) \to H^n(\mathbf{E}_\xi; F_\xi) \to H^n(\mathbf{E}_{\mathbf{B}_{\geq x}}; F_{\mathbf{B}_{\geq x}}) \oplus H^n(\mathbf{E}_{\mathbf{B}_{\not\geq x}}; F_{\mathbf{B}_{\not\geq x}}) \to \cdots$$

Where possible, we will use x's to refer to objects in $\mathbf{E}_{\mathbf{B}_{\not\geq x}}$ and z's to refer to objects of $\mathbf{E}_{\mathbf{B}_{\geq x}}$. We can write down explicitly what $\mathcal{S}^n(\mathbf{E}_\xi; F_\xi)$, $\mathcal{S}^n(\mathbf{E}_{\mathbf{B}_{\geq x}}; F_{\mathbf{B}_{\geq x}})$, and $\mathcal{S}^n(\mathbf{E}_{\mathbf{B}_{\not\geq x}}; F_{\mathbf{B}_{\not\geq x}})$ are:

$$\mathcal{S}^n(\mathbf{E}_\xi; F_\xi) = \prod_{x_0 \leq \cdots \leq x_n \in \mathbf{E}_\xi} F_\xi(x_0)$$
$$\mathcal{S}^n(\mathbf{E}_{\mathbf{B}_{\geq x}}; F_{\mathbf{B}_{\geq x}}) = \prod_{z_0 \leq \cdots \leq z_n \in \mathbf{E}_{\mathbf{B}_{\geq x}}} F_\xi(x_0)$$
$$\mathcal{S}^n(\mathbf{E}_{\mathbf{B}_{\not\geq x}}; F_{\mathbf{B}_{\not\geq x}}) = \prod_{x_0 \leq \cdots \leq x_n \in \mathbf{E}_{\mathbf{B}_{\not\geq x}}} F_\xi(x_0)$$

Define another quotient map

$$\rho : \mathcal{S}^n(\mathbf{E}_\xi; F_\xi) \to \mathcal{S}^n(\mathbf{E}_{\mathbf{B}_{\geq x}}; F_{\mathbf{B}_{\geq x}}) \oplus \mathcal{S}^n(\mathbf{E}_{\mathbf{B}_{\not\geq x}}; F_{\mathbf{B}_{\not\geq x}})$$

by setting to 0 any coordinate corresponding to a sequence $x_0 \leq \cdots \leq x_n$ in $\mathbf{E}_\xi$ that has objects in both $\mathbf{E}_{\mathbf{B}_{\geq x}}$ and $\mathbf{E}_{\mathbf{B}_{\not\geq x}}$. This is a chain map by an analogous argument to the one for the quotient before Proposition 7.11.

The map ρ is clearly surjective, so we have an short exact sequence

$$0 \to N^\bullet \to \mathcal{S}^\bullet(\mathbf{E}_\xi; F_\xi) \to \mathcal{S}^n(\mathbf{E}_{\mathbf{B}_{\geq x}}; F_{\mathbf{B}_{\geq x}}) \oplus \mathcal{S}^n(\mathbf{E}_{\mathbf{B}_{\not\geq x}}; F_{\mathbf{B}_{\not\geq x}}) \to 0$$

for a particular chain complex $N^\bullet$.

We describe $N^\bullet$ explicitly:

$$N^n = \prod_{x_0 \leq \cdots \leq x_n} F_\xi(x_0),$$

where $x_0 \in \mathbf{E}_{\mathbf{B}_{\ngeq x}}$, $x_n \in \mathbf{E}_{\mathbf{B}_{\geq x}}$.

We can rewrite $N^\bullet$ to pay attention to how many of the x_i's are in $\mathbf{E}_{\mathbf{B}_{\ngeq x}}$ and how many are in $\mathbf{E}_{\mathbf{B}_{\geq x}}$:

$$N^n = \prod_{x_0 \leq \cdots \leq x_p \leq z_0 \leq \cdots \leq z_{n-p-1}} F_\xi(x_0),$$

where $x_i \in \mathbf{E}_{\mathbf{B}_{\ngeq x}}$, $z_i \in \mathbf{E}_{\mathbf{B}_{\geq x}}$, $p \geq 0, n-p \geq 1$.

Proposition 7.13. *Let $\xi : \mathbf{B} \to \mathbf{Sh}$ be a poset bundle of sheaves with $\mathbf{B}$ an admissible poset for $x > 0$. If $N^\bullet$ is as above, there is a chain map*

$$\varphi_2 : \mathcal{S}^{n-1}(\mathbf{E}_{\mathbf{B}_{\ngeq x}}; F_{\mathbf{B}_{\ngeq x}}) \to N^n$$

that induces an isomorphism in cohomology.

Proof. We define a filtration $\mathcal{J}$ of $N^\bullet$:

$$\mathcal{J}^p N^n = \{u \in N^n : u|_\sigma \neq 0 \Rightarrow \sigma = x_0 \leq \cdots \leq x_s \leq z_0 \leq \cdots \leq z_{n-s-1}, \text{ with } s \geq p\}.$$

The proof that this is a filtration is analogous to the proofs of the filtrations from Proposition 7.11.

Let E be the spectral sequence associated to the filtration $\mathcal{J}$ of N. We have

$$E_0^{p+q} = \frac{\mathcal{J}^p N^{p+q}}{\mathcal{J}^{p+1} N^{p+q}} = \{u \in B^n : u|_\sigma \neq 0 \Rightarrow \sigma = x_0 \leq \cdots \leq x_p \leq z_0 \leq \cdots \leq z_{q-1}\}.$$

The vertical differentials in E_0 are given by

$$du|_{x_0 \leq \cdots \leq x_p \leq z_0 \leq \cdots \leq z_{q-1}} = (-1)^{p+1} \sum_{i=0}^{q-1} (-1)^i u|_{x_0 \leq \cdots \leq x_p \leq z_0 \leq \cdots \leq \hat{z}_i \leq \cdots \leq z_{q-1}}.$$

We can thus write

$$E_0^{p,\bullet} = \prod_{x_0 \leq \cdots \leq x_p} (-1)^{p+1} \mathcal{S}^{\bullet-1}(\{z \in \mathbf{E}_{\mathbf{B}_{\geq x}} \mid z \geq x_p\}, \Delta F_\xi(x_0)).$$

But the $\mathcal{S}$ complex on the right is of a poset with a constant sheaf. By Lemma 7.5 the underlying poset has a unique minimum, so

$$E_1^{p,q} = H^q E_0^{p,\bullet} = \begin{cases} \prod_{x_0 \leq \cdots \leq x_p} (-1)^{p+1} F_\xi(x_0) & \text{if } q = 1, \\ 0 & \text{otherwise.} \end{cases}$$
$$= \begin{cases} (-1)^n \mathcal{S}^{n-1}(\mathbf{E}_{\mathbf{B}_{\ngeq x}}; F_{\mathbf{B}_{\ngeq x}}) & \text{if } q = 1, \\ 0 & \text{otherwise.} \end{cases}$$

So on the E_1 page we have the single $q = 1$ row

$$\cdots \to (-1)^n \mathcal{S}^{n-1}(\mathbf{E}_{\mathbf{B}_{\ngeq x}}; F_{\mathbf{B}_{\ngeq x}}) \to (-1)^{n+1} \mathcal{S}^n(\mathbf{E}_{\mathbf{B}_{\ngeq x}}; F_{\mathbf{B}_{\ngeq x}}) \to \cdots.$$

The differential on this page is induced by the differential

$$du|_{x_0\le\cdots\le x_p\le z_0\le\cdots\le z_{q-1}} = \sum_{i=0}^{p}(-1)^i u|_{x_0\le\cdots\le\hat{x}_i\le\cdots\le x_p\le z_0\le\cdots\le z_{q-1}},$$

which, since it keeps the z-sequence constant, induces the following differential on the above row on the E_1 page:

$$du|_{x_0\le\cdots\le x_p} = \sum_{i=0}^{p}(-1)^i u|_{x_0\le\cdots\le\hat{x}_i\le\cdots\le x_p}.$$

Since $d(-d) = (-d)d = 0$, $\ker(-d) = \ker d$, and $\operatorname{im}(-d) = \operatorname{im} d$, we have that the E_2 page is

$$E_2^{p,q} \cong \begin{cases} H^{p+q-1}\mathcal{S}^\bullet(\mathbf{E}_{\mathbf{B}_{\not\geq x}}; F_{\mathbf{B}_{\not\geq x}}) & \text{if } q = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Then $E_2^{p,q} \cong E_\infty^{p,q}$ and so

$$E \Rightarrow H^{n-1}\mathcal{S}^\bullet(\mathbf{E}_{\mathbf{B}_{\not\geq x}}; F_{\mathbf{B}_{\not\geq x}}) \cong N^n.$$

In particular, this isomorphism is witnessed by a similar quasi-isomorphism to that in Proposition 7.11, namely $\varphi_2 : \mathcal{S}^{n-1}(\mathbf{E}_{\mathbf{B}_{\not\geq x}}; F_{\mathbf{B}_{\not\geq x}}) \to N^n$ defined by

$$\varphi_2 u|_{x_0\le\cdots\le x_n} = \begin{cases} u|_{x_0\le\cdots\le x_{n-1}} & \text{if } x_{n-1} \in \mathbf{E}_{\mathbf{B}_{\not\geq x}}, x_n \in \mathbf{E}_{\mathbf{B}_{\geq x}}, \\ 0 & \text{otherwise.} \end{cases}$$ □

We can now, again, easily prove the headlined theorem.

Proof of Theorem 7.12. We have the short exact sequence from before

$$0 \to N^\bullet \to \mathcal{S}^\bullet(\mathbf{E}_\xi; F_\xi) \to \mathcal{S}^\bullet(\mathbf{E}_{\mathbf{B}_{\not\geq x}}; F_{\mathbf{B}_{\not\geq x}}) \oplus \mathcal{S}^\bullet(\mathbf{E}_{\mathbf{B}_{\geq x}}; F_{\mathbf{B}_{\geq x}}) \to 0,$$

from which we get a long exact sequence in homology

$$\cdots \to H^{n-1}(\mathbf{E}_{\mathbf{B}_{\geq x}}; F_{\mathbf{B}_{\geq x}}) \oplus H^{n-1}(\mathbf{E}_{\mathbf{B}_{\not\geq x}}; F_{\mathbf{B}_{\not\geq x}}) \to H^n N^\bullet \to H^n(\mathbf{E}_\xi; F_\xi) \to$$
$$\to H^n(\mathbf{E}_{\mathbf{B}_{\geq x}}; F_{\mathbf{B}_{\geq x}}) \oplus H^n(\mathbf{E}_{\mathbf{B}_{\not\geq x}}; F_{\mathbf{B}_{\not\geq x}}) \to H^{n+1}N^\bullet \to \cdots$$

Replacing the occurrences of $H^n N^\bullet$ with $H^{n-1}(\mathbf{E}_{\mathbf{B}_{\not\geq x}}; F_{\mathbf{B}_{\not\geq x}})$ and the maps around those occurrences with the appropriate compositions with $\varphi_2^\bullet$ and $\varphi_2^{\bullet-1}$ gives the required long exact sequence. □

7.5 The bicomplex and the total sheaf

We have all the necessary prerequisites to prove the main theorem:

Theorem 7.14. *Let $\xi : \mathbf{B} \to \mathbf{Sh}$ be a poset bundle of sheaves with $\mathbf{B}$ a recursively admissible finite poset, and $(\mathbf{E}_\xi; F_\xi)$ the associated total sheaf. Then there is a spectral sequence*

$$E_2^{p,q} = H^p(\mathbf{B}; \mathcal{H}^q_{fib}(\xi)) \Rightarrow H^\bullet(\mathbf{E}_\xi; F_\xi).$$

Proof. Proposition 6.10 gives us

$$E_2^{p,q} = H^p(\mathbf{B}; \mathcal{H}^q_{fib}(\xi)) \Rightarrow H^\bullet T^\bullet_\xi,$$

so it is enough to show that $H^\bullet T^\bullet_\xi \cong H^\bullet(\mathbf{E}_\xi, F_\xi)$. We will do this by induction on the cardinality of $\mathbf{B}$. Recall the chain map $\omega : \mathcal{S}^\bullet(\mathbf{E}_\xi; F_\xi) \to T^\bullet_\xi$ from Section 7.2:

$$\omega u|_{\sigma,\tau} = (-1)^{\varsigma(q)} \sum_z (-1)^{m(z)} u|_z,$$

where the sum is taken over all traversals z of the grid of (σ, τ). We have two short exact sequences from Theorems 7.11 and 7.13. The map ω gives a morphism of these short exact sequences

$$\begin{array}{ccccccccc}
0 & \longrightarrow & N^n & \xrightarrow{\varepsilon} & \mathcal{S}^n(\mathbf{E}_\xi; F_\xi) & \xrightarrow{\pi} & \mathcal{S}^n(\mathbf{E}_{\mathbf{B}_{\not\geq x}}; F_{\mathbf{B}_{\not\geq x}}) \oplus \mathcal{S}^n(\mathbf{E}_{\mathbf{B}_{\geq x}}; F_{\mathbf{B}_{\geq x}}) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \omega'} & & \downarrow{\scriptstyle \omega} & & \downarrow{\scriptstyle \omega \oplus \omega} & & \\
0 & \longrightarrow & M^n & \xrightarrow{\varepsilon} & T^n_\xi & \xrightarrow{\pi} & T^n_{\mathbf{B}_{\not\geq x}} \oplus T^n_{\mathbf{B}_{\geq x}} & \longrightarrow & 0
\end{array}$$

where the maps ε are the injections and the maps π the projections of the respective modules. The map ω' is the restriction of ω to the subcomplexes N^n and M^n. We need to check the commutativity of the two squares.

(Left square). The maps ε are just injections, so we have

$$\begin{aligned}
\varepsilon\omega u|_{\sigma,\tau} = \omega u|_{\sigma,\tau} &= (-1)^{\varsigma(q)} \sum_z (-1)^{m(z)} u|_z \\
&= (-1)^{\varsigma(q)} \sum_z (-1)^{m(z)} \varepsilon u|_z = \omega\varepsilon u|_{\sigma,\tau}.
\end{aligned}$$

(Right square). Similarly, the maps π are projections, so

$$\begin{aligned}
\pi\omega u|_{\sigma,\tau} = \omega u|_{\sigma,\tau} &= (-1)^{\varsigma(q)} \sum_z (-1)^{m(z)} u|_z \\
&= (-1)^{\varsigma(q)} \sum_z (-1)^{m(z)} \pi u|_z = \omega\pi u|_{\sigma,\tau}.
\end{aligned}$$

The naturality of the homology functor then gives a morphism of long exact sequences, which contains the commutative diagram in Figure 7.6.

Recall from Propositions 7.11 and 7.13 the quasi-isomorphisms

$$\varphi_1 : T^{n-1}_{\mathbf{B}_{\not\geq x}} \to M^n \text{ and } \varphi_2 : \mathcal{S}^{n-1}(\mathbf{E}_{\mathbf{B}_{\not\geq x}}; F_{\mathbf{B}_{\not\geq x}}) \to M^n.$$

$$
\begin{array}{ccc}
H^{n+1}N^{\bullet} & \xrightarrow{\ \omega'^{\bullet}\ } & H^{n+1}M^{\bullet} \\
\uparrow{\scriptstyle \delta} & & \uparrow{\scriptstyle \delta} \\
H^{n}(\mathbf{E}_{\mathbf{B}_{\not\geq x}};F_{\mathbf{B}_{\not\geq x}}) \oplus H^{n}(\mathbf{E}_{\mathbf{B}_{\geq x}};F_{\mathbf{B}_{\geq x}}) & \xrightarrow{\ \omega^{\bullet}\oplus\omega^{\bullet}\ } & H^{n}T^{\bullet}_{\mathbf{B}_{\not\geq x}} \oplus H^{n}T^{\bullet}_{\mathbf{B}_{\geq x}} \\
\uparrow{\scriptstyle \pi^{\bullet}} & & \uparrow{\scriptstyle \pi^{\bullet}} \\
H^{n}(\mathbf{E}_{\xi},F_{\xi}) & \xrightarrow{\ \omega^{\bullet}\ } & H^{n}T^{\bullet}_{\xi} \\
\uparrow{\scriptstyle \varepsilon^{\bullet}} & & \uparrow{\scriptstyle \varepsilon^{\bullet}} \\
H^{n}N^{\bullet} & \xrightarrow{\ \omega'^{\bullet}\ } & H^{n}M^{\bullet} \\
\uparrow{\scriptstyle \delta} & & \uparrow{\scriptstyle \delta} \\
H^{n-1}(\mathbf{E}_{\mathbf{B}_{\not\geq x}};F_{\mathbf{B}_{\not\geq x}}) \oplus H^{n-1}(\mathbf{E}_{\mathbf{B}_{\geq x}};F_{\mathbf{B}_{\geq x}}) & \xrightarrow{\ \omega^{\bullet}\oplus\omega^{\bullet}\ } & H^{n-1}T^{\bullet}_{\mathbf{B}_{\not\geq x}} \oplus H^{n-1}T^{\bullet}_{\mathbf{B}_{\geq x}}
\end{array}
$$

Fig. 7.6: A portion of the commutative diagram given by the morphism of short exact sequences.

Claim. The following diagram commutes

$$
\begin{array}{ccc}
\mathcal{S}^{n-1}(\mathbf{E}_{\mathbf{B}_{\not\geq x}};F_{\mathbf{B}_{\not\geq x}}) & \xrightarrow{\ \varphi_2\ } & N^{n} \\
\downarrow{\scriptstyle \omega} & & \downarrow{\scriptstyle \omega'} \\
T^{n-1}_{\mathbf{B}_{\not\geq x}} & \xrightarrow{\ \varphi_1\ } & M^{n}
\end{array}
$$

Proof of claim. Let $u \in \mathcal{S}^{n-1}(\mathbf{E}_{\mathbf{B}_{\not\geq x}};F_{\mathbf{B}_{\not\geq x}})$. Suppose

$$\sigma = x_0 \leq \cdots \leq x_s \leq z_0 \leq \cdots \leq z_{t-1},\ \tau = y_0 \leq \cdots \leq y_q$$

with $s+t+q=n$. If $t>1$, it is clear that

$$\varphi_1 \omega u|_{\sigma,\tau} = 0 = \omega' \varphi_2 u|_{\sigma,\tau},$$

since each summand of $\omega'\varphi_2 u|_{\sigma,\tau}$ is 0 under φ_2.

If $t=1$, let $\sigma' = x_0 \leq \cdots \leq x_s$. Then we have

$$\omega'\varphi_2 u|_{\sigma,\tau} = (-1)^{\varsigma(q)} \sum_{z'} (-1)^{m(z')} \varphi_2 u|_{z'},$$

where the sum is taken over the traversals z' of (σ, τ).

Pick a traversal z' of (σ, τ). We zoom in on the top right of the grid of (σ, τ).

$$\begin{array}{ccc} \cdots & y'_1 \longrightarrow & y'_2 \\ & & \uparrow \\ & \iddots & y'_0 \\ & & \vdots \end{array}$$

Note that $y'_0, y'_2 \in \mathbf{E}_{z_0}$. If z' passes through y'_0, then $\varphi_2 u|_{z'} = 0$. If z' passes through y'_1, then $\varphi_2 u|_{z'} = u|_z$, for a particular traversal z of (σ', τ). Moreover, in this second case there are exactly q many squares in the rightmost column that are in the count for $m(z')$, so $m(z') = q + m(z)$. Therefore we have

$$\begin{aligned} \omega' \varphi_2 u|_{\sigma,\tau} &= (-1)^{\varsigma(q)} \sum_{z'} (-1)^{m(z')} \varphi_2 u|_{z'} = (-1)^{\varsigma(q)} \sum_{z} (-1)^{m(z)+q} u|_z \\ &= (-1)^q (-1)^{\varsigma(q)} \sum_z (-1)^{m(z)} u|_z = (-1)^q \omega u_{\sigma',\tau} = \varphi_1 \omega u|_{\sigma,\tau}. \end{aligned}$$ □

We can then form the augmented commutative diagram in Figure 7.7.

The two columns are exact since, by Propositions 7.11 and 7.13, the maps $\varphi_1^\bullet$ and $\varphi_2^\bullet$ are isomorphisms. The squares commute by the commutativity of the diagram from the morphism of long exact sequences and the claim.

We finish the proof by induction on the cardinality of $\mathbf{B}$. If $|\operatorname{Obj}\mathbf{B}| = 1$, then

$$T_\xi^n = \mathcal{S}^0(\mathbf{B}; \mathcal{S}^n) = \prod_{x \in \mathbf{B}} \mathcal{S}^n(\mathbf{E}_x; F_x) = \mathcal{S}^n(\mathbf{E}_\xi; F_\xi),$$

and $\omega = (-1)^{\varsigma(q)} \operatorname{id}$, so ω is a quasi-isomorphism.

If $\omega : \mathcal{S}^n(\mathbf{E}_\xi; F_\xi) \to T_\xi^n$ is a quasi-isomorphism for $|\operatorname{Obj}\mathbf{B}| < i$, then we can form the commutative diagram in Figure 7.7 for $|\operatorname{Obj}\mathbf{B}| = i$. Each row other than the middle one contains an instance of the inductive hypothesis, since both $\mathbf{B}_{\not\geq x}$ and $\mathbf{B}_{\geq x}$ have fewer objects than $\mathbf{B}$; and $\mathbf{B}$ is recursively admissible. Therefore, by the Five Lemma 2.35, the middle row is an isomorphism and thus ω is a quasi-isomorphism. This completes the induction and the proof of the theorem. □

7.6 A bundle over a non-poset base

The restriction to poset bundles over a recursively admissible base in this chapter has been dictated by the techniques in the proof of Theorem 7.14. It is possible, however, to find examples that do not satisfy this requirement, but for which the theorem still holds. In this section we describe a bundle ξ over the category $\mathbf{C}_{\mathbb{Z}\setminus 2\mathbb{Z}}$ (recall Example 6.2) and explicitly construct an isomorphism $\vartheta : T_\xi \to \mathcal{S}(\mathbf{E}_\xi; F_\xi)$.

$$
\begin{array}{ccc}
H^n(\mathbf{E}_{\mathbf{B}_{\ngeq x}}; F_{\mathbf{B}_{\ngeq x}}) & \xrightarrow{\varphi_1^{\bullet -1}\omega'^{\bullet}\varphi_2^{\bullet}} & H^n T^{\bullet}_{\mathbf{B}_{\ngeq x}} \\
\uparrow{\scriptstyle \varphi_2^{\bullet -1}\delta} & & \uparrow{\scriptstyle \varphi_1^{\bullet -1}\delta} \\
H^n(\mathbf{E}_{\mathbf{B}_{\ngeq x}}; F_{\mathbf{B}_{\ngeq x}}) \oplus H^n(\mathbf{E}_{\mathbf{B}_{\geq x}}; F_{\mathbf{B}_{\geq x}}) & \xrightarrow{\omega^{\bullet} \oplus \omega^{\bullet}} & H^n T^{\bullet}_{\mathbf{B}_{\ngeq x}} \oplus H^n T^{\bullet}_{\mathbf{B}_{\geq x}} \\
\uparrow{\scriptstyle \pi^{\bullet}} & & \uparrow{\scriptstyle \pi^{\bullet}} \\
H^n(\mathbf{E}_{\xi}; F_{\xi}) & \xrightarrow{\omega^{\bullet}} & H^n T^{\bullet}_{\xi} \\
\uparrow{\scriptstyle \varepsilon^{\bullet}\varphi_2^{\bullet}} & & \uparrow{\scriptstyle \varepsilon^{\bullet}\varphi_1^{\bullet}} \\
H^{n-1}(\mathbf{E}_{\mathbf{B}_{\ngeq x}}; F_{\mathbf{B}_{\ngeq x}}) & \xrightarrow{\varphi_1^{\bullet -1}\omega'^{\bullet}\varphi_2^{\bullet}} & H^{n-1} T^{\bullet}_{\mathbf{B}_{\ngeq x}} \\
\uparrow{\scriptstyle \varphi_2^{\bullet -1}\delta} & & \uparrow{\scriptstyle \varphi_1^{\bullet -1}\delta} \\
H^{n-1}(\mathbf{E}_{\mathbf{B}_{\ngeq x}}; F_{\mathbf{B}_{\ngeq x}}) \oplus H^{n-1}(\mathbf{E}_{\mathbf{B}_{\geq x}}; F_{\mathbf{B}_{\geq x}}) & \xrightarrow{\omega^{\bullet} \oplus \omega^{\bullet}} & H^{n-1} T^{\bullet}_{\mathbf{B}_{\ngeq x}} \oplus H^{n-1} T^{\bullet}_{\mathbf{B}_{\geq x}}
\end{array}
$$

Fig. 7.7: Augmented commutative diagram, where the instances of $H^{\bullet}B^{\bullet}$ and $H^{\bullet}A^{\bullet}$ are replaced.

Together with Proposition 6.10 this implies that the claim of Theorem 7.14 is true for this non-poset bundle ξ.

Let $\mathbf{B} = \mathbf{C}_{\mathbb{Z}/2\mathbb{Z}}$ with its only object denoted by $\circ$, and let $\mathbf{C}$ be the category with two objects x and y and no non-identity arrows. Define $F : \mathbf{C} \to \mathbf{Ab}$ by $F(x) = F(y) = \mathbb{Z}$. Let g be the unique non-identity arrow in $\mathbf{B}$. To describe the bundle $\xi : \mathbf{B} \to \mathbf{Sh}$ we set $\xi(\circ) = (\mathbf{C}, F)$ and give the sheaf morphism $\xi(g) = \gamma$ (also see Figure 7.8):

$$\gamma_1(x) = y \quad \gamma_1(y) = x \quad \gamma_2(m|_y) = m|_x \quad \gamma_2(m|_x) = m|_y.$$

The total sheaf $(\mathbf{E}_{\xi}, F)$ is then as follows

$$\mathbf{E}_{\xi} : \quad x \rightleftarrows y \qquad\qquad F_{\xi} : \quad \mathbb{Z} \underset{\mathrm{id}}{\overset{\mathrm{id}}{\leftrightarrows}} \mathbb{Z}$$

Let $C_1^{\bullet} = \mathcal{S}^{\bullet}(\mathbf{E}_{\xi}; F_{\xi})$. Since between any two objects of $\mathbf{E}_{\xi}$ there is a unique arrow, we can describe an n-simplex of $N\mathbf{E}_{\xi}$ by just a string $w_0 w_1 \ldots w_n$ of $n+1$ objects of $\mathbf{E}_{\xi}$. Explicitly,

$$C_1^n = \bigoplus_{w_0 \ldots w_n} \mathbb{Z}.$$

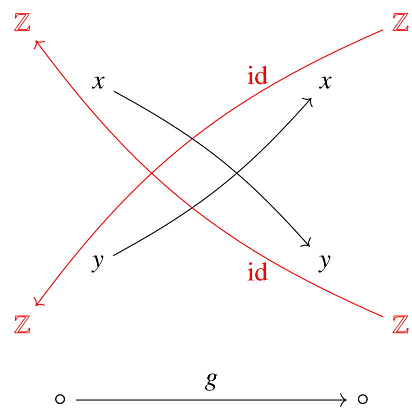


Fig. 7.8: The definition of the sheaf morphism $\xi(g)$. The elements in red are the modules in the sheaf.

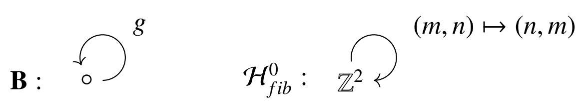
Now, let E be the spectral sequence associated with ξ and consider

$$E_1^{p,q} = \mathcal{S}^p(\mathbf{B}, \mathcal{H}^q_{fib}).$$

The category $\mathbf{B}$ has only one object and we have

$$\mathcal{H}^q_{fib}(\circ) = H^q(\mathbf{C}; F) = \begin{cases} \mathbb{Z}^2, & \text{if } q = 0, \\ 0, & \text{otherwise,} \end{cases}$$

since $\mathbf{C}$ has no non-identity arrows. Furthermore, $\mathcal{H}^0_{fib}(g) : (m, n) \mapsto (n, m)$. Then $(\mathbf{B}, \mathcal{H}^0_{fib})$ is given by



Let $C_2^\bullet = \mathcal{S}^\bullet(\mathbf{B}; \mathcal{H}^0_{fib})$. Since there is only one object in $\mathbf{B}$, we can describe an n-simplex of $N\mathbf{B}$ by a string $f_1 f_2 \ldots f_n$ of n arrows in $\mathbf{B}$. Explicitly,

$$C_2^n = \bigoplus_{f_1 \ldots f_n} \mathbb{Z}^2.$$

We will informally associate the first coordinate of $\mathbb{Z}^2$ above with x and the second with y.

We now construct a chain isomorphism $\vartheta : C_2^\bullet \to C_1^\bullet$. To do that we introduce some notation. Let $\_ \dagger \_ : \mathrm{Obj}(\mathbf{E}_\xi)^2 \to \mathbf{B}(\circ, \circ)$ be a set function defined by

$$w_0 \dagger w_1 = \begin{cases} g, & \text{if } w_0 \neq w_1, \\ \mathrm{id}, & \text{if } w_0 = w_1. \end{cases}$$

Then let $h : N\mathbf{E}_\xi \to N\mathbf{B}$ be defined by

$$h(w_0 w_1 \dots w_n) = \{w_0 \dagger w_1\}\{w_1 \dagger w_2\} \dots \{w_{n-1} \dagger w_n\}.$$

Finally, if $\tau \in N\mathbf{B}$ is an n-simplex and $u \in \mathcal{S}(\mathbf{B}; \mathcal{H}^0_{fib})$ with $u|_\tau = (m, n)$, we will write $(u|_\tau)_1 = m$ and $(u|_\tau)_2 = n$.

For an n-simplex $\sigma = w_0 \dots w_n \in N\mathbf{E}_\xi$ and $u \in \mathcal{S}^\bullet(\mathbf{E}_\xi; F_\xi)$, we define

$$\vartheta u|_\sigma = \begin{cases} (u|_{h(\sigma)})_1, & \text{if } w_0 = x, \\ (u|_{h(\sigma)})_2, & \text{if } w_0 = y, \end{cases}$$

Consider the set function h again. Each n-simplex in $N\mathbf{B}$ determines two n-simplices in $N\mathbf{E}_\xi$ – one starting with x and one starting with y. For example,

$$h(xyxx) = h(yxyy) = gg\,\mathrm{id}\,.$$

This means that h is two-to-one and therefore $\vartheta : C_2^n \to C_1^n$ is an isomorphism for each n. Remains to show that ϑ is also a chain map. We want the following diagram to commute

$$\begin{array}{ccc} C_1^n & \xrightarrow{d_1} & C_1^{n+1} \\ \uparrow \vartheta & & \uparrow \vartheta \\ C_2^n & \xrightarrow{d_2} & C_2^{n+1} \end{array}$$

Recall from Section 4.1 that if $\sigma = x_0 \to x_1 \to \cdots \to x_n$ is a simplex, for $j \in \{0, \dots, n\}$ we write

$$\sigma_j = x_0 \to \cdots \to x_{j-1} \to x_{j+1} \to \cdots \to x_n,$$

where the arrow $x_{j-1} \to x_{j+1}$ is the composition $x_{j-1} \to x_j \to x_{j+1}$.

Now, if $\sigma = w_0 w_1 \dots w_n \in N\mathbf{E}_\xi$, we claim that $h(\sigma_j) = h(\sigma)_j$. If $j = 0$ or n, this is clear from the definition of h. Otherwise, we have

$$h(\sigma_j) = \quad \cdots \longrightarrow \circ \xrightarrow{w_{j-1} \dagger w_{j+1}} \circ \longrightarrow \cdots$$

$$h(\sigma)_j = \quad \cdots \longrightarrow \circ \xrightarrow{\{w_{j-1} \dagger w_j\} \circ \{w_j \dagger w_{j+1}\}} \circ \longrightarrow \cdots$$

But $gg = \mathrm{id}$, so these are the same arrow.

Let $u \in C_2^n$ and $\sigma = x w_1 \dots w_{n+1} \in N\mathbf{B}$. We have

$$(d_1 \vartheta u)|_\sigma = \vartheta u|_{\sigma_0} + \sum_{j=1}^{n+1} (-1)^j \vartheta u|_{\sigma_j} = \vartheta u|_{\sigma_0} + \sum_{j=1}^{n+1} (-1)^j (u|_{h(\sigma_j)})_1$$

and

$$(\vartheta d_2 u)|_\sigma = (d_2 u|_{h(\sigma)})_1 = (\mathcal{H}^0_{fib}(x \dagger w_1)(u|_{h(\sigma)_0}))_1 + \sum_{j=1}^{n+1}(-1)^j (u|_{h(\sigma)_j})_1.$$

Since we established that $h(\sigma_j) = h(\sigma)_j$, we only need to consider the first summands of each expression:

$$\left(\mathcal{H}^0_{fib}(x \dagger w_1)(u|_{h(\sigma)_0})\right)_1 = \left\{\begin{matrix} (u|_{h(\sigma)_0})_1, & \text{if } w_1 = x, \\ (u|_{h(\sigma)_0})_2, & \text{if } w_1 = y, \end{matrix}\right\} = \vartheta u|_{\sigma_0}.$$

The argument goes through analogously if σ starts with y. Therefore $\vartheta : C_2^\bullet \to C_1^\bullet$ is a chain isomorphism and $H^\bullet C_2^\bullet \cong H^\bullet C_1^\bullet$. Returning to where $C_2^\bullet$ came from, we have that

$$E_2^{p,q} = \begin{cases} H^p C_2^\bullet, & \text{if } q = 0, \\ 0, & \text{otherwise.} \end{cases}$$

And since there is only one non-zero row on the E_2 page, the spectral sequence collapses and

$$H^\bullet T_\xi^\bullet \cong H^\bullet C_2^\bullet \cong H^\bullet C_1^\bullet = H^\bullet(\mathbf{E}_\xi; F_\xi).$$

This confirms the claim of Theorem 7.14 for this bundle $\xi : \mathbf{C}_{\mathbb{Z}/2\mathbb{Z}} \to \mathbf{Sh}$.

8

Applications

The statement of 7.14 closely resembles that of [ET12, Theorem 5.1]. Despite this, the reframing of the result in terms of sheaf cohomology, as opposed to coloured poset homology, leads to applications that are quite different from those of the coloured poset version. The key difference, explored in this chapter, is that while the theorem in [ET12] models complex interactions between the homologies of the fibers of a bundle of coloured posets (seen in the application to Khovanov homology), the main theorem of this thesis implies that if $\xi : \mathbf{B} \to \mathbf{Sh}$ is a poset bundle of sheaves with $\mathbf{B}$ recursively admissible, then it is only the cohomology of the sheaf at the maximum of $\mathbf{B}$ that contributes to the cohomology of the total sheaf of ξ.

By the end of this chapter, we will be able to conclude that, for example, the cohomology of a sheaf on the poset in Figure 8.1 can only be non-zero in degrees 0 and 1.

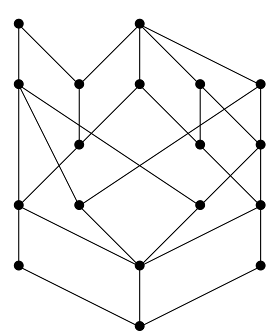


Fig. 8.1: The cohomology of any sheaf on this poset is zero in all degrees $\neq 0, 1$. Convention is that arrows go up.

It turns out that the restriction to recursively admissible posets means that we only deal with posets with 1.

Proposition 8.1. *Let* $\mathbf{B}$ *be a recursively admissible poset. Then* $\mathbf{B}$ *has a unique maximum.*

Proof. This follows from the recursive definition (Definition 7.3): the poset $\mathbf{B}$ is either Boolean of rank 1, so it has a unique maximum, or all its maximums are contained in $\mathbf{B}_{\geq x}$ for some $x \succ 0$, since $\mathbf{B}_{\geq x}^{\geq y} \neq \emptyset$ for all $y \in \mathbf{B}_{\not\geq x}$. Equivalently, the statement follows by induction on the size of $\mathbf{B}$. □

The admissibility property provides a kind of 'factorisation' for posets into bundles. The simplest way to do this is to turn an admissible poset into a bundle over $\mathbb{B}_1$. Note that Boolean lattices are recursively admissible, so we can later apply Theorem 7.14.

Lemma 8.2. *Let* $\mathbf{E}$ *be an admissible poset for* $\mathbf{E}', \mathbf{E}''$ *and* $(\mathbf{E}, F) \in \mathbf{Sh}$*. Then there is a poset bundle of sheaves* $\xi : \mathbb{B}_1 \to \mathbf{Sh}$ *such that* $(\mathbf{E}_\xi, F_\xi) = (\mathbf{E}, F)$ *(recall the construction of the total sheaf* $(\mathbf{E}_\xi, F_\xi)$*, Definition 6.3).*

Proof. We need to specify $\xi(0)$, $\xi(1)$, and $\xi(0 \leq 1)$.

- $\xi(0) = (\mathbf{E}', F)$,
- $\xi(1) = (\mathbf{E}'', F)$,
- the sheaf morphism $\gamma = \xi(0 \leq 1)$ consists of a covariant functor (or just a poset map in this setting) $\gamma_1 : \mathbf{E}' \to \mathbf{E}''$ and a natural transformation $\gamma_2 : F\gamma_1 \to F$:
  - Let $\gamma_1(x)$ be the unique minimum of $\{y \in \mathbf{E}'' \mid x \leq y\}$. Then if $x \leq x'$ in $\mathbf{E}'$, we have $\{y \in \mathbf{E}'' \mid x \leq y\} \supseteq \{y \in \mathbf{E}'' \mid x' \leq y\}$ and so $\gamma_1(x) \leq \gamma_1(x')$.
  - Since $x \leq \gamma_1(x)$, we have a morphism $F(x) \leftarrow F(\gamma_1(x))$ from $(\mathbf{E}, F)$. Set $\gamma_{2,x}$ to be this morphism.

Remains to show that $(\mathbf{E}, F) = (\mathbf{E}_\xi, F_\xi)$. It is enough to show that $\mathbf{E} = \mathbf{E}_\xi$ by the construction of F_ξ. If $x \leq y$ in $\mathbf{E}$ and either $x, y \in \mathbf{E}'$ or $x, y \in \mathbf{E}''$, then clearly $x \leq y$ in $\mathbf{E}_\xi$ (as an arrow of type a)). Suppose $x \leq y$ in $\mathbf{E}$ and $x \in \mathbf{E}'$, $y \in \mathbf{E}''$. Then $x \leq \gamma_1(x) \leq y$, so $x \leq y$ in $\mathbf{E}_\xi$. Conversely, the set of arrows in $\mathbf{E}_\xi$ is generated by inequalities that hold in $\mathbf{E}$. Therefore, $x \leq y$ in $\mathbf{E}$ if and only if $x \leq y$ in $\mathbf{E}_\xi$. □

We can also 'factorise' a poset into a bundle over a more complicated base.

Proposition 8.3. *Let* $\mathbf{E}$ *and* $\mathbf{B}$ *be posets, let* $(\mathbf{E}, F) \in \mathbf{Sh}$*, and let* $\pi : \mathbf{E} \to \mathbf{B}$ *be an onto poset map, such that for all* $x < y$ *in* $\mathbf{B}$*, the subposet* $\pi^{-1}(x) \cup \pi^{-1}(y)$ *of* $\mathbf{E}$ *is admissible for* $\pi^{-1}(x), \pi^{-1}(y)$*. Then there is a poset bundle of sheaves* $\xi : \mathbf{B} \to \mathbf{Sh}$ *such that* $(\mathbf{E}, F) = (\mathbf{E}_\xi, F_\xi)$*.*

Proof. Following the approach from the previous proposition, set $\xi(x) = (\pi^{-1}(x), F)$ and if $x < y$ in $\mathbf{B}$, then $\xi_1(x < y)$ sends $z \in \pi^{-1}(x)$ to the minimum of the subposet $\{w \in \pi^{-1}(y) \mid z \leq w\}$.

Now suppose $z < w$ in $\mathbf{E}$ and $z \in \pi^{-1}(x)$, $w \in \pi^{-1}(y)$. Since π is a poset map, $x < y$ in $\mathbf{B}$ and $z < \xi_1(x < y)(z) \leq w$ in $\mathbf{E}_\xi$.

If $z < w$ in $\mathbf{E}_\xi$ is an arrow of type b) or a composition arrow, then by Proposition 6.5 there is a $v \in \pi^{-1}(\pi(w))$, such that $z < v < w$ in $\mathbf{E}_\xi$, where $z < v$ and $v < w$ are arrows of type b) and a), respectively. But both those arrows exist in $\mathbf{E}$, so $z < w$ in $\mathbf{E}$. □

The following is a consequence of recursively admissible posets' having a unique maximum (or final object).

Proposition 8.4. *Let* $\mathbf{B}$ *be a recursively admissible poset and let* $\xi : \mathbf{B} \to \mathbf{Sh}$ *be a poset bundle of sheaves. If* $1 \in \mathbf{B}$ *is the unique maximal object, then*

$$H^\bullet(\mathbf{E}_\xi, F_\xi) \cong H^\bullet(\xi(1)).$$

Proof. Let E be the spectral sequence associated with ξ. We know that

$$E_2^{p,q} = H^p(\mathbf{B}, \mathcal{H}^q_{fib}).$$

Now, $\mathbf{B}$ has a unique maximum 1 (Proposition 8.1), so the functors $\varprojlim_{\mathbf{B}}$ and the 'evaluation at 1' functor $\_(1) : \mathbf{Sh}(\mathbf{B}) \to {}_R Mod$ are naturally isomorphic (recall Example 3.5). But by Proposition 3.9 we know that evaluation functors are exact. Therefore

$$H^p(\mathbf{B}; \mathcal{H}^q_{fib}) = \begin{cases} H^q(\xi(1)), & \text{if } p = 0, \\ 0, & \text{otherwise.} \end{cases}$$

Thus the spectral sequence collapses, we get $H^n(T^\bullet_\xi) \cong H^n(\xi(1))$, and since $\mathbf{B}$ is recursively admissible, Theorem 7.14 applies. This means we have

$$H^\bullet(\xi(1)) \cong H^\bullet(T^\bullet_\xi) \cong H^\bullet(\mathbf{E}_\xi, F_\xi).$$ □

We can now package the discussion into the following self-contained application.

Theorem 8.5. *Let* $\mathbf{E}$ *and* $\mathbf{B}$ *be posets, with* $\mathbf{B}$ *recursively admissible. Suppose that* $\pi : \mathbf{E} \to \mathbf{B}$ *is an onto poset map such that for all* $x < y$ *in* $\mathbf{B}$*, the subposet* $\pi^{-1}(x) \cup \pi^{-1}(y)$ *of* $\mathbf{E}$ *is admissible for* $\pi^{-1}(x), \pi^{-1}(y)$*. Then*

$$H^\bullet(\mathbf{E}; F) \cong H^\bullet(\pi^{-1}(1); F)$$

for all $F \in \mathbf{Sh}(\mathbf{E})$*, where* 1 *is the unique maximum of* $\mathbf{B}$*.*

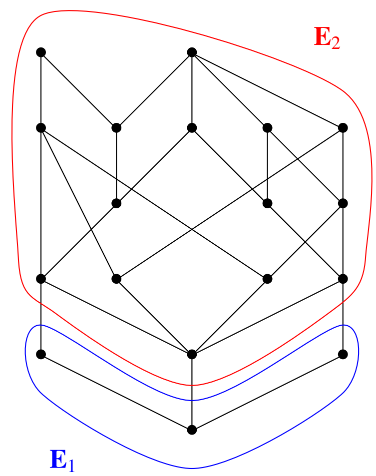
Remark 8.6. The above recipe can be applied repeatedly. Indeed, one can imagine cases where a poset $\mathbf{E}$ is admissible for $\mathbf{E}_1, \mathbf{E}_2$, and $\mathbf{E}_2$ is admissible for $\mathbf{E}_3, \mathbf{E}_4$, but $\mathbf{E}_1$ is not admissible, so the poset map $\pi : \mathbf{E} \to \mathbb{B}_2$ required for the above theorem does not exist. Despite this, we can apply the theorem twice with $\mathbf{B} = \mathbb{B}_1$ and deduce that

$$H^\bullet(\mathbf{E}; F) \cong H^\bullet(\mathbf{E}_4; F),$$

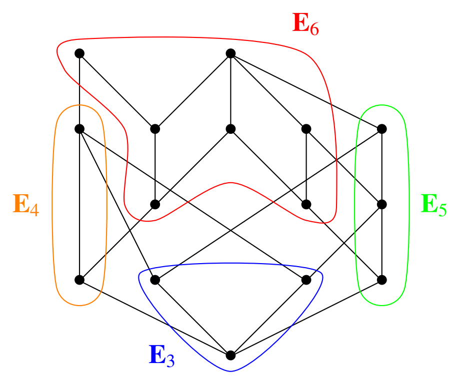
for any $F \in \mathbf{Sh}(\mathbf{E})$.

Conversely, if the required poset map $\pi : \mathbf{E} \to \mathbf{B}$ exists for some recursively admissible $\mathbf{B}$, we can instead repeatedly apply Theorem 8.5 for $\mathbb{B}_1$, at each step applying the recursive definition. The upshot is that replacing the recursively admissible $\mathbf{B}$ with the concrete $\mathbb{B}_1$ in the above theorem results in an equivalent statement.

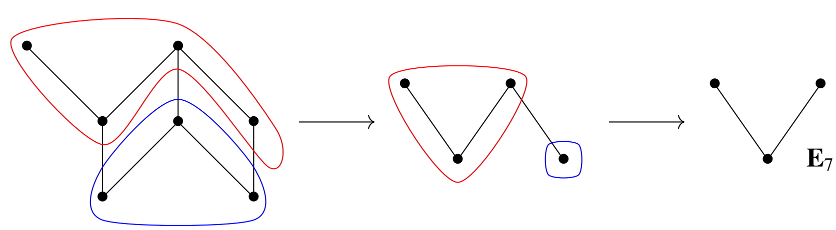
Example 8.7. We can now examine the explicit poset given at the start of the chapter (with arrowheads omitted, but always pointing up). Let **E** be the poset in Figure 8.1 and choose an $F \in \mathbf{Sh}(\mathbf{E})$. First, **E** is admissible for $\mathbf{E}_1, \mathbf{E}_2$ by inspection of the following diagram.



Thus, Theorem 8.5 implies that $H^\bullet(\mathbf{E}; F) \cong H^\bullet(\mathbf{E}_2; F)$. We can apply the theorem again, this time with $\mathbf{B} = \mathbb{B}_2$, giving $H^\bullet(\mathbf{E}; F) \cong H^\bullet(\mathbf{E}_6; F)$:



Another two applications of Theorem 8.5 with $\mathbf{B} = \mathbb{B}_1$ reduce the poset even further.



We thus have that $H^\bullet(\mathbf{E}; F) \cong H^\bullet(\mathbf{E}_7; F)$. To see that the cohomology of $(\mathbf{E}_7; F)$ is zero for all degrees ≥ 2, we can use the chain complex

$$\mathcal{T}^\bullet(\mathbf{E}_7; F) := \mathcal{S}^\bullet(\mathbf{E}_7; F)/D^\bullet,$$

where $D^\bullet$ is the subcomplex consisting of the degenerate simplices in $\mathbf{E}_7$, i.e. the simplices that involve an identity arrow. This new chain complex $\mathcal{T}^\bullet$ is homotopy equivalent to $\mathcal{S}^\bullet$ ([ET15, p.138]) and since it only involves non-degenerate simplices, its cohomology is clearly trivial at degrees ≥ 2.

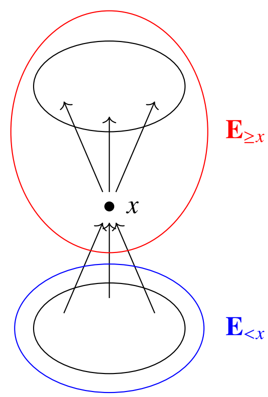
There is also a more general example that we can apply our theorem to.

Proposition 8.8. *Let* $\mathbf{E}$ *be a poset and let* $x \in \mathbf{E}$ *be a total point, i.e. for all* $y \in \mathbf{E}$*, either* $x \leq y$ *or* $y \leq x$*. Then*

$$H^\bullet(\mathbf{E}; F) \cong H^\bullet(\mathbf{E}_{\geq x}; F)$$

for any $F \in \mathbf{Sh}(\mathbf{E})$*.*

Proof. If $\mathbf{E}_{<x} = \emptyset$, then $\mathbf{E} = \mathbf{E}_{\geq x}$ and the statement of the proposition is trivial. Otherwise, consider the subposets $\mathbf{E}_{\geq x}$ and $\mathbf{E}_{<x}$:



For any $y \in \mathbf{E}_{<x}$, we have $\min \mathbf{E}_{\geq x}^{\geq y} = x$ and so $\mathbf{E}$ is admissible for $\mathbf{E}_{<x}, \mathbf{E}_{\geq x}$. Applying Theorem 8.5 gives the required result. □

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