Around Exponential-Algebraic Closedness

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‘Todder, you can trust math.’

Freer said ‘You heard it here first.’

Pemulis compulsively zipped and unzipped one of the covers. ‘Take a breather, Keith. Todd, trust math. As in Matics, Math E. First-order predicate logic. Never fail you. Quantities and their relation. Rates of change. The vital statistics of God or equivalent. When all else fails. When the boulder’s slid all the way back to the bottom. When the headless are blaming. When you do not know your way about. You can fall back and regroup around math. Whose truth is deductive truth. Independent of sense or emotionality. The syllogism. The identity. Modus Tollens. Transitivity. Heaven’s theme song. The nightlight on life’s dark wall, late at night. Heaven’s recipe book. The hydrogen spiral. The methane, ammonia, H_2O. Nucleic acids. A and G, T and C. The creeping inevibatility. Caius is mortal. Math is not mortal. What it is is: listen: it’s true.’

‘This from a man on academic probation for who knows the length.’

David Foster Wallace, *Infinite Jest*

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The contributions of many other people were not exactly mathematical, but still made a difference.

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Abstract

We present some results related to Zilber's *Exponential-Algebraic Closedness Conjecture*, showing that various systems of equations involving algebraic operations and certain analytic functions admit solutions in the complex numbers. These results are inspired by Zilber's theorems on *Raising to Powers*.

We show that algebraic varieties which split as a product of a linear subspace of an additive group and an algebraic subvariety of a multiplicative group intersect the graph of the exponential function, provided that they satisfy Zilber's *freeness* and *rotundity* conditions, using techniques from tropical geometry.

We then move on to prove a similar theorem, establishing that varieties which split as a product of a linear subspace and a subvariety of an abelian variety A intersect the graph of the exponential map of A (again under the analogues of the freeness and rotundity conditions). The proof uses homology and cohomology of manifolds.

Finally, we show that the graph of the modular j-function intersects varieties which satisfy *freeness* and *broadness* and split as a product of a *Möbius subvariety* of a power of the upper-half plane and a complex algebraic variety, using Ratner's orbit closure theorem to study the images under j of Möbius varieties.

Keywords: abelian varieties, algebraic groups, Exponential-Algebraic Closedness, exponential function, modular j-function, quasiminimality.

Mathematics Subject Classification: Primary: 03C65. Secondary: 11F03, 11L99, 14K12.

Contents

Notation

$\mathbb{C}^\times$	The set of non-zero complex numbers, $\mathbb{C} \setminus \{0\}$.
$\mathbb{S}_1$	The unit circle $\{z \in \mathbb{C} \mid \lvert z\rvert = 1\}$.
$\mathbb{R}_{\geq 0}$	The non-negative real numbers.
ω	The natural numbers.
$\mathbb{H}$	The upper-half plane (complex numbers with positive imaginary part).
$\langle z, w\rangle$	For $z, w \in \mathbb{C}^n$, the usual Hermitian product $z_1\overline{w}_1 + \cdots + z_n\overline{w}_n$.
exp	We use the same symbol to denote the exponential map $\exp : \mathbb{C} \to \mathbb{C}^\times$ and any of its Cartesian powers $\exp : \mathbb{C}^n \to (\mathbb{C}^\times)^n$.
j	Similarly, denotes the modular j-invariant $j : \mathbb{H} \to \mathbb{C}$ and its Cartesian powers $j : \mathbb{H}^n \to \mathbb{C}^n$.
$\exp_A$	The exponential map of the abelian variety A.
Γ_f	The graph of the function f.
$\operatorname{trdeg}(z_1, \ldots, z_n)$	The transcendence degree of the tuple $z = (z_1, \ldots, z_n)$ over $\mathbb{Q}$
$\operatorname{ldim}_K(z_1, \ldots, z_n)$	The linear dimension over the field K of the elements $z_1, \ldots, z_n$ of a K-vector space.
$V^\vee$	When V is a K-vector space, its dual (linear functions from V to K).
$\operatorname{aff}(\tau)$	The affine span of a polyhedron $\tau \subseteq \mathbb{R}^n$.
$\tau_\mathbb{C}$	The complexification $\operatorname{aff}(\tau) \otimes \mathbb{C}$ of $\operatorname{aff}(\tau)$.
$\operatorname{Trop}(W)$	The tropicalization of the algebraic variety $W \subseteq (\mathbb{C}^\times)^n$
Y_Σ	The toric variety associated to the polyhedral complex Σ.
$\Sigma_1 \cap_{st} \Sigma_2$	The stable intersection of the polyhedral complexes Σ_1 and Σ_2.
T_xM	The tangent space at the point x of the manifold M.
LA	The Lie algebra (tangent space at identity) of the abelian variety A.
$\dim S$	Dimension of the set S, usually in the sense of complex analytic geometry.
$\dim_\mathbb{R} S$	Dimension of the set S in the sense of real analytic geometry.
$\mathcal{A}_W$	The amoeba of the algebraic variety $W \subseteq (\mathbb{C}^\times)^n$.
W^{Reg}	The set of regular points of the variety W.

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Chapter 1

Introduction

1.1 Hand-Waving Introduction

A general theme in model theory, ever since its early development, has been that the "best" structures in an axiomatizable class are the *existentially closed* structures: those in which we can solve any equation that is solvable in some extension.

So, for example, the rational numbers $\mathbb{Q}$ are a great example of linear order, because however we take $x < y$ we can find elements that are smaller than x, larger than y, or that sit in between x and y: we say that $(\mathbb{Q}, <)$ is a *dense linear order without endpoints*, and we know that it has excellent model-theoretic properties.

However, if we add the field operations $+$ and $\cdot$ to the language, then $\mathbb{Q}$ stops being a nice structure: suddenly, it lacks elements such as the square root of 2 which "should" be there, in the sense that we can find extensions of the rational field which contain them. Of course, we can solve this problem by moving to the real algebraic numbers, or to the reals: these are *real closed fields*, they contain solutions to all polynomial equations of odd degree and to all polynomial equations of even degree which can have one. Again, we have an object with good model-theoretic properties.

If, on the other hand, we consider the rationals as a pure field, without the ordering, there is no longer any reason why $x^2 + 1 = 0$ should have no solution, and therefore we should add elements to the field to solve all polynomial equations; on top of the real algebraic numbers, we need to add one element (the square root of -1) and then close under the field operations to end up with a field in which all non-constant polynomials are solvable, an *algebraically closed field*. If we add the square root of -1 to the reals, we obtain the familiar complex numbers.

As a consequence of the existential closedness property, the structures $(\mathbb{Q}, <)$, $(\mathbb{R}, +, \cdot, <)$ and $(\mathbb{C}, +, \cdot)$ turn out to have well-behaved definable sets: if we can define a set in one of these structures, in the relative language, then it is not going to be a complicated set. For example, any subset of the reals that is definable in the ordered field language is a finite union of points and intervals: we will not be able to define the rationals, or a Cantor set, or even the integers.

In a sense, these examples say that these three very familiar mathematical structures ($\mathbb{Q}, \mathbb{R}$, and $\mathbb{C}$) are "the best" among some given classes (linear orders, ordered fields, and fields respectively). In general, analysing classes of structures it often happens that the best structures are also the more natural ones: why should there be an ordered field with better properties than the real numbers?

This poses a bit of a chicken-and-egg dilemma: are the natural structures better for some intrinsic reason, or do they seem natural to us because the human-driven evolution of mathematics has been built around them? It is true that the real numbers are the best ordered field, but this is at least partly due to the fact that the notion of ordered field has been built around them, and perhaps there is a universe in which nobody cares about ordered fields and there is some mathematical object unknown to us of the utmost importance. This seems like a very interesting topic for a PhD thesis in the philosophy of maths. This thesis is not that thesis, and we leave it to the reader to decide whether this is a good or a bad thing.

Whatever the case may be, the fundamental question of this thesis can be summarized as asking whether something similar can be said about the complex exponential function. An *exponential field* is a field endowed with a group homomorphism from its additive group to its multiplicative group. There is an exponential field which is the "best" exponential field, in the sense that its definable sets are well-behaved. Is that (isomorphic to) the field of complex numbers, equipped with the usual exponential map? To answer this question, we need to understand where the complex numbers sit with respect to existential closedness: can we find complex solutions to all the equations in polynomials and exponentials which should have a solution?

This thesis aims to make a contribution towards the solution of this problem: we find solutions to some systems of equations, and we do it because we want to make sure that some functions are good in the sense that they do not define bad sets. The way in which we do it is geometric: many people learn in high school that solving a 2-by-2 system of linear equations is the same thing as intersecting two lines in the plane, and what we do in this thesis is solving systems of complex analytic equations by intersecting complex analytic varieties.

1.2 Slightly Less Hand-Waving Introduction

This thesis is concerned with Zilber's *Exponential-Algebraic Closedness Conjecture* and with some related problems.

The Exponential-Algebraic Closedness Conjecture predicts sharp sufficient conditions for some systems of equations in polynomials and exponentials to be solvable in the complex numbers. While the problem is essentially a problem in complex geometry (we look for solutions to the equations by intersecting complex analytic sets) its motivation is model theoretic: a theorem of Bays and Kirby, building on work of Zilber, says that if the conjecture holds then the complex exponential field $\mathbb{C}_{\exp}$ is *quasiminimal*, meaning that all the subsets of $\mathbb{C}$ which can be defined using polynomials and exponentials are either countable or cocountable. Quasiminimality of $\mathbb{C}_{\exp}$ would be a major result in the model theory of analytic functions, with ties to the (now disproved) Trichotomy Conjecture, hence the interest of model theorists in this question.

As the work on the complex exponential function progressed, several people noticed the similarity of exp with other complex analytic functions, most notably the exponential maps of semiabelian varieties and the modular j-invariant. There are model-theoretically interesting questions that can be asked about these functions too, and conjectures have been posed asking about the solution of systems of equations which combine polynomials and these analytic functions. Some of these systems will also be treated.

Let us remark that the original form of the conjecture, sometimes referred to as Strong Exponential Algebraic-Closedness, is concerned with the existence of solutions that are “sufficiently generic”. Another feature of the theorem of Bays and Kirby is that genericity properties are not required for quasiminimality of $\mathbb{C}_{\exp}$. As a result, we will not worry about genericity of solutions, but focus just on their existence.

1.3 Statements of the Main Results

The main results of this thesis are Theorems 3.7.8, 4.4.1 and 5.5.7. They all have the same form: they say that the graph of a certain geometric function f intersects an algebraic variety of the form $L \times W$, where L is a subset of the domain of f with some geometric property related to f, W is an algebraic subvariety of the codomain of f, and the product $L \times W$ satisfies some conditions.

These results were inspired by a result of Zilber, from [Zil02] and [Zil15], that Theorem 3.7.8 is an improvement of. We give the statement of the theorems, without defining the terminology, and provide a brief indication of their proofs.

Theorem (Theorem 3.7.8)**.** *Let $L \times W$ be a free rotund subvariety of $\mathbb{C}^n \times (\mathbb{C}^\times)^n$ such that $L \leq \mathbb{C}^n$ is a linear subspace and $W \subseteq (\mathbb{C}^\times)^n$ is an algebraic variety. Then $L \times W \cap \Gamma_{\exp} \neq \varnothing$.*

The proof of this theorem uses tropical geometry and the theory of amoebas, distinguishing between the case in which L is defined over the reals and the case in which it is not. We will see that if L is defined over the reals then it is sufficient to find an intersection between $L \times W$ and the *blurring* of the graph $\Gamma_{\exp}$ by the unit circle, in the sense of [Kir19b], and it will provide a sufficiently good approximation; while if L is not defined over the reals then we need to consider the behaviour of W near 0 and ∞ to find sufficiently nice approximate solutions. This theorem is a major improvement on Zilber's [Zil02, Theorem 5] and [Zil15, Theorem 7.2], which had much stronger assumptions on the linear subspace L.

It should be noted, moreover, that Zilber's work on this kind of statement was motivated by model-theoretic considerations other than quasiminimality of $\mathbb{C}_{\exp}$, and that this result has some model-theoretic consequences of its own. Further discussion of this can be found in Sections 2.4 and 3.8.

Theorem (Theorem 4.4.1)**.** *Let A be an abelian variety of dimension g,* $\exp_A$:

$\mathbb{C}^g \to A$ its exponential map, $L \leq \mathbb{C}^g$ a linear subspace and $W \subseteq A$ an algebraic variety such that the variety $L \times W$ is free and rotund.

Then $L \times W \cap \Gamma_{\exp_A} \neq \varnothing$.

The proof of this theorem uses some facts from the homology of compact manifolds, and intersection homology in particular: closed subgroups and algebraic subvarieties of the abelian variety A can both be seen as homological cycles, and we will see that under the assumptions in the statement of the theorem the intersection between the closure of $\exp_A(L)$ and W "generically" provides good approximations. Using o-minimality, we are able to show that these approximations actually exist everywhere, and not just generically. This theorem implies a older results of Ax, [Ax72b, Theorem 1 and Corollary]: in work that predates Exponential-Algebraic Closedness, he proved Theorem 4.4.1 in the cases in which $\dim L = 1$ or A is simple.

Theorem (Theorem 5.5.7)**.** *Let $L \times W$ be a free broad subvariety of $\mathbb{H}^n \times \mathbb{C}^n$ with L a Möbius subvariety of $\mathbb{H}^n$ and W an algebraic variety in $\mathbb{C}^n$.*

Then $L \times W \cap \Gamma_j \neq \varnothing$.

The last main theorem concerns the modular j-function, and its proof uses ergodic theory, and Ratner's theorem in particular, to show that if L is a free Möbius subvariety of $\mathbb{H}^n$ then $j(L)$ is dense in $\mathbb{C}^n$; with some complex analysis we then conclude that $L \times W$ intersects the graph of j. After this theorem appeared in the preprint [Gal21], a more general result in the context of Shimura varieties appeared in [EZ21].

1.4 Structure of the Thesis

We conclude this introduction with an overview of the topics of each chapter.

The first part of Chapter 2 introduces the problem of Exponential-Algebraic Closedness from the model-theoretic point of view: we briefly discuss Zilber's Trichotomy Conjecture and the way in which its refutation led to the study of the model theory of the exponential function; we describe, albeit informally, both the axiomatization of Zilber's exponential field $\mathbb{B}$ and of the first-order theory of K-powered fields for some field $K \subseteq \mathbb{C}$. In the final section of the chapter we state various results that will be applied later on in the thesis: the Ax-Schanuel Theorem, the Open Mapping Theorem and the Proper Mapping

Theorem.

Chapter 3 contains the proof of the first main theorem, Theorem 3.7.8. We introduce amoebas and tropical geometry and use them to prove the theorem; we also discuss the model-theoretic consequences of this result.

Chapter 4 deals with the second main theorem, Theorem 4.4.1. We introduce complex abelian varieties more or less from scratch, discuss their homology and cohomology, introduce the Abelian Exponential-Algebraic Closedness Conjecture and use everything to give a proof of the main theorem: we first deal with the easier case in which $\exp_A(L)$ is dense in A, and then tackle the more general question. We again discuss the model-theoretic consequences and describe a future line of work.

Chapter 5 is about the modular j-function: again, we gather all the basic facts concerning this function before moving on to discuss its similarities and differences with exp, state the j-algebraic closedness conjecture and prove the special case Theorem 5.5.7, the third main theorem of the thesis. In the last section of the chapter we also sketch the proof of a partial result for the first derivative of j.

Finally, Chapter 6 presents some ideas on how this work should progress, from both the model-theoretic and the complex-geometric point of view.

The chapters of the thesis are quite independent of one another, with a few exceptions here and there.

Subsection 2.3.4 and Section 2.5 are probably the only parts of Chapter 2 that contain material which is needed to understand the results and proofs of the latter chapters, while the rest of the chapter provides motivation and context. It may be skipped by the reader who is already familiar with the problem, or who for some other reason is very enthusiastic about finding complex solutions to systems of equations involving analytic functions and does not need any further context.

Some of the proofs in Section 4.3, which deal with abelian varieties, are similar to their counterparts for the multiplicative group in Section 3.2, and therefore it might be helpful to read Section 3.2 first; similarly, Sections 2.4, 3.8 and 4.5 are rather sequential and it would be confusing to mix them up; it should however be pointed out that Chapters 3 and 4 make perfect sense even without Sections 3.8 and 4.5.

The goal of Section 5.3 is to explain why it makes sense to treat the j-function and exp along similar model-theoretic lines, and therefore it would probably feel a bit strange to read it without being familiar with the model theory of exp.

Chapter 6 is about future work and therefore we recommend not reading it before having had at least a look at the past and present work in the first five chapters.

Chapter 2

Background

2.1 Introduction

The main aim of this chapter is to introduce the topic of the thesis, the Exponential-Algebraic Closedness Conjecture. In particular, in this chapter we care about the model-theoretic motivation for the problem: while most of the rest of the thesis will involve very little model theory, it is important to keep in mind that the question is originally about definability and classification.

Therefore, the first two sections of this chapter will follow the historical path of the problem: we will start in Section 2.2 by revisiting the ideas behind Zilber's Trichotomy Conjecture and Hrushovski's counterexample, and move on in Section 2.3 to sketch a description of the theory of the exponential field $\mathbb{B}$. This theory is not first-order, but it is uncountably categorical, and the definable subsets of its models satisfy good geometric properties. Conjecturally, its continuum-sized model $\mathbb{B}$ is isomorphic to the exponential field $\mathbb{C}_{\exp}$, and it will hopefully be clear in what sense a positive answer to this question would "save the spirit" of the Trichotomy Conjecture.

In Section 2.4 we will move to a slightly different topic, and discuss Zilber's work on a first-order class of structures with different model-theoretic properties which should still contain a structure whose underlying set is the complex field $\mathbb{C}$, expanded by the appropriate predicates. The results of this thesis actually provide new structures that fit in this class.

Finally, we will use Section 2.5 to list some miscellaneous results of geometric flavour which will be used across the thesis: most notably, the Ax-Schanuel Theorem and Remmert's Proper and Open Mapping Theorems from complex analysis.

This chapter is written with the model-theoretically-inclined reader in mind, and therefore it tends to take the most basic notions from model theory (such as language, theory, structure, model) for granted and focus on the bigger picture. The reader who for any reason has stumbled upon this thesis with little interest in model theory, but wanting to learn about the complex-geometric side of the problem, is very welcome and invited to focus on Subsections 2.3.3 and 2.3.4 and then move on to the next chapters. The reader who wants to learn model theory is also very welcome, but encouraged to do so from one of the excellent textbooks in the area such as [Hod93], [Mar06b], [TZ12], [Kir19a].

2.2 Zilber's Trichotomy Conjecture

The prologue to this story is Zilber's Trichotomy Conjecture on uncountably categorical structures.

Definition 2.2.1. Let κ be an infinite cardinal. A theory T is κ-categorical if it has exactly one model of cardinality κ up to isomorphism.

A structure M is κ-categorical if the full first-order theory of M is κ-categorical.

Morley's celebrated theorem on uncountably categorical theories is regarded as the origin of modern model theory.

Theorem 2.2.2 ([Mor65, Theorem 5.6])**.** *Let T be a countable first-order theory. If T is κ-categorical for some $\kappa > \aleph_0$, then it is κ-categorical for all $\kappa > \aleph_0$.*

Example 2.2.3. There are three main examples of uncountably categorical theories:

1. The theory of infinite sets in the language $\mathcal{L} = \{=\}$ of pure equality. In this theory, models are determined up to isomorphism by their cardinality, and thus the theory is κ-categorical for every infinite cardinal κ, including $\aleph_0$.

2. The theory of vector spaces over $\mathbb{Q}$, in the language of abelian groups expanded by a function symbol for multiplication by every scalar in $\mathbb{Q}$.

Models of this theory are determined by the cardinality of a base of the vector space, and thus there are countably many non-isomorphic countable models (one for each $n \in \omega + 1$) and one model for every uncountable cardinal.

3. The theory of algebraically closed fields of a fixed characteristic in the language of rings. These theory behaves similarly to the theory of vector spaces, with transcendence degree playing a role similar to that of linear dimension in vector spaces: there is one countable algebraically closed field of transcendence degree n for every $n \in \omega + 1$, and one algebraically closed field of transcendence degree κ and size κ for every uncountable κ.

Zilber investigated uncountably categorical theories with the aim of determining what are the features which give a theory this extremely strong property. He gave the following answer:

> The key factor is measurability by a dimension and high homogeneity of the structures. ([Zil01, p. 2]).

It is easy to see how these features present themselves in the structures described in Example 2.2.3: each of the structures has a clear candidate for "dimension" (respectively cardinality, linear dimension, transcendence degree), and the structures are clearly homogeneous in the sense that given two sufficiently generic elements it is easy to cook up an automorphism of the structure which swaps them.

Recall that uncountable categoricity has strong ties to *strong minimality.*

Definition 2.2.4. Let M be a first-order structure. An infinite definable subset $S \subseteq M$ is *minimal* if for every definable subset D of M, $S \cap D$ or $S \setminus D$ is finite.

S is *strongly minimal* if this property holds in every elementary extension of M.

M is a minimal structure if it is a minimal subset of itself (every definable set is finite or cofinite) and it is strongly minimal if all its elementary extensions are minimal.

The following fact establishes the connection between uncountably categorical theories and strongly minimal structures.

Fact 2.2.5 ([BL71, Theorem 1 and Theorem 2])**.** *Every strongly minimal structure is uncountably categorical.*

Every model of an uncountably categorical theory contains a strongly minimal set.

Zilber's famous Trichotomy Conjecture was an attempt to classify strongly minimal theories. We state the conjecture although we do not introduce all the terminology involved; we will only explain what its guiding principles were.

Conjecture 2.2.6 ([Zil84], Trichotomy Conjecture, disproved by Hrushovski)**.** *The geometry of every strongly minimal structure M is either:*

1. *Trivial;*
2. *Non-trivial and locally modular;*
3. *Isomorphic to the geometry of an algebraically closed field K definable in M.*

We should note that Conjecture 2.2.6 does not actually appear in this form in [Zil84], but it is the natural consequence of Theorem 3.1 and Conjecture B of that paper.

Even the reader who is not familiar with model theory, but is familiar with the natural numbers up to 3, will have noticed that Conjecture 2.2.6 suggests a trichotomy and that the examples presented in Example 2.2.3 were three: in fact, the geometry of infinite sets is trivial, the geometry of vector spaces is locally modular and the geometry of an algebraically closed field is obviously isomorphic to the geometry of an algebraically closed field. In fact, the spirit of the trichotomy conjecture was that, while the examples discussed above are not the only examples of strongly minimal uncountably categorical structures, there should be a sense in which every other example fits into a classification in which those three are the main specimens of each class. To use Zilber's own words,

> [the conjecture] was based on the belief that logically perfect structures could not be overlooked in the natural progression of mathematics. ([Zil01, p. 3]).

As we mentioned above, Conjecture 2.2.6 was disproved in [Hru93] by Hrushovski, who modified the classical Fraïssé amalgamation construction to come up with a strongly minimal set which does not fit into this classification. A very approachable account of this construction is given by Ziegler in [Zie13].

Let us note, however, that the conjecture turned out to be true for many classes of strongly minimal structures: Hrushovski and Zilber proved the conjecture in the setting of so-called *Zariski geometries* (see [HZ96]), Peterzil and Starchenko for o-minimal structures (see [PS98]), Hyttinen and Kangas for quasiminimal classes (see [HK16]), Eleftheriou, Hasson and Peterzil for strongly minimal expansions of 2-dimensional groups definable in o-minimal structures (see [EHP21]), and many other authors are still proposing finer classifications as in Baldwin and Verbovskiy's recent preprint [BV21]. While the conjecture as it was stated was false, it proved very influential in shaping the way in which the community thinks about model theory, and its spirit was widely accepted as correct.

2.3 Zilber's Exponential Field

This section will be concerned with *exponential fields.*

Definition 2.3.1. An *exponential field* is a field F equipped with a group homomorphism $\exp : F \to F^{\times}$.

Sometimes (e.g. in [Kir13]) exponential fields are defined in slightly different ways, for example with exp defined on a $\mathbb{Q}$-vector subspace of F rather than on all of F, or with the additional requirement that exp is surjective. We will take this definition as it seems the most natural one, it encompasses both $\mathbb{C}$ and $\mathbb{R}$ with their usual exponentials, and it does everything we need it to.

Zilber defined *quasiminimal structures* as structures in which every definable subset is countable or cocountable: they thus represent a generalization of strongly minimal structures, although they obviously have rather different features. For example, for easy model-theoretic reasons it does not make sense to talk about quasiminimal *theories*, at least in a first-order setting.

Moreover, it is not completely clear what *definable* should mean here, as we will see that non-first-order theories are quite important in this work. We overcome this ambiguity by accepting the following definition of quasiminimality (see for example [Kir19b, Definition 1.2]).

Definition 2.3.2. A structure M is *quasiminimal* if for every countable subset

A of M and every subset $S \subseteq M$, if S is invariant under $\mathrm{Aut}(M/A)$ then S is countable or cocountable.

Zilber's Quasiminimality Conjecture is that the complex exponential field is such a structure - that every set that we can define in the complex numbers using polynomials and exponentials is either countable or cocountable.

Conjecture 2.3.3 (Zilber's Quasiminimality Conjecture, [Zil97]). *The structure* $\mathbb{C}_{\exp} := (\mathbb{C}, +, -, \cdot, 0, 1, \exp)$ *is quasiminimal.*

The idea of the Quasiminimality Conjecture is that, just as we can get from strong minimality of algebraically closed fields the good geometric theory of definable subsets in higher dimension (namely algebraic varieties), if quasiminimality holds then the *exponential-algebraic varieties* which we can define in $\mathbb{C}^n$ using polynomials and exponentials are well-behaved objects from a geometric point of view. A failure of quasiminimality would very likely mean that the real numbers are definable in $\mathbb{C}_{\exp}$, and therefore that the structure interprets full second-order arithmetic - which would make its model theory about as wild as it can possibly be, with no hope for a good geometry of definable sets.

As one approach to prove this conjecture, in [Zil05b] Zilber worked with the Hrushovski amalgamation construction to obtain an exponential field $\mathbb{B}$, called the *pseudo-exponential field*, which is the unique model of cardinality $2^{\aleph_0}$ of a (non-first-order) uncountably categorical theory T all of whose models are quasiminimal; Zilber's work was then refined and improved by Bays and Kirby, most notably in [BK18]. The natural conjecture, then, is that $\mathbb{C}_{\exp}$ is a model of T, and therefore by categoricity it is isomorphic to $\mathbb{B}$.

The philosophical point which would emerge from a positive answer to this question is that while Hrushovski's counterexample does change the landscape in the theory of strong minimality, it is still similar to an object that was not "overlooked in the natural progression of mathematics", the exponential field: the problem was that first-order logic lacked the expressive power to describe it. To use Zilber's words once again,

> Based on the analysis of pseudo-exponentiation, one would like to conclude hypothetically that basic Hrushovski structures have analytic prototypes. ([Zil01, p. 8]).

To know more about the comparison between the exponential field and the Hrushovski counterexample the reader should consult [Zil01].

In this section we will introduce the axioms of the pseudo-exponential field $\mathbb{B}$ and comment on their status in the exponential field $\mathbb{C}_{\exp}$.

2.3.1 The Language

As we mentioned, the theory of $\mathbb{B}$ is not a first-order theory.

Definition 2.3.4. Let $\mathcal{L}_{\omega_1,\omega}(Q)$ denote the language $\mathcal{L}_{\omega_1,\omega}$ (which allows for infinite conjunctions and disjunctions and finite quantifications) expanded by the quantifier Q, where $Qx\varphi(x)$ is interpreted as "there exist uncountably many x such that $\varphi(x)$".

We will denote by $\mathcal{L}_{\exp}$ the language of exponential rings (so the language of rings expanded by one symbol for the exponential function) in the infinitary logic $\mathcal{L}_{\omega_1,\omega}(Q)$.

We will introduce our axioms as axioms in this language, referring to an abstract structure $(F, +, -, \cdot, 0, 1, \exp)$.

2.3.2 The Easy Axioms

The axioms of $\mathbb{B}$ can be gathered into six families, the first three of which are the easiest to check. (Arguably, $\mathbb{B}$ has only one axiom since working in an infinitary logic allows us to consider the conjunction of all the axioms as a single $\mathcal{L}_{\omega_1,\omega}(Q)$-sentence, but we still consider them as separate for clarity).

Definition 2.3.5. Denote by ACF_0 the set of axioms which state that F is an algebraically closed field of characteristic 0.

Definition 2.3.6. Denote by SE (Surjective Exponential) the axiom which states that $\exp : F \to F^\times$ is a surjective group homomorphism.

There is little to say about these two in the complex numbers: obviously $\mathbb{C}$, forgetting about the exponential, is an algebraically closed field of characteristic 0, and obviously $\exp : \mathbb{C} \to \mathbb{C}^\times$ is a surjective group homomorphism. Note, moreover, that so far we are still in first-order logic.

Definition 2.3.7. Denote by SK (Standard Kernel) the axiom which states that $\ker(\exp)$ is an infinite cyclic group, generated by a transcendental element.

This is the first step we take out of the first-order realm: we need $\mathcal{L}_{\omega_1,\omega}$ to require that

$$\exists z\left(\exp(z)=1\wedge\left(\forall x\left(\exp(x)=1\rightarrow\bigvee_{n\in\mathbb{Z}}x=nz\right)\right)\right)$$

and that the generator is transcendental. However, there is still no problem in showing that the axiom holds in the complex numbers: the kernel of the complex exponential is $2\pi i\mathbb{Z}$, and $2\pi i$ is clearly transcendental because π is.

2.3.3 The Schanuel Property

Schanuel's Conjecture is a long-standing open problem in transcendental number theory; roughly, it poses a lower bound for transcendence of the exponential function, predicting that n complex numbers $z_1,\dots,z_n$ and their exponentials should never lie on an algebraic variety defined over $\mathbb{Q}$ of dimension less than n, unless $z_1,\dots,z_n$ satisfy a $\mathbb{Q}$-linear relation.

Conjecture 2.3.8 (Schanuel, see [Lan66, p. 30])**.** *Let $z_1,\dots,z_n\in\mathbb{C}$ be complex numbers. Then*

$$\operatorname{trdeg}(z_1,\dots,z_n,\exp(z_1),\dots\exp(z_n))-\operatorname{ldim}_{\mathbb{Q}}(z_1,\dots,z_n)\geq 0.$$

Some easy consequences of Schanuel's conjecture are that e is transcendental ($\operatorname{trdeg}(1,e)\geq 1$) and that π is transcendental ($\operatorname{trdeg}(i\pi,-1)\geq 1$), which of course are both classically known. However, if Schanuel's conjecture were true then we would also obtain

$$\operatorname{trdeg}(e,\pi)=\operatorname{trdeg}(1,i\pi,e,-1)\geq\operatorname{ldim}_{\mathbb{Q}}(1,i\pi)=2$$

i.e. algebraic independence of e and π, which is not known and considered an extremely hard problem in itself. As even what looks like an easy consequence of Schanuel's conjecture is far from being solved, number theorists believe that a solution to Schanuel's conjecture is still decades away.

A known case of the conjecture is the Lindemann-Weierstrass Theorem, which amounts to Schanuel's conjecture for algebraic numbers.

Theorem 2.3.9 (Lindemann-Weierstrass, [Lin82], [Wei85])**.** *Let $z_1,\dots,z_n\in\mathbb{C}$*

be algebraic. Then

$$\operatorname{trdeg}(\exp(z_1), \ldots, \exp(z_n)) \geq \operatorname{ldim}_{\mathbb{Q}}(z_1, \ldots, z_n).$$

We will later discuss the geometric version of this statement, the *Ax-Lindemann-Weierstrass Theorem*, which establishes that the Zariski closure of the exponential of an algebraic variety is a translate of an algebraic subgroup of $(\mathbb{C}^\times)^n$.

A corollary of the Lindemann-Weierstrass Theorem is Schanuel's Conjecture for $n = 1$.

Corollary 2.3.10. *Let $z \in \mathbb{C}$, $z \neq 0$. Then* $\operatorname{trdeg}(z, \exp(z)) \geq 1$.

Proof. If z is transcendental then it is obvious. If z is algebraic then it follows from Theorem 2.3.9. □

The transcendence axiom in the theory of $\mathbb{B}$ has the form of Schanuel's Conjecture.

Definition 2.3.11. Denote by SP (Schanuel Property) the set of axioms in the language of exponential fields which state that for all n,

$$\forall z_1, \ldots, z_n \in F \left(\operatorname{trdeg}(z_1, \ldots, z_n, \exp(z_1), \ldots, \exp(z_n)) \geq \operatorname{ldim}_{\mathbb{Q}}(z_1, \ldots, z_n)\right).$$

As we mentioned, it is very unlikely that it will be proved in the near future that $\mathbb{C}_{\exp}$ satisfies SP, and thus that $\mathbb{C}_{\exp} \cong \mathbb{B}$. However, not all hope to get something model-theoretically meaningful is lost, as we see in the next subsection.

2.3.4 Exponential-Algebraic Closedness

We come now to Exponential-Algebraic Closedness, the main question that this thesis is concerned with. As we now need to start discussing the role of algebraic varieties, let us note here that all the algebraic varieties discussed in this thesis will be irreducible unless explicitly stated otherwise.

Exponential-Algebraic Closedness is (a generalization of) a dual version of Schanuel's Conjecture, in the following sense. As we mentioned, Schanuel's Conjecture predicts that points of the form $(z_1, \ldots, z_n, \exp(z_1), \ldots, \exp(z_n))$

do not lie on algebraic subvarieties of $\mathbb{C}^n \times (\mathbb{C}^\times)^n$ defined over $\mathbb{Q}$ of dimension less than n, unless $(z_1, \ldots, z_n)$ lies in some rational subspace of $\mathbb{C}^n$.

One might therefore ask whether given an algebraic variety of dimension larger than n it does contain points of the form $(z_1, \ldots, z_n, \exp(z_1), \ldots, \exp(z_n))$.

Definition 2.3.12. We will use $\Gamma_{\exp}$ to denote the graph of any Cartesian power of the exponential function, so

$$\Gamma_{\exp} := \{(z_1, \ldots, z_n, w_1, \ldots, w_n) \in \mathbb{C}^n \times (\mathbb{C}^\times)^n \mid w_j = \exp(z_j)\, \forall j\}.$$

As $\Gamma_{\exp}$ has dimension n, a naive guess might be that in fact it must intersect all algebraic varieties of dimension at least n. This however soon proves not to be the case.

Example 2.3.13. Let $V \subseteq \mathbb{C}^2 \times (\mathbb{C}^\times)^2$ be the algebraic variety defined as

$$V := \left\{(z_1, z_2, w_1, w_2) \in \mathbb{C}^2 \times (\mathbb{C}^\times)^2 \mid z_1 - z_2 = 0,\ w_1 w_2^{-1} + 1 = 0\right\}.$$

V has dimension 2. Assume there exists a point $(z_1, z_2, w_1, w_2) \in V \cap \Gamma_{\exp}$. Then by definition we have

$$\begin{cases} z_1 = z_2 \\ w_1 = -w_2 \\ \exp(z_1, z_2) = (w_1, w_2) \end{cases}$$

which is clearly impossible, as it implies that w_1 is equal to both w_2 and $-w_2$ without being 0.

It is quite clear what the problem in Example 2.3.13 is: since the variety V splits as the product of two cosets of algebraic groups $V_1 \leq \mathbb{C}^2$ and $V_2 \leq (\mathbb{C}^\times)^n$, and $\exp(V_1)$ and V_2 are in fact cosets of the same group, the required intersection has to be empty. More generally, it seems plausible that if a variety $V \subseteq \mathbb{C}^n \times (\mathbb{C}^\times)^n$ is contained in a product of cosets of algebraic groups of $\mathbb{C}^n$ and $(\mathbb{C}^\times)^n$, at least one of which is proper, then the behaviour of the exponential map on V cannot really be expected to be "generic".

All this is made precise by the notion of *freeness*. We introduce it in the generality of exponential fields.

Definition 2.3.14. Let $V \subseteq F^n \times (F^\times)^n$ be an algebraic variety. Denote by $\pi_1 : F^n \times (F^\times)^n \twoheadrightarrow F^n$ and $\pi_2 : F^n \times (F^\times)^n \twoheadrightarrow (F^\times)^n$ the projections on the domain and codomain of (the n-th Cartesian power of) exp.

We say that V is *free* if $\pi_1(V)$ is not contained in any affine $\mathbb{Q}$-linear subspace of F^n and $\pi_2(V)$ is not contained in any coset of an algebraic subgroup of $(F^\times)^n$.

We note that algebraic subgroups of the multiplicative group $(F^\times)^n$ are exactly those defined by finitely many equations of the form $w_1^{k_1} \cdots w_n^{k_n} = 1$ for $k_1, \ldots, k_n \in \mathbb{Z}$. These are the images under the exponential map of $\mathbb{Q}$-linear subspaces of F^n.

Another condition, *rotundity*, ensures that not only the variety V is "big" in the sense of dimensions, but that this property is preserved under taking algebraic quotients.

Definition 2.3.15. Let $V \subseteq F^n \times (F^\times)^n$ be an algebraic variety.

Let $L \leq F^n$ be a rational vector subspace, and let $\exp(L)$ be its image under exp, an algebraic subgroup of $(F^\times)^n$.

Denote by $\pi_L : F^n \times (F^\times)^n \twoheadrightarrow F^n/L \times (F^\times)^n/\exp(L)$ the algebraic quotient map.

We say that V is *rotund* if for every such L,

$$\dim(\pi_L(V)) \geq n - \dim L.$$

For example, then, rotundity is asking that $\dim V \geq n$ (obtained for $L = \langle 0 \rangle$).

We are now in a position to state the Strong Exponential-Algebraic Closedness axiom. We first state the *strong* version.

Definition 2.3.16. Denote by SEAC (Strong Exponential-Algebraic Closedness) the set of axioms in the language of exponential fields which states that for all n, for every free and rotund variety $V \subseteq F^n \times (F^\times)^n$ defined over a subfield F_0, there is a point in $V \cap \Gamma_{\exp}$ that is generic in V over F_0.

While SEAC is necessary to define the pseudo-exponential field, we will not really be concerned with genericity questions in this thesis. Hence, we also state the weaker version that we will work on.

Definition 2.3.17. Denote by EAC (Exponential-Algebraic Closedness) the set of axioms in the language of exponential fields which states that for all n, for every free and rotund variety $V \subseteq F^n \times (F^\times)^n$,

$$V \cap \Gamma_{\exp} \neq \varnothing.$$

From the model-theoretic point of view, these are conditions of existential closedness: they single out systems of equations that are "not obviously contradictory" and stipulate that they must all be solvable in the structure.

The weak axiom became particularly relevant after the following theorem of Bays and Kirby was established.

Theorem 2.3.18 ([BK18, Theorem 1.5])**.** *If* $\mathbb{C}_{\exp}$ *satisfies* EAC*, then it is quasiminimal.*

In light of what we discussed in Subsection 2.3.3, this is an extremely helpful result: before this theorem was established Schanuel's Conjecture seemed to be an obstruction towards the proof of quasiminimality of $\mathbb{C}_{\exp}$ unlikely to be overcome; most people would probably have been happy with a result of the form "if Schanuel's Conjecture holds, then $\mathbb{C}_{\exp}$ is quasiminimal". Theorem 2.3.18, on the other hand, removes the dependence of quasiminimality on Schanuel's Conjecture: while proving that $\mathbb{C}_{\exp}$ satisfies EAC seems by no means easy, it is a problem on which it seems far simpler to make some progress.

A few special cases of EAC for the complex numbers are already known.

Theorem 2.3.19. *Let* $V \subseteq \mathbb{C} \times \mathbb{C}^\times$ *be an algebraic curve whose projections to* $\mathbb{C}$ *and* $\mathbb{C}^\times$ *are both infinite.*

Then $V \cap \Gamma_{\exp} \neq \varnothing$*.*

This is a classical result, of which we show an easy proof to help the reader familiarize with these objects.

Proof. Let $f(z_1, z_2) : \mathbb{C} \times \mathbb{C}^\times \to \mathbb{C}$ be the polynomial function which defines V. To find an intersection between V and $\Gamma_{\exp}$ we then need to find a zero of the function $z \mapsto f(z, \exp(z))$.

If $f(z, \exp(z)) \neq 0$ for every z, then there is an affine function g such that $f(z, \exp(z)) = \exp(g(z))$ for every z (see [Mar06a, Theorem 2.2]).

This is only possible if the affine function is multiplication by a natural number m, and $f(z, \exp(z)) = \exp(mz)$. Therefore $f(z_1, z_2) = z_2^m$, and the function z_2^m has no zeros on $\mathbb{C} \times \mathbb{C}^\times$ - thus V is empty. $\square$

In [Mar06a] Marker takes this forward and shows that if V is defined over $\mathbb{Q}$ then $V \cap \Gamma_{\exp}$ contains infinitely many points. By Schanuel's Conjecture for $n = 1$ (Corollary 2.3.10) this implies that there are solutions which are generic in V, hence this is really a result about Strong Exponential-Algebraic Closedness. In [MZ16] Mantova removes the assumption on the field of definition, proving this theorem for any $V \subseteq \mathbb{C} \times \mathbb{C}^\times$.

By $\pi_1 : \mathbb{C}^n \times (\mathbb{C}^\times)^n \to \mathbb{C}^n$ we again denote the projection to the first block of n coordinates.

Theorem 2.3.20 (Brownawell-Masser, D'Aquino-Fornasiero-Terzo, Aslanyan-Kirby-Mantova)**.** *Let $V \subseteq \mathbb{C}^n \times (\mathbb{C}^\times)^n$ be an algebraic variety, and suppose* $\dim(\pi_1(V)) = n$. *Then $V \cap \Gamma_{\exp} \neq \varnothing$.*

Note that the condition on the dimension of the projection implies that V is rotund and that it satisfies the first half of the definition of freeness. The theorem allows for varieties V such that $\pi_2(V)$ is contained in a subgroup of $(\mathbb{C}^\times)^n$.

As is noticeable from the multiple attribution, this theorem has been proved in different forms and in different ways. The proofs of Brownawell-Masser ([BM17, Proposition 2]) and D'Aquino-Fornasiero-Terzo ([DFT21, Theorem 3.6]) used the Newton-Kantorovich approximation theorem, and therefore methods from the theory of real analytic functions. The proof of Aslanyan-Kirby-Mantova ([AKM22, Theorem 1.5]) is based on methods from complex analysis, which seem more natural.

Theorem 2.3.21 (Mantova-Masser)**.** *Let $V \subseteq \mathbb{C}^n \times (\mathbb{C}^\times)^n$ be a free rotund algebraic variety, and suppose* $\dim(\pi_1(V)) = 1$. *Then $V \cap \Gamma_{\exp} \neq \varnothing$.*

This unpublished theorem is the "opposite" of Theorem 2.3.20: there, the first projection of V needs to be dimensionally large and cover almost all of $\mathbb{C}^n$, while here it is as small as it can be (if it had dimension 0 then V could not be free). The proof uses methods from the geometry of Riemann surfaces (hence the need for the dimension of the projection to be 1).

These results are quite different from the main theorems of this thesis, which are concerned with algebraic varieties which split as products of a subvariety of $\mathbb{C}^n$ and one of $(\mathbb{C}^\times)^n$. They are, however, a generalization of the following result of Zilber, which we will come back to and discuss in more detail later on.

Theorem 2.3.22 (Zilber). *Let $V = L \times W$ where $L \leq \mathbb{C}^n$ is a linear space and $W \subseteq (\mathbb{C}^\times)^n$ is an algebraic variety.*

If V is free and rotund, and L is:

1. *Defined over $\mathbb{R}$, assuming a certain Diophantine conjecture, or;*
2. *Defined over a "generic" subfield K of the reals;*

then $V \cap \Gamma_{\exp} \neq \varnothing$.

The theorem under assumption 1. is [Zil02, Theorem 5]; under assumption 2. it is [Zil15, Theorem 7.2]. We postpone a discussion of these assumptions to Subsection 2.4.3.

2.3.5 The Countable Closure Property

Finally, we need to deal with the Countable Closure Property. This is not too hard to grasp intuitively, especially in the case of the complex numbers in which it basically says that given a finite set of parameters X the set of isolated solutions to systems of equations in polynomials and exponentials with parameters in X is countable. However, this relies on a notion of *isolated* which does not have an immediate analogue in the context of exponential fields where there is no topology. This obstacle can be overcome by introducing a notion of differentiability and looking at solutions of systems for which the Jacobian determinant of the system does not vanish.

Definition 2.3.23. Let $(F, \exp_F)$ be an exponential field.

The ring $F[X_1, \dots, X_n]^E$ of *exponential polynomials* in n variables over F is the smallest ring that contains $F[X_1, \dots, X_n]$ and is closed under a formal operation exp which extends $\exp_F$ on the constants and such that $\exp(f(X) + g(X)) = \exp(f(X))\exp(g(X))$ for every $f, g \in F[X_1, \dots, X_n]^E$.

See [Kir10a, Section 2] for a justification of the existence of this object.

On the ring of exponential polynomials we can define formal derivations $\frac{\partial}{\partial X_j}$ by extending the formal derivations on polynomials by the rule $\frac{\partial}{\partial X_j}(\exp(X_j)) = \exp(X_j)$. Since every element of the ring is a composition of exponentials and algebraic operation, this is sufficient to define $\frac{\partial}{\partial X_j}$ on $F[X_1, \ldots, X_n]^E$ using the chain rule.

The goal of this construction is to introduce the notion of a *Khovanskii system* of exponential polynomial equations.

Definition 2.3.24. Let F be an exponential field. A Khovanskii system over F is a system of equations of the form

$$\begin{cases} f_1(z_1, \ldots, z_n) = 0 \\ f_2(z_1, \ldots, z_n) = 0 \\ \vdots \\ f_n(z_1, \ldots, z_n) = 0 \end{cases}$$

where each $f_j \in F[X_1, \ldots, X_n]^E$, together with the inequality

$$\begin{vmatrix} \frac{\partial f_1}{\partial X_1} & \cdots & \frac{\partial f_1}{\partial X_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial X_1} & \cdots & \frac{\partial f_n}{\partial X_n} \end{vmatrix} (z_1, \ldots, z_n) \neq 0.$$

Khovanskii systems are, in turn, necessary to define the notion of exponential algebraicity.

Definition 2.3.25. Let $X \subseteq F$ be a subset, and suppose $a \in F$. We say that a is *exponentially algebraic* over X if there are $x_1, \ldots, x_{n-1}$ such that $(a, x_1, \ldots, x_{n-1})$ is a solution to a Khovanskii system over F.

The set of the elements that are exponentially algebraic over X is the *exponential algebraic closure* of X, denoted $\mathrm{ecl}(X)$.

Exponential algebraic closure is a *pregeometry*, a closure operator with good model-theoretic properties similar to algebraic closure in algebraically closed fields and linear span in vector spaces.

In particular, for example, sets that coincide with their exponential algebraic closures will be algebraically closed fields with an exponential.

Definition 2.3.26. Denote by CCP (Countable Closure Property) the axiom which says that for every finite subset $X \subseteq F$, $\mathrm{ecl}(X)$ is countable.

Note that to state the axiom CCP we need the quantifier Q introduced in Subsection 2.3.1.

We define a notion of exponential-algebraic independence which will be needed later.

Definition 2.3.27. We will say that a tuple $\lambda = (\lambda_1, \ldots, \lambda_n) \in F^n$ is *exponentially-algebraically independent* over the set X if for every j, $\lambda_j \notin \mathrm{ecl}(\{\lambda_1, \ldots, \lambda_{j-1}\} \cup X)$.

If a tuple is exponentially-algebraically independent over $\varnothing$ then we simply say it exponentially-algebraically independent.

As we mentioned above, in $\mathbb{C}_{\exp}$, if a tuple $(a, x_1, \ldots, x_{n-1})$ solves a Khovanskii system then it is an isolated solution of the system (although it should be noted that the converse does not hold, as there are topologically isolated solutions whose Jacobian determinant is 0, but that does not pose an issue). Given that any system of exponential-polynomial equations in the complex numbers has at most countably many isolated solution by analyticity of the functions involved, and that there are only countably many Khovanskii systems over a finite set, the CCP holds for $\mathbb{C}_{\exp}$ (this is [Kir10a, Remark 3.4]).

2.3.6 The Axioms

We conclude by giving the full list of axioms.

Definition 2.3.28. Let ECF denote the following list of $\mathcal{L}_{\omega_1,\omega}(Q)$-axioms in the language of exponential fields:

(ACF_0) F is an algebraically closed field of characteristic 0;

(SE) $\exp : F \to F^\times$ is a surjective group homomorphism;

(SK) The kernel of exp is an infinite cyclic group, generated by a transcendental element;

(SP) For all $z_1, \ldots, z_n \in F$ that are linearly independent over $\mathbb{Q}$,

$$\mathrm{trdeg}(z_1, \ldots, z_n, \exp(z_1), \ldots, \exp(z_n)) \geq n;$$

(SEAC) Every free and rotund algebraic variety $V \subseteq F^n \times (F^\times)^n$ defined over a subfield F_0 intersects the graph of the exponential in a point that is generic over F_0;

(CCP) For every finite subset X of F, the exponential algebraic closure of X is countable.

The statement most commonly referred to as Zilber's Conjecture at the moment is then the following.

Conjecture 2.3.29. $\mathbb{C}_{\exp}$ *satisfies the axiomatization* ECF*.*

In particular, a positive answer to Conjecture 2.3.29 would imply a positive answer to Conjecture 2.3.3.

2.4 Raising to Powers

2.4.1 Model-Theoretic Framework

We devote this section to a more careful discussion of Theorem 2.3.22, as the main theorems of this thesis are at least inspired by it.

Part of the motivation for this theorem comes from the fact that it is possible to give a first-order axiomatization of a class of structures for which the existential closedness statement takes the form of Theorem 2.3.22. This is done in [Zil03] and [Zil15].

More precisely, Zilber fixes a field $K \subseteq \mathbb{C}$ of finite transcendence degree and defines a language $\mathcal{L}_K$ which expands the language of $\mathbb{Q}$-vector spaces by:

1. A binary equivalence relation E;
2. An n-ary predicate L for any subspace L of $\mathbb{C}^n$ defined over K;
3. An n-ary predicate EW for every algebraic subvariety W of $(\mathbb{C}^\times)^n$ definable over $\mathbb{Q}$.

The interpretation of these symbols in $\mathbb{C}$ is that $E(z_1, z_2)$ means $\exp(z_1) = \exp(z_2)$ (so $z_1 - z_2 \in 2\pi i\mathbb{Z}$), $L(z_1, \dots, z_n)$ means that $(z_1, \dots, z_n) \in L$ and $EW(z_1, \dots, z_n)$ means that $(\exp(z_1), \dots, \exp(z_n)) \in W$.

One can then define a first-order class of structures. Denoting the structure by D, we require it to satisfy the axioms for a *powered field with exponents in* K, or K*-powered field*, which we will denote by PK:

1. D is an infinite-dimensional vector space over K;

2. For every predicate EW, $\forall(x_1,\dots,x_n),(y_1,\dots,y_n),(\bigwedge_{i=1}^n E(x_i,y_i)) \to (EW(x_1,\dots,x_n) \to EW(y_1,\dots,y_n))$;
3. The quotient D/E is the multiplicative group of an algebraically closed field of characteristic 0, and the predicates EW define its algebraic varieties over $\mathbb{Q}$.

We denote by $\mathbb{C}^K$ the structure on the complex numbers which interprets these predicates in the way described above.

We are interested in completions of the common theory of these structures.

Zilber gives such a completion in a way that is overall similar to the axiomatization given in Definition 2.3.28, but which is first-order: to the axioms PK that are described above, one adds a transcendence statement and an existential closedness statement. The theory turns out to be *superstable* - a property that model theorists will be familiar with and which is not needed in the rest of the thesis, so we do not define it here.

We do not get here into the details of the abstract forms of the axioms, which are discussed in detail in Sections 3, 4, and 5 of [Zil15]; we simply sum up the main result of that paper in the following statements.

Theorem 2.4.1 ([Zil15, Theorem 6.9])**.** *For any field $K \subseteq \mathbb{C}$ of finite transcendence degree there is a complete superstable theory T_K of powered fields with exponents in K.*

The axioms of these theory are the axioms PK*, a transcendence statement and an existential closedness statement.*

Theorem 2.4.2 ([Zil15, Theorem 7.2])**.** *The structure $\mathbb{C}^K$ is a model of T_K if:*

1. *There is $a \in \mathbb{C}^m$ (possibly $m = 0$) such that for all $z_1,\dots,z_n$,*

$$\operatorname{ldim}_K(z_1,\dots,z_n/a) + \operatorname{trdeg}(\exp(z_1),\dots,\exp(z_n)/\exp(a)) \geq$$

$$\geq \operatorname{ldim}_{\mathbb{Q}}(z_1,\dots,z_n/a);$$

2. *For every free rotund variety of the form $L \times W$ where $L \leq \mathbb{C}^n$ is a linear space defined over the field K, $\exp(L) \cap W \neq \varnothing$.*

2.4.2 Bays-Kirby-Wilkie Property

In this subsection we focus on 1. in Theorem 2.4.2, the transcendence statement.

It is a statement similar to Schanuel's Conjecture. Bays, Kirby and Wilkie studied something similar in [BKW10] and proved the following.

Theorem 2.4.3 ([BKW10, Theorem 1.3]). *Let F be an exponential field,* ker *the kernel of its exponential map, C an* ecl*-closed subset of F and $\lambda \in F^m$ a tuple that is exponentially-algebraically independent over C.*

Then for all $z \in F^n$,

$$\mathrm{ldim}_{\mathbb{Q}(\lambda)}(z/\ker) + \mathrm{trdeg}(\exp(z)/C, \lambda) - \mathrm{ldim}_{\mathbb{Q}}(z/\ker) \geq 0.$$

In [Zil15, Section 7], Zilber applies Theorem 2.4.3 with $F = \mathbb{C}$, $C = \mathrm{ecl}(\varnothing)$ and $\lambda \in \mathbb{C}^m$ a tuple that is exponentially-algebraically independent independent over $\mathrm{ecl}(\varnothing)$. Then, writing $K = \mathbb{Q}(\lambda)$, one gets that for all $z \in \mathbb{C}^n$,

$$\mathrm{ldim}_K(z/2\pi i) + \mathrm{trdeg}(\exp(z)/C, \lambda) - \mathrm{ldim}_{\mathbb{Q}}(z/2\pi i) \geq 0$$

and therefore

$$\mathrm{ldim}_K(z/2\pi i) + \mathrm{trdeg}(\exp(z)) - \mathrm{ldim}_{\mathbb{Q}}(z/2\pi i) \geq 0.$$

We note that, by the Countable Closure Property for $\mathbb{C}_{\exp}$, $\mathrm{ecl}(\varnothing)$ is countable, and therefore many tuples $\lambda \in \mathbb{C}^m$ which are exponentially-algebraically independent over $\mathrm{ecl}(\varnothing)$ exist (although no explicit example is known).

Thus there are infinitely many examples of a field K for which the first clause of Theorem 2.4.2 is satisfied.

2.4.3 Exponential-Algebraic Closedness

Let us now focus on the second clause of Theorem 2.4.2.

The statement is a weaker version of the Exponential-Algebraic Closedness axiom for exponential fields (Definition 2.3.17): it does not ask for intersections between $\Gamma_{\exp}$ and any variety V, but just with varieties which split as a product of the form $L \times W$.

Example 2.4.4. Suppose L is defined by finitely many equations of the form $\lambda_{j,1}z_1 + \cdots + \lambda_{j,n}z_n = 0$ for $j = 1, \ldots, d$, with $\lambda_{j,h} \in K$ for every j, h, and that W is defined by polynomial equations $f_1(w) = \cdots = f_{n-d}(w) = 0$.

If $(z, w) \in L \times W \cap \Gamma_{\exp}$, then $z \in L$, $w \in W$ and $w = \exp(z)$; hence, $w \in \exp(L) \cap W$. Conversely, if $w \in \exp(L) \cap W$ then $w = \exp(z)$ for some $z \in L$, and thus $(z, w) \in L \times W \cap \Gamma_{\exp}$. Showing that $L \times W \cap \Gamma_{\exp} \neq \varnothing$, then, is equivalent to showing that $\exp(L) \cap W \neq \varnothing$; we try to determine what the set $\exp(L)$ looks like.

If $w \in \exp(L)$, then there is $z \in L$ such that $\exp(z) = w$; if $z \in L$ then it satisfies finitely many linear equations. Let $\lambda_1 z_1 + \cdots + \lambda_n z_n = 0$ be one such equation: if that holds, then

$$\exp(\lambda_1 z_1 + \cdots + \lambda_n z_n) = 1$$

and since exp is a group homomorphism

$$\exp(\lambda_1 z_1) \cdots \exp(\lambda_n z_n) = 1.$$

We might then be tempted to write

$$(\exp(z_1))^{\lambda_1} \cdots (\exp(z_n))^{\lambda_n} = 1,$$

but this only makes sense a priori if $\lambda_j \in \mathbb{N}$ for every j: if that is not the case, then $\exp(z)^\lambda$ may denote any determination of

$$\exp(\lambda \log(\exp(z))),$$

so every complex number of the form

$$\exp(\lambda z + 2\pi k i \lambda)$$

for $k \in \mathbb{Z}$. Note that if $\lambda \in \mathbb{R} \setminus \mathbb{Q}$ then $\{2\pi i k \lambda + 2\pi i \mathbb{Z} \mid k \in \mathbb{Z}\}$ is a dense subset of $\mathbb{R}/2\pi i \mathbb{Z}$.

Therefore, raising to the power λ for $\lambda \notin \mathbb{Z}$ is a multivalued operator: if we accept that, then it actually makes sense to write

$$\exp(L) = \{w \in (\mathbb{C}^\times)^n \mid w_1^{\lambda_{j,1}} \cdots w_n^{\lambda_{j,n}} = 1 \,\forall j = 1, \ldots, d\}.$$

The topology of this space can vary quite a lot as we vary the coefficients λ in $\mathbb{C}$ - for example, it can be closed or it can be dense in a semialgebraic subset of the real algebraic group $(\mathbb{C}^{\times})^n$. In Chapter 3 we will see examples of spaces of this kind.

Zilber's theorems from [Zil02] and [Zil15] (see Theorem 2.3.22 above) show that $L \times W \cap \Gamma_{\exp} \neq \varnothing$ if we impose some additional conditions on L.

In [Zil02], a Diophantine conjecture is assumed, the *Conjecture on Intersection with Tori* or CIT. This is a primitive form of what is now known as the Zilber-Pink Conjecture, and reads as follows.

Conjecture 2.4.5 ([Zil02, Conjecture 1])**.** *Let $W \subseteq (\mathbb{C}^{\times})^n$ be an algebraic variety.*

Then there is a finite set $\mathcal{T}$ of translate of algebraic subgroups of $(\mathbb{C}^{\times})^n$ such that if S is an algebraic subgroup of $(\mathbb{C}^{\times})^n$ and K is an irreducible component of $S \cap W$ with

$$\dim K > \dim S + \dim W - n$$

then there is $T \in \mathcal{T}$ such that K is contained in T.

In other words, this conjecture says that *atypical intersections* between algebraic varieties and algebraic subgroups are controlled by finitely many algebraic subgroups. The Zilber-Pink Conjecture, which is now considered one of the most important open problems in Diophantine geometry, is a generalisation of this problem, asking to study atypical intersections in more general arithmetic varieties than the multiplicative group. Zilber also investigated the relation between Conjecture 2.4.5 and Schanuel's Conjecture (see [Zil02, Proposition 5]).

Zilber's result was the following.

Theorem 2.4.6 ([Zil02, Theorem 5])**.** *Assume Conjecture 2.4.5. Let $L \times W$ be a free rotund subvariety of $\mathbb{C}^n \times (\mathbb{C}^{\times})^n$, with $L \leq \mathbb{C}^n$ defined over $\mathbb{R}$ and $W \subseteq (\mathbb{C}^{\times})^n$ an algebraic variety.*

Then $L \times W \cap \Gamma_{\exp} \neq \varnothing$.

We give a rough sketch of the proof strategy.

1. A theorem of Khovanskii is used to show that $L \times W \cap \Gamma_{\exp} \neq \varnothing$ for "almost all" choices of W, in the sense that given any $L \times W$ there is a family of varieties $\mathcal{W} = \{W(a) \mid a \in A\}$, where A is a constructible set,

such that $W \in \mathcal{W}$ and the set $\{a \in A \mid L \times W(a) \cap \Gamma_{\exp} \neq \varnothing\}$ is dense in A (in the Euclidean topology).

2. Given $a \in A$ such that $W = W(a)$, there are then sequences $\{w_j\}_{j \in \omega} \subseteq (\mathbb{C}^\times)^n$ and $\{a_j\}_{j \in \omega}$ such that $w_j \in \exp(L) \cap W(a_j)$ for all j.

3. Using Conjecture 2.4.5, it is shown that there is a finite set ρ depending on L and $\mathcal{W}$ of algebraic varieties such that the sequence $\{w_j\}_{j \in \omega}$ from point 2. is convergent if it stays uniformly away from all the varieties in ρ.

4. The particular form of the theorem of Khovanskii mentioned in 1. allows to find approximating solutions in the "right" places: this means that these sequences of approximating solutions converge.

Of course, Conjecture 2.4.5 is quite a big assumption to make. For this reason, Zilber gave a second version of his result in [Zil15].

Theorem 2.4.7 ([Zil15, Theorem 7.2])**.** *Let $\lambda \in \mathbb{R}^n \subseteq \mathbb{C}^n$ be a tuple that is exponentially-algebraically independent.*

Let $L \times W$ be a free rotund variety, where $L \leq \mathbb{C}^n$ is a linear space that is defined over $\mathbb{Q}(\lambda)$ and $W \subseteq (\mathbb{C}^\times)^n$ is an algebraic variety.

Then $L \times W \cap \Gamma_{\exp} \neq \varnothing$.

The proof of this theorem is similar to the proof of Theorem 2.4.6, but the use of Conjecture 2.4.5 is replaced by a combination of a theorem of Laurent from Diophantine geometry (the Mordell-Lang Conjecture for the multiplicative group, [Lau83, Theorème 1]) and Theorem 2.4.3, the transcendence statement from the previous section.

Theorems 2.4.6 and 2.4.7, together with Theorem 2.4.3, establish then that for many fields K the structure $\mathbb{C}^K$ satisfies the complete superstable theory mentioned in Theorem 2.4.2.

The results in Chapter 3 of this thesis expand the class of fields with this property, showing that the structure $\mathbb{C}^K$ satisfies the second clause of Theorem 2.4.2 for every field K of finite transcendence degree.

2.5 Some Results from Geometry

We conclude this chapter by gathering some results which will be used in the rest of the thesis. As they are used across different chapters, it seems a good idea to list them all here and refer to them later on.

2.5.1 Ax-Schanuel Theorems

Ax-Schanuel type statements measure the transcendence of some geometric functions. The original statement was proved by Ax in [Ax71] for formal differential fields and in [Ax72a] for exponential maps of algebraic groups: the spirit of the result is that algebraic varieties and analytic subgroups of algebraic groups "tend to be in general position". These statements are two sides of the same coin: the differential algebra side and the analytic geometry side of an exponential coin.

We give the differential-algebraic version of the statement first.

Theorem 2.5.1 ([Ax71, Theorem 3])**.** *Let $\mathbb{Q} \subseteq C \subseteq F$ be fields and Δ a set of derivations on F such that $C \subseteq \bigcap_{D\in\Delta} \ker(D)$.*

Let $y_1, \dots, y_n, z_1, \dots, z_n \in F^\times$ be such that for all $j = 1, \dots, n$ and every $D \in \Delta$ $Dy_j = \frac{Dz_j}{z_j}$ and one of the following holds:

a. There are no $k_1, \dots, k_n \in \mathbb{Z}$ such that $z_1^{k_1} \cdots z_n^{k_n} \in C$, or,

b. $y_1, \dots, y_n$ are $\mathbb{Q}$-linearly independent modulo C.

Then

$$\operatorname{trdeg}_C(y_1, \dots, y_n, z_1, \dots, z_n) \geq n + \operatorname{rk}(Dy_j)_{D\in\Delta, j=1,\dots,n}.$$

Theorem 2.5.1 applies, for example, to a field of complex analytic functions with linearly independent elements $y_1, \dots, y_n, \exp(y_1), \dots, \exp(y_n)$ and $\Delta = \{D\}$ the usual derivation $\frac{d}{dz}$. Then $D(\exp(y_j)) = \exp(y_j)Dy_j$, as we wanted.

We are more interested in the geometric version of the statement.

Theorem 2.5.2 ([Ax72a, Theorem 1])**.** *Let G be an algebraic group defined over $\mathbb{C}$ and identified with the Lie group of its $\mathbb{C}$-points.*

Let A be an analytic subgroup of G, K an analytic subvariety of A.

Let W be the Zariski closure of K. Then there exists an analytic subgroup B

of G such that $W, A \subseteq B$ and

$$\dim K \leq \dim W + \dim A - \dim B.$$

We will need a consequence of this theorem for *complex semiabelian varieties.*

Definition 2.5.3. A *semiabelian variety S* is a commutative algebraic group which is an extension of an abelian variety A by a torus T, so for which there is an exact sequence

$$1 \to T \to S \to A \to 0.$$

Definition 2.5.3 should be commented on. By a *torus* we mean a power of the multiplicative group: as we will only deal with complex semiabelian varieties, the reader is free to think of algebraic tori simply as complex algebraic groups isomorphic to $(\mathbb{C}^\times)^n$. An *abelian variety* is a projective algebraic group: these will be defined and studied in detail in Chapter 4. The reader who is not familiar with complex abelian varieties, and who does not want to read ahead before coming back to this chapter, is free to think of semiabelian varieties as algebraic tori for the moment, and add the abelian varieties later.

An important feature of semiabelian varieties is that every complex semiabelian variety S of dimension n has an *exponential map* $\exp_S : \mathbb{C}^n \to S$: this is a group homomorphism, an analytic universal covering map, and it satisfies a certain differential equation. Most notable, of course, is the case $S = \mathbb{C}^\times$, in which the exponential is the usual complex exponential exp and the differential equation is $\frac{d}{dz}\exp(z) = \exp(z)$.

The graph of $\exp_S$ is an analytic subgroup of $\mathbb{C}^n \times S$, and therefore it makes sense to apply Theorem 2.5.2 to it. In particular, Theorem 2.5.2 has the following important consequence.

Corollary 2.5.4 ([Kir06, Theorem 8.1])**.** *Let S be a semiabelian variety, $\exp_S : \mathbb{C}^n \to S$ its exponential map, $\Gamma_{\exp_S}$ the graph of the exponential.*

Let $V \subseteq \mathbb{C}^n \times S$ be an algebraic variety, and K a positive dimensional irreducible analytic component of $V \cap \Gamma_{\exp_S}$ such that $\dim K > \dim V - n$.

Then there is $L \leq \mathbb{C}^n$ a linear subspace such that $\exp(L)$ *is an algebraic subgroup of S and K is contained in a translate of $L \times \exp(L)$.*

Moreover, if C is some constructible set and $\{V(c) \mid c \in C\}$ is a family of *varieties, then there are finitely many linear spaces* $L_1, \dots, L_k$ *such that any positive dimensional irreducible analytic component* K *of* $V(c) \cap \Gamma_{\exp_S}$ *with* $\dim K > \dim V(c) - n$*, for* $c \in C$*, is contained in a translate of one of the algebraic subgroups* $L_j \times \exp(L_j)$*.*

A component with $\dim K > \dim V - n$, as in the statement of this Corollary, is called an *atypical component* of the intersection. A component is *typical* if it is not atypical. A careful reader will have noticed that this problem is connected to Conjecture 2.4.5. We can derive an additional consequence of Corollary 2.5.4:

Corollary 2.5.5 (See [Ax72a, Corollary 2])**.** *In the set-up of Corollary 2.5.4, let* $\gamma \in \mathbb{C}^n \times S$ *be the point such that* $K \subseteq \gamma \cdot (L \times \exp_S(L))$*. Then* K *is a typical component of* $(V \cap (\gamma \cdot L \times \exp_S(L))) \cap (\Gamma_{\exp_{|L}})$

The take-home message from this kind of statement is that algebraic varieties tend to intersect $\Gamma_{\exp_S}$ in a typical way, and in fact the only case in which this does not happen is when something prevents it - namely, when the component is typical with respect to intersection in a smaller ambient space.

Finally, we recall the Ax-Lindemann-Weierstrass Theorem, a consequence of the Ax-Schanuel Theorem which says that the Zariski closure of the exponential of an algebraic variety is an algebraic group.

A proof is given for example in [PZ08, Theorem 2.1] for abelian varieties, but the same holds for general semiabelian varieties.

Theorem 2.5.6 (Ax-Lindemann-Weierstrass Theorem)**.** *Let* $V \subseteq \mathbb{C}^n$ *be an algebraic variety,* S *a semiabelian variety of dimension* n*. Then* $\exp_S(V)$ *is Zariski-dense in an algebraic subgroup of* S*.*

2.5.2 Some Complex Analysis

Finally, we list some facts about complex analytic sets and maps between them. Our standard reference on complex analytic sets is [Chi12], which we invite the reader to consult for any doubts on the matter (especially Chapter 1 and Appendix 2).

Definition 2.5.7. A continuous map $f : X \to Y$ of topological spaces is *proper* if for every compact set $K \subseteq Y$, $f^{-1}(K)$ is compact.

It is interesting to see when the subset of a Cartesian product has proper projection on one of the coordinates.

Proposition 2.5.8 ([Chi12, 3.1])**.** *Let X and Y be locally compact, Hausdorff topological spaces, $D \subseteq X$ and $G \subseteq Y$ open subsets with $\overline{G}$ compact.*

Let A be a relatively closed subset of $D \times G$. The projection $\pi : A \to D$, $(x, y) \mapsto x$, is proper if and only if A does not have limit points on $D \times \partial G$.

Proof. ($\Rightarrow$) Suppose the projection is proper, and there is a sequence $\{a_j\}_{j\in\omega} \subseteq A$ such that $a = \lim_j a_j \in D \times \partial G$. Since D and G are open, $a \notin D \times G$ (and therefore $a \notin A$); however, since $a \in D \times \partial G$ the sequence $\{\pi(a_j)\}_{j\in\omega}$ converges to some $b \in D$.

Then consider a compact neighbourhood U of b in D. We have that $\pi^{-1}(U)$ is not compact, because $a \notin \pi^{-1}(U)$ (it is not in the domain A of π) but it is the limit of the a_j's, which eventually lie in $\pi^{-1}(U)$.

($\Leftarrow$) Suppose A does not have limit points on $D \times \partial G$, and let $K \subseteq D$ be a compact set. Then $\pi^{-1}(K)$ is compact because any sequence in it that has a limit in D has a limit in A, given that A is closed in $D \times G$ without limit points on $D \times \partial G$. $\square$

Proper maps are important in complex analysis because they preserve analyticity of complex analytic sets. As usual, the dimension of the fibres of a holomorphic map is relevant when we want to study the image of the map.

Definition 2.5.9. Let $f : A \to B$ be a holomorphic map between complex analytic sets. For any $z \in A$, let $\dim_z f$ denote the codimension of the fibre

$$\dim_z f := \dim A - \dim f^{-1}(f(z))$$

and $\dim f$ the maximal such value,

$$\dim f := \max_{z\in A} \dim_z f.$$

Theorem 2.5.10 (Remmert's Proper Mapping Theorem, [Chi12, 5.8])**.** *Let A be a complex analytic set and Y a complex manifold, and suppose $f : A \to Y$ is proper.*

Then $f(A)$ is an analytic subset of Y, and

$$\dim f(A) = \dim f$$

When the map is not proper there is not much that can be said about its image; the best one can hope for is the following result.

Proposition 2.5.11 ([Chi12, 3.8])**.** *Let A be a complex analytic set and Y a complex manifold, and suppose $f : A \to Y$ is holomorphic.*

Then $f(A)$ is contained in a countable union of analytic subsets of Y, of dimension not exceeding $\dim f$*.*

Of a similar flavour to the Proper Mapping Theorem is the Open Mapping Theorem, which says that if the fibres of a map have the "right" dimension then the map is open. The reader may recall the one variable version of this fact from a basic course in complex analysis: in that case, the statement is that any non-constant holomorphic map $f : U \subseteq \mathbb{C} \to \mathbb{C}$ is open.

Theorem 2.5.12 (Open Mapping Theorem, [Chi12, Appendix 2, Theorem 2])**.** *Let A be a complex analytic set and Y a complex manifold. A holomorphic map $f : A \to Y$ is open if and only if* $\dim f^{-1}(z) = \dim A - \dim Y$ *for every $z \in f(A)$.*

An immediate corollary of the Open Mapping Theorem is the following.

Corollary 2.5.13. *Let A be a complex analytic set, Y a complex manifold, $f : A \to Y$ holomorphic and suppose $f^{-1}(f(z))$ has dimension* $\dim A - \dim Y$*.*

Then there is an open neighbourhood U of z such that f is open on z.

Chapter 3

Complex Exponential

3.1 Introduction

This chapter is devoted to the first result we want to show: a new version of Theorem 2.3.22, which establishes Exponential-Algebraic Closedness for all varieties of the form $L \times W$ where $L \leq \mathbb{C}^n$ is linear and $W \subseteq (\mathbb{C}^\times)^n$ is algebraic.

Theorem (Theorem 3.7.8)**.** *Let $L \times W$ be a free rotund subvariety of $\mathbb{C}^n \times (\mathbb{C}^\times)^n$ such that $L \leq \mathbb{C}^n$ is a linear space and $W \subseteq (\mathbb{C}^\times)^n$ is an algebraic variety. Then $L \times W \cap \Gamma_{\exp} \neq \varnothing$.*

The proof uses *tropical geometry*, a relatively young branch of mathematics which studies subvarieties of $(\mathbb{C}^\times)^n$, or more generally of toric varieties, and in particular their behaviour near 0 and ∞. The main idea of tropical geometry, which was already present in old work of Bergman ([Ber71]), is that this behaviour can be described completely using finitely many semilinear subspaces of $\mathbb{R}^n$, which intuitively correspond to the "directions" in which points on the variety approach 0 and ∞. This results in an object called the *tropicalization* of a variety (Definition 3.3.12), a finite union of semilinear spaces.

Tropical geometry is closely related to the study of *amoebas*, the images of algebraic varieties under the coordinatewise logarithm of the absolute value. These are semianalytic subsets of $\mathbb{R}^n$; there is a precise sense in which if we "look at an amoeba from far away" what we end up seeing is not that different from a tropicalization. This will be explained in Section 3.3.

Concerning the linear space L, there is an important distinction to be made between spaces that are defined over the reals and spaces that are not. In fact, if we consider the coordinatewise real part map $\mathrm{Re} : \mathbb{C}^n \to \mathbb{R}^n$ defined as

$$(x_1 + iy_1, \ldots, x_n + iy_n) \mapsto (x_1, \ldots, x_n)$$

then we have that if L is defined over the reals it satisfies $L = \mathrm{Re}(L) + i\mathrm{Re}(L)$; if it is not, then it satisfies $\mathrm{Re}(L) \lneq \mathrm{Re}(L) + i\mathrm{Re}(L)$. Moreover, if L is defined over the reals and not contained in any rational space, and we denote by $\mathbb{S}_1$ the unit circle $\{z \in \mathbb{C} \mid |z| = 1\}$ in the complex plane, then $\exp(L)$ is dense in $\exp(L) \cdot \mathbb{S}_1^n$; if L is not defined over the reals, on the other hand, its exponential might even be a closed analytic subset of $(\mathbb{C}^\times)^n$ (an example of this is the subspace of $\mathbb{C}^2$ defined by $z_2 = iz_1$, which is discussed in several occasions in this chapter). We will see that this easy dichotomy has meaningful consequences, and the proofs of Theorem 3.7.8 in the two cases are quite different.

This chapter has something of a "back-and-forth" structure, as it moves between sections on tropical geometry, sections on Exponential-Algebraic Closedness, and sections which combine these things to get the main proofs. More precisely, the structure of the chapter is as follows.

In Section 3.2 we make a few remarks on specific features of this form of the Exponential-Algebraic Closedness problem.

Section 3.3 will be dedicated to the basic properties of amoebas and tropical geometry.

Section 3.4 moves back to the study of systems of equations, explaining how the systems corresponding to varieties of the form $L \times W$ can be traced back to the so-called systems of "exponential sums equations".

In Section 3.5 we see how to use the material from the previous sections to get the main theorem when L is defined over the reals. These first five sections form a self-contained exposition of the proof of this first instance of the theorem.

Section 3.6 takes things further, by introducing some results due to Kazarnovskii which describe the interaction of complex spaces with tropical geometry.

In Section 3.7 we prove that $L \times W \cap \Gamma_{\exp} \neq \varnothing$ even when L is not defined over the reals, thus completing the proof of the main result of this chapter.

Finally, in Section 3.8 we tie the results of this chapter with those of Section 2.4, discussing how Theorem 3.7.8 implies that whether or not the structure $\mathbb{C}^K$ is a model of the theory T_K depends only on the transcendence statement.

The main results of this chapter have already appeared in the preprint [Gal22a].

3.2 Exponential-Algebraic Closedness

As already outlined in the Introduction, our goal in this chapter is to establish a stronger result than Theorem 2.3.22, which does not need any additional assumptions on free rotund varieties of the form $L \times W$ in which L is linear.

We want to show that if $L \times W$ is free and rotund, with $L \leq \mathbb{C}^n$ and $W \subseteq (\mathbb{C}^\times)^n$, then $L \times W \cap \Gamma_{\exp} \neq \varnothing$. As we have already seen in Example 2.4.4, $L \times W \cap \Gamma_{\exp} \neq \varnothing$ if and only if $W \cap \exp(L) \neq \varnothing$, and to prove this we aim to understand what $\exp(L)$ looks like as a subset of $(\mathbb{C}^\times)^n$. We begin by examining two examples.

Example 3.2.1. Let

$$L_{\sqrt{2}} := \left\{(z_1, z_2) \mid z_2 = \sqrt{2} z_1\right\},$$

$$L_i := \{(z_1, z_2) \mid z_2 = i z_1\}$$

and

$$W := \{(w_1, w_2) \in (\mathbb{C}^\times)^2 \mid w_1 + w_2 + 1 = 0\}.$$

The varieties $L_{\sqrt{2}} \times W$ and $L_i \times W$ will serve as our recurring examples for this chapter.

It is clear that both varieties are free: $L_{\sqrt{2}}$ and L_i are not contained in $\mathbb{Q}$-linear subspaces of $\mathbb{C}^2$, and W is not contained in an algebraic subgroup of $(\mathbb{C}^\times)^2$. They are also both rotund (in fact, for subvarieties of $\mathbb{C}^2 \times (\mathbb{C}^\times)^2$, freeness implies rotundity).

Suppose we want to find an intersection between $L_{\sqrt{2}} \times W$ and $\Gamma_{\exp}$, the graph of the exponential. Then we want to find a point $(z_1, z_2, \exp(z_1), \exp(z_2))$ in $L \times W$, which thus has to satisfy $z_2 = \sqrt{2} z_1$ and $\exp(z_1) + \exp(z_2) = 1$; this is equivalent to finding a complex number $z \in \mathbb{C}$ such that $\exp(z) + \exp(\sqrt{2} z) + 1 = 0$. By abusing notation, and writing $w^{\sqrt{2}}$ to mean any determination of $\exp(\sqrt{2}(\log w))$, we can see this as looking for a point $w \in \mathbb{C}^\times$ such that

$w + w^{\sqrt{2}} + 1 = 0$.

The same reasoning, of course, can be applied to $L_i \times W$, and thus we can see intersecting that with $\Gamma_{\exp}$ to be equivalent to solving $w + w^i + 1 = 0$.

The two situations are quite different from a geometric perspective, as we now see. For a given number $w \in \mathbb{C}^\times$, all the determinations of $w^{\sqrt{2}}$ form a dense subset of the set $\{z \in \mathbb{C}^\times \mid |z| = |w|^{\sqrt{2}}\}$. In fact, if $w = \rho(\cos\theta + i\sin\theta) \in \mathbb{C}^\times$ for some $\rho \in \mathbb{R}^{>0}$ and $\theta \in [0, 2\pi[$ then

$$\exp^{-1}(w) = \{x + iy \in \mathbb{C} \mid e^x = \rho \wedge y \in \theta + 2\pi\mathbb{Z}\}.$$

Since $\sqrt{2}\mathbb{Z} + \mathbb{Z}$ is dense in $\mathbb{R}/\mathbb{Z}$, the set

$$\exp(\sqrt{2}\exp^{-1}(w)) = \{\exp(\sqrt{2}x + i\sqrt{2}y) \in \mathbb{C}^\times \mid e^x = \rho \wedge y \in \theta + 2\pi\mathbb{Z}\} =$$

$$= \left\{\rho^{\sqrt{2}}(\cos(\sqrt{2}(\theta + 2k\pi)) + i\sin(\sqrt{2}(\theta + 2k\pi))) \in \mathbb{C}^\times \mid k \in \mathbb{Z}\right\}$$

is dense in $\{z \in \mathbb{C}^\times \mid |z| = |w|^{\sqrt{2}} = \rho^{\sqrt{2}}\}$ as we wanted. We see then that $\exp(L_{\sqrt{2}})$ is dense in

$$\exp(L_{\sqrt{2}}) \cdot \mathbb{S}_1^2 = \left\{(w_1, w_2) \in (\mathbb{C}^\times)^2 \mid |w_2| = |w_1|^{\sqrt{2}}\right\}$$

where $\mathbb{S}_1$ denotes the unit circle $\{z \in \mathbb{C} \mid |z| = 1\}$ and the operation $|w_1|^{\sqrt{2}}$ is well-defined, not multivalued, as it is a real power of a real number.

On the other hand,

$$\exp(i\exp^{-1}(w)) = \{\exp(-y + ix) \in \mathbb{C}^\times \mid e^x = \rho \wedge y \in \theta + 2\pi\mathbb{Z}\} =$$

$$= \left\{e^{\theta + 2k\pi}(\cos(\rho) + i\sin(\rho)) \in \mathbb{C}^\times \mid k \in \mathbb{Z}\right\}$$

is an infinite discrete subset of $\mathbb{C}^\times$. Thus $\exp(L_i)$ is a closed, complex analytic subgroup of $(\mathbb{C}^\times)^2$.

An important ingredient in our proofs will be a result by Kirby which combines the Ax-Schanuel Theorem 2.5.2 with the Remmert open mapping theorem 2.5.12. We are going to apply it to the following function.

Definition 3.2.2 (δ-map of a variety)**.** Let V be an algebraic subvariety of

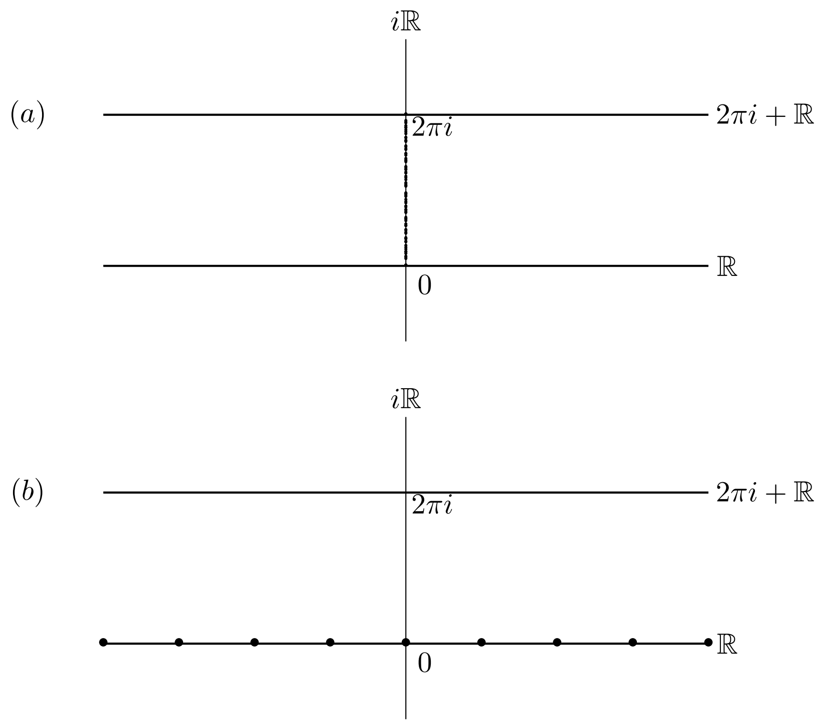

Figure 3.1: The exponential map identifies the multiplicative group $\mathbb{C}^\times$ with the strip $\{z \in \mathbb{C} \mid 0 \leq \mathrm{Im}(z) \leq 2\pi\}$, glued with itself joining the outer lines: in other words, the multiplicative group is a cylinder. With this interpretation, the sets of all determinations of $1^{\sqrt{2}}$ and 1^i are shown in the figure: in (a) we see the dense sets of points obtained as $(2\pi i\mathbb{Z}) \cdot \sqrt{2} + \mathbb{Z}$, in (b) the discrete set $-2\pi\mathbb{Z}$.

$\mathbb{C}^n \times (\mathbb{C}^\times)^n$. The *$\delta$-map of V* is the function

$$\delta : V \to (\mathbb{C}^\times)^n$$

which maps $(v_1, v_2) \in V$ to $\frac{v_2}{\exp(v_1)}$.

The following fact was established by Kirby in the proof of in [Kir19b, Proposition 6.2 and Remark 6.3].

Fact 3.2.3. *Suppose the variety V is free and rotund. Then there is a Zariski-open dense subset $V^\circ \subseteq V$ such that the δ-map of V is open on V°.*

For varieties of the form $L \times W$, actually, something stronger holds and we can say more about the structure of the set V°.

Proposition 3.2.4. *Suppose $L \times W$ is a free rotund algebraic subvariety of $\mathbb{C}^n \times (\mathbb{C}^\times)^n$, with L a linear subspace of $\mathbb{C}^n$ and W an algebraic subvariety of $(\mathbb{C}^\times)^n$. Then there is a Zariski-open dense subset $W^\circ \subseteq W$ such that the δ-map of $L \times W$ is open on $L \times W^\circ$.*

Proof. Suppose $(l_0, w_0) \in L \times W$ is a point around which δ is open; let U_L be a neighbourhood of l_0 and U_W be a neighbourhood of w_0 such that $\delta_{|U_L \times U_W}$ is open. Let l be any point in L, and $V_L = (l - l_0) + U_L$ a neighbourhood of l that is a translate of U_L. Then any open subset O_V of $V_L \times U_W$ is a translate by $((l - l_0), 1)$ of an open subset O_U of $U_L \times U_W$. This implies that $\delta(O_V)$ is a translate, by $\exp(l - l_0)$, of $\delta(O_U)$, so an open set.

Therefore if δ is open around (l_0, w_0), then it is open around any point of $L \times \{w_0\}$. Thus the Zariski-open dense set of Fact 3.2.3 must be of the form $L \times W^\circ$ for some Zariski-open dense subset W° of W. $\square$

Moreover, we have that openness of the δ-map at a single point is sufficient to prove rotundity.

Proposition 3.2.5. *Let $L \times W$ be an algebraic subvariety of $\mathbb{C}^n \times (\mathbb{C}^\times)^n$, with L a linear subspace of $\mathbb{C}^n$ and W an algebraic subvariety of $(\mathbb{C}^\times)^n$.*

If there is a point at which the δ-map of $L \times W$ is open, then the variety is rotund.

Proof. Suppose Q is a linear subspace of $\mathbb{C}^n$. Let $\pi_Q : \mathbb{C}^n \twoheadrightarrow \mathbb{C}^n/Q$ and $\pi_{\exp(Q)} : (\mathbb{C}^\times)^n \twoheadrightarrow (\mathbb{C})^\times/\exp(Q)$ denote the projections; δ the δ-map of $L \times W$;

δ_Q the δ-map of $\pi_Q(L) \times \pi_{\exp(Q)}(W)$.

Let $(l, w) \in L \times W$. We see that

$$\delta_Q(\pi_Q(l), \pi_{\exp(Q)}(w)) = \frac{\pi_{\exp(Q)}(w)}{\exp(\pi_Q(l))} =$$

$$= \frac{w \cdot \exp(Q)}{\exp(l) \cdot \exp(Q)} = \frac{w}{\exp(l)} \cdot \exp(Q) = \pi_{\exp(Q)}(\delta(l, w))$$

and therefore the square

$$\begin{tikzcd}
L \times W \arrow[r, "\delta"] \arrow[d, "\pi_Q \times \pi_{\exp(Q)}"] & (\mathbb{C}^\times)^n \arrow[d, "\pi_{\exp(Q)}"] \\
\pi_Q(L) \times \pi_{\exp(Q)}(W) \arrow[r, "\delta_Q"] & (\mathbb{C}^\times)^n / \exp(Q)
\end{tikzcd}$$

commutes.

Now let $(l_0, w_0) \in L \times W$ be the point at which δ is open, so that there are neighbourhoods $U_L \subseteq L$ of l_0 and $U_W \subseteq W$ of w_0 such that $\delta_{|U_L \times U_W}$ is an open map. Then we see that

$$\delta_Q(\pi_Q(U_L) \times \pi_{\exp(Q)}(U_W)) = \pi_{\exp(Q)}(\delta(U_L \times U_W))$$

and the set on the right-hand side is the projection of an open set, and thus it has to be open. Thus the set on the left-hand side is open too, and that is only possible if $\dim \pi_Q(L) + \dim \pi_{\exp_Q}(W) \geq n - \dim Q$ as we wanted. $\square$

Using this fact together with a common procedure known as the Rabinovich trick we can make a very useful reduction, proving that we can assume without loss of generality that if $L \times W$ is free and rotund then the δ-map is open everywhere.

Lemma 3.2.6. *Suppose $L \times W$ is a free rotund algebraic variety, with L a linear subspace of $\mathbb{C}^n$ and W an algebraic subvariety of $(\mathbb{C}^\times)^n$. Then there is a free rotund subvariety $L' \times W'$ of $\mathbb{C}^{n+1} \times (\mathbb{C}^\times)^{n+1}$ such that the δ-map of $L' \times W'$ is open on all the domain, and if $\exp(L') \cap W' \neq \varnothing$ then $\exp(L) \cap W \neq \varnothing$.*

Proof. Since the set of points where δ is open is of the form $L \times W^\circ$, there is an algebraic function $F : W \to \mathbb{C}$ such that δ is open around each point (l, w) for which $F(w) \neq 0$. Let $L' := L \times \mathbb{C}$ and

$$W' := \left\{(w_1, \ldots, w_{n+1}) \in (\mathbb{C}^\times)^n \mid (w_1, \ldots, w_n) \in W \wedge F(w_1, \ldots, w_n) = w_{n+1}\right\}.$$

Note that F must have at least one zero; if it does not, then δ is already open on $L \times W$.

Consider then the variety $L' \times W'$. It is clear that L' is free. If W' were not free, then its points would identically satisfy a relation of the form $w_1^{k_1} \cdots w_{n+1}^{k_{n+1}} = c$ for integers $k_1, \ldots, k_{n+1}$ and $c \in \mathbb{C}$. In particular, k_{n+1} must be non-zero (otherwise W would not be free). Thus we have that $w_{n+1} = (c^{-1} w_1^{-k_1} \cdots w_n^{-k_n})^{-\frac{1}{k_{n+1}}}$; this would mean that F does not have any zeros on W, contradiction.

For rotundity we use Proposition 3.2.5. Given a point in $L' \times W'$, which therefore has the form $(l_1, \ldots, l_{n+1}, w_1, \ldots, w_{n+1})$ there are neighbourhoods $U_L \subseteq L$ and $U_W \subseteq W$ of $(l_1, \ldots, l_n)$ and $(w_1, \ldots, w_n)$ respectively such that the δ-map of $L \times W$ is open on $U_L \times U_W$. Let $U_{L'} := U_L \times \mathbb{C}$, and

$$U_{W'} := \{(w_1, \ldots, w_{n+1}) \in W' \mid (w_1, \ldots, w_n) \in U_W\}.$$

The image of $U_{L'} \times U_{W'}$ under the δ-map of $L' \times W'$ is then the Cartesian product of an open subset of $(\mathbb{C}^\times)^n$ by $\mathbb{C}$, i.e. an open subset of $(\mathbb{C}^\times)^{n+1}$: therefore the δ-map of $L' \times W'$ is open at all of its points, and therefore the variety must be rotund.

Finally, it is clear that if $(w_1, \ldots, w_{n+1}) \in \exp(L') \cap W'$ then $(w_1, \ldots, w_n) \in \exp(L) \cap W$. $\square$

We introduce another assumption which simplifies the proofs: we may take $\dim L = \operatorname{codim} W$, so that $\dim L \times W = n$ when $L \times W \subseteq \mathbb{C}^n \times (\mathbb{C}^\times)^n$.

Lemma 3.2.7. *Let $L \times W$ be a free rotund variety in $\mathbb{C}^n \times (\mathbb{C}^\times)^n$. Then there is a space $L'' \subseteq L$ such that $L'' \times W$ is free and rotund, and* $\dim L'' + \dim W = n$.

Proof. Let δ be the δ-map of $L \times W$; by rotundity, δ is open around some point $(0, w) \in L \times W$. This implies that there is a point $z \in \log W$, with $\exp(z) = w$, which lies in an irreducible component of the set $\log W \cap z + L$ of dimension $\dim L + \dim W - n$, by Theorem 2.5.12. For a sufficiently generic subspace

$H \leq \mathbb{C}^n$, of dimension $2n - \dim L - \dim W$, we will have then that

$$\dim(\log W \cap z + L \cap z + H) = \dim L + \dim W - n + 2n - \dim L - \dim W - n = 0$$

and thus z is an isolated point in it. Therefore, the δ-map of $(L \cap H) \times W$ is open at the point $(0, w)$: the variety $(L \cap H) \times W$ is then rotund by Proposition 3.2.5 and since we H is generic we may also take it to be free. Therefore $L'' := L \cap H$ satisfies the lemma. □

Thus in what follows we will, when necessary, assume freely that $\dim L = \mathrm{codim} W$. This will not affect the generality of our statements.

3.3 Amoebas and Tropical Geometry

In this section we introduce the basics on amoebas and tropical geometry, focusing on the interaction between the two and on the notion of stable intersection. In the next sections we will show how this ties to the exponential-algebraic closedness question for varieties of the form $L \times W$. Amoebas will be more important in the case in which L is defined over the reals, as then we will see that finding a point in $\exp(L) \cap W$ is as hard as finding a point in the intersection of the real part of L with the amoeba of W. Tropicalizations will play a more important role in the case in which L is not defined over the reals, as then we will need a precise understanding of the behaviour of W as its points approach 0 or ∞, which will be given by tropical geometry.

3.3.1 Amoebas

Amoebas were introduced in [GKZ94, Chapter 6] as a tool to analyse the behaviour near 0 and ∞ of subvarieties of $(\mathbb{C}^\times)^n$. A good survey on their properties is [Mik04a].

Denote by $\mathrm{Log} : (\mathbb{C}^\times)^n \to \mathbb{R}^n$ the map

$$(z_1, \ldots, z_n) \mapsto (\log |z_1|, \ldots, \log |z_n|) .$$

Definition 3.3.1. Let $W \subseteq (\mathbb{C}^\times)^n$ be an algebraic variety. The *amoeba* $\mathcal{A}_W$ of W is the image of W under the map Log.

Figure 3.2 shows a picture of an amoeba.

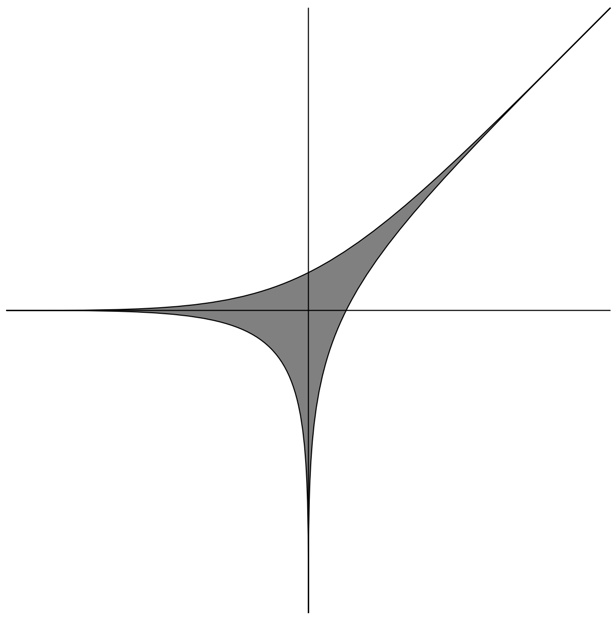

Figure 3.2: The amoeba of the algebraic variety W defined by $w_1 + w_2 + 1 = 0$. We see that the amoeba has three "tentacles": the diagonal one corresponds to the behaviour of W when w_1 and w_2 are both very big, and thus their absolute values are roughly the same; the vertical one corresponds to points for which w_2 is very close to 0 (and thus its logarithm to $-\infty$) and w_1 to -1; the horizontal one to points with w_1 close to 0 and w_2 to -1.

An amoeba is a closed proper subset of $\mathbb{R}^n$. Much of the theory of amoebas has been carried out for amoebas of hypersurfaces; however, we will use amoebas of varieties of arbitrary codimension. For now, we only state the following theorem, which will help us later on to establish the tie between amoebas and tropical varieties.

Theorem 3.3.2 ([Pur08, Corollary 5.2])**.** *Let $W \subseteq (\mathbb{C}^\times)^n$ be an algebraic variety, and I be the ideal of polynomials which vanish on W. Then*

$$\mathcal{A}_W = \bigcap_{f \in I} \mathcal{A}_f$$

where $\mathcal{A}_f$ denotes the amoeba of the hypersurface cut out by f.

3.3.2 Polyhedral Geometry

In this subsection we review a few basic facts on polyhedral geometry, so that we have all the tools to discuss tropical varieties later on.

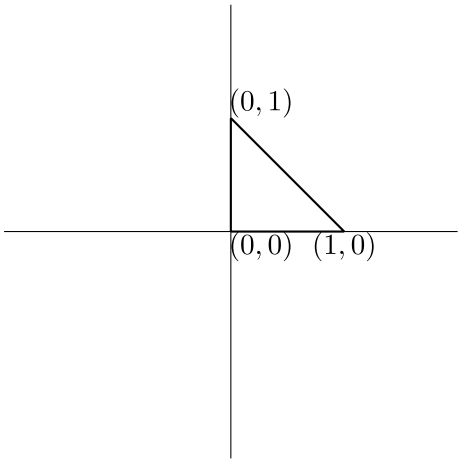

Figure 3.3: This triangle is a polyhedron defined by $x_1 + x_2 \leq 1$, $x_1 \geq 0$ and $x_2 \geq 0$. The three sides are faces induced by the vectors $(1, 1)$, $(-1, 0)$ and $(0, -1)$.

We start by recalling some of the definitions, starting with the basic notion of polyhedron.

Definition 3.3.3. A *polyhedron* is a subset of $\mathbb{R}^n$ of the form

$$\{x \in \mathbb{R}^n \mid Ax \leq b\}$$

where A is a $d \times n$ matrix with real entries and $b \in \mathbb{R}^d$. If the matrix has entries in $\mathbb{Q}$ and $b \in \mathbb{Q}^d$ we will say the polyhedron is *rational.*

A bounded polyhedron is called a *polytope.*

A *face* of a polyhedron P with respect to some $w \in \mathbb{R}^n$ is a subset of P of the form

$$\text{face}_w(P) := \{x \in P \mid w \cdot x \geq w \cdot y \ \ \forall y \in P\}.$$

We will denote by $\text{aff}(\tau)$ the affine span of a polyhedron.

We will deal with sets of polyhedra that enjoy good coherence properties.

Definition 3.3.4. A *polyhedral complex* Σ is a set of polyhedra such that:

1. If $P \in \Sigma$ then every face of P is in Σ;
2. If $P_1, P_2 \in \Sigma$, then $P_1 \cap P_2$ is either empty or a face of both (and thus an element of Σ).

The union of all polyhedra of Σ is called the *support* of Σ and denoted $|\Sigma|$. We will often abuse notation and write $x \in \Sigma$ for $x \in \mathbb{R}^n$ to mean that $x \in |\Sigma|$, so there is $P \in \Sigma$ such that $x \in P$; when P is a polyhedron, $P \in \Sigma$ will literally mean that P is one of the polyhedra of Σ.

The polyhedra in a polyhedral complex are called the *cells* of the complex; those which are not contained in any larger polyhedra are the *facets* of the complex and the faces of a facet which are not contained in any larger polyhedron in the complex (other than the facet itself) are called the *ridges* of the complex. Obviously, by a *rational polyhedral complex* we will mean a polyhedral complex all of whose polyhedra are rational.

The type of polyhedral complex that we will mostly be interested in is the *normal fan* of a polytope.

Definition 3.3.5. A *cone* in $\mathbb{R}^n$ is a polyhedron P for which there exist $v_1, \ldots, v_k \in \mathbb{R}^n$ such that

$$P = \left\{ \sum_{i=1}^{k} \lambda_i v_i \mid \lambda_1, \ldots, \lambda_k \in \mathbb{R}_{\geq 0} \right\}.$$

A *fan* is a polyhedral complex Σ such that all polyhedra in Σ are cones.

Definition 3.3.6. Let P be a polytope. The *normal fan* of P is the polyhedral complex which contains for every face F of P the polyhedron obtained as the closure (in the Euclidean topology on $\mathbb{R}^n$) of the set

$$\{w \in \mathbb{R}^n \mid F = \mathrm{face}_w(P)\}.$$

The $(n-1)$-skeleton of the normal fan is the polyhedral complex obtained by removing the polyhedra of dimension n from the the normal fan.

We will see in the next subsection that tropicalizations of algebraic varieties are polyhedral complexes which enjoy strong structural properties.

We are also interested in the notion of *mixed volume* of a collection of polytopes. First we recall how to perform standard operations on subsets of Euclidean space.

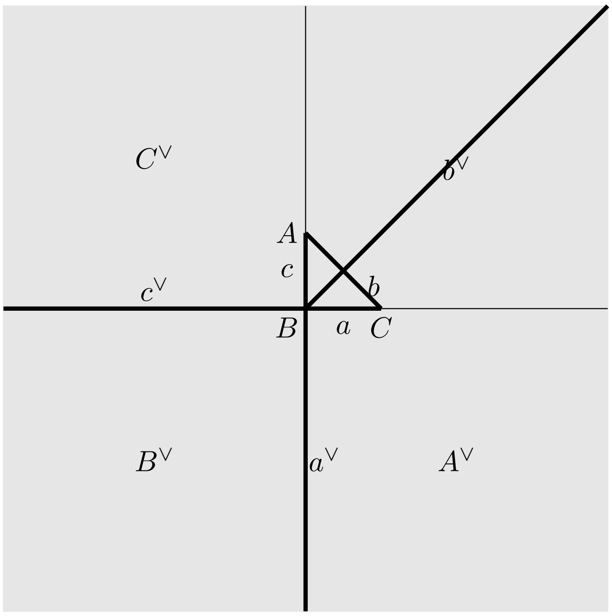

Figure 3.4: (The support of) the normal fan of the polytope of Figure 3.3: the normal cone to the full triangle, which is the face of $(0, 0)$, is the origin; the normal cones to the sides of the triangles are the three half-lines; and the normal cones to the three vertices of the triangles are the portions of space in between the half-lines. Each of the normal cones (except the origin) has been labelled to show which face of the polytope it is normal to. Considering only the origin and the three half-lines, we obtain the 1-skeleton of the complex.

Definition 3.3.7. Let $A, B \subseteq \mathbb{R}^n$. The *Minkowski sum* of A and B is the set

$$A + B := \{a + b \in \mathbb{R}^n \mid a \in A,\, b \in B\}.$$

If $\lambda \in \mathbb{R}$, then

$$\lambda A := \{\lambda a \in \mathbb{R}^n \mid a \in A\}.$$

We define the *normalized volume* of a polytope $P \subseteq \mathbb{R}^n$ to be the standard Euclidean volume multiplied by $n!$. This is so that the smallest simplex with integer vertices in $\mathbb{R}^n$ has volume 1.

We then use the following fact. It is proved in [MS15, Proposition 4.6.3] under the stronger assumption that the polytopes have integer vertices to draw the stronger conclusion that the resulting polynomial has integer coefficients; however, the same proof will yield the statement that we give here.

Proposition 3.3.8. *Let $P_1, \dots, P_r$ be polytopes in $\mathbb{R}^n$. The normalized volume of the polytope $\lambda_1 P_1 + \dots + \lambda_r P_r$ is a homogeneous polynomial in $\lambda_1, \dots, \lambda_r$ of degree n.*

This allows us to give the following definition of mixed volume.

Definition 3.3.9. Let $P_1, \dots, P_n$ be polytopes in $\mathbb{R}^n$. The mixed volume of the polytopes, denoted $\mathrm{MV}(P_1, \dots, P_n)$ is the coefficient of the unique square-free monomial $\lambda_1 \cdots \lambda_n$ in the polynomial obtained in Proposition 3.3.8.

We will see later on how to relate the mixed volumes of a collection of polytopes to the intersection of the $(n-1)$-skeletons of their normal fans.

3.3.3 Tropicalizations

Tropical geometry is the heart of this section. Here we will introduce tropical varieties and show how to interpret them as limits of amoebas; we will also state part of the *Structure Theorem*, a fundamental result in tropical geometry which establishes structural properties of tropical varieties.

There are several equivalent ways to define tropical varieties. We choose the approach based on *initial forms* of polynomials, as it is going to be the most convenient one to discuss tropical compactifications later on (in Section 3.7).

The initial forms of a Laurent polynomial with complex coefficients can be thought of as limit forms that the polynomial takes when it is evaluated at some point $w \in (\mathbb{C}^\times)^n$, some of whose coordinates approach 0 or ∞.

In the next definition we use multi-index notation: for $k \in \mathbb{Z}^n$, we use w^k to mean the monomial $w_1^{k_1} \cdots w_n^{k_n}$.

Definition 3.3.10. Let $x \in \mathbb{R}$, and let f be a Laurent polynomial in n variables, so $f \in \mathbb{C}[w_1^{\pm 1}, \ldots, w_n^{\pm 1}]$. Write f as

$$f := \sum_{k \in S} c_k w^k$$

for some finite subset $S \subseteq \mathbb{Z}^n$. The *initial form of f with respect to x* is the polynomial

$$\mathrm{in}_x(f) := \sum_{k \in K_x} c_k w^k$$

where K_x is the set

$$K_x := \{k \in K \mid x \cdot k \geq y \cdot k \ \ \forall y \in K\}.$$

If I is an ideal in the ring of Laurent polynomials, then $\mathrm{in}_x(I)$ denotes the set of initial forms of polynomials in f.

Example 3.3.11. To see an example of how to take initial forms, consider the polynomial $f := w_1 + w_2 + 1$. For this, the set K is the set $\{(1,0), (0,1), (0,0)\}$. Hence, it is easy to see that:

1. If x has $x_1, x_2 < 0$, then $x \cdot (0,0)$ is bigger than $x \cdot (1,0)$ and $x \cdot (0,1)$; thus the initial form of f is 1.

2. If $x_1 - x_2 > 0$ and $x_1 > 0$, then the largest scalar product is $x \cdot (1,0)$; thus the initial form of f is w_1.

3. Similarly, if $x_1 - x_2 < 0$ and $x_2 > 0$ then we obtain w_2 as an initial form.

4. If $x_1 = x_2$, and both are positive, the maximum is obtained twice as $x \cdot (1,0) = x \cdot (0,1)$. Therefore the initial form is $w_1 + w_2$.

5. If $x_1 = 0$ and $x_2 < 0$ then $x \cdot (1,0) = x \cdot (0,0)$ and both are larger than $x \cdot (0,1)$, so the initial form is $w_1 + 1$.

6. In the same way, if $x_2 = 0$ and $x_1 < 0$ then the initial form is $w_2 + 1$.

7. Finally, if $x = (0,0)$ then the initial form of f is f itself as all scalar products are the same.

It is clear that the regions we chose form a partition of $\mathbb{R}^2$, and that they are in fact the relative interiors of the polyhedra that we see in Figure 3.4.

The idea of initial forms is that if we plug in very large or very small values for some of the variables of a polynomial, then the value of the polynomial has no hope of being zero unless there are at least two monomials that are roughly of the same size. Thus, most of the time, the value of the polynomial will be decided simply by the fact that one of the monomials takes a much larger value than the others. This motivates the following definition of tropical variety.

Definition 3.3.12. Let $W \subseteq (\mathbb{C}^\times)^n$ be an algebraic variety, and let I be the ideal of Laurent polynomials which vanish on W. Then the *tropicalization* $\operatorname{Trop}(W)$ of W is the subset of $\mathbb{R}^n$ defined as

$$\{x \in \mathbb{R}^n \mid \operatorname{in}_x(I) \neq \langle 1 \rangle\}.$$

Remark 3.3.13. As the ideal is taken in the ring of Laurent polynomials, an ideal is the whole ring if and only if it is generated by monomials, as these are the invertible elements there. Hence the tropical variety of W can be defined as the set of x's for which the initial ideal of I is not a monomial ideal.

Example 3.3.14. It is clear at this point that the tropicalization of the algebraic variety

$$W := \{(w_1, w_2) \in (\mathbb{C}^\times)^2 \mid w_1 + w_2 + 1 = 0\}$$

is the $(n-1)$-skeleton of the polyhedral complex shown in Figure 3.4.

As we mentioned above, there are several ways to define tropical varieties, the equivalence of which is known as the *Fundamental Theorem of Tropical Algebraic Geometry* ([MS15, Theorem 3.2.5]). Here we state just part of it.

Theorem 3.3.15 ([MS15, Theorem 3.2.5])**.** *Let $W \subseteq (\mathbb{C}^\times)^n$ be an algebraic variety, I the ideal of Laurent polynomials which vanish on W. Then*

$$\operatorname{Trop}(W) = \bigcap_{f \in I} \operatorname{Trop}(V(f))$$

where $V(f)$ denotes the hypersurface cut out in $(\mathbb{C}^\times)^n$ by f.

Theorem 3.3.15 is obviously very similar to the corresponding result for amoebas (Theorem 3.3.2). The comparison between the two allows us to make precise the idea of tropical varieties being “nonstandard amoebas”, in the sense that they can be thought of as amoebas where the base of the logarithm is infinite.

First, we recall what we mean by *Hausdorff metric.*

Definition 3.3.16. Let $A, B \subseteq \mathbb{R}^n$ be closed sets, and let d denote the standard Euclidean metric on $\mathbb{R}^n$. The *Hausdorff distance* between A and B is defined as

$$d_{\mathrm{Haus}}(A, B) := \max\left\{\sup_{a\in A} d(a, B), \sup_{b\in B} d(b, A)\right\}$$

where $d(x, Y)$ denotes the usual Euclidean distance between the point x and the set Y.

We will call *Hausdorff topology* the topology induced on the space of closed sets of $\mathbb{R}^n$ by this metric, and we will say that a set S is the *Hausdorff limit* of a sequence $\{S_j\}_{j\in\omega}$ if the sequence converges to S in the Hausdorff topology. It is clear by the definition that d_{Haus} is a metric, not a pseudometric, and therefore the topology is T_2: Hausdorff limits are unique.

For $t \in \mathbb{R}$, denote by $\mathrm{Log}_t : (\mathbb{C}^\times)^n \to \mathbb{R}^n$ the map

$$(w_1, \ldots, w_n) \mapsto (\log_t |w_1|, \ldots, \log_t |w_n|)$$

and for an algebraic variety W denote by $\mathcal{A}^t_W$ the image of W under Log_t (so that the usual map Log coincides with Log_e and the usual amoeba $\mathcal{A}_W$ with $\mathcal{A}^e_W$).

The limit of the amoebas in base t of a variety, for t going to infinity, is the tropicalization.

Theorem 3.3.17. *For $t \to \infty$, the sets $\mathcal{A}^t_W$ converge to* $\mathrm{Trop}(W)$ *in the Hausdorff metric.*

Proof. An easy proof for hypersurfaces is given in [Mik04b, Corollar 6.4]. For varieties of arbitrary codimension it is a harder problem, see [Jon16, Theorem A]. □

We finish this section by reviewing a few more results in tropical geometry which describe structural properties of tropical varieties.

Definition 3.3.18. A polyhedral complex is *pure* if all its facets (i.e. all the polyhedra which are not faces of larger polyhedra in the complex) have the same dimension.

Thus for pure complexes it makes sense to talk about the *dimension of the complex*, meaning the dimension of the facets. Part of the Structure Theorem asserts that the tropicalization of an algebraic variety of (complex) dimension d is a pure polyhedral complex of (real) dimension d.

Theorem 3.3.19 (Part of the Structure Theorem, [MS15, Theorem 3.3.6])**.** *Let* $W \subseteq (\mathbb{C}^\times)^n$ *be an algebraic variety of dimension* d*. Then* $\mathrm{Trop}(W)$ *is a pure polyhedral complex of dimension* d*.*

Definition 3.3.20. Let Σ be a polyhedral complex, τ a polyhedron in Σ. The *star* $\mathrm{star}_\tau(\Sigma)$ is the polyhedral complex formed by the polyhedra of the form

$$\sigma' := \{\lambda(x-y) \mid \lambda \geq 0, x \in \tau, y \in \sigma\}$$

for each polyhedron $\sigma \in \Sigma$ that τ is a face of (including $\tau = \mathrm{face}_0(\tau)$).

Example 3.3.21. If Σ is the 1-skeleton of the polyhedral complex in Figure 3.4, then the star of each half-line is the line that contains it.

We recall the result which states that the star of a cell in $\mathrm{Trop}(W)$ is the tropicalization of an initial variety of W and review some of its consequences.

Lemma 3.3.22 ([MS15, Lemma 3.3.7])**.** *Let* Σ *be the polyhedral fan supported on* $\mathrm{Trop}(W)$*. Suppose* $\tau \in \Sigma$ *is a face, and* $w \in \mathrm{relint}(\tau)$*. Then* $\mathrm{star}_\tau(\Sigma) = \mathrm{Trop}(\mathrm{in}_w(W))$*.*

As the star depends on the face, and not on the point, this implies that the initial variety $\mathrm{in}_w(W)$ is constant as w varies in the relative interior of a face. Therefore, we can give the following definition.

Definition 3.3.23. Let Σ be the polyhedral fan supported on $\mathrm{Trop}(W)$, $\tau \in \Sigma$. We denote by W_τ the initial variety $\mathrm{in}_w(W)$ for every $w \in \mathrm{relint}(\tau)$.

We should remark that initial varieties of irreducible varieties are not necessarily irreducible.

Note that if τ is a facet of Σ then by definition the star of τ in Σ is a linear space (the linear span of τ) and therefore the tropicalization of the initial variety W_τ is a subspace of $\mathbb{R}^n$. This implies that W_τ is a finite union of cosets of an algebraic subgroup of $(\mathbb{C}^\times)^n$.

Example 3.3.24. Thinking back about Example 3.3.11, we see how in that case the four non-monomial initial forms of the polynomial correspond to the four polyhedra in the 1-skeleton of the fan (three half-lines and the origin).

Finally, we consider the following proposition, according to which the initial variety of $W \subseteq (\mathbb{C}^\times)^n$ with respect to the face τ is invariant under multiplication by elements in the image under exp of the complex space generated by τ.

Proposition 3.3.25. *Let $W \subseteq (\mathbb{C}^\times)^n$ be an algebraic variety, $\tau \in \operatorname{Trop}(W)$ a face of its tropicalization, $\tau_\mathbb{C}$ the complex subspace of $\mathbb{C}^n$ generated by τ (seen as a subset of $\mathbb{R}^n \subseteq \mathbb{C}^n$).*

Then W_τ is invariant under translation by elements of $\exp(\tau_\mathbb{C})$.

Proof. This is a consequence of [MS15, Corollary 3.2.13], which introduces a notion of tropicalization for monomial maps and shows that monomial maps commute with tropicalizations. Since quotienting by an algebraic subgroup of $(\mathbb{C}^\times)^n$ can be seen as a monomial map, this result implies that the dimension of $W_\tau/\exp(\tau_\mathbb{C})$ is equal to the dimension of W_τ, which is therefore invariant under translation by elements of $\exp(\tau_\mathbb{C})$. $\square$

3.3.4 Stable Intersections

Finally, we introduce *stable intersections*, intersections between polyhedral complexes which are preserved under small perturbations. Our approach is slightly different from the one in [MS15, Section 3.6], as they also care about the *multiplicities* of the polyhedra in a complex, which we have not defined and will not need.

Hence, our definition of stable intersection will be as follows.

Definition 3.3.26. Let Σ_1, Σ_2 be polyhedral complexes in $\mathbb{R}^n$. The *stable intersection* of Σ_1 and Σ_2, denoted $\Sigma_1 \cap_{st} \Sigma_2$, is the polyhedral complex consisting of polyhedra of the form $\sigma_1 \cap \sigma_2$, where $\sigma_i \in \Sigma_i$ for $i = 1, 2$ and $\dim(\sigma_1 + \sigma_2) = n$.

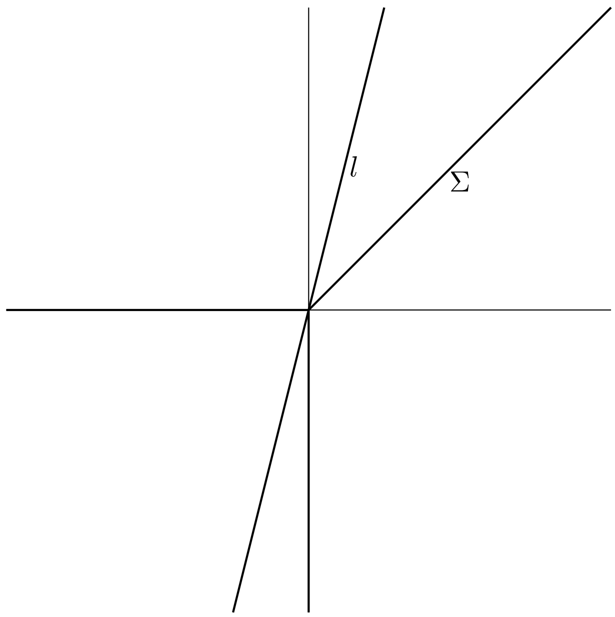

Figure 3.5: The stable intersection of the usual polyhedral complex Σ with the line l is $\{0\}$: each facet intersects the line transversely. The stable intersection of Σ with, say, the x-axis is still $\{0\}$: although the x-axis intersects the horizontal facet, the Minkowski sum of the facet and the axis does not have dimension 2, and so the intersection does not count towards the stable intersection.

In "classical" tropical geometry, the main interest in stable intersections comes from this fact, which ties the combinatorial definition to its geometric meaning.

Fact 3.3.27 ([MS15, Theorem 3.6.1])**.** *Let* $W_1, W_2 \subseteq (\mathbb{C}^\times)^n$ *be algebraic varieties. Then* $\mathrm{Trop}(W_1) \cap_{st} \mathrm{Trop}(W_2) \neq \varnothing$ *if and only if the set*

$$\{w \in (\mathbb{C}^\times)^n \mid w \cdot W_1 \cap W_2 \neq \varnothing\}$$

is Zariski-open dense in $(\mathbb{C}^\times)^n$*.*

We will see how to obtain a version of this fact for intersections of the form $\exp(L) \cap W$ with $L \leq \mathbb{C}^n$, rather than $W_1 \cap W_2$.

To do so, we are going to need to connect the notion of stable intersection to the mixed volume of a collection of polytopes. Thus, we conclude this section by stating a theorem which goes in that direction.

Theorem 3.3.28 ([MS15, Theorem 4.6.9])**.** *Suppose* $P_1, \dots, P_r$ *are polytopes in* $\mathbb{R}^n$ *with integer vertices and* $\Sigma_1, \dots, \Sigma_r$ *are the* $(n-1)$*-skeletons of their normal fans. Fix* $w \in \mathbb{R}^n$*, and let* $Q_i := \mathrm{face}_w(P_i)$*.*

Then $w \in \Sigma_1 \cap_{st} \dots \cap_{st} \Sigma_r$ *if and only if for every* $J \subseteq \{1, \dots, r\}$ *we have* $\dim\left(\sum_{j \in J} Q_j\right) \geq |J|$*, if and only if the* r*-dimensional mixed volume of the faces* $\mathrm{MV}(Q_1, \dots, Q_r)$ *is non-zero.*

3.4 Exponential Sums Equations

In this section we introduce the framework of exponential sums equations. This is a natural way to interpret the question of exponential-algebraic closedness for varieties of the form $L \times W$, and allows us to talk about Newton polytopes and related objects.

Let

$$(\mathbb{C}^n)^\vee := \{\varphi : \mathbb{C}^n \to \mathbb{C} \mid \varphi \text{ is linear}\}$$

denote the usual dual space of the complex vector space $\mathbb{C}^n$. Note that we can easily talk about convexity in this space: given two functions φ_1, φ_2, the segment between φ_1 and φ_2 is the set $\{(1-t)\varphi_1 + t\varphi_2 \mid t \in [0,1]\}$, and a set is convex if it contains the segment between any two of its elements. As usual, the *convex hull* of a set is the smallest convex set which contains it.

Definition 3.4.1. An *exponential sum* is a function $f : \mathbb{C}^n \to \mathbb{C}$ of the form

$$z \mapsto \sum_{\varphi \in S} c_\varphi \exp(\varphi(z))$$

where $S \subseteq (\mathbb{C}^n)^\vee$ is a finite set, and $c_\varphi \in \mathbb{C}$ for each φ.

The *Newton polytope* of the exponential sum f is the convex hull of S in $(\mathbb{C}^n)^\vee$.

Of course systems of exponential sums equations can take very different forms, but we are only interested in the ones we can attach to varieties of the form $L \times W$.

A variety $W \subseteq (\mathbb{C}^\times)^n$ is defined by a system of Laurent polynomial equations, so equations of the form

$$\sum_{j \in S} c_j w^j = 0$$

where $S \subseteq \mathbb{Z}^n$ is a finite subset. It is clear that any Laurent polynomial can be seen as an exponential sum of the form

$$\sum_{j \in S} c_j \exp(j \cdot z) = 0$$

where $j \cdot z = j_1 z_1 + \cdots + j_n z_n$ denote the usual scalar product. The system of exponential sums obtained from the Laurent polynomials which define W clearly defines the complex analytic subset $\log W$ of $\mathbb{C}^n$.

As for the linear space $L \leq \mathbb{C}^n$, this is defined by linear equations such as $\lambda_1 z_1 + \cdots + \lambda_n z_n = 0$. Clearly, the function $\varphi_\lambda : (z_1, \ldots, z_n) \mapsto \lambda_1 z_1 + \cdots + \lambda_n z_n$ is an element of $(\mathbb{C}^n)^\vee$, and therefore, L is the unique irreducible component containing 0 of the complex analytic set defined by the exponential sums

$$\exp(\varphi_\lambda(z)) - 1 = 0.$$

This larger set is a countable union of translates of L.

Definition 3.4.2. Let $L \leq \mathbb{C}^n$ be a linear space defined by equations $\varphi_\lambda(z) = 0$, $W \subseteq (\mathbb{C}^\times)^n$ an algebraic variety defined by equations $\sum_{j \in S} c_j w^j$.

The *system of exponential sums attached to* $L \times W$ is the system defined by the corresponding equations of the form $\exp(\varphi_\lambda(z)) = 1$ and $\sum_{j \in S} c_j \exp(j \cdot z) = 0$.

It is useful to associate a system of exponential sums to the variety $L \times W$

because it allows us to give a characterization of rotundity of the pair in terms of the Newton polytope of the system; this is similar to [Zil02, Lemma 3] (with its converse) but we state it in a slightly different way.

Definition 3.4.3. Let V be a vector space of dimension n, and let $A_1, \ldots, A_n$ be finite subsets of V. We say that $A_1, \ldots, A_n$ satisfy the *Rado property* if there is a basis $\{v_1, \ldots, v_n\}$ of V such that $v_j \in A_j$ for each j.

This property is named after the classical result known as *Rado's Theorem on Independent Transversals*:

Theorem 3.4.4 ([Rad42, Theorem 1])**.** *Let $A_1, \ldots A_n$ be finite subsets of a vector space V of dimension n. Then the span of $\bigcup_{j \in J} A_j$ has dimension at least $|J|$ for each subset $J \subseteq \{1, \ldots, n\}$ if and only if $A_1, \ldots, A_n$ satisfy the Rado property.*

For a subspace $L \leq \mathbb{C}^n$, we denote as usual by $L^\perp$ the subspace of the dual space defined as

$$L^\perp := \{\varphi \in (\mathbb{C}^n)^\vee \mid \varphi(z) = 0 \, \forall z \in L\}.$$

For a polytope $P \subseteq (\mathbb{C}^n)^\vee$, we denote by $v(P)$ the finite set of its vertices (its 0-dimensional faces).

Lemma 3.4.5. *Let $L \leq \mathbb{C}^n$ be a linear space of dimension d, and $W \subseteq (\mathbb{C}^\times)^n$ a variety of codimension d defined by polynomials $f_1, \ldots, f_d$ with Newton polytopes $P_1, \ldots, P_d$. Let $\pi_{L^\perp} : (\mathbb{C}^n)^\vee \twoheadrightarrow (\mathbb{C}^n)^\vee / L^\perp$ denote the projection.*

Then the variety $L \times W$ is rotund if and only if $\pi_{L^\perp}(v(P_1)), \ldots \pi_{L^\perp}(v(P_d))$ satisfy the Rado property.

Proof. Suppose the projections of the vertex sets do not satisfy the Rado property. Then by Theorem 3.4.4, after renumbering the polytopes there is $k \leq d$ such that $\pi_{L^\perp}(P_1), \ldots, \pi_{L^\perp}(P_k)$ are contained in a subspace $T \leq (\mathbb{C}^n)^\vee / L^\perp$ of dimension $\dim T < k$. We may take T to be $\pi_{L^\perp}(S)$, with S the span of $P_1, \ldots, P_k$ in $(\mathbb{C}^n)^\vee$. Since all the sets $v(P_j)$ are sets of integer vectors, S is defined over $\mathbb{Q}$.

Therefore, as $\dim(\pi_{L^\perp}(S)) < k$, we must have $\dim S - \dim(L^\perp \cap S) < k$. As is well-known, taking the annihilator of subspaces in $(\mathbb{C}^n)^\vee$ yields spaces which are isomorphic to spaces in $\mathbb{C}^n$ - namely, we canonically have that $(L^\perp)^\perp \cong L$

and $S^\perp$ is isomorphic to (and will then be identified with) a rational subspace of $\mathbb{C}^n$. Under these identifications we have $(L^\perp \cap S)^\perp = L + S^\perp$. Then we have

$$\dim(\pi_{S^\perp}(L)) = \dim L - \dim(L \cap S^\perp) =$$
$$= n - \dim L^\perp - (n - \dim(L^\perp + S)) =$$
$$= \dim L^\perp + \dim S - \dim(L^\perp \cap S) - \dim L^\perp =$$
$$= \dim S - \dim(L^\perp \cap S) = \dim(\pi_{L^\perp}(S)).$$

Hence we have $\dim(\pi_{S^\perp}(L)) = \dim(\pi_{L^\perp}(S)) < k$.

Consider now the algebraic subgroup $\exp(S^\perp)$ of $(\mathbb{C}^\times)^n$. Since S is the span of polytopes $P_1, \ldots, P_k$, the quotient of the variety W under $\exp(S^\perp)$ must have codimension at least k, as the quotient of each of the hypersurfaces cut out by $f_1, \ldots, f_k$ is still a hypersurface (cut out by a polynomial with Newton polytope $\pi_{S^\perp}(P_j)$). Therefore,

$$\dim(\pi_{\exp(S^\perp)}(W)) \leq \dim S - k.$$

Hence, we find that there is a rational subspace $S^\perp \leq \mathbb{C}^n$ such that

$$\dim(\pi_{S^\perp}(L)) + \dim(\pi_{\exp(S^\perp)}(W)) < k + \dim S - k = n - \dim S^\perp$$

which contradicts the definition of rotundity. Hence if the polytope projections do not satisfy the Rado property then $L \times W$ is not rotund, establishing one direction of the lemma.

Now assume the projections of the vertex sets of the polytopes satisfy the Rado property, and let $S^\perp \leq \mathbb{C}^n$ be the annihilator of some rational subspace S of $(\mathbb{C}^n)^\vee$ (we introduce $S^\perp$ as an annihilator rather than by itself to maintain consistency in the proof - this way, in both directions $S^\perp$ is a subspace of $\mathbb{C}^n$ and S of $(\mathbb{C}^n)^\vee$). Consider the variety $\pi_{\exp(S^\perp)}(W)$: this has dimension $\dim S - k$ for some $k \geq 0$.

After renumbering the polytopes we may assume that $\pi_{S^\perp}(P_1), \ldots, \pi_{S^\perp}(P_k)$ are Newton polytopes for $\pi_{\exp(S^\perp)}(W)$. Replacing, if necessary, the polynomials $f_1, \ldots, f_k$ by other elements of the ideal of polynomials which vanish on W we may assume without loss of generality that $P_1, \ldots, P_k \subseteq S$: hence, by the Rado property of the sets $\pi_{L^\perp}(v(P_1)), \ldots, \pi_{L^\perp}(v(P_k))$, we have $\dim(\pi_{L^\perp}(S)) \geq k$.

We have proved above that $\dim(\pi_{L^\perp}(S)) = \dim(\pi_{S^\perp}(L))$, and as a consequence $\dim(\pi_{S^\perp}(L)) \geq k$. Thus,

$$\dim \pi_{S^\perp}(L) + \dim \pi_{\exp(S^\perp)}(W) \geq k + \dim S - k = \dim S = n - \dim S^\perp.$$

As $S^\perp$ was arbitrary, this establishes rotundity of the variety. $\square$

Corollary 3.4.6. *Let $L \times W \subseteq \mathbb{C}^n \times (\mathbb{C}^\times)^n$ be a variety with L linear and $\dim L + \dim W = n$.*

Then $L \times W$ is rotund if and only if the Newton polytopes of the associated system of exponential sums satisfy the Rado property. If L is defined over the reals, this is equivalent to the polytope having non-zero mixed volume.

Proof. The first assertion is an immediate consequence of Lemma 3.4.5. For the second one apply [MS15, Lemma 4.6.6], which ties mixed volume to the Rado property. $\square$

3.5 Raising to Real Powers

In this section we establish the main result of the chapter for varieties of the form $L \times W$, where L is defined over the real numbers; we are going to do so using a density property of sets of the form $\exp(L)$. Throughout this section, even when not explicitly stated, we assume that the space L is defined over $\mathbb{R}$ (we say it is $\mathbb{R}$-linear, and more generally given a field $K \subseteq \mathbb{C}$, by a K-linear subspace of $\mathbb{C}^n$ we mean a subspace of $\mathbb{C}^n$ defined over K).

Recall that $\mathbb{S}_1$ denotes the unit circle $\{z \in \mathbb{C} \mid |z| = 1\}$.

Proposition 3.5.1. *Let $L \leq \mathbb{C}^n$ be an $\mathbb{R}$-linear space that is not contained in any $\mathbb{Q}$-linear space. Then $\exp(L)$ is dense in $\exp(L) \cdot \mathbb{S}_1^n$.*

Proof. Let $\mathrm{Re}(L)$ denote the set $\{\mathrm{Re}(l) \in \mathbb{R}^n \mid l \in L\}$. Since L is $\mathbb{R}$-linear, we have $L = \mathrm{Re}(L) + i\mathrm{Re}(L)$.

Since L is not contained in any $\mathbb{Q}$-linear subspace of $\mathbb{C}^n$, $\mathrm{Re}(L)$ is not contained in any $\mathbb{Q}$-linear subspace of $\mathbb{R}^n$. Therefore, $\exp(i\mathrm{Re}(L))$ is dense in $\mathbb{S}_1^n$: to see this, consider $\mathbb{S}_1^n$ as $\mathbb{R}^n/2\pi i\mathbb{Z}^n$, and $\exp(i\mathrm{Re}(L))$ as $\mathrm{Re}(L) + 2\pi i\mathbb{Z}^n$: as $\mathrm{Re}(L)$

is not contained in any rational subspace, this is not contained in any proper closed subgroup, and is therefore dense. Thus the lemma holds:

$$\exp(L) = \exp(\mathrm{Re}(L)) \cdot \exp(i(\mathrm{Re}(L)))$$

which is dense in $\exp(L) \cdot \mathbb{S}_1^n$. □

The set $\exp(L) \cdot \mathbb{S}_1^n$ can easily be related to the material discussed in the previous section.

Proposition 3.5.2. *Let L be an $\mathbb{R}$-linear subspace of $\mathbb{C}^n$. Then:*

1. $\exp(L) \cdot \mathbb{S}_1^n = \mathrm{Log}^{-1}(\mathrm{Re}(L))$.
2. *If W is an algebraic subvariety of $(\mathbb{C}^\times)^n$, then $\mathcal{A}_W \cap \mathrm{Re}(L) \neq \varnothing$ if and only if $W \cap \exp(L) \cdot \mathbb{S}_1^n \neq \varnothing$.*

Proof. Part 1 is straightforward: $w \in \exp(L) \cdot \mathbb{S}_1^n$ if and only if there is $l \in L$ such that

$$(|w_1|, \ldots, |w_n|) = (|\exp(l_1)|, \ldots, |\exp(l_n)|) = \left(e^{\mathrm{Re}(l_1)}, \ldots, e^{\mathrm{Re}(l_n)}\right)$$

if and only if $\mathrm{Log}(w) \in \mathrm{Re}(L)$.

Part 2 follows by a similar argument: both statements are true if and only if there is $(l, w) \in L \times W$ such that $\frac{w}{\exp(l)} \in \mathbb{S}_1^n$. □

We will now see that thanks to Lemma 3.2.6 it is sufficient to intersect $\mathcal{A}_W$ and $\mathrm{Re}(L)$ to find an intersection between $\exp(L)$ and W.

Let $(*)$ denote the following assumption: for every free rotund variety of the form $L \times W$, $\mathrm{Re}(L) \cap \mathcal{A}_W \neq \varnothing$.

Lemma 3.5.3. *Suppose assumption $(*)$ holds.*

Then for every free rotund variety of the form $L \times W$, $\exp(L) \cap W \neq \varnothing$.

Proof. By Lemma 3.2.6, we can assume without loss of generality that the δ-map of $L \times W$ is open. By Proposition 3.5.2 and the assumption $(*)$, we know that $\exp(L) \cdot \mathbb{S}_1^n \cap W \neq \varnothing$. Clearly, this implies that the image $\mathrm{im}(\delta)$ of the δ-map intersects $\mathbb{S}_1^n$. As $\mathrm{im}(\delta)$ is open, this means that actually there is an open subset of $\mathbb{S}_1^n$ contained in $\mathrm{im}(\delta)$: thus, as $\exp(i\mathrm{Re}(L))$ is dense in $\mathbb{S}_1^n$, it must be the case that $\mathrm{im}(\delta) \cap \exp(i\mathrm{Re}(L)) \neq \varnothing$.

Hence there is a point $(l, w) \in L \times W$ such that $\frac{w}{\exp(l)} \in \exp(i\mathrm{Re}(L)) \leq \exp(L)$: this implies that $w \in \exp(L)$. $\square$

Hence, Lemma 3.5.3 means that we only need to prove that the assumption $(*)$ holds to establish the result.

Remark 3.5.4. The reader who is familiar with [Kir19b] will notice that assumption $(*)$ says, in the language of that paper, that all free rotund varieties $L \times W$ intersect the *blurring* of $\Gamma_{\exp}$ by $\mathbb{S}_1^n$, and Lemma 3.5.3 says that intersecting these varieties with the blurred graph is equivalent to intersecting them with the actual graph.

To intersect $\mathrm{Re}(L)$ and $\mathcal{A}_W$ we will use a classical result of Khovanskii to show that given any free rotund pair $L \times W$ we can find W' such that $\exp(L) \cap W' \neq \varnothing$ and $\mathcal{A}_{W'} = \mathcal{A}_W$.

Recall from Definition 3.4.2 that if $L \leq \mathbb{C}^n$ is a linear space and $W \subseteq (\mathbb{C}^\times)^n$ an algebraic variety, the system of exponential sums attached to $L \times W$ is the system of equations:

$$\begin{cases} \exp(\lambda_{1,1} z_1) \cdots \exp(\lambda_{1,n} z_n) = 1 \\ \vdots \\ \exp(\lambda_{d,1} z_1) \cdots \exp(\lambda_{d,n} z_n) = 1 \\ \sum_{j \in S_1} c_j \exp(j \cdot z) = 0 \\ \vdots \\ \sum_{j \in S_{n-d}} \exp(j \cdot z) = 0 \end{cases}$$

We recall the definition of a *coherent set of faces* in a collection of polyhedra, and of a *shortening* of a system. These can be found in [Kho91, Section 3.13] (the set of faces is called *concordant* there) and in [Zil02, Section 6].

Definition 3.5.5. Let $P_1, \ldots, P_k \subseteq \mathbb{R}^n$ be convex polytopes. A *coherent set of faces* of $P_1, \ldots, P_k$ is a set of polytopes $Q_1, \ldots, Q_k$ for which there is $w \in \mathbb{R}^n$ such that

$$Q_j = \mathrm{face}_w(P_j)$$

for each j.

Definition 3.5.6. Let $f_1 = \cdots = f_k = 0$ be a system of exponential sums equations. Let P_j be the Newton polytope of f_j for each j, and suppose $Q_1, \ldots, Q_k$ is a coherent set of faces of $P_1, \ldots, P_k$.

The *shortening* of the system associated to $Q_1, \ldots, Q_k$ is the system $g_1 = \cdots = g_k = 0$, where

$$g_j = \sum_{\varphi \in S \cap Q_j} c_\varphi \exp(\varphi(z))$$

for each j.

Note that by definition the Newton polytope of the exponential sum f_j' is Q_j for each j.

Example 3.5.7. A shortening describes the behaviour of the system of exponential sums as some of the variables approach infinity. As an example, consider the exponential sums

$$f_1(z_1, z_2) = \exp(z_1) + \exp(z_2)$$

$$f_2(z_1, z_2) = \exp(z_1) + \exp(z_2) + 1.$$

Then $P_1 = \mathrm{conv}\{(1,0),(0,1)\}$ and $P_2 = \mathrm{conv}\{(1,0),(0,1),(0,0)\}$. P_1 then has three faces (the two points $(1,0)$ and $(0,1)$ and P_1 itself), which are obtained as $\mathrm{face}_w(P_1)$ for w equal to $(-1,0)$, $(0,-1)$ or $(1,1)$ respectively.

It is easy to see that each of these induces a face of P_2: $\mathrm{face}_{(-1,0)}(P_2) = \mathrm{conv}\{(0,0),(0,1)\}$, $\mathrm{face}_{(1,1)}(P_2) = P_1$, and $\mathrm{face}_{(0,-1)}(P_2) = \mathrm{conv}\{(0,0),(1,0)\}$. This means that there are three coherent sets of faces of P_1 and P_2, and thus the system has three shortenings:

$$\begin{cases} \exp(z_1) = 0 \\ \exp(z_1) + 1 = 0 \end{cases}$$

and

$$\begin{cases} \exp(z_2) = 0 \\ \exp(z_2) + 1 = 0 \end{cases}$$

which are obviously inconsistent as $0 \notin \mathrm{im}(\exp)$, and

$$\begin{cases} \exp(z_1) + \exp(z_2) = 0 \\ \exp(z_1) + \exp(z_2) = 0 \end{cases}$$

which defines a curve.

Definition 3.5.8. Let $G \subseteq \mathbb{R}^n$ be an open set. A system of exponential sums is *non-degenerate at infinity* in the domain $\mathbb{R}^n + iG$ if:

1. All solutions of the system in the domain are isolated;
2. All shortenings of the system do not have solutions in $\mathbb{R}^n + iG$.

Solvability of non-degenerate at infinity systems of exponential sums has been established long ago. In particular, we have the following result of Zilber.

Theorem 3.5.9 ([Zil02, Theorem 4] and subsequent discussion; see also [Kho91, Theorem 3.13.1])**.** *Let $f_1 = \cdots = f_n = 0$ be a system of exponential sums. If there are arbitrarily large balls $G \subseteq \mathbb{R}^n$ such that the system is non-degenerate at infinity in $\mathbb{R}^n + iG$, then the system has a solution.*

We are going to show that given a variety $L \times W$ whose δ-map is open, there is $s \in \mathbb{S}_1^n$ such that:

1. $\mathcal{A}_{s \cdot W} = \mathcal{A}_W$;
2. The system of exponential sums associated to $L \times s \cdot W$ is non-degenerate at infinity in $\mathbb{C}^n$.

This will allow us to use Theorem 3.5.9 to find the intersection between $\mathrm{Re}(L)$ and $\mathcal{A}_W$ that we are looking for.

First of all, for a change, we notice a feature of varieties that are *not* rotund. For this, we are going to use a simple property of holomorphic functions in several variables.

Proposition 3.5.10. *Let $U \subseteq \mathbb{C}^n$ be an open set such that $U \cap \mathbb{R}^n \neq \varnothing$, and assume $f : U \to \mathbb{C}$ is a non-zero holomorphic function. Then f does not vanish on $U \cap \mathbb{R}^n$.*

Proof. By induction on n. If $n = 1$, then it follows from the fact that zeros of holomorphic functions are isolated.

If $n > 1$, assume f vanishes on every point of $U \cap \mathbb{R}^n$. Let $\pi : U \to \mathbb{C}$ be the projection on the last coordinate, and $\pi' : U :\to \mathbb{C}^{n-1}$ the projection on the first $n-1$ coordinates. For $r \in \pi(U \cap \mathbb{R}^n)$ consider the function $f_r : \pi'(\pi^{-1}(r)) \to \mathbb{C}$ defined by $(z_1, \ldots, z_{n-1}) \mapsto f(z_1, \ldots, z_n, r)$. Since f vanishes on $U \cap \mathbb{R}^n$, f_r vanishes on $\pi'(\pi^{-1}(r)) \cap \mathbb{R}^{n-1}$ for all r, and therefore by the inductive hypothesis all functions f_r are identically zero.

Therefore f vanishes on the set $\bigcup_{r \in \pi(U \cap \mathbb{R}^n)} \pi^{-1}(r)$. As this has real codimension 1 in U, and the zero-locus of f must be a complex analytic set, f is identically zero. $\square$

Corollary 3.5.11. *Let $f : (\mathbb{C}^\times)^n \to \mathbb{C}$ be a non-zero holomorphic function. Then f does not vanish on any open subset of $\mathbb{S}_1^n$.*

Proof. Apply the previous proposition to the function $f(\exp(iz)) : U \to \mathbb{C}$ for every $U \subseteq \mathbb{C}^n$ which intersects $\mathbb{R}^n$. $\square$

Lemma 3.5.12. *Let $\{L_j \times W_j \mid j \in \omega\}$ be a countable set of non-rotund varieties, and let δ_j be the δ-map of $L_j \times W_j$ for each j. Then $\mathbb{S}_1^n \nsubseteq \bigcup_{j \in \omega} \operatorname{im}(\delta_j)$.*

Proof. By Proposition 2.5.11, the image of a complex analytic function $f : A \to B$ is contained in a countable union of complex analytic subsets of B, of dimension at most $\dim A - \min\{\dim f^{-1}(b) \mid b \in B\}$.

In this case then consider $\delta_j : L_j \times W_j \to (\mathbb{C}^\times)^n$. Since $L_j \times W_j$ is not rotund, its image has empty interior, and the fibres of δ_j need to have positive dimension. Thus, the image of each δ_j must be contained in a countable union of analytic subsets of $(\mathbb{C}^\times)^n$ of positive codimension. By Corollary 3.5.11, all of these function do not vanish on open subsets of $\mathbb{S}_1^n$: therefore the union of their images cannot cover the whole set. This stays true if we consider countably many δ's at once. $\square$

We recall that for this kind of variety, $w \notin \operatorname{im}(\delta)$ is equivalent to $\exp(L) \cap w^{-1} \cdot W = \varnothing$.

We now examine shortenings of systems associated to varieties of the form $L \times W$.

Lemma 3.5.13. *Let $f_1 = \cdots = f_n = 0$ be a system of exponential sums equations associated to a variety of the form $L \times W$, so for all j we have that*

f_j can be rewritten as a Laurent polynomial (all exponents are integers) or it has only two terms.

Then for any shortening of the system, one of the following holds:

1. *There is j for which the shortened equation f'_j has the form $\exp(\varphi(z)) = 0$, and is thus inconsistent;*
2. *The shortened system is associated to $L \times W_\tau$ for an initial variety W_τ of W such that $L \times W_\tau$ is not rotund.*

Proof. Assume the first condition does not hold; then all faces in the coherent set $Q_1, \ldots, Q_n$ which defines the shortening are positive dimensional. In particular, all polytopes of the equations defining L appear among the Q_j's, and so the system defines $L \times W'$ for some initial variety W' of W.

However, all the faces are contained in translates of the same hyperplane, and therefore the mixed volume $\mathrm{MV}(Q_1, \ldots, Q_n)$ has to be zero by [MS15, Lemma 4.6.6]. So the variety is not rotund by Corollary 3.4.6. ☐

Thus we may prove the following result.

Lemma 3.5.14. *Let $L \times W$ be a free rotund variety whose δ-map is open. Then there is $s \in \mathbb{S}_1^n$ such that the system defining $L \times s \cdot W$ is non-degenerate at infinity in $\mathbb{C}^n$.*

Proof. Since the δ-map is open, all of its fibres are discrete: therefore all intersections $L \cap z + \log W$ are isolated, and the first condition in the definition of non-degeneracy at infinity is satisfied for all systems associated to varieties of the form $L \times w \cdot W$ for $w \in (\mathbb{C}^\times)^n$.

Consider now all shortenings of the system corresponding to $L \times W$. We saw in Lemma 3.5.13 that there are two kinds; let us focus on the second kind, namely the one associated to $L \times W_\tau$ for some initial variety W_τ of W.

This system defines

$$\left(\bigcup_{t \in T} t + L\right) \cap \log W_\tau$$

for a countable set T. Each translate $t + L$ intersects $\log W_\tau$ if and only if $\exp(t) \in \mathrm{im}(\delta_\tau)$ where δ_τ is the δ-map of $L \times W_\tau$.

By Lemma 3.5.12, there is $s \in \mathbb{S}_1^n$ such that for each $t \in T$, $s \cdot \exp(t)$ does not lie in the image of δ_τ for each shortening of this kind: therefore, $\exp(t)$ does not lie in the image of the δ-map of $L \times s^{-1} \cdot W_\tau$ for every $t \in T$ and for every shortening.

Since $s^{-1} \cdot W_\tau = (s^{-1} \cdot W)_\tau$, this means that all shortenings of the second kind of the system are inconsistent. The shortenings of the first kind are always inconsistent, as they equate the exponential of something to 0, and therefore the system associated to $L \times s^{-1} \cdot W$ is non-degenerate at infinity in $\mathbb{C}^n$. □

It is immediate that if $s \in \mathbb{S}_1^n$ then $\mathcal{A}_{s \cdot W} = \mathcal{A}_W$. Therefore, we obtain the desired point in $\mathrm{Re}(L) \cap \mathcal{A}_W$.

Lemma 3.5.15. *Let $L \times W$ be a variety whose δ-map is open. Then* $\mathrm{Re}(L) \cap \mathcal{A}_W \neq \varnothing$*.*

Proof. By Lemma 3.5.14, there is $s \in \mathbb{S}_1^n$ such that the system associated to $L \times s \cdot W$ is non-degenerate at infinity in $\mathbb{C}^n$; therefore by Theorem 3.5.9 it has a solution.

If a point z solves this system, then $\mathrm{Re}(z) \in \mathrm{Re}(L)$: the exponential sums associated to L take the form $\exp(\varphi_\lambda(z)) = 1$, and if φ_λ is a real function then it needs to be the case that $\lambda \cdot \mathrm{Re}(z) = 0$. We have already noticed that $\mathrm{Re}(z) \in \mathcal{A}_{s \cdot W} = \mathcal{A}_W$, and therefore $\mathrm{Re}(z) \in \mathrm{Re}(L) \cap \mathcal{A}_W$. □

Theorem 3.5.16. *Let $L \times W$ be a free rotund variety, $L \leq \mathbb{C}^n$ linear defined over the reals. Then* $\exp(L) \cap W \neq \varnothing$*.*

Proof. It follows from the combination of Lemmas 3.5.3 and 3.5.15. □

The Rabinovich trick (which we have already used to show that the δ-map can be assumed to be open) actually implies that the intersection between $\exp(L)$ and W is Zariski-dense in W.

Corollary 3.5.17. *Let $L \times W$ be a free rotund variety, $L \leq \mathbb{C}^n$ defined over the reals. Then* $\exp(L) \cap W$ *is Zariski-dense in W.*

Proof. Let $F : W \to \mathbb{C}$ be an algebraic function: then $\{w \in W \mid F(w) \neq 0\}$ is a Zariski-open subset of W. Define

$$W' := \{(w_1, \dots, w_{n+1}) \in \mathbb{C}^n \mid (w_1, \dots, w_n) \in \mathbb{C}^n \wedge w_{n+1} = F(w_1, \dots, w_n)\}.$$

Then $(L \times \mathbb{C}) \times W'$ is a free rotund subvariety of $\mathbb{C}^{n+1} \times (\mathbb{C}^\times)^{n+1}$, and therefore there is a point in $\exp(L \times \mathbb{C}) \cap W'$, i.e. a point (w, w_{n+1}) with $w \in \exp(L) \cap W$ and $0 \neq w_{n+1} = F(w)$. Hence, $w \in \exp(L) \cap \{w \in W \mid F(w) \neq 0\}$, as we wanted. □

3.6 Complex Tropical Geometry

If we try to extend our result to all varieties of the form $L \times W$ with L linear (so without any assumptions on the field of definition of L) we run into some obvious issues - the first of which is the fact that, for such an L, $\exp(L)$ is not dense in $\exp(L) \cdot \mathbb{S}_1^n$. The easiest example of this is the space $L = \{(z_1, z_2) \in \mathbb{C}^2 \mid z_2 = iz_1\}$: in this case $\exp(L)$ is a closed 1-dimensional analytic subgroup of $(\mathbb{C}^\times)^2$, while $\exp(L) \cdot \mathbb{S}_1^2 = (\mathbb{C}^\times)^2$. This obviously makes it harder to find good approximate solutions.

However, a second aspect to be taken into consideration is the fact that if L is not defined over $\mathbb{R}$ then $L \lneq \mathrm{Re}(L) + i\mathrm{Re}(L)$; and in particular, $\dim_{\mathbb{C}} L < \dim_{\mathbb{R}} \mathrm{Re}(L)$. Thus, if we are given a free rotund variety $L \times W$, with $\dim L = \mathrm{codim} W$, then we find that $\mathrm{Re}(L) \cap_{st} \mathrm{Trop}(W)$ must be a positive dimensional polyhedral complex. Vaguely, this can be interpreted to mean that “there are ways to make $\exp(L)$ and W approach infinity in the same direction”: in such a direction, W will resemble one of its initial varieties W_τ, and if $L \times W_\tau$ is itself rotund (and therefore satisfies the open mapping property) then we have good chances to use points in $\exp(L) \cap W_\tau$ as approximate solutions of our system.

To make this precise we need to generalize some notions from tropical geometry so that they also handle this kind of variety: this has been done by Kazarnovskii in [Kaz14b] and [Kaz14a]. In this section we review some of his notions, and use them to derive the following result which will be used later on.

Theorem 3.6.1. *Let $L \times W$ be a free rotund variety, with L linear not defined over $\mathbb{R}$. Then there is $\tau \in \mathrm{Trop}(W)$, $\dim \tau > 0$, such that:*

1. *$L \times W_\tau$ is rotund;*

2. $(\mathrm{Supp}(\mathrm{Re}(L) \cap_{st} \mathrm{Trop}(W)) \cap \mathrm{relint}\tau \neq \varnothing$.

This result will be given a geometric interpretation in the next section.

In what follows, let $\langle z, w \rangle$ denote the usual Hermitian product on $\mathbb{C}^n$, so that if $z = (z_1, \dots, z_n)$ and $w = (w_1, \dots, w_n)$ then $\langle z, w \rangle = z_1 \overline{w}_1 + \dots + z_n \overline{w}_n$.

Remark 3.6.2. Identify $z = (z_1, \dots, z_n) = (x_1 + iy_1, \dots, x_n + iy_n)$ with the real vector $(x_1, y_1, \dots, x_n, y_n) \in \mathbb{R}^{2n}$. Then we have that $\mathrm{Re}\,(\langle z, z' \rangle)$ coincides with the real scalar product in $\mathbb{R}^{2n}$, as

$$\mathrm{Re}(\langle (x_1 + iy_1, \dots, x_n + iy_n), (x'_1 + iy'_1, \dots, x'_n + iy'_n) \rangle) =$$

$$= x_1 x'_1 + y_1 y'_1 + \dots + x_n x'_n + y_n y'_n.$$

Definition 3.6.3. A *piecewise-linear function* on $\mathbb{C}^n$ is a function $h : \mathbb{C}^n \to \mathbb{R}$ for which there are polyhedra $P_1, \dots, P_k \subseteq \mathbb{C}^n \cong \mathbb{R}^{2n}$ and vectors $a_1, \dots, a_k \in \mathbb{C}^n$ such that $\bigcup_{j=1}^k P_j = \mathbb{C}^n$ and h coincides with the function $z \mapsto \mathrm{Re}(\langle z_j, a_j \rangle)$ on each polyhedron P_j .

Any $\mathbb{R}$-linear function $\mathbb{C}^n \to \mathbb{R}$ is clearly piecewise-linear.

Definition 3.6.4. Let A be a non-empty closed convex subset of $\mathbb{C}^n$. The *support function* of A is the function $h_A : \mathbb{C}^n \to \mathbb{R}$ defined by

$$h_A(z) = \sup\{\mathrm{Re}(\langle z, a \rangle) \mid a \in A\}.$$

The support function of a convex polyhedron is then clearly a piecewise-linear function, whose domains of linearity form a polyhedral complex which coincides with the normal fan to the polyhedron (and whose locus of non-differentiability coincides with the $(n-1)-$skeleton of such a fan).

In [Kaz14a] Kazarnovskii introduced a numerical invariant for collections of polytopes in $\mathbb{C}^n$, the *pseudo-mixed volume*, and used it to study tropical properties of systems of exponential sums with complex exponents, with a particular attention to the behaviour of stable intersections. He also proved that non-vanishing of the pseudo-mixed volume is equivalent to a certain occurrence of the Rado property - we will show in a bit how to tie this to rotundity using Lemma 3.4.5.

Given a Laurent polynomial $f \in \mathbb{C}[w_1^{\pm 1}, \dots, w_n^{\pm n}]$, consider its Newton polytope

P. We have seen how this is a subset of the dual space $(\mathbb{C}^n)^\vee$: we can therefore interpret the support function $h_P : (\mathbb{C}^n)^\vee \to \mathbb{R}$ as

$$h_P(z) = \{\max \operatorname{Re}(\varphi(z)) \mid \varphi \in P\}.$$

As the exponents of f are integer (and therefore real) numbers for any function in P we will have $\varphi(z + iv) = \varphi(z)$ for all $v \in \mathbb{R}^n$: therefore, the domains of linearity of h_P are subsets of the form $D + i\mathbb{R}^n$ for some polyhedra $D \subseteq \mathbb{R}^n$.

On the other hand, consider the function $z \mapsto l_1 z_1 + \cdots + l_n z_n$ on $\mathbb{C}^n$: this is a point in $(\mathbb{C}^n)^\vee$. Consider the convex hull of $\{0, (l_1, \ldots, l_n)\}$: this is a segment in $(\mathbb{C}^n)^\vee$, orthogonal to the real hyperplane

$$H := \{z \in \mathbb{C}^n \mid \operatorname{Re}(\langle z, \bar{l}\rangle) = 0\}.$$

If $(l_1, \ldots, l_n) \in \mathbb{R}^n$, then H itself is invariant under translation by elements in $i\mathbb{R}^n$.

Consider a linear space $L \leq \mathbb{C}^n$. Let $\operatorname{Re}(L)_\mathbb{C} := \operatorname{Re}(L) \otimes \mathbb{C}$ be the complexification of its real part (more generally, for every polyhedron $\tau \subseteq \mathbb{R}^n$ we will denote by $\tau_\mathbb{C}$ the complexification $\operatorname{aff}(\tau) \otimes \mathbb{C}$ of its span). Every space L may be written as $L = \operatorname{Re}(L)_\mathbb{C} \cap H$ for some $H \subseteq \mathbb{C}^n$ with $\operatorname{Re}(H) = \mathbb{R}^n$. Thus, the equations defining L may be assumed to take the form

$$\begin{cases} r_{1,1} z_1 + \cdots + r_{1,n} z_n = 0 \\ \vdots \\ r_{k,1} z_1 + \cdots + r_{k,n} z_n = 0 \\ \lambda_{1,1} z_1 + \cdots + \lambda_{1,n} z_n = 0 \\ \vdots \\ \lambda_{n-d-k,1} z_1 + \cdots + \lambda_{n-d-k,n} z_n = 0 \end{cases}$$

where the vectors $(r_{j,1}, \ldots, r_{j,n})$ are in $\mathbb{R}^n$ and the corresponding equations define $\operatorname{Re}(L)_\mathbb{C}$ and the vectors $(\lambda_{j,1}, \ldots, \lambda_{j,n})$ are in $\mathbb{C}^n \setminus \mathbb{R}^n$. We denote by h_{r_j} the function $z \mapsto \langle z, r_j \rangle$ and by h_{λ_j} the function $z \mapsto \langle z, \overline{\lambda_j} \rangle$.

Consider the system of exponential sums associated to the variety $L \times W$, in the sense of Definition 3.4.2. We know how to attach to this system a set of convex polytopes. For each of these polytopes, the locus of non-differentiability of the support function is a polyhedral complex. In particular, the corner locus of h_{P_j} will be $\Sigma_j + i\mathbb{R}^n$ where Σ_j denotes the tropicalization of the hypersurface defined by the polynomial f_j; the corner locus of h_{r_j} (resp. h_{λ_j}) will be the hyperplane defined by $\mathrm{Re}(\langle z, r_j\rangle) = 0$ (resp. $\mathrm{Re}(\langle z, \overline{\lambda}_j\rangle))$. We refer to these as the *collection of polyhedral complexes* associated to the variety $L \times W$.

Non-emptiness of the stable intersection of these complexes can be characterized using the *mixed Monge-Ampère operator*. This is an operator which associates to a tuple of k piecewise-linear functions on $\mathbb{C}^n$ a *current*, i.e. a linear functional on the space of smooth compactly supported differential forms. We do not get into the details, but rather refer the reader to [Kaz14a, Section 3]; the main idea, which will be used in the upcoming proof, is that non-vanishing of the mixed Monge-Ampère operator for the support functions of n convex polytopes in $\mathbb{C}^n$ is equivalent to non-emptiness of the stable intersections of the corresponding polyhedral complexes. This is essentially [Kaz14a, Theorem 3.1], which establishes an isomorphism of rings between a ring of currents and a ring of equivalence classes of polyhedral complexes, in which the product is given by stable intersections.

Lemma 3.6.5. *The stable intersection of the collection of polyhedral complexes associated to the variety $L \times W$ is non-empty if and only if the variety $L \times W$ is rotund.*

Proof. By Lemma 3.4.5, rotundity is equivalent to the Rado property for the collection of convex polytopes associated to $L \times W$. By [Kaz14a, Corollary 3.3], this collection has the Rado property if and only if the mixed pseudo-volume of the polytopes is non-zero; by Theorem 3.5 in the same paper this is equivalent to non-vanishing of the mixed Monge-Ampère operator on the collection of support functions of the polytopes, and finally by [Kaz14a, Proposition 3.1] this is equivalent to non-emptiness of the stable intersection. $\square$

As an aside, we note that going back to the case in which L is defined over the reals this says that $L \times W$ is rotund if and only if $\mathrm{Re}(L) \cap_{st} \mathrm{Trop}(W) \neq \varnothing$: therefore a version of Fact 3.3.27 holds for this kind of variety, establishing that the stable intersection between $\mathrm{Trop}(W)$ and $\mathrm{Re}(L)$ (which acts as a sort of "$\mathrm{Trop}(\exp(L))$") can be lifted to an actual intersection between W and $\exp(L)$.

This allows us to prove the main result of this section, Theorem 3.6.1.

Proof of Theorem 3.6.1. As $L \times W$ is rotund, Lemma 3.6.5 guarantees that the collection of polyhedral complexes associated to the variety has non-empty stable intersection. This stable intersection is obviously a subset of the stable intersection of any subset of the collection of polyhedral complexes: therefore, it needs to be contained in the stable intersection of those complexes which are loci of non-differentiability of $\mathbb{R}$-linear functions, that is the stable intersections of the tropical hypersurfaces $\Sigma_j + i\mathbb{R}^n$ and the hyperplanes defined by $\mathrm{Re}\langle r_j, z\rangle$ for $r_j \in \mathbb{R}^n$. If L were defined over the reals then these would be all the complexes, and the stable intersection would turn out to be exactly $i\mathbb{R}^n$.

Since L is not defined over the reals, there is at least one complex linear equation (and so at least one hyperplane not defined over $\mathbb{R}$ in the collection of polyhedral complexes associated to $L \times W$). This does not contain $i\mathbb{R}^n$, and therefore the stable intersection of the complexes cannot be of the form iT with T an n-dimensional polyhedral complex in $\mathbb{R}^n$. So let τ_0 be any maximal cell of the stable intersection: this contains, in its relative interior, a point of the form $x + iy \in \mathbb{C}^n$ with $x \neq 0$. Therefore x is in the support of $\mathrm{Re}(L) \cap_{st} \mathrm{Trop}(W)$; let $\tau \in \mathrm{Trop}(W)$ be the cell such that $x \in \mathrm{relint}(\tau)$.

Consider the variety $L \times W_\tau$. As W_τ is an initial variety, its tropicalization is equal to $\mathrm{star}_\tau(\mathrm{Trop}(W))$ (by Lemma 3.3.22); in other words it contains cells with the same affine spans as the cells of $\mathrm{Trop}(W)$ which contain τ. Therefore, if the collection of complexes associated to $L \times W$ has non-empty stable intersection, then so must the collection associated to $L \times W_\tau$: by the converse implication of Lemma 3.6.5, then, $L \times W_\tau$ is rotund.

So $L \times W_\tau$ satisfies the statement. $\square$

3.7 Raising to Complex Powers

In this section we establish our main result, the existence of solutions to all systems of equations associated to free rotund varieties of the form $L \times W$ with L linear.

To do so we are going to use Theorem 3.6.1 to show that in a suitable compactification of $(\mathbb{C}^\times)^n$, W and $\exp(L)$ have got sequences of points with the same limit, and that these sequences can be taken to avoid phenomena of asymptoticity. To make this clear, the first step is to give a geometric interpretation of Theorem 3.6.1, explaining the second clause of the statement (the consequences of $\mathrm{Re}(L) \cap_{st} \mathrm{Trop}(W)$ intersecting the relative interior of τ).

We will study compactifications of W which live in *toric varieties*.

Definition 3.7.1. A *complex toric variety* is a complex algebraic variety Y for which there exists an embedding $\iota : (\mathbb{C}^\times)^n \hookrightarrow Y$ for some n, such that:

1. The image of ι is an open dense subset of Y;
2. There is a continuous action of $(\mathbb{C}^\times)^n$ on Y which coincides with the usual multiplication on the image of ι.

The three easiest examples of complex toric varieties are the torus $\mathbb{C}^\times$ itself, with just one orbit of the multiplicative action, the affine line $\mathbb{A}^1(\mathbb{C})$ with the two orbits $\{0\}$ and $\mathbb{A}^1(\mathbb{C}) \setminus \{0\}$ and the projective line $\mathbb{P}^1(\mathbb{C})$ with orbits $\{0\}, \{\infty\}$ and $\mathbb{P}^1(\mathbb{C}) \setminus \{0, \infty\}$. The theory of toric varieties is classical and several books on the subject exist, such as [Ful93] and [CLS11].

A standard construction associates a toric variety to a polyhedral fan. We refer the reader to [MS15, Section 6.1] or [NS13, Section 4] for details on this construction and just recall that given the polyhedral fan $\Sigma \subseteq \mathbb{R}^n$ there is a toric variety Y_Σ such that the action of $(\mathbb{C}^\times)^n$ on Y_Σ has precisely one orbit $\mathcal{O}_\tau$ for each polyhedron $\tau \in \Sigma$. Denoting by $\mathbb{T}_\tau$ the algebraic subgroup of $(\mathbb{C}^\times)^n$ obtained as $\exp(\tau_\mathbb{C})$, we have that the orbit $\mathcal{O}_\tau$ is itself a torus, isomorphic to $(\mathbb{C}^\times)^n/\mathbb{T}_\tau$: each point of $\mathcal{O}_\tau$ is a point at infinity of a translate $w \cdot \mathbb{T}_\tau$, and different translates have different points at infinity.

Example 3.7.2. Consider once again the curve $W = \{(w_1, w_2) \in (\mathbb{C}^\times)^2 \mid w_1 + w_2 + 1 = 0\}$: we know that $\Sigma = \mathrm{Trop}(W)$ is a polyhedral complex consisting of three half-lines and the origin. In this case, thus Y_Σ needs to have four orbits. The orbit $\mathcal{O}_0$ coincides with $(\mathbb{C}^\times)^2$. The other three are attached to the algebraic subgroups of $(\mathbb{C}^\times)^2$ defined by $w_1 w_2^{-1} = 1$, $w_1 = 1$ and $w_2 = 1$ respectively. Thus, for example, Y_Σ contains a point at infinity for each translate of the group defined by $w_1 = 1$, of the form (c, ∞) for some $c \in \mathbb{C}^\times$.

We are particularly interested, of course, in the case in which Σ is the tropicalization of a variety W. In that case Tevelev has defined and studied *tropical compactifications* of varieties: the tropical compactification of W is a compact subset of $Y_{\mathrm{Trop}(W)}$. Again, we refer the reader to [Tev07], [MS15, Section 6.4] or [NS13, Section 4] for details and just recall the following properties of tropical compactifications.

Theorem 3.7.3 (Tevelev, see [NS13, Lemma 12 and Corollary 13])**.** *Let $W \subseteq (\mathbb{C}^\times)^n$ be an algebraic variety, and let $\Sigma = \mathrm{Trop}(W)$. Then there is a subvariety $\overline{W} \subseteq Y_\Sigma$ such that:*

1. *$\overline{W}$ is complete;*
2. *$\overline{W} \cap \mathcal{O}_0 = W$;*
3. *For every $\tau \in \Sigma$, $\mathbb{T}_\tau$ acts by translation on the initial variety W_τ, and $W_\tau/\mathbb{T}_\tau = \overline{W} \cap \mathcal{O}_\tau$ under the isomorphism $(\mathbb{C}^\times)^n/\mathbb{T}_\tau \cong \mathcal{O}_\tau$.*

In other words, limits of sequences in W belong to the orbits $\mathcal{O}_\tau$, and any such sequence can be approximated by a sequence on the initial variety W_τ that lies in just one translate of $\mathbb{T}_\tau$ and has the same limit in Y_Σ.

As we mentioned above, we are going to use these objects to give a geometric interpretation of Theorem 3.6.1.

Lemma 3.7.4. *Let $L \times W$ be a free rotund variety, L linear not defined over the reals. Let $\Sigma = \mathrm{Trop}(W)$, and Y_Σ be the toric variety associated to the fan. Then there is $\tau \in \Sigma$, $\dim(\tau) > 0$, such that:*

1. *$L \times W_\tau$ is rotund;*
2. *There exist $s \in \mathbb{S}_1^n$ and a sequence $\{s \cdot \exp(l_j)\}_{j\in\omega} \subseteq (s \cdot \exp(L)) \cap W_\tau$ such that $\lim_j s \cdot \exp(l_j) \in \mathcal{O}_\tau$ and the δ-map of $L \times W_\tau$ is open at $(0, s \cdot \exp(l_j))$ for each $j \in \omega$.*

Proof. Let $L \times W$ be a free rotund variety, and take $\tau \in \mathrm{Trop}(W)$ to be the polyhedron whose existence is granted by Theorem 3.6.1, so that $L \times W_\tau$ is rotund and $\mathrm{Supp}(\mathrm{Re}(L) \cap_{st} \mathrm{Trop}(W)) \cap \mathrm{relint}(\tau) \neq \varnothing$.

Consider the initial variety W_τ. By Proposition 3.3.25, it is invariant under translation by elements of $\exp(\tau_\mathbb{C})$. The variety $L \times W_\tau$ is not necessarily free as W_τ may be contained in a translate of an algebraic subgroup (in fact, it may even be such a translate) but we may assume that it is by restricting the ambient space: if W_τ is contained in a translate $a \cdot \exp(Q)$, where Q is a $\mathbb{Q}$-linear subspace of $\mathbb{C}^n$, then we consider the variety $(L \cap Q) \times W_\tau$. Up to a translation, an isomorphism of algebraic groups will take this to a free rotund subvariety of $\mathbb{C}^{\dim Q} \times (\mathbb{C}^\times)^{\dim Q}$, so we can treat $L \times W_\tau$ as if it were free itself.

By Proposition 3.2.4, there is a Zariski-open dense subset $(W_\tau)^\circ$ of W_τ such that the δ-map of $L \times W_\tau$ is open on $L \times (W_\tau)^\circ$. By Corollary 3.5.17, $\overline{\exp(\mathrm{Re}(L)_\mathbb{C})} = \exp(L) \cdot \mathbb{S}_1^n$ intersects W_τ in a Zariski-dense subset, and therefore it must in particular intersect $(W_\tau)^\circ$; then, let $w \in (W_\tau)^\circ$ be a point in this intersection, so that $w \in s_0 \cdot \exp(L)$ for some $s_0 \in \mathbb{S}_1^n$. The δ-map, as we said, is open at $(0, w)$, and therefore there are neighbourhoods V of 0 in L and U of w in W such that $\frac{U}{\exp(V)} \subseteq \mathrm{im}(\delta)$ contains an open ball around w.

Since $\mathrm{Re}(L) \cap \mathrm{relint}(\tau) \neq \varnothing$, we may consider a sequence $\{t_j\}_{j\in\omega} \subseteq \mathrm{Re}(L)_\mathbb{C} \cap \tau_\mathbb{C}$ such that $\lim_{j\in\omega} \exp(t_j) \in \mathcal{O}_\tau$ (this follows from Theorem 3.7.3, since $\exp(t_j) \in \mathbb{T}_\tau$ for each j). Hence, $\lim_{j\in\omega} w \cdot \exp(t_j) = w \cdot \lim_{j\in\omega} \exp(t_j) \in \mathcal{O}_\tau$ as well, since the multiplicative action of $(\mathbb{C}^\times)^n$ on Y_Σ is continuous and $\mathcal{O}_\tau$ is an orbit. Also, $\exp(t_j) \cdot U \subseteq W_\tau$ for each j, given that W_τ is invariant under translation by $\exp(\tau_\mathbb{C})$.

Moreover, as $\{\exp(t_j)\}_{j\in\omega} \subseteq \exp(L) \cdot \mathbb{S}_1^n$, for each j there is $l_j^0 \in L$ such that $\frac{\exp(t_j)\cdot w}{\exp\left(l_j^0\right)} \in \mathbb{S}_1^n$. By compactness of $\mathbb{S}_1^n$, we may assume that the sequence $\left\{\frac{\exp(t_j)\cdot w}{\exp\left(l_j^0\right)}\right\}_{j\in\omega}$ has a limit, call it $s \in \mathbb{S}_1^n$.

Consider for each j the open set $\frac{\exp(t_j)\cdot U}{\exp\left(l_j^0\right)\cdot\exp(V)} = \frac{\exp(t_j)}{\exp\left(l_j^0\right)} \cdot \frac{U}{\exp(V)}$. As the set $\frac{U}{\exp(V)}$ was built to contain an open ball around w, for j sufficiently large we must have $s \in \frac{\exp(t_j)}{\exp\left(l_j^0\right)} \cdot \frac{U}{\exp(V)}$.

Therefore we can take a sequence $\{s \cdot \exp(l_j)\}_{j\in\omega}$, where each $\exp(l_j)$ lies in $\exp\left(l_j^0\right) \cdot \exp(V) \subseteq \exp(L)$ and each $s \cdot \exp(l_j)$ belongs to $\exp(t_j) \cdot U \subseteq W_\tau$; in other words, so that $\{s \cdot \exp(l_j)\}_{j\in\omega} \subseteq s \cdot \exp(L) \cap W_\tau$, as required. $\square$

Corollary 3.7.5. *Let $L \times W$ be a free rotund variety, L linear not defined over the reals, $\tau \in \mathrm{Trop}(W)$ as given by Lemma 3.7.4, U a neighbourhood of the identity in $(\mathbb{C}^\times)^n$. Then there is a sequence $\{t_j\}_{j\in\omega} \subseteq \tau_\mathbb{C}$ such that $\lim_j \exp(t_j) \in \mathcal{O}_\tau$ and $\exp(t_j) \cdot U \cap \exp(L) \neq \varnothing$ for all $j \in \omega$.*

Proof. By replacing U by $U \cap U^{-1}$ if necessary we assume that U is symmetric.

Consider the sequence $\{s \cdot \exp(l_j)\}_{j\in\omega}$ given by Lemma 3.7.4.

As $\lim_j s \cdot \exp(l_j) \in \mathcal{O}_\tau \cong (\mathbb{C}^\times)^n/\mathbb{T}_\tau$, there is a translate $w \cdot \mathbb{T}_\tau$ such that $\lim_j s \cdot \exp(l_j) \cdot \mathbb{T}_\tau = w \cdot \mathbb{T}_\tau$. Then we may assume that $s \cdot \exp(l_j) \cdot U \cap w \cdot \mathbb{T}_\tau \neq \varnothing$ for all $j \in \omega$. Therefore, we automatically get that for all sufficiently large j's,

$$\frac{s \cdot \exp(l_j)}{w} \cdot U \cap \mathbb{T}_\tau \neq \varnothing$$

$$\frac{s}{w} \cdot \exp(l_j) \cdot U \cap \mathbb{T}_\tau \neq \varnothing$$

and this implies that taking a subsequence if necessary we may assume

$$\exp(l_j) \cdot U \cap \mathbb{T}_\tau \neq \varnothing.$$

Hence, we can extract from each $\exp(l_j) \cdot U \cap \mathbb{T}_\tau$ a point $a_j \in \mathbb{T}_\tau$. Taking a sequence $\{t_j\}_{j\in\omega}$, where for each j we have $\exp(t_j) = a_j$, we prove the corollary. $\square$

Next we show that if we can take s in the statement of Lemma 3.7.4 to be the identity $(1, \ldots, 1)$, then we can use the points in the sequence in $\exp(L) \cap W_\tau$ as approximations for points in $\exp(L) \cap W$.

Lemma 3.7.6. *Let $L \times W$ be a free rotund variety, L linear not defined over the reals. Let $\Sigma = \mathrm{Trop}(W)$, and Y_Σ be the toric variety associated to the fan. Suppose there is $\tau \in \Sigma$, $\dim(\tau) > 0$, such that:*

1. *$L \times W_\tau$ is rotund;*
2. *$\exp(L) \cap W_\tau$ contains a point w such that the δ-map of $L \times W_\tau$ is open at $(0, w)$.*

Then $\exp(L) \cap W \neq \varnothing$.

Proof. Let w be the point given by Assumption 2. Take $z \in \exp^{-1}(w)$.

Let L^* be any subspace of $\mathbb{C}^n$ such that $L \oplus L^* = \mathbb{C}^n$; let $\pi_L : \mathbb{C}^n \to \mathbb{C}^n/L \cong L^*$ denote the composition of the projection on the quotient with the isomorphism between $\mathbb{C}^n/L$ and L^*. Since the δ-map of $L \times W_\tau$ is open at $(0, w)$, the point z is isolated in $z + L \cap \log W_\tau$: hence, there is a ball B_1 around 0 in L such that $z + \partial B_1 \cap \log W_\tau = \varnothing$. Given B_1, there must be a ball B_2 around 0 in L^* such that for all $x \in z + B_1 + B_2 \cap \log W_\tau$, $x + \partial B_1 \cap \log W_\tau = \varnothing$: this is because $z + \partial B_1$ is compact and $\log W_\tau$ is closed, therefore if their intersection is empty then it must be empty for all translates of $z + \partial B_1$ that are sufficiently close to z. Let U denote the bounded open neighbourhood of 0 in $\mathbb{C}^n$ given by $B_1 + B_2$.

By Proposition 2.5.8, then, the projection π_L is proper as a map from $z + U \cap \log W_\tau$ to $\pi_L(z) + B_2$: this is because $z + U \cap \log W_\tau$ has no limit points on $z + \partial B_1 + B_2 \cap \log W_\tau$. We already know the projection is open on this domain, and therefore it has to be surjective. In other words, $(z + U \cap \log W_\tau) + L = z + B_2 + L$.

By Corollary 3.7.5, there is a sequence $\{t_j\}_{j\in\omega} \subseteq \tau_{\mathbb{C}}$ such that $z + t_j + U \cap L \neq \varnothing$ for all $j \in \omega$.

It is easy to see that, since $\log W_\tau$ is invariant under translation by $\tau_{\mathbb{C}}$, $(t_j + z + U) \cap \log W_\tau = t_j + (z + U \cap \log W_\tau)$ for each j. Hence, all the sets $U \cap -(z + t_j) + \log W_\tau$ are equal to a set K. Note that, by the choice of the sets B_1 and B_2, $\overline{K} \cap (\partial B_1 + B_2) = \varnothing$.

On the other hand, as $\log W$ is not invariant under translation by $\tau_{\mathbb{C}}$, the sets $K_j := U \cap -(z + t_j) + \log W$ may not be all equal; nevertheless, their closures $\overline{K_j}$ converge in the Hausdorff distance to $\overline{K}$. This means that for sufficiently large j's, $\overline{K_j} \cap (\partial B_1 + B_2) = \varnothing$: if this were not the case, there would be a point in $K \cap (\partial B_1 + B_2)$. Therefore, for sufficiently large j's, the map π_L is open and proper (again by Proposition 2.5.8) as a map from K_j into B_2, and therefore $K_j + L = B_2 + L$: we assume this holds for all j.

Since $K_j \subseteq -(z + t_j) + \log W$, $z + t_j + K_j \subseteq \log W$. Also, $z + t_j + K_j + L = z + t_j + B_2 + L$ for all j's. As $(z + t_j + B_2 + L) \cap L \neq \varnothing$ (it contains $z + t_j + U$, and thus a point which lies in $\log W_\tau \cap L$), we may thus conclude: there are $z' \in z + t_j + K_j \subseteq \log W$ and $l \in L$ such that $z' + l \in L$, and thus $z' \in L$. $\square$

Finally we set out to prove that the intersection with the original variety exists. This is done by induction, exploiting first the simple geometric structure of initial varieties of curves and then the fact that initial varieties are invariant under translation by subgroups.

Proposition 3.7.7. *Let $L \times W$ be a free rotund variety with L linear not defined over the reals and* $\mathrm{codim} L = \dim W = 1$*. Then there is $\tau \in \mathrm{Trop}(W)$,* $\dim(\tau) > 0$*, such that:*

1. *$L \times W_\tau$ is rotund;*

2. There is $w \in \exp(L) \cap W_\tau$ such that the δ-map of $L \times W_\tau$ is open at $(0, w)$.

Proof. Let $\tau \in \operatorname{Trop}(W)$ be the polyhedron given by Lemma 3.7.4. The initial variety W_τ is then a finite union of cosets of the subgroup $\mathbb{T}_\tau$. By rotundity of $L \times W_\tau$, the δ-map of this variety is open (as W_τ has dimension 1, if it is open on a Zariski-open dense subset then it is open everywhere) and therefore its image is a subgroup of $(\mathbb{C}^\times)^n$ with non-empty interior - that is, it is $(\mathbb{C}^\times)^n$. Then $\exp(L) \cap W_\tau \neq \varnothing$, and the δ-map is open everywhere. $\square$

This allows us to establish our main result.

Theorem 3.7.8. *Let $L \times W$ be a free rotund variety, L a linear space. Then $\exp(L) \cap W \neq \varnothing$.*

Proof. We prove this by induction on the dimension of W.

Assume $\dim W = 1$. If L is defined over the reals, then this follows from Theorem 3.5.16. Otherwise, by Proposition 3.7.7 there is a point $w \in \exp(L) \cap W_\tau$ with $L \times W_\tau$ rotund, such that the δ-map is open around $(0, w)$. Hence by Lemma 3.7.6, $\exp(L) \cap W \neq \varnothing$.

Now assume $\dim W > 1$. Again, if L is defined over the reals then we apply Theorem 3.5.16. Otherwise, consider the polyhedron $\tau \in \operatorname{Trop}(W)$ given by Lemma 3.7.4: the initial variety W_τ is then invariant under translation by the algebraic group $\mathbb{T}_\tau$. Consider then the quotients $\pi_{\tau_\mathbb{C}}(L)$ and $\pi_{\mathbb{T}_\tau}(W_\tau)$: their product is a rotund variety, and $\dim(\pi_{\mathbb{T}_\tau}(W_\tau)) < \dim W$. It is not necessarily free, as $\pi_{\mathbb{T}_\tau}(W_\tau)$ might be contained in a translate of some algebraic subgroup $\exp(Q)$ with Q a $\mathbb{Q}$-linear space, but that is not an issue as then up to a translation we may consider $(\pi_{\tau_\mathbb{C}}(L) \cap Q) \times \pi_{\mathbb{T}_\tau}(W_\tau)$ which is free and rotund in the ambient space $Q \times \exp(Q)$ (similarly to what we did in the proof of Lemma 3.7.4). By the induction hypothesis, $\exp(\pi_{\tau_\mathbb{C}}(L)) \cap \pi_{\mathbb{T}_\tau}(W) \neq \varnothing$. By applying the Rabinovich trick if necessary (as that does not change the dimension of $\pi_{\mathbb{T}_\tau}(W)$) we may further assume that there is a point w in this intersection such that the relevant δ-map is open at $(0, w)$.

Therefore, there is $(l, w) \in L \times W_\tau$ such that the δ-map of $L \times W_\tau$ is open at (l, w), and $\frac{w}{\exp(l)} \in \mathbb{T}_\tau$. Since W_τ is invariant under translation by $\mathbb{T}_\tau$, this implies that actually $\frac{w}{t} \in \exp(L) \cap W_\tau$ for some $t \in \mathbb{T}_\tau$, and the δ-map is open at $\left(0, \frac{w}{t}\right)$ - this is again implied by invariance of W_τ under $\mathbb{T}_\tau$, as if the δ-map is open at $(0, w)$ it must be open at every point of $\{0\} \times (w \cdot \mathbb{T}_\tau)$ by definition. Hence, by Lemma 3.7.6, $\exp(L) \cap W \neq \varnothing$. □

Example 3.7.9. We conclude by testing this argument in the example

$$\begin{cases} iz_1 - z_2 = 0 \\ w_1 + w_2 + 1 = 0 \end{cases}$$

In this case obviously $\mathrm{Re}(L) = \mathbb{R}^2$, and for every initial variety W_τ we have that $L \times W_\tau$ is rotund (the fact that the dimension is low makes it very easy to check: rotundity of $L \times W_\tau$ just means that $\log W_\tau$ and L are not parallel as complex lines).

So let us fix an initial variety, say W_τ for τ the positive half-line spanned by $(1, 1)$. The initial variety W_τ is then defined by $w_1 + w_2 = 0$, and

$$\log W_\tau = \{(z_1, z_2) \in \mathbb{C}^2 \mid z_1 - z_2 \in (2\mathbb{Z} + 1)\pi i \ \}.$$

The points of $L \cap \log W_\tau$ are easy to find: solving

$$\begin{cases} iz_1 - z_2 = 0 \\ z_1 - z_2 = ik\pi \end{cases}$$

for all odd $k \in \mathbb{Z}$, one gets the points

$$\frac{2h+1}{2}(\pi - i\pi, \pi + i\pi)$$

for all $h \in \mathbb{Z}$. Considering those with $h \gg 0$, the points will be very close to points in $\log W$, and by "openness at infinity" of the δ-map this induces intersections between $\log W$ and L.

Finally, let us note that the intersection is again Zariski-dense.

Corollary 3.7.10. *Let $L \times W$ be a free rotund variety, L a linear space. Then $\exp(L) \cap W$ is Zariski-dense in W.*

Proof. By the same proof as Corollary 3.5.17. □

3.8 Model-Theoretic Consequences

We conclude this chapter by indicating the model-theoretic relevance of Theorem 3.7.8.

In Chapter 2, with Theorems 2.4.1 and 2.4.2, we discussed Zilber's results on raising to powers. In particular, we saw how given a field K of finite transcendence degree, the structure $\mathbb{C}^K$ falls in a first-order axiomatizable class of structures as long as it satisfies a transcendence statement and an existential closedness statement.

While transcendence statements remain hard problems in general, and it is complicated to say anything about them, Theorem 3.7.8 establishes the existential closedness clause of Theorem 2.4.2 for every field $K \subseteq \mathbb{C}$ of finite transcendence degree.

Therefore, we obtain the following.

Theorem 3.8.1. *Let $K \subseteq \mathbb{C}$ be a field of finite transcendence degree, and T_K the theory associated to K by Theorem 2.4.1.*

If there is $a \in \mathbb{C}^m$ (possibly $m = 0$) such that for all $z_1, \ldots, z_n$,

$$\mathrm{ldim}_K(z_1, \ldots, z_n/a) + \mathrm{trdeg}(\exp(z_1), \ldots, \exp(z_n)/\exp(a)) \geq$$

$$\geq \mathrm{ldim}_{\mathbb{Q}}(z_1, \ldots, z_n/a)$$

then the structure $\mathbb{C}^K$ is a model of T_K.

In particular, if $K = \mathbb{Q}(\lambda)$ where $\lambda \in \mathbb{C}^n$ is exponentially-algebraically independent over $\mathrm{ecl}(\varnothing)$, then $\mathbb{C}^K$ is a model of T_K.

Proof. The first part is the immediate consequence of Theorems 2.4.2 and 3.7.8: the former says that $\mathbb{C}^K$ is a model of T_K if it satisfies two statements, and the latter proves the second statement for every K.

The second part follows from Theorem 2.4.3, which proves the transcendence statement for fields of the form $\mathbb{Q}(\lambda)$ with λ an exponentially-algebraically independent tuple. $\square$

Chapter 4

Abelian Varieties

4.1 Introduction

In the previous chapters we have introduced Exponential-Algebraic Closedness as a problem which concerns the complex exponential function. However, exp shares many features with other functions in complex geometry; most notably, if we consider $\mathbb{C}^\times$ as a complex Lie group then exp is its exponential map in the sense of Lie groups (a holomorphic, surjective group homomorphism from the tangent space to $\mathbb{C}^\times$ at identity into $\mathbb{C}^\times$).

As such, it is worth asking whether it makes sense to ask similar questions for other exponential maps of complex Lie groups. In this chapter we focus on the exponential maps of *complex abelian varieties.*

Abelian varieties are projective algebraic groups; complex abelian varieties are then compact complex Lie groups. This chapter discusses the Exponential-Algebraic Closedness problem for abelian varieties, focusing on a specific case that is inspired by Theorem 3.7.8. The structure of the chapter is as follows.

In Section 4.2 we introduce complex abelian varieties from scratch, reviewing the basic results from their general theory, and take the chance to discuss homology and cohomology in this setting, focusing on the role of homology in intersection theory. We also devote a short subsection to an introduction to o-minimality, a well-known subfield of model theory that we need some results from. This section does not feature any new results, but we felt it was important to fix all the notation and conventions; unfortunately, it was not possible to do this for everything, but we hope the references we point to are sufficient.

In Section 4.3 we state the Exponential-Algebraic Closedness conjecture for abelian varieties and present some examples. We take a short detour in Subsection 4.3.1 to present some old but relatively little-known related results.

Section 4.4 contains the proof of the main result of this chapter.

Theorem (Theorem 4.4.1)**.** *Let A be an abelian variety of dimension g,* $\exp_A : \mathbb{C}^g \to A$ *its exponential map,* $L \leq \mathbb{C}^g$ *a linear subspace and* $W \subseteq A$ *an algebraic variety such that the variety* $L \times W$ *is free and rotund.*

Then $L \times W \cap \Gamma_{\exp_A} \neq \varnothing$*.*

All the terminology will be introduced in the first two sections, but even without knowing it the reader will notice the similarity with the main result of the previous chapter. We conclude the section with an example of intersection between a variety of the form $L \times W$ and the graph of the exponential of a product of non-isogenous elliptic curves.

In Section 4.5 we compare this theorem to Theorem 3.7.8 from a model-theoretic perspective: we saw in Chapter 2 and in Section 3.8 that from Theorem 3.7.8 one can extract the existential closedness statement of a certain first-order theory and that, together with a transcendence statement, this implies that a certain structure on the complex numbers is a model of this theory. It had already been noted by Zilber in [Zil15] that model-theoretically speaking the main results of that paper could be translated to the context of abelian varieties; while we do not give the details of this translation we briefly comment on how it works, leaving the full elaboration to future work.

Finally, in Section 4.6 we will discuss some work in progress towards a possible improvement on the main theorem, aiming to replace the linear space L by an algebraic variety V.

The main result of this chapter appears in the preprint [Gal22b]. The partial result Theorem 4.4.7 appears in Section 2 of the preprint [Gal21].

4.2 Geometric Preliminaries

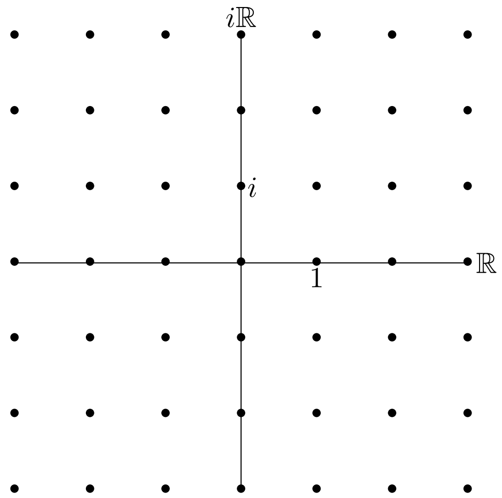

Figure 4.1: The lattice $\mathbb{Z} + i\mathbb{Z}$ in $\mathbb{C}$. Each square with vertices of the form $m+in$ and side length 1, with only one horizontal and one vertical side included, is a fundamental domain for $\mathbb{Z} + i\mathbb{Z}$.

4.2.1 Complex Abelian Varieties

In this subsection we recall the basic facts on the geometry of complex abelian varieties.

A *lattice* in the vector space $\mathbb{R}^n$ is a discrete subgroup of rank n. A *compact torus* is a real Lie group of the form $\mathbb{R}^n/\Lambda$, where Λ is a lattice in $\mathbb{R}^n$.

The compact tori we are interested in are complex, so rather than in a space of the form $\mathbb{R}^n$ we take our lattices in a complex space $\mathbb{C}^g$.

Definition 4.2.1. A *complex torus* is a complex Lie group of the form $\mathbb{C}^g/\Lambda$, where Λ is a lattice in $\mathbb{C}^g \cong \mathbb{R}^{2g}$.

Given any complex torus $\mathbb{C}^g/\Lambda$, a *fundamental domain* for Λ will be any connected region $\mathcal{F}$ of $\mathbb{C}^g$ such that for every $z \in \mathbb{C}^g$ there is a unique $v \in \mathcal{F}$ such that $z - v \in \Lambda$.

An easy example of a complex torus is the one-dimensional torus $\mathbb{C}/\mathbb{Z} + i\mathbb{Z}$, which as a real Lie group is isomorphic to $\mathbb{R}^2/\mathbb{Z}^2$. A fundamental domain for the lattice $\mathbb{Z} + i\mathbb{Z}$ is the square

$$\{a + ib \in \mathbb{C} \mid 0 \leq a < 1,\, 0 \leq b < 1\}.$$

Definition 4.2.2. An *abelian variety* is a projective algebraic variety A which admits a regular group structure, i.e. which admits a group operation $+ : A \times A \to A$ that is described locally by polynomial maps.

Abelian varieties over the field of complex numbers are complex tori; in fact, they can be characterized as the complex tori which admit an embedding into a projective space $\mathbb{P}^n(\mathbb{C})$. This is the content of the following theorem, which goes back to Riemann.

Theorem 4.2.3 (Riemann)**.** *Let A be an abelian variety. Then there is a complex torus $T = \mathbb{C}^g/\Lambda$ such that $T \cong A$ as complex Lie groups.*

We do not delve here into the beautiful theory surrounding this result, which moves forward to characterize the complex tori which admit such an embedding into projective space, and refer the reader to Section A.5 of [HS00, Section A.5] for a proof of this result which omits some of the technical details, and to the first four chapters of [BL04] for a more precise account. We are going to focus on the map which witnesses the isomorphism between the abelian variety and the complex torus, the *exponential* of A.

Definition 4.2.4. Let A be an abelian variety. The surjective holomorphic group homomorphism $\exp_A : \mathbb{C}^g \to A$ with kernel a lattice Λ is the *exponential* of A.

The exponential is a universal covering map and a group homomorphism; by definition it has discrete kernel, and it satisfies a differential equation (see [Mar00] and [Kir09] for details) hence it shares the main features of the complex exponential function.

Moreover, $\mathbb{C}^g$ can be considered as the tangent space to A at identity. Since abelian varieties are homogeneous spaces, we can embed the tangent at any point of any variety in the tangent space at identity: more precisely, given an algebraic subvariety $W \subseteq A$ and a regular point $w \in W$, we will write T_wW for the tangent space to W at w and embed it into $\mathbb{C}^g$ by identifying it with the tangent space at 0 to the variety $-w + W$. Therefore, we will always refer to tangent spaces to points of W as linear subspaces of $\mathbb{C}^g$.

We will often consider subvarieties of the tangent bundle of A, which by the above is isomorphic to $\mathbb{C}^g \times A$. We will use the same symbols $(+, -, 0)$ to talk about the group operations and the identity element on $\mathbb{C}^g$, A, and $\mathbb{C}^g \times A$: it should be clear from context where we are performing the operation.

Definition 4.2.5. An *abelian subvariety* of the abelian variety A is an algebraic subvariety B of A which is also an (algebraic) subgroup of A.

In fact, if an algebraic subvariety $B \subseteq A$ is a subgroup then the group structure on B is automatically algebraic. The tangent space at identity to B, T_0B, is a subspace of $\mathbb{C}^g$ which serves as the domain for the exponential map $\exp_B$: we will denote it by LB, the *Lie algebra* of B.

Finally, we give the definition of a *simple* abelian variety (we are often going to use it to discuss some easy cases of the statements we consider) and state the Poincaré Complete Reducibility Theorem to stress the importance of this notion.

Definition 4.2.6. Let A be an abelian variety. We say that A is *simple* if it does not have any proper non-zero abelian subvarieties.

In other words, simple abelian varieties have no non-trivial algebraic subgroups. The reader who is familiar with Chow's Theorem will immediately see that this implies that simple abelian varieties, being projective objects, do not have complex analytic subgroups.

The Poincaré Complete Reducibility Theorem says that simple abelian varieties can be used as "building blocks" for all abelian varieties, through the notion of *isogeny.*

Definition 4.2.7. Let A and B be abelian varieties. An *isogeny* $f : A \to B$ is a surjective morphism of algebraic groups with finite kernel.

Theorem 4.2.8 (Poincaré Complete Reducibility Theorem, [HS00, CorollaryA.5.1.8])**.** *Any abelian variety is isogenous to a product of powers of distinct, pairwise non-isogenous simple abelian varieties.*

4.2.2 Homology and Cohomology

We use this subsection to fix notation and conventions for the rest of the chapter.

The homology theory we are going to use is singular homology. Thus, by *standard simplex* in $\mathbb{R}^n$ we mean the set

$$\Delta := \left\{ (x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_1, \ldots, x_n \geq 0 \text{ and } \sum_{i=1}^{n} x_i = 1 \right\}$$

while a simplex in the topological space X is a continuous map $\sigma : \Delta \to X$; *chains* are formal sums of simplices. A *simplicial complex* is a polyhedral complex all of whose polyhedra are affinely isomorphic to the standard simplex. Simplicial complexes are used to *triangulate* topological spaces.

Definition 4.2.9. A *triangulation* of the topological space X is a homeomorphism between X and a simplicial complex.

Given a simplex $\sigma : \Delta \to X$ of dimension n, the *boundary* of σ is the chain obtained as the formal sum of the simplices $\sigma' : \Delta' \to X$, where each Δ' is a non-trivial, oriented face of the simplex Δ. The boundary map is defined on chains by extending this function by linearity. A *cycle* is a chain with boundary 0. Two cycles C_1 and C_2 are homologous if there is a chain D whose boundary is $C_1 - C_2$, and the n-th homology group is the group of cycles of dimension n modulo this equivalence relation. Of course there are many texts where this is discussed in great detail, for example [Hat02, Chapter 2].

In the case of abelian varieties (and in general in the case of tori), the structure of the homology groups is particularly easy to compute: in an abelian variety A of dimension g, the n-th homology group is a free abelian group of rank $\binom{2g}{n}$. [BL04, Excercise 1.7] describes how to obtain a basis of each homology group starting from the elements of the lattice; informally, we may say that the n-th homology group is generated by the classes of some cycles that are supported on each of the subgroups generated by n lattice elements. A formal description of this requires the notion of *Pontryagin product*, which we do not deal with.

The easiest case is of course the first homology group $H_1(A) \cong \mathbb{Z}^{2g}$: given a basis $\{\lambda_1, \dots, \lambda_{2g}\}$ of the lattice consider the simplices $\sigma_j : [0,1] \to A$ defined as $\sigma_j(t) = \exp_A(t\lambda_j)$. The homology classes of these form a basis for $H_1(A)$.

As is the case in general with algebraic varieties, it is possible to triangulate complex algebraic subvarieties of an abelian varieties. More precisely, we have the following classical theorem (see for example [Hir75, Section 2] for a proof).

Theorem 4.2.10. *Every algebraic set admits a triangulation.*

We will therefore treat algebraic varieties as cycles: given a variety W and a triangulation $\mathcal{T}$, we can identify the variety with the cycle $\sum_{\sigma \in \mathcal{T}} \sigma$. The choice of triangulation is only unique up to homological equivalence, but this is sufficient for our purposes.

The situation concerning cohomology is a bit more delicate, as we are going to consider differential cohomology. We recall the definition of complex differential form. We will follow the approach presented in [Wel80, Section I.3].

Recall that given a manifold of even dimension M with tangent bundle TM, an *almost complex structure* on M is a vector bundle automorphism $J : TM \to TM$ such that $J^2 = -\mathbb{I}$ where $\mathbb{I}$ is the identity. If we equip M with an almost complex structure, we call it an *almost complex manifold.*

We start by examining complex-valued differential forms on real manifolds; we will then see how to modify this notion to handle the case of almost complex structures. Let M be a (real) differentiable manifold. Consider the complexification TM_c of its tangent bundle, so that $TM_c := TM \otimes_{\mathbb{R}} \mathbb{C}$. We define the space of complex-valued differential forms of total degree r on M as the space of infinitely differentiable sections of the r-th exterior power of the cotangent bundle of M. In other words, a complex-valued differential form ω on M of degree r associates to each point $x \in M$ a $\mathbb{C}$-linear function $\omega_x : \bigwedge^r T_x M_c \to \mathbb{C}$.

Now assume (M, J) is an almost complex manifold. We may extend J to a $\mathbb{C}$-linear bundle automorphism on TM_c: for each x, the fibre of J is a linear map $J_x : T_x M \to T_x M$, and we may extend this to

$$J_{x,c} : T_x M \otimes_{\mathbb{R}} \mathbb{C} \to T_x M \otimes_{\mathbb{R}} \mathbb{C}$$

defined by $J_{x,c}(z \otimes \alpha) = J_x(z) \otimes \alpha$. It is then still the case that $J_c^2 = -\mathbb{I}$, but since J_c is $\mathbb{C}$-linear it has eigenvalues i and $-i$.

We can thus define the bundles of eigenspaces: let $TM^{1,0}$ be the bundle of eigenspaces for i and $TM^{0,1}$ the bundle of eigenspaces for $-i$. This yields a decomposition of the complexified tangent bundle, so that $TM_c = TM^{1,0} \oplus TM^{0,1}$.

We can define fibrewise conjugation too:

$$Q_x : T_x M \otimes_{\mathbb{R}} \mathbb{C} \to T_x M \otimes_{\mathbb{R}} \mathbb{C}$$

is defined by $Q_x(z \otimes \alpha) = z \otimes \overline{\alpha}$.

Proposition 4.2.11. *The bundles $T^{1,0}M$ and $T^{0,1}M$ are isomorphic under Q.*

Proof. We check this fibrewise, so let $x \in M$ be a point. If $z \otimes \alpha \in T_x^{1,0}M$ then it is an eigenvector for i of $J_{c,x}$, so

$$J_{x,c}(z \otimes \alpha) = J_x(z) \otimes \alpha = i(z \otimes \alpha) = z \otimes i\alpha.$$

Therefore,

$$J_{c,x}(Q_x(z \otimes \alpha)) = J_{c,x}(z \otimes \overline{\alpha}) = J_x(z) \otimes \overline{\alpha} =$$

$$= \overline{J_x(z) \otimes \alpha} = \overline{z \otimes i\alpha} = z \otimes (-i)\overline{\alpha} = (-i)(z \otimes \overline{\alpha})$$

so $Q_x(z \otimes \alpha)$ is an eigenvector for $-i$ of $J_{c,x}$. By linearity, since there is a basis of $T_x^{1,0}M$ of elements of the form $z \otimes \alpha$, we may extend this to all elements of $T_x^{1,0}M$.

With a similar argument we can check that the square of Q_x is the identity (on $T_x^{1,0}M$): it swaps the eigenspaces of J_c. □

In fact, considering T_xM as a complex vector space using the almost complex structure J (so $i \cdot z = J(z)$ by definition, for all $z \in T_xM$) we have that $T_xM \cong_{\mathbb{C}} T_x^{1,0}M$, the latter being a complex vector space because it is the eigenspace of $J_{x,c}$ for i, so that $i \cdot z \otimes \alpha = J_{x,c}(z \otimes \alpha) = z \otimes i\alpha$ for all $z \otimes \alpha \in T_x^{1,0}M$. These two spaces will thus often be identified, with $T_x^{0,1}M$ identified with $Q(T_xM)$.

It is clear by the definition that $Q_x(Q_x(z \otimes \alpha)) = z \otimes \overline{\overline{\alpha}} = z \otimes \alpha$, so as we should expect conjugation has order 2.

The decomposition into eigenspaces of TM_c transfers over to a decomposition of the complexified cotangent bundle, which we can use to study differential forms.

Definition 4.2.12. A *complex-valued differential form of type* (p, q) (or (p, q)-*form* for short) on the almost complex manifold M is a $\mathcal{C}^\infty$ section of the bundle

$$\bigwedge^p T^*M^{1,0} \wedge \bigwedge^q T^*M^{0,1}$$

i.e. an infinitely differentiable function which assigns to each point in x a

$\mathbb{C}$-linear function

$$\omega_x : \bigwedge^p T_x M^{1,0} \wedge \bigwedge^q T_x M^{0,1} \to \mathbb{C}.$$

The *total degree* of a form of type (p, q) is $p + q$.

Every form ω of type (p, q) has a *complex conjugate* $\overline{\omega}$ of type (q, p), defined by

$$\overline{\omega}_x(v_q \wedge v_p) = \omega_x(Q_x(v_p) \wedge Q_x(v_q)).$$

The wedge product of differential forms is defined locally as

$$(\omega_{1,x} \wedge \omega_{2,x}) \left(\bigwedge_{j=1}^{p_1+p_2} v_j \wedge \bigwedge_{h=1}^{q_1+q_2} w_h \right) =$$

$$= k \sum_{\sigma \in S_{p_1+p_2}, \rho \in S_{q_1+q_2}} \operatorname{sgn}(\sigma)\operatorname{sgn}(\rho)\omega_{1,x}(v_{\sigma,p_1} \wedge w_{\rho,q_1})\omega_{2,x}(v_{\sigma,p_2} \wedge w_{q,\rho})$$

where by v_{σ,p_1} we mean that we apply the permutation σ to $(1, 2, \ldots, p_1 + p_2)$ and then take the wedge product $\bigwedge_{j=1}^{p_1} v_{\sigma(j)}$; the analogous convention is used for v_{σ,p_2}, w_{ρ,q_1}, and w_{ρ,q_2}; k is a positive rational coefficient which depends only on p_1, p_2, q_1, q_2 and not on the differential forms.

Example 4.2.13. Given a complex manifold M, and a holomorphic function $f : M \to \mathbb{C}$, the differential $(1, 0)$-form df is defined locally by $(df)_x(v) = \nabla f(x) \cdot v$.

Of course, the most common way to use a differential form is to integrate it. Let M be a complex manifold, $U \subseteq M$ and $V \subseteq \mathbb{C}^n$ open subsets, $\varphi : V \to U$ a biholomorphism. Given coordinate functions $z_1, \ldots, z_n$ on U, we may consider the differentials dz_j, differential forms on U, and $d(z_j \circ \varphi)$, differential forms on V (defined as in Example 4.2.13).

Locally on U, a differential ω can be described by

$$\omega_z = f(z)dz_1 \wedge \cdots \wedge dz_n \wedge \cdots \wedge d\overline{z}_1 \wedge \cdots \wedge d\overline{z}_n$$

for some holomorphic function $f : U \to \mathbb{C}$. The (n, n)-form $\bigwedge_j dz_j \wedge d\overline{z}_j$ is usually referred to as the *volume element* of M. Therefore, we may define $\int_U \omega$ as

$$\int_U \omega = \int_U f(z)dz_1 \wedge \cdots \wedge dz_n \wedge d\overline{z}_1 \wedge \cdots \wedge d\overline{z}_n =$$

$$= \int_V (f \circ \varphi) d(z_1 \circ \varphi) \wedge \cdots \wedge d(z_n \circ \varphi) \wedge d(\overline{z}_1 \circ \varphi) \wedge \cdots \wedge d(\overline{z}_n \circ \varphi)$$

where the integral in the last term is the usual integral of a holomorphic function in $\mathbb{C}^n$.

The next proposition is an example of a useful property of $(d, 0)$-forms which we will use later on.

Proposition 4.2.14. *Let M be a complex manifold of dimension d, ω a non-zero complex differential form of type $(d, 0)$.*

Then, if it is defined,

$$\int_M \omega \wedge \overline{\omega} \neq 0.$$

Proof. Cover M by open subsets. On any of these subsets U, fix coordinate functions $z_1, \ldots, z_d$: then there is a holomorphic function $f : U \to \mathbb{C}$ $\omega_x = f(x) dz_1 \wedge \cdots \wedge dz_d$ for all $x \in U$.

Then, $\omega_x \wedge \overline{\omega}_x$ is equal, up to multiplication by some positive rational (see the convention for the wedge product above), to

$$f(x)\overline{f(x)} dz_1 \wedge \cdots \wedge dz_d \wedge d\overline{z}_1 \wedge \cdots \wedge d\overline{z}_d$$

and therefore integrating ω on U is the same as integrating a positive function times the volume element $\bigwedge_j (dz_j \wedge d\overline{z}_j)$. This gives a non-zero number. Summing over the open sets U will give the total integral on M: this is non-zero, as all the contribution are equal to a positive real multiplied by the same constant. $\square$

Integration can be easily extended to an operation on chains: we integrate a form of total degree d on a d-dimensional simplex by integrating it on the relative interior of its image, and extend this by linearity to the set of all chains.

As usual, there are notions of exterior derivative and of closed and exact differential form. We will not get into detail on what these are precisely; we are just interested in the following statement, which is obtained from the de Rham isomorphism theorem.

Proposition 4.2.15 ([BL04, Proposition 1.3.5])**.** *Let ω be any differential form on A. Then there is a form ω' invariant under translation by elements A*

such that for each cycle $C \subseteq A$

$$\int_C \omega = \int_C \omega'.$$

Recall that a differential form ω of total degree d is *decomposable* if there exist forms $\omega_1, \dots, \omega_d$ of degree 1 such that $\omega = \bigwedge_{j=1}^d \omega_j$.

We shall now see that translation-invariant differential forms on abelian varieties admit a basis which consists of decomposable forms.

Suppose $\mathcal{B} = \{e_1, \dots, e_g\}$ is a basis of $\mathbb{C}^g$. Then the corresponding coordinate functions are defined by taking $v_j(v)$ to be the coefficient of e_j when we write $v = \sum_{j=1}^g v_j(v)e_j$; $\overline{v_j}$ is the complex conjugate of v_j.

The differentials of the coordinate functions, dv_j and $d\overline{v}_j$, are then differential forms of type $(1,0)$ or $(0,1)$ on $\mathbb{C}^g$.

Proposition 4.2.16 ([BL04, Section 1.4]). *Let $v_1, \dots, v_g$ be complex coordinate functions on $\mathbb{C}^g$. Then:*

1. *The differentials $dv_1, \dots, dv_g, d\overline{v}_1, \dots, d\overline{v}_g$ form a basis of the space of complex-valued invariant differential 1-forms on A;*
2. *More generally, for every $0 \leq n \leq 2g$ the differentials $\bigwedge dz_I \wedge d\overline{z}_J$ where I and J are ordered subsets of $\{1, \dots, 2g\}$ such that $|I| + |J| = n$ form a basis of the space of complex-valued invariant differential forms of total degree n on A.*

In the following example we see some of these differential forms at work.

Example 4.2.17. Consider the elliptic curve $A \cong \mathbb{C}/\mathbb{Z} + i\mathbb{Z}$. On $\mathbb{C}$ we have complex differential forms $dz = dx + idy \in H^{1,0}(A; \mathbb{C})$ and $d\overline{z} = dx - idy \in H^{0,1}(A; \mathbb{C})$.

Let $C := \exp([0,1])$ where denote the image under the exponential map of the horizontal segment in the complex plane. Then $\int_C dz = \int_C dx + i \int_C dy = 1 = \int_C d\overline{z}$. If on the other hand we take the vertical segment $C' := \exp(i[0,1])$, then $\int_{C'} dz = i$ while $\int_{C'} d\overline{z} = -i$.

Now let C'' be $\exp(S)$ where S is the circle of centre $\frac{1}{2} + \frac{i}{2}$ and radius $\frac{1}{2}$. If we parametrize C' using a curve $\gamma : [0,1] \to S$ with $\gamma(0) = \gamma(1) = \frac{i}{2}$ which winds around C'' once, we have that

$$\int_{C''} dz = \int_0^1 \mathrm{Re}(\gamma(z))dx + i\int_0^1 \mathrm{Im}(\gamma(z))dy = 0 + i0 = 0.$$

Finally, if $U \subseteq A$ is an open subset, for example $\exp(S)$ where S is the square with vertices 0, $\frac{1}{2}$, $\frac{i}{2}$ and $\frac{1}{2} + \frac{i}{2}$ then

$$\int_U dz \wedge d\overline{z} = \int_U (-2idx \wedge dy) = -\frac{i}{2}$$

(recall that this could not be 0, by Proposition 4.2.14).

We can use differential forms "dually" to vector subspaces of $\mathbb{C}^g$, using them to intuitively give a measure of oriented volumes of cycles.

Any real vector space $T \subseteq \mathbb{C}^g$ can be decomposed into the direct sum of a (unique) complex vector space $T^{\mathbb{C}} := T \cap iT$ and a (not unique) *totally real space* $T^{\mathbb{R}}$ (so that $T^{\mathbb{R}} \cap iT^{\mathbb{R}} = \langle 0 \rangle$).

Fix a base of T of the form $\{t_1, it_1, \ldots, t_p, it_p, s_1, \ldots, s_q\}$. For a point $a \in A$, see T as a subspace of $T_aA \cong \mathbb{C}^g$, and let v denote the element

$$\bigwedge_{j=1}^{p} t_j \wedge \bigwedge_{k=1}^{q} s_k \wedge \bigwedge_{j=1}^{p} Q_a(t_j) \in \bigwedge^{p+q} T \wedge \bigwedge^{p} Q_a(T).$$

We can thus associate to T a complex differential form ω_T of degree $(2g - p - q, 2g - p)$. Consider, again for a fixed $a \in A$, a mapping

$$\lambda_a : \bigwedge^{2g-p-q} T_a^{1,0}A \wedge \bigwedge^{2g-p} T_a^{0,1}A \to \bigwedge^{2g} T_a^{1,0}A \wedge \bigwedge^{2g} T_a^{0,1}A$$

defined by

$$\bigwedge_{j=1}^{2g-p-q} z_j \wedge \bigwedge_{k=1}^{2g-p} w_k \mapsto v \wedge \bigwedge_{j=1}^{2g-p-q} z_j \wedge \bigwedge_{k=1}^{2g-p} w_k.$$

Fix a basis of $\mathbb{C}^g$ of the form $\{c_1, c_2, \ldots, c_g, ic_1, \ldots, ic_g\}$. Then $\omega_{T,a}$ maps every vector u in the domain of λ_a to the unique complex number $\omega_{T,a}(u)$ such that $\lambda_a(u) = \omega_{T,a}(u) \bigwedge_{j=1}^{g} c_j \wedge \bigwedge_{k=1}^{g} Q_a(c_j)$. It is clear that, in the way we have set up things, the form depends on the choice of basis, and changing basis will give a scalar multiple of the form.

The complex conjugate $\overline{\omega}_T$ is a form of degree $(2g - p, 2g - p - q)$ defined analogously but with the basis element v defined as

$$\bigwedge_{j=1}^{p} t_j \wedge \bigwedge_{k=1}^{q} Q_a(s_k) \wedge \bigwedge_{j=1}^{p} Q_a(t_j) \in \bigwedge^{p} T_a^{1,0}A \wedge \bigwedge^{p+q} T_a^{1,0}A.$$

It is then clear that for all a, and for all v in the appropriate symmetric power of the tangent space T_aA,

$$\omega_{T,a}(v) = \overline{\omega}_{T,a}(Q_a(v)).$$

Of course the fact that using this convention the degree of ω_T is (d_1, d_2) with $d_1 \leq d_2$, while for $\overline{\omega}_T$ we have $d_2 \leq d_1$, is purely coincidental and we could have set up things the other way around.

Note that if $T = T^{\mathbb{C}}$, so if T is a complex space, $q = 0$ and thus $\omega_T = \overline{\omega}_T$.

It is worth asking what these forms look like as products of 1-forms, using the bases from Proposition 4.2.16.

Definition 4.2.18. Let $\mathcal{B} := \{v_1, iv_1, \ldots, v_g, iv_g\}$ be a real basis of $\mathbb{C}^g$, and $\mathcal{B}' \subseteq \mathcal{B}$. We then define

$$d\mathcal{B}' := \bigwedge_{j | v_j \in \mathcal{B}'} dv_j \wedge \bigwedge_{j | iv_j \in \mathcal{B}'} d\overline{v}_j.$$

We call this the *basis wedge* of $\mathcal{B}'$.

Proposition 4.2.19. *Let $T \leq \mathbb{C}^g$ be a real vector space, and suppose $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2$ is a real basis of $\mathbb{C}^g$ of the form $\{v_1, iv_1, v_2, iv_2, \ldots, v_g, iv_g\}$ and $\mathcal{B}_1$ is a basis of T.*

Then in the corresponding basis of the space of invariant forms, up to scalar multiplication $\omega_T = d\mathcal{B}_2$.

Proof. We prove this by induction on the real codimension of T. If $\mathrm{codim}_{\mathbb{R}} T = 1$, then we may write $\mathcal{B} = \{v_1, iv_1, \ldots, v_g, iv_g\}$ with $\mathcal{B} \setminus \{iv_g\}$ a real basis of T.

Then clearly for any $a \in A$, $d(\overline{v}_g)_a(v) \neq 0$ if and only if $Q_a(v) \notin T$. Thus dv_g needs to be a scalar multiple of the form ω_T. The same of course holds for dv_g if the basis of T is $\mathcal{B} \setminus \{v_g\}$.

Suppose we know this for T of codimension $n < 2g$, and consider T of codimension $n+1$. We write $\mathcal{B}$ as $\mathcal{B}_1 \cup \mathcal{B}_2$.

Let $b \in \mathcal{B}_2$, and denote by T' the space generated by T and b: then $\omega_{T'} = d(\mathcal{B}_2 \setminus \{b\})$. Up to scalar multiplication, then, $\omega_T = \omega_{T'} \wedge db = d\mathcal{B}_2$, as can be verified directly by the definition of wedge product. □

Let us now consider what these forms do when we integrate them on cycles in the abelian variety A.

Proposition 4.2.20. *Let $T \leq \mathbb{C}^g$ be a real vector subspace of dimension $2g-d$ with associated differential form ω_T, and $C \subseteq A$ be a cycle of (real) dimension d.*

If $\int_C \omega_T \neq 0$, then there is a smooth point $c \in C$ such that $T_cC \cap T = \langle 0 \rangle$.

Proof. If $\int_C \omega_T \neq 0$ then there is a point $c \in C$ such that $\omega_{T,c} \neq 0$. The linear function $\omega_{T,c}$ is defined on

$$\bigwedge_{j=1}^{2g-p-q} T_cC \wedge \bigwedge_{k=1}^{2g-p} Q_a(T_cC)$$

as the wedge between an element of the domain and a basis wedge for some basis $\mathcal{B}$ of T. Since the form is non-zero, there is an element of the domain on which it is non-zero, which implies that $T_cC + T = \mathbb{C}^g$. □

Proposition 4.2.20 has a partial converse in the case in which the cycle C is a complex subvariety of A and the space T is complex.

Proposition 4.2.21. *Let $L \leq \mathbb{C}^g$ be a complex vector subspace of (real) dimension $2g-2d$ with associated differential form $\omega_L = \overline{\omega}_L$, and $W \subseteq A$ a complex algebraic subvariety of (complex) dimension d.*

If there is one point $w \in W$ with $L \cap T_wW = \langle 0 \rangle$, then $\int_W \omega_L \neq 0$.

Proof. Since W is a complex variety, at each of its smooth points the tangent space T_wW is a complex vector space of complex dimension d. As we noted, the form ω_L is a (d,d) form.

If there is one point $w \in W$ with $L \cap T_wW = \langle 0 \rangle$, then the pullback of ω_L to W is a non-zero complex (d,d)-form on a complex projective variety of dimension

d. Since there is a basis $\mathcal{B}$ of L that has the form $\{v_1, iv_1, \ldots, v_{g-d}, iv_{g-d}\}$, we may assume that ω_L is the wedge product of a $(d,0)$-form and its complex conjugate. As pullbacks preserve wedge products, the same holds for the pullback of ω_L to W.

Therefore its integral needs to be non-zero, by Proposition 4.2.14. □

Finally, let us recall the role of the *cup product*.

Definition 4.2.22. The cup product is the pairing on cohomology groups

$$\cup : H^j(M) \times H^i(M) \to H^{i+j}(M)$$

defined on forms ω_1 and ω_2 in cohomology classes $[\omega_1]$ and $[\omega_2]$ as $[\omega_1 \cup \omega_2] = [\omega_1] \cup [\omega_2]$.

There are more general ways to define the cup product, but all we will need is its duality with intersections, described in the next subsection. For proofs that the cup product as we introduced it is well-defined see [Bre13, Section VI.4] or [Hat02, Section 3.2].

4.2.3 Transversality and Intersections

In this subsection we gather some notions on transversality which we will need later. In particular, we will focus on the Transversality Theorem and its connections with intersection theory in homology. Even when not explicitly stated, by "manifold" we always mean "smooth manifold".

Definition 4.2.23 (Transversality for manifolds)**.** Let M be a manifold, N_1 and N_2 submanifolds of M.

We say that N_1 and N_2 *intersect transversely* if for every $x \in N_1 \cap N_2$, $T_x N_1 + T_x N_2 = T_x M$.

Definition 4.2.24 (Transversality for maps)**.** Let M_1 and M_2 be manifolds, $f : M_1 \to M_2$ a smooth map, N a submanifold of M_2.

We say that f is *transverse to* N, denoted $f \pitchfork N$, if for every $x \in f^{-1}(N)$ we have

$$T_{f(x)}N + \operatorname{im}(df_x) = T_x M_2.$$

Note that both these definitions allow for degenerate "empty" cases: if N_1 and

N_2 are disjoint then they intersect transversely, and if $N \cap \mathrm{im}(f) = \varnothing$ then $f \pitchfork N$.

In a torus (or more generally in a real Lie group), where we have a continuous operation available, it is easy to relate the two notions of transversality.

Proposition 4.2.25. *Let T be a real torus of dimension n, M and N submanifolds of T. For a fixed $t \in T$, let $f : M \to T$ be the function $m \mapsto m - t$.*

Then M and $t + N$ intersect transversely if and only if $f \pitchfork N$.

Proof. Just by unpacking the definitions: M and $t + N$ intersect transversely if and only if for every $x \in M \cap t + N$, $T_xM + T_x(t + N) = \mathbb{R}^n$.

On the other hand, since $f^{-1}(N) = M \cap t + N$, this is the same condition for $f \pitchfork N$: for every $x \in M \cap t + N$, $T_xN + \mathrm{im}(df_x) = T_xN + T_xM$, so it spans the whole space if and only if M and $t + N$ intersect transversely. □

We now state the powerful *Transversality Theorem.*

Theorem 4.2.26 (Transversality Theorem, [Hir76, Theorem 3.2.1])**.** *Let M, S, N be manifolds, and consider a smooth map $F : M \times S \to N$. We assume all these manifolds are without boundary.*

For $s \in S$, denote by $f_s : M \to N$ the map $f_s(m) := F(m, s)$.

If F is transverse to a submanifold $N' \subseteq N$, then for all s in an open dense subset of S, f_s is transverse to N'.

Note that all the manifolds we work with are without boundary.

We want to use this statement to obtain information on genericity of transverse intersections of submanifolds.

Corollary 4.2.27. *Let M, N be submanifolds of a torus T. There is an open dense subset $O \subseteq T$ such that for all $x \in O$, M and $x+N$ intersect transversely.*

Proof. Consider the map $F : M \times T \to T$ which maps (m, t) to $m - t$. The result then follows from the combination of Theorem 4.2.26, which ensures that $f_t \pitchfork N$ for t generic in T, and Proposition 4.2.25 which compares transversality of maps to transversality of manifolds. □

The importance of transverse intersections lies in their connection with homology theory. Namely, transversality represents a good notion of *general position*, by which we mean that given homology classes α and β, the homology class of the intersection of a representative of α and one of β does not depend on the choice of representative, as long as they intersect transversely. More precisely:

Theorem 4.2.28 ([Bre13, Theorem VI.11.9])**.** *Let M be a manifold, α and β homology classes in $H_*(M)$.*

Then there is a homology class γ such that if A and B are smooth submanifolds of M, lying in α and β respectively, which intersect transversely, then $A \cap B \in \gamma$.

This, however, poses a problem as not all homology classes are represented by submanifolds. We cannot ignore this issue as we are going to work with the homology classes of algebraic subvarieties of abelian varieties, and it may be possible that such algebraic subvarieties are singular.

However, when working with complex varieties we do have more tools at hand. For example, we know that complex algebraic varieties are *stratified*, in the sense that we can decompose them as unions of smooth sets of increasing dimension.

We recall the following famous result of Whitney. It is stated there as every complex variety having a regular stratification, here we expand on that.

Theorem 4.2.29 ([Whi15, Theorem 19.2])**.** *Let W be a complex algebraic variety of dimension d. Then there exist $W_0, \dots, W_d$ such that:*

1. *Each W_j is either empty or a smooth complex constructible set of dimension j;*
2. $W = W_0 \cup \dots \cup W_d$.

In simple words, Theorem 4.2.29 says that we can decompose any algebraic variety in a union of smooth (not necessarily closed) varieties of increasing dimension: the singular locus of the variety is still well-behaved from an algebraic and differential point of view, even though it consists of points where the variety has some local irregularity.

We are going to use stratifications together with the notion of *dimensional transversality* due to Goresky and Macpherson. This is weaker than usual transversality, but we will see how it is enough for our purposes.

Definition 4.2.30 ([GM83, Section 2.1])**.** Let M be a smooth manifold, and suppose A and B are cycles in M.

We say that A and B are *dimensionally transverse* if $\dim(A \cap B) \leq \dim A + \dim B - \dim M$.

Remark 4.2.31. The definition in the Goresky-Macpherson paper is actually more general than this, as it is developed in a setting in which the ambient space is allowed to have singularities. Therefore, they define dimensional transversality by requiring that the intersection is *allowable*, which roughly means that not only the intersection has the expected dimension, but moreover it interacts well with the singularities of the ambient space. Since we are only going to consider subvarieties of abelian varieties, and abelian varieties are smooth, this is not going to be a problem for us. Therefore we will stick with Definition 4.2.30 when discussing dimensional transversality.

The interest of dimensional transversality is, as the reader might by now expect, that if two cycles are dimensionally transverse then their intersection lies in a homology class which only depends on the homology classes of the cycles. This is the content of the following theorem.

Theorem 4.2.32 ([GM83, Theorem 1])**.** *Let M be a manifold of dimension n. There is a unique pairing*

$$\cdot : H_i(M) \times H_j(M) \to H_{i+j-n}(M)$$

such that if C and D are dimensionally transverse cycles with $C \in \gamma$ and $D \in \delta$, $C \cap D \in \gamma \cdot \delta$.

An interesting feature of this intersection product is that it corresponds to cup product under Poincaré duality. We do not define Poincaré duality here, but just recall that it is an isomorphism between homology and cohomology groups of a compact manifold. The interested reader may consult [Hat02, Section 3.3] for more details.

Lemma 4.2.33 ([GM83, Section 2.4])**.** *Let M be a manifold of dimension n, and suppose $P : H_j(M) \to H^{n-j}(M)$ denotes the Poincaré duality isomorphism.*

If $\alpha \in H_i(M)$ and $\beta \in H_j(M)$ are homology classes, then $P(\alpha \cdot \beta) = P(\alpha) \cup P(\beta)$.

We are going to use this in the setting of abelian varieties, and in particular in a situation in which the two cycles we are interested in are a subgroup and an algebraic subvariety.

Lemma 4.2.34. *Let A be an abelian variety of dimension g, $\mathbb{T}$ a closed subgroup of A, $W \subseteq A$ an algebraic variety of (complex) dimension d.*

Then there is an open dense subset O of A such that for all $a \in A$, $\mathbb{T}$ and $a + W$ are dimensionally transverse.

Proof. Consider a stratification $W = W_0 \cup \cdots \cup W_d$ of d. Since each W_j is smooth, by Corollary 4.2.27 there are open dense subsets $O_1, \ldots, O_d$ such that for each $a \in O_j$, $\mathbb{T}$ and $a + W_j$ are transverse.

Then their intersection has dimension $\dim_{\mathbb{R}} \mathbb{T} + 2j - 2g$ if it is non-empty (and then $\dim_{\mathbb{R}} \mathbb{T} + 2j - 2g \geq 0$). Since the W_j's are disjoint, it follows that if $a \in \bigcap_{j=0}^{d} O_j$

$$\dim(\mathbb{T} \cap a + W) = \dim \left(\bigcup_{j=0}^{d} \mathbb{T} \cap a + W_j \right) \leq \dim(\mathbb{T} \cap a + W_d) \leq \dim_{\mathbb{R}} \mathbb{T} + 2d - 2g$$

as we wanted.

Therefore we can take $O := \bigcap_{j=0}^{d} O_j$. □

4.2.4 Definability and O-Minimality

We conclude this section by introducing some background on o-minimal geometry. We are going to use these facts in the proof of the main result of this chapter, which needs definability of Hausdorff limits. As it is sort of out of tune with the rest of the work, and we expect the average reader of this thesis to have a background in model theory and thus be probably already familiar with o-minimality, we will not get too much into detail. The reader who wants to know more about o-minimal structures, not just on the real numbers, should consult the book [Dri98].

We follow Pila's approach from [Pil11] to introduce o-minimal structures on the reals.

Definition 4.2.35. An *o-minimal structure on $\mathbb{R}$ expanding the real field* is a collection $\{\mathcal{S}_n \mid n \in \omega\}$, where each $\mathcal{S}_n$ is a set of subsets of $\mathbb{R}^n$, which satisfies the following conditions:

1. Each $\mathcal{S}_n$ is a boolean algebra;

2. $\mathcal{S}_n$ contains every semi-algebraic subset of $\mathbb{R}^n$;

3. If $A \in \mathcal{S}_n$ and $B \in \mathcal{S}_m$, then $A \times B \in \mathcal{S}_{n+m}$;

4. If $m \geq n$, $\pi : \mathbb{R}^m \to \mathbb{R}^n$ is the projection on the first n coordinates, and $A \in \mathcal{S}_m$, then $\pi(A) \in \mathcal{S}_n$;

5. The boundary of every subset of $\mathcal{S}_1$ is finite.

O-minimal structures have played an important role in pure and applied model theory over the past thirty years: for example, in the aforementioned paper [Pil11] they were used to prove the André-Oort Conjecture. The structure which we will use is the structure $\mathbb{R}_{\text{an}}$.

Definition 4.2.36. The structure $\mathbb{R}_{\text{an}}$ is the smallest o-minimal structure on $\mathbb{R}$ expanding the real field which contains every globally subanalytic subset of $\mathbb{R}^n$ for each n.

As this is the only structure we are going to refer to, we will say that a set is *definable*, rather than $\mathbb{R}_{\text{an}}$-definable, if it lies in the structure.

The fact that this structure exists is due to Denef and van den Dries ([DD88, Section 4]). It is obviously a structure which is suitable to talk about complex abelian varieties, as $\exp_A$ is an analytic map on each neighbourhood of a fundamental domain and hence definable there. Note that then, in particular, we can treat closed subgroups of abelian varieties A as definable sets, as they are images under $\exp_A$ of the intersection of a linear space with finitely many fundamental domains.

Another important feature of definable set is that they admit triangulations. The following, which is in a sense a generalization of 4.2.10, is known as the *Triangulation Theorem* for definable sets.

Theorem 4.2.37 ([Dri98, Theorem 8.7.1])**.** *Every definable set admits a triangulation.*

Therefore we may treat definable sets as cycles, and integrate differential forms on them.

We are interested in Hausdorff limits of definable families of sets.

Definition 4.2.38. A definable family of definable sets is a family of sets of the form

$$\{S(x) \mid x \in X\}$$

where $S \subseteq \mathbb{R}^{m+n}$, $X \subseteq \mathbb{R}^n$ and

$$S(x) := \{s \in \mathbb{R}^m \mid (s, x) \in S\}$$

are all definable sets.

The tameness properties of o-minimal structures allow us to avoid pathological situations when taking Hausdorff limits of such definable collections. In particular, such limits do not "blow up" in the sense that the dimension does not increase. Recall that Hausdorff limits and Hausdorff distance have been introduced in Chapter 3 (see Definition 3.3.16). We slightly generalize that definition to take limits along definable curves.

Definition 4.2.39. Let $\mathcal{S} := \{S(x) \mid x \in X\}$ be a definable family of definable sets. Suppose $\gamma : [0, 1) \to X$ is a definable function. The *limit of* $\mathcal{S}$ along γ, if it exists, is a set S such that for any sequence $\{t_j\}_{j\in\omega} \subseteq [0, 1)$ converging to 1,

$$S = \lim_{j\in\omega} S(\gamma(t_j)).$$

The following result follows from a theorem of Marker and Steinhorn on definable types in o-minimal theories ([MS94, Theorem 2.1]), which was strengthened by Pillay in [Pil94, Corollary 2.4]. A version for semialgebraic sets was established by Bröcker in [Brö92, Corollary 2.8], while a direct geometric proof appears in [KPV14, Corollary 2]. We refer the reader also to [LS04, Theorem] and [Dri05, Theorem 3.1 and Proposition 3.2].

Theorem 4.2.40. *Let $\{S(x) \mid x \in X\}$ be a definable family of definable compact sets over the set $X \subseteq \mathbb{R}^n$. Suppose $\gamma : [0, 1) \to X$ is a definable function. Then*

$$S(1) := \lim_{t\to 1} S(\gamma(t))$$

exists, is a compact definable set, and

$$\dim S(1) \leq \lim_{t\to 1} \dim S(\gamma(t)).$$

We use Theorem 4.2.40 to establish that in the o-minimal setting intersection homology works even better than it usually does.

Lemma 4.2.41. *Let A be an abelian variety, $W \subseteq A$ an algebraic subvariety and $\mathbb{T}$ a closed subgroup of A.*

Then $W \cap \mathbb{T}$ contains a cycle which lies in the homology class $\{W\} \cdot \{\mathbb{T}\}$.

Proof. If W and $\mathbb{T}$ are dimensionally transverse (which in the smooth case coincides with the usual notion of transversality) then the lemma holds by the Goresky-Macpherson definition of the intersection pairing, Theorem 4.2.32.

By Lemma 4.2.34, this is the generic situation: there is an open dense subset O of A such that for every $a \in O$ the intersection $a + \mathbb{T} \cap W$ is dimensionally transverse.

Let $\gamma : [0, 1] \to A$ be a definable function with $\gamma([0, 1)) \subseteq O$ and $\gamma(1) = 0_A$. We can choose γ so that $\gamma(c_1) + \mathbb{T} \neq \gamma(c_2) + \mathbb{T}$ for all $c_1 \neq c_2 \in O$: it suffices to take $\gamma : [0, 1] \to O + \mathbb{T} \subseteq A/\mathbb{T}$, which is still a dense open set, and then lift it to a curve in the original space.

For any $c \in (0, 1)$, let

$$S_c := \bigcup_{0 \leq t \leq c} (\gamma(t) + \mathbb{T}) \cap W.$$

Since all intersections $(\gamma(t) + \mathbb{T}) \cap W$ are dimensionally transverse and pairwise disjoint, it is clear that

$$\dim_{\mathbb{R}} S_c = \dim_{\mathbb{R}} \mathbb{T} + \dim_{\mathbb{R}} W - 2g + 1$$

for every $c \in (0, 1)$, and that denoting by B_c the intersection $\gamma(c) + \mathbb{T} \cap W$ for $c \in [0, 1)$, $\partial S_c = B_0 - B_c$ as a chain.

Let now $S_1 = \lim_{c \to 1} S_c$ and $B_1 = \lim_{c \to 1} B_c$. These are both Hausdorff limits of definable families, and therefore by Theorem 4.2.40 the dimensions do not increase: thus $\dim_{\mathbb{R}} S_1 \leq \dim_{\mathbb{R}} S_c$, and as S_1 contains every S_c the dimensions need to be equal; $\dim_{\mathbb{R}} B_1 \leq \dim_{\mathbb{R}} B_c$, and a priori it could be that $\dim_{\mathbb{R}} B_1 < \dim_{\mathbb{R}} B_c$ (although a consequence of this lemma is that this is only possible if the B_c's are homologically trivial). Both S_1 and B_1 admit a triangulation by Theorem 4.2.37.

Claim: $\partial S_1 = B_0 - B_1$.

Proof of Claim: Consider the definable set

$$\{(s,c) \mid s \in S_c\}$$

whose fibre above each c is S_c. By a consequence of the Trivialization Theorem for o-minimal structures (see [Dri98, 9.2.1]) there is a simplicial complex $\mathcal{K}$ such that (after reparametrizing γ to change the starting point if necessary) for all $c \in (0,1)$, S_c is definably homeomorphic to $\mathcal{K}$. Consider the resulting simplices $\sigma_{j,c} : \Delta_j \to S_c$ (so each Δ_j is a face of the standard simplex and each $\sigma_{j,c}$ is a continuous function).

The boundary map on simplices preserves limits, in the sense that $\lim_{c\to 1} \partial\sigma_{j,c} = \partial\sigma_{j,1}$: this is clear as Δ_j is compact and $\sigma_{j,c}$ is continuous. Therefore,

$$\partial S_1 = \sum_{j=1}^{M} \partial\sigma_{j,1} = \lim_{c\to 1} \sum_{j=1}^{M} \partial\sigma_{j,c} = \lim_{c\to 1} (B_0 - B_c) = B_0 - B_1$$

so we are done.

This proves the claim.

Then B_1 is homologous to B_0, which is a dimensionally transverse intersection and therefore lies in the correct homology class. $\square$

Remark 4.2.42. Lemma 4.2.41 fails in general: after [Bre13, Theorem VI.11.10] an example is discussed in which the intersection of cycles A and B does not contain any cycle which lies in the product homology class, although every neighbourhood of the intersection does. In that case a curve of $\sin\left(\frac{1}{x}\right)$-type is used: this is a highly non-definable object.

4.3 Abelian Exponential-Algebraic Closedness

In this section we describe the problem of exponential-algebraic closedness in the setting of abelian varieties.

As in the case of the complex exponential, the question is whether it is possible to solve certain systems of equations involving polynomials and the exponential map of an abelian variety. The geometric interpretation is given again through notions of *freeness* and *rotundity* which are different in form, but not in spirit, from the corresponding notions in the case of complex exp.

Definition 4.3.1. Let A be an abelian variety of dimension g, $V \subseteq \mathbb{C}^g \times A$ an algebraic variety.

Let π_1 and π_2 denote the projections of $\mathbb{C}^g \times A$ on $\mathbb{C}^g$ and A respectively.

The variety V is *free* if $\pi_1(V)$ is not contained in a translate of the Lie algebra of a non-trivial abelian subvariety of A and $\pi_2(V)$ is not contained in a translate of a non-trivial abelian subvariety of A.

Given an abelian subvariety $B \leq A$, denote by π_B the projection $\pi_B : \mathbb{C}^g \times A \to \mathbb{C}^{g-\dim B} \times A/B$.

The variety V is *rotund* if $\dim \pi_B(V) \geq \dim A/B$ for every abelian subvariety B.

Note that since this definition quantifies over abelian subvarieties it becomes almost trivial in the case of a simple abelian variety. If A is simple, then $V \subseteq \mathbb{C}^g \times A$ is free if and only if $\pi_1(V)$ and $\pi_2(V)$ are both positive-dimensional, and it is rotund if and only if $\dim V \geq g$.

The conjecture is then that every free rotund algebraic variety intersects the graph of the exponential function. While it is not stated there in this form, it is implicit in [BK18, Sections 8 and 9].

Conjecture 4.3.2 (Abelian Exponential-Algebraic Closedness)**.** *Let A be an abelian variety of dimension g, $\exp_A : \mathbb{C}^g \to A$ its exponential map, $\Gamma_{\exp_A}$ the graph of $\exp_A$.*

For every free and rotund algebraic variety $V \subseteq \mathbb{C}^g \times A$, $V \cap \Gamma_{\exp_A} \neq \varnothing$.

Part of the motivation for this conjecture ties back to the Quasiminimality Conjecture: just as in the case of the complex exponential, we are interested in this conjecture because if established it would imply that a certain structure on the complex numbers is quasiminimal. In fact, Bays and Kirby have treated this matter in a uniform way, using the construction of Γ-*fields* (see [BK18, Section 3]): these are fields together with a predicate for a certain module which mimics the behaviour of the graph of a transcendental group homomorphism such as exp or the exponential of an abelian variety. One of the main results of [BK18] (Theorem 1.7 there) is that Γ-fields give rise to categorical quasiminimal structures, and the goal is to show that the corresponding structures on the complex numbers are isomorphic to Γ-fields.

We do not get into the details of this construction, but simply state its main consequence.

Theorem 4.3.3 ([BK18, Sections 8 and 9])**.** *Let A be a simple abelian variety. If Conjecture 4.3.2 holds for all powers of A, then the structure $(\mathbb{C}, +, \cdot, \Gamma_{\exp_A})$ is quasiminimal.*

Therefore, just like Exponential-Algebraic Closedness, Conjecture 4.3.2 has a model-theoretic motivation which ties it to the theory of quasiminimal structures.

As in the case of the complex exponential function, rotundity has a direct consequence on the geometry of the variety.

Definition 4.3.4. Let $V \subseteq \mathbb{C}^g \times A$ be an algebraic variety. The *δ-map* of the variety V is the function $\delta : V \to A$ defined by $(v_1, v_2) \mapsto v_2 - \exp_A(v_1)$.

As in the case of the complex exponential, we then have the following:

Lemma 4.3.5. *Let $V \subseteq \mathbb{C}^g \times A$ be a free rotund variety, and let $\delta : V \to A$ map $v = (v_1, v_2)$ to $v_2 - \exp_A(v_1)$.*

Then there is a Zariski-open dense subset $V^\circ \subseteq V$ such that for every $v \in V$, δ is open at v.

The proof of this result is quite similar to Kirby's proof of Fact 3.2.3, mentioned in the previous chapter.

Proof. Let $V \subseteq \mathbb{C}^g \times A$ and δ be as in the statement. δ is clearly a complex analytic map, and therefore its fibres are complex analytic sets: if there is $a \in A$ such that $\delta^{-1}(a)$ is a complex analytic subset of V of dimension $\dim V - g$, then the image of δ contains, around a, a g-dimensional complex analytic set, and in particular the map is open on some neighbourhood of any point $v \in \delta^{-1}(a)$.

The fibre $\delta^{-1}(a)$ coincides with the set

$$\{v \in V \mid \delta(v) = a\} = \{(v_1, v_2) \in V \mid v_2 - \exp_A(v_1) = a\} =$$

$$= \{(v_1, v_2) \in V \mid \exp(v_1) = v_2 - a\}$$

and therefore it is a translate of the intersection $\Gamma_{\exp_A} \cap (V - (0,a))$.

Consider the family of varieties $\mathcal{V} := \{V - (0,a) \mid a \in A\}$. By Kirby's Uniform Ax-Schanuel Theorem (Corollary 2.5.4 in this thesis) there is a finite collection of abelian subvarieties $\mathcal{C}$ such that any irreducible component of any intersection $(V-(0,a)) \cap \Gamma_{\exp_A}$ of dimension larger than $\dim V - g$ is contained in a translate of the tangent bundle of some $B \in \mathcal{C}$, $\gamma + (LB \times B)$.

If B is minimal with this property, then

$$\dim(\gamma + (LB \times B) \cap (V - (0,a))) > \dim V - g + \dim B.$$

To see this, consider the variety $V_B := (LB \times B) \cap (-\gamma + V - (0,a))$: as a subvariety of $LB \times B$, and by minimality of B, it must satisfy

$$\dim(V_B \cap \Gamma_{\exp_B}) = \dim V_B - \dim B$$

as if it did not we could apply the uniform Ax-Schanuel Theorem in B and obtain a smaller abelian subvariety. As $\dim(V_B \cap \Gamma_{\exp_B}) > \dim V - g$, we have $\dim V_B > \dim V - g + \dim B$.

By rotundity, if π_B denotes the quotient $\mathbb{C}^g \times A \twoheadrightarrow \mathbb{C}^{g - \dim B} \times A/B$ we must have

$$\dim \pi_B(V) \geq g - \dim B$$

and therefore for almost all $\gamma \in \mathbb{C}^g \times A$,

$$\dim(\gamma + (LB \times B) \cap (V - (0,a))) = \dim V - g + \dim B.$$

Therefore after removing a Zariski-closed proper subset of V we obtain a set $V^\circ \subseteq V$, Zariski-open dense in V, such that for every point in V and every abelian subvariety B we must have that

$$\dim(\gamma + (LB \times B) \cap (V - (0,a))) = \dim V - g + \dim B.$$

By the argument above, this implies that the δ-map of V is open at every point of V°: if it were not, then the dimension equality would not be satisfied for some abelian subvariety $B \subseteq A$. $\square$

As in the case of the complex exponential, when the variety has the form $L \times W$

we know more about the structure of the set V°.

Proposition 4.3.6. *If $L \leq \mathbb{C}^g$ is a linear space, $W \subseteq A$ is algebraic, and $L \times W$ is free and rotund, then there is a Zariski-open dense subset W° of W such that δ is open at every point of $L \times W^\circ$.*

Proof. It is clear that if δ is open at (l, w), then it is open at (l', w) for every $l' \in L$ (by translation along L, as in the proof of Proposition 3.2.4). Since we know the set of points where the map is open has to be Zariski-open, it can only take the form $L \times W^\circ$. □

As in the exponential case, Lemma 4.3.5 has a converse statement.

Lemma 4.3.7. *Let $L \leq \mathbb{C}^g$ be a linear space, $W \subseteq A$ an algebraic variety and δ the δ-map of $L \times W$. If there is a point $(l, w) \in L \times W$ such that δ is open at (l, w), then the variety $L \times W$ is rotund.*

Proof. Let $B \leq A$ be an abelian subvariety, and let $LB \cong \mathbb{C}^{g-\dim B} \leq \mathbb{C}^g$ be its tangent space at identity.

Let $\pi_B : \mathbb{C}^g \times A \twoheadrightarrow \mathbb{C}^{g-\dim B} \times A/B$ denote the quotient map. Consider also the partial quotients $p : \mathbb{C}^g \twoheadrightarrow \mathbb{C}^g/LB$ and $q : A \twoheadrightarrow B$. We need to show that $\dim \pi_B(L \times W) \geq \dim A - \dim B$.

If $U_L \subseteq L$ and $U_W \subseteq W$ are open subsets such that δ is open on $U_L \times U_W$, then we have that $\delta(U_L \times U_W)$ is an open subset of A; thus $q(\delta(U_L \times U_W))$ is an open subset of A/B.

Let δ_Q denote the δ-map of the variety $p(L) \times q(W)$. It is clear that $\delta_Q \circ \pi_B = q \circ \delta$, as in the proof of Proposition 3.2.5: therefore, $\delta_Q(\pi_B(U_L \times U_W))$ is an open subset of A/B. This implies that $\dim(\pi_B(U_L \times U_W)) \geq \dim(A/B)$, because the image of an analytic map cannot have larger dimension than its domain. So rotundity holds. □

In Lemma 3.2.6 we made a useful reduction, showing that we could assume in that context that the δ-map is open, without losing any generality. The technique in the proof of that lemma, the Rabinovich trick, is based on the fact that $\mathbb{C}^\times$ has a "hole" where 0 should be: using that, we were able to hide the whole subset of the variety on which δ was not open. Abelian varieties, however, are compact and in particular they do not have any "holes" in that sense: therefore there is no hope of repeating the same procedure, and we have to allow for varieties whose δ-map is open only locally.

We can however assume that $\dim L + \dim W = g$.

Lemma 4.3.8. *Let $L \times W$ be a free rotund variety in $\mathbb{C}^g \times A$. Then there is a space $L'' \subseteq L$ such that $L'' \times W$ is free and rotund, and* $\dim L'' + \dim W = g$.

Proof. The proof is exactly the same as the proof of Lemma 3.2.7, substituting $\exp_A$ for exp and its inverse for log. □

Therefore we will, in this chapter too, freely assume that $\dim L = \text{codim} W$ when we need it.

Note that we can use the characterization of rotundity in simple varieties to easily establish a partial result in that setting.

Proposition 4.3.9. *Let A be a simple abelian variety, $L \times W \subseteq \mathbb{C}^g \times A$ a free rotund variety, with $L \leq \mathbb{C}^g$ a linear space and $W \subseteq A$ a smooth algebraic variety.*

Then $\exp_A(L) \cap W \neq \varnothing$.

Proof. Let $\mathbb{T} = \overline{\exp_A(L)}$ (we will say more in Subsection 4.4.1 on what this subgroup has to look like). Let δ_L denote the δ-map of $L \times W$, and $\alpha_{\mathbb{T}} : \mathbb{T} \times W \to A$ the similar map $(t, w) \mapsto w - t$.

Since $\mathbb{T}$ and W are compact subsets of A, the image of $\alpha_{\mathbb{T}}$ is compact and hence closed; since A is simple and W is smooth, the Zariski-open subset of W on which δ_L is open coincides with W. This is because the set on which δ_L is not open was obtained in the proof of Lemma 4.3.5 as a finite union of sets, each corresponding to an abelian subvariety of A; but A does not have any non-trivial abelian subvarieties in this case.

If $a \in \text{im}(\alpha_{\mathbb{T}})$ then there are $(t, w) \in \mathbb{T} \times W$ such that $w - t = a$. Consider the variety $(t + L) \times W$: the δ-map of this variety is open around (t, w), and therefore its image contains a neighbourhood of a; this image is contained in $\text{im}(\alpha_{\mathbb{T}})$, which therefore also contains a neighbourhood of a. Therefore $\text{im}(\delta_{\mathbb{T}})$ is open and closed, so the map is surjective, and thus there is $w \in W \cap \mathbb{T}$.

Since δ_L is open around $(0, w)$, there are open neighbourhoods U_0 of 0 in L and U_w of w in W such that $\delta_L(U_0 \times U_w)$ is an open neighbourhood of w in A. Thus its intersection with $\mathbb{T}$ is an open neighbourhood of w in $\mathbb{T}$, and it contains points of $\exp_A(L)$ (as it is dense in $\mathbb{T}$). Therefore there are points $(l', w') \in L \times W$ such that $w' - \exp_A(l') \in \exp_A(L)$, and thus $w' \in W \cap \exp_A(L)$. □

There are not many results on this topic in the literature. The main one is due to Aslanyan, Kirby and Mantova, and it is an analogue of Theorem 2.3.20 in this context.

Theorem 4.3.10 ([AKM22, Theorem 1.4])**.** *Let A be an abelian variety of dimension g, $V \subseteq \mathbb{C}^g \times A$ an algebraic subvariety such that the projection of V to $\mathbb{C}^g$ has dimension g.*

Then $V \cap \Gamma_{\exp_A} \neq \varnothing$.

4.3.1 Ancient History

In this subsection we gather some old notions and results which predate exponential-algebraic closedeness, but are directly related to it.

Geometrically Non-Degenerate Subvarieties

First we discuss two notions of *non-degeneracy*, due to Ran, for subvarieties of abelian varieties. This material is from [Ran80].

We denote by $G(d, g)$ the *Grassmannian*, the set of all d-dimensional subspaces of $\mathbb{C}^g$. It is well-known that this can be given the structure of a complex projective variety.

Definition 4.3.11. Let A be an abelian variety of dimension g, $W \subseteq A$ an algebraic subvariety of dimension d.

The *Gauss map* of W is the map $\gamma : W^{\mathrm{Reg}} \to G(d, g)$ which maps each point $w \in W$ to the tangent space $T_w W$.

Definition 4.3.12. Let $W \subseteq A$ be an algebraic variety. Let ν be the standard embedding of the Grassmannian in a projective space. W is *non-degenerate* if the image $\nu(\gamma(W))$ of its Gauss map is not contained in any hyperplane.

It is useful to characterize non-degeneracy in terms of differential forms.

Lemma 4.3.13 ([Ran80, Lemma II.1])**.** *Let $W \subseteq A$ be an algebraic subvariety of an abelian variety, with embedding $\iota : W \hookrightarrow A$. Then W is non-degenerate* *if and only if the pullback mapping* $\iota^* : H^{d,0}(A) \to H^{d,0}(W)$ *on forms of degree* $(d,0)$ *is injective.*

We are interested in a stronger notion of non-degeneracy, with a different constraint on the kernel of the pullback on forms.

Definition 4.3.14. A variety $W \subseteq A$ is *geometrically non-degenerate* if the kernel of the pullback $\iota^* : H^{d,0}(A) \to H^{d,0}(W)$ does not contain any decomposable form.

The following characterization of geometrical non-degeneracy suggests that this notion is related to rotundity.

Lemma 4.3.15 ([Ran80, Lemma II.12])**.** *A variety* $W \subseteq A$ *is geometrically non-generate if and only if for every abelian subvariety* B *of* A*, with quotient map* $\pi_B : A \to A/B$*,*

$$\dim \pi_B(W) = \min\{\dim W, \dim A/B\}.$$

We will make this connection explicit using another characterization of geometrical non-degeneracy. Ran states it without proof, but we prefer to expand on it for clarity.

Proposition 4.3.16. *A variety* $W \subseteq A$ *is geometrically non-degenerate if and only if there is no subspace* $L \leq \mathbb{C}^g$ *with* $\dim L = \mathrm{codim} W$ *such that* $L \cap T_w W \neq \langle 0 \rangle$ *for every* $w \in W^{\mathrm{Reg}}$*.*

Proof. Suppose $W \subseteq A$ is geometrically non-degenerate, and let $L \leq \mathbb{C}^g$ be a subspace with $\dim L = \mathrm{codim} W$. We know how to associate to L the form ω_L, which is a decomposable (d,d)-form; after multiplying it by a scalar if necessary, we can write it as $\omega \wedge \overline{\omega}$ for some $(d,0)$-form ω. Since W is geometrically non-degenerate, the pullback $\iota^*(\omega)$ of ω to W is non-zero and therefore, by Proposition 4.2.14, $\int_W \omega_L = \int_W \iota^*(\omega) \wedge \iota^*(\overline{\omega}) \neq 0$, which implies by Proposition 4.2.21 that L intersects non-trivially some tangent space to W.

For the converse, assume that any space L of dimension equal to $\mathrm{codim} W$ intersects transversely at least one tangent space to W. Then given any decomposable holomorphic $(d,0)$-form ω, we write it as $dz_1 \wedge \cdots \wedge dz_d$ for some basis vectors $z_1, \ldots, z_d$. The space L generated by $\{z_1, \ldots, z_d\}$ then has to intersect W transversely at at least one point, and therefore the pullback of the form is non-zero. □

If L is as described in Proposition 4.3.16, we say it *witnesses geometrical degeneracy* of W.

We can then give two separate proofs of the following statement.

Lemma 4.3.17. *Let $L \leq \mathbb{C}^g$, $W \subseteq A$ with $\dim L = \mathrm{codim} W$. Then:*

1. *If W is geometrically non-degenerate, $L \times W$ is rotund;*
2. *If $L \times W$ is rotund then L does not witness geometrical degeneracy of W.*

Proof. The first proof uses Lemma 4.3.7: if W is geometrically non-degenerate, then there is a smooth point $w \in W$ such that $L + T_w W = \mathbb{C}^g$. Clearly for such a point the δ-map of $L \times W$ is open at $(0, w)$, and therefore the variety $L \times W$ needs to be rotund.

For the second statement, we have that if rotundity holds then for almost every $w \in W$ the δ-map is open at $(0, w)$, and thus there must be at least one for which $T_w W$ and L intersect transversely, as we want.

The second proof only uses the definition of rotundity with dimensional inequalities and Ran's work. In fact, in Footnote 2 of [Ran80] he notices how if W is geometrically degenerate then there must be B such that $\dim B - \dim(L \cap T_0 B) < \dim W - \dim \pi(W)$, where $\pi : A \to A/B$ is the projection. But then consider the following:

$$\dim L + \dim B - \dim(L \cap T_0 B) < \dim L + \dim W - \dim \pi(W)$$

$$\dim L - \dim(L \cap T_0 B) + \dim \pi(W) < \dim L + \dim W - \dim B$$

$$\dim \pi_B(L \times W) < \dim A - \dim B$$

which contradicts the definition of rotundity. □

This shows that while rotundity originated in Zilber's work as a model-theoretic notion, similar concepts were already discussed in the literature on the geometry of subvarieties of abelian varieties.

Zeros of Theta Functions

We now move to a different topic and discuss a result of Ax, from [Ax72b].

We recall that a *theta function* for the lattice Λ is an entire function $\theta : \mathbb{C}^g \to \mathbb{C}$ which satisfies a functional equation of the form

$$\theta(z+\lambda) = \exp(g_\lambda(z))\theta(z)$$

for every $\lambda \in \Lambda$, where exp denotes the complex exponential function (**not** the exponential map of the abelian variety isomorphic to $\mathbb{C}^g/\Lambda$) and g_λ is an affine function.

The relevance of theta functions lies in the fact that for any such θ, by the functional equation the zero locus

$$Z_\theta := \{z \in \mathbb{C}^g \mid \theta(z) = 0\}$$

is invariant under translation by points in the lattice. This means that $\exp_A(Z_\theta)$ is a closed analytic subset of the abelian variety $A \cong \mathbb{C}^g/\Lambda$: as abelian varieties are projective, it is therefore an algebraic hypersurface in A. More is actually true: all algebraic hypersurfaces in A can be represented this way.

Theorem 4.3.18 (Poincaré, [HS00, Theorem A.5.2.2]). *For every algebraic hypersurface $H \subseteq A$ there is a theta function θ such that $H = \exp_A(Z_\theta)$.*

Poincaré's theorem is actually more general - it applies to analytic divisors on complex tori, with no need for algebraicity.

This means, in particular, that if an algebraic subvariety $W \subseteq A$ is a complete intersection (the intersection of $\operatorname{codim} W$ hypersurfaces) then $\log W$ is the intersection of $\operatorname{codim} W$ zero loci of theta functions.

This ties the exponential-algebraic closedness problem to the following results of Ax.

Theorem 4.3.19 ([Ax72b, Theorem 1]). *Let θ be a reduced (see [Wei58]) theta function for the lattice Λ, L a complex subspace of $\mathbb{C}^g$ of dimension 1. If the restriction of θ to L is not constant, then it has infinitely many zeros.*

More generally:

Theorem 4.3.20 ([Ax72b, Corollary]). *Let $\theta_1, \dots, \theta_d$ be theta functions with* *respect to a lattice Λ for which the variety $A \cong \mathbb{C}^g/\Lambda$ is a simple abelian variety.*

Then $\theta_1, \ldots, \theta_d$ have infinitely many common zeros on each linear subspace of $\mathbb{C}^g$ of dimension d.

Clearly, Theorems 4.3.19 and 4.3.20 are related to exponential-algebraic closedness.

Theorem 4.3.19 can be seen as studying sets of the form $\exp_A(L) \cap W$, where L is a one-dimensional subspace of $\mathbb{C}^g$ and W a hypersurface. If θ is constant on L, then L is contained in a translate of $\exp_A^{-1}(W)$: this means that $L \times W$ is not rotund (actually, the Ax-Lindemann-Weierstrass Theorem 2.5.6 for abelian varieties implies that $\exp(L)$ is contained in an abelian subvariety of A and therefore the variety is also not free). The theorem thus states that the only possible obstruction to finding zeros of a theta function on a one-dimensional linear space amounts to a failure of freeness and rotundity, as we want.

As for Theorem 4.3.20, it essentially says that in simple varieties it is easy to establish exponential-algebraic closedness for varieties of the form $L \times W$ (although it only deals with the case in which W is a complete intersection). This is similar to what we did in Proposition 4.3.9.

4.4 Abelian E.A.C. for Varieties of the form $L \times W$

In this section we prove the main result of the chapter.

Theorem 4.4.1. *Let A be an abelian variety of dimension g, $L \times W$ a free rotund variety with $L \leq \mathbb{C}^g$ a linear space and $W \subseteq A$ an algebraic variety.*

Then $\exp_A(L) \cap W \neq \varnothing$.

We will prove Theorem 4.4.1 in full generality in Subsection 4.4.2. Before we get there, we will give a simpler proof for a special case: the reader may guess that if $\exp_A(L)$ is dense in A then it is somehow easier to find intersections between it and W, as there are points of $\exp_A(L)$ in any open subset of A. We will see that this is in fact the case.

4.4.1 The Dense Case

We begin by characterizing which subspaces of $\mathbb{C}^g$ have dense exponential. This characterization is not new as it appears implicitly for example in [UY18a, Section 3], but we work out the precise statements.

As in Chapter 3, we endow $\mathbb{C}^g$ with the Hermitian product defined for $z = (z_1, \dots, z_g)$ and $w = (w_1, \dots, w_g)$ as $\langle z, w\rangle = z_1\overline{w}_1 + \cdots + z_g\overline{w}_g$. As in the previous chapter (see Remark 3.6.2) we note that $\mathrm{Re}\,(\langle z, w\rangle)$ coincides with the scalar product on $\mathbb{R}^{2g} \cong \mathbb{C}^g$, under the usual identification of $(x_1+iy_1, \dots, x_g+iy_g)$ with $(x_1, y_1, \dots, x_n, y_n)$.

We use this to define the dual of a lattice.

Definition 4.4.2. Let Λ be a lattice of rank $2g$ in $\mathbb{C}^g$. The *dual lattice* Λ^* is the lattice defined by

$$\Lambda^* := \{\theta \in \mathbb{C}^g | \mathrm{Re}\,(\langle \lambda, \theta\rangle) \in \mathbb{Z}\ \forall \lambda \in \Lambda\}.$$

We use the dual lattice to describe the real hyperplanes of $\mathbb{C}^g$ whose image in $\mathbb{C}^g/\Lambda$ is closed.

Lemma 4.4.3 (See [UY18a, Section 3])**.** *Let H be a real hyperplane in $\mathbb{C}^g$. Then $H + \Lambda$ is closed in $\mathbb{C}^g/\Lambda$ if and only if H can be defined by an equation of the form* $\mathrm{Re}\left(\sum_{i=1}^g \overline{\theta}_i z_i\right) = 0$ *for $\theta = (\theta_1, \dots, \theta_g) \in \Lambda^*$.*

Proof. As a real torus, $\mathbb{C}^g/\Lambda$ is isomorphic to $\mathbb{R}^{2g}/\mathbb{Z}^{2g}$. It is well-known that the hyperplanes H of $\mathbb{R}^{2g}$ which have closed image in the quotient are those defined by a $\mathbb{Q}$-linear equation, as these have a set of generators in $\mathbb{Z}^{2g}$; in other words, $H \cap \mathbb{Z}^{2g}$ is a lattice in H. Hence, in the setting of $\mathbb{C}^g/\Lambda$ we must characterize the spaces H for which $H \cap \Lambda$ is a lattice in H.

So suppose $H \cap \Lambda$ is a lattice in H. Then H contains $2g-1$ linearly independent elements of Λ, and there is an element of Λ^* which is orthogonal to each of those for the real scalar product. Then by Remark 3.6.2 we have the claim.

Conversely, suppose H is defined by an equation of the form above. Then H is orthogonal to a space defined by an element of the dual, and so it is generated by points in the lattice. ☐

This property easily translates to complex hyperplanes.

Corollary 4.4.4. *Suppose L is a complex hyperplane of $\mathbb{C}^g$ which cannot be defined by an equation of the form $\sum_{i=1}^g \overline{\theta}_i z_i = 0$ for $\theta = (\theta_1, \dots, \theta_g) \in \Lambda^*$ (we* *say such an hyperplane is not defined over the conjugate of the dual lattice). Then $L + \Lambda$ is dense in $\mathbb{C}^g/\Lambda$.*

Proof. If a complex linear space satisfies $\mathrm{Re}\left(\sum_{i=1}^{g} \overline{\theta}_i z_i\right) = 0$ for coefficients in Λ^*, then it must satisfy $\mathrm{Im}\left(\sum_{i=1}^{g} \overline{\theta}_i z_i\right) = 0$, and hence the full equation $\sum_{i=1}^{g} \overline{\theta}_i z_i = 0$. Hence if a complex linear hyperplane does not have dense exponential, it is defined over the conjugate of the dual lattice. □

Thus it is trivial to obtain the following lemma, which clears the picture for general complex linear spaces.

Lemma 4.4.5. *Let L be a complex linear subspace of $\mathbb{C}^g$. Then $L + \Lambda$ is dense in $\mathbb{C}^g/\Lambda$ if and only if L is not contained in a complex hyperplane defined over the conjugate of Λ^*.*

We provide explicit computations in the case of powers of an elliptic curve $E \cong \mathbb{C}/\Lambda$, where Λ is a lattice of the form $\mathbb{Z} + \tau\mathbb{Z}$ for some complex number $\tau = a + ib$.

Identifying $\mathbb{C}$ with $\mathbb{R}^2$, the lattice Λ is generated by the points $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} a \\ b \end{pmatrix}$. Clearly then a pair of generators of Λ^* is given by $\begin{pmatrix} 0 \\ \frac{1}{b} \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -\frac{a}{b} \end{pmatrix}$. These correspond to the complex numbers $-\frac{i}{b}$ and $1 + \frac{ia}{b}$. Note that, multiplying both for ib, we obtain 1 and $-a + ib = -\overline{\tau}$, which generate the conjugate of Λ. Hence, in the case of powers of an elliptic curve, the conjugate of the dual lattice is actually the original lattice up to multiplication by a scalar, and Lemma 4.4.5 takes the following form.

Corollary 4.4.6. *Let $E \cong \mathbb{C}/\Lambda$ be an elliptic curve, $L \leq \mathbb{C}^g$ a complex linear space,* $\exp$ *the exponential map of the abelian variety E^g. Then* $\exp(L)$ *is dense in E^g if and only if L is not contained in an hyperplane defined over Λ.*

Recall that an elliptic curve E is said to have *Complex Multiplication* when its endomorphism ring is strictly larger than $\mathbb{Z}$, and that these are characterized as the curves whose period lattice has the form $\mathbb{Z}+\tau\mathbb{Z}$ for τ an imaginary quadratic number (see [HS00, Example A.5.1.3]). Corollary 4.4.6 has a particularly nice interpretation in this case. In fact, if E has CM, linear spaces defined over the lattice are Lie algebras of abelian subvarieties of E^g, and the corollary can be seen as stating that all free complex hyperplanes have dense exponentials.

We now move to the proof of Theorem 4.4.1 for spaces which dense exponential.

Theorem 4.4.7. *Let A, L, W be as in Theorem 4.4.1, and assume moreover that L is not contained in hyperplanes defined over the conjugate of the dual lattice.*

Then $\exp_A(L) \cap W \neq \varnothing$.

Proof. Let $\delta : L \times W \to A$ be the δ-map of $L \times W$. By Lemma 4.3.5, there is $(l_0, w_0) \in L \times W$ such that δ is open at (l_0, w_0). Then the image of δ contains an open set U; since $\exp_A(L)$ is dense there is $l \in L$ such that $\exp_A(l) \in \operatorname{im}(\delta) \cap \exp_A(L)$.

Thus we may find $(l_1, w_1) \in L \times W$ such that $w_1 - \exp_A(l_1) = \exp_A(l)$. But then $w_1 = \exp_A(l) + \exp_A(l_1) \in \exp_A(L)$. □

4.4.2 The General Case

In this subsection we prove Theorem 4.4.1 in full generality. Therefore, for the rest of the subsection, even when not explicitly specified the following notations will be used:

1. A is an abelian variety of dimension g with exponential map $\exp_A : \mathbb{C}^g \to A$;
2. $L \times W$ is a free rotund subvariety of $\mathbb{C}^g \times A$, with $L \leq \mathbb{C}^g$ linear and $W \subseteq A$ an algebraic subvariety;
3. $\mathbb{T} = \overline{\exp_A(L)}$ is the closure of the exponential of L in A; it is a closed subgroup of A, and T denotes the real subspace of $\mathbb{C}^g$ such that $\exp_A(T) = \mathbb{T}$;
4. $\delta : L \times W \to A$ is the δ-map of $L \times W$, which maps (l, w) to $w - \exp_A(l)$;
5. The differential forms ω_L and ω_T are attached to the spaces L and T respectively as in Subsection 4.2.2

We need to combine the results from Sections 4.2 and 4.3.

Proposition 4.4.8. *Let $L \times W$ be a free rotund variety, and assume* $\dim L + \dim W = g$. *Then* $\int_W \omega_L \neq 0$.

Proof. By Lemma 4.3.5, the δ-map of $L \times W$ is open on a set of the form $L \times W^\circ$ with $W^\circ \subseteq W$ Zariski-open dense. This implies that there exists at least one point $w \in W$ such that $T_w W + L = \mathbb{C}^g$.

Therefore, by Proposition 4.2.21, $\int_W \omega_L \neq 0$. □

We now study the interaction between the differential forms ω_T and ω_L. Letting $d = \dim W = \operatorname{codim} L$, we know that ω_L is a form of degree (d, d), and since $L \leq T$ the degrees of T need to be larger than those of L.

Proposition 4.4.9. *There is a real vector subspace $T' \leq \mathbb{C}^g$ such that $L = T \cap T'$, and $\omega_L = \omega_T \wedge \overline{\omega}_{T'}$.*

Proof. One may write a base $\{v_1, \dots, v_g\}$ of $\mathbb{C}^g$ such that the corresponding real basis $\mathcal{B} = \{v_1, iv_1, \dots, v_g, iv_g\}$ consists of bases $\mathcal{B}_1 \subseteq \mathcal{B}_2 \subseteq \mathcal{B}$ of L, T, and $\mathbb{C}^g$ respectively, and such that $\mathcal{B}_1 \cup (\mathcal{B} \setminus \mathcal{B}_2)$ is a basis of T'.

This way, using Proposition 4.2.19 one sees that ω_L is $d(\mathcal{B} \setminus \mathcal{B}_1)$, while $\omega_T = d(\mathcal{B} \setminus \mathcal{B}_2)$ and $\overline{\omega}_{T'} = d(\mathcal{B} \setminus (\mathcal{B}_1 \cup (\mathcal{B} \setminus \mathcal{B}_2)))$ (to see it is the form $\overline{\omega}_{T'}$ rather than $\omega_{T'}$ one has to compare the degrees).

Since clearly

$$(\mathcal{B} \setminus \mathcal{B}_2) \cup (\mathcal{B} \setminus (\mathcal{B}_1 \cup (\mathcal{B} \setminus \mathcal{B}_2))) = \mathcal{B} \setminus \mathcal{B}_1$$

we have that then $\omega_T \wedge \overline{\omega}_{T'} = \omega_L$. □

Moreover, we note that the cohomology class of ω_T and the homology class of $\mathbb{T}$ are essentially Poincaré duals.

Proposition 4.4.10. *The cohomology class of ω_T is a scalar multiple of the Poincaré dual of the homology class of $\mathbb{T}$.*

Proof. This is essentially [BL04, Lemma 4.10.2]. □

We are now ready to examine the consequences of freeness and rotundity on the homological level.

Lemma 4.4.11. *Let $L \times W$ be a free rotund variety, and suppose the intersection $\mathbb{T} \cap W$ is dimensionally transverse.*

Then $\int_{\mathbb{T} \cap W} \overline{\omega}_{T'} \neq 0$.

Proof. By the cup product-intersection duality, we know that (up to multiplying be a scalar if necessary)

$$\int_{\mathbb{T}\cap W} \overline{\omega}_{T'} = \int_A [W] \wedge \omega_T \wedge \overline{\omega}_{T'} = \int_W \omega_L \neq 0$$

where $[W]$ is used to denote any form lying in the cohomology class that is dual to the homology class of W. We have used Proposition 4.4.10 to note that $[W] \wedge \omega_T = [W \cap \mathbb{T}]$. □

By Lemma 4.2.41, even when $\mathbb{T} \cap W$ is not a dimensionally transverse intersection it contains a cycle which lies in the product homology class. Therefore we actually have the following.

Proposition 4.4.12. *Let $L \times W$ be a free rotund variety. Then there are a cycle $C \subseteq \mathbb{T} \cap W$ and a point $c \in C$ such that $T_c C \cap T' = \langle 0 \rangle$.*

Proof. By Lemmas 4.4.11 and 4.2.41, $W \cap \mathbb{T}$ contains a cycle C such that $\int_C \overline{\omega}_{T'} \neq 0$. By Proposition 4.2.20, this implies that there is $c \in C$ such that $T_c C$ and T' intersect transversely. □

We are then ready to prove the main theorem of this chapter, Theorem 4.4.1.

Proof of Theorem 4.4.1. By Lemma 4.3.8, we may assume without loss of generality that $\dim L + \dim W = g$.

Since $L \times W$ is free and rotund, by Proposition 4.4.12 there are a cycle $C \subseteq \mathbb{T} \cap W$ and a point $c_0 \in C$ such that $T_{c_0} C + T' = \mathbb{C}^g$. Note that

$$\dim_{\mathbb{R}} C = \dim_{\mathbb{R}} T + \dim_{\mathbb{R}} W - 2g$$

and

$$\dim_{\mathbb{R}} T' = 2g - \dim_{\mathbb{R}} T + \dim_{\mathbb{R}} L$$

therefore

$$\dim_{\mathbb{R}} T_{c_0} C + \dim_{\mathbb{R}} T' = \dim_{\mathbb{R}} W + \dim_{\mathbb{R}} L = 2g.$$

Thus, $T_{c_0} C \cap T' = \langle 0 \rangle$. Since $L \leq T'$, this implies in particular that $T_{c_0} C \cap L = \langle 0 \rangle$.

Therefore,

$$\dim_{\mathbb{R}} (T_{c_0} C + L) = \dim_{\mathbb{R}} T_{c_0} C + \dim_{\mathbb{R}} L =$$

$$= \dim_{\mathbb{R}} T + \dim_{\mathbb{R}} W - 2g + \dim_{\mathbb{R}} L = \dim_{\mathbb{R}} T.$$

As both spaces are contained in T, then, we have that $T_{c_0}C + L = T$.

This means that there is a neighbourhood $U \subseteq C$ such that $U + \exp_A(L)$ is open in $\mathbb{T}$. Therefore, since $\exp_A(L)$ is dense in T, there is $(l, c) \in L \times C$ such that $c - \exp_A(l) \in \exp_A(L)$; and therefore $c \in \exp_A(L) \cap C \subseteq \exp_A(L) \cap W$. $\square$

Finally, we improve the result to show that actually the intersection between $\exp_A(L)$ and W is Zariski-dense in W.

To do this, we are going to need a few preliminary results.

Proposition 4.4.13. *Let $L \times W$ be a free rotund variety with* $\dim L + \dim W = g$. *Then there is $w \in \exp_A(L) \cap W$ such that $T_w W + L = \mathbb{C}^g$.*

Proof. Assume again, using Lemma 4.3.8, that $\dim L + \dim W = g$.

Theorem 4.4.1 implies that there is $w_0 \in \exp_A(L) \cap W$ such that δ is open around $(0, w_0)$. Let $U \subseteq W$ be a neighbourhood of w_0 such that δ is open on $L \times U$. In every neighbourhood of w_0 contained in U there are points of $\exp_A(L)$, by density of $\exp_A(L)$ in its closure: therefore, $\exp_A(L) \cap U$ is a countable set with no isolated points.

Assume that for every point $w \in \exp_A(L) \cap U$, $L + T_w W \neq \mathbb{C}^g$. Clearly the set of points $w \in W$ for which $L + T_w W \neq \mathbb{C}^g$ is closed: thus, since $\exp_A(L) \cap U$ is countably infinite, w_0 lies in a positive-dimensional set of points with this property, and therefore δ cannot have discrete fibres and be open around $(0, w_0)$. Therefore there must be a point w arbitrarily close to w_0 with $T_w W + L = \mathbb{C}^g$: in other words, if $T_{w_0} W + L \neq \mathbb{C}^g$ but δ is open at $(0, w_0)$, then w_0 is isolated in the set of points with this property. $\square$

Recall that a subspace $S \leq \mathbb{C}^n$ is *totally real* if $S \cap iS = \langle 0 \rangle$, and that a submanifold N of a complex manifold M is totally real if $T_n N$ is totally real in $T_n M$ for every $n \in N$.

We then consider the following result on complex manifolds. It can be seen as a geometric version of Proposition 3.5.10, which was concerned with holomorphic functions which vanish on a specific totally real submanifold of $\mathbb{C}^n$, i.e. $\mathbb{R}^n$.

Proposition 4.4.14. *Let M be a complex manifold with $\dim M = n$, $N \subseteq M$ a totally real submanifold with $\dim_{\mathbb{R}} N = n$, and $f : M \to \mathbb{C}$ a holomorphic function.*

If f vanishes on N then $f \equiv 0$ on M.

Proof. Suppose f vanishes on N. Consider the complex submanifold $M' \subseteq M$ defined by $\{z \in M \mid f(z) = 0\}$

Then for every $n \in N$, $T_n N \subseteq T_n M'$. Since $T_n N$ is a totally real subspace of real dimension n, it cannot be contained in a proper complex subspace of $T_n M$: therefore, $T_n M' = T_n M$ and as a consequence $M' = M$. □

Lemma 4.4.15. *Let $L \times W$ be a free rotund variety. Suppose T is the real subspace of $\mathbb{C}^g$ such that $\exp_A(T) = \overline{\exp_A(L)}$, and let $L := T + iT$ be the smallest complex subspace of $\mathbb{C}^g$ which contains T.*

Then every holomorphic function $f : W \to A$ which vanishes on $\exp_A(L) \cap W$ vanishes on $\exp_A(L_1) \cap W$.

Proof. As usual we assume for simplicity that $\dim L + \dim W = g$: if it is not, we intersect L with generic hyperplanes to lower its dimension.

Let $c := \dim_{\mathbb{R}}(T \cap iT) - \dim_{\mathbb{R}} L$. Then

$$\dim_{\mathbb{R}} T = \frac{\dim_{\mathbb{R}} L_1 + c + \dim_{\mathbb{R}} L}{2}$$

and therefore at a point $w \in W \cap \exp_A(L)$ where $T_w W + L = \mathbb{C}^g$ (as given by Proposition 4.4.13) we have

$$\dim_{\mathbb{R}} T_w W \cap T = \dim_{\mathbb{R}} W + \dim L_1 + \frac{c + \dim_{\mathbb{R}} L}{2} - 2g =$$

$$= \frac{\dim_{\mathbb{R}} W}{2} + \frac{c}{2} + \dim L_1 - g.$$

Note that

$$\dim_{\mathbb{R}}(T_w W \cap T \cap iT) = \dim_{\mathbb{R}} T_w W + \dim_{\mathbb{R}}(T \cap iT) - 2g =$$

$$= \dim_{\mathbb{R}} T_w W + c + \dim_{\mathbb{R}} L - 2g = c$$

and

$$\dim_{\mathbb{R}}(T_wW \cap T) + i(T_wW \cap T) = \dim_{\mathbb{R}} W + c + \dim_{\mathbb{R}} L_1 - 2g - c =$$

$$= \dim_{\mathbb{R}} W + \dim_{\mathbb{R}} L_1 - 2g = \dim_{\mathbb{R}}(T_wW \cap L_1)$$

so $T_wW \cap T$ is a totally real subspace of $T_wW \cap L_1$ of half its dimension, so it is not contained in any proper complex subspace. Therefore there is a neighbourhood $U \subseteq L_1$ of 0 such that $\exp_A(U)$ is complex manifold, and $W \cap \mathbb{T}$ is at least locally a totally real submanifold of $W \cap \exp_A(U)$, of dimension $\frac{\dim_{\mathbb{R}}(W \cap U)}{2}$ ("locally" in the sense that there is a neighbourhood $U' \subseteq W$ of w such that $U' \cap \mathbb{T}$ is a totally real submanifold of the complex manifold $U' \cap \exp_A(U)$). Thus by Proposition 4.4.14, there is no holomorphic function which vanishes on $W \cap \mathbb{T}$ unless it vanishes on $W \cap U$, and thus on $W \cap \exp_A(L_1)$. $\square$

Lemma 4.4.15 is sufficient to establish Zariski-density of $\exp_A(L) \cap W$ in W in the case in which $T + iT = \mathbb{C}^g$. To prove the more general case we need an inductive argument.

Theorem 4.4.16. *Let $L \times W$ be a free rotund variety. Then $\exp_A(L) \cap W$ is Zariski-dense in W.*

Proof. If $T + iT = L_1 \lneq \mathbb{C}^g$, then let $k = k(L)$ be the length of the following chain of inclusions:

$$L = L_0 \leq T = T_0 \leq L_1 \leq T_1 \leq \cdots \leq L_{k-1} \leq T_{k-1} \leq L_k = \mathbb{C}^g$$

where each $L_{j+1} = T_j + iT_j$ is the smallest complex subspace of $\mathbb{C}^g$ that contains T_j and each T_{j+1} is the real subspace of $\mathbb{C}^g$ such that $\overline{\exp_A(L_j)} = \exp_A(T_{j+1})$. Note that all the inclusions are proper, except possibly the last one: if there is a j such that $T_{j-1} = L_j$ then necessarily $T_{j-1} = \mathbb{C}^g$ or the variety $L \times W$ would not be free.

Lemma 4.4.15 then shows that every holomorphic function which vanishes on $W \cap \exp_A(L)$ vanishes on $W \cap \exp_A(L_1)$: this is the $k = 1$ case. Since k is finite, repeated applications of the lemma show that no algebraic function can vanish on $\exp_A(L) \cap W$, which is therefore Zariski-dense in W. $\square$

4.4.3 An Example

We present an example of a variety of the form $L \times W$; as it is necessary to introduce more machinery than usual, we devote a whole subsection to this.

Recall (for example from [Hid13, Section 2.3.4]) that for every elliptic curve $E \cong \mathbb{C}/\Lambda$ there is a meromorphic function $\wp : \mathbb{C} \setminus \Lambda \to \mathbb{C}$, Λ-periodic with double poles at each point of Λ such that E may be embedded in the projective space $\mathbb{P}^2(\mathbb{C})$ as the set of all points of the form $\begin{bmatrix} 1 \\ \wp(z) \\ \wp'(z) \end{bmatrix}$ for $z \in \mathbb{C} \setminus \Lambda$ together with the point at infinity $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. In other terms, the exponential map of E has the form

$$z \mapsto \begin{bmatrix} 1 \\ \wp(z) \\ \wp'(z) \end{bmatrix}$$

(the fact that it is a group homomorphism is [Hid13, Theorem 2.5.1]). The function $\wp$ is called a *Weierstrass $\wp$-function* for E.

Consider then the lattices $\Lambda_1 = \mathbb{Z} + i\sqrt{2}\mathbb{Z}$ and $\Lambda_2 = \mathbb{Z} + i\sqrt{5}\mathbb{Z}$. As $\sqrt{2}$ does not lie in the $\mathbb{Q}$-linear span of $\sqrt{5}$, the elliptic curves $E_1 \cong \mathbb{C}/\Lambda_1$ and $E_2 \cong \mathbb{C}/\Lambda_2$ are not isogenous. Let $\wp_1$ and $\wp_2$ denote the relative Weierstrass $\wp$-functions, so that for $j = 1, 2$ a point in E_j has the form $\begin{bmatrix} 1 \\ \wp_j(z) \\ \wp_j'(z) \end{bmatrix}$.

We can embed the product of the two elliptic curves in $\mathbb{P}^8(\mathbb{C})$ using the classical Segre embedding for product of projective spaces (see for example [HS00, Example A.1.2.6(b)]) so that a point in $E_1 \times E_2 \subseteq \mathbb{P}^8(\mathbb{C})$ has the form

$$\begin{bmatrix} 1 \\ \wp_2(z_2) \\ \wp_2'(z_2) \\ \wp_1(z_1) \\ \wp_1(z_1)\wp_2(z_2) \\ \wp_1(z_1)\wp_2'(z_2) \\ \wp_1'(z_1) \\ \wp_1'(z_1)\wp_2(z_2) \\ \wp_1'(z_1)\wp_2(z_2) \end{bmatrix}.$$

Consider a smooth curve W contained in $E_1 \times E_2$ that is the intersection of $E_1 \times E_2$ with a hypersurface cut out in $\mathbb{P}^8(\mathbb{C})$ by a polynomial $F \in \mathbb{C}[Z_0, \ldots, Z_8]$. Let moreover

$$L := \{(z_1, z_2) \in \mathbb{C}^2 \mid z_1 = z_2\}.$$

Suppose we want to find a point in $L \times W \cap \Gamma_{E_1 \times E_2}$: it corresponds to finding $z \in \mathbb{C}$ such that the corresponding point in $E_1 \times E_2$ (which we may write using the $\wp$-functions) satisfies the polynomial equation $F = 0$.

To do so, let us understand what $\overline{\exp_{E_1 \times E_2}(L)}$ looks like. Since, as we already noted, $\sqrt{2}$ does not lie in the $\mathbb{Q}$-linear span of $\sqrt{5}$, we have that for every fixed $z \in \mathbb{C}$ the set $z + i\sqrt{2}\mathbb{Z}$ has closed image under the exponential of E_1, while its image under the exponential of E_2 looks like the set $z + i\sqrt{2}\mathbb{Z} + i\sqrt{5}\mathbb{Z}$, which is dense in $\mathrm{Re}(z) + i\mathbb{R}$. Therefore the closure of $\exp_{E_1 \times E_2}(L)$ is the set

$$\exp(\{(z_1, z_2) \in \mathbb{C}^2 \mid \mathrm{Re}(z_1) = \mathrm{Re}(z_2)\}).$$

The homology argument then implies that $\overline{\exp_{E_1 \times E_2}(L)} \cap W \neq \varnothing$: in algebraic terms, there are $x+iy_1$ and $x+iy_2$ in $\mathbb{C}$ such that $F(\exp_{E_1 \times E_2}(x+iy_1, x+iy_2)) = 0$, simply because $\overline{\exp_{E_1 \times E_2}(L)}$ and W are closed subsets of $E_1 \times E_2$ and the intersection product of their homology classes is non-zero.

Then we may conclude by density: by openness of the δ-map (which in this case holds everywhere on $L \times W$, because they are both spaces of dimension and codimension 1) we find $(l, w) \in L \times W$ (with w arbitrarily close to $(\exp_{E_1 \times E_2}(x + iy_1, x + iy_2)))$ such that $w - \exp_{E_1 \times E_2}(l) \in \exp_{E_1 \times E_2}(L)$, which

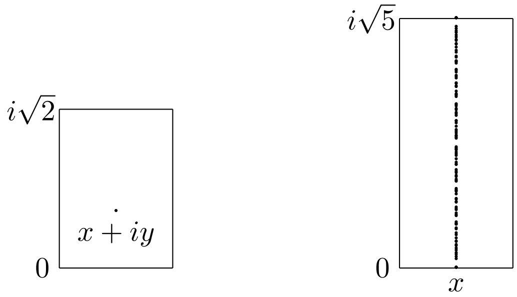

Figure 4.2: As in the example, let $\Lambda_1 = \mathbb{Z} + i\sqrt{2}\mathbb{Z}$ and $\Lambda_2 = \mathbb{Z} + i\sqrt{5}\mathbb{Z}$. Given a point $z = x + iy \in \mathbb{C}$, the set of images under the exponential of points of the form $(z, z + \lambda)$ for $\lambda \in \Lambda_2$ is dense in the set of images of points $(z, \mathrm{Re}(z) + y)$ for $y \in \mathbb{R}$. This is shown in this figure: for a fixed $z \in \mathbb{C}/\Lambda_1$ on the left, there are infinitely many determinations on the right, densely filling the vertical line.

as usual implies that $w \in \exp_{E_1 \times E_2}(L)$ to begin with. Therefore, there is $z \in \mathbb{C}$ such that $w = \exp_{E_1 \times E_2}(z, z)$, giving a zero of the system of equations under discussion.

4.5 Model-Theoretic Consequences

Theorem 3.7.8 had its roots in a model-theoretic result (Theorems 2.4.1 and 2.4.2), so it is natural to wonder what is the model-theoretic relevance of Theorem 4.4.1.

Zilber mentions this in the Introduction to [Zil15]:

> [...] one can easily replace [the multiplicative group of a field] $F^\times$ by any semiabelian variety A and carry out the same construction and axiomatisation since also a corresponding analogue of Ax's Theorem and its corollaries is available. ([Zil15, p. 4]).

We do not include here the full construction of the first-order theory that corresponds to the theory T_K of Theorem 2.4.1, and leave the elaboration to future work; however, we point out its main features.

Fix a simple complex abelian variety A and a field $K \subseteq \mathbb{C}$ of finite transcendence degree, and consider the ring $\mathrm{End}(A)$ of endomorphisms of A.

The language of the theory is then an expansion of the language of $\mathrm{End}(A) \otimes \mathbb{Q}$-vector spaces, which includes a predicate L for any K-definable subsets L of

$(\mathbb{C}^g)^n$, a predicate EW for any algebraic subvariety W of A^n defined over the field of definition of A, and a binary relation E. Just as in the case of exp, the interpretation of these symbols in the structure on $\mathbb{C}^g$ is:

1. $\forall x, y \in \mathbb{C}^g,\ E(x, y) \leftrightarrow \exp_A(x) = \exp_A(y)$ ($\leftrightarrow x - y$ is in the lattice Λ s.t. $A \cong \mathbb{C}^g/\Lambda$);
2. For L a definable subset of $(\mathbb{C}^g)^n$, $\forall z_1, \dots, z_n \in \mathbb{C}^g,\ L(z_1, \dots, z_n) \leftrightarrow (z_1, \dots, z_n) \in L$;
3. For W an algebraic subvariety of A^n, $\forall z_1, \dots, z_n \in \mathbb{C}^g$, $EW(z_1, \dots, z_n) \leftrightarrow (\exp_A(z_1), \dots, \exp_A(z_n)) \in W$.

We denote the structure by $(\mathbb{C}^g)^{A,K}$.

It is worth stressing that the we do not take a predicate L for every *subspace* of $(\mathbb{C}^g)^n$, but rather for every definable subset of $(\mathbb{C}^g)^n$ in the structure $\mathbb{C}^g$ as a vector space. So for example, if $g = 2$, then we do not take a predicate for

$$\{(z_1, z_2) \in \mathbb{C}^2 \mid z_1 = z_2\},$$

while we do take one for

$$\{(z_1, z_2, z_3, z_4) \in \mathbb{C}^4 \mid (z_3, z_4) = (z_1, z_2)\}.$$

This is because the theory is supposed to mimic the "raising to powers" idea, and therefore the spaces that we add are supposed to work like the graphs of "multivalued endomorphisms" of $\mathrm{End}(A)$.

One may then define a theory by giving axioms on a structure D, requiring that it is a K-vector space, and that D/E has the same first-order theory of the abelian variety A in the language with the quotients under E of the predicates EW.

Again, one finds a completion $T_{A,K}$ of this theory that can be described by a transcendence statement and an existential closedness statement. In particular, the latter will take the by-now familiar form of "for all free rotund varieties $L \times W$, there is $(z_1, \dots, z_n) \in D$ s.t. $L(z_1, \dots, z_n) \wedge EW(z_1, \dots, z_n)$". If we take as D the structure on the space $\mathbb{C}^g$ the existential closedness statement will amount to the following proposition.

Proposition 4.5.1. *Let $L \leq (\mathbb{C}^g)^n$ be a K-definable subset, $W \subseteq A^n$ an* *algebraic variety such that $L \times W$ is free rotund (as a subvariety of $\mathbb{C}^{gn} \times A$). Then $L \times W \cap \Gamma_{\exp_{A^n}} \neq \varnothing$.*

This is obviously a corollary of Theorem 4.4.1, and therefore the structure $(\mathbb{C}^g)^{A,K}$ is a model of the theory $T_{A,K}$ if and only if it satisfies a transcendence statement. It seems very likely that a theorem similar to Theorem 2.4.3 can be established in the context of abelian varieties, and that therefore we can show that $(\mathbb{C}^g)^{A,K}$ is a model of the theory $T_{A,K}$ for a sufficiently generic choice of K; again, we leave working out the full details to future work.

4.6 Future Work: The $V \times W$ Case

We conclude this chapter by presenting some considerations towards the extension of Theorem 4.4.1 to varieties of the form $V \times W$, where $V \subseteq \mathbb{C}^g$ is an algebraic variety and not necessarily a linear space. While we do not have definitive results for this question, there are some preliminary remarks which might be helpful to solve it in the future.

The idea is that given an affine algebraic variety $V \subseteq \mathbb{C}^g$, neighbourhoods of large points on the variety resemble open subsets of linear spaces.

Example 4.6.1. Two examples suggest that this intuition is correct.

If $V = \{(z_1, z_2) \in \mathbb{C}^2 \mid z_1 z_2 = 1\}$, then the "large" points take the form $\left(z, \frac{1}{z}\right)$ where one of the coordinates is very large and thus the other one is close to 0. If we take the first one to be large, then it is clear that

$$\lim_{z \to \infty} B(0, R) \cap \left(-\left(z, \frac{1}{z}\right)\right) + V = B(0, R) \cap \{(z_1, z_2) \in \mathbb{C}^2 \mid z_2 = 0\}.$$

In plain terms, locally this variety stays close to the given line.

Similarly, if we consider $V = \{(z_1, z_2) \in \mathbb{C}^2 \mid z_2 = z_1^2\}$, in points of large absolute value the second coordinate varies much faster than the first one. This means that

$$\lim_{z \to \infty} B(0, R) \cap \left(-(z, z^2) + V\right) = B(0, R) \cap \{(z_1, z_2) \in \mathbb{C}^2 \mid z_1 = 0\}.$$

Of course, these two examples are fundamentally different in that in the first case the "horizontal line" defined by $z_2 = 0$ acts as a "global asymptote" for V, in the sense that the points on the variety are actually approaching the line. In the second case, on the other hand, the asymptote only works "locally": there is no vertical line that V approaches without intersecting it.

This idea, of approximating the behaviour of V at infinity by linear spaces, is made precise by the following theorem.

Theorem 4.6.2 (Peterzil-Starchenko, [PS18, Theorem 6.3])**.** *Let A be an abelian variety of dimension g with exponential map $\exp_A : \mathbb{C}^g \to A$, and $V \subseteq \mathbb{C}^g$ an algebraic variety.*

Then there exist complex algebraic varieties $C_1, \dots, C_k$ and linear spaces $L_1, \dots, L_k$ in $\mathbb{C}^g$ such that

$$\overline{\exp_A(V)} = \exp_A(V) \cup \bigcup_{j=1}^{k} \overline{\exp_A(C_j + L_j)}$$

and moreover:

1. $\dim C_j < \dim V$ *for every j;*
2. *If L_j is maximal for inclusion among $L_1, \dots, L_k$, then* $\dim C_j = 0$.

We refer to the linear spaces $L_1, \dots, L_k$ as the *asymptotes* of V.

Ullmo and Yafaev conjectured in [UY18a] a version of this theorem without the C_j's, according to which the behaviour of $\exp_A(V)$ is determined by finitely many linear asymptotes, and proved it in the case in which V is a curve (note that in Theorem 4.6.2 the C_j's have dimension smaller than V, and thus they are zero dimensional for V a curve). However, an example in Section 8 of [PS18] shows that when the dimension of V is larger than 1 then the C_j's are necessary.

While the proof of Peterzil and Starchenko is model-theoretic and uses non-standard models of the theory of algebraically closed valued fields, a slightly more general result has been proved by Dinh and Vu in [DV20] by complex-analytic methods.

We also note that the structure of the limit set (finitely many varieties which are invariant under translations by linear spaces, the maximal ones of which are exactly translates of linear spaces) closely resembles the structure of the tropicalization of an algebraic subvariety of $(\mathbb{C}^\times)^n$ (finitely many varieties which are invariant under translations by algebraic subgroups, the maximal ones of which are exactly translates of algebraic subgroups). While there are also some obvious differences (for example the initial varieties in tropical geometry have the same dimension as the original variety, while in the Peterzil-Starchenko theorem there are examples of varieties with asymptotes of larger dimension) it is possible that this similarity has some meaning, and we leave it to future work to investigate it.

We would like to use Theorem 4.6.2 to prove the Abelian Exponential-Algebraic Closedness Conjecture in the case in which the subvariety of $\mathbb{C}^g \times A$ has the form $V \times W$, where $V \subseteq \mathbb{C}^g$ and $W \subseteq A$ are algebraic varieties. A first attempt would be to try to show that if the variety $V \times W$ is rotund, then there is at least one asymptote L of V such that $L \times W$ is rotund and thus $\exp_A(L) \cap W$ is non-empty, and then try to lift the intersection to a point in $\exp_A(V) \cap W$.

However, this approach is trickier than it looks, as the asymptotes encode information on the global behaviour of the variety, rather than its local aspect.

We explain what we mean with an example.

Example 4.6.3. Let A be the square of the elliptic curve $E \cong \mathbb{C}/\mathbb{Z} + i\mathbb{Z}$, and $V := \{(z_1, z_2) \in \mathbb{C}^2 \mid z_2 = z_1^2\}$ the parabola in $\mathbb{C}^2$. Take as W the abelian subvariety of A obtained as the exponential of the space

$$\{(z_1, z_2) \in \mathbb{C}^2 \mid z_1 = 0\}.$$

The variety $V \times W$ is not free, because W is an abelian subvariety, but it is rotund which is enough to make our point here.

We have already discussed how, if we take points $(z_1, z_2) \in V$ of large absolute value, neighbourhoods of (z_1, z_2) in V resemble vertical lines: a slight perturbation of z_1 might cause z_2 to move by a comparatively large amount.

[PS18, Example 4.6(2)] describes how this variety has only one asymptote: the whole complex space $\mathbb{C}^2$. This asymptote is in particular rotund, but it does not take an expert in exponential-algebraic closedness to notice that $\exp_A(\mathbb{C}^2) \cap W = W \neq \varnothing$.

Thus if we take any point $w \in W$ there is a sequence $\{(z_j, z_j^2)\}_{j\in\omega} \subseteq V$ such that $\exp_A(z_j, z_j^2)$ converges to w; equivalently, the sequence $\{w - \exp_A(z_j, z_j^2)\}_{j\in\omega}$ converges to $0 \in A$.

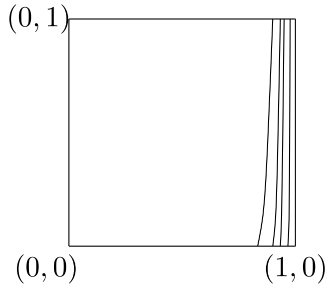

Figure 4.3: This picture, although referred to the real torus $\mathbb{R}^2/\mathbb{Z}^2$ rather than to a complex one, gives a good description of what is going on in Example 4.6.3: while locally on each single piece of the exponential of the parabola the projection on the horizontal axis is open, it is clear that as the coordinates become larger the curve looks more and more like a vertical line, and therefore it "loses rotundity" at infinity.

However, if we suppose that $(z_j, z_j^2) \to \infty$ as j goes to infinity, we fix a neighbourhood U of 0 in $\mathbb{C}^g$ and consider the sequence of open subsets of V $\{V_j\}_{j\in\omega}$, where $V_j := (z_j, z_j^2) + U \cap V$, then we see that although $\exp_A(V_j) - W$ is an open subset of A for every A, as j goes to infinity the set shrinks. At the limit, it will coincide with W, and therefore there is no indication that the approximating sets contain 0 as well.

In this case, obviously, we know how to get around this problem: since we can pick the first coordinate of a point in V arbitrarily, we see that every point of the form $\exp_A(m + in, m^2 - n^2 + 2imn)$ lies in W. However, for more general V and W the fact that V "loses rotundity locally", even though the asymptote maintains it, may pose a more substantial issue.

A partial solution to the problem described in Example 4.6.3 is given by the notion of *stabilizer* given by Peterzil and Starchenko. We do not delve into the definition of stabilizer, which uses model-theoretic notions ([PS18, Definition 3.12]) but we just note its most important feature, which we essentially treat as the definition.

Fact 4.6.4 ([PS18, Section 3])**.** *Let $V \subseteq \mathbb{C}^g$ be an algebraic variety. A stabilizer for V in the sense of Peterzil-Starchenko is a linear subspace $H \leq \mathbb{C}^g$ for which there exist a sequence $\{v_j\}_{j\in\omega}$ of points in V and a neighbourhood U of 0 such that*

$$\lim_{j\to\infty} d((v_j + U \cap V), (v_j + H)) = 0$$

where d denotes the usual Euclidean distance

Lemma 4.6.5 ([PS18, Lemma 4.10])**.** *Every asymptote of V contains a stabilizer for V.*

Stabilizers are more useful than asymptotes when it comes to exponential-algebraic closedness.

Theorem 4.6.6. *Let $V \subseteq \mathbb{C}^g$ and $W \subseteq A$ be algebraic varieties such that $V \times W$ is free and rotund and there exists at least one stabilizer H for V such that $H \times W$ is rotund.*

Then $\exp_A(V) \cap W \neq \varnothing$.

As this theorem is by no means definitive, and we hope to improve it in the future, we only give a sketch of the proof.

Proof sketch. Since $H \times W$ is rotund, $\exp_A(H) \cap W \neq \varnothing$ and it is Zariski dense in the intersection of W with the Zariski closure of $\exp_A(H)$.

So let $w \in \exp_A(H) \cap W$ be such that the δ-map of $H \times W$ is open on $H \times U_W$ for some neighbourhood $U_W \subseteq W$ of w.

By definition of stabilizer, then there are a positive real R and a sequence $\{v_j\}_{j\in\omega}$ such that $\exp_A(v_j)$ converges to w and the sequence $\{U_{V,j}\}_{j\in\omega} = \{v_j - (B(v_j, R) \cap V)\}_{j\in\omega}$ converges in the Hausdorff metric to an open subset $H_0 = B(0, R) \cap H$ of H.

By choosing R appropriately, we can make sure that all the images of the δ-map of $V \times W$ restricted to the sets $U_{V,j} \times U_w$ contain an open ball of fixed radius around the point $w - \exp_A(v_j)$. Therefore, as $\lim_j w - \exp_A(v_j) = 0$, 0 will eventually be contained in the image: this will give a point in $W \cap \exp_A(V)$. $\square$

Chapter 5

The j-Function

5.1 Introduction

The third main result of this thesis is a theorem on the modular j-function. This is an important function in number theory, a holomorphic function on the upper half-plane $\mathbb{H} := \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$ which classifies elliptic curves. In fact, as we have seen in Chapter 4, every elliptic curve is isomorphic to a complex torus, so a group of the form $\mathbb{C}/\Lambda$; the lattice Λ can be taken to have the form $\mathbb{Z} + \tau\mathbb{Z}$ for $\tau \in \mathbb{H}$. We denote by E_τ the elliptic curve isomorphic to $\mathbb{C}/\mathbb{Z} + \tau\mathbb{Z}$.

It turns out that two elliptic curves E_{τ_1} and E_{τ_2} are isomorphic (as complex algebraic groups) if and only if the two complex numbers τ_1 and τ_2 satisfy a certain arithmetic relation (there are $a, b, c, d \in \mathbb{Z}$ such that $ad - bc = 1$, and $\frac{a\tau_1+b}{c\tau_1+d} = \tau_2$). The j-function has the property that for all $\tau_1, \tau_2 \in \mathbb{H}$, $j(\tau_1) = j(\tau_2)$ if and only if the elliptic curves are isomorphic: therefore, the behaviour of the j-function is somehow influenced by the arithmetic relations of elliptic curves. We will discuss this in more detail in Section 5.2, but hopefully the reader can gather from this introduction that the j-function is a remarkable object which sits at the crossroad of complex analysis, number theory, and algebraic geometry.

The j-function is in some aspects similar to the complex exponential: it is a transcendental function with countable, discrete fibres; a version of Schanuel's conjecture has been formulated for it.

Conjecture 5.1.1 ([Ber02, Conjecture Modulaire]). *Let $z_1, \ldots, z_n \in \mathbb{H}$ be non-special points which lie in distinct $\mathrm{GL}_2(\mathbb{Q})^+$-orbits.*

Then

$$\operatorname{trdeg}(z_1, \ldots, z_n, j(z_1), \ldots, j(z_n)) \geq n.$$

The definitions of *special point* and of the action of $\mathrm{GL}_2(\mathbb{Q})^+$ on $\mathbb{H}$ will be given below. However, the spirit of this conjecture is that, just as in the exp case, the lower bound for transcendence of the function is given exactly by the arithmetic relations that the function needs to preserve.

The similarities with the exponential function have led to many questions for exp to be asked for j as well; of particular interest for us is the question of *j-Algebraic Closedness*, an analogue of Exponential-Algebraic Closedness which aims to determine sufficient conditions for systems that involve polynomials and the j-function to be solvable in the complex numbers. This chapter contains an introduction to this conjecture and a solution to a special case of it.

Theorem (Theorem 5.5.7). *Let $L \times W$ be a free broad subvariety of $\mathbb{H}^n \times \mathbb{C}^n$ with L a Möbius subvariety of $\mathbb{H}^n$ and W an algebraic variety in $\mathbb{C}^n$.*

Then $L \times W \cap \Gamma_j \neq \varnothing$.

The notions of *freeness*, *broadness* and *Möbius variety* will be introduced in Sections 5.2 and 5.3. Γ_j denotes the graph of (the n-th Cartesian power of) the j-function; since we will use Γ to denote the arithmetic group $\mathrm{SL}_2(\mathbb{Z})$, however, we prefer to avoid referring to Γ_j, and in fact the statement of the theorem below is given in a slightly different equivalent way (which references $j(L) \cap W$ rather than $L \times W \cap \Gamma_j$). We gave the statement in this way in the introduction to highlight the similarity with the main theorems of Chapters 3 and 4: just as in those cases, the theorem establishes that the graph of the relevant transcendental function intersects a variety which splits as a product $L \times W$, where W is any algebraic subvariety of the codomain and L has a nice geometric structure. We will say more about why this result is akin to the "raising to powers" idea in the next section.

The structure of the chapter is as follows.

In Section 5.2 we introduce the modular j-function, describing some of its properties such as its interplay with the action of $\mathrm{SL}_2(\mathbb{R})$ on $\mathbb{H}$ by Möbius transformation. We also introduce *Möbius* and *weakly special* subvarieties, objects of fundamental importance.

This will help us, in Section 5.3, to introduce the j-Algebraic Closedness conjecture. We first take a short detour to discuss some similarities between exp and j, and then state the conjecture.

In Section 5.4 we use Ratner's theorem from ergodic theory to show that the image under the j-function of a Möbius subvariety which is not contained in any weakly special subvariety of $\mathbb{H}^n$ is dense in $\mathbb{C}^n$ in the Euclidean topology.

In Section 5.5 we prove the main result of this chapter.

Finally, in Section 5.6 we sketch the proof of a partial result which concerns solutions of systems of equations that involve algebraic operations and the first derivative of the j-function.

The work in this chapter has been greatly helped by Gareth Jones's suggestion to use Ratner's Theorem to tackle Lemma 5.4.2. The author wishes to express his gratitude.

The main results of this chapter have already appeared in Sections 3 and 4 of the preprint [Gal21].

5.2 Geometric Preliminaries

Let $\mathbb{H}$ denote the complex upper half plane, i.e. the set $\{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$. The special linear group $\mathrm{SL}_2(\mathbb{R})$ acts on $\mathbb{H}$ by Möbius transformations, that is, the matrix $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ acts on $z \in \mathbb{H}$ by mapping it to $\frac{az+b}{cz+d}$.

Remark 5.2.1. This action can actually be seen as a restriction of the action by Möbius transformations of a larger group on a larger set, that of $\mathrm{GL}_2(\mathbb{C})$ on the Riemann sphere $\mathbb{P}^1(\mathbb{C})$. This has two consequences that are of interest for us.

The first one is that we can consider $\mathrm{SL}_2(\mathbb{R})$ acting not just on $\mathbb{H}$, but also on $\mathbb{R} \cup \{\infty\}$. This will be needed to give the definition of cusp.

The second one is that, while the action can be defined for any matrix in $\mathrm{GL}_2(\mathbb{C})$, it is clearly invariant by scalar multiplication, meaning that given $\lambda \in \mathbb{C}$ and $h \in \mathrm{GL}_2(\mathbb{C})$, $hz = \lambda hz$ for all $z \in \mathbb{H}$; hence this can be seen as an action of $\mathrm{SL}_2(\mathbb{C})$. The situation is not exactly the same for real matrices:

let $\mathrm{GL}_2(\mathbb{R})^+ := \{h \in \mathrm{GL}_2(\mathbb{R}) \mid \det(h) > 0\}$. The action of $\mathrm{SL}_2(\mathbb{R})$ can be seen as the restriction of the action on the same space of $\mathrm{GL}_2(\mathbb{R})^+$. We will be especially concerned with matrices which up to scalar multiplication have rational entries, as these define the only algebraic relations on $\mathbb{H}$ whose algebraicity is preserved under the j-function. We will freely abuse notation and consider the action of $\mathrm{GL}_2(\mathbb{Q})^+$ on $\mathbb{H}$, claiming that a matrix $h \in \mathrm{SL}_2(\mathbb{R})$ is in $\mathrm{GL}_2(\mathbb{Q})^+$ when actually what we mean is that it is a scalar multiple of such a matrix.

We introduce the notion of cusp.

Definition 5.2.2. Let Γ be a discrete subgroup of $\mathrm{SL}_2(\mathbb{R})$. A *cusp* for Γ is a point $x \in \mathbb{R} \cup \{\infty\}$ such that $\gamma x = x$ for every $\gamma \in \Gamma$ of the form $\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$ for some $t \in \mathbb{R}$ (such elements of $\mathrm{SL}_2(\mathbb{R})$ are called *parabolic*).

For the rest of this section, let $G = \mathrm{SL}_2(\mathbb{R})$ and $\Gamma = \mathrm{SL}_2(\mathbb{Z})$.

Recall the following definition of commensurability of subgroups and the characterization for commensurability of conjugates of Γ in G.

Definition 5.2.3. Two subgroups H and K of a group are *commensurable* if $H \cap K$ has finite index in both H and K.

Fact 5.2.4 ([Mil97, Lemmas 5.29 and 5.30])**.** *Let $g \in G$. Then $g\Gamma g^{-1}$ and Γ are commensurable if and only if $g \in \mathrm{GL}_2(\mathbb{Q})^+$.*

We recall the definition of the j-function, following [Mil97, Section I.4].

Definition 5.2.5. We say a function $f : \mathbb{H} \to \mathbb{C}$ is Γ-*invariant* if for any $\gamma \in \Gamma$ and $z \in \mathbb{H}$, $f(\gamma z) = f(z)$.

According to Definition 5.2.2, the only cusp for Γ is ∞. As the Γ-orbit of ∞ can easily be checked to be $\mathbb{Q}$, studying the behaviour of a Γ-invariant function at infinity is the same as studying it at any rational point.

Definition 5.2.6. Let f be a Γ-invariant function on $\mathbb{H}$, $\mathbb{D}$ denote the open unit disk. Then we can define $f^* : \mathbb{D} \setminus \{0\} \to \mathbb{C}$ by $f^*(\exp(2\pi i z)) = f(z)$; this is well-defined, because if $z \in \mathbb{H}$ then $\exp(2\pi i z) \in \mathbb{D} \setminus \{0\}$ and if $\exp(2\pi i z_1) = \exp(2\pi i z_2)$ then $z_1 - z_2 \in \mathbb{Z}$ and thus $f(z_1) = f(z_2)$ by Γ-invariance. We say f is meromorphic (resp. has a pole of order n) at the cusp if f^* is meromorphic (resp. has a pole of order n) at 0.

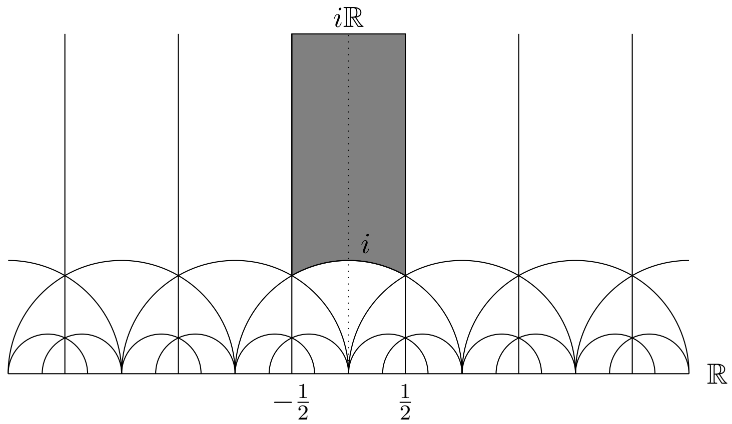

Figure 5.1: Some of the fundamental domains for the action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{H}$: the curved triangular shapes in the picture (including the degenerate ones with one vertex at infinity) contain exactly one point for each $\mathrm{SL}_2(\mathbb{Z})$-orbit. There are other fundamental domains, curved triangular shapes cut out by infinitely many half-circles of smaller and smaller radius centred on the rational numbers. The shaded area, included between the half-circle of centre 0 and radius 1 and the half-lines defined by $\mathrm{Re}(z) = \frac{1}{2}$ and $\mathrm{Re}(z) = -\frac{1}{2}$, is often referred to as *the* fundamental domain for this action.

Now we have everything that is needed to define the j-function.

Definition 5.2.7. The modular invariant j is the unique function $j : \mathbb{H} \to \mathbb{C}$ that is holomorphic, Γ-invariant, with a simple pole at the cusp and such that $j(i) = 1728$ and $j(\exp(\frac{i\pi}{3})) = 0$.

It is well-known that j establishes an analytic isomorphism between $\Gamma\backslash\mathbb{H}$ (as the action is by left multiplication, the quotient is usually written on the left) and $\mathbb{C}$. We will be studying intersections between images through j of certain algebraic subvarieties of $\mathbb{H}^n$ and algebraic subvarieties of $\mathbb{C}^n$.

Remark 5.2.8. It should be noted that technically the phrase "algebraic subvarieties of $\mathbb{H}^n$" is incorrect: the upper half-plane is not a ring, and therefore we do not talk about polynomials whose indeterminates take values in $\mathbb{H}$. We abuse terminology and say that V is an algebraic subvariety of $\mathbb{H}^n$ to mean that V is an irreducible component of $W \cap \mathbb{H}^n$ for some algebraic subvariety $W \subseteq \mathbb{C}^n$.

The varieties we will mostly be concerned with are Möbius varieties.

Definition 5.2.9. Let L be an algebraic subvariety of $\mathbb{H}^n$. We say L is a

Möbius subvariety if L can be defined just by using conditions of the form $z_k = gz_i$ for $g \in G, i, k \leq n$ and $z_i = \tau$ for $\tau \in \mathbb{H}, i \leq n$.

Example 5.2.10. The variety $\{(z_1, z_2) \in \mathbb{H}^2 | z_2 = z_1 + \sqrt{2}\}$ is a Möbius variety, defined by $z_2 = gz_1$ for $g = \begin{pmatrix} 1 & \sqrt{2} \\ 0 & 1 \end{pmatrix}$.

Among Möbius varieties lie weakly special varieties.

Definition 5.2.11. An algebraic variety $V \subseteq \mathbb{H}^n$ is *weakly special* if it is a Möbius subvariety of $\mathbb{H}^n$ such that for all $i, j \leq n$ such that an equation of the form $z_j = gz_i$ holds on V, we can assume g to be a matrix in $\mathrm{GL}_2(\mathbb{Q})^+$.

As the name suggest, this is a generalization of a notion of *special* variety.

Definition 5.2.12. A *special point* in $\mathbb{H}^n$ is a point $z = (z_1, \ldots, z_n) \in \mathbb{H}^n$ such that each z_i is an imaginary quadratic element.

A theorem of Schneider establishes the relevance of special points with respect to the j-function.

Theorem 5.2.13 ([Sch37])**.** *Let $\tau \in \mathbb{H}$. Then τ and $j(\tau)$ are both algebraic numbers if and only if τ is imaginary quadratic.*

Example 5.2.14. From Definition 5.2.7 we know that $j(i) = 1728$ and $j(\exp(\frac{\pi i}{3})) = 0$, in accordance with Schneider's Theorem.

Definition 5.2.15. A weakly special subvariety $V \subseteq \mathbb{H}^n$ is *special* if for every $i \leq n$ such that V satisfies an equation of the form $z_i = c$, c is an imaginary quadratic point.

Equivalently, a weakly special variety is special if and only if it contains a special point.

Just like special points are "bi-algebraic", in the sense that they are the only algebraic points in $\mathbb{H}^n$ whose algebraicity is preserved by j, weakly special and special varieties are interesting because they are the only algebraic subvarieties of $\mathbb{H}^n$ whose image under j is still an algebraic variety. This is the content of a theorem of Ullmo and Yafaev.

Theorem 5.2.16 ([UY11, Theorem 1.2])**.** *Let V be an algebraic subvariety of $\mathbb{H}^n$. Then $j(V)$ is an algebraic subvariety of $\mathbb{C}^n$ if and only if V is weakly special, and $j(V)$ is defined over $\mathbb{Q}$ if and only if V is special.*

This does not explain what the image of a weakly special subvariety of $\mathbb{H}^n$ looks like. For that, we need to introduce *modular polynomials*.

Lemma 5.2.17 ([Lan87, Theorems 5.2 and 5.3])**.** *For each $N \in \mathbb{N}^{>0}$ there is an irreducible polynomial $\Phi_N \in \mathbb{Z}[X, Y]$ such that:*

1. *For every $z_1, z_2 \in \mathbb{H}$ such that $\Phi_N(j(z_1), j(z_2)) = 0$, there is $g \in \mathrm{GL}_2(\mathbb{Q})^+$ with coprime integer entries and determinant N such that $z_2 = gz_1$;*
2. *$\Phi_1(X, Y) = X - Y$, and for $N \geq 2$, Φ_N is symmetric of total degree at least $2N$.*

Definition 5.2.18. For each N, the polynomial Φ_N of Lemma 5.2.17 is known as the N-th *modular polynomial.*

Varieties defined by modular polynomials and constant coordinates are then obviously the counterparts in $\mathbb{C}^n$ of (weakly) special subvarieties of $\mathbb{H}^n$.

Definition 5.2.19. An algebraic variety $V \subseteq \mathbb{C}^n$ is *weakly special* if it is defined only by equations of the form $\Phi_N(z_i, z_j) = 0$ where Φ_N is a modular polynomial and $z_k = c$ for some $c \in \mathbb{C}$.

A weakly special variety is *special* if it contains the image under j of a special point.

It is clear from the definition and from Lemma 5.2.17 that (weakly) special subvarieties of $\mathbb{C}^n$ are the images under j of (weakly) special subvarieties of $\mathbb{H}^n$.

5.3 j-Algebraic Closedness

5.3.1 Why j?

While there are evident reasons why the exponential maps of abelian and semiabelian varieties were studied along the same lines as the complex exponential (the similar geometric behaviour, the differential equation, the fact that exp is itself the covering map of the semiabelian variety $\mathbb{C}^\times$), a non-expert might think that the j-function is quite different from exp and not see why one should think that it obeys similar laws. That non-expert would not be too far off the mark: the most obvious difference, for example, is that j is not a group homomorphism, so the best algebraic properties of exp are lost.

We therefore take this subsection, before we move on to state and analyse the j-Algebraic Closedness Conjecture, to have a look at some parallels between j and exp, to try to convince the skeptical non-expert that it is not a stretch to regard this problem as part of the same family as Exponential-Algebraic Closedness.

Arguably, the push that set this in motion was Pila and Tsimerman's Ax-Schanuel Theorem for j.

Theorem 5.3.1 (Ax-Schanuel for j, [PT16, Theorem 1.1]). *Let $V \subseteq \mathbb{H}^n \times \mathbb{C}^n$ be an algebraic variety, Γ_j the graph of j, and let $U \subseteq V \cap \Gamma_j$ be an irreducible component with $\dim U > \dim V - n$.*

Then the projection of U to $\mathbb{C}^n$ is contained in a weakly special subvariety.

This is an Ax-Schanuel Theorem in the sense that it does for j what Theorem 2.5.4 does for semiabelian varieties: it establishes that atypical intersections between algebraic varieties and the graph of j are governed by modular relations, just as atypical intersections between algebraic varieties and the graph of exp are governed by group relations.

As we mentioned in Section 5.2, this should be considered in connection with *bi-algebraic geometry*, in the sense of Klingler, Ullmo and Yafaev (see [KUY18] for an exhaustive treatment of this topic). Weakly special subvarieties of $\mathbb{H}^n$ are the only algebraic varieties whose image through the j-function is algebraic; similarly, in a semiabelian variety of dimension n the image of an algebraic subvariety of $\mathbb{C}^n$ through the exponential map is algebraic if and only if the variety is the Lie algebra of a semiabelian subvariety. Hence, the existence of varieties of this kind allows for the existence of atypical intersections; the point of Ax-Schanuel statements is that there is no other reason why atypical intersections exist.

As is the case for the Ax-Schanuel Theorem for exp, there are more general versions of this statement which say a bit more about the atypical intersections between algebraic varieties in $\mathbb{C}^n$ and special varieties; these are related to the Zilber-Pink Conjecture.

Definition 5.3.2. Let $W \subseteq \mathbb{C}^n$ be an algebraic variety. An *atypical subvariety* of W is an algebraic variety $V \subseteq W$ which is an irreducible component of an intersection $W \cap S$, where $S \subseteq \mathbb{C}^n$ is a special variety, such that $\dim V > \dim W + \dim S - n$.

We mentioned the Zilber-Pink Conjecture in Section 2.4, in connection with the Conjecture on Intersections with Tori (Conjecture 2.4.5) as a statement which predicts a tame behaviour of atypical intersections between algebraic subvarieties and algebraic subgroups of $(\mathbb{C}^\times)^n$. Similarly, the conjecture for the j-function predicts that there are finitely many atypical subvarieties in any algebraic subvariety of $\mathbb{C}^n$. While the conjecture is a very difficult open problem, Theorem 5.3.1 leads to some partial results.

Theorem 5.3.3 (Weak Modular Zilber-Pink, [PT16, Theorem 7.1]). *Let $W \subseteq \mathbb{C}^n$ be an algebraic variety. Then W contains finitely many atypical subvarieties with no constant coordinates.*

Theorem 5.3.4 (Weak Modular Zilber-Pink for Parametric Families, [Asl21, Theorem 7.9]). *Let $W \subseteq \mathbb{C}^{n+l}$ be an algebraic variety; for $w_0 \in \mathbb{C}^l$, denote by W_{w_0} the variety $\{w \in \mathbb{C}^n \mid (w, w_0) \in W\}$.*

There is a finite set $\mathcal{T}$ of special subvarieties of $\mathbb{C}^n$ such that for every $w \in \mathbb{C}^l$ and every atypical subvariety with no constant coordinates V of W_w there is $T \in \mathcal{T}$ such that $V \subseteq W \cap T$.

These theorems, and Theorem 5.3.4 in particular, should be compared to Corollary 2.5.4 for semiabelian varieties, which said that atypical intersections in a semiabelian variety can be traced back to the influence of finitely many algebraic subgroups.

Theorem 5.3.1 led model theorists to consider the extent of the similarities between the theory of the exponential and that of the j-function; the initial idea to build a *pseudo-j-function*, similar to Zilber's pseudoexponential, was soon abandoned, but many analogies remained. In [Asl22], Aslanyan studied the first-order theory of differential fields equipped with a function that behaves like j, similarly to Kirby's work on the exponential differential equation in [Kir06] and [Kir09].

In particular, he defined notions of *freeness* and *broadness* for algebraic varieties in these differential fields (similar to freeness and rotundity in fields with an exponential) and conjectured that all free broad varieties intersect the graph of the j-like function of the differential field. This conjecture was then established by Aslanyan, Eterović, and Kirby ([AEK21, Theorem 1.1]). It was natural to ask whether the same could be said about the "actual" j-function on the complex numbers. This will be the topic of the next subsection.

Let us note, before we move on, that analogues of Theorem 5.3.1 have been proved in more and more general settings, most notably Shimura varieties ([MPT19, Theorem 1.1]) and variations of Hodge structures ([BT19, Theorem 1.1]); Psapas has a version for general linear groups ([Pap19, Theorem]); Blázquez-Sanz, Casale, Freitag, and Nagloo have proved in [Blá+21] some similar statements in the differential-algebraic setting. There is a case to be made for questions similar to Exponential-Algebraic Closedness to be considered for all these structures; Eterović and Zhao have began to do so for Shimura varieties in [EZ21]. There they state the conjecture in this context and generalize to Shimura varieties results that were known for the j-function, including the main theorem of this chapter that had already appeared in [Gal21].

5.3.2 The j-Algebraic Closedness Conjecture

As we mentioned, j-Algebraic Closedness was first considered in a setting of differential fields; it was first stated for the complex j-function in [AK21], where definition of freeness and broadness for subvarieties of $\mathbb{H}^n \times \mathbb{C}^n$ were given, although similar questions were addressed in the earlier work [EH21] as the "motivating questions" of the paper.

Definition 5.3.5. Let $V \subseteq \mathbb{H}^n \times \mathbb{C}^n$ be an algebraic variety; denote by $\pi_1 : \mathbb{H}^n \times \mathbb{C}^n \twoheadrightarrow \mathbb{H}^n$ and by $\pi_2 : \mathbb{H}^n \times \mathbb{C}^n \twoheadrightarrow \mathbb{C}^n$ the projection maps.

We say that V is *free* if $\pi_1(V)$ is not contained in a proper weakly special subvariety of $\mathbb{H}^n$ and $\pi_2(V)$ is not contained in a proper weakly special subvariety of $\mathbb{C}^n$.

While reading this, the reader should keep in mind the definitions of freeness from Chapter 2 for the complex exponential and from Chapter 4 for the exponentials of abelian varieties, together with the discussion in Subsection 5.3.1: freeness there stipulated that the projections of the variety are not contained in translates of algebraic subgroups or their Lie algebras, and we know that these are exactly the "bi-algebraic" varieties for exp; this definition does the same thing, considering Theorem 5.2.16.

As for the analogue of rotundity, *broadness*, it turns out to be more manageable.

Definition 5.3.6. For any ordered subtuple $I = (i_1, \ldots, i_k)$ of $(1, \ldots, n)$

(ordered in the sense that $i_1 < i_2 < \cdots < i_k$), let

$$\pi_I : \mathbb{H}^n \times \mathbb{C}^n \twoheadrightarrow \mathbb{H}^k \times \mathbb{C}^k$$

denote the coordinate projection

$$(z_1, \ldots, z_n, w_1, \ldots, w_n) \mapsto (z_{i_1}, \ldots, z_{i_k}, w_{i_1}, \ldots, w_{i_k}).$$

A subvariety $V \subseteq \mathbb{C}^n \times \mathbb{H}^n$ is *broad* if for all ordered subtuples I,

$$\dim(\pi_I(V)) \geq k.$$

Rotundity asks that a variety satisfies countably many dimension inequalities (except in the degenerate case of a simple abelian variety) as there are countably many algebraic subgroups of a complex torus or of a non-simple complex abelian varieties. Broadness, on the other hand, is the conjunction of only finitely many conditions: this is because the bi-algebraic relations for j are the modular ones and, model-theoretically speaking, their geometry is trivial and therefore described by binary relations. This simplifies things a great deal.

Comparing things with the exponential function once again, we have that just as free rotund varieties are expected to intersect the graph of exp, free broad varieties are expected to intersect the graph of j. This is the content of the next conjecture.

Conjecture 5.3.7 ([AK21, Conjecture 1.2])**.** *Let $V \subseteq \mathbb{H}^n \times \mathbb{C}^n$ be a free broad algebraic variety.*

Then $V \cap \Gamma_j \neq \varnothing$.

The first result which was established in this setting was in [EH21].

Theorem 5.3.8 ([EH21, Theorem 1.1])**.** *Let $V \subseteq \mathbb{H}^n \times \mathbb{C}^n$ be an algebraic variety with $\pi_1(V)$ a Zariski-dense subset of $\mathbb{H}^n$.*

Then $V \cap \Gamma_j \neq \varnothing$.

This was proved using Rouché's Theorem from complex analysis, and it is an analogue for j of Theorems 2.3.20 for exp and 4.3.10 for abelian varieties.

5.3.3 Raising to Powers for j

In this subsection we will explain why the main theorem of this chapter, Theorem 5.5.7, is akin to Theorem 3.7.8, the main theorem of Chapter 3. This subsection is not necessary for the rest of the chapter, but we feel it adds a new angle to the j-exp comparison which can help with the general understanding of the problem.

We saw in Example 2.4.4 that given a linear subspace L of $\mathbb{C}^n$, the set $\exp(L)$ can be interpreted as the intersection of some graphs of multivalued functions; as the easiest example, if $L = \{(z_1, z_2) \in \mathbb{C} \mid z_2 = \sqrt{2}z_1\}$ then $\exp(L)$ is the set of all pairs of the form $(w, w^{\sqrt{2}})$ where $w^{\sqrt{2}}$ denotes any determination of $\exp(\sqrt{2}\log w)$. In the following example we study what happens to the same space (more correctly to its intersection with $\mathbb{H}^2$) under the j-function.

Example 5.3.9. First of all, consider the matrix $g = \begin{pmatrix} 2^{\frac{1}{4}} & 0 \\ 0 & 2^{-\frac{1}{4}} \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R})$. Then for all $z \in \mathbb{H}$,

$$gz = \frac{2^{\frac{1}{4}}z}{2^{-\frac{1}{4}}} = \left(2^{\frac{1}{4}} \cdot 2^{\frac{1}{4}}\right) z = \sqrt{2}z.$$

This means that the variety $L := \{(z_1, z_2) \in \mathbb{H} \mid z_2 = \sqrt{2}z_1\}$ is a Möbius variety.

We want to study $j(L)$. If $(w_1, w_2) \in j(L)$, then there is $(z_1, z_2) \in L$ such that $j(z_1, z_2) = (w_1, w_2)$; therefore there is $z \in \mathbb{H}$ such that $j(z) = w_1$ and $j(\sqrt{2}z) = w_2$. We know that the fibre $j^{-1}(w_1)$ is equal to the $\mathrm{SL}_2(\mathbb{Z})$-orbit of z: however, for all non-identity $\gamma \in \mathrm{SL}_2(\mathbb{Z})$, $g\gamma \neq \gamma g$ and therefore $j(gz) = j(\gamma gz) \neq j(g\gamma z)$. This means that there are countably many different points of the form

$$(j(z), j(g\gamma z))$$

for $\gamma \in \mathrm{SL}_2(\mathbb{Z})$. Therefore, $j(L)$ is also the graph of a multivalued function which associates to w all the determinations of $j(g(j^{-1}(w)))$, as $j^{-1}(w)$ varies in some $\mathrm{SL}_2(\mathbb{Z})$-orbit.

Obviously, the argument in Example 5.3.9 applies to any Möbius variety: therefore, images of Möbius varieties are intersections of graphs of multivalued functions which are obtained by looking at different determinations of j^{-1}. From this perspective, the similarity between the main theorem of this chapter and Theorem 3.7.8 should be clear.

5.4 Density of Images of Möbius Subvarieties

The goal of this chapter is to show that free broad varieties of the form $L \times W$, with L a Möbius variety, intersect the graph of j; as usual, this amounts to showing that $j(L) \cap W \neq \varnothing$, and the first step to be taken is understanding what $j(L)$ looks like.

Freeness of $L \times W$ implies in particular that L is not contained in any weakly special subvariety of $\mathbb{H}^n$. As in this section we will be concerned with Möbius varieties, and we could for the moment forget about the subvariety $W \subseteq \mathbb{C}^n$, we give a definition of freeness for Möbius variety for convenience (so that we can say "let L be a free Möbius variety" rather than "let $L \times W$ be a free variety" when we have no use for W).

Definition 5.4.1. A Möbius variety $L \subseteq \mathbb{H}^n$ is *free* if it is not contained in any weakly special variety, i.e., if no coordinate is constant on L and for no $i, k \leq n$ there is $h \in \mathrm{GL}_2(\mathbb{Q})^+$ such that $z_k = hz_i$ for all $z \in L$.

The goal of this subsection is to establish the following.

Lemma 5.4.2. *Let L be a free Möbius variety. Then $j(L)$ is dense in $\mathbb{C}^n$ in the Euclidean topology.*

Lemma 5.4.2 is an application of Ratner's theorem on unipotent flows, first proved by Ratner in [Rat91]. We will provide a proof for completeness; the reader who is interested in this topic is referred to the exposition in [Mor05]. We also remark that a more general result, which applies to higher dimensional Shimura varieties, has been obtained by Ullmo and Yafaev in [UY18b]; this is relevant in generalizations of this work to the Shimura setting, as in [EZ21, Theorem 1.3].

Recall that on a Lie group we can always define a left-invariant measure μ, the *Haar measure*, which is unique up to multiplication by scalars. Among other things, this is used to define lattices.

Definition 5.4.3. Let Γ be a subgroup of a Lie group G. A *fundamental domain* for Γ is a measurable subset $\mathcal{F}$ of G such that $\Gamma\mathcal{F} = G$ and that $\gamma\mathcal{F} \cap \mathcal{F}$ has measure zero for all non-identity $\gamma \in \Gamma$.

Definition 5.4.4. A subgroup Γ of a Lie group G is a *lattice* in G if it is discrete and it has a fundamental domain of finite measure.

There is an important link between the notions of lattice and commensurability which will be needed later.

Proposition 5.4.5. *Let Γ and Γ' be lattices in G, Δ be the diagonal of G^2, i.e. the group $\{(g, g) \in G^2 | g \in G\}$. Then the following are equivalent:*

1. $\Gamma \cap \Gamma'$ *is a lattice in* G*;*

2. $(\Gamma \times \Gamma') \cap \Delta$ *is a lattice in* Δ*;*

3. Γ *and* Γ' *are commensurable.*

Proof. That 1 and 2 are equivalent is clear once we observe that $(\Gamma \times \Gamma') \cap \Delta$ is the set $\{(\gamma, \gamma) \in \Delta | \gamma \in \Gamma \cap \Gamma'\}$.

For the equivalence of 1 and 3 we have the more general property that a subgroup L' of a lattice L is a lattice if and only if $[L : L']$ is finite. To see this, observe that a fundamental domain for L' is given by the union of $[L : L']$ fundamental domains for L; and hence it has finite measure if and only if $[L : L']$ is finite. □

We can now state (a form of) Ratner's theorem. The following statement is the combination of Theorem 1.1.14, Remark 1.1.15 and Remark 1.1.19 from [Mor05].

Theorem 5.4.6 (Ratner's Orbit Closure Theorem)**.** *Let G be a Lie group, Γ a lattice in G, H a subgroup generated by unipotent elements.*

Then for every $x \in G$, there is a closed subgroup S of G with the following properties:

a. $H \subseteq S$*;*

b. *Every connected component of S contains an element of H;*

c. $x^{-1}\Gamma x \cap S$ *is a lattice in* S*;*

d. *The double coset $\Gamma x H$ is dense in $\Gamma x S$.*

We will first apply the theorem to the case of $G = \mathrm{SL}_2(\mathbb{R})$ and $\Gamma = H = \mathrm{SL}_2(\mathbb{Z})$, and then use this to extend this to powers of $\mathrm{SL}_2(\mathbb{R})$.

Lemma 5.4.7 (See [Mor05, Exercise 1.27])**.** *Let $g \in G$. Then the double coset $\Gamma g \Gamma$ is discrete if and only if $g \in \mathrm{GL}_2(\mathbb{Q})^+$, and it is dense otherwise.*

Proof. First note that Γ is generated by the matrices $s = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $t = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$, which are unipotent. Hence we consider the action of Γ on $\Gamma \backslash G$ by right multiplication. Fix $g \in G$; by Ratner's theorem there is a subgroup S satisfying the properties listed in Theorem 5.4.6. Consider the connected component of the identity S°; this is a normal subgroup of S and, because $\Gamma \subseteq S$, S° is normalized by Γ.

Since G is a simple Lie group (it does not have non-trivial connected normal subgroups) and Γ is Zariski-dense in G as a consequence of the Borel density theorem ([Mor05, Theorem 4.7.1]), the only connected subgroups of G that are normalized by Γ are the trivial subgroup and G. Suppose $S^\circ = \{\mathbb{I}_2\}$; then, because every connected component of S contains an element of Γ, $S = \Gamma$. But then $g^{-1}\Gamma g \cap \Gamma$ is a lattice in Γ, which amounts to say that $g^{-1}\Gamma g$ and Γ are commensurable by Proposition 5.4.5. By Fact 5.2.4 this is the case if and only if $g \in \mathrm{GL}_2(\mathbb{Q})^+$, so we must have $S = G$; then $\Gamma g \Gamma$ is dense in $\Gamma g G = G$. $\square$

We now describe the full setting to which we want to apply the theorem. Fix $n \geq 2$ and let $G = \mathrm{SL}_2(\mathbb{R})^{n-1}$ and $\Gamma = \mathrm{SL}_2(\mathbb{Z})^{n-1}$. Consider now the subgroup $H = \Delta(\Gamma)$, where by Δ we mean the diagonal, i.e. the subgroup $\{(\gamma_2, \dots, \gamma_n) \in \Gamma \mid \gamma_2 = \dots = \gamma_n\}$. (It will hopefully become clear later why we start the enumeration at 2).

We can see H as a subgroup generated by unipotent elements by identifying G with the subgroup of $\mathrm{SL}_{2n-2}(\mathbb{R})$ of all matrices of the form

$$\begin{pmatrix} k_2 & \mathbb{O}_2 & \cdots & \mathbb{O}_2 \\ \mathbb{O}_2 & k_3 & \cdots & \mathbb{O}_2 \\ \vdots & \ddots & \cdots & \vdots \\ \mathbb{O}_2 & \cdots & \cdots & k_n \end{pmatrix}$$

with each $k_n \in \mathrm{SL}_2(\mathbb{R})$, where each $\mathbb{O}_2$ denotes a 2 by 2 block of zeros, and Γ and H with the corresponding subgroups. Then, as $\mathrm{SL}_2(\mathbb{Z})$ is generated by the unipotent elements $s = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $t = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$, H is generated by the matrices

$$\begin{pmatrix} s & \mathbb{O}_2 & \cdots & \mathbb{O}_2 \\ \mathbb{O}_2 & s & \cdots & \mathbb{O}_2 \\ \vdots & \ddots & \cdots & \vdots \\ \mathbb{O}_2 & \cdots & \cdots & s \end{pmatrix} \text{ and } \begin{pmatrix} t & \mathbb{O}_2 & \cdots & \mathbb{O}_2 \\ \mathbb{O}_2 & t & \cdots & \mathbb{O}_2 \\ \vdots & \ddots & \cdots & \vdots \\ \mathbb{O}_2 & \cdots & \cdots & t \end{pmatrix}$$

and so it makes sense to apply the theorem on cosets of groups generated by unipotent elements.

While representing Γ, G and H this way fits better with Ratner's theorem, we will stick with the previous definitions because they are equivalent and easier to work with.

For the general case we are going to need two group-theoretic facts: Goursat's lemma and a proposition on normal subgroups of powers of simple Lie groups.

Fact 5.4.8 (Goursat's Lemma, [Lan05, Exercise 1.5])**.** *Let G_1 and G_2 be groups, T a subgroup of $G_1 \times G_2$ such that the projections $\pi_i : T \to G_i$ are surjective for $i = 1, 2$. Identify the kernel N_1 of π_1 with a normal subgroup of G_2, and the kernel N_2 of π_2 with a normal subgroup of G_1.*

Then the image of T in $G_1/N_2 \times G_2/N_1$ is the graph of an isomorphism.

We want to study subgroups of $\mathrm{SL}_2(\mathbb{R})^{n-1}$, in order to understand which subgroups are associated to elements of the group by Ratner's theorem. First we remark that closed normal subgroups are products of trivial subgroups and copies of the group. Then we move on to study general connected subgroups, and see that graphs of inner automorphisms are the only additional possibility.

Proposition 5.4.9. *Let G be a simple Lie group, $N \trianglelefteq G^n$ a normal connected subgroup. Then N is a product of trivial groups and copies of G.*

Proof. Observe that if the k-th coordinate of an element is not the identity, then it can be moved to any element of the group by conjugating by an element that is the identity of all the coordinates except the k-th one. □

This suggests that in order to move forward we need an understanding of the group af automorphisms of $\mathrm{SL}_2(\mathbb{R})$. This has a very clear description as follows.

Proposition 5.4.10. *The Lie group automorphisms of $\mathrm{SL}_2(\mathbb{R})$ are exactly the automorphisms of the form $g \mapsto hgh^{-1}$ for matrices $h \in \mathrm{GL}_2(\mathbb{R})$.*

Proof. It is clear that every conjugation induces an isomorphism. To see that this is actually all of them we appeal to Section XV in Chapter IV of [Che46], where it is shown that the group of automorphisms of a Lie group embeds in the group of automorphisms of its Lie algebras, and Theorem 5 in Chapter 9 of [Jac79], where it is proved that all automorphisms of the Lie algebra of $\mathrm{SL}_2(\mathbb{R})$ are induced by conjugation by an element of $\mathrm{GL}_2(\mathbb{R})$. □

Proposition 5.4.11. *Let S be a closed connected subgroup of $G = \mathrm{SL}_2(\mathbb{R})^{n-1}$ such that all the projections $\pi_i : S \to \mathrm{SL}_2(\mathbb{R})$ are surjective. Then if $S \neq G$ there are $i, k \leq n-1$ and an element $h \in \mathrm{GL}_2(\mathbb{R})$ such that for all $g \in S$, $g_k = h g_i h^{-1}$.*

Proof. We do this by induction on n, with the base case $n = 3$.

So suppose $S \leq \mathrm{SL}_2(\mathbb{R})^2$ is a proper Lie subgroup with surjective projections. Then by Goursat's lemma there are normal subgroups (the kernels of the projections) N_1 and N_2 of $\mathrm{SL}_2(\mathbb{R})$ such that the image of S in $\mathrm{SL}_2(\mathbb{R})/N_2 \times \mathrm{SL}_2(\mathbb{R})/N_1$ is the graph of an isomorphism. If the kernels N_1 and N_2 are trivial, then S is the graph of an automorphism of $\mathrm{SL}_2(\mathbb{R})$. Since S is closed, it is a Lie subgroup, and hence it must be the graph of a continuous automorphism; by Proposition 5.4.10, these are given by conjugation by elements of $\mathrm{GL}_2(\mathbb{R})$. The only other non-trivial normal subgroup of $\mathrm{SL}_2(\mathbb{R})$ is the group $\{\mathbb{I}_2, -\mathbb{I}_2\}$, and in that case S would be given by the union of the graph of an automorphism φ and the set of elements of the form $(g, -\varphi(g))$. However, this would contradict the assumption that S is connected, so we are done.

Now suppose $n > 3$, and the result holds for $n-1$. If S is a subgroup of G, see it as a subgroup of $\mathrm{SL}_2(\mathbb{R})^{n-2} \times \mathrm{SL}_2(\mathbb{R})$. If the projection of S on $\mathrm{SL}_2(\mathbb{R})^{n-2}$ is not surjective, then it is a proper subgroup and we can conclude by the inductive hypothesis, so suppose it is surjective. Note that in this case any two of the first $n-2$ coordinates are not the graph of an automorphism.

We can now apply Goursat's lemma to S as a subgroup of $\mathrm{SL}_2(\mathbb{R})^{n-2} \times \mathrm{SL}_2(\mathbb{R})$, since both projections are surjective, and find normal subgroups N_1 and N_2 such that the image of S in $\mathrm{SL}_2(\mathbb{R})^{n-2}/N_2 \times \mathrm{SL}_2(\mathbb{R})/N_1$ is the graph of an isomorphism; again, N_1 must be trivial (for the same reasons why it had to be trivial in the base case: if it is $\mathrm{SL}_2(\mathbb{R})$ then the projection is not surjective, and if it is $\{\mathbb{I}_2, -\mathbb{I}_2\}$ then the resulting subgroup is not connected) and N_2 is a proper subgroup of $\mathrm{SL}_2(\mathbb{R})^{n-2}$. We focus on N_2: this is a normal subgroup of $\mathrm{SL}_2(\mathbb{R})^{n-2}$ such that the quotient $\mathrm{SL}_2(\mathbb{R})^{n-2}/N_2$ is isomorphic to $\mathrm{SL}_2(\mathbb{R})$, so by Proposition 5.4.9, and the assumption that S is connected, it must be a product of $n-3$ copies of $\mathrm{SL}_2(\mathbb{R})$ and a trivial group, say at coordinate k. Then $S = \{g \in \mathrm{SL}_2(\mathbb{R})^{n-1} | g_{n-1} = hg_kh^{-1}\}$ for some $h \in \mathrm{GL}_2(\mathbb{R})$. □

In the proof of Lemma 5.4.7 we used the fact that connected subgroups of $\mathrm{SL}_2(\mathbb{R})$ that are normalized by Γ are indeed normal, so they cannot be non-trivial as $\mathrm{SL}_2(\mathbb{R})$ is a simple Lie group. The corresponding statement for powers of $\mathrm{SL}_2(\mathbb{R})$ and the diagonal action we are considering is then the following.

Proposition 5.4.12. *Let S be a closed connected subgroup of $\mathrm{SL}_2(\mathbb{R})^n$, and suppose S is normalized by $H = \{(\gamma_2, \dots, \gamma_n) \in \mathrm{SL}_2(\mathbb{Z})^{n-1} \mid \gamma_2 = \dots = \gamma_n\}$. Then either S is normal, or there are $i, k \leq n$ such that $g_i = g_k$ for every $g \in S$.*

Proof. Let S be normalized by H, and suppose it is not normal. By restricting to a subgroup of $\mathrm{SL}_2(\mathbb{R})^m$ for some $m < n$ if necessary we can assume without loss of generality that the projections are surjective. Then by Proposition 5.4.11 there are $i, k \leq n$ and $h \in \mathrm{GL}_2(\mathbb{R})$ such that $g_k = hg_ih^{-1}$ for every $g \in S$. Hence it's enough to prove this when $n = 2$, and the rest will follow inductively.

So suppose S is a closed connected subgroup of $\mathrm{SL}_2(\mathbb{R})^2$; in this case, H is the diagonal of $\mathrm{SL}_2(\mathbb{Z})^2$, and S is the graph of conjugation by some matrix $h \in \mathrm{GL}_2(\mathbb{R})$. Hence,

$$S = \{(g, hgh^{-1}) \in \mathrm{SL}_2(\mathbb{R})^2 \mid g \in \mathrm{SL}_2(\mathbb{R})\}.$$

Since S is normalized by H, for every $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ and $g \in \mathrm{SL}_2(\mathbb{R})$ the point

$$(\gamma g \gamma^{-1}, \gamma hgh^{-1}\gamma^{-1})$$

belongs to S. However, as S is the graph of conjugation by h, it contains exactly one point whose first coordinate is $\gamma g \gamma^{-1}$, i.e.

$$(\gamma g \gamma^{-1}, h\gamma g \gamma^{-1}h^{-1}).$$

As this holds for all g and all γ, we can conclude that h commutes with all $\gamma \in \Gamma$.

But then, by Zariski density of $\mathrm{SL}_2(\mathbb{Z})$, it commutes with all of $\mathrm{SL}_2(\mathbb{R})$, and thus it is a scalar matrix; but then S is the diagonal of $\mathrm{SL}_2(\mathbb{R})$, as required. $\square$

We are now in a position to prove the following lemma.

Lemma 5.4.13. *Fix $(g_2, \ldots, g_n) \in G$. If the Möbius subvariety $L \subseteq \mathbb{H}^n$ defined by the conditions $z_i = g_i z_1$ for each $i = 2, \ldots, n$ is free, then the subgroup S associated to $(g_2, \ldots, g_n)$ by Ratner's theorem is G.*

Hence, with notations as above for Γ and H, $\Gamma(g_2, \ldots, g_n)H$ is dense in G.

Proof. We do this by induction on $n \in \mathbb{N} \setminus \{0, 1\}$, so the base case is $n = 2$. In that case the variety is free if and only if g_2 is not a scalar multiple of a rational matrix, so we just need to apply Lemma 5.4.7.

Now assume $n > 2$, and suppose S is a proper subgroup of G. Consider the connected component of the identity, S°. This is a normal subgroup of S, and since S contains H each coordinate of S° is normalised by Γ. Hence the projections of S° are either $\mathrm{SL}_2(\mathbb{R})$ or the trivial group. If the projection of S° to the k-th component is trivial then, because every connected component of S contains an element of H, $\pi_k(S) = \mathrm{SL}_2(\mathbb{Z})$. As $g^{-1}\Gamma g \cap \Gamma$ is a lattice in Γ, this implies that $g_k^{-1} \cdot \mathrm{SL}_2(\mathbb{Z}) \cdot g_k$ and $\mathrm{SL}_2(\mathbb{Z})$ are commensurable. Hence $g_k \in \mathrm{GL}_2(\mathbb{Q})^+$, contradicting the assumption that L was free.

Therefore we may assume that all the projections are surjective. By Proposition 5.4.11, then, there are two coordinates on which S is the graph of an automorphism. Therefore it suffices to prove the Lemma for $n = 3$, as any other case can be reduced to this one.

Suppose then that $n = 3$, so that $G = \mathrm{SL}_2(\mathbb{R})^2$ and $\Gamma = \mathrm{SL}_2(\mathbb{Z})^2$, and S is a proper subgroup of $G = \mathrm{SL}_2(\mathbb{R})^2$. Then S° must be the graph of the identity by Proposition 5.4.12. Since every connected component of S must contain an element of H, and $H \subseteq \Delta(\mathbb{R})$, $S = \Delta(\mathbb{R})$. By property *c.* in Ratner's theorem, $g^{-1}\Gamma g \cap \Delta(\mathbb{R})$ is a lattice in $\Delta(\mathbb{R})$: but by Proposition 5.4.5 this implies that $g_2^{-1} \cdot \mathrm{SL}_2(\mathbb{Z}) \cdot g_2$ and $g_3^{-1} \cdot \mathrm{SL}_2(\mathbb{Z}) \cdot g_3$ are commensurable. Hence, $\mathrm{SL}_2(\mathbb{Z})$ and $g_2 g_3^{-1} \cdot \mathrm{SL}_2(\mathbb{Z}) \cdot g_3 g_2^{-1}$ are commensurable and therefore $g_2 g_3^{-1}$ is a scalar multiple of a rational matrix, contradicting the assumption that the Möbius variety L was free. Hence, if L is free, then $\Gamma(g_2, g_3)H$ is dense in $\Gamma(g_2, g_3)G = G$, as required. $\square$

We get density of $j(L)$ as a corollary of Lemma 5.4.13. Note that, since free Möbius varieties can be seen as products of one-dimensional Möbius varieties, we just need to take care of this case and the extension will be immediate as products of dense sets are dense.

Proof of Lemma 5.4.2. Suppose that

$$L = \{z \in \mathbb{H}^n | z_i = g_i z_1 \text{ for } i = 2, \dots, n\}$$

and let U be an open subset in $\mathbb{C}^n$. Take the first projection $U_1 = \pi_1(U)$, and let $z \in \mathbb{H}$ be any point such that $j(z) \in U_1$. Because j is surjective and the action of $\mathrm{SL}_2(\mathbb{R})$ on $\mathbb{H}$ is transitive and continuous, there is an open subset O of $\mathrm{SL}_2(\mathbb{R})^{n-1}$ such that $(j(z), j(h_2 z), \dots, j(h_n z)) \in U$ if and only if $(h_2, \dots, h_n) \in O$. O is open, so by Lemma 5.4.13 there are $\gamma, \gamma_2, \dots, \gamma_n$ such that $(\gamma_2 g\gamma, \dots, \gamma_n g\gamma) \in O$. Then

$$(j(z), j(\gamma_2 g\gamma z), \dots, j(\gamma_n g\gamma z)) = (j(\gamma z), j(g_2 \gamma z), \dots, j(g_n \gamma z)) \in j(L) \cap U.$$

$\square$

5.5 Intersections

We are now ready to prove the existence of intersections between images of free Möbius subvarieties and algebraic subvarieties of $\mathbb{C}^n$. The first easy, but crucial, remark concerns the set of matrices that join two points in the upper half plane via Möbius transformations.

Proposition 5.5.1. *Let $z_1, z_2 \in \mathbb{H}$, with $z_l = x_l + iy_l$ for $l = 1, 2$. Then for any $c, d \in \mathbb{R}$ such that $|cz_1 + d|^2 = \frac{y_1}{y_2}$, there are $a, b \in \mathbb{R}$ such that if g is the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R})$, $gz_1 = z_2$.*

Proof. Direct calculations show that $\begin{pmatrix} \sqrt{y} & \frac{x}{\sqrt{y}} \\ 0 & \frac{1}{\sqrt{y}} \end{pmatrix} i = x + iy$, for all x and y. Therefore, since the stabilizer of i is $\mathrm{SO}_2(\mathbb{R})$, we find that for all θ:

$$\begin{pmatrix} \sqrt{y_2} & \frac{x_2}{\sqrt{y_2}} \\ 0 & \frac{1}{\sqrt{y_2}} \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{y_1}} & -\frac{x_1}{\sqrt{y_1}} \\ 0 & \sqrt{y_1} \end{pmatrix} z_1 = z_2$$

and that the lower entries of the product matrix are $-\frac{\cos\theta}{\sqrt{y_1 y_2}}$ and $x_1\frac{\cos\theta}{\sqrt{y_1 y_2}} + (\sin\theta)\sqrt{\frac{y_1}{y_2}}$. Therefore,

$$cz_1 + d = \sqrt{\frac{y_1}{y_2}}(\sin\theta - i\cos\theta).$$

□

This together with density of images of Möbius varieties is enough to prove the existence of intersections in the case $\dim L = 1$.

Lemma 5.5.2. *Let $L \times W$ be a subvariety of $\mathbb{H}^n \times \mathbb{C}^n$ such that L is a free Möbius variety of dimension* 1*, W is an algebraic variety, and* $\dim L + \dim W \geq n$*. Then W has a dense subset of points of $j(L)$.*

We will obtain this as a corollary of a stronger result, which does not require W to be an algebraic variety.

We recall the well-known *argument principle* from complex analysis, as we will use it in our proof.

Theorem 5.5.3 (Argument Principle, [Ahl66, Theorem 5.20])**.** *Let $U \subseteq \mathbb{C}$ be a bounded open set, f a function that is holomorphic on $\overline{U}$ with no zeros on ∂U.*

Then the number of zeros of f in U is equal to

$$\frac{1}{2\pi i}\int_{\partial U} \frac{f'(z)}{f(z)} dz.$$

Lemma 5.5.4. *Let $g_2, \ldots, g_n \in \mathrm{SL}_2(\mathbb{R})$; V an open subset of $\mathbb{C}^n$, $f : V \to \mathbb{C}$ a holomorphic function. Denote by W the zero locus of f. Then W has a dense subset of points of the form $(j(z_1), j(g_2 z_1), \ldots, j(g_n z_1))$.*

Proof. Let $L = \{(z_1, \ldots, z_n) \mid z_i = g_i z_1 \text{ for } i = 2, \ldots, n\}$.

If f is of the form $X_i - c$ for some $c \in \mathbb{C}$, without loss of generality we can assume $i = 1$, and then the result follows directly from density of $j(L)$: take z_1 to be any element such that $j(z_1) = c$, and if O is any open subset of $\mathbb{C}^{n-1}$, then we find $\gamma_2 g_2 \gamma, \ldots, \gamma_n g_n \gamma$ such that

$$(j(\gamma_2 g_2 \gamma z_1), \ldots, j(\gamma_n g_n \gamma z_1)) \in O.$$

Then the point $(j(\gamma z), j(g_2\gamma z), \dots, j(g_n\gamma z))$ belongs to the open subset $\{c\}\times O$ of W. Hence from now on we assume W has no constant coordinates.

Let $w \in W$ be a regular point such that no coordinate of w is 0 or 1728. Then find a point $z \in \mathbb{H}$ such that $j(z) = w$, and find $h_2, \dots, h_n \in \mathrm{SL}_2(\mathbb{R})$ such that $z_i = h_i z_1$ for $i = 2, \dots, n$; we denote by h the tuple $(1, h_2, \dots, h_n)$. Then consider the function $F : \mathbb{H} \to \mathbb{C}$, mapping z to $f(j(z), j(h_2 z), \dots, j(h_n z))$: by construction $F(z_1) = 0$, so consider a small compact neighbourhood U of z_1.

By density, there is a sequence $\{g^i\}_{i\in\mathbb{N}}$, converging to h, such that each g^i is a tuple of the form

$$(g_2^i, \dots, g_n^i) = (\gamma_2^i g_2 \gamma^i, \dots, \gamma_n^i g_n \gamma^i)$$

for some $\gamma^i, \gamma_2^i, \dots, \gamma_n^i \in \mathrm{SL}_2(\mathbb{Z})$, where $g_2, \dots, g_n$ are the matrices defining L.

Then consider the sequence of functions $\{F_i\}_{i\in\mathbb{N}}$, where each F_i is defined on U as $F_i(z) = f(j(z), j(g_2^i z), \dots, j(g_n^i z))$. It is then clear that

$$\lim_{i\in\mathbb{N}} F_i(z_1) = F(z_1) = 0.$$

Now consider the derivative of F. Writing $j(hz_1)$ for the tuple

$$(j(z_1), j(h_2 z_1), \dots, j(h_n z_1)),$$

we have that

$$\frac{d}{dz}F(z_1) =$$

$$= \frac{\partial f}{\partial Y_1}(j(hz_1))j'(z_1) + \frac{\partial f}{\partial Y_2}(j(hz_1))(j(h_2 z_1))' + \dots + \frac{\partial f}{\partial Y_n}(j(hz_1))(j(h_n z_1))'$$

where if $h_i = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $(j(h_i z_1))' = \frac{j'(h_i z_1)}{(cz_1+d)^2}$.

It is clear that $\left\{\frac{d}{dz}F_i(z_1)\right\} \to_{i\in\mathbb{N}} \frac{d}{dz}F(z_1)$. Therefore if $\frac{d}{dz}F(z_1) \neq 0$, then F is not constant and thus for sufficiently large i the functions $F_i(z_1)$ are not constant either. Thus we may assume neither F nor the F_i's have zeros on ∂U. Since $F(z_1) = 0$, by the argument principle we have

$$\int_{\partial U} \frac{F'(z)}{F(z)} dz \neq 0$$

and therefore, by uniform convergence,

$$\int_{\partial U} \frac{F_i'(z)}{F_i(z)} dz \neq 0$$

for sufficiently large i. This implies that F_i has a zero in U.

Claim: Without loss of generality we can assume that $\frac{d}{dz}F(z_1) \neq 0$.

Proof of Claim: Suppose $\frac{d}{dz}F(z_1) = 0$. Because $w = j(hz_1)$ is a regular point in W and no coordinate of w is 0 or 1728, the summands in $\frac{d}{dz}F(z_1)$ are not all zero. Then in particular at least two of them are not zero, and so there is $l > 1$ such that $\frac{\partial f}{\partial Y_l}(j(hz_l))\frac{j'(h_l z_1)}{(cz_1+d)^2} \neq 0$. But then it is enough to change the matrix h_l, which we are free to do by Proposition 5.5.1: there is a matrix h_l' such that $h_l' z_1 = h_l z_1$, but $(j(h_l' z_1))' \neq (j(h_l z_1))'$. This proves the claim.

Then we are done: for sufficiently large i apply the argument principle and find z_0 close to z_1 such that $F_i(z_0) = 0$, i.e., such that

$$f(j(z_0), j(\gamma_2 g_2 \gamma z_0), \ldots, j(\gamma_n g_n \gamma z_0)) = 0$$

for some $\gamma, \gamma_2, \ldots, \gamma_n$. Then $(j(\gamma z_0), j(g_2\gamma z_0), \ldots, j(g_n \gamma z_0)) \in j(L) \cap W$. □

Proof of Lemma 5.5.2. Apply Lemma 5.5.4 in the case where $V = \mathbb{C}^n$ and f is a polynomial. □

The following technical lemma will be used to prove the main result.

Lemma 5.5.5. *Let $W \subseteq \mathbb{C}^n$ be an irreducible algebraic variety. There is a non-empty Zariski open subset W' of W such that for any Möbius variety $L \subseteq \mathbb{H}^n$ such that $L \times W$ is free broad, any intersection between $L \times W'$ and the graph Γ_j of j is typical (i.e., $\dim((L \times W') \cap \Gamma_j) = \dim L \times W' - n$).*

Proof. Let $I = (i_1, \ldots, i_k)$ be a tuple of coordinates and consider as usual the projection π_I; by abuse of notation we use π_I to denote the projections on both $\mathbb{H}^n$, and $\mathbb{C}^n$ as it is clear from context where we are using it.

For any such projection there is a non-empty Zariski-open subset $W_I \subseteq W$ such that for every $w \in W_I$, $\dim(\pi_I^{-1}(\pi_I(w))) = \dim W - \dim \pi_I(W)$.

By the weak modular Zilber-Pink theorem for parametric families, Theorem

5.3.4, if we consider the family

$$\{W_{w_I} \mid w_I \in \pi_I(W)\},$$

there are finitely many weakly special subvarieties with no constant coordinates of $\mathbb{C}^n$ (and hence a Zariski-closed subset S_I of W) such that any maximal atypical component of one of the varieties W_{w_I} is contained in S_I. Thus, define

$$W' := W \setminus \bigcup_{I \subseteq [n]} (S_I \cup W_I).$$

Now let L be a Möbius variety such that $L \times W$ is free broad. Suppose U is a bounded open subset of L whose image under j is analytic, that C is an irreducible component of the intersection $j(U) \cap W'$, and that

$$\dim C > \dim L + \dim W - n \geq 0.$$

Now let I be the maximal subset of $\{1, \ldots, n\}$ for which coordinates with the corresponding indices are constant on C; then consider its projection π_I.

Now C is contained in a single fibre of π_I; consider the fibres L_c and $W_{j(c)}$, for some $c \in \mathbb{H}^{n_I}$, and the complement I_0 of the set I. The projection π_{I_0} has zero-dimensional fibres on L_c and on $W_{j(c)}$, by maximality of I, so it preserves dimensions. The component $\pi_{I_0}(C)$ of the intersection

$$j(\pi_{I_0}(U_c)) \cap \pi_{I_0}(W_c)$$

must have typical dimension: if it did not, it would give rise to an atypical component of the intersection of $L \times W$ with the graph of j. By the Ax-Schanuel Theorem for j (Theorem 5.3.1) it would then have to be contained in a weakly special subvariety, but it cannot have constant coordinates, by maximality of I, and it does not satisfy modular relations because we assumed that C is not contained in $W \setminus W'$. Thus,

$$\dim C = \dim \pi_{I_0}(C) = \dim L_c + \dim W_{j(c)} - (n - n_I) > \dim L + \dim W - n$$

$$\dim W_{j(c)} > \dim L - \dim L_c + \dim W - n_I.$$

However, the fibre dimension theorem and broadness of the variety imply that

$$\dim W_{j(c)} = \dim W - \dim \pi_I(W) \leq \dim W - n_I + \dim \pi_I(L).$$

Comparing these, we obtain:

$$\dim W - n_I + \dim \pi_I(L) > \dim L - \dim L_c + \dim W - n_I$$

$$\dim L_c + \dim \pi_I(L) > \dim L$$

which cannot hold as L is a Möbius variety. Therefore the component C cannot be atypical, and any atypical component must be contained in $W \setminus W'$. $\square$

Finally, we note that just as in the case of the exponential function (Lemma 3.2.7) there is no harm in assuming that given a free broad subvariety $L \times W \subseteq \mathbb{H}^n \times \mathbb{C}^n$, $\dim L + \dim W = n$.

Lemma 5.5.6. *Let $L \times W$ be a free broad variety. Then there is an algebraic subvariety $W' \subseteq W$ such that $L \times W'$ is free and broad and* $\dim L + \dim W' = n$.

Proof. Let $I = (i_1, \ldots, i_k)$ be a tuple of coordinates and consider as usual the projection π_I; again, it denotes the projections on both $\mathbb{H}^n$ and $\mathbb{C}^n$.

By rotundity, $\dim \pi_I(L) + \dim \pi_I(W) \geq k$. Therefore, there is a Zariski-open subset W° of W such that locally around every point of W° the restriction of π_I has image of dimension at least $k - \dim \pi_I(L)$. Intersecting W with a generic hyperplane which passes through one such point we can clearly maintain freeness of the variety and the dimension inequality.

As there are only finitely many projections to check, we pick a hyperplane which maintains the dimension inequality for each projection associated to an ordered subtuple of $(1, \ldots, n)$. This brings down the dimension of W by one; repeating this finitely many times we obtain a W' of dimension $n - \dim L$. $\square$

We can now prove the main result.

Theorem 5.5.7. *Let $L \times W$ be a free broad subvariety of $\mathbb{H}^n \times \mathbb{C}^n$ with L a Möbius variety. Then W contains a subset of points of $j(L)$ which is dense in the Euclidean topology.*

Proof. We do this by induction on $d = \dim L$. The case $d = 1$ is Lemma 5.5.2, so suppose the theorem holds for d, L has dimension $d+1$, and W has dimension $n-d-1$ (which we may do by Lemma 5.5.6).

By definition of Möbius subvariety, after reordering the coordinates if necessary we can write L as a product $L_1 \times \cdots \times L_{d+1}$, where each L_i is a one-dimensional Möbius subvariety. There are numbers n_1, n_2 such that $n_1 + n_2 = n$, $L' := L_1 \times \cdots \times L_d$ is a d-dimensional Möbius subvariety of $\mathbb{H}^{n_1}$ and L_{d+1} is a 1-dimensional Möbius subvariety of $\mathbb{H}^{n_2}$; let $\pi_i : \mathbb{C}^n \to \mathbb{C}^{n_i}$, for $i = 1, 2$ denote the corresponding projections on the codomain. By broadness, $\pi_1(W)$ has dimension at least $n_1 - d$, and hence by the inductive hypothesis it contains a dense subset of points of $j(L')$; now there are two cases.

If $\dim(\pi_1(W)) = n_1 - d$, then by the fibre dimension theorem any point $w_1 \in \pi_1(W) \cap j(L')$ has a fibre W_{w_1} of dimension at least $\dim W - (n_1 - d) = n_2 - 1$. Therefore by Lemma 5.5.4 $\pi_2(W_{w_1})$ has a dense subset of points of $j(L_{d+1})$, and we are done.

If $\dim(\pi_1(W)) = n_1 - d + k$ for some positive k, then for a generic point $w \in \pi_1(W)$ the fibre W_w has dimension $n_2 - 1 - k$, so we cannot argue as in the previous case.

As the variety $L' \times \pi_1(W)$ is broad, by the inductive hypothesis and Lemma 5.5.6, we may assume $L' \times \pi_1(W)$ intersects the graph of j in an analytic set of dimension $k' \geq k$. Then, denoting by Γ_j the graph of j,

$$\pi_1^{-1}(L' \times \pi_1(W) \cap \Gamma_j) = (L' \times \mathbb{C}^{n_2}) \times W \cap \Gamma_j$$

has dimension $k' + n_2 - 1 - k \geq n_2 - 1$.

Now let U be a small open ball in L', so that $(j(U) \times \mathbb{C}^{n_2}) \cap W$ is an analytic set in $j(U) \times \mathbb{C}^{n_2}$, and π_{res} denote the restriction of the second projection π_2 to the set $(j(U) \times \mathbb{C}^{n_2}) \cap W$.

Suppose that π_{res} is finite: then it is proper. We prove this using Proposition 2.5.8. Given that π_{res} is finite, there is a ball $B \subseteq j(U)$ such that $W_{w_2} \cap j(U) \times \mathbb{C}^{n_2}$ does not intersect the set $\partial B \times \{w_2\}$. As $W \cap j(U) \times \mathbb{C}^{n_2}$ is closed in $j(U) \times \mathbb{C}^{n_2}$ and π_{res} is finite and hence open, this property actually holds in a neighbourhood of w_2; therefore, by Proposition 2.5.8, by taking U sufficiently small we can make sure that the map π_{res} is proper.

So, under the assumption that π_{res} is finite, we can apply the Proper Mapping Theorem (Theorem 2.5.10), which states that the image of π_{res} is an analytic set. Since we proved $j(U) \times \mathbb{C}^{n_2} \cap W$ has dimension at least $n_2 - 1$, the image of π_{res} is either an open subset of $\mathbb{C}^{n_2}$, or an analytic set in $\mathbb{C}^{n_2}$ of codimension 1; either way, using density of $j(L_{d+1})$ in the first case and Lemma 5.5.4 otherwise, we can find a point $w \in W \cap j(U) \times \mathbb{C}^{n_2}$ such that $\pi_2(w) \in j(L_{d+1})$; therefore $w \in j(L) \cap W$, as we wanted.

Therefore, it remains to show that if we choose the ball U appropriately, then the map π_{res} is finite. Consider a generic point $w_2 \in \pi_2(W)$, and its fibre

$$W_{w_2} := \{w \in W \mid \pi_2(w) = w_2\}.$$

Using once again broadness and the fibre dimension theorem, we know that

$$\dim W_{w_2} = \dim W - \dim \pi_2(W) \leq n - d - 1 - n_2 + 1 = n_1 - d.$$

Hence,

$$\dim W_{w_2} + \dim(j(L') \times \mathbb{C}^{n_2}) \leq n_1 - d + d + n_2 = n.$$

Therefore any positive dimensional intersection between these varieties is of atypical dimension; by Lemma 5.5.5, it suffices to make sure that $j(U) \times \mathbb{C}^{n_2} \cap W$ is contained in the Zariski-open non-empty subset W' of W for this property to be verified. This set clearly contains points of $j(L') \times \mathbb{C}^{n_2}$, because $\pi_1(W) \cap j(L')$ is dense in $\pi_1(W)$, so we are done. $\square$

Example 5.5.8. Let $L = \{(z_1, z_2) \in \mathbb{H} \mid z_2 = \sqrt{2} z_1\}$ (we saw in Example 5.3.9 that this is a Möbius variety) and $W = \{(w_1, w_2) \in \mathbb{C}^2 \mid w_1 + w_2 + 1 = 0\}$.

Let $(w_1, w_2) \in W$: for example, we might take $w_1 = w_2 = -\frac{1}{2}$. We then find $z \in \mathbb{H}$ such that $j(z) = -\frac{1}{2}$. We may take it so that $j'(z) \neq -\frac{1}{2}$: if it is equal, we take some γz for which $j'(\gamma z)$ is different.

Ratner's theorem says that we can find a sequence $\{g^i\}_{i \in \omega}$ in the double coset $\Gamma g \Gamma$, with $g = \begin{pmatrix} 2^{\frac{1}{4}} & 0 \\ 0 & 2^{-\frac{1}{4}} \end{pmatrix}$ that converges to the identity.

It is then clear that $\{j(z) + j(g^i z) + 1\}_{i \in \omega}$ converges to 0, and the sequence of its derivatives converges to $2j'(z) + 1 \neq 0$ (because $j'(z) \neq -\frac{1}{2}$). Therefore by applying the argument principle we may find some $\gamma_1 g \gamma_2$ and some z_0 close to

z such that $j(z_0) + j(\gamma_1 g \gamma_2 z_0) + 1 = 0$. Therefore, $j(\gamma_2 z_0) + j(g\gamma_2 z_0) + 1 = 0$, and

$$(\gamma_2 z_0, g\gamma_2 z_0, j(\gamma_2 z_0), j(g\gamma_2 z_0)) \in L \times W$$

as we wanted.

5.6 Derivatives of the j-Function

5.6.1 Background and Notation

We conclude with some remarks on extensions of the results in the previous section to the first derivative of the j-function. Recall that j, j' and j'' are algebraically independent, and therefore many results in this area (for example the Ax-Schanuel Theorem) tend to consider them simultaneously. The methods in this chapter seem to be insufficient to address the problems of systems of equations which involve all functions simultaneously; however, they can be employed to approximately solve systems of equations involving j'. We take the chance to illustrate a slightly different technique in the proof.

First of all we remark that while j is a modular function, j' and j'' are not: j' is a modular form of weight 2, and the transformation law for j'' under the action of $\mathrm{SL}_2(\mathbb{Z})$ is more complicated. This means that the methods in the previous section do not apply directly if we look for intersections of the form $j'(L) \cap W$ for L a Möbius subvariety of $\mathbb{H}^n$.

Therefore to get more general results we need to work in jet spaces. The action of $\mathrm{SL}_2(\mathbb{R})$ on $\mathbb{H}$ induces an action on $J_2\mathbb{H}$, $J_2\mathcal{F}$ is a fundamental domain for the action of $\mathrm{SL}_2(\mathbb{Z})$ and $J_2 j$ is invariant under the action of $\mathrm{SL}_2(\mathbb{Z})$, so this seems like a better framework to work in.

We recall some general facts about jet spaces, and provide explicit computations for the facts stated above.

Definition 5.6.1. Let M be a complex analytic manifold. The *k-th jet space* of M for a natural number k is the space of equivalence classes of holomorphic functions from a small neighbourhood of $0 \in \mathbb{C}$ into M, identifying maps that are equal up to order k.

We will only be interested in second jets, so we assume $k = 2$ in the following.

An element in $J_2\mathbb{H}$ is then a triple (z, r, s), where $z \in \mathbb{H}$, $r, s \in \mathbb{C}$, that corresponds to the function $f : U \to \mathbb{H}$ taking w to $z + rw + s\frac{w^2}{2}$.

Jets are a functorial construction: given a map $\varphi : M \to N$, there is an induced map $J_k\varphi : J_kM \to J_kN$, that takes the equivalence class of the function $f : U \to M$ to that of $\varphi \circ f$. Therefore, if for a fixed $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R})$ we consider the map $g \cdot (-) : \mathbb{H} \to \mathbb{H}$, we can see what the action induced on $J_2\mathbb{H}$ is:

$$g \cdot (z, r, s) = \left(\frac{az + b}{cz + d}, \frac{r}{(cz + d)^2}, \frac{s}{(cz + d)^2} - \frac{2cr^2}{(cz + d)^3} \right)$$

Similarly we have to consider the second jet of the j-function itself, which is obtained as:

$$J_2j(z, r_1, r_2) = \Big(j(z), j'(z)r, j''(z)r^2 + j'(z)s \Big)$$

so that in particular for example $J_2j(z, 1, 0) = (j(z), j'(z), j''(z))$.

Using the transformation laws for j' and j'', i.e., for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$,

$$j'(\gamma z) = (cz + d)^2 j'(z)$$

and

$$j''(\gamma z) = (cz + d)^4 j''(z) + 2c(cz + d)^3 j'(z)$$

one can prove that $J_2j(\gamma \cdot (z, r, s)) = J_2j(z, r, s)$.

5.6.2 Intersections for j'

For simplicity, we deal with the case of a subvariety $L \times W$ of $\mathbb{H}^2 \times \mathbb{C}^2$, as this implies the general case.

The fact that j' is not $\mathrm{SL}_2(\mathbb{Z})$-invariant leads us to consider a blurring of a Möbius subvariety. This is similar to what is done in [Kir19b] for the exponential function and in [AK21] for j, but note that here the blurring is not fixed, and it depends on the variety that we are considering.

Definition 5.6.2. Let L be the Möbius subvariety of $\mathbb{H}^2$ of points of the form

(z, gz), for some $g = \begin{pmatrix} a_g & b_g \\ c_g & d_g \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R})$. Consider the tangent variety

$$L' := \left\{ (z_1, z_2, z_3, z_4) \in J_1\mathbb{H}^2 \mid z_2 = gz_1, z_3 = 1, z_4 = \frac{1}{(c_g z_1 + d_g)^2} \right\}.$$

The *blurring* of L' by $\mathrm{SL}_2(\mathbb{Z})$ is the set L^*, defined as

$$\left\{ (z_1, gz_1, z_3, z_4) \in J_1\mathbb{H}^2 \mid z_3 = \frac{1}{(c_\gamma(\gamma^{-1}z_1) + d_\gamma)^2}, z_4 = \frac{1}{(c_{g\gamma}(\gamma^{-1}z_1) + d_{g\gamma})^2} \right\}$$

where $\gamma = \begin{pmatrix} a_\gamma & b_\gamma \\ c_\gamma & d_\gamma \end{pmatrix}$ varies in $\mathrm{SL}_2(\mathbb{Z})$ and $g\gamma = \begin{pmatrix} a_{g\gamma} & b_{g\gamma} \\ c_{g\gamma} & d_{g\gamma} \end{pmatrix}$.

Definition 5.6.2 needs to be explained. A point in the first jet of $\mathbb{H}^2$ carries two bits of information: the first two coordinates give a point in $\mathbb{H}^2$, while the second two coordinates refer to the direction from which the point is approached. Hence, the variety L' is the variety of points in L, where to each point we associate the natural direction of the parametrization $z \mapsto (z, gz)$. When we move to the blurred version L^*, its projection to the first two coordinates is still the original Möbius variety L; however, each of these points is considered together with the directions associated to the parametrizations $z \mapsto (\gamma z, g\gamma z)$, for all $\gamma \in \mathrm{SL}_2(\mathbb{Z})$.

Let $T_1 j : J_2\mathbb{H}^2 \to \mathbb{C}^2$ denote the composition $\pi \circ J_1 j$, where $\pi : J_2\mathbb{C}^2 \cong \mathbb{C}^4 \to \mathbb{C}^2$ is the projection on the third and fourth coordinate. Hence,

$$T_1(j)(z_1, z_2, r_1, r_2) = \left(j'(z_1)r_1, j'(z_2)r_2\right).$$

This allows to prove the following proposition:

Proposition 5.6.3. *Let L be a free Möbius subvariety of $\mathbb{H}^2$. Then $T_1 j(L^*)$ is dense in $\mathbb{C}^2$.*

Proof. Let (w_1, w_2) be a point in $\mathbb{C}^2$. As j' is surjective, find z_1 such that $j'(z_1) = w_1$, and $h = \begin{pmatrix} a_h & b_h \\ c_h & d_h \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R})$ such that $\frac{j'(hz_1)}{(c_h z_1 + d_h)^2} = w_2$. Then, as in the proof of Lemma 5.4.2, we can consider a sequence of matrices in $\Gamma g \Gamma$ that tends to h; denote a specific element of this sequence by $\gamma_1 g \gamma$. Then we just need to prove that $\left(j'(z_1), \frac{j'(\gamma_1 g\gamma z_1)}{(c_{\gamma_1 g\gamma} z_1 + d_{\gamma_1 g\gamma})^2}\right)$ is in $T_1 j(L^*)$. Consider the point

$$z = \left(z_1, \gamma_1 g\gamma z_1, 1, \frac{1}{(c_{\gamma_1 g\gamma} z_1 + d_{\gamma_1 g\gamma})^2}\right).$$

It's clear that

$$J_1 j(z) = \left(j(z_1), j(\gamma_1 g\gamma z), j'(z_1), \frac{j'(\gamma_1 g\gamma z_1)}{(c_{\gamma_1 g\gamma} z_1 + d_{\gamma_1 g\gamma})^2}\right)$$

Since $J_1 j$ is invariant under the action of $\mathrm{SL}_2(\mathbb{Z})$, it is then sufficient to find an element $\overline{\gamma} \in \mathrm{SL}_2(\mathbb{Z})^2$ such that $\overline{\gamma} \cdot z \in L^*$. Then of course we consider $\overline{\gamma} = (\gamma, \gamma_1^{-1})$.

Claim: $\overline{\gamma} \cdot z = \left(\gamma z_1, g\gamma z_1, \frac{1}{(c_\gamma z_1 + d_\gamma)^2}, \frac{1}{(c_{g\gamma} z_1 + d_{g\gamma})^2}\right)$.

Proof of Claim: It is immediate for the first three coordinates, the only issue is on the fourth one. Using the well-known cocylce relation for automorphy factors we obtain that

$$(c_{g\gamma} z + d_{g\gamma z})^2 = (c_{\gamma_1 g\gamma} z + d_{\gamma_1 g\gamma})^2 (c_{\gamma_1^{-1}}(\gamma_1 g\gamma z) + d_{\gamma_1^{-1}})^2.$$

But the fourth coordinate of $\overline{\gamma} \cdot z$ is

$$\frac{1}{(c_{\gamma_1 g\gamma} z + d_{\gamma_1 g\gamma})^2 (c_{\gamma_1^{-1}}(\gamma_1 g\gamma z) + d_{\gamma_1^{-1}})^2}$$

which is then equal to

$$\frac{1}{(c_{g\gamma} z + d_{g\gamma z})^2}$$

proving the claim.

It remains to show that $\overline{\gamma} \cdot z$ is indeed a point in L^*. The condition $z_2 = g z_1$ is clearly satisfied. The second condition in the definition of the blurring is given by $\frac{1}{(c_\gamma(\gamma^{-1}\gamma z_1) + d_\gamma)^2} = \frac{1}{(c_\gamma z_1 + d_\gamma)^2} = z_3$. Similarly the last condition is satisfied.

This concludes the proof: the sequence $T_1 j((g_i) z_1)$ approximates w_2 and hence $T_1 j(L^*)$ is dense in $\mathbb{C}^2$. $\square$

Remark 5.6.4. The crucial piece of information is that while j' is not a modular function, it is a modular form of weight 2 for $\mathrm{SL}_2(\mathbb{Z})$, and hence it is easy to come up with the blurring that makes the density argument work. This leads to speculations about similar results for modular forms in more generality - although we are not aware of any framework that could motivate such an investigation.

We now close the argument, showing that given an open neighbourhood U inside a complex algebraic variety $W \subseteq \mathbb{C}^2$, the intersection $U \cap T_1 j(L^*)$ is non-empty. While in the previous section we used transversality, that approach is more complicated here as it requires explicit, hands-on computations involving j' and j'' which are hard to solve. Instead, we go down a different road and, recalling that the restrictions of j and its derivatives to fundamental domains are definable in the o-minimal structure $\mathbb{R}_{\text{an,exp}}$ by well-known results of Peterzil-Starchenko (see [PS04, Theorem 4.1]), we use o-minimality. In particular, we will need the following result of Johns.

Theorem 5.6.5 ([Joh01, Theorem])**.** *Let $R = (R, <, ...)$ be an o-minimal structure, $U \subseteq R^n$ an open set, $f : U \to R^n$ a continuous, definable, injective function. Then f is open.*

We will also need an analogue for j' of Proposition 5.5.1, giving that the fibre of certain maps are one-dimensional subsets of $\mathrm{SL}_2(\mathbb{R})$.

Proposition 5.6.6. *Let $(w_1, w_2) \in W$. Then given z such that $j'(z) = w_1$, there is a definable one-dimensional subset H of $\mathrm{SL}_2(\mathbb{R})$ such that for every $h = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in H$, $\frac{j'(hz)}{(cz+d)^2} = w_2$.*

Proof. Fix w and z as in the statement. First we note that, as $\frac{1}{|cz+d|^2} = \frac{\mathrm{Im}(hz)}{\mathrm{Im}(z)}$, $\frac{j'(hz)}{(cz+d)^2} = w_2$ implies

$$|w_2| = \left| \frac{j'(hz)}{(cz+d)^2} \right| = \left| j'(hz) \frac{\mathrm{Im}(hz)}{\mathrm{Im}(z)} \right|.$$

Hence, for h to satisfy $(j(hz))' = \frac{j'(hz)}{(cz+d)^2} = w_2$, it is necessary that

$$|j'(hz)| \mathrm{Im}(hz) = |w_2| \mathrm{Im}(z).$$

Now let us consider the function $\varphi : w \mapsto j'(w)\mathrm{Im}(w)$. By using the Fourier series expansion of j,

$$j(z) = \sum_{n=-1}^{\infty} c_n \exp(2\pi n i z)$$

where all c_n's are positive integers (see for example [Rad38]) we see that j' takes imaginary values on $i\mathbb{R}_{\geq 1}$ and that its absolute value is strictly increasing; a direct computation shows that the Jacobian of φ as a function $\mathbb{R} \times \mathbb{R}_{>0} \to \mathbb{R}^2$ is nonsingular on points of $\{0\} \times \mathbb{R}_{\geq 1}$. Hence, as $j'(i) = 0$ and j' has a simple pole at infinity, for every $x \in \mathbb{R}_{\geq 0}$ there is a $z \in i\mathbb{R}_{\geq 1}$ such that $|\varphi(z)| = x$ and φ is open around z.

Then the definable set of w's such that $|j'(w)|\mathrm{Im}(w) = |w_2|\mathrm{Im}(z)$ is non-empty and one-dimensional. For each of these w's, by Proposition 5.5.1, there is a unique $h \in \mathrm{SL}_2(\mathbb{R})$ such that $hz = w$ and $j(hz)' = w_2$, so we obtain H as required. □

Lemma 5.6.7. *Let $W \subseteq \mathbb{C}^2$ be a one-dimensional algebraic variety not contained in the image through $T_1 j$ of a weakly special variety, $U \subseteq W$ an open subset. Then there is an open subset $O \subseteq \mathrm{SL}_2(\mathbb{R})$ such that for every $h \in O$, there is a point in U of the form $(j'(z), j(hz)')$.*

In particular, by the results in Section 5.4, $O \cap \Gamma g \Gamma \neq \varnothing$ for every g that is not a multiple of a rational matrix.

Proof. Let f be the polynomial defining W, and let S be a connected component of the set

$$\{(z,h) \in \mathbb{C} \times \mathrm{SL}_2(\mathbb{R}) \mid f(j'(z), (j(hz))') = 0\}.$$

Denote by $\pi_1(S)$ the projection to $\mathbb{C}$ and by $\pi_2(S)$ the projection to $\mathrm{SL}_2(\mathbb{R})$.

S is definable in $\mathbb{R}_{\text{an, exp}}$. Since Skolem functions are definable in o-minimal structures, there is a definable function $F : \pi_1(S) \to \mathrm{SL}_2(\mathbb{R})$ that satisfies $(z, F(z)) \in S$ for each $z \in \pi_1(S)$. We can assume F is injective, as W is not contained in the image of a weakly special variety, and therefore by Theorem 5.6.5 it is open on its image. Every point $F(z)$ in the image has a small one-dimensional neighbourhood V such that actually $(z, v) \in S$ for every $v \in V$, by Proposition 5.6.6. Therefore $\pi_2(S)$ has non-empty interior, and we are done. □

Now the following analogue of Theorem 5.5.7 for j' can be obtained by applying the same argument.

Theorem 5.6.8. *Let $L \times W$ be a free broad subvariety of $\mathbb{H}^n \times \mathbb{C}^n$, with L a Möbius subvariety of $\mathbb{H}^n$ and W an algebraic variety. Then W has a dense* *subset of points of* $T_1(j(L^*))$.

Chapter 6

Future Directions

6.1 Introduction

We conclude the thesis by mentioning some of the many open problems in the area. The main end goal, of course, remains proving quasiminimality of $\mathbb{C}_{\exp}$ by way of Exponential-Algebraic Closedness: therefore, the next step is to generalize the results of this thesis, and in particular those of Chapter 3, to establish that the conjecture holds for larger and larger classes of algebraic varieties. At the same time, we have seen the importance of existential-closedness-type conjectures for other functions, and of course one would like to make progress on those conjectures as well.

On the other hand, it should not be forgotten that the motivation for these problems is model-theoretical: other than quasiminimality of $\mathbb{C}_{\exp}$ there are other questions that float around, concerning the search for other quasiminimal structures and the model-theoretic analysis of transcendental functions.

In this chapter we collect some questions which we hope to make progress on over the next few years. In particular, Section 6.2 focuses on questions of model-theoretic flavours, while Section 6.3 lists some problems around Exponential-Algebraic Closedness that should be treatable with the tools we have now.

6.2 Model-Theoretic Directions

6.2.1 Raising to Powers and Quasiminimality

The results of this thesis lean towards the complex-analytic side of the problem, the concrete question of establishing the existence of solutions.

However, there are still many interesting model-theoretic aspects to be considered. The first question that comes to mind is the following: we have seen how Theorem 3.7.8 is the existential closedness statement for some first-order classes of structures, axiomatised by the theories T_K as K ranges over the finite transcendence degree subfields of $\mathbb{C}$.

It should not be forgotten, though, that the natural setting for the model theory of the exponential function is not first order logic: it would be interesting to see whether Theorem 3.7.8 can be the existential closedness statement of a theory in $\mathcal{L}_{\omega_1,\omega}(Q)$ with quasiminimal models. This possibility was mentioned by Bays and Kirby in [BK18, Remark 3.12], where they comment on the fact that they tried to incorporate this kind of structure in their work but ended up taking another route and leaving it to future work. The model theory of quasiminimal structures, after an initial promising phase of development (see [Zil05a], [Kir10b], [Bay+14]) has lost some momentum in the last few years, also due to the lack of meaningful examples. While the end goal is still to prove that $\mathbb{C}_{\exp}$ is quasiminimal, establishing the existence of a quasiminimal raising to powers structure on $\mathbb{C}$ would be a meaningful stepping stone.

Question 6.2.1. Is Theorem 3.7.8 the existential closedness statement of a theory of raising to powers in an infinitary logic and an appropriate language, with quasiminimal models, satisfied by the complex numbers?

6.2.2 Raising to Powers for j

The reader might have noticed that while both Chapter 3 and 4 contain a section called "model-theoretic consequences", Chapter 5 does not, although its main theorem is supposedly similar to the main theorems of the other chapter.

The reason for this is that so far we have been unable to find a convincing model-theoretic framework to discuss raising to powers for j, i.e. we have been unable to define a first-order theory which has Theorem 5.5.7 as its existential closedness statement. The main obstacle can be explained as follows. In the case of exp, the kernel of the exponential function is somehow an "internal" object in models of the theory: we may define ker(exp) as $\{z \in \mathbb{C} \mid E(z, 0)\}$

(recall that in the language of T_K $E(z_1, z_2)$ is interpreted as $\exp(z_1) = \exp(z_2)$) and given any $z_0 \in \mathbb{C}$, the set $\{z \in \mathbb{C} \mid E(z_0, z)\}$ is equal to $z_0 + \ker(\exp)$. This means that all E-equivalence classes (i.e. all preimages of exp) are translates of each other.

If we try to develop an analogue for j we see that the situation is different: the obvious language for the structure is an expansion of the language of sets with a $\mathrm{GL}_2(\mathbb{Q})^+$-action which features, among others, an equivalence relation $J(z_1, z_2)$ which we interpret as $j(z_1) = j(z_2)$. Since the action is then part of the language, it is possible that the J-equivalence classes are different among them (for example they might have different cardinalities). It is not clear what the consequences of this fact are.

It is also worth noting that Eterović has established a transcendence result is the style of Theorem 2.4.3 for "j-transcendental" numbers ([Ete18, Theorem 6.7]) and therefore if we had the theory it would probably be quite easy to show that the complex structure is a model.

Question 6.2.2. Is there a first-order theory of "raising to powers" for j?

It should be noted that a model-theoretic analysis of j in similar terms but with a different goal has been carried out in [Har13], where a categoricity result is established for a two-sorted theory involving the j-function in an infinitary logic.

6.3 Complex-Analytic Directions

On the other hand, there is still a lot to do in the purely complex-analytic setting - that is, there are many cases of Exponential-Algebraic Closedness which are yet to be solved. In this section we introduce some of the questions which seem, to various extents, within reach.

6.3.1 The Product Case for exp

The first one is similar to what we described in Section 4.6.

Question 6.3.1. Let $V \subseteq \mathbb{C}^n$, $W \subseteq (\mathbb{C}^\times)^n$ be algebraic varieties, and suppose the product $V \times W$ is a free rotund variety of $\mathbb{C}^n \times (\mathbb{C}^\times)^n$.

Is $V \times W \cap \Gamma_{\exp} \neq \varnothing$?

To tackle this problem we should first address the following question of Dinh and Vu.

Question 6.3.2 ([DV20, Section 1])**.** Let $V \subseteq \mathbb{C}^n$ be an algebraic variety.

Is it true that there are real semi-algebraic sets $C_1, \dots, C_k$ and real vector subspaces $L_1, \dots, L_k$ such that

$$\overline{\exp(V)} = \exp(V) \cup \bigcup_{j=1}^{k} \exp(L_j + C_j)?$$

The reader will probably guess that this question springs out of an attempt to generalize Theorem 4.6.2 to this setting: in fact, at the end of [DV20, Section 4] a counterexample is given to the literal translation of Theorem 4.6.2 (with the C_j's complex algebraic varieties and the L_j's complex linear subspaces) to the complex exponential function. Hopefully, it is possible to prove Question 6.3.2, perhaps using the o-minimal method of Peterzil and Starchenko from [PS18] rather than the complex analytic approach of [DV20], and then use it together with Theorem 3.7.8 to answer Question 6.3.1.

6.3.2 Linear First Projection

The second question was suggested by Martin Hils when the author gave a talk in the Münster model theory seminar.

Question 6.3.3. Let $V \subseteq \mathbb{C}^n \times (\mathbb{C}^\times)^n$ be a free rotund algebraic variety. Let, as usual,

$$\pi_1 : \mathbb{C}^n \times (\mathbb{C}^\times)^n \twoheadrightarrow \mathbb{C}^n$$

denote the projection to the first block of coordinates.

Suppose $\pi_1(V)$ is (Zariski-dense in) a linear subspace of $\mathbb{C}^n$.

Is $V \cap \Gamma_{\exp} \neq \varnothing$?

As an example, one may consider the variety $V \subseteq \mathbb{C}^3 \times (\mathbb{C}^\times)^3$ consisting of points (z, w) which satisfy the system:

$$\begin{cases} z_2 - \lambda z_1 = 0 \\ f(z_1, z_2, z_3, w_1, w_2, w_3) = 0 \\ g(z_1, z_2, z_3, w_1, w_2, w_3) = 0 \end{cases}$$

where f and g are polynomials which depend on all six variables and $\lambda \in\notin \mathbb{Q}$. Assuming f and g are chosen appropriately, the variety V has projection to $\mathbb{C}^3$ of dimension 2, and does not split as a product - therefore it falls outside of all the available E.A.C. results.

However, it seems plausible that since $\pi_1(V)$ will be a Zariski-dense subset of the linear space L that is defined by $z_2 = \lambda z_1$, and we know how to find intersections between $\exp(L)$ and $\pi_2(V)$, we may perhaps find intersections between $\Gamma_{\exp}$ and V. For instance, if $\lambda \in \mathbb{R} \setminus \mathbb{Q}$, then a good starting point would be to study the geometry of the set

$$\pi_2(V) \cap \{w \in (\mathbb{C}^\times)^3 \mid |w_2| = |w_1|^\lambda\}$$

as the second projection of any intersection needs to lie in this set.

6.3.3 Semiabelian Varieties

As we saw in Chapter 2, the Ax-Schanuel Theorem can be stated in the generality of semiabelian varieties (Corollary 2.5.4). In fact, the E.A.C. conjecture can be stated for general semiabelian varieties (this has been done for example in [AKM22, Conjecture 1.3]), although this case has hardly been addressed in the literature so far: [AKM22, Theorem 6.1] shows that the main theorem of that paper (Theorem 2.3.20 in this thesis) holds when we consider the exponential map of a *split semiabelian variety*, i.e. a semiabelian variety of the form $A \times (\mathbb{C}^\times)^n$ where A is an abelian variety, and it is mentioned that it is likely that the method can be extended to general semiabelian varieties.

Similarly, it would be interested to see if the results of this thesis can be extended to the semiabelian setting: since Theorems 3.7.8 and 4.4.1 have a similar formulation, but for algebraic tori and abelian varieties respectively, it is natural to ask the following question.

Question 6.3.4. Let S be a complex semiabelian variety with exponential map $\exp_S : \mathbb{C}^n \to S$, $L \times W$ a free rotund subvariety of $\mathbb{C}^n \times S$ with $L \leq \mathbb{C}^n$ a linear subspace and $W \subseteq S$ an algebraic variety.

Is $L \times W \cap \Gamma_{\exp_S} \neq \varnothing$?

The problem, of course, is that the methods of proof for Theorems 3.7.8 and 4.4.1 were completely different, and neither goes through in the semiabelian setting: we cannot use tropical geometry, because even its known generalizations only deal with toric varieties and semiabelian varieties are not toric; and we cannot use a homology argument because semiabelian varieties are not compact. Therefore, new ideas are needed to deal with this version of the problem.

As an example, consider the semiabelian variety $S := \mathbb{C}^\times \times E$, where E is an elliptic curve; the exponential map is then $\exp : \mathbb{C}^2 \to \mathbb{C}^\times \times E$, which maps (z_1, z_2) to $(\exp(z_1), \exp_E(z_2))$.

Given a non-constant algebraic function $f : S \to \mathbb{C}$, let

$$W := \{w \in S \mid f(w) = 0\}$$

and let moreover

$$L := \{(z_1, z_2) \in \mathbb{C}^2 \mid z_1 = z_2\}.$$

As usual, the first step to show that $\exp_S(L) \cap W \neq \varnothing$ is to study the closure $\overline{\exp_S(L)}$. This can take different forms, depending on the elliptic curve E; we give an example of this.

Let $\Lambda \subseteq \mathbb{C}$ be a lattice such that $E \cong \mathbb{C}/\Lambda$; we may assume that Λ has the form $\mathbb{Z} + \tau\mathbb{Z}$ for some $\tau \in \mathbb{H}$.

Proposition 6.3.5. *If $\tau = 2\pi i$, then $\exp_S(L)$ is closed.*

Proof. Suppose $\{l_j\}_{j\in\omega}$ is a sequence in L such that $\{\exp_S(l_j)\}_{j\in\omega}$ has a limit in S. Each l_j has the form (z_j, z_j) for some $z_j \in \mathbb{C}$. Since the sequence is convergent, the real parts of the z_j's must converge to some $x \in \mathbb{R}$ (otherwise the first coordinate of $\exp_S(l_j)$ would diverge). We thus may substitute the sequence $\{l_j\}_{j\in\omega}$ by a sequence of the form $\{(x + iy_j, x + iy_j)\}_{j\in\omega}$ if necessary.

Now we know that $\{(iy_j, iy_j) + 2\pi i\mathbb{Z}^2\}_{j\in\omega}$ must converge in $\mathbb{R}^2/2\pi i\mathbb{Z}^2$; clearly, the limit is $(y, y) + 2\pi i\mathbb{Z}^2$ for some $y \in \mathbb{R}$. Therefore $\{\exp_S(l_j)\}_{j\in\omega}$ converges to $\exp_S(x + iy, x + iy) \in \exp_S(L)$, as we wanted. $\square$

It is possible, however, that $\exp_S(L)$ is not closed.

Proposition 6.3.6. *If $\tau \in i\mathbb{R}^{>0}$, but $\tau \notin 2\pi i\mathbb{Q}$, then $\exp_S(L)$ is not closed.*

Proof. Let $\tau = it$ for some positive $t \in \mathbb{R}$. Consider the sequence $\{(x + nit, x +$

$nit)\}_{n\in\omega} \subseteq L$: this sequence is dense in the set

$$\exp_S\left(\left\{(z_1, z_2) \in \mathbb{C}^2 \mid \mathrm{Re}(z_1) = \mathrm{Re}(z_2)\right\}\right),$$

which is a closed set of real dimension 3. □

Therefore, as we have seen, there are subtleties even in the split semiabelian case that we do not know how to deal with yet. This will be the object of future work.

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