# Conformal Symmetry and Chirality in Perturbative Algebraic Quantum Field Theory

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PHD

UNIVERSITY OF YORK
MATHEMATICS

December 2021

# Abstract

The concepts of conformal covariance and chirality in 2 spacetime dimensions are formulated and examined within the perturbative algebraic quantum field theory framework. Firstly the qualitative features of the massless scalar field in 2 dimensions are examined, with a particular focus on the properties which are of general significance in the study of 2-dimensional conformal field theories. A general condition for the extension of covariance under local isometries to conformal covariance is then formulated for classical field theories, which is shown to quantise naturally for non-interacting theories. Features such as primary fields are identified and discussed, leading to a generalisation of the transformation law for the stress-energy tensor of the massless scalar field. Finally, the topic of chirality is discussed. In particular, an emphasis is placed on constructing chiral algebras as natural sub-theories of 2-dimensional conformal field theories on globally hyperbolic Lorentzian manifolds. Cauchy surfaces are used as a natural model for the co-dimension 1 spaces upon which chiral field configurations are defined, until in the final chapter we propose a method by which these algebras may be described without such auxiliary data and in a model-independent way.

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# Acknowledgements

This thesis has been produced during the most unusual of times, which makes the help and support I was fortunate to receive along the way all the more invaluable. I would like to begin by thanking my parents for providing food, shelter and wonderful company throughout several stages of the pandemic. As a teenager who was always so eager to fly the nest, I would never have believed how glad I was to return. Though we were unable to spend much time together, my sister also never failed to send her own love and support by any means necessary, for which she has my deepest gratitude.

At York, I must thank the entirety of the mathematics PhD network, as well as a large number of the staff, for making the department such a warm and welcoming place to be. In particular, I would like to thank Maximilian Ruep and Berend Visser, for many enlightening and enjoyable discussions about the wonders of AQFT; Matthew Sparkes, Jenny Winstone, and Cordelia Webb, for essential constitutional walks around the beautiful campus of York; and Mihail Hurmazov, Ross Ratcliffe, and David Serrano, for countless hours of enjoyment over a healthy mixture of pints and boardgames.

I must also give a special thanks to Georgia Schneider. Although her mathematics, coupled with her enthusiasm for it, frequently proved sufficiently enticing to endanger progress on my own, that is more than forgiven by the unwavering love and support she has given me, particularly through the final stages of this thesis.

Last but not least, I would like to thank my supervisors Kasia Rejzner and Benoît Vicedo. You have always been ready to challenge me when necessary, yet it speaks volumes of the warmth of our relationship that our meetings were often mistaken to be phone-calls to old friends. With your help I have been able to travel farther and wider, geographically as well as mathematically, than I could have possibly imagined, and I look forward to the next stage in our journey.

# Author's declaration

I declare that the work presented in this thesis, except where otherwise stated, is based on my own research carried out at the University of York and has not been submitted previously for any degree at this or any other university. Sources are acknowledged by explicit references.

I confirm that the work submitted is my own, except where work which has formed part of jointly-authored publications has been included. My contribution and those of the other authors to this work have been explicitly indicated below. I confirm that appropriate credit has been given within this thesis where reference has been made to the work of others.

Chapter 2 from section 2.5 onwards, chapters 3 through 6, and the appendices have a substantial overlap with [CRV21], which was published jointly with my supervisors Kasia Rezjner and Benoît Vicedo. Since the original submission of this work a preprint [CRV22] has also been made available which is based predominantly upon part II of this thesis. Any parts of those works that also appear in this thesis were written by myself.

## — 1 —

# Introduction

*Quantum field theory* (QFT) is a cornerstone of modern theoretical physics. Its predictions have been validated to greater precision than any other theory in science, and it is widely regarded as our most fundamental description of nature.

In spite of this significance, QFT is still famously lacking a complete, mathematically rigorous formulation. Around the mid 1960s, two distinct approaches were taken to address this issue, to which almost all later attempts may trace a lineage. On the one hand there is the description in terms of *nets of local algebras* due to Haag and Kastler (of which the standard reference is [Haa96]). Alternatively, there is the construction in terms of *operator-valued distributions* pioneered by Wightman (which one may learn more about in [SW00]).

An umbrella term for successors to the Haag-Kastler axiomatisation is *algebraic quantum field theory* (AQFT). Frequently, such successors adopt the perspective of *locally covariant qft*, where one begins with a collection of spacetimes then assigns to each an algebra of observables in a systematic manner. The crucial insight which connects this idea to the net of local observables is that one should consider a suitable region $\mathcal{O}$ within a spacetime $\mathcal{M}$ as a spacetime in its own right.

Early successes of local quantum physics include an axiomatic renormalisation of the stress-energy tensor [Wal77], leading to a conceptually clear account of the origins of the *Casimir effect* in QFT [Kay79]. It also soon became clear that AQFT was particularly well suited to the study of QFT on *curved spacetimes* [BFV03; HW01]. This provides a semi-classical approximation to any hypothetically unified theory of quantum gravity and leading to predictions such as the *Unruh effect* which states that, to a non-inertial observer, the vacuum appears to be filled with thermal radiation. A particular consequence of this general principle being *Hawking radiation*.

For all the insights AQFT provides, it seems presently that the price to pay is a dearth of constructive examples. Indeed, there are currently no known examples of interacting AQFTs in 4 spacetime dimensions. (Though the same could be said of any rigorous description of QFT.)

Some of the more recent developments in this field include *perturbative algebraic quantum field theory* (pAQFT) [Rej16; BDF09], which addresses the issue of construction by providing a general mathematical formulation of the techniques of perturbation theory employed in QFT by theoretical physicists. Whilst in general this construction does not yield a fully-fledged AQFT, in the case of the sine-Gordon model [BR18], it has been shown that non-perturbative results can be recovered by pAQFT.

Where constructive, non-perturbative models appear most frequently is in spacetimes of dimension 2 and in the presence of conformal symmetries. It is in this setting where the Wightman formulation of QFT has thrived. Physically, *conformal field theories* (CFTs) occur in the description of phase transitions in condensed matter theory, as the fixed points of Wilsonian renormalisation flow, and in string theory in the form of sigma models. Of all the mathematical tools used in the study of 2D CFT, *vertex operator algebras* (VOAs) have had particular success. In addition to providing a rich description of CFT, VOAs have also found uses outside of mathematical physics, such as in the proof of the *monstrous moonshine conjecture* [Bor92], as well as in the *geometric Langlands correspondence* [Fre05].

Starting instead from the Haag-Kastler axioms, one arrives at a formulation of 2DCFT in terms of *conformal nets*. In general, the extent of the relationship between the Wightman and Haag-Kastler settings is currently not fully understood. However, by restricting to more specific contexts one is typically able to make more meaningful comparisons, and 2DCFT is certainly no exception. Early results include the work of Fredenhagen and Jörß [FJ96], in which it was shown that Wightman fields may be constructed out of the observables of a conformal net. Later, Carpi, Kawahigashi, Longo and Weiner [CKLW18] were able to show that VOAs satisfying an additional property of *strong locality* (which includes most common examples) may be used to generate a conformal net, from which the methods of Fredenhagen and JörSS can recover the original VOA. This correspondence has since been used to transfer results regarding sub-theories in the VOA setting to conformal nets [CGH19].

It is our intention in this thesis to provide a new perspective on the world of 2DCFT through the lens of pAQFT. We shall begin with a detailed account of the classical field theory necessary in constructing any pAQFT. Next, we restrict to the theory we shall be focusing on for the majority of the thesis: the massless scalar field in two dimensions. After quantising the classical theory, we discuss some of the distinctive characteristics of this AQFT, such as the sub-algebras satisfying the *Heisenberg* and *Virasoro* relations.

We then proceed to construct a general framework with which we can describe conformal symmetry in pAQFT. Firstly, we provide a condition under which the standard symmetries (or more accurately *covariance*) of a classical field theory may be extended to *conformal covariance*, and show that for non-interacting theories this property is preserved by quantisation.

This extended covariance then enables us to classify fields in our theories in terms of how they respond to conformal transformations. From here we are able to recover well-known results from 2DCFT, such as the transformation law for the stress-energy tensor of the massless scalar field.

Part II of the thesis is concerned with the formulation, in the framework of pAQFT, of a phenomenon unique to 2DCFT where the space of physical (*on-shell*) fields naturally decouples into two distinct *chiral sectors*. Each sector possesses trivial dynamics, and can be effectively described as being localised to a lower dimensional submanifold of the original spacetime. After defining a chiral sector explicitly, we proceed to construct its algebras of observables. Having done this, we are then able to see how these algebras naturally describe a subtheory of the massless scalar field. Moreover, we will be able to show that the algebraic operations on chiral fields are tightly constrained by the conformal symmetry.

Finally, we use what we have learned of chirality in pAQFT to propose a definition of chiral subalgebras which is independent of whichever model one uses to construct a conformally covariant AQFT. We show how our prior construction fits into this abstract definition, and we are also able to use it to prove a model-independent result about causality in chiral algebras.

## — 2 —

# Mathematical Preliminaries

In this section, we provide an account of the constructions of pAQFT relevant to our discussion. For a more thorough exposition, the reader is directed towards [Rej16].

In particular, whilst we may, from time to time, discuss the possibility of interactions in the classical theory, all of our quantum constructions shall be specific to the free scalar field. In light of this, for the purpose of this thesis, the reader may interpret the 'p' in pAQFT as either referring to the particular use of $\hbar$ as a *formal* parameter when quantising in section 2.8, or more generally to our use of techniques and concepts central to the development of pAQFT.

After reviewing some of the motivation and mathematical tools behind pAQFT, we begin our construction with the kinematics (i.e. states and observables) of the classical theory. Due to our use of deformation quantisation, this will also establish the observables of the quantum theory. Next, we address in Section 2.6 the matter of imposing suitable dynamics on the system, using the concept of *generalised Lagrangians*. For an appropriately chosen Lagrangian, we are then able to endow our space of observables with a Poisson structure.

At this point, the algebra is decidedly 'off-shell', as the field configurations we consider include those which do not satisfy the equations of motion. Therefore, in Section 2.7, we make a detour to examine how, in the case of the free scalar field, our construction does indeed recover the canonical (i.e. 'equal-time') Poisson bracket on-shell. Here we also briefly explore the *dg perspective* of QFT, where the algebra we assign to each spacetime is instead a cochain complex such that the usual algebra of observables is recovered as its cohomology in degree zero. This approach is at the heart of the Costello-Gwilliam formalism [CG16] as well as descriptions of 'higher' QFT as outlined in, for example, [BPSW21].

Satisfied with our choice of Poisson structure, we then use it in Section 2.8 to deform the pointwise product of functionals into an associative product $\star$, which is analogous to the composition of operators in canonical quantisation. Once the quantum algebra has been established, we discuss the comparison between classical and quantum observables. The difficulty in 'quantising' classical observables is traditionally known as the *ordering problem*. In an attempt to find the most natural solution to this problem, we then introduce in Section 2.9 the concept of *local covariance*, where we require our theory to be defined in a coherent manner across multiple spacetimes. This is so that we may be sure our ordering prescription is not dependent on the global geometry of any particular spacetime (which local algebras should in principle be unaware of).

As a somewhat gentle introduction to many of the concepts we shall be using in this thesis, we shall first discuss the difference between conventional constructions of quantum mechanical/field theories and the approach we shall be taking.

To keep a common thread, we shall be focusing on the example of the *simple harmonic oscillator* (SHO) throughout the first few sections, where already we will be able to see many concepts which survive mostly intact in the transition to QFT.

## 2.1 WHY ALGEBRAIC?

Arguably the most fundamental distinction between the formulation of quantum theories employed in this thesis and the more common approach in theoretical physics is that we focus on algebras of observables, rather than (Hilbert) spaces of states. Whilst both are necessary for a complete description of a quantum theory in any formulation, there are several arguments for why the algebras should have the priority.

- Many common constructions of quantum theories begin by defining *algebraic* structures, before seeking a faithful representation of those structures on a suitable vector space. For instance, many students are first taught the relation $[\widehat{q}, \widehat{p}] = i\hbar$ before being told that $\widehat{q}\psi(q) = q\psi(q)$, $\widehat{p} = -i\hbar\partial_q$ is a valid realisation of this relation on $L^2(\mathbb{R})$.
- In quantum mechanics we are usually comfortable working with Hilbert spaces which decompose into a direct sum of invariant *superselection sectors* (for example, the Clebsh-Gordan decomposition of a pair of $j = 1/2$ representations of $\mathfrak{so}(3)$). However, in the typical Fock space construction of QFT on Minkowski spacetime, one's observables can only act on the superselection sector of the vacuum state. By keeping the algebra separate from its representations (reducible

or otherwise), it is much easier to study the behaviour of a theory in alternative superselection sectors.

- Again in quantum mechanical systems, one is often saved from the nuances of an algebra of observables' representations by the *Stone von-Neumann theorem*, which (roughly speaking) states that all 'nice' representations of the canonical commutation relations[1] above are unitarily equivalent to the wavefunction representation on $L^2(\mathbb{R})$. However, this is *not true* in quantum field theory. In this setting, particularly in curved spacetimes, it is more often the case that one has infinitely many equally good, yet inequivalent representations to choose from.

Compared to the Hilbert-space approach, one has to determine which properties an algebra should exhibit intrinsically, and which are merely products of a particular choice of representation.

Ideally, one would only work with $C^*$-algebras, a special class of Banach *-algebras with an involution satisfying $||a||^2 = ||a^*a||$. The ur-example is the space $\mathfrak{B}(\mathcal{H})$ of bounded operators on a Hilbert space, where $*$ is the adjoint operation.

However, it is often the case in concrete examples (particularly in the setting of deformation quantisation, which we shall discuss next) that such boundedness is difficult to satisfy. As such, the minimum one typically asks for is a $*$-algebra. This is an algebra $\mathcal{A}$ (almost always over $\mathbb{C}$ in the context of QFT) equipped with an involution ${}^* : \mathcal{A} \to \mathcal{A}$ and satisfying all the usual algebraic relations one would expect. In our case, we will also usually assume that such $*$-algebras have a topology, with respect to which, scaling, addition, multiplication and the involution are all continuous.

In the algebraic approach, rather than observables acting on states, states act on observables. In particular we have

**Definition 2.1.1.** An *algebraic state* on a $*$-algebra $\mathcal{A}$ with multiplicative identity $\mathbb{1}$ is a linear map $\omega : \mathcal{A} \to \mathbb{C}$ such that

1. $\omega(\mathbb{1}) = 1$
2. $\omega(a^*a) \geq 0 \, \forall \, a \in \mathcal{A}$

For $\mathcal{A} = \mathfrak{B}(\mathcal{H})$, each vector $\psi \in \mathcal{H}$ with $||\psi|| = 1$ defines an algebraic state $A \mapsto \langle \psi, A\psi \rangle$. Moreover, a density matrix $\rho$ also defines an algebraic state by $A \mapsto \mathrm{Tr}(\rho A)$.

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[1] Specifically, irreducible and strongly-continuous representations of the *Weyl relations*, which are a formal exponentiation of the canonical commutation relations

As such, for an observable quantity (i.e. self-adjoint) $a \in \mathcal{A}$, we think of $\omega(a)$ as the *expectation value* of $a$ in the state $\omega$.

## 2.2 DEFORMATION QUANTISATION: OVERVIEW

Now that we have discussed what it is we want to construct, namely a $*$-algebra of quantum observables, we shall look at a simple example of the techniques we shall be using to do so.

In the beginning, one typically has a *phase space*. Abstractly, this represents the space of all possible initial conditions for whatever system we are studying. Concretely, this is typically realised as the *cotangent bundle* $T^*X$ for some smooth manifold $X$. A point $q \in X$ representing the initial position of a particle (or the collective positions of many particles), and a covector $p \in T^*_q X$ describing the initial momentum.

Consider the canonical Poisson structure on $T^*X$, which in a suitable choice of coordinates may be expressed as

$$\begin{aligned} \{\cdot,\cdot\} : \mathcal{C}^\infty(T^*X) \times \mathcal{C}^\infty(T^*X) \to \mathbb{R}, \\ f, g \mapsto \sum_{i=1}^{n} (\frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i}). \end{aligned} \tag{2.1}$$

Combined with a choice of Hamiltonian $H \in \mathcal{C}^\infty(T^*X)$, it produces the equations of motion $\dot{f} = \{H, f\}$, which one can solve uniquely for any initial data $(q,p) \in T^*X$. One of the best-understood examples of such a system is the *simple harmonic oscillator* which, choosing suitable coordinates and parameters, has the Hamiltonian

$$H(q,p) = \frac{1}{2}\sum_{i=1}^{n}\left(q_i^2 + p_i^2\right). \tag{2.2}$$

Ever since Dirac's seminal work on the subject [Dir26], the Poisson bracket has been a favoured starting point in the quantisation of classical theories. In the typical telling, one 'promotes' the coordinates $q_i, p_i$ to *linear operators* $\hat{q}_i, \hat{p}_i$ on some Hilbert space $\mathcal{H}$ such that

$$\hat{q}_i\hat{p}_j - \hat{p}_j\hat{q}_i = [\hat{q}_i, \hat{p}_j] = i\hbar\widehat{\{q_i, p_j\}} = i\hbar\delta_{ij}. \tag{2.3}$$

We have already seen the *Schrödinger representation* where, for $X = \mathbb{R}^n$, $\mathcal{H} = L^2(\mathbb{R}^n)$, $\hat{q}_i$ is the multiplicative operator $\psi(q) \mapsto q_i\psi(q)$, and $\hat{p}_i = -i\hbar\partial_{q_i}$ (each defined on a suitable domain, such as compactly supported smooth functions). In principle, the process of *Dirac quantisation* would involve extending this to an assignment of a (possibly unbounded) operator $\hat{f}$ for every $f \in \mathcal{C}^\infty(T^*X)$ such that

$$[\hat{f}, \hat{g}] = i\hbar\widehat{\{f, g\}}. \tag{2.4}$$

However, here we run into the first issue with this standard approach. A classic result [Gro46] states that, under certain physically reasonable assumptions, (2.4) cannot hold for all $f, g$. In fact, inconsistencies occur as soon as one considers observables such as $q^2p$.

Part of the problem is readily apparent. Classically, there is no difference between $q^2p, qpq$ or $pq^2$, yet the same is not true for their quantum counterparts. Heuristically, one needs to pick an *order* in which to place the quantised observables.

It turns out that a condition sufficient to specify an ordering map is that, $\forall\, a, b \in \mathbb{R}^n, m \in \mathbb{N}$

$$(a \cdot \widehat{q + b} \cdot p)^m = (a \cdot \hat{q} + b \cdot \hat{p})^m.$$

We can even simplify this to the single rule $e^{i(aq+bp)} \mapsto e^{i(a\hat{q}+b\hat{p})}$. One can show that this prescription, known as the *Weyl ordering*, yields the average over all possible orderings of a monomial $q^n p^m$, e.g. $q^2p \mapsto \frac{1}{3}\left(\hat{q}^2\hat{p} + \hat{q}\hat{p}\hat{q} + \hat{p}\hat{q}^2\right)$.

By identifying the vector $(a, b) \in \mathbb{R}^{2n}$ with the linear observable $(q, p) \mapsto a \cdot q + b \cdot p$, we can define the space of classical, *polynomial* observables as the symmetric algebra over $\mathbb{R}^{2n}$, which we denote $\mathrm{Sym}(\mathbb{R}^{2n})$.

One can show that the Weyl ordering then defines a linear isomorphism between $\mathrm{Sym}(\mathbb{R}^{2n})^{\mathbb{C}}$, the complexified classical polynomials and the subalgebra of polynomial quantum observables. [2] This means that we can also use the quantum operator product to define a new product on $\mathrm{Sym}(\mathbb{R}^{2n})$. The operator $\star$ is defined such that $\widehat{f \star g} = \hat{f}\hat{g}$. Defining the commutator $[\cdot, \cdot]_\star$ with respect to $\star$ yields a Poisson bracket on the classical algebra known as the *Moyal bracket*. Note that, contrary to the objective of Dirac quantisation, this is not proportional to our original Poisson bracket, but instead satisfies

$$[f, g]_\star = i\hbar\,\{f, g\} + \mathcal{O}(\hbar^2). \tag{2.5}$$

We shall call any associative product $\star$ satisfying (2.5) a *deformation* of the Poisson bracket $\{\cdot, \cdot\}$. One can show from our definition that the highest power of $\hbar$ appearing

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[2] The Weyl transform can also be defined for more general observables, we only restrict to polynomials for clarity.

in $f \star g$ is equal to the smallest of $\deg f$ and $\deg g$, and is hence always finite. If we wish to include non-polynomial observables, then in general we may obtain an infinite series in $\hbar$. If we can show that every such series converges, then the deformation is said to be *strict*. Alternatively, we can assume that $\hbar$ is a *formal* parameter, in which case, if the vector space of classical observables is $\mathsf{Obs}_{\mathrm{cl}}$, then the vector space for the quantum algebra of observables is $\mathsf{Obs}_{\mathrm{q}} = \mathsf{Obs}_{\mathrm{cl}}[[\hbar]]$. We then say that the deformation $\star$ is *formal*.

## 2.3 OFF-SHELL FORMALISM

We now show a very different way of constructing the algebra of observables for the simple harmonic oscillator. For simplicity we shall only consider the 1-dimensional case. Recall that, after specifying a Hamiltonian, we established a correspondence between points in the phase space $T^*X \simeq \mathbb{R}^2$ and physical trajectories $\mathbb{R} \to \mathbb{R}$.

Let us now begin by considering the space of all conceivable trajectories. We define the space $\mathcal{E}$ to comprise all the *smooth* maps $\mathbb{R} \to \mathbb{R}$. We can distinguish the physical trajectories from the non-physical via the *principle of least action*. For the simple harmonic oscillator, the relevant quantity is the *Lagrangian*

$$L : \mathcal{E} \to C^\infty(\mathbb{R}), \tag{2.6}$$

$$q \mapsto \frac{1}{2}(\dot{q}(t)^2 - q(t)^2). \tag{2.7}$$

The integral of the resulting function is, formally, the *action* of the trajectory $q$. However, it is important to note that in general, the resulting function is *not* integrable over $\mathbb{R}$. We can remedy this by multiplying $L[q] \in C^\infty(\mathbb{R})$ by an arbitrary function $f$ which is *compactly supported* to ensure the integral converges. The action is then formally recovered in the limit where $f \equiv 1$ on an arbitrary large time interval. We denote the approximate actions by

$$\mathcal{L}(f)[q] := \int_{t=-\infty}^{\infty} \frac{1}{2}\left(\dot{q}(t)^2 - q(t)^2\right) f(t)\mathrm{d}t. \tag{2.8}$$

Even though we are unable to define a single action functional, we can still define a variation principle unambiguously, we say that a trajectory $q \in \mathcal{E}$ is *critical* if

$$\delta S[q, \mathring{q}] := \frac{d}{d\epsilon}\mathcal{L}(f)[q + \epsilon\mathring{q}]|_{\epsilon=0} \tag{2.9}$$

vanishes for all loops $\mathring{q} \in \mathcal{E}$ such that $\mathring{q} \equiv 0$ outside some finite time interval $[t_0, t_1] \subset \mathbb{R}$, where $f$ is chosen such that $f \equiv 1$ on $[t_0, t_1]$. (One can quickly check that the particular choice of $f$ doesn't matter once this constraint is met.) In essence, we consider only perturbations of a trajectory which have fixed end-points $t_0$ and $t_1$, but we consider all pairs of end-points simultaneously.

This condition yields the familiar equation of motion $\ddot{q} + q = 0$. However, we now want to consider the case where an extra term is added, namely $\mathcal{L}_{\text{int}}[q] := \int_{-\infty}^{\infty} q(t)g(t)\mathrm{d}t$ for some fixed $g \in \mathcal{C}_c^\infty(\mathbb{R})$. Adding this to $\mathcal{L}(f)$ leads to the inhomogeneous equation of motion

$$\ddot{q}(t) + q(t) := Pq = g(t). \tag{2.10}$$

Physically, if, for example, $\operatorname{supp} g = [t_i, t_f]$, then this equation describes the situation where our simple harmonic operator is coupled to a driving force, which is switched on at time $t_i$, then later switched off at $t_f$. If we assume that the oscillator is initially at rest, a physically reasonable solution to this equation ought to satisfy $q(t) = 0$, $\forall\, t < t_i$.

There exists a distinguished Green's function $\Delta^R(t, t')$ such that

$$q(t) = \int_{t'=-\infty}^{\infty} \Delta^R(t, t')g(t')\mathrm{d}t, \tag{2.11}$$

is the desired solution. By Fourier methods, or otherwise, one can deduce that this Green's function is of the form $\Delta^R(t, t') = \theta(t - t') \sin(t - t')$, where $\theta$ is the Heaviside step function.

This function is referred to as the *retarded propagator* of our theory. Its transpose is the *advanced propagator* $\Delta^A(t, t') = \Delta^R(t', t) = \Delta^R(t, t') + \sin(t - t')$, which corresponds to the scenario where the oscillator begins with some non-trivial motion that is later arrested perfectly by the driving force at time $t_f$.

The remarkable insight of Rudolph Peierls [Pei52] was that such advanced and retarded responses to external forces can actually be used to define a Poisson structure. We can identify $\mathcal{C}_c^\infty(\mathbb{R})$ with a family of linear observables on $\mathcal{E}$ where

$$f \leftrightsquigarrow \left(q \mapsto \int_{\mathbb{R}} f(t)q(t)\mathrm{d}t\right).$$

We then define the *Peierls bracket* of two such observables as

$$\{f, g\} := \int_{\mathbb{R}^2} f(t)\Delta(t - t')g(t')\mathrm{d}t\mathrm{d}t', \tag{2.12}$$

where $\Delta = \Delta^R - \Delta^A = -\sin(t - t')$.

The Peierls bracket can then be extended to the symmetric algebra $\operatorname{Sym}(\mathcal{C}_c^\infty(\mathbb{R}))$ by imposing the Leibniz rule $\{f, gh\} := \{f, g\}\, h + g\, \{f, h\}$. One can then confirm that this yields a well-defined Poisson algebra.

This algebra is degenerate, however, which ultimately stems from the fact that the map

$$\Delta g(t) := \int_{\mathbb{R}} \Delta(t,t')g(t')\,\mathrm{d}t', \tag{2.13}$$

from compactly supported functions to solutions, is not injective. It turns out that the kernel of this map is precisely functions of the form $Pg$ for $g \in \mathcal{C}_c^\infty(\mathbb{R})$. The ideal in $\mathrm{Sym}(\mathcal{C}_c^\infty(\mathbb{R}))$ generated by functions of this form is also an ideal of the Poisson bracket. If we denote this ideal by $\mathfrak{I}$, then the quotient space $\mathrm{Sym}(\mathcal{C}_c^\infty(\mathbb{R}))/\mathfrak{I}$, inherits a Poisson structure from the Peierls bracket.

**Theorem 2.3.1.** *There is an isomorphism of Poisson algebras between* $\mathrm{Sym}(\mathcal{C}_c^\infty(\mathbb{R}))/\mathfrak{I}$ *and* $\mathrm{Sym}(\mathbb{R}^2)$*, where the former is equipped with the quotient Peierls bracket and the latter is equipped with the canonical bracket.*

*Proof.* It is sufficient to define a pair of maps $\mathcal{C}_c^\infty(\mathbb{R}) \leftrightarrow \mathbb{R}^2$ such that the kernel of the forward map and the cokernel of the reverse map are both $\mathrm{Ker}\Delta$.

The forward map is the easiest to define. For some fixed time $t_0 \in \mathbb{R}$, we send $f \in \mathcal{C}_c^\infty(\mathbb{R})$ to the pair $(\Delta f(t_0), \dot{\Delta f}(t_0))$. Given that this map depends only on $f$ through $\Delta f$, it is clear that its kernel contains $\mathrm{Ker}\Delta$. The fact that this is the *entirety* of the kernel follows from the uniqueness of solutions to the second-order ODE $Pq(t) = 0$.

To see that this map yields a Poisson algebra homomorphism, we begin with the canonical bracket on the image:

$$\{(\Delta f(t_0), \dot{\Delta f}(t_0)), (\Delta g(t_0), \dot{\Delta g}(t_0))\}_{\mathrm{can}} = \Delta f(t_0)\dot{\Delta g}(t_0) - \dot{\Delta f}(t_0)\Delta g(t_0). \tag{2.14}$$

Expanding out the integrals and evaluating the derivatives, we have

$$\begin{aligned} &\Delta f(t_0)\dot{\Delta g}(t_0) - \dot{\Delta f}(t_0)\Delta g(t_0) = \\ &\qquad \int_{\mathbb{R}^2} \left(\sin(t_0 - t')\cos(t_0 - t'') - \cos(t_0 - t')\sin(t_0 - t'')\right) f(t')g(t'')\mathrm{d}t'\mathrm{d}t''. \end{aligned} \tag{2.15}$$

By use of a standard trigonometric identity we can then simplify the integrand to $-\sin(t' - t'')f(t')g(t'')$, which is precisely the integrand of the Peierls bracket, hence

$$\{f, g\}_{\mathrm{Pei}} = \Big\{(\Delta f(t_0), \dot{\Delta f}(t_0)), (\Delta g(t_0), \dot{\Delta g}(t_0))\Big\}_{\mathrm{can}} \tag{2.16}$$

The other direction is a little trickier, as there is no unique $f$ corresponding to a physical trajectory $q(t)$. Firstly, given initial data $(q_0, p_0)$ at a time $t_0$, the corresponding trajectory is given by

$$q(t) = q_0\cos(t - t_0) + p_0\sin(t - t_0). \tag{2.17}$$

We then need to find $f \in \mathcal{C}_c^\infty(\mathbb{R})$ such that $\Delta f(t) = q(t)$. Suppose we have some $f_0 \in \mathcal{C}_c^\infty(\mathbb{R})$ such that $\Delta f_0$ is not zero everywhere. This is guaranteed to exist, as we can take for example $f_0$ to be everywhere non-negative, with $\operatorname{supp} f_0 \subset (0, \pi)$, which would imply that $\Delta f_0(0) > 0$. Given $\Delta f_0 \neq 0$, there must exist some $g_0 \in \mathcal{C}_c^\infty(\mathbb{R})$ such that $\int_{\mathbb{R}} g_0(t) \Delta f_0(t)\, \mathrm{d}t \neq 0$. By rescaling $f_0, g_0$ as appropriate, we can then ensure that $\{f_0, g_0\}_{\text{Pei}} = 1$. By skew-symmetry of $\Delta$, we also have that $\Delta f_0$ and $\Delta g_0$ are linearly independent (if $g_0 = \lambda f_0 + h$ for $h \in \text{Ker}\Delta$, then $\{f_0, g_0\}_{\text{Pei}} = \lambda \{f_0, f_0\} = 0$), hence span the space of solutions to $P$. Thus, the map $(a, b) \mapsto a f_0 + b g_0$ is surjective up to $\text{Ker}\Delta$, and preserves Poisson brackets.

□

*Remark* 2.3.2. Suppose we take $f_1, g_1$ to be linear combinations of $f_0, g_0$ from the above proof such that $\Delta f_1 = \cos$, $\Delta g_1 = \sin$. We can then arrange all of the maps in the following diagram

$$\begin{tikzcd}
 & 0 \arrow[r] \arrow[d] & \mathbb{R}^2 \arrow[r, "\Omega_2"] \arrow[d, "\alpha^+"] & \mathbb{R}^2 \arrow[r] \arrow[d, "\mathrm{Sol}"] & 0 \arrow[d] & \\
0 \arrow[r] & \mathcal{C}_c^\infty(\mathbb{R}) \arrow[r, "P"] \arrow[d] & \mathcal{C}_c^\infty(\mathbb{R}) \arrow[r, "\Delta"] \arrow[d, "\alpha"] & \mathcal{C}^\infty(\mathbb{R}) \arrow[r, "P"] \arrow[d, "\Pi"] & \mathcal{C}^\infty(\mathbb{R}) \arrow[r] \arrow[d] & 0 \\
 & 0 \arrow[r] & \mathbb{R}^2 \arrow[r, "\Omega_2"] & \mathbb{R}^2 \arrow[r] & 0 &
\end{tikzcd} \tag{2.18}$$

Where $\mathrm{Sol}$ is the map (2.17), $\Pi : q(t) \mapsto (q(t_0), \dot{q}(t_0))$, $\Omega_2 : (q, p) \mapsto (-p, q)$, and $\alpha, \alpha^+$ are (up to twisting by $\Omega_2$) the maps we needed to find for the above theorem. These maps all depend on the choice of functions $f_1, g_1$. More precisely, $\alpha, \mathrm{Sol}$, and $\Pi$, depend on their equivalence class modulo $\text{Ker}\,\Delta$, whereas $\alpha^+$ depends on the particular choice of representatives. The choice made at the beginning of this remark is particularly nice, as $\mathrm{Sol}$ then carries the interpretation of solving an initial value problem at $t = 0$, but other choices are equally valid This diagram is exact along rows (the kernel of any horizontal map is the image of the one preceding it), and the composition of any two vertical maps is an identity. In other words, this diagram constitutes a retract of exact sequences.

After finding an appropriate $\alpha^+$ and establishing the resulting diagram commutes, showing the Poisson brackets are preserved reduces to showing that

$$\int_{\mathbb{R}} f(t) q(t) = (\alpha f) \cdot (\Pi q), \tag{2.19}$$

for any $q(t) \in \text{Ker}P$.

As for the quantisation, it is possible to define a product $\star$ on $\mathrm{Sym}(\mathcal{C}_c^\infty(\mathbb{R}))$ such that $f \star g = fg + \frac{i\hbar}{2}\{f, g\}$ for $f, g \in \mathcal{C}_c^\infty(\mathbb{R})$. However, by this point it is more useful to go straight to the product we actually use in QFT, which we shall introduce in section 2.8.

## 2.4 A PRIMER ON DISTRIBUTIONS

Distributions, sometimes referred to as *generalised functions*, arise naturally in any attempt to study quantum field theory. As such, any rigorous treatment requires an understanding of what operations are possible on distributions and why. In this section we provide a brief overview of the methods of manipulating distributions that we shall make use of frequently throughout this work. We shall start by defining what a distribution actually is, before studying methods of quantifying how distributions 'fail' to be regular functions. This shall lead us the the concept of the *wavefront set*, which we will then use to provide sufficient conditions for operations such as the pullback and pointwise multiplication of distributions to be well-defined.

As a motivating example to keep in mind, we shall use the techniques developed in this section to show that the generalised function $u(x) = \frac{1}{x+i0^+}$ can be squared, even though distributions such as the Dirac delta $\delta(x)$ cannot.

Many of the technical details that we shall gloss over here may be found in [Hör15], alternatively, a recent pedagogical review is provided in [BDH14]. We shall also point out relevant sections of these and other texts when the reader may wish for more details than we provide in this overview.

### 2.4.1 Topologies on the Spaces of Functions

We have already been introduced to the space $\mathcal{C}^\infty(\mathbb{R}^n)$ of smooth functions, as well as its subspace $\mathcal{C}_c^\infty(\mathbb{R}^n)$ of functions with compact support. From now on, we shall use the notation due to Schwartz [Sch57] and denote these spaces by $\mathfrak{E}(\mathbb{R}^n)$ and $\mathfrak{D}(\mathbb{R}^n)$ respectively.

For any open $U \subseteq \mathbb{R}^n$ we can equip $\mathfrak{E}(U)$ with a topology by defining the family of *semi-norms*

$$||\varphi||_{K,m} := \sum_{|\alpha| \le m} \sup_{x \in K} |\partial^\alpha \varphi(x)|, \tag{2.20}$$

where $K \subset U$ is compact, $m \in \mathbb{N}$, $\alpha \in \mathbb{N}^n$ is a multi-index (i.e. $\partial^\alpha = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n}$), and $|\alpha| := \alpha_1 + \cdots + \alpha_n$. Spelling it out, a sequence $(\varphi_j)_{j\in\mathbb{N}} \subset \mathfrak{E}(U)$ converges to $\varphi$ in this topology if and only if

$$||\varphi_j - \varphi||_{K,m} \to 0$$

for every pair $K, m$. The topology thus obtained is *Fréchet* [Rud91, §1.46], i.e. it is metrisable and complete. This will prove useful later, as the Fréchet property provides a nice definition of a derivative, with which we shall be defining many operations on our algebra of observables.

The definition of the topology on $\mathfrak{D}(U)$ is a little more involved. Firstly, for any compact set $K \subset U$, we shall define the topological space $\mathfrak{D}_U(K)$ as the set of smooth functions $f \in \mathfrak{E}(U)$ such that $\operatorname{supp} f \subseteq K$, equipped with the subspace topology from the inclusion $\mathfrak{D}_U(K) \subset \mathfrak{E}(U)$. Let $\mathcal{K} = \{K_n\}_{n\in\mathbb{N}}$ be a collection of compact sets such that

1. $K_n \subset \operatorname{Int}(K_{n+1})$,
2. $\bigcup_{n\in\mathbb{N}} K_n = U$.

We can define the topology on $\mathfrak{D}(U)$ as the final topology under the inclusions $\mathfrak{D}_U(K_n) \subset \mathfrak{D}(U)$. To be precise, the topology on $\mathfrak{D}(U)$ is the finest *locally convex* topology such that each inclusion map is continuous.[3] Equipped with this topology, we refer to $\mathfrak{D}(U)$ as the space of *test functions* on $U$.

We can now define the space of *distributions* on $U$ as the topological dual of $\mathfrak{D}(U)$, that is, the space of continuous linear maps $\mathfrak{D}(U) \to \mathbb{R}$. We shall denote the space of distributions by $\mathfrak{D}'(U)$, and we shall denote the evaluation of a distribution $u \in \mathfrak{D}'(U)$ on a test function $f \in \mathfrak{D}(U)$ by $\langle u, f\rangle$.

The above definition of the topology of $\mathfrak{D}(U)$ is precise, albeit somewhat abstract. Indeed it is entirely absent from [Hör15], where instead the following property of distributions is taken as their definition (a proof of their equivalence may be found in [Rud91, Theorem 6.8]).

**Proposition 2.4.1.** *A linear map $u : \mathfrak{D}(U) \to \mathbb{R}$ is a distribution if and only if for all compact subsets $K \subset U$, there exists constants $C_K \in \mathbb{R}_{>0}$ and $k \in \mathbb{N}$ such that $\forall\, f \in \mathfrak{D}_U(K)$*

$$|\langle u, f\rangle| \leq C_K \sum_{|\alpha|\leq k} ||f||_{K,\alpha}, \tag{2.21}$$

*where $|\alpha| := \sum_{i=1}^{n} \alpha_i$.*

---

[3] A topology on a vector space $V$ is *locally convex* if (a) translations and scalings are continuous and (b) the origin $0 \in V$ admits a neighbourhood basis comprising only convex sets.

A corollary of this result is that every element of $\mathfrak{E}(U)$ defines a distribution by integration:

$$u : f \mapsto \int_U u(x) f(x) d^n x. \tag{2.22}$$

This follows from proposition 2.4.1 by setting $k = 0$ and $C_K = \int_K u(x) d^n x$. By an abuse of notation, we typically use $u$ to denote both the function and the distribution. We call any distributions which arise in this way *regular*.

The fact that $\mathfrak{E}(U)$ embeds into $\mathfrak{D}'(U)$ is one reason we define distributions as dual to functions with compact support. Whilst it might seem unusual to consider the dual of $\mathfrak{D}(U)$ rather than of $\mathfrak{E}(U)$, we shall soon see that the dual space of $\mathfrak{E}(U)$ is *also* a subspace of $\mathfrak{D}(U)$. We can also use this result to prove another, which demonstrates why we must use such an unusual topology for $\mathfrak{D}(U)$:

**Proposition 2.4.2.** *Let $f \in \mathfrak{D}(\mathbb{R})$ be any test function such that $\int_{\mathbb{R}} f \mathrm{d}x = 1$, the sequence of functions $f_n(x) := n^{-1} f(n^{-1} x) \in \mathfrak{D}(\mathbb{R})$ for $n \geq 1$ converges to zero in the topology of $\mathfrak{E}(\mathbb{R})$, but not in the topology of $\mathfrak{D}(\mathbb{R})$.*

*Proof.* As $f$ is compactly supported $|\partial^\alpha f|$ is bounded for each $\alpha \in \mathbb{N}$. If $|\partial^\alpha f(x)| \leq C_\alpha$, then clearly, for any compact $K \subset \mathbb{R}$, $|f_n|_{K,\alpha} \leq n^{-(\alpha+1)} C_\alpha$, and hence $|f_n|_{K,\alpha} \to 0$ for every $K, \alpha$. Thus $f_n \to 0$ in the topology of $\mathfrak{E}(U)$. To show this sequence does not converge in the topology of $\mathfrak{D}(\mathbb{R})$, consider the map $u : f \mapsto \int_{\mathbb{R}} f(x) \mathrm{d}x$. By proposition 2.4.2, this is a regular distribution corresponding to the constant function $u(x) = 1$. Clearly $\int_{\mathbb{R}} f_n \mathrm{d}x = \int_{\mathbb{R}} f \mathrm{d}x = 1$. Thus $f_n$ cannot converge to 0 in the topology of $\mathfrak{D}(U)$, as the continuity of $u$ would necessarily imply $\lim_{n\to\infty} \langle u, f_n \rangle = 0$, which is not the case. ☐

### 2.4.2 Localisation of Distributions

It is often necessary to ask 'where' a given distribution is non-trivial. In the case of a regular distribution $u \in \mathfrak{E}(U)$, the answer should be the *support* $\operatorname{supp} u$, which is defined as the closure of the set $\{x \in U \mid u(x) \neq 0\}$. We will now extend this notion to regular distributions. Firstly, given an open subset $V \subset U$, we can define an inclusion $\iota_{V,U} : \mathfrak{D}(V) \hookrightarrow \mathfrak{D}(U)$ such that $\iota_{V,U}(f)(x)$ is $f(x)$ for $x \in V$ and 0 otherwise. We can then define a restriction map $r_{U,V} : \mathfrak{D}'(U) \to \mathfrak{D}'(V)$ dual to this, i.e. $\langle r_{U,V}(u), f \rangle = \langle u, \iota_{V,U}(f) \rangle$.

**Definition 2.4.3.** The *support* of a distribution $u \in \mathfrak{D}'(U)$ is the **complement** of the set of points $x \in U$ such that there exists a neighbourhood $V$ of $x$ for which $r_{U,V} u = 0$.

As desired, this definition coincides with the usual notion of support for regular distributions. For an example of a singular distribution, the Dirac delta $\delta : f \mapsto f(0)$ has support $\{0\}$, which also matches the intuitive notion that the support comprises all the points where a distribution is sensitive to the definition of the input function. More generally, one can quickly show that, if $\operatorname{supp} u \cap \operatorname{supp} f = \emptyset$, then $\langle u, f \rangle = 0$.

A nice consequence of this definition is the following:

**Proposition 2.4.4.** *Let $\mathfrak{E}'(U)$ denote the continuous dual of $\mathfrak{E}(U)$ with respect to its Fréchet topology. There is a linear isomorphism between $\mathfrak{E}'(U)$ and the subspace of $\mathfrak{D}'(U)$ comprising distributions $u$ such that $\operatorname{supp} u$ is compact.*

*Proof.* See [Hör15, §2.3]. □

### 2.4.3 Defining the Wavefront Set

We will now work our way towards answering the question: 'when can one multiply a given pair of distributions?' There is already a partial answer we can give without too much work. Namely, if $u \in \mathfrak{D}'(U)$ is a regular distribution, then we can define its product with any $v \in \mathfrak{D}'(U)$ to be $\langle uv, f \rangle := \langle v, uf \rangle$.

Note that, in a similar fashion, we can also define the *derivative* of a distribution by $\langle \partial_{x_i} u, f \rangle := -\langle u, \partial_{x_i} f \rangle$. (The minus sign is present so that this definition is consistent with (2.22) when $u$ is a regular distribution.)

For $U = \mathbb{R}^n$, we can also define the Fourier transform for a certain class of distributions. The space $\mathfrak{S}(\mathbb{R}^n)$ of *Schwartz* functions (also called *Schwartz space*) is the subspace of functions $f \in \mathcal{C}^\infty(\mathbb{R}^n)$ such that

$$\sup_{x \in \mathbb{R}^n} |x^\alpha \partial^\beta f(x)| =: ||f||_{\alpha,\beta} < \infty, \tag{2.23}$$

where $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ for $\alpha \in \mathbb{N}^n$ and $\partial^\beta$ is defined similarly. The Fourier transform of a Schwartz function is again Schwartz. We can then recall some relevant facts about the Fourier transform of functions:

**Proposition 2.4.5.** *Let $\hat{f}$ denote the Fourier transform of a Schwartz function $f \in \mathfrak{S}(\mathbb{R}^n)$. We use the convention that*

$$\hat{f}(\xi) := \int_{\mathbb{R}^n} f(x) e^{-i\xi \cdot x} \mathrm{d}^n x. \tag{2.24}$$

*Then the following equations hold.*

1. $\widehat{f * g} = \hat{f} \cdot \hat{g}$,
2. $\widehat{fg} = (2\pi)^{-n} \hat{f} * \hat{g}$,
3. $\int_{\mathbb{R}^n} \hat{f}\, g \,\mathrm{d}^n x = \int_{\mathbb{R}^n} f\, \hat{g} \,\mathrm{d}^n x$,
4. $\int_{\mathbb{R}^n} f\, \bar{g} \,\mathrm{d}^n x = (2\pi)^{-n} \int_{\mathbb{R}^n} \hat{f}\, \bar{\hat{g}} \,\mathrm{d}^n x$,

*where* $*$ *denotes the convolution* $f * g(x) = \int_{\mathbb{R}^n} f(y)g(x-y)\mathrm{d}^n y$*, and the first two equalities hold* almost everywhere (*i.e. they differ on a set of measure zero*).

The first two properties suggest that products may be exchanged for convolutions under the Fourier transform. This provides an intuitive picture for the Hörmander product we shall be introducing shortly.

The third property motivates the definition of the Fourier transform of a distribution as $\langle \hat{u}, f \rangle = \langle u, \hat{f} \rangle$.

By using the semi-norms (2.23), we can equip $\mathfrak{S}(\mathbb{R}^n)$ with a topology such that the inclusions $\mathfrak{D}(\mathbb{R}^n) \subset \mathfrak{S}(\mathbb{R}^n) \subset \mathfrak{E}(\mathbb{R}^n)$ are continuous. We can also define the continuous dual space $\mathfrak{S}'(\mathbb{R}^n)$, elements of which are referred to as *tempered* distributions. It is then clear that the dual relations to the above inclusions hold, namely $\mathfrak{E}'(\mathbb{R}^n) \subset \mathfrak{S}'(\mathbb{R}^n) \subset \mathfrak{D}'(\mathbb{R}^n)$.

Fourier transforms behave as well as one could hope in Schwartz space. The Fourier transform is a linear homeomorphism of $\mathfrak{S}(\mathbb{R}^n)$ [Hör15, Theorem 7.1.5], which one can use to show that the transform on tempered distributions by the relation following Proposition 2.4.5 is also a linear homeomorphism of $\mathfrak{S}'(\mathbb{R}^n)$ (where the topology is the weak dual topology induced by $\mathfrak{S}(\mathbb{R}^n)$). In fact, it is in the space of tempered distributions that the well-known formula

$$\frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{ix\xi} \mathrm{d}^n \xi = \delta(x) \tag{2.25}$$

can be given rigorous meaning.

A more surprising result is that, for a compactly supported distribution $u \in \mathfrak{E}(\mathbb{R}^n)$, the Fourier transform of $u$ is regular. If we define the family of functions $e_\xi(x) := e^{-i\xi \cdot x}$, then $\hat{u}(\xi) = \langle u, e_\xi \rangle$ is a regular function which agrees with the definition of $\hat{u}$ as a tempered distribution. Moreover, $\hat{u}$ extends to an entire function on $\mathbb{C}$, whose growth rate is determined by 'how singular' $u$ is.

All of these facts and more are collected in the *Payley-Weiner-Schwartz theorem*:

**Theorem 2.4.6.** *There is a one-to-one correspondence between entire functions $F$ on $\mathbb{C}^n$ such that $\forall\, z \in \mathbb{C}^n$, there exists constants $C, N, B \in \mathbb{R}$ such that*

$$|F(z)| \leq C(1+|z|)^N e^{B|\mathrm{Im}(z)|}, \tag{2.26}$$

*and $u \in \mathfrak{E}'(\mathbb{R}^n)$ such that $F = \hat{u}$.*

*Moreover, $u$ is regular if and only if for every $N \in \mathbb{N}$ there exists $C_N \in \mathbb{R}_{>0}$ such that*

$$|F(z)| \leq C_N(1+|z|)^{-N} e^{B|\mathrm{Im}(z)|}. \tag{2.27}$$

A statement with tighter estimates and a proof of this theorem can be found in [Hör15, Theorem 7.3.1].

The connection between the smoothness of a distribution and the growth rate of its Fourier transform is the key to providing a precise mathematical formulation of singular structure. We shall demonstrate this with an example: firstly, the Fourier transform of the Dirac delta is $\hat{\delta}(\xi) = 1$, which extends trivially to an entire function. However, this function fails to decay rapidly, which is a consequence of the fact that $\delta$ is singular. Consider the distribution $u = \frac{1}{x+i0^+}$ introduced in the beginning, which we can define more precisely as the map

$$\langle u, f\rangle := \lim_{\epsilon \searrow 0} \int_{\mathbb{R}} \frac{f(x)}{x+i\epsilon} \mathrm{d}x. \tag{2.28}$$

This distribution is tempered. Roughly speaking this is because $u$ decays as $1/x$ as $x \to \infty$. As such, one can compute its Fourier transform, by contour integration or otherwise, to be

$$\hat{u}(\xi) = -2\pi i \theta(\xi). \tag{2.29}$$

This function is not itself smooth (because $u$ does not decay rapidly), nevertheless we can still observe that $\hat{u}$ fails to decay rapidly as $\xi \to \infty$, *but not as* $\xi \to -\infty$. It is precisely this asymmetry that enables us to define the pointwise product of $u$ with itself, where we cannot with $\delta$. We can compute the convolution of $\hat{u}$ with itself as

$$\begin{aligned}
(\hat{u} * \hat{u})(\xi) &= -(2\pi)^2 \int_{\mathbb{R}} \theta(\eta)\theta(\xi-\eta)\mathrm{d}\eta, \\
&= -(2\pi)^2 \int_{\eta=0}^{\xi} \mathrm{d}\eta, \\
&= -(2\pi)^2 \xi \theta(\xi) \\
&= -(2\pi)\widehat{u'}(\xi).
\end{aligned}$$

From which we may conclude, using proposition 2.4.5, that $u^2(x) = -u'(x)$. Note that a similar attempt to define $\hat{\delta} * \hat{\delta}$ would fail, as the relevant integral is clearly divergent.

This provides the intuitive picture for how distributions can be multiplied. However, in order to formulate a more precise condition for multiplication, we need a slightly different approach. For any pair of functions $\varphi, \psi \in \mathfrak{E}(\mathbb{R}^n)$, we can define their *tensor product* in $\mathfrak{E}(\mathbb{R}^{2n})$ as the function $(\varphi \otimes \psi)(x, y) := \varphi(x)\psi(y)$. The pointwise product $\varphi\psi$ is then given by the *pullback* of $\varphi \otimes \psi$ along the embedding $\iota_\Delta : x \mapsto (x, x)$.

It turns out that one can similarly define the tensor product of *distributions* as a map $\otimes : \mathfrak{D}'(U) \times \mathfrak{D}'(V) \to \mathfrak{D}'(U \times V)$ such that

$$\langle u \otimes v, f \otimes g \rangle_{U \times V} = \langle u, f \rangle_U \langle v, g \rangle_V .$$

As $\mathfrak{D}(U) \otimes \mathfrak{D}(V)$ is sequentially dense in $\mathfrak{D}(U \times V)$, with a little effort one can show that this defined the tensor product entirely.

Thus, we can replace the problem of defining multiplication with the problem of defining pullbacks of distributions along embeddings. To skip ahead a little, it turns out that this is possible whenever the 'directions' of the singularities of the distribution are not normal to the image of the embedding.

To make sense of this statement, we must define the direction of a singularity. Following [BDH14], and similarly to how we defined supports, we to make our definition backwards to ensure that the set of singular directions is closed. We also need to make an auxiliary definition which effectively defines the topology on the space of directions.

**Definition 2.4.7.** A *conic neighbourhood* of the point $x \in \mathbb{R}^n \setminus \{0\}$ is a subset $V \subseteq \mathbb{R} \setminus \{0\}$ which is closed under multiplication by positive scalars, and which contains an open set containing $x$.

**Definition 2.4.8.** For a tempered distribution $u \in \mathfrak{S}'(\mathbb{R}^n)$, we say $\xi \in \mathbb{R}^n \setminus \{0\}$ is a *regular direction* if there exists a conic neighbourhood $V$ of $\xi$ such that $\hat{u}|_V$ decays rapidly. More precisely this means that for every $N \in \mathbb{N}$ there exists $C_N$ such that, for every $\eta \in V$

$$|\hat{u}(\eta)| \leq C_N(1 + |\eta|)^{-N}$$

Conversely, $\xi \in \mathbb{R}^n \setminus \{0\}$ is a *singular direction* if it is not a regular direction. We denote by $\Sigma(u)$ the set of all singular directions of $u$. (This is sometimes referred to as the *frequency set* of $u$.)

**Example 2.4.9.** For $\delta$ every direction is singular, hence $\Sigma(\delta) = \mathbb{R} \setminus \{0\}$. Whereas, for $u(x) = \frac{1}{x+i0^+}$, every $\xi > 0$ is regular, as $\hat{u}$ vanishes there, and every $\xi < 0$ is singular. For any regular distribution $v$, by theorem 2.4.6, we clearly have $\Sigma(v) = \emptyset$.

As well as the direction of singularities, we also need to pay attention to their locations. It straightforward to show that, for any $f \in \mathfrak{D}(U)$, $f(x)\delta(x) = f(0)\delta(x)$. As such,

$$\Sigma(f\delta) = \begin{cases} \mathbb{R} \setminus \{0\} & \text{if } f(0) \neq 0, \\ \emptyset & \text{else.} \end{cases}$$

The general form of this statement is that $\Sigma(fu) \subseteq \Sigma(u)$, where this inclusion may be proper if $f$ vanishes 'where' $u$ is singular. This allows us to localise the frequency sets by introducing

$$\Sigma_x(u) := \bigcap_{f(x) \neq 0} \Sigma(fu). \tag{2.30}$$

A useful side-effect of this definition is that, even though $\Sigma(u)$ cannot be defined for an arbitrary distribution $u \in \mathfrak{D}'(\mathbb{R}^n)$, for any $f \in \mathfrak{D}(\mathbb{R}^n)$, $fu \in \mathfrak{E}(\mathbb{R}^n)$, hence $\Sigma_x(u)$ is well defined for every $x \in \mathbb{R}^n$. In fact, we can also define $\Sigma_x(u)$ for $u \in \mathfrak{D}'(U)$, $x \in U$, by using inclusion $\mathfrak{D}'(U) \hookrightarrow \mathfrak{D}'(\mathbb{R}^n)$.

We are finally ready to define the wavefront set:

**Definition 2.4.10.** Let $U \subseteq \mathbb{R}^n$ be open. The *wavefront set* of a distribution $u \in \mathfrak{D}'(U)$ is the subset of $U \times \mathbb{R}^n \setminus \{0\}$ given by

$$\mathrm{WF}(u) := \bigcup_{x \in U} \{x\} \times \Sigma_x(u). \tag{2.31}$$

The wavefront set allows us to distinguish different subspaces of $\mathfrak{D}'(U)$ by imposing constraints on their wavefront sets.

**Definition 2.4.11.** Let $U \subseteq \mathbb{R}^n$ be open, and let $\Gamma \subseteq U \times (\mathbb{R}^n \setminus \{0\})$ be a closed conic set[4]. We then define the space

$$\mathfrak{D}'_\Gamma(U) = \{u \in \mathfrak{D}'(U) \,|\, \mathrm{WF}(u) \subseteq \Gamma\}. \tag{2.32}$$

This space is given a *pseudo-topology* (i.e. a definition of convergent sequences), where a sequence $(u_j)_{j\in\mathbb{N}} \subset \mathfrak{D}'_\Gamma(U)$ converges to $u \in \mathfrak{D}'_\Gamma(U)$ if and only if

---

[4] A set $\Gamma \subseteq U \times (\mathbb{R}^n \setminus \{0\})$ is called *conic* if $(x, \lambda\xi) \in \Gamma$, $\forall\,(x,\xi) \in \Gamma, \lambda > 0$. A conic set $\Gamma$ is open if every point has a conic neighbourhood $\Gamma' \subseteq \Gamma$, and closed if its complement is open.

- $|\langle u - u_j, \varphi\rangle| \to 0$ as $j \to \infty$ for every $\varphi \in \mathfrak{D}(U)$,
- $\sup_{\xi\in V} |\xi|^N |\widehat{\varphi u}(\xi) - \widehat{\varphi u_j}(\xi)| \to 0$ as $j \to \infty$ whenever $N > 0$, $\varphi \in \mathfrak{D}(U)$, and $V \subseteq \mathbb{R}^n \setminus \{0\}$ such that $(\operatorname{supp}\varphi \times V) \cap \Gamma = \emptyset$.

### *2.4.4 Operations on Distributions*

The main reason for introducing this notion of convergence is [Hör15, Theorem 8.2.3], which states that, for every $u \in \mathfrak{D}'_\Gamma$, there is a sequence $(u_j)_{j\in\mathbb{N}}$ of test functions which converges to $u$ in $\mathfrak{D}'_\Gamma(U)$. Given that the pullback of each $u_j$ along any smooth map $U \to V$ is well-defined, this gives a potential definition for the pullback of $u$. This potential is realised by the following [Hör15, p. 8.2.4]

**Theorem 2.4.12.** *Let $U$ and $V$ be open subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively, and let $f : U \to V$ be a smooth map. Denote the* normal set *of $f$ by*

$$N_f = \left\{(f(x), \eta) \in V \times \mathbb{R}^m \mid df_x^t \eta = 0\right\}.$$

*Then, for every conic set $\Gamma \subseteq V \times (\mathbb{R}^m \setminus \{0\})$ such that*

$$\Gamma \cap N_f = \emptyset,$$

*one can define a map $f^* : \mathfrak{D}'_\Gamma(V) \to \mathfrak{D}'_{f^*\Gamma}(U)$, where*

$$f^*\Gamma := \left\{(x, df_x^t \eta) \mid (f(x), \eta) \in \Gamma\right\}.$$

*Moreover, this map is sequentially continuous in the sense of definition 2.4.11, and for $u \in \mathfrak{E}(V)$ we have that $f^*u = u \circ f$.*

*Proof.* One can of course consult Hörmander. Alternatively, for a proof at a more relaxed pace one can instead see [BF09, §4.3]. □

There are many things we can learn from this result. Firstly, note that if $f$ is a diffeomorphism then $N_f = V \times \{0\}$, hence $\Gamma \cap N_f = \emptyset$ for all $\Gamma$. From this we learn that wavefront sets transform *covariantly* under a change of coordinates, hence we could think of $U \times \mathbb{R}^n \setminus \{0\}$ as $\dot{T}^*U$, the *cotangent bundle* of $U$ with the zero section removed. This allows one to define the wavefront set without a preferred choice of coordinates, hence we can also define the wavefront set of a distribution on a manifold by use of charts.

In order to determine when the pullback of a tensor product of distributions $u, v \in \mathfrak{D}'(U)$ along $f(x) = (x, x)$ is well defined, we need to be able to estimate the wavefront set of $u \otimes v$. This is done by [Hör15, Theorem 8.2.9], which we shall also state here.

**Theorem 2.4.13.** *For $u \in \mathfrak{D}'(U)$, $v \in \mathfrak{D}'(V)$, the tensor product $u \otimes v \in \mathfrak{D}'(U \times V)$ satisfies*

$$\mathrm{WF}(u \otimes v) \subseteq (\mathrm{WF}(u) \cup \underline{0}_U) \times (\mathrm{WF}(v) \cup \underline{0}_V) \setminus \underline{0}_{U \times V}.$$

With all of the necessary results in place, we shall now demonstrate how the wavefront set can be used to define multiplication.

**Theorem 2.4.14.** *Let $\Gamma_1, \Gamma_2 \subseteq \dot{T}^*U$ be two conic sets and denote*

$$-\Gamma = \left\{(x, \xi) \in \dot{T}^*U \mid (x, -\xi) \in \Gamma\right\}.$$

*If $\Gamma_1 \cap -\Gamma_2 = \emptyset$, then one can define the pointwise multiplication map $\mathfrak{D}'_{\Gamma_1}(U) \otimes \mathfrak{D}'(\Gamma_2)(U) \to \mathfrak{D}'_{\Gamma}$, where $\Gamma = \left\{(x, \xi + \eta) \in \dot{T}^*U \mid (x, \xi) \in (\Gamma_1 \cup \underline{0}_U), (x, \eta) \in (\Gamma_2 \cup \underline{0}_U)\right\}$. This coincides with the usual pointwise product of functions when restricted to $\mathfrak{E}(U) \subseteq \mathfrak{D}'_{\Gamma_1}(U) \cap \mathfrak{D}'_{\Gamma_2}(U)$.*

*Proof.* As we have already mentioned, the strategy is to define the pointwise product as the pullback of $u \otimes v$ along the diagonal embedding $f : x \mapsto (x, x)$. The conic set containing $\mathrm{WF}(u \otimes v)$ is estimated by theorem 2.4.13, hence in particular we have, for $u \in \mathfrak{D}'_{\Gamma_1}(U)$, $v \in \mathfrak{D}'_{\Gamma_2}(U)$

$$\mathrm{WF}(u \otimes v) \subseteq \Gamma_3 := (\Gamma_1 \cup \underline{0}_U) \times (\Gamma_2 \cup \underline{0}_U) \setminus \underline{0}_{U^2}.$$

Now, rather than showing that $N_f \cap \Gamma_3 = \emptyset$, we can in fact skip ahead and go straight to computing $f^*\Gamma_3$, as one can quickly show that $f^*\Gamma_3 \cap \underline{0}_U = \emptyset \Rightarrow N_f \cap \Gamma_3 = \emptyset$.

We clearly have that, for $(\xi, \eta) \in \mathbb{R}^{2n}$, $df^t_x(\xi, \eta) = \xi + \eta$, hence

$$f^*\Gamma_3 = \{(x, \xi + \eta) \in T^*U \mid (x, x, \xi, \eta) \in \Gamma_3\}$$

This set intersects $\underline{0}_U$ if and only if there exists $(x, \xi) \in \Gamma_1$ such that $(x, -\xi) \in \Gamma_2$, i.e. $(x, \xi) \in \Gamma_1 \cap -\Gamma_2$. As we have assumed this set to be empty, this cannot be the case. Hence $f^*\Gamma_3 = \Gamma$ and the product is well-defined. ☐

This is consistent with our earlier computation of $u^2$ for $u(x) = \frac{1}{x+i0^+}$, as in this case one can show that $\mathrm{WF}(u) = \{0\} \times \mathbb{R}_{>0}$, hence $\mathrm{WF}(u) \cap -\mathrm{WF}(u) = \emptyset$.

As a final fact, to recall the intuition behind defining our operations on distributions using Fourier transforms, we state [DB14, Lemma 3], which can be seen as a generalisation of Parseval's theorem.

**Lemma 2.4.15.** *Let* $\Gamma$ *be a closed conic set in* $\dot{T}^*U$*, and let* $\Lambda = \left\{(x,\xi) \in \dot{T}^*U \mid (x,-\xi) \notin \Gamma\right\}$*. Further, denote* $\mathfrak{E}'_\Lambda(U) = \{v \in \mathfrak{E}'(U) \mid \mathrm{WF}(v) \subseteq \Lambda\}$*, then the pairing*

$$\langle u, v\rangle = \frac{1}{2\pi^n}\int_{\mathbb{R}^n} \widehat{u\varphi}(\xi)\hat{v}(-\xi)\mathrm{d}^n\xi \tag{2.33}$$

*is well-defined, where* $u \in \mathfrak{D}'_\Gamma(U)$*,* $v \in \mathfrak{E}'_\Lambda$*, and* $\varphi$ *is any function in* $\mathfrak{D}(U)$ *such that* $\varphi \equiv 1$ *on* $\operatorname{supp} v$*.*

## 2.5 CLASSICAL KINEMATICS

Let $M$ be a smooth manifold (we shall specify dimension and topological constraints later). For the theory of a real scalar field, we take our configuration space, $\mathfrak{E}(M)$, to be the space of smooth real-valued functions on $M$. (Note that the choice $M = \mathbb{R}$ corresponds precisely to the configuration space of the simple harmonic oscillator from section 2.3.)

More generally, we might consider the space of smooth sections of some vector bundle $E \xrightarrow{\pi} M$, to which the following constructions can be readily generalised. Note that this space is 'off-shell' in the sense that it includes field configurations which may not satisfy any equations of motions later imposed by the dynamics.

Classically, observables are maps $\mathcal{F} : \mathfrak{E}(M) \to \mathbb{C}$. Typically, we also assume them to be smooth, with respect to an appropriate notion of smoothness which we shall introduce shortly. The derivative of a functional at a point $\varphi \in \mathfrak{E}(M)$ and in a direction $h \in \mathfrak{E}(M)$ is defined in the obvious way as

$$\left\langle \mathcal{F}^{(1)}[\varphi], h\right\rangle := \lim_{\epsilon\to 0} \frac{\mathcal{F}[\varphi + \epsilon h] - \mathcal{F}[\varphi]}{\epsilon}, \tag{2.34}$$

whenever this limit exists. If it exists for all $\varphi, h \in \mathfrak{E}(M)$, and the map

$$\mathcal{F}^{(1)} : (\varphi, h) \mapsto \left\langle \mathcal{F}^{(1)}[\varphi], h\right\rangle$$

is continuous with respect to the product topology on $\mathfrak{E}(M)^2$ then we say $\mathcal{F}$ is $\mathcal{C}^1$.

Higher derivatives of $\mathcal{F}$ are defined similarly by

$$\left\langle \mathcal{F}^{(n)}[\varphi], h_1 \otimes \cdots \otimes h_n\right\rangle := \frac{\partial^n \mathcal{F}[\varphi + \epsilon_1 h_1 + \cdots + \epsilon_n h_n]}{\partial\epsilon_1 \cdots \partial\epsilon_n}\Big|_{\epsilon_1 = \cdots = \epsilon_n = 0}, \tag{2.35}$$

wherever these limits exist. If $\forall\, n \in \mathbb{N}$ and $\varphi \in \mathfrak{E}(M)$, $\mathcal{F}^{(n)}[\varphi]$ exists, and the maps

$$\begin{aligned} \mathcal{F}^{(n)} : \mathfrak{E}(M) \times \mathfrak{E}(M^n) &\to \mathbb{C} \\ (\varphi, h_1 \otimes \cdots \otimes h_n) &\mapsto \left\langle \mathcal{F}^{(n)}[\varphi], h_1 \otimes \cdots \otimes h_n\right\rangle \end{aligned}$$

are all continuous then we say $\mathcal{F}$ is *Bastiani* smooth as discussed in, for example [BDLR18, §II]. For $\mathcal{F}$ a Bastiani smooth functional, following [BDLR18, Definition III.1], we may define its *spacetime support* as

$$\operatorname{supp}\mathcal{F} := \{x \in M \mid \forall\, U \ni x \text{ open }, \exists\, \varphi \in \mathfrak{E}(M), \psi \in \mathfrak{D}(U) \text{ s.t. } \mathcal{F}[\varphi+\psi] \neq \mathcal{F}[\varphi]\} \tag{2.36}$$

We shall denote by $\mathfrak{F}(M)$ the space of Bastiani smooth functionals of the real scalar field over $M$ with compact spacetime support.

Various pieces of notation are commonly used when discussing functional derivatives. For clarity, we collect some of them here. A consequence of the above definition is that, for $\mathcal{F}$ a $\mathcal{C}^1$ functional, $\mathcal{F}^{(1)}[\varphi]$ is an element of $\mathfrak{E}'(M)^{\mathbb{C}}$ [BDLR18, §III], where the superscript $\mathbb{C}$ denotes complexification. Hence the bracket $\langle\cdot,\cdot\rangle$ in (2.34) can be seen as denoting the canonical pairing, as defined in section 2.4. If $M$ is equipped with a preferred volume form $\mathcal{F}^{(1)}[\varphi]$ may be given an integral kernel, typically written as[5]

$$\left\langle \mathcal{F}^{(1)}[\varphi], h \right\rangle = \int_M \frac{\delta \mathcal{F}[\varphi]}{\delta\varphi(x)} h(x)\, \mathrm{d}V_M. \tag{2.37}$$

Finally, we introduce the map, for a $\mathcal{C}^1$ functional $\mathcal{F}$, $\frac{\delta}{\delta\varphi} : \mathcal{F} \mapsto \mathcal{F}^{(1)}$.

Similarly to the $n=1$ case, for a Bastiani smooth functional $\mathcal{F} \in \mathfrak{F}(M)$, $\mathcal{F}^{(n)}[\varphi]$ will in general be a complex-valued, compactly-supported distribution of $n$ variables [BDLR18, proposition III.4], i.e. $\mathcal{F}^{(n)}[\varphi] \in \mathfrak{E}'(M^n)^{\mathbb{C}}$. Recall that we say this distribution is *regular* if there exists $f \in \mathfrak{D}(M^n)$ such that $\forall\, h \in \mathfrak{E}(M^n)$

$$\left\langle \mathcal{F}^{(n)}[\varphi], h \right\rangle = \int_{M^n} f(x_1,\ldots,x_n)h(x_1,\ldots,x_n)\mathrm{d}V_M^n.$$

If $\mathcal{F}^{(n)}[\varphi]$ is a regular distribution for every $n \in \mathbb{N}$ and $\varphi \in \mathfrak{E}(\mathcal{M})$, then we say that $\mathcal{F}$ is a *regular functional*, and we denote the space of regular functionals $\mathfrak{F}_{\mathrm{reg}}(M)$.

Regular functionals are particularly convenient to work with, as we shall see when defining the Poisson bracket and $\star$ product later. However, they exclude many functionals of physical interest, such as components of the stress-energy tensor in the case of the scalar field. Thus, we next consider the subspace of $\mathfrak{F}(M)$ consisting of *local* functionals.

Following [Rej16], we define a functional $\mathcal{F}$ to be *local* if there exists an open cover $\bigcup_{\alpha\in\mathcal{A}} U_\alpha = \mathfrak{E}(M)$ such that, for $\varphi \in U_\alpha$

$$\mathcal{F}[\varphi] = \int_M f_\alpha\left(j_x^k\varphi\right)\mathrm{d}V_M, \tag{2.38}$$

---

[5] On Lorentzian manifolds, we always have the metric volume form. However, when we discuss chiral fields later, we shall have to revisit this shortcut.

where $j_x^k\varphi$ is the $k^{\text{th}}$ jet prolongation of $\varphi$ at $x$, which is heuristically characterised by

$$j_x^k\varphi = (\varphi(x), \nabla\varphi(x), \dots, \nabla^k\varphi(x)),$$

i.e. the equivalence class of functions whose first $k$ derivatives agree with $\varphi$ at $x$, and $f_\alpha$ is some smooth, compactly-supported function on the $k^{\text{th}}$ jet bundle of $M$, whose fibres are the spaces of the aforementioned equivalence classes. We denote by $\mathfrak{F}_{\text{loc}}(M)$ the space of local functionals on $M$, and by $\mathfrak{F}_{\text{mloc}}(M)$ the space of *multilocal* functionals the algebraic completion of $\mathfrak{F}_{\text{loc}}(M)$ under the pointwise product of functionals.

Whilst this is the best definition of a local functional, rather than getting deep into the details of the definition, we simply state the properties which locality implies that are most relevant for our purposes. Perhaps the most important such property [Rej16, Remark 3.2] is that, for every $n \in \mathbb{N}$, $\varphi \in \mathfrak{E}(M)$, the support of $\mathcal{F}^{(n)}[\varphi]$ (in the sense of definition 2.4.3) is contained within the thin diagonal

$$\Delta_n = \{(x, \dots, x) \in M^n\}_{x \in M},$$

and that its wavefront set is orthogonal to $T\Delta_n$.

Notably this implies that, for $n \geq 2$, these derivatives must either vanish or fail to be regular. In other words, the intersection $\mathfrak{F}_{\text{reg}}(M) \cap \mathfrak{F}_{\text{loc}}(M)$ of functionals which are both regular and local comprises only linear functionals of the form

$$\Phi(f) : \varphi \mapsto \int_M f(x)\varphi(x)\,\mathrm{d}V_M,$$

for $f \in \mathfrak{D}(M)$.

Whilst it is possible to perform our classical and quantum operations on local functionals, the result is typically not local. As such, we need a space of functionals which is algebraically convenient, like $\mathfrak{F}_{\text{reg}}(M)$, but which also contains the physically important subspace $\mathfrak{F}_{\text{loc}}(M)$. The space of *microcausal* functionals accomplishes this. However, unlike the previous classes of functionals, it cannot be defined on an arbitrary manifold. Instead we require the structure of a *spacetime*, which we define in accordance with [FV12, §2.1] as follows:

**Definition 2.5.1.** A *spacetime* is a tuple $\mathcal{M} = (M, g, \mathfrak{o}, \mathfrak{t})$ such that $(M, g)$ is an orientable Lorentzian manifold (with metric signature $(+, -, -\cdots)$) of some fixed dimension $d$, $\mathfrak{o} \subset \Omega^d(M)$ is an equivalence class of nowhere-vanishing volume forms, defining an orientation, and $\mathfrak{t} \subset \mathfrak{X}(M)$ is an equivalence class of timelike vector fields, where $t \sim t' \Leftrightarrow g_x(t_x, t'_x) > 0 \,\forall\, x \in M$.

We will typically write $\mathfrak{F}(\mathcal{M})$, $\mathfrak{F}_{\text{reg}}(\mathcal{M})$, and $\mathfrak{F}_{\text{loc}}(\mathcal{M})$ to refer to the respective spaces of functionals associated to the underlying manifold of $\mathcal{M}$.

For any point $x$ in a spacetime $\mathcal{M}$, we can define the closed past/future lightcone of the cotangent space $\overline{V}_{\pm}(x) \subset T_x^*M$ as comprising covectors $k$ for which $\hat{g}_x(k,k) \geq 0$ and $\pm k(t_x) \geq 0$, for any $t \in \mathfrak{t}$, where $\hat{g}_x$ is the metric induced on $T_x^*M$ by $g$. We can then define the sub-fibre bundles $\overline{V}_{\pm}$ such that their fibres at $x$ are $\overline{V}_{\pm}(x)$ respectively.

Using this, we call a functional $\mathcal{F} \in \mathfrak{F}(\mathcal{M})$ *microcausal* if it satisfies the *microcausal spectral condition*

$$\mathrm{WF}(\mathcal{F}^{(n)}[\varphi]) \cap \left(\overline{V}_+^n \cup \overline{V}_-^n\right) = \emptyset, \tag{2.39}$$

Where the wavefront set of $\mathcal{F}^{(n)}[\varphi]$ is just as we defined in section 2.4. The space of microcausal functionals is denoted $\mathfrak{F}_{\mu c}(\mathcal{M})$, and contains all regular and local functionals [BFR19, Proposition 3.3].

The characteristic features of these spaces, as well as the relations between them, are summarised in the following diagram.

$$\begin{tikzcd}
& \boxed{\mathfrak{F} \;\Big|\; \mathcal{F}^{(n)}[\varphi] \text{ exists } \forall\, n, \varphi} & \\
& \boxed{\mathfrak{F}_{\mu c} \;\Big|\; \mathrm{WF}\left(\mathcal{F}^{(n)}[\varphi]\right) \cap \left(\overline{V}_+^n \cup \overline{V}_-^n\right) = \emptyset} \arrow[u, hook] & \\
\boxed{\mathfrak{F}_{\text{loc}} \;\Big|\; \operatorname{supp}\left(\mathcal{F}^{(n)}[\varphi]\right) \subseteq \Delta_n} \arrow[ur, hook] & & \boxed{\mathfrak{F}_{\text{reg}} \;\Big|\; \mathrm{WF}\left(\mathcal{F}^{(n)}\right) = \emptyset} \arrow[ul, hook]
\end{tikzcd}$$

## 2.6 CLASSICAL DYNAMICS

We shall now impose the dynamics in much the same way as was done for the simple harmonic oscillator in section 2.3. Recall that the foundational idea of this approach, due to Peierls [Pei52], is the formulation of a Poisson structure in terms of the advanced and retarded responses of a field to perturbation. A construction of the classical algebra of observables using the Peierls bracket was set forth in [DF03], and developed in detail in [BFR19] More recent overviews may be found in, e.g. [Rej16, §4] or [FR15, §5.1].

This approach has the advantage of being independent of any particular reference frame, and hence covariant under local isometries, (as will be explored further in section 2.9) whilst still endowing our space of observables with a Poisson structure,

The existence of this Poisson bracket is indeed contrary to a common notion that such a structure requires one to split a spacetime into 'space' and 'time'.

As we saw in section 2.3, the problem with an action functional as is typically written is that their region of integration must be restricted to a compact subset of spacetime in order to guarantee a finite value is returned. A convenient way to achieve this is to define a map $\mathcal{L} : \mathfrak{D}(\mathcal{M}) \to \mathfrak{F}_{\text{loc}}(\mathcal{M})$, where the functional $\mathcal{L}(f)$ is interpreted as the action functional with an introduced cutoff function $f$. Not every such map is suitable however, the necessary criteria are outlined in the following definition (after [Rej16, §4.1]).

**Definition 2.6.1.** A map $\mathcal{L} : \mathfrak{D}(\mathcal{M}) \to \mathfrak{F}_{\text{loc}}(\mathcal{M})$ is called a *generalised Lagrangian* if it satisfies the following conditions:

1. If $f, g, h \in \mathfrak{D}(\mathcal{M})$ such that $\operatorname{supp} f \cap \operatorname{supp} h = \emptyset$ then
$$\mathcal{L}(f+g+h) = \mathcal{L}(f+g) - \mathcal{L}(g) + \mathcal{L}(g+h). \qquad (\textit{Additivity})$$
2. $\operatorname{supp} \mathcal{L}(f) \subseteq \operatorname{supp} f$. (*Support*)
3. If $\beta$ is an isometry of $(M, g)$ which preserves orientation and time-orientation, then for $f \in \mathfrak{D}(\mathcal{M})$ and $\varphi \in \mathfrak{E}(\mathcal{M})$,
$$\mathcal{L}(f)[\beta^*\varphi] = \mathcal{L}(\beta_* f)[\varphi]. \qquad (\textit{Covariance})$$

*Remark* 2.6.2. The additivity property is a weaker version of linearity, which still captures the concept that $\mathcal{L}$ depends only locally upon $f$. We will only make explicit use of Lagrangians which are linear, but the more general definition may be necessary, for example, when considering Yang-Mills theories or when following the Epstein-Glaser renormalisation procedure, where $f$ plays the role of a coupling constant, as well as cutoff.

Additionally, we note that this definition refers to the *spacetime support*, $\operatorname{supp} \mathcal{F}$ for a functional $\mathcal{F}$. This is the *closure* of the set of points $x \in \mathcal{M}$ such that, for all $\varphi \in \mathfrak{E}(\mathcal{M})$, there exists some perturbation localised to a neighbourhood of $x$, say $\psi \in \mathfrak{D}(U)$ for some $U \ni x$, which changes the output of $\mathcal{F}$, i.e. $\mathcal{F}[\varphi + \psi] \neq \mathcal{F}[\varphi]$. Distributions are in particular linear functionals, and in this case the spacetime support coincides with the support of a distribution we defined in section 2.4.

The generalised Lagrangian we shall focus on is that of the Klein-Gordon field on $d$-dimensional Minkowski space $\mathbb{M}_d$, which is given by

$$\mathcal{L}(f)[\varphi] := \frac{1}{2}\int_{\mathbb{M}_d} f\left[\partial_\mu\varphi\partial^\mu\varphi - m^2\varphi^2\right] \mathrm{d}^d x. \tag{2.40}$$

Heuristically, one may think of the limit of $\mathcal{L}(f)$ as $f$ tends to a Dirac delta $\delta_x$ as describing the Lagrangian density at $x$ and, if $f$ instead tends to the constant function $\mathbf{1}$, then $\mathcal{L}(f)$ becomes the action functional $S$. However one must bear in mind that, in general, these limits may not (and typically *will* not) yield well-defined local functionals.

**Definition 2.6.3.** Given a generalised Lagrangian $\mathcal{L}$, we define the *Euler-Lagrange derivative* (*EL derivative* for short) at a point $\varphi \in \mathfrak{E}(\mathcal{M})$ as the distribution $S'[\varphi]$ such that

$$\left\langle \mathcal{L}(f)^{(1)}[\varphi], h \right\rangle =: \langle S'[\varphi], h\rangle . \tag{2.41}$$

where, $h \in \mathfrak{D}(\mathcal{M})$ and $f \in \mathfrak{D}(\mathcal{M})$ is chosen such that $f^{-1}\{1\}$ contains a neighbourhood of $\operatorname{supp} h$ [6].

One can use the additivity and support properties to verify that $S'[\varphi]$ is well-defined (i.e. (2.41) is independent of the choice of $f$). A field configuration $\varphi \in \mathfrak{E}(\mathcal{M})$ is called *on-shell* if it's *EL* derivative $S'[\varphi]$ vanishes as a distribution.

Different choices of generalised Lagrangian may yield the same *EL* derivative. If a generalised Lagrangian $\mathcal{L}_0$ satisfies $\operatorname{supp} \mathcal{L}_0(f) \subseteq \operatorname{supp} df$, then clearly its *EL* derivative vanishes for all $\varphi \in \mathfrak{E}(\mathcal{M})$. In such a case, we describe $\mathcal{L}_0$ as *null*. Clearly, adding $\mathcal{L}_0$ to an arbitrary generalised Lagrangian would not change its *EL* derivative. Given this, we say that two generalised Lagrangians, $\mathcal{L}$ and $\mathcal{L}'$ define the same *action* if their difference is null, we denote this fact by $[\mathcal{L}] = [\mathcal{L}'] =: S$.

In the case where $S$ is a quadratic action, i.e. it may be represented by a Lagrangian $\mathcal{L}$ such that $\mathcal{L}(f)$ is a quadratic functional for all $f$, then the map $\varphi \mapsto \langle S'[\varphi], h\rangle$ is linear in $\varphi$. We assume that this functional can be expressed in the form $\varphi \mapsto \langle P\varphi, h\rangle$, where $-P$ is a *normally hyperbolic* differential operator, i.e. $P$ is a second order differential operator of the form $\nabla^a\nabla_a+$ *lower order terms*. A more precise definition of normally hyperbolic differential operators can be found in, e.g. [BGP07, §1.5]. As an example, given the free field Lagrangian (2.40), $P$ is simply the Klein-Gordon operator $-(\Box + m^2)$.

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[6] We opt for a slightly stronger condition on $f$ than found in, for example [BFR19, Definition 3.2]. This is ultimately insignificant, but it makes it easier to show that null Lagrangians (defined below) have vanishing Euler-Lagrange derivatives.

For interacting theories, one must take a further functional derivative, defining

$$\left\langle \mathcal{L}(f)^{(2)}[\varphi], h \otimes g \right\rangle =: \langle S''[\varphi], h \otimes g \rangle \,, \tag{2.42}$$

where $f^{-1}\{1\}$ contains a neighbourhood of $\operatorname{supp} h$ or $\operatorname{supp} g$ (or both). By the *Schwartz kernel theorem* [Hör15, Theorem 5.2.1], we may then express this in terms of an operator $P_\varphi : \mathfrak{D}(\mathcal{M}) \to \mathfrak{D}'(\mathcal{M})$, for each $\varphi \in \mathfrak{E}(\mathcal{M})$

$$\langle S''[\varphi], h \otimes g \rangle = \langle P_\varphi g, h \rangle \,. \tag{2.43}$$

For a broad class of physically relevant actions, $P_\varphi$ is a self-adjoint[7], normally hyperbolic differential operator.

**Definition 2.6.4.** We refer to the equation $P_\varphi \phi = 0$ for $\phi \in \mathfrak{E}(\mathcal{M})$ as the *linearised equations of motion* at the configuration $\varphi$ and, if such an operator exists for every $\varphi \in \mathfrak{E}(\mathcal{M})$, we say that the action satisfies the *linearisation hypothesis*. If $\varphi$ is an on-shell configuration, then $\operatorname{Ker} P_\varphi$ can be thought of as the tangent space at $\varphi$ to the manifold of on-shell configurations. Note that for a free action, $P$ coincides with $P_\varphi$ for every $\varphi \in \mathfrak{E}(\mathcal{M})$.

In the study of PDEs on Lorentzian manifolds, the global structure is often relevant. In particular, for physically relevant PDEs, where initial data cannot propagate faster than the speed of light, one often wishes to place constraints upon the causal structure of the spacetime. (It might be perfectly acceptable for a cannonball to wrap around the universe and hit the back of the cannon from which it was fired, but perhaps the impact shouldn't happen *before* it was fired.) Indeed, there is an entire *causal hierarchy* [MS08] of conditions which make a spacetime more causally well-behaved. We shall choose to sit right atop the hierarchy, and consider only those spacetimes which are *globally hyperbolic*.

**Definition 2.6.5.** A Lorentzian manifold $\mathcal{M} = (M, g)$ is *globally hyperbolic* if it posesses a *Cauchy surface*, i.e. a subset $\Sigma \subset M$ such that every inextensible timelike curve in $\mathcal{M}$ intersects $\Sigma$ precisely once.

One can think of globally hyperbolic spacetimes as those which admit global, albeit *non-canonical*, decompositions into "space" and "time". This is best exemplified by the following result [BS03, Theorem 1.1].

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[7] To be precise, we mean that $P_\varphi$ is *formally* self-adjoint, i.e. $\int_{\mathcal{M}} f P_\varphi g \mathrm{d}V_{\mathcal{M}} = \int_{\mathcal{M}} g P_\varphi f \mathrm{d}V_{\mathcal{M}}$ for every $f, g \in \mathfrak{D}(\mathcal{M})$.

**Theorem 2.6.6.** *For any globally hyperbolic spacetime $\mathcal{M}$, there exists a smooth, spacelike Cauchy surface $\Sigma$ of $\mathcal{M}$ and an isomorphism $\mathcal{M} \simeq \mathbb{R}\times\Sigma$, where the latter manifold is equipped with the product metric $\tilde{g} = \beta \mathrm{d}t^2 - h$, where $t : \mathbb{R}\times\Sigma \to \mathbb{R}$ is the time function corresponding to projection onto the first factor, and $h$ is a symmetric tensor field on $\mathbb{R}\times\Sigma$ which restricts to a Riemannian metric on each hypersurface $\{t_0\}\times\Sigma$.*

The key feature of globally hyperbolic spacetimes is the existence of *Green hyperbolic* differential operators $P$, characterised by the property that the equation $P\varphi = 0$ admits special fundamental solutions, called the *advanced and retarded propagators* $E^{R/A} : \mathfrak{D}(\mathcal{M}) \to \mathfrak{E}(\mathcal{M})$. These maps are similar to the maps $\Delta^{R/A}$ from section 2.3, and are uniquely distinguished by the fact that, for any $f \in \mathfrak{D}(\mathcal{M})$

$$PE^{R/A}f = E^{R/A}Pf = f, \tag{2.44}$$

$$\mathrm{supp}\left(E^{R/A}f\right) \subseteq \mathfrak{J}^{\pm}(\mathrm{supp}(f)). \tag{2.45}$$

Here $\mathfrak{J}^{\pm}(K)$ denotes the causal future/past of $K$, i.e. the set of all points connected to some point $x \in K$ by a causal future/past directed curve respectively. For detailed exposition of the theory of normally hyperbolic and Green hyperbolic differential operators, we refer the reader to [Bär15] and [BGP07].

Each propagator is formally adjoint to the other in the sense that, for all $f, g \in \mathfrak{D}(\mathcal{M})$

$$\left\langle f, E^R g\right\rangle = \left\langle g, E^A f\right\rangle. \tag{2.46}$$

Their difference $E = E^R - E^A$ defines a map, known as the *Pauli-Jordan function*, from $\mathfrak{D}(\mathcal{M})$ to the space of solutions of $P\phi = 0$, which we shall use to define our Poisson structure.

Note that we shall mostly be considering a free theory, governed by the single linear equation $P\varphi = 0$. However, to generalise to the interacting case, one need only replace $P$ with the linearised operator $P_\varphi$ defined by (2.43), and note that the fundamental solutions are then defined relative to this linearised operator.

Recall that the phase space of a free field theory is simply the space $\mathrm{Ker}\,P$ of solutions to the equations of motion. Traditionally, we identify this with the space of Cauchy data on some fixed surface, i.e. the field strength and canonically conjugate momentum at some fixed time. [BGP07, Proposition 3.4.7] states that *all* solutions with spacelike-compact support may be expressed as $Ef$ for some $f \in \mathfrak{D}(\mathcal{M})$ and also that the kernel of this map is precisely $P(\mathfrak{D}(\mathcal{M}))$. In other words, we can identify the space of physical field configurations with the quotient $\mathfrak{D}(\mathcal{M})/P(\mathfrak{D}(\mathcal{M}))$. One *could* then define the algebra of observables on $\mathcal{M}$ to be the space of smooth maps from this space to $\mathbb{C}$, which $E$ naturally equips with a non-degenerate Poisson bracket. This is not, however, the approach that we shall take, which we outline below.

Given two regular functionals $\mathcal{F}, \mathcal{G} \in \mathfrak{F}_{\mathrm{reg}}(\mathcal{M})$, we can use $E$ to define a new functional

$$\{\mathcal{F}, \mathcal{G}\}[\varphi] := \left\langle \mathcal{F}^{(1)}[\varphi], E\mathcal{G}^{(1)}[\varphi] \right\rangle \tag{2.47}$$

called the *Peierls bracket* of $\mathcal{F}$ and $\mathcal{G}$, where we recall that $\mathcal{F}^{(1)}[\varphi]$ and $\mathcal{G}^{(1)}[\varphi]$ may be identified with smooth test functions when $\mathcal{F}$ and $\mathcal{G}$ are regular. *Local* functionals also possess this property, hence we can define the Peierls bracket of local functionals. However, the result of this operation is not in general local, i.e. $\mathfrak{F}_{\mathrm{loc}}(\mathcal{M})$ is not closed under the Peierls bracket.

To obtain a closed algebra, we extend the domain of the Pauli-Jordan function to include a suitable class of distributions. As shown in Appendix B, the pairing $\langle f, Eg \rangle$ is well defined if $f$ and $g$ are compactly-supported distributions satisfying the $(n = 1)$ wavefront set spectral condition (2.39). In particular, this means (2.47) is well defined for $\mathcal{F}, \mathcal{G} \in \mathfrak{F}_{\mu c}(\mathcal{M})$, and one can show (Appendix B) that the result is again a microcausal functional. Once it is established that $\{\cdot, \cdot\}$ is also a derivation over the pointwise product of functionals, we may conclude that $(\mathfrak{F}_{\mu c}(\mathcal{M}), \cdot, \{\cdot, \cdot\})$ is a Poisson algebra [BFR19, Theorem 4.1], which we shall denote $\mathfrak{P}(\mathcal{M})$. This is our classical algebra of observables, which we shall quantise by deformation in section 2.8.

Once again, we shall point out that this Poisson algebra is *off-shell*, in the sense that the underlying space, $\mathfrak{F}_{\mu c}(\mathcal{M})$, comprises functionals defined for all conceivable field configurations $\varphi$, not only the critical points of the action. This is intentional, and in the following section we shall see how it is possible from here to both recover the on-shell algebra in a natural way, and in the same stroke describe any potential gauge symmetries a theory may possess.

## 2.7 GOING ON-SHELL

A well-known result [Kim93, (2.6)] states that, given a manifold $X$ with some closed submanifold $Y \subseteq X$, there is an isomorphism

$$\mathcal{C}^{\infty}(Y) \simeq \mathcal{C}^{\infty}(X)/\mathcal{I}(Y), \tag{2.48}$$

where $\mathcal{I}(Y) \subseteq \mathcal{C}^{\infty}(X)$ is the ideal of functions vanishing on $Y$. The construction of the Poisson algebra of on-shell observables may be regarded as an infinite-dimensional analogue of this isomorphism, where $\mathcal{C}^\infty(X)$ is replaced with $\mathfrak{F}_{\mu c}(\mathcal{M})$. We define the ideal $\mathfrak{I}_S \subseteq \mathfrak{F}_{\mu c}(\mathcal{M})$ to be the set of functionals which vanish for all on-shell configurations, i.e. $\forall \mathcal{F} \in \mathfrak{I}_S, P\varphi = 0 \Rightarrow \mathcal{F}[\varphi] = 0$. The following fact then tells us how to obtain the dynamics on this subspace.

**Proposition 2.7.1.** *The space $\mathfrak{I}_S$ is also an ideal with respect to $\{\cdot,\cdot\}$.*

*Proof.* (Sketch) This can be proved from (2.47) because, if $\varphi$ is a solution, $\mathcal{F} \in \mathfrak{I}_S$, and $\mathcal{G} \in \mathfrak{F}_{\mu c}(\mathcal{M})$ then $\varphi + \epsilon E\mathcal{G}^{(1)}[\varphi]$ is also a solution for any $\epsilon > 0$, hence

$$\mathcal{F}[\varphi + \epsilon E\mathcal{G}^{(1)}[\varphi]] = 0, \tag{2.49}$$

i.e. $\{\mathcal{F},\mathcal{G}\}[\varphi] = \left\langle \mathcal{F}^{(1)}[\varphi], E\mathcal{G}^{(1)}[\varphi]\right\rangle = 0$, indicating that $\{\mathfrak{F}_{\mu c}(\mathcal{M}), \mathfrak{I}_S\} \subseteq \mathfrak{I}_S$ as desired. For a more detailed proof, see [BFR19, Proposition 4.2]. □

We can thus construct the on-shell algebra as follows.

**Definition 2.7.2.** Given a globally hyperbolic spacetime $\mathcal{M}$ with an action $S$ satisfying the linearisation hypothesis, the *on-shell algebra of observables* $\mathfrak{P}_{\mathrm{on}}(\mathcal{M})$ is defined as the space $\mathfrak{F}_{\mu c}(\mathcal{M})/\mathfrak{I}_S$ with the Poisson bracket given by

$$\{[\mathcal{F}],[\mathcal{G}]\} := [\{\mathcal{F},\mathcal{G}\}]\,. \tag{2.50}$$

One advantage of characterising the on-shell observables in this way is that it involves linear, algebraic structures (namely $\mathfrak{F}_{\mu c}(\mathcal{M})$ and $\mathfrak{I}_S$) even when the underlying space of on-shell field configurations may be complicated. It is also a first step towards an alternative formulation of classical/quantum field theory in terms of *differential, graded algebras* (dg-algebras), where the space of *non-trivial* symmetries of a theory is described in much the same way as the space of on-shell observables. This perspective is particularly useful when quantising gauge theories [Rej16, Chapter 7].

As a brief aside, we shall provide here a rough outline of this dg reformulation of pAQFT. However, it is important to note that this formulation has not been fully realised for the microcausal functionals we have been working with. As such, we follow [GR19, §5.1] and work instead with the space $\mathfrak{F}_{\mathrm{reg}}(\mathcal{M})$ of *regular* functionals defined in section 2.5.

Recall that we called a field configuration $\varphi$ *on-shell* precisely when $\langle S'[\varphi], h\rangle$, vanished for all $h \in \mathfrak{D}(\mathcal{M})$. We would like to find a suitable generalisation of the maps

$\langle S'[-], h\rangle$ such that *every* element of $\mathfrak{I}_S(\mathcal{M})$ can be expressed in this way. We can think of $h$ as defining a *vector field* on $\mathfrak{E}(\mathcal{M})$ by

$$X_h = \int_{\mathcal{M}} h(x) \frac{\delta}{\delta\varphi(x)} dV_{\mathcal{M}}. \tag{2.51}$$

We can then express the above functionals as

$$\langle S'[-], h\rangle = X_h \cdot \mathcal{L}(f), \tag{2.52}$$

where $f \equiv 1$ on $\operatorname{supp} h$, and $(X_h \cdot \mathcal{F})[\varphi] := \left\langle \mathcal{F}^{(1)}[\varphi], h \right\rangle$ is the natural action of the vector field on a functional by derivation. The key idea for a dg formulation of pAQFT is that it should be possible to obtain the entire ideal $\mathfrak{I}_S$ in this way, given the right notion of a vector field.

For a *rough* overview, we begin by noting that, sections of the tangent bundle, using the identification $T\mathfrak{E}(\mathcal{M}) \simeq \mathfrak{E}(\mathcal{M}) \oplus \mathfrak{E}(\mathcal{M})$, are Bastiani smooth maps $X : \mathfrak{E}(\mathcal{M}) \to \mathfrak{E}(\mathcal{M})$. In [Rej16], conditions were given for such maps to behave as graded analogues of microcausal functionals. The definition is too complex to give here. Suffice it to say that the functional derivatives $X^{(n)}(\varphi) : \mathfrak{E}(\mathcal{M}^n) \to \mathfrak{E}(\mathcal{M})$ are well defined, and can be canonically identified with elements of $\mathfrak{E}'(\mathcal{M}^{n+1})$ which have wavefront sets disjoint from $\overline{V}_+^{n+1} \cup \overline{V}_-^{n+1}$ as in (2.39).

As with vector fields on finite-dimensional manifolds, wedge products of microcausal vector fields are then defined which, after taking a topological completion, produces the complex of *microcausal polyvector fields*, which we shall denote $\bigwedge^\bullet \mathfrak{V}_{\mu c}(\mathcal{M})$. Importantly, one property of microcausal vector fields is that

$$\operatorname{supp} X := \overline{\bigcup_{\varphi \in \mathfrak{E}(\mathcal{M})} \operatorname{supp} X^{(1)}(\varphi)} \cup \overline{\bigcup_{\varphi \in \mathfrak{E}(\mathcal{M})} \operatorname{supp} X(\varphi)}$$

is compact, where $X^{(1)}(\varphi) : \mathfrak{E}(\mathcal{M}) \to \mathfrak{E}'(\mathcal{M})$ has been identified with an element of $\mathfrak{E}'(\mathcal{M}^2)$ (again, refer to [Rej16] for details).

We can define a map $\delta_S : \mathfrak{V}_{\mu c}(\mathcal{M}) \to \mathfrak{F}_{\mu c}(\mathcal{M})$ by $\delta_S(X)[\varphi] := \left\langle \mathcal{L}(f)^{(1)}[\varphi], X(\varphi) \right\rangle$, where $f \equiv 1$ on a neighbourhood of the $\operatorname{supp} X$. We call $\delta_S(X)$ the *variation of the action with respect to* $X$.

The principle of critical action for $\varphi \in \mathfrak{E}(\mathcal{M})$ can be expressed as the condition that, $\delta_S(X)[\varphi] \equiv 0, \forall X \in \mathfrak{V}_{\mu c}(\mathcal{M})$. Hence, it is clear that all functionals which arise as a variation of the action under a vector field must vanish on-shell, in other words, $\delta_S(\mathfrak{V}_{\mu c}(\mathcal{M})) \subseteq \mathfrak{I}_S(\mathcal{M})$.

We can now begin to see aspects of the BV formalism (see, e.g. [Rej16, §4.3] for more details) appearing if we extend $\delta_S : \mathfrak{V}_{\mu c}(\mathcal{M}) \to \mathfrak{F}_{\mu c}(\mathcal{M})$ to form a cochain complex:

$$\cdots \xrightarrow{\delta_S} \bigwedge^3 \mathfrak{V}_{\mu c}(\mathcal{M}) \xrightarrow{\delta_S} \bigwedge^2 \mathfrak{V}_{\mu c}(\mathcal{M}) \xrightarrow{\delta_S} \mathfrak{V}_{\mu c}(\mathcal{M}) \xrightarrow{\delta_S} \mathfrak{F}_{\mu c}(\mathcal{M}) \longrightarrow 0, \tag{2.53}$$

where $\delta_S$ is defined in lower degrees via the graded Leibniz rule: for example, a homogeneous element $X \wedge Y \in \bigwedge^2 \mathfrak{V}_{\mu c}(\mathcal{M})$ is mapped to $\delta_S(X \wedge Y) = \delta_S(X)Y - \delta_S(Y)X$. We call this the *Koszul complex associated to* $\delta_S$, denoted $\mathfrak{K}(\delta_S)$.

It is, at this point, natural to ask whether or not there exists a physical interpretation of $H^{-1}(\mathfrak{K}(\delta_S))$, or the cohomology in yet lower degrees. To answer the first, note that for a vector field $X$, $\delta_S(X) = 0$ implies that the infinitesimal transformation $\varphi \mapsto \varphi + \epsilon X[\varphi]$ leaves the action invariant to first order in $\epsilon$. As such, the kernel of $\delta_S$ in degree $-1$ comprises infinitesimal generators of *gauge symmetries*[8]. The image of $\delta_S$ in degree $-1$ contains vector fields of the form $\delta_S(X \wedge Y) = \delta_S(X)Y - \delta_S(Y)X$. In the physics literature these are referred to as *trivial gauge symmetries*. They are, in a sense, less insightful because they are defined the same way regardless of the action in question, and also because they act trivially on shell. As such, we can regard $H^{-1}(\mathfrak{K}(\delta_S))$ as the space of *non-trivial gauge symmetries* [9].

The above discussion motivates us to consider the complex $\bigwedge^\bullet \mathfrak{V}_{\mu c}$ as the primary kinematical object of a physical theory, with $\delta_S$ representing the choice of dynamics. This perspective is advantageous both in describing conformally covariant field theories (where the generalised Lagrangian formalism proves inconvenient) as well as in the formulation of chiral sectors of a 2D CFT, where one may require choices of $\delta_S$ which cannot arise from a generalised Lagrangian.

Finally, as an aside now that we have constructed our on-shell algebra, it is informative to make a comparison to the 'canonical' bracket defined relative to some choice of Cauchy surface $\Sigma$.

**Definition 2.7.3** (Canonical Poisson Algebra)**.** Let $\Sigma \subset \mathcal{M}$ be a Cauchy surface, we define the associated *canonical Poisson algebra* as follows: The underlying vector space $\mathfrak{F}_{\text{can}}(\Sigma)$ consists of functionals $F : \mathfrak{D}(\Sigma) \times \mathfrak{D}(\Sigma) \to \mathbb{C}$ which are Bastiani smooth, the arguments of this functional represent the initial field strength and momentum on $\Sigma$ of some on-shell field configuration. Given a pair $F, G$ of such functionals, their

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[8] Gauge in the sense that the perturbation of $\varphi$ is always localised in some compact region. Other definitions of gauge symmetry are available.

[9] In principle, one can go further [CG16, Introduction §3.2], interpreting elements of $H^{-2}(\mathfrak{K}(\delta_S))$ as 'symmetries between symmetries', however, such notions are tricky to formulate precisely and are well beyond the scope of this thesis.

canonical bracket is then defined as

$$\{F, G\}_{\text{can}}[\phi, \pi] := \int_{\Sigma} \left[ \frac{\delta F[\phi, \pi]}{\delta \phi(x)} \frac{\delta G[\phi, \pi]}{\delta \pi(x)} - \frac{\delta G[\phi, \pi]}{\delta \phi(x)} \frac{\delta F[\phi, \pi]}{\delta \pi(x)} \right] \mathrm{d}V_{\Sigma}. \tag{2.54}$$

It is not immediately obvious why the Peierls bracket should be related to this canonical bracket, other than because $E$ parametrises the space of on-shell field configurations. Especially as the canonical bracket requires a particular Cauchy surface to be specified, a manifestly Lorentz non-covariant choice. However, by sending the initial data $(\phi, \pi) \in \mathfrak{E}(\Sigma) \times \mathfrak{E}(\Sigma)$, to their corresponding solution, one can construct a map $\mathfrak{F}_{\mu c}(\mathcal{M}) \to \mathfrak{F}_{\text{can}}(\Sigma)$ which in turn yields a Poisson algebra homomorphism from the on-shell Peierls bracket to the canonical [FR15, §3.2].

## 2.8 DEFORMATION QUANTISATION

Having established our Poisson structure, the next step is to deform it to construct our *quantum* algebra of observables. Here we take an approach that is analogous to Moyal-Weyl quantisation, though the fact that our configuration space is infinite dimensional will present extra difficulties particular to the quantisation of field theories. In particular, as is common in perturbative QFT, our deformation shall be formal, meaning that quantised products will be formal power series in $\hbar$, allowing us to ignore the issue of proving convergence of our formulae.

For *regular* functionals $\mathcal{F}, \mathcal{G} \in \mathfrak{F}_{\text{reg}}(\mathcal{M})$ we can define the *star product* of $\mathcal{F}$ and $\mathcal{G}$ directly as

$$(\mathcal{F} \star \mathcal{G})[\varphi] = \mathcal{F}[\varphi]\mathcal{G}[\varphi] + \sum_{n \geq 1} \left( \frac{i\hbar}{2} \right)^n \frac{1}{n!} \left\langle E^{\otimes n}, \mathcal{F}^{(n)}[\varphi] \otimes \mathcal{G}^{(n)}[\varphi] \right\rangle. \tag{2.55}$$

We may write this formula more concisely as

$$\mathcal{F} \star \mathcal{G} := m \circ e^{\frac{i\hbar}{2} \left\langle E, \frac{\delta}{\delta \varphi} \otimes \frac{\delta}{\delta \varphi} \right\rangle} (\mathcal{F} \otimes \mathcal{G}), \tag{2.56}$$

where $m$ is the pointwise multiplication map $m(\mathcal{F} \otimes \mathcal{G})[\varphi] := (\mathcal{F} \otimes \mathcal{G})[\varphi \otimes \varphi] = \mathcal{F}[\varphi] \cdot \mathcal{G}[\varphi]$. A general result [HR19, Proposition 4.5] states that this exponential form guarantees $\star$ is associative. As mentioned, this deformation is formal, meaning we have actually defined a map $\star : \mathfrak{F}_{\text{reg}}(\mathcal{M}) \otimes \mathfrak{F}_{\text{reg}}(\mathcal{M}) \to \mathfrak{F}_{\text{reg}}(\mathcal{M})[[\hbar]]$. We can then define the $\star$ product on $\mathfrak{F}_{\text{reg}}(\mathcal{M})[[\hbar]]$ by linearity in the formal parameter.

Writing the first few terms explicitly, we see $\mathcal{F} \star \mathcal{G} = \mathcal{F} \cdot \mathcal{G} + \frac{i\hbar}{2} \{\mathcal{F}, \mathcal{G}\} + \mathcal{O}(\hbar^2)$. Thus the classical term of $\star$ (i.e. the coefficient of $\hbar^0$) is simply the pointwise product. The

Dirac quantisation rule also holds modulo terms of order $\hbar^2$, hence $\star$ is a deformation of the classical product in the sense of section 2.2. However, if we wished to apply (2.56) to local functionals, divergences would appear. Consider for example the family of quadratic functionals, for $f \in \mathfrak{D}(\mathcal{M})$

$$\Phi^2(f)[\varphi] := \int_{\mathcal{M}} f(x)\varphi^2(x)\,\mathrm{d}V_x. \tag{2.57}$$

A naïve computation of the star product for two such functionals would yield

$$\begin{aligned}\Phi^2(f) \star \Phi^2(g) \text{ ``}=\text{'' } &\Phi^2(f)\cdot\Phi^2(g) + \tfrac{i\hbar}{2}\left\{\Phi^2(f), \Phi^2(g)\right\} \\ &- \frac{\hbar^2}{2}\int_{\mathcal{M}^2} f(x)E^2(x;y)g(y)\,\mathrm{d}V_x\,\mathrm{d}V_y.\end{aligned} \tag{2.58}$$

In general, the $\mathcal{O}(\hbar^2)$ term of this product is ill-defined if $\mathrm{supp} f \cap \mathrm{supp} g \neq \emptyset$. Recalling section 2.4, we can describe the problem as the fact that there exists $(x,\xi) \in \dot{T}^*\mathcal{M}^2$ such that $(x,\xi),(x,-\xi) \in \mathrm{WF}(E)$.

The solution is to make use of a *Hadamard distribution*. Historically [KW91], Hadamard distributions were characterised locally as having a singular part analogous to that of 2-point function of the Minkowski vacuum (see (2.78)). It was later discovered by Radzikowski [Rad96] that the Hadamard condition could be more elegantly expressed in terms of wavefront sets. We make this notion precise with the following definition.

**Definition 2.8.1.** A complex-valued distribution $W \in \mathfrak{D}'(\mathcal{M}^2)^{\mathbb{C}}$ is *Hadamard* if it satisfies the following properties [Rej16, §5.1]

**H0** The wavefront set of $W$ satisfies

$$\mathrm{WF}(W) = \left\{(x,y;\xi,\eta) \in \mathrm{WF}(E) \,|\, (x;\xi) \in \overline{V}_+\right\} \tag{2.59}$$

**H1** $W = \frac{i}{2}E + H$, where $H$ is a symmetric, real distribution.

**H2** $W$ is a weak bi-solution to $P$.

**H3** $W$ is positive semi-definite in the sense that, $\forall\, f \in \mathfrak{D}(\mathcal{M};\mathbb{C})$ $\left\langle W, \bar{f}\otimes f\right\rangle \geq 0$.

For our present purposes, the key consequence of this definition is that $\mathrm{WF}(W) \subseteq \overline{V}_+ \times \overline{V}_-$. Given that $\mathrm{WF}(\mathcal{F}^{(n)}[\varphi]) \subseteq \dot{T}^*\mathcal{M}^n \setminus \left(\overline{V}_+^n \cup \overline{V}_-^n\right)$ for any $\mathcal{F} \in \mathfrak{F}_{\mu c}(\mathcal{M})$, we can then use lemma 2.4.15 (generalised to manifolds) to show that the pairing

$$\left\langle W^{\otimes n}, \mathcal{F}^{(n)}[\varphi] \otimes \mathcal{G}^{(n)}[\varphi]\right\rangle$$

is always well-defined for any pair $\mathcal{F},\mathcal{G} \in \mathfrak{F}_{\mu c}(\mathcal{M})$, and $\varphi \in \mathfrak{E}(\mathcal{M})$.

A choice of Hadamard distribution yields a corresponding star product by

$$\mathcal{F} \star_H \mathcal{G} := m \circ e^{\left\langle \hbar W, \frac{\delta}{\delta\varphi} \otimes \frac{\delta}{\delta\varphi} \right\rangle} (\mathcal{F} \otimes \mathcal{G}) . \tag{2.60}$$

In contrast to the Peierls bracket, one cannot easily extend this formula to an interacting theory by replacing $W$ with a Hadamard distribution of the linearised theory. In short this is because repeated multiplications would involve taking derivatives $\frac{\delta}{\delta\varphi} W$, which means that the product will fail in general to be associative.

Property **H1** ensures that $\star_H$ is a deformation in the sense of eq. (2.5)[10]. Moreover, it implies that any freedom in the choice of a Hadamard state $W$ lies solely in the choice of its symmetric part $H$. As such, we shall denote by $\mathrm{Had}(\mathcal{M})$ the set of bi-distributions $H$ such that $\frac{i}{2}E + H$ is a Hadamard distribution as per the above definition.

For every choice $H \in \mathrm{Had}(\mathcal{M})$, $\star_H$ is an associative product on $\mathfrak{F}_{\mathrm{reg}}(\mathcal{M})$. Moreover, on this space it is isomorphic to $\star$: if we define the map $\alpha_H : \mathfrak{F}_{\mathrm{reg}}(\mathcal{M}) \to \mathfrak{F}_{\mathrm{reg}}(\mathcal{M})$ by

$$\alpha_H \mathcal{F} = e^{\frac{\hbar}{2} \left\langle H, \frac{\delta^2}{\delta\varphi^2} \right\rangle} \mathcal{F}, \tag{2.61}$$

then $\alpha_H \left(\mathcal{F} \star \mathcal{G}\right) = (\alpha_H \mathcal{F}) \star_H (\alpha_H \mathcal{G})$, for any $\mathcal{F}, \mathcal{G} \in \mathfrak{F}_{\mathrm{reg}}(\mathcal{M})$ and the inverse of this map is simply $\alpha_{-H}$. Where $\star$ and $\star_H$ differ, however, is that the latter can also be extended to a well defined product on $\mathfrak{F}_{\mu c}(\mathcal{M})$, as shown in appendix B.

On a generic globally hyperbolic spacetime, it is well-known [FNW81] that there exist infinitely many Hadamard distributions, thus we need never fear that $\mathrm{Had}(\mathcal{M})$ is empty. However, there is usually no natural way of selecting *which* $H \in \mathrm{Had}(\mathcal{M})$ to use. Thus, whilst we can always construct a well defined algebra

$$(\mathfrak{F}_{\mu c}(\mathcal{M})[[\hbar]], \star_H) =: \mathfrak{A}^H(\mathcal{M}) \tag{2.62}$$

for an arbitrary globally hyperbolic spacetime $\mathcal{M}$, it would be unnatural to define the quantum algebra to be any particular such choice. Fortunately, the algebraic structure of $\mathfrak{A}^H(\mathcal{M})$ is independent of the Hadamard distribution selected. If $H, H' \in \mathrm{Had}(\mathcal{M})$, then

$$\alpha_{H-H'} \left(\mathcal{F} \star_{H'} \mathcal{G}\right) = \left(\alpha_{H-H'} \mathcal{F}\right) \star_H \left(\alpha_{H-H'} \mathcal{G}\right), \tag{2.63}$$

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[10] As a brief overview of the other properties: **H0** allows the star product to be defined for local observables, **H2** ensures that $\mathfrak{I}_S(\mathcal{M})$ is a 2-sided ideal with respect to $\star_H$, and **H3** ensures that $W$ defines an algebraic state (once the quantum algebra of observables is constructed). For a more complete account, see [Rej16].

where $\alpha_{H-H'} : \mathfrak{A}^{H'}(\mathcal{M}) \to \mathfrak{A}^{H}(\mathcal{M})$ is defined just as in (2.61). As one might expect, the inverse of this map is $\alpha_{H'-H}$, hence all of our candidate algebras are in fact *isomorphic* to one another. One way in which we can define the quantum algebra without any undue preference to a particular Hadamard distribution is as follows:

**Definition 2.8.2.** The *quantum algebra of the free field theory*, denoted $\mathfrak{A}(\mathcal{M})$, is a unital, associative $*$-algebra whose elements are tuples $(\mathcal{F}_H)_{H\in\mathrm{Had}(\mathcal{M})}$ such that $\mathcal{F}_H \in \mathfrak{F}_{\mu c}(\mathcal{M})$ and subject to the compatibility criterion

$$\mathcal{F}_{H'} = \alpha_{H'-H}\mathcal{F}_H, \tag{2.64}$$

with a product defined by

$$(\mathcal{F}_H)_{H\in\mathrm{Had}(\mathcal{M})} \star (\mathcal{G}_H)_{H\in\mathrm{Had}(\mathcal{M})} := (\mathcal{F}_H \star_H \mathcal{G}_H)_{H\in\mathrm{Had}(\mathcal{M})} \,. \tag{2.65}$$

Yet again, this algebra is intentionally off-shell. We define the on-shell theory more-or-less the same way as we did in the classical case.

**Definition 2.8.3.** The *on-shell quantum algebra* of the free field is defined as the space

$$\mathfrak{A}_{\mathrm{on}}(\mathcal{M}) := \left\{ ([\mathcal{F}]_H)_{H\in\mathrm{Had}(\mathcal{M})} \mid (\mathcal{F}_H)_{H\in\mathrm{Had}(\mathcal{M})} \in \mathfrak{A}(\mathcal{M}) \right\}, \tag{2.66}$$

where $[\mathcal{F}]$ denotes the equivalence class of $\mathcal{F}$ with respect to the equivalence relation $\mathcal{F} \sim \mathcal{F} + \mathfrak{I}_S[[\hbar]]$, and where the product on $\mathfrak{A}_{\mathrm{on}}(\mathcal{M})$ is given by

$$([\mathcal{F}]_H)_{H\in\mathrm{Had}(\mathcal{M})} \star ([\mathcal{G}]_H)_{H\in\mathrm{Had}(\mathcal{M})} = ([\mathcal{F} \star_H \mathcal{G}]_H)_{H\in\mathrm{Had}(\mathcal{M})} \,. \tag{2.67}$$

The underlying vector space is well-defined because $\alpha_{H'-H} : \mathfrak{I}_S[[\hbar]] \longrightarrow \mathfrak{I}_S[[\hbar]]$ for every pair $H', H \in \mathrm{Had}(\mathcal{M})$, and the product is well defined as $\mathfrak{I}_S[[\hbar]]$ is an ideal with respect to $\star_H$ for every $H$. Both of these facts are a consequence of property **H2** of Hadamard distributions.

It is important to bear in mind that, whilst we have deformed the classical algebra $\mathfrak{F}_{\mu c}(\mathcal{M})$ into a quantum algebra $\mathfrak{A}(\mathcal{M})$, we have not yet specified a *quantisation map*, embedding classical observables into the quantum algebra. We will need to establish such a map before computing commutation relations for the quantum stress energy tensor in section 3.3. However, before considering what this map may be, it is instructive to study how the construction we have just outlined varies as we change the underlying spacetime $\mathcal{M}$.

## 2.9 LOCAL COVARIANCE AND NORMAL ORDERING

We have deliberately said little about Poincaré covariance in the construction above. The reason being that we take the perspective that covariance under any symmetries a particular spacetime may enjoy is just a special case of a broader property we wish to implement: namely *local covariance*. The concept of local covariance, introduced in [HW01] and [BFV03], unites the representation of spacetime symmetries as automorphisms of the algebra of observables with the principle that an observable localised to a region $\mathcal{O} \subset \mathcal{M}$ of a spacetime should be 'unaware' of the structure of the spacetime beyond this region.

The foundational idea is that, if there exists a 'suitable' embedding of a spacetime $\mathcal{M}$ into a spacetime $\mathcal{N}$, then there should be a corresponding embedding (more precisely, an injective *-homomorphism) of observables $\mathfrak{A}(\mathcal{M}) \to \mathfrak{A}(\mathcal{N})$. A spacetime symmetry is just a suitable embedding of $\mathcal{M}$ into itself which also admits an inverse. If the corresponding algebra homomorphism is similarly invertible, then we would have, in particular, an action of the isometry group of $\mathcal{M}$ on $\mathfrak{A}(\mathcal{M})$ as desired.

To formulate local covariance more precisely, it is convenient to invoke the language of category theory. (For an introduction to this language, see for example Chapter 1 of [Lan98].)

To begin with, by specifying the suitable embeddings of spacetimes, we promote the collection of globally hyperbolic spacetimes to a category, which is denoted Loc and defined as follows:

- An object of Loc is a *spacetime* $\mathcal{M}$, as specified in definition 2.5.1, of a fixed dimension $d$.
- For a pair of spacetimes $\mathcal{M} = (M, g, \mathfrak{o}, \mathfrak{t})$ and $\mathcal{N} = (N, g', \mathfrak{o}', \mathfrak{t}')$, a morphism $\chi : \mathcal{M} \to \mathcal{N}$ is a smooth isometric embedding $\chi : M \hookrightarrow N$ which is an isometry, i.e. $\chi^* g' = g$, and *admissible* in the sense that $\chi(M)$ is causally convex in $N$, $\mathfrak{o} = \chi^* \mathfrak{o}'$, and $\mathfrak{t} = \chi^* \mathfrak{t}'$.

Given a smooth map $\chi : M \to N$, there is a natural map $\mathfrak{F}(M) \to \mathfrak{F}(N)$ defined by $\mathcal{F} \mapsto \chi_* \mathcal{F} := \mathcal{F} \circ \chi^*$. Note this map is well-defined because the map $\chi^* : \mathfrak{E}(N) \to \mathfrak{E}(M)$

is continuous with respect to the Fréchet topology of each space[11], hence

$$(\chi_*\mathcal{F})^{(n)} = \mathcal{F}^{(n)} \circ \left(\chi^*, (\chi^{\otimes n})^*\right) : \mathfrak{E}(N) \times \mathfrak{E}(N^n) \to \mathbb{C} \tag{2.68}$$

exists and is continuous whenever $\mathcal{F}^{(n)}$ is, thus $\chi_*\mathcal{F}$ is Bastiani smooth.

**Proposition 2.9.1.** *Given a smooth embedding $\chi : M \to N$, the map $\chi_* : \mathfrak{F}(M) \to \mathfrak{F}(N)$ described above is injective.*

*Proof.* Suppose that $\mathcal{F}, \mathcal{G} \in \mathfrak{F}(M)$ satisfy $\chi_*\mathcal{F} = \chi_*\mathcal{G}$. As $\mathrm{supp}\mathcal{F} \cup \mathrm{supp}\mathcal{G}$ is compact, we may find some $f \in \mathfrak{D}(M)$ such that $f|_{\mathrm{supp}\mathcal{F}\cup\mathrm{supp}\mathcal{G}} \equiv 1$. For any $\varphi \in \mathfrak{E}(M)$, we then have $\psi := \chi_*(f\varphi) \in \mathfrak{D}(N) \subseteq \mathfrak{E}(N)$ such that $\chi^*\psi$ coincides with $\varphi$ on both $\mathrm{supp}\mathcal{F}$ and $\mathrm{supp}\mathcal{G}$. From this we may conclude $\mathcal{F}[\varphi] = \mathcal{F}[\chi^*\psi] = \mathcal{G}[\chi^*\psi] = \mathcal{G}[\varphi]$ as required. □

Each map $\chi_* : \mathfrak{F}(M) \to \mathfrak{F}(N)$ is an algebra homomorphism with respect to the pointwise product of functionals. Moreover it is clear that, for $\chi = \mathbb{1}_M$, $\chi_* = \mathbb{1}_{\mathfrak{F}(M)}$, and $(\chi \circ \rho)_*\mathcal{F} = \mathcal{F} \circ (\chi \circ \rho)^* = \mathcal{F} \circ \rho^* \circ \chi^* = \chi_*(\rho_*\mathcal{F})$, hence we say that $\mathfrak{F}$ is a *functor* from the category $\mathsf{Loc}$ to $\mathsf{Alg}$, the category whose objects are algebras over $\mathbb{C}$, and whose morphisms are algebra homomorphisms.

We show later in Section 4.2 that even if $\chi$ preserves the metric only up to a scale, then $\chi_*\mathcal{F}$ is still microcausal whenever $\mathcal{F}$ is, hence $\chi_*\left(\mathfrak{F}_{\mu c}(\mathcal{M})\right) \subset \mathfrak{F}_{\mu c}(\mathcal{N})$ for all $\mathsf{Loc}$ morphisms $\chi : \mathcal{M} \to \mathcal{N}$. In fact, all of the different spaces of functionals specified in section 2.5 are each preserved under the map $\chi_*$, and thus may be considered functors from $\mathsf{Loc}$ to a suitable category of observables, for instance the category of vector spaces over $\mathbb{C}$.

Next, we need to find a way to specify dynamics in a coherent way across all spacetimes. This involves extending our definition of a generalised Lagrangian to that of a *natural Lagrangian*. In categorical language, we can define a natural Lagrangian as a natural transformation $\mathcal{L} : \mathfrak{D} \Rightarrow \mathfrak{F}_{\mathrm{loc}}$, such that for each $\mathcal{M} \in \mathsf{Loc}$, $\mathcal{L}_{\mathcal{M}}$ is a generalised Lagrangian as per definition 2.6.1. Here, $\mathfrak{D}$ is the functor assigning each spacetime its space of compactly-supported test functions, and to each morphism $\chi : \mathcal{M} \to \mathcal{N}$ the map $\chi_* : \mathfrak{D}(\mathcal{M}) \to \mathfrak{D}(\mathcal{N})$ defined by

$$\chi_*f(y) = \begin{cases} f(\chi^{-1}(y)) & \text{if } y \in \chi(\mathcal{M}), \\ 0 & \text{else.} \end{cases} \tag{2.69}$$

---

[11] To see this in the case where $M, N$ are open subsets of $\mathbb{R}^n$, note that, by repeated application of the chain rule, we have $\partial_x^\alpha \varphi(\chi(x)) = \sum_{\beta\leq|\alpha|}(\partial_y^\beta\varphi)(\chi(x))P_{|\alpha|,\chi}(x)$, where $P_{|\alpha|,\chi}$ are some polynomials in the components of $\chi$ and their derivatives of order at most $|\alpha|$. This means there exists some $C_{K,m,\chi} > 0$ such that $||\chi^*\varphi||_{K,m} \leq C_{K,m,\chi}||\varphi||_{\chi(K),m}$, hence the map is continuous.

Spelling this out, the naturality condition reduces to the condition that, for every morphism of spacetimes $\chi : \mathcal{M} \to \mathcal{N}$, $f \in \mathfrak{D}(\mathcal{M})$ and $\varphi \in \mathfrak{E}(\mathcal{N})$

$$\mathcal{L}_{\mathcal{N}}(\chi_* f)[\varphi] = \mathcal{L}_{\mathcal{M}}(f)[\chi^* \varphi], \tag{2.70}$$

which is essentially a generalisation of the covariance condition appearing in definition 2.6.1. As an example, this condition is satisfied by the Klein-Gordon Lagrangian

$$\mathcal{L}_{\mathcal{M}}(f)[\varphi] := \frac{1}{2} \int_{\mathcal{M}} f \left[ g(\nabla \varphi, \nabla \varphi) - m \varphi^2 \right] \mathrm{d}V_{\mathcal{M}} \tag{2.71}$$

where $\nabla$ is the gradient operator associated to the metric $g$ of $\mathcal{M}$.

The Euler-Lagrange derivatives of $\mathcal{L}_{\mathcal{M}}$ and $\mathcal{L}_{\mathcal{N}}$ satisfy the condition that, $\forall \varphi \in \mathfrak{E}(\mathcal{N}), h \in \mathfrak{D}(\mathcal{M})$

$$\langle S'_{\mathcal{N}}[\varphi], \chi_* h \rangle = \langle S'_{\mathcal{M}}[\chi^* \varphi], h \rangle \,, \tag{2.72}$$

which is obtained simply by combining (2.70) with the definition (2.41) of the Euler-Lagrange derivative.

In the case of the free scalar field, $S'_{\mathcal{N}}[\varphi] = P_{\mathcal{N}} \varphi$, hence (2.72) implies $\chi^* P_{\mathcal{N}} = P_{\mathcal{M}} \chi^*$. In turn this means that, if we consider $E_{\mathcal{N}}^{R/A}$ as maps $\mathfrak{D}(\mathcal{N}) \to \mathfrak{E}(\mathcal{N})$ (and likewise $E_{\mathcal{M}}^{R/A}$), then

$$E_{\mathcal{M}}^{R/A} = \chi^* E_{\mathcal{N}}^{R/A} \chi_*.$$

(See the proof of proposition 4.2.3 for an argument in the context of *conformal* embeddings, which may also be applied essentially unchanged to the isometric case.) From here, it can explicitly verified (using (2.47) and (2.68)) that $\chi_* : \mathfrak{F}_{\mu c}(\mathcal{M}) \to \mathfrak{F}_{\mu c}(\mathcal{N})$ is a Poisson algebra homomorphism where each space is equipped with its respective Peierls bracket. Thus, the assignment $\mathfrak{P} : \mathsf{Loc} \to \mathsf{Poi}$ outlined in the above section is locally covariant. A similar argument in the case of conformal embeddings is also given later in section 4.2.

We shall occasionally use the generic designation Obs to denote the category our observables (either classical or quantum) belong to. Choices of Obs relevant to our discussion include

- Vec, whose objects are vector spaces over $\mathbb{C}$, and whose morphisms are linear maps. This is the most generic space generally considered, and is appropriate when one wishes to treat classical and quantum theories on an equal footing.

- Poi the category of Poisson algebras and Poisson algebra homomorphisms. This is the primary category of observables for classical theories.
- $*$-Alg, the category of topological $*$-algebras with *continuous* $*$-homomorphisms between objects. We choose this as the target category of quantum theories, as the perturbative nature of our construction requires us to consider unbounded operators, else we would use instead the category of $C^*$-algebras.
- In each of the above cases, we may add a *dg-structure*, i.e. if Obs is any of the above categories, Ch(Obs) comprises cochain complexes which in each degree take values in Obs. Such categories are at the heart of the BV formalism in both the classical and quantum case [GR19], [CG16].

A *locally covariant field theory* (classical or quantum) is then defined simply as a functor from Loc $\to$ Obs. Already this captures a lot of important features, such as the representation of spacetime symmetries as automorphisms of the algebra of observables. Whilst one can go further by imposing additional axioms for such a functor to satisfy, this general definition will suffice for our purposes. In particular, it is typically required that all the morphisms in the aforementioned categories (with the exception of Ch(Obs)) are taken to be *injective*. Equivalently, one instead require only those morphisms in the image of the functor to be injective, thus this can be seen as a condition either on the category or the functor.

We have already shown proposition 2.9.1 that this is the case for the classical functor $\mathfrak{P}$, from which the injectivity of the homomorphisms for $\mathfrak{A}$ readily follows when they are introduced shortly.

The BV formalism outlined in the previous section can also be made locally covariant. Just like $\mathfrak{F}_{\mu c}$, we can promote $\mathfrak{V}_{\mu c}$ to a functor Loc $\to$ Vec by defining the morphism

$$(\chi_* X)[\varphi] := \chi_*(X[\chi^*\varphi]),$$

where $\chi_*$ on the right-hand side denotes the pushforward of test functions. (This is another reason that we need to consider vector fields with compact support.) A choice of natural Lagrangian then yields a natural transformation $\delta_S : \mathfrak{V}_{\mu c} \Rightarrow \mathfrak{F}_{\mu c}$. From this it follows that the construction of the Koszul complex $\mathfrak{K}(\delta_S)$ itself defines a functor Loc $\to$ Ch(Poi).

We have already sketched an explanation of how our construction of the classical theory may be made locally covariant. If $H_0 \in \mathrm{Had}(\mathcal{N})$, then one can show that $\chi^* H_0 \in$

$\mathrm{Had}(\mathcal{M})$, thus we can define a map $\mathfrak{A}^{(\chi^* H_0)}(\mathcal{M}) \to \mathfrak{A}^{H_0}(\mathcal{N})$ as just the linear extension of $\chi_* : \mathfrak{F}_{\mu c}(\mathcal{M}) \to \mathfrak{F}_{\mu c}(\mathcal{N})$ to formal power series in $\hbar$. This map satisfies

$$\chi_*(\mathcal{F} \star_{(\chi^* H_0)} \mathcal{G}) = \chi_*\mathcal{F} \star_{H_0} \chi_*\mathcal{G} \tag{2.73}$$

thus it defines a $*$-algebra homomorphism. The map $\mathfrak{A}\chi : \mathfrak{A}(\mathcal{M}) \to \mathfrak{A}(\mathcal{N})$ is then given by

$$\left(\mathfrak{A}\chi \left(\mathcal{F}_H\right)_{H \in \mathrm{Had}(\mathcal{M})}\right)_{H_0} = \chi_*\mathcal{F}_{(\chi^* H_0)}, \tag{2.74}$$

which can be shown to satisfy the criterion (2.64), making the map well-defined. With these morphisms, we can then declare $\mathfrak{A} : \mathsf{Loc} \to \mathsf{Obs}$ to be a *locally covariant quantum field theory*.

Next, we turn to the topic of *normal ordering*. Recall from section 2.2 that an ordering prescription was a procedure for assigning classical observables their quantum counterparts. In our case, we seek a map $: - :_{\mathcal{M}} : \mathfrak{F}_{\mathrm{loc}}(\mathcal{M}) \to \mathfrak{A}(\mathcal{M})$, such that the $\hbar^0$ coefficient of $:\mathcal{F}:_{\mathcal{M}}$ is $\mathcal{F}$. Given our somewhat indirect definition of $\mathfrak{A}(\mathcal{M})$, it is helpful to outline here the general strategy for defining a normal ordering prescription, before we turn our attention to any particular maps.

It is easiest to define a normal ordering prescription as a choice of map $\mathfrak{F}_{\mathrm{loc}}(\mathcal{M}) \to \mathfrak{A}^H(\mathcal{M})$ for every $H \in \mathrm{Had}(\mathcal{M})$. Suppose we denote each map by $\mathcal{F} \mapsto (:\mathcal{F}:)_H$. Collectively, they define a map $\mathfrak{F}_{\mathrm{loc}}(\mathcal{M}) \to \mathfrak{A}(\mathcal{M})$ if, for every $H, H' \in \mathrm{Had}(\mathcal{M})$ and $\mathcal{F} \in \mathfrak{F}_{\mathrm{loc}}(\mathcal{M})$

$$(:\mathcal{F}:)_H = \alpha_{H-H'}(:\mathcal{F}:)_{H'}. \tag{2.75}$$

By choosing a fixed Hadamard state $H_0 \in \mathrm{Had}(\mathcal{M})$, we can define a quantisation map which has the physical interpretation of normal ordering 'with respect to' that state. As indicated above, we first define a map $\mathfrak{F}_{\mathrm{loc}}(\mathcal{M}) \to \mathfrak{A}^H(\mathcal{M})$ by

$$\mathcal{F} \mapsto \alpha_{H-H_0}\mathcal{F} =: (\overset{\circ}{\circ}\mathcal{F}\overset{\circ}{\circ}_{H_0})_H. \tag{2.76}$$

This clearly satisfies the criterion (2.75) above, and hence is a valid normal ordering prescription. We may also characterise this prescription as the only consistent choice such that the map $\mathfrak{F}_{\mathrm{loc}}(\mathcal{M}) \to \mathfrak{A}^{H_0}(\mathcal{M})$ is the inclusion of $\mathfrak{F}_{\mathrm{loc}}(\mathcal{M})$ into $\mathfrak{F}_{\mu c}(\mathcal{M})[[\hbar]]$, the underlying vector space of $\mathfrak{A}^{H_0}(\mathcal{M})$.

Similar to our definition of a natural Lagrangian, a *locally covariant ordering prescription* is defined to be a natural transformation from $\mathfrak{F}_{\mathrm{loc}}$ to $\mathfrak{A}$. (Note that we must assume that the target category of each functor is Vec, as normal ordering is linear, but not a homomorphism.) Explicitly, this naturality condition is realised by the equation, for every admissible embedding $\chi : \mathcal{M} \to \mathcal{N}$,

$$:\chi_* \mathcal{F}:_{\mathcal{N}} = \mathfrak{A}\chi\left(:\mathcal{F}:_{\mathcal{M}}\right). \tag{2.77}$$

It is tempting to believe that a covariant prescription across all spacetimes can be found by making a covariant choice of Hadamard state for each spacetime. However, it is now a well-established fact that such a choice cannot be made consistently across all spacetimes. (See the remarks following [HW01, definition 3.2] for a discussion relevant to the scalar field, and [FV12, §6.3] for a more general result.)

The solution is to instead define an ordering prescription which depends upon the Hadamard *parametrix* of the spacetime in question. Before the characterisation via wavefront sets, due to [Rad96] and used in (2.59), Hadamard states were defined by the ability to express them locally (i.e. in some neighbourhood of the thin diagonal $\Delta \subset \mathcal{M}^2$) in what is known as *local Hadamard form*. A precise description of the local Hadamard condition for 4-dimensional spacetimes may be found in [KW91, §3.3].[12] In dimension $2$, a state with 2-point function $W(x, y)$ is said to be locally Hadamard if there exists some neighbourhood $\Delta \subset U \subseteq \mathcal{M}^2$ such that, $\forall\, N \in \mathbb{N}$,

$$W(x; y) := -\frac{1}{4\pi} \lim_{\epsilon \searrow 0} \left( V_N(x, y) \log\left(\frac{\sigma_\epsilon(x; y)}{\lambda^2}\right) + w_N(x; y) \right), \tag{2.78}$$

where $\sigma(x; y)$ is the world function, defined as half the squared geodesic distance between $x$ and $y$, $t$ is some choice of a time function (i.e. level sets of $t$ are Cauchy surfaces), $\sigma_\epsilon$ is defined by

$$\sigma_\epsilon(x; y) := \sigma(x; y) + 2i\epsilon\,(t(x) - t(y)) + \epsilon^2, \tag{2.79}$$

$w_N$ is some $2N + 1$ times continuously differentiable function, and $V_N$ is a smooth function which depends only on the metric of $\mathcal{M}$. We have omitted some subtleties in the definition regarding geodesic completeness (i.e. the true domain of $\sigma$), for which we again refer the readers to the precise definition given in the above reference.

The series of distributions $W_N^{\text{sing}} := W - w_N$, for $N \in \mathbb{N}$, together constitute the *Hadamard parametrix*, which is independent of the choice of state. The parametrix defines a normal ordering prescription, first as a map $\mathfrak{F}_{\text{loc}}(\mathcal{M}) \to \mathfrak{A}^H(\mathcal{M})$

$$(:\mathcal{F}:_{\mathcal{M}})_H = \lim_{N \to \infty} \alpha_{H - H_N^{\text{sing}}} \mathcal{F}, \tag{2.80}$$

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[12] It has recently been discovered by Moretti [Mor21] that the definition given in this reference is in need of slight modification. However, it is believed that none of the established results depending on this definition are invalidated by this.

where $H_N^{\text{sing}} = W_N^{\text{sing}} - \frac{i}{2}E$. On analytic spacetimes, one can prove that the series $W_N^{\text{sing}}$ converges to a distribution, albeit defined only on some neighbourhood of the diagonal $\Delta \subset \mathcal{M}^2$, necessitating the restriction to $\mathfrak{F}_{\text{loc}}$. However, even on non-analytic spacetimes this map is well-defined for any local functional $\mathcal{F}$ because the order $N$ at which we must truncate the series in (2.78) depends only on the order of the functional $\mathcal{F}$. This corresponds to the highest order derivative of a field configuration $\varphi$ which enters into the definition of $\mathcal{F}[\varphi]$, and is guaranteed to be finite [Rej16, §6.2.2]. For instance, if $\mathcal{F}$ has order $n$, then $\alpha_{H-H_N^{\text{sing}}}\mathcal{F} = \alpha_{H-H_n^{\text{sing}}}\mathcal{F}$ for all $N \geq n$, thus the sequence always converges after finitely many steps. From now on we shall suppress both the truncation of the series, as well as the limit in (2.80). Instead we shall write $(:\mathcal{F}:_{\mathcal{M}})_H = \alpha_{H-H^{\text{sing}}}\mathcal{F}$, where one may interpret $H^{\text{sing}}$ as $H_N^{\text{sing}}$ for a sufficiently large $N$.

We can then verify that, for $H, H' \in \text{Had}(\mathcal{M})$

$$(:\mathcal{F}:_{\mathcal{M}})_H = \alpha_{H-H^{\text{sing}}}\mathcal{F} = \alpha_{H-H'} \circ \alpha_{H'-H^{\text{sing}}}\mathcal{F} = \alpha_{H-H'}\left(:\mathcal{F}:_{\mathcal{M}}\right)_{H'}, \tag{2.81}$$

i.e. the family of functionals $\Big((:\mathcal{F}:_{\mathcal{M}})_H\Big)_{H\in\text{Had}(\mathcal{M})}$ satisfies the compatibility criterion (2.64), hence the map $:-:_{\mathcal{M}} : \mathfrak{F}_{\text{loc}}(\mathcal{M}) \to \mathfrak{A}(\mathcal{M})$ is well defined.

Crucially, the Hadamard parametrix is also locally covariant. If $H_{\mathcal{M}/\mathcal{N}}^{\text{sing}}$ are the (symmetrised) Hadamard parametrices for two spacetimes $\mathcal{M}$, $\mathcal{N}$, related by a Loc morphism $\chi : \mathcal{M} \to \mathcal{N}$, then $\chi^* H_{\mathcal{N}}^{\text{sing}} = H_{\mathcal{M}}^{\text{sing}}$. [13] Thus, we can use the fact that $(\chi_*\mathcal{F})^{(n)}[\varphi] = (\chi_*)^{\otimes n}\mathcal{F}^{(n)}[\chi^*\varphi]$, to show

$$\alpha_{H-H_{\mathcal{N}}^{\text{sing}}}(\chi_*\mathcal{F}) = \chi_*\left(\alpha_{\chi^*(H-H_{\mathcal{N}}^{\text{sing}})}\mathcal{F}\right). \tag{2.82}$$

On the left hand side, we have simply $(:\chi_*\mathcal{F}:_{\mathcal{N}})_H$, whereas on the right hand side, once we note that $\alpha_{\chi^*(H-H_{\mathcal{M}}^{\text{sing}})}\mathcal{F} = \alpha_{\chi^*H-H_{\mathcal{M}}^{\text{sing}}}\mathcal{F} = (:\mathcal{F}:_{\mathcal{M}})_{\chi^*H}$, we see that this is $(\mathfrak{A}\chi{:}\mathcal{F}{:}_{\mathcal{M}})_H$ as required.

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[13] This is a direct consequence of the fact that $\chi^* : \text{Had}(\mathcal{N}) \to \text{Had}(\mathcal{M})$.

# Part I

# Conformal Field Theory in pAQFT

### — 3 —

# The Massless Scalar Field

Now that we have constructed both a classical and quantum algebra of observables, and introduced several ordering maps between them, we may study their finer details in an explicit example. As our ultimate goal is to understand conformal field theory from the perspective of pAQFT, the massless scalar field is the obvious place to begin. Moreover, owing to its flat geometry and compact Cauchy surfaces, the Einstein cylinder $\mathcal{E}$, defined as the image of 2D Minkowski space $\mathbb{M}^2$ under the identification $(t, x) \sim (t, x + 2\pi)$, provides a natural and convenient setting in which to explore the chiral aspects of the massless scalar field within the pAQFT framework.

In this section, we shall see how the quantum algebra of observables for the massless scalar field contains a pair of Heisenberg algebras and a pair of Virasoro algebras, one each for the left and right null-derivatives of the field. In the construction of the Virasoro algebra, we shall also see that the principle of local covariance outlined in Section 2.9 is necessary to recover the 'radially-ordered' form of the Virasoro algebra. The argument involved in this re-ordering constitutes a mathematically rigorous form of the known trick of identifying $1 + 2 + 3 + \cdots = \zeta(-1)$.

## 3.1 MINKOWSKI SPACE

We begin by finding the causal propagator for the massless scalar field in Minkowski space. From this we shall later obtain the propagator for the cylinder, and hence the Poisson algebra $\mathfrak{P}(\mathcal{E})$. Moreover we shall begin to see how the classical Poisson algebra of the massless scalar field naturally contains two chiral subalgebras, which we explore further in part II.

The equation of motion for the massless scalar field on Minkowski space is simply

$$\left(\partial_t^2 - \partial_x^2\right) \varphi = 0. \tag{3.1}$$

This is easiest to solve if we adopt null coordinates $u = t-x, v = t+x$. The fundamental solutions $E^{R/A}$ to (3.1) must then satisfy

$$4\frac{\partial}{\partial u}\frac{\partial}{\partial v}E^{R/A}(u,v;u',v') = 2\delta(u-u')\delta(v-v'). \tag{3.2}$$

By inspection one can then deduce that the distributions

$$E^{R/A}(u,v;u',v') = -\frac{1}{2}\theta(\pm(u-u'))\theta(\pm(v-v')) \tag{3.3}$$

both satisfy (3.2) and have the desired supports. Taking their difference we find the Pauli-Jordan function to be

$$E(u,v;u',v') = -\frac{1}{2}\left[\theta(u-u')\theta(v-v') - \theta(u'-u)\theta(v'-v)\right]. \tag{3.4}$$

We can rewrite this propagator in the form

$$E(u,v;u',v') = -\frac{1}{4}\left[\operatorname{sgn}(u-u') + \operatorname{sgn}(v-v')\right], \tag{3.5}$$

where $\operatorname{sgn}(x) = \theta(x) - \theta(-x)$. In other words, we can decouple the $u$-dependent terms from the $v$-dependent, defining the summands

$$E = E^{\ell} + E^{r}, \tag{3.6}$$

such that $E^{\ell}$ (resp. $E^{r}$) does not depend on $v$ (resp. $u$).

This split is significant for functionals which depend on the field configuration $\varphi$ only through its left/right null derivative. If we indicate the action of the differential operator $\partial_u$ on a functional $\mathcal{F}$ by $(\partial_u^*\mathcal{F})[\varphi] := \mathcal{F}[\partial_u\varphi]$, then the functional derivative of $\partial_u^*\mathcal{F}$ is given by

$$(\partial_u^*\mathcal{F})^{(1)}[\varphi] = -\partial_u\mathcal{F}^{(1)}[\partial_u\varphi], \tag{3.7}$$

where the first $\partial_u$ on the right-hand side is the derivative in the sense of distributions [Hör15, §3.1]. Consequently, the Peierls bracket of two such functionals is

$$\{\partial_u^*\mathcal{F}, \partial_u^*\mathcal{G}\}[\varphi] = \left\langle(\partial_u\otimes\partial_u)E, \mathcal{F}^{(1)}[\partial_u\varphi]\otimes\mathcal{G}^{(1)}[\partial_u\varphi]\right\rangle. \tag{3.8}$$

This equality motivates the construction of a new Poisson algebra, outlined in the following proposition:

**Proposition 3.1.1.** *The space $\mathfrak{F}_{\mu c}(\mathbb{M}_2)$, equipped with the pointwise product, and the bracket*

$$\{\mathcal{F},\mathcal{G}\}_{\ell}[\varphi] := \left\langle(\partial_u\otimes\partial_u)E, \mathcal{F}^{(1)}[\varphi]\otimes\mathcal{G}^{(1)}[\varphi]\right\rangle \tag{3.9}$$

*is a Poisson algebra, which we denote $\mathfrak{P}_{\ell}(\mathbb{M}_2)$. Furthermore, the map $\partial_u^* : \mathfrak{F}_{\mu c}(\mathbb{M}_2) \to \mathfrak{F}_{\mu c}(\mathbb{M}_2)$ yields a Poisson algebra homomorphism $\mathfrak{P}_{\ell}(\mathbb{M}_2) \to \mathfrak{P}(\mathbb{M}_2)$.*

*Proof.* Firstly, note that (3.9) is well defined for $\mathcal{F}, \mathcal{G} \in \mathfrak{F}_{\mu c}(\mathcal{M})$ because $\mathrm{WF}((\partial_u \otimes \partial_u)E) \subseteq \mathrm{WF}(E)$. Similarly, we also have that that all the estimates of $\mathrm{WF}\Big( \{\mathcal{F}, \mathcal{G}\}^{(n)} \Big)$ given in the proof of Proposition B.1 also hold for $\mathrm{WF}\Big( \{\mathcal{F}, \mathcal{G}\}^{(n)}_\ell \Big)$. Thus, $\{\mathcal{F}, \mathcal{G}\}_\ell$ is microcausal for the same reason that $\{\mathcal{F}, \mathcal{G}\}$ is.

Next, we address the relation of this new bracket to the Peierls bracket. By replacing $\varphi$ with $\partial_u \varphi$ in (3.9) and comparing with (3.8), it then follows that

$$(\partial_u^* \{\mathcal{F}, \mathcal{G}\}_\ell)\, [\varphi] = \{\partial_u^* \mathcal{F}, \partial_u^* \mathcal{G}\}\, [\varphi].$$

Note the right-hand side of this equation is well defined as

$$\mathrm{WF}((\partial_u^* \mathcal{F})^{(n)}[\varphi]) = \mathrm{WF}((-1)^n {\partial_u}^{\otimes n} \mathcal{F}^{(n)}[\partial_u \varphi]) \subseteq \mathrm{WF}(\mathcal{F}^{(n)}[\partial_u \varphi]), \tag{3.10}$$

hence $\mathfrak{F}_{\mu c}(\mathcal{M})$ is preserved by $\partial_u^*$.

Finally, we must show that $\{\cdot, \cdot\}_\ell$ satisfies the Jacobi identity. Let $\mathcal{F}, \mathcal{G}$, and $\mathcal{H}$ all be microcausal functionals. Consider

$$\partial_u^* \left( \left\{ \mathcal{F}, \left\{ \mathcal{G}, \mathcal{H} \right\}_\ell \right\}_\ell + \cdots \right) = \left\{ \partial_u^* \mathcal{F}, \left\{ \partial_u^* \mathcal{G}, \partial_u^* \mathcal{H} \right\} \right\} + \cdots ,$$

where $\cdots$ includes both remaining even permutations of $\mathcal{F}, \mathcal{G}$, and $\mathcal{H}$. The right-hand side of this vanishes as the Peierls bracket satisfies the Jacobi identity. Because $\partial_u$ is surjective on $\mathfrak{E}(\mathcal{M})$, $\partial_u^*$ is injective on $\mathfrak{F}(\mathcal{M})$. From this we may conclude that $\{\mathcal{F}, \{\mathcal{G}, \mathcal{H}\}_\ell\}_\ell + \cdots$ also vanishes, hence $\{\cdot, \cdot\}$ satisfies the Jacobi identity. □

Note that, $(\partial_u \otimes \partial_u)E^r = 0$, hence the integral kernel of the differentiated propagator is

$$\partial_u \partial_{u'} E(u, v; u', v') = \partial_u \partial_{u'} E^\ell(u, v; u', v') = \tfrac{1}{2} \delta'(u - u'). \tag{3.11}$$

This form of the commutator can be seen as an example of the *mutual locality* of chiral fields, [Kac98, Definition 2.3], a concept central to many theorems in the VOA framework We shall henceforth refer to $\{\cdot, \cdot\}_\ell$ as the *chiral* bracket, and the analogously defined $\{\cdot, \cdot\}_r$ as the *anti-chiral* bracket.

In part II, we shall examine these chiral algebraic structures in detail, in particular, we shall demonstrate how they can be placed on a one-dimensional space, close to the notion that chiral fields 'live' on a single light-ray.

## 3.2 THE HEISENBERG ALGEBRA ON THE CYLINDER

We shall now find the advanced and retarded propagators for the Einstein cylinder $\mathcal{E}$. If $(u, v)$ denotes the null coordinates of a point in $\mathbb{M}_2$, then we define an equivalence relation on $\mathbb{M}_2$ by $(u, v) \sim (u + 2\pi, v - 2\pi)$. The Einstein cylinder is then defined as the quotient space $\mathcal{E} = \mathbb{M}_2/\sim$, with the unique metric such that the covering map $\pi : \mathbb{M}_2 \to \mathcal{E}$ is a local isometry. We will write points in $\mathcal{E}$ as equivalence classes $[u, v] \subset \mathbb{M}_2$, where $[u, v] = [u + 2\pi, v - 2\pi]$.

The causal propagator for the cylinder may be obtained from the advanced and retarded propagators of Minkowski spacetime using the method of images. Firstly, note there is an isomorphism between $\mathfrak{E}(\mathbb{M}_2)^{\mathbb{Z}} = \{f \in \mathfrak{E}(\mathbb{M}_2) \mid f \circ T_n \equiv f, \forall\, n \in \mathbb{Z}\}$ and $\mathfrak{E}(\mathcal{E})$. In the direction $\mathfrak{E}(\mathcal{E}) \to \mathfrak{E}(\mathbb{M}_2)$, this map is simply the corestriction of $\pi^*$ to the space of $\mathbb{Z}$ invariants. If we denote the inverse of this isomorphism by $\pi_*$, then we claim the retarded and advanced propagators on the cylinder are given by

$$E_{\mathcal{E}}^{R/A} = \pi_* E^{R/A} \pi^*. \tag{3.12}$$

For this map to be well defined, amongst other details, we must show that the domain of $E^{R/A}$ can be extended to the image $\pi^*\left(\mathfrak{D}(\mathcal{E})\right)$, and that the output of $E^{R/A}\pi^*$ contains only $\mathbb{Z}$ invariants. Proof of which can be found in appendix A.

That these maps are then the desired propagators follows from the relationship between the equations of motion on the cylinder and Minkowski. Let $U \subseteq \mathbb{M}_2$ be a sub-spacetime of $\mathbb{M}_2$ and let $\iota_U : U \hookrightarrow \mathbb{M}_2$ be its inclusion into $\mathbb{M}_2$. If $U$ is small enough that $\pi \circ \iota_U : U \to \mathcal{E}$ is an embedding, then we can show from (2.72) that

$$(\pi \circ \iota_U)^* P_{\mathcal{E}} = P_U (\pi \circ \iota_U)^*. \tag{3.13}$$

Furthermore, $\iota_U$ is itself an isometric embedding, hence

$$\iota_U^* P_{\mathbb{M}_2} = P_U \iota_U^*. \tag{3.14}$$

Combining these equations, we find

$$\iota_U^* \pi^* P_{\mathcal{E}} = \iota_U^* P_{\mathbb{M}_2} \pi^*. \tag{3.15}$$

One can then show that $\mathbb{M}_2$ is covered by open sets $U$ for which (3.15) holds, and hence that $\pi^* P_{\mathcal{E}} = P_{\mathbb{M}_2} \pi^*$. By acting on the left-hand side of (3.12) with $\pi^* P_{\mathcal{E}}$ and the right-hand side with $P_{\mathbb{M}_2} \pi^*$, we are then able to see why these maps are fundamental solutions to $P_{\mathcal{E}}$.

Throughout this section we shall use the chart $(\pi|_U)^{-1} : \mathcal{E} \to U$, where $U = (0, 2\pi) \times \mathbb{R} \subset \mathbb{R}^2$, in order to define integrals on $\mathcal{E}$. As $\pi(U)$ covers $\mathcal{E}$ up to a set of measure zero, we have that, $\forall\, f \in \mathfrak{D}(\mathcal{E})$

$$\int_{\mathcal{E}} f \mathrm{dV}_{\mathcal{E}} = \int_U \pi^* f \mathrm{dV}_{\mathbb{M}^2}.$$

Using the method of images (theorem A.2), which in this case states that $E_{\mathbb{M}^2}\pi^* : \mathfrak{D}(\mathcal{E}) \to \mathfrak{E}(\mathbb{M}^2)$ is well-defined and coincides with $\pi^* E_{\mathcal{E}}$, we can then write, for $f, g \in \mathfrak{D}(\mathcal{E})$

$$\begin{aligned}
\int_{\mathcal{E}} f(E_{\mathcal{E}} g)\mathrm{dV}_{\mathcal{E}} &= \int_U \pi^* f(\pi^* E_{\mathcal{E}} g)\mathrm{dV}_{\mathbb{M}^2} \\
&= \int_U \pi^* f(E_{\mathbb{M}^2}\pi^* g)\mathrm{dV}_{\mathbb{M}^2} \\
&= \int_{U\times\mathbb{M}^2} \pi^* f(u,v) E_{\mathbb{M}^2}(u,v;u',v')\pi^* g(u',v')\mathrm{dV}^2_{\mathbb{M}^2} \\
&= \sum_{k\in\mathbb{Z}} \int_{U\times U} \Big[\pi^* f(u,v) E_{\mathbb{M}^2}(u,v;u'-2\pi k, v'+2\pi k) \\
&\qquad\qquad \pi^* g(u'-2\pi k, v'+2\pi k)\Big]\mathrm{dV}^2_{\mathbb{M}^2} \\
&= \int_{U\times U} \pi^* f(u,v) \left(\sum_{k\in\mathbb{Z}} E_{\mathbb{M}^2}(u-2\pi k, v+2\pi k; u',v')\right)\pi^* g(u',v')\mathrm{dV}^2_{\mathbb{M}^2}
\end{aligned} \tag{3.16}$$

Note that the infinite series converges because, for $(u,v) \in U, (u',v') \in U \cap \operatorname{supp}\pi^* g$, there are only finitely many translates $(u-2\pi k, v+2\pi k)$ which are causally separated from $(u',v')$, beyond which we have $E(u-2\pi k, v+2\pi k; u', v') = 0$.

We can interpret the final line as an integral kernel for $E_{\mathcal{E}}$. Explicitly, this is

$$\begin{aligned}
E_{\mathcal{E}}(u,v;u',v') &= \sum_{k\in\mathbb{Z}} E_{\mathbb{M}^2}(u-2\pi k, v+2\pi k; u', v'), \\
&= -\frac{1}{2}\left(\left\lfloor \frac{u-u'}{2\pi}\right\rfloor + \left\lfloor \frac{v-v'}{2\pi}\right\rfloor + 1\right).
\end{aligned} \tag{3.17}$$

Once again, we see the characteristic splitting of the $u$-dependent and $v$-dependent terms of $E_{\mathcal{E}}$, which we write $E_{\mathcal{E}} = E^{\ell}_{\mathcal{E}} + E^{r}_{\mathcal{E}}$, just as before.

Just as with Proposition 3.1.1, we can define a chiral bracket $\{\cdot,\cdot\}_\ell$ on $\mathfrak{F}_{\mu c}(\mathcal{E})$ using $(\partial_u \otimes \partial_u)\, E_{\mathcal{E}}$ instead of $E_{\mathcal{E}}$, yielding the chiral Poisson algebra $\mathfrak{P}_\ell(\mathcal{E})$. The proof that $\mathfrak{P}_\ell(\mathcal{E})$ is a Poisson algebra and that $\partial_u^* : \mathfrak{F}_{\mu c}(\mathcal{E}) \to \mathfrak{F}_{\mu c}(\mathcal{E})$ is a Poisson algebra homomorphism carries over essentially unchanged from $\mathbb{M}_2$. Recall that $(\partial_u \otimes \partial_u)E(u,v;u',v') = 0$ whenever $u \neq u'$, hence if we apply the same operator to (3.17), only the $k = 0$ term survives:

$$(\partial_u \otimes \partial_u)\, E_{\mathcal{E}}(u,v;u',v') = (\partial_u \otimes \partial_u)\, E^{\ell}(u,v;u',v') = \frac{1}{2}\delta'(u-u'). \tag{3.18}$$

We shall perform our next set of calculations using $\{\cdot,\cdot\}_\ell$. In an effort to avoid confusion, when we are working in $\mathfrak{P}_\ell(\mathcal{E})$, we shall denote the field configuration input to the functional by $\psi$. We think of $\psi$ as $\partial_u\varphi$ which is realised when we apply the algebra homomorphism $(\partial_u^*\mathcal{F})[\varphi] = \mathcal{F}[\partial_u\varphi] = \mathcal{F}[\psi]$.

We first define the family of functionals $\{A_n\}_{n\in\mathbb{Z}} \subset \mathfrak{F}(\mathcal{E})$ by

$$A_n[\psi] := \frac{1}{\sqrt{\pi}}\int_{u=0}^{2\pi} e^{inu}\psi(u,-u)\,\mathrm{d}u. \tag{3.19}$$

Their derivatives are given by

$$\left\langle A_n^{(1)}[\psi], h\right\rangle = \frac{1}{\sqrt{\pi}}\int_{u=0}^{2\pi} e^{inu}h(u,-u)\,\mathrm{d}u, \tag{3.20}$$

for $h \in \mathfrak{D}(\mathcal{E})$.

For $\psi \in \mathfrak{E}(\mathcal{E})$, $A_n[\psi]$ is simply the $n^{\text{th}}$ Fourier mode of $\psi$ restricted to the $t=0$ Cauchy surface $\Sigma_0$ if we wind around the surface *clockwise*. These functionals are not local (despite being linear) nor even microcausal because, one can show [BDH14, §4] the wavefront set of $A_n^{(1)}[\psi]$ is the co-normal bundle to $\Sigma_0$ and hence contains timelike covectors in violation of the microcausal spectral condition.

A direct computation of the chiral bracket yields

$$\begin{aligned}\left\{A_n, A_m\right\}_\ell[\psi] &= \frac{1}{\pi}\int_{u=0}^{2\pi}\int_{u'=0}^{2\pi} e^{i(nu+mu')}(\partial_u\partial_{u'}E_{\mathcal{E}}^\ell)(u,-u;u',-u')\,\mathrm{d}u\mathrm{d}u' \\ &= \frac{1}{2\pi}\int_{u=0}^{2\pi}\int_{u'=0}^{2\pi} e^{i(nu+mu')}\delta'(u-u')\,\mathrm{d}u'\mathrm{d}u' \\ &= -in\delta_{n+m,0},\end{aligned} \tag{3.21}$$

hence

$$\left\{A_n, A_m\right\}_\ell = -in\delta_{n+m,0}, \tag{3.22}$$

where we suppress the constant functional for convenience.

This demonstrates that the Lie algebra generated by the $A_n$ with the Lie bracket $\{\cdot,\cdot\}_\ell$ is isomorphic to the Heisenberg algebra (also known as the *oscillator algebra* [Kac98, §2.5]): this is the infinite dimensional Lie algebra over $\mathbb{C}$ with generators $\{a_n\}_{n\in\mathbb{Z}}$ , $K$ subject to the relations

$$[a_n, a_m] = n\delta_{n+m,0}K, \quad [a_n, K] = 0.$$

Moreover, as $\partial_u^*$ is a Poisson algebra homomorphism, we see that the algebra generated by $\mathcal{A}_n := \partial_u^* A_n$ with the Peierls bracket is also isomorphic to the Heisenberg algebra.

Quantising this family of functionals is relatively simple. Let $H \in \mathrm{Had}(\mathcal{E})$ be some Hadamard distribution. As the functionals $\mathcal{A}_n$ are linear, the definition of the $\star_H$ product implies the Dirac quantisation rule is valid:

$$\left[\mathcal{A}_n, \mathcal{A}_m\right]_{\star_H} = i\hbar\left\{\mathcal{A}_n, \mathcal{A}_m\right\} = \hbar n \delta_{n+m,0}. \tag{3.23}$$

Furthermore, $\alpha_{H'-H}$ acts by identity on linear functionals, hence this result is independent of our choice of a Hadamard state $H$.

Of course, there is nothing particularly special about the choice of $\Sigma_0$ as the Cauchy surface. As we shall see in Part I, the functionals $\mathcal{A}_n$ are just one possible realisation of a subalgebra of *chiral* observables of $\mathfrak{A}(\mathcal{E})$, with every other Cauchy surface $\Sigma \subset \mathcal{E}$ giving rise to an isomorphic subalgebra.

### 3.3 THE VIRASORO ALGEBRA

As the Virasoro algebra arises from quadratic functionals, the ordering ambiguities we could previously disregard become relevant, and we cannot so easily carry computations from Minkowski space over to the cylinder. To start, the classical functionals are defined analogously to the $A_n$ functionals. Again, we begin by defining a family $\{L_n\}_{n\in\mathbb{Z}} \subset \mathfrak{F}(\mathcal{E})$, by

$$L_n[\psi] := \int_{u=0}^{2\pi} e^{inu}\psi^2(u,-u)\,\mathrm{d}u.$$

Physically, the functionals $L_n$ correspond to a mode decomposition of the $uu$ component of the stress-energy tensor, restricted to the $t = 0$ Cauchy surface. As before, we shall compute the chiral bracket of $L_n$ with $L_m$ in order to obtain the Peierls bracket for the functionals $\mathcal{L}_n := \partial_u^* L_n$ (not to be confused with the natural Lagrangian).

For future reference, the functional derivatives of $L_n$ are

$$\left\langle L_n^{(1)}[\psi], g\right\rangle = 2\int_{u=0}^{2\pi} e^{inu}\psi(u,-u)g(u,-u)\,\mathrm{d}u, \tag{3.24a}$$

$$\left\langle L_n^{(2)}[\psi], g\otimes h\right\rangle = 2\int_{u=0}^{2\pi} e^{inu}g(u,-u)h(u,-u)\,\mathrm{d}u. \tag{3.24b}$$

Here again, the wavefront set of $L_n^{(1)}[\psi]$ is contained within the conormal bundle of $\Sigma_0$ and hence $L_n$ is not microcausal. Moreover, we see that, like $A_n$, these functionals are additive, which means that the support of $L_n^{(2)}$, and hence that of $\mathcal{L}_n^{(2)}$, is contained within the thin diagonal $\Delta_2 \subset \mathcal{E}^2$. This will be vital when we later apply the locally covariant Wick ordering prescription outlined in Section 2.9 to these functionals.

The chiral bracket of $L_n$ with $L_m$ is given by

$$\begin{aligned}
\{L_n, L_m\}_\ell[\psi] &= 2\int_{u=0}^{2\pi}\int_{u'=0}^{2\pi} \delta'(u-u')e^{inu}\psi(u,-u)\cdot e^{imu'}\psi(u',-u')\,\mathrm{d}u\,\mathrm{d}u' \\
&= -2\int_{u=0}^{2\pi}\Big[in\psi(u,-u)+(\partial_u\psi)(u,-u)\Big]\psi(u,-u)e^{i(n+m)u}\,\mathrm{d}u, \\
&= -i(n-m)\int_{u=0}^{2\pi} e^{i(n+m)u}\psi^2(u,-u)\,\mathrm{d}u \\
&= -i(n-m)L_{n+m}[\psi], \qquad (3.25)
\end{aligned}$$

where the move from the second to the third line can be made by exploiting the skew-symmetry of the equation under the interchange of $n$ with $m$. Hence, we can already see that the $L_n$ under the chiral bracket generate a copy of the Witt algebra.

Next, we shall quantise the $\mathcal{L}_n$ observables. Using (3.25), we can immediately note that the $\mathcal{O}(\hbar)$ term of $[{:}\mathcal{L}_n{:}, {:}\mathcal{L}_m{:}]$ must be $\hbar(n-m){:}\mathcal{L}_{n+m}{:}$, regardless of the quantisation map used. In order to determine the $\mathcal{O}(\hbar^2)$ term though, we must decide on a particular choice of prescription.

As explained in Section 2.8, it is inconvenient to work directly with $\mathfrak{A}(\mathcal{E})$. Instead, we perform our computations in $\mathfrak{A}^H(\mathcal{E})$ for some suitable choice of Hadamard distribution $H$. The simplest choice is to take $H = W_{\mathcal{E}} - \frac{i}{2}E_{\mathcal{E}}$, where $W_{\mathcal{E}}$ is defined by the integral kernel

$$W_{\mathcal{E}}(u,v;u',v') = \frac{1}{4\pi}\sum_{k\in\mathbb{Z}^*}\frac{1}{k}\left(e^{-ik(u-u')}+e^{-ik(v-v')}\right). \qquad (3.26)$$

Unlike the analogous (i.e. time-translation invariant) distribution for the massive scalar field, $W_{\mathcal{E}}$ does not define a *vacuum* state, owing to the presence of zero mode solutions to the massless Klein-Gordon equation. However, this is no issue in the algebraic approach to QFT, as the construction of our algebra of observables is independent of any choice of ground state and, hence, of any way in which we may choose to handle the problem of zero modes.

Moreover, we are concerned with the $\star$ products of functionals which depend on the field configuration $\varphi$ only through one of its null derivatives. In effect, this means we only depend on $W_{\mathcal{E}}$ to define the 2-point function for the derivative field

$$(\partial_u\otimes\partial_u)W_{\mathcal{E}}(\mathbf{x};\mathbf{y}) = \Big\langle(\partial_u\varphi)(\mathbf{x})(\partial_u\varphi)(\mathbf{y})\Big\rangle_\omega. \qquad (3.27)$$

Taking this derivative annihilates any zero-modes, thus there is no ambiguity in defining the integral kernel of $(\partial_u\otimes\partial_u)W_{\mathcal{E}}$.

If we consider the $\star_H$ product of two functionals of the form $\partial_u^*\mathcal{F}$, we find

$$((\partial_u^*\mathcal{F}) \star_H (\partial_u^*\mathcal{G}))\,[\varphi] = \sum_{n=0}^{\infty} \frac{\hbar^n}{n!} \left\langle [(\partial_u \otimes \partial_u)\, W_{\mathcal{E}}]^{\otimes n}, \mathcal{F}^{(n)}[\partial_u\varphi] \otimes \mathcal{G}^{(n)}[\partial_u\varphi] \right\rangle. \tag{3.28}$$

Analogously to Proposition 3.1.1, we can hence define a chiral subalgebra of $\star_H$ via the following:

**Proposition 3.3.1.** *The space $\mathfrak{F}_{\mu c}(\mathcal{E})[[\hbar]]$, equipped with the associative product $\star_{H,\ell}$ defined by*

$$(\mathcal{F} \star_{H,\ell} \mathcal{G})[\varphi] := \sum_{n\in\mathbb{N}} \frac{\hbar^n}{n!} \left\langle [(\partial_u \otimes \partial_u) W_{\mathcal{E}}]^{\otimes n}, \mathcal{F}^{(n)}[\varphi] \otimes \mathcal{G}^{(n)}[\varphi] \right\rangle, \tag{3.29}$$

*is a $*$-algebra, which we denote by $\mathfrak{A}_\ell^H(\mathcal{E})$. Moreover, the linear extension of $\partial_u^*$ – defined in Proposition 3.1.1 – to $\mathfrak{F}_{\mu c}(\mathcal{E})[[\hbar]]$ yields a $*$-algebra homomorphism $\mathfrak{A}_\ell^H(\mathcal{E}) \to \mathfrak{A}^H(\mathcal{E})$.*

*Proof.* Just as in the classical case, because $\mathrm{WF}((\partial_u \otimes \partial_u)\, W) \subseteq \mathrm{WF}(W)$, the closure of $\mathfrak{F}_{\mu c}(\mathcal{E})[[\hbar]]$ under $\star_{H,\ell}$ is proved in exactly the same way as for $\star_H$, as spelled out in proposition B.2. That $\partial_u^*$ intertwines $\star_{H,\ell}$ with $\star_H$ is verified by (3.28). Associativity follows from injectivity of $\partial_u^*$ and the associativity of $\star_H$, because

$$\begin{aligned} \partial_u^*(F_1 \star_{H,\ell} (F_2 \star_{H,\ell} F_3)) &= \partial_u^* F_1 \star_H \partial_u^*(F_2 \star_{H,\ell} F_3) \\ &= (\partial_u^* F_1) \star_H \partial_u^* F_2 \star_H \partial_u^* F_3 = \partial_u^*((F_1 \star_{H,\ell} F_2) \star_{H,\ell} F_3) \end{aligned}$$

□

We may now compute the product $L_n \star_{H_{\mathcal{E}},\ell} L_m$. In the abstract algebra, this amounts to computing $\mathopen{:}\mathcal{L}_n \mathclose{:}_{H_{\mathcal{E}}} \star \mathopen{:} \mathcal{L}_m \mathclose{:}_{H_{\mathcal{E}}}$ . Later, we shall compare this to the product of the *covariantly* ordered $\mathcal{L}_n$.

As the $L_n$ functionals are quadratic, the power series for their star product truncates at $\mathcal{O}(\hbar^2)$. Thus, it may be written in full as

$$\begin{aligned} L_n \star_{H_{\mathcal{E}},\ell} L_m = L_n \cdot L_m &+ \hbar \left\langle [(\partial_u \otimes \partial_u)\, W_{\mathcal{E}}], L_n^{(1)}[\psi] \otimes L_m^{(1)}[\psi] \right\rangle \\ &+ \frac{\hbar^2}{2} \left\langle [(\partial_u \otimes \partial_u)\, W_{\mathcal{E}}]^{\otimes 2}, L_n^{(2)}[\psi] \otimes L_m^{(2)}[\psi] \right\rangle. \end{aligned} \tag{3.30}$$

First, let us consider the $\mathcal{O}(\hbar)$ term

$$\begin{aligned} &\left\langle [(\partial_u \otimes \partial_u)\, W_{\mathcal{E}}], L_n^{(1)}[\psi] \otimes L_m^{(1)}[\psi] \right\rangle = \\ &\qquad \sum_{k\in\mathbb{N}} \frac{1}{\pi} \int_{u=0}^{2\pi} \int_{u'=0}^{2\pi} k e^{-ik(u-u')} \cdot e^{inu} \psi(u,-u) \cdot e^{imu'} \psi(u',-u')\, \mathrm{d}u \mathrm{d}u'. \end{aligned} \tag{3.31}$$

We can simplify this slightly by reintroducing the $A_n$ functionals. Upon doing so, we find

$$\left\langle \left[ (\partial_u \otimes \partial_u) W \right], L_n^{(1)}[\psi] \otimes L_m^{(1)}[\psi] \right\rangle = \sum_{k=1}^{\infty} k A_{n-k}[\psi] A_{m+k}[\psi]. \tag{3.32}$$

(Note that for any function $\psi$ the above series is absolutely convergent as the smoothness of $\psi$ guarantees $|A_n[\psi]|$ decays rapidly in $n$.)

For the commutator, we need only the anti-symmetric part of (3.32), which is markedly simpler. For now, however, we proceed to compute the $\mathcal{O}(\hbar^2)$ term. To do this, we need the following form of the squared propagator:

$$\left\langle \left[ (\partial_u \otimes \partial_u) W_{\mathcal{E}} \right]^2, f \right\rangle = \frac{1}{16\pi^2} \sum_{k=0}^{\infty} \sum_{l=0}^{k} l(k-l) \int_{\mathcal{E}^2} e^{-ik(u-u')} f(u,v,u',v') \, \mathrm{d}V^2. \tag{3.33}$$

This can be obtained naïvely by just squaring (3.26) and applying the Cauchy product formula. For a proof that this indeed converges to the correct distribution, see Appendix C. We then find

$$\begin{aligned} &\frac{1}{2}\left\langle \left[ (\partial_u \otimes \partial_u) W_{\mathcal{E}} \right]^{\otimes 2}, L_n^{(2)}[\psi] \otimes L_m^{(2)}[\psi] \right\rangle \\ &\qquad = \frac{1}{8\pi^2} \sum_{k \in \mathbb{N}} \sum_{l=0}^{k} l(k-l) \int_{u=0}^{2\pi} \int_{u'=0}^{2\pi} e^{-ik(u-u')} e^{inu} e^{imu'} \, \mathrm{d}u \mathrm{d}u', \\ &\qquad = \frac{1}{2} \sum_{k \in \mathbb{N}} \sum_{l=0}^{k} l(k-l) \delta_{n-k,0} \delta_{m+k,0}, \\ &\qquad = \frac{n(n^2-1)}{12} \theta(n) \delta_{n+m,0}. \end{aligned} \tag{3.34}$$

Hence, altogether we have

$$(L_n \star_{H_{\mathcal{E}},\ell} L_m) = L_n \cdot L_m + \hbar \sum_{k=1}^{\infty} k A_{n-k} \cdot A_{m+k} + \frac{\hbar^2}{12} n^2(n-1) \theta(n) \delta_{n+m,0}. \tag{3.35}$$

Next, we compute the commutator $[L_n, L_m]_{\star_{H_{\mathcal{E}},\ell}}$ Taking the anti-symmetric part of the $\mathcal{O}(\hbar^2)$ term is straightforward: simply drop the $\theta(n)$. For (3.32), note that we can write

$$\sum_{k=1} k A_{n-k} A_{m+k} = \frac{1}{2} \left( \sum_{k \in \mathbb{Z}} k A_{n-k} A_{m+k} + \sum_{k \in \mathbb{Z}} |k| A_{n-k} A_{m+k} \right). \tag{3.36}$$

The first series is anti-symmetric under an interchange of $n$ and $m$, whereas the latter is symmetric and can thus be disregarded. Next, we take two copies of the anti-symmetric series, for the first copy we make the change of variables $k \mapsto (n-k)$, and for the second we choose $k \mapsto (k-m)$. Recombining these two copies we find

$$\sum_{k\in\mathbb{Z}} kA_{n-k}A_{m+k} = \frac{1}{2}(n-m)\sum_{k\in\mathbb{Z}} A_k A_{n+m-k}. \tag{3.37}$$

By the second convolution theorem, this final series converges (up to a constant factor) to the $(n+m)^{\text{th}}$ Fourier mode of $\psi^2$. Thus, (3.37) is equal to $(n-m)L_{n+m}$, agreeing with our earlier calculation using the chiral bracket $\{\cdot,\cdot\}_\ell$. Combining this with the $\mathcal{O}(\hbar^2)$ term (3.34), we arrive at the Virasoro relations

$$[L_n, L_m]_{\star_{H_\mathcal{E},\ell}} = \hbar(n-m)L_{n+m} + \frac{\hbar^2}{12}n(n^2-1)\delta_{n+m,0}. \tag{3.38}$$

Using the $*$-algebra homomorphism $\partial_u^*$ from Proposition 3.3.1, we can then conclude that

$$[\mathcal{L}_n, \mathcal{L}_m]_{\star_{H_\mathcal{E}}} = \hbar(n-m)\mathcal{L}_{n+m} + \frac{\hbar^2}{12}n(n^2-1)\delta_{n+m,0}. \tag{3.39}$$

Finally, applying $\alpha_{H-H_{\text{cyl}}}$ and using the identity (2.63) we obtain the commutation relation

$$[{}^{\circ}_{\circ}\mathcal{L}_n{}^{\circ}_{\circ}{}_{H_\mathcal{E}}, {}^{\circ}_{\circ}\mathcal{L}_m{}^{\circ}_{\circ}{}_{H_\mathcal{E}}] = \hbar(n-m)\ {}^{\circ}_{\circ}\ \mathcal{L}_{n+m}\ {}^{\circ}_{\circ}{}_{H_\mathcal{E}} + \frac{\hbar^2}{12}n(n^2-1)\delta_{n+m,0} \tag{3.40}$$

in $\mathfrak{A}(\mathcal{E})$, recalling that $\left({}^{\circ}_{\circ}\ \mathcal{L}_n\ {}^{\circ}_{\circ}{}_{H_\mathcal{E}}\right)_H = \alpha_{H-H_\mathcal{E}}\mathcal{L}_n$.

It is curious that at this stage we have commutators recognisable as what one might call the 'planar' Virasoro relations (for example [Kac98, (2.6.6)]) for a central charge $c=1$, despite the fact that all the functionals in question belong on the cylinder. We will now compute the correction to these relations which occurs when adopting the locally covariant Wick ordering prescription. In doing so, we shall see the result is the 'radially ordered' Virasoro relations.

Recall from section 2.9 that heuristically, locally covariant Wick ordering means normal ordering with respect to the Hadamard parametrix. In the case of the Minkowski cylinder, the Hadamard parametrix (2.78) is particularly simple. Locally the cylinder is isometric to Minkowski space, hence the parametrix of the cylinder coincides with that of Minkowski. For an arbitrary choice of length scale $\lambda$, the singular part of a Hadamard distribution for the *undifferentiated* field $\varphi$ is [BR18, (3.18)]

$$H_{\text{sing}}(u,v;u',v') = \lim_{\epsilon\searrow 0} -\frac{1}{4\pi}\log\left(\frac{|(u-u')(v-v')|}{\lambda^2}\right). \tag{3.41}$$

Here it is clear that the parametrix exists only locally, as $H_{\text{sing}}$ is not spacelike periodic. Passing over to the differentiated field $\psi$, the singular term becomes

$$\partial_u\partial_{u'}H_{\text{sing}}(u;u') = -\frac{1}{4\pi}\frac{1}{(u-u')^2}. \tag{3.42}$$

For the cylindrical vacuum, we have

$$\partial_u \partial'_u H_{\mathcal{E}}(u; u') = \frac{1}{4\pi} \sum_{k \in \mathbb{N}} k e^{-ik(u-u')}. \tag{3.43}$$

We can think of the above series formally as the derivative of a geometric series. Replacing $u - u'$ with $z_\epsilon = u - u' - i\epsilon$ makes this series absolutely convergent for $\epsilon > 0$, thus we can write the 2-point function as

$$\partial_u \partial_{u'} H_{\mathcal{E}}(u; u') = \frac{1}{4\pi} \lim_{\epsilon \searrow 0} \frac{e^{iz_\epsilon}}{(1 - e^{iz_\epsilon})^2}. \tag{3.44}$$

Performing an asymptotic expansion of this function near the coincidence limit $u - u' = 0$, we find

$$\partial_u \partial_{u'} H_{\mathcal{E}}(u; u') \approx -\frac{1}{4\pi} \frac{1}{(u-u')^2} - \frac{1}{4\pi} \frac{1}{12} + \mathcal{O}\left((u-u')^2\right). \tag{3.45}$$

Which provides an explicit verification that the vacuum state differs from the parametrix only by the addition of a smooth, symmetric function. Moreover, this allows us to calculate $:\mathcal{L}_n:_{\mathcal{E}}$. As we are working in $\mathfrak{A}^{H_{\mathcal{E}}}(\mathcal{E})$, we need only compute the functional $(:\mathcal{L}_n:_{\mathcal{E}})_{H_{\mathcal{E}}}$, which is given by

$$\begin{aligned}
(:\mathcal{L}_n:_{\mathcal{E}})_{H_{\mathcal{E}}} &= \alpha_{H_{\mathcal{E}} - H_{\text{sing}}} \mathcal{L}_n \\
&= \mathcal{L}_n + \frac{\hbar}{2} \left\langle H_{\mathcal{E}} - H_{\text{sing}}, \mathcal{L}_n^{(2)} \right\rangle \\
&= \mathcal{L}_n + \frac{\hbar}{2} \left\langle [(\partial_u \otimes \partial_u)(H_{\mathcal{E}} - H_{\text{sing}})], L_n^{(2)} \right\rangle \\
&= \mathcal{L}_n + \hbar \int_{u=0}^{2\pi} e^{inu} \left[ (\partial_u \partial_{u'} H_{\mathcal{E}}) - (\partial_u \partial_{u'} H_{\text{sing}}) \right] (u, -u; u, -u) \, \mathrm{d}u \\
&= \mathcal{L}_n - \frac{\hbar}{24} \delta_{n,0}.
\end{aligned} \tag{3.46}$$

For a generic Hadamard state $H \in \mathrm{Had}(\mathcal{E})$ we then have

$$\begin{aligned}
(:\mathcal{L}_n:_{\mathcal{E}})_H &= \alpha_{H - H_{\text{sing}}} \mathcal{L}_n = \alpha_{H - H_{\mathcal{E}}} \left( \alpha_{H_{\mathcal{E}} - H_{\text{sing}}} \mathcal{L}_n \right) \\
&= \alpha_{H - H_{\mathcal{E}}} \mathcal{L}_n - \frac{\hbar}{24} \delta_{n,0} = \left( {}^{\circ}_{\circ} \mathcal{L}_n {}^{\circ}_{\circ}{}_{H_{\mathcal{E}}} \right)_H - \frac{\hbar}{24} \delta_{n,0}.
\end{aligned} \tag{3.47}$$

In other words, the quantum observables $:\mathcal{L}_n:_{\mathcal{E}}$ and ${}^{\circ}_{\circ} \mathcal{L}_n {}^{\circ}_{\circ}{}_{H_{\mathcal{E}}}$ in $\mathfrak{A}(\mathcal{E})$ defined, respectively, as the locally covariant Wick ordering and the normal ordering with respect to the vacuum $H_{\text{cyl}}$ of the classical functionals $\mathcal{L}_n$, are related by a shift

$$:\mathcal{L}_n:_{\mathcal{E}} = {}^{\circ}_{\circ} \mathcal{L}_n {}^{\circ}_{\circ}{}_{H_{\mathcal{E}}} - \frac{\hbar}{24} \delta_{n,0}. \tag{3.48}$$

With this shift we find, as expected, that the commutation relations of $:\mathcal{L}_n:_{\mathcal{E}}$ are

$$[:\mathcal{L}_n:_{\mathcal{E}}, :\mathcal{L}_m:_{\mathcal{E}}] = \hbar(n-m) :\mathcal{L}_{n+m}:_{\mathcal{E}} + \frac{\hbar^2}{12} n^3 \delta_{n+m,0}. \tag{3.49}$$

Recall that $\overset{\circ}{\circ} - \overset{\circ}{\circ}_{H_{\mathcal{E}}}$ can be interpreted as normal ordering with respect to the vacuum $H_{\mathcal{E}}$. Moreover, we established the Hadamard parametrix $H_{\text{sing}}$ of the cylinder is effectively the 2-point function of the Minkowski vacuum, embedded into some suitable neighbourhood of $\Delta \subset \mathcal{E}^2$. Accordingly, (3.38) computes the commutation relations for Fourier modes of the stress-energy tensor normally ordered with respect to $H_{\mathcal{E}}$, and (3.49) the same but ordered with respect to the Minkowski vacuum.

We note here that the procedure we have just outlined is in effect the derivation of the Casimir effect given by Kay in [Kay79]. There, Wald's axiomatic approach to renormalising the expectation value of the stress-energy tensor [Wal77] is applied to the Klein-Gordon model on the Einstein cylinder, which then produces the normal ordering formula (2.80), specifically for the components $T_{\mu\nu}$ of the stress-energy tensor.

In the standard approach to CFT in two dimensions, one typically imposes (3.38) as the standard commutation relations for Laurent modes of the stress energy tensor, here understood as a field over the complex plane in some precise sense. Then, mapping the plane to the 'cylinder' via the map $z \mapsto e^{iz}$, one may obtain the radially ordered commutation relations, concordant with (3.49). However, in our framework, it does not make much sense to speak of a Virasoro algebra for the plane, as there is no suitable notion of mode expansion for the stress-energy tensor when considering the constraint that functionals must be compactly supported (see the remark preceding chapter 6). In fact, arguably the most significant differences between our approach and the VOA framework is that the latter relies on mode decomposition in order to analyse the singularity structure of quantum fields, whereas we instead use tools from microlocal analysis.

## 3.4 CONNECTION TO ZETA REGULARISATION

There is a well known trick in the physicists' literature to explain (3.48). Firstly, recall that we can write a given $\mathcal{L}_n$ functional as an infinite series over $\mathcal{A}_m$ functionals (which is point-wise convergent) as:

$$\mathcal{L}_n = \frac{1}{2} \sum_{k \in \mathbb{Z}} \mathcal{A}_k \cdot \mathcal{A}_{n-k}. \tag{3.50}$$

The $\star_{H_{\mathcal{E}}}$ product of two such functionals is

$$\mathcal{A}_k \star_{H_{\mathcal{E}}} \mathcal{A}_{n-k} = \mathcal{A}_k \cdot \mathcal{A}_{n-k} + \hbar k \theta(k) \delta_{n,0}. \tag{3.51}$$

In particular, for $n \neq 0$ this means that $\mathcal{A}_k \cdot \mathcal{A}_{n-k} = \mathcal{A}_k \star_{H_{\mathcal{E}}} \mathcal{A}_{n-k}$. Hence, we can define a family of observables $\{(\widetilde{\mathcal{L}}_n)\}_{n \in \mathbb{Z}^*} \subset \mathfrak{A}(\mathcal{E})$ by replacing the classical, pointwise product in (3.50) with $\star$. This family would then coincide with $\{\mathbin{⁞}\mathcal{L}_n\mathbin{⁞}_{H_\mathcal{E}}\}_{n\in\mathbb{Z}^*}$. For $n = 0$, we may still replace the pointwise product with $\star_{H_\mathcal{E}}$, but the ordering of the functionals is now significant. Naïvely replacing the classical pointwise product $\cdot$ in (3.50) for $n = 0$ by the $\star$ product yields the quantum observable

$$\widetilde{\mathcal{L}}_0 = \frac{1}{2}\sum_{k=1}^{\infty} \mathcal{A}_{-k} \star_{H_\mathcal{E}} \mathcal{A}_k + \frac{1}{2}\sum_{k=-\infty}^{0} \mathcal{A}_{-k} \star_{H_\mathcal{E}} \mathcal{A}_k. \tag{3.52}$$

Casting rigour aside, we could then 'reorder' $\widetilde{\mathcal{L}}_0$ by moving every $\mathcal{A}_k$ in the second series to the left hand side of the product, which would produce the infamous divergent series

$$\widetilde{\mathcal{L}}_0 = \frac{1}{2}\mathcal{A}_0 \star_{H_\mathcal{E}} \mathcal{A}_0 + \sum_{k=1}^{\infty} \mathcal{A}_{-k} \star_{H_\mathcal{E}} \mathcal{A}_k + \frac{\hbar}{2}\sum_{k\in\mathbb{N}} k. \tag{3.53}$$

The rigorous and covariant way of reordering $\mathcal{L}_0$, as we saw in the previous section, is to apply the map $\alpha_{H_\mathcal{E}-H_{\mathrm{sing}}}$. If we define $w_\mathcal{E}(u) := (\partial_u \otimes \partial_u)\big[H_\mathcal{E}(u;0) - H_{\mathrm{sing}}(u;0)\big]$, where we exploit translation invariance to write $w_\mathcal{E}$ as a function of a single variable, then we can write the normally ordered form of $\mathcal{L}_0$ as

$$\alpha_{H_\mathcal{E}-H_{\mathrm{sing}}}\mathcal{L}_0 = \mathcal{L}_0 + \frac{\hbar}{2}\lim_{u\to 0} w_\mathcal{E}(u). \tag{3.54}$$

By approximating both $H_\mathcal{E}$ and $H_{\mathrm{sing}}$ by smooth functions, we can write

$$\begin{aligned}
w_\mathcal{E}(u) &= \lim_{\epsilon\searrow 0}\left[\sum_{n=0}^{\infty} n e^{-inu}e^{-n\epsilon} - \int_{p=0}^{\infty} p e^{-ipu}e^{-p\epsilon}\,\mathrm{d}p\right] &(3.55)\\
&= \lim_{z\to -iu}\frac{d}{dz}\left[\frac{1}{1-e^z} + \frac{1}{z}\right] &(3.56)\\
&= \lim_{z\to -iu}\frac{d}{dz}\left[-\sum_{k=0}^{\infty}\frac{B_k}{k!}z^{k-1} + z^{-1}\right] &(3.57)\\
&= \lim_{z\to -iu}\frac{d}{dz}\left[\sum_{k=0}^{\infty}\frac{\zeta(-k)}{k!}z^{k}\right], &(3.58)
\end{aligned}$$

where here $B_k$ denotes the $k^{\text{th}}$ Bernoulli number. This explains the appearance of $\zeta(-1)$ as $u \to 0$ in the normal ordering of $\mathcal{L}_0$ without any recourse to intermediate divergent series.

To close out this section, we make a brief remark about how our notion of normal ordering corresponds to the procedure of shuffling creation operators past annihilators, or similarly the normally ordered products of chiral fields [Kac98, (2.3.5)].

Consider the classical product of a collection of $\mathcal{A}_{m_i}$, the functional derivative of this may be written

$$(\mathcal{A}_{m_1}\cdots\mathcal{A}_{m_k})^{(1)} = \sum_{i=1}^{k}(\mathcal{A}_{m_1}\cdots\widehat{\mathcal{A}_{m_i}}\cdots\mathcal{A}_{m_k})\mathcal{A}^{(1)}_{m_i}, \tag{3.59}$$

where $\widehat{-}$ indicates omission. From this we may compute that

$$\begin{aligned}(\mathcal{A}_{m_1}\cdots\mathcal{A}_{m_k}) \star_{H_{\mathcal{E}}} \mathcal{A}_n& \\ = \mathcal{A}_{m_1}\cdots\mathcal{A}_{m_k}\cdot\mathcal{A}_n + \hbar\sum_{i=1}^{k}&(\mathcal{A}_{m_1}\cdots\widehat{\mathcal{A}_{m_i}}\cdots\mathcal{A}_{m_k})m_i\theta(-m_i)\delta_{m_i+n,0}.\end{aligned} \tag{3.60}$$

Note that the $i^{\text{th}}$ term in the sum vanishes if $n \leq m_i$. If $n \leq m_i$ for every $i \in \{1,\dots k\}$, then we are only left with the $\hbar^0$ term on the right hand side. Moving to the abstract algebra $\mathfrak{A}(\mathcal{E})$ by applying the formal map $\alpha^{-1}_{H_{\mathcal{E}}}$, we then have

$$\overset{\circ}{\circ}\mathcal{A}_{m_1}\cdots\mathcal{A}_{m_k}\cdot\mathcal{A}_n\overset{\circ}{\circ}_{H_{\mathcal{E}}} = \overset{\circ}{\circ}\mathcal{A}_{m_1}\cdots\mathcal{A}_{m_k}\overset{\circ}{\circ}_{H_{\mathcal{E}}}\star\mathcal{A}_n, \tag{3.61}$$

where we make use of the fact that we can canonically identify linear classical observables with their quantum counterparts. Applying this procedure iteratively, if we assume that the sequence $i \mapsto m_i$ is monotonically decreasing, then we can write

$$\overset{\circ}{\circ}\mathcal{A}_{m_1}\cdots\mathcal{A}_{m_k}\overset{\circ}{\circ}_{H_{\mathcal{E}}} = \mathcal{A}_{m_1}\star\cdots\star\mathcal{A}_{m_k}, \qquad m_i \leq m_{i+1}. \tag{3.62}$$

Given that $[\mathcal{A}_m,\mathcal{A}_n] = 0$ whenever $m$ and $n$ are either both negative or both positive, we have recovered the familiar result that normal ordering moves $\mathcal{A}_m$ 'to the right' if $m \leq 0$ and 'to the left' if $m > 0$.

## — 4 —

# Conformally Covariant Field Theory

So far, our classical and quantum algebras of observables are insensitive to any conformal symmetries a given theory may possess. This is because the morphisms in Loc are isometric embeddings, required to preserve the metric exactly. To study the conditions for and consequences of conformal covariance, we must relax this condition to allow *conformally* admissible embeddings.

**Definition 4.0.1.** Let $\mathcal{M} = (M, g, \mathfrak{o}, \mathfrak{t})$ and $\mathcal{N} = (N, g', \mathfrak{o}', \mathfrak{t}')$ be a pair of spacetimes (i.e. objects of Loc). A smooth embedding $\chi : M \hookrightarrow N$ is *conformally admissible* if $\chi^*\mathfrak{o}' = \mathfrak{o}$, $\chi^*\mathfrak{t}' = \mathfrak{t}$, and $\chi^*g' = \Omega^2 g$, where $\Omega \in \mathfrak{C}(\mathcal{M})$ is some nowhere-vanishing function known as the *conformal factor*.

The category CLoc – first introduced by Pinamonti in [Pin09] – is the natural setting for the study of conformal field theories. It comprises the same objects as Loc, but enlarges the collection of morphisms to conformally admissible embeddings. As one might expect, we upgrade the concept of locally covariant field theory to locally *conformally* covariant field theory simply by replacing the category Loc with CLoc. In the next section, we show explicitly how this may be done for a large class of classical theories, and for the conformally coupled scalar field in the quantum case.

It is worth noting that although in this thesis we focus primarily on the 1+1 dimensional case, the discussion which follows in §4.1 is applicable to spacetimes of arbitrary dimension.

### 4.1 CONFORMAL LAGRANGIANS

In this section we shall outline the language necessary to identify a particular Lagrangian (more precisely, its corresponding action) as being conformally covariant. In order to do so we must first introduce some notation.

**Definition 4.1.1.** Let $\chi : \mathcal{M} \hookrightarrow \mathcal{N}$ be a conformally admissible embedding with conformal factor $\Omega^2$. Given $\Delta \in \mathbb{R}$, the *weighted pushforward* with respect to $\Delta$ is defined by

$$\begin{aligned} \chi_*^{(\Delta)} : \mathfrak{D}(\mathcal{M}) &\to \mathfrak{D}(\mathcal{N}), \\ f &\mapsto \chi_*\left(\Omega^{-\Delta} f\right), \end{aligned} \tag{4.1}$$

where $\chi_*$ denotes the standard pushforward of test functions (2.69). Similarly, we define the *weighted pullback* with respect to $\Delta$ by

$$\begin{aligned} \chi_{(\Delta)}^* : \mathfrak{E}(\mathcal{N}) &\to \mathfrak{E}(\mathcal{M}), \\ \varphi &\mapsto \Omega^{\Delta}\chi^*\varphi. \end{aligned} \tag{4.2}$$

In the following proposition, we collect some useful properties of these maps.

**Proposition 4.1.2.** *Let* $\chi \in \mathrm{Hom}_{\mathsf{CLoc}}(\mathcal{M};\mathcal{N})$, *and* $\rho \in \mathrm{Hom}_{\mathsf{CLoc}}(\mathcal{N};\mathcal{O})$. *Then,*

1. $\rho_*^{(\Delta)} \circ \chi_*^{(\Delta)} = (\rho \circ \chi)_*^{(\Delta)}$
2. $\chi_{(\Delta)}^* \circ \rho_{(\Delta)}^* = (\rho \circ \chi)_{(\Delta)}^*$
3. *For* $\varphi \in \mathfrak{E}(\mathcal{N})$, $f \in \mathfrak{D}(\mathcal{M})$

   $$\int_{\mathcal{N}} \varphi\left(\chi_*^{(\Delta)} f\right) \mathrm{d}V_{\mathcal{N}} = \int_{\mathcal{M}} \left(\chi_{(d-\Delta)}^*\varphi\right) f \mathrm{d}V_{\mathcal{M}},$$

   *where* $d = \mathrm{Dim}(\mathcal{M}) = \mathrm{Dim}(\mathcal{N})$.

*Proof.* The first of these results is easiest to see as a consequence of the other two, thus we defer its proof until the end.

Result 2 can be obtained by a direct computation. Firstly, note that if $\chi^* g_{\mathcal{N}} = \Omega_\chi^2 g_{\mathcal{M}}$, and $\rho^* g_{\mathcal{O}} = \Omega_\rho^2 g_{\mathcal{N}}$, then the conformal factor for $\rho \circ \chi$ is given by $(\rho \circ \chi)^* g_{\mathcal{O}} = (\Omega_\chi \cdot \chi^*\Omega_\rho)^2 g_{\mathcal{M}}$. If we select some arbitrary $\varphi \in \mathfrak{E}(\mathcal{O})$, then

$$\begin{aligned} \chi_{(\Delta)}^*\left(\rho_{(\Delta)}^*\varphi\right) &= \chi_{(\Delta)}^*\left(\Omega_\rho^\Delta \rho^*\varphi\right) \\ &= \left(\Omega_\chi \cdot (\chi^*\Omega_\rho)\right)^\Delta (\chi^*\rho^*\varphi) \\ &= (\rho \circ \chi)_{(\Delta)}^* \varphi. \end{aligned}$$

To prove 3, first note that, because $\mathrm{supp}\left(\chi_*^{(\Delta)} f\right) \subseteq \chi(\mathcal{M})$, we may restrict the first integral to $\chi(\mathcal{M})$, where we may consider $\chi$ to be a diffeomorphism. Next, recall that a standard result for conformal transformations states $\chi^*(\mathrm{d}V_{\mathcal{N}}) = \Omega^d \mathrm{d}V_{\mathcal{M}}$. From this we find

$$\begin{aligned}\chi^*\left(\varphi \cdot (\chi_*^{(\Delta)} f) \cdot \mathrm{d}V_{\mathcal{N}}\right) &= (\chi^*\varphi) \cdot \left(\Omega^{-\Delta} f\right) \cdot \left(\Omega^d \mathrm{d}V_{\mathcal{M}}\right) \\ &= \left(\chi^*_{(d-\Delta)}\varphi\right) f \, \mathrm{d}V_{\mathcal{M}}.\end{aligned}$$

Finally, to prove 1, let $f \in \mathfrak{D}(\mathcal{M})$ and take some arbitrary test function $h \in \mathfrak{D}(\mathcal{O})$. Then, consider $\int_{\mathcal{O}} h \left(\rho_*^{(\Delta)} \chi_*^{(\Delta)} f\right) \mathrm{d}V_{\mathcal{O}}$. Using the two results we have just established, we see that

$$\begin{aligned}\int_{\mathcal{O}} h \left(\rho_*^{(\Delta)} \chi_*^{(\Delta)} f\right) \mathrm{d}V_{\mathcal{O}} &= \int_{\mathcal{M}} \left(\chi^*_{(d-\Delta)} \rho^*_{(d-\Delta)} h\right) f \, \mathrm{d}V_{\mathcal{M}} \\ &= \int_{\mathcal{M}} \left((\rho \circ \chi)^*_{(d-\Delta)} h\right) f \, \mathrm{d}V_{\mathcal{M}} \\ &= \int_{\mathcal{O}} h \left((\rho_\circ \chi)_*^{(\Delta)} f\right) \mathrm{d}V_{\mathcal{O}}.\end{aligned}$$

Thus, as this holds for every choice of $h \in \mathfrak{D}(\mathcal{O})$, we can conclude that $\rho_*^{(\Delta)} \chi_*^{(\Delta)} f = (\rho_\circ \chi)_*^{(\Delta)} f$. □

Using these definitions, we can then state the condition required for the theory arising from a natural Lagrangian $\mathcal{L}$ to be conformally covariant.

**Definition 4.1.3** (Conformal Natural Lagrangian). Let $\mathcal{L} : \mathfrak{D} \Rightarrow \mathfrak{F}_{\mathrm{loc}}$ be a natural Lagrangian as per Section 2.9. Suppose there exists $\Delta \in \mathbb{R}$ such that, for every conformally admissible embedding $\chi \in \mathrm{Hom}_{\mathsf{CLoc}}(\mathcal{M}; \mathcal{N})$, every $\varphi \in \mathfrak{E}(\mathcal{N})$, and every $f \in \mathfrak{D}(\mathcal{M})$

$$\left\langle S'_{\mathcal{M}}[\chi^*_{(\Delta)}\varphi], f\right\rangle = \left\langle S'_{\mathcal{N}}[\varphi], \chi_*^{(\Delta)} f\right\rangle, \tag{4.3}$$

where $S'_{\mathcal{M}}$ is the Euler-Lagrange derivative of $\mathcal{L}_{\mathcal{M}}$ as defined in (2.41). In this case, we call $\mathcal{L}$ a *conformal natural Lagrangian*.

We can state this condition more elegantly by once again taking the BV perspective where, instead of focusing on the natural Lagrangian $\mathcal{L}$, we use its associated differential $\delta_S : \mathfrak{V}_{\mu c} \Rightarrow \mathfrak{F}_{\mu c}$.

Firstly, we can use the weighted pullback to define a modification of the functor assigning a spacetime its classical observables, $\mathfrak{F}_{\mu c}$. For $\Delta \in \mathbb{R}$, let $\mathfrak{F}_{\mu c}^{(\Delta)}$ be a functor $\mathsf{CLoc} \to \mathsf{Vec}$ which assigns to each spacetime $\mathcal{M}$ its microcausal observables $\mathfrak{F}_{\mu c}(\mathcal{M})$ as usual, but assigns to $\chi \in \mathrm{Hom}_{\mathsf{CLoc}}(\mathcal{M}; \mathcal{N})$ the morphism

$$(\mathfrak{F}_{\mu c}^{(\Delta)} \chi \mathcal{F})[\varphi] := \mathcal{F}[\chi^*_{(\Delta)}\varphi]. \tag{4.4}$$

Proposition 4.1.2 assures us these morphisms compose as they should. Moreover, by using

$$\left(\mathfrak{F}_{\mu c}^{(\Delta)}\chi\mathcal{F}\right)^{(n)}[\varphi] = \left(\chi_*^{(d-\Delta)}\right)^{\otimes n}\mathcal{F}^{(n)}[\chi^*_{(\Delta)}\varphi], \tag{4.5}$$

we can see that the wavefront sets of functional derivatives are independent of the choice of $\Delta$. Then, by noting that the joint future/past lightcones $\overline{V}^n_\pm$ are preserved under pullback by $\chi$, and are both preserved under pushforward by a conformal embedding, the wavefront set spectral condition (2.39) is also preserved. Hence $\mathfrak{F}_{\mu c}^{(\Delta)}\chi : \mathfrak{F}_{\mu c}(\mathcal{M}) \to \mathfrak{F}_{\mu c}(\mathcal{N})$ as desired.

Similarly to $\mathfrak{F}_{\mu c}$, for any choice of weight $\Delta$, we can define an extension $\mathfrak{V}_{\mu c}^{(\Delta)} : \mathsf{CLoc} \to \mathsf{Vec}$ by

$$\left(\mathfrak{V}_{\mu c}^{(\Delta)}\chi X\right)[\varphi] = \chi_*^{(\Delta)}(X[\chi^*_{(\Delta)}\varphi]),$$

where $\chi_*^{(\Delta)}$ is again the weighted pushforward of test functions. Recall that we defined local covariance in the BV formalism as the condition that $\delta_S$ is a natural transformation $\mathfrak{V}_{\mu c} \Rightarrow \mathfrak{F}_{\mu c}$, where each is a functor $\mathsf{Loc} \to \mathsf{Vec}$. Similarly, (4.3) simply states that such a theory is conformally covariant if the same collection of maps comprising $\delta_S$ also define a natural transformation $\delta_S : \mathfrak{V}_{\mu c}^{(\Delta)} \Rightarrow \mathfrak{F}_{\mu c}^{(\Delta)}$, where each is now a functor $\mathsf{CLoc} \to \mathsf{Vec}$.

## 4.2 CONFORMALLY COVARIANT CLASSICAL FIELD THEORY

We can now see how the criterion for conformal covariance that has just been outlined gives rise to classical dynamical structures which vary as one would expect under conformal transformations. The first result compares the linearised equations of motion on two spacetimes related by a conformally admissible embedding.

**Proposition 4.2.1.** *Let $\mathcal{L}$ be a conformal natural Lagrangian which satisfies the linearisation hypothesis* (2.43)*. If $\chi \in \mathrm{Hom}_{\mathsf{CLoc}}(\mathcal{M};\mathcal{N})$ and $\varphi \in \mathfrak{E}(\mathcal{N})$, then*

$$\chi_*^{(d-\Delta)}P_{\mathcal{M}}[\chi^*_{(\Delta)}\varphi] = P_{\mathcal{N}}[\varphi]\chi_*^{(\Delta)}, \tag{4.6}$$

*where each differential operator has been implicitly restricted to the space of* test *functions of the appropriate spacetime.*

*Proof.* The proof is effectively a direct computation. Let $g \in \mathfrak{D}(\mathcal{M})$ and $h \in \mathfrak{D}(\mathcal{N})$. Recall from the definition of $P_{\mathcal{N}}$ that

$$\left\langle P_{\mathcal{N}}[\varphi]\chi_*^{(\Delta)}g, h\right\rangle_{\mathcal{N}} = \left\langle S''_{\mathcal{N}}[\varphi], \left(\chi_*^{(\Delta)}g\right)\otimes h\right\rangle_{\mathcal{N}}. \tag{4.7}$$

This then allows us to employ (4.3) as

$$\begin{aligned}\left\langle S''_{\mathcal{N}}[\varphi], \left(\chi_*^{(\Delta)} g\right) \otimes h\right\rangle_{\mathcal{N}} &= \left\langle \mathcal{L}_{\mathcal{N}}(f)^{(2)}[\varphi], \left(\chi_*^{(\Delta)} g\right) \otimes h\right\rangle \\ &= \frac{d}{d\epsilon} \left\langle \mathcal{L}_{\mathcal{N}}(f)^{(1)}[\varphi + \epsilon h], \chi_*^{(\Delta)} g\right\rangle \bigg|_{\epsilon=0} \\ &= \frac{d}{d\epsilon} \left\langle S'_{\mathcal{N}}\left[\varphi + \epsilon h\right], \chi_*^{(\Delta)} g\right\rangle_{\mathcal{N}} \bigg|_{\epsilon=0} \\ &= \frac{d}{d\epsilon} \left\langle S'_{\mathcal{M}}\left[\chi^*_{(\Delta)}\varphi + \epsilon \chi^*_{(\Delta)} h\right], g\right\rangle_{\mathcal{M}} \bigg|_{\epsilon=0} \\ &= \left\langle P_{\mathcal{M}}\left[\chi^*_{(\Delta)}\varphi\right] g, \chi^*_{(\Delta)} h\right\rangle_{\mathcal{M}} \\ &= \left\langle \chi_*^{(d-\Delta)} P_{\mathcal{M}}[\chi^*_{(\Delta)}\varphi] g, h\right\rangle_{\mathcal{N}}, \end{aligned} \tag{4.8}$$

where $f^{-1}\{1\}$ contains a neighbourhood of $\mathrm{supp}\,(\chi_*^{(\Delta)} g)$. Note that in the third line we use (4.3) and, for the final equality, we use the fact that $\chi_*^{(d-\Delta)}$ is the adjoint of $\chi^*_{(\Delta)}$. As the choice of $h$ is arbitrary, we may then conclude that the two operators coincide. □

*Remark* 4.2.2. As $P_{\mathcal{N}}[\varphi]$ and $P_{\mathcal{M}}[\chi^*_{(\Delta)}\varphi]$ are both self-adjoint, we can write an equivalent form of (4.6) for linear maps $\mathfrak{E}(\mathcal{N})$, namely

$$P_{\mathcal{M}}[\chi^*_{(\Delta)}\varphi]\chi^*_{(\Delta)} = \chi^*_{(d-\Delta)} P_{\mathcal{N}}[\varphi]. \tag{4.9}$$

Using this equation, we can immediately see that the solution spaces for these two operators are closely related: if $\psi$ is a solution to $P_{\mathcal{N}}[\varphi]$, then $\chi^*_{(\Delta)}\psi$ is a solution to $P_{\mathcal{M}}[\chi^*_{(\Delta)}\varphi]$.

Moreover if, for $\lambda > 0$, we take $\mathcal{N} = (M, \lambda^2 g_{\mathcal{M}}, \mathfrak{o}_{\mathcal{M}}, \mathfrak{t}_{\mathcal{M}})$, i.e. just $\mathcal{M}$ with the metric scaled by some factor $\lambda^2$ and $\chi = \mathrm{Id}_M$, then $\chi^*_{(\Delta)}\psi = \lambda^{\Delta}\psi$. This indicates that $\Delta$ is what is typically referred to in the literature as the *scaling dimension* of the field $\varphi$.

When a pair of normally-hyperbolic differential operators are related in the above manner, we can similarly relate their fundamental solutions. The following proposition, which reduces to [Pin09, Lemma 2.2] in the particular case of the conformally coupled Klein-Gordon field in 4D, establishes the conformal covariance of the Pauli-Jordan function arising from a suitable conformal natural Lagrangian. To simplify notation, we shall refer only to a single differential operator on each spacetime, i.e. we suppress the dependence on an initial field configuration $\varphi$ or $\chi^*_{(\Delta)}\varphi$, though this does not mean that the scope of the result is limited to free theories.

**Proposition 4.2.3.** *Let* $\chi \in \mathrm{Hom}_{\mathsf{CLoc}}(\mathcal{M}; \mathcal{N})$*, and let* $P_{\mathcal{M}}$*,* $P_{\mathcal{N}}$ *be a pair of symmetric, normally hyperbolic differential operators on* $\mathcal{M}$ *and* $\mathcal{N}$ *respectively such that*

$$P_{\mathcal{M}}\chi^*_{(\Delta)} = \chi^*_{(d-\Delta)} P_{\mathcal{N}}. \tag{4.10}$$

*If $E^{R/A}_{\mathcal{M}/\mathcal{N}}$ denotes the advanced/retarded propagator for $P_{\mathcal{M}/\mathcal{N}}$ as appropriate, then*

$$E^{R/A}_{\mathcal{M}} = \chi^*_{(\Delta)} E^{R/A}_{\mathcal{N}} \chi^{(d-\Delta)}_*. \tag{4.11}$$

*Proof.* Recall that the advanced and retarded propagators of $P_{\mathcal{M}}$ are uniquely determined by their composition with $P_{\mathcal{M}}$ and their support properties. As such, we simply need to establish that the operator on the right-hand side of (4.11) satisfies the relevant criteria (2.44) and (2.45).

Firstly, if we act on this operator with $P_{\mathcal{M}}$ we see

$$P_{\mathcal{M}}\, \chi^*_{(\Delta)} E^{R/A}_{\mathcal{N}} \chi^{(d-\Delta)}_* = \chi^*_{(d-\Delta)} P_{\mathcal{N}} E^{R/A}_{\mathcal{N}} \chi^{(d-\Delta)}_*.$$

By definition, $P_{\mathcal{N}} \circ E^{R/A}_{\mathcal{N}} = \mathbb{1}_{\mathfrak{D}(\mathcal{N})}$, and clearly $\chi^*_{(d-\Delta)} \chi^{(d-\Delta)}_* = \mathbb{1}_{\mathfrak{D}(\mathcal{M})}$, hence

$$P_{\mathcal{M}} \left( \chi^*_{(\Delta)} E^{R/A}_{\mathcal{N}} \chi^{(d-\Delta)}_* \right) = \mathbb{1}_{\mathfrak{D}(\mathcal{M})}. \tag{4.12}$$

If we denote by $P^c_{\mathcal{M}}$ the restriction of $P_{\mathcal{M}}$ to $\mathfrak{D}(\mathcal{M})$, and likewise $P^c_{\mathcal{N}}$, by the symmetry of these operators, we have that

$$\chi^{(d-\Delta)}_* P^c_{\mathcal{M}} = P^c_{\mathcal{N}} \chi^{(\Delta)}_*.$$

Thus, acting on $P^c_{\mathcal{M}}$ with our candidate propagator yields

$$\chi^*_{(\Delta)} E^{R/A}_{\mathcal{N}} \chi^{(d-\Delta)}_* P^c_{\mathcal{M}} = \chi^*_{(\Delta)} E^{R/A}_{\mathcal{N}} P^c_{\mathcal{N}} \chi^{(\Delta)}_*,$$

which is again simply $\mathbb{1}_{\mathfrak{D}(\mathcal{M})}$.

Finally, we must determine the supports of these functions. Let $f \in \mathfrak{D}(\mathcal{M})$. Note that $\operatorname{supp}(\chi^{(d-\Delta)}_* f) = \chi(\operatorname{supp} f)$, hence, using the support property of $E^{R/A}$

$$\operatorname{supp}\left( E^{R/A}_{\mathcal{N}} \chi^{(d-\Delta)}_* f \right) \subseteq \mathfrak{J}^{\pm}_{\mathcal{N}} \left( \chi \left( \operatorname{supp} f \right) \right).$$

Pulling this back to $\mathcal{M}$, we have

$$\operatorname{supp}\left( \chi^*_{(\Delta)} E^{R/A}_{\mathcal{N}} \chi^{(d-\Delta)}_* f \right) \subseteq \chi^{-1} \left( \mathfrak{J}^{\pm}_{\mathcal{N}} \left( \chi(\operatorname{supp} f) \right) \right).$$

Conformally admissible embeddings preserve causal structure. In particular, if $\gamma : [0,1] \to \mathcal{M}$ is a causal, future/past-directed curve, then $\chi \circ \gamma$ is also causal and future/past-directed. This means that $\chi \left( \mathfrak{J}^{\pm}_{\mathcal{M}}(\operatorname{supp} f) \right) = \mathfrak{J}^{\pm}_{\mathcal{N}} \left( \chi(\operatorname{supp} f) \right)$. Hence our candidate propagators also meet the desired support criteria, and must genuinely be the advanced and retarded propagators for $P_{\mathcal{M}}$ as required. □

One can show that conformal invariance as defined in appendix D of [Wal10] implies (4.10), given our assumption that $P_{\mathcal{M}}$ and $P_{\mathcal{N}}$ are both *symmetric* in the sense that $\langle f, P_{\mathcal{M}}\varphi\rangle_{\mathcal{M}} = \langle P_{\mathcal{M}} f, \varphi\rangle_{\mathcal{M}}$ for all $f \in \mathfrak{D}(\mathcal{M})$, $\varphi \in \mathfrak{E}(\mathcal{M})$.

Similar to the case of (isometric) local covariance, the consequence of proposition 4.2.3 is that we can define a symplectomorphism from the solution space of $P_{\mathcal{M}}$ to that of $P_{\mathcal{N}}$. Recall that we can identify the space of solutions to $P_{\mathcal{M}}$ with $\mathfrak{D}(\mathcal{M})/P_{\mathcal{M}}\left(\mathfrak{D}(\mathcal{M})\right)$. If $f, g \in \mathfrak{D}(\mathcal{M})$, then

$$\langle f, E_{\mathcal{M}} g\rangle = \left\langle \chi_*^{(d-\Delta)} f, E_{\mathcal{N}}\left(\chi_*^{(d-\Delta)} g\right)\right\rangle. \tag{4.13}$$

Moreover, from (4.6), it follows that $\chi_*^{(d-\Delta)}\left(P_{\mathcal{M}}\left(\mathfrak{D}(\mathcal{M})\right)\right) \subseteq P_{\mathcal{N}}\left(\mathfrak{D}(\mathcal{N})\right)$, hence $\chi_*^{(\Delta)}$ yields a well-defined map between the quotient spaces $\mathfrak{D}(\mathcal{M})/P_{\mathcal{M}}\left(\mathfrak{D}(\mathcal{M})\right) \to \mathfrak{D}(\mathcal{N})/P_{\mathcal{N}}\left(\mathfrak{D}(\mathcal{N})\right)$.

As was the case in Section 2.9, this symplectomorphism of solution spaces in turn gives rise to a Poisson algebra homomorphism relating the Peierls brackets for each spacetime.

A quick calculation shows that the map $\mathfrak{F}_{\mu c}^{(\Delta)}\chi$ defined in (4.4) is a Poisson algebra homomorphism: for $\mathcal{F}, \mathcal{G} \in \mathfrak{F}_{\mu c}(\mathcal{M})$, $\varphi \in \mathfrak{E}(\mathcal{N})$ we have that

$$\begin{aligned}
\left\{\mathfrak{F}_{\mu c}^{(\Delta)}\chi\mathcal{F}, \mathfrak{F}_{\mu c}^{(\Delta)}\chi\mathcal{G}\right\}_{\mathcal{N}}[\varphi] &= \left\langle \left(\mathfrak{F}_{\mu c}^{(\Delta)}\chi\mathcal{F}\right)^{(1)}[\varphi], E_{\mathcal{N}}(\varphi)\left(\mathfrak{F}_{\mu c}^{(\Delta)}\chi\mathcal{G}\right)^{(1)}[\varphi]\right\rangle_{\mathcal{N}} \\
&= \left\langle \chi_*^{(d-\Delta)}\mathcal{F}^{(1)}[\chi_{(\Delta)}^*\varphi], E_{\mathcal{N}}(\varphi)\chi_*^{(d-\Delta)}\mathcal{G}^{(1)}[\chi_{(\Delta)}^*\varphi]\right\rangle_{\mathcal{N}} \\
&= \left\langle \mathcal{F}^{(1)}[\chi_{(\Delta)}^*\varphi], E_{\mathcal{M}}(\chi_{(\Delta)}^*\varphi)\mathcal{G}^{(1)}[\chi_{(\Delta)}^*\varphi]\right\rangle_{\mathcal{M}} \\
&= \left(\mathfrak{F}_{\mu c}^{(\Delta)}\chi\left\{\mathcal{F}, \mathcal{G}\right\}_{\mathcal{M}}\right)[\varphi].
\end{aligned}$$

We may summarise the above results as ensuring that the following is well-defined:

**Definition 4.2.4** (Locally Conformally Covariant Classical Field Theory)**.** For some $\Delta \in \mathbb{R}$, let $\mathcal{L}$ be a conformal natural Lagrangian of weight $\Delta$. The *locally conformally covariant classical field theory* associated to $\mathcal{L}$ is a functor $\mathfrak{P} : \mathsf{CLoc} \to \mathsf{Poi}$, which assigns

- To every spacetime $\mathcal{M} \in \mathsf{CLoc}$, the algebra $\mathfrak{F}_{\mu c}(\mathcal{M})$ equipped with the Peierls bracket $\{\cdot, \cdot\}_{\mathcal{M}}$ associated to the generalised Lagrangian $\mathcal{L}_{\mathcal{M}}$.
- To every morphism $\chi \in \mathrm{Hom}_{\mathsf{CLoc}}(\mathcal{M}; \mathcal{N})$, the Poisson algebra homomorphism $\mathfrak{F}_{\mu c}^{(\Delta)}\chi$.

**Example 4.2.5** (The Conformally Coupled Scalar Field)**.** The simplest example of a conformal natural Lagrangian is that of the conformally coupled scalar field. For spacetimes of dimension $d$, this is given by, for $\mathcal{M} \in \mathsf{CLoc}$, $f \in \mathfrak{D}(\mathcal{M})$, $\varphi \in \mathfrak{E}(\mathcal{M})$

$$\mathcal{L}_{\mathcal{M}}(f)[\varphi] := \frac{1}{2}\int_{\mathcal{M}} f\left[g_{\mathcal{M}}\left(\nabla\varphi, \nabla\varphi\right) - \xi_d R_{\mathcal{M}}\varphi^2\right] \mathrm{d}V_{\mathcal{M}}, \tag{4.14}$$

where $R_{\mathcal{M}}$ is the scalar curvature function for the spacetime $\mathcal{M}$ and $\xi_d = \frac{d-2}{4(d-1)}$ is the conformal coupling constant.

In this case, we can see that the Euler-Lagrange derivative satisfies the desired covariance property with $\Delta = \frac{(d-2)}{2}$.

Even in this example we see the necessity of phrasing (4.3) in terms of variations of the action. Naïvely, we may have assumed conformal covariance to be given by $\mathcal{L}_{\mathcal{M}}(f)[\chi^*_{(\Delta)}\varphi] = \mathcal{L}_{\mathcal{N}}(\chi^{(\Delta)}_* f)[\varphi]$. However, the presence of the test function $f$ in the above Lagrangian prevents the integration by parts necessary for this equation to hold.

## 4.3 CONFORMALLY COVARIANT QUANTUM FIELD THEORY

In order to discuss quantisation, we must return our attention to free field theories. In doing so we can once again refer unambiguously to a single operator $P_{\mathcal{M}}$ producing the equations of motion on $\mathcal{M}$.

We saw in section 2.8 that quantisation of a free field theory is achieved through the use of arbitrarily selected Hadamard distributions for each $P_{\mathcal{M}}$. The covariance of the quantum algebras was thus dependent on the fact that, given an admissible embedding $\chi : \mathcal{M} \to \mathcal{N}$, the pullback of a Hadamard distribution on $\mathcal{N}$ by $\chi$ is again a Hadamard distribution on $\mathcal{N}$. We have already seen that the *weighted* pullback of the causal propagator on $\mathcal{N}$ is the causal propagator on $\mathcal{M}$. The following proof, again adapted from [Pin09], gives the corresponding result for Hadamard distributions.

**Proposition 4.3.1.** *Let $\chi \in \mathrm{Hom}_{\mathsf{CLoc}}(\mathcal{M}; \mathcal{N})$ be a conformally admissible embedding with conformal factor $\Omega$, and let $P_{\mathcal{M}}, P_{\mathcal{N}}$ be a pair of normally hyperbolic differential operators satisfying*

$$P_{\mathcal{M}}\chi^*_{(\Delta)} = \chi^*_{(d-\Delta)}P_{\mathcal{N}}.$$

*If $W_{\mathcal{N}} : \mathfrak{D}(\mathcal{N})^{\mathbb{C}} \to \mathfrak{E}(\mathcal{N})^{\mathbb{C}}$ is a Hadamard distribution for $P_{\mathcal{N}}$, then*

$$W_{\mathcal{M}} := \chi^*_{(\Delta)}W_{\mathcal{N}}\chi^{(d-\Delta)}_* \tag{4.15}$$

*is a Hadamard distribution for $P_{\mathcal{M}}$.*

*Proof.* Firstly, (4.13) ensures that the anti-symmetric part of $W_{\mathcal{M}}$ is $\frac{i}{2}E_{\mathcal{M}}$. Secondly, by a direct computation, we can see that $P_{\mathcal{M}}W_{\mathcal{M}} \equiv 0$, hence $W_{\mathcal{M}}$ is a distributional solution to $P_{\mathcal{M}}$. Thirdly we clearly have that $\chi_*^{(d-\Delta)}\overline{f} = \overline{\left(\chi_*^{(d-\Delta)}f\right)}$, hence positivity of $W_{\mathcal{M}}$ follows directly from that of $W_{\mathcal{N}}$.

Thus, all that remains to be shown is that $W_{\mathcal{M}}$ has the appropriate wavefront set:

Considered as a distribution in $\mathfrak{D}'(\mathcal{M}^2)^{\mathbb{C}}$, $W_{\mathcal{M}}$ is defined on the dense subspace $\mathfrak{D}(\mathcal{M})^{\otimes 2} \subset \mathfrak{D}(\mathcal{M}^2)$ by

$$\langle W_{\mathcal{M}}, f \otimes g\rangle = \left\langle W_{\mathcal{N}}, \chi_*^{(d-\Delta)}f \otimes \chi_*^{(d-\Delta)}g\right\rangle. \tag{4.16}$$

This differs from the usual pullback $\chi^* W_{\mathcal{N}}$ only in the multiplication by the smooth function $\Omega^{d-\Delta} \otimes \Omega^{d-\Delta}$, hence $\mathrm{WF}(W_{\mathcal{M}}) = \mathrm{WF}\left((\chi^*)^{\otimes 2}W_{\mathcal{N}}\right)$.

At this point it is convenient to regard $\chi(\mathcal{M})$ as a spacetime in its own right, with all the relevant data being that inherited from $\mathcal{N}$ by restriction. We then observe that $\chi$ factorises as $\iota \circ \xi$, where the inclusion $\iota : \chi(\mathcal{M}) \hookrightarrow \mathcal{N}$ is an isometric embedding, and $\xi : \mathcal{M} \to \chi(\mathcal{M})$ is a conformal *diffeomorphism*. With this, we write $\chi^* W_{\mathcal{N}} = \xi^*\left(\iota^* W_{\mathcal{N}}\right)$. As $\xi$ is a diffeomorphism, we know that $\mathrm{WF}\left(\xi^*\left(\iota^* W_{\mathcal{N}}\right)\right) = \xi^*\mathrm{WF}(\iota^* W_{\mathcal{N}})$, and, since $\iota$ is an isometric admissible embedding $\mathrm{WF}(\iota^* W_{\mathcal{N}}) = \Gamma_{\chi(\mathcal{M})}$, where $\Gamma_{\mathcal{M}} = \mathrm{WF}(W)$ for any (and hence every) Hadamard distribution $W$ on $\mathcal{M}$.

It is only left for us to show that $\xi^*\Gamma_{\chi(\mathcal{M})} = \Gamma_{\mathcal{M}}$. Let $(y_1, y_2; \eta_1, \eta_2) \in \Gamma_{\chi(\mathcal{M})}$, and let $\gamma : (-\epsilon, 1+\epsilon)$ be a null geodesic satisfying $\gamma(0) = y_1$, $\gamma(1) = y_2$, $\dot{\gamma}^\flat(0) = \eta_1$, $\dot{\gamma}^\flat(1) = \eta_2$. It is then readily verified that $\xi^{-1} \circ \gamma$ is a null geodesic segment which demonstrates $(x_1, x_2; k_1, k_2) \in \Gamma_{\mathcal{M}}$, where $y_i = \xi(x_i)$, and $k_i = \eta_i \circ d\xi|_{x_i}$. Thus we see that $\xi^*\Gamma_{\chi(\mathcal{M})} \subseteq \Gamma_{\mathcal{M}}$. Similarly, if $\tilde{\gamma}$ is a null geodesic segment demonstrating that $(x_1, x_2; k_1, k_2) \in \Gamma_{\mathcal{M}}$, then $\gamma := \xi \circ \tilde{\gamma}$ shows that $(y_1, y_2; \eta_1, \eta_2) \in \Gamma_{\chi(\mathcal{M})}$. From this we can conclude that $\mathrm{WF}(W_{\mathcal{M}}) = \mathrm{WF}(\chi^* W_{\mathcal{N}}) = \Gamma_{\mathcal{M}}$, hence $W_{\mathcal{M}}$ is indeed a Hadamard distribution for $P_{\mathcal{M}}$. □

If we, by a slight abuse of notation, write $W_{\mathcal{M}} = \chi_{(\Delta)}^* W_{\mathcal{N}}$, then the above proposition can be expressed as $\chi_{(\Delta)}^* : \mathrm{Had}(\mathcal{N}) \to \mathrm{Had}(\mathcal{M})$. This map, together with the map $\mathfrak{F}_{\mu c}^{(\Delta)}\chi$ defined in the previous section, creates the algebra homomorphism required to make the quantum theory conformally covariant.

Firstly we observe that, if $H_{\mathcal{M}}$ is the symmetric part of $W_{\mathcal{M}}$ *etc*, then a quick computation confirms that

$$\left(\mathfrak{F}_{\mu c}^{(\Delta)}\chi\mathcal{F}\right) \star_{H_{\mathcal{N}}} \left(\mathfrak{F}_{\mu c}^{(\Delta)}\chi\mathcal{G}\right) = \mathfrak{F}_{\mu c}^{(\Delta)}\chi\left(\mathcal{F} \star_{H_{\mathcal{M}}} \mathcal{G}\right).$$

In other words, for a Hadamard distribution $H_{\mathcal{N}} \in \mathrm{Had}(\mathcal{N})$, $\mathfrak{F}^{(\Delta)}_{\mu c}\chi$ defines a $*$-algebra homomorphism $\mathfrak{A}^{H_{\mathcal{M}}}(\mathcal{M}) \to \mathfrak{A}^{H_{\mathcal{N}}}(\mathcal{N})$, using the notation introduced in (2.62).

To see that these maps define a homomorphism $\mathfrak{A}(\mathcal{M}) \to \mathfrak{A}(\mathcal{N})$ note that, if $H'_{\mathcal{N}} \in \mathrm{Had}(\mathcal{N})$ and $H'_{\mathcal{M}} := \chi^*_{(\Delta)} H'_{\mathcal{N}}$ then, using (4.5), one can show that

$$\alpha_{H'_{\mathcal{N}}-H_{\mathcal{N}}} \circ \mathfrak{F}^{(\Delta)}_{\mu c}\chi = \mathfrak{F}^{(\Delta)}_{\mu c}\chi \circ \alpha_{H'_{\mathcal{M}}-H_{\mathcal{M}}}, \tag{4.17}$$

hence our homomorphisms are compatible with the isomorphisms between different concrete realisations of $\mathfrak{A}(\mathcal{N})$ as required.

Thus we have shown that the following definition makes sense.

**Definition 4.3.2** (The Quantum Massless Scalar Field)**.** Let $\mathcal{L} : \mathfrak{D} \Rightarrow \mathfrak{F}_{\mathrm{loc}}$ be the conformal natural Lagrangian of the massless scalar field in spacetime dimension $d$, given by (4.14). The *locally conformally covariant quantum field theory* associated to $\mathcal{L}$ is a functor $\mathfrak{A} : \mathsf{CLoc} \to *\text{-}\mathsf{Alg}$, which assigns

- To every spacetime $\mathcal{M} \in \mathsf{CLoc}$, the algebra $\mathfrak{A}(\mathcal{M})$ defined in Section 2.8.
- To every morphism $\chi \in \mathrm{Hom}_{\mathsf{CLoc}}(\mathcal{M}; \mathcal{N})$, the $*$-algebra homomorphism defined, for $\mathcal{F} = (\mathcal{F}_H)_{H \in \mathrm{Had}(\mathcal{M})} \in \mathfrak{A}(\mathcal{M})$ and $H_{\mathcal{N}} \in \mathrm{Had}(\mathcal{N})$, by

  $$(\mathfrak{A}\chi\mathcal{F})_{H_{\mathcal{N}}} := \mathfrak{F}^{(\Delta)}_{\mu c}\chi \left(\mathcal{F}_{\chi^*_{(\Delta)} H_{\mathcal{N}}}\right),$$

  where $\Delta = \frac{d-2}{2}$.

## — 5 —

# Primary and Homogeneously Scaling Fields

### 5.1 FRAMED SPACETIMES

Now that we have constructed the quantum theory of the massless scalar field, we can begin comparing our formalism to the standard CFT literature. In formulations of CFT descended from the Osterwalder-Schrader axioms, one defines a field $\varphi(z, \bar{z})$, to be primary with conformal weights $(h, \widetilde{h}) \in \mathbb{R}^2$ if, for a holomorphic function $z \mapsto w(z)$

$$\varphi(z, \bar{z}) \mapsto \left(\frac{\partial w}{\partial z}\right)^h \left(\frac{\partial \bar{w}}{\partial \bar{z}}\right)^{\widetilde{h}} \varphi(w(z), \bar{w}(\bar{z})). \tag{5.1}$$

In order to reach an analogous definition of a primary field within the AQFT framework, we must equip our spacetimes with frames. As a motivating example, Minkowski space is naturally equipped with the frame (in null coordinates) $(du, dv)$. The Minkowski metric is then simply $ds^2 = du \odot dv$, where $\odot$ denotes the symmetrised tensor product. A general conformal automorphism, $\chi$, of Minkowski space can be written in the form

$$\chi : (u, v) \mapsto (\mu(u), \nu(v)), \tag{5.2}$$

where either $\mu, \nu \in \mathrm{Diff}_+(\mathbb{R})$ or $\mathrm{Diff}_-(\mathbb{R})$. This is readily shown to be conformal as, for any $(u, v) \in \mathbb{M}_2$

$$\chi^*(du \odot dv)_{(u,v)} = \mu'(u)\nu'(v)(du \odot dv)_{(u,v)}. \tag{5.3}$$

Hence the conformal factor is the product $\Omega^2(u, v) = \mu'(u)\nu'(v)$. To generalise this splitting of the conformal factor to arbitrary globally-hyperbolic spacetimes, we introduce a new category, which combines the conformal covariance we have just described with the idea of augmenting each spacetime with a *frame*, as may be found in, for example, [Few18].

**Definition 5.1.1.** The category CFLoc, consists of objects that are tuples $\mathcal{M} = (M, (e^\ell, e^r))$, where $M$ is a 2-manifold, and $e^\ell, e^r$ are a pair of 1-forms such that, $\forall p \in M$, $\{{e^\ell}_p, {e^r}_p\}$ spans $T^*_p M$, subject to the condition that the map

$$(M, (e^\ell, e^r)) \mapsto (M, e^\ell \odot e^r, [e^\ell \wedge e^r], [e^\ell + e^r]) \tag{5.4}$$

sends objects in CFLoc to objects in Loc.

A morphism $\chi : (M, (e^\ell, e^r)) \to (N, (\tilde{e}^\ell, \tilde{e}^r)))$ is a smooth embedding $\chi : M \hookrightarrow N$ such that if $\mathcal{M}$ and $\mathcal{N}$ are the spacetimes obtained in the above manner from $(M, (e^\ell, e^r))$ and $(N, (\tilde{e}^\ell, \tilde{e}^r)))$ respectively, then $\chi \in \mathrm{Hom}_{\mathsf{CLoc}}(\mathcal{M}; \mathcal{N})$ . In other words, $\chi$ is a conformally admissible embedding of $M$ into $N$ with respect to the metrics and orientations induced by their coframes.

*Remark* 5.1.2. This definition is intentionally reminiscent of the category FLoc introduced in [Few18]. However, we have not imposed the same rigidity condition that $\chi^* \tilde{e}^{\ell/r} = e^{\ell/r}$. One reason is that this would mean we only have isometric embeddings of the corresponding spacetimes, where instead we want *conformally admissible* embeddings. Moreover, as we shall see in the following proposition, the requirement that the induced embedding of spacetimes be conformally admissible is sufficient strong enough to imply a similar, albeit less rigid, action of $\chi$ on the frames.

As every 2D globally hyperbolic spacetime is parallelisable, each may be expressed as the spacetime induced by some object of CFLoc, i.e. the map (5.4) is surjective. Furthermore, from the definition of the morphisms in CFLoc, it is evident that this map extends to a fully faithful functor $\mathfrak{p}$ : CFLoc $\to$ CLoc, hence we have an *equivalence* between the two in the sense of category theory.

Rather than relying solely on this equivalence, however, the following proposition provides a test of whether an embedding $\chi : M \hookrightarrow N$ is conformally admissible with respect to the spacetime structure induced by the frames $(e^\ell, e^r)$ and $(\tilde{e}^\ell, \tilde{e}^r)$.

**Proposition 5.1.3.** *Let* $\mathcal{M} = (M, (e^\ell, e^r))$, $\mathcal{N} = (N, (\tilde{e}^\ell, \tilde{e}^r)))$ *be two objects in* CFLoc, *a smooth embedding* $\chi : M \hookrightarrow N$ *is then a* CFLoc *morphism between* $\mathcal{M}$ *and* $\mathcal{N}$ *if and only if there exists a pair of smooth, everywhere-positive functions* $\omega_\ell, \omega_r \in \mathfrak{C}_{>0}(M)$ *such that*

$$\chi^* \tilde{e}^{\ell/r} = \omega_{\ell/r} e^{\ell/r}. \tag{5.5}$$

*Proof.* Suppose first that the embedding $\chi$ satisfies (5.5), then it is clearly conformal, as

$$\chi^*(\tilde{e}^\ell \odot \tilde{e}^r) = \Omega^2(e^\ell \odot e^r), \tag{5.6}$$

where the conformal factor is $\Omega^2 = \omega_\ell \omega_r$. To show it is admissible, consider first

$$\chi^*[\tilde{e}^\ell \wedge \tilde{e}^r] := [\chi^*(\tilde{e}^\ell \wedge \tilde{e}^r)] = [\omega_\ell \omega_r (e^\ell \wedge e^r)] = [e^\ell \wedge e^r], \tag{5.7}$$

where the final equality comes from the fact that the product $\omega_\ell \omega_r$ is everywhere positive. Hence, $\omega_\ell \omega_r (e^\ell \wedge e^r)$ defines the same orientation as $e^\ell \wedge e^r$, establishing that $\chi$ is orientation preserving.

Next, to show $\chi$ preserves time orientation, consider

$$\chi^*(\tilde{e}^\ell + \tilde{e}^r) = \omega_\ell e^\ell + \omega_r e^r. \tag{5.8}$$

For this 1-form to define the same time orientation as $e^\ell + e^r$, first we need to prove it is timelike. Let $g = e^\ell \odot e^r$, then

$$g(\omega_\ell e^\ell + \omega_r e^r, \omega_\ell e^\ell + \omega_r e^r) = 2\omega_\ell \omega_r > 0, \tag{5.9}$$

hence it is everywhere timelike. Next, we need to show it is compatible with the original orientation:

$$g(\omega_\ell e^\ell + \omega_r e^r, e^\ell + e^r) = \omega_\ell + \omega_r > 0. \tag{5.10}$$

Thus (5.5) is a sufficient condition for $\chi$ to be a conformally admissible embedding.

Conversely, let us now assume that $\chi$ is conformally admissible. Let $\tilde{e}^{\ell/r}|_{\chi(M)}$ denote the restriction of $\tilde{e}^{\ell/r}$ to the image of $M$ under $\chi$. As $\chi$ is conformal, the pullback of each of these 1-forms must be a null 1-form on $M$ with respect to the induced metric. At every point $p \in M$, this tells us that $\chi^* \tilde{e}^\ell|_{\chi(M)}(p)$ must be colinear with either $e^\ell(p)$ or $e^r(p)$. That it must be colinear with $e^\ell(p)$ in particular is due to the fact that $\chi$ preserves orientation; a similar argument can then be made for $\tilde{e}^r$. Thus we have two functions $\omega_{\ell/r} \in \mathfrak{E}_{>0}(M)$ such that $\chi^* \tilde{e}^\ell|_{\chi(M)}(p) = \omega_{\ell/r} e^{\ell/r}$. Their product is the conformal factor of $\chi$ and hence must be positive. Finally, for $\chi$ to preserve time orientation, $\omega_\ell$ and $\omega_r$ must satisfy (5.10), thus each function must be everywhere-positive. □

## 5.2 DEFINITION OF PRIMARY FIELDS

Using these frames, we can define a modified pushforward, similar to (4.1), except now with a *pair* of weights $(\lambda, \tilde{\lambda}) \in \mathbb{R}^2$ specified. The weighted pushforward of a test function $f \in \mathfrak{D}(M)$ under a morphism $\chi : \mathcal{M} \to \mathcal{N}$ with left/right conformal factors $\omega_{\ell/r}$ is given by

$$\chi_*^{(\lambda,\tilde{\lambda})} f = \chi_* \left( \omega_\ell^{-\lambda} \omega_r^{-\tilde{\lambda}} f \right). \tag{5.11}$$

We then construct the functor $\mathfrak{D}^{(h,\widetilde{h})} : \mathsf{CFLoc} \to \mathsf{Vec}$, for $(h,\widetilde{h}) \in \mathbb{R}^2$ as follows: for an object $\mathcal{M} \in \mathsf{CFLoc}$, define $\mathfrak{D}^{(h,\widetilde{h})}(\mathcal{M}) = \mathfrak{D}(M)$, and for a morphism $\chi : \mathcal{M} \to \mathcal{N}$:

$$\mathfrak{D}^{(h,\widetilde{h})}\chi(f) = \chi_*^{(1-h,1-\widetilde{h})} f. \tag{5.12}$$

With this functor, we can finally define a *primary field of weight* $(h,\widetilde{h})$ to be a natural transformation $\Phi : \mathfrak{D}^{(h,\widetilde{h})} \Rightarrow \mathfrak{A}$, where $\mathfrak{A} : \mathsf{CFLoc} \to \mathsf{Vec}$ is a locally covariant QFT, which may or may not be the 'pullback' $\widetilde{\mathfrak{A}} \circ \mathfrak{p}$ of some theory $\widetilde{\mathfrak{A}} : \mathsf{CLoc} \to \mathsf{Vec}$. Explicitly, this means that, if $\mathscr{M}$ is the spacetime constructed from $\mathcal{M} \in \mathsf{CFLoc}$ according to (5.4), and likewise $\mathscr{N}$ arises from $\mathcal{N} \in \mathsf{CFLoc}$, then we have a pair of linear maps $\Phi_{\mathcal{M}/\mathcal{N}}$ such that, for any $\chi \in \mathrm{Hom}_{\mathsf{CFLoc}}(\mathcal{M};\mathcal{N})$, the following diagram commutes

$$\begin{tikzcd}
\mathfrak{D}(\mathscr{M}) \arrow[r, "\mathfrak{D}^{(h,\widetilde{h})}\chi"] \arrow[d, "\Phi_{\mathcal{M}}"] & \mathfrak{D}(\mathscr{N}) \arrow[d, "\Phi_{\mathcal{N}}"] \\
\widetilde{\mathfrak{A}}(\mathscr{M}) \arrow[r, "\widetilde{\mathfrak{A}}\chi"] & \widetilde{\mathfrak{A}}(\mathscr{N})
\end{tikzcd} \tag{5.13}$$

Heuristically, we can see how this definition relates to (5.1) by taking the 'limit' of $\Phi_{\mathcal{M}}(f)$ as $f \to \delta_x$, the Dirac delta distribution localised at $x \in M$. Whilst there is no guarantee that $\Phi_{\mathcal{M}}(f)$ converges in this limit, (5.12) *does* converge in the weak-$*$ topology to $\omega_\ell(x)^h \omega_r(x)^{\widetilde{h}} \delta_{\chi(x)}$. If we imagine for a moment that $\Phi_{\mathcal{M}}(x) := \lim_{f\to\delta_x} \Phi_{\mathcal{M}}(f)$ is well-defined, the statement that $\Phi$ is primary with weights $(h,\widetilde{h})$ implies

$$\mathfrak{A}\chi\Phi_{\mathcal{M}}(x) = \lim_{f\to\delta_x} \Phi_{\mathcal{N}}\left(\mathfrak{D}^{(h,\widetilde{h})}\chi f\right) = \omega_\ell(x)^h \omega_r(x)^{\widetilde{h}} \Phi_{\mathcal{N}}(\chi(x)). \tag{5.14}$$

Recalling that, if $\chi : \mathbb{M}_2 \to \mathbb{M}_2$ is expressed in null coordinates as $\chi(u,v) = (\mu(u),\nu(v))$, then $\omega_\ell = d\mu/du$ and $\omega_r = d\nu/dv$, we see that we have recovered a Lorentzian signature analogue of (5.1) as desired.

We can also recover the physical interpretations of the sum and difference of $h$ and $\widetilde{h}$, referred to as the *scaling dimension* $\Delta$ and *spin* $s$ of the field respectively. For the scalar field, we have already encountered the scaling dimension as the number $\Delta$ appearing in, for example, Definition 4.2.4. If we consider a field with spin $s = 0$, the action of the corresponding $\mathfrak{D}$ functor is

$$\mathfrak{D}^{(\Delta/2,\Delta/2)} f = \chi_*^{(2-\Delta)} f. \tag{5.15}$$

The right hand side of which is precisely the action of the functor $\mathfrak{D}^{(\Delta)}$ as defined in [Pin09]. Hence, any primary field *à la* Pinamonti's definition $\Phi : \mathfrak{D}^{(\Delta)} \Rightarrow \mathfrak{A}$ defines a primary field of spin $0$ in our description: $\widetilde{\Phi} : \mathfrak{D}^{(\Delta/2,\Delta/2)} \Rightarrow \mathfrak{A} \circ \mathfrak{p}$ where $\widetilde{\Phi}_{\mathcal{M}} := \Phi_{\mathscr{M}}$.

Conversely, a choice of spin 0 primary field $\widetilde{\Phi} : \mathfrak{D}^{(\Delta/2,\Delta/2)} \Rightarrow \mathfrak{A} \circ \mathfrak{p}$ unambiguously defines a natural transformation $\Phi : \mathfrak{D}^{(\Delta)} \Rightarrow \mathfrak{A}$. To see this, note that if $\mathcal{M}$ and $\widetilde{\mathcal{M}}$ represent different frames for the same spacetime $\mathcal{M} = \mathfrak{p}(\mathcal{M}) = \mathfrak{p}(\widetilde{\mathcal{M}})$, then the identity morphism of the underlying manifold constitutes a $\mathsf{CFLoc}$ morphism $\mathcal{M} \to \widetilde{\mathcal{M}}$, hence we can deduce from (5.13) that $\widetilde{\Phi}_{\mathcal{M}} \equiv \widetilde{\Phi}_{\widetilde{\mathcal{M}}}$. In other words, a non-vanishing spin represents an *obstruction* to a primary field in our sense being expressed in the frame-independent manner of [Pin09].

**Example 5.2.1.** The null derivative of the scalar field defines a map $\partial\Phi_{\mathcal{M}} : \mathfrak{D}(\mathcal{M}) \to \mathfrak{F}_{\mu c}(\mathcal{M})$

$$\partial\Phi_{\mathcal{M}}(f)[\varphi] = \int_{\mathcal{M}} f(x)(e_\ell\varphi)e^\ell \wedge e^r,$$

where $e_\ell$ is the vector field dual to $e^r$. To see that this is a primary field consider the upper-right path through the diagram (5.13):

$$\begin{aligned}\partial\Phi_{\mathcal{N}}\left(\mathfrak{D}^{(h,\widetilde{h})}\chi(f)\right)[\varphi] &= \int_{\chi(M)} \left(\chi^{-1}\right)^* \left(\omega_\ell^{h-1}\omega_r^{\widetilde{h}-1} f\right) \cdot (\tilde{e}_\ell\varphi)\,\tilde{e}^\ell \wedge \tilde{e}^r, \\ &= \int_M \left(\omega_\ell^{h-1}\omega_r^{\widetilde{h}-1} f\right) \cdot \chi^*(\tilde{e}_\ell\varphi)\,\left(\omega_\ell\,\omega_r e^\ell \wedge e^r\right).\end{aligned}$$

Next, using $\chi^*(\tilde{e}_\ell\varphi) = (\chi^*\tilde{e}_\ell)(\chi^*\varphi) = \omega_\ell^{-1}(e_\ell\chi^*\varphi)$ we have

$$\partial\Phi_{\mathcal{N}}\left(\mathfrak{D}^{(h,\widetilde{h})}\chi(f)\right)[\varphi] = \int_M \omega_\ell(x)^{h-1}\omega_r(x)^{\widetilde{h}} f(x)(e_\ell(\chi^*\varphi))e^\ell \wedge e^r.$$

To compare this with the lower-left path, we first observe that the algebra isomorphisms $\alpha_{\chi^*H'-H}$ all act by identity on linear functionals, thus if $\mathcal{F}$ is linear, $\mathfrak{A}\chi(\mathcal{F})[\varphi] = \mathcal{F}[\chi^*\varphi]$. Hence the observable we obtain in this way is

$$\mathfrak{A}\chi(\partial\Phi_{\mathcal{M}}(f))[\varphi] = \int_M f(x)(e_\ell(\chi^*\varphi))e^\ell \wedge e^r.$$

By fixing $(h, \widetilde{h})$ such that the diagram commutes, we can therefore conclude that $\partial\Phi$ is a primary field of weight $(1, 0)$. Similarly, if we consider the field $\bar{\partial}\Phi$, obtained by acting with $e_r$ instead of $e_\ell$, we would obtain a primary field of weight $(0, 1)$.

We can also consider the wide subcategory $\mathsf{CFLoc}_0$ comprising all the same spacetimes, but only those embeddings for which the conformal factors $\omega_\ell, \omega_r$ are constant. This category contains all the morphisms of $\mathsf{FLoc}$, which correspond to $\omega_\ell = \omega_r = 1$. The additional morphisms are generated by the *boosts and dilations*, defined, for $\Lambda \in \mathbb{R}_{>0}$ by

$$\begin{aligned}&b_\Lambda : (M, (e^\ell, e^r)) \mapsto (M, (\Lambda^{-1}e^\ell, \Lambda e^r)),\\ &d_\Lambda : (M, (e^\ell, e^r)) \mapsto (M, (\Lambda e^\ell, \Lambda e^r)),\end{aligned}$$

where in each case, the smooth embedding inducing the morphism is simply $\mathrm{Id}_M$. A *homogeneously scaling field of weight* $(h, \tilde{h})$ is then a natural transformation $\Phi : \mathfrak{D}^{(h,\tilde{h})}|_{\mathsf{CFLoc}_0} \Rightarrow \mathfrak{A}|_{\mathsf{CFLoc}_0}$. In other words, $\Phi$ responds to boosts and dilations in the same way a primary field would.

Given the underlying manifold is unchanged, both $\mathfrak{D}^{(h,\tilde{h})}(b_\Lambda(\mathcal{M}))$ and $\mathfrak{D}^{(h,\tilde{h})}(\mathcal{M})$, are simply $\mathfrak{D}(M)$. Upon making this identification, we have that $\mathfrak{D}^{(h,\tilde{h})}b_\Lambda \simeq \Lambda^{-(h-\tilde{h})}\mathbb{1}_{\mathfrak{D}(M)}$ and $\mathfrak{D}^{(h,\tilde{h})}d_\Lambda \simeq \Lambda^{h+\tilde{h}-2}\mathbb{1}_{\mathfrak{D}(M)}$. Similarly, $\mathfrak{A}b_\Lambda \simeq \mathfrak{A}d_\Lambda \simeq \mathbb{1}_{\mathfrak{A}(\mathcal{M})}$, where $\mathcal{M}$ is the spacetime corresponding to $\mathcal{M}$. This reduces the test for a field $\Phi$ to scale homogeneously to the equations

$$\Phi_{b_\Lambda(\mathcal{M})}(\Lambda^{-(h-\tilde{h})}f) = \Phi_{\mathcal{M}}(f), \quad \Phi_{d_\Lambda(\mathcal{M})}(\Lambda^{h+\tilde{h}-2}f) = \Phi_{\mathcal{M}}(f). \tag{5.16}$$

This concept is very similar to the concept of a *quasi-primary field*. However, one should note that the group of $\mathsf{CFLoc}_0$ automorphisms of $\mathbb{M}^2$ comprises only the proper, orthochronous Poincaré transformations and dilations. This is strictly less than the full group of Möbius transformations, $\mathrm{PSL}(2,\mathbb{R}) \times \mathrm{PSL}(2,\mathbb{R})$ under which quasi-primary fields transform nicely.

In order to describe the action of these Möbius transformations, note that the conformal compactification $\mathbb{M}^2 \to S^1 \times S^1$ is described in our framework by a conformally admissible embedding $\mathbb{M}^2 \hookrightarrow \mathcal{E}$, where the coordinate $\tilde{u}$ on the cylinder is the complex argument of $\frac{1+iu}{1-iu}$, the image of the corresponding coordinate on Minkowski under the Cayley map. Once this identification is made, Möbius transformations defined on the projective line $\mathbb{R} \cup \{\infty\}$ by

$$u \mapsto \frac{au+b}{cu+d}, \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{R}),$$

then yield well-defined $\mathsf{CLoc}$ automorphisms of $\mathcal{E}$. However, even a transformation as simple as $u \mapsto u + c$ for $c \in \mathbb{R}$ becomes highly non-trivial as an automorphism of the cylinder.

As such, our concept of a homogeneously scaling field is strictly weaker than that of a quasi-primary field. The concept still has some utility in its ability to specify the spin and scaling dimension of field. The former, amongst other things, can quantify the frame-dependence of a field, whilst the latter, with additional assumptions, can be used to impose constraints on the Poisson brackets/commutators of pairs of fields. It is likely that, one may be able to identify a subcategory of $\mathsf{CLoc}$ or $\mathsf{CFLoc}$ such that the restricted automorphism group of $\mathcal{E}$ is the full group of Möbius transformations one would expect. However, we shall not explore the issue further.

## 5.3 EXAMPLES OF PRIMARY FIELDS

For the massless scalar field, we identify several notable examples of primary and homogeneously scaling fields below:

1. As demonstrated in the above example, the derivative fields $\partial\Phi$ and $\bar{\partial}\Phi$ are both primary. Taking higher derivatives will produce homogeneously scaling fields of increasing weight: $\partial^n\bar{\partial}^m\Phi$ is homogeneously scaling with weight $(n, m)$, though note that if both $n$ and $m$ are non-zero, this field vanishes on-shell.
2. Higher powers of primary fields are again primary classically, but when quantised, they may fail to be even homogeneously scaling in general. The stress-energy tensor is a special case, which we discuss in the remark below.
3. The (smeared) vertex operator $e^{ia\Phi}_{\mathcal{M}}(f)$ defined, for $f \in \mathfrak{D}(\mathcal{M})$, $a \in \mathbb{R}$ by

   $$e^{ia\Phi}_{\mathcal{M}}(f)[\varphi] := \int_M f(x)e^{ia\varphi(x)}\,\mathrm{dVol},$$

   classically is neither primary nor homogeneously scaling. However, the covariantly normal-ordered field $:e^{ia\Phi}:$ is a quantum primary with spin $0$ and scaling dimension $\frac{\hbar a^2}{2\pi}$

   To see this, consider the lower-left path of (5.13). For $f \in \mathfrak{D}(\mathcal{M})$, $\varphi \in \mathfrak{E}(\mathcal{N})$, $H \in \mathrm{Had}(\mathcal{M})$, and $H' \in \mathrm{Had}(\mathcal{N})$, we have

   $$\mathfrak{A}\chi\left(:e^{ia\Phi}(f):_{\mathcal{M}}\right)_{H'}[\varphi] = \sum_{n=0}^{\infty}\left(\frac{\hbar}{2}\right)^n\frac{1}{n!}\left\langle\left(\chi^*H' - H^{\mathrm{sing}}_{\mathcal{M}}\right)^{\otimes n}, e^{ia\Phi}_{\mathcal{M}}(f)^{(2n)}[\chi^*\varphi]\right\rangle. \tag{5.17}$$

   The functional derivatives of $e^{ia\Phi}_{\mathcal{M}}$ can be calculated straightforwardly, and yield, for any $n \in \mathbb{N}$

   $$\begin{aligned}\left\langle\left(\chi^*H' - H^{\mathrm{sing}}_{\mathcal{M}}\right)^{\otimes n}, e^{ia\Phi}_{\mathcal{M}}(f)^{(2n)}[\chi^*\varphi]\right\rangle = \\ (-a^2)^n\int_M e^{ia\chi^*\varphi}f(x)\Big(\lim_{y\to x}\chi^*H'(x;y) - H^{\mathrm{sing}}_{\mathcal{M}}(x;y)\Big)^n\,\mathrm{dVol} \\ = \int_M e^{ia\chi^*\varphi}f(x)\Big(-a^2\chi^*h'(x;x) + \frac{a^2}{4\pi}\log(\Omega(x))\Big)^n\,\mathrm{dVol}.\end{aligned} \tag{5.18}$$

   Here, $h'$ is the smooth part of $H'$, and the $\log(\Omega(x))$ term arises from the difference in the local Hadamard form (2.78) of $\chi^*H'$ and $H^{\mathrm{sing}}_{\mathcal{M}}$ (see the following remark for details). We can then express the action of the morphism $\mathfrak{A}\chi$ as

   $$\mathfrak{A}\chi\left(:e^{ia\Phi}(f):_{\mathcal{M}}\right)_{H'}[\varphi] = e^{ia\Phi}_{\mathcal{M}}\left(fe^{\left(-\hbar\frac{a^2}{2}(\iota_\Delta\circ\chi)^*h'\right)}\Omega^{\hbar\frac{a^2}{4\pi}}\right)[\chi^*\varphi], \tag{5.19}$$

where $\iota_\Delta(x) = (x, x)$, and we are using the linearity of $e^{ia\Phi}_{\mathcal{M}}$ in the test function to extend it [1] to a map $\mathfrak{D}(\mathcal{M})[[\hbar]] \to \mathfrak{F}_{\mu c}(\mathcal{M})[[\hbar]]$.

We can compare this to $:e^{ia\Phi}_{\mathcal{N}}:$, where we have, for $g \in \mathfrak{D}(\mathcal{N})$

$$\begin{aligned} :e^{ia\Phi}_{\mathcal{N}}(g):_{H'}[\varphi] &= \sum_{n=0}^{\infty} \left(\frac{\hbar}{2}\right)^n \frac{1}{n!} \left\langle h'^{\otimes n}, e^{ia\Phi}_{\mathcal{N}}(g)^{(2n)}[\varphi] \right\rangle, \\ &= \left\langle e^{-\hbar \frac{a^2}{2} h'_\Delta}, e^{ia\Phi}_{\mathcal{N}}(g) \right\rangle, \\ &= e^{ia\Phi}_{\mathcal{N}} \left( g e^{-\hbar \frac{a^2}{2} h'_\Delta} \right) [\varphi]. \end{aligned}$$

As $e^{ia\Phi}$ is a classical primary field of scaling dimension $0$, we have $e^{ia\Phi}_{\mathcal{M}}(f)[\chi^*\varphi] = e^{ia\Phi}_{\mathcal{N}}(\chi_* \Omega^{-d} f)[\varphi]$, hence

$$\mathfrak{A}\chi \left( :e^{ia\Phi}_{\mathcal{M}}(f): \right)_{H'} [\varphi] = :e^{ia\Phi}_{\mathcal{N}} \left( \mathfrak{D}^{\left(\frac{\hbar a^2}{4\pi}\right)}(f) \right) :_{H'}[\varphi]$$

as required.

*Remark* 5.3.1. The prefactor $V$ in (2.78) is a little tricky. In order to analyse it effectively, we can use 'special double null coordinates' [Bor98; Kay01] $u', v'$ such that

$$ds^2 = (1 + Au'^2 + Bu'v' + Cv'^2 + \mathcal{O}(3))du'dv', \tag{5.20}$$

where $\mathcal{O}(3)$ denotes terms of order at least $3$ in $u'$ and $v'$. In this system, one can then express $V_N$ for $N \geq 3$ and mass $m$ as

$$V_N(u'_1, v'_1; u'_2, v'_2) = 1 - \frac{m^2}{2}(u'_2 - u'_1)(v'_2 - v'_1) + \mathcal{O}(3). \tag{5.21}$$

In any case we clearly see that the coincidence limit of $V$ appearing when testing the naturality of $:e^{ia\Phi}:$ is $1$. Moreover, we can use this form to prove that the normally ordered stress-energy tensor $:T:$ is homogeneously scaling. We already know the necessary weights from the fact that $T$ is a classical primary of weight $(2, 0)$, hence for a dilation $d_\Lambda$, we must show that

$$:T:_{d_\Lambda(\mathcal{M})}(\Lambda^{h+\tilde{h}-2} f) = :T:_{\mathcal{M}}(f). \tag{5.22}$$

We already know the classical terms agree, thus we need only check the $\mathcal{O}(\hbar)$ term, which reduces to the condition that

$$\left\langle H^{\text{sing}}_{d_\Lambda(\mathcal{M})} - H^{\text{sing}}_{\mathcal{M}}, T_{\mathcal{M}}(f)^{(2)} \right\rangle = 0. \tag{5.23}$$

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[1] In doing so, we avoid any necessity to prove summation and integration may be interchanged, or that $\mathrm{Exp}(\hbar(A + B \log C)) = \mathrm{Exp}(\hbar A) C^{\hbar B}$. If one is not comfortable with such manipulations of formal series, reassurance may be found in the fact that, if the field configuration $\varphi$ is held fixed, and $\hbar$ is chosen to be any positive number, then the series (5.17) converges absolutely, as a series of complex numbers, to the right hand side of (5.19).

Using the Hadamard recurrence relations [DB60], one can deduce that $V_N$ is invariant under constant scalings, hence

$$H^{\mathrm{sing}}_{d_\Lambda(\mathcal{M})} - H^{\mathrm{sing}}_{\mathcal{M}} = V_N(x, y) \log(\Lambda^2). \tag{5.24}$$

Given that $e_\ell = e_{\ell,u'}(u', v')\partial_{u'}$ for some $e_{\ell,u'} \in \mathfrak{E}(M)$, we can then use the above form for $V_N$ to show that $\lim_{x\to y}(e_\ell \otimes e_\ell)V_N(x, y) = 0$, and hence that (5.23) holds $\forall\, f \in \mathfrak{D}(M)$.

# — 6 —

# The Stress-Energy Tensor of the Massless Scalar Field

A well known feature of chiral CFTs is the transformation law for the stress-energy tensor, constrained by the famous Lüscher-Mack theorem [LM76] Here we shall show explicitly that, for the free scalar field in 2D Minkowski space, the stress-energy tensor satisfies precisely this transformation law. Moreover, we shall also see that there exist analogous transformation laws on arbitrary globally-hyperbolic spacetimes.

The $uu$ component of the stress-energy tensor[1] on a framed spacetime $\mathcal{M} = (M, (e^\ell, e^r)) \in$ CFLoc, is a distribution valued in $\mathfrak{F}_{\text{loc}}(M)$ defined, for $f \in \mathfrak{D}(M)$, $\varphi \in \mathfrak{E}(M)$ by

$$T_{\mathcal{M}}(f)[\varphi] := \frac{1}{2} \int_M f \cdot (e_\ell \varphi)^2 e^\ell \wedge e^r. \tag{6.1}$$

Note that we can replace the test function $f$ with a compactly supported distribution, so long as its singularity structure is compatible with the constraint that $T_{\mathcal{M}}(f)$ is a microcausal distribution. In particular, the generators of the Virasoro algebra $B_n$ from section 3.3 can be expressed as $T_{\mathcal{E}}(f_n)$, where the integral kernel of $f_n$ is $e^{inu}\delta(u + v)$ in the null-coordinates for the cylinder.

Classically, $T$ is a primary field with conformal weight $(2, 0)$, i.e. $T : \mathfrak{D}^{(2,0)} \Rightarrow \mathfrak{P} \circ \mathfrak{p}$, where $\mathfrak{P}$ is the classical theory for the massless scalar field, as given in definition 4.2.4.

However, when quantised, $:T:$ picks up obstructions which prevent the necessary diagram from commuting in general.

Before we study the transformation properties of the stress-energy tensor restricted to Minkowski space, we are now in a position to address a comment made earlier about finding generators of the Virasoro algebra on Minkowski space. On Minkowski space, one is often able to consider a broader class of test functions with which to

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[1] We may also refer to $T_{uu}$ as the *chiral* component of $T$, in which case $T_{vv}$ would be the *anti-chiral* component. For ease of notation, we consider only the chiral component, dropping the subscript.

smear quantum fields. For instance, in Wightman field theory, one typically uses the space of Schwartz functions $\mathfrak{S}(\mathbb{M}^n)$ defined above from proposition 2.4.5.

In particular, if one is able to extend the domain of $T_{\mathbb{M}^2} : \mathfrak{D}(\mathbb{M}^2) \to \mathfrak{F}_{\mathrm{loc}}(\mathbb{M}^2)$ to include functions of the form $(1+iu)^{n-1}(1-iu)^{-n-1}$ for $n \in \mathbb{Z}$, then one would expect [FST89, §2.3] the resulting observables to commute according to the Virasoro relations (after quantisation). However, if we focus on the classical algebra, we can quickly see that such observables are in fact simply the generators of the Einstein cylinder, pulled back to $\mathbb{M}^2$, adding further justification to our claim that the Einstein cylinder is the natural choice of spacetime to focus on in our framework.

Consider the Cayley map $\mathbb{R} \to S^1 \subset \mathbb{C}$ defined by $u \mapsto \left(\frac{u-i}{u+i}\right)$. Taking the complex argument of this number, and applying the same map to $v$, we define a conformal embedding $\mathbb{M}^2 \hookrightarrow \mathcal{E}$

$$\chi(u,v) = \left[\arg\left(\frac{1+iu}{1-iu}\right), \arg\left(\frac{1+iv}{1-iv}\right)\right].$$

The image of this map is a maximal simply-connected causal diamond, containing all but a singular point of the $t = 0$ Cauchy surface in $\mathcal{E}$. Its conformal factors are $\omega_\ell(u,v) = \omega_r(v,u) = \partial_u\left(\arctan\left(\frac{-2u}{u^2-1}\right)\right) = \frac{2}{(1+iu)(1-iu)}$.

We shall discuss how precisely to place $T$ 'on a null-ray' in part II. For now, it shall suffice to say that we may identify $T(f)$ with $T(g)$ if $\int_{-\infty}^{\infty} f(u,v)dv \equiv \int_{-\infty}^{\infty} g(u,v)dv$. In particular, we define a family $f_n \in \mathfrak{E}(\mathbb{M}^2)$ by

$$f_n(u,v) = \frac{4\pi}{v^2+1}(1+iu)^{n-1}(1-iu)^{-n-1}. \tag{6.2}$$

This is equivalent in this new sense to the modes given above, and if we take its weighted pushforward, we see that

$$\mathfrak{D}^{(2,0)}\chi f_n(u,v) = e^{inu}. \tag{6.3}$$

If we assume that $T$ is still natural under this expanded set of test functions, we may then conclude that, up to equivalence $\mathfrak{P}\chi T_{\mathbb{M}^2}(f_n)$ coincides with $B_n$.

In order to make our analysis more concrete, we restrict our attention to the subcategory of CFLoc containing the single object $\mathbb{M}_2$. Here, the locally covariant normal ordering prescription $: - :_{\mathbb{M}_2}$ is simply $\overset{\circ}{\circ} - \overset{\circ}{\circ}_{H_{\mathbb{M}}}$, where $H_{\mathbb{M}}$ is the symmetric part of the Minkowski vacuum. Hence, if we work in the concrete algebra $\mathfrak{A}^{H_{\mathbb{M}}}(\mathbb{M})$ we can identify $T_{\mathbb{M}_2}(f)$ directly with its quantum counterpart with no modification.

Given a CFLoc morphism $\chi : \mathbb{M}_2 \to \mathbb{M}_2$, if the covariantly ordered field $:T:$ was primary, we would expect in particular that $\mathfrak{A}\chi\left(:T:_{\mathbb{M}_2}(f)\right) - :T:_{\mathbb{M}_2}\left(\mathfrak{D}^{(2,0)}\chi f\right)$ would vanish. Upon making the identification $\mathfrak{A}(\mathbb{M}_2) \simeq \mathfrak{A}^{H_\mathbb{M}}(\mathbb{M}_2)$ this term becomes

$$\alpha_{\chi^* H_\mathbb{M} - H_\mathbb{M}} \left(T_{\mathbb{M}_2}(f)\right) - T_{\mathbb{M}_2}\left(\mathfrak{D}^{(2,0)}\chi f\right). \tag{6.4}$$

We already know that this vanishes in the classical limit $\hbar \to 0$, hence we only need to compute the $\mathcal{O}(\hbar)$ term. Recall that in null coordinates we can express a $\mathsf{CFLoc}$ morphism $\mathbb{M}_2 \to \mathbb{M}_2$ using a pair of functions $\mu, \nu \in \mathrm{Diff}_+(\mathbb{R})$ by $\chi(u, v) = (\mu(u), \nu(v))$. Upon doing so we see

$$\begin{aligned}\left\langle (\chi^* H_{\mathbb{M}_2} - H_{\mathbb{M}_2}), T_{\mathbb{M}_2}(f)^{(2)} \right\rangle &= \\ &\int_{\mathbb{R}^2} \partial_u \partial_{u'} \Big[ H_{\mathbb{M}_2}(\mu(u); \mu(u')) - H_{\mathbb{M}_2}(u; u') \Big] f(u) \delta(u - u') \, \mathrm{d}u \mathrm{d}u',\end{aligned} \tag{6.5}$$

where we have integrated out $v$ and $v'$ and defined $f(u) := \int_\mathbb{R} f(u, v) \, \mathrm{d}v$. It only remains to determine

$$\begin{aligned}\lim_{u' \to u} \Big[ \mu'(u)\mu'(u')(H_{\mathbb{M}_2})_{uu'}(\mu(u); \mu(u')) - (H_{\mathbb{M}_2})_{uu'}(u; u') \Big] &= \\ \lim_{u' \to u} \left[ \frac{\mu(u)\mu(u')}{(\mu(u) - \mu(u'))^2} - \frac{1}{(u - u')^2} \right].\end{aligned} \tag{6.6}$$

By Taylor expanding $\mu(u')$ around $u$, one eventually finds that the limit exists and is equal to

$$\frac{1}{6}\left( \frac{\mu'''(u)}{\mu'(u)} - \frac{3}{2}\left( \frac{\mu''(u)}{\mu'(u)} \right)^2 \right) =: \frac{1}{6} S(\mu)(u), \tag{6.7}$$

where $S(\mu)$ denotes the Schwarzian derivative of the function $\mu$. From this it is clear that $:T:$ is not primary, as

$$\mathfrak{A}\chi\left(:T:_{\mathbb{M}_2}(f)\right) = :T:_{\mathbb{M}_2}\left(\mathfrak{D}^{(2,0)}\chi(f)\right) - \frac{1}{4\pi}\frac{\hbar}{12}\left\langle S(\mu), f \right\rangle. \tag{6.8}$$

Thus we recover the well-known result that, on Minkowski spacetime, the quantum stress-energy tensor transforms almost as a primary of weight $(2, 0)$, but is obstructed by an $\mathcal{O}(\hbar)$ correction proportional to the Schwarzian derivative of the transformation. We can now use our framework to generalise this result to any globally hyperbolic spacetime. The failure for (5.13) to commute for $\chi \in \mathrm{Hom}_{\mathsf{CFLoc}}(\mathcal{M}; \mathcal{N})$ is

$$\left\langle \widetilde{S}(\chi), f \right\rangle = \mathfrak{A}\chi\left(:T:_\mathcal{M}(f)\right) - :T:_\mathcal{N}\left(\mathfrak{D}^{(2,0)}\chi(f)\right). \tag{6.9}$$

Whilst the right hand side of this equation requires an arbitrary choice of $H' \in \mathrm{Had}(\mathcal{N})$ and $\varphi \in \mathfrak{E}(\mathcal{N})$, $\widetilde{S}$ is actually independent of both of these choices. As in Minkowski space, the classical term cancels and we are left to compute

$$\left\langle \widetilde{S}(\chi), f \right\rangle = \frac{\hbar}{2}\left[ \left\langle \chi^* H' - H^{\mathrm{sing}}_{\mathcal{M}}, T_\mathcal{M}(f)^{(2)} \right\rangle - \left\langle H' - H^{\mathrm{sing}}_{\mathcal{N}}, T_\mathcal{N}\left(\mathfrak{D}^{(2,0)}\chi(f)\right)^{(2)} \right\rangle \right],$$

where the choice of configuration $\varphi$ has been suppressed as no remaining terms depend on it. If we define $h' = H' - H^{\text{sing}}_{\mathcal{N}}$, then one can show that $\left\langle h', T_{\mathbb{N}}\left(\mathfrak{D}^{(2,0)}\chi(f)\right)^{(2)}\right\rangle = \left\langle \chi^* h', T_{\mathbb{M}}(f)^{(2)}\right\rangle$, which cancels with the smooth part of $\chi^* H'$, and hence

$$\widetilde{S}(\chi) = \frac{\hbar}{2}\iota^*_{\Delta}\left((e_\ell \otimes e_\ell)\left(\chi^* H^{\text{sing}}_{\mathcal{N}} - H^{\text{sing}}_{\mathcal{M}}\right)\right), \tag{6.10}$$

where we are again using the embedding $\iota_\Delta : x \mapsto (x, x) \in \mathcal{M}^2$. If we take $\chi : \mathbb{M}_2 \to \mathbb{M}_2$ to be as above, we then see that $\widetilde{S}(\chi) = S(\mu)$, hence the original Schwarzian derivative is recovered.

Note that the right-hand side of (6.9) can be defined for *any* confomally covariant QFT. A *Lüscher-Mack theorem* for pAQFT would then imply that, as a distribution, this is equal to (6.10) up to multiplication by some constant, which we could then interpret as the central charge of the theory. We stress that such a result has not yet been found, however we intend to return to this issue in future work.

# Part II

# Chirality

## — 7 —

# Where do Chiral Fields Live?

### 7.1 THE GEOMETRY OF CHIRAL FIELDS

An almost universal assumption in the study of conformal field theory in 2 dimensions is that chiral fields 'live' on Riemann surfaces. This is a powerful assumption, as it enables one to use the tools of complex analysis to tackle certain problems. For instance, the tricky functional analysis of distributions is replaced by either complex analysis or the algebraic manipulation of Laurent distributions.

The use of Riemann surfaces in chiral field theory is typically justified in one of two ways: One can perform a Wick rotation from 2D Minkowski space $\mathbb{M}^2$ to Euclidean space. By introducing the complex variable $z = x + it$, this space is then identified with the complex plane. Conformal symmetry of the original theory suggests that only the *conformal class* of the Riemannian metric $dzd\bar{z}$ matters. Thus, a more general theory might be defined on an arbitrary Riemann surface. The fact that this Wick rotated theory still describes what we began with is then a consequence of the Osterwalder-Schrader theorem [OS73], which gives sufficient conditions for the $n$-point correlator functions of the Wick rotated theory to be analytically continued back to the original spacetime.

An alternative justification, found, for example in [Kac98, Chapter 1], begins by describing a spacetime event using the null coordinates $(u, v) = (t - x, t + x)$. One may complexify each of these coordinates independently. This is possible in a conformally symmetric Wightman QFT because the null momentum operators $P_u, P_v$ have a jointly positive spectrum, allowing the translation operator $e^{i(qP_u + rP_v)}$ to be defined for values of $q$ and $r$ lying in the upper complex half-plane $\mathbb{H} \subset \mathbb{C}$. This leads to an embedding of $\mathbb{M}^2$ as the boundary of $\mathbb{H}^2 \subset \mathbb{C}^2$. We then identify the chiral/anti-chiral fields as those which depend on only one copy of $\mathbb{H}$. By restricting attention to one or the other, the result is a conformally symmetric theory defined over $\mathbb{H}$.

Since this story was developed, it has emerged that Lorentzian signature QFT is far from restricted to Minkowski spacetime. As we have seen, so long as a spacetime $\mathcal{M}$ is *globally hyperbolic*, there exists a general method for constructing a $*$-algebra of quantum observables $\mathfrak{A}(\mathcal{M})$. (Even this requirement may be relaxed in certain circumstances, e.g. if the spacetime is *locally* globally hyperbolic in a particular sense [Kay92].)

A crucial feature of AQFT is that the algebra of observables is constructed *before* a Hilbert space of states. This is preferable because, although for most physically reasonable theories 'vacuum-like' states can always be found on globally hyperbolic spacetimes, they cannot be obtained in a systematic manner [Few15, Theorem 5.4] and such states ultimately depend on global geometric features, of which local observables ought to be ignorant. However, this does mean that our approach must necessarily be quite different to that which is usual in the study of 2DCFT.

In this part of the thesis, we give a characterisation of chirality on any two-dimensional globally hyperbolic Lorentzian manifold. To begin, we consider why the two stories above are ill-suited for this generalisation.

The idea of Wick rotating the time coordinate seems appealing, however it is well known that there is no analogue of the Osterwalder-Schrader theorem on curved spacetimes. Moreover, as Wick rotation is defined on the level of correlator functions, it is currently unknown how this procedure may be carried out in a purely algebraic setting.

An attempt to analytically continue the translation operators also suffers from the lack of a preferred state on curved spacetimes. In fact, by employing the joint spectrum of momentum operators, this approach assumes not only to the existence of a preferred state, but in particular the existence of a *highly symmetric* preferred state.

However, there is one aspect of this second approach that *can* be generalised to an arbitrary spacetime: as we shall see, it is possible to identify a chiral/anti-chiral sector of a conformally covariant QFT which is defined on a one-dimensional manifold, without the use of a preferred system of coordinates.

Our first example of a chiral field arises from the theory of a massless scalar field on 2D Minkowski spacetime. Recall that, in null coordinates, the equation of motion of this theory is simply

$$\partial_u \partial_v \varphi =: P\varphi = 0. \tag{7.1}$$

The solution of this equation is famously the sum of two terms: one which depends only on the $u$ coordinate and one solely dependent on $v$. To understand the geometry of the spaces the decoupled coordinates describe, consider the Einstein cylinder $\mathcal{E}$, which we recall is defined as the quotient of $\mathbb{M}^2$ under the equivalence relation

$$(t, x) \sim (t, x + 2\pi). \tag{7.2}$$

In null coordinates, all functions on the cylinder satisfy the periodicity condition $\varphi(u - 2\pi, v + 2\pi) = \varphi(u, v)$. We can write a general solution of (7.1) as

$$\varphi(u, v) = \varphi_\ell(u) + \varphi_r(v) + \frac{p}{2\pi}(u + v), \tag{7.3}$$

where $p \in \mathbb{R}$ is a constant and each term $\varphi_{\ell/r}$ must be $2\pi$ periodic. In other words, a general solution to the wave equation on the cylinder is determined by a pair of functions $\varphi_{\ell/r} \in \mathfrak{E}(S^1)$ and the constant $p$.

We can describe this phenomenon without coordinates in the following manner: consider the map $\pi_\ell$ which sends an event $(u, v) \in \mathbb{M}^2$ to the line $\{(u, v') \in \mathbb{M}^2\}_{v' \in \mathbb{R}}$. Each such line is determined uniquely by the choice of $u$, and the space $\mathbb{M}_\ell$ of all such lines is then diffeomorphic to $\mathbb{R}$. Defining the analogous map on the cylinder, we again have that $\pi_\ell[u, v] = \pi_\ell[u, v']$ for all $v, v'$, but we have a further identification that $\pi_\ell[u, v] = \pi_\ell[u + 2\pi, v]$, which tells us that the space $\mathcal{E}_\ell$ of all such lines is actually one-to-one with $\mathbb{R}/2\pi\mathbb{Z} \simeq S^1$.

Thus we can describe solutions to the wave equation on either Minkowski space or the cylinder in the same language: for $\mathcal{M} \in \{\mathbb{M}^2, \mathcal{E}\}$, let $\mathcal{M}_{\ell/r}$ denote respectively the spaces of right / left moving null rays, for instance

$$\mathbb{M}^2_\ell \ni \gamma = \Big\{(u_0, v) \in \mathbb{M}^2 \,|\, v \in \mathbb{R}\Big\} \tag{7.4}$$

for some fixed $u \in \mathbb{R}$. We also denote by $\pi_{\ell/r} : \mathcal{M} \to \mathcal{M}_{\ell/r}$ the obvious surjections onto these spaces. We can then write a class of solutions to the wave equation on $\mathcal{M}$ as $\varphi = \pi_\ell^* \varphi_\ell + \pi_r^* \varphi_r$. [1]

We can define these spaces also on arbitrary spacetimes. Recall from definition 2.5.1 that a spacetime is a tuple $\mathcal{M} = (M, g, \mathfrak{o}, \mathfrak{t})$ comprising, in order, a manifold $M$, a

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[1] Note that for $\mathcal{M} = \mathcal{E}$, this method only generates the solutions for $p = 0$. As we are only hoping to find subalgebras of observables, it is not necessary to generate *all* the solutions to the equation of motion. However, the missing solutions would have to be found if one wished to reconstruct the full spacetimes' observables from the chiral subalgebras.

Lorentzian metric $g$ on $M$, an orientation $\mathfrak{o}$ (i.e. an equivalence class of nowhere-vanishing 2-forms) and a time-orientation $\mathfrak{t}$ (which may be defined as an equivalence class of timelike vector fields which are all future-pointing).

**Definition 7.1.1.** Consider an inextensible null geodesic $\gamma : \mathbb{R} \to \mathcal{M}$ which is everywhere future-directed according to $\mathfrak{t}$. We can call $\gamma$ *left-moving* if, for $o \in \mathfrak{o}$ and $t \in \mathfrak{t}$, $o(t \otimes \dot{\gamma}) < 0$, otherwise we call $\gamma$ *right-moving*.

Note, we never have $o(t \otimes \dot{\gamma}) = 0$, because $\gamma$ being null implies $\dot{\gamma}$ is never colinear with $t$, and $o$ is nowhere-vanishing, and hence non-degenerate.

If we then identify reparametrisations (i.e. we consider only the image of a geodesic), we can define a set $\mathcal{M}_\ell$ of *right*-moving, inextensible null geodesics and similarly $\mathcal{M}_r$, the space of *left*-moving geodesics. Note the apparent mismatch is so that elements of $\mathcal{M}_\ell$ are identified by 'left-moving' coordinates on $\mathcal{M}$ and *vice-versa*. For $\mathcal{M} \in \{\mathbb{M}^2, \mathcal{E}\}$, these spaces are clearly analogous to the spaces of null rays introduced above. By a slight abuse of the terminology, we shall refer to elements of $\mathcal{M}_{\ell/r}$ as 'geodesics'. The maps $\pi_{\ell/r}$ from before generalise to this setting by defining $\pi_{\ell/r}(x) \in \mathcal{M}_{\ell/r}$ as the equivalence class of the inextensible right/left moving null geodesic $\gamma$ such that $\gamma(0) = x$.

As we are studying conformal field theories, it is necessary to know how these spaces behave under the conformally admissible embeddings from definition 4.0.1. A useful characterisation of conformal embeddings is that they preserve null geodesics. The admissibility (i.e. orientation-preserving) property further indicates that left-moving geodesics are mapped to left-moving geodesics and right to right. More precisely, we have the following proposition:

**Proposition 7.1.2.** *Let $\chi : \mathcal{M} \to \widetilde{\mathcal{M}}$ be a conformally admissible embedding, then there exist natural maps $\chi_{\ell/r} : \mathcal{M}_{\ell/r} \to \widetilde{\mathcal{M}}_{\ell/r}$ such that*

$$\widetilde{\pi}_{\ell/r} \circ \chi = \chi_{\ell/r} \circ \pi_{\ell/r}. \tag{7.5}$$

*Hence the maps $\pi_{\ell/r}$ define a functor $\mathsf{CLoc} \to \mathsf{Man}^1_+$, where $\mathsf{Man}^1_+$ is the category of smooth, oriented 1-manifolds with smooth, oriented embeddings as morphisms.*

*Proof.* From the definition of $\pi_{\ell/r}$, we have that $\pi_{\ell/r}(x) = \pi_{\ell/r}(x')$ if and only if there is a right/left-moving null geodesic $\gamma$ connecting $x$ with $x'$. As $\chi$ is conformally admissible, $\chi \circ \gamma$ will also be a right/left-moving null geodesic connecting $\chi(x)$ with $\chi(x')$ hence $\widetilde{\pi}_{\ell/r} \circ \chi(x) = \widetilde{\pi}_{\ell/r} \circ \chi(x')$. This means we can define $\chi_{\ell/r}(\gamma)$ to be $\pi_{\ell/r} \circ \chi(x)$ for any $x$

such that $\pi_{\ell/r}(x) = \gamma$, as this definition does not depend on our choice of $x$. Clearly this definition makes $\chi_{\ell/r}$ injective, and satisfies (7.5).

Statements about smoothness and orientability can be verified once we have defined the smooth structure and orientation on $\mathcal{M}_{\ell/r}$ in the following discussion. For now, we shall conclude by showing that, for a sequence of conformally admissible embeddings $\mathcal{M}_1 \overset{\chi_1}{\hookrightarrow} \mathcal{M}_2 \overset{\chi_2}{\hookrightarrow} \mathcal{M}_3$, if we denote the corresponding maps to null geodesics by $\pi_{i,\ell/r} : \mathcal{M}_i \to (\mathcal{M}_i)_{\ell/r}$, and the maps we have just defined by $\chi_{i,\ell/r} : (\mathcal{M}_i)_{\ell/r} \to (\mathcal{M}_{i+1})_{\ell/r}$, then $(\chi_2 \circ \chi_1)_{\ell/r} = \chi_{2,\ell/r} \circ \chi_{1,\ell/r}$.

Firstly, by (7.5), we have that $\pi_{3,\ell/r} \circ (\chi_2 \circ \chi_1) = (\chi_2 \circ \chi_1)_{\ell/r} \circ \pi_{1,\ell/r}$. Secondly, from the commutativity of the following diagram, we have that $\pi_{3,\ell/r} \circ (\chi_2 \circ \chi_1) = (\chi_{2,\ell/r} \circ \chi_{1,\ell/r}) \circ \pi_{1,\ell/r}$.

$$\begin{tikzcd}
\mathcal{M}_1 \arrow[r, "\chi_1"] \arrow[d, two heads, "\pi_{1,\ell/r}"] & \mathcal{M}_2 \arrow[r, "\chi_2"] \arrow[d, two heads, "\pi_{2,\ell/r}"] & \mathcal{M}_3 \arrow[d, two heads, "\pi_{3,\ell/r}"] \\
(\mathcal{M}_1)_{\ell/r} \arrow[r, "\chi_{1,\ell/r}"] & (\mathcal{M}_2)_{\ell/r} \arrow[r, "\chi_{2,\ell/r}"] & (\mathcal{M}_3)_{\ell/r}
\end{tikzcd}$$

As $\pi_{i,\ell/r}$ is surjective, it is also right-cancelative. Hence we may deduce from $(\chi_2 \circ \chi_1)_{\ell/r} \circ \pi_{1,\ell/r} = (\chi_{2,\ell/r} \circ \chi_{1,\ell/r}) \circ \pi_{1,\ell/r}$ the desired equality. □

The spaces $\mathcal{M}_{\ell/r}$ are somewhat awkward to work with directly. Instead it is easier to note that, given a spacelike Cauchy surface $\Sigma \subset \mathcal{M}$, the restriction $\pi_{\ell/r}|_\Sigma$ becomes a bijection. Because the elements of $\mathcal{M}_{\ell/r}$ are in particular inextensible causal curves this is true even on an arbitrary globally hyperbolic spacetime[2].

These maps endow $\mathcal{M}_{\ell/r}$ with differentiable structures as well as orientations, independent of the choice of $\Sigma$. This is well-defined, as any pair of Cauchy surfaces $\Sigma, \Sigma' \subset \mathcal{M}$ of the same spacetime are diffeomorphic to one another, as a consequence of [BS03, Theorem 1.1]. In particular, we shall actually define the orientation of $\mathcal{M}_\ell$ such that each diffeomorphism $\pi_\ell|_\Sigma : \Sigma \overset{\simeq}{\to} \mathcal{M}_\ell$ *reverses* orientation, this is motivated by the example of the $t = 0$ Cauchy surface $\Sigma_0 \subset \mathbb{M}^2$, where the spatial coordinate $x \in \Sigma_0$ corresponds to the $u = -x$ null ray in $\mathbb{M}^2_\ell$. Given $\chi : \mathcal{M} \to \widetilde{\mathcal{M}}$ from before we

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[2] For the remainder of this thesis, we shall assume implicitly that all Cauchy surfaces are both smooth and spacelike. (Recall that [BS03, Theorem 1.1] guarantees the existence of such Cauchy surfaces for every globally hyperbolic spacetime.)

have that, for any Cauchy surface $\Sigma \subset \mathcal{M}$

$$\chi_{\ell/r} = \widetilde{\pi}_{\ell/r} \circ \chi \circ \pi_{\ell/r}|_{\Sigma}^{-1}. \tag{7.6}$$

From which we can deduce that each map is smooth and oriented.

However, despite each $\Sigma$ inheriting a Riemannian metric from $\mathcal{M}$, it is clear that different Cauchy surfaces would yield different metrics on $\mathcal{M}_{\ell/r}$, hence we must be able to show that any algebra constructed on a Cauchy surface $\Sigma$ is in some sense independent of this metric.

In the sections that follow, we shall build a configuration space, and hence both classical and quantum algebras of observables, using arbitrarily selected Cauchy surfaces as the underlying space. Clearly, it will be important to establish that our constructions do not depend in any significant way upon this choice.

Not only should we expect diffeomorphic Cauchy surfaces to yield isomorphic algebras, but we should also expect a 'reparametrisation invariance', where any algebra constructed over a surface $\Sigma$ should carry an action of $\mathrm{Diff}_+(\Sigma)$ by automorphisms.

It is well known that every 2D Lorentzian manifold is conformally flat, i.e. they are locally conformally isometric to $\mathbb{M}^2$. However, this does not mean that every spacetime is 'the same as' Minkowski from the perspective of a CFT. A recent work by Benini, Giorgetti and Schenkel [BGS21] explains in detail the manner in which the category CLoc can be replaced by a *skeletal* category, the objects of which are just $\mathbb{M}^2$ and $\mathcal{E}$.

Central to this discussion is the extension of the conformal flatness result, which shows that all *globally hyperbolic* 2D Lorentzian manifolds can be embedded into one of these two spacetimes in a particular way.

**Theorem 7.1.3.** *Let $\Sigma_0 \subset \mathcal{M}_0$ denote the $t = 0$ Cauchy surface of either 2D Minkowski space or the Einstein cylinder (i.e. $\mathcal{M}_0 \in \{\mathbb{M}^2, \mathcal{E}\}$). Then, for any orientation-preserving diffeomorphism $\Sigma \xrightarrow{\sim} \Sigma_0$ where $\Sigma \subset \mathcal{M}$ is a Cauchy surface of a 2D globally hyperbolic spacetime, there exists a* CLoc *morphism $\mathcal{M} \to \mathcal{M}_0$ such that the following diagram commutes.*

$$\begin{tikzcd} \Sigma \arrow[r, "\sim"] \arrow[d, hook] & \Sigma_0 \arrow[d, hook] \\ \mathcal{M} \arrow[r] & \mathcal{M}_0 \end{tikzcd} \tag{7.7}$$

*Proof.* In [BGS21, Theorem 3.3], it was shown, using [FM16, Theorem 4.4] and [Mon14, Theorem 2.2] for planar and cylindrical spacetimes respectively, that there exist CLoc morphisms $\chi_0 : \mathcal{M} \to \mathcal{M}_0$. In particular, in [FM16] an arbitrary Cauchy surface $\Sigma \subset \mathcal{M}$ is selected such that $\Sigma \to \Sigma_{(0,1)} := \{(-x, x) \in \mathbb{M}^2 \,|\, x \in (0,1)\}$, expressed in null coordinates. As the image of $\mathcal{M}$ must be causally convex, it must be contained within the diamond $U = (-1, 0) \times (0, 1)$. Given any choice of oriented diffeomorphism $\rho_1 : (0,1) \xrightarrow{\sim} \mathbb{R}$, one can construct a map $\chi_1 : U \to \mathbb{M}^2$ by sending $(u, v) \mapsto (-\rho_1(-u), \rho_1(v))$. This map is clearly conformally admissible, as both $\rho_1(v)$ and $-\rho_1(-u)$ are orientation preserving diffeomorphisms. A point of the form $(-x, x)$ is mapped to $(-\rho_1(x), \rho_1(x))$, hence $\Sigma_{(0,1)} \xrightarrow{\sim} \Sigma_0$. Going back to our original Cauchy surface $\Sigma$, if we are given an arbitrary diffeomorphism $\rho : \Sigma \xrightarrow{\sim} \Sigma_0$, then we can construct $\chi_1$ from $\rho_1 = \rho \circ \chi_0|^{-1}_{\Sigma_{(0,1)}}$. The desired embedding $\mathcal{M} \to \mathbb{M}^2$ is then simply $\chi_1 \circ \chi_0$.

An important property of this embedding is that it can be expressed entirely in terms of $\rho$. We can define maps $\pi^{\Sigma}_{\ell/r} : \mathcal{M} \to \Sigma$ such that $\Sigma \cap \pi_{\ell/r}(x) = \{\pi^{\Sigma}_{\ell/r}(x)\}$. One then has that

$$\chi(x) = (-\rho \circ \pi^{\Sigma}_{\ell}(x), \rho \circ \pi^{\Sigma}_{r}(x)), \tag{7.8}$$

using null coordinates on $\mathbb{M}^2$. This is because $\chi(\Sigma \cap \pi_{\ell/r}(x)) = \rho(\Sigma) \cap \tilde{\pi}_{\ell/r}(\chi(x))$, where $\tilde{\pi}_{\ell}(u, v) = \{(u, v') \in \mathbb{M}^2\}_{v' \in \mathbb{R}}$ *etc.* This correspondence is represented visually in fig. 7.1.

We now consider the case where $\rho : \Sigma \xrightarrow{\sim} \Sigma_0 \subset \mathcal{E}$ is a compact Cauchy surface of a spacetime $\mathcal{M}$. Let $p_\Sigma : \overline{\Sigma} \to \Sigma$ be the universal cover of $\Sigma$. Because $\mathcal{M} \simeq \Sigma \times \mathbb{R}$, this also defines a universal cover $p : \overline{\mathcal{M}} \to \mathcal{M}$ which, given the pullback metric along $p$, is also globally hyperbolic, with $\overline{\Sigma}$ as a Cauchy surface. If we now denote by $\overline{\Sigma}_0$ the $t = 0$ Cauchy surface of Minkowski space, which is also the universal cover of $\Sigma_0 \subset \mathcal{E}$, and by $p_0 : \mathbb{M}^2 \to \mathcal{E}$ and $p_{\Sigma_0} : \overline{\Sigma}_0 \to \Sigma_0$ the canonical projections, we can construct the commutative diagram

$$\begin{tikzcd}
\overline{\Sigma} \arrow[r, two heads, "p_\Sigma"'] \arrow[d, hook] \arrow[rrr, dashed, bend left, "\bar{\rho}"] & \Sigma \arrow[r, "\rho"] \arrow[d, hook] & \Sigma_0 \arrow[d, hook] & \overline{\Sigma}_0 \arrow[l, two heads, "p_{\Sigma_0}"] \arrow[d, hook] \\
\overline{\mathcal{M}} \arrow[r, two heads, "p"] \arrow[rrr, dashed, bend right, "\overline{\chi}"'] & \mathcal{M} & \mathcal{E} & \mathbb{M}^2 \arrow[l, two heads, "p_0"']
\end{tikzcd} \tag{7.9}$$

where $\bar{\rho}$ is a lift of $\rho$, and $\overline{\chi}$ is the CLoc morphism corresponding to $\bar{\rho}$ by the previous argument. We would like to define $\chi : \mathcal{M} \to \mathcal{E}$ by $\chi(x) = p_0 \circ \overline{\chi}(\bar{x})$ for some $p(\bar{x}) = x$. Clearly this map is well defined if and only if $\overline{\chi}$ is equivariant with respect to the automorphisms of the covering maps, i.e. for every $d : \overline{\mathcal{M}} \to \overline{\mathcal{M}}$ such that $p \circ d = p$, there must be $d_0 : \mathbb{M}^2 \to \mathbb{M}^2$ such that $p_0 \circ d_0 = p_0$, and $\overline{\chi} \circ d = d_0 \circ \overline{\chi}$.

By definition $\bar{\rho}$ is equivariant with respect to the automorphisms of $p_\Sigma$ and $p_{\Sigma_0}$, where for both covering maps, such automorphisms are restrictions of the aforementioned $d$ and $d_0$. Spelling it out, this means that, for every $d_\Sigma : \overline{\Sigma} \to \overline{\Sigma}$ such that $p_\Sigma d_\Sigma = p_\Sigma$ there exists $d_0 : \overline{\Sigma}_0 \to \overline{\Sigma}_0$ such that $p_{\Sigma_0} d_{\Sigma_0} = p_{\Sigma_0}$ and

$$\bar{\rho} \circ d_\Sigma = d_{\Sigma_0} \circ \bar{\rho},$$

moreover, $d_\Sigma$ extends to a map $d : \overline{\mathcal{M}} \to \overline{\mathcal{M}}$ and likewise for $d_{\Sigma_0}$. (In particular $d_0(u,v) = (u - 2\pi n, v + 2\pi n)$, hence $d_{\Sigma_0}(x) = x + 2\pi n$ for some $n \in \mathbb{Z}$.) Moreover, the manner in which $\overline{\mathcal{M}}$ is constructed means that $d$ is always an isometry preserving $\Sigma$, hence $\pi^{\overline{\Sigma}}_{\ell/r} \circ d = d_\Sigma \circ \pi^{\overline{\Sigma}}_{\ell/r}$. We can then use (7.8) to show that

$$\begin{aligned} \chi \circ d(x) &= (-\bar{\rho} \circ \pi^{\overline{\Sigma}}_{\ell} \circ d(x), \bar{\rho} \circ \pi^{\overline{\Sigma}}_{r} \circ d(x)) \\ &= (-d_{\Sigma_0} \circ \bar{\rho} \circ \pi^{\overline{\Sigma}}_{\ell}(x), d_{\Sigma_0} \circ \bar{\rho} \circ \pi^{\overline{\Sigma}}_{r}(x)) \\ &= d_0 \circ \chi(x), \end{aligned} \tag{7.10}$$

as required. □

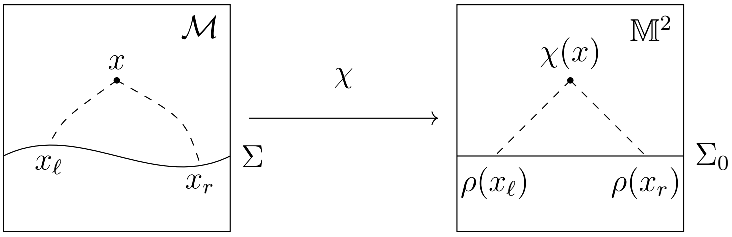

Figure 7.1: A diagrammatic representation of (7.8), where we have used the shorthand $x_{\ell/r} = \pi^{\Sigma}_{\ell/r}(x)$. The dashed lines represent null geodesics, which are necessarily preserved by the conformally admissible embedding $\chi$.

One way of phrasing this result is that, any diffeomorphism $\Sigma \xrightarrow{\sim} \Sigma_0$, where $\Sigma_0$ is a Cauchy surface of $\mathbb{M}^2$ of $\mathcal{E}$, can be extended such that its domain is the entirety of $\mathcal{M}$. If we instead have a diffeomorphism $\Sigma \xrightarrow{\sim} \tilde{\Sigma}$ to a Cauchy surface of some other spacetime $\widetilde{\mathcal{M}}$, then we can use theorem 7.1.3 to prove a weaker, but more general extension as follows.

**Corollary 7.1.4.** *Let $\mathcal{M}, \widetilde{\mathcal{M}}$ be a pair of 2D globally hyperbolic spacetimes with Cauchy surfaces $\Sigma, \tilde{\Sigma}$ respectively. For any orientation-preserving embedding $\rho : \Sigma \hookrightarrow \tilde{\Sigma}$, there exists an open, causally convex subset $\mathcal{N} \subseteq \mathcal{M}$ such that $\Sigma$ is also a Cauchy surface of $\mathcal{N}$ and $\rho$ extends to a CLoc morphism $\chi : \mathcal{N} \to \widetilde{\mathcal{M}}$ such that the following diagram commutes.*

$$\begin{tikzcd} \Sigma \arrow[r, "\rho", "\sim"'] \arrow[d, hook] & \tilde{\Sigma} \arrow[d, hook] \\ \mathcal{N} \arrow[r, "\chi"] & \widetilde{\mathcal{M}} \end{tikzcd} \tag{7.11}$$

*Moreover, if $\rho$ is a diffeomorphism, then $\chi$ is Cauchy.*

*Proof.* Suppose first that $\rho$ is invertible. Choose a diffeomorphism $\rho_0 : \widetilde{\Sigma} \overset{\sim}{\to} \Sigma_0$, where $\Sigma_0$ is a Cauchy surface for the appropriate choice of $\mathcal{M}_0 \in \{\mathbb{M}^2, \mathcal{E}\}$ (it is implicit that this and all following diffeomorphisms are orientation-preserving). This also provides a diffeomorphism $\rho_1 = \rho_0 \circ \rho : \Sigma \overset{\sim}{\to} \Sigma_0$. Applying theorem 7.1.3 to both, we obtain $\chi_1, \chi_0$ in the following diagram.

$$\begin{tikzcd}
\Sigma \arrow[r, "\rho", "\sim"'] \arrow[dd, hook] & \widetilde{\Sigma} \arrow[r, "\rho_0"] \arrow[d, hook] & \Sigma_0 \arrow[d, hook] \\
& \widetilde{\mathcal{M}} \arrow[r, "\chi_0"] & \mathcal{M}_0 \\
\mathcal{M} \arrow[urr, bend right, "\chi_1"'] & &
\end{tikzcd} \tag{7.12}$$

We then consider the space $\mathcal{N} = \chi_1^{-1}(\chi_0(\widetilde{\mathcal{M}}) \cap \chi(\mathcal{M}))$. This space is open and causally convex as both $\chi_1(\mathcal{M})$ and $\chi_0(\widetilde{\mathcal{M}})$ are, and each property is preserved by intersection. We can also see that $\mathcal{N}$ contains $\Sigma$, as $\chi_1(\Sigma) = \rho_1(\Sigma) = \rho_0(\widetilde{\Sigma}) \subset \chi_0(\widetilde{\mathcal{M}})$. Given that $\chi_1(\mathcal{N}) \subseteq \chi_0(\widetilde{\mathcal{M}})$, we can also define the map $\chi := \chi_0^{-1} \circ \chi_1 : \mathcal{N} \to \widetilde{\mathcal{M}}$, because all CLoc morphisms are diffeomorphisms onto their images. Adding this into the above diagram, we obtain

$$\begin{tikzcd}
\Sigma \arrow[r, "\rho"] \arrow[d, hook] & \widetilde{\Sigma} \arrow[r, "\rho_0"] \arrow[d, hook] & \Sigma_0 \arrow[d, hook] \\
\mathcal{N} \arrow[r, "\chi"] \arrow[d, hook] & \widetilde{\mathcal{M}} \arrow[r, "\chi_0"] & \mathbb{M}^2 \\
\mathcal{M} \arrow[urr, bend right, "\chi_1"'] & &
\end{tikzcd} \tag{7.13}$$

the commutativity of which demonstrates that $\chi$ is indeed a Cauchy morphism $\mathcal{N} \to \widetilde{\mathcal{M}}$.

In the case where $\rho$ is only an embedding, take $\widetilde{\mathcal{N}} \subseteq \widetilde{\mathcal{M}}$ to be the Cauchy development of $\rho(\Sigma)$, which is the set of points $\tilde{x} \in \widetilde{\mathcal{M}}$ such that every inextensible causal curve through $\tilde{x}$ intersects $\rho(\Sigma)$. Clearly $\widetilde{\mathcal{N}}$ is open and causally convex, and hence is a sub-spacetime, with $\rho(\Sigma)$ as a Cauchy surface. Then the preceding argument applies by replacing $\widetilde{\mathcal{M}}$ with $\widetilde{\mathcal{N}}$. $\square$

The reason that diffeomorphisms $\mathcal{M} \supset \Sigma \overset{\sim}{\to} \Sigma_0 \subset \mathbb{M}^2$ can be extended to the entirety of $\mathcal{M}$ is that, for any pair of points $x, y \in \Sigma_0$, one can produce a null geodesic from each such that they intersect precisely once in $\mathbb{M}^2$. For example, if we express $\Sigma_0$ in null coordinates as the set $\Sigma_0 = \{(-x, x) \in \mathbb{M}^2\}_{x\in\mathbb{R}}$, then a right-moving null geodesic from $x$ will intersect a left-moving geodesic from $y$ at the point $(-x, y) \in \mathbb{M}^2$. If we truncate $\mathbb{M}^2$ to events in the past of some Cauchy surface, say $t = T$ for some $T > 0$, then this is no longer the case. If we consider $\Sigma_0$ to also be a Cauchy surface of the truncated Minkowski spacetime $\mathbb{M}^2_{t<T}$, the above theorem can extend the identity map in one direction, resulting in the inclusion $\mathbb{M}^2_{t<T} \to \mathbb{M}^2$, but there exists *no* conformally admissible embedding $\mathbb{M}^2 \to \mathbb{M}^2_{t<T}$ which acts as identity on the $t = 0$ Cauchy surface, as demonstrated by fig. 7.2.

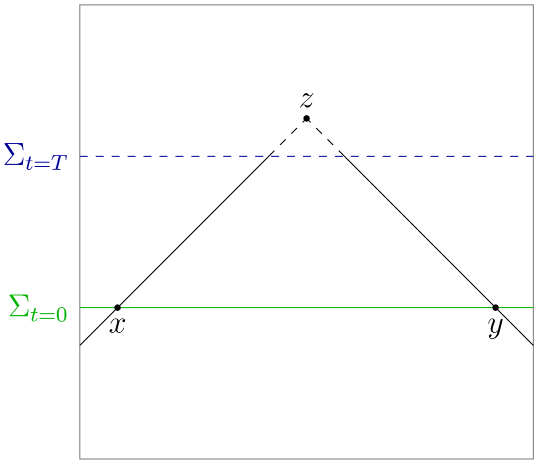

Figure 7.2: The null geodesics originating from $x$ and $y$ intersect in $\mathbb{M}^2$, but not in $\mathbb{M}^2_{t<T}$. Thus any map $\mathbb{M}^2 \to \mathbb{M}^2_{t<T}$ which restricts to the identity on $\Sigma_{t=0}$ cannot preserve null geodesics, and hence cannot be conformal.

We have thus established that oriented diffeomorphisms $\rho : \Sigma \xrightarrow{\sim} \tilde{\Sigma}$ extend at least partially to CLoc morphisms $\mathcal{N} \to \mathcal{M}$. Notably, these morphisms are always *Cauchy*, as the image of $\mathcal{N}$ always contains $\tilde{\Sigma}$. This is significant, because it means that for theories satisfying the time-slice axiom, diffeomorphisms of Cauchy surfaces yield *isomorphisms* of the corresponding algebras of observables. In the special cases of $\mathbb{M}^2$ and $\mathcal{E}$, the time-slice axiom is not even necessary, as diffeomorphisms of Cauchy surfaces extend fully to CLoc isomorphisms of each spacetime. Finally, by considering the group $\mathrm{Diff}_+(\Sigma)$, we shall later see that, by associating a chiral subalgebra to a particular Cauchy surface $\Sigma$ of a given spacetime, invariance of the algebra under *reparametrisations* of $\Sigma$ comes as a natural consequence of conformal covariance of the full spacetime algebra.

In the following, we shall be constructing algebras on Cauchy surfaces which we claim to capture a chiral (or anti-chiral) sector of the full algebra in question. In order to show that our constructions are 'natural', we will effectively have to show that for every diagram of the form (7.11), there is a corresponding diagram of algebras. We can lay the groundwork for a precise formulation of this principle with the following definition.

**Definition 7.1.5.** The category $\mathsf{CCauchy}$ has as

- *Objects:* Pairs $(\Sigma, \mathcal{M})$ such that $\Sigma$ is a Cauchy surface of $\mathcal{M} \in \mathsf{CLoc}$.
- *Morphisms:* Pairs $(\rho, \chi)$ such that $\rho : \Sigma \to \widetilde{\Sigma}$ is a smooth oriented embedding, $\chi \in \mathsf{CLoc}(\mathcal{M}, \widetilde{\mathcal{M}})$ such that (7.11) commutes.

*Remark* 7.1.6. There are two natural functors out of $\mathsf{CCauchy}$. Clearly we have $\Pi_2 : \mathsf{CCauchy} \longrightarrow \mathsf{CLoc}$, which sends $(\Sigma, \mathcal{M})$ to $\mathcal{M}$ and $(\rho, \chi)$ to $\chi$. The target category for $\Pi_1$, which forgets about $\mathcal{M}$ and $\chi$, is $\mathsf{CRie}_1$, the category of Riemannian 1-manifolds with smooth oriented embeddings (which are necessarily conformal) as morphisms. Note that *both* functors are surjective. This is obvious for $\Pi_2$, as every globally hyperbolic spacetime possesses a Cauchy surface. Given a Riemannian 1-manifold $\Sigma$, we can also easily construct a globally hyperbolic spacetime $\mathcal{M}_\Sigma := \Sigma \times \mathbb{R}$ with orientations defined in the obvious way and a metric $ds^2 = dt^2 - dx^2$, where $dt$ is the coordinate one-form on $\mathbb{R}$, and $dx$ is the metric volume form of $\Sigma$.

## 7.2 CHIRAL CONFIGURATION SPACES

In this section we shall construct a model for the configuration space of chiral fields. Similarly to the full spacetime, we refer to elements of this space as *chiral field configurations*. The term 'chiral field' will instead be used for objects bearing a closer resemblance to the locally covariant fields of part I which we shall discuss in section 8.4.

Given a field configuration $\varphi \in \mathfrak{E}(\mathbb{M}^2)$ satisfying the equation of motion $\partial_u \partial_v \varphi = 0$, if we take its derivative with respect to $u$, we obtain a function that is independent of $v$ and *vice-versa*. Not only does this allow us to separate the left-moving term of d'Alembert's solution from the right-moving term, building our configuration space from derivatives of scalar field on $\mathbb{M}^2$ allows us to avoid the well-known problems which arise when trying to find a vacuum state for the massless scalar field [BFR17].

We would like to formulate statements such as '$\partial_u \varphi$ depends only on u' without explicit reference to our choice of coordinates. The first issue is that the operator $\partial_u$ depends on a choice of frame. In chapter 5 we included such a choice as part of our background spacetime data. However, the additional requirement that the resulting function has only one independent variable allows us to define our chiral algebras without the need to make any such choice.

The Lorentzian metric on a spacetime $\mathcal{M}$ causes the cotangent bundle $T^*\mathcal{M}$ to naturally decompose as $T^*\mathcal{M} = T^*_\ell\mathcal{M} \oplus T^*_r\mathcal{M}$, where, in null coordinates $u, v$, the fibres of $T^*_\ell\mathcal{M}$ and $T^*_r\mathcal{M}$ are spanned by $du$ and $dv$ respectively. Let $\Pi_{\ell/r} : T^*\mathcal{M} \to T^*_{\ell/r}\mathcal{M}$ be the projections onto each subbundle, the operation $\Pi_\ell d$ then sends $\varphi \mapsto \partial_u\varphi du$. Note that, even though we have described this splitting in terms of a frame, the splitting itself depends only on the conformal class of the metric $g_{\mathcal{M}}$.

We now have a 1-form on $\mathcal{M}$, but we would like a function on $\mathcal{M}_\ell$. Recalling our discussion in the previous section, we may use a Cauchy surface $\Sigma \subset \mathcal{M}$ as a proxy for $\mathcal{M}_\ell$. Restricting $\Pi_\ell d\varphi$ to $\Sigma$, we may then map this to a smooth function by using the Hodge star $*_\Sigma$ associated to the Riemannian metric on $\Sigma$. Finally, noting that $\Sigma$ has the opposite orientation to $\mathcal{M}_\ell$, we multiply the resulting function by a factor of $-1$ to account for this (see the example below).

Thus, altogether we have a map

$$\partial_\Sigma := (-1) \cdot *_\Sigma i^*_\Sigma \Pi_\ell d, \tag{7.14}$$

where $i_\Sigma : \Sigma \hookrightarrow \mathcal{M}$ is the inclusion map. We shall henceforth refer to $\partial_\Sigma$ as the *chiral derivative* corresponding to the Cauchy surface $\Sigma \subset \mathcal{M}$. Similarly, we may also define the *anti-chiral derivative* $\bar{\partial}_\Sigma = *_\Sigma i^*_\Sigma \Pi_r d$, though we will rarely use this, as most statements concerning $\partial_\Sigma$ are readily generalised.

As an example, consider the Cauchy surface in Minkowski space expressed in null coordinates as $\Sigma = \{(-s, \gamma(s))\}_{s\in\mathbb{R}}$ for some $\gamma \in \mathrm{Diff}_+(\mathbb{R})$. As we noted above, given an arbitrary configuration $\varphi \in \mathfrak{E}(\mathbb{M}^2)$, we have that $\Pi_\ell d\varphi = (\partial_u\varphi)du$. After a quick computation one can verify that, using the parametrisation of $\Sigma$ given above, we have that $i^*_\Sigma du = -ds$, and the induced volume form on $\Sigma$ may be expressed as $\mathrm{d}V_\Sigma = \sqrt{\gamma'(s)}ds$. Thus, altogether we have

$$(-1) \cdot *_\Sigma i^*_\Sigma \Pi_\ell d\varphi(s) = (\partial_\Sigma\varphi)(s) = \frac{1}{\sqrt{\gamma'(s)}}(\partial_u\varphi)(-s, \gamma(s)). \tag{7.15}$$

From this one can also show that, at least for $\Sigma \subset \mathbb{M}^2$, $\partial_\Sigma$ is surjective. We shall soon see that this is true for arbitrary $\mathcal{M}$, however we shall first address the question of how these maps interact with our preferred spacetime transformations.

In part I, we discussed how the inclusion of *conformal* isometries in our collection of allowable morphisms necessitated more complicated transformations than the standard pullback of smooth functions/sections of a fibre bundle. Interestingly, even though we are now considering one-dimensional Riemannian manifolds, the definition 4.1.1 of a *weighted pullback* carries over unchanged. Given a CCauchy morphism $(\rho, \chi)$ from $(\Sigma, \mathcal{M})$ to $(\widetilde{\Sigma}, \widetilde{\mathcal{M}})$ such that $\chi^*\widetilde{g} = \Omega^2 g$, the restriction $\rho$ has conformal factor $(\Omega|_\Sigma)^2$, in the sense that $\rho^*\widetilde{g}|_{\widetilde{\Sigma}} = (\Omega|_\Sigma)^2\, g|_\Sigma$.

**Definition 7.2.1.** For $\rho \in \mathsf{CRie}_1(\Sigma, \widetilde{\Sigma})$ such that $\rho^* g_{\widetilde{\Sigma}} = \omega^2 g_\Sigma$, we define the *weighted pullback* $\rho^*_{(\mu)} : \mathfrak{E}(\widetilde{\Sigma}) \longrightarrow \mathfrak{E}(\Sigma)$ by

$$\rho^*_{(\mu)}\psi := \omega^\mu \rho^*\psi \tag{7.16}$$

It turns out that, for $\mu = 1$, these are precisely the maps required to preserve the images of the $\partial_\Sigma$ operators, as demonstrated by the following proposition.

**Proposition 7.2.2.** *Define the contravariant functor* $\mathfrak{E}_{(1)} : \mathsf{CRie}_1 \to \mathsf{Vec}$ *such that* $\mathfrak{E}_{(1)}(\Sigma) = \mathfrak{E}(\widetilde{\Sigma})$*, and* $\mathfrak{E}_{(1)}\rho = \rho^*_{(1)} : \mathfrak{E}(\widetilde{\Sigma}) \to \mathfrak{E}(\Sigma)$*. Then the morphisms* $\partial_\Sigma : \mathfrak{E}(\mathcal{M}) \to \mathfrak{E}(\Sigma)$ *constitute a natural transformation* $\partial : \mathfrak{E} \circ \Pi_2 \Rightarrow \mathfrak{E}_{(1)} \circ \Pi_1$*, i.e. for every* CCauchy *morphism* $(\rho, \chi) : (\Sigma, \mathcal{M}) \to (\widetilde{\Sigma}, \widetilde{\mathcal{M}})$ *the following diagram commutes.*

$$\begin{tikzcd}
\mathfrak{E}(\Sigma) & \mathfrak{E}(\widetilde{\Sigma}) \arrow[l, "\rho^*_{(1)}"'] \\
\mathfrak{E}(\mathcal{M}) \arrow[u, "\partial_\Sigma"] & \mathfrak{E}(\widetilde{\mathcal{M}}) \arrow[l, "\chi^*"] \arrow[u, "\partial_{\widetilde{\Sigma}}"']
\end{tikzcd} \tag{7.17}$$

*Proof.* The exterior derivative $d$ commutes with the pullback along any smooth map, and, by the definition of a CCauchy morphism, $i^*_{\widetilde{\Sigma}}\chi^* = \rho^* i^*_\Sigma$. Similarly, $\Pi_\ell$ commutes with all conformally admissible embeddings, leaving only the Hodge dual to check. If $\rho : X \to Y$ is a conformal embedding of Riemannian $n$-manifolds with $\rho^* g_Y = \Omega^2 g_X$, then the Hodge operator on $p$-forms behaves as $*_X \circ \rho^* = \Omega^{2p-n}\rho^* \circ *_Y$ [Bes87, Theorem 1.159 h)]. For $p = n = 1$ we then get the necessary factor to make the diagram commute. □

*Remark* 7.2.3. Just as in section 5.2, one can consider the case where $\widetilde{\mathcal{M}} = \Lambda\mathcal{M}$, i.e. the underlying manifold is held fixed and the metric is scaled by some constant $\Lambda^2 \in \mathbb{R}_{>0}$. We may then take $\chi$ to be the identity map of the underlying manifold, whereupon the map $\mathfrak{E}(\widetilde{\Sigma}) \to \mathfrak{E}(\Sigma)$ in the above proposition becomes $\psi \mapsto \Lambda\psi$. As such, the physical interpretation of the above proposition is that the chiral boson $\partial\varphi$ has a *scaling dimension* of 1.

Finally, so that we may be sure that the chiral configuration space is not a proper subspace of $\mathfrak{E}(\Sigma)$, we have the following result.

**Proposition 7.2.4.** *For every* $(\Sigma, \mathcal{M}) \in \mathsf{CCauchy}$*,* $\partial_\Sigma$ *is surjective. Moreover, for every* $\psi \in \mathfrak{E}(\Sigma)$*, there is a* solution $\varphi \in \mathrm{Ker} P_{\mathcal{M}} \subset \mathfrak{E}(\mathcal{M})$ *such that* $\partial_\Sigma \varphi = \psi$*.*

*Proof.* For $\overline{\Sigma}_0 \subset \mathbb{M}^2$, we can write the solution of $\partial_{\overline{\Sigma}_0}\varphi = \psi$ explicitly as

$$\varphi(u, v) = \int_0^u \psi(-u')\,\mathrm{d}u'. \tag{7.18}$$

As $\varphi$ only depends on $u$, this is clearly a solution to the equations of motion. By using $\mathbb{M}^2$ as the universal covering space of $\mathcal{E}$, we get the corresponding result for the Cauchy surface $\Sigma_0 \subset \mathcal{E}$, however, we must add an extra step. If $\psi$ in (7.18) is $2\pi$-periodic (i.e. it corresponds to a function in $\mathfrak{E}(\Sigma_0)$) then $\varphi(u - 2\pi, v + 2\pi) = \varphi(u, v) + \int_0^{2\pi} \psi(x)dx$. In other words, $\varphi$ only defines a function on $\mathcal{E}$ if $\psi$ is exact.

To solve this, we choose for the solution of $\partial_{\Sigma_0}\varphi = \psi$ on the cylinder

$$\varphi(u, v) = \int_0^u \psi_0(-u')\,\mathrm{d}u' + \frac{1}{2\pi}\left(\int_0^{2\pi} \psi(x)\,\mathrm{d}x\right)(u + v), \tag{7.19}$$

where $\psi_0(x) = \psi(x) - \frac{1}{2\pi}\int_0^{2\pi} \psi(x')\,\mathrm{d}x'$ is the 'exact part' of $\psi$. This is then clearly in the form (7.3) of a general solution to the wave equation on a cylinder.

For an arbitrary element $(\Sigma, \mathcal{M}) \in \mathsf{CCauchy}$, we take a diffeomorphism $\rho : \Sigma \xrightarrow{\sim} \Sigma_0$, where as before $\Sigma_0$ is the $t = 0$ Cauchy surface of $\mathcal{M}_0 \in \{\mathbb{M}^2, \mathcal{E}\}$ as appropriate. We can then solve $\partial_\Sigma \varphi = \psi$ for any $\psi \in \mathfrak{E}(\Sigma)$ using the corresponding embedding $\chi : \mathcal{M} \to \mathcal{M}_0$. In particular, suppose that $\partial_{\Sigma_0}\varphi_0 = (\rho^{-1})^*_{(1)}\psi$, where $\psi$ is one of the solutions constructed above, then

$$\partial_\Sigma \chi^* \varphi_0 = \rho^*_{(1)} \partial_{\Sigma_0} \varphi_0 = \psi. \tag{7.20}$$

Moreover, as $\chi^*$ maps $\mathrm{Ker}\, P_{\mathcal{M}_0} \to \mathrm{Ker}\, P_{\mathcal{M}}$ (c.f. remark 4.2.2), $\varphi$ is also a solution to the equations of motion as desired. □

*Remark* 7.2.5. What we have obtained in this section is a *covariant* surjection $\partial_\Sigma : \mathfrak{E}(\mathcal{M}) \to \mathfrak{E}(\Sigma)$. Equations eq. (7.18) and eq. (7.19) tell us that we may interpret the image of this space as corresponding to *chiral* solutions to the wave equation. With the caveats that (i) constant solutions cannot be obtained from any splitting of $\partial_\Sigma$ and (ii) that, on the cylinder, one cannot completely separate solutions depending only on $u$ from those depending only on $v$.

One reason for working with $\mathfrak{E}(\Sigma)$ in spite of these caveats is that it is, topologically speaking, a very "nice" space to work with. In particular, as we shall see in the following chapter, it is relatively easy to define spaces of functionals on $\mathfrak{E}(\Sigma)$ analogous to the microcausal functionals over $\mathfrak{E}(\mathcal{M})$. We can then give these functionals a Poisson bracket, and see how the resulting algebra compares to the algebra $\mathfrak{P}(\mathcal{M})$ we constructed in section 2.6. With a little work, our covariant surjection $\partial_\Sigma$ will then dualise to a covariant injection of the Poisson algebra defined on $\Sigma$ into $\mathfrak{P}_{\text{on}}(\mathcal{M})$.

## — 8 —

# Classical Observables

Now that we have identified our configuration space, and how it transforms under appropriate morphisms, we can begin to discuss the algebras of observables, and from there the dynamics, of the massless scalar field.

### 8.1 CLASSICAL CHIRAL ALGEBRA

There are many ways we could begin to construct the chiral algebra of observables. We shall begin by looking at a space common to all definitions: namely the regular, linear observables.

We shall denote by $\{\Psi_\Sigma(f) \,|\, f \in \mathfrak{D}(\mathcal{M})\}$ the family of linear observables

$$\mathfrak{E}(\Sigma) \to \mathbb{R}, \quad \psi \mapsto \int_\Sigma f\psi \mathrm{d}V_\Sigma. \tag{8.1}$$

Naturally, we can also push these forward to maps $\partial_\Sigma^* \Psi_\Sigma(f) : \mathfrak{E}(\mathcal{M}) \to \mathbb{R}$, by $\varphi \mapsto \Psi_\Sigma(f)[\partial_\Sigma \varphi]$. It is easy to see that $\partial_\Sigma^* \Psi_\Sigma(f)$ is both linear and continuous on $\mathfrak{E}(\mathcal{M})$, i.e. it is a distribution, thus we may attempt to compute a commutator of $\partial_\Sigma^* \Psi_\Sigma(f)$ with $\partial_\Sigma^* \Psi_\Sigma(g)$. However, even though $\Psi_\Sigma(f)$ is regular, $\partial_\Sigma^* \Psi_\Sigma(f)$ fails to be even microcausal: because its support lies entirely within $\Sigma$, its wavefront set lies normal to $\Sigma$, which in particular means that it contains timelike covectors.

It shall be the purpose of later sections in this chapter to address the singularity of these distributions. For now, we shall ignore this issue and attempt a naïve calculation of $\langle E, \partial_\Sigma^* \Psi_\Sigma(f) \otimes \partial_\Sigma^* \Psi_\Sigma(g) \rangle$. (Recall that Hörmander's results enabling computations with distributions are sufficient, but not necessary.) Fortunately, it turns out this pairing *is* well-defined, in brief because the differential operator in $\partial_\Sigma$ annihilates the $v$ dependent terms of $E$, yielding a more amenable wavefront set.

If we attempt to compute the Peierls bracket, we obtain

$$\{\partial_\Sigma^* \Psi(f), \partial_\Sigma^* \Psi(g)\} = \langle (\partial_\Sigma \otimes \partial_\Sigma) E, f \otimes g \rangle_{\Sigma^2}. \tag{8.2}$$

In other words, the commutator function for chiral observables is simply $(\partial_\Sigma \otimes \partial_\Sigma)E$, provided this is a well-defined distribution in $\mathfrak{D}'(\Sigma^2)$, which we shall henceforth denote by the shorthand $E_\Sigma$.

On Minkowski space, the integral kernel of the Pauli-Jordan function may be expressed in null coordinates as

$$E(u, v, u', v') = -\frac{1}{4}\left(\operatorname{sgn}(u - u') + \operatorname{sgn}(v - v')\right). \tag{8.3}$$

Taking $\Sigma_0$ to be the $t = 0$ Cauchy surface, we may use (7.15) to compute the chiral commutator function as

$$E_{\Sigma_0}(s, s') = \frac{1}{2}\delta'(s - s'), \tag{8.4}$$

which is clearly well-defined. This agrees with (3.11), though now the reduced number of coordinates is expressed in a more geometric manner by taking the pullback of $(\partial_u \otimes \partial_u)E$ along an embedding of the 1-manifold $\Sigma$ into the 2-manifold $\mathbb{M}^2$. Because we also have an explicit definition for the causal propagator (3.17), we can similarly show that its chiral derivative is well-defined.

We can then transfer the result from these two explicit examples to arbitrary spacetimes. Let $(\Sigma, \mathcal{M}) \in \mathsf{CCauchy}$, and choose a diffeomorphism $\rho : \Sigma \xrightarrow{\sim} \Sigma_0 \subset \mathcal{M}_0 \in \{\mathbb{M}^2, \mathcal{E}\}$. From proposition 7.2.2, we have that $\rho^*_{(1)}\partial_{\Sigma_0} = \partial_\Sigma \chi^*$. Given the conformal covariance of the causal propagator (proposition 4.2.3), this implies

$$\begin{aligned}(\rho^*_{(1)} \otimes \rho^*_{(1)})E_{\Sigma_0} &= (\rho^*_{(1)} \otimes \rho^*_{(1)})(\partial_{\Sigma_0} \otimes \partial_{\Sigma_0})E_{\mathcal{M}_0} \\ &= (\partial_\Sigma \otimes \partial_\Sigma)(\chi^* \otimes \chi^*)E_{\mathcal{M}_0} \\ &=: E_\Sigma. \end{aligned} \tag{8.5}$$

This tells us that $E_\Sigma$ is a well-defined distribution in $\mathfrak{D}'(\Sigma^2)$ as desired, and that $\mathrm{WF}(E_\Sigma) = (\rho^{\otimes 2})^*\mathrm{WF}(E_{\Sigma_0})$.

Now that we have a bi-distribution on $\Sigma$, we can define a binary operation, for $F, G \in \mathfrak{F}_{\mathrm{reg}}(\Sigma)$, and $\psi \in \mathfrak{E}(\Sigma)$ by

$$\{F, G\}^\Sigma_\ell [\psi] = \left\langle E_\Sigma, F^{(1)}[\psi] \otimes G^{(1)}[\psi] \right\rangle. \tag{8.6}$$

Similarly to the construction of $\mathfrak{P}(\mathcal{M})$, we may now ask if there exists a space of functionals which is closed under this operation.

**Proposition 8.1.1.** *Let $\mathfrak{F}_c(\Sigma)$ be the space comprising functionals $F : \mathfrak{E}(\Sigma) \to \mathbb{R}$ such that*

1. *$F$ is Bastiani smooth with respect to the Fréchet topology on $\mathfrak{E}(\Sigma)$*

2. $\mathrm{WF}(F^{(n)}[\psi]) \cap \left(\Xi^n_+ \cup \Xi^n_-\right) = \emptyset$ *where*
$$\Xi^n_\pm = \{(s_1,\dots,s_n;\xi_1,\dots,\xi_n) \in T^*\Sigma^n \mid \pm\,\xi_i \geq 0, 0 \leq i \leq 1\},$$
*and the sign of a covector is defined with respect to an arbitrary oriented coordinate on* $\Sigma$.

*then* $\{\cdot,\cdot\}^\Sigma_\ell$ *is a Poisson bracket on* $\mathfrak{F}_c(\Sigma)$. *We denote the resulting Poisson algebra* $\mathfrak{P}_\ell(\Sigma,\mathcal{M})$

*Proof.* Note that $\Xi^1_+ \cap \Xi^1_- = \dot{T}^*\Sigma$, hence $F^{(1)}[\psi]$ is a regular distribution $\forall\, F \in \mathfrak{F}_c(\Sigma), \psi \in \mathfrak{E}(\Sigma)$. This means that, for $F, G \in \mathfrak{F}_c(\Sigma)$, the bracket (8.6) is well-defined. As such, it remains to show that $\mathfrak{F}_c(\Sigma)$ is closed under these operations, and that $\{\cdot,\cdot\}^\Sigma_\ell$ has all the properties of a Poisson bracket.

The fact that $\{\cdot,\cdot\}^\Sigma_\ell$ is a skew-symmetric bilinear form follows immediately from the definition, as does the fact that it is a derivation in each of its arguments. Rather than directly proving that the Jacobi identity is satisfied, we shall later prove that there is an injective homomorphism $\mathfrak{P}_\ell(\Sigma,\mathbb{M}^2) \to \mathfrak{P}(\mathcal{M})$, thus the Jacobi identity on $\mathfrak{P}_\ell(\Sigma,\mathbb{M}^2)$ follows from the same identity on $\mathfrak{P}(\mathcal{M})$.

Thus all that remains is to show $\mathfrak{F}_c(\Sigma)$ is closed under $\{\cdot,\cdot\}^\Sigma_\ell$. The argument here proceeds along the same lines as the closure proof in appendix B, but we shall outline the steps explicitly here.

It is sufficient to show that, $\forall\, F, G \in \mathfrak{F}_c(\Sigma), \psi \in \mathfrak{E}(\Sigma)$, and $k, m \in \mathbb{N}$,
$$\mathrm{WF}\left(\left\langle E_\Sigma, F^{(k+1)}[\psi] \otimes G^{(m+1)}[\psi]\right\rangle\right) \cap (\Xi^{(k+m)}_+ \cup \Xi^{(k+m)}_-) = \emptyset, \tag{8.7}$$
where $\langle\cdot,\cdot\rangle$ pairs the first variable of $E_\Sigma$ with the first variable of $F^{(k+1)}[\psi]$ and the second variable of $E_\Sigma$ with the first variable of $G^{(m+1)}[\psi]$ according to [Hör15, Theorem 8.2.14].

For simplicity, we will suppress the $\psi$ dependence of $F^{(k+1)}[\psi]$ and $G^{(m+1)}[\psi]$ for the rest of this proof, and we shall also restrict our attention to $\Sigma_0 \subset \mathbb{M}^2$. We will also use $(\underline{s}_F;\underline{\xi}_F)$ as short hand for an element of $T^*\Sigma^{k+1} \simeq \mathbb{R}^{2(k+1)}$ *etc.* As $F$ and $G$ are Bastiani smooth, $F^{(k+1)} \otimes G^{(m+1)}$ is compactly supported, we just have to consider the set
$$\begin{aligned}&\mathrm{WF}(F^{(k+1)} \otimes G^{(m+1)}) \bullet \overline{\mathrm{WF}(E_\Sigma)} := \\ &\quad\Big\{(\underline{s}_F,\underline{s}_G;\underline{\xi}_F,\underline{\xi}_G) \in T^*\Sigma^{k+m} \mid \exists\,(r_1,r_2;\eta_1,\eta_2) \in \overline{\mathrm{WF}(E_\Sigma)}, \\ &\qquad\qquad (r_1,\underline{s}_F,r_2,\underline{s}_G;\eta_1,\underline{\xi}_F,\eta_2,\underline{\xi}_G) \in \mathrm{WF}(F^{(k+1)} \otimes G^{(m+1)})\Big\},\end{aligned}$$

where $\overline{\mathrm{WF}(E_\Sigma)} := \mathrm{WF}(E_\Sigma) \cup \underline{0}_{\Sigma^2}$, and $\underline{0}_X$ denotes the zero section of $T^*X$. Firstly, if this set avoids $\underline{0}_{\Sigma^{k+m}}$, then the distribution in (8.7) is well-defined, in which case $\mathrm{WF}(F^{(k+1)} \otimes G^{(m+1)}) \bullet \overline{\mathrm{WF}(E_\Sigma)}$ contains its wavefront set.

We need firstly the estimate [Hör15, Theorem 8.2.9]

$$\mathrm{WF}(F^{(k+1)} \otimes G^{(m+1)}) \subseteq \overline{\mathrm{WF}(F^{(k+1)})} \times \overline{\mathrm{WF}(G^{(m+1)})} \setminus \underline{0}_{\Sigma^{k+m+2}}$$

and secondly the wavefront set of our commutator function

$$\mathrm{WF}(E_\Sigma) = \{(r, r; \eta, -\eta) \in \dot{T}\Sigma^2\},$$

which is readily obtained by inspection of (8.4). Suppose

$$(\underline{s}_F, \underline{s}_G; \underline{\xi}_F, \underline{\xi}_G) \in \Xi^{k+n}_{\pm} \cap \left(\mathrm{WF}(F^{(k+1)} \otimes G^{(m+1)}) \bullet \overline{\mathrm{WF}(E_\Sigma)}\right)$$

Unpacking the notation, this means there exists some $(r_1, r_2; \eta, -\eta) \in \overline{\mathrm{WF}(E_\Sigma)}$ such that

$$(r_1, \underline{s}_F; \eta, \underline{\xi}_F) \in \overline{\mathrm{WF}(F^{(k+1)})}, \quad (r_2, \underline{s}_G; -\eta, \underline{\xi}_G) \in \overline{\mathrm{WF}(G^{(m+1)})}$$

with at least one of these covectors being non-zero. Suppose in particular that $(\underline{s}_F, \underline{s}_G; \underline{\xi}_F, \underline{\xi}_G) \in \Xi^{k+n}_+$ Because $(r_1, \underline{s}_F; \eta, \underline{\xi}_F) \notin \Xi^{k+1}_+ \setminus \underline{0}_{\Sigma^{k+1}}$, we see that $\eta \leq 0$. But then $(r_2, \underline{s}_G; -\eta, \underline{\xi}_G) \notin \mathrm{WF}(G^{(m+1)})$, hence $(r_2, \underline{s}_G, -\eta, \underline{\xi}_G) \in \underline{0}_{\Sigma^{m+1}}$. This in turn implies that $(r_1, \underline{s}_F; \eta, \underline{\xi}_F)$ cannot belong to $\mathrm{WF}(F^{(k+1)})$, hence $(r_1, \underline{s}_F, r_2, \underline{s}_G; \eta_1, \underline{\xi}_F, \eta_2, \underline{\xi}_G) \in \underline{0}_{\Sigma^{k+m+2}}$, which is disjoint from $\mathrm{WF}(F^{(k+1)} \otimes G^{(m+1)}) \bullet \overline{\mathrm{WF}(E_\Sigma)}$. As $\underline{0}_{\Sigma^{k+m}} \subset \Xi^{k+m}_+$, this automatically tells us that Hörmander's criterion is satisfied, hence the distribution in (8.7) is well-defined. Similar reasoning to the above shows that $\Xi^{k+m}_-$ is also disjoint from $\mathrm{WF}(F^{(k+1)} \otimes G^{(m+1)}) \circ \overline{\mathrm{WF}(E_\Sigma)}$, hence (8.7) holds. □

*Remark* 8.1.2. The equation (8.6) can be expressed without coordinates as

$$\langle (\partial_\Sigma \otimes \partial_\Sigma) E, f \otimes g \rangle = -\frac{1}{2} \int_\Sigma f(*d_\Sigma g)\, \mathrm{d}V_\Sigma. \tag{8.8}$$

Hence we may instead express the commutator function for the chiral bracket as $\frac{1}{2} * d_\Sigma$. This is consistent with (8.5) which, due to the non-uniqueness of $\rho : \Sigma \xrightarrow{\sim} \Sigma_0$, implies that $E_\Sigma$ must be invariant under $\mathrm{Diff}_+(\Sigma)$.

Finally, we shall examine how the algebras we have defined behave under conformally admissible embeddings.

**Proposition 8.1.3.** *Let* $(\rho, \chi) : \mathcal{M} \to \widetilde{\mathcal{M}}$ *be a* CCauchy *morphism. Define*

$$\begin{aligned}\mathfrak{P}_\ell(\rho, \chi) : \mathfrak{P}_\ell(\Sigma, \mathcal{M}) &\to \mathfrak{P}_\ell(\widetilde{\Sigma}, \widetilde{\mathcal{M}}) \\ F &\mapsto F \circ \rho^*_{(1)}. \end{aligned} \tag{8.9}$$

*Then* $\mathfrak{P}_\ell$ *defines a functor* CCauchy $\to$ Poi.

*Proof.* As $\rho$ is an oriented embedding, one can fairly quickly convince themselves that the sets $\Xi^n_\pm$ are preserved by the induced maps $\rho_* : T^*\Sigma^n \to T^*\widetilde{\Sigma}^n$, thus $\mathfrak{F}_c(\Sigma) \to \mathfrak{F}_c(\widetilde{\Sigma})$ under this map.

To show this map is a Poisson algebra homomorphism, consider, for $F, G \in \mathfrak{F}_c(\Sigma)$, $\widetilde{\psi} \in \mathfrak{E}(\widetilde{\Sigma})$

$$\{\mathfrak{P}_\ell \chi F, \mathfrak{P}_\ell \chi G\}^{\widetilde{\Sigma}}_\ell [\widetilde{\psi}] = \left\langle E_{\widetilde{\Sigma}}, (\mathfrak{P}_\ell \chi F)^{(1)} [\widetilde{\psi}] \otimes (\mathfrak{P}_\ell \chi G)^{(1)} [\widetilde{\psi}] \right\rangle . \tag{8.10}$$

A quick calculation shows that $\left\langle (\mathfrak{P}_\ell \chi F)^{(1)} [\widetilde{\psi}], f \right\rangle = \left\langle F^{(1)}[\rho^*_{(1)}\widetilde{\psi}], \rho^*_{(1)} f \right\rangle$. Then, recalling Proposition 7.2.2, we have $\rho^*_{(1)}\partial_{\widetilde{\Sigma}} = \partial_\Sigma \chi^*$, from which we can deduce that

$$(\rho^*_{(1)} \otimes \rho^*_{(1)}) E_{\widetilde{\Sigma}} = (\partial_\Sigma \otimes \partial_\Sigma) E_{\mathcal{M}} = E_\Sigma, \tag{8.11}$$

thus (8.10) becomes simply $\{F, G\}^{\widetilde{\Sigma}}_\ell [\rho^*_{(1)}\widetilde{\psi}]$ as desired. □

## 8.2 COMPARISON WITH PEIERLS ALGEBRA

As we have already seen, direct comparison between chiral observables and observables of the full spacetime algebra is complicated by the fact that $\partial^*_\Sigma : \mathfrak{D}'(\Sigma) \to \mathfrak{D}'(\mathcal{M})$ fails to send regular distributions on $\Sigma$ to microcausal distributions on $\mathcal{M}$. Roughly speaking, this is due to the fact that the restriction $i^*_\Sigma : \Omega^1(\mathcal{M}) \to \Omega^1(\Sigma)$ in (7.14) is 'too sharp'.

In order to make comparisons, we therefore wish to find a more regular map, which coincides with $\partial_\Sigma$ on-shell. We shall again begin with the example of the $t = 0$ Cauchy surface, $\Sigma_0 \subset \mathbb{M}^2$. In null coordinates, and for $\epsilon > 0$ we define the family of maps

$$\begin{aligned}\partial_{\Sigma_0,\epsilon} : \mathfrak{E}(\mathbb{M}^2) &\to \mathfrak{E}(\Sigma_0) \\ \varphi &\mapsto \int_{\mathbb{R}} (\partial_u \varphi)(-s, v) \delta_\epsilon(\tfrac{-s+v}{2}) \, \mathrm{d}v \end{aligned}$$

where the family $\{\delta_\epsilon\}_{\epsilon > 0}$ constitute a *nascent delta*, i.e. each function is smooth, integrates to 1, and satisfies $\operatorname{supp} \delta_\epsilon = [-\epsilon, \epsilon]$. In the limit as $\epsilon \to 0$, these maps weakly converge to $\partial_{\Sigma_0}$ in the sense that, for all $f \in \mathfrak{D}(\Sigma)$, $\langle \partial_{\Sigma_0,\epsilon} \varphi, f \rangle_{\Sigma_0} \to \langle \partial_{\Sigma_0} \varphi, f \rangle_{\Sigma_0}$. Moreover, if $\varphi \in \operatorname{Ker} P$, then $\partial_{\Sigma_0,\epsilon}\varphi = \partial_\Sigma \varphi$ for every $\epsilon > 0$. If we define the natural transformation $\Pi_{\text{on}} : \mathfrak{P} \Rightarrow \mathfrak{P}_{\text{on}}$ by

$$(\Pi_{\text{on}})_{\mathcal{M}} : \mathfrak{F}_{\mu c}(\mathcal{M}) \to \mathfrak{F}_{\mu c}(\mathcal{M})/\mathfrak{I}_S(\mathcal{M}) \tag{8.12}$$

$$\mathcal{F} \mapsto [\mathcal{F}], \tag{8.13}$$

then $(\Pi_{\text{on}})_{\mathbb{M}^2} \circ \partial^*_{\Sigma_0,\epsilon} = (\Pi_{\text{on}})_{\mathbb{M}^2} \circ \partial^*_{\Sigma_0,\epsilon'}$, $\forall\, \epsilon, \epsilon' > 0$.

**Proposition 8.2.1.** *For every $\epsilon > 0$, the map $\partial^*_{\Sigma_0,\epsilon} : \mathfrak{F}(\Sigma_0) \to \mathfrak{F}(\mathbb{M}^2)$ defined such that $\partial^*_{\Sigma_0,\epsilon}F[\varphi] = F[\partial_{\Sigma_0,\epsilon}\varphi]$ yields an injective Poisson algebra homomorphism $\mathfrak{P}_\ell(\Sigma_0, \mathbb{M}^2) \to \mathfrak{P}(\mathbb{M}^2)$.*

*Proof.* Firstly, we must show that the image of $\mathfrak{F}_c(\Sigma_0)$ under $\partial^*_{\Sigma_0,\epsilon}$ lies within $\mathfrak{F}_{\mu c}(\mathbb{M}^2)$. We initially restrict our attention to the linear, regular observables on $\Sigma$, which may be identified with $\mathfrak{D}(\Sigma_0)$. Linearity and continuity are preserved by $\partial^*_{\Sigma_0,\epsilon}$, hence we have a map $\partial^*_{\Sigma_0,\epsilon} : \mathfrak{D}(\Sigma_0) \to \mathfrak{D}'(\mathbb{M}^2)$. This map has an associated Schwartz kernel $K \in \mathfrak{D}'(\mathbb{M}^2 \times \Sigma_0)$. We can then use this kernel to compute the wavefront sets of functionals in the image of $\partial^*_{\Sigma_0,\epsilon}$, as $\forall\, \varphi \in \mathfrak{E}(\mathbb{M}^2), h \in \mathfrak{D}((\mathbb{M}^2)^n)$

$$\left\langle (\partial^*_{\Sigma_0,\epsilon}F)^{(n)}[\varphi], h \right\rangle = \left\langle K^{\otimes n}, h \otimes F^{(n)}[\partial_{\Sigma_0,\epsilon}\varphi] \right\rangle \tag{8.14}$$

where the first variable of each copy of $K$ is paired with a variable of $h$ and so the rest with $F^{(n)}[\partial_{\Sigma_0,\epsilon}\varphi]$. We can then use [Hör15, Theorem 8.2.14] once again to estimate $\mathrm{WF}((\partial^*_{\Sigma_0,\epsilon}F)^{(n)}[\varphi])$ given estimates for $\mathrm{WF}(K^{\otimes n})$ and $\mathrm{WF}(F^{(n)}[\partial_{\Sigma_0,\epsilon}\varphi])$.

By inspection, and using the appropriate coordinates, the integral kernel $K$ may be written as

$$K(u, v, s) = -\partial_u \left( \delta(u + s)\delta_\epsilon(\tfrac{u+v}{2}) \right), \tag{8.15}$$

from which we may deduce that

$$\mathrm{WF}(K) = \left\{ (u, v, -u; \xi, 0, \xi) \in \dot{T}^*(\Sigma_0 \times \mathbb{M}^2) \right\}. \tag{8.16}$$

After some work, the corresponding estimate is then

$$\begin{aligned} \mathrm{WF}((\partial^*_{\Sigma_0,\epsilon}F)^{(n)}[\varphi]) \subseteq \Big\{ &(u_1, v_1, \ldots, u_n, v_n; \xi_1, 0, \ldots, \xi_n, 0) \in \dot{T}^*(\mathbb{M}^2)^n \mid \\ &(-u_1, \ldots, -u_n; \xi_1, \ldots, \xi_n) \in \mathrm{WF}(F^{(n)}[\partial_{\Sigma_0,\epsilon}\varphi]) \Big\}. \end{aligned} \tag{8.17}$$

The wavefront set condition on $F^{(n)}[\partial_{\Sigma_0,\epsilon}\varphi]$ then precludes the option that $\xi_i$ all have the same sign, which is precisely what we need to conclude that $\mathrm{WF}((\partial^*_{\Sigma_0,\epsilon}F)^{(n)}[\varphi]) \cap (V^n_+ \cup V^n_-) = \emptyset$, i.e. $\partial^*_{\Sigma_0,\epsilon} : \mathfrak{F}_c(\Sigma_0) \to \mathfrak{F}_{\mu c}(\mathbb{M}^2)$.

Having established this, the more straightforward task is to show that this map preserves Poisson brackets. Spelling it out, we need to demonstrate that $\forall\, F, G \in \mathfrak{F}_c(\Sigma_0), \varphi \in \mathfrak{E}(\mathbb{M}^2)$

$$\left\langle E, (\partial^*_{\Sigma_0,\epsilon}F)^{(1)}[\varphi] \otimes (\partial^*_{\Sigma_0,\epsilon}G)^{(1)}[\varphi]\right\rangle = \left\langle E_{\Sigma_0}, F^{(1)}[\partial_{\Sigma_0,\epsilon}\varphi] \otimes G^{(1)}[\partial_{\Sigma_0,\epsilon}\varphi]\right\rangle . \tag{8.18}$$

This simply amounts to the statement that $(\partial_{\Sigma_0,\epsilon} \otimes \partial_{\Sigma_0,\epsilon})E = (\partial_{\Sigma_0} \otimes \partial_{\Sigma_0})E$. This is easily verified by looking at the precise form (8.3) of $E$. However, note also that, as a map, the image of $E$ is in the kernel of the wave-operator $P$, hence $\partial_{\Sigma_0,\epsilon}E = \partial_{\Sigma_0}E$. By skew-symmetry, we can also verify that acting on the second argument of $E$ with $\partial_{\Sigma_0,\epsilon}$ behaves the same way.

Lastly, the fact that this map is injective is a direct consequence of the fact that $\partial_{\Sigma_0,\epsilon}$ is surjective. □

Finally, we can use our embedding theorems to create analogous embeddings for the chiral algebra of an arbitrary element $(\Sigma, \mathcal{M}) \in$ CCauchy. Suppose $(\rho, \chi) : (\Sigma, \mathcal{M}) \to (\Sigma_0, \mathbb{M}^2)$ is a CCauchy morphism. Starting from the equation $\partial_\Sigma \chi^* = \rho^*_{(1)}\partial_{\Sigma_0}$, we might try to define a regularised chiral derivative for $\Sigma$ by $\partial_{\Sigma,\epsilon} = \rho^*_{(1)}\partial_{\Sigma_0}\chi_*$, where we make sense of the pushforward by only asking for $\chi_*\varphi(x) = \varphi(\chi^{-1}(x))$ to hold in some neighbourhood of $\Sigma_0$. As $\partial_{\Sigma_0}\varphi_0$ only depends on $d\varphi_0|_{\Sigma_0}$, we see that $\partial_{\Sigma,\epsilon}$ is well defined. However, by smoothing out $\partial_{\Sigma_0}$ to $\partial_{\Sigma_0,\epsilon}$, we increased the region it is sensitive to. In order to make sense of $\partial_{\Sigma_0,\epsilon}\chi_*$, we need to make sure that the support of $\partial_{\Sigma_0,\epsilon}$ is contained within the image of $\chi$. For the particular way we have constructed $\partial_{\Sigma_0,\epsilon}$, this means that $\mathrm{Img}\chi$ must contain the $t = \pm\epsilon$ Cauchy surfaces $\Sigma_{\pm\epsilon}$. In effect this is because $\partial_{\Sigma_0,\epsilon}$ defines a map $\mathfrak{E}(U) \to \mathfrak{E}(\Sigma_0)$ for any open, causally-convex neighbourhood $U$ of $\mathcal{J}^-(\Sigma_\epsilon) \cap \mathcal{J}^+(\Sigma_{-\epsilon})$. Hence, for any embedding $\chi : \mathcal{M} \to \mathbb{M}^2$ such that $\chi(\mathcal{M}) \supset U$, we can define $\partial_{\Sigma_0,\epsilon} \circ \chi_* := \partial_{\Sigma_0,\epsilon}(\chi^{-1})^*$, where $\chi^{-1} : \chi(\mathcal{M}) \xrightarrow{\sim} \mathcal{M}$.

This might seem simple, as we can make $\epsilon$ arbitrarily small. However, if we consider the case where $\mathcal{M} \subset \mathbb{M}^2$ is the space in between the Cauchy surfaces expressed in $(t, x)$ coordinates as $\Sigma_\pm = \{(\pm e^{-x^2}, x)\}_{x\in\mathbb{R}}$, then clearly there is no $\epsilon > 0$ such that $\Sigma_{\pm\epsilon} \subset \mathcal{M}$.

The solution in this case is to find a new conformal embedding $\mathcal{M} \hookrightarrow \mathbb{M}^2$, which 'expands' $\mathcal{M}$ to contain these Cauchy surfaces. We accomplish this using the following proposition. Note that in the following, the relation $\prec$ between subsets of a spacetime $\mathcal{M}$ is defined such that $U \prec V$ if and only if there is no future-directed causal curve in $\mathcal{M}$ from $v \in V$ to $u \in U$, in which case we say that $U$ *is no later than* V. For Cauchy surfaces $\Sigma \prec \Sigma'$, this necessarily implies that every point $\Sigma$ is either in the past, or is spacelike separated from, every point in $\Sigma'$.

**Lemma 8.2.2.** *Let $\mathcal{M} \subset \mathbb{M}^2$ be an open, causally-convex neighbourhood of the $t = 0$ Cauchy surface $\Sigma_0$, and let $\Sigma_\pm$ be a pair of Cauchy surfaces of $\mathcal{M}$ such that $\Sigma_- \prec \Sigma_0 \prec \Sigma_+$. Then there exists a* $\mathsf{CLoc}$ *morphism $\chi : \mathcal{M} \to \mathbb{M}^2$, such that $\chi(\Sigma_0) = \Sigma_0$ and $\chi(\Sigma_-) \prec \Sigma_{-1} \prec \Sigma_1 \prec \chi(\Sigma_+)$, where $\Sigma_{\pm 1}$ are the $t = \pm 1$ Cauchy surfaces of $\mathbb{M}^2$.*

*Proof.* For the purposes of this proof, it will be convenient to work in $(t, x)$ coordinates rather than our usual null $(u, v)$ system. Given this, we can express each Cauchy surface as $\Sigma_\pm = \{(\pm t_\pm(x), x)\}_{x\in\mathbb{R}}$ for a pair of smooth functions $t_\pm$ satisfying $t_\pm(x) > 0, |t'_\pm(x)| < 1 \, \forall \, x \in \mathbb{R}$.

The goal is to find some $\rho \in \mathrm{Diff}_+(\Sigma_0)$ such that its extension to an embedding $\chi : \mathcal{M} \to \mathbb{M}^2$ takes $\Sigma_+$ to the future of $\Sigma_1$ and $\Sigma_-$ to the past of $\Sigma_{-1}$. If we consider only $\Sigma_+$ (i.e. we look at the special case where $t_+ = t_-$), then we need to show that $t > 1$ for every $(t, x) \in \chi(\Sigma_+)$. Using (7.8), and noting that in our case $\pi^{\Sigma_0}_{\ell/r}(t, x) = (0, x \pm t)$, we can formulate the equivalent condition for $\rho$ as

$$\rho(x + t_+(x)) - \rho(x - t_+(x)) > 2. \tag{8.19}$$

One choice of $\rho$ which satisfies this inequality is

$$\rho(x) := 2 \int_{x'=0}^{x} \frac{\mathrm{d}x'}{t_+(x')}. \tag{8.20}$$

Note that $\rho$ is well defined as we always have that $t_+(x) > 0$, and from $\rho'(x) = \frac{2}{t_+(x)}$, we see that $\rho \in \mathrm{Diff}_+(\Sigma_0)$. To see that $\rho$ satisfies the inequality (8.19), we substitute our choice into the above expression to find

$$\rho(x + t_+(x)) - \rho(x - t_+(x)) = 2 \int_{x-t_+(x)}^{x+t_+(x)} \frac{\mathrm{d}x'}{t_+(x')} = \frac{4t_+(x)}{t_+(x + c)}, \tag{8.21}$$

for some $c \in (-t_+(x), t_+(x))$, using the intermediate value theorem. Finally, using $|t'_+(x)| < 1$, we see that $t_+(x+c) \in (t_+(x)-c, t_+(x)+c) \subset (0, 2t_+(x))$, hence $4t_+(x)/t_+(x+c) > 2$ as required.

In the case where $t_- \neq t_+$, one can use similar arguments to show that

$$\rho(x) = 2 \int_{x'=0}^{x} \left( \frac{1}{t_+(x)} + \frac{1}{t_-(x)} \right) \mathrm{d}x' \tag{8.22}$$

is an element of $\mathrm{Diff}_+(\mathbb{R})$ which satisfies both the necessary inequalities: $|\rho(x+t_\pm(x)) - \rho(x - t_\pm(x))| > 2$. $\square$

With that, we are finally able to establish how $\mathfrak{P}_\ell(\Sigma, \mathcal{M})$ embeds into $\mathfrak{P}(\mathcal{M})$ in the general case.

**Theorem 8.2.3.** *For every* $(\Sigma, \mathcal{M}) \in \mathsf{CCauchy}$*, there exists an injective Poisson algebra homomorphism* $\partial^*_{\Sigma,\epsilon} : \mathfrak{P}_\ell(\Sigma, \mathcal{M}) \to \mathfrak{P}(\mathcal{M})$*. Moreover, by selecting such a map for each object in* $\mathsf{CCauchy}$*, we obtain a natural transformation*

$$\partial^*_\epsilon : \mathfrak{P}_\ell \Rightarrow \mathfrak{P}_{\mathrm{on}} \circ \Pi_2. \tag{8.23}$$

*Proof.* To first define the maps $\partial_{\Sigma,\epsilon}$, we must treat topologically planar spacetimes separately from cylindrical spacetimes. So let us first suppose that $\Sigma \simeq \mathbb{R}$. We can define a *time function* on $\mathcal{M}$ such that $\Sigma = t^{-1}\{0\}$. Using this, we then specify a pair of Cauchy surfaces $\Sigma_\pm \subset \mathcal{M}$ by $t^{-1}\{\pm 1\}$. By taking a diffeomorphism $\rho_0 : \Sigma \xrightarrow{\sim} \Sigma_0 \subset \mathbb{M}^2$, we then obtain a $\mathsf{CLoc}$ morphism $\mathcal{M} \to \mathbb{M}^2$ by theorem 7.1.3. Let $\Sigma_{t_0}$ denote the $t = t_0$ Cauchy surface in $\mathbb{M}^2$. If there exists some $\epsilon > 0$ such that $\mathcal{J}^+(\Sigma_{-\epsilon}) \cap \mathcal{J}^-(\Sigma_\epsilon) \subset \chi_0(\mathcal{M})$, then already we can define the map $\partial_{\Sigma_0,\epsilon}\chi_{0*}$ by the argument preceding lemma 8.2.2, hence we can set $\partial_{\Sigma,\epsilon} := \rho^*_{(1)}\partial_{\Sigma_0,\epsilon}\chi_* : \mathfrak{E}(\mathcal{M}) \to \mathfrak{E}(\Sigma)$.

If this is not the case, we simply apply lemma 8.2.2 to $\chi_0(\mathcal{M})$ with Cauchy surfaces $\chi_0(\Sigma_\pm)$, to obtain a new embedding $\chi : \mathcal{M} \to \mathbb{M}^2$, whereupon we can use any value $\epsilon \in (0, 1)$.

For $\Sigma \simeq S^1$, we proceed along similar lines to before, defining a pair of Cauchy surfaces $\Sigma_- \prec \Sigma \prec \Sigma_+$ and a diffeomorphism $\Sigma \xrightarrow{\sim} \Sigma_0 \subset \mathcal{E}$ extending to a $\mathsf{CLoc}$ morphism $\chi : \mathcal{M} \to \mathcal{E}$. This time, however, due to the compactness of $S^1$ we can deduce that, if $\chi(\Sigma_\pm) = \{(\pm t_\pm(x), x)\}_{x\in\mathbb{R}}$, then both $t_\pm(x)$ are bounded from below, as $\Sigma \cap \Sigma_\pm = \emptyset$. In particular, there exists some $\epsilon > 0$ such that $\mathrm{Img}\, t_+ \cup \mathrm{Img}\, t_- \subset (\epsilon, \infty)$, hence $\chi(\Sigma_-) \prec \Sigma_{-\epsilon} \prec \Sigma_0 \prec \Sigma_\epsilon \prec \chi(\Sigma_+)$, and we may define $\partial_{\Sigma,\epsilon} := \rho^*_{(1)}\partial_{\Sigma_0,\epsilon}\chi_* : \mathfrak{E}(\mathcal{M}) \to \mathfrak{E}(\Sigma)$.

Now that we have maps between configuration spaces, we must now show that they are Poisson algebra homomorphisms, and that they satisfy the desired naturality condition. The fact that $\partial^*_{\Sigma,\epsilon}F \in \mathfrak{F}_{\mu c}(\mathcal{M})$ follows from the fact that $(\chi^{-1})_* : \mathfrak{F}_{\mu c}(\mathcal{M}) \to \mathfrak{F}_{\mu c}(\chi(\mathcal{M}))$. To show the map is Poisson, by following similar arguments to proposition 8.1.3 it suffices to show that $(\partial_{\Sigma,\epsilon} \otimes \partial_{\Sigma,\epsilon})E_\mathcal{M} = E_\Sigma$. This can be shown readily as

$$\begin{aligned}(\partial_{\Sigma,\epsilon} \otimes \partial_{\Sigma,\epsilon})E_{\mathcal{M}} &= (\rho_1^* \otimes \rho_1^*)(\partial_{\Sigma_0,\epsilon} \otimes \partial_{\Sigma_0,\epsilon})E_{\chi(\mathcal{M})} \\ &= (\rho_1^* \otimes \rho_1^*)(\partial_{\Sigma_0,\epsilon} \otimes \partial_{\Sigma_0,\epsilon})E_{\mathcal{M}_0} \\ &= (\rho_1^* \otimes \rho_1^*)E_{\Sigma_0} \\ &= E_{\Sigma}. \end{aligned} \tag{8.24}$$

where in the second line we have used the fact that the support of $\partial_{\Sigma_0,\epsilon}$ is within the image of $\chi(\mathcal{M})$, where $E_{\chi(\mathcal{M})}$ coincides with $E_{\mathcal{M}_0}$.

Finally we consider the naturality. Let $(\rho, \chi) : (\Sigma, \mathcal{M}) \to (\widetilde{\Sigma}, \widetilde{\mathcal{M}})$ be a CCauchy morphism. Suppose that we have constructed $\partial_{\Sigma,\epsilon}$ and $\partial_{\widetilde{\Sigma},\widetilde{\epsilon}}$. For simplicity we shall also use $\partial^*_{\Sigma,\epsilon}$ to denote the map $\mathfrak{P}_\ell(\Sigma, \mathcal{M}) \to \mathfrak{P}_{\text{on}}(\mathcal{M})$. To show that

$$\begin{tikzcd}\mathfrak{P}_\ell(\Sigma, \mathcal{M}) \arrow[r, "\partial^*_{\Sigma,\epsilon}"] \arrow[d, "\mathfrak{P}_\ell(\rho,\chi)"] & \mathfrak{P}_{\text{on}}(\mathcal{M}) \arrow[d, "\mathfrak{P}_{\text{on}}\chi"] \\ \mathfrak{P}_\ell(\widetilde{\Sigma}, \widetilde{\mathcal{M}}) \arrow[r, "\partial^*_{\widetilde{\Sigma},\widetilde{\epsilon}}"] & \mathfrak{P}_{\text{on}}(\widetilde{\mathcal{M}})\end{tikzcd} \tag{8.25}$$

commutes, we need only show that $F[\partial_{\Sigma,\epsilon}\chi^*\varphi] = F[\rho^*_{(1)}\partial_{\widetilde{\Sigma},\widetilde{\epsilon}}\varphi]$, for every $F \in \mathfrak{P}_\ell(\Sigma, \mathcal{M})$ and $\varphi \in \operatorname{Ker} P_{\widetilde{\mathcal{M}}}$. Immediately we have that $\partial_{\widetilde{\Sigma},\widetilde{\epsilon}}\varphi = \partial_{\widetilde{\Sigma}}\varphi$, and also $\chi^* : \operatorname{Ker} P_{\widetilde{\mathcal{M}}} \to \operatorname{Ker} P_{\mathcal{M}}$, hence $\partial_{\Sigma,\epsilon}\chi^*\varphi = \partial_\Sigma\chi^*\varphi$. Thus, both functionals are equal to $F[\partial_\Sigma\chi^*\varphi]$ and the diagram commutes. □

*Remark* 8.2.4. There are several comments to make at this point.

Firstly, it is important to note that all of the arbitrary decision making that goes into the definition of $\partial^*_{\Sigma,\epsilon}$, namely the choice of diffeomorphism $\rho : \Sigma \xrightarrow{\sim} \Sigma_0$ and the corresponding choice of $\partial_{\Sigma_0,\epsilon}$ is ultimately irrelevant as all choices result in the same map $\mathfrak{P}_\ell(\Sigma, \mathcal{M}) \to \mathfrak{P}_{\text{on}}(\mathcal{M})$ once we take the quotient by $\mathfrak{I}_S(\mathcal{M})$.

Secondly, even though we also include the full spacetime $\mathcal{M}$ in our definition of the algebra $\mathfrak{P}_\ell(\Sigma, \mathcal{M})$, it is important to note that this is mostly a bookkeeping device. For instance, if $\widetilde{\Sigma} = \Sigma$, and $\chi$ is simply the inclusion for $\mathcal{M} \subset \widetilde{\mathcal{M}}$, then $\rho$ is the identity and $\mathfrak{P}_\ell(\rho, \chi)$ is the identity on $\mathfrak{F}_c(\Sigma)$.

Thus we have found a theory of lower dimensionality which still embeds naturally into the full *on-shell* spacetime algebra. Of course, we would not expect a sensible embedding into the off-shell algebra, as $\mathfrak{P}_\ell(\Sigma, \mathcal{M})$ is an algebra on the initial data for solutions in $\mathcal{M}$, hence it is intrinsically on-shell.

Moreover, unlike the canonical algebra of Cauchy data which we discuss in the following section, this embedding is done without the inclusion of an auxiliary field (namely the conjugate momentum in the canonical algebra). Roughly speaking, this is because one is able to write a solution to the wave equation as a sum of two solutions to *first order* PDEs. In other words, it is a definitively chiral phenomenon.

## 8.3 COMPARISON TO EQUAL TIME COMMUTATION RELATIONS

An alternative way of deciding the form of the chiral bracket is by comparison to the *canonical* or *equal-time* Poisson bracket. In terms of integral kernels, and with respect to a Cauchy surface $\Sigma \subset \mathcal{M}$, this might be typically written as

$$\{\Phi(\mathbf{x}), \Pi(\mathbf{y})\}^{\Sigma}_{\text{can}} = \delta_\Sigma(\mathbf{x}, \mathbf{y}), \tag{8.26}$$

where $\mathbf{x}, \mathbf{y} \in \Sigma$ and $\delta_\Sigma \in \mathfrak{D}'(\Sigma^2)$ is the Dirac delta with support on the diagonal of $\Sigma^2$.

To make this more precise, we define the space of Cauchy data on $\Sigma$ to be $\mathcal{C}(\Sigma) = \mathfrak{D}(\Sigma, \mathbb{R}^2)$, i.e. pairs of smooth functions on $\Sigma$ with compact support. For each $f \in \mathfrak{D}(\Sigma)$, can then denote the regular linear observables on $\mathcal{C}(\Sigma)$ by

$$\Phi(f)[\psi, \pi] = \int_\Sigma f\psi \, \mathrm{d}V_\Sigma, \qquad \Pi(f)[\psi, \pi] = \int_\Sigma f\pi \, \mathrm{d}V_\Sigma. \tag{8.27}$$

On which we define the canonical Poisson bracket

$$\{\Phi(f), \Pi(g)\}^{\Sigma}_{\text{can}} = \int_\Sigma fg \, \mathrm{d}V_\Sigma. \tag{8.28}$$

For the massless scalar field, if $(\phi, \pi)$ are the Cauchy data for a solution $\varphi$, then $\phi = \varphi|_\Sigma$ and $\pi = \dot{\varphi}|_\Sigma$, where $\dot{\varphi}$ is the derivative of $\varphi$ along the future-directed normal vector to $\Sigma$. This suggests the map $\rho_\pm : \mathcal{C}(\Sigma) \to \mathfrak{E}(\Sigma)$, defined by $\rho_\pm(\phi, \pi) = \frac{1}{2}(\pi \mp {*d\varphi})$, sends Cauchy data to the associated chiral configuration .

If consider the chiral boson $\Psi$ from (8.1), then the pullback of $\Psi_\Sigma(f)$ along $\rho_+$ is

$$(\rho_+^* \Psi_\Sigma(f))[\phi, \pi] = \frac{1}{2} \int_\Sigma (f\pi - f \, {*d\varphi}) \, \mathrm{d}V_\Sigma.$$

Noting that $\int_\Sigma f(*d\varphi)\mathrm{d}V_\Sigma = \int_\Sigma f d\varphi = -\int_\Sigma \varphi df = -\int_\Sigma \varphi(*df)\mathrm{d}V_\Sigma$, we can write this pullback in terms of the observables in (8.27) as

$$\rho_+^* \Psi_\Sigma(f) = \frac{1}{2} \left( \Pi(f) + \Phi(*df) \right). \tag{8.29}$$

We can then consider the pullback of the canonical Poisson bracket along this map, which we may then express as

$$\begin{aligned}\{\Psi_\Sigma(f),\Psi_\Sigma(g)\}_\ell^\Sigma &= \left\{\rho_+^*\Psi_\Sigma(f),\rho_+^*\Psi_\Sigma(g)\right\}_{\text{can}}^\Sigma\\ &= \frac{1}{4}\left\{\Pi(f)+\Phi(*df),\Pi(g)+\Phi(*dg)\right\}_{\text{can}}^\Sigma\\ &= \frac{1}{2}\left\{\Phi(*df),\Pi(g)\right\}_{\text{can}}^\Sigma\\ &= -\frac{1}{2}\int_\Sigma f dg.\end{aligned}$$

Which agrees exactly with eq. (8.8).

## 8.4 CHIRAL PRIMARY FIELDS

We have already discussed in chapter 5, *fields* are a central aspect of any approach to quantum field theory, however the precise definition of what a field *is* varies considerably. Previously, we gave a definition of fields rooted in the principle of local covariance, where theories are functors Loc $\to$ Obs for some suitable category of observables, and fields are natural transformations from the functor of test functions to the functor describing the theory. By defining new spacetime categories which accounted for conformal isometries, and suitably modifying the functor assigning spaces their test functions, we could say that a field was primary if the naturality condition held for this expanded set of morphisms.

In [Sch08, Chapter 9], Schottenloher provides a characterisation of primary fields in 2D Euclidean CFT which we summarise below. Firstly, similarly to Wightman QFTS, a field is defined as a tempered distribution over $\mathbb{C}$ with values in unbounded operators on some Hilbert space. The condition for such a field to be primary (with weight $\mu$) can be *formally* expressed as the condition that, for every holomorphic map $z \mapsto w(z)$

$$U(w)\Phi(z)U(w)^{-1} = \left(\frac{dw}{dz}\right)^\mu \Phi(w(z)) \tag{8.30}$$

where $U$ is a unitary representation of the holomorphic transformations on $\mathcal{H}$. To obtain the precise definition, one considers the infinitesimal transformation $w(z) = z + \epsilon w_0(z)$, for a holomorphic map $w_0$. Differentiating each side with respect to $\epsilon$ and evaluating at $\epsilon = 0$ then generates the correct equation. In particular, on the left one obtains an action of holomorphic functions on unbounded operators by derivations. This is assumed (as one of the axioms) to be generated by brackets/commutators with the stress-energy tensor, which we explore in section 8.6. Recalling that $\Phi(x)$ is really the integral kernel of a distribution, we can integrate both sides of (8.30) with some

$f \in \mathfrak{S}(\mathbb{C})$ to obtain

$$U(w)\Phi(f)U(w)^{-1} = \Phi(w_*^{(\mu-1)}f), \tag{8.31}$$

where $\Phi(f) = \int_{\mathbb{C}} \Phi(z)f(z)\,\mathrm{d}z$ and

$$w_*^{(\mu-1)}f := \left[\left(\frac{dw}{dz}\right)^{\mu-1} \cdot f\right] \circ w^{-1} \tag{8.32}$$

is a map $\mathfrak{S}(\mathbb{C}) \to \mathfrak{S}(\mathbb{C})$. (To make this more precise, for $w(z) = z + \epsilon w_0(z)$, we may take $w^{-1}(z) = z - \epsilon w_0(z)$ in order to generate the correct infinitesimal action.)

Forgetting the trouble with extending these relations beyond the infinitesimal generators, we can use this relation to characterise primary fields as *equivariant maps* between two representations of the 'conformal group' of holomorphic functions. One representation acting on $\mathfrak{B}(\mathcal{H})$, the algebra of observables, the other acting on $\mathfrak{S}(\mathbb{C})$, the space of test functions.

We should now be clear and state that we do not have a complete definition of a chiral primary field which unites these two perspectives. However, we can infer several qualitative features, and by restricting the scope of our locally covariant construction, we can obtain a partial definition of a chiral primary field.

We begin by noting how the characterisation of primary fields as equivariant maps is very close to the definition of locally covariant fields as natural transformations.

Recall that, given any object $c$ of a category $\mathcal{C}$, there is a group $\mathrm{Aut}_{\mathcal{C}}(c)$ comprising the invertible morphisms $c \to c$. If we restrict our attention to this subcategory, a functor $F : \mathrm{Aut}_{\mathcal{C}}(c) \to \mathsf{Vec}$ is simply a representation of $\mathrm{Aut}_{\mathcal{C}}(c)$ on the space $F(c)$, and a natural transformation $F \Rightarrow G$ between two functors is a single map $F(c) \to G(c)$ which is equivariant with respect to the two representations of $\mathrm{Aut}_{\mathcal{C}}(c)$. Hence if we consider just one framed spacetime $\mathcal{M} = (M, (e^\ell, e^r)) \in \mathsf{CFLoc}$ as in definition 5.1.1, and take the group $\mathrm{Aut}(\mathcal{M})$ of conformally admissible diffeomorphisms $\mathcal{M} \to \mathcal{M}$, then $\mathfrak{D}^{(\mu,\widetilde{\mu})}$ defines a representation of $\mathrm{Aut}(\mathcal{M})$ on $\mathfrak{D}(M)$, any functor $\mathfrak{A} : \mathsf{CFLoc} \to \mathsf{Obs}$ defines a representation of $\mathrm{Aut}(\mathcal{M})$ on $\mathfrak{A}(\mathcal{M})$, and a primary field of weight $(\mu, \widetilde{\mu})$ defines an equivariant map between the two.

Notably, we had to go from $\mathsf{CLoc}$ to a new category, $\mathsf{CFLoc}$ which assigns additional data to each spacetime in the form of a global frame. In the spirit of [Few18], this is so that tensor fields such as the stress-energy tensor may be separated into scalar components. As we do not wish for the theory to depend on this additional data, we assumed that the functor $\mathsf{CFLoc} \to \mathsf{Obs}$ factorised into the composition of the surjective functor $\mathfrak{p} : \mathsf{CFLoc} \to \mathsf{CLoc}$, which forgets about the additional frame data, and a functor $\mathsf{CLoc} \to \mathsf{Obs}$ defining the theory.

It is at present unclear how this procedure generalises to the chiral setting for an arbitrary globally hyperbolic spacetime. However, if we restrict our attention to Minkowski spacetime, then we can continue our analysis further. We define the category $\mathsf{Cauchy}(\mathbb{M}^2)$ as the full subcategory of $\mathsf{CCauchy}$ comprising objects $(\Sigma, U)$ where $U \subseteq \mathbb{M}^2$ is open and causally convex.

We can implement the weighted pushforwards as a family of functors $\mathfrak{D}_\ell^{(\mu)} : \mathsf{Cauchy}(\mathbb{M}^2) \to \mathsf{TVec}$ such that $\mathfrak{D}_\ell^{(\mu)}(\Sigma, U) = \mathfrak{D}(\Sigma)$, and for a morphism $(\rho, \chi) : (\Sigma, U) \to (\widetilde{\Sigma}, \widetilde{U})$ $\mathfrak{D}_\ell^{(\mu)}(\rho, \chi)(f) := \rho_*(\omega_\ell|_\Sigma^{\mu-1} f)$, where $\chi^* du = \omega_\ell du$. Lastly, we define the functor $\mathfrak{U} : \mathsf{Cauchy}(\mathbb{M}^2) \to \mathsf{CFLoc}$ by $\mathfrak{U}(\Sigma, U) = U$, equipped with the canonical frame it inherits from $\mathbb{M}^2$. We can then relate our weighted representations using the following result

**Proposition 8.4.1.** *For an object* $(\Sigma, U) \in \mathsf{Cauchy}(\mathbb{M}^2)$*, define the map* $\eta_{(\Sigma,U)} : \mathfrak{D}(U) \to \mathfrak{D}(\Sigma)$ *by*

$$(\eta_{(\Sigma,U)} h)(s) := \int_{v \in \mathbb{R}} h(-s, v) \mathrm{d}v, \tag{8.33}$$

*where the coordinate* $s$ *on* $\Sigma$ *is obtained as the restriction of the map* $\mathbb{M}^2 \to \mathbb{R}; (u, v) \mapsto -u$*. Then* $\eta$ *defines a natural transformation* $\mathfrak{D}^{(\mu,0)} \circ \mathfrak{U} \Rightarrow \mathfrak{D}_\ell^{(\mu)}$*, i.e. for every morphism* $(\rho, \chi) : (\Sigma, U) \to (\widetilde{\Sigma}, \widetilde{U})$*, the following diagram commutes.*

$$\begin{tikzcd}
\mathfrak{D}(U) \arrow[r, "\eta_{(\Sigma,U)}"] \arrow[d, "\mathfrak{D}^{(\mu,0)}\chi"] & \mathfrak{D}(\Sigma) \arrow[d, "\mathfrak{D}^{(\mu)}(\rho,\chi)"] \\
\mathfrak{D}(\widetilde{U}) \arrow[r, "\eta_{(\widetilde{\Sigma},\widetilde{U})}"] & \mathfrak{D}(\widetilde{\Sigma})
\end{tikzcd} \tag{8.34}$$

*Proof.* Suppose that $\Sigma$ is sent to the interval $\mathcal{I} \subseteq \mathbb{R}$ under the map $(u, v) \mapsto -u$. We can then express $\Sigma$ as the graph $\{(-s, \gamma(s)) \in U\}_{s \in \mathcal{I}}$ for some smooth embedding $\gamma : \mathcal{I} \hookrightarrow \mathbb{R}$. We can similarly express $\widetilde{\Sigma}$ in terms of some $\widetilde{\gamma} : \widetilde{\mathcal{I}} \hookrightarrow \mathbb{R}$. Using (7.8), we can write $\chi$ explicitly in null-coordinates as

$$\chi(u, v) = (-\rho(-u), \widetilde{\gamma}\rho\gamma^{-1}(v)), \tag{8.35}$$

where $\gamma^{-1} : \gamma(\mathcal{I}) \to \mathcal{I}$, and we have identified $\rho$ with a smooth embedding $\mathcal{I} \hookrightarrow \widetilde{\mathcal{I}}$ using our coordinate system. From this, we may calculate the conformal factors of $\chi$ as

$$\omega_\ell(u, v) = \omega_\ell(u) = \rho'(-u), \qquad \omega_r(u, v) = \omega_r(v) = \frac{\widetilde{\gamma}'(\rho\gamma^{-1}(v))}{\gamma'(\gamma^{-1}(v))} \rho'(\gamma^{-1}(v)). \tag{8.36}$$

In particular this means that $\omega_\ell(-s, v) = \omega_\ell|_\Sigma(s) = \rho'(s)$.

With our conformal factors suitably equated, we may now compute each path around the above diagram. Let $h \in \mathfrak{D}(\mathbb{M}^2)$, then

$$\begin{aligned} &(\rho_*^{(1-\mu)} \circ \eta(h))(s) = \omega_\ell|_\Sigma^{\mu-1}(s) \int_{v\in\mathbb{R}} h(-\rho^{-1}(s), v)\,\mathrm{d}v \\ &(\eta \circ \chi_*^{(1-\mu,1)}(h))(s) = \\ &\int_{v'\in\mathbb{R}} \omega_\ell(-\rho^{-1}(s))^{\mu-1}\omega_r(\gamma\rho^{-1}\widetilde{\gamma}^{-1}(v'))^{-1}h(-\rho^{-1}(s), \gamma\rho^{-1}\widetilde{\gamma}^{-1}(v'))\,\mathrm{d}v'. \end{aligned} \tag{8.37}$$

We then make the change of variables for the second integral $v = \gamma\rho\widetilde{\gamma}^{-1}(v')$, noting that then $dv = \omega_r(v)^{-1}dv'$ hence the two integrals are in fact equal. □

*Remark* 8.4.2. What made this diagram commute were the facts that, firstly $\omega_\ell$ depended only on $u$, and hence could be taken out of the integral and secondly that $\omega_r$ depended only on $v$ and naturally accounted for the change of variables necessary to relate the integrals. Without a generalisation of these statements for a general morphism of CFLoc, this argument cannot be readily generalised.

Given that Cauchy($\mathbb{M}^2$) is a subcategory of CCauchy, we already have a classical theory defined on it, $\mathfrak{P}_\ell|_{\mathbb{M}^2}$ : Cauchy($\mathbb{M}^2$) → Poi. Thus, for any natural transformation $\Psi : \mathfrak{D}_\ell^{(\mu)} \Rightarrow \mathfrak{P}_\ell|_{\mathbb{M}^2}$, the maps $\eta$ from the proposition 8.4.1, along with the natural transformation $\partial_\epsilon^* : \mathfrak{P}_\ell \Rightarrow \mathfrak{P}_{\text{on}} \circ \Pi_2$ from theorem 8.2.3 can be composed horizontally with $\Psi$ to define a natural transformation

$$\partial_\epsilon^* \circ \Psi \circ \eta : \mathfrak{D}^{(\mu,0)} \circ \mathfrak{U} \Rightarrow \mathfrak{P}_{\text{on}} \circ \mathfrak{p} \circ \mathfrak{U}. \tag{8.38}$$

Unpacking the definition, this means that, for every conformally admissible embedding $\chi : U \longrightarrow \widetilde{U}$ which restricts to a map $\rho : \Sigma \to \widetilde{\Sigma}$ between Cauchy surfaces, the following diagram commutes.

$$\begin{tikzcd} \mathfrak{D}(U) \arrow[r, "\eta_{(\Sigma,U)}"] \arrow[d, "\mathfrak{D}^{(\mu,0)}\chi"] & \mathfrak{D}(\Sigma) \arrow[r, "\Psi_{(\Sigma,U)}"] \arrow[d, "\mathfrak{D}_\ell^{(\mu)}(\rho{,}\chi)"] & \mathfrak{P}_\ell(\Sigma, U) \arrow[r, "\partial_{\Sigma,\epsilon}^*"] \arrow[d, "\mathfrak{P}_\ell(\rho{,}\chi)"] & \mathfrak{P}_{\text{on}}(U) \arrow[d, "\mathfrak{P}_{\text{on}}\chi"] \\ \mathfrak{D}(\widetilde{U}) \arrow[r, "\eta_{(\widetilde{\Sigma},\widetilde{U})}"] & \mathfrak{D}(\widetilde{\Sigma}) \arrow[r, "\Psi_{(\widetilde{\Sigma},\widetilde{U})}"] & \mathfrak{P}_\ell(\widetilde{\Sigma}, \widetilde{U}) \arrow[r, "\partial_{\widetilde{\Sigma},\widetilde{\epsilon}}^*"] & \mathfrak{P}_{\text{on}}(\widetilde{U}) \end{tikzcd} \tag{8.39}$$

If one can then further show that, for any pair of Cauchy surfaces $\Sigma, \widetilde{\Sigma} \subset U$,

$$\partial_{\Sigma,\epsilon}^* \circ \Psi_{(\Sigma,U)} \circ \eta_{(\Sigma,U)} = \partial_{\widetilde{\Sigma},\widetilde{\epsilon}}^* \circ \Psi_{(\widetilde{\Sigma},U)} \circ \eta_{(\widetilde{\Sigma},U)}, \tag{8.40}$$

then it is clear that we in fact have a natural transformation $\mathfrak{D}^{(\mu,0)} \Rightarrow \mathfrak{P}_{\text{on}} \circ \mathfrak{p}$, i.e. a primary field of weight $(\mu, 0)$ in the sense of section 5.2.

As an example where this is the case, we can consider the *chiral boson*

$$\Psi_{(\Sigma,U)}(f)[\psi] := \int_{\Sigma} f\psi \mathrm{d}V_{\Sigma}. \tag{8.41}$$

As we are working on-shell, (8.40) is satisfied if, $\forall\, \varphi \in \operatorname{Ker} P_{\mathbb{M}^2}, h \in \mathfrak{D}(U)$

$$\Psi_{(\Sigma,U)}(\eta_{(\Sigma,U)}h)[\partial_{\Sigma,\epsilon}\varphi] = \Psi_{(\widetilde{\Sigma},U)}(\eta_{(\widetilde{\Sigma},U)}h)[\partial_{\widetilde{\Sigma},\widetilde{\epsilon}}\varphi]. \tag{8.42}$$

By expanding out each definition, one can eventually show that these maps indeed coincide, and that in fact they are equal to $\partial\Phi_{\mathbb{M}^2}(h)[\varphi]$, the null derivative field from example 5.2.1.

With this, we make our partial definition of a chiral primary field.

**Definition 8.4.3.** Let $\mathfrak{A}_\ell : \mathsf{Cauchy}(\mathbb{M}^2) \to \mathsf{TVec}$ be a functor describing some locally covariant theory. A *chiral primary field of weight* $\mu$ *on* $\mathbb{M}^2$ *with values in* $\mathfrak{A}_\ell$ is then defined as a natural transformation $\mathfrak{D}_\ell^{(\mu)} \Rightarrow \mathfrak{A}_\ell$.

We can now show that, at least classically, monomials in the field strength satisfy this definition, which is in agreement with section 5.3.

**Example 8.4.4.** For $n \in \mathbb{N}$, and $(\Sigma, U) \in \mathsf{Cauchy}(\mathbb{M}^2)$ the maps $\Psi^n : \mathfrak{D}(\Sigma) \to \mathfrak{P}_\ell(\Sigma, U)$ defined by

$$\Psi^n_{(\Sigma,U)}(f)[\psi] := \int_{\mathcal{I}} f(s)\psi^n(s)\gamma'(s)^{\frac{n}{2}}\mathrm{d}s \tag{8.43}$$

constitute a chiral primary field of weight $n$. Moreover $\partial^* \circ \Psi^n \circ \eta = \Pi_{\mathrm{on}} \circ \partial\Phi^n$.

To see this we must show that, for every commuting square

$$\begin{tikzcd} \Sigma \arrow[r, "\rho"] \arrow[d] & \widetilde{\Sigma} \arrow[d] \\ U \arrow[r, "\chi"] & \widetilde{U} \end{tikzcd} \tag{8.44}$$

and every pair $f \in \mathfrak{D}(\Sigma)$, $\psi \in \mathfrak{E}(\widetilde{\Sigma})$

$$\Psi^n_{(\Sigma,U)}(f)[\rho^*_{(1)}\psi] = \Psi^n_{(\widetilde{\Sigma},\widetilde{U})}(\rho_*(\omega_\ell|_{\Sigma}^{\mu-1}f))[\psi]. \tag{8.45}$$

Using the same coordinate system as in the proof or proposition 8.4.1, we can express the conformal factors in terms of (8.36) and hence write each side of this equation explicitly as

$$\Psi^n_{(\Sigma,U)}(f)[\rho^*_{(1)}\psi] = \int_{\mathcal{I}} f(s)(\rho^*\psi)^n(s)\left(\frac{\widetilde{\gamma}'(\rho(s))}{\gamma'(s)}\right)^{\frac{n}{2}} \rho'(s)^n\gamma'(s)^{\frac{n}{2}}\mathrm{d}s, \tag{8.46}$$

$$\Psi^n_{(\widetilde{\Sigma},\widetilde{U})}(\rho_*(\omega_\ell|_{\Sigma}^{n-1}f))[\psi] = \int_{\widetilde{\mathcal{I}}} \rho_*((\rho')^{n-1}f)(\tilde{s})\psi^n(\tilde{s})\widetilde{\gamma}'(\tilde{s})^{\frac{n}{2}}\mathrm{d}\tilde{s}, \tag{8.47}$$

from which we can clearly see the two expressions coincide.

Finally, if we take $\varphi \in \mathrm{Ker} P_U$, then

$$\begin{aligned}\partial\Phi^n_U(f)[\varphi] &= \int_U f(u,v)(\partial_u\varphi)^n(u)\,\mathrm{d}u\mathrm{d}v \\ &= \int_{\mathcal{I}} \left(\int_{\mathbb{R}} f(u,v)\,\mathrm{d}v\right)(\partial_u\varphi)^n(u)\,\mathrm{d}u \\ &= \Psi^n(\eta f)[\partial_\Sigma\varphi] \qquad (8.48)\end{aligned}$$

where we have used the facts that $\partial_\Sigma\varphi(s) = \frac{1}{\sqrt{\gamma'(s)}}(\partial_u\varphi)(-s,\gamma(s))$, and $\mathrm{supp}\,(\eta f) \subseteq \mathcal{I}$. This demonstrates that $\partial\Phi^n$ and $\partial^* \circ \Psi^n \circ \eta$ define the same on-shell observable as required.

However, we shall usually be able to work with a weaker definition, where this behaviour is only observed with respect to isometries and dilations.

**Definition 8.4.5.** Define the subcategory $\mathsf{Cauchy}(\mathbb{M}^2)_0$ of $\mathsf{Cauchy}(\mathbb{M}^2)$ which contains the same objects, but only those morphisms for which $\omega_\ell$ is constant. Let $\mathfrak{A}_\ell : \mathsf{Cauchy}(\mathbb{M}^2) \to \mathsf{TVec}$ as before. A *homogeneously scaling locally covariant field of weight $\mu$ on $\mathbb{M}^2$ with values in $\mathfrak{A}_\ell$* is a natural transformation $\mathfrak{D}^{(\mu)}_\ell|_{\mathsf{Cauchy}(\mathbb{M}^2)_0} \Rightarrow \mathfrak{A}_\ell|_{\mathsf{Cauchy}(\mathbb{M}^2)_0}$

*Remark* 8.4.6. What is typically missing from the locally covariant description (by design) is the continuity that an algebra-valued distribution enjoys. We have accounted for this by making the target category of both $\mathfrak{D}^{(\mu)}_\ell$ and the functor describing the theory (e.g. $\mathfrak{P}_\ell$) the category $\mathsf{TVec}$ of topological vector spaces. As we shall see in the following section, adding in this requirement will allow us to apply our algebraic operations, such as Poisson brackets, $\star$-products and commutators, to fields in order to produce ordinary, $\mathbb{C}$-valued distribution. The transformation properties of the fields will then descend to the level of these distributions, allowing us to impose tight constraints.

To conclude this section, we shall demonstrate a property of fields satisfying the above definition that shall be useful in later proofs.

**Lemma 8.4.7.** *Let $\Psi : \mathfrak{D}^{(\mu)}_\ell|_{\mathsf{Cauchy}(\mathbb{M})_0} \Rightarrow \mathfrak{P}_\ell|_{\mathsf{Cauchy}(\mathbb{M}^2)_0}$ be a locally covariant field on $\mathbb{M}^2$ with values in $\mathfrak{P}_\ell|_{\mathsf{Cauchy}(\mathbb{M}^2)_0}$, then $\forall\, f \in \mathfrak{D}(\Sigma)$, $\mathrm{supp}\,\Psi_{(\Sigma,U)}(f) \subseteq \mathrm{supp}\, f$.*

*Proof.* Due to the linearity of $\Psi_{(\Sigma,U)}$, we may assume that $\mathrm{supp}\, f$ is connected, (otherwise $f$ is a finite sum of $f_i \in \mathfrak{D}(\Sigma)$ which have connected supports). Let $\Sigma_f \subseteq \Sigma$ be any open (in $\Sigma$), connected neighbourhood of $\mathrm{supp}\, f$, then there exists an open, causally-convex neighbourhood $\Sigma_f \subset U_f \subseteq U$ (which can be seen by taking the intersection of any open, causally convex neighbourhood of $\Sigma_f$ with $U$). Let us denote by $(i, i_U)$ the inclusion morphism $(\Sigma_f, U_f) \to (\Sigma, U)$, then we can clearly see that $\mathfrak{D}_\ell^{(\mu)}(i, i_U) : \mathfrak{D}(\Sigma_f) \to \mathfrak{D}(\Sigma)$ is simply the pushforwards along the inclusion $i$, hence $f = \mathfrak{D}^{(\mu)}(i, i_U)(f|_\Sigma) = i_* f|_\Sigma$.

Using the naturality of $\Psi$, we may then write that

$$\begin{aligned}\Psi_{(\Sigma,U)}(f) &= \Psi_{(\Sigma,U)}\left(\mathfrak{D}^{(\mu)}(i, i_U)(f|_\Sigma)\right)\\ &= \mathfrak{P}_\ell(i, i_U)\left(\Psi_{(\Sigma_f,U_f)}(f|_\Sigma)\right).\end{aligned} \tag{8.49}$$

From the definition of the morphism $\mathfrak{P}_\ell(i, i_U)$, it is then clear that $\operatorname{supp}\Psi_{(\Sigma,U)}(f) \subseteq \Sigma_f$. Given that this holds for any open, connected neighbourhood of $\operatorname{supp} f$, we must conclude that $\operatorname{supp}\Psi_{(\Sigma,U)}(f) \subseteq \operatorname{supp} f$. □

*Remark* 8.4.8. Note that we only needed inclusion morphisms for this result. However, as we shall not be dealing with any explicit examples of fields which do not scale homogeneously, there is no need to introduce yet more notation in order to state the most efficient version of this lemma.

## 8.5 CONSTRAINTS ON CHIRAL BRACKETS

We have now seen several ways the Poisson bi-vector of the chiral algebra may be obtained. In this section, we see that some generic assumptions about a conformal field theory can yield tight constraints on the Poisson structure.

Other than the conformal covariance property, the key feature we shall be employing is *Einstein causality*. A classical theory $\mathfrak{P} : \mathsf{Loc} \to \mathsf{Poi}$ (or a quantum theory $\mathfrak{A} : \mathsf{Loc} \to \mathsf{Alg}$) satisfies Einstein causality if, for any pair $\mathcal{N}, \mathcal{N}' \in \mathcal{M}$ of casually convex open sets which are spacelike separated, the Poisson bracket (*resp.* commutator) of any pair $F \in \mathfrak{P}(\mathcal{N}), G \in \mathfrak{P}(\mathcal{N}')$ vanishes.

It is in this section that we take advantage of our alternative definition of locally covariant fields in section 8.4, as it enables us to use results from the theory of distributions in our analysis. If we define a *field with values in* $\mathfrak{P}_\ell(\Sigma, \mathcal{M})$ (with no assumptions on covariance) as a linear continuous map $\Psi^i : \mathfrak{D}(\Sigma) \to \mathfrak{P}_\ell(\Sigma, \mathcal{M})$ satisfying $\operatorname{supp}\Psi^i_\Sigma(f) \subseteq \operatorname{supp} f$ then we may show the following.

**Proposition 8.5.1.** *Let $\Psi^i$ and $\Psi^j$ be a pair of fields with values in $\mathfrak{P}_\ell(\Sigma, \mathcal{M})$ such that, for any $\psi \in \mathfrak{E}(\Sigma)$, the Schwartz kernel $K_\psi^{i/j} \in \mathfrak{D}'(\mathcal{M}^2)$ associated to the map $f \mapsto \Psi^i(f)^{(1)}[\psi]$*

*satisfies* $\mathrm{WF}(K_\psi^{i/j}) \cap \left\{(x, y, \xi, 0) \in \dot{T}^*\mathcal{M}^2\right\} = \emptyset$. *Then the map*

$$f \otimes g \mapsto \left\{\Psi_\Sigma^i(f), \Psi_\Sigma^j(g)\right\}_\ell^\Sigma [\psi] \tag{8.50}$$

*defines a distribution* $E_\psi^{ij} \in \mathfrak{D}'(\Sigma^2)$.

*Proof.* We can equip the underlying space $\mathfrak{F}_c(\Sigma)$ of $\mathfrak{P}_\ell(\Sigma, \mathcal{M})$ with the topology $\tau_{BDF}$, which is the initial topology with respect to the maps

$$\begin{aligned} \mathfrak{F}_c(\Sigma^n) &\to \mathfrak{E}'_{\Xi_n}(\Sigma^n) \\ F &\mapsto F^{(n)}[\psi] \end{aligned}$$

where the topology on $\mathfrak{E}'_{\Xi_n}(\Sigma^n) = \{u \in \mathfrak{E}(\Sigma^n) \,|\, \mathrm{WF}(u) \in \Xi_n\}$ is the *Hörmander topology* [BDH16, p2], and the cones $\Xi_n = \Xi_+^n \cup \Xi_-^n$ are defined in proposition 8.1.1. In particular, this combined with the assumption that $f \mapsto \Psi_\Sigma^i(f)$ is also continuous, implies that

$$f \mapsto \psi_\Sigma^i(f)^{(1)}[\psi] \tag{8.51}$$

is a linear, continuous map $\mathfrak{D}(\Sigma) \to \mathfrak{D}(\Sigma) \subset \mathfrak{E}'(\Sigma)$.

By the Schwartz kernel theorem, we may therefore identify the map (8.51) with an element $K_\psi^i \in \mathfrak{D}'(\Sigma^2)$. We then claim that the desired distribution has the integral kernel

$$\left\{\Psi^i(z_1), \Psi^j(z_2)\right\}[\psi] := \int_{\Sigma^2} E(y_1, y_2) K_\psi^i(z_1, y_1) K_\psi^j(z_2, y_2)\, \mathrm{d}y_1 \mathrm{d}y_2 \tag{8.52}$$

where $K_\psi^j$ the corresponding distribution from $\Psi_\Sigma^j$. To show that this integral kernel is well defined, we use [Hör15, Theorem 8.2.14], for which the necessary conditions are

1. The map $\mathrm{supp}\,(K_\psi^i \otimes K_\psi^j) \ni (z_1, y_1, z_2, y_2) \mapsto (z_1, z_2)$ is proper, i.e. the pre-image of any compact set is compact.
2. $\left\{(y_1, y_2; -\eta_1, -\eta_2) \in \mathrm{WF}(E) \,|\, \exists\, (z_1, y_1, z_2, y_2; 0, \eta_1, 0, \eta_2) \in \mathrm{WF}(K_\psi^i \otimes K_\psi^j)\right\} = \emptyset$.

The first of these follows from the fact that $\mathrm{supp}\,\Psi^i(f) \subseteq \mathrm{supp}\, f$, from which we may deduce that $\mathrm{supp}\,(K_\psi^i \otimes K_\psi^j) \subseteq \{(z_1, z_1, z_2, z_2) \in \Sigma^4\}_{(z_1,z_2)\in\Sigma^2}$, hence the projection map is clearly proper. The second is then a consequence of the restriction on $\mathrm{WF}(K_\psi^{i/j})$ made in the hypothesis and [Hör15, Theorem 8.2.9]. □

*Remark* 8.5.2. The technical condition on $\mathrm{WF}(K_\psi^{i/j})$ may appear restrictive. However, if one considers the motivating example of such fields,

$$\Psi^P(f)[\psi] = \int_{\mathbb{R}} f(x)P(\varphi(x), \partial_x\varphi(x), \ldots, \partial_x^n\varphi(x))\,\mathrm{d}x, \tag{8.53}$$

where $P$ is some polynomial with coefficients in $\mathfrak{E}(\mathbb{R})$, then $K_\psi^P(x,y)$ is a polynomial in $\delta(x-y)$ and its derivatives (with coefficients in $\mathfrak{E}(\mathbb{R})$). This means $\mathrm{WF}(K_\psi^P)$ is orthogonal to the tangent bundle of $\Delta_2 \subset \mathcal{M}^2$, hence, in particular it satisfies the condition set out in proposition 8.5.1.

For the remainder of this section, we shall assume that $\Sigma = \Sigma_0$, the $t = 0$ Cauchy surface of $\mathbb{M}^2$. This allows us to use both translation as well as dilation morphisms. We do this primarily for convenience, as we can then easily formulate the condition of homogeneous scaling. To generalise, we may either use the embedding results such at theorem 7.1.3, or we could adopt a more geometric approach using the *microlocal scaling degree* [BF00, §6] of the relevant distributions.

We can think of the map $\mathfrak{P}_\ell(\Sigma, \mathcal{M}) \to \mathbb{R}$ given by evaluation at a fixed $\psi \in \mathfrak{E}(\Sigma)$ as a classical 'state'. One of these states, namely $\psi \equiv 0$ is special in that it is invariant under the action of $\mathrm{Diff}_+(\Sigma) = \mathrm{Aut}_{\mathsf{CCauchy}}(\Sigma)$. This is a classical realisation of the notion that the vacuum state is invariant under conformal transformations. As such, one can consider the following results as statements about the 'vacuum expectation values' of the corresponding observables.

**Proposition 8.5.3.** *Let* $E_0^{ij}(x,x') = \left\{\Psi_\Sigma^i(x), \Psi_\Sigma^j(x')\right\}_\ell^\Sigma [0]$. *This distribution is translation invariant, i.e. there exists a distribution in* $\mathfrak{D}(\Sigma_0)$ (*which we shall also denote* $E_0^{ij}$ *by an abuse of notation*) *such that* $E_0^{ij}(x,x') = E_0^{ij}(x-x')$.

*Proof.* Let $t_c : x \mapsto x + c$ be a translation operator on $\Sigma_0$, it suffices to show that

$$\left\langle E_0^{ij}, t_{c*}f \otimes t_{c*}g\right\rangle = \left\langle E_0^{ij}, f \otimes g\right\rangle \tag{8.54}$$

for all $f, g \in \mathfrak{D}(\Sigma_0)$. As we are only considering translations, we can ignore the conformal weights for the time being. Hence the covariance property of our fields reads

$$\Psi_{\Sigma_0}^i(t_{c*}f) = \mathfrak{P}_\ell t_c \Psi_{\Sigma_0}^i(f). \tag{8.55}$$

expanding out the left-hand side of (8.54), and making use of the fact that $\mathfrak{P}_\ell t_{c*}$ is a Poisson algebra homomorphism, we find

$$\left\langle E_0^{ij}, t_{c*}f \otimes t_{c*}g\right\rangle = \left(\mathfrak{P}_\ell t_c \left\{\Psi_{\Sigma_0}^i(f), \Psi_{\Sigma_0}^i(g)\right\}_\ell^{\Sigma_0}\right)[0]. \tag{8.56}$$

Recall that these homomorphisms were defined by $(\mathfrak{P}_\ell \rho F)[\psi] := F[\rho^*_{(1)}\psi]$. However, in this case our choice of configuration is invariant under the action of all such morphisms, hence we arrive at the desired equation. □

**Proposition 8.5.4.** *$E_0^{ij}$ is supported on the diagonal $\{(x,x)\}_{x\in\Sigma_0} \subset \Sigma_0^2$, hence it is of the form*

$$E_0^{ij}(x,x') = \sum_{k=0}^{n} a_k \left(\frac{\partial}{\partial x}\right)^k \delta(x-x'), \tag{8.57}$$

*for some $n \in \mathbb{N}$ and $a_k \in \mathbb{R}$.*

The first statement is actually a consequence of Einstein causality in the full theory, as well as the fact that $\operatorname{supp} \Psi^i_\Sigma(f) \subseteq \operatorname{supp} f$, as was shown in lemma 8.4.7. As such, we shall save the proof of this until theorem 10.2.3 The fact that a distribution on the diagonal is necessarily of the form (8.57) is [Hör15, Theorem 2.3.4].

Now we have taken full advantage of the translation morphisms, we introduce the dilation morphisms, for $\Lambda > 0$, $m_\Lambda : \mathbb{M}^2 \to \mathbb{M}^2; x \mapsto \Lambda \cdot x$. Clearly these preserve $\Sigma_0$, and we shall denote their restriction/co-restriction to $\Sigma_0$ also by $m_\Lambda$. The next result is the first that utilises *conformal* covariance, which is why it is only now relevant whether or not the fields $\Psi^{i/j}$ are *primary*.

First, we need to briefly introduce a new definition:

**Definition 8.5.5.** A distribution $u \in \mathfrak{D}'(\mathbb{R}^n)$ *scales homogeneously with degree $\mu \in \mathbb{R}$* if, $\forall\, \Lambda > 0$, $m_\Lambda^* u = \Lambda^{-\mu} u$ or, in terms of integral kernels, $u(\Lambda x) = \Lambda^{-\mu} u(x)$.

**Proposition 8.5.6.** *If $\Psi^i$ and $\Psi^j$ are homogeneously scaling with weights $\mu_i, \mu_j \in \mathbb{N}$ respectively, then $E_0^{ij}$ scales homogeneously with degree $\mu_i + \mu_j$, hence*

$$E_0^{ij}(x,x') = a \left(\frac{\partial}{\partial x}\right)^{\mu_i+\mu_j-1} \delta(x-x'). \tag{8.58}$$

*Proof.* First we consider the claim of homogeneous scaling. Similarly to the translation invariance, it will suffice to show that

$$\left\langle m_\Lambda^* E_0^{ij}, f \otimes g \right\rangle \equiv \Lambda^{-2} \left\langle E_0^{ij}, (m_{\Lambda *} f) \otimes (m_{\Lambda *} g) \right\rangle = \left\langle E_0^{ij}, f \otimes g \right\rangle \tag{8.59}$$

For the dilation morphisms introduced above, the conformal factor $\omega_\ell$ is the constant $\Lambda^{-1}$, hence the naturality condition implies

$$\Lambda^{\mu_i - 1} \Psi^i_{\Sigma_0}\left(m_{\Lambda *}(f)\right) = \mathfrak{P}_\ell m_\Lambda \Psi^i_{\Sigma_0}(f), \tag{8.60}$$

where, since $\Lambda^{\mu_i-1}$ is constant, we can simply pull it outside $\Psi^i_{\Sigma_0}$. Bringing these factors over to the right-hand side, we find

$$\left\langle E_0^{ij}, (m_{\Lambda *}f) \otimes (m_{\Lambda *}g) \right\rangle = \Lambda^{2-\mu_i+\mu_j} \left( \mathfrak{P}_\ell m_\Lambda \left\{ \Psi^i_{\Sigma_0}(f), \Psi^i_{\Sigma_0}(g) \right\}_\ell^{\Sigma_0} \right) [0]. \tag{8.61}$$

Again, noting that our 'state' $F \mapsto F[0]$ is invariant under the $\mathfrak{P}_\ell m_\Lambda$ morphisms, we arrive at the desired equation.

A quick calculation shows that that the distribution $(\partial/\partial x)^k \delta(x)$ scales homogeneously with degree $1+k$. [1] As we have already established $E_0^{ij}$ to be of the form (8.57), we see that all but one of these terms must vanish, leaving us with (8.58). ☐

Finally, applying this result when $\Psi^i = \Psi^j$ is the chiral boson, we get.

**Corollary 8.5.7.** *The commutator of the chiral boson on $\Sigma_0$ is proportional to $\delta'(x - x')$.*

Naturally, we already know this to be the case, but it is nevertheless significant that conformal covariance alone determines everything except the constant of proportionality. We shall see in section 9.3 that, for the chiral boson, even this constant can determined using partial knowledge of the OPE of $\Psi$ with itself.

*Remark* 8.5.8. Note that $\delta^{(\mu_i+\mu_j-1)}$ is skew-symmetric precisely when $\mu_i + \mu_j$ is even. This is clearly satisfied for the bracket of a field with itself given that its weight is a natural number. Moreover, we can easily see how a similar result would look for fermionic fields. If the Poisson bracket was suitably graded, then the bracket of a fermionic field with itself would instead be a symmetric distribution supported on the diagonal and would hence vanish unless the weight $\mu$ was a half-integer.

## 8.6 CHIRAL STRESS-ENERGY TENSOR AND CONFORMAL SYMMETRY

Another important example of a chiral primary field is the (chiral component of the) stress-energy tensor. This is simply half the square of the chiral boson defined above,

$$T_\Sigma(f)[\psi] := \frac{1}{2} \int_\Sigma f \psi^2 \mathrm{d}V_\Sigma. \tag{8.62}$$

Note that the corresponding observable in the full algebra, given as an integral kernel, is then $\partial^*_\Sigma T_\Sigma(x)[\varphi] = \frac{1}{2}(\partial_\Sigma \varphi)^2(x)$, so this is indeed the left-moving component of the stress-energy tensor for the massless scalar field.

---

[1] The general result is that for $\delta \in \mathfrak{D}'(\mathbb{R}^n)$, and $\alpha \in \mathbb{N}^n$ a multi-index, the distribution $\partial^\alpha \delta$ scales homogeneously with degree $n + |\alpha|$

It is well-understood in the physical literature that spacetime symmetries are generated infinitesimally by the stress-energy tensor: either with the Poisson bracket in the classical theory, or the commutator in the quantum theory. In the framework of locally covariant QFT, this fact is encapsulated by the principle of *relative Cauchy evolution*. The concept is a little more subtle than in the Wightman framework, as a generic spacetime does not posess translation symmetries (which by Noether's theorem would then be associated to momentum operators).

In relative Cauchy evolution, rather than considering infinitesimal translations, one instead perturbs the spacetime metric slightly, $g \mapsto g + \epsilon h$ for some *compactly supported* symmetric tensor $h$. One then compares how the time-evolution of an observable $\mathcal{O}$ localised in the past of $\operatorname{supp} h$ proceeds in the perturbed and unperturbed spacetimes, the discrepancy is what we call the relative Cauchy evolution of $\mathcal{O}$ with respect to $\epsilon h$. For more details, see [BFV03, §4.1]. Notably, in many examples, it has been shown that relative Cauchy evolution is generated infinitesimally by the stress-energy tensor.

In the present framework, we can demonstrate this explicitly with the following result

**Proposition 8.6.1.** *Let $\Sigma \subset \mathcal{E}$ be a Cauchy surface of the Einstein cylinder $\mathcal{E}$ and let $h \in \mathfrak{D}(\Sigma)$ such that the flow $\rho^{(t)} \in \mathrm{Diff}(\Sigma)$ generated by the vector field $h\,\mathrm{d}/\mathrm{d}x$ is orientation preserving at every $t$ where it is defined. Let $\Psi$ be the chiral boson* (8.1) *and $T$ be the chiral stress energy tensor. Then*

$$\{T_\Sigma(h), \Psi_\Sigma(f)\}_\ell^\Sigma = -\Psi_\Sigma(hf') = \frac{d}{dt}\left(\mathfrak{P}_\ell \rho^{(t)} \Psi_\Sigma(f)\right)\Big|_{t=0}. \tag{8.63}$$

*In other words, the $\mathrm{Diff}_+(\Sigma)$ covariance of $\Psi$ is generated by taking the Peierls bracket with $T$.*

*Proof.* For the flow $\{\rho^{(t)} \in \mathrm{Diff}_+(\Sigma)\}_{t\in(-\epsilon,\epsilon)}$ of a vector field $X \in \mathfrak{X}(\Sigma)$, we can write $\rho_*^{(t)} f = (\rho^{(-t)})^* f = f - t\mathcal{L}_X f + \mathcal{O}(t^2)$ where $\mathcal{L}_X$ denotes the Lie derivative along $X$. By linearity of $\Psi_\Sigma$ in the test function, we then have that

$$\mathfrak{P}_\ell \rho^{(t)} \Psi_\Sigma(f) = \Psi_\Sigma(\rho_*^{(t)} f) = \Psi_\Sigma(f) - t\Psi_\Sigma(\mathcal{L}_X f) + \mathcal{O}(t^2). \tag{8.64}$$

setting $X = h\frac{d}{dx}$, we see that the second and third terms of (8.63) are equal.

To establish the first equation we just compute the chiral Poisson bracket explicitly:

$$\begin{aligned}\{T(h), \Psi(f)\}_{\ell}^{\Sigma}[\phi] &= -\int_{\Sigma} T(h)^{(1)}[\phi](x)\frac{d\Psi(f)^{(1)}[\varphi](x)}{dx}\,\mathrm{d}x \\ &= -\int_{\Sigma} h(x)\phi(x)\frac{df}{dx}(x)\,\mathrm{d}x \\ &= -\Psi\left(hf'\right)[\phi].\end{aligned}$$

□

In the future, it would be interesting to see if a more general result could be obtained using relative Cauchy evolution. In the CFT literature, it is common (e.g. [Sch08, Definition 9.7]) to *define* primary fields in terms of their commutator (Poisson bracket in our case, as we are still classical) with the stress-energy tensor. Given that a primary field ought to respond in a predictable way to a metric perturbation of the form $g_{\mathcal{M}} \mapsto g_{\mathcal{M}}(1+\epsilon h)$ for $h \in \mathfrak{D}(\mathcal{M})$, one ought to be able to find constraints on its relative Cauchy evolution. However, we shall not explore this issue further.

Now that we have studied the classical algebra in detail, in the following chapter, we shall see that many of our constructions require only minimal adjustments to obtain the analogous quantum constructions.

## — 9 —

# Quantisation of the Chiral Algebra

Now we turn our attention to the quantisation of the algebra constructed in the previous chapter. As one might expect, we start by finding a deformation of the algebras $\mathfrak{P}_\ell(\Sigma, \mathcal{M})$. We then show how these algebras embed naturally into $\mathfrak{A}(\mathcal{M})$ from part I. Finally, we discuss how this chiral algebra can compute the *operator product expansions* of both the chiral boson and the stress energy tensor, and comment on how the form of these OPEs is constrained by scaling invariance.

### 9.1 THE QUANTUM CHIRAL ALGEBRA

We shall start as we did when constructing the classical chiral algebra, namely by considering a pair of linear functionals $\Psi(f), \Psi(g)$ for $f, g \in \mathfrak{D}_{\ell/r}(\Sigma)$. Firstly, ignoring wavefront sets, $\partial_\Sigma^* \Psi(f)$ is a linear observable on $\mathcal{M}$, thus we may attempt to compute its $\star_H$ product, resulting in

$$\partial_\Sigma^* \Psi(f) \star_H \partial_\Sigma^* \Psi(g) = \partial_\Sigma^* \Psi(f) \cdot \partial_\Sigma^* \Psi(g) + \hbar \left\langle (\partial_\Sigma \otimes \partial_\Sigma)[\tfrac{i}{2} E + H], f \otimes g \right\rangle_{\Sigma^2}. \tag{9.1}$$

Thus, similarly to the case of the chiral Poisson bracket, the product of linear observables is computed by the bi-distribution $(\partial_\Sigma \otimes \partial_\Sigma)[\frac{i}{2} E + H] =: W_\Sigma \in \mathfrak{D}'(\Sigma^2)$, where again we can verify this distribution is well-defined by combining proposition 4.3.1 and proposition 7.2.2 to obtain $W_\Sigma = (\rho_{(1)}^* \otimes \rho_{(1)}^*) W_{\Sigma_0}$, for a suitable choice $\Sigma \overset{\sim}{\to} \Sigma_0 \subset \mathcal{M}_0 \in \{\mathbb{M}^2, \mathcal{E}\}$. Moreover, precisely the same wavefront set condition that caused $\mathfrak{F}_c(\Sigma)$ to be closed as a Poisson algebra allows us to define a deformation quantisation of that algebra via the following proposition:

**Proposition 9.1.1.** *Let $\Sigma \subset \mathcal{M}$ be a Cauchy surface of some globally hyperbolic spacetime, and let $\mathfrak{F}_c(\Sigma)$ denote the space of functionals defined in proposition 8.1.1. Then, for any $H \in \mathrm{Had}(\mathcal{M})$, the space $\mathfrak{F}_c(\Sigma)[[\hbar]]$ equipped with the product*

$$F \star_{H,\ell} G[\psi] = \sum_{n=0}^{\infty} \frac{\hbar^n}{n!} \left\langle W_\Sigma^{\otimes n}, F^{(n)}[\psi] \otimes G^{(n)}[\psi] \right\rangle \tag{9.2}$$

*is a closed $*$-algebra, with involution given by pointwise complex conjugation.*

*Proof.* The proof is comparable to that of Proposition 8.1.1. We won't prove the more elementary properties of a $*$-algebra, though we shall point out that associativity follows from the particular form of (9.2) [HR19, Proposition 4.5].

For well-definedness and closure we must show that, for every $n, m > k \in \mathbb{N}$, the map $\mathfrak{D}(\Sigma^n) \otimes \mathfrak{D}(\Sigma^m) \to \mathfrak{D}'(\Sigma^{n+m-2k})$ defined by

$$\begin{aligned} f \otimes g \mapsto \int_{\Sigma^{2k}} & W_\Sigma(u_1, u_{k+1}) \cdots W_\Sigma(u_k, u_{2k}) \\ & f(u_1, \ldots, u_k, u_1', \ldots u_{n-k}') g(u_{k+1}, \ldots, u_{2k}, u_1'', \ldots u_{m-k}'') \, \mathrm{d}u_1 \cdots \mathrm{d}u_{2k} \end{aligned} \tag{9.3}$$

extends to a map

$$\mathfrak{E}'_{\Gamma_n}(\Sigma^n) \otimes \mathfrak{E}'_{\Gamma_m}(\Sigma^m) \to \mathfrak{D}'_{\Gamma_{n+m-2k}}(\Sigma^{n+m-2k}) \tag{9.4}$$

where $\Gamma_n = T^*\Sigma^n \backslash (\Xi_+^n \cup \Xi_-^n)$ is the cone of allowable wavefronts from Proposition 8.1.1.

Looking in particular at $W_{\Sigma_0}$, or in general by applying Hörmander's pullback theorem to $(\Pi_\ell d \otimes \Pi_\ell d)W$ along the embedding $\Sigma \times \Sigma \hookrightarrow \mathcal{M} \times \mathcal{M}$, we see that

$$\mathrm{WF}(W_\Sigma) = \left\{ (r, r; \xi, -\xi) \in \dot{T}^*\Sigma^2 \,|\, \xi > 0 \right\}, \tag{9.5}$$

where the sign of $\xi$ is defined with respect to an arbitrary oriented chart on $\Sigma$. We must now consider the set

$$\begin{aligned} (\overline{\Gamma}_{n+k} \times \overline{\Gamma}_{m+k} \setminus \underline{0}_{\Sigma^{n+m+2k}}) \bullet \overline{\mathrm{WF}(W_\Sigma^{\otimes k})} := \Big\{ & (\underline{s}_F, \underline{s}_G; \underline{\xi}_F, \underline{\xi}_G) \in T^*\Sigma^{n+m} \,| \\ & \exists\, (\underline{r}_1, \underline{r}_2; \underline{\eta}, -\underline{\eta}) \in T^*\Sigma^{2k}, (r_{11}, r_{12}, \ldots, r_{k1}, r_{k2}; \eta_1, -\eta_1, \ldots, \eta_k, -\eta_k) \in \overline{\mathrm{WF}(W_\Sigma^{\otimes k})}, \\ & (\underline{r}_1, \underline{s}_F, \underline{r}_2, \underline{s}_G; \underline{\eta}, \underline{\xi}_F, -\underline{\eta}\underline{\xi}_G) \in (\overline{\Gamma}_{n+m} \times \overline{\Gamma}_{m+k} \setminus \underline{0}_{\Sigma^{n+m+2k}}) \Big\}. \end{aligned}$$

If this set is disjoint from $\underline{0}_{\Sigma^{n+m}}$, then the domain of (9.3) can be extended to the set in (9.4), and if the set is disjoint from $\Xi_\pm^{n+m}$, then its codomain is also where we need it. We only spell out the argument that the existence of a covector

$$(\underline{s}_F, \underline{s}_G; \underline{\xi}_F, \underline{\xi}_G) \in \Xi_+^{n+m} \cap (\overline{\Gamma}_{n+k} \times \overline{\Gamma}_{m+k} \setminus \underline{0}_{\Sigma^{n+m+2k}}) \bullet \overline{\mathrm{WF}(W_\Sigma^{\otimes k})}$$

leads to a contradiction. The remaining case $(\underline{s}_F, \underline{s}_G; \underline{\xi}_F, \underline{\xi}_G) \in \Xi_-^{n+m}$ then follows from an analogous argument.

Suppose $(\underline{r}_1, \underline{r}_2; \underline{\eta}, -\underline{\eta}) \in \overline{\mathrm{WF}(W_\Sigma^{\otimes k})}$ is the witness for the statement $(\underline{s}_F, \underline{s}_G; \underline{\xi}_F, \underline{\xi}_G) \in (\overline{\Gamma}_{n+k} \times \overline{\Gamma}_{m+k} \setminus \underline{0}_{\Sigma^{n+m+2k}}) \bullet \overline{\mathrm{WF}(W_\Sigma^{\otimes k})}$. In particular, this means that $(\underline{r}_1, \underline{s}_F; \underline{\eta}, \underline{\xi}_F) \in \overline{\Gamma}_{n+k}$. However, as $\eta_i \geq 0, \forall\, 1 \leq i \leq k$, this is only consistent with $(\underline{s}_F, \underline{s}_G; \underline{\xi}_F, \underline{\xi}_G) \in \Xi_+^{n+m}$ if both $\underline{\xi}_F$ and $\underline{\eta}$ vanish, the latter of which implies $\underline{\xi}_G$ also vanishes, leading to the contradiction. □

A routine calculation then shows that, for $H, H' \in \mathrm{Had}$, one can define maps $\beta_{H'-H} : \mathfrak{F}_c(\Sigma)[[\hbar]] \to \mathfrak{F}_c(\Sigma)[[\hbar]]$ analogously to (2.63),

$$\beta_{H'-H} F = \sum_{n=0}^{\infty} \frac{\hbar^n}{2^n n!} \left\langle (H'_\Sigma - H_\Sigma)^{\otimes n}, F^{(2n)} \right\rangle \tag{9.6}$$

which intertwine $\star_{H,\ell}$ with $\star_{H',\ell}$. We then define the quantum chiral algebra as follows:

**Definition 9.1.2** (Quantum Chiral Algebra)**.** Let $\Sigma$ be a Cauchy surface of some globally hyperbolic spacetime $\mathcal{M}$, the *quantum chiral algebra* on $\Sigma$ is the $*$-algebra defined by

$$\mathfrak{A}_\ell(\Sigma, \mathcal{M}) = \left\{ (F_H)_{H \in \mathrm{Had}(\mathcal{M})} \subset \mathfrak{F}_c(\Sigma)[[\hbar]] \,|\, \beta_{H'-H} F_H = F_{H'} \right\} \tag{9.7}$$

with product

$$(F_H)_{H \in \mathrm{Had}(\mathcal{M})} \star_\ell (G_H)_{H \in \mathrm{Had}(\mathcal{M})} = (F_H \star_{H,\ell} G_H)_{H \in \mathrm{Had}(\mathcal{M})}. \tag{9.8}$$

To a given CCauchy morphism $(\rho, \chi) : (\Sigma, \mathcal{M}) \to (\widetilde{\Sigma}, \widetilde{\mathcal{M}})$, we assign the map defined, for $\widetilde{H} \in \mathrm{Had}(\widetilde{\mathcal{M}})$ by

$$(\mathfrak{A}_\ell(\rho, \chi) F)_{\widetilde{H}} = F_{\chi^* \widetilde{H}} \circ \rho^*_{(1)}. \tag{9.9}$$

*Remark* 9.1.3. Note that the map (9.9) is well-defined because: firstly, we have already seen that $F \mapsto F \circ \rho^*_{(1)}$ is a well-defined map $\mathfrak{F}_c(\Sigma) \mapsto \mathfrak{F}_c(\widetilde{\Sigma})$; secondly, the consistency condition is satisfied, because

$$\begin{aligned} \left\langle (\widetilde{H}'_\Sigma - \widetilde{H}_\Sigma)^{\otimes n}, (F_{\chi^* \widetilde{H}} \circ \rho^*_{(1)})^{(2n)} \right\rangle &= \left\langle (\rho^*_{(1)})^{\otimes 2n} (\widetilde{H}'_\Sigma - \widetilde{H}_\Sigma), F^{(2n)}_{\chi^* \widetilde{H}} \circ \rho^*_{(1)} \right\rangle, \\ &= 2^n \left( \frac{d^n}{d\hbar^n} \beta_{\chi^* \widetilde{H}' - \chi^* \widetilde{H}} F|_{\hbar=0} \right) \circ \rho^*_{(1)}; \end{aligned}$$

and lastly, using a similar equation to the above we can verify that $\mathfrak{A}_\ell(\rho, \chi)$ is a homomorphism with respect to $\star_\ell$.

Having defined the quantum chiral algebra, we must ask both how these algebras vary as we change Cauchy surfaces, and how they relate to the full algebras $\mathfrak{A}(\mathcal{M})$. Both of these are addressed by the following theorem.

**Theorem 9.1.4.** *For an object $(\Sigma, \mathcal{M}) \in$ CCauchy, any choice of map $\partial_{\Sigma,\epsilon} : \mathfrak{E}(\mathcal{M}) \to \mathfrak{E}(\Sigma)$ as defined in theorem 8.2.3 defines a map $\theta_{\Sigma,\epsilon} : \mathfrak{A}_\ell(\Sigma, \mathcal{M}) \to \mathfrak{A}(\mathcal{M})$ by*

$$\theta_{\Sigma,\epsilon} \left( F_H \right)_{H \in \mathrm{Had}(\mathcal{M})} := \left( \partial^*_{\Sigma,\epsilon} F_H \right)_{H \in \mathrm{Had}(\mathcal{M})} \tag{9.10}$$

*Moreover, choosing such a map for every pair $(\Sigma, \mathcal{M})$ yields a natural transformation $\theta : \mathfrak{A}_\ell \Rightarrow \mathfrak{A}_{\mathrm{on}} \circ \Pi_2$.*

*Proof.* Firstly, this map is well-defined because

$$\alpha_{H'-H}\partial^*_{\Sigma,\epsilon} = \partial^*_{\Sigma,\epsilon}\beta_{H'-H}, \tag{9.11}$$

as can be verified by a direct computation using property **H2** from definition 2.8.1 of Hadamard distributions to conclude that $(\partial_{\Sigma,\epsilon}\otimes\partial_{\Sigma,\epsilon})H = (\partial_\Sigma\otimes\partial_\Sigma)H$. Given this, most of what remains to be shown follows directly from the corresponding classical result theorem 8.2.3. For the first statement all that we really need to check is that each $\theta_{\Sigma,\epsilon}$ is a homomorphism of $\star$ products. Looking at the coefficient of $\hbar^n$, this requires one to show that for every $H \in \mathrm{Had}(\mathcal{M})$ and $n \in \mathbb{N}$

$$\begin{aligned}&\left\langle(\tfrac{i}{2}E{+}H)^{\otimes n}, (\partial_{\Sigma,\epsilon}F)^{(n)}[\varphi]\otimes(\partial_{\Sigma,\epsilon}G)^{(n)}[\varphi]\right\rangle\\ &\qquad = \left\langle[(\partial_\Sigma\otimes\partial_\Sigma)^{\otimes n}(\tfrac{i}{2}E+H)]^{\otimes n}, F^{(n)}[\partial_{\Sigma,\epsilon}\varphi]\otimes G^{(n)}[\partial_{\Sigma,\epsilon}\varphi]\right\rangle.\end{aligned} \tag{9.12}$$

Similarly to before, we can show that, for $u \in \mathfrak{D}'(\mathcal{M}^{2n})$ for any $n \in \mathbb{N}$,

$$\begin{aligned}&\left\langle u, (\partial_{\Sigma,\epsilon}F)^{(n)}[\varphi]\otimes(\partial_{\Sigma,\epsilon}G)^{(n)}[\varphi]\right\rangle\\ &= \left\langle(\partial_{\Sigma,\epsilon}\otimes\partial_{\Sigma,\epsilon})^{\otimes n}u, F^{(n)}[\partial_{\Sigma,\epsilon}\varphi]\otimes G^{(n)}[\partial_{\Sigma,\epsilon}\varphi]\right\rangle.\end{aligned} \tag{9.13}$$

which once again follows from the fact that both $E$ and $H$ are bi-solutions to the equations of motion.

Naturality is also essentially unchanged from the classical result. Let $(\rho,\chi) : (\Sigma,\mathcal{M}) \to (\widetilde{\Sigma},\widetilde{\mathcal{M}})$ be a CCauchy morphism. For the appropriate *on-shell* diagram to commute, we need to show that, $\forall\,(F_H) \in \mathfrak{A}_\ell(\Sigma,\mathcal{M})$

$$\mathfrak{A}\chi\theta_{\Sigma,\epsilon}(F_H) = \theta_{\widetilde{\Sigma},\widetilde{\epsilon}}\mathfrak{A}_\ell(\rho,\chi)(F_H) + \mathfrak{I}_S(\widetilde{\mathcal{M}}), \tag{9.14}$$

where we have suppressed the index set $\mathrm{Had}(\mathcal{M})$, and we have used the fact that the ideal $\mathfrak{I}_S(\mathcal{M})$ can be identified unambiguously as a subspace of $\mathfrak{A}(\widetilde{\mathcal{M}})$. Using the definition of each of these maps, this means we need that, for every $\widetilde{H} \in \mathrm{Had}(\widetilde{\mathcal{M}})$, $\varphi \in \mathrm{Ker}\,P_{\widetilde{\mathcal{M}}}$

$$F_H[\partial_{\Sigma,\epsilon}\chi^*\varphi] = F_H[\rho^*_{(1)}\partial_{\widetilde{\Sigma},\widetilde{\epsilon}}\varphi],$$

where $H = \chi^*\widetilde{H}$, which is precisely the same equation as in theorem 8.2.3. □

## 9.2 OPERATOR PRODUCT EXPANSIONS

We now turn our attention to one of the central features of 2DCFT: the *operator product expansion*. We shall begin by summarising how they arise in Euclidean signature, then continue to demonstrate how several of these equations arise in the Lorentzian setting.

There exists a very powerful axiomatisation of the chiral sector of a conformally invariant Wightman QFT build upon the mathematical structure of *vertex operator algebras*. For a comprehensive account of this framework, we refer the reader to [Kac98] and [FB04]. However, we shall give a brief account here of the concepts of fields and their OPES, which are of central significance in this approach.

A *field* in a vertex operator algebra is defined as formal power-series $a(z) = \sum_{n\in\mathbb{Z}} a_n z^{-n}$, where $a_n \in \mathrm{End}(V)$ for some complex vector space $V$ (the space of states of the theory) and subject to the condition that, for every $v \in V$, there exists $N \in \mathbb{Z}$ such that $a_n v = 0 \,\forall\, n \geq N$.

This definition can be connected to our previous notions of a field. If we express a Laurent *polynomial* $f \in \mathbb{C}[z, z^{-1}]$ as $\sum_{n\in\mathbb{Z}} f_n z^{-n}$ where only finitely many $f_n \in \mathbb{C}$ are non-zero, then one can think of $a(z)$ as a distribution on this space with values in $\mathrm{End}(V)$, where the pairing is given by

$$\langle a, f\rangle := \sum_{n\in\mathbb{Z}} a_n f_{-n}. \tag{9.15}$$

Given a pair, $a(z), b(w)$ of such fields, one is often interested in whether they are *mutually local*. We can define the composition of $a(z)$ with $b(w)$ as $\sum_{n,m\in\mathbb{Z}} a_n b_m z^{-n} w^{-m} \in \mathrm{End}(V)[[z, z^{-1}, w, w^{-1}]]$. The fields are then said to be mutually local if, for some $N \in \mathbb{N}$

$$(z-w)^N [a(z), b(w)] = 0. \tag{9.16}$$

One can then show [Kac98, Corollary 2.2] that this commutator is of the form

$$[a(z), b(w)] = \sum_{j=0}^{N-1} \partial_w^{(j)} \delta(z-w) c^j(w), \tag{9.17}$$

for some $c^j(w) \in \mathrm{End}(V)[[w, w^{-1}]]$. The distributions $\partial_w^{(j)}$ are then expressed as the difference of boundary values of two holomorphic functions on open subsets of $\mathbb{C}^2$. Both functions are written as $(z-w)^{-(j+1)}$, however the domain of the first instance is taken to be the region $|z| > |w|$, whereas the domain of the second is $|w| < |z|$. The *OPE* of $a$ and $b$ in either of these domains is then obtained by replacing each $\partial_w^{(j)} \delta(z-w)$ with the appropriate holomorphic function in (9.17). This is typically written

$$a(z)b(w) \sim \sum_{j=0}^{N-1} \frac{c^j(w)}{(z-w)^{j+1}}. \tag{9.18}$$

This series differs from the *actual* product $a(z)b(w)$ by a term referred to as the *normally ordered product* $:a(z)b(w):$, which can be computed by a decomposition of $a(z)$ and $b(z)$ into positive and negative frequency modes.

We have already seen a property similar to (9.16) in the classical theory. Recall that the chiral bracket of the chiral boson with itself may be considered a bi-distribution on $\Sigma^2$ for any Cauchy surface $\Sigma$. In the theory of distributions used in pAQFT, the analogue of this equation is the combined statement that a distribution $u \in \mathfrak{D}'(\mathbb{R}^2)$ is supported on the diagonal $\{x,x\}_{x\in\mathbb{R}} \subset \mathbb{R}^2$, and that it has a scaling degree $N$ with respect to the diagonal. Even generically, the axiom of Einstein causality, when restricted to a chiral algebra on a Cauchy surface, implies that the Peierls bracket/commutator of two fields has this property in a Lorentzian AQFT.

We now turn our attention to the problem of finding an analogue of (9.18) in our current framework. We shall discuss the general case shortly, however we shall begin by computing several examples.

In order to be concrete, we restrict once more to spacetimes which are open, causally convex subsets of Minkowski space. Suppose that $\Sigma$ is a Cauchy surface of such a subset $U \subseteq \mathbb{M}^2$. From now on, we shall always assume our Hadamard distribution $W$ is the restriction of

$$W_{\mathbb{M}^2}(u,v,u',v') = \lim_{\epsilon\searrow 0} \frac{-1}{4\pi} \ln\left(\frac{-(u-u')(v-v')+i\epsilon t}{\Lambda^2}\right). \tag{9.19}$$

For a Cauchy surface $\Sigma = \{(-s,\gamma(s))\}_{s\in\mathbb{R}}$ of $\mathbb{M}^2$, we can write the chiral derivative of $W_{\mathbb{M}^2}$ as

$$W_\Sigma(s,s') = \frac{-1}{4\pi}\frac{1}{\sqrt{\gamma'(s)\gamma'(s')}}\left(-\frac{d}{ds}\mathrm{PV}\left(\frac{1}{s-s'}\right) + i\delta'(s-s')\right), \tag{9.20}$$

where PV denotes the Cauchy principal value defined by

$$\left\langle \mathrm{PV}\left(\tfrac{1}{x}\right), f\right\rangle = \lim_{\epsilon\searrow 0}\int_{\mathbb{R}\setminus(-\epsilon,\epsilon)} \frac{f(x)}{x}\,\mathrm{d}x.$$

We shall now adopt the shorthand

$$\frac{1}{(s-s')^{n+1}} := \frac{1}{n!}\left(-\frac{d}{ds}\right)^n \mathrm{PV}\left(\frac{1}{s-s'}\right)$$

for the remainder of this section.

From (9.20) we can also get the formula for $\Sigma \subset U \subseteq \mathbb{M}^2$ by a suitable restriction, as every Cauchy surface of $U$ can be extended to a Cauchy surface of $\mathbb{M}^2$. This Hadamard distribution gives us a concrete realisation of $\mathfrak{A}_\ell(\Sigma, U)$ as $(\mathfrak{F}_c(\Sigma)[[\hbar]], \star_{H,\ell})$ Let us take a pair $f, g \in \mathfrak{D}(\Sigma)$ with disjoint support. Using this realisation, the $\star$ product of the chiral boson evaluated on each test function is

$$(\Psi_\Sigma(f) \star_{H,\ell} \Psi_\Sigma(g))[\psi] = (\Psi_\Sigma(f) \cdot \Psi_\Sigma(g))[\psi] - \frac{\hbar}{4\pi} \int_{\mathbb{R}^2} \frac{f(s)g(s')}{(s-s')^2} \,\mathrm{d}s\,\mathrm{d}s', \tag{9.21}$$

where the disjoint support allows us to drop the imaginary part of the $\mathcal{O}(\hbar)$ term. In the above expression there is nothing to stop us from taking a limit where $f$ and $g$ converge to Dirac deltas with supports at a pair of fixed points $s, s'$. Doing so, we can write

$$(\Psi_\Sigma(s) \star_{H,\ell} \Psi_\Sigma(s'))[\psi] = \psi(s)\psi(s') - \frac{\hbar}{4\pi} \frac{1}{(s-s')^2}. \tag{9.22}$$

This is a smooth function on $\Sigma^2 \setminus \Delta$. If we restrict our attention to the term which is singular as we approach the diagonal, we could then write

$$(\Psi_\Sigma(s) \star_{H,\ell} \Psi_\Sigma(s'))[\psi] \sim -\frac{\hbar}{4\pi} \frac{1}{(s-s')^2} \tag{9.23}$$

As another example, we may define the chiral stress energy tensor by

$$T_\Sigma(f)[\psi] := \frac{1}{2} \int_\Sigma f(s)\psi(s)^2 \sqrt{\gamma'(s)} \,\mathrm{d}s. \tag{9.24}$$

Applying the same procedure to this field, we find

$$\begin{aligned}(T_\Sigma(f) \star_{H,\ell} T_\Sigma(g))\,[\psi] = T_\Sigma(f) \cdot T_\Sigma(g)[\psi] &+ \frac{\hbar}{4\pi} \int_{\mathbb{R}^2} \frac{f(s)\psi(s)g(s')\psi(s')}{(s-s')^2} \,\mathrm{d}s\,\mathrm{d}s' \\ &+ \frac{\hbar^2}{32\pi^2} \int_{\mathbb{R}^4} \frac{f(s_1)\delta(s_1-s_2)g(s_3)\delta(s_3-s_4)}{(s_1-s_3)^2(s_2-s_4)^2} \,\mathrm{d}s_1 \cdots \mathrm{d}s_4.\end{aligned} \tag{9.25}$$

Where we once again can allow $f, g$ to approach deltas to obtain

$$(T_\Sigma(s) \star_{H,\ell} T_\Sigma(s'))\,[\psi] = \frac{1}{4}\psi(s)^2\psi(s')^2 + \frac{\hbar}{4\pi}\frac{\psi(s)\psi(s')}{(s-s')^2} + \frac{\hbar^2}{32\pi^2}\frac{1}{(s-s')^4}. \tag{9.26}$$

Taylor expanding $\varphi(s)$ around $s'$, we then obtain a well-known OPE

$$\omega_{H_{\mathbb{M}^2},\psi}\left(T_\Sigma(s) \star_{H,\ell} T_\Sigma(s')\right) \sim \frac{\hbar^2}{32\pi^2}\frac{1}{(s-s')^4} + \frac{\hbar}{4\pi}\frac{2T_\Sigma(s')[\psi]}{(s-s')^2} + \frac{\hbar}{4\pi}\frac{T'_\Sigma(s')[\psi]}{s-s'}, \tag{9.27}$$

where $T'_\Sigma(g) := -T_\Sigma(*_\Sigma d_\Sigma g)$.

Now that we have seen some explicit examples, we can make some comments about the general connection. In the VOA formalism, the analysis of the products of fields is enabled by the existence of a product

$$\mathrm{End}(V)[[z, z^{-1}]] \otimes \mathrm{End}(V)[[w, w^{-1}]] \to \mathrm{End}(V)[[z, z^{-1}, w, w^{-1}]] \tag{9.28}$$

from operator-valued distributions to operator valued *bi*-distributions. The purpose of the following section is to explore the extent to which such an analysis can be reproduced using the tools of pAQFT.

Recall that in the classical theory we had a topology on our algebra $\mathfrak{P}_\ell(\Sigma, \mathcal{M})$ which was initial with respect to the differentiation maps $\mathfrak{F}_c(\Sigma) \to \mathfrak{E}'_{\Xi_n}(\Sigma^n)$. This topology naturally extends to each of the spaces $\mathfrak{F}_c(\Sigma)[[\hbar]]$, and we can further show that the maps $\beta_{H'-H} : \mathfrak{F}_c(\Sigma)[[\hbar]] \to \mathfrak{F}_c(\Sigma)[[\hbar]]$ are homeomorphisms with respect to this topology [BDF09, §3.1], hence if we equip $\mathfrak{A}_\ell(\Sigma, \mathcal{M})$ with the initial topology with respect to the maps

$$(F_H)_{H \in \mathrm{Had}(\mathcal{M})} \mapsto F_{H_0}$$

for each $H_0 \in \mathrm{Had}(\mathcal{M})$, then these maps are also homeomorphisms. In other words, to check that a map into or out of $\mathfrak{A}_\ell(\Sigma, \mathcal{M})$ is continuous, one need only show that this is the case for any of its concrete realisations. This fact is particularly convenient when considering states of the form we define below.

**Definition 9.2.1.** A *Gaussian state* is a map $\omega_{H_0,\psi} : \mathfrak{A}_\ell(\Sigma, \mathcal{M}) \to \mathbb{C}[[\hbar]]$, where $H_0 \in \mathrm{Had}(\mathcal{M})$, $\psi \in \mathfrak{E}(\Sigma)$, defined by

$$\omega_{H_0,\psi}(F_H)_{H \in \mathrm{Had}(\mathcal{M})} \mapsto F_{H_0}[\psi]. \tag{9.29}$$

*Remark* 9.2.2. As we have already noted, evaluation maps are continuous in the Bastiani topology, hence the map $\omega_{H_0,\psi} : \mathfrak{A}_\ell(\Sigma, \mathcal{M}) \to \mathbb{C}[[\hbar]]$ is also continuous.

Now that we have identified a suitable topology, we may define a *field* on $\Sigma$ (without any mention of local covariance) as any continuous, linear map $\Psi^i : \mathfrak{D}(\Sigma) \to \mathfrak{A}_\ell(\Sigma, \mathcal{M})$. For reasons we shall see in the following proposition, we shall also add the constraint that

$$\mathrm{supp}\, \Psi^i(f) \subseteq \mathrm{supp}\, f, \tag{9.30}$$

where the support of an observable $(F_H)_{H \in \mathrm{Had}(\mathcal{M})} \in \mathfrak{A}_\ell(\Sigma, \mathcal{M})$ is defined as the union of the supports of each coefficient of $\hbar^n$ of $F_H$ for any $H \in \mathrm{Had}(\mathcal{M})$. (This definition makes sense as $\beta_{H'-H}$ does not affect the support.)

**Proposition 9.2.3.** *Let $\omega_{H,\psi} : \mathfrak{A}_\ell(\Sigma, \mathcal{M}) \to \mathbb{C}$ be a Gaussian state, and let $\Psi^i, \Psi^j : \mathfrak{D}(\Sigma) \to \mathfrak{A}_\ell(\Sigma, \mathcal{M})$ be a pair of fields. Then the map*

$$f \otimes g \mapsto \omega_{H,\psi}\left(\Psi^i(f) \star \Psi^j(g)\right) \tag{9.31}$$

*defines a distribution on $\Sigma^2$.*

*Proof.* Given the topology we have defined on $\mathfrak{A}_\ell(\Sigma, \mathcal{M})$ all we must show is that, for each $n \in \mathbb{N}$, the map

$$f \otimes g \mapsto \left\langle (\tfrac{i}{2}E_\Sigma + H_\Sigma)^{\otimes n}, \Psi^i(f)^{(n)}[\psi] \otimes \Psi^j(g)^{(n)}[\psi] \right\rangle \tag{9.32}$$

defines a distribution in $\mathfrak{D}'(\Sigma^2)$.

The argument follows a similar line to the classical case in section 8.5. Firstly, we see that $\forall\, H \in \mathrm{Had}(\mathcal{M})$ the map $f \mapsto (\Psi^i(f))^{(n)}_H[\psi]$ is a linear, continuous map $\mathfrak{D}(\Sigma) \to \mathfrak{E}'_{\Xi_n}(\Sigma^n)$. Hence, there is associated to it by the Schwartz kernel theorem a distribution $K^i_{n,\psi} \in \mathfrak{D}'(\Sigma^{n+1})$. The distribution corresponding to the coefficient of $\hbar^n$ in the $\star$ product is then

$$\begin{aligned}\frac{d^n}{d\hbar^n}\omega_{H,\psi}(\Psi^i(z_1) \star \Psi^j(z_2))|_{\hbar=0} = \int_{\Sigma^{2n}} \Big[&W_\Sigma(y_1, y_2) \cdots W_\Sigma(y_{2n-1}, y_{2n}) \\ &K^i_{n,\psi}(z, y_1, \ldots, y_{2n-1}) K^j_{n,\psi}(z, y_2, \ldots, y_{2n})\Big]\, \mathrm{d}y_1 \cdots \mathrm{d}y_{2n}.\end{aligned}$$

where $W_\Sigma = \frac{i}{2}E_\Sigma + H_\Sigma$.

We must then check the same conditions for this composition as we did in section 8.5. From $\operatorname{supp}\Psi^i(f) \subseteq \operatorname{supp} f$, we know $\Psi^i(f)^{(n)}[\psi] \subseteq (\operatorname{supp} f)^{\times n}$ and hence

$$\left\{(z, y_1, \ldots y_n) \in \operatorname{supp} K^i_{n,\psi} \,|\, z \in U\right\} \subseteq U^{n+1},$$

which shows that the required projection is proper.

For the wavefront sets, we have that

$$(z, y_1, \ldots, y_n; 0, \eta_1, \ldots, \eta_n) \in \mathrm{WF}(K^i_{n,\psi}) \Rightarrow (y_1, \ldots, y_n; \eta_1, \ldots, \eta_n) \in \Xi_n,$$

which in turn implies

$$\begin{aligned}&(z_1, z_2, y_1, \ldots, y_{2n}; 0, 0, \eta_1, \ldots \eta_{2n}) \in \mathrm{WF}(K^i_{n,\psi} \otimes K^j_{n,\psi}) \Rightarrow \\ &\qquad (y_1, \ldots, y_{2n}; \eta_1, \ldots \eta_{2n}) \in \Xi_{2n} \subseteq \dot{T}^*\Sigma^{2n} \setminus -\mathrm{WF}(W_\Sigma^{\otimes n}).\end{aligned}$$

From which we may conclude that each composition is a well-defined distribution coinciding with the coefficient of $\hbar^n$ in (9.32). □

We shall denote this distribution using its integral kernel

$$\omega_{H,\psi}\left(\Psi^i(s) \star \Psi^j(s')\right).$$

**Proposition 9.2.4.** *The anti-symmetric part of the above distribution can be written as*

$$\omega_{H,\psi}\left(\left[\Psi^i(s), \Psi^j(s')\right]_\star\right)$$

*and is supported on the diagonal* $\Delta \subset \Sigma^2$*.*

*Proof.* The fact that we can express the anti-symmetric part of the distribution in terms of the commutator is just a consequence of the linearity of $\omega_{H,\psi}$. To show that it is supported on the diagonal, we use the condition (9.30) and note that any $h \in \mathfrak{D}(\Sigma^2 \setminus \Delta)$ can be approximated by a series $f \otimes g$ such that $\operatorname{supp} f \cap \operatorname{supp} g = \emptyset$. We can then use the chiral version of Einstein causality (which we prove in a general setting in theorem 10.2.3) to show that $[\Psi^i(f), \Psi^j(g)]_\star = 0$ for each such $f, g$, hence the distribution must send $h$ to 0. □

We can now provide an analysis similar to that of section 8.5 by limiting our attention to homogeneously scaling chiral fields on $\mathbb{M}^2$. As before, we shall consider the $\Sigma_0$ Cauchy surface, and the action of the dilation operators $\{m_\Lambda : \Sigma_0 \to \Sigma_0\}_{\Lambda>0}$ and translation operators $\{t_c : \Sigma_0 \to \Sigma_0\}_{c\in\mathbb{R}}$. Once again, we shall use $t_c, m_\Lambda$ to refer to both the automorphisms of $\Sigma_0$ as well as the full spacetime $\mathbb{M}^2$.

If $\Psi^i : \mathfrak{D}^{(\mu)}_\ell|_{\mathsf{Cauchy}(\mathbb{M}^2)_0} \Rightarrow \mathfrak{A}_\ell|_{\mathsf{Cauchy}(\mathbb{M}^2)_0}$ be a homogeneously scaling locally covariant field on $\mathbb{M}^2$ with values in $\mathfrak{A}_\ell$, then in particular it responds to scalings and translations as

$$\Psi^i_{\Sigma_0}(\Lambda s) = \Lambda^{\mu-1}\mathfrak{A}_\ell m_\Lambda(\Psi^i_{\Sigma_0}(s)), \tag{9.33}$$

$$\Psi^i_{\Sigma_0}(s+c) = \mathfrak{A}_\ell t_c(\Psi^i_{\Sigma_0}(s)). \tag{9.34}$$

**Proposition 9.2.5.** *The Gaussian state* $\omega_{H_{\mathbb{M}^2},0} : \mathfrak{A}_\ell(\Sigma_0, \mathbb{M}^2) \to \mathbb{C}[[\hbar]]$ *is invariant with respect to the action of the scaling and translation morphisms.*

*Proof.* Translation invariance is trivial. Explicitly, the equation for scaling invariance is satisfied if

$$F_{m^*_\Lambda H_{\mathbb{M}^2}}[0] = F_{H_{\mathbb{M}^2}}[0]. \tag{9.35}$$

This is satisfied for every $F$ because $\beta_{H'-H}$ depends on the Hadamard distributions only through their chiral derivatives. Given that $m^*_\Lambda H_{\mathbb{M}^2} - H_{\mathbb{M}^2} = -\frac{1}{2\pi}\ln(\Lambda)$, the chiral derivative of this term vanishes, hence $\beta_{m^*_\Lambda H_{\mathbb{M}^2} - H_{\mathbb{M}^2}}$ acts as the identity. □

*Remark* 9.2.6. It is worth noting that it was necessary for us to use the fact that chiral observables are all defined in terms of the derivative field $\partial_\Sigma\varphi$.

We may now state the quantum analogue of proposition 8.5.6, the proof of which can be directly adapted from that of the classical result.

**Proposition 9.2.7.** *Let $\Psi^i, \Psi^j$ be a pair of homogeneously scaling locally covariant fields of weights $\mu_i, \mu_j$ on $\mathbb{M}^2$ with values in $\mathfrak{A}_\ell$. Then*

$$\omega_{H_{\mathbb{M}^2},0}\left(\left[\Psi^i_{\Sigma_0}(s), \Psi^j_{\Sigma_0}(s')\right]_\star\right) \propto \delta^{(\mu_i+\mu_j-1)}(s-s'). \tag{9.36}$$

For the chiral boson, this reduces to the classical result we have already seen. However, we can also show that the chiral stress-energy tensor defined above is locally covariant and homogeneously scaling with degree $2$ on $\mathbb{M}^2$, (recall from Part I that the quantised stress energy tensor is not primary). Hence this result also holds for $\Psi^i = \Psi^j = T$, where we find that the expectation value of the commutator is proportional to $\delta'''(s-s')$. This is a well-known result in the VOA formalism, where the constant of proportionality is known as the *central charge*, and is a vital parameter in the characterisation of a 2DCFT.

As before, none of this tells us anything we did not already know. Instead, it is the style of argument we wish to emphasise. There is a generalisation of the notion of a distribution scaling homogeneously with degree $\mu$. The *Steinmann scaling degree* of a distribution $u \in \mathfrak{D}'(U)$, where $U \subseteq \mathbb{R}^n$ is an open subset such that $\lambda x \in U \,\forall\, x \in U, \lambda > 0$, is defined as

$$\mathrm{sd}(u) = \inf\left\{\delta \in \mathbb{R} \mid \lim_{\lambda\to 0} \lambda^\delta u(\lambda x) = 0\right\}. \tag{9.37}$$

Note that if $u$ scales homogeneously with degree $\mu$, then $\mathrm{sd}(u) = \mu$, and if $u$ is a regular distribution, then by Taylor expanding $u$ around $x = 0$ one finds that $\mathrm{sd}(u) \leq 0$. Whilst we found in our two examples that the highest order term in $\hbar$ (equivalently the term which did not vanish when evaluating at $\psi = 0$) was homogeneously scaling with degree given by the degrees of our fields each term in the expansion of the OPEs has a well-defined scaling degree which counts the power of $(s-s')$ appearing in the denominator.

This suggest that in the future it may be possible to decompose the product of two fields into a series of bi-distributions with decreasing scaling degrees. Truncating this series at $\mathrm{sd} = 0$ would then give the usual form of the OPE modulo smooth terms.

## 9.3 RECONSTRUCTING THE BOUNDARY TERM

To close out this section, we shall briefly comment on how the constants of proportionality that appeared in our arguments from homogeneous scaling can be fixed by constraints on wavefront sets.

**Proposition 9.3.1.** *Let $u_n \in \mathfrak{D}'(\mathbb{R} \setminus \{0\})$ be the regular distribution defined by the function $\frac{1}{x^{n+1}}$. If $\overline{u}_n \in \mathfrak{D}'(\mathbb{R})^{\mathbb{C}}$ is an extension of $u_n$ which also scales homogeneously with degree $n+1$, then*

$$\overline{u}_n(x) = \frac{(-1)^n}{n!} \frac{d^n}{dx^n} \left[ \alpha \delta(x) + \mathrm{PV}\left(\frac{1}{x}\right) \right] \tag{9.38}$$

*for some $\alpha \in \mathbb{C}$.*

*Moreover, the wavefront set of $\overline{u}_n$ is*

$$\mathrm{WF}(\overline{u}_n) = \begin{cases} \{0\} \times \mathbb{R}_{<0} & : \quad \alpha = i\pi \\ \{0\} \times \mathbb{R}_{>0} & : \quad \alpha = -i\pi \\ \{0\} \times \mathbb{R} \setminus \{0\} & : \quad \textit{else} \end{cases} \tag{9.39}$$

*and for the values $\alpha = \pm i\pi$, we may write $\overline{u}_n$ as*

$$\overline{u}_n(x) = \frac{(-1)^n}{n!} \frac{d^n}{dx^n} \lim_{\epsilon \searrow 0} \frac{1}{x \mp i\epsilon}. \tag{9.40}$$

*Proof.* Firstly, note that $\frac{(-1)^n}{n!} \frac{d^n}{dx^n} \mathrm{PV}\left(\frac{1}{x}\right)$ is a well defined distribution on all of $\mathbb{R}$ which coincides with $u_n$ on the complement of $\{0\}$. It also scales homogeneously with degree $n+1$, hence the difference between this distribution and any other extension of $u_n$ must be proportional to $\delta^{(n)}(x)$.

As $\bar{u}_n$ is a tempered distribution, we can compute its wavefront set by simply taking its Fourier transform, which is

$$\hat{\overline{u}}_n(\xi) = \frac{(-i\xi)^n}{n!} \left[ \alpha - i\pi \,\mathrm{sgn}(\xi) \right]. \tag{9.41}$$

From this we can clearly see which values of $\alpha$ correspond to which wavefront set in (9.39). □

Similarly to our embedding $\partial^*_{\Sigma_0,\epsilon} : \mathfrak{F}_c(\Sigma_0) \to \mathfrak{F}_{\mu c}(\mathbb{M}^2)$, we can use $\partial^*_{\Sigma_0,\epsilon}$ to map $\overline{u}_n(s - s') \in \mathfrak{D}'(\Sigma_0^2)$ to a distribution $(\partial^*_{\Sigma_0,\epsilon})^{\otimes 2} \overline{u}_n \in \mathfrak{D}'((\mathbb{M}^2)^2)$. Again, we can compute the wavefront set of this new distribution by considering its Fourier transform as a

Schwartz distribution, which is

$$\begin{aligned}\widehat{(\partial^*_{\Sigma_0,\epsilon})^{\otimes 2}}\overline{u}_n(\xi,\eta) =& -4\xi_u\eta_u\hat{\delta}_\epsilon(2\xi_v)\hat{\delta}_\epsilon(2\eta_v)\int_{\mathbb{R}^2}\overline{u}_n(s-s')e^{i(\xi_u-\xi_v)s}e^{i(\eta_u-\eta_v)s}\,\mathrm{d}s\,\mathrm{d}s', \qquad (9.42)\\ =& -8\pi\xi_u\eta_u\frac{[-i(\xi_u-\xi_v)]^n}{n!}\hat{\delta}_\epsilon(2\xi_v)\hat{\delta}_\epsilon(2\eta_v)\cdot\\ & [\alpha - i\pi\mathrm{sgn}(\xi_u-\xi_v)]\delta(\xi_u-\xi_v+\eta_u-\eta_v) \qquad (9.43)\end{aligned}$$

The presence of the $\hat{\delta}_\epsilon$ terms means that this function decays rapidly for all neighbourhoods of directions in which $\xi_v, \eta_v$ are non-zero. Next, if we consider $(\xi_0, 0, \eta_0, 0) \in \dot{T}^*(\mathbb{M}^2)^2$ for $\xi_0 + \eta_0 > 0$, then we can find a conic neighbourhood $\Gamma$ such that $\xi_u - \xi_v + \eta_u - \eta_v > 0$, $\forall\,(\xi,\eta) \in \Gamma$, hence the $\delta$ term ensures rapid decay. An analogous argument holds for $\xi_0 + \eta_0 < 0$. Finally, we may find a conic neighbourhood $\Gamma$ of $(\xi_0, 0, -\xi_0, 0)$ where $\xi_0 > 0$ such that $\mathrm{sgn}(\xi_u - \xi_v) = 1$, $\forall\,(\xi,\eta) \in \Gamma$ and hence we have rapid decay precisely when $\alpha = i\pi$. Once again, a similar argument implies rapid decay in the case $\xi_0 < 0$ precisely when $\alpha = -i\pi$.

From this, we may define a coordinate-free condition to uniquely determine $\alpha$, namely

$$\mathrm{WF}((\partial^*_{\Sigma_0,\epsilon})^{\otimes 2}\overline{u}_n) \subseteq \mathrm{WF}(W) \qquad (9.44)$$

where $W$ is a Hadamard distribution, if and only if $\alpha = -i\pi$.

Going the other way, the Schwartz kernel $K \in \mathfrak{D}'(\Sigma_0 \times \mathbb{M}^2)$ associated to $\partial_{\Sigma,\epsilon}$ allows us to define a correspondence between subsets of $\dot{T}^*\mathbb{M}^2$ and subsets of $\dot{T}^*\Sigma_0$

$$\begin{aligned}\Gamma \mapsto \Big\{&(s,\xi_s) \in \dot{T}^*\Sigma_0 \,|\\ &\exists\,(u,v;\xi_u,\xi_v) \in \Gamma, (s,u,v;\xi_s,\xi_u,\xi_v,) \in \mathrm{WF}(K)\Big\}\end{aligned} \qquad (9.45)$$

In two dimensions, the wavefront set of a Hadamard distribution factorises into two disjoint components

$$\mathrm{WF}(W) = \Gamma_\ell \sqcup \Gamma_r, \qquad (9.46)$$

where, in $\mathbb{M}^2$ we can write $\Gamma_\ell = \{(u,v,u,v';\xi_u,0,-\xi_u,0)\,|\,\xi_u > 0\}$ and a similar expression for $\Gamma_r$. Under the correspondence associated to $K$, $\Gamma_\ell$ leads to a spectral condition on $\Sigma_0$

$$\mathrm{WF}(u) = \Big\{(s,s;\xi_s,-\xi_s) \in \dot{T}^*\Sigma_0^2\,|\,\xi_s > 0\Big\}. \qquad (9.47)$$

It is this wavefront set condition which uniquely determines the constant of proportionality $\alpha$.

It is interesting to note that the wavefront set spectral condition may be interpreted as a 'positivity of energy' condition which is valid even on curved spacetimes. What we have seen is that, given only knowledge of the "singular part" of the right hand side of expressions such as (9.22) and (9.26), the analogous condition for states on the chiral algebra is enough to uniquely determine their extensions as homogeneously scaling distributions on $\mathfrak{D}'(\Sigma_0^2 \setminus \Delta)$. One might argue that this is a somewhat artificial scenario. However, in the formulation of QFTS using *factorisation algebras* [CG16], the product of observables (known as the *factorisation product*) is defined only for observables localised to disjoint regions, hence one would expect such distributions to arise naturally in this setting.

## — 10 —

# Towards a Model Independent Definition of Chiral Algebras

To conclude, we shall discuss how one might in general define the 'chiral sector' of a generic AQFT. For an on-shell algebra, we are able to provide a precise definition. We shall see that our classical algebra fits this definition, and also that locality in the sense of commutativity of chiral algebras localised in disjoint regions is implied by Einstein causality in the full algebra.

## 10.1 DEFINITION

We need a category whose objects are connected sets of null geodesics of spacetimes. It is similar in definition and purpose to the category CCauchy introduced earlier. However, this time we will be able to give a simpler definition by realising this structure as a *comma category*.

**Definition 10.1.1.** Let $\mathcal{A} \xrightarrow{S} \mathcal{B} \xleftarrow{T} \mathcal{C}$ be a pair of functors with common target. The *comma category* $(S \downarrow T)$ is the category such that

- Objects are triples $(a, h, c)$ such that $h \in \mathcal{B}(S(a), T(c))$.
- Morphisms $(a, h, c) \to (a', h', c')$ are pairs $(f, g) \in \mathcal{A}(a, a') \times \mathcal{C}(c, c')$ such that the following diagram commutes.

$$\begin{tikzcd}
S(a) \arrow[r, "Sf"] \arrow[d, "h"] & S(a') \arrow[d, "h'"] \\
T(c) \arrow[r, "Tg"] & T(c')
\end{tikzcd} \tag{10.1}$$

Recall that in proposition 7.1.2 we established that the map $\pi_\ell : \mathcal{M} \to \mathcal{M}_\ell$ projecting a spacetime $\mathcal{M}$ onto its space of *right-moving* null geodesics was 'functorial' in the sense that, for any $\chi \in \mathsf{CLoc}(\mathcal{M}, \widetilde{\mathcal{M}})$, there was a map $\chi_\ell$ such that $\tilde{\pi}_\ell \chi = \chi_\ell \pi_\ell$, where $\tilde{\pi}_\ell$ is the corresponding projection on $\widetilde{\mathcal{M}}$. In particular $\pi_\ell$ defines a functor $\mathsf{CLoc} \to \mathsf{Man}_1^+$ where the latter category comprises connected, oriented 1-manifolds with orientation-preserving smooth embeddings as morphisms.

One of the categories we shall use to define chiral algebras in a more generic setting is the comma category $(\mathrm{Id}_{\mathsf{Man}_1^+} \downarrow \pi_\ell)$ where $\mathrm{Id}_{\mathsf{Man}_1^+}$ denotes the identity functor. Unpacking the definition, this means that an object of this category is defined by a choice $\mathcal{I} \in \mathsf{Man}_1^+$, a spacetime $\mathcal{M} \in \mathsf{CLoc}$ and an embedding $i : \mathcal{I} \hookrightarrow \pi_\ell(\mathcal{M})$. Similarly, a morphism is a pair, $\mathcal{I} \xrightarrow{\rho} \mathcal{J}$, $\mathcal{M} \xrightarrow{\chi} \mathcal{N}$, of a smooth embedding and admissible embedding respectively such that the following diagram commutes.

$$\begin{tikzcd}
\mathcal{I} \arrow[r, "i"] \arrow[d, "\rho"] & \pi_\ell(\mathcal{M}) \arrow[d, "\chi"] \\
\mathcal{J} \arrow[r, "j"] & \pi_\ell(\mathcal{N})
\end{tikzcd} \tag{10.2}$$

**Definition 10.1.2.** Let $\mathfrak{A} : \mathsf{CLoc} \to \mathsf{Obs}$ be an AQFT with values in some suitable category $\mathsf{Obs}$. A *chiral subalgebra* of $\mathfrak{A}$ is a functor $\mathfrak{A}_c : \mathsf{Man}_1^+ \to *\text{-}\mathsf{Alg}$ and a functor $N : (\mathrm{Id}_{\mathsf{Man}_1^+} \downarrow \pi_\ell) \to (\mathfrak{A}_c \downarrow \mathfrak{A})$.

Explicitly, this means that for every square of the form (10.2), there is a commuting diagram in $\mathsf{Obs}$ as follows.

$$\begin{tikzcd}
\mathfrak{A}_c(\mathcal{I}) \arrow[r, "Ni"] \arrow[d, "\mathfrak{A}_c\rho"] & \mathfrak{A}(\mathcal{M}) \arrow[d, "\mathfrak{A}\chi"] \\
\mathfrak{A}_c(\mathcal{J}) \arrow[r, "Nj"] & \mathfrak{A}(\mathcal{N})
\end{tikzcd} \tag{10.3}$$

*Remark* 10.1.3. There is a degree of redundancy in this definition. If we asked only for a functor $(\mathrm{Id}_{\mathsf{Man}_1^+} \downarrow \pi_\ell) \to (\mathrm{Id}_{\mathsf{Obs}} \downarrow \mathfrak{A})$, then this would automatically contain all the information necessary to describe $\mathfrak{A}_c$.

For a comma category $(S \downarrow T)$ we define the projection functors $\Pi_{\mathcal{A}} : (S \downarrow T) \to \mathcal{A}$, $\Pi_{\mathcal{C}} : (S \downarrow T) \to \mathcal{C}$. Given a functor $N : (\mathrm{Id}_{\mathsf{Man}_1^+} \downarrow \pi_\ell) \to (\mathrm{Id}_{\mathsf{Obs}} \downarrow \mathfrak{A})$, we can then define $\mathfrak{A}_c := \Pi_{\mathsf{Obs}} \circ N$.

There is another equivalent characterisation of this definition. Given a diagram

$$\begin{tikzcd}
 & \mathcal{B} & \\
\mathcal{A} \arrow[ur, "S"] \arrow[dr, "F"'] & & \mathcal{C} \arrow[ul, "T"'] \arrow[dl, "G"] \\
 & \mathcal{D} &
\end{tikzcd}$$

there is a one-to-one correspondence between natural transformations $F \circ \Pi_{\mathcal{A}} \Rightarrow G \circ \Pi_{\mathcal{C}}$ and functors $(S \downarrow T) \to (F \downarrow G)$. Applying this correspondence to definition 10.1.2, the functor $N$ defines a natural transformation $\mathfrak{A}_c \circ \Pi_{\mathsf{Man}_1^+} \Rightarrow \mathfrak{A} \circ \Pi_{\mathsf{CLoc}}$, which bears a close resemblance to definition 9.1.2 of a chiral algebra as a functor $\mathfrak{A}_\ell : \mathsf{CCauchy} \to *{-}\mathsf{Alg}$ for which we then found in theorem 9.1.4 a natural transformation $\mathfrak{A}_\ell \Rightarrow \mathfrak{A}_{\mathrm{on}} \circ \Pi_2$, where $\Pi_2 : \mathsf{CCauchy} \to \mathsf{CLoc}$.

Secondly, this definition always admits a trivial subalgebra by taking $\mathfrak{A}_c(\mathcal{I}) = 0$ for every $\mathcal{I}$. In principle, one might define a chiral subalgebra as *maximal* if, in addition to the above, it possesses the universal property that, for every alternative choice $N'$ : $(\mathrm{Id}_{\mathsf{Man}_1^+} \downarrow \pi_\ell) \to (\mathfrak{A}'_c \downarrow \mathfrak{A})$ there exists a unique functor $I : (\mathfrak{A}'_c \downarrow \mathfrak{A}) \to (\mathfrak{A}_c \downarrow \mathfrak{A})$ such that $N' = N \circ I$. However, we shall not explore this idea further.

## 10.2 CAUSALITY IN CHIRAL SUBALGEBRAS

We now make use of this definition by proving that causality in the chiral sense is guaranteed for any chiral subalgebra of an AQFT satisfying Einstein causality. To make this precise, we define each of these properties as follows.

**Definition 10.2.1.** An AQFT $\mathfrak{A} : \mathsf{CLoc} \to *\text{-}\mathsf{Alg}$ satisfies the *Einstein causality* axiom if, for every diagram of $\mathsf{CLoc}$ morphisms $\mathcal{M}_1 \xrightarrow{\chi_1} \mathcal{M} \xleftarrow{\chi_2} \mathcal{M}_2$ such that $\chi_1(\mathcal{M}_1)$ is spacelike separated from $\chi_2(\mathcal{M}_2)$,

$$[\mathfrak{A}\chi_1\left(\mathfrak{A}(\mathcal{M}_1)\right), \mathfrak{A}\chi_2\left(\mathfrak{A}(\mathcal{M}_2)\right)] = 0.$$

**Definition 10.2.2.** A functor $\mathfrak{A}_c : \mathsf{Man}_1^+ \to *\text{-}\mathsf{Alg}$ is *mutually local* if for every $\mathcal{I}_1 \xrightarrow{i_1} \mathcal{I} \xleftarrow{i_2} \mathcal{I}_2$ such that $i_1(\mathcal{I}_1) \cap i_2(\mathcal{I}_2) = \emptyset$,

$$[\mathfrak{A}_c i_1(\mathfrak{A}_c(\mathcal{I}_1)), \mathfrak{A}_c i_2(\mathfrak{A}_c(\mathcal{I}_2))] = 0.$$

**Theorem 10.2.3.** *Let* $\mathfrak{A} : \mathsf{CLoc} \to *\text{-}\mathsf{Alg}$ *be an* AQFT *such that* $\mathfrak{A}\chi$ *is injective for every* $\mathsf{CLoc}$ *morphism* $\chi$, *and let* $N : (\mathrm{Id}_{\mathsf{Man}_1^+} \downarrow \pi_\ell) \to (\mathfrak{A}_c \downarrow \mathfrak{A})$ *be a chiral subalgebra of* $\mathfrak{A}$, *then* $\mathfrak{A}_c$ *is mutually local if* $\mathfrak{A}$ *satisfies the Einstein causality axiom.*

*Proof.* Given $\mathcal{I}_1 \xrightarrow{i_1} \mathcal{I} \xleftarrow{i_2} \mathcal{I}_2$ as above, we shall construct a diagram $\mathcal{M}_1 \xrightarrow{\chi_1} \mathcal{M} \xleftarrow{\chi_2} \mathcal{M}_2$ satisfying the conditions of definition 10.2.1 along with maps $\rho : \mathcal{I} \to \pi_\ell(\mathcal{M})$ and $\rho_i : \mathcal{I} \to \pi_\ell(\mathcal{M}_i)$ such that the following commutes.

$$\begin{tikzcd}
\mathcal{I}_1 \arrow[r, "i_1"] \arrow[d, "\rho_1"] & \mathcal{I} \arrow[d, "\rho"] & \mathcal{I}_2 \arrow[l, "i_2"'] \arrow[d, "\rho_2"] \\
\pi_\ell(\mathcal{M}_1) \arrow[r, "\pi_\ell\chi_1"'] & \pi_\ell(\mathcal{M}) & \pi_\ell(\mathcal{M}_2) \arrow[l, "\pi_\ell\chi_2"]
\end{tikzcd} \tag{10.4}$$

We first construct $\mathcal{M}$ by choosing an arbitrary metric on $\mathcal{I}$ compatible with the pre-existing orientation. $\mathcal{M}$ is then given by the manifold $\mathcal{I} \times \mathbb{R}$ with metric $g_{\mathcal{M}} = du \odot dv$, where $du$ is the volume form on $\mathcal{I}$ induced by the metric, and $dv$ is the canonical one-form on $\mathbb{R}$. By construction, there is a canonical isomorphism $\rho : \pi_\ell(\mathcal{M}) \xrightarrow{\simeq} \mathcal{I}$ given by $u \mapsto [\{(u,v)\}_{v \in \mathbb{R}}]$.

Next, we take an arbitrary Cauchy surface $\Sigma \subset \mathcal{M}$ and define $\Sigma_i = \Sigma \cap \pi_\ell^{-1}(\rho \circ i_i(\mathcal{I}_i))$ for $i \in \{1,2\}$, where by $\pi_\ell^{-1}$ we mean in particular the preimage under the map $\pi_\ell : \mathcal{M} \to \rho(\mathcal{I})$. The requisite spacetime $\mathcal{M}_i$ may then be obtained as the *Cauchy development* $D(\Sigma_i)$ (with the embedding $\chi_i$ being simply the inclusion map). This is the set of events $x \in \mathcal{M}$ such that every inextensible causal curve intersecting $x$ also intersects $\Sigma_i$. A sketch of this situation is provided in fig. 10.1.

Note that $\Sigma_1 \cap \Sigma_2 = \emptyset$ means that $D(\Sigma_1)$ and $D(\Sigma_2)$ must be causally disjoint: suppose we have $x_1 \in D(\Sigma_1)$, $x_2 \in D(\Sigma_2)$ such that there is a causal curve $\gamma$ connecting the two. If we maximally extend $\gamma$, then it must intersect both $\Sigma_1$ and $\Sigma_2$ by definition of the Cauchy developments. However, as $\Sigma$ is a Cauchy surface, each inextensible causal curve intersects it *precisely* once, thus $x_1$ and $x_2$ cannot be connected by any causal curve.

We can also see that $\pi_\ell(\mathcal{M}_i) = \pi_\ell(\Sigma_i) = \rho \circ i_i(\mathcal{I}_i)$. This means that if $\rho_i = \rho \circ i_i|^{\pi_\ell(\mathcal{M}_i)}$, then $\pi_\ell \chi_i \circ \rho_i = \rho \circ i_i$ as required, as $\pi_\ell \chi_i$ is simply the inclusion map of $\pi_\ell(\mathcal{M}_i) \subseteq \pi_\ell(\mathcal{M})$.

We then apply the functor $N$ to the above diagram, which gives

$$\begin{tikzcd}
\mathfrak{A}_c(\mathcal{I}_1) \arrow[r, "\mathfrak{A}_c i_1"] \arrow[d, "N\rho_1"] & \mathfrak{A}_c(\mathcal{I}) \arrow[d, "N\rho"] & \mathfrak{A}_c(\mathcal{I}_2) \arrow[l, "\mathfrak{A}_c i_2"'] \arrow[d, "N\rho_2"] \\
\mathfrak{A}(\mathcal{M}_1) \arrow[r, "\mathfrak{A}\chi_1"'] & \mathfrak{A}(\mathcal{M}) & \mathfrak{A}(\mathcal{M}_2) \arrow[l, "\mathfrak{A}\chi_2"]
\end{tikzcd} \tag{10.5}$$

Finally, from Einstein causality, we see that, for $F \in \mathfrak{A}_c(\mathcal{I}_1)$, $G \in \mathfrak{A}_c(\mathcal{I}_2)$,

$$\begin{aligned} N\rho([\mathfrak{A}_c i_1(F), \mathfrak{A}_c i_2(G)]_{\mathfrak{A}_c(\mathcal{I})}) &= [\mathfrak{A}\chi_1 \circ N\rho_1(F), \mathfrak{A}\chi_2 \circ N\rho_2(G)]_{\mathfrak{A}(\mathcal{M})} \\ &= 0. \end{aligned}$$

Given that the morphisms $\mathfrak{A}\chi$ are injective by hypothesis, we may also conclude that $[\mathfrak{A}_c i_1(F), \mathfrak{A}_c i_2(G)]_{\mathfrak{A}_c(\mathcal{I})} = 0$ as required. □

*Remark* 10.2.4. This definition does not actually depend on the conformal symmetry of $\mathfrak{A}$. In fact, a similar argument may be used in arbitrary dimensions to show that the canonical algebra associated to a Cauchy surface $\Sigma$, defined as the limit of the inverse

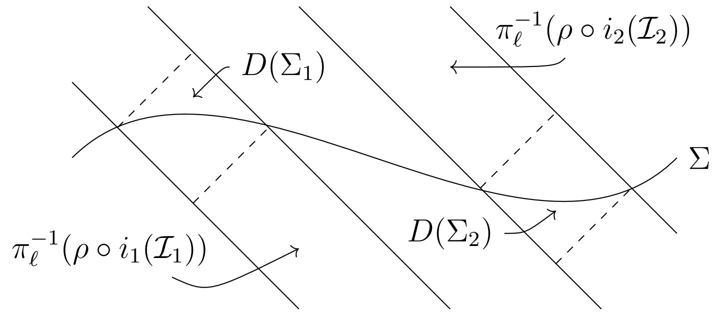

Figure 10.1: A sketch of $\Sigma_1$, $\Sigma_2$ and their Cauchy developments.

system of neighbourhoods of $\Sigma$, is local in much the same way, provided such limits exist and can be assigned functorially.

It is also worth noting that the analogous proof for $\mathfrak{A}_\ell : \mathsf{CCauchy} \to \mathsf{Obs}$ is even simpler, as the necessary hypothesis amounts to the existence of a diagram of the form

$$\begin{tikzcd}
\Sigma_1 \arrow[r, "\rho_1"] \arrow[d, hook] & \Sigma \arrow[d, hook] & \Sigma_2 \arrow[l, "\rho_2"'] \arrow[d, hook] \\
\mathcal{M}_1 \arrow[r, "\chi_1"] & \mathcal{M} & \mathcal{M}_2 \arrow[l, "\chi_2"']
\end{tikzcd} \tag{10.6}$$

such that $\rho_1(\Sigma_1) \cap \rho_2(\Sigma_2) = \emptyset$, which already implies that $\chi_1(\mathcal{M}_1)$ is causally disjoint from $\chi_2(\mathcal{M}_2)$.

**Proposition 10.2.5.** *There exists a chiral subalgebra* $(\mathfrak{P}_c, N)$ *of* $\mathfrak{P}_{\mathrm{on}}$.

*Proof.* Firstly, we define the algebra on $\mathcal{I} \in \mathsf{Man}_1^+$ such that the underlying space is the space of maps $F : \Omega^1(\mathcal{I}) \to \mathbb{C}$ which are Bastiani smooth and satisfy the wavefront set condition specified in proposition 8.1.1. Note that both the definition of Bastiani smoothness as well as the wavefront set are not affected by changing domains to $\Omega^1(\mathcal{I})$, as both are defined with respect to a chart on $\mathcal{I}$, which also provides an identification $\Omega^1(\mathcal{I}) \simeq \mathfrak{E}(\mathcal{I})$. To avoid confusion, we shall use an alternative notation for the functional derivatives of this algebra, namely

$$\frac{d}{d\epsilon} F[j + \epsilon k]|_{\epsilon=0} =: \int_{\mathcal{I}} \frac{\delta F}{\delta j}[j] \wedge k. \tag{10.7}$$

Given our wavefront set condition, this means that $\frac{\delta F}{\delta j}[j] \in \mathfrak{D}(\mathcal{I})$.

The Poisson bracket of this algebra is then defined by

$$\{F, G\}_{\mathcal{I}}[j] := \int_{\mathcal{I}} \frac{\delta F}{\delta j}[j] \wedge d\left(\frac{\delta G}{\delta j}[j]\right). \tag{10.8}$$

It is then straightforward to verify that, for a smooth, oriented embedding $f : \mathcal{I} \to \mathcal{J}$, $\frac{\delta(f_*F)}{\delta j}[j] = f_*\left(\frac{\delta F}{\delta j}[f^*j]\right)$, and hence that

$$f_*\{F, G\}_{\mathcal{I}} = \{f_*F, f_*G\}_{\mathcal{J}}\,.$$

We must now define the functor $(\mathrm{Id}_{\mathsf{Man}_1^+} \downarrow \pi_\ell) \to (\mathfrak{P}_c \downarrow \mathfrak{P}_{\mathrm{on}})$. For $\mathcal{I} \xrightarrow{\rho} \mathcal{M}$, we take an arbitrary Cauchy surface $\Sigma \subset \mathcal{M}$, and a regularised chiral derivative $\partial_{\Sigma,\epsilon} : \mathfrak{E}(\mathcal{M}) \to \mathfrak{E}(\Sigma)$. Note that each $\Sigma$ is also an object in $\mathsf{Man}_1^+$, thus we can produce a sequence of maps

$$\mathfrak{P}_c(\mathcal{I}) \xrightarrow{\mathfrak{P}_c(\pi_\ell|_\Sigma \circ \rho)} \mathfrak{P}_c(\Sigma) \longrightarrow \mathfrak{P}_\ell(\Sigma, \mathcal{M}) \xrightarrow{\partial^*_{\Sigma,\epsilon}} \mathfrak{P}_{\mathrm{on}}(\mathcal{M}), \tag{10.9}$$

where the central map is defined by $F[j] \mapsto (\psi \mapsto F[\psi \mathrm{d}V_\Sigma])$. The composition is overall independent of our choices $\Sigma$ and $\partial_{\Sigma,\epsilon}$ if, for every other choice $\Sigma'$, $\partial_{\Sigma',\epsilon'}$, the diagram

$$\begin{tikzcd}
& \mathfrak{P}_c(\Sigma) \arrow[r] \arrow[dd] & \mathfrak{P}_\ell(\Sigma, \mathcal{M}) \arrow[dr] \arrow[dd] & \\
\mathfrak{P}_c(\mathcal{I}) \arrow[ur] \arrow[dr] \arrow[rrr, dashed, "N\rho"] & & & \mathfrak{P}_{\mathrm{on}}(\mathcal{M}) \\
& \mathfrak{P}_c(\Sigma') \arrow[r] & \mathfrak{P}_\ell(\Sigma', \mathcal{M}) \arrow[ur] &
\end{tikzcd} \tag{10.10}$$

commutes (defining the arrow $N\rho$), where the vertical arrows are the respective homomorphisms corresponding to $(\pi_\ell|_{\Sigma'}^{-1} \circ \pi_\ell|_\Sigma) : \Sigma \overset{\sim}{\to} \Sigma'$. This is readily verified: the left-hand triangle commutes by functoriality of $\mathfrak{P}_c$ and the right-hand triangle commutes by theorem 8.2.3.

We verify the commutativity of the inner square explicitly. If we take $F \in \mathfrak{P}_c(\Sigma)$, $\psi \in \mathfrak{E}(\Sigma')$, then applying the upper-right maps to $F$ and evaluating at $\psi$ we get $F\left[\{(\pi_\ell|_{\Sigma'}^{-1} \circ \pi_\ell|_\Sigma)^*_{(1)}\psi\}\mathrm{dV}_\Sigma\right]$. Going instead through the lower-left corner of the square results in $F\left[(\pi_\ell|_{\Sigma'}^{-1} \circ \pi_\ell|_\Sigma)^*(\psi \mathrm{dV}_{\Sigma'})\right]$. Recalling that $(\pi_\ell|_{\Sigma'}^{-1} \circ \pi_\ell|_\Sigma)^*_{(1)}\psi = \omega_\Sigma(\pi_\ell|_{\Sigma'}^{-1} \circ \pi_\ell|_\Sigma)^*\psi$, where $(\pi_\ell|_{\Sigma'}^{-1} \circ \pi_\ell|_\Sigma)^*\mathrm{dV}_{\Sigma'} = \omega_\Sigma \mathrm{dV}_\Sigma$, we see that these evaluations coincide, concluding the proof. □

# Conclusion

The purpose of this thesis was to show that the general constructive toolkit of pAQFT could be readily modified to describe conformal field theories in general and 2DCFT in particular. In light of this, the fact that many of our results should already be familiar to anyone fluent in 2DCFT is quite reassuring.

The novelty in these results lie both in their origins and the methods used in their proof, which emphasise a geometric perspective that is also compatible with the causal structure of the full spacetime. A good example of this is the derivation of the Schwarzian derivative term in chapter 6, where we in fact found a transformation law valid across all spacetimes, which happened to coincide with the Schwarzian term when we restricted our morphisms to Minkowski space. This lies in contrast to many standard proofs which typically rely on the assumption that the underlying space one is working with is a compactified null-ray in Minkowski space.

Even where our results were restricted to Minkowski, primarily in the discussion of how one can constrain the Poisson brackets/commutators of homogeneously scaling fields, many of the tools we used have well understood analogues on general spacetimes, so it is not unreasonable to expect that the results too may have general analogues.

Fields played a central role at many stages of our analysis. This is necessary if one wishes to make any meaningful connections to formalisms which are based on the Wightman axioms such at VOAS. Indeed, learning more about fields and their role in pAQFT stands to be one of the key consequences of the work we have done here. Our results are mostly preliminary, but they point towards a more general structure. For instance, we were able to show in section 8.5 (and later section 9.3) that one of the standard topologies placed on the classical (hence also the quantum) algebra allows one to create distributions out of fields using the Poisson bracket ($*$-product), with transformation properties of the fields descending to analogous properties of the distributions.

In this case, we were able to simplify the topological analysis involved by always holding in place a fixed field configuration. If, in the future, one wished to make sense of statements such as 'the product of two fields is a distribution in two-variables with values in the algebra of observables', inspired by the product of fields in VOAs $\mathrm{End}(V)[[z,z^{-1}]] \times \mathrm{End}(V)[[w,w^{-1}]] \to \mathrm{End}(V)[[z,z^{-1},w,w^{-1}]]$, then it is likely that one would need to use a space of functionals with better topological properties than those used in this thesis. A good candidate in this regard might be the the space $\mathfrak{F}_{D\mu c}(\mathcal{M})$ of functionals with controls on both the wavefront set and the *dual* wavefront set which was introduced by Dabrowski in [Dab14]. It is in a setting such as this that one could then begin to formulate more general statements about the OPEs of fields in pAQFT.

Another simplification we made in our study of fields occurred when we restricted our definition of chiral primary fields to open subsets of Minkowski space. Even this case we were able to formulate, and take advantage of, local covariance. However, to obtain a more general description one would need to determine what is the minimal amount of auxiliary data required to specify a field as a map $\mathfrak{D}(\Sigma) \to \mathfrak{P}_\ell(\Sigma,\mathcal{M})$, as well as a coordinate independent analogue of the map $h \mapsto \int_{v\in\mathbb{R}} h(u,v)dv$. In any case, perhaps more desirable would be an appropriate notion of a primary field for the model-independent algebras we introduced in the final chapter.

As a final comment, we would like to discuss the way in which we described the on-shell physics. Previously we noted that one should not expect the chiral algebra to embed naturally into the off-shell algebra of the full spacetime, as the chiral algebra is already on-shell. We resolved this matter by simply taking the appropriate quotient. However, as we briefly remarked upon earlier in the thesis, there is extra data encoded in the off-shell algebra, particularly when one considers a theory with non-trivial symmetries.

Our definition of the chiral algebras $\mathfrak{A}_\ell(\Sigma,\mathcal{M})$ trivially extend to a dg variant (i.e. with values in $\mathsf{Ch}(\mathsf{Alg})$) by taking the cochain complex which is $\mathfrak{A}_\ell(\Sigma,\mathcal{M})$ in degree zero, and which vanishes in all other degrees. From this perspective, we could then say that the embedding $\mathfrak{A}_\ell(\Sigma,\mathcal{M}) \to \mathfrak{A}(\mathcal{M})$ is natural *modulo exact terms*. We would expect this formulation to be beneficial, for example, when treating theories with gauge-symmetries, such as the WZW model.

On the topic of gauge symmetries, whilst we were able to find chiral subalgebras which embedded naturally into the full algebra, we were unable to show that a combination of the chiral and anti-chiral subalgebras was isomorphic to the on-shell algebra. A simple argument for why this is the case involves the undifferentiated field $\Phi(f) \in \mathfrak{F}_{\mathrm{reg}}(\mathcal{M})$. If we denote by $\mathbf{1} \in \mathfrak{E}(\mathcal{M})$ the constant function, then $\Phi(f)[\varphi + c\mathbf{1}] = \Phi(f) + c\int_{\mathcal{M}} f \mathrm{d}V_{\mathcal{M}}$. However, both the chiral and anti-chiral derivatives are insensitive to the addition of constants, hence any functional built from chiral observables must be invariant under this action. If one treated $\varphi \mapsto \varphi + c\mathbf{1}$ as a *gauge symmetry*, then it may be possible to construct an isomorphism between a combination of chiral + anti-chiral algebras and the space of on-shell *gauge invariant* observables in the full spacetime algebra. This perspective would fit naturally into the BV formalism in pAQFT, as one could simply form the Chevalley-Eilenberg complex with respect to the derivation which encodes the infinitesimal action of this symmetry $D_c \cdot \mathcal{F}[\varphi] = \frac{d}{d\epsilon}\mathcal{F}[\varphi + \epsilon c\mathbf{1}] =: cD \cdot \mathcal{F}[\varphi]$. The $0^{\mathrm{th}}$ cohomology of the resulting BV complex would then be the desired space of on-shell invariants. Hence we find yet more motivation for a dg reformulation of our framework.

# Part III

# Appendices

## — A —

# Method of Images

It is well-known that if a space $Y$ can be expressed as the quotient of some other space $X$ under the action of some group (satisfying certain properties), then we can use this relation in order to build Green's functions on $Y$ out of Green's functions. Here we give a coordinate-free account of some of the necessary results, then explain how this method may be used to construct the retarded/advanced propagators of the cylinder from those of Minkowski space.

**Lemma A.1.** *Let $P$ be a differential operator on a smooth manifold $\mathcal{M}$ and let $G : \mathfrak{D}(\mathcal{M}) \to \mathfrak{E}(\mathcal{M})$ be a fundamental solution to $P$, i.e. $PGf = GPf = f$ for all $f \in \mathfrak{D}(\mathcal{M})$. For $U \subset \mathcal{M}$ open, define*

$$\mathcal{M} \setminus \operatorname{supp}_U G = \bigcup \{V \subset \mathcal{M} \text{ open} \,|\, \operatorname{supp} f \subset V \Rightarrow (Gf)|_U \equiv 0\} . \tag{A.1}$$

*Let $\varphi \in \mathfrak{E}(\mathcal{M})$, if there exists an open cover $\bigcup_{\alpha \in \mathcal{A}} U_\alpha = \mathcal{M}$ such that, $\forall\, \alpha \in \mathcal{A}$, $\operatorname{supp} \varphi \cap \operatorname{supp}_{U_\alpha} G$ is compact, then one can define a function $G\varphi \in \mathfrak{E}(\mathcal{M})$ such that $PG\varphi = GP\varphi = \varphi$.*

*Proof.* We claim that the local definitions

$$G\varphi|_{U_\alpha} := G(\rho_\alpha \varphi)|_{U_\alpha},$$

where $\rho_\alpha \in \mathfrak{D}(\mathcal{M})$ such that $\rho_\alpha \equiv 1$ on $\operatorname{supp} \varphi \cap \operatorname{supp}_{U_\alpha} G$ can be glued together to form the desired map. Suppose $\alpha, \beta \in \mathcal{A}$ such that $U_{\alpha\beta} = U_\alpha \cap U_\beta \neq \emptyset$. One can quickly verify that $\operatorname{supp}_{U_{\alpha\beta}} G \subseteq \operatorname{supp}_{U_\alpha} G \cap \operatorname{supp}_{U_\beta} G$, hence $\rho_\alpha \varphi|_{U_{\alpha\beta}} = \rho_\beta \varphi|_{U_{\alpha\beta}}$. In particular this means that $\operatorname{supp}\left((\rho_\alpha - \rho_\beta)\varphi\right) \subset \mathcal{M} \setminus \operatorname{supp}_{U_{\alpha\beta}} G$ and hence $G(\rho_\alpha \varphi)|_{U_{\alpha\beta}} = G(\rho_\beta \varphi)|_{U_{\alpha\beta}}$, thus $G\varphi$ is a well-defined function.

Next we note that, for every open $U \subseteq \mathcal{M}$, we have $U \subset \operatorname{supp}_U G$. To see this, let $V = \mathcal{M} \setminus \operatorname{supp}_U G$ hence, by definition, $Gf|_U \equiv 0$, $\forall\, f \in \mathfrak{D}(V)$. If there is an element

$x \in U \cap V$, there exists $f \in \mathfrak{D}(V)$ such that $f(x) = 1$. However, this would in turn imply that $f(x) = PGf(x) = P(Gf|_U)(x) = 0$. This contradiction implies $U \cap V = \emptyset$, hence $U \subset \operatorname{supp}_U G$. As such, we may assume that the cover $\{U_\alpha\}_{\alpha \in \mathcal{A}}$ satisfies $U_\alpha \subset \operatorname{supp}_{U_\alpha} G$ for every $\alpha$.

In the above argument, we used the locality of differential operators, namely that $(P\psi)|_U = (P\psi|_U)|_U$ for any $\psi \in \mathfrak{E}(\mathcal{M})$. This also means that $(PG\varphi)|_{U_\alpha} = (\rho_\alpha \varphi)|_{U_\alpha}$. As $U_\alpha \subset \operatorname{supp}_{U_\alpha} G$, for any $x \in U_\alpha$ we must either have $x \in \operatorname{supp} \varphi$, in which case $\rho_\alpha(x) = 1$, or $\varphi(x) = 0$. In both cases, we have $\rho_\alpha(x)\varphi(x) = \varphi(x)$, hence $(PG\varphi)|_{U_\alpha} = \varphi|_{U_\alpha}$. For the same reasons, we have that $(\rho_\alpha(P\varphi))|_{U_\alpha} = (P(\rho_\alpha \varphi))|_{U_\alpha}$ and hence $(GP\varphi)|_{U_\alpha} = \varphi|_{U_\alpha}$ concluding the proof. □

**Theorem A.2** (The Method of Images). *Let $\pi : \widetilde{\mathcal{M}} \to \mathcal{M}$ be a regular covering of $\mathcal{M}$ by $\widetilde{\mathcal{M}}$. Further, let $P$ and $\widetilde{P}$ be a pair of differential operators for $\mathcal{M}$ and $\widetilde{\mathcal{M}}$ respectively, such that $\pi^* P = \widetilde{P} \pi^*$. Further, let $\widetilde{G}$ be a fundamental solution to $\widetilde{P}$ such that*

1. *There exists a covering $\bigcup_{\alpha \in \mathcal{A}} U_\alpha = \widetilde{\mathcal{M}}$ such that, $\forall K \subset \mathcal{M}$ compact, $\pi^{-1}(K) \cap \operatorname{supp}_{U_\alpha} \widetilde{G}$ is compact,*
2. *$\forall \rho \in \operatorname{Aut}(\pi)$, $\rho^* \widetilde{G} = \widetilde{G} \rho^*$.*

*Then there exists a fundamental solution $G$ for $P$ such that $\pi^* G = \widetilde{G} \pi^*$*

*Proof.* Because $\operatorname{supp} \pi^* f = \pi^{-1}(\operatorname{supp} f)$, condition 1 tells us that $\widetilde{G}\pi^* f$ is well defined and satisfies $\widetilde{P}\widetilde{G}\pi^* f = \widetilde{G}\widetilde{P}\pi^* f = \pi^* f$

Next, 2 ensures that for any $\rho \in \operatorname{Aut}(\pi)$

$$\rho^* \widetilde{G} \pi^* f = \widetilde{G} \rho^* \pi^* f = \widetilde{G}(\pi \circ \rho)^* f = \widetilde{G} \pi^* f, \tag{A.2}$$

i.e. $\widetilde{G}\pi^* f$ is a $\operatorname{Aut}(\pi)$ invariant, and hence can be expressed as $\pi^* F$ for some $F \in \mathfrak{E}(\mathcal{M})$. As our choice of $f$ was arbitrary, this defines a map $f \mapsto F$, which is clearly linear. As such we denote it $G : \mathfrak{D}(\mathcal{M}) \to \mathfrak{E}(\mathcal{M})$.

To show that $G$ is then a fundamental solution for $P$ is a fairly mechanical process:

$$\pi^* PGf = \widetilde{P} \pi^* Gf = \widetilde{P} \widetilde{G} \pi^* f = \pi^* f. \tag{A.3}$$

From the injectivity of $\pi^*$, we may then conclude $PGf = f$. Next, using the same trick

$$\pi^* GPf = \widetilde{G} \pi^* Pf = \widetilde{G} \widetilde{P} \pi^* f = \pi^* f, \tag{A.4}$$

which again shows $GPf = f$. $\square$

The following lemma shows how this applies to the equations of motion of a locally covariant (classical) field theory.

**Lemma A.3.** *Let* $\mathcal{L} : \mathfrak{D} \Rightarrow \mathfrak{F}_{\text{loc}}$ *be a natural Lagrangian such that, for any* $\mathcal{M} \in \mathsf{Loc}$, $\varphi \in \mathfrak{E}(\mathcal{M})$, $\langle S''_{\mathcal{M}}[\varphi], h \otimes g\rangle = \langle P_{\mathcal{M}}[\varphi]h, g\rangle$ *where* $P_{\mathcal{M}}[\varphi]$ *is some linear differential operator. If* $\widetilde{\mathcal{M}}, \mathcal{M} \in \mathsf{Loc}$ *and* $\pi : \widetilde{\mathcal{M}} \to \mathcal{M}$ *is a surjective map such that for every* $x \in \widetilde{\mathcal{M}}$, *there exists a subspacetime*[1] $\mathcal{N} \ni x$ *such that* $\pi|_{\mathcal{N}}$ *is an admissible embedding, then*

$$\pi^* P_{\mathcal{M}}[\varphi] = P_{\widetilde{\mathcal{M}}}[\pi^*\varphi]\pi^*. \tag{A.5}$$

*Proof.* Recall that the naturality of $\mathcal{L}$ implies that, for every admissible embedding $\chi : \mathcal{M} \hookrightarrow \mathcal{N}$, $\chi^* P_{\mathcal{N}}[\varphi] = P_{\mathcal{M}}[\chi^*\varphi]\chi^*$. Applying this to the composed map $\pi|_{\mathcal{N}} = \pi \circ \iota$ and then to the inclusion $\iota : \mathcal{N} \hookrightarrow \mathcal{M}$, we have, for $\varphi \in \mathfrak{E}(\mathcal{M})$ and $g \in \mathfrak{D}(\mathcal{M})$

$$\begin{aligned}(\pi^*(P_{\mathcal{M}}[\varphi]g))|_{\mathcal{N}} &= P_{\mathcal{N}}[(\pi^*\varphi)|_{\mathcal{N}}](\pi^*g)|_{\mathcal{N}} \\ &= (P_{\widetilde{\mathcal{M}}}[\pi^*\varphi]\,\pi^*g)|_{\mathcal{N}}.\end{aligned}$$

Given that $\widetilde{\mathcal{M}}$ is covered by $\mathcal{N} \subseteq \widetilde{\mathcal{M}}$ for which this holds, we may conclude $\pi^*(P_{\mathcal{M}}[\varphi]\,g) = P_{\widetilde{\mathcal{M}}}[\pi^*\varphi]\pi^*g$ as desired. $\square$

Given that the equations of motion are related in this way, we can now show that the propagators are as well. For any $K \subset \mathcal{M}$ compact, there exist Cauchy surfaces $\Sigma_{\pm} \subset \mathcal{M}$ such that $\Sigma_- \prec K \prec \Sigma_+$. This in turn implies $\pi^{-1}(\Sigma_-) \prec \pi^{-1}(K) \prec \pi^{-1}(\Sigma_+)$. For a covering $\pi : \widetilde{\mathcal{M}} \to \mathcal{M}$ of globally hyperbolic spacetimes, the preimage of any Cauchy surface $\Sigma \subset \mathcal{M}$ is again a Cauchy surface $\pi^{-1}(\Sigma) \subset \widetilde{\mathcal{M}}$, as we will now show. Let $\gamma : \mathbb{R} \to \widetilde{\mathcal{M}}$ be an inextendible timelike curve. Note that $\pi\left(\mathrm{Img}(\gamma) \cap \pi^{-1}\Sigma\right) = \mathrm{Img}(\pi \circ \gamma) \cap \Sigma$. As $\pi \circ \gamma$ is also an inextendible timelike curve, there exists precicely one $t_0 \in \mathbb{R}$ such that $\pi \circ \gamma(t_0) \in \Sigma$, hence $\gamma(t_0) \in \pi^{-1}(\Sigma)$. Given that $\gamma(t) \in \pi^{-1}(\Sigma)$ *only* when $\pi \circ \gamma(t) \in \Sigma$, we see that all inextensible timelike curves intersect $\pi^{-1}(\Sigma)$ precisely once, indicating it is a Cauchy surface. Using a known result [San13], this means that $\operatorname{supp} \pi^{-1}K \cap \mathcal{J}(L)$ is compact for every compact set $L \subset \mathcal{M}$.

Finally, if $E^{R/A}_{\widetilde{\mathcal{M}}}$ are the retarded/advanced propagators for $P_{\widetilde{\mathcal{M}}}$, the support properties (2.45) of $E^{R/A}_{\widetilde{\mathcal{M}}}$ imply that $\operatorname{supp}_U E^{R/A}_{\widetilde{\mathcal{M}}} = \mathcal{J}^{\mp}(\overline{U})$, where $\overline{U}$ is the closure of $U$. Thus, if we take a cover of $\widetilde{\mathcal{M}}$ by precompact sets $U_\alpha$, then $\operatorname{supp} \pi^{-1}(K) \cap \operatorname{supp}_{U_\alpha} E^{R/A}_{\widetilde{\mathcal{M}}}$ is compact for each $U_\alpha$ as required by 1 of Theorem A.2.

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[1] i.e. the inclusion $\mathcal{N} \hookrightarrow \widetilde{\mathcal{M}}$ is an admissible embedding of spacetimes

Next, as $g_{\widetilde{\mathcal{M}}} = \pi^* g_{\mathcal{M}}$, $\mathrm{Aut}(\pi)$ comprises isometries of $\widetilde{\mathcal{M}}$, we also have $\rho^* E^{R/A}_{\widetilde{\mathcal{M}}} = E^{R/A}_{\widetilde{\mathcal{M}}} \rho^*$ (see the discussion in the proof of proposition 4.2.3), satisfying condition 2 of Theorem A.2.

We thus have a pair of propagators $E^{R/A}_{\mathcal{M}} : \mathfrak{D}(\mathcal{M}) \to \mathfrak{E}(\mathcal{M})$ which satisfy

$$\pi^* E^{R/A}_{\mathcal{M}} = E^{R/A}_{\widetilde{\mathcal{M}}} \pi^*. \tag{A.6}$$

It is straightforward to verify that these satisfy the support criteria (2.45), hence they are *the* retarded/advanced propagators for $\mathcal{M}$.

# — B —

# Closure Proofs for Microcausal Functionals

**Proposition B.1.** *Let $\mathcal{M}$ be a globally hyperbolic spacetime, let $S$ be a quadratic action on $\mathcal{M}$, then $\{\cdot,\cdot\}_S : \mathfrak{F}_{\mu c}(\mathcal{M}) \times \mathfrak{F}_{\mu c}(\mathcal{M}) \to \mathfrak{F}_{\mu c}(\mathcal{M})$.*

*Proof.* We shall only prove this fact for $\mathcal{M} \subseteq \mathbb{R}^d$, but it is possible to 'patch together' the results over an atlas for a more general $\mathcal{M}$. We begin by rephrasing Theorem 8.2.13 of [Hör15]:

Suppose that $X \subseteq \mathbb{R}^n$, and $Y \subseteq \mathbb{R}^m$. Let $K \in \mathfrak{D}'(X \times Y)$ and $u \in \mathfrak{E}'(Y)$. Theorem 8.2.13 allows us to define a new distribution $K \circ u$, with integral kernel

$$(K \circ u)(x) = \int_Y K(x,y)u(y)\,\mathrm{d}y, \tag{B.1}$$

and estimate its wavefront set. Namely, $K \circ u$ exists whenever $\mathrm{WF}'(K)_Y \cap \mathrm{WF}(u) = \emptyset$, where

$$\mathrm{WF}'(K)_Y := \{(y;\eta) \in T^*Y \setminus \underline{0}_Y \mid \exists\, x \in X, (x,y;0,-\eta) \in \mathrm{WF}(K)\},$$

is the wavefront set of $K$ *twisted with respect to* $Y$ (and $\underline{0}_Y$ denotes the zero section of $T^*Y$).

Moreover, whenever $K \circ u$ does exist, we have

$$\mathrm{WF}(K \circ u) \subseteq \{(x,\xi) \in T^*X \mid \exists\,(y,\eta) \in \mathrm{WF}(u) \cup \underline{0}_Y, (x,y;\xi,\eta) \in \mathrm{WF}(K)\} \tag{B.2}$$

Let $\mathcal{F},\mathcal{G} \in \mathfrak{F}_{\mu c}(\mathcal{M})$, the $m^{\text{th}}$ functional derivative of their Peierls bracket can be written, omitting the dependence on a field configuration $\varphi \in \mathfrak{E}(\mathcal{M})$, as follows:

$$(\{\mathcal{F},\mathcal{G}\}_S)^{(m)} = \sum_{\{J_1,J_2\} \in P_m} \left[\left(\mathcal{F}^{(|J_1|+1)} \otimes \mathcal{G}^{(|J_2|+1)}\right) \circ E\right] s_{J_1,J_2}, \tag{B.3}$$

where the sum runs over partitions $J_1 \sqcup J_2 = \{1, \ldots, m\}$, $\circ$ is the operation described above, and $s_{J_1,J_2} : \mathfrak{D}(\mathcal{M}^m) \to \mathfrak{D}(\mathcal{M}^m)$ is an operation permuting the variables of a given test function according to a permutation $\sigma_{J_1,J_2} \in S_m$ such that $i \in J_1 \Rightarrow \sigma_{J_1,J_2}(i) \leq |J_1|$. (As $\mathcal{F}^{(m)}$ is permutation invariant as a distribution, this is a sufficient characterisation of $\sigma_{J_1,J_2}$.) In fact, as we are only testing for microcausality, the only property we need of these distributions is that, for $0 \leq k \leq m$, the wavefront set of $\left(\mathcal{F}^{(k+1)} \otimes \mathcal{G}^{(m-k+1)}\right) \circ E$ is disjoint from the cones $\overline{V}^m_{\pm}$, defined by

$$\overline{V}^m_+ = \left\{(x_1, \ldots, x_m; \xi_1, \ldots, \xi_m) \in T^*\mathcal{M} \,|\, \xi_i \in \overline{V}_+(x_i) \,\forall\, i \leq m\right\}, \tag{B.4}$$

where $\overline{V}_+(x)$ denotes the closed future/past lightcone in $T^*_x\mathcal{M}$, and similar for $\overline{V}^m_-$.

We set $X = \mathcal{M}^n$, $Y = \mathcal{M}^2$, $K = \mathcal{F}^{(k+1)} \otimes \mathcal{G}^{(m-k+1)}$, and $u = E$. Using [Hör15, Theorem 8.2.9], we can estimate $\mathrm{WF}(\mathcal{F}^{(k+1)} \otimes \mathcal{G}^{(m-k+1)})$ by

$$\mathrm{WF}\left(\mathcal{F}^{(k+1)} \otimes \mathcal{G}^{(m-k+1)}\right) \subseteq \left(\mathrm{WF}(\mathcal{F}^{(k+1)}) \cup 0_{\mathcal{M}^{k+1}}\right) \times \left(\mathrm{WF}(\mathcal{G}^{(m-k+1)}) \cup 0_{\mathcal{M}^{m-k+1}}\right), \tag{B.5}$$

where $0_{\mathcal{M}} = \mathcal{M} \times \{0\} \subseteq T^*\mathcal{M}$ denotes the zero section of $T^*\mathcal{M}$ *etc.* Let $(y_{\mathcal{F}}, y_{\mathcal{G}}; \eta_{\mathcal{F}}, \eta_{\mathcal{G}}) \in T^*Y \setminus 0_Y$.

The wavefront set of the causal propagator, as may be found in [Rej16, §4.4.1], can be written as

$$\mathrm{WF}(E) = \left\{(x, y; \xi, \eta) \in T^*\mathcal{M}^2 \,|\, (x, \xi) \in \overline{V}_+ \cup \overline{V}_-, (x, \xi) \sim (y, -\eta)\right\}, \tag{B.6}$$

where the relation $(x, \xi) \sim (y, \eta)$ means there exists a null geodesic $\gamma : (0-\epsilon, 1+\epsilon) \to \mathcal{M}$ (for some $\epsilon > 0$) such that $(\gamma(0), \dot{\gamma}^\flat(0)) = (x, \xi)$, and $(\gamma(1), \dot{\gamma}^\flat(1)) = (y, \eta)$, where $v^\flat := g_{\mathcal{M}}(v, \cdot)$. However, for our purposes, we can use the much simpler estimate

$$\mathrm{WF}(E) \subset (\overline{V}_+ \times \overline{V}_-) \cup (\overline{V}_- \times \overline{V}_+), \tag{B.7}$$

i.e. if $(x, y; \xi, \eta) \in \mathrm{WF}(E)$ then either $(x, \xi) \in \overline{V}_+$ and $(y, \eta) \in \overline{V}_-$, *or* $(x, \xi) \in \overline{V}_-$ and $(y, \eta) \in \overline{V}_+$.

Suppose there exists $\underline{x}_{\mathcal{F}} \in \mathcal{M}^k$ and $\underline{x}_{\mathcal{G}} \in \mathcal{M}^{n-k}$ such that

$$(\underline{x}_{\mathcal{F}}, y_{\mathcal{F}}, \underline{x}_{\mathcal{G}}, y_{\mathcal{G}}; 0, -\eta_{\mathcal{F}}, 0, -\eta_{\mathcal{G}}) \in \mathrm{WF}'\left(\mathcal{F}^{(k+1)} \otimes \mathcal{G}^{(m-k+1)}\right)_Y,$$

then either $\eta_{\mathcal{F}} = 0$, or $(y_{\mathcal{F}}; \eta_{\mathcal{F}}) \notin \overline{V}_{\pm}$. The same is also true of $(y_{\mathcal{G}}; \eta_{\mathcal{G}})$, though at least one of $\eta_{\mathcal{F}}$ and $\eta_{\mathcal{G}}$ must be non-zero. Thus we see that the intersection of $\mathrm{WF}(\mathcal{F}^{(k+1)} \otimes \mathcal{G}^{(m-k+1)})$ with $\mathrm{WF}(E)$ must be trivial, as $(y_{\mathcal{F}}, y_{\mathcal{G}}; \eta_{\mathcal{F}}, \eta_{\mathcal{G}}) \in \mathrm{WF}(E) \Rightarrow (y_{\mathcal{F}}; \eta_{\mathcal{F}}), (y_{\mathcal{G}}; \eta_{\mathcal{G}}) \in (\overline{V}_+ \cup \overline{V}_-) \setminus 0_{\mathcal{M}}$.

Thus we can apply theorem 8.2.13 and conclude not only that $(\mathcal{F}^{(k+1)} \otimes \mathcal{G}^{(m-k+1)}) \circ E$ is well defined, but also that its wavefront set has trivial intersection with both $\overline{V}^m_+$ and $\overline{V}^m_-$. To see this, let $(\underline{x}_{\mathcal{F}}, \underline{x}_{\mathcal{G}}; \underline{\xi}_{\mathcal{F}}, \underline{\xi}_{\mathcal{G}}) \in \overline{V}^m_+$. Any $(y_{\mathcal{F}}, y_{\mathcal{G}}; \eta_{\mathcal{F}}, \eta_{\mathcal{G}}) \in \mathrm{WF}(E) \cup 0_Y$ necessarily belongs also to either $\overline{V}_+ \times \overline{V}_-$ or $\overline{V}_- \times \overline{V}_+$. Suppose it is the former, then, by microcausality, $(\underline{x}_{\mathcal{G}}, y_{\mathcal{G}}; \underline{\xi}_{\mathcal{G}}, -\eta_{\mathcal{G}}) \notin \mathrm{WF}(\mathcal{G}^{(m-k+1)})$. Recalling (B.5), this means there is only a chance that $(\underline{x}_{\mathcal{F}}, y_{\mathcal{F}}, \underline{x}_{\mathcal{G}}, y_{\mathcal{G}}; \underline{\xi}_{\mathcal{F}}, \eta_{\mathcal{F}}, \underline{\xi}_{\mathcal{G}}, \eta_{\mathcal{G}}) \in \mathrm{WF}(\mathcal{F}^{(k+1)} \otimes \mathcal{G}^{(m-k+1)})$ if $\underline{\xi}_{\mathcal{G}}$ and $\eta_{\mathcal{G}}$ are both zero. However, this still fails, as $\eta_{\mathcal{G}} = 0 \Rightarrow \eta_{\mathcal{F}} = 0$, which in turn implies that $(\underline{x}_{\mathcal{F}}, y_{\mathcal{F}}; \underline{\xi}_{\mathcal{F}}, -\eta_{\mathcal{F}}) \notin \mathrm{WF}(\mathcal{F}^{(k+1)})$. The wavefront set estimate from 8.2.13 then allows us to conclude that $(\underline{x}_{\mathcal{F}}, \underline{x}_{\mathcal{G}}; \underline{\xi}_{\mathcal{F}}, \underline{\xi}_{\mathcal{G}}) \notin \mathrm{WF}\left((\mathcal{F}^{(k+1)} \otimes \mathcal{G}^{(m-k+1)}) \circ E\right)$. Applying the corresponding argument to $\Gamma^m_-$, we see that all derivatives of $\{\mathcal{F}, \mathcal{G}\}_S$ satisfy the requisite wavefront set condition to be declared microcausal. ☐

**Proposition B.2.** *Let $\mathcal{M}$ be a globally hyperbolic spacetime, $P$ a normally hyperbolic operator on $\mathcal{M}$, and $W = \frac{i}{2}E + H$ a Hadamard distribution for $P$, then $\mathfrak{F}_{\mu c}(\mathcal{M})[[\hbar]]$ is closed under $\star_H$.*

*Proof.* Let $\mathcal{F}, \mathcal{G} \in \mathfrak{F}_{\mu c}(\mathcal{M})$, the $m^{\text{th}}$ derivative of the $\mathcal{O}(\hbar^n)$ term of $\mathcal{F} \star_H \mathcal{G}$ is,

$$\left(\tfrac{d^n}{d\hbar^n}(\mathcal{F} \star_H \mathcal{G})|_{\hbar=0}\right)^{(m)} = \sum_{\{J_1, J_2\} \in P_m} \left[\left(\mathcal{F}^{(|J_1|+n)} \otimes \mathcal{G}^{(|J_2|+n)}\right) \circ W^{\otimes n}\right] s_{J_1, J_2}, \tag{B.8}$$

where all notation is the same as in the previous proof, and the contraction $\circ$ is computed in the expected way, namely

$$\begin{aligned} &\left[\left(\mathcal{F}^{(|J_1|+n)} \otimes \mathcal{G}^{(|J_2|+n)}\right) \circ W^{\otimes n}\right](x_1, \dots x_m) \\ &= \int_{\mathcal{M}^{2n}} \Big[\mathcal{F}^{(|J_1|+n)}(x_1, \dots x_{|J_1|}, y_1, \dots y_n) \mathcal{G}^{(|J_2|+n)}(x_{|J_1|+1}, \dots x_m, y_{n+1}, \dots y_{2n}) \\ &\qquad\qquad W(y_1, y_{n+1}) \cdots W(y_n, y_{2n})\Big] \, \mathrm{d}y_1 \cdots \mathrm{d}y_{2n}. \end{aligned}$$

In order to apply theorem 8.2.13 to $\left(\mathcal{F}^{(k+n)} \otimes \mathcal{G}^{(m-k+n)}\right) \circ (\chi W)^{\otimes n}$ for $0 \leq k \leq m$, we must show that

$$\mathrm{WF}'\left(\mathcal{F}^{(k+n)} \otimes \mathcal{G}^{(m-k+n)}\right)_Y \cap \mathrm{WF}((\chi W)^{\otimes n}) = \emptyset,$$

where $Y = \mathcal{M}^{2n}$ comprises the $y_i$ variables in the above integral. The justification of this proceeds similarly to before. Firstly, we note the following estimate, obtained by repeated application of 8.2.9 from [Hör15]

$$\mathrm{WF}(W^{\otimes n}) \subseteq (\mathrm{WF}(W) \cup 0_{\mathcal{M}^2})^n \setminus 0_{\mathcal{M}^{2n}}.$$

Hence, if $(y_1, \dots, y_{2n}; \eta_1, \dots, \eta_{2n}) \in \mathrm{WF}((\chi W)^{\otimes n})$, then for each $i \in \{1, \dots, n\}$, either $\eta_i$ and $\eta_{n+i}$ are both zero, or $(y_i; \eta_i) \in \overline{V}_+$ and $(y_{n+i}; \eta_{n+i}) \in \overline{V}_-$, moreover, $\eta_i$ must be non-zero for at least one $i$. Denote $\underline{y}_{\mathcal{F}} = (y_i)_{i=1}^{n}$ and $\underline{y}_{\mathcal{G}} = (y_i)_{i=n+1}^{2n}$, and similarly $\underline{\eta}_{\mathcal{F}}$ and $\underline{\eta}_{\mathcal{G}}$. Then we have that $(\underline{y}_{\mathcal{F}}; \underline{\eta}_{\mathcal{F}}) \in \overline{V}_{+}^{n}$, and $(\underline{y}_{\mathcal{G}}; \underline{\eta}_{\mathcal{G}}) \in \overline{V}_{-}^{n}$, hence neither can $(\underline{x}_{\mathcal{F}}, \underline{y}_{\mathcal{F}}; 0, -\underline{\eta}_{\mathcal{F}})$ belong to $\mathrm{WF}(\mathcal{F}^{(k+n)})$, for any $\underline{x}_{\mathcal{F}} \in \mathcal{M}^k$, nor $(\underline{x}_{\mathcal{G}}, \underline{y}_{\mathcal{G}}; 0, -\underline{\eta}_{\mathcal{G}})$ belong to $\mathrm{WF}(\mathcal{G}^{(m-k+n)})$, for any $\underline{x}_{\mathcal{F}} \in \mathcal{M}^{m-k}$. [1]

Now we must show that 8.2.13 precludes $\overline{V}_{\pm}^{m}$ from $\mathrm{WF}((\mathcal{F}^{(k+n)} \otimes \mathcal{G}^{(m-k+n)}) \circ (\chi W)^{\otimes n})$. Let $(\underline{x}_{\mathcal{F}}, \underline{x}_{\mathcal{G}}; \underline{\xi}_{\mathcal{F}}, \underline{\xi}_{\mathcal{G}}) \in \overline{V}_{+}^{m} \cap \mathrm{WF}((\mathcal{F}^{(k+n)} \otimes \mathcal{G}^{(m-k+n)}) \circ (\chi W)^{\otimes n})$, then we must have some $(\underline{y}_{\mathcal{F}}, \underline{\eta}_{\mathcal{F}}), (\underline{y}_{\mathcal{G}}, \underline{\eta}_{\mathcal{G}}) \in \dot{T}^*\mathcal{M}^n$ such that $(\underline{y}_{\mathcal{F}}, \underline{y}_{\mathcal{G}}; \underline{\eta}_{\mathcal{F}}, \underline{\eta}_{\mathcal{G}}) \in (\mathrm{WF}((\chi W)^{\otimes n}) \cup 0_{\mathcal{M}^{2n}})$ and

- $(\underline{x}_{\mathcal{F}}, \underline{y}_{\mathcal{F}}; \underline{\xi}_{\mathcal{F}}, \underline{\eta}_{\mathcal{F}}) \in (\mathrm{WF}(\mathcal{F}^{(k+n)}) \cup 0_{\mathcal{M}^{k+n}})$,
- $(\underline{x}_{\mathcal{G}}, \underline{y}_{\mathcal{G}}; \underline{\xi}_{\mathcal{G}}, \underline{\eta}_{\mathcal{G}}) \in (\mathrm{WF}(\mathcal{G}^{(m-k+n)}) \cup 0_{\mathcal{M}^{m-k+n}})$,
- $(\underline{x}_{\mathcal{F}}, \underline{x}_{\mathcal{G}}; \underline{\xi}_{\mathcal{F}}, \underline{\xi}_{\mathcal{G}}) \notin 0_{\mathcal{M}^m}$.

However, we have that $(\underline{y}_{\mathcal{F}}, \underline{y}_{\mathcal{G}}; \underline{\eta}_{\mathcal{F}}, \underline{\eta}_{\mathcal{G}}) \in \overline{V}_{+}^{n} \times \overline{V}_{-}^{n}$, hence $(\underline{x}_{\mathcal{G}}, \underline{y}_{\mathcal{G}}; \underline{\xi}_{\mathcal{G}}, -\underline{\eta}_{\mathcal{G}}) \in \overline{V}_{+}^{m-k+n}$, so this covector must belong to the zero section. But then $(\underline{y}_{\mathcal{F}}, \underline{y}_{\mathcal{G}}; \underline{\eta}_{\mathcal{F}}, \underline{\eta}_{\mathcal{G}}) \in \mathrm{WF}((\chi W)^{\otimes n}) \cup 0_{\mathcal{M}^{2n}}$ implies $\underline{\eta}_{\mathcal{F}}$ also vanishes, hence $(\underline{x}_{\mathcal{F}}, \underline{y}_{\mathcal{F}}; \underline{\xi}_{\mathcal{F}}, -\underline{\eta}_{\mathcal{F}}) \in \overline{V}_{+}^{k+n}$. Thus we cannot satisfy all three conditions simultaneously, and are forced to conclude that

$$\overline{V}_{+}^{m} \cap \mathrm{WF}((\mathcal{F}^{(k+n)} \otimes \mathcal{G}^{(m-k+n)}) \circ (\chi W)^{\otimes n}) = \emptyset.$$

To carry out the analogous argument for $\overline{V}_{-}^{m}$, one instead starts with the observation that

$$(\underline{x}_{\mathcal{F}}, \underline{x}_{\mathcal{G}}; \underline{\xi}_{\mathcal{F}}, \underline{\xi}_{\mathcal{G}}) \in \overline{V}_{-}^{m} \text{ and } (\underline{y}_{\mathcal{F}}, \underline{y}_{\mathcal{G}}; \underline{\eta}_{\mathcal{F}}, \underline{\eta}_{\mathcal{G}}) \in \Big(\mathrm{WF}((\chi W)^{\otimes n}) \cup 0_Y\Big)$$
$$\Rightarrow (\underline{x}_{\mathcal{F}}, \underline{y}_{\mathcal{F}}; \underline{\xi}_{\mathcal{F}}, -\underline{\eta}_{\mathcal{F}}) \in \overline{V}_{-}^{k+n}$$

and proceeds accordingly.

This proves

$$\mathrm{WF}\left(\left(\tfrac{d^n}{d\hbar^n}(\mathcal{F} \star_H \mathcal{G})|_{\hbar=0}\right)^{(m)}\right) \cap \overline{V}_{\pm}^{m} = \emptyset,$$

thus each coefficient of $\mathcal{F} \star_H \mathcal{G}$ is a microcausal functional. □

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[1] Note that here we required the tighter restriction on $\mathrm{WF}(W)$ relative to $E$: if we had covectors $(y_i; \eta_i) \in \overline{V}_{+}$ and $(y_j; \eta_j) \in \overline{V}_{-}$, for $i, j \in \{1, \ldots, n\}$, then it might be possible to find $(\underline{x}_{\mathcal{F}}, \underline{y}_{\mathcal{F}}; 0, -\underline{\eta}_{\mathcal{F}}) \in \mathrm{WF}(\mathcal{F}^{(k+n)})$, hence the above intersection would in general be non-empty, preventing us from proceeding any further.

## — C —

# Squaring the Propagator

In this section, we explain in detail why the expression (3.33) for $[(\partial_u \otimes \partial_u)W_{\mathcal{E}}]^2$ is valid. To simplify notation, we shall write $(\partial_u \otimes \partial_u)W_{\mathcal{E}} =: w$, and denote by $w_N$ the truncation of the series defining $w$ to the first $N$ terms.

Theorem 8.2.4 of [Hör15] gives the necessary conditions for the square of a distribution to exist. However, it does not provide a convenient integral kernel with which to evaluate such products on test functions. A good starting point to this end may be found in [Hör71, Theorem 2.5.10], where it is stated that for any pair of cones $\Gamma_a, \Gamma_b \subseteq \dot{T}^*\mathcal{M}$ such that $\Gamma_a \cap -\Gamma_b = \emptyset$, the multiplication of distributions, considered as a map $\mathfrak{D}'_{\Gamma_a}(\mathcal{M}) \times \mathfrak{D}'_{\Gamma_b}(\mathcal{M}) \to \mathfrak{D}'(\mathcal{M})$ is sequentially continuous in each of its arguments. In other words, if we take some fixed $u \in \mathfrak{D}'_{\Gamma_a}(\mathcal{M})$, and a sequence $v_n$ converging to $v$ in the sense of $\mathfrak{D}'_{\Gamma_b}(\mathcal{M})$, then $u \cdot v_n$ weakly converges to $u \cdot v$, and *vice versa* for a sequence in $\mathfrak{D}'_{\Gamma_a}(\mathcal{M})$.

Let $\Gamma \subseteq \dot{T}^*\mathcal{E}^2$ be a cone which both contains $\mathrm{WF}(w)$ and satisfies $\Gamma \cap -\Gamma = \emptyset$. We can show that the smooth distributions $w_N$ obtained by truncating the sum appearing in (3.26) converge to $w$ in $\mathfrak{D}'_\Gamma$.

Firstly, we shall pick an open subset $U \subset \mathcal{E}^2$ which can be identified with an open subset of $\mathbb{R}^4$. We shall only prove convergence for the restriction of $w_N$ to $U$, though the full result follows from this with little trouble. Following [Hör15, Definition 8.2.2] for sequential convergence, we must show that, for all $\chi \in \mathfrak{D}(U)$ and conic $V \subseteq \mathbb{R}^4$ such that $\operatorname{supp}\chi \times V \cap \Gamma = \emptyset$,

$$\sup_{\xi \in V} \left| (1 + |\xi|)^k \left( \widehat{\chi w}(\xi) - \widehat{\chi w_N}(\xi) \right) \right| \to 0 \text{ as } N \to \infty.$$

If we choose our coordinates for $U$ appropriately, we can express this Fourier transform as

$$\widehat{\chi w}(\xi) - \widehat{\chi w_N}(\xi) = \sum_{n=N+1}^{\infty} n \int_U \chi(x) e^{-in(\underline{u},x)} e^{-i(\xi,x)}\, \mathrm{d}x, \tag{C.1}$$

where $\underline{u} = (1, 0, -1, 0)$ is a constant vector. If we set $F(x) := -(\underline{u}, x)$, then each integral appearing in (C.1) can be expressed as $T_\chi(n, \xi)$ using the notation in [BF09, §4.3.2]. For any $\eta > 0$, $(x, 0, y, 0; -\eta, 0, \eta, 0) \in \mathrm{WF}(w)$, hence we cannot have $dF^*_x n = \xi$ for $\xi \in V$, else this would violate the assumption that $\operatorname{supp}\chi \times V \cap \Gamma = \emptyset$. This means the conditions are met for the second estimate of corollary 2 from [BF09, §4.3.2], i.e. for any $k \in \mathbb{N}$

$$|T_\chi(n, \xi)| \leq C_{\chi,V,k}(1 + n + |\xi|)^{-2k} \leq C'_{\chi,V,k}(1 + n)^{-k}(1 + |\xi|)^{-k},$$

for some appropriate choice of positive constants. This allows us to uniformly bound the original expression in $\xi$ as

$$\sup_{\xi\in V} \left|(1 + |\xi|)^k \left(\widehat{\chi w}(\xi) - \widehat{\chi w_N}(\xi)\right)\right| \leq C'_{\chi,V,k} \sum_{n=N+1}^{\infty} (1 + n)^{1-k}.$$

The series on the right converges for every $k \geq 3$, and the upper bound for $k = 3$ also provides an upper bound for $k = 1$ or $2$.

Thus we can write, for $f \in \mathfrak{D}(\mathcal{E}^2)$

$$\left\langle w^2, f\right\rangle = \lim_{N\to\infty} \langle w_N \cdot w, f\rangle,$$

which allows us to bring all summation outside of the integrals arising from the duality pairing. Noting that $w_N$ is a smooth function for all finite $N$, we can hence evaluate this pairing directly as

$$\begin{aligned}\left\langle w^2, f\right\rangle &= \lim_{N\to\infty} \sum_{m=0}^{\infty} m \int_{\mathcal{E}^2} e^{-im(u-u')} \left[\sum_{n=0}^{N} n e^{-in(u-u')} f(u, v, u', v')\right] \mathrm{d}V^2 \\ &= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} nm \int_{\mathcal{E}^2} e^{-i(n+m)(u-u')} f(u, v, u', v')\, \mathrm{d}V^2,\end{aligned}$$

where, *a priori*, the sum over $m$ must be performed first.

As $f$ is smooth, the integral is rapidly decaying as a function of $n + m$, hence the sum is absolutely convergent. Rearranging the double sum accordingly, it is then clear that the sequence of partial sums

$$w^2_N(u, v, u', v') := \sum_{k=0}^{N}\sum_{l=0}^{k} l(k - l)e^{-ik(u-u')} \tag{C.2}$$

converges to $w^2$ in the weak topology of $\mathfrak{D}'(\mathcal{E}^2)$.

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