Stochastic Differential Equations in a Scale of Hilbert Spaces

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Abstract

This thesis consists of two main sections.

Section II is motivated by studies of stochastic differential equations in infinite dimensional spaces. Here we consider an SDE with coefficients defined in a scale of Hilbert spaces and prove existence, uniqueness and path-continuity of infinite-time solutions using a variation of Ovsjannikov's method. Markov property and several norm estimates are also established. Our findings are then applied to a system of equations describing non-equilibrium stochastic dynamics of (real-valued) spins of an infinite particle system on a typical realization of a Poisson or Gibbs point process in $\mathbb{R}^n$. This section improves upon the work of [10] where finite-time solutions were considered.

Section III is motivated by studies of stochastic systems describing non-equilibrium dynamics of (real-valued) spins of an infinite particle system in $\mathbb{R}^n$. Here we consider a row-finite system of stochastic differential equations with dissipative drift. The existence and uniqueness of infinite-time solutions is proved via finite volume approximations and a version of Ovsjannikov's method. This section improves upon the work of [1, 2] and [11] by considering a multiplicative noise and a more general configuration in a stochastic setting.

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Author's declaration

I declare that this thesis is a presentation of original work and I am the sole author. This work has not previously been presented for an award at this, or any other, University. All sources are acknowledged in the references section.

List of Symbols and Terminology

Part II - SDEs in a Scale of Hilbert Spaces

Part III - Row-finite systems of SDEs with dissipative drift

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I Introduction

This thesis studies an application of the so-called Ovsjannikov's method to stochastic differential equations in infinite dimensional spaces and extends a couple of existing publications. A natural question may arise: what is Ovsjannikov's method? This question, for now, can be answered by saying that Ovsjannikov's method is a process of fining a solution which lives in an intersection $\mathcal{U}$ of a certain family of Banach spaces called a scale (another concept that will be defined in Section II). In this approach certain assumptions are made on how coefficients of an SDE are acting on the chosen scale and by construction we will see that there exists an important relationship between $\mathcal{U}$ and a space from which an initial condition is selected. Let us now continue this introduction with short subsection explaining the history behind Ovsjannikov's method.

I.1 History

A common consensus is that Ovsjannikov's method was first introduced to a large audience in 1958 by [27]. This method was invented to tackle evolution equations of the form

$$\frac{d}{dt}\phi(t) = A\phi(t), \quad \phi(0) = \mathfrak{x}, \quad t \in [0, T], \tag{I.1}$$

arising from studies of various natural phenomena, where it is not obvious how to realise a linear operator A as an operator in a Banach space. A simple example showing that such a situation is far from impossible can be illustrated by considering a separable Hilbert space of weighted real sequences

$$l^2(\omega) := \left\{ z \in \mathbb{R}^{\mathbb{N}} \;\middle|\; \|z\|_{l^2(\omega)} := \sqrt{\sum_{n\in\mathbb{N}} \omega_n |z_n|^2} < \infty \right\}$$

and observing that a diagonal matrix $A := \{a_{ij}\}_{i,j\in\mathbb{N}}$ is a bounded linear operator in $l^2(\omega)$ if and only if $\{a_{ii}\}_{i\in\mathbb{N}}$ is a bounded sequence. In [27] an alternative approach was proposed. In particular it was proposed to consider a family of continuously embedded normed linear spaces $\mathbf{X} := \{X_{\mathfrak{a}}\}_{\mathfrak{a}\in\mathcal{A}}$ such that for all $\alpha < \beta \in \mathcal{A}$ we have

$$A : X_\alpha \to X_\beta \quad \text{and} \quad \|A\|_\alpha^\beta \le \frac{C}{\beta - \alpha}.$$

Under these conditions it was shown that there exists a finite time solution of equation (I.1) such that if $\phi(0) \in X_\gamma$ then $\phi \in X_\mathfrak{a}$ for all $\gamma < \mathfrak{a}$. This method remained somewhat unnoticed until 1965 when it was reintroduced by L. V. Ovsjannikov (see [47]) who formulated his ideas around the concept of a scale of Banach spaces and outlined an application to the Cauchy-Kovalevska problem. In 1968 a large text [61] was published with a significant part devoted to the method of Ovsjannikov and its applications. In [61] a term "Ovsjannikov theorem" was introduced and proceedings revolved around the following Cauchy problem

$$\frac{d}{dt}\phi(t) = A\phi(t) + f(t), \quad \phi(0) = \mathfrak{x}, \quad t \in [0,T] \tag{I.2}$$

where f is a continuous bounded function and A was allowed to depend on t. Subsequently Cauchy problem (I.1) and (I.2) were generalised to a non-linear case first by [62] with additional work published by [46, 44]. Later T. Nishida (see [42]) reflected upon the work of [44] and introduced a simplification. Another work considering weighted Banach spaces was published later on by [59]. A further generalisation of a linear case to the following Cauchy problem

$$\frac{d}{dt}\phi(t) = A\phi(t) + f(t, \phi(t)), \quad \phi(0) = \mathfrak{x}, \quad t \in [0,T] \tag{I.3}$$

was considered in the book by [18]. It was shown in particular that equation (I.3) admits a solution under an assumption that f is bounded, uniformly continuous map with a Lipschitz type condition on the second variable. Subsequently [5] further improved upon the work of [18] showing existence under even weaker conditions on f. Another generalization of Ovsjannikov's method can be found in [66]. For more recent developments related to Ovsjannikov's method one can consult for example the following references [6, 23, 24] and also [25]. For the purpose of this thesis we will be particularly interested in a couple of publication two of which are recent. One of the publications that is of a particular importance to us is a recent work (see [11]) by A. Daletskii and D. Finkelshtein. In [11] Ovsjannikov's method is used to study an infinite system of first order differential equations in $\mathbb{R}^d$

$$\frac{d}{dt}q_x(t) = F_x(\bar{q}(t)), \quad q_x(0) = \mathfrak{x}_x, \quad x \in \gamma, \quad t \in [0,T]$$

where $\gamma \subset \mathbb{R}^d$ is countable, $\bar{q} \equiv \{q_x\}_{x\in\gamma}$ and F_x depends only on a finite number of components of the vector $\bar{q}$ and satisfies certain dissipative type conditions.

I.2 An Outline of Section II

Let us suppose that we are given some suitably filtered probability space and a separable Hilbert space H. Let W be a cylinder Wiener process on H and $L_2(H,H)$ be a space of Hilbert-Schmidt operators on H. One can now proceed to study the following stochastic differential equations

$$dX(t) = F(X(t))dt + \Phi(X(t))dW(t), \quad t \in [0,T] \tag{I.4}$$

where $F : H \to H$ and $\Phi : H \to L_2$. Equations like (I.4) arise from studies of various phenomena among which are diffusion processes, infinite particle systems, environmental pollution and transportation. Academic literature covering (I.4) is very extensive however its roots can be traced back to several texts among which are [28, 63, 58] and [14]. Classical theory (see [39, 32]) guaranties existence of a strong solution of equation (I.4) under the assumption that both F and Φ satisfy Lipschitz conditions on bounded sets that is $\forall n \in \mathbb{N}$, $\forall x, y \in H$, $\exists C_n$ such that

$$||x||< n \text{ and } ||y||< n \implies ||F(x) - F(y)||+||\Phi(x) - \Phi(y)||_{L_2}\leq C_n||x - y||.$$

If one is willing to use semigroup approach then a more general evolution SDE can be considered

$$dX(t) = AX(t)dt + F(X(t))dt + \Phi(X(t))dW(t), \quad t \in [0,T] \tag{I.5}$$

and existence of a solution is once again guaranteed [15] under for example suitable Lipschitz assumptions on F and Φ and an assumption that A generates a C_0-semigroup in H.

In this thesis in general and in Section II in particular we would like to study an extension of the classical theory and using Ovsjannikov's method solve equation (I.4) in a suitable scale of Hilbert spaces. In our work we will be following in footsteps of A. Daletskii who recently extended Ovsjannikov's method to a certain class of SDEs (see [10]) proving existence of finite time solutions. We shall now briefly outline the progress achieved in [10]. We begin by fixing a positive real interval $\mathcal{A} := [\underline{\mathfrak{a}}, \overline{\mathfrak{a}}]$ and assuming that we have a scale $\{X_{\mathfrak{a}}\}_{\mathfrak{a}\in\mathcal{A}}$ of separable Hilbert spaces that is

$$X_\alpha \subset X_\beta \text{ and } ||u||_\beta \leq ||u||_\alpha \text{ if } \alpha < \beta \in \mathcal{A} \text{ and } u \in X_\alpha.$$

Moreover we fix another scale $\{H_{\mathfrak{a}}\}_{\mathfrak{a}\in\mathcal{A}}$ of separable Hilbert spaces such that

$$H_{\mathfrak{a}} := \{\text{Space of Hilbert-Schmidt operators from } H \text{ to } X_{\mathfrak{a}}\}, \quad \forall(\mathfrak{a} \in \mathcal{A})$$

and impose the following Lipschitz type conditions on F and Φ. That is for all $\alpha < \beta \in \mathcal{A}$ and $u, v \in X_\alpha$ we assume that

$$||F(u) - F(v)||_\beta \leq \frac{L}{(\beta-\alpha)^{\frac{1}{2}}}||u - v||_\alpha$$

$$||\Phi(u) - \Phi(v)||_{H_\beta} \leq \frac{L}{(\beta-\alpha)^{\frac{1}{2}}}||u - v||_{H_\alpha}.$$

Now under these conditions one can prove that there exists a constant $\bar{b}$ such that for all $b \in (0, \bar{b})$ equation (I.4) admits a unique solution in the space M_b^2. Where M_b^2 is a Banach space of square-integrable progressively measurable processes ξ such that $\xi : [0, (\overline{\mathfrak{a}}-\underline{\mathfrak{a}})b) \to X_{\overline{\mathfrak{a}}}$, $\xi(t) \in X_{\mathfrak{a}}$ whenever $t > (\mathfrak{a} - \underline{\mathfrak{a}})b$ and

$$||\xi||_{M_b^2} := \sup\left\{\left(\mathbb{E}||\xi(t)||_{\mathfrak{a}}^2 p_b(\mathfrak{a}, t)\right)^{\frac{1}{2}} : \mathfrak{a} \in (\underline{\mathfrak{a}}, \overline{\mathfrak{a}}],\ t \in (\mathfrak{a} - \underline{\mathfrak{a}})b\right\}$$

where p is a certain special function. In Section II we will see that under a suitable modification of Lipschitz type conditions on F and Φ one can prove existence of global solutions of equation (I.4) living in $M_b^{\mathfrak{p}}$ for all $\mathfrak{p} \geq 2$. In Section II we will also see that M spaces can be simplified.

I.3 An Outline of Section III

The study of properties of various physical phenomena has led to consideration of systems of infinitely many coupled finite dimensional stochastic differential equations. Such systems are known as lattice models with certain conditions on the so-called “spin variables”, which are being modelled by the SDEs. Term "stochastic dynamics" is also often used to describe such systems in general and in particular SDEs that model the time dependence of spin variables. Origins of this terminology can be found in [51] and additional mathematical framework can be found, for example, in [4] and [37]. Questions concerning existence and uniqueness of solutions of such systems have also been studied in [22] and [57].

In recent decades studies of physical phenomena pertaining to non-crystalline (amorphous) substances and ferrofluids and amorphous magnets has led to an increased interest in studying countable systems of particles randomly distributed in $\mathbb{R}^d$. Characterisation of each particle in such a system by an internal real or vector valued "spin" parameter naturally leads to the consideration of a lattice model based on a fixed configuration $\gamma \subset \mathbb{R}^d$ of particle positions. Instances when $\gamma \equiv \mathbb{Z}^d$ are well studied and have an extensive literature, see for example [21, 38] and [33]. However, as described in [11] there are instances when the configuration γ of particle positions doesn't have a regular structure but instead lends itself as a locally finite subset of $\mathbb{R}^d$ where a typical number of "neighbour variables" of a particle located at $x \in \gamma$ is proportional to $\log|x|$ for large $|x|$.

In Section II we saw an extension of work by [10]. This extension showed, under a suitable choice of coefficients, how to construct a unique strong solution of a stochastic differential equation, driven by a cylinder Wiener process, in a separable Hilbert space

$$d\xi(t) = F(\xi(t))dt + \Phi(\xi(t))dW(t), \quad t \in [0,T] \tag{I.6}$$

using Ovsjannikov's method. The end result was a strong solution that takes values in an intersection of a suitably chosen scale of Hilbert spaces. This general theory was subsequently used to extend the work of [11] [*in a sense of considering a stochastic version*] by considering a lattice system on a locally finite subset $\gamma \subset \mathbb{R}^d$ such that the spin variables q_x and q_y are allowed to interact via a pair potential if the distance between $x, y \in \gamma$ is no more than a fixed and positive interaction radius r, that is, they are neighbours in the geometric graph defined by γ and r. Precisely speaking we considered a system

$$d\xi_x(t) = \phi_x(\Xi(t))dt + \Psi_x(\Xi(t))dW_x(t), \; x \in \gamma, \; t \in [0,T] \tag{I.7}$$

where ϕ_x and Ψ_x were required to satisfy the so-called "finite range " and "uniform Lipschitz continuity" conditions and showed that system $(I.7)$ can be realised in a suitable scale of separable Hilbert spaces and hence studied using Ovsjannikov's method.

In Section III, we would like to further build upon results of [10, 11] and [2, 1] and consider a lattice system of the form

$$d\xi_x(t) = \Phi_x(\xi_x(t), \Xi(t))dt + \Psi_x(\xi_x(t), \Xi(t))dW_x(t),\ \ x \in \gamma,\ \ t \in [0, T] \tag{I.8}$$

where $\Phi_x(a, b) \equiv V(a) + \phi_x(b)$, where V is a real valued one particle potential satisfying the dissipativity condition, and Ψ_x is Lipschitz. In our approach we will assume, as in [11], that configuration of particles $\gamma \subset \mathbb{R}^d$ is a locally finite subset of $\mathbb{R}^d$ distributed according to a Poisson or, more generally, Gibbs measure with a superstable low regular interaction energy, so that for all $x \in \gamma$ a number of particles in a certain compact vicinity of x is proportional to $\log|x|$ for large $|x|$.

Unfortunately, system (I.8) doesn't lend itself for an immediate and straightforward application of Ovsjannikov's method. Hence in this section we opt for an approach that was used in [2] and consider a so-called sequence of "finite volume approximations" of the system (I.8). Precisely speaking a sequence of finite volume approximations is a sequence of solutions of truncated systems of the following form

$$\begin{aligned} &\xi^n_{x,t} = \zeta_x + \int_0^t \Phi_x(\xi^n_{x,s}, \Xi^n_s)ds + \int_0^t \Psi_x(\xi^n_{x,s}, \Xi^n_s)dW_x(s), \quad \forall(x \in \Lambda_n \wedge t \in [0, T]) \\ &\xi^n_{x,t} = \zeta_x, \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \forall(x \notin \Lambda_n \wedge t \in [0, T]) \end{aligned} \tag{I.9}$$

where $\gamma \supset \Lambda_n \uparrow \gamma$ are finite. Using a comparison Theorem III.20, which builds upon the method of Ovsjannikov, we ultimately show that the sequence of finite volume approximations converges to a unique strong solution of the system (I.8) in a certain scale of Banach spaces.

II SDEs in a Scale of Hilbert Spaces

II.1 Summary

We begin this section by fixing some common notation and a couple of special definitions including a definition of a scale and an Ovsjannikov map. We continue with an outline of our probability space as well as a number of important measure and measurable spaces. Subsequently we introduce a certain family $\mathbb{Y}$ (see Definition II.26) of stochastic processes and prove that if fact this family is a scale. We conclude the first subsection by exhibiting our main SDE, defining what we mean by a strong solution and featuring, without a proof, our main existence Theorem II.33.

Then we move on to the next subsection containing a number of auxiliary results. In particular we define a certain integral map (see Definition II.37) and prove that in fact it is an Ovsjannikov map on $\mathbb{Y}$ (see Theorem II.38). We also establish convergence of a certain infinite sum and using a collection of these result we conclude this subsection by proving a Cauchy like estimate, see Theorem II.42.

Next we use Theorem II.42 to prove existence and uniqueness (see subsection II.4) and establish various norm estimates. Finally we consider an application of our general theory to a system of equations describing non-equilibrium stochastic dynamics of (real-valued) spins of an infinite particle system in $\mathbb{R}^n$.

II.2 Main Framework

II.2.1 General Notation

In our framework all vector spaces will be over $\mathbb{R}$ and the cardinal number of any given set A will always be denoted by $\#A$. Hence if A is a finite set then naturally $\#A$ will stand for the number of elements in A. We now start this subsection by introducing the following sets that will be frequently used throughout this text:

$$\mathbb{R}_+ := (0,\infty), \quad \mathbb{R}_0 := [0,\infty), \quad \mathbb{R}_1 := [1,\infty), \quad \mathbb{R}_2 := [2,\infty), \quad \mathbb{N}_0 := \mathbb{N}\cup\{0\}. \tag{II.1}$$

We also introduce constants $T,\ \underline{\mathfrak{a}},\ \overline{\mathfrak{a}} \in \mathbb{R}_+$, $\mathfrak{p} \in \mathbb{R}_1$ and a special notation for the following closed intervals:

$$\mathcal{A} := [\underline{\mathfrak{a}}, \overline{\mathfrak{a}}],$$

$$\mathcal{T} := [0, T].$$

Given two normed vector spaces A and B we fix the following compact notation

$$A \prec B \iff \begin{cases} \text{A is a subspace of B} \\ ||x||_B \leq \ ||x||_A, \quad \forall (x \in A). \end{cases} \tag{II.2}$$

and agree that given any two sets X and Y the symbol X^Y will be understood as an infinite Cartesian product, that is

$$X^Y = \bigtimes_{y\in Y} X = \Big\{ \{z_y\}_{y\in Y} \ \Big| \ z_y \in X \text{ for all } y \in Y \Big\}.$$

Remark II.1. *Sometimes we will call X^Y the set of all maps from Y to X.*

Moreover given a family of sets $\mathbf{X} := \{X_{\mathfrak{a}}\}_{\mathfrak{a}\in\mathcal{A}}$ we introduce the following notation:

$$\uparrow\!\mathbf{X} := \bigcup_{\mathfrak{a}\in(\underline{\mathfrak{a}},\overline{\mathfrak{a}})} X_{\mathfrak{a}}, \quad \downarrow\!\mathbf{X} := \bigcap_{\mathfrak{a}\in(\underline{\mathfrak{a}},\overline{\mathfrak{a}})} X_{\mathfrak{a}}.$$

II.2.2 Scales and Ovsjannikov Maps

We now proceed to introduce several important definitions.

Definition II.2. *A family* $\mathbf{X} \coloneqq \{X_{\mathfrak{a}}\}_{\mathfrak{a}\in\mathcal{A}}$ *of Banach spaces is called a scale if* $X_\alpha \prec X_\beta$ *for all* $\alpha < \beta \in \mathcal{A}$*.*

Remark II.3. *It is perhaps important to note at this point that within the context of Definition II.2 above all Banach spaces in the scale* $\mathbf{X}$ *have the same zero vector. Moreover when* $\mathbf{X}$ *is a scale we see that the following equality holds:*

$$ {}^{\uparrow}\mathbf{X} = \bigcup_{\mathfrak{a}\in[\underline{\mathfrak{a}},\overline{\mathfrak{a}})} X_{\mathfrak{a}}, $$

$$ {}_{\downarrow}\mathbf{X} = \bigcap_{\mathfrak{a}\in(\underline{\mathfrak{a}},\overline{\mathfrak{a}}]} X_{\mathfrak{a}}. $$

Definition II.4. *Let* $\mathbf{X}$ *be a scale and* $\mathbf{Z} \coloneqq \{Z_{\mathfrak{a}}\}_{\mathfrak{a}\in\mathcal{A}}$ *be a family of Banach spaces. Moreover let* $q \in \mathbb{R}_0$*. Then*

$$ G : {}^{\uparrow}\mathbf{X} \to Z_{\overline{\mathfrak{a}}} $$

is called an Ovsjannikov map of order q *from* $\mathbf{X}$ *to* $\mathbf{Z}$ *if*

$$ \left.\begin{array}{l} \text{(1) } G\Big|_{X_\alpha} : X_\alpha \to Z_\beta \\ \text{(2) } ||G(x) - G(y)||_{Z_\beta} \leq \frac{L}{(\beta-\alpha)^q} ||x-y||_{X_\alpha} \end{array}\right\} \exists (L \in \mathbb{R}_+)\ \forall (\alpha < \beta \in \mathcal{A}\ \wedge\ x, y \in X_\alpha). \quad \text{(II.3)} $$

Definition II.5. *Suppose* $\mathbf{X}$ *is a scale and* $\mathbf{Z} \coloneqq \{Z_{\mathfrak{a}}\}_{\mathfrak{a}\in\mathcal{A}}$ *is a family of Banach spaces. Let us define the following spaces of Ovsjannikov maps:*

$$ \mathcal{O}(\mathbf{X}, \mathbf{Z}, q) \coloneqq \{\textit{space of Ovsjannikov maps of order } q \textit{ from } \mathbf{X} \textit{ to } \mathbf{Z}\}, $$

$$ \mathcal{O}(\mathbf{X}, q) \coloneqq \{\textit{space of Ovsjannikov maps of order } q \textit{ from } \mathbf{X} \textit{ to } \mathbf{X}\}. $$

Remark. Usually we will deal with situations when both $\mathbf{X}$ and $\mathbf{Z}$ are scales.

Definition II.6. *We now take a moment to fix in place the following notation that will be frequently used throughout this text:*

(1) Let $\mathbb{X} := \{\mathbb{X}_{\mathfrak{a}}\}_{\mathfrak{a}\in\mathcal{A}}$ *be a scale of separable Hilbert spaces,*

(2) Let $\mathcal{H}$ *be a separable Hilbert space,*

(3) Let $\mathbb{H} := \{\mathbb{H}_{\mathfrak{a}}\}_{\mathfrak{a}\in\mathcal{A}}$ *be a family of sets such that for all* $\mathfrak{a} \in \mathcal{A}$, $\mathbb{H}_{\mathfrak{a}}$ *is the space of Hilbert-Schmidt operators from* $\mathcal{H}$ *to* $\mathbb{X}_{\mathfrak{a}}$. *Precisely speaking for all* $\mathfrak{a} \in \mathcal{A}$ *we have*

$$\mathbb{H}_{\mathfrak{a}} := \left\{ A \in L(\mathcal{H}, \mathbb{X}_{\mathfrak{a}}) \;\middle|\; \begin{array}{l} \|A\|_{\mathbb{H}_{\mathfrak{a}}} := \left(\sum_{n\in\mathbb{N}} ||A(\mathfrak{e}_n)||^2_{\mathbb{X}_{\mathfrak{a}}} \right)^{\frac{1}{2}} < \infty, \\ \mathfrak{e} := \{\mathfrak{e}_n\}_{n\in\mathbb{N}} \text{ is an orthonormal basis of } \mathcal{H} \end{array} \right\}. \tag{II.4}$$

Remark II.7. *It can be shown that family* $\mathbb{H}$ *is the family of separable Hilbert spaces and for all* $\mathfrak{a} \in \mathcal{A}$ *the norm of* $\mathbb{H}_{\mathfrak{a}}$ *is independent of the choice of the orthonormal basis for* $\mathcal{H}$. *Details of this classical result can be found in [52].*
Moreover since $\mathbb{X}$ *is a scale we see from the Definition II.2 that for all* $\alpha < \beta \in \mathcal{A}$ *we have the following:*

$$A \in L(\mathcal{H}, \mathbb{X}_{\alpha}) \implies A \in L(\mathcal{H}, \mathbb{X}_{\beta}),$$

$$\sum_{n\in\mathbb{N}} ||A(\mathfrak{e}_n)||^2_{\mathbb{X}_{\beta}} \leq \sum_{n\in\mathbb{N}} ||A(\mathfrak{e}_n)||^2_{\mathbb{X}_{\alpha}}.$$

Therefore it follows that $\mathbb{H}$ *is a scale.*

Let us now move on to the discussion of the underlying probability space, on which this section will be subsequently based.

II.2.3 Probability and Measure Spaces

We shall now proceed to describe the probability space and also a couple of important spaces of measurable maps and stochastic processes, that will become important in the main body of this text. Let us begin with a couple of auxiliary definitions.

Definition II.8. *A probability space* $(\Omega, \mathcal{F}, \mathbb{P})$ *is said to be complete if* $G \subset \mathcal{F}$ *where*

$$G \equiv \left\{ A \subset \Omega \;\middle|\; \textit{for some } F \ (A \subset F \textit{ and } \mathbb{P}(F) = 0) \right\}.$$

Remark II.9. *Collection* G *above sometimes called the collection of all null-sets of* $\mathcal{F}$.

Definition II.10. *In a filtered probability space a filtration* $\mathbb{F} := \{\mathcal{F}_t\}_{t\in\mathcal{T}}$ *is called normal if*

(1) $H \subset \mathcal{F}_0$

(2) $\mathcal{F}_t = \mathcal{F}_{t+}$ *for all* $t \in \mathcal{T}$

$$\textit{where} \quad H \equiv \left\{ A \in \mathcal{F} \;\middle|\; \mathbb{P}(A) = 0 \right\} \quad \textit{and} \quad \mathcal{F}_{t+} := \bigcap_{s\in(t,T]} \mathcal{F}_s.$$

Let us now introduce the following assumptions. There will also be a couple of additional assumption introduced at the end of this subsection.

(1) Let us agree in the first place that all probability and measure spaces in our subsequent discussion in this section are complete.

(2) Now we fix a filtered probability space

$$\mathbf{P} := (\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F}) \tag{II.5}$$

on which all of our subsequent work will be based. Moreover we assume that filtration $\mathbb{F} := \{\mathcal{F}_t\}_{t\in\mathcal{T}}$ is normal in a sence of Definition II.10.

(3) We fix a measure space $\mathbf{M} := (\mathcal{T}, \mathscr{B}(\mathcal{T}), dt)$ where dt is a Lebesgue measure and $\mathscr{B}(\mathcal{T})$ is a Borel $\sigma-$algebra.

Remark II.11. *Lebesgue measure* dt *in this text will sometimes be denoted by* ds *or* $d\tau$. *In general we will use a Riemann integral notation for all Bochner-Lebesgue integrals in this text.*

(4) We agree to work on a fixed product measure space

$$\mathbf{MP} := (\overline{\Omega} := \mathcal{T} \otimes \Omega, \overline{\mathcal{F}} := \mathscr{B}(\mathcal{T}) \otimes \mathcal{F}, \overline{\mathbb{P}} := dt \otimes \mathbb{P}). \tag{II.6}$$

(5) For all $t \in \mathcal{T}$ we will sometimes refer to the following fixed measurable spaces:

$$\begin{aligned} \mathbf{P}_t &:= (\Omega, \mathcal{F}_t), \\ \mathbf{M}_t &:= ([0,t], \mathscr{B}([0,t])), \\ \mathbf{MP}_t &:= ([0,t] \otimes \Omega, \mathscr{B}([0,t]) \otimes \mathcal{F}_t). \end{aligned} \tag{II.7}$$

(6) Given two measurable spaces $\mathbf{A}$ and $\mathbf{B}$ we denote by $\mathcal{M}(\mathbf{A}, \mathbf{B})$ the space of all measurable maps from $\mathbf{A}$ to $\mathbf{B}$.

(7) Moreover we now fix notation for the following measurable spaces that will be frequently mentioned throughout this text:

$$\begin{aligned} &\left.\begin{aligned} &\text{(a)} \quad \mathbf{M}^{\mathbb{X}_{\mathfrak{a}}} := (\mathbb{X}_{\mathfrak{a}}, \mathscr{B}(\mathbb{X}_{\mathfrak{a}})) \\ &\text{(b)} \quad \mathbf{M}^{\mathbb{H}_{\mathfrak{a}}} := (\mathbb{H}_{\mathfrak{a}}, \mathscr{B}(\mathbb{H}_{\mathfrak{a}})) \end{aligned}\right\} \forall (\mathfrak{a} \in \mathcal{A}), \\ &\text{(c)} \quad \mathbf{M}^{\mathbb{R}} := (\mathbb{R}, \mathscr{B}(\mathbb{R})). \end{aligned} \tag{II.8}$$

Now, the following definition fixes how we understand, denote and refer to stochastic processes in this section.

Definition II.12. *Let Y be a Banach space and $\mathbf{Y} := (Y, \mathscr{B}(Y))$ be a measurable space. A measurable stochastic process is an element of $\mathcal{M}(\mathbf{MP}_T, \mathbf{Y})$ and the set of all measurable stochastic processes from $\mathbf{MP}_T$ to $\mathbf{Y}$ is denoted by $\mathcal{S}(Y)$.*

Remark II.13. *From a classical measure theory, see for example [7, 54, 60], it follows that if $\xi \in \mathcal{S}(Y)$ then for all $t \in \mathcal{T}$ and all $\omega \in \Omega$ we have the following:*

$$\mathcal{M}(\mathbf{M}, \mathbf{Y}) \ni \xi_{\cdot,\omega} : \mathcal{T} \to Y,$$

$$\mathcal{M}(\mathbf{P}, \mathbf{Y}) \ni \xi_{t,\cdot} : \Omega \to Y.$$

Let us now introduce a couple of important classifications of stochastic processes.

Definition II.14. *Let Y be a Banach space, $\mathbf{Y} := (Y, \mathscr{B}(Y))$ be a measurable space and suppose that $\xi \in \mathcal{S}(Y)$. Moreover for all $t \in \mathcal{T}$ let ξ^t be a restriction of ξ to $[0,t] \times \Omega$. That is*

$$\xi^t := \xi\Big|_{[0,t] \times \Omega} \quad \text{for all } t \in \mathcal{T}.$$

A stochastic Process ξ is called progressively measurable if for all $t \in \mathcal{T}$ a restriction process ξ^t is an element of $\mathcal{M}(\mathbf{MP}_t, \mathbf{Y})$.

Definition II.15. *Let Y be a Banach space and suppose that $\xi, \zeta \in \mathcal{S}(Y)$. We would like to define the following notation*

$$\xi \approx \zeta \iff \forall (t \in \mathcal{T})\ \mathbb{P}\{\omega \in \Omega \mid \xi_{t,\omega} \neq \zeta_{t,\omega}\} = 0$$

$$\iff \forall (t \in \mathcal{T})\ \xi_t = \zeta_t,\ \mathbb{P} - a.s.$$

Definition II.16. *Let Y be a Banach space and suppose that $\xi \in \mathcal{S}(Y)$. A modification of ξ is a stochastic process $\tilde{\xi} \in \mathcal{S}(Y)$ such that $\xi \approx \tilde{\xi}$.*

Theorem II.17. *Let Y be a Banach space and $\mathbf{Y} := (Y, \mathscr{B}(Y))$ be a measurable space. If every sample path of $\xi \in \mathcal{S}(Y)$ is continuous and ξ is adapted to $\mathbb{F}$ then ξ is progressively measurable.*

Proof. We fix $t \in \mathcal{T}$, define ζ to be the restriction of ξ to $[0,t] \times \Omega$ and conclude the proof by showing that $\zeta \in \mathcal{M}(\mathbf{MP}_t, \mathbf{Y})$. As a first step, observe the following:

(1) For any fixed $\alpha \in [0,t]$ the map $\zeta_\alpha : [0,t] \times \Omega \to Y$, defined in the following way

$$[0,t] \times \Omega \ni (s,\omega) \xrightarrow{\zeta_\alpha} \xi_{\alpha,\omega} \in Y,$$

is $\mathscr{B}([0,t]) \times \mathcal{F}_t$ measurable because ξ is adapted to $\mathbb{F}$ and for all $A \in \mathscr{B}(Y)$ the inverse image $\zeta_\alpha^{-1}(A) = [0,t] \times \xi_\alpha^{-1}(A)$.

(2) For any fixed $\beta < \gamma \in [0,t]$ the set $[\beta, \gamma] \times \Omega$ is $\mathscr{B}([0,t]) \times \mathcal{F}_t$ measurable. Moreover, because for any fixed $\alpha \in [0,t]$ the map ζ_α is $\mathscr{B}([0,t]) \times \mathcal{F}_t$ measurable we conclude that

the product map

$$\mathbb{1}_{[\beta,\gamma]\times\Omega}\zeta_\alpha$$

is also $\mathscr{B}([0,t]) \times \mathcal{F}_t$ measurable.

Now for all $n \in \mathbb{N}$ we consider a partition $\{\psi_i\}_{i=0}^n$ of $[0,t]$ into n intervals of equal length such that $\psi_0 = 0$ and $\psi_n = t$. Moreover we define a process $\zeta^n : [0,t] \times \Omega \to Y$ in the following way

$$\zeta^n := \sum_{i=1}^{n} \mathbb{1}_{[\psi_{i-1},\psi_i]\times\Omega}\zeta_{\psi_i}.$$

Now from (1) and (2) above it clear that ζ^n is $\mathscr{B}([0,t]) \times \mathcal{F}_t$ measurable. Moreover by considering an arbitrary pair $(s,\omega) \in [0,t] \times \Omega$ we see that $\zeta^n_{s,\omega} = \zeta_{\psi_j,\omega}$ for some $0 \leq j \leq n$ such that $s \in [\psi_{j-1}, \psi_j]$. Because $|s - \psi_j| \leq \frac{t}{n}$ and every sample path of ξ is continuous we conclude that

$$\lim_{n\to\infty} \zeta^n_{s,\omega} = \zeta_{s,\omega}.$$

Finally by Theorem IV.4 we conclude that ζ is $\mathscr{B}([0,t]) \times \mathcal{F}_t$ measurable map and the proof is complete. □

Theorem II.18. *Let Y be a Banach space and $\mathbf{Y} := (Y, \mathscr{B}(Y))$ be a measurable space. If $\xi \in \mathcal{S}(Y)$ is continuous and adapted to $\mathbb{F}$ then ξ is progressively measurable.*

Proof. It is important to recall that we are working with a complete probability spaces in this section. Since ξ is continuous it follows that there exists $N \in \mathcal{F}$ such that $\mathbb{P}(N) = 0$ and for all $\omega \in \Omega - N$ the trajectory $\xi_{\cdot,\omega}$ is continuous. Let us now define the following process

$$\zeta_{t,\omega} := \begin{cases} \xi_{t,\omega} & \forall (t \in [0,T] \wedge \omega \in \Omega - N), \\ 0 & \forall (t \in [0,T] \wedge \omega \in N). \end{cases}$$

Hence we see that every sample path of $\zeta \in \mathcal{S}(Y)$ is continuous. In addition we see that for all $t \in \mathcal{T}$ we have the following relation

$$\zeta_t = \xi_t, \ \mathbb{P} - a.s.$$

Hence by Theorem IV.5 we conclude that ζ is adapted to $\mathbb{F}$. Therefore, Theorem II.17 tells us that ζ is progressively measurable. Finally using the fact that the set $[0,T] \times N$ is a measurable rectangle we get

$$\overline{\mathbb{P}}([0,T] \times N) = dt([0,T])\mathbb{P}(N) = 0$$

and using the definition of ζ we thereby arrive at the following conclusion

$$\zeta = \xi, \ \overline{\mathbb{P}} - a.s.$$

Hence if we define $\overline{dt \times \mathbb{P}}$ to be the restriction of the product measure $\overline{\mathbb{P}}$ to $\mathscr{B}([0,t]) \times \mathcal{F}_t$ and equip the measurable space $\mathbf{MP}_t$ with $\overline{dt \times \mathbb{P}}$ then we see that

$$\zeta\Big|_{[0,t] \times \Omega} = \xi\Big|_{[0,t] \times \Omega}, \ \overline{dt \times \mathbb{P}} - a.s.$$

Now because ζ is progressively measurable we conclude by Theorem IV.5 that ξ is also progressively measurable hence the proof is complete. □

Remark II.19. *Theorem II.18 is a more general version of Theorem II.17 (see [30] for additional details). Moreover from [30] one can learn that in fact a more general results then Theorem II.18 holds. In particular one can drop an assumption of continuity and only require sample paths of ξ to be càdlàg.*

Definition II.20. *Let $\mathscr{X} := (X, \mathcal{A}, \eta)$ be a measure space, Y be a Banach space, with norm denoted by $\|\cdot\|_Y$, and $\mathscr{Y} := (Y, \mathscr{B}(Y))$ be a measurable space. For all $p \in \mathbb{R}_1$ we define the following Banach spaces*

$$\mathcal{L}^p(\mathscr{X}, \mathscr{Y}) := \left\{ f : X \to Y \ \middle| \ \begin{array}{l} \|f\|_{\mathcal{L}^p(\mathscr{X},\mathscr{Y})} := \left(\int_X \|f\|_Y^p d\eta \right)^{\frac{1}{p}} < \infty, \\ f \in \mathcal{M}(\mathscr{X}, \mathscr{Y}) \end{array} \right\}. \tag{II.9}$$

Remark II.21. *As it is often done in academic literature, we will not consider explicitly the dependence of $\mathcal{L}^p(\cdot,\cdot)$ spaces on equivalence classes. We will work directly with the Definition II.20 and when necessary acknowledge any issues arising from such dependence.*

Theorem II.22. *Let $\mathbf{X} := \{X_\mathfrak{a}\}_{\mathfrak{a}\in\mathcal{A}}$ be a scale and $p \in \mathbb{R}_1$. Moreover for all $\mathfrak{a} \in \mathcal{A}$ define a measurable space $\mathbf{M}^{X_\mathfrak{a}} := (X_\mathfrak{a}, \mathscr{B}(X_\mathfrak{a}))$. Then $\mathbf{L} := \{\mathcal{L}^p(\mathbf{P}, \mathbf{M}^{X_\mathfrak{a}})\}_{\mathfrak{a}\in\mathcal{A}}$ is a scale.*

Proof. The fact that $\mathbf{L}$ is a family of Banach spaces is a standart result from functional analysis, see for example [13, 36, 52]. Therefore to finish the proof it remains to verify that conditions (1) and (2) of the Definition II.2 are satisfied. To this end let us start by fixing $\alpha < \beta \in \mathcal{A}$ and $f \in \mathcal{L}^p(\mathbf{P}, \mathbf{M}^{X_\alpha})$. By Definition II.20 it follows that $f \in \mathcal{M}(\mathbf{P}, \mathbf{M}^{X_\alpha})$. Because $\mathbf{X}$ is a scale we conclude that $f \in \mathcal{M}(\mathbf{P}, \mathbf{M}^{X_\beta})$ and $\|f\|^p_{X_\beta} \leq \|f\|^p_{X_\alpha}$. From Theorem IV.9 we therefore see that

$$\int_\Omega \|f\|^p_{X_\beta} d\mathbb{P} \;\leq\; \int_\Omega \|f\|^p_{X_\alpha} d\mathbb{P}.$$

Now, it follows that $f \in \mathcal{L}^p(\mathbf{P}, \mathbf{M}^{X_\beta})$ and

$$\|f\|_{\mathcal{L}^p(\mathbf{P},\mathbf{M}^{X_\beta})} \leq \|f\|_{\mathcal{L}^p(\mathbf{P},\mathbf{M}^{X_\alpha})},$$

hence the proof is complete. □

Remark II.23. *It follows from Theorem II.22 above that for all $p \in \mathbb{R}_1$ the family of Banach spaces*

$$\mathbb{L}^p := \{\mathcal{L}^p(\mathbf{P}, \mathbf{M}^{\mathbb{X}_\mathfrak{a}})\}_{\mathfrak{a}\in\mathcal{A}}$$

is in fact a scale.

Let now state the final assumption on which, going forward, this section will also be based.

(8) We define a cylindrical Wiener process W in $\mathcal{H}$ (see Definition II.6 and Remark II.24 bellow) and assume that filtration $\mathbb{F} := \{\mathcal{F}_t\}_{t\in\mathcal{T}}$ on our probability space satisfies the following standard conditions:

(a) $W(t)$ is $\mathcal{F}_t$ measurable, for all $t \in \mathcal{T}$,

(b) $W(t) - W(s)$ is independent of $\mathcal{F}_s$, For all $s \leq t \in \mathcal{T}$.

Remark II.24. *Based on [15] we now sketch the construction of a cylindrical Wiener process in $\mathcal{H}$. One begins this construction with a linear, self-adjoint and positive definite operator $Q : \mathcal{H} \to \mathcal{H}$ and chooses $\mathcal{H} \subset \mathcal{H}_1$ so that $\mathcal{H}_0 := Q^{\frac{1}{2}}(\mathcal{H})$ is embedded into $\mathcal{H}_1$ via Hillbert–Schmidt embedding. One then proceeds to prove that*

$$\widehat{W}(t) = \sum_{n=1}^{\infty} Q^{\frac{1}{2}} e_n w_n(t), \quad \forall (t \in \mathcal{T}),$$

where $\{e_n\}_{n\in\mathbb{N}}$ is a complete orthonormal basis for $\mathcal{H}$ and $\{w_n\}_{n\in\mathbb{N}}$ is a family of independent standard real-valued Wiener processes, is a classical (is a sense of [15]) Wiener process on $\mathcal{H}_1$. Finally, one calls $\widehat{W}$ a cylindrical Wiener process in $\mathcal{H}$ when $Q \equiv I$. Hence for some complete orthonormal basis $\{\mathfrak{q}_n\}_{n\in\mathbb{N}}$ of $\mathcal{H}$

$$W(t) = \sum_{n=1}^{\infty} \mathfrak{q}_n w_n(t), \quad \forall (t \in \mathcal{T}).$$

Finally we define spaces of stochastic processes that can be integrated with respect to W.

Definition II.25. *For all $\mathfrak{a} \in \mathcal{A}$ we define the following space*

$$\mathcal{N}_W^{\mathfrak{a}} := \left\{ \xi \in \mathcal{S}(\mathbb{H}_{\mathfrak{a}}) \;\middle|\; \begin{array}{l} \mathbb{E}\left[\int_0^T \|\xi(s)\|_{\mathbb{H}_{\mathfrak{a}}}^2 ds\right] < \infty, \\ \xi \textit{ is progressively measurable} \end{array} \right\}. \tag{II.10}$$

Now we conclude this subsection by noting that, stochastic integration in this text follows the approach of [15, 48]. In particular it is known (see subsection IV.2) that if $\xi \in \mathcal{N}_W^{\mathfrak{a}}$ for some $\mathfrak{a} \in \mathcal{A}$ then an integral process

$$\int_0^t \xi(s) dW(s), \ t \in \mathcal{T}$$

is well defined and represents a square integrable $\mathbb{X}_{\mathfrak{a}}$ (see Definition II.6) valued martingale with respect to $\mathbb{F}$ with almost surely continuous trajectories.

II.2.4 $\mathbb{Y}$ spaces

Let us now introduce a family of normed linear spaces of stochastic processes that from now on will be at the centre or our attention.

Definition II.26. *For all $p \in \mathbb{R}_1$ and all $\mathfrak{a} \in \mathcal{A}$ let*

$$\mathbb{Y}^p_{\mathfrak{a}} := \left\{ \xi \in \mathcal{S}(\mathbb{X}_{\mathfrak{a}}) \;\middle|\; \begin{array}{l} \|\xi\|_{\mathbb{Y}^p_{\mathfrak{a}}} := \left(\sup\left\{ \mathbb{E}\left[\|\xi(t)\|^p_{\mathbb{X}_{\mathfrak{a}}} \right] \;\middle|\; t \in \mathcal{T} \right\} \right)^{\frac{1}{p}} < \infty, \\ \xi \text{ is progressively measurable} \end{array} \right\}, \tag{II.11}$$

$$\mathbb{Y}^p := \{\mathbb{Y}^p_{\mathfrak{a}}\}_{\mathfrak{a} \in \mathcal{A}}. \tag{II.12}$$

be, respectively, a normed linear space of $\mathbb{X}_{\mathfrak{a}}$ valued progressively measurable processes and a family of such spaces.

Remark II.27. *Let us fix some $p \in \mathbb{R}_1$ and $\mathfrak{a} \in \mathcal{A}$. Now, strictly speaking $\|\cdot\|_{\mathbb{Y}^p_{\mathfrak{a}}}$ is a seminorm and $\mathbb{Y}^p_{\mathfrak{a}}$ should be defined and understood as a space of equivalence classes, in the same way as traditional $\mathcal{L}$ spaces are understood. In line with an academic literature, we will however make no attempt to explicitly deal with equivalence classes beyond this remark and shall treat $\mathbb{Y}^p_{\mathfrak{a}}$ in the same way as $\mathcal{L}$ spaces are often treated. One fact that nevertheless needs to be remembered/agreed is that any two precesses $\xi^1, \xi^2 \in \mathbb{Y}^p_{\mathfrak{a}}$ will be called equal if and only if $\|\xi^1 - \xi^2\|_{\mathbb{Y}^p_{\mathfrak{a}}} = 0$. However, from the definition of a seminorm $\|\cdot\|_{\mathbb{Y}^p_{\mathfrak{a}}}$ we can see that given any two equal processes $\xi^1, \xi^2 \in \mathbb{Y}^p_{\mathfrak{a}}$ it is still possible that for all $t \in \mathcal{T}$ we have $\xi^1_t \neq \xi^2_t$ on some subset of Ω of measure zero. In other words $\xi^1 = \xi^2$, in $\mathbb{Y}^p_{\mathfrak{a}}$, if and only if $\xi^1 \approx \xi^2$ (see Definition II.15 and II.16).*

Theorem II.28. *Let $p \in \mathbb{R}_1$, X be a Banach space and let $\mathbf{X} := (X, \mathscr{B}(X))$ be a measurable space. Moreover let*

$$Z^p := \left\{ \xi \in \mathcal{S}(X) \;\middle|\; \begin{array}{l} \|\xi\|_{Z^p} := \left(\sup\left\{ \mathbb{E}\left[\|\xi(t)\|^p_X \right] \;\middle|\; t \in \mathcal{T} \right\} \right)^{\frac{1}{p}} < \infty, \\ \xi \text{ is progressively measurable} \end{array} \right\}.$$

Then Z^p is a Banach space.

Proof. According to the Definition II.26 we need to show that Z^p is complete. Therefore let us start by assuming that $\mathscr{X} := \{\xi^n\}_{n \in \mathbb{N}}$ is a Cauchy sequence in Z^p and defining for all $t \in \mathcal{T}$

the following sequence $\mathscr{X}_t := \{\xi_t^n\}_{n\in\mathbb{N}}$. Now, using the Definition II.26 once again we see that for all $t \in \mathcal{T}$ the sequence $\mathscr{X}_t$ is Cauchy in $\mathcal{L}^p(\mathbf{P}, \mathbf{X})$ and

$$\lim_{n,m\to\infty} \|\xi_t^n - \xi_t^m\|_{\mathcal{L}^p(\mathbf{P},\mathbf{X})} = 0, \text{ uniformly on } \mathcal{T}. \tag{II.13}$$

Hence let us define a map $\xi : \overline{\Omega} \to X$ in the following way

$$\xi(t,\omega) := \overbrace{\left[\lim_{n\to\infty} \xi_t^n\right]}^{\text{in } \mathcal{L}^p(\mathbf{P},\mathbf{X})}(\omega).$$

Now using equation (II.13) it is clear that

$$\lim_{n\to\infty} \|\xi_t^n - \xi_t\|_{\mathcal{L}^p(\mathbf{P},\mathbf{X})} = 0, \text{ uniformly on } \mathcal{T}.$$

Therefore we conclude that

$$\lim_{n\to\infty} \|\xi^n - \xi\|_{Z^p} = 0,$$

and to finish the proof it remains to show that $\xi \in Z^p$. To this end observe that for all $t \in \mathcal{T}$

$$\left| \mathbb{E}\left[\|\xi_t^n\|_X^p\right] - \mathbb{E}\left[\|\xi_t\|_X^p\right] \right| \leq \|\xi^n - \xi\|_{Z^p}^p,$$

which shows that

$$\lim_{n\to\infty} \mathbb{E}\left[\|\xi_t^n\|_X^p\right] = \mathbb{E}\left[\|\xi_t\|_X^p\right], \text{ uniformly on } \mathcal{T}. \tag{II.14}$$

Because each element of $\mathscr{X}$ is in $\mathcal{M}(\mathbf{MP}, \mathbf{X})$ we conclude that the map

$$\mathcal{T} \ni t \longrightarrow \mathbb{E}\left[\|\xi_t^n\|_X^p\right] \in \mathbb{R}$$

is $\mathscr{B}(\mathcal{T})$ measurable. Hence using Theorem IV.4 and equation (II.14) we conclude that the map

$$\mathcal{T} \ni t \longrightarrow \mathbb{E}\left[\|\xi_t\|_X^p\right] \in \mathbb{R}$$

is also $\mathscr{B}(\mathcal{T})$ measurable. Moreover from the Definition II.26 and equation (II.14) above we also see that there exist a constant $\bar{k} \in \mathbb{N}$ such that

$$\mathbb{E}\Big[\|\xi_t\|_X^p\Big] \le \|\xi^{\bar{k}}\|_{Z^p}^p + 1 \text{ for all } t \in \mathcal{T},$$

which shows, according to Theorem IV.9 and IV.18, that $\xi \in \mathcal{L}^p(\mathbf{MP}, \mathbf{X})$. Now for all $n \in \mathbb{N}$, knowing that $\xi \in \mathcal{S}(X)$, we apply identical arguments to $\mathbb{E}[\|\xi_t^n - \xi_t\|_X^p]$ and conclude that

$$\lim_{n\to\infty} \|\xi^n - \xi\|_{\mathcal{L}^p(\mathbf{MP},\mathbf{X})} = 0.$$

Hence by Theorem IV.12 we see that there exist a subsequence ρ such that

$$\lim_{n\to\infty} \xi^{\rho(n)} = \xi,\ \overline{\mathbb{P}} - a.s.$$

Hence if we fix $t \in \mathcal{T}$ and define $\overline{dt \times \mathbb{P}}$ to be the restriction of the product measure $\overline{\mathbb{P}}$ to $\mathscr{B}([0,t]) \times \mathcal{F}_t$ and equip the measurable space $\mathbf{MP}_t$ with $\overline{dt \times \mathbb{P}}$ then recalling that for all $n \in \mathbb{N}$ the process $\xi^{\rho(n)}$ is progressively measurable we see that

$$\lim_{n\to\infty} \xi^{\rho(n)}\Big|_{[0,t]\times\Omega} = \xi\Big|_{[0,t]\times\Omega},\ \overline{dt \times \mathbb{P}} - a.s.$$

Hence by Theorem IV.4 we conclude that ξ is progressively measurable. Finally it is now clear that for some $\bar{m} \in \mathbb{N}$ we have $\xi^{\bar{m}} - \xi \in Z^p$ and $\xi^{\bar{m}} \in Z^p$. Since Z^p is a vector space we conclude that $\xi \in Z^p$ and the proof is complete. □

Theorem II.29. *Suppose that $p \in \mathbb{R}_1$. Then $\mathbb{Y}^p$ is the scale.*

Proof. From Theorem II.28 we already know that $\mathbb{Y}^p$ is a family of Banach spaces so to conclude the proof it only remains to show that conditions (1) and (2) of the Definition II.2 are satisfied.

Let us begin by fixing $\alpha < \beta \in \mathcal{A}$ and $\xi \in \mathbb{Y}_\alpha^p$. By Definition II.26 we see that $\xi \in \mathcal{S}(\mathbb{X}_\alpha)$. Because $\mathbb{X}$ is a scale we conclude that $\xi \in \mathcal{S}(\mathbb{X}_\beta)$ and $\|\xi\|_{\mathbb{X}_\beta}^p \le \|\xi\|_{\mathbb{X}_\alpha}^p$. From Theorem IV.9

we see that for all $t \in \mathcal{T}$ we have the following inequality

$$\int_\Omega \|\xi_t\|_{\mathbb{X}_\beta}^p d\mathbb{P} \;\leq\; \int_\Omega \|\xi_t\|_{\mathbb{X}_\alpha}^p d\mathbb{P},$$

which shows that

$$\|\xi\|_{\mathbb{Y}_\beta} \leq \; \|\xi\|_{\mathbb{Y}_\alpha}.$$

Finally, since ξ is progressively measurable we conclude that $\xi \in \mathbb{Y}_\beta^p$ and the proof is complete.

□

II.2.5 A Strong Solution of an SDE in a Scale

Let us start this subsection by introducing another constant $\mathfrak{q} \in [0, \frac{1}{2\mathfrak{p}})$ and defining the following two Ovsjannikov maps

$$\Phi \in \mathcal{O}(\mathbb{X}, \mathbb{H}, \mathfrak{q}), \tag{II.15}$$

$$F \in \mathcal{O}(\mathbb{X}, \mathfrak{q}). \tag{II.16}$$

Remark II.30. *Without loss of generality let us assume that both Φ and F share the same constant L (see Definition II.4) which is from now on fixed.*

Observe now that according to the Definition II.4 for all $\alpha < \beta \in \mathcal{A}$

$$F\Big|_{\mathbb{X}_\alpha} : \mathbb{X}_\alpha \to \mathbb{X}_\beta \quad \text{and} \quad \Phi\Big|_{\mathbb{X}_\alpha} : \mathbb{X}_\alpha \to \mathbb{H}_\beta$$

are continuous maps. Therefore if $p \in \mathbb{R}_2$ and $\xi \in \mathbb{Y}^p_\alpha$ then

(1) $F \circ \xi$ is in $\mathcal{S}(\mathbb{X}_\beta)$ and progressively measurable. Moreover for all $x \in \mathbb{X}_\alpha$ we have

$$\begin{aligned}
\|F(x)\|_{\mathbb{X}_\beta} &= \|F(x) + F(0) - F(0)\|_{\mathbb{X}_\beta} \\
&\leq \|F(x) - F(0)\|_{\mathbb{X}_\beta} + \|F(0)\|_{\mathbb{X}_\beta} \\
&\leq \frac{L}{(\beta-\alpha)^{\mathfrak{q}}}\|x\|_{\mathbb{X}_\alpha} + \|F(0)\|_{\mathbb{X}_\beta} \\
&\leq \frac{L}{(\beta-\alpha)^{\mathfrak{q}}}\Big(M + \|x\|_{\mathbb{X}_\alpha}\Big),
\end{aligned}$$

where

$$M := \frac{\|F(0)\|_{\mathbb{X}_{\underline{\mathfrak{a}}}}(\overline{\mathfrak{a}} - \underline{\mathfrak{a}})^{\mathfrak{q}}}{L}. \tag{II.17}$$

Hence we see that

$$\|F(\xi)\|^p_{\mathbb{X}_\beta} \leq \left(\frac{L}{(\beta-\alpha)^{\mathfrak{q}}}\right)^p 2^{p-1}\Big(M^p + \|\xi\|^p_{\mathbb{X}_\alpha}\Big),$$

and thereby conclude by Theorem IV.9 that $F \circ \xi$ is in $\mathcal{L}^p(\mathbf{MP}, \mathbf{M}^{\mathbb{X}_\beta})$.

(2) $\Phi \circ \xi$ is in $\mathcal{S}(\mathbb{H}_\beta)$ and progressively measurable. Moreover Theorem IV.2 and calculations nearly identical to the ones we have done above show that

$$\mathbb{E}\left[\int_0^T \|\Phi(\xi(s))\|^2_{\mathbb{H}_\beta} ds\right] < \infty.$$

Hence we conclude that for all $t \in \mathcal{T}$

$$\int_0^t \Phi(\xi(s)) dW(s) \tag{II.18}$$

is well defined and, treated as a process, represents a square integrable $\mathbb{X}_\beta$ (see Definition II.6) valued martingale with respect to $\mathbb{F}$ with almost surely continuous trajectories.

Remark II.31. *Let us return to an integral (II.18) above and clarify that it depends on β only up to a modification. To this end let us fix $\alpha < \beta < \gamma \in \mathcal{A}$ and*

$$\eta_\beta := \int_0^t \Phi_\beta(\xi(s)) dW(s), \quad t \in \mathcal{T}$$
$$\eta_\gamma := \int_0^t \Phi_\gamma(\xi(s)) dW(s), \quad t \in \mathcal{T}$$

considered respectively as $\mathbb{X}_\beta$ and $\mathbb{X}_\gamma$ valued random variables. Using Îto isometry and the fact that $\Phi \in \mathcal{O}(\mathbb{X}, \mathbb{H}, \mathfrak{q})$ observe now that

$$\begin{aligned}\mathbb{E}\Big[\|\eta_\beta - \eta_\gamma\|^2_{\mathbb{X}_\gamma}\Big] &\leq \mathbb{E}\Big[\int_0^t \|\Phi_\beta(\xi(s)) - \Phi_\gamma(\xi(s))\|^2_{\mathbb{H}_\gamma} ds\Big], \quad t \in \mathcal{T} \\ &\leq \frac{L^2}{(\gamma-\alpha)^{2\mathfrak{q}}}\mathbb{E}\Big[\int_0^t \|\xi(s) - \xi(s)\|^2_{\mathbb{X}_\alpha} ds\Big], \quad t \in \mathcal{T} \\ &\leq 0,\end{aligned}$$

hence we see that $\eta_\beta \approx \eta_\gamma$.

For the remainder of this section our focus shall be fixed on finding a solution for an SDE of the following form

$$d\xi(t) = F(\xi(t))dt + \Phi(\xi(t))dW(t), \ t \in \mathcal{T}.$$

Speaking more precisely we shall in fact be mainly concerned with an equivalent problem. That is our goal is to find a unique strong solution of the following stochastic integral equation

$$\xi(t) = \zeta_{\underline{\mathfrak{a}}} + \int_0^t F(\xi(s))ds + \int_0^t \Phi(\xi(s))dW(s), \ t \in \mathcal{T} \tag{II.19}$$

where $\zeta_{\underline{\mathfrak{a}}}$ is an element of $\mathbb{X}_{\underline{\mathfrak{a}}}$. In order to achieve our goal we first need to agree on the type of a stochastic process shall be accepted as a strong solution of the equation (II.19) above. Now, keeping in mind that our strong solution has to somehow make use of scales that we have previously outlined we put forward the following definition.

Definition II.32. *A stochastic process ξ is called a strong solution of the equation (II.19) if*

$$\xi \in \downarrow\mathbb{Y}^{2\mathfrak{p}}$$

and

$$\xi \approx \zeta_{\underline{\mathfrak{a}}} + \int_0^{\cdot} F(\xi(s))ds + \int_0^{\cdot} \Phi(\xi(s))dW(s).$$

This subsection will now be concluded by stating the main existence and uniqueness result of this section, which will be proved gradually with the final argument given in subsection II.4.

Theorem II.33. *Suppose that $\mathfrak{p} \in \mathbb{R}_1$ and $\mathfrak{q} \in [0, \frac{1}{2\mathfrak{p}})$. Moreover let $\Phi \in \mathcal{O}(\mathbb{X}, \mathbb{H}, \mathfrak{q})$ and $F \in \mathcal{O}(\mathbb{X}, \mathfrak{q})$. Then for all $\zeta_{\underline{\mathfrak{a}}} \in \mathbb{X}_{\underline{\mathfrak{a}}}$ there exists a unique strong solution (in a sense of Definition II.32) of the stochastic integral equation (II.19).*

Remark II.34. *Uniqueness of the strong solution will be understood in line with the argument given in the Remark II.27 following the Definition II.26. In particular we shall say that ξ is the unique strong solution if given any other strong solution η we have*

$$\|\xi - \eta\|_{\mathbb{Y}^{\mathfrak{p}}_{\mathfrak{a}}} = 0,$$

for all $\mathfrak{a} \in (\underline{\mathfrak{a}}, \overline{\mathfrak{a}})$.

II.3 Auxiliary Results

II.3.1 Ovsjannikov Map on $\mathbb{Y}$

We begin this subsection with a result that will be needed later on.

Theorem II.35. *Suppose that $A, B, k \in \mathbb{R}_+$ and $q \in [0, \frac{1}{k})$. Then*

$$\sum_{n=0}^{\infty} \frac{A^n}{B^{qn}} \frac{n^{qn}}{\sqrt[k]{n!}} < \infty. \tag{II.20}$$

Proof. By analysing a ratio of terms of the series (II.20) above we see that

$$\begin{aligned}
\frac{A^{n+1}}{B^{q(n+1)}} \frac{(n+1)^{q(n+1)}}{\sqrt[k]{(n+1)!}} \bigg/ \frac{A^n}{B^{qn}} \frac{n^{qn}}{\sqrt[k]{n!}} &= \frac{A}{B^q}(n+1)^{qn+q-\frac{1}{k}} \frac{1}{n^{qn}} \\
&= \frac{A}{B^q} \frac{1}{(n+1)^{\frac{1}{k}-q}} \left(1 + \frac{1}{n}\right)^{qn}.
\end{aligned}$$

Now since

$$\begin{aligned}
\lim_{n\to\infty} \frac{A}{B^q} \frac{1}{(n+1)^{\frac{1}{k}-q}} \left(1 + \frac{1}{n}\right)^{qn} &= \frac{A}{B^q} \left(\lim_{n\to\infty} \frac{1}{(n+1)^{\frac{1}{k}-q}}\right) \left(\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^{qn}\right) \\
&= \frac{A}{B^q}(0)(e^q) \\
&= 0
\end{aligned}$$

we conclude by ratio test that the series (II.20) converges and the proof is complete. □

Theorem II.36. *Suppose that $C, k \in \mathbb{R}_+$. Then*

$$\sum_{n=0}^{\infty} \frac{C^n}{\sqrt[k]{n!}} < \infty. \tag{II.21}$$

Proof. By analysing a ratio of terms of the series (II.21) above we see that

$$\lim_{n\to\infty} \left(\frac{C^{n+1}}{\sqrt[k]{(n+1)!}} \bigg/ \frac{C^n}{\sqrt[k]{n!}} \right) = \lim_{n\to\infty} \frac{C}{\sqrt[k]{n+1}} = 0.$$

Hence the proof is complete. □

Now using machinery of subsection II.2.5 we would like to introduce the following definition.

Definition II.37. *We define a map* $\mathfrak{I} : {}^{\uparrow}\mathbb{Y}^{2\mathfrak{p}} \to \mathbb{Y}^{2\mathfrak{p}}_{\underline{\mathfrak{a}}}$ *by letting for all* $t \in \mathcal{T}$ *and all* $\xi \in {}^{\uparrow}\mathbb{Y}^{2\mathfrak{p}}$

$$\mathfrak{I}(\xi)(t) := \zeta_{\underline{\mathfrak{a}}} + \int_0^t F(\xi(s))ds + \int_0^t \Phi(\xi(s))dW(s). \tag{II.22}$$

Theorem II.38. *Map* $\mathfrak{I}$ *from the Definition II.37 is Ovsjannikov. That is* $\mathfrak{I} \in \mathcal{O}(\mathbb{Y}^{2\mathfrak{p}}, \mathfrak{q})$.

Proof. Fix $\alpha < \beta \in \mathcal{A}$ and processes $\xi, \eta \in \mathbb{Y}^{2\mathfrak{p}}_{\alpha}$. We now check that the integral map $\mathfrak{I}$ satisfies the Definition II.4. We begin by showing that $\mathfrak{I}\Big|_{\mathbb{Y}^{2\mathfrak{p}}_{\alpha}} : \mathbb{Y}^{2\mathfrak{p}}_{\alpha} \to \mathbb{Y}^{2\mathfrak{p}}_{\beta}$. To this end we recall from subsection II.2.5 the following information:

(1) $\zeta_{\underline{\mathfrak{a}}} \in \mathbb{X}_{\underline{\mathfrak{a}}}$,

(2) $F(\xi)$ is in $\mathcal{L}^{2\mathfrak{p}}(\mathbf{MP}, \mathbf{M}^{\mathbb{X}_\beta})$,

(3) $\Phi(\xi)$ is in $\mathcal{N}^{\beta}_{W}$ and $\mathbb{E}\left[\int_0^T \|\Phi(\xi(s))\|^2_{\mathbb{H}_\beta} ds\right] < \infty$.

Now using this information, we conclude via an application of Theorem IV.18, IV.23 and IV.27, that $\mathfrak{I}(\xi) \in \mathcal{S}(\mathbb{X}_\beta)$ and $\|\mathfrak{I}(\xi)\|_{\mathbb{Y}^{2\mathfrak{p}}_{\beta}} < \infty$. Moreover using Theorem IV.23 we see that $\mathfrak{I}(\xi)$ is continuous and $\mathbb{F}$ adapted process. Hence we conclude by Theorem II.18 that $\mathfrak{I}(\xi)$ is progressively measurable. Now, from the collection of all preceding arguments we can conclude that $\mathfrak{I}\Big|_{\mathbb{Y}^{2\mathfrak{p}}_{\alpha}} : \mathbb{Y}^{2\mathfrak{p}}_{\alpha} \to \mathbb{Y}^{2\mathfrak{p}}_{\beta}$ establishing condition (1) of the Definition II.4.

Let us now show that condition (2) of the Definition II.4 also holds. We begin by defining the following maps

$$\left.\begin{aligned} \bar{F}(t) &:= F(\xi(t)) - F(\eta(t)) \\ \bar{\Phi}(t) &:= \Phi(\xi(t)) - \Phi(\eta(t)) \\ \bar{\mathfrak{I}}(t) &:= ||\mathfrak{I}(\xi)(t) - \mathfrak{I}(\eta)(t)||^{2\mathfrak{p}}_{\mathbb{X}_\beta} \end{aligned}\right\} \forall (t \in \mathcal{T}) \tag{II.23}$$

and establishing the following inequality for all $t \in \mathcal{T}$.

$$\begin{aligned} \mathbb{E}\Big[||\mathfrak{I}(\xi)(t) - \mathfrak{I}(\eta)(t)||^{2\mathfrak{p}}_{\mathbb{X}_\beta}\Big] &= \mathbb{E}\Big[\Big\| \int_0^t \bar{F}(s)ds + \int_0^t \bar{\Phi}(s)dW(s)\Big\|^{2\mathfrak{p}}_{\mathbb{X}_\beta}\Big] \\ &\leq 2^{\mathfrak{p}-1}\mathbb{E}\Big[\Big\| \int_0^t \bar{F}(s)ds\Big\|^{2\mathfrak{p}}_{\mathbb{X}_\beta}\Big] + 2^{\mathfrak{p}-1}\mathbb{E}\Big[\Big\| \int_0^t \bar{\Phi}(s)dW(s)\Big\|^{2\mathfrak{p}}_{\mathbb{X}_\beta}\Big]. \end{aligned} \tag{II.24}$$

For the rest of this proof let us now fix some $t \in \mathcal{T}$. From inequality (II.24) above we get

$$\mathbb{E}\Big[||\mathcal{I}(\xi)(t) - \mathcal{I}(\eta)(t)||^{2\mathfrak{p}}_{\mathbb{X}_\beta}\Big] \leq 2^{\mathfrak{p}}\mathbb{E}\Big[\Big(\int_0^t \|\bar{F}(s)\|_{\mathbb{X}_\beta} ds\Big)^{2\mathfrak{p}}\Big] + 2^{\mathfrak{p}}\mathbb{E}\Big[\Big\|\int_0^t \bar{\Phi}(s)dW(s)\Big\|^{2\mathfrak{p}}_{\mathbb{X}_\beta}\Big]. \quad \text{(II.25)}$$

Now using Hölder Inequality IV.10 we see that

$$\Big(\int_0^t ||\bar{F}(s)||_{\mathbb{X}_\beta} ds\Big)^{2\mathfrak{p}} \leq t^{2\mathfrak{p}-1}\int_0^t ||\bar{F}(s)||^{2\mathfrak{p}}_{\mathbb{X}_\beta} ds.$$

Moreover, using Theorem IV.21 and IV.27 we see that (see subsection IV.5.2 for details)

$$\mathbb{E}\Big[\Big\|\int_0^t \bar{\Phi}(s)dW(s)\Big\|^{2\mathfrak{p}}_{\mathbb{X}_\beta}\Big] \leq \Big(\frac{2\bar{\mathfrak{p}}^3}{\bar{\mathfrak{p}}-1}\Big)^{\mathfrak{p}} T^{\frac{\mathfrak{p}-1}{\mathfrak{p}}} \int_0^t \mathbb{E}\Big[\|\bar{\Phi}(s)\|^{2\mathfrak{p}}_{\mathbb{H}_\beta}\Big] ds,$$

where $\bar{\mathfrak{p}} > \mathfrak{p}$. Therefore letting

$$\bar{L}_1 \equiv \bar{L}_1(T, \mathfrak{p}) := \Big(\frac{\bar{\mathfrak{p}}^3}{\bar{\mathfrak{p}}-1}\Big)^{\mathfrak{p}} T^{\frac{\mathfrak{p}-1}{\mathfrak{p}}}$$

and using Fubini Theorem IV.18 together with inequality (II.25) above we see that

$$\mathbb{E}\Big[||\mathcal{I}(\xi)(t) - \mathcal{I}(\eta)(t)||^{2\mathfrak{p}}_{\mathbb{X}_\beta}\Big] \leq 4^{\mathfrak{p}}T^{2\mathfrak{p}-1}\int_0^t \mathbb{E}\Big[||\bar{F}(s)||^{2\mathfrak{p}}_{\mathbb{X}_\beta}\Big] ds + 4^{\mathfrak{p}}\bar{L}_1 \int_0^t \mathbb{E}\Big[\|\bar{\Phi}(s)\|^{2\mathfrak{p}}_{\mathbb{H}_\beta}\Big] ds. \quad \text{(II.26)}$$

Moreover, combining the definition (II.23) together with the fact that $\Phi \in \mathcal{O}(\mathbb{X}, \mathbb{H}, \mathfrak{q})$ and $F \in \mathcal{O}(\mathbb{X}, \mathfrak{q})$ we see that

$$\left.\begin{aligned} ||\bar{F}(s)||^{2\mathfrak{p}}_{\mathbb{X}_\beta} &\leq \Big(\tfrac{L}{(\beta-\alpha)^{\mathfrak{q}}}\Big)^{2\mathfrak{p}} ||\xi(s) - \eta(s)||^{2\mathfrak{p}}_{\mathbb{X}_\alpha} \\ \|\bar{\Phi}(s)\|^{2\mathfrak{p}}_{\mathbb{H}_\beta} &\leq \Big(\tfrac{L}{(\beta-\alpha)^{\mathfrak{q}}}\Big)^{2\mathfrak{p}} ||\xi(s) - \eta(s)||^{2\mathfrak{p}}_{\mathbb{X}_\alpha} \end{aligned}\right\} \forall (s \in [0, t]). \quad \text{(II.27)}$$

Therefore, returning to inequality (II.26) above we see that

$$\mathbb{E}[\bar{\mathcal{I}}(t)] \leq (4^{\mathfrak{p}}T^{2\mathfrak{p}} + 4^{\mathfrak{p}}T\bar{L}_1)\Big(\frac{L}{(\beta-\alpha)^{\mathfrak{q}}}\Big)^{2\mathfrak{p}} \sup\Big\{\mathbb{E}\Big[||\xi(t) - \eta(t)||^{2\mathfrak{p}}_{\mathbb{X}_\alpha}\Big] : t \in \mathcal{T}\Big\}. \quad \text{(II.28)}$$

Now letting

$$\bar{L} := L(4^{\mathfrak{p}} + 4^{\mathfrak{p}}T^{1-2\mathfrak{p}}\bar{L}_1)^{\frac{1}{2\mathfrak{p}}}, \quad \text{(II.29)}$$

we finally see that

$$||\mathcal{J}(\xi) - \mathcal{J}(\eta)||_{\mathbb{Y}^{2\mathfrak{p}}_{\beta}} \leq \frac{\bar{L}T}{(\beta-\alpha)^{\mathfrak{q}}} ||\xi - \eta||_{\mathbb{Y}^{2\mathfrak{p}}_{\alpha}} \tag{II.30}$$

and the proof is complete. □

Remark II.39. *At this point we would like to point out that although $\bar{L}$ depends on T we also have the following relation*

$$\lim_{T\to\infty} \bar{L} = 2L.$$

Using Theorem II.38 above we are now in position to define something that we will be called an iterated or a composite map. That is for all $n \in \mathbb{N}$ we define

$$\mathcal{J}^n := \overbrace{\mathcal{J} \circ \mathcal{J} \circ \cdots \circ \mathcal{J}}^{n \text{ times}}, \tag{II.31}$$

and let $\mathcal{J}^0$ be the identity map from $\mathbb{Y}^{2\mathfrak{p}}_{\underline{\mathfrak{a}}}$ to $\mathbb{Y}^{2\mathfrak{p}}_{\underline{\mathfrak{a}}}$. Suppose that $n \in \mathbb{N}_0$, our next theorem provides a useful representation of the iterated map $\mathcal{J}^n$. Precisely speaking we have the following result.

Theorem II.40. *For all $n \in \mathbb{N}_0$*

$$\mathcal{J}^n : \mathbb{Y}^{2\mathfrak{p}}_{\underline{\mathfrak{a}}} \to {\downarrow}\mathbb{Y}^{2\mathfrak{p}}. \tag{II.32}$$

Proof. We prove this statement by induction. For $n = 0$ the statement (II.32) is trivially true since Theorem II.29 established that $\mathbb{Y}^{2\mathfrak{p}}$ is a scale hence $\mathbb{Y}^{2\mathfrak{p}}_{\underline{\mathfrak{a}}} \subset {\downarrow}\mathbb{Y}^{2\mathfrak{p}}$. Now suppose that induction hypothesis holds for some $n \geq 0$. Fix arbitrary $\mathfrak{a} \in (\underline{\mathfrak{a}}, \overline{\mathfrak{a}})$ and $\delta \in (\underline{\mathfrak{a}}, \mathfrak{a})$. Observe that induction hypothesis implies that $\mathcal{J}^n : \mathbb{Y}^{2\mathfrak{p}}_{\underline{\mathfrak{a}}} \to \mathbb{Y}^{2\mathfrak{p}}_{\delta}$. However because $\mathcal{J} \in \mathcal{O}(\mathbb{Y}^{2\mathfrak{p}}, \mathfrak{q})$ we know that

$$\mathcal{J}\Big|_{\mathbb{Y}^{2\mathfrak{p}}_{\delta}} : \mathbb{Y}^{2\mathfrak{p}}_{\delta} \to \mathbb{Y}^{2\mathfrak{p}}_{\mathfrak{a}}$$

hence by composition $\mathcal{J} \circ \mathcal{J}^n \equiv \mathcal{J}^{n+1}$ and so it follows that $\mathcal{J}^{n+1} : \mathbb{Y}^{2\mathfrak{p}}_{\underline{\mathfrak{a}}} \to \mathbb{Y}^{2\mathfrak{p}}_{\mathfrak{a}}$. Finally because $\mathfrak{a} \in (\underline{\mathfrak{a}}, \overline{\mathfrak{a}})$ was arbitrary we see that $\mathcal{J}^{n+1} : \mathbb{Y}^{2\mathfrak{p}}_{\underline{\mathfrak{a}}} \to {\downarrow}\mathbb{Y}^{2\mathfrak{p}}$ and the proof is complete. □

Remark II.41. *Observe that Theorem II.40 shows that if* $\xi \in \mathbb{Y}^{2\mathfrak{p}}_{\underline{\mathfrak{a}}}$ *then the sequence* $\{\mathcal{J}^n(\xi)\}_{n=0}^{\infty}$ *belongs to* $\mathbb{Y}^{2\mathfrak{p}}_{\mathfrak{a}}$ *for all* $\mathfrak{a} \in (\underline{\mathfrak{a}}, \overline{\mathfrak{a}})$.

II.3.2 Discussion

In this subsection we would like to bring to light an important observation. However, before we proceed let us pause for a moment to fix the following constants:

$$t_0 \in \mathcal{T},$$

$$\alpha < \beta \in (\underline{\mathfrak{a}}, \overline{\mathfrak{a}}).$$

Moreover let temporary consider a fixed stochastic process $\xi \in \mathbb{Y}^{2\mathfrak{p}}_{\underline{\mathfrak{a}}}$.

<u>**Observation**</u>

Let us consider here an arbitrary $n \in \mathbb{N}$ and a partition $\{\psi_i\}_{i=0}^{n}$ of $[\alpha, \beta]$ into n intervals of equal length. That is $\psi_0 = \alpha$, $\psi_n = \beta$ and $\psi_{i+1} - \psi_i = \frac{\beta-\alpha}{n}$ for all $0 \leq i \leq n-1$. Moreover let us fix in place the following temporary notation:

$$K_n^{n+1}(t) := \mathcal{J}^n(\xi)(t) - \mathcal{J}^{n+1}(\xi)(t) \ \forall (t \in [0, t_0]),$$

$$\mathbf{N} := 4^{\mathfrak{p}} T^{2\mathfrak{p}-1} + 4^{\mathfrak{p}} \bar{L}_1,$$

$$\star := \mathbb{E}\Big[||K_n^{n+1}(t)||^{2\mathfrak{p}}_{\mathbb{X}_{\psi_n}} \Big]$$

and deduce from the definition (II.29) of $\bar{L}$ that

$$\mathbf{N} = \left(\frac{\bar{L}}{L} \right)^{2\mathfrak{p}} T^{2\mathfrak{p}-1}.$$

Now combining Theorem II.38 with inequality (II.26) and (II.27) we see that

$$\star \leq \mathbf{N} \left(\frac{L}{(\psi_n - \psi_{n-1})^{\mathfrak{q}}} \right)^{2\mathfrak{p}} \int_0^{t_0} \mathbb{E}\Big[||K_{n-1}^{n}(t_1)||^{2\mathfrak{p}}_{\mathbb{X}_{\psi_{n-1}}} \Big] dt_1. \tag{II.33}$$

Expanding inequality (II.33) above further we see that

$$\begin{aligned}
\star &\leq \mathbf{N}^2\left(\frac{L}{(\psi_n-\psi_{n-1})^{\mathfrak{q}}}\frac{L}{(\psi_{n-1}-\psi_{n-2})^{\mathfrak{q}}}\right)^{2\mathfrak{p}}\int_0^{t_0}\int_0^{t_1}\mathbb{E}\Big[||K_{n-2}^{n-1}(t_2)||^{2\mathfrak{p}}_{\mathbb{X}_{\psi_{n-2}}}\Big]dt_2dt_1,\\
&\leq \mathbf{N}^n\left(\frac{L}{(\psi_n-\psi_{n-1})^{\mathfrak{q}}}\cdots\frac{L}{(\psi_1-\psi_0)^{\mathfrak{q}}}\right)^{2\mathfrak{p}}\int_0^{t_0}\cdots\int_0^{t_{n-1}}\mathbb{E}\Big[||K_0^1(t_n)||^{2\mathfrak{p}}_{\mathbb{X}_{\psi_0}}\Big]dt_n\cdots dt_1,\\
&\leq \mathbf{N}^n\left(\frac{L}{(\psi_n-\psi_{n-1})^{\mathfrak{q}}}\cdots\frac{L}{(\psi_1-\psi_0)^{\mathfrak{q}}}\right)^{2\mathfrak{p}}\Big\|K_0^1\Big\|^{2\mathfrak{p}}_{\mathbb{Y}^{2\mathfrak{p}}_{\psi_0}}\int_0^{t_0}\cdots\int_0^{t_{n-1}}dt_n\cdots dt_1,\\
&\leq \mathbf{N}^n\left(\frac{L}{(\psi_n-\psi_{n-1})^{\mathfrak{q}}}\cdots\frac{L}{(\psi_1-\psi_0)^{\mathfrak{q}}}\right)^{2\mathfrak{p}}\Big\|K_0^1\Big\|^{2\mathfrak{p}}_{\mathbb{Y}^{2\mathfrak{p}}_{\psi_0}}\frac{T^n}{n!} \qquad \text{(II.34)}
\end{aligned}$$

Now observing that the following relation holds

$$\mathbf{N}^nT^n=\left(\frac{\bar{L}^nT^n}{L^n}\right)^{2\mathfrak{p}}$$

we can, using the fact that $\psi_0=\alpha$, $\psi_n=\beta$, therefore establish the following inequality

$$\begin{aligned}
\|K_n^{n+1}\|_{\mathbb{Y}^{2\mathfrak{p}}_{\beta}} &\leq \frac{\bar{L}^nT^n}{(\psi_n-\psi_{n-1})^{\mathfrak{q}}\cdots(\psi_1-\psi_0)^{\mathfrak{q}}}\frac{1}{\sqrt[2\mathfrak{p}]{n!}}\|K_0^1\|_{\mathbb{Y}^{2\mathfrak{p}}_{\alpha}},\\
&\leq \frac{\bar{L}^nT^n}{(\beta-\alpha)^{\mathfrak{q}n}}\frac{n^{\mathfrak{q}n}}{\sqrt[2\mathfrak{p}]{n!}}\|K_0^1\|_{\mathbb{Y}^{2\mathfrak{p}}_{\alpha}}. \qquad \text{(II.35)}
\end{aligned}$$

II.3.3 A Cauchy Type Estimate

Let us start this subsection by defining recursively maps $\mathcal{K}^n:\mathcal{L}^1(\mathbf{M},\mathbf{M}^{\mathbb{R}})\to\mathcal{L}^1(\mathbf{M},\mathbf{M}^{\mathbb{R}})$, for all $n\in\mathbb{N}_0$, via the following formula

$$\mathcal{K}^n(t,f):=\begin{cases} f(t) & n=0\\ \int_0^t f(s)ds & n=1\\ \int_0^t \mathcal{K}^{n-1}(s,f)ds & n>1.\end{cases} \qquad \text{(II.36)}$$

Now using inequality (II.35) established earlier we can see that in addition the following theorem can also be formulated and proved.

Theorem II.42. *Suppose $\alpha < \beta \in (\underline{\mathfrak{a}}, \overline{\mathfrak{a}})$ and $\xi, \eta \in \mathbb{Y}^{2\mathfrak{p}}_{\underline{\mathfrak{a}}}$. Then for all $n \in \mathbb{N}$*

$$||\mathcal{J}^n(\xi) - \mathcal{J}^{n+1}(\eta)||_{\mathbb{Y}^{2\mathfrak{p}}_\beta} \leq \frac{\bar{L}^n T^n}{(\beta-\alpha)^{\mathfrak{q}n}} \frac{n^{\mathfrak{q}n}}{\sqrt[2\mathfrak{p}]{n!}} ||\xi - \mathcal{I}(\eta)||_{\mathbb{Y}^{2\mathfrak{p}}_\alpha}. \tag{II.37}$$

Proof. Fixing $t \in \mathcal{T}$ we prove by induction that

$$\mathbb{E}\Big[||\mathcal{J}^n(\xi)(t) - \mathcal{J}^{n+1}(\eta)(t)||^{2\mathfrak{p}}_{\mathbb{X}_\beta}\Big] \leq \mathbf{N}^n \left(\frac{L^n n^{\mathfrak{q}n}}{(\beta-\alpha)^{\mathfrak{q}n}}\right)^{2\mathfrak{p}} \mathcal{K}^n\Big(t, \mathbb{E}\Big[||\xi - \mathcal{I}(\eta)||^{2\mathfrak{p}}_{\mathbb{X}_\alpha}\Big]\Big)$$

from where the inequality (II.37) follows at once because by the definition of map $\mathcal{K}^n$ we can see that the following relation holds

$$\mathcal{K}^n\Big(t, \mathbb{E}\Big[||\xi - \mathcal{I}(\eta)||^{2\mathfrak{p}}_{\mathbb{X}_\alpha}\Big]\Big) = \int_0^t \int_0^{t_1} \cdots \int_0^{t_{n-1}} \mathbb{E}\Big[||\xi(t_n) - \mathcal{I}(\eta)(t_n)||^{2\mathfrak{p}}_{\mathbb{X}_\alpha}\Big] dt_n dt_{n-1} \cdots dt,$$

$$\leq ||\xi - \mathcal{I}(\eta)||^{2\mathfrak{p}}_{\mathbb{Y}^{2\mathfrak{p}}_\alpha} \int_0^t \int_0^{t_1} \cdots \int_0^{t_{n-1}} dt_n dt_{n-1} \cdots dt.$$

Now back to the proof. Clearly the case $n = 1$ follows immediately from the Theorem II.38. Precisely speaking a passage from an inequality (II.26) to an inequality (II.28) shows that

$$\mathbb{E}\Big[||\mathcal{I}(\xi)(t) - \mathcal{I}^2(\eta)(t)||^{2\mathfrak{p}}_{\mathbb{X}_\beta}\Big] \leq (4^{\mathfrak{p}} T^{2\mathfrak{p}} + 4^{\mathfrak{p}} T \bar{L}_1)\left(\frac{L}{(\beta-\alpha)^{\mathfrak{q}}}\right)^{2\mathfrak{p}} \int_0^t \mathbb{E}\Big[||\xi(s) - \mathcal{I}(\eta)(s)||^{2\mathfrak{p}}_{\mathbb{X}_\alpha}\Big] ds$$

$$= \mathbf{N}\left(\frac{L}{(\beta-\alpha)^{\mathfrak{q}}}\right)^{2\mathfrak{p}} \mathcal{K}^1\Big(t, \mathbb{E}\Big[||\xi - \mathcal{I}(\eta)||^{2\mathfrak{p}}_{\mathbb{X}_\alpha}\Big]\Big).$$

Now, suppose that the induction hypothesis holds for some $n \geq 1$. Choosing $\psi \in (\alpha, \beta)$ such that $\beta - \psi = \frac{\beta-\alpha}{n+1}$ we see, using Theorem II.38, that

$$\mathbb{E}\Big[||\mathcal{J}^{n+1}(\xi)(t) - \mathcal{J}^{n+2}(\eta)(t)||^{2\mathfrak{p}}_{\mathbb{X}_\beta}\Big] \leq \mathbf{N}\left(\frac{L}{(\beta-\psi)^{\mathfrak{q}}}\right)^{2\mathfrak{p}} \int_0^t \mathbb{E}\Big[||\mathcal{J}^n(\xi)(s) - \mathcal{J}^{n+1}(\eta)(s)||^{2\mathfrak{p}}_{\mathbb{X}_\psi}\Big] ds.$$

Hence letting

$$\mathbf{A} := \mathbb{E}\Big[||\xi - \mathcal{I}(\eta)||^{2\mathfrak{p}}_{\mathbb{X}_\alpha}\Big]$$

and applying the induction hypothesis we get

$$\mathbb{E}\Big[||\mathcal{I}(\xi)(t)^{n+1} - \mathcal{I}^{n+2}(\eta)(t)||^{2\mathfrak{p}}_{\mathbb{X}_\beta}\Big] \leq \mathbf{N}\left(\frac{L}{(\beta-\psi)^{\mathfrak{q}}}\right)^{2\mathfrak{p}} \mathbf{N}^n\left(\frac{L^n n^{\mathfrak{q}n}}{(\psi-\alpha)^{\mathfrak{q}n}}\right)^{2\mathfrak{p}} \int_0^t \mathcal{K}^n(s,\mathbf{A})ds$$

$$\leq \mathbf{N}^{n+1}\left(\frac{L^{n+1}n^{\mathfrak{q}n}}{(\beta-\psi)^{\mathfrak{q}}(\psi-\alpha)^{\mathfrak{q}n}}\right)^{2\mathfrak{p}} \mathcal{K}^{n+1}(s,\mathbf{A}).$$

Moreover we see that

$$\frac{L^{n+1}}{(\beta-\psi)^{\mathfrak{q}}(\psi-\alpha)^{\mathfrak{q}n}} n^{\mathfrak{q}n} = L^{n+1}\left(\frac{\beta-\alpha}{n+1}\right)^{-\mathfrak{q}}\left(\frac{n(\beta-\alpha)}{n+1}\right)^{-\mathfrak{q}n} n^{\mathfrak{q}n}$$

$$= \frac{L^{n+1}}{(\beta-\alpha)^{\mathfrak{q}(n+1)}} \frac{(n+1)^{\mathfrak{q}(n+1)}}{n^{\mathfrak{q}n}} n^{\mathfrak{q}n}$$

$$= \frac{L^{n+1}(n+1)^{\mathfrak{q}(n+1)}}{(\beta-\alpha)^{\mathfrak{q}(n+1)}}.$$

Hence we now conclude that

$$\mathbb{E}\Big[||\mathcal{I}(\xi)(t)^{n+1} - \mathcal{I}^{n+2}(\eta)(t)||^{2\mathfrak{p}}_{\mathbb{X}_\beta}\Big] \leq \mathbf{N}^{n+1}\left(\frac{L^{n+1}(n+1)^{\mathfrak{q}(n+1)}}{(\beta-\alpha)^{\mathfrak{q}(n+1)}}\right)^{2\mathfrak{p}} \mathcal{K}^{n+1}(s,\mathbf{A})$$

and the proof is complete. □

Remark II.43. *It is clear from the definition of the composite map $\mathcal{I}^n$ that the Theorem II.42 is trivially true for $n = 0$. Moreover it is essential that $\alpha \in (\underline{\mathfrak{a}}, \overline{\mathfrak{a}})$ because it is possible that $\mathcal{I}(\eta)$ does not belong to $\mathbb{Y}^{2\mathfrak{p}}_{\underline{\mathfrak{a}}}$.*

II.4 Existence and Uniqueness

We now prove an important result which will immediately allow us to establish Theorem II.33.

Theorem II.44. *There exists a unique element* $\phi \in_{\downarrow} \mathbb{Y}^{2\mathfrak{p}}$ *such that* $\mathcal{I}(\phi) \approx \phi$*. Moreover if* $\mathfrak{a} \in (\underline{\mathfrak{a}}, \overline{\mathfrak{a}})$ *and* $\xi \in \mathbb{Y}^{2\mathfrak{p}}_{\underline{\mathfrak{a}}}$ *then*

$$\overbrace{\lim_{n\to\infty} \mathcal{I}^n(\xi)}^{in\ \mathbb{Y}^{2\mathfrak{p}}_{\mathfrak{a}}} \approx \phi.$$

Proof. Fix $\xi \in \mathbb{Y}^{2\mathfrak{p}}_{\underline{\mathfrak{a}}}$ and $\mathfrak{a} \in (\underline{\mathfrak{a}}, \overline{\mathfrak{a}})$. Fix also an arbitrary $\gamma \in (\underline{\mathfrak{a}}, \mathfrak{a})$ and using theorem II.42 observe that for all $m \geq n \in \mathbb{N}$ we have

$$\begin{aligned}
||\mathcal{I}^n(\xi) - \mathcal{I}^m(\xi)||_{\mathbb{Y}^{2\mathfrak{p}}_{\mathfrak{a}}} &\leq \sum_{k=n}^{m-1} ||\mathcal{I}^k(\xi) - \mathcal{I}^{k+1}(\xi)||_{\mathbb{Y}^{2\mathfrak{p}}_{\gamma}} \\
&\leq \sum_{k=n}^{m-1} \frac{\bar{L}^k T^k}{(\mathfrak{a}-\gamma)^{\mathfrak{q}k}} \frac{k^{\mathfrak{q}k}}{\sqrt[2\mathfrak{p}]{k!}}\ ||\xi - \mathcal{I}(\xi)||_{\mathbb{Y}^{2\mathfrak{p}}_{\gamma}} \\
&\leq \sum_{k=n}^{\infty} \frac{\bar{L}^k T^k}{(\mathfrak{a}-\gamma)^{\mathfrak{q}k}} \frac{k^{\mathfrak{q}k}}{\sqrt[2\mathfrak{p}]{k!}}\ ||\xi - \mathcal{I}(\xi)||_{\mathbb{Y}^{2\mathfrak{p}}_{\gamma}}. \qquad \text{(II.38)}
\end{aligned}$$

Since $\mathfrak{q} \in (0, \frac{1}{2\mathfrak{p}})$ we see from the Theorem II.35 that the right hand side of inequality (II.38) above is a remainder of a convergent series. Therefore we conclude that the sequence $\{\mathcal{I}^n(\xi)\}_{n\in\mathbb{N}}$ is Cauchy in $\mathbb{Y}^{2\mathfrak{p}}_{\mathfrak{a}}$. Since $\mathfrak{a}$ is arbitrary, let us now consider $\alpha < \beta \in (\underline{\mathfrak{a}}, \overline{\mathfrak{a}})$ and the following processes:

$$\phi_\alpha := \overbrace{\lim_{n\to\infty} \mathcal{I}^n(\xi)}^{\text{in } \mathbb{Y}^{2\mathfrak{p}}_{\alpha}},$$
$$\phi_\beta := \overbrace{\lim_{n\to\infty} \mathcal{I}^n(\xi)}^{\text{in } \mathbb{Y}^{2\mathfrak{p}}_{\beta}}.$$

Because $\mathbb{Y}^{2\mathfrak{p}}_{\alpha} \prec \mathbb{Y}^{2\mathfrak{p}}_{\beta}$ we see that for all $n \in \mathbb{N}$ we have

$$\begin{aligned}
\|\phi_\beta - \phi_\alpha\|_{\mathbb{Y}^{2\mathfrak{p}}_{\beta}} &\leq \|\phi_\beta - \mathcal{I}^n(\xi)\|_{\mathbb{Y}^{2\mathfrak{p}}_{\beta}} + \|\mathcal{I}^n(\xi) - \phi_\alpha\|_{\mathbb{Y}^{2\mathfrak{p}}_{\beta}} \\
&\leq \|\phi_\beta - \mathcal{I}^n(\xi)\|_{\mathbb{Y}^{2\mathfrak{p}}_{\beta}} + \|\mathcal{I}^n(\xi) - \phi_\alpha\|_{\mathbb{Y}^{2\mathfrak{p}}_{\alpha}}
\end{aligned}$$

which shows that

$$\|\phi_\beta - \phi_\alpha\|_{\mathbb{Y}_\beta^{2\mathfrak{p}}} = 0. \tag{II.39}$$

Therefore, from equation (II.39) above we now see that $\phi_\beta \approx \phi_\alpha$. Hence letting

$$\phi_\alpha =: \phi := \phi_\beta,$$

we conclude that $\phi \in_{\downarrow} \mathbb{Y}^{2\mathfrak{p}}$ and

$$\overbrace{\lim_{n\to\infty} \mathcal{J}^n(\xi)}^{\text{in } \mathbb{Y}_{\mathfrak{a}}^{2\mathfrak{p}}} \approx \phi.$$

Now, from Theorem II.38 it follows that $\mathcal{J}$ is a continuous map from $\mathbb{Y}_\gamma^{2\mathfrak{p}}$ into $\mathbb{Y}_{\mathfrak{a}}^{2\mathfrak{p}}$. Moreover we have just established that $\phi \in \mathbb{Y}_\gamma^{2\mathfrak{p}}$ hence we see that the following is true

$$\overbrace{\lim_{n\to\infty} \mathcal{J}^{n+1}(\xi)}^{\text{in } \mathbb{Y}_{\mathfrak{a}}^{2\mathfrak{p}}} \approx \phi,$$

$$\overbrace{\lim_{n\to\infty} \mathcal{J}^{n+1}(\xi)}^{\text{in } \mathbb{Y}_{\mathfrak{a}}^{2\mathfrak{p}}} = \mathcal{J}\left(\overbrace{\lim_{n\to\infty} \mathcal{J}^{n}(\xi)}^{\text{in } \mathbb{Y}_{\gamma}^{2\mathfrak{p}}} \right) \approx \mathcal{J}(\phi),$$

which shows that $\mathcal{J}(\phi) \approx \phi$.

Finally suppose that there exists another $\psi \in_{\downarrow} \mathbb{Y}^{2\mathfrak{p}}$ such that $\mathcal{J}(\psi) \approx \psi$. In this case it is clear that the following equality holds

$$||\mathcal{J}^n(\phi) - \mathcal{J}^{n+1}(\psi)||_{\mathbb{Y}_{\mathfrak{a}}^{2\mathfrak{p}}} = ||\phi - \psi||_{\mathbb{Y}_{\mathfrak{a}}^{2\mathfrak{p}}}.$$

However from the Theorem II.42 we can also infer that

$$\begin{aligned} ||\mathcal{J}^n(\phi) - \mathcal{J}^{n+1}(\psi)||_{\mathbb{Y}_{\mathfrak{a}}^{2\mathfrak{p}}} &\leq \frac{\bar{L}^n T^n}{(\mathfrak{a}-\gamma)^{\mathfrak{q}n}} \frac{n^{\mathfrak{q}n}}{\sqrt[2\mathfrak{p}]{n!}} ||\phi - \mathcal{J}(\psi)||_{\mathbb{Y}_{\gamma}^{2\mathfrak{p}}} \\ &= \frac{\bar{L}^n T^n}{(\mathfrak{a}-\gamma)^{\mathfrak{q}n}} \frac{n^{\mathfrak{q}n}}{\sqrt[2\mathfrak{p}]{n!}} ||\phi - \psi||_{\mathbb{Y}_{\gamma}^{2\mathfrak{p}}}. \end{aligned} \tag{II.40}$$

Since $\mathfrak{q} \in (0, \frac{1}{2\mathfrak{p}})$, by Theorem II.35, the right hand side of inequality (II.40) above tends to zero hence we conclude that $||\phi - \psi||_{\mathbb{Y}_{\mathfrak{a}}^{2\mathfrak{p}}} = 0$. Therefore we now see that $\phi \approx \psi$, which shows that ϕ is unique (see Remark II.27 and II.34) hence the proof is complete. □

We are now in a position to prove the main existence Theorem of this section. That is we prove Theorem II.33 that was outlined earlier in subsection II.2.5.

Theorem II.45. *For all $\zeta_{\underline{\mathfrak{a}}} \in \mathbb{X}_{\underline{\mathfrak{a}}}$ there exists a unique strong solution of the stochastic integral equation (II.19).*

Proof. Let ϕ be the process found by the Theorem II.44 above. Then we see that

$$\phi \in_{\downarrow} \mathbb{Y}^{2\mathfrak{p}}$$

and

$$\phi \approx \zeta_{\underline{\mathfrak{a}}} + \int_0^{\cdot} F(\phi(s))ds + \int_0^{\cdot} \Phi(\phi(s))dW(s).$$

Therefore we can conclude that ϕ satisfies the Definition II.32 of a strong solution and the proof is complete. □

Remark II.46. *From Theorem IV.2 we can in fact deduce now that $\phi \in_{\downarrow} \mathbb{Y}^{\mathfrak{p}}$.*

Corollary II.47. *Let ξ be a unique strong solution of the stochastic integral equation (II.19). Moreover let $\mathfrak{a} \in (\underline{\mathfrak{a}}, \overline{\mathfrak{a}})$ and $\eta \in \mathbb{Y}_{\underline{\mathfrak{a}}}^{2\mathfrak{p}}$. Then*

$$\overbrace{\lim_{n\to\infty} \mathcal{J}^n(\eta)}^{in\ \mathbb{Y}_{\mathfrak{a}}^{2\mathfrak{p}}} \approx \xi$$

$$\overbrace{\lim_{n\to\infty} \mathcal{J}^n(\eta)}^{in\ \mathbb{Y}_{\mathfrak{a}}^{\mathfrak{p}}} \approx \xi.$$

Proof. From Theorem II.44 above we deduce immediately that $\xi \approx \phi$ and so using Theorem II.44 once again we see that

$$\overbrace{\lim_{n\to\infty} \mathcal{J}^n(\eta)}^{\text{in } \mathbb{Y}_{\mathfrak{a}}^{2\mathfrak{p}}} \approx \phi \approx \xi.$$

Finally using in addition Theorem IV.2 we can deduce that

$$\|\mathcal{J}^n(\eta) - \xi\|_{\mathbb{Y}_{\mathfrak{a}}^{\mathfrak{p}}} = \|\mathcal{J}^n(\eta) - \phi\|_{\mathbb{Y}_{\mathfrak{a}}^{\mathfrak{p}}}$$

$$\leq \|\mathcal{J}^n(\eta) - \phi\|_{\mathbb{Y}_{\mathfrak{a}}^{2\mathfrak{p}}}$$

which completes the proof. □

Finally we note that the dependence of Theorem II.45 above on $\zeta_{\underline{\mathfrak{a}}} \in \mathbb{X}_{\underline{\mathfrak{a}}}$ is illusory because we can always reduce the index set $[\underline{\mathfrak{a}}, \overline{\mathfrak{a}}]$ for our scales to accommodate for almost (i.e. except for the final space) any choice of the initial condition. In other words our work shows that the following result is true.

Theorem II.48. *For all $\alpha \in (\underline{\mathfrak{a}}, \overline{\mathfrak{a}})$ and $\zeta_\alpha \in \mathbb{X}_\alpha$ there exists a stochastic process γ such that*

$$\gamma \in \bigcap_{\mathfrak{a}\in(\alpha,\overline{\mathfrak{a}})} \mathbb{Y}_{\mathfrak{a}}^{\mathfrak{p}} \tag{II.41}$$

and

$$\gamma \approx \zeta_\alpha + \int_0^\cdot F(\gamma(s))ds + \int_0^\cdot \Phi(\gamma(s))dW(s). \tag{II.42}$$

Moreover if ξ is another stochastic process satisfying condition (II.41) and (II.42) above then for all $\mathfrak{a} \in (\alpha, \overline{\mathfrak{a}})$ we have

$$\|\gamma - \xi\|_{\mathbb{Y}_{\mathfrak{a}}^{\mathfrak{p}}} = 0.$$

II.5 Estimates of the Solution

II.5.1 Norm Estimates

In this subsection we will denote by ξ the unique strong solution of the equation (II.19) and establish, relying on Theorems II.42, II.44 ,II.45 and Corollary II.47 a number of norm related inequalities. We begin with a few preliminary observations.

<u>Observation I</u>

Repeating a calculation from subsection II.2.5 we see that for all $\alpha < \beta \in \mathcal{A}$ and all $x \in \mathbb{X}_\alpha$

$$\begin{aligned}\|F(x)\|_{\mathbb{X}_\beta} &= \|F(x) + F(0) - F(0)\|_{\mathbb{X}_\beta} \\ &\leq \|F(x) - F(0)\|_{\mathbb{X}_\beta} + \|F(0)\|_{\mathbb{X}_\beta} \\ &\leq \frac{L}{(\beta-\alpha)^{\mathfrak{q}}}\left(M + \|x\|_{\mathbb{X}_\alpha}\right)\end{aligned}$$

where

$$M := \frac{\|F(0)\|_{\mathbb{X}_{\underline{\mathfrak{a}}}}(\overline{\mathfrak{a}} - \underline{\mathfrak{a}})^{\mathfrak{q}}}{L}. \tag{II.43}$$

Hence we see that

$$\|F(x)\|^{2\mathfrak{p}}_{\mathbb{X}_\beta} \leq \left(\frac{L}{(\beta-\alpha)^{\mathfrak{q}}}\right)^{2\mathfrak{p}} 2^{2\mathfrak{p}-1}\left(M^{2\mathfrak{p}} + \|x\|^{2\mathfrak{p}}_{\mathbb{X}_\alpha}\right). \tag{II.44}$$

Moreover letting

$$N := \frac{\|\Phi(0)\|_{\mathbb{H}_{\underline{\mathfrak{a}}}}(\overline{\mathfrak{a}} - \underline{\mathfrak{a}})^{\mathfrak{q}}}{L} \tag{II.45}$$

we similarly conclude that

$$\begin{aligned}\|\Phi(x)\|_{\mathbb{H}_\beta} &\leq \frac{L}{(\beta-\alpha)^{\mathfrak{q}}}\left(N + \|x\|_{\mathbb{X}_\alpha}\right), \\ \|\Phi(x)\|^{2\mathfrak{p}}_{\mathbb{H}_\beta} &\leq \left(\frac{L}{(\beta-\alpha)^{\mathfrak{q}}}\right)^{2\mathfrak{p}} 2^{2\mathfrak{p}-1}\left(N^{2\mathfrak{p}} + \|x\|^{2\mathfrak{p}}_{\mathbb{X}_\alpha}\right).\end{aligned} \tag{II.46}$$

Observation II

Suppose now that $\alpha < \beta \in \mathcal{A}$, $t \in \mathcal{T}$ and $\zeta_{\underline{\mathfrak{a}}} \in \mathbb{X}_{\underline{\mathfrak{a}}}$. Moreover using **observation I** define

$$\mathbf{K} \coloneqq \max[M, N]. \tag{II.47}$$

Now using **observation I**, Theorem II.38 and inequality (II.26) we see that

$$\mathbb{E}\Bigg[||\zeta_{\underline{\mathfrak{a}}} - \mathcal{I}(\zeta_{\underline{\mathfrak{a}}})(t)||^{2\mathfrak{p}}_{\mathbb{X}_\beta}\Bigg] \leq 4^{\mathfrak{p}} T^{2\mathfrak{p}-1} \int_0^t \mathbb{E}\Bigg[||F(\zeta_{\underline{\mathfrak{a}}})||^{2\mathfrak{p}}_{\mathbb{X}_\beta}\Bigg] ds + 4^{\mathfrak{p}} \bar{L}_1 \int_0^t \mathbb{E}\Bigg[\|\Phi(\zeta_{\underline{\mathfrak{a}}})\|^{2\mathfrak{p}}_{\mathbb{H}_\beta}\Bigg] ds \tag{II.48}$$

$$\leq \Bigg(4^{\mathfrak{p}} T^{2\mathfrak{p}-1} + 4^{\mathfrak{p}} \bar{L}_1\Bigg)\Bigg(\frac{L}{(\beta-\alpha)^{\mathfrak{q}}}\Bigg)^{2\mathfrak{p}} \int_0^t \Bigg(\mathbf{K} + \|\zeta_{\underline{\mathfrak{a}}}\|_{\mathbb{X}_{\underline{\mathfrak{a}}}}\Bigg)^{2\mathfrak{p}} ds$$

$$= \mathbf{N}\Bigg(\frac{L}{(\beta-\alpha)^{\mathfrak{q}}}\Bigg)^{2\mathfrak{p}} \int_0^t \Bigg(\mathbf{K} + \|\zeta_{\underline{\mathfrak{a}}}\|_{\mathbb{X}_{\underline{\mathfrak{a}}}}\Bigg)^{2\mathfrak{p}} ds. \tag{II.49}$$

Observation III

Finally, Suppose $\mathfrak{a} \in (\underline{\mathfrak{a}}, \overline{\mathfrak{a}})$, $t \in \mathcal{T}$ and $\zeta_{\underline{\mathfrak{a}}} \in \mathbb{X}_{\underline{\mathfrak{a}}}$. Moreover consider a partition $\{\psi_i\}_{i=0}^{n+1}$ of $[\underline{\mathfrak{a}}, \mathfrak{a}]$ into $n+1$ intervals of equal length. That is $\psi_0 = \underline{\mathfrak{a}}$, $\psi_{n+1} = \mathfrak{a}$ and $\psi_{i+1} - \psi_i = \frac{\mathfrak{a}-\underline{\mathfrak{a}}}{n+1}$ for all $0 \leq i \leq n$. Now, from Theorem II.42 we see that for all $n \in \mathbb{N}_0$ we have

$$\mathbb{E}\Bigg[||\mathcal{I}^n(\zeta_{\underline{\mathfrak{a}}})(t) - \mathcal{I}^{n+1}(\zeta_{\underline{\mathfrak{a}}})(t)||^{2\mathfrak{p}}_{\mathbb{X}_{\mathfrak{a}}}\Bigg] \leq \mathbf{N}^n \Bigg(\frac{L^n n^{\mathfrak{q}n}}{(\mathfrak{a}-\psi_n)^{\mathfrak{q}n}}\Bigg)^{2\mathfrak{p}} \mathcal{K}^n\Bigg(t, \mathbb{E}\Bigg[||\zeta_{\underline{\mathfrak{a}}} - \mathcal{I}(\zeta_{\underline{\mathfrak{a}}})||^{2\mathfrak{p}}_{\mathbb{X}_{\psi_n}}\Bigg]\Bigg)$$

$$\leq \mathbf{N}^{n+1} \Bigg(\frac{L^n n^{\mathfrak{q}n}}{(\mathfrak{a}-\psi_n)^{\mathfrak{q}n}}\Bigg)^{2\mathfrak{p}} \Bigg(\frac{L}{(\psi_n - \underline{\mathfrak{a}})^{\mathfrak{q}}}\Bigg)^{2\mathfrak{p}} \mathcal{K}^{n+1}\Bigg(t, \Bigg(\mathbf{K} + \|\zeta_{\underline{\mathfrak{a}}}\|_{\mathbb{X}_{\underline{\mathfrak{a}}}}\Bigg)^{2\mathfrak{p}}\Bigg)$$

$$\leq \mathbf{N}^{n+1} \Bigg(\frac{L^n n^{\mathfrak{q}n}}{(\mathfrak{a}-\psi_n)^{\mathfrak{q}n}}\Bigg)^{2\mathfrak{p}} \Bigg(\frac{L}{(\psi_n - \underline{\mathfrak{a}})^{\mathfrak{q}}}\Bigg)^{2\mathfrak{p}} \frac{T^{n+1}}{(n+1)!}\Bigg(\mathbf{K} + \|\zeta_{\underline{\mathfrak{a}}}\|_{\mathbb{X}_{\underline{\mathfrak{a}}}}\Bigg)^{2\mathfrak{p}}.$$

Moreover we see that

$$\frac{L^{n+1}}{(\mathfrak{a}-\psi_n)^{\mathfrak{q}}(\psi_n - \underline{\mathfrak{a}})^{\mathfrak{q}n}} n^{\mathfrak{q}n} = L^{n+1}\Bigg(\frac{\mathfrak{a}-\underline{\mathfrak{a}}}{n+1}\Bigg)^{-\mathfrak{q}} \Bigg(\frac{n(\mathfrak{a}-\underline{\mathfrak{a}})}{n+1}\Bigg)^{-\mathfrak{q}n} n^{\mathfrak{q}n}$$

$$= \frac{L^{n+1}}{(\mathfrak{a}-\underline{\mathfrak{a}})^{\mathfrak{q}(n+1)}} \frac{(n+1)^{\mathfrak{q}(n+1)}}{n^{\mathfrak{q}n}} n^{\mathfrak{q}n}$$

$$= \frac{L^{n+1}(n+1)^{\mathfrak{q}(n+1)}}{(\mathfrak{a}-\underline{\mathfrak{a}})^{\mathfrak{q}(n+1)}}.$$

Hence we now conclude that

$$\mathbb{E}\left[||\mathcal{I}^n(\zeta_{\underline{\mathfrak{a}}})(t) - \mathcal{I}^{n+1}(\zeta_{\underline{\mathfrak{a}}})(t)||^{2\mathfrak{p}}_{\mathbb{X}_{\mathfrak{a}}}\right] \leq \mathbf{N}^{n+1}\left(\frac{L^{n+1}(n+1)^{\mathfrak{q}(n+1)}}{(\mathfrak{a}-\underline{\mathfrak{a}})^{\mathfrak{q}(n+1)}}\right)^{2\mathfrak{p}} \frac{T^{n+1}}{(n+1)!}\left(\mathbf{K} + \|\zeta_{\underline{\mathfrak{a}}}\|_{\mathbb{X}_{\underline{\mathfrak{a}}}}\right)^{2\mathfrak{p}}$$

and so we see that

$$||\mathcal{I}^n(\zeta_{\underline{\mathfrak{a}}}) - \mathcal{I}^{n+1}(\zeta_{\underline{\mathfrak{a}}})||_{\mathbb{Y}^{2\mathfrak{p}}_{\mathfrak{a}}} \leq \frac{\bar{L}^{n+1}T^{n+1}}{(\mathfrak{a}-\underline{\mathfrak{a}})^{\mathfrak{q}(n+1)}} \frac{(n+1)^{\mathfrak{q}(n+1)}}{\sqrt[2\mathfrak{p}]{(n+1)!}}\left(\mathbf{K} + \|\zeta_{\underline{\mathfrak{a}}}\|_{\mathbb{X}_{\underline{\mathfrak{a}}}}\right). \tag{II.50}$$

We now obtain a promised earlier norm estimate.

Theorem II.49. *Suppose that* $\mathfrak{a} \in (\underline{\mathfrak{a}}, \overline{\mathfrak{a}})$ *and* $\zeta_{\underline{\mathfrak{a}}} \in \mathbb{X}_{\underline{\mathfrak{a}}}$. *Moreover suppose that* ξ *is the unique strong solution of the equation (II.19). Then*

$$||\xi||_{\mathbb{Y}^{2\mathfrak{p}}_{\mathfrak{a}}} \leq \sum_{n=0}^{\infty} \frac{\bar{L}^nT^n}{(\mathfrak{a}-\underline{\mathfrak{a}})^{\mathfrak{q}n}} \frac{n^{\mathfrak{q}n}}{\sqrt[2\mathfrak{p}]{n!}}\left(\mathbf{K} + \|\zeta_{\underline{\mathfrak{a}}}\|_{\mathbb{X}_{\underline{\mathfrak{a}}}}\right). \tag{II.51}$$

Proof. We begin this proof by observing that a constant $\zeta_{\underline{\mathfrak{a}}} \in \mathbb{X}_{\underline{\mathfrak{a}}}$ considered as a process is an element of $\mathbb{Y}^{2\mathfrak{p}}_{\underline{\mathfrak{a}}}$. Therefore from Corollary II.47 we see that

$$\lim_{n\to\infty} ||\mathcal{I}^n(\zeta_{\underline{\mathfrak{a}}})||_{\mathbb{Y}^{2\mathfrak{p}}_{\mathfrak{a}}} = ||\xi||_{\mathbb{Y}^{2\mathfrak{p}}_{\mathfrak{a}}}.$$

Now using an estimate (II.50) observe that for all $n \in \mathbb{N}$ the following inequalities hold:

$$\begin{aligned}
\|\mathcal{I}^n(\zeta_{\underline{\mathfrak{a}}})\|_{\mathbb{Y}^{2\mathfrak{p}}_{\mathfrak{a}}} - \|\mathcal{I}^0(\zeta_{\underline{\mathfrak{a}}})\|_{\mathbb{Y}^{2\mathfrak{p}}_{\mathfrak{a}}} &= \sum_{k=1}^{n} \|\mathcal{I}^k(\zeta_{\underline{\mathfrak{a}}})\|_{\mathbb{Y}^{2\mathfrak{p}}_{\mathfrak{a}}} - \|\mathcal{I}^{k-1}(\zeta_{\underline{\mathfrak{a}}})\|_{\mathbb{Y}^{2\mathfrak{p}}_{\mathfrak{a}}}, \\
&\leq \sum_{k=1}^{n} \|\mathcal{I}^{k-1}(\zeta_{\underline{\mathfrak{a}}}) - \mathcal{I}^k(\zeta_{\underline{\mathfrak{a}}})\|_{\mathbb{Y}^{2\mathfrak{p}}_{\mathfrak{a}}}, \\
&\leq \sum_{k=1}^{n} \frac{\bar{L}^kT^k}{(\mathfrak{a}-\underline{\mathfrak{a}})^{\mathfrak{q}k}} \frac{k^{\mathfrak{q}k}}{\sqrt[2\mathfrak{p}]{k!}}\left(\mathbf{K} + \|\zeta_{\underline{\mathfrak{a}}}\|_{\mathbb{X}_{\underline{\mathfrak{a}}}}\right). \qquad \text{(II.52)}
\end{aligned}$$

Hence for all $n \in \mathbb{N}$ we can see that

$$\|\mathcal{I}^n(\zeta_{\underline{\mathfrak{a}}})\|_{\mathbb{Y}^{2\mathfrak{p}}_{\mathfrak{a}}} \leq \|\mathcal{I}^0(\zeta_{\underline{\mathfrak{a}}})\|_{\mathbb{Y}^{2\mathfrak{p}}_{\mathfrak{a}}} + \sum_{k=1}^{n} \frac{\bar{L}^kT^k}{(\mathfrak{a}-\underline{\mathfrak{a}})^{\mathfrak{q}k}} \frac{k^{\mathfrak{q}k}}{\sqrt[2\mathfrak{p}]{k!}}\left(\mathbf{K} + \|\zeta_{\underline{\mathfrak{a}}}\|_{\mathbb{X}_{\underline{\mathfrak{a}}}}\right). \tag{II.53}$$

Now refining inequality (II.53) further we see that

$$\begin{aligned}
\|\mathcal{J}^n(\zeta_{\underline{\mathfrak{a}}})\|_{\mathbb{Y}^{2\mathfrak{p}}_{\mathfrak{a}}} &\leq \mathbf{K} + \|\zeta_{\underline{\mathfrak{a}}}\|_{\mathbb{X}_{\underline{\mathfrak{a}}}} + \sum_{k=1}^{n} \frac{\bar{L}^k T^k}{(\mathfrak{a}-\underline{\mathfrak{a}})^{\mathfrak{q}k}} \frac{k^{\mathfrak{q}k}}{\sqrt[2\mathfrak{p}]{k!}} \Big(\mathbf{K} + \|\zeta_{\underline{\mathfrak{a}}}\|_{\mathbb{X}_{\underline{\mathfrak{a}}}} \Big) \\
&\leq \Big(1 + \sum_{k=1}^{n} \frac{\bar{L}^k T^k}{(\mathfrak{a}-\underline{\mathfrak{a}})^{\mathfrak{q}k}} \frac{k^{\mathfrak{q}k}}{\sqrt[2\mathfrak{p}]{k!}} \Big) \Big(\mathbf{K} + \|\zeta_{\underline{\mathfrak{a}}}\|_{\mathbb{X}_{\underline{\mathfrak{a}}}} \Big) \\
&= \sum_{k=0}^{n} \frac{\bar{L}^k T^k}{(\mathfrak{a}-\underline{\mathfrak{a}})^{\mathfrak{q}k}} \frac{k^{\mathfrak{q}k}}{\sqrt[2\mathfrak{p}]{k!}} \Big(\mathbf{K} + \|\zeta_{\underline{\mathfrak{a}}}\|_{\mathbb{X}_{\underline{\mathfrak{a}}}} \Big). \qquad \text{(II.54)}
\end{aligned}$$

Finally taking the limit on both sides of an inequality (II.54) above we see that an equation (II.51) holds hence the proof is complete. □

Remark II.50. *It is clear from the definition (II.43) and (II.45) that if F and Φ are linear maps then $\mathbf{K} \equiv 0$ hence in this case from the Theorem II.49 we see that for all $\mathfrak{a} \in (\underline{\mathfrak{a}}, \overline{\mathfrak{a}})$ we have the following norm estimate*

$$||\xi||_{\mathbb{Y}^{2\mathfrak{p}}_{\mathfrak{a}}} \leq \sum_{n=0}^{\infty} \frac{\bar{L}^n T^n}{(\mathfrak{a}-\underline{\mathfrak{a}})^{\mathfrak{q}n}} \frac{n^{\mathfrak{q}n}}{\sqrt[2\mathfrak{p}]{n!}} \|\zeta_{\underline{\mathfrak{a}}}\|_{\mathbb{X}_{\underline{\mathfrak{a}}}}.$$

Theorem II.51. *Let $R \in \mathbb{R}_+$ such that $\|\zeta_{\underline{\mathfrak{a}}}\|_{\mathbb{X}_{\underline{\mathfrak{a}}}} \leq R$ and suppose that ξ is the unique strong solution of the equation (II.19). Then there exists a constant $\kappa \in \mathbb{R}_+$ for all $\mathfrak{a} \in (\underline{\mathfrak{a}}, \overline{\mathfrak{a}})$ such that the following is true*

$$\bar{L}T \leq \kappa \implies ||\xi||_{\mathbb{Y}^{2\mathfrak{p}}_{\mathfrak{a}}} \leq 2R.$$

Proof. We begin by fixing $\mathfrak{a} \in (\underline{\mathfrak{a}}, \overline{\mathfrak{a}})$ and letting $\zeta = 0$ that is ζ is the common zero vector in the scale $\mathbb{X}$. Moreover we fix $\mathfrak{a}_2 < \mathfrak{a}_1 \in (\underline{\mathfrak{a}}, \mathfrak{a})$ and suppose that $\kappa \leq \frac{R(\mathfrak{a}-\mathfrak{a}_1)^{\mathfrak{q}}}{4\mathfrak{p}(\mathbf{K}-R)}$. From Corollary II.47 we conclude that

$$\lim_{n\to\infty} ||\mathcal{J}^n(\zeta)||_{\mathbb{Y}^{2\mathfrak{p}}_{\mathfrak{a}}} = ||\xi||_{\mathbb{Y}^{2\mathfrak{p}}_{\mathfrak{a}}}.$$

Hence, to conclude the proof we show by induction that for all $n \in \mathbb{N}_0$ we have the following inequality $||\mathcal{J}^n(\zeta)||_{\mathbb{Y}^{2\mathfrak{p}}_{\mathfrak{a}}} \leq 2R$. The base case $n = 0$ simply follows from the choice $\zeta = 0$. Hence suppose that induction hypothesis holds for some $n \geq 0$. Now combining Theorem II.38 with inequalities (II.48) - (II.49) we can deduce that the following inequality holds

$$||\mathfrak{I}^{n+1}(\zeta)||_{\mathbb{Y}^{2\mathfrak{p}}_{\mathfrak{a}}} \leq \|\zeta_{\underline{\mathfrak{a}}}\|_{\mathbb{X}_{\underline{\mathfrak{a}}}} + \frac{4^{\mathfrak{p}}\bar{L}T}{(\mathfrak{a}-\mathfrak{a}_2)^{\mathfrak{q}}}\bigg(\mathbf{K} + \|\mathfrak{I}^n(\zeta)\|_{\mathbb{Y}^{2\mathfrak{p}}_{\mathfrak{a}_2}}\bigg).$$

Hence using the hypothesis we see that

$$\begin{aligned}
||\mathfrak{I}^{n+1}(\zeta)||_{\mathbb{Z}^{2\mathfrak{p}}_{\mathfrak{a}}} &\leq R + \frac{4^{\mathfrak{p}}\bar{L}T}{(\mathfrak{a}-\mathfrak{a}_2)^{\mathfrak{q}}}(\mathbf{K}+R) \\
&\leq R + \frac{4^{\mathfrak{p}}\kappa}{(\mathfrak{a}-\mathfrak{a}_2)^{\mathfrak{q}}}(\mathbf{K}+R) \\
&\leq R + \frac{(\mathfrak{a}-\mathfrak{a}_1)^{\mathfrak{q}}}{(\mathfrak{a}-\mathfrak{a}_2)^{\mathfrak{q}}}R. \\
&\leq 2R.
\end{aligned}$$

Hence it follows that $||\mathfrak{I}^{n+1}(\zeta)||_{\mathbb{Y}^{2\mathfrak{p}}_{\mathfrak{a}}} \leq R$ and the proof is complete. □

II.5.2 Continuity

In this subsection we will continue to denote by ξ a strong solution of an equation (II.19). Moreover we shall now show, using Kolmogorov Theorem IV.26, that stochastic process ξ has a continuous modification. We proceed to formulate and prove the following theorem.

Theorem II.52. *Let ξ be a strong solution of (II.19). Then ξ has a continuous modification.*

Proof. Fix arbitrary $\gamma < \mathfrak{a} \in (\underline{\mathfrak{a}}, \overline{\mathfrak{a}})$ and also fix in place some $s \leq t \in \mathfrak{T}$. Using Theorems II.38, II.42 and II.44 we see that for all $n \in \mathbb{N}$ we have the following chain of inequalities

$$\begin{aligned}
\mathbb{E}\bigg[||\xi(s) - \mathfrak{I}^n(\zeta_{\underline{\mathfrak{a}}})(t)||^{2\mathfrak{p}}_{\mathbb{X}_{\mathfrak{a}}}\bigg] &\leq \bigg(\sum_{k=1}^{n} \frac{\bar{L}^k(t-s)^k}{(\mathfrak{a}-\gamma)^{\mathfrak{q}k}}\frac{k^{\mathfrak{q}k}}{\sqrt[2\mathfrak{p}]{k!}}\bigg)^{2\mathfrak{p}} ||\xi - \zeta_{\underline{\mathfrak{a}}}||^{2\mathfrak{p}}_{\mathbb{Y}^{2\mathfrak{p}}_{\gamma}} \\
&\leq |t-s|^{2\mathfrak{p}}\bigg(\sum_{k=1}^{n} \frac{\bar{L}^k T^{k-1}}{(\mathfrak{a}-\gamma)^{\mathfrak{q}k}}\frac{k^{\mathfrak{q}k}}{\sqrt[2\mathfrak{p}]{k!}}\bigg)^{2\mathfrak{p}} ||\xi - \zeta_{\underline{\mathfrak{a}}}||^{2\mathfrak{p}}_{\mathbb{Y}^{2\mathfrak{p}}_{\gamma}} \\
&\leq |t-s|^{2\mathfrak{p}}\bigg(\sum_{k=1}^{\infty} \frac{\bar{L}^k T^{k-1}}{(\mathfrak{a}-\gamma)^{\mathfrak{q}k}}\frac{k^{\mathfrak{q}k}}{\sqrt[2\mathfrak{p}]{k!}}\bigg)^{2\mathfrak{p}} ||\xi - \zeta_{\underline{\mathfrak{a}}}||^{2\mathfrak{p}}_{\mathbb{Y}^{2\mathfrak{p}}_{\gamma}}. \qquad \text{(II.55)}
\end{aligned}$$

Now we would like to find a way to use Kolmogorov Theorem IV.26 to conclude this proof.

Before this can be done let us recall that the following equality holds

$$\lim_{n\to\infty} \mathbb{E}\Bigg[||\xi(s) - \mathcal{I}^n(\zeta_{\underline{\mathfrak{a}}})(t)||^{2\mathfrak{p}}_{\mathbb{X}_\mathfrak{a}}\Bigg] = \mathbb{E}\Bigg[||\xi(s) - \xi(t)||^{2\mathfrak{p}}_{\mathbb{X}_\mathfrak{a}}\Bigg].$$

Therefore we now can see that

$$\mathbb{E}\Bigg[||\xi(s) - \xi(t)||^{2\mathfrak{p}}_{\mathbb{X}_\mathfrak{a}}\Bigg] \le |t-s|^{2\mathfrak{p}} \Bigg(\sum_{k=1}^{\infty} \frac{\bar{L}^k T^{k-1}}{(\mathfrak{a}-\gamma)^{\mathfrak{q}k}} \frac{k^{\mathfrak{q}k}}{\sqrt[2\mathfrak{p}]{k!}} \Bigg)^{2\mathfrak{p}} \Bigg(||\xi||_{\mathbb{Y}^{2\mathfrak{p}}_\gamma} + ||\zeta_{\underline{\mathfrak{a}}}||_{\mathbb{Y}^{2\mathfrak{p}}_\gamma} \Bigg)^{2\mathfrak{p}}$$

and conclude, by Kolmogorov Theorem IV.26, that stochastic process ξ has a continuous modification hence the proof is complete. □

Remark II.53. *Now let $\bar{\xi}$ be a continuous modification of ξ. According to Theorem II.38 we have, for all $\mathfrak{a} < \beta \in \mathcal{A}$, the following estimate*

$$\begin{aligned} ||\mathcal{I}(\xi) - \mathcal{I}(\bar{\xi})||_{\mathbb{Y}^{2\mathfrak{p}}_\beta} &\le \frac{\bar{L}T}{(\mathfrak{a}-\gamma)^{\mathfrak{q}}} ||\xi - \bar{\xi}||_{\mathbb{Y}^{2\mathfrak{p}}_\mathfrak{a}} \\ &= 0 \end{aligned}$$

which shows that $\mathcal{I}(\xi) \approx \mathcal{I}(\bar{\xi})$ and since $\xi \approx \bar{\xi}$ we conclude that

$$\bar{\xi} \approx \zeta_{\underline{\mathfrak{a}}} + \int_0^\cdot F(\bar{\xi}(s))ds + \int_0^\cdot \Phi(\bar{\xi}(s))dW(s).$$

Remark II.54. *Looking at the proof of Theorem II.52 one may wonder if $\bar{\xi}$, a continuous modification ξ, depends on the choice of $\mathfrak{a} \in (\underline{\mathfrak{a}}, \overline{\mathfrak{a}})$ in a significant way. Using Remark II.53 it is simple to show that such dependence is only up to a modification. Indeed suppose that $\mathfrak{a}_1, \mathfrak{a}_2 \in (\underline{\mathfrak{a}}, \overline{\mathfrak{a}})$ then Remark II.53 above shows that $\xi \approx \bar{\xi}_{\mathfrak{a}_1}$ and $\xi \approx \bar{\xi}_{\mathfrak{a}_2}$ hence we see that $\bar{\xi}_{\mathfrak{a}_1} \approx \bar{\xi}_{\mathfrak{a}_2}$.*

II.5.3 Markov Property

In this subsection we assume that $\mathfrak{q} \in [0, \frac{1}{4\mathfrak{p}})$. Let us now begin by additionally assuming that we have $\alpha < \beta \in \mathcal{A}$ and some temporary fixed $s \in \mathcal{T}$. Then our work in the previous part of this section implies that for any $\mathcal{F}_s$ measurable and $\mathbb{X}_\alpha$ random variables ζ there exists a unique (up to a modification) $\mathbb{X}_\beta$ valued strong solution of the following SDE

$$\xi(t,s,\zeta) \coloneqq \zeta + \int_s^t F(\xi(\tau,s,\zeta))d\tau + \int_s^t \Phi(\xi(\tau,s,\zeta))dW(\tau), \quad \forall (s \le t \in \mathcal{T}).$$

Moreover a strong solution $\xi(\cdot,s,\zeta)$ is adapted to $\mathbb{F}|_{[s,T]}$. The main goal of this subsection is to show that $\xi(\cdot,s,\zeta)$ has a Markov property. However, before we begin we pause to make a number of preliminary observations.

Observation I

For this observation let us keep $s \in \mathcal{T}$ fixed and recall from our previous work that we have $\xi(\cdot,s,\zeta) \in \mathbb{Y}_\beta^{2\mathfrak{p}}$. Therefore we see that

$$\mathbb{E}\Big[\|\xi(t,s,\zeta)\|_{\mathbb{X}_\beta}^2\Big] < \infty, \quad \forall (s \le t \in \mathcal{T})$$

and so in particular we can say that for all $s \le t \in \mathcal{T}$ we have $\xi(t,s,\zeta) \in \mathcal{L}^2(\mathbf{P}, \mathbf{M}^{\mathbb{X}_\beta})$.

Observation II

For this observation let us assume that $u \le s \le t \in \mathcal{T}$ and $\alpha < \delta < \beta \in \mathcal{A}$ are fixed. Then it follows that

$$\begin{aligned}\xi(t,u,\zeta) &= \zeta + \int_u^t F(\xi(\tau,u,\zeta))d\tau + \int_u^t \Phi(\xi(\tau,u,\zeta))dW(\tau), \ \mathbb{P}-a.s. \\ &= \xi(s,u,\zeta) + \int_s^t F(\xi(\tau,u,\zeta))d\tau + \int_s^t \Phi(\xi(\tau,u,\zeta))dW(\tau), \ \mathbb{P}-a.s. \qquad \text{(II.56)}\end{aligned}$$

Now since $\xi(s,u,\zeta) \in \mathbb{Y}_\delta^{2\mathfrak{p}}$ is $\mathcal{F}_s$ measurable and equation (II.56) has a unique (up to a modification) solution we conclude that

$$\xi(t,u,\zeta) = \xi(t,s,\xi(s,u,\zeta)), \ \mathbb{P}-a.s.$$

Now if $\zeta \in \mathbb{X}_\alpha$ is a constant then (see [15]) $\xi(t,s,\zeta)$ is independent of $\mathcal{F}_s$. Therefore if $\phi \in C_b(\mathbb{X}_\beta)$, where $C_b(\mathbb{X}_\beta)$ a space of continuous and bounded maps on $\mathbb{X}_\beta$, then

$$\begin{aligned} A \in \sigma(\phi \circ \xi(t,s,\zeta)) &\implies A = \xi(t,s,\zeta)^{-1}\Big(\phi^{-1}(B)\Big),\ \exists (B \in \mathscr{B}(\mathbb{X}_\beta)) \\ &\implies A \in \sigma(\xi(t,s,\zeta)). \end{aligned} \tag{II.57}$$

which shows that $\phi(\xi(t,s,\zeta))$ is also independent of $\mathcal{F}_s$. In particular we have

$$\mathbb{E}\Big[\phi(\xi(t,s,\zeta))\Big|\mathcal{F}_s\Big] = \mathbb{E}\Big[\phi(\xi(t,s,\zeta))\Big]. \tag{II.58}$$

Observation III

Fix $t \in \mathcal{T}$ and suppose that $\phi \in C_b(\mathbb{X}_\beta)$. Moreover suppose that η and $\{\eta_n\}_{n\in\mathbb{N}}$ are elements of $\mathcal{L}^2(\mathbf{P}, \mathbf{M}^{\mathbb{X}_\beta})$. Then

$$\overbrace{\lim_{n\to\infty} \eta_n}^{\mathcal{L}^2(\mathbf{P},\mathbf{M}^{\mathbb{X}_\beta})} = \eta \quad \Longrightarrow \quad \overbrace{\lim_{n\to\infty} \phi(\eta_n)}^{\mathcal{L}^2(\mathbf{P},\mathbf{M}^{\mathbb{X}_\beta})} = \phi(\eta). \tag{II.59}$$

This fact follows from the dominated convergence Theorem IV.13 in the following way. By Theorem IV.12 we chose an arbitrary subsequence $\{\eta_{\sigma(n)}\}_{n\in\mathbb{N}}$ such that

$$\|\eta_{\sigma(n)} - \eta\|^2_{\mathbb{X}_\beta} \to 0, \ \mathbb{P} - a.s.$$

Hence we see that

$$\|\phi(\eta_{\sigma(n)}) - \phi(\eta)\|^2_{\mathbb{X}_\beta} \to 0, \ \mathbb{P} - a.s.$$

Moreover there exists a constant B such that $\|\phi\|_{\mathbb{X}_\beta} \leq B$. Therefore we see that for all $n \in \mathbb{N}$

$$\|\phi(\eta_{\sigma(n)}) - \phi(\eta)\|^2_{\mathbb{X}_\beta} \leq 4B^2.$$

Finally, since $4B^2 \in \mathcal{L}^2(\mathbf{P}, \mathbf{M}^{\mathbb{R}})$ we conclude by Theorem IV.13 that

$$\overbrace{\lim_{n\to\infty} \phi(\eta_{\sigma(n)})}^{\mathcal{L}^2(\mathbf{P}_t,\mathbf{M}^{\mathbb{X}_\beta})} = \phi(\eta)$$

and because subsequence $\{\eta_{\sigma(n)}\}_{n\in\mathbb{N}}$ was arbitrary we see that implication (II.59) holds.

Let us now prove that ξ is Markov in the following sense.

Theorem II.55. *Fix $u \leq s \leq t \in \mathcal{T}$, $\alpha < \delta < \beta \in \mathcal{A}$ and $\phi \in C_b(\mathbb{X}_\beta)$. Then*

$$\mathbb{E}\Big[\phi(\xi(t,u,\zeta))\Big|\mathcal{F}_s\Big] = \mathbb{E}\Big[\phi(\xi(t,s,\xi(s,u,\zeta)))\Big], \ \mathbb{P} - a.s. \tag{II.60}$$

Proof. First we take a note that, by continuity of ϕ it is clear that both sides in the equation (II.60) above are well defined.

Now using **Observation II** we see that equation (II.60) can be established by showing that

$$\mathbb{E}\Big[\phi(\xi(t,s,\xi(s,u,\zeta)))\Big|\mathcal{F}_s\Big] = \mathbb{E}\Big[\phi(\xi(t,s,\xi(s,u,\zeta)))\Big],\ \mathbb{P}-a.s. \tag{II.61}$$

To this end, using also **observation I**, let us therefore show that equation (II.61) above is true for any $\eta \in \mathcal{L}^2(\mathbf{P}, \mathbf{M}^{\mathbb{X}_\delta})$. Now if η is almost surely a constant then we can immediately conclude from **observation II** that

$$\mathbb{E}\Big[\phi(\xi(t,s,\eta))\Big|\mathcal{F}_s\Big] = \mathbb{E}\Big[\phi(\xi(t,s,\eta))\Big],\ \mathbb{P}-a.s. \tag{II.62}$$

Continuing the proof, if η is a simple map that is

$$\eta = \sum_{i=1}^{m} x_i \mathbb{1}_{F_i}$$

for some partition $F_1, \ldots, F_m \subset \mathcal{F}_s$ of Ω and constants $x_1, \ldots, x_m$ in $\mathbb{X}_\beta$ then we see that

$$\xi(t,s,\eta) = \sum_{i=1}^{m} \xi(t,s,x_i)\mathbb{1}_{F_i},\ \mathbb{P}-a.s. \tag{II.63}$$

Therefore we see that the following equality holds

$$\begin{aligned}
\mathbb{E}\Big[\phi(\xi(t,s,\eta))\Big|\mathcal{F}_s\Big] &= \mathbb{E}\Big[\phi\Big(\sum_{i=1}^{m}\xi(t,s,x_i)\mathbb{1}_{F_i}\Big)\Big|\mathcal{F}_s\Big],\ \mathbb{P}-a.s.\\
&= \mathbb{E}\Big[\sum_{i=1}^{m}\phi(\xi(t,s,x_i))\mathbb{1}_{F_i}\Big|\mathcal{F}_s\Big],\ \mathbb{P}-a.s.\\
&= \sum_{i=1}^{m}\mathbb{E}\Big[\phi(\xi(t,s,x_i))\mathbb{1}_{F_i}\Big|\mathcal{F}_s\Big],\ \mathbb{P}-a.s.
\end{aligned} \tag{II.64}$$

Now from **observation II** we recall two facts; namely, that for all $i \in \{1, \ldots, m\}$

(1) A random variable $\xi(t,s,x_i)$ is independent of $\mathcal{F}_s$,

(2) $F_1, \ldots, F_m \subset \mathcal{F}_s$ is a partition of Ω and an indicator map $\mathbb{1}_{F_i}$ is $\mathcal{F}_s$ measurable.

Hence we see that

$$\mathbb{E}\Big[\phi(\xi(t,s,\eta))\Big|\mathcal{F}_s\Big] = \sum_{i=1}^{m}\mathbb{E}\Big[\phi(\xi(t,s,x_i))\mathbb{1}_{F_i}\Big],\ \mathbb{P}-a.s.$$

$$= \mathbb{E}\Big[\phi(\xi(t,s,\eta))\Big],\ \mathbb{P}-a.s. \tag{II.65}$$

Finally suppose that $\eta \in \mathcal{L}^2(\mathbf{P}, \mathbf{M}^{\mathbb{X}_\delta})$ is arbitrary and find a sequence $\{\eta_{\sigma(n)}\}_{n\in\mathbb{N}}$ of simple maps that converges to η. Moreover using Theorem IV.12 one can chose this sequence such that, almost surely, pointwise convergence also holds. Precisely speaking we have the following

$$\overbrace{\lim_{n\to\infty}\eta_{\sigma(n)}}^{\mathcal{L}^2(\mathbf{P},\mathbf{M}^{\mathbb{X}_\delta})} = \eta$$

$$\|\eta_{\sigma(n)} - \eta\|^2_{\mathbb{X}_\delta} \to 0,\ \mathbb{P}-a.s.$$

Hence using Theorem IV.48 we see that

$$\overbrace{\lim_{n\to\infty}\xi(t,s,\eta_{\sigma(n)})}^{\mathcal{L}^2(\mathbf{P},\mathbf{M}^{\mathbb{X}_\beta})} = \xi(t,s,\eta)$$

$$\|\xi(t,s,\eta_{\sigma(n)}) - \xi(t,s,\eta)\|^2_{\mathbb{X}_\beta} \to 0,\ \mathbb{P}-a.s. \tag{II.66}$$

Now from our previous calculation (i.e. the case of a simple map) we see that

$$\mathbb{E}\Big[\phi(\xi(t,s,\eta_{\sigma(n)}))\Big|\mathcal{F}_s\Big] = \mathbb{E}\Big[\phi(\xi(t,s,\eta_{\sigma(n)}))\Big],\ \mathbb{P}-a.s.$$

Finally using convergence (II.66) and the fact that ϕ is continuous and bounded we can invoke a standard conditional dominated convergence Theorem to conclude that

$$\lim_{n\to\infty}\mathbb{E}\Big[\phi(\xi(t,s,\eta_{\sigma(n)}))\Big|\mathcal{F}_s\Big] = \mathbb{E}\Big[\phi(\xi(t,s,\eta))\Big|\mathcal{F}_s\Big]. \tag{II.67}$$

Moreover <u>**observation III**</u> now shows that

$$\lim_{n\to\infty}\mathbb{E}\Big[\phi(\xi(t,s,\eta_{\sigma(n)}))\Big] = \mathbb{E}\Big[\phi(\xi(t,s,\eta))\Big]. \tag{II.68}$$

hence combining equation (II.67) and (II.68) above we finally see that

$$\mathbb{E}\left[\phi(\xi(t,s,\eta))\middle|\mathcal{F}_s\right] = \mathbb{E}\left[\phi(\xi(t,s,\eta))\right],\ \mathbb{P}-a.s.$$

and proof is complete. □

II.6 Stochastic Spin Dynamics of a Quenched Particle System

II.6.1 Setting

In the first part of this text we saw an extension of work by [10]. This extension showed, under a suitable choice of coefficients, how to construct a unique strong solution of a stochastic differential equation (II.69), driven by a cylinder Wiener process, in a separable Hilbert space

$$d\xi(t) = F(\xi(t))dt + \Phi(\xi(t))dW(t), \; t \in \mathcal{T} \tag{II.69}$$

using the method of Ovsjannikov. The end result (see subsection II.4) was a strong solution that takes values in an intersection of a suitably chosen scale of Hilbert spaces. In this subsection we put the general theory to use by considering a practical example.

Our example (see also [10, 11]) is motivated by the study of stochastic dynamics of interacting particle systems. In order to outline an application of the general theory we need to fix some new terminology first. Hence for some $d \in \mathbb{N}$ we now let γ be a locally finite subset of $\mathbb{R}^d$ and $|\cdot|$ be the Euclidean norm in $\mathbb{R}^d$. Moreover we fix $\rho \in \mathbb{R}_+$ and introduce notation for the following sets:

$$\overline{B(x,\rho)} \coloneqq \{y \in \mathbb{R}^d \mid |x-y| \leq \rho\},$$

$$B_x \coloneqq \gamma \cap \overline{B(x,\rho)} \qquad \forall x \in \gamma,$$

$$n_x \coloneqq \# B_x \equiv \text{ number of elements in } B_x \;\; \forall x \in \gamma.$$

We observe that, although the number n_x is finite for each $x \in \gamma$, in general $n \coloneqq \{n_x\}_{x\in\gamma}$ is unbounded. However we assume that there exists $\mathcal{N} \in \mathbb{R}_+$ such that n satisfies the following regularity condition.

$$n_x \leq \mathcal{N}(1 + \log(1 + |x|)) \; \forall (x \in \gamma). \tag{II.70}$$

Remark II.56. *Condition (II.70) holds if γ is a typical realization of a Poisson or Gibbs (Ruelle) point process in $\mathbb{R}^d$. For details see [35, 56].*

Finally we will also need an access to two families of measurable maps $\phi_{\cdot,\cdot}$ and $\psi_{\cdot,\cdot}$ defined as

$$\left.\begin{array}{l}\phi_{x,y} : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \\ \\ \psi_{x,y} : \mathbb{R} \times \mathbb{R} \to \mathbb{R}\end{array}\right\} \forall (x, y \in \gamma).$$

We shall also assume that the following conditions are fulfilled

(1) Finite range. That is, for all $x, y \in \gamma$

$$|x - y| \geq \rho \implies \phi_{x,y} \equiv 0 \equiv \psi_{x,y}$$

(2) Uniform Lipschitz continuity. That is, there exist $C > 0$ such that for all $x, y \in \gamma$ and all $a, b, c, d \in \mathbb{R}$

$$|\phi_{x,y}(a, b) - \phi_{x,y}(c, d)| \leq C(|a - b| + |c - d|)$$

$$|\psi_{x,y}(a, b) - \psi_{x,y}(c, d)| \leq C(|a - b| + |c - d|)$$

II.6.2 Particle System

Let us suppose that each particle with position $x \in \gamma$ is characterized by an internal parameter (spin) process $\sigma_x : \overline{\Omega} \to \mathbb{R}$. We are now interested in studying a lattice system describing the time evolution of all spin parameters. That is we want to study the following system of stochastic differential equations

$$d\sigma_x(t) = \Upsilon_x(\bar{\sigma})dt + \Psi_x(\bar{\sigma})dW_x(t),\ x \in \gamma,\ t \in \mathfrak{T} \tag{II.71}$$

where we assume the following:

(1) $\bar{\sigma} := \{\sigma_x\}_{x\in\gamma}$ and $\overline{W} := \{W_x\}_{x\in\gamma}$ is a family of independent Wiener processes in $\mathbb{R}$,

(2) $\Upsilon : \mathbb{R}^\gamma \to \mathbb{R}^\gamma$ and $\Psi : \mathbb{R}^\gamma \to \mathbb{R}^\gamma$ are defined as follows

$$\left.\begin{array}{l}\Upsilon_x(z) := \sum_{y\in\gamma} \phi_{x,y}(z_x, z_y) \\ \\ \Psi_x(z) := \sum_{y\in\gamma} \psi_{x,y}(z_x, z_y)\end{array}\right\} \forall (x \in \gamma \wedge z \in \mathbb{R}^\gamma).$$

Remark II.57. *Systems like (II.71) are well-studied is case $\gamma \equiv \mathbb{Z}^d$ see [1, 2]. Particle systems where γ is random are studied in a research paper by [19].*

Our aim now is to show that it is possible to realise system (II.71) as an equation in a suitable scale of Hilbert spaces which will consequently allow us to apply theory developed in previous sections in order to find its strong solutions.

II.6.3 Existence of the Dynamics

The process by which we shall find a suitable scale of separable Hilbert spaces in which our dynamics will live starts from looking back at the Definition II.6 from subsection II.2. We see that what is required is to move away from an abstract $\mathbb{X}$ and $\mathcal{H}$ to something more concrete and suitable to the problem at hand. Hence for the rest of this subsection we propose to make the following re-definition:

$$\mathbb{X}_{\mathfrak{a}} := \left\{ z \in \mathbb{R}^{\gamma} \;\middle|\; \|z\|_{\mathbb{X}_{\mathfrak{a}}} := \left(\sum_{x \in \gamma} e^{-\mathfrak{a}|x|} |z_x|^2 \right)^{\frac{1}{2}} < \infty \right\},\ \forall (\mathfrak{a} \in \mathcal{A})$$

$$\mathcal{H} := \left\{ z \in \mathbb{R}^{\gamma} \;\middle|\; \|z\|_{\mathcal{H}} := \left(\sum_{x \in \gamma} |z_x|^2 \right)^{\frac{1}{2}} < \infty \right\}.$$

Moreover let us also recall that $\mathbb{H} := \{\mathbb{H}_{\mathfrak{a}}\}_{\mathfrak{a} \in \mathcal{A}}$ is defined as follows

$$\mathbb{H}_{\mathfrak{a}} := \left\{ A \in L(\mathcal{H}, \mathbb{X}_{\mathfrak{a}}) \;\middle|\; \begin{array}{l} \|A\|_{\mathbb{H}_{\mathfrak{a}}} := \left(\sum_{z \in \gamma} ||A(\mathfrak{e}_z)||^2_{\mathbb{X}_{\mathfrak{a}}} \right)^{\frac{1}{2}} < \infty, \\ \mathfrak{e} := \{\mathfrak{e}_z\}_{z \in \gamma} \text{ is an orthonormal basis of } \mathcal{H} \end{array} \right\}. \tag{II.72}$$

We will also make use of the map $\overline{\Psi} : \mathbb{R}^{\gamma} \to (\mathbb{R}^{\gamma})^{\mathcal{H}}$ defined for all $z \in \mathbb{R}^{\gamma}$ in the following way

$$\overline{\Psi}(z)(q) := \Psi(z) \cdot q,\ \forall (q \in \mathcal{H})$$

where $\cdot$ is the usual dot product i.e. $(\Psi(z) \cdot q)_x = \Psi_x(z) q_x$ for all $x \in \gamma$.

Now before we proceed, suppose for a moment that we have shown that there exists some $L \in \mathbb{R}_+$ such that $\Upsilon \in \mathcal{O}(\mathbb{X}, \mathfrak{q})$ and $\overline{\Psi} \in \mathcal{O}(\mathbb{X}, \mathbb{H}, \mathfrak{q})$ (see Definition II.4 and II.5) then because $\overline{W} := \{W_x\}_{x \in \gamma}$ is a cylindrical Wiener process in $\mathcal{H}$ (see for example [32]) we can see, by rewriting system (II.71) in the following way

$$d\sigma_x(t) = \Upsilon(\bar{\sigma})dt + \overline{\Psi}(\bar{\sigma})d\overline{W}(t)\ t \in \mathfrak{T}, \tag{II.73}$$

that we can study its integral counterpart by means of the general theory established in the previous subsections. In particular existence and uniqueness of a strong solution follows immediately from subsection II.4. Let us formulate this result in a theorem bellow.

Theorem II.58. *System (II.71) admits a unique strong solution.*

Proof. This result follows from rearranging system (II.71) into the form of equation (II.73), subsection II.4 and most importantly Theorem II.59 below. □

We now conclude this subsection with the following important theorem.

Theorem II.59. *There exists $L \in \mathbb{R}_+$ such that $\Upsilon \in \mathcal{O}(\mathbb{X}, \mathfrak{q})$ and $\overline{\Psi} \in \mathcal{O}(\mathbb{X}, \mathbb{H}, \mathfrak{q})$.*

Proof. First we show that $\Upsilon \in \mathcal{O}(\mathbb{X}, \mathfrak{q})$. Looking at the Definition II.4 we see that to accomplish this task it is sufficient to show that there exists $L \in \mathbb{R}_+$ such that for any fixed $\alpha < \beta \in \mathcal{A}$ and fixed $\mathfrak{w}, \mathfrak{u} \in \mathbb{X}_\alpha$ we have the following inequality

$$\|\Upsilon(\mathfrak{w}) - \Upsilon(\mathfrak{u})\|_{\mathbb{X}_\beta} \leq \frac{L}{(\beta - \alpha)^{\mathfrak{q}}}\|\mathfrak{w} - \mathfrak{u}\|_{\mathbb{X}_\alpha}.$$

Hence let us start by observing that

$$\|\Upsilon(\mathfrak{w}) - \Upsilon(\mathfrak{u})\|^2_{\mathbb{X}_\beta} = \sum_{x\in\gamma}\left|\sum_{y\in\gamma}\phi_{xy}(\mathfrak{w}_x, \mathfrak{w}_y) - \phi_{xy}(\mathfrak{u}_x, \mathfrak{u}_y)\right|^2 e^{-\beta|x|}.$$

Therefore we see that

$$\begin{aligned}\|\Upsilon(\mathfrak{w}) - \Upsilon(\mathfrak{u})\|^2_{\mathbb{X}_\beta} &\leq C\sum_{x\in\gamma}\left|\sum_{y\in B_x} |\mathfrak{w}_x - \mathfrak{u}_x| + |\mathfrak{w}_y - \mathfrak{u}_y|\right|^2 e^{-\beta|x|} \\ &\leq 2C\sum_{x\in\gamma}\sum_{y\in B_x} n_x(|\mathfrak{w}_x - \mathfrak{u}_x|^2 + |\mathfrak{w}_y - \mathfrak{u}_y|^2)e^{-\beta|x|} \\ &\leq 2C(\mathbf{A} + \mathbf{B})\end{aligned} \tag{II.74}$$

where we have used the following abbreviations:

$$\mathbf{A} \coloneqq \sum_{x\in\gamma}\sum_{y\in B_x} n_x|\mathfrak{w}_x - \mathfrak{u}_x|^2 e^{-\beta|x|}, \tag{II.75}$$

$$\mathbf{B} \coloneqq \sum_{x\in\gamma}\sum_{y\in B_x} n_x|\mathfrak{w}_y - \mathfrak{u}_y|^2 e^{-\beta|x|}. \tag{II.76}$$

Hence it remains to show that $\mathbf{A} < \infty$ and $\mathbf{B} < \infty$. Now since

$$x \in \gamma \wedge y \in B_x \iff x, y \in \gamma \wedge |x - y| < \rho \iff y \in \gamma \wedge x \in B_y$$

we see that the following estimate holds

$$\begin{aligned}\mathbf{B} &= \sum_{x\in\gamma}\sum_{y\in B_x} n_x|\mathfrak{w}_y - \mathfrak{u}_y|^2 e^{-\beta|x|} \\ &\le e^{\beta\rho}\sum_{y\in\gamma}\sum_{x\in B_y} n_x|\mathfrak{w}_y - \mathfrak{u}_y|^2 e^{-(\beta-\alpha)|y|}e^{-\alpha|y|}, \\ &\le e^{\bar{\mathfrak{a}}\rho}K_1\|\mathfrak{w} - \mathfrak{u}\|^2_{\mathbb{X}_\alpha} \end{aligned} \tag{II.77}$$

where we have used the following abbreviation

$$K_1 \coloneqq \sup\left\{\sum_{x\in B_y} n_x e^{-(\beta-\alpha)|y|} \;\middle|\; y \in \gamma\right\}. \tag{II.78}$$

Now using inequality assumption (II.70) on the logarithmic growth of components of n we can estimate K_1 by observing first that there exists $M \in \mathbb{N}$ such that

$$M < |x| \implies n_x \le \mathcal{N}|x|^{\mathfrak{q}}.$$

Moreover because γ is a locally finite subset of $\mathbb{R}^d$ we know that $\overline{B(0,M)} \coloneqq \overline{B(0,M)} \cap \gamma$ has only a finite number of elements. Hence we can define the following constant

$$K_2 \coloneqq \sum_{x\in\overline{B(0,M)}} n_x < \infty$$

and now observe that all $y \in \gamma$ we have

$$\sum_{x\in B_y} n_x e^{-(\beta-\alpha)|y|} \le K_2 + \mathcal{N}\sum_{x\in B_y} |x|^{\mathfrak{q}} e^{-(\beta-\alpha)|y|}$$

$$\le K_2 + \mathcal{N}\sum_{x\in B_y} (|y|^{\mathfrak{q}}+\rho^{\mathfrak{q}}) e^{-(\beta-\alpha)|y|}$$

$$\le K_2 + \left(K_3 := \sum_{x\in\gamma} \mathcal{N}\rho^{\mathfrak{q}} e^{-(\beta-\alpha)|x|}\right) + \mathcal{N}\sum_{x\in B_y} |y|^{\mathfrak{q}} e^{-(\beta-\alpha)|y|}$$

$$\le K_2 + K_3 + \mathcal{N} n_y |y|^{\mathfrak{q}} e^{-(\beta-\alpha)|y|}.$$

Remark II.60. *The fact that*

$$K_3 := \sum_{x\in\gamma} \mathcal{N}\rho^{\mathfrak{q}} e^{-(\beta-\alpha)|x|} < \infty$$

follows directly from the Theorem III.4 which is located in the next section on this document. See also Remark III.5 and inequality (III.3) therein.

hence we see that for all $y \in \gamma$ such that $M < |y|$ we have

$$\sum_{x\in B_y} n_x e^{-(\beta-\alpha)|y|} \le K_2 + K_3 + \mathcal{N}^2 |y|^{2\mathfrak{q}} e^{-(\beta-\alpha)|y|}.$$

Now returning to the definition (II.78) of K_1 we see once again that, because $\overline{B(0,M)}$ has only a finite number of elements, we can without loss of generality (see also Theorem III.20) consider the following estimate:

$$\begin{aligned} K_1 &\le K_2 + K_3 + \mathcal{N}^2 \sup\left\{ |y|^{2\mathfrak{q}} e^{-(\beta-\alpha)|y|} \;\middle|\; y \in \gamma \right\} \\ &\le K_2 + K_3 + \mathcal{N}^2 \sup\left\{ h^{2\mathfrak{q}} e^{-(\beta-\alpha)h} \;\middle|\; h > 0 \right\} \\ &\le K_2 + K_3 + \mathcal{N}^2 \sup\left\{ \left(h e^{\frac{-(\beta-\alpha)h}{2\mathfrak{q}}} \right)^{2\mathfrak{q}} \;\middle|\; h > 0 \right\} \\ &\le K_2 + K_3 + \mathcal{N}^2 \left(\sup\left\{ h e^{\frac{-(\beta-\alpha)h}{2\mathfrak{q}}} \;\middle|\; h > 0 \right\} \right)^{2\mathfrak{q}}. \end{aligned} \tag{II.79}$$

Now, we can deduce that function $he^{\frac{-(\beta-\alpha)h}{2\mathfrak{q}}} : (0,\infty) \to \mathbb{R}$ attains its supremum when $\frac{d}{dh}he^{\frac{-(\beta-\alpha)h}{2\mathfrak{q}}} = 0$ that is when $h = \frac{2\mathfrak{q}}{(\beta-\alpha)}$. Hence it follows from inequality (II.79) that

$$K_1 \leq \frac{(\overline{\mathfrak{a}}-\underline{\mathfrak{a}})(K_2+K_3)+4\mathfrak{q}^{2\mathfrak{q}}\mathcal{N}^2}{(\beta-\alpha)^{2\mathfrak{q}}}$$

$$\mathbf{B} \leq e^{\overline{\mathfrak{a}}\rho}\frac{(\overline{\mathfrak{a}}-\underline{\mathfrak{a}})(K_2+K_3)+4\mathfrak{q}^{2\mathfrak{q}}\mathcal{N}^2}{(\beta-\alpha)^{2\mathfrak{q}}}\|\mathfrak{w}-\mathfrak{u}\|^2_{\mathbb{X}_\alpha}.$$

Now returning to the definition (II.75) we also see that

$$\begin{aligned}\mathbf{A} &\leq \sum_{x\in\gamma} n_x^2|\mathfrak{w}_x-\mathfrak{u}_x|^2e^{-\beta|x|} \\ &\leq \sum_{x\in\gamma} n_x^2|\mathfrak{w}_x-\mathfrak{u}_x|^2e^{-(\beta-\alpha)|x|}e^{-\alpha|x|} \\ &\leq \sup\left\{n_x^2e^{-(\beta-\alpha)|x|}\;\middle|\; x\in\gamma\right\}\|\mathfrak{w}-\mathfrak{u}\|^2_{\mathbb{X}_\alpha}. \end{aligned} \tag{II.80}$$

Now relying on our previous calculations we see that

$$\mathbf{A} \leq \frac{4\mathfrak{q}^{2\mathfrak{q}}\mathcal{N}^2}{(\beta-\alpha)^{2\mathfrak{q}}}\|\mathfrak{w}-\mathfrak{u}\|^2_{\mathbb{X}_\alpha}.$$

Finally looking back at the inequality (II.74) we see that

$$\begin{aligned}\|\Upsilon(\mathfrak{w})-\Upsilon(\mathfrak{u})\|^2_{\mathbb{X}_\beta} &\leq 2C(\mathbf{A}+\mathbf{B}) \\ &\leq \frac{L_1^2}{(\beta-\alpha)^{2\mathfrak{q}}}\|\mathfrak{w}-\mathfrak{u}\|^2_{\mathbb{X}_\alpha}\end{aligned}$$

where we let $L_1^2 \coloneqq 2Ce^{\overline{\mathfrak{a}}\rho}(\overline{\mathfrak{a}}-\underline{\mathfrak{a}})(K_2+K_3)+8\mathfrak{q}^{2\mathfrak{q}}\mathcal{N}^2$. Hence it follows that

$$\|\Upsilon(\mathfrak{w})-\Upsilon(\mathfrak{u})\|_{\mathbb{X}_\beta} \leq \frac{L_1}{(\beta-\alpha)^{\mathfrak{q}}}\|\mathfrak{w}-\mathfrak{u}\|_{\mathbb{X}_\alpha}.$$

To finish the proof it still remains to show that $\overline{\Psi} \in \mathcal{O}(\mathbb{X},\mathbb{H},\mathfrak{q})$. Once again Looking at the Definition II.4 we see that to accomplish this task it is sufficient at this point to show that

$$\|\overline{\Psi}(\mathfrak{w})\|_{\mathbb{H}_\beta} \leq \|\Psi(\mathfrak{w})\|_{\mathbb{X}_\beta}.$$

From definition (II.72) we see that with the choice of canonical orthonormal basis $\mathfrak{e}$ for $\mathcal{H}$ we have the following result

$$\begin{aligned}\|\overline{\Psi}(\mathfrak{w})\|^2_{\mathbb{H}_\beta} &\leq \sum_{z\in\gamma}\|\Psi(\mathfrak{w})(\mathfrak{e}_z)\|^2_{\mathbb{X}_\beta}\\ &\leq \sum_{z\in\gamma}\sum_{x\in\gamma} e^{-\beta|x|}|\Psi_x(\mathfrak{w})\mathfrak{e}_{z,x}|^2.\end{aligned}$$

Hence by the choice of orthonormal basis we see that

$$\begin{aligned}\|\overline{\Psi}(\mathfrak{w})\|_{\mathbb{H}_\beta} &\leq \left(\sum_{x\in\gamma} e^{-\beta|x|}\left|\Psi_x(\mathfrak{w})\mathfrak{e}_{x,x}\right|^2\right)^{\frac{1}{2}}\\ &= \left(\sum_{x\in\gamma} e^{-\beta|x|}\left|\Psi_x(\mathfrak{w})\right|^2\right)^{\frac{1}{2}}\\ &= \|\Psi(\mathfrak{w})\|_{\mathbb{X}_\beta}\end{aligned}$$

hence the proof is complete. □

III Row-finite systems of SDEs with dissipative drift

III.1 Summary

We begin this section by fixing some additional notation and a couple of new measure and measurable spaces. In addition we fix a scale of infinite sequences (see Definition III.3) and slightly redefine the scale $\mathbb{Y}$ from the previous section. Subsequently we introduce our main stochastic system see (III.12) together with all assumptions that we place on the coefficients. Moreover we slightly redefined the concept of a strong solution, see Definition III.17.

Then we move on to the next subsection containing a number of auxiliary results. In particular using subsection IV.4 from the Appendix we prove a variation of an infinite dimensional Gronwall's inequality which in this thesis we call a comparison Theorem III.23. We also prove an important Corollary III.24 to Theorem III.23 which we subsequently use to study truncated systems in subsection III.4 and establish a key estimate via Theorem III.30.

Finally in subsection III.5 we solve a one dimensional SDE (see Theorem III.34) which we then use to establish existence in subsection III.6. This section is then concluded with the proof of uniqueness.

III.2 Main Framework

III.2.1 General Notation

In this section we continue to assume that all vector spaces will be over $\mathbb{R}$ and denote by $\#A$ the cardinal number of any given set A. We shall also reuse most of the notation from the previous section. In particular notation (II.1) will be often used.

We shall continue to work with constants $\rho \in \mathbb{R}_+,\ d \in \mathbb{N}$ defined in subsection II.6 and the following closed intervals:

$$\mathcal{A} := [\underline{\mathfrak{a}}, \overline{\mathfrak{a}}],$$

$$\mathcal{T} := [0, T].$$

From subsection II.6 we remember that γ is a locally finite subset of $\mathbb{R}^d$. Moreover we shall now agree that $|\cdot|,\ |\cdot|_S$ will respectively denote the Euclidean and supremum norm in $\mathbb{R}^d$. We also recall the following notation, that will be used frequently in this section:

$$B(x,\rho) := \{y \in \mathbb{R}^d \mid |x-y| < \rho\},$$

$$\overline{B(x,\rho)} := \{y \in \mathbb{R}^d \mid |x-y| \le \rho\},$$

$$B_x := \gamma \cap \overline{B(x,\rho)} \qquad \forall x \in \gamma,$$

$$n_x := \#B_x, \qquad \forall x \in \gamma.$$

Remark III.1. *The fact that γ is a locally finite subset of $\mathbb{R}^d$ means that $\gamma \cap X$ is finite if $X \subset \mathbb{R}^d$ is compact and also implies that γ is a countable subset of $\mathbb{R}^d$.*

Next, we fix in place a real valued function $a : \mathbb{R}^d \to \mathbb{R}^+$ and make the following assumptions:

(**A**) $a(x) \le \bar{a}$ for some constant $\bar{a} \in \mathbb{R}^+$,

(**B**) $n_x \le \mathcal{N}(1 + \log(1 + |x|))$ for some constant $\mathcal{N} \in \mathbb{R}^+$ and all $x \in \gamma$.

See Remark II.56 for an additional explanation of the assumption (**B**).

Now suppose that $\mathbf{X} := \{X_\mathfrak{a}\}_{\mathfrak{a}\in\mathcal{A}}$ is a family of sets. Let us also recall, from subsection II.2, the following notation:

$$\uparrow\mathbf{X} := \bigcup_{\mathfrak{a}\in(\underline{\mathfrak{a}},\overline{\mathfrak{a}})} X_\mathfrak{a}, \quad \downarrow\mathbf{X} := \bigcap_{\mathfrak{a}\in(\underline{\mathfrak{a}},\overline{\mathfrak{a}})} X_\mathfrak{a}.$$

Finally given two vector spaces A and B let us remember that in addition we agreed to use the following shorthand notation

$$A \prec B \iff \text{A is a subspace of B.}$$

III.2.2 Scales and Ovsjannikov Maps

We now proceed to introduce several important definitions. First of all we note that the Definition II.2 of a scale and the Definition II.4 of an Ovsjannikov Map will play an important role in this section. For convenience let us also restate bellow the following definition from subsection II.2.2

Definition III.2. *Suppose* $\mathbf{X}$ *is a scale and* $\mathbf{Z} := \{Z_\mathfrak{a}\}_{\mathfrak{a}\in\mathcal{A}}$ *is a family of Banach spaces. Let us define the following spaces of Ovsjannikov maps:*

$$\mathcal{O}(\mathbf{X},\mathbf{Z},q) := \{\textit{space of Ovsjannikov maps of order } q \textit{ from } \mathbf{X} \textit{ to } \mathbf{Z}\},$$

$$\mathcal{O}(\mathbf{X},q) := \{\textit{space of Ovsjannikov maps of order } q \textit{ from } \mathbf{X} \textit{ to } \mathbf{X}\}.$$

Definition III.3. *For all* $p \in \mathbb{R}_1$ *and all* $\mathfrak{a} \in \mathcal{A}$ *let*

$$l^p_\mathfrak{a} := \left\{ z \in \mathbb{R}^\gamma \;\middle|\; \|z\|_{l^p_\mathfrak{a}} := \left(\sum_{x\in\gamma} e^{-\mathfrak{a}|x|} |z_x|^p \right)^{\frac{1}{p}} < \infty \right\}$$

$$\mathscr{L}^p := \{l^p_\mathfrak{a}\}_{\mathfrak{a}\in\mathcal{A}}$$

be, respectively, a normed linear space of weighted real sequences and a family of such spaces.

Let us now direct our effort towards proving the following result which we shall reuse in this section a number of times.

Theorem III.4. *Recall that in subsection II.6 we defined* $n := \{n_x\}_{x\in\gamma}$. *It follows that* $n \in l^1_{\underline{\mathfrak{a}}}$.

Proof. Observe that assumption (**B**) implies that there exists $M \in \mathbb{N}$ such that

$$M < |x| \implies n_x \leq \mathcal{N}|x|.$$

Moreover since $\overline{B(0,M)} \cap \gamma$ has only finite number of elements hence without loss of generality we can assume that the following inequality holds

$$\sum_{x \in \gamma} e^{-\underline{\mathfrak{a}}|x|}|n_x| \leq \mathcal{N} \sum_{x \in \gamma} e^{-\underline{\mathfrak{a}}|x|}|x|.$$

Hence to conclude this proof we need to show that

$$\sum_{x \in \gamma} e^{-\underline{\mathfrak{a}}|x|}|x| < \infty. \tag{III.1}$$

To accomplish this task we start by making a couple of preliminary observations and definitions. Hence let us start by fixing a suitable $k \in \mathbb{N}$ such that $\sqrt{d}\frac{1}{2^k} < \rho$ and considering the following k^{th} grid-partition or $\mathbb{R}^d$

$$\mathcal{R}^k := \{\mathcal{R}^k_z\}_{z \in \mathbb{Z}^d}$$

$$\mathcal{R}^k_z := \left\{ x \in \mathbb{R}^d \;\middle|\; \frac{z_i - 1}{2^k} \leq x_i \leq \frac{z_i}{2^k} \right\}.$$

We shall refer to a member of the family $\mathcal{R}^k$ by calling it a $k^{\text{th}}-$rectangle. Observe moreover that for all $z \in \mathbb{Z}^d$ the following equality holds

$$Diam(\mathcal{R}^k_z) := \sup\{|x - y|_S \mid x, y \in \mathcal{R}^k_z\} = \frac{1}{2^k}.$$

Now we introduce the following sets:

$$I_n := \left\{ x \in \mathbb{R}^d \;\middle|\; |x|_S \leq \frac{1}{2} n \right\} \; \forall n \in \mathbb{N}_0,$$

$$J_n := I_n - I_{n-1} \qquad \forall n \in \mathbb{N}.$$

Consider also the real function $e^{-\underline{\mathfrak{a}}x}x : [0, \infty) \to \mathbb{R}$. We see that $\frac{d}{dx}e^{-\underline{\mathfrak{a}}x}x = e^{-\underline{\mathfrak{a}}x}(1 - \underline{\mathfrak{a}}x)$ and so it follows that $\frac{d}{dx}e^{-\underline{\mathfrak{a}}x}x < 0$ if $x > \frac{1}{\underline{\mathfrak{a}}}$. Therefore letting $m \in \mathbb{N}$ be the smallest natural number such that $\max\{\frac{1}{\underline{\mathfrak{a}}}, 2\} \leq m$ we see that $e^{-\underline{\mathfrak{a}}x}x : (m, \infty) \to \mathbb{R}$ is a decreasing function. Finally observe that the following statements are true:

(1) I_1 contains exactly 2^{k+1} of $k^{\text{th}}-$rectangles.

(2) J_n contains fewer then $n2^{k+2}$ of $k^{\text{th}}-$rectangles.

(3) For all $n \in \mathbb{N}$, if $x \in \gamma \cap J_n$ then $|x| \geq n - 1$.

(4) Suppose that $n \in \mathbb{N}$ and $z \in \mathbb{Z}^d$. Consider $x, y \in \gamma \cap \mathcal{R}_z^k \subset J_n$. It follows that

$$\begin{aligned} |x - y| &\leq \sqrt{d}|x - y|_S \\ &\leq \sqrt{d}Diam(\mathcal{R}_z^k) \\ &\leq \sqrt{d}\frac{1}{2^k} \\ &\leq \rho. \end{aligned}$$

Hence we see that $y \in B_x$ and so from the assumption (**B**) we see that

$$\begin{aligned} \#\gamma \cap \mathcal{R}_z^k &\leq n_x \\ &\leq \mathcal{N}|x| \\ &\leq \mathcal{N}n. \end{aligned}$$

Therefore we conclude that for all $n \in \mathbb{N}$, $\#\gamma \cap J_n \leq \mathcal{N}n^2 2^{k+2}$.

Returning now to the series (III.1) we see that because J_m is compact and γ is locally finite we can let

$$B := \sum_{x \in \gamma \cap J_m} e^{-\underline{\mathfrak{a}}|x|}|x|$$

and observe that

$$\begin{aligned} \sum_{x \in \gamma} e^{-\underline{\mathfrak{a}}|x|}|x| &\leq B + \sum_{\substack{n \in \mathbb{N} \\ n > m}} \sum_{x \in \gamma \cap J_n} e^{-\underline{\mathfrak{a}}|x|}|x| \\ &\leq B + \mathcal{N}2^{k+2} \sum_{\substack{n \in \mathbb{N} \\ n > m}} e^{-\underline{\mathfrak{a}}(n-1)}(n-1)n^2. \end{aligned}$$

Hence letting $\mathcal{K} := \frac{m-1}{m}$ we see that

$$\sum_{x\in\gamma} e^{-\underline{\mathfrak{a}}|x|}|x| \leq B + \mathcal{N}2^{k+2} \sum_{\substack{n\in\mathbb{N}\\ n>m}} e^{-\mathcal{K}\underline{\mathfrak{a}}n} n^3. \tag{III.2}$$

Now, one can show via a simple calculation involving the integral test (for details see [43]) that the right hand side of the inequality (III.2) above is finite hence the proof is complete. □

Remark III.5. *From Theorem III.4 above it is clear that*

$$\sum_{x\in\gamma} e^{-\underline{\mathfrak{a}}|x|} < \infty. \tag{III.3}$$

Theorem III.6. *Suppose that $p \in \mathbb{R}_1$. Then $\mathscr{L}^p$ is a scale.*

Proof. It is clear from the Definition III.3 that $\mathscr{L}^p$ is a family of normed linear spaces. Moreover conditions (1) and (2) of the Definition II.2 follow immediately from the simple fact that if $\alpha < \beta \in \mathcal{A}$ then $e^{-\alpha} > e^{-\beta}$. Hence to conclude the proof we fix $\mathfrak{a} \in \mathcal{A}$ and show that $l^p_{\mathfrak{a}}$ is a Banach space.

Let us begin by assuming that $\{z^n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $l^p_{\mathfrak{a}}$. Now fix an arbitrary $\epsilon > 0$ and a suitable constant $N_\epsilon \in \mathbb{N}$ such that for all $n, m > N_\epsilon$ we have

$$\left(\sum_{x\in\gamma} e^{-\mathfrak{a}|x|}|z^n_x - z^m_x|^p\right)^{\frac{1}{p}} < \epsilon. \tag{III.4}$$

Because ϵ is arbitrary we see from inequality (III.4) above that for all $x \in \gamma$ sequence $\{z^n_x\}_{n\in\mathbb{N}}$ is Cauchy in $\mathbb{R}$. Hence, it follows that we can define a new sequence $\mathbf{z} := \{\mathbf{z}_x\}_{x\in\gamma}$ in $\mathbb{R}^\gamma$ as follows

$$\mathbf{z}_x := \lim_{n\to\infty} z^n_x \ \forall x \in \gamma.$$

Now we complete the proof by showing that $\mathbf{z} \in l^p_{\mathfrak{a}}$ and $\overbrace{\lim_{n\to\infty} z^n}^{\text{in } l^p_{\mathfrak{a}}} = \mathbf{z}$. To begin, we fix an arbitrary finite subset A of γ. Now for all $n, m > N_\epsilon$ we see from inequality (III.4) that

$$\sum_{x\in A} e^{-\mathfrak{a}|x|}|z^n_x - z^m_x|^p < \epsilon^p. \tag{III.5}$$

Hence we can deduce that for all $n > N_\epsilon$

$$\lim_{m\to\infty} \sum_{x\in A} e^{-\mathfrak{a}|x|}|z^n_x - z^m_x|^p =$$

$$= \sum_{x\in A} e^{-\mathfrak{a}|x|}|z^n_x - \lim_{m\to\infty} z^m_x|^p = \tag{III.6}$$

$$= \sum_{x\in A} e^{-\mathfrak{a}|x|}|z^n_x - \mathbf{z}_x|^p \leq \epsilon^p.$$

Since $A \subset \gamma$ is arbitrary we see from inequality (III.6) above that for all $n > N_\epsilon$

$$\sum_{x\in\gamma} e^{-\mathfrak{a}|x|}|z^n_x - \mathbf{z}_x|^p \leq \epsilon^p.$$

Because ϵ is also arbitrary we conclude that $\overbrace{\lim_{n\to\infty} z^n}^{\text{in } l^p_{\mathfrak{a}}} = \mathbf{z}$. Moreover we see that if $n > N_\epsilon$ then $z^n - \mathbf{z} \in l^p_{\mathfrak{a}}$. Since $l^p_{\mathfrak{a}}$ is a vector space we conclude that $\mathbf{z} \in l^p_{\mathfrak{a}}$ hence the proof is complete. □

III.2.3 Probability and Measure Spaces

We continue working on the same probability space as described in subsection II.2.3 with a few important changes that we shall outline bellow.

(1) Given two measurable spaces **A** and **B** we continue to denote by $\mathcal{M}(\mathbf{A}, \mathbf{B})$ the space of all measurable maps from **A** to **B**. The notion of a stochastic process in this section will understood in line with the Definition II.12. Moreover, in addition to the measurable spaces fixed in subsection II.2.3 we also fix here the following measurable spaces

$$\mathbf{M}^p_{\mathfrak{a}} := (l^p_{\mathfrak{a}}, \mathscr{B}(l^p_{\mathfrak{a}}))\ \forall (a \in \mathcal{A} \wedge p \in \mathbb{R}_1).$$

(2) We redefine our notation for W. In this section W will stand for the family of independent real valued Wiener processes on **MP**. That is we let $W := \{W_x\}_{x\in\gamma}$. Moreover we also require our filtration $\mathbb{F} := \{\mathcal{F}_t\}_{t\in\mathcal{T}}$ to satisfy the following standard properties:

   (a) For all $t \in \mathcal{T}$ and all $x \in \gamma$, $W_x(t)$ is $\mathcal{F}_t$ measurable

   (b) For all $s \leq t \in \mathcal{T}$ and all $x \in \gamma$ $W_x(t) - W_x(s)$ is independent of $\mathcal{F}_s$.

For convenience let us recall here the following definition from subsection II.2.3.

Definition III.7. *Let* $\mathscr{X} := (X, \mathcal{A}, \eta)$ *be a measure space,* Y *be a Banach space, with norm denoted by* $\|\cdot\|_Y$*, and* $\mathscr{Y} := (Y, \mathscr{B}(Y))$ *be a measurable space. For all* $p \in \mathbb{R}_1$ *we define the following Banach spaces*

$$\mathcal{L}^p(\mathscr{X}, \mathscr{Y}) := \left\{ f : X \to Y \;\middle|\; \begin{array}{l} \|f\|_{\mathcal{L}^p(\mathscr{X},\mathscr{Y})} := \left(\int_X \|f\|_Y^p d\eta \right)^{\frac{1}{p}} < \infty, \\ f \in \mathcal{M}(\mathscr{X}, \mathscr{Y}) \end{array} \right\}. \tag{III.7}$$

Remark III.8. *As it is often done in academic literature, we will not consider explicitly the dependence of* $\mathcal{L}^p(\cdot,\cdot)$ *spaces on equivalence classes. We will work directly with the Definition III.7 and when necessary acknowledge any issues arising from such dependence.*

Definition III.9. *For all* $p \in \mathbb{R}_1$ *we introduce the following spaces of stochastic processes.*

$$L^p_{ad} := \{\xi \in \mathcal{L}^p(\mathbf{MP}, \mathbf{M}^{\mathbb{R}}) \mid \xi \textit{ is adapted to } \mathbb{F}\}. \tag{III.8}$$

Remark III.10. *Suppose that* $p \geq 2$ *and* $\xi \in L^p_{ad}$*. Then* $\xi \in L^2_{ad}$ *by Theorem IV.2 and by Fubini Theorem IV.18 we also see that*

$$\int_0^T \mathbb{E}\left[|\xi(t)|^2\right] dt < \infty. \tag{III.9}$$

This fact allows us to conclude that if $p \geq 2$ *then every process in* L^p_{ad} *can be stochastically integrated with respect to the standard Wiener proces. See [29] and section IV.3 for more details.*

III.2.4 $\mathbb{Y}$ spaces

In the previous section we worked with an abstract scale $\mathbb{X}$. In this section however we will be working with the scale $\mathscr{L}^p$ and so we redefine accordingly our definition of the scale $\mathbb{Y}^p$ which first appeared in subsection II.2.4. In this section we will be working with the following definition.

Definition III.11. *For all $p \in \mathbb{R}_1$ and all $\mathfrak{a} \in \mathcal{A}$ let*

$$\mathbb{Y}^p_{\mathfrak{a}} := \left\{ \xi \in \mathcal{S}(l^p_{\mathfrak{a}}) \;\middle|\; \begin{array}{l} \|\xi\|_{\mathbb{Y}^p_{\mathfrak{a}}} := \left(\sup \left\{ \mathbb{E}\left[\|\xi\|^p_{l^p_{\mathfrak{a}}} \right] \;\middle|\; t \in \mathcal{T} \right\} \right)^{\frac{1}{p}} < \infty, \\ \xi_x \textit{ is adapted to } \mathbb{F} \textit{ for all } x \in \gamma. \end{array} \right\} \tag{III.10}$$

$$\mathbb{Y}^p := \{\mathbb{Y}^p_{\mathfrak{a}}\}_{\mathfrak{a} \in \mathcal{A}} \tag{III.11}$$

be, respectively, a normed linear space of $l^p_{\mathfrak{a}}$ valued processes and a family of such spaces.

Remark III.12. *See also Remark II.27 from the previous section with addresses equivalence classes and additional questions which may arise from the Definition III.11 above.*

Theorem III.13. *Let $p \in \mathbb{R}_1$, $\mathfrak{a} \in \mathcal{A}$ and suppose that $\xi \in \mathbb{Y}^p_{\mathfrak{a}}$. Then $\xi_x \in L^p_{ad}$ for all $x \in \gamma$.*

Proof. From Definition III.9 and III.11 we see that to complete the proof we need to show that for all $x \in \gamma$ we have $\xi_x \in \mathcal{L}^p(\mathbf{MP}, \mathbf{M}^{\mathbb{R}})$. Let us begin by fixing $x \in \gamma$ and establishing that $\xi_x \in \mathcal{M}(\mathbf{MP}, \mathbf{M}^{\mathbb{R}})$. To this end we define maps:

$$\mathscr{I}^x : l^p_{\mathfrak{a}} \to l^p_{\mathfrak{a}}, \quad \mathscr{R}^x : \mathbb{R} \to l^p_{\mathfrak{a}}, \quad \xi|_x : \overline{\Omega} \to l^p_{\mathfrak{a}}$$

using the following formulae

$$\mathscr{I}^x(\psi)_y := \left\{ \begin{array}{l|l} \psi_y & y \in \gamma \wedge y = x \\ 0 & y \in \gamma \wedge y \neq x \end{array} \right.$$

$$\mathscr{R}^x(z)_y := \left\{ \begin{array}{l|l} z & y \in \gamma \wedge y = x \\ 0 & y \in \gamma \wedge y \neq x \end{array} \right.$$

$$\xi|_x := \mathscr{I}^x(\xi).$$

Now observe that for each $x \in \gamma$ map $\mathscr{I}^x$ is continuous which implies that $\xi|_x \in \mathcal{M}(\mathbf{MP}, \mathbf{M}^p_{\mathfrak{a}})$. Moreover observe that each $x \in \gamma$ map $\mathscr{R}^x$ is continuous and $\xi|_x = \mathscr{R}^x \circ \xi_x$. Consider now arbitary $A := [a, b] \subset \mathbb{R}$ and $x \in \gamma$. By continuity $B := \mathscr{R}^x([a, b])$ is compact and so

$B \in \mathscr{B}(l^p_{\mathfrak{a}})$. Since $\xi|_x \in \mathcal{M}(\mathbf{MP}, \mathbf{M}^p_{\mathfrak{a}})$ it follows that $(\xi|_x)^{-1}(B) \in \overline{\mathcal{F}}$. However

$$(\xi|_x)^{-1}(B) = (\xi_x)^{-1} \circ (\mathscr{R}^x)^{-1}(B) = (\xi_x^{-1})(A)$$

which establishes that $\xi_x \in \mathcal{M}(\mathbf{MP}, \mathbf{M}^{\mathbb{R}})$ for all $x \in \gamma$.

Finally since for all $x \in \gamma$ we have $|\xi_x| \leq e^{\frac{\mathfrak{a}}{p}|x|} \|\xi\|_{l^p_{\mathfrak{a}}}$ we may now conclude using Theorem IV.9 that $\xi_x \in \mathcal{L}^p(\mathbf{MP}, \mathbf{M}^{\mathbb{R}})$ for all $x \in \gamma$ and the proof is complete. □

Remark III.14. *In simple terms, Theorem III.13 above shows that, for all $p \in \mathbb{R}_1$ and $\mathfrak{a} \in \mathcal{A}$, component processes of each $\xi \in \mathbb{Y}^p_{\mathfrak{a}}$ can be stochastically integrated with respect to the standard Wiener process.*

Theorem III.15. *Let $p \in \mathbb{R}_1$ and $\mathfrak{a} \in \mathcal{A}$. Then $\mathbb{Y}^p_{\mathfrak{a}}$ is a Banach space.*

Proof. Suppose that $\mathscr{X} := \{\xi^n\}_{n \in \mathbb{N}}$ is a Cauchy sequence in $\mathbb{Y}^p_{\mathfrak{a}}$. From Theorem II.28 we know that there exists $\xi \in \mathcal{S}(l^p_{\mathfrak{a}})$ such that

$$\xi = \overbrace{\lim_{n \to \infty} \xi^n}^{\text{in } \mathbb{Y}^p_{\mathfrak{a}}}.$$

Hence to conclude this proof we fix $x \in \gamma$ and show that ξ_x is adapted to $\mathbb{F}$. To this end we also fix $t \in \mathcal{T}$ and observe that

$$\lim_{n \to \infty} \|\xi^n_{x,t} - \xi_{x,t}\|_{\mathcal{L}^p(\mathbf{P}, \mathbf{M}^{\mathbb{R}})} = 0.$$

Therefore using Theorem IV.12 we find a subsequence σ such that $\xi^{\sigma(n)}_{x,t} \to \xi_{x,t}$ almost surely as $n \to \infty$. Since $\xi^{\sigma(n)}_{x,t}$ is $\mathcal{F}_t$ measurable for all $n \in \mathbb{N}$ we conclude by Theorem IV.4 that $\xi_{x,t}$ is also $\mathcal{F}_t$ measurable and the proof is complete. □

Theorem III.16. *Suppose that $p \in \mathbb{R}_1$. Then $\mathbb{Y}^p$ is the scale.*

Proof. Follows immediately from Theorem II.29 in the previous section. □

III.2.5 Stochastic System

Let us start this subsection my making the following redefinition of the constant $\mathfrak{p}$ from the previous section. That is for the rest of this section we shall agree that $\mathfrak{p} \in \mathbb{R}_2$ is fixed.

Now, for the remainder of this section our focus shall be fixed on finding a solution for a system of SDEs of the following form

$$d\xi_{x,t} = \Phi_x(\xi_{x,t}, \Xi_t)dt + \Psi_x(\xi_{x,t}, \Xi_t)dW_x(t), \quad x \in \gamma, \quad t \in \mathfrak{T}.$$

Speaking more precisely we shall in fact be mainly concerned, as in the previous section, with an equivalent problem. That is our goal is to find a unique strong solution of the following system stochastic integral equations

$$\xi_{x,t} = \zeta_x + \int_0^t \Phi_x(\xi_{x,s}, \Xi_s)ds + \int_0^t \Psi_x(\xi_{x,s}, \Xi_s)dW_x(s), \quad x \in \gamma, \quad t \in \mathfrak{T} \tag{III.12}$$

under the following conditions and additional assumptions:

(1) We assume that $\zeta \in l^{\mathfrak{p}}_{\mathfrak{a}}$.

(2) We let V in $C(\mathbb{R})$ and assume that for all $x \in \gamma$ maps $\Phi_x : \mathbb{R} \times l^{\mathfrak{p}}_{\mathfrak{a}} \to \mathbb{R}$ are measurable and defined in the following way

$$\Phi_x(q, \{z_y\}_{y\in\gamma}) := V(q) + \sum_{y \in B_x} a(x-y)z_y \tag{III.13}$$

for all $q \in \mathbb{R}$ and all $\{z_y\}_{y\in\gamma} \in l^{\mathfrak{p}}_{\mathfrak{a}}$, where function a was defined by assumption (**A**).

(**C**) There exists $c \in \mathbb{R}_0$ and $R \leq \mathfrak{p}$ such that for all $q \in \mathbb{R}$ and all $x \in \gamma$

$$|\Phi_x(q, 0)| \leq c(1 + |q|^R). \tag{III.14}$$

(**D**) There exists $b \in \mathbb{R}$ such that for all $q_1, q_2 \in \mathbb{R}$ and all $x \in \gamma$

$$(q_1 - q_2)(\Phi_x(q_1, 0) - \Phi_x(q_2, 0)) \leq b(q_1 - q_2)^2. \tag{III.15}$$

(3) We assume that for all $x \in \gamma$ maps $\Psi_x : \mathbb{R} \times l^{\mathfrak{p}}_{\mathfrak{a}} \to \mathbb{R}$ are measurable.

(E) There exists $M_1, M_2 \in \mathbb{R}$ such that for all $q_1, q_2 \in \mathbb{R}$, $Z_1, Z_2 \in l^{\mathfrak{p}}_{\mathfrak{a}}$ and all $x \in \gamma$

$$|\Psi_x(q_1, Z_1) - \Psi_x(q_2, Z_2)| \leq M_1|q_1 - q_2| + M_2 n_x \sum_{y \in B_x} |z_{1,y} - z_{2,y}| \tag{III.16}$$

$$|\Psi_x(0,0)| \leq c. \tag{III.17}$$

The main goal of this document is to show that stochastic system (III.12) admits a unique strong solution. In order to achieve this goal we need to agree on the definition of a strong solution. Bellow we propose a similar definition to the one given in the previous section.

Definition III.17. *A stochastic process Ξ is called a strong solution of the system (III.12) if*

$$\Xi \in_{\downarrow} \mathbb{Y}^{\mathfrak{p}}$$

and

$$\xi_x \approx \zeta_x + \int_0^\cdot \Phi_x(\xi_{x,s}, \Xi_s)ds + \int_0^\cdot \Psi_x(\xi_{x,s}, \Xi_s)dW_x(s), \ \forall (x \in \gamma).$$

We now conclude this subsection with the following theorem. Existence and uniqueness of a strong solution of system (III.12) will be proved over the couple of subsequent subsections.

Theorem III.18. *Suppose that $q_1, q_2 \in \mathbb{R}$ and $Z_1, Z_2 \in l^{\mathfrak{p}}_{\mathfrak{a}}$. Moreover for all $x \in \gamma$ let*

$$\tilde{a}_x = \left(\sum_{y \in B_x} a^2(x-y) \right)^{\frac{1}{2}}.$$

Then for all $x \in \gamma$ we have the following two inequalities:

$$|\Phi_x(q_1, Z_1)| \leq c(1 + |q_1|^R) + \tilde{a}_x \left(\sum_{y \in B_x} z_{1,y}^2 \right)^{\frac{1}{2}}, \tag{III.18}$$

$$(q_1 - q_2)(\Phi_x(q_1, Z_1) - \Phi_x(q_2, Z_2)) \leq (b + \frac{1}{2})(q_1 - q_2)^2 + \frac{1}{2}\tilde{a}_x^2 \sum_{y \in B_x} (z_{1,y} - z_{2,y})^2. \tag{III.19}$$

Proof. First we fix some $x \in \gamma$ and prove inequality (III.18). Using the definition of Φ_x we begin by observing that

$$|\Phi_x(q_1, Z_1)| = |\frac{1}{2}V(q_1) - \sum_{y \in B_x} a(x-y) z_{1,y}|.$$

Hence it follows that we have the following estimate

$$|\Phi_x(q_1, Z_1)| = |\Phi_x(q_1, 0) - \sum_{y \in B_x} a(x-y) z_{1,y}| \tag{III.20}$$

$$\leq |\Phi_x(q_1, 0)| + |\sum_{y \in B_x} a(x-y) z_{1,y}|.$$

Therefore using assumption (**C**) we see that

$$|\Phi_x(q_1, Z_1)| \leq c(1 + |q_1|^R) + \sum_{y \in B_x} |a(x-y) z_{1,y}|$$

$$\leq c(1 + |q_1|^R) + \left(\sum_{y \in B_x} a^2(x-y) \right)^{\frac{1}{2}} \left(\sum_{y \in B_x} z_{1,y}^2 \right)^{\frac{1}{2}}.$$

Hence using the definition of $\tilde{a}_x$ above we see that

$$|\Phi_x(q_1, Z_1)| \leq c(1 + |q_1|^R) + \tilde{a}_x \left(\sum_{y \in B_x} z_{1,y}^2 \right)^{\frac{1}{2}}$$

which establishes that inequality (III.18) is true. Now, keeping $x \in \gamma$ fixed, we show that inequality (III.19) above is also true. Let us start by defining the following two abbreviations

$$U_x := \Phi_x(q_1, Z_1) - \Phi_x(q_2, Z_2), \quad \forall (x \in \gamma),$$

$$W_x := \sum_{y \in B_x} a(x-y)(z_{1,y} - z_{2,y}), \ \forall (x \in \gamma).$$

Now we observe from equation (III.13) and (III.20) that

$$(q_1 - q_2) U_x = (q_1 - q_2)(\Phi_x(q_1, 0) - \Phi_x(q_2, 0)) + (q_1 - q_2) W_x.$$

Hence using assumption (**D**) we see that

$$(q_1 - q_2) U_x \leq b(q_1 - q_2)^2 + \frac{1}{2}(q_1 - q_2)^2 + \frac{1}{2}\left(\sum_{y \in B_x} a(x-y)(z_{1,y} - z_{2,y}) \right)^2$$

$$\leq (b + \frac{1}{2})(q_1 - q_2)^2 + \frac{1}{2} \sum_{y \in B_x} a^2(x-y) \sum_{y \in B_x} (z_{1,y} - z_{2,y})^2.$$

Finally using, once again, the definition of $\tilde{a}_x$ above we see that

$$(q_1 - q_2)(\Phi_x(q_1, Z_1) - \Phi_x(q_2, Z_2)) \leq (b + \frac{1}{2})(q_1 - q_2)^2 + \frac{1}{2}\tilde{a}_x^2 \sum_{y \in B_x} (z_{1,y} - z_{2,y})^2$$

and the proof is complete. □

III.3 Auxiliary Results

III.3.1 Ovsjannikov map on $\mathcal{L}^1$

In this section we prove two results that will be used later on to show that stochastic system (III.12) admits a unique strong solution.

Theorem III.19. *Suppose that* $\mathfrak{a} \in \mathcal{A}$ *and let* $\Xi := \{\xi_x\}_{x\in\gamma}$ *be an element in* $\mathbb{Y}^{\mathfrak{p}}_{\mathfrak{a}}$. *Then for all* $x \in \gamma$ *we have* $\Phi_x(\xi_x, \Xi) \in L^1_{ad}$ *and* $\Psi_x(\xi_x, \Xi) \in L^2_{ad}$.

Proof. We combine Theorems III.13 and IV.2 to conclude that for all $x \in \gamma$ we have

$$\xi_x \in L^{\mathfrak{p}}_{ad} \subset L^2_{ad} \subset L^1_{ad}.$$

Since composition of measurable maps is measurable we conclude that $x \in \gamma$ we have

$$\Phi_x(\xi_x, \Xi), \Psi_x(\xi_x, \Xi) \in \mathcal{M}(\mathbf{MP}, \mathbf{M}^{\mathbb{R}})$$

and adapted to $\mathbb{F}$. Now according to the definition (III.13) and the assumption (**C**) we have for all $x \in \gamma$ the following inequality

$$|\Phi_x(\xi_x, \Xi)| \le |c|(1 + |\xi_x|^R) + \sum_{y\in B_x} a(x-y)|\xi_y|.$$

Moreover, because $R \le \mathfrak{p}$ we can use Theorem IV.9 to conclude that $\Phi_x(\xi_x, \Xi) \in L^1_{ad}$. Finally we combine Theorem IV.16 with the assumption (**E**) to conclude that for all $x \in \gamma$ we have

$$|\Psi_x(\xi_x, \Xi)|^2 \le 4|\Psi_x(0,0)|^2 + 4M_1^2|\xi_x|^2 + 4M_2^2 n_x^3 \sum_{y\in B_x} |\xi_x|^2. \tag{III.21}$$

Now, once again applying Theorem IV.9 to the inequality (III.21) above we conclude that $\Psi_x(\xi_x, \Xi) \in L^2_{ad}$ hence the proof is complete. □

Theorem III.20. *Suppose that* $\mathfrak{q} \in (0, 1)$ *and let* $Q := \{Q_{x,y}\}_{x,y\in\gamma}$ *be an infinite real matrix such that for all* $x, y \in \gamma$ *we have the following implication*

$$x \notin B_y \iff Q_{x,y} = 0 \iff y \notin B_x. \tag{III.22}$$

Moreover assume that for all $x, y \in \gamma$ there exist $C \in \mathbb{R}_0$ and $q \in \mathbb{R}_1$ such that

$$|Q_{x,y}| \leq C n_x^q. \tag{III.23}$$

Then $Q \in \mathcal{O}(\mathscr{L}^1, \mathfrak{q})$. That is Q is an Ovsjannikov map of order $\mathfrak{q}$ on $\mathscr{L}^1$.

Proof. Consider arbitrary $\alpha < \beta \in \mathcal{A}$ and fix $z \in l^1_\alpha$. We will complete this proof by showing that the following inequality holds

$$\|Qz\|_\beta \leq \frac{L}{(\beta - \alpha)^{\mathfrak{q}}} \|z\|_\alpha. \tag{III.24}$$

Since Q is linear, inequality (III.24) above automatically verifies conditions (1) and (2) of the Definition II.4 and also shows that $Q : ^{\uparrow}\mathscr{L}^1 \to l^1_{\mathfrak{a}}$.

Remark III.21. *Using assumption (**B**) we see that there exists $M, \overline{\mathfrak{N}} \in \mathbb{N}$ such that*

$$M < |x| \implies n_x^q \leq \overline{\mathfrak{N}} |x|^{\frac{\mathfrak{q}}{2}}.$$

Moreover because γ is a locally finite subset of $\mathbb{R}^d$ we know that $\overline{B(0, M)} \cap \gamma$ has only a finite number of elements. Hence in this proof we can assume without loss of generality that $M < |x|$ for all $x \in \gamma$.

Consider now the following equation

$$\|Qz\|_\beta = \sum_{x \in \gamma} e^{-\beta |x|} \left| \sum_{y \in \gamma} Q_{x,y} z_y \right|. \tag{III.25}$$

Moreover, for all $x \in \gamma$ we will make use of the following facts

I. $x \notin B_y \vee y \notin B_x \implies Q_{x,y} = 0.$

II. $y \in B_x \implies -|x| \leq -|y| + \rho.$

III. $x \in B_y \implies |x|^{\frac{\mathfrak{q}}{2}} \leq |y|^{\frac{\mathfrak{q}}{2}} + \rho^{\frac{\mathfrak{q}}{2}}.$

Now, using equation (III.25) together with the facts **I** and **II** we see that

$$\begin{aligned}\|Qz\|_\beta &\le \sum_{x\in\gamma}\sum_{y\in\gamma}|Q_{x,y}|e^{-\beta|x|}|z_y| \\ &\le e^{\beta\rho}\sum_{x\in\gamma}\sum_{y\in B_x}|Q_{x,y}|e^{-\beta|y|}|z_y| \\ &\le e^{\beta\rho}\sum_{x\in\gamma}\sum_{y\in B_x}|Q_{x,y}|e^{-(\beta-\alpha)|y|}e^{-\alpha|y|}|z_y|. \qquad \text{(III.26)}\end{aligned}$$

Hence from inequality (III.26) we see that

$$\begin{aligned}\|Qz\|_\beta &\le e^{\beta\rho}\sum_{x\in\gamma}\sum_{y\in\gamma}|Q_{x,y}|e^{-(\beta-\alpha)|y|}e^{-\alpha|y|}|z_y| \\ &= e^{\beta\rho}\sum_{y\in\gamma}\sum_{x\in\gamma}|Q_{x,y}|e^{-(\beta-\alpha)|y|}e^{-\alpha|y|}|z_y| \\ &\le e^{\bar{\mathfrak{a}}\rho}K\|z\|_\alpha, \qquad \text{(III.27)}\end{aligned}$$

where

$$K := \sup\left\{\sum_{x\in\gamma}|Q_{x,y}|e^{-(\beta-\alpha)|y|}\ \middle|\ y\in\gamma\right\}. \qquad \text{(III.28)}$$

We now estimate the value of supremum in the definition (III.28) above. Hence using condition (III.23) together with the fact **I** we see that for all $y\in\gamma$

$$\begin{aligned}\sum_{x\in\gamma}|Q_{x,y}|e^{-(\beta-\alpha)|y|} &= \sum_{x\in B_y}|Q_{x,y}|e^{-(\beta-\alpha)|y|} \\ &\le C\sum_{x\in B_y}n_x^q e^{-(\beta-\alpha)|y|}.\end{aligned}$$

Using now assumption (**B**) together with the fact **III** and Remark III.21 we see that

$$\sum_{x\in\gamma}|Q_{x,y}|e^{-(\beta-\alpha)|y|} \le C\sum_{x\in B_y}\overline{\mathfrak{N}}|x|^{\frac{\mathfrak{q}}{2}}e^{-(\beta-\alpha)|y|}.$$

Hence we get the following chain of inequalities

$$\begin{aligned}\sum_{x\in\gamma}|Q_{x,y}|e^{-(\beta-\alpha)|y|} &\leq C\overline{\mathfrak{N}}\sum_{x\in B_y}(|y|^{\frac{\mathfrak{q}}{2}}+\rho^{\frac{\mathfrak{q}}{2}})e^{-(\beta-\alpha)|y|}\\ &\leq C\overline{\mathfrak{N}}n_y|y|^{\frac{\mathfrak{q}}{2}}e^{-(\beta-\alpha)|y|}+C\overline{\mathfrak{N}}\rho^{\frac{\mathfrak{q}}{2}}\sum_{x\in\gamma}n_xe^{-(\beta-\alpha)|x|}\\ &\leq C\overline{\mathfrak{N}}|y|^{\frac{\mathfrak{q}}{2}}|y|^{\frac{\mathfrak{q}}{2}}e^{-(\beta-\alpha)|y|}+B\\ &\leq C\overline{\mathfrak{N}}|y|^{\mathfrak{q}}e^{-(\beta-\alpha)|y|}+B\end{aligned}$$

where $B := C\overline{\mathfrak{N}}\rho^{\frac{\mathfrak{q}}{2}}\sum_{x\in\gamma}n_xe^{-(\beta-\alpha)|x|}$ can be defined using a slight variation of Theorem III.4.

Now returning to equation (III.28) we see that

$$\begin{aligned}K &\leq B+C\overline{\mathfrak{N}}\sup\left\{|y|^{\mathfrak{q}}e^{-(\beta-\alpha)|y|}\;\middle|\;y\in\gamma\right\}\\ &\leq B+C\overline{\mathfrak{N}}\sup\left\{h^{\mathfrak{q}}e^{-(\beta-\alpha)h}\;\middle|\;h>0\right\}\\ &\leq B+C\overline{\mathfrak{N}}\sup\left\{\left(he^{-\frac{\beta-\alpha}{\mathfrak{q}}h}\right)^{\mathfrak{q}}\;\middle|\;h>0\right\}\\ &\leq B+C\overline{\mathfrak{N}}\left(\sup\left\{he^{-\frac{\beta-\alpha}{\mathfrak{q}}h}\;\middle|\;h>0\right\}\right)^{\mathfrak{q}}.\end{aligned} \tag{III.29}$$

Now, we can deduce that function $he^{-\frac{\beta-\alpha}{\mathfrak{q}}h} : (0,\infty)\to\mathbb{R}$ attains its supremum when $\frac{d}{dh}he^{-\frac{\beta-\alpha}{\mathfrak{q}}h}=0$ that is when $h=\frac{\mathfrak{q}}{(\beta-\alpha)}$. Hence it follows from inequality (III.29) that

$$K \leq \frac{B(\overline{\mathfrak{a}}-\underline{\mathfrak{a}})^{\mathfrak{q}}+C\overline{\mathfrak{N}}\mathfrak{q}^{\mathfrak{q}}}{(\beta-\alpha)^{\mathfrak{q}}}.$$

Now, continuing from equation (III.27) we finally see that

$$\begin{aligned}\|Qz\|_\beta &\leq e^{\underline{\mathfrak{a}}\rho}K\|z\|_\alpha\\ &\leq \frac{4e^{\underline{\mathfrak{a}}\rho}(B(\overline{\mathfrak{a}}-\underline{\mathfrak{a}})^{\mathfrak{q}}+C\overline{\mathfrak{N}}\mathfrak{q}^{\mathfrak{q}})}{(\beta-\alpha)^{\mathfrak{q}}}\|z\|_\alpha\end{aligned} \tag{III.30}$$

hence the proof is complete. □

III.3.2 Comparison Theorem

Remark III.22. *In the following Theorem we will describe an equation of the form*

$$f(t) = z_{\underline{\mathfrak{a}}} + \int_0^t Q(f(s))ds, \quad t \in \mathcal{T} \tag{III.31}$$

and rely on our work in subsection IV.4 to conclude, with the choice

$$\mathbb{X} \equiv \mathscr{L}^1 \quad and \quad F \equiv Q,$$

that equation (III.31) has a unique solution, in the context of Theorem IV.43.

Theorem III.23 (Comparison Theorem)**.**
Suppose $z_{\underline{\mathfrak{a}}} \in l^1_{\underline{\mathfrak{a}}}$, $q < 1$ *and matrix* $Q := \{Q_{x,y}\}_{x,y\in\gamma}$ *is an element of* $\mathcal{O}(\mathscr{L}^1, q)$. *Moreover suppose that* $Q_{x,y} \geq 0$ *for all* $x, y \in \gamma$ *and, in the context of Theorem (IV.43), let* f *be the unique solution of the integral equation*

$$f(t) = z_{\underline{\mathfrak{a}}} + \int_0^t Q(f(s))ds, \quad t \in \mathcal{T}. \tag{III.32}$$

Finally, suppose that $g : \mathcal{T} \to l^1_{\underline{\mathfrak{a}}}$ *is a bounded map such that for all* $x \in \gamma$

$$g_x(t) \leq z_{\underline{\mathfrak{a}},x} + \left[\int_0^t Q(g(s))ds\right]_x, \quad t \in \mathcal{T}. \tag{III.33}$$

Then for all $t \in \mathcal{T}$ *and all* $x \in \gamma$

$$g_x(t) \leq f_x(t). \tag{III.34}$$

Proof. For all $\mathfrak{a} \in \mathcal{A}$ let $H_{\mathfrak{a}} = \mathcal{B}(\mathcal{T}, l^1_{\mathfrak{a}})$ (see Remark IV.36) and define the following family $\mathbf{H} := \{H_{\mathfrak{a}}\}_{\mathfrak{a}\in\mathcal{A}}$. It follows from subsection IV.4 that $\mathbf{H}$ is a scale. Moreover from Theorem IV.37 we know that map $\mathcal{I} : \overset{\uparrow}{\mathbf{H}} \to H_{\overline{\mathfrak{a}}}$ defined for all $t \in \mathcal{T}$ and all $\kappa \in H_\alpha$ via formula

$$\mathcal{I}(\kappa)(t) := z_{\underline{\mathfrak{a}}} + \int_0^t Q(\kappa(s))ds$$

is an Ovsjannikov map of order q on $\mathbf{H}$. That is $\mathcal{I} \in \mathcal{O}(\mathbf{H}, q)$.

Therefore, using Theorem IV.42, we see that if $\underline{\mathfrak{a}} < \beta \in \mathcal{A}$ then the sequence $\{\mathcal{I}^n(g)\}_{n\in\mathbb{N}}$ defined recursively in the following way

$$\left.\begin{aligned} \mathcal{I}^1(g)(t) &:= z_{\underline{\mathfrak{a}}} + \textstyle\int_0^t Q(g(s))ds \\ &\vdots \\ \mathcal{I}^{n+1}(g)(t) &:= \mathcal{I}(\mathcal{I}^n(g))(t) \end{aligned}\right\}, \ \forall (t \in \mathcal{T})$$

is such that

$$\overbrace{\left[\lim_{n\to\infty} \mathcal{I}^n(g)\right]}^{\text{in } \mathcal{B}([0,T],l^1_\beta)} = f.$$

Convergence in the supremum norm therefore implies that $\lim_{n\to\infty} \mathcal{I}^n_x(g)(t) = f_x(t)$ for all $x \in \gamma$ and all $t \in \mathcal{T}$. Hence to conclude the proof it is sufficient to fix $x \in \gamma$ and $t \in \mathcal{T}$ and prove by induction that

$$g_x(t) \leq \mathcal{I}^n_x(g)(t), \ \forall (n \in \mathbb{N}). \tag{III.35}$$

Case $n = 1$ is satisfied by the initial assumption on g, so let us now assume that the induction hypothesis (III.35) is true for some $n \geq 1$ and proceed by considering the following chain of inequalities

$$\begin{aligned} \mathcal{I}^{n+1}_x(g)(t) &= \mathcal{I}_x(\mathcal{I}^n(g))(t) & \text{(III.36)} \\ &= z_{\underline{\mathfrak{a}},x} + \left[\int_0^t Q(\mathcal{I}^n(g)(s))ds\right]_x \\ &= z_{\underline{\mathfrak{a}},x} + \sum_{y\in\gamma} Q_{x,y} \int_0^t \mathcal{I}^n_y(g)(s)ds \\ &\geq z_{\underline{\mathfrak{a}},x} + \sum_{y\in p} Q_{x,y} \int_0^t g_y(s)ds \\ &= z_{\underline{\mathfrak{a}},x} + \left[\int_0^t Q(g(s))ds\right]_x \\ &\geq g_x(t). & \text{(III.37)} \end{aligned}$$

Finally from inequalities (III.36) - (III.37) above we conclude that inequality (III.35) also holds hence the proof is complete. □

Corollary III.24. *Suppose that $z_{\underline{\mathfrak{a}},x} \geq 0$ for all $x \in \gamma$. Moreover assume that components of g are non-negative functions, that is $g_x(t) \geq 0$ for all $x \in \gamma$ and all $t \in \mathfrak{T}$. Then for all $\beta > \alpha \in \mathcal{A}$ there exists a constant $K(\alpha, \beta) \in \mathbb{R}$ such that*

$$\sum_{x\in\gamma} e^{-\beta|x|} \sup_{t\in\mathfrak{T}} g_x(t) \leq K(\alpha,\beta) \sum_{x\in\gamma} e^{-\alpha|x|} z_{\underline{\mathfrak{a}},x}. \tag{III.38}$$

Proof. Using Theorem III.23, we begin this proof by making an observation that for all $x \in \gamma$ and all $t \in \mathfrak{T}$ the following inequality holds

$$\begin{aligned} g_x(t) &\leq z_{\underline{\mathfrak{a}},x} + \left[\int_0^t Q(g(s))ds \right]_x \\ &\leq z_{\underline{\mathfrak{a}},x} + \left[\int_0^t Q(f(s))ds \right]_x . \end{aligned}$$

Therefore we see that for all $x \in \gamma$

$$\begin{aligned} \sup_{t\in\mathfrak{T}} g_x(t) &\leq z_{\underline{\mathfrak{a}},x} + \left[\int_0^T Q(f(s))ds \right]_x \\ &= f_x(T). \end{aligned} \tag{III.39}$$

Hence it follows that

$$\begin{aligned} \sum_{x\in\gamma} e^{-\beta|x|} \sup_{t\in\mathfrak{T}} g_x(t) &\leq \sum_{x\in\gamma} e^{-\beta|x|} f_x(T) \\ &\leq \|f(T)\|_{l^1_\beta}. \end{aligned} \tag{III.40}$$

Norm in the inequality (III.40) above can be estimated using Theorem IV.45 and remark that proceeds it. In particular we get

$$\|f(T)\|_{l^1_\beta} \leq \sum_{n=0}^{\infty} \frac{L^n T^n}{(\beta-\alpha)^q} \frac{n^q}{n!} \|z_{\underline{\mathfrak{a}}}\|_{l^1_\alpha}.$$

Finally letting $K(\alpha,\beta)=\sum_{n=0}^{\infty}\frac{L^nT^n}{(\beta-\alpha)^q}\frac{n^q}{n!}$ we see that

$$\sum_{x\in\gamma} e^{-\beta|x|}\sup_{t\in\mathfrak{T}} g_x(t)\leq K(\alpha,\beta)\|z_{\underline{\mathfrak{a}}}\|_{l^1_\alpha}$$

hence the proof is complete. $\square$

III.4 Truncated Systems

We now start working with a sequence $\{\Lambda_n\}_{n\in\mathbb{N}}$ of finite subsets of γ such that $\Lambda_n \uparrow \gamma$ as $n \to \infty$. Moreover for each $n \in \mathbb{N}$ we now wish to introduce and study the following system of stochastic integral equations

$$\begin{aligned} \xi^n_{x,t} &= \zeta_x + \int_0^t \Phi_x(\xi^n_{x,s}, \Xi^n_s)ds + \int_0^t \Psi_x(\xi^n_{x,s}, \Xi^n_s)dW_x(s), \quad \forall x \in \Lambda_n \wedge t \in \mathfrak{T} \\ \xi^n_{x,t} &= \zeta_x, \qquad \forall x \notin \Lambda_n \wedge t \in \mathfrak{T} \end{aligned} \tag{III.41}$$

We would like to note that for each $n \in \mathbb{N}$ stochastic system (III.41) is a stopped/truncated version of our original stochastic system (III.12), which was described in subsection III.2.5.

In this section our goal is to prove two important results concerning system (III.41). In the subsequent sections these two results will help us establish that system (III.12) admits a unique strong solution. Now, relying on [3, 20] and in particular on [32] we state our next result without a proof.

Theorem III.25. *For all $n \in \mathbb{N}$ and $\zeta \in l^{\mathfrak{p}}_{\underline{\mathfrak{a}}}$ system (III.41) has a solution $\Xi^n \in \mathbb{Y}^{\mathfrak{p}}_{\underline{\mathfrak{a}}}$.*

Remark III.26. *A term solution in the Theorem III.25 above is to be understood in the same sense as explained in the Definition II.32 except we do not require Ξ^n to be an element of $\downarrow\mathbb{Y}^{\mathfrak{p}}$.*

Remark III.27. *Combining Theorems (III.25) and (III.19) with the Definition (IV.34) we see that ξ^n_x in an Itô process for all $n \in \mathbb{N}$ and $x \in \gamma$.*

In the next two sections of this document it will be shown that the sequence $\{\Xi^n\}_{n\in\mathbb{N}}$ converges to the unique strong solution of the system (III.12). However before this can be achieved we need to establish the following two theorems.

Theorem III.28. *Suppose that $n \in \mathbb{N}$ and let Ξ^n be the process defined by Theorem III.25. Moreover all $x \in \gamma$ let ξ^n_x be components of Ξ^n. Then for all $\underline{\mathfrak{a}} < \alpha \in \mathcal{A}$ we have*

$$\sum_{x\in\gamma} e^{-\alpha|x|} \sup_{n\in\mathbb{N}} \sup_{t\in\mathfrak{T}} \mathbb{E}\Big[|\xi^n_{x,t}|^{\mathfrak{p}} \Big] < \infty. \tag{III.42}$$

Proof. Let us start by recalling that

$$\xi^n_{x,t} = \zeta_x + \int_0^t \Phi_x(\xi^n_{x,s}, \Xi^n_s)ds + \int_0^t \Psi_x(\xi^n_{x,s}, \Xi^n_s)dW_x(s), \quad \forall x \in \Lambda_n \wedge t \in \mathfrak{T}$$

$$\xi^n_{x,t} = \zeta_x, \qquad \forall x \notin \Lambda_n \wedge t \in \mathfrak{T}.$$

Hence using Itô Lemma IV.35 we see that if $x \in \Lambda_n$ then for all $t \in \mathfrak{T}$

$$|\xi^n_{x,t}|^{\mathfrak{p}} = |\zeta_x|^{\mathfrak{p}} + \int_0^t \mathfrak{p}(\xi^n_{x,s})^{\mathfrak{p}-1}\Phi_x(\xi^n_{x,s}, \Xi^n_s)ds + \\ + \int_0^t \frac{(\mathfrak{p}-1)\mathfrak{p}}{2}(\xi^n_{x,s})^{\mathfrak{p}-2}(\Psi_x(\xi^n_{x,s}, \Xi^n_s))^2 ds + \\ + \int_0^t \mathfrak{p}(\xi^n_{x,s})^{\mathfrak{p}-1}\Psi_x(\xi^n_{x,s}, \Xi^n_s)dW_x(s).$$

Now from assumptions (**C**), (**D**) and Theorem III.18 we can deduce that for all $t \in \mathfrak{T}$

$$\begin{aligned}(\xi^n_{x,t})^{\mathfrak{p}-1}\Phi_x(\xi^n_{x,t}, \Xi^n_t) &= (\xi^n_{x,t})^{\mathfrak{p}-2}(\xi^n_{x,t})\Phi_x(\xi^n_{x,t}, \Xi^n_t) \\ &\le (\xi^n_{x,t})^{\mathfrak{p}-2}\left[(b+\frac{1}{2})|\xi^n_{x,t}|^2 + \frac{1}{2}\tilde{a}_x^2 \sum_{y\in B_x}|\xi^n_{y,t}|^2 + \xi^n_{x,t}\Phi_x(0,0)\right] \\ &\le (\xi^n_{x,t})^{\mathfrak{p}-2}\left[(b+1)|\xi^n_{x,t}|^2 + \tilde{a}_x^2 \sum_{y\in B_x}|\xi^n_{y,t}|^2 + c^2\right] \\ &\le (b+1)|\xi^n_{x,t}|^{\mathfrak{p}} + \tilde{a}_x^2|\xi^n_{x,t}|^{\mathfrak{p}-2}\sum_{y\in B_x}|\xi^n_{y,t}|^2 + |\xi^n_{x,t}|^{\mathfrak{p}-2}c^2 \\ &\le (b+1)|\xi^n_{x,t}|^{\mathfrak{p}} + \tilde{a}_x^2 n_x \max_{y\in B_x}|\xi^n_{y,t}|^{\mathfrak{p}-2}\max_{y\in B_x}|\xi^n_{y,t}|^2 + |\xi^n_{x,t}|^{\mathfrak{p}-2}c^2 \\ &\le (b+1)|\xi^n_{x,t}|^{\mathfrak{p}} + \tilde{a}_x^2 n_x \max_{y\in B_x}|\xi^n_{y,t}|^{\mathfrak{p}} + (1+|\xi^n_{x,t}|)^{\mathfrak{p}}c^2.\end{aligned}$$

Now using in addition Theorem IV.16 we see that for all $t \in \mathfrak{T}$ we have

$$\begin{aligned}(\xi^n_{x,t})^{\mathfrak{p}-1}\Phi_x(\xi^n_{x,t}, \Xi^n_t) &\le (b+1)|\xi^n_{x,t}|^{\mathfrak{p}} + \tilde{a}_x^2 n_x \sum_{y\in B_x}|\xi^n_{y,t}|^{\mathfrak{p}} + 2^{\mathfrak{p}-1}c^2 + 2^{\mathfrak{p}-1}c^2|\xi^n_{x,t}|^{\mathfrak{p}} \\ &\le (b+1+2^{\mathfrak{p}-1}c^2)|\xi^n_{x,t}|^{\mathfrak{p}} + \tilde{a}_x^2 n_x \sum_{y\in B_x}|\xi^n_{y,t}|^{\mathfrak{p}} + 2^{\mathfrak{p}-1}c^2.\end{aligned}$$

Hence we arrive at the following estimate

$$(\xi^n_{x,t})^{\mathfrak{p}-1}\Phi_x(\xi^n_{x,t},\Xi^n_t) \leq (b+1+2^{\mathfrak{p}-1}c^2)|\xi^n_{x,t}|^{\mathfrak{p}} + \bar{a}^2 n_x^3 \sum_{y\in B_x} |\xi^n_{y,t}|^{\mathfrak{p}} + 2^{\mathfrak{p}-1}c^2. \tag{III.43}$$

Moreover from assumption (**E**) we know that for all $t \in \mathfrak{T}$

$$\begin{aligned}
(\xi^n_{x,s})^{\mathfrak{p}-2}(\Psi_x(\xi^n_{x,s},\Xi^n_s))^2 &\leq (\xi^n_{x,s})^{\mathfrak{p}-2}\Bigg[4M_1^2|\xi^n_{x,t}|^2 + 4M_2^2 n_x^2\Bigg(\sum_{y\in B_x}|\xi^n_{y,t}|\Bigg)^2 + 4|\Psi_x(0,0)|^2\Bigg] \\
&\leq (\xi^n_{x,s})^{\mathfrak{p}-2}\Bigg[4M_1^2|\xi^n_{x,t}|^2 + 4M_2^2 n_x^3 \sum_{y\in B_x}|\xi^n_{y,t}|^2 + 4c^2\Bigg] \\
&\leq 4M_1^2|\xi^n_{x,t}|^{\mathfrak{p}} + 4M_2^2 n_x^4 \max_{y\in B_x}|\xi^n_{y,t}|^{\mathfrak{p}-2}\max_{y\in B_x}|\xi^n_{y,t}|^2 + 4c^2|\xi^n_{x,s}|^{\mathfrak{p}-2} \\
&\leq 4M_1^2|\xi^n_{x,t}|^{\mathfrak{p}} + 4M_2^2 n_x^4 \sum_{y\in B_x}|\xi^n_{y,t}|^{\mathfrak{p}} + 4c^2 2^{\mathfrak{p}-1}(1+|\xi^n_{x,s}|^{\mathfrak{p}}) \\
&\leq (4M_1^2 + 4c^2 2^{\mathfrak{p}-1})|\xi^n_{x,t}|^{\mathfrak{p}} + 4M_2^2 n_x^4 \sum_{y\in B_x}|\xi^n_{y,t}|^{\mathfrak{p}} + 4c^2 2^{\mathfrak{p}-1}. \qquad \text{(III.44)}
\end{aligned}$$

Before proceeding further it is convenient to fix the following notation:

$$A_1 := (b+1+2^{\mathfrak{p}-1}c^2), \tag{III.45}$$

$$A_2 := (4M_1^2 + 4c^2 2^{\mathfrak{p}-1}), \tag{III.46}$$

$$A_3 := (\mathfrak{p}\bar{a}^2 + \mathfrak{p}^2 4M_2^2), \tag{III.47}$$

$$A_4 := 5\mathfrak{p}^2 2^{\mathfrak{p}} c^2 T \tag{III.48}$$

and observe from inequalities (III.43) and (III.44) that for all $x \in \Lambda_n$ and all $t \in \mathfrak{T}$ we have

$$\mathbb{E}\Big[|\xi^n_{x,t}|^{\mathfrak{p}}\Big] \leq \mathfrak{p}^2(A_1+A_2)\int_0^t \mathbb{E}\Big[|\xi^n_{x,s}|^{\mathfrak{p}}\Big]ds + + A_3 n_x^4 \sum_{y\in B_x}\int_0^t \mathbb{E}\Big[|\xi^n_{y,s}|^{\mathfrak{p}}\Big]ds + A_4. \tag{III.49}$$

Now using Definition III.11 together with Theorem III.25 and IV.18 we would like to define a measurable map $\eta^n : \mathfrak{T} \to l^1_{\underline{\mathfrak{a}}}$, that is a map $\eta^n \in \mathcal{M}(\mathbf{M}, \mathbf{M}^1_{\underline{\mathfrak{a}}})$, via the following formula

$$\eta^n_x(t) := \max_{m\leq n}\ \mathbb{E}\Big[|\xi^m_{x,t}|^{\mathfrak{p}}\Big], \quad \forall (t \in \mathfrak{T}).$$

Hence we deduce from the inequality (III.49) and from the system (III.41) that for all $x \in \gamma$

$$\eta_x^n(t) \leq \sum_{y \in \gamma} Q_{x,y} \int_0^t \eta_y^n(s)ds + A_x,\ t \in \mathfrak{T}. \tag{III.50}$$

where

$$Q_{x,y} = \begin{cases} \mathfrak{p}^2(A_1 + A_2) + A_3 n_x^4, & x = y, \\ A_3 n_x^4, & 0 < |x - y| < \rho, \\ 0, & |x - y| > \rho. \end{cases} \tag{III.51}$$

and

$$A_x = |\zeta_x|^p + A_4. \tag{III.52}$$

Moreover the following facts can now also be deduced from (III.50), (III.51) and (III.52).

(1) $A \in l^1_{\underline{\mathfrak{a}}}$ as a result of Theorem III.4 and the choice $\zeta \in l^{\mathfrak{p}}_{\underline{\mathfrak{a}}}$.

(2) By definition of η^n it is clear that for all $t \in \mathfrak{T}$ we have

$$\|\eta^n(t)\|_{l^1_{\underline{\mathfrak{a}}}} \leq \sum_{m \leq n} \|\Xi^m\|^{\mathfrak{p}}_{\mathbb{Y}^{\mathfrak{p}}_{\underline{\mathfrak{a}}}} + \|\zeta\|^{\mathfrak{p}}_{l^{\mathfrak{p}}_{\underline{\mathfrak{a}}}}.$$

Hence we see that $\eta^n \in \mathcal{B}(\mathfrak{T}, l^1_{\underline{\mathfrak{a}}})$.

(3) From equation (III.51) we see that there exists a constant C such that $|Q_{x,y}| \leq C n_x^4$. Therefore using Theorem III.20 we conclude that for some $q \in (0,1)$ matrix Q is the Ovsjannikov operator of order q on $\mathcal{L}^1$.

Now since $n \in \mathbb{N}$ is arbitrary, an application of Theorem III.23 and Corollary III.24 to the inequality (III.50) above tells us that for all $n \in \mathbb{N}$ we have

$$\sum_{x \in \gamma} e^{-\alpha|x|} \sup_{t \in \mathfrak{T}} \eta_x^n(t) \leq K(\underline{\mathfrak{a}}, \alpha) \sum_{x \in \gamma} e^{-\underline{\mathfrak{a}}|x|} |A_x|.$$

Hence we see that

$$\sum_{x\in\gamma} e^{-\alpha|x|} \sup_{t\in\mathfrak{T}} \max_{m\le n} \mathbb{E}\Big[|\xi^m_{x,t}|^{\mathfrak{p}}\Big] \le K(\underline{\mathfrak{a}},\alpha) \sum_{x\in\gamma} e^{-\underline{\mathfrak{a}}|x|}|A_x|.$$

Therefore

$$\sup_{n\in\mathbb{N}} \Big\{ \sum_{x\in\gamma} e^{-\alpha|x|} \sup_{t\in\mathfrak{T}} \max_{m\le n} \mathbb{E}\Big[|\xi^m_{x,t}|^{\mathfrak{p}}\Big]\Big\} \le K(\underline{\mathfrak{a}},\alpha) \sum_{x\in\gamma} e^{-\underline{\mathfrak{a}}|x|}|A_x|. \tag{III.53}$$

Remark III.29. *Consider now arbitrary $x \in \gamma$. It is clear that*

$$\sup_{n\in\mathbb{N}} \Big(\max_{m\le n} \sup_{t\in\mathfrak{T}} \mathbb{E}\Big[|\xi^m_{x,t}|^{\mathfrak{p}}\Big]\Big) \le \sup_{n\in\mathbb{N}} \sup_{t\in\mathfrak{T}} \mathbb{E}\Big[|\xi^n_{x,t}|^{\mathfrak{p}}\Big].$$

Moreover for all $\epsilon > 0$ there exists $k \in \mathbb{N}$ such that

$$\begin{aligned}\sup_{n\in\mathbb{N}} \sup_{t\in\mathfrak{T}} \mathbb{E}\Big[|\xi^n_{x,t}|^{\mathfrak{p}}\Big] - \epsilon &\le \sup_{t\in\mathfrak{T}} \mathbb{E}\Big[|\xi^k_{x,t}|^{\mathfrak{p}}\Big] \\ &\le \max_{m\le k} \sup_{t\in\mathfrak{T}} \mathbb{E}\Big[|\xi^m_{x,t}|^{\mathfrak{p}}\Big] \\ &\le \sup_{n\in\mathbb{N}} \Big(\max_{m\le n} \sup_{t\in\mathfrak{T}} \mathbb{E}\Big[|\xi^m_{x,t}|^{\mathfrak{p}}\Big]\Big).\end{aligned}$$

Since ϵ is arbitrary It follows that

$$\begin{aligned}\sup_{t\in\mathfrak{T}} \sup_{n\in\mathbb{N}} \mathbb{E}\Big[|\xi^n_{x,t}|^{\mathfrak{p}}\Big] &= \sup_{n\in\mathbb{N}} \Big(\max_{m\le n} \sup_{t\in\mathfrak{T}} \mathbb{E}\Big[|\xi^m_{x,t}|^{\mathfrak{p}}\Big]\Big) \\ &= \sup_{n\in\mathbb{N}} \Big(\sup_{t\in\mathfrak{T}} \max_{m\le n} \mathbb{E}\Big[|\xi^m_{x,t}|^{\mathfrak{p}}\Big]\Big).\end{aligned}$$

Remark above shows that if an arbitrary set $A \subset \gamma$ is finite then

$$\sup_{n\in\mathbb{N}} \Big\{ \sum_{x\in A} e^{-\alpha|x|} \sup_{t\in\mathfrak{T}} \max_{m\le n} \mathbb{E}\Big[|\xi^m_{x,t}|^{\mathfrak{p}}\Big]\Big\} = \sum_{x\in A} e^{-\alpha|x|} \sup_{t\in\mathfrak{T}} \sup_{n\in\mathbb{N}} \mathbb{E}\Big[|\xi^m_{x,t}|^{\mathfrak{p}}\Big].$$

Hence from inequality (III.53) we finally learn that

$$\sum_{x\in\gamma} e^{-\alpha|x|} \sup_{t\in\mathcal{T}} \sup_{n\in\mathbb{N}} \mathbb{E}\Big[|\xi^n_{x,t}|^{\mathfrak{p}}\Big] \leq K(\underline{\mathfrak{a}},\alpha) \sum_{x\in\gamma} e^{-\underline{\mathfrak{a}}|x|}|A_x| \tag{III.54}$$

and the proof is complete. □

III.4.1 A Cauchy Estimate

Theorem III.30. *If* $\underline{\mathfrak{a}} < \alpha \in \mathcal{A}$ *then* $\{\Xi^n\}_{n\in\mathbb{N}}$ *(defined by Theorem III.25) is Cauchy in* $\mathbb{Y}^{\mathfrak{p}}_{\alpha}$*.*

Proof. We begin this proof by fixing $n, m \in \mathbb{N}$ and assuming that ξ^n_x, ξ^m_x are respectively components of Ξ^n, Ξ^m for all $x \in \gamma$. We also let $\bar{\Xi}^{n,m} := \Xi^n - \Xi^m$ and assume, without loss of generality, that $\Lambda_n \subset \Lambda_m$. For all $x \in \gamma$ we shall now estimate components $\bar{\xi}^{n,m}_x$ of $\bar{\Xi}^{n,m}$ by considering three separate cases namely: $x \notin \Lambda_m$, $x \in \Lambda_n$ and $x \in \Lambda_m - \Lambda_n$.

First of all, from the definition of the system (III.41) we see that if $x \notin \Lambda_m$ then we have

$$\bar{\xi}^{n,m}_{x,t} = 0, \ \forall (t \in \mathcal{T}). \tag{III.55}$$

Let us now define for all $x \in \gamma$ and all $t \in \mathcal{T}$ the following processes

$$\Phi^{n,m}_x(t) := \Phi_x(\xi^n_{x,t}, \Xi^n_t) - \Phi_x(\xi^m_{x,t}, \Xi^m_t) \tag{III.56}$$

$$\Psi^{n,m}_x(t) := \Psi_x(\xi^n_{x,t}, \Xi^n_t) - \Psi_x(\xi^m_{x,t}, \Xi^m_t) \tag{III.57}$$

and consider the situation when $x \in \Lambda_n$. In this case we have

$$\bar{\xi}^{n,m}_{x,t} = \int_0^t \Phi^{n,m}_x(s)ds + \int_0^t \Psi^{n,m}_x(s)dW_x(s), \ t \in \mathcal{T}. \tag{III.58}$$

Hence using Itô Lemma IV.35 we see that if $x \in \Lambda_n$ then for all $t \in \mathcal{T}$

$$\begin{aligned} |\bar{\xi}^{n,m}_{x,t}|^{\mathfrak{p}} = & \int_0^t \mathfrak{p}(\bar{\xi}^{n,m}_{x,s})^{\mathfrak{p}-1}\Phi^{n,m}_x(s)ds + \\ & + \int_0^t \frac{\mathfrak{p}(\mathfrak{p}-1)}{2}(\bar{\xi}^{n,m}_{x,s})^{\mathfrak{p}-2}(\Psi^{n,m}_x(s))^2 ds + \\ & + \int_0^t \mathfrak{p}(\bar{\xi}^{n,m}_{x,t})^{\mathfrak{p}-1}\Psi^{n,m}_x(s)dW_x(s). \end{aligned} \tag{III.59}$$

Now, from Theorem III.18 we can see that for all $t \in \mathcal{T}$ we have

$$
\begin{aligned}
(\bar{\xi}^{n,m}_{x,t})^{\mathfrak{p}-1}\Phi^{n,m}_x(t) &= (\bar{\xi}^{n,m}_{x,t})^{\mathfrak{p}-2}\bar{\xi}^{n,m}_{x,t}\Big(\Phi_x(\xi^n_{x,t},\Xi^n_t) - \Phi_x(\xi^m_{x,t},\Xi^m_t)\Big)\\
&\le (\bar{\xi}^{n,m}_{x,t})^{\mathfrak{p}-2}\Big((b+\frac{1}{2})(\xi^n_{x,t}-\xi^m_{x,t})^2 + \frac{1}{2}\tilde{a}^2_x\sum_{y\in B_x}(\xi^n_{y,t}-\xi^m_{y,t})^2\Big)\\
&\le (b+1)|\bar{\xi}^{n,m}_{x,t}|^{\mathfrak{p}} + \max_{y\in B_x}|\bar{\xi}^{n,m}_{x,t}|^{\mathfrak{p}-2}\Big(\tilde{a}^2_x n_x \max_{y\in B_x}|\bar{\xi}^{n,m}_{x,t}|^2\Big)\\
&\le (b+1)|\bar{\xi}^{n,m}_{x,t}|^{\mathfrak{p}} + \tilde{a}^2_x n_x \max_{y\in B_x}|\bar{\xi}^{n,m}_{x,t}|^{\mathfrak{p}}\\
&\le (b+1)|\bar{\xi}^{n,m}_{x,t}|^{\mathfrak{p}} + \tilde{a}^2_x n_x \sum_{y\in B_x}|\bar{\xi}^{n,m}_{x,t}|^{\mathfrak{p}}\\
&\le (b+1)|\bar{\xi}^{n,m}_{x,t}|^{\mathfrak{p}} + \bar{a}^2 n^3_x \sum_{y\in B_x}|\bar{\xi}^{n,m}_{x,t}|^{\mathfrak{p}}. \qquad \text{(III.60)}
\end{aligned}
$$

Moreover, using assumption (**E**) we can see that for all $t \in \mathcal{T}$ we also have

$$
\begin{aligned}
(\bar{\xi}^{n,m}_{x,t})^{\mathfrak{p}-2}(\Psi^{n,m}_x(t))^2 &= (\bar{\xi}^{n,m}_{x,t})^{\mathfrak{p}-2}\Big(\Phi_x(\xi^n_{x,t},\Xi^n_t) - \Phi_x(\xi^m_{x,t},\Xi^m_t)\Big)^2\\
&\le (\bar{\xi}^{n,m}_{x,t})^{\mathfrak{p}-2}\Big(2M_1^2(\xi^n_{x,t}-\xi^m_{x,t})^2 + 2M_2^2 n^3_x\sum_{y\in B_x}(\xi^n_{y,t}-\xi^m_{y,t})^2\Big)\\
&\le 2M_1^2|\bar{\xi}^{n,m}_{x,t}|^{\mathfrak{p}} + \max_{y\in B_x}|\bar{\xi}^{n,m}_{x,t}|^{\mathfrak{p}-2}\Big(2M_2^2 n^4_x \max_{y\in B_x}|\bar{\xi}^{n,m}_{x,t}|^2\Big)\\
&\le 2M_1^2|\bar{\xi}^{n,m}_{x,t}|^{\mathfrak{p}} + 2M_2^2 n^4_x \max_{y\in B_x}|\bar{\xi}^{n,m}_{x,t}|^{\mathfrak{p}}\\
&\le 2M_1^2|\bar{\xi}^{n,m}_{x,t}|^{\mathfrak{p}} + 2M_2^2 n^4_x \sum_{y\in B_x}|\bar{\xi}^{n,m}_{x,t}|^{\mathfrak{p}}. \qquad \text{(III.61)}
\end{aligned}
$$

Therefore letting

$$B_1 := (b+1+2M_1^2) \quad \text{and} \quad B_2 := (\mathfrak{p}\bar{a}^2 + 2\mathfrak{p}^2 M_2^2)$$

we can deduce from equation (III.59) that if $x \in \Lambda_n$ then

$$\mathbb{E}\Big[|\bar{\xi}^{n,m}_{x,t}|^{\mathfrak{p}}\Big] \le \mathfrak{p}^2 B_1\int_0^t \mathbb{E}\Big[|\bar{\xi}^{n,m}_{x,s}|^{\mathfrak{p}}\Big]ds + B_2 n^4_x \sum_{y\in B_x}\int_0^t \mathbb{E}\Big[|\bar{\xi}^{n,m}_{y,s}|^{\mathfrak{p}}\Big]ds, \ t \in \mathcal{T}. \qquad \text{(III.62)}$$

Finally, when $x \in \Lambda_m - \Lambda_n$ we see using Theorem IV.16 that for all $t \in \mathfrak{T}$

$$|\bar{\xi}^{n,m}_{x,t}|^{\mathfrak{p}} \leq (|\xi^n_{x,t}|+|\xi^m_{x,t}|)^{\mathfrak{p}}$$

$$\leq 2^{\mathfrak{p}-1}|\xi^n_{x,t}|^{\mathfrak{p}}+2^{\mathfrak{p}-1}|\xi^m_{x,t}|^{\mathfrak{p}}.$$

Therefore, using Theorem III.28 and equation (III.55), we see now that if $x \in \Lambda_m - \Lambda_n$ then for all $t \in \mathfrak{T}$ we have

$$\mathbb{E}\Big[|\bar{\xi}^{n,m}_{x,t}|^2\Big] \leq 2^{\mathfrak{p}} \sup_{n\in\mathbb{N}} \mathbb{E}\Big[|\xi^n_{x,t}|^{\mathfrak{p}}\Big]$$

$$\leq 2^{\mathfrak{p}} \mathbb{1}_{\Lambda_m-\Lambda_n}(x) \sup_{n\in\mathbb{N}} \sup_{t\in\mathfrak{T}} \mathbb{E}\Big[|\xi^n_{x,t}|^{\mathfrak{p}}\Big]. \tag{III.63}$$

Therefore we can finally deduce, combining equations (III.55), (III.62) and (III.63), that for all $x \in \gamma$ and for all $t \in \mathfrak{T}$ we have

$$\mathbb{E}\Big[|\bar{\xi}^{n,m}_{x,t}|^{\mathfrak{p}}\Big] \leq \mathfrak{p}^2 B_1 \int_0^t \mathbb{E}\Big[|\bar{\xi}^{n,m}_{x,s}|^{\mathfrak{p}}\Big]ds+$$

$$+ B_2 n_x^4 \sum_{y\in B_x} \int_0^t \mathbb{E}\Big[|\bar{\xi}^{n,m}_{y,s}|^{\mathfrak{p}}\Big]ds+ \tag{III.64}$$

$$+ 2^{\mathfrak{p}} \mathbb{1}_{\Lambda_m-\Lambda_n}(x) \sup_{n\in\mathbb{N}} \sup_{t\in\mathfrak{T}} \mathbb{E}\Big[|\xi^n_{x,t}|^{\mathfrak{p}}\Big].$$

Now we shall continue this proof by applying the same reasoning, as in our proof of the Theorem III.28, to an infinite system of inequalities (III.64).

To begin we define, relying on the inequality (III.64) a measurable map $\varrho^{n,m} : \mathfrak{T} \to l^1_{\underline{\mathfrak{a}}}$, that is a map $\varrho^{n,m} \in \mathcal{M}(\mathbf{M}, \mathbf{M}^1_{\underline{\mathfrak{a}}})$, via the following formula

$$\varrho^{n,m}_x(t) := \mathbb{E}\Big[|\bar{\xi}^{n,m}_{x,t}|^{\mathfrak{p}}\Big], \quad \forall (t \in \mathfrak{T}) \tag{III.65}$$

and deduce from inequality (III.55), (III.62) and (III.63) that for all $x \in \gamma$

$$\varrho^{n,m}_x(t) \leq \sum_{y\in\gamma} Q_{x,y} \int_0^t \varrho^{n,m}_y(s)ds + A_x,\ t \in \mathfrak{T}$$

where

$$Q_{x,y} = \begin{cases} \mathfrak{p}^2 B_1 + B_2 n_x^4, & x = y, \\ B_2 n_x^4, & 0 < |x-y| < \rho, \\ 0, & |x-y| > \rho. \end{cases} \tag{III.66}$$

and

$$A_x = 2^{\mathfrak{p}} \mathbb{1}_{\Lambda_m - \Lambda_n}(x) \sup_{n\in\mathbb{N}} \sup_{t\in\mathcal{T}} \mathbb{E}\left[|\xi_{x,t}^n|^{\mathfrak{p}} \right]. \tag{III.67}$$

Now, fixing $\underline{\mathfrak{a}} < \tilde{\alpha} < \alpha \in \mathcal{A}$ we can deduce from (III.65), (III.66) and (III.67) that

(1) $A \in l^1_{\tilde{\alpha}}$ as a result of Theorem III.28.

(2) Identical arguments as in Theorem III.28 show that $\varrho^{n,m} \in \mathcal{B}(\mathcal{T}, l^1_{\tilde{\alpha}})$.

(3) From equation (III.66) we see that there exists a constant D such that $|Q_{x,y}| \leq D n_x^4$. Therefore using Theorem III.20 we conclude that for some $q \in (0,1)$ matrix Q is the Ovsjannikov operator of order q on $\mathcal{L}^1$.

Therefore we can now use Theorem III.23 and Corollary III.24 to conclude that

$$\sum_{x\in\gamma} e^{-\alpha|x|} \sup_{t\in\mathcal{T}} \varrho_x^{n,m}(t) \leq K(\tilde{\alpha}, \alpha) \sum_{x\in\gamma} e^{-\tilde{\alpha}|x|} |A_x|. \tag{III.68}$$

From equation (III.68) and definition (III.11) we therefore see that we have the following estimate

$$\begin{aligned} \|\Xi^n - \Xi^m\|^{\mathfrak{p}}_{\mathbb{Y}^{\mathfrak{p}}_{\alpha}} &= \sup_{t\in\mathcal{T}} \mathbb{E}\left[\|\Xi^n_t - \Xi^m_t\|^{\mathfrak{p}}_{l^{\mathfrak{p}}_{\alpha}} \right] \\ &= \sup_{t\in\mathcal{T}} \mathbb{E}\left[\sum_{x\in\gamma} e^{-\alpha|x|} |\xi^n_{x,t} - \xi^m_{x,t}|^{\mathfrak{p}} \right] \\ &\leq \sum_{x\in\gamma} e^{-\alpha|x|} \sup_{t\in\mathcal{T}} \varrho_x^{n,m}(t) \\ &\leq K(\tilde{\alpha}, \alpha) \sum_{x\in\gamma} e^{-\tilde{\alpha}|x|} |A_x|. \end{aligned}$$

Simplifying further we arrive at

$$\begin{aligned}
\|\Xi^n - \Xi^m\|^{\mathfrak{p}}_{\mathbb{Y}^{\mathfrak{p}}_\alpha} &\leq K(\tilde{\alpha},\alpha)\sum_{x\in\gamma} e^{-\tilde{\alpha}|x|}2^{\mathfrak{p}}\mathbb{1}_{\Lambda_m-\Lambda_n}(x)\sup_{n\in\mathbb{N}}\sup_{t\in\mathfrak{J}}\mathbb{E}\Big[|\xi^n_{x,t}|^{\mathfrak{p}}\Big] \\
&\leq 2^{\mathfrak{p}}K(\tilde{\alpha},\alpha)\sum_{x\in\Lambda_m-\Lambda_n} e^{-\tilde{\alpha}|x|}\sup_{n\in\mathbb{N}}\sup_{t\in\mathfrak{J}}\mathbb{E}\Big[|\xi^n_{x,t}|^{\mathfrak{p}}\Big]. \qquad \text{(III.69)}
\end{aligned}$$

Estimate above implies that the right hand side of equation (III.69) is the remainder of a convergent series hence the proof is complete. □

III.5 One Dimensional Special Case

Let us begin this section with the following definition, which complements Definition III.9.

Definition III.31. *For all $p \in \mathbb{R}_1$ we introduce the following spaces of stochastic processes.*

$$L^p_{ad}(t) := \{\xi \in \mathcal{L}^p(\mathbf{MP}_t, \mathbf{M}^{\mathbb{R}}) \mid \xi \textit{ is adapted to } \mathbb{F}|_{[0,t]}\}.$$

Suppose that $\underline{\mathfrak{a}} < \alpha \in \mathcal{A}$ and for all $n \in \mathbb{N}$ let Ξ^n be a solution of the truncated system (III.41). Using Theorem III.30 we recall that the sequence $\{\Xi^n\}_{n\in\mathbb{N}}$ is Cauchy in $\mathbb{Y}^{\mathfrak{p}}_{\alpha}$. Now since $\mathbb{Y}^{\mathfrak{p}}_{\alpha}$ is a Banach space, by Theorem III.15, we are now in a position to define the following process

$$\overbrace{\Xi := \lim_{n\to\infty} \Xi^n}^{\text{in } \mathbb{Y}^{\mathfrak{p}}_{\alpha}}. \tag{III.70}$$

Consider now an arbitrary $x \in \gamma$. The main goal of this section is to prove that the following stochastic integral equation

$$\eta_{x,t} = \zeta_x + \int_0^t \Phi_x(\eta_{x,s}, \Xi_s)ds + \int_0^t \Psi_x(\eta_{x,s}, \Xi_s)dW_x(s), \ t \in \mathfrak{T} \tag{III.71}$$

has a strong solution, in a usual sense, in $L^{\mathfrak{p}}_{ad}(T) \equiv L^{\mathfrak{p}}_{ad}$.

Remark III.32. *Note that, for a fixed $x \in \gamma$, the principal difference between equation (III.71) and (III.12) is that the process Ξ is fixed in (III.71) and defined by the limit (III.70). As for ζ_x, Φ_x and Ψ_x we continue with the same assumptions (see definition (III.13), assumptions (**C**), (**D**), (**C**), and Theorem III.18).*

In order to establish existence of a strong solution of equation (III.71) (see Theorem III.34) we need the following auxiliary result.

Theorem III.33. *Let $x \in \gamma$ and ξ_x be an $x-$component of Ξ (see definition (III.70)). Then*

$$\mathbb{E}\left[\sup_{t\in\mathfrak{T}} |\xi_{x,t}|^{\mathfrak{p}}\right] < \infty.$$

Proof. We shall prove this theorem by showing that for all $\epsilon > 0$ there exist $N \in \mathbb{N}$ such that for all $n, m \geq N$ we have

$$\mathbb{E}\left[\sup_{t\in\mathfrak{T}}|\xi^n_{x,t} - \xi^m_{x,t}|^{\mathfrak{p}}\right] < \epsilon$$

where ξ^n_x, ξ^m_x are respectively components of Ξ^n, Ξ^m.

Since $\Lambda_n \uparrow \gamma$ we begin by finding some $\bar{N} \in \mathbb{N}$ such that $x \in \Lambda_{\bar{N}}$ and temporary fixing some $n, m \geq \bar{N}$. Moreover let us assume, without loss of generality, that $n < m$ so that $x \in \Lambda_n \subset \Lambda_m$ and we define

$$\bar{\xi}^{n,m}_{x,t} := \xi^n_{x,t} - \xi^m_{x,t}, \quad \forall(t \in \mathfrak{T}).$$

Now we recalling, from Theorem III.30, definitions (III.56), (III.57) and an equation (III.58) we see, again via Itô Lemma, that for all $t \in \mathfrak{T}$

$$\begin{aligned}|\bar{\xi}^{n,m}_{x,t}|^{\mathfrak{p}} = &\int_0^t \mathfrak{p}(\bar{\xi}^{n,m}_{x,s})^{\mathfrak{p}-1}\Phi^{n,m}_x(s)ds + \\ &+\int_0^t \frac{\mathfrak{p}(\mathfrak{p}-1)}{2}(\bar{\xi}^{n,m}_{x,s})^{\mathfrak{p}-2}(\Psi^{n,m}_x(s))^2 ds + \\ &+\int_0^t \mathfrak{p}(\bar{\xi}^{n,m}_{x,t})^{\mathfrak{p}-1}\Psi^{n,m}_x(s)dW_x(s).\end{aligned} \tag{III.72}$$

Therefore we see from equation (III.72) above that

$$\begin{aligned}\sup_{t\in\mathfrak{T}}|\bar{\xi}^{n,m}_{x,t}|^{\mathfrak{p}} = &\int_0^T \mathfrak{p}(\bar{\xi}^{n,m}_{x,s})^{\mathfrak{p}-1}\Phi^{n,m}_x(s)ds + \\ &+\int_0^T \frac{\mathfrak{p}(\mathfrak{p}-1)}{2}(\bar{\xi}^{n,m}_{x,s})^{\mathfrak{p}-2}(\Psi^{n,m}_x(s))^2 ds + \\ &+\sup_{t\in\mathfrak{T}}\int_0^t \mathfrak{p}(\bar{\xi}^{n,m}_{x,t})^{\mathfrak{p}-1}\Psi^{n,m}_x(s)dW_x(s).\end{aligned} \tag{III.73}$$

Moreover from inequality (III.60) and (III.61) we see that

$$(\bar{\xi}^{n,m}_{x,t})^{\mathfrak{p}-1}\Phi^{n,m}_x(t) \leq (b+1)|\bar{\xi}^{n,m}_{x,t}|^{\mathfrak{p}} + \bar{a}^2 n_x^3 \sum_{y\in B_x}|\bar{\xi}^{n,m}_{x,t}|^{\mathfrak{p}}. \tag{III.74}$$

and

$$(\bar{\xi}^{n,m}_{x,t})^{\mathfrak{p}-2}(\Psi^{n,m}_x(t))^2 \leq 2M_1^2|\bar{\xi}^{n,m}_{x,t}|^{\mathfrak{p}}+2M_2^2 n_x^4 \sum_{y\in B_x}|\bar{\xi}^{n,m}_{x,t}|^{\mathfrak{p}}. \tag{III.75}$$

Now combining an equation (III.73) with inequality (III.74) and (III.75) above we see that the following inequality holds

$$\mathbb{E}\bigg[\sup_{t\in\mathcal{T}}|\bar{\xi}^{n,m}_{x,t}|^{\mathfrak{p}}\bigg] \leq K + \mathbb{E}\bigg[\sup_{t\in\mathcal{T}}\int_0^t \mathfrak{p}(\bar{\xi}^{n,m}_{x,s})^{\mathfrak{p}-1}\Psi^{n,m}_x(s)dW_x(s)\bigg] \tag{III.76}$$

where

$$C_1 := \mathfrak{p}^2(b+1+2M_1^2)$$

$$C_2 := n_x^4(\mathfrak{p}\bar{a}^2 + 2\mathfrak{p}^2 M_2^2)$$

$$K := C_1\int_0^T \mathbb{E}\bigg[|\bar{\xi}^{n,m}_{x,s}|^{\mathfrak{p}}\bigg]ds + C_2\sum_{y\in B_x}\int_0^T \mathbb{E}\bigg[|\bar{\xi}^{n,m}_{y,s}|^{\mathfrak{p}}\bigg]ds.$$

Now using Burkholder-Davis-Gundy inequality IV.33 together with Jensen inequality IV.21 we see that the following estimate on the stochastic term from the inequality (III.76) holds.

$$\begin{aligned}\mathbb{E}\bigg[\sup_{t\in\mathcal{T}}\int_0^t \mathfrak{p}(\bar{\xi}^{n,m}_{x,s})^{\mathfrak{p}-1}\Psi^{n,m}_x(s)dW_x(s)\bigg] &\leq \mathbb{E}\bigg[\bigg(\int_0^t\Big(\mathfrak{p}(\bar{\xi}^{n,m}_{x,s})^{\mathfrak{p}-1}\Psi^{n,m}_x(s)\Big)^2 ds\bigg)^{\frac{1}{2}}\bigg]\\ &\leq \bigg(\mathbb{E}\bigg[\int_0^t\Big(\mathfrak{p}(\bar{\xi}^{n,m}_{x,s})^{\mathfrak{p}-1}\Psi^{n,m}_x(s)\Big)^2 ds\bigg]\bigg)^{\frac{1}{2}}.\end{aligned} \tag{III.77}$$

To simplify inequality (III.77) we note that according to the assumption (**E**) the following estimate holds for all $t\in\mathcal{T}$

$$\begin{aligned}\Big((\bar{\xi}^{n,m}_{x,t})^{\mathfrak{p}-1}\Psi^{n,m}_x(t)\Big)^2 &= (\bar{\xi}^{n,m}_{x,t})^{2\mathfrak{p}-2}\bigg(M_1|\bar{\xi}^{n,m}_{x,t}|+M_2 n_x\sum_{y\in B_x}|\bar{\xi}^{n,m}_{y,t}|\bigg)^2\\ &\leq (\bar{\xi}^{n,m}_{x,t})^{2\mathfrak{p}-2}\bigg(2M_1^2|\bar{\xi}^{n,m}_{x,t}|^2+2M_2^2 n_x^3\sum_{y\in B_x}|\bar{\xi}^{n,m}_{y,t}|^2\bigg)\\ &\leq 2M_1^2|\bar{\xi}^{n,m}_{x,t}|^{2\mathfrak{p}}+\max_{y\in B_x}|\bar{\xi}^{n,m}_{y,t}|^{2\mathfrak{p}-2}\bigg(2M_2^2 n_x^4\max_{y\in B_x}|\bar{\xi}^{n,m}_{y,t}|^2\bigg),\end{aligned}$$

which shows that

$$\Big((\bar{\xi}^{n,m}_{x,t})^{\mathfrak{p}-1}\Psi^{n,m}_x(t)\Big)^2 \le 2M_1^2|\bar{\xi}^{n,m}_{x,t}|^{2\mathfrak{p}} + 2M_2^2 n_x^4 \sum_{y\in B_x}|\bar{\xi}^{n,m}_{y,t}|^{2\mathfrak{p}}.$$

Now by letting

$$C_3 \coloneqq 2\mathfrak{p}^2M_1^2T \quad \text{and} \quad C_4 \coloneqq 2\mathfrak{p}^2M_2^2n_x^4T$$

it follows now that inequality (III.77) can be written in the following way

$$\mathbb{E}\bigg[\sup_{t\in\mathfrak{T}}\int_0^t \mathfrak{p}(\bar{\xi}^{n,m}_{x,s})^{\mathfrak{p}-1}\Psi^{n,m}_x(s)dW_x(s)\bigg] \le C_3\sup_{t\in\mathfrak{T}}\mathbb{E}\bigg[|\bar{\xi}^{n,m}_{x,t}|^{2\mathfrak{p}}\bigg] + C_4\sum_{y\in B_x}\sup_{t\in\mathfrak{T}}\mathbb{E}\bigg[|\bar{\xi}^{n,m}_{x,t}|^{2\mathfrak{p}}\bigg].$$

Therefore returning to the inequality (III.76) we see that

$$\begin{aligned}\mathbb{E}\bigg[\sup_{t\in\mathfrak{T}}|\bar{\xi}^{n,m}_{x,t}|^{\mathfrak{p}}\bigg] \le TC_1\sup_{t\in\mathfrak{T}}\mathbb{E}\bigg[|\bar{\xi}^{n,m}_{x,t}|^{\mathfrak{p}}\bigg]+ \\ + TC_2\sum_{y\in B_x}\sup_{t\in\mathfrak{T}}\mathbb{E}\bigg[|\bar{\xi}^{n,m}_{y,t}|^{\mathfrak{p}}\bigg]+ \\ +C_3\sup_{t\in\mathfrak{T}}\mathbb{E}\bigg[|\bar{\xi}^{n,m}_{x,t}|^{2\mathfrak{p}}\bigg]+ \\ + C_4\sum_{y\in B_x}\sup_{t\in\mathfrak{T}}\mathbb{E}\bigg[|\bar{\xi}^{n,m}_{x,t}|^{2\mathfrak{p}}\bigg].\end{aligned} \tag{III.78}$$

Since B_x is finite we can now use Theorem III.30 to conclude that, with a suitable choice of $n, m \in \mathbb{N}$, the right hand side of the inequality (III.78) above can be made arbitrary small hence the proof is complete. □

III.5.1 Existence

Theorem III.34. *Equation (III.71) admits a unique strong solution.*

Proof. Relying on [3] we conclude that equation (III.71) admits a unique local maximal solution η_x such that for all $t \in [0,\infty)$

$$\eta_{x,t\wedge\tau_n} = \zeta_x + \int_0^{t\wedge\tau_n}\Phi_x(\eta_{x,s\wedge\tau_n},\Xi_{s\wedge\tau_n})ds + \int_0^{t\wedge\tau_n}\Psi_x(\eta_{x,s\wedge\tau_n},\Xi_{s\wedge\tau_n})dW_x(s), \tag{III.79}$$

where by construction, for all $n \in \mathbb{N}$, stopping time τ_n is the first exit time of η_x from the interval (-n, n). Hence to complete the proof we will now show that η_x is in fact a global solution. That is we are going to establish that almost surely $\lim_{n\to\infty} \tau_n = \infty$.

We begin by applying an Itô Lemma IV.35 to an equation (III.79) to establish that for all $t \in [0, \infty)$ we have the following

$$
\begin{aligned}
|\eta_{x,t\wedge\tau_n}|^{\mathfrak{p}} = &\int_0^{t\wedge\tau_n} \mathfrak{p}(\eta_{x,s\wedge\tau_n})^{\mathfrak{p}-1}\Phi_x(\eta_{x,s\wedge\tau_n}, \Xi_{s\wedge\tau_n})ds + \\
&+ \int_0^{t\wedge\tau_n} \frac{\mathfrak{p}(\mathfrak{p}-1)}{2}(\eta_{x,s\wedge\tau_n})^{\mathfrak{p}-2}(\Psi_x(\eta_{x,s\wedge\tau_n}, \Xi_{s\wedge\tau_n}))^2 ds + \\
&+ \int_0^{t\wedge\tau_n} \mathfrak{p}(\eta_{x,s\wedge\tau_n})^{\mathfrak{p}-1}\Psi_x(\eta_{x,s\wedge\tau_n}, \Xi_{s\wedge\tau_n})dW_x(s).
\end{aligned}
\tag{III.80}
$$

Now by letting

$$
\bar{\Phi}_x^{\mathfrak{p}}(\eta, t) \coloneqq (\eta_{x,t\wedge\tau_n})^{\mathfrak{p}-1}\Phi_x(\eta_{x,t\wedge\tau_n}, \Xi_{t\wedge\tau_n}), \ \forall(t \in [0, \infty)) \tag{III.81}
$$

we see from inequalities (III.43), (III.44) and definitions (III.45) - (III.48) that for all $t \in [0, \infty)$

$$
\begin{aligned}
\Phi_x^{\mathfrak{p}}(\eta, t) &\leq A_1|\eta_{x,t\wedge\tau_n}|^{\mathfrak{p}} + \bar{a}^2 n_x^3 \sum_{y\in B_x} |\xi_{y,t\wedge\tau_n}|^{\mathfrak{p}} + 2^{\mathfrak{p}-1}c^2 \\
&\leq A_1|\eta_{x,t\wedge\tau_n}|^{\mathfrak{p}} + \bar{a}^2 n_x^3 \sum_{y\in B_x} \sup_{t\in\mathfrak{T}} |\xi_{y,t\wedge\tau_n}|^{\mathfrak{p}} + 2^{\mathfrak{p}-1}c^2 \\
&\leq A_1|\eta_{x,t\wedge\tau_n}|^{\mathfrak{p}} + \bar{a}^2 n_x^3 \sum_{y\in B_x} \sup_{t\in\mathfrak{T}} |\xi_{y,t}|^{\mathfrak{p}} + 2^{\mathfrak{p}-1}c^2
\end{aligned}
\tag{III.82}
$$

and

$$
\begin{aligned}
(\eta_{x,s\wedge\tau_n})^{\mathfrak{p}-2}(\Psi_x(\eta_{x,s\wedge\tau_n}, \Xi_{s\wedge\tau_n}))^2 &\leq A_2|\eta_{x,t\wedge\tau_n}|^{\mathfrak{p}} + 4M_2^2 n_x^4 \sum_{y\in B_x} |\xi_{y,t\wedge\tau_n}|^{\mathfrak{p}} + 4c^2 2^{\mathfrak{p}-1} \\
&\leq A_2|\eta_{x,t\wedge\tau_n}|^{\mathfrak{p}} + 4M_2^2 n_x^4 \sum_{y\in B_x} \sup_{t\in\mathfrak{T}} |\xi_{y,t}|^{\mathfrak{p}} + 4c^2 2^{\mathfrak{p}-1}.
\end{aligned}
\tag{III.83}
$$

Therefore combining inequality (III.82) and (III.83) together with inequality (III.80) we see that for all $t \in [0,\infty)$ we have

$$\mathbb{E}\Big[|\eta_{x,t\wedge\tau_n}|^{\mathfrak{p}}\Big] \leq \mathfrak{p}^2(A_1+A_2)\int_0^t \mathbb{E}\Big[|\eta_{x,s\wedge\tau_n}|^{\mathfrak{p}}\Big]ds + Tn_x^4 A_3 \sum_{y\in B_x} \mathbb{E}\Big[\sup_{t\in\mathfrak{T}}|\xi_{y,t}^n|^{\mathfrak{p}}\Big] + A_4$$

$$\leq D\int_0^t \mathbb{E}\Big[|\eta_{x,s\wedge\tau_n}|^{\mathfrak{p}}\Big]ds + K(x). \tag{III.84}$$

Where

$$D := \mathfrak{p}^2(A_1+A_2)$$

$$K(x) := Tn_x^4 A_3 \sum_{y\in B_x} \mathbb{E}\Big[\sup_{t\in\mathfrak{T}}|\xi_{y,t}^n|^{\mathfrak{p}}\Big] + A_4.$$

Now using Gronwall's inequality IV.20 together with the inequality (III.84) above we see that for all $t \in [0,\infty)$ we have

$$\mathbb{E}\Big[|\eta_{x,t\wedge\tau_n}|^{\mathfrak{p}}\Big] \leq K(x)e^{Dt}. \tag{III.85}$$

However using the definition of τ_n we see that for all $n \in \mathbb{N}$ we have $|\eta_{x,\tau_n}| \geq n$. Moreover, because $\mathbb{P}(\tau_n < t) = \mathbb{E}\Big[\mathbb{1}_{\{\tau_n<t\}}\Big]$ we also see that for all $t \in [0,\infty)$

$$n^{\mathfrak{p}}\mathbb{P}(\tau_n < t) \leq \mathbb{E}\Big[|\eta_{x,\tau_n}|^{\mathfrak{p}}\mathbb{1}_{\{\tau_n<t\}}\Big]$$

$$\leq \mathbb{E}\Big[|\eta_{x,\tau_n}|^{\mathfrak{p}}\mathbb{1}_{\{\tau_n<t\}}\Big] + \mathbb{E}\Big[|\eta_{x,\tau_n}|^{\mathfrak{p}}\mathbb{1}_{\{\tau_n\geq t\}}\Big],$$

$$= \mathbb{E}\Big[|\eta_{x,t\wedge\tau_n}|^{\mathfrak{p}}\mathbb{1}_{\{\tau_n<t\}}\Big] + \mathbb{E}\Big[|\eta_{x,t\wedge\tau_n}|^{\mathfrak{p}}\mathbb{1}_{\{\tau_n\geq t\}}\Big]$$

$$= \mathbb{E}\Big[|\eta_{x,t\wedge\tau_n}|^{\mathfrak{p}}\Big]. \tag{III.86}$$

Therefore using inequality (III.85) and (III.86) above we see that for all $n \in \mathbb{N}$ and for all $t \in [0,\infty)$ we have

$$\mathbb{P}(\tau_n < t) \leq \frac{1}{n^{\mathfrak{p}}}K(x)e^{Dt}.$$

Hence for all $t \in [0, \infty)$ we have

$$\lim_{n \to \infty} \mathbb{P}(\tau_n < t) = 0. \tag{III.87}$$

Now convergence in probability and the fact that $\{\tau_n\}_{n \in \mathbb{N}}$ is an increasing sequence imply that almost surely $\lim_{n \to \infty} \tau_n = \infty$ hence the proof is complete. $\square$

III.6 Existence and Uniqueness

In this subsection we will learn that system (III.12) admits a unique strong solution. We shall start by showing existence.

Theorem III.35. *Suppose that $\mathfrak{p} \in \mathbb{R}_2$ and for all $x \in \gamma$ maps Φ_x and Ψ_x satisfy conditions (III.13) - (III.17). Then for all $\zeta \in l^{\mathfrak{p}}_{\underline{\mathfrak{a}}}$ stochastic system (III.12) admits a strong solution.*

Proof. Let us start by fixing some $\underline{\mathfrak{a}} < \alpha \in \mathcal{A}$. Now, according to the Theorem III.30 sequence $\{\Xi^n\}_{n\in\mathbb{N}}$ converges in $\mathbb{Y}^{\mathfrak{p}}_{\alpha}$. Therefore, this proof can be completed by letting

$$\overbrace{\Xi := \lim_{n\to\infty} \Xi^n}^{\text{in } \mathbb{Y}^{\mathfrak{p}}_{\alpha}}$$

and showing that $\Xi \equiv \{\xi_x\}_{x\in\gamma}$ is also a strong solution of the system (III.12). However because Ξ in $\mathbb{Y}^{\mathfrak{p}}_{\alpha}$ we see from the Definition III.17 that to complete the proof it only remains to show that for all $x \in \gamma$ and all $t \in \mathfrak{T}$ we have

$$\xi_{x,t} = \zeta_x + \int_0^t \Phi_x(\xi_{x,s}, \Xi_s)ds + \int_0^t \Psi_x(\xi_{x,s}, \Xi_s)dW_x(s), \quad \mathbb{P} - a.s. \tag{III.88}$$

Using our work in the previous section III.5, in particular using Theorem III.34 we begin by defining a family of processes $H := \{\eta_x\}_{x\in\gamma}$ such that for all $x \in \gamma$ and all $t \in \mathfrak{T}$ we have

$$\eta_{x,t} = \zeta_x + \int_0^t \Phi_x(\eta_{x,s}, \Xi_s)ds + \int_0^t \Psi_x(\eta_{x,s}, \Xi_s)dW_x(s), \quad \mathbb{P} - a.s. \tag{III.89}$$

Now, if $n \in \mathbb{N}$ then we also recall from the Theorem III.25 and the Definition, of the truncated system, (III.41) that for all $x \in \gamma$ and all $t \in \mathfrak{T}$ we have

$$\left.\begin{aligned} &\xi^n_{x,t} = \zeta_x + \int_0^t \Phi_x(\xi^n_{x,s}, \Xi^n_s)ds + \int_0^t \Psi_x(\xi^n_{x,s}, \Xi^n_s)dW_x(s) && \forall x \in \Lambda_n \\ &\xi^n_{x,t} = \zeta_x && \forall x \notin \Lambda_n \end{aligned}\right\}, \ \mathbb{P} - a.s.$$

Moreover convergence $\overbrace{\Xi = \lim_{n\to\infty} \Xi^n}^{\text{in } \mathbb{Y}^{\mathfrak{p}}_{\alpha}}$ in particular implies that

$$\underset{n\to\infty}{lim} \ \underset{t\in\mathfrak{T}}{sup} \mathbb{E}\Big[\sum_{x\in\gamma} e^{-\alpha|x|} |\xi^n_{x,t} - \xi_{x,t}|^{\mathfrak{p}} \Big] = 0. \tag{III.90}$$

Let us now fix some $x \in \gamma$ and show that for all $t \in \mathcal{T}$ equation (III.88) above holds. To begin we observe from equation (III.90) above and Theorem IV.2 that uniformly on $\mathcal{T}$ we have the following result

$$\lim_{n\to\infty} \mathbb{E}\Big[|\xi^n_{x,t} - \xi_{x,t}|\Big] = 0.$$

Therefore in order to conclude this proof it remains to show that uniformly on $\mathcal{T}$ we have

$$\lim_{n\to\infty} \mathbb{E}\Big[|\xi^n_{x,t} - \eta_{x,t}|\Big] = 0. \tag{III.91}$$

Remark III.36. *From equation (III.91) it would follow that $\xi_x \approx \eta_x$ and so equation (III.88) can be obtained via similar techniques as seen previously in this section.*

Now since $\Lambda_n \uparrow \gamma$ as $n \to \infty$ let us assume that for some $n \in \mathbb{N}$ we have $x \in \Lambda_n \subset \gamma$. Moreover we define the following processes

$$\Phi^n_x(t) := \Phi_x(\xi^n_{x,t}, \Xi^n_t) - \Phi_x(\eta_{x,t}, \Xi_t)$$

$$\Psi^n_x(t) := \Psi_x(\xi^n_{x,t}, \Xi^n_t) - \Psi_x(\eta_{x,t}, \Xi_t)$$

$$\mathscr{X}^n_{x,t} := \xi^n_{x,t} - \eta_{x,t}.$$

Hence using Itô Lemma we begin observing that for all $t \in \mathcal{T}$ we have

$$\begin{aligned} |\mathscr{X}^n_{x,t}|^{\mathfrak{p}} = \int_0^t \mathfrak{p}(\mathscr{X}^n_{x,t})^{\mathfrak{p}-1}\Phi^{n,m}_x(s)ds \,+ \\ + \int_0^t \frac{\mathfrak{p}(\mathfrak{p}-1)}{2}(\mathscr{X}^n_{x,t})^{\mathfrak{p}-2}(\Psi^{n,m}_x(s))^2 ds + \\ + \int_0^t \mathfrak{p}(\mathscr{X}^n_{x,t})^{\mathfrak{p}-1}\Psi^{n,m}_x(s)dW_x(s). \end{aligned} \tag{III.92}$$

Therefore, from inequality (III.60) and (III.61) we can see that for all $t \in \mathcal{T}$ we have

$$(\xi^n_{x,t} - \eta_{x,t})^{\mathfrak{p}-1}\Phi^n_x(t) \le (b+1)(\xi^n_{x,t} - \eta_{x,t})^{\mathfrak{p}} + \bar{a}^2 n^3_x \sum_{y\in B_x} (\xi^n_{y,t} - \xi_{y,t})^{\mathfrak{p}} \tag{III.93}$$

and

$$(\xi^n_{x,t} - \eta_{x,t})^{\mathfrak{p}-2}\bigg(\Psi^n_x(t)\bigg)^2 \leq 2M_1^2(\xi^n_{x,t} - \eta_{x,t})^{\mathfrak{p}} + 2M_2^2 n_x^4 \sum_{y\in B_x}(\xi^n_{y,t} - \xi_{y,t})^{\mathfrak{p}}. \tag{III.94}$$

Now, because B_x is finite it is clear from equation (III.90) that

$$\mathbb{E}\bigg[\sum_{y\in B_x}(\xi^n_{y,t} - \xi_{y,t})^{\mathfrak{p}}\bigg]$$

can be made arbitrary small uniformly on $\mathcal{T}$ by taking $n \in \mathbb{N}$ sufficiently large. Therefore from inequality (III.93) and (III.94) above we see that for all $t \in \mathcal{T}$

$$\mathbb{E}\bigg[(\xi^n_{x,t} - \eta_{x,t})^{\mathfrak{p}-1}\Phi^n_x(t)\bigg] \leq (b+1)\mathbb{E}\bigg[(\xi^n_{x,t} - \eta_{x,t})^{\mathfrak{p}}\bigg] + A^n_x \tag{III.95}$$

and

$$\mathbb{E}\bigg[(\xi^n_{x,t} - \eta_{x,t})^{\mathfrak{p}-2}\bigg(\Psi^n_x(t)\bigg)^2\bigg] \leq 2M_1^2\mathbb{E}\bigg[(\xi^n_{x,t} - \eta_{x,t})^{\mathfrak{p}}\bigg] + A^n_x \tag{III.96}$$

where

$$A^n_x := \max\{\bar{a}^2 n_x^3,\ 2M_2^2 n_x^4\}\mathbb{E}\bigg[\sum_{y\in B_x}(\xi^n_{y,t} - \xi_{y,t})^{\mathfrak{p}}\bigg].$$

Moreover $A^n_x \to 0$ uniformly on $\mathcal{T}$ as $n \to \infty$. Therefore using inequality (III.95) and (III.96) above we can conclude from equation (III.92) that for all $x \in \gamma$ and all $t \in \mathcal{T}$ we have

$$\mathbb{E}\bigg[|\xi^n_{x,t} - \eta_{x,t}|^{\mathfrak{p}}\bigg] \leq C\int_0^t \mathbb{E}\bigg[|\xi^n_{x,s} - \eta_{x,s}|^{\mathfrak{p}}\bigg]ds + \bar{A}^n_x \tag{III.97}$$

where

$$C := \mathfrak{p}^2(b + 1 + 2M_1^2), \tag{III.98}$$

$$\bar{A}^n_x := 2\mathfrak{p}^2 T A^n_x. \tag{III.99}$$

Finally using Gronwall inequality IV.20 we see that for all $t \in \mathcal{T}$ we have

$$\mathbb{E}\Bigg[|\xi^n_{x,t} - \eta_{x,t}|^{\mathfrak{p}}\Bigg] \leq A^n_x e^{CT}$$

which shows that for all $x \in \gamma$ and uniformly on $\mathcal{T}$

$$\lim_{n\to\infty} \mathbb{E}\Bigg[|\xi^n_{x,t} - \eta_{x,t}|^{\mathfrak{p}}\Bigg] = 0.$$

Equation (III.91) now follows via application of Therorem IV.2 hence the proof is complete.

□

In the following theorem we now address uniqueness.

Theorem III.37. *Suppose $\zeta \in l^{\mathfrak{p}}_{\underline{\mathfrak{a}}}$ and $\underline{\mathfrak{a}} < \alpha \in \mathcal{A}$. Then stochastic system (III.12) admits a unique strong solution Ξ in $\mathbb{Y}^{\mathfrak{p}}_{\alpha}$.*

Proof. For contradiction, using Theorem III.35, suppose that Ξ^1 and Ξ^2 are distinct strong solutions of the system (III.12) in $\mathbb{Y}^{\mathfrak{p}}_{\alpha}$. Now let us define a map $\bar{\Xi} \in \mathbb{Y}^{\mathfrak{p}}_{\alpha}$ via the following formula

$$\bar{\Xi}_t := \Xi^1_t - \Xi^2_t.$$

We see that for all $t \in \mathcal{T}$ we have

$$\bar{\xi}_{x,t} = \int_0^t \Phi_x(\xi^1_{x,s}, \Xi^1_s) - \Phi_x(\xi^2_{x,s}, \Xi^2_s)ds + \int_0^t \Psi_x(\xi^1_{x,s}, \Xi^1_s) - \Psi_x(\xi^2_{x,s}, \Xi^2_s)dW_x(s), \quad \mathbb{P} - a.s.$$

Now as in the proof of Theorem III.30 we deduce, using Ito Lemma, that

$$|\bar{\xi}_{x,t}|^{\mathfrak{p}} = \int_0^t \mathfrak{p}(\bar{\xi}_{x,s})^{\mathfrak{p}-1}\Phi^{1,2}_x(s)ds + \\ + \int_0^t \frac{\mathfrak{p}(\mathfrak{p}-1)}{2}(\bar{\xi}_{x,s})^{\mathfrak{p}-2}(\Psi^{1,2}_x(s))^2 ds + \\ + \int_0^t \mathfrak{p}(\bar{\xi}_{x,s})^{\mathfrak{p}-1}\Psi^{1,2}_x(s)dW_x(s).$$

where we have chosen for all $t \in \mathcal{T}$ to let

$$\Phi_x^{1,2}(t) := \Phi_x(\xi_{x,t}^1, \Xi_t^1) - \Phi_x(\xi_{x,t}^2, \Xi_t^2)$$
$$\Psi_x^{1,2}(t) := \Psi_x(\xi_{x,t}^1, \Xi_t^1) - \Psi_x(\xi_{x,t}^2, \Xi_t^2).$$

Therefore we see that

$$\mathbb{E}\Big[|\bar{\xi}_{x,t}|^{\mathfrak{p}}\Big] \leq B_1(\mathfrak{p}, b, c, M_1) \int_0^t \mathbb{E}\Big[|\bar{\xi}_{x,s}|^{\mathfrak{p}}\Big] ds + B_2(x, \mathfrak{p}, M_2) \sum_{y \in B_x} \int_0^t \mathbb{E}\Big[|\bar{\xi}_{y,s}|^{\mathfrak{p}}\Big] ds \tag{III.100}$$

where

$$B_1(\mathfrak{p}, b, c, M_1) := \mathfrak{p}b + \frac{\mathfrak{p}}{2} + M_1^2 \mathfrak{p}(\mathfrak{p} - 1)$$
$$B_2(x, \mathfrak{p}, M_2) := \mathfrak{p}\tilde{a}_x^2 + n_x^4(\mathfrak{p} + M_2^2 \mathfrak{p}(\mathfrak{p} - 1)).$$

Let us now fix $\underline{\mathfrak{a}} < \tilde{\alpha} \leq \alpha \in \mathcal{A}$ and use inequality (III.100) to define a measurable map $\kappa : \mathcal{T} \to l_{\tilde{\alpha}}^1$ via the following formula

$$\kappa_x(t) := \mathbb{E}\Big[|\bar{\xi}_{x,t}|^{\mathfrak{p}}\Big].$$

Hence we now deduce from inequality (III.100) that

$$\kappa_x(t) \leq \sum_{y \in \gamma} Q_{x,y} \int_0^t \kappa_y(s) ds$$

where for all $x, y \in \gamma$ we have

$$Q_{x,y} = \begin{cases} B_1(\mathfrak{p}, b, c, M_1) + B_2(x, \mathfrak{p}, M_2), & x = y, \\ B_2(x, \mathfrak{p}, M_2), & 0 < |x - y| \leq \rho, \\ 0, & |x - y| > \rho. \end{cases} \tag{III.101}$$

Moreover we see that the following facts are also true

(1) By construction (see Theorem III.35) $\kappa \in \mathcal{B}(\mathcal{T}, l_{\tilde{\alpha}}^1)$.

(2) From equation (III.101) we see that there exists a constant C such that $|Q_{x,y}| \leq C n_x^4$.

Therefore, by Theorem III.20, for some $q \in (0,1)$ matrix Q is Ovsiannikov map on $\mathcal{L}^1$.

Therefore we can now use Theorem III.23 and Corollary III.24 to conclude that

$$\sum_{x\in\gamma} e^{-\alpha|x|} \sup_{t\in\mathfrak{T}} \kappa_x(t) \leq K(\tilde{\alpha},\alpha) \sum_{x\in\gamma} e^{-\tilde{\alpha}|x|}|A_x|$$

where A_x is a zero sequence in $l^1_{\tilde{\alpha}}$. Therefore we establish that

$$\sup_{t\in\mathfrak{T}} \mathbb{E}\Bigg[\sum_{x\in\gamma} e^{-\alpha|x|}|\bar{\xi}_{x,t}|^{\mathfrak{p}}\Bigg] = 0. \tag{III.102}$$

Hence

$$||\Xi^1 - \Xi^2||_{\mathbb{Y}^{\mathfrak{p}}_{\alpha}} = 0 \tag{III.103}$$

and the proof is complete. □

IV Appendices

IV.1 Expectation, Measurability and Related Inequalities

This subsection of the appendix is based on [7, 12, 17, 36, 60] and outlines a number of theorems that are used throughout the main body of the text. We assume here that we are working on the probability space and with definitions described in subsection II.2.1 and II.2.3.

In this subsection of the appendix we will find it convenient to make the following definition.

Definition IV.1. *Suppose that $\mathbf{X} := (X, \mathcal{A}, \mu)$ is a complete measure spaces. Moreover, let E be a separable Banach space and $\mathbf{M}^E := (E, \mathscr{B}(E))$ be a measurable space. For all $p \in \mathbb{R}_1$ we make the following definitions:*

$$\mathcal{L}^p := \mathcal{L}^p(\mathbf{X}, \mathbf{M}^{\mathbb{R}}), \tag{IV.1}$$

$$\mathcal{L}^p(E) := \mathcal{L}^p(\mathbf{X}, \mathbf{M}^E), \tag{IV.2}$$

$$\mathcal{L}^p_+ := \{f \in \mathcal{L}^p | f \geq 0 \text{ almost surely}\}. \tag{IV.3}$$

Theorem IV.2. *Suppose that $\mathbf{X}$ is a finite measure spaces. Moreover suppose that we have in addition real numbers $1 \leq q \leq p$ and $f \in \mathcal{L}^p$. Then*

$$\mathcal{L}^p \subset \mathcal{L}^q,$$

$$\|f\|_{\mathcal{L}^q} \leq \mu(X)^{\left(\frac{1}{q} - \frac{1}{p}\right)} \|f\|_{\mathcal{L}^p}.$$

Theorem IV.3 (Borel–Cantelli Theorem)**.**
Let $\{A_i\}_{i\in\mathbb{N}}$ be a sequence of measurable subsets in a measure space $\mathbf{X}$. Then

$$\sum_{i=0}^{\infty} \mu(A_i) < \infty \implies \mu\left(\bigcap_{j=0}^{\infty} \bigcup_{i=j}^{\infty} A_i \right) = 0.$$

Theorem IV.4. *Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence in $\mathcal{M}(\mathbf{X}, \mathbf{M}^E)$. Suppose that we have a map $f : X \to E$ such that $\lim_{n\to\infty} f_n(x) = f(x)$ almost everywhere. Then f is also a measurable map, that is $f \in \mathcal{M}(\mathbf{X}, \mathbf{M}^E)$.*

Theorem IV.5. *Suppose that* $f \in \mathcal{M}(\mathbf{X}, \mathbf{M}^E)$ *and we have a map* $g : X \to E$ *such that* $f = g$ *almost everywhere. Then* $g \in \mathcal{M}(\mathbf{X}, \mathbf{M}^E)$*.*

Theorem IV.6. *Let* $F \coloneqq \{f_n\}_{n\in\mathbb{N}}$ *be a sequence in* $\mathcal{M}(\mathbf{X}, \mathbf{M}^E)$ *and suppose that* F *is Cauchy almost everywhere. Then there exists a measurable map* $f : X \to E$*, that is* $f \in \mathcal{M}(\mathbf{X}, \mathbf{M}^E)$*, such that* $f_n \to f$ *almost everywhere.*

Theorem IV.7 (Egoroff Theorem)**.**
Suppose that $\mathbf{X}$ *is a finite measure space. Moreover suppose that* $\{f_n\}_{n\in\mathbb{N}}$ *and* f *are elements of* $\mathcal{M}(\mathbf{X}, \mathbf{M}^E)$*. If*

$$\lim_{n\to\infty} f_n(x) = f(x) \tag{IV.4}$$

almost everywhere then given any $\delta > 0$ *there exists a measurable set* F *such that* $\mu(F) \leq \delta$ *and convergence (IV.4) holds uniformly on* $X - F$*.*

Theorem IV.8 (Riesz-Weyl Theorem)**.**
Let $F \coloneqq \{f_n\}_{n\in\mathbb{N}}$ *be a sequence in* $\mathcal{M}(\mathbf{X}, \mathbf{M}^E)$ *and suppose that* F *is Cauchy in* μ*. Then*

(1) There exists $f \in \mathcal{M}(\mathbf{X}, \mathbf{M}^E)$ *such that* f *is unique almost everywhere and*

$$f_n \xrightarrow{\mu} f \text{ as } n \to \infty.$$

(2) There exists a subsequence of F *that converges to* f *almost uniformly.*

Theorem IV.9. *Let* $f \in \mathcal{M}(\mathbf{X}, \mathbf{M}^E)$ *and let* $g \in \mathcal{L}^p$*. If* $\|f\|_E \leq g$*,* $\mu - a.e.$ *then* $f \in \mathcal{L}^p(E)$*.*

Theorem IV.10 (Holder inequality)**.**
Suppose that $p, q > 1$ *and* $\frac{1}{p} + \frac{1}{q} = 1$*. Moreover suppose that* $f \in \mathcal{L}^p(E)$ *and* $g \in \mathcal{L}^q(E)$*. Then* $fg \in \mathcal{L}^1(E)$ *and*

$$\|fg\|_{\mathcal{L}^1(E)} \leq \|f\|_{\mathcal{L}^p(E)} \|g\|_{\mathcal{L}^q(E)}.$$

Theorem IV.11 (Minkowski inequality)**.**
If $f, g \in \mathcal{L}^p(E)$*. Then* $f + g \in \mathcal{L}^p(E)$ *and*

$$\|f + g\|_{\mathcal{L}^p(E)} \leq \|f\|_{\mathcal{L}^p(E)} + \|g\|_{\mathcal{L}^p(E)}.$$

Theorem IV.12. *Let* $F \coloneqq \{f_n\}_{n\in\mathbb{N}}$ *be a sequence in* $\mathcal{L}^p(E)$ *and let* $f \in \mathcal{L}^p(E)$. *If*

$$\lim_{n\to\infty} \|f_n - f\|_{\mathcal{L}^p(E)} = 0$$

then there exists a subsequence $\{f_{\sigma(n)}\}_{n\in\mathbb{N}}$ *of* F, *which converges to* f *almost everywhere.*

Theorem IV.13. *Let* $F \coloneqq \{f_n\}_{n\in\mathbb{N}}$ *be a sequence in* $\mathcal{L}^p(E)$ *and let* $f \in \mathcal{M}(\mathbf{X}, \mathbf{M}^E)$. *If*

$$\lim_{n\to\infty} f_n(x) = f(x) \ \mu - a.e.$$

and there exists $g \in \mathcal{L}^p$ *such that for all* $n \in \mathbb{N}$ *we have* $\|f_n\|_E \leq g$ *almost everywhere then*

$$f \in \mathcal{L}^p(E) \quad \textit{and} \quad \lim_{n\to\infty} \|f_n - f\|_{\mathcal{L}^p(E)} = 0.$$

Theorem IV.14.

Let $F \coloneqq \{f_n\}_{n\in\mathbb{N}}$ *be a sequence in* $\mathcal{L}^p(E)$ *and let* $f \in \mathcal{L}^p(E)$. *If*

$$\lim_{n\to\infty} f_n(x) = f(x) \ \mu - a.e.$$

then

$$\lim_{n\to\infty} \|f_n - f\|_{\mathcal{L}^p(E)} = 0 \quad \Longleftrightarrow \quad \lim_{n\to\infty} \|f_n\|_{\mathcal{L}^p(E)} = \|f\|_{\mathcal{L}^p(E)}.$$

Theorem IV.15. *Let* $F \coloneqq \{f_n\}_{n\in\mathbb{N}}$ *be a sequence in* $\mathcal{L}^p(E)$ *and let* $f \in \mathcal{L}^p(E)$. *Then*

(1) $\|f_n - f\|_{\mathcal{L}^p(E)} \to 0$ *as* $n \to \infty \quad \Longrightarrow \quad f_n \xrightarrow{\mu} f$ *as* $n \to \infty$,

(2) $\{f_n\}_{n\in\mathbb{N}}$ *is Cauchy in* $\mathcal{L}^p(E) \quad \Longrightarrow \quad \{f_n\}_{n\in\mathbb{N}}$ *is Cauchy in* μ.

Theorem IV.16. *For some* $n \in \mathbb{N}$, *suppose that* $x_k \geq 0$ *for all* $1 \leq k \leq n$ *and* $p \geq 1$. *Then*

$$\left(\sum_{k=1}^{n} x_k\right)^p \leq n^{p-1} \sum_{k=1}^{n} x_k^p.$$

Theorem IV.17 (Young Inequality)**.**

Suppose that $p, q \in (1, \infty)$ *are such that* $\frac{1}{p} + \frac{1}{q} = 1$ *and* $x, y \in \mathbb{R}^+$. *Then*

$$xy \leq \frac{x^p}{p} + \frac{y^q}{q}. \tag{IV.5}$$

Moreover equality in (IV.5) above occurs if and only if $y = x^{p-1}$.

Theorem IV.18 (Fubini Theorem)**.**

Suppose that $\mathbf{X}$ *is a* $\sigma -$ *finite measure spaces and let* $\mathbf{Y} := (Y, \mathcal{B}, \eta)$ *be another* $\sigma -$ *finite measure spaces. Moreover let*

$$\mathbf{XY} := (X \times Y, \mathcal{A} \times \mathcal{B}, \mu \times \eta)$$

be a product measure space and let $u : X \times Y \to E$ *be an* $\mathcal{A} \times \mathcal{B}$ *measurable map. If at least one of the following integrals is finite:*

$$\int_{X \times Y} \|u\|_E d(\mu \times \eta), \quad \int_X \int_Y \|u\|_E d\mu d\eta, \quad \int_Y \int_X \|u\|_E d\eta d\mu$$

then all three integrals are finite, $u \in \mathcal{L}^1(\mathbf{XY}, \mathbf{M}^E)$ *and the following statements are true:*

(1) $x \to u(x, y) \in \mathcal{L}^1(\mathbf{X}, \mathbf{M}^E)$ $\eta -$ *almost everywhere,*

(2) $y \to u(x, y) \in \mathcal{L}^1(\mathbf{Y}, \mathbf{M}^E)$ $\mu -$ *almost everywhere,*

(3) $y \to \int_X u(x, y) d\mu(x) \in \mathcal{L}^1(\mathbf{Y}, \mathbf{M}^E)$,

(4) $x \to \int_Y u(x, y) d\eta(y) \in \mathcal{L}^1(\mathbf{X}, \mathbf{M}^E)$,

(5) $\int_{X \times Y} \|u\|_E d(\mu \times \eta) = \int_X \int_Y \|u\|_E d\mu d\eta = \int_Y \int_X \|u\|_E d\eta d\mu$.

Remark IV.19. *It follows that if* $f \in \mathcal{L}^1(\mathbf{MP}, \mathbf{M}^{\mathbb{R}})$ *then by Theorem IV.18 function*

$$t \to \int_\Omega f(t) d\mathbb{P}$$

is $\mathscr{B}(\mathcal{T})$ *measurable.*

Theorem IV.20 (Grönwall Inequality)**.**

Suppose that $\alpha \in \mathbb{R}$*,* $\beta \in \mathbb{R}_+$ *and* $f \in \mathcal{L}^1(\mathbf{M}, \mathbf{M}^{\mathbb{R}})$ *satisfies the following inequality*

$$f(t) \leq \alpha + \beta \int_0^t f(s)ds, \ \forall (t \in \mathfrak{T}).$$

Then

$$f(t) \leq \alpha e^{\beta t}, \ (\forall t \in \mathfrak{T}).$$

Theorem IV.21 (Jensen Inequality)**.**

Let $\Lambda : \mathbb{R}^+ \to \mathbb{R}^+$ *and* $V : \mathbb{R}^+ \to \mathbb{R}^+$ *be a concave and a convex function respectively. Suppose that* $w, u \in \mathcal{L}^1_+$ *and* $uw \in \mathcal{L}^1$*. Then* $\Lambda(u)w \in \mathcal{L}^1$*,* $V(u)w \in \mathcal{L}^1$ *and the following two inequalities hold:*

$$\frac{\int_X \Lambda(u) w d\mu}{\int_X w d\mu} \leq \Lambda\left(\frac{\int_X uwd\mu}{\int_X wd\mu}\right) \quad \textit{and} \quad V\left(\frac{\int_X uwd\mu}{\int_X wd\mu}\right) \leq \frac{\int_X V(u)wd\mu}{\int_X wd\mu}.$$

IV.2 Wiener Process in a Hilbert Space

This subsection of the appendix is based on [15, 17, 48, 52] and outlines a number of theorems that are used throughout the main body of the text. We assume here that we are working on the probability space and with definitions described in Subsection II.2.1 and II.2.3.

Let us however recall for convenience that we are working with a fixed separable Hilbert space $\mathcal{H}$ from the Definition II.6 and a fixed cylindrical Wiener process W (see Remark II.24) in $\mathcal{H}$. We also assume that filtration $\mathbb{F} := \{\mathcal{F}_t\}_{t\in\mathcal{T}}$ on our probability space satisfies the following standard conditions:

(1) $W(t)$ is $\mathcal{F}_t$ measurable, for all $t \in \mathcal{T}$,

(2) $W(t) - W(s)$ is independent of $\mathcal{F}_s$, for all $s \le t \in \mathcal{T}$.

Let us assume in addition that we have another separable Hilbert space X and a (separable Hilbert) space H of Hilbert-Schmidt operators from $\mathcal{H}$ to X. That is

$$H := \left\{ A \in L(\mathcal{H}, X) \;\middle|\; \begin{array}{l} \|A\|_H := \left(\sum_{n\in\mathbb{N}} ||A(\mathfrak{e}_n)||_X^2 \right)^{\frac{1}{2}} < \infty, \\ \mathfrak{e} := \{\mathfrak{e}_n\}_{n\in\mathbb{N}} \text{ is an orthonormal basis of } \mathcal{H} \end{array} \right\}. \tag{IV.6}$$

Moreover let us define the following space

$$\mathcal{N}_W := \left\{ \xi \in \mathcal{S}(H) \;\middle|\; \begin{array}{l} \mathbb{E}\left[\int_0^T \|\xi(s)\|_H^2 ds \right] < \infty, \\ \xi \text{ is progressively measurable} \end{array} \right\}. \tag{IV.7}$$

Definition IV.22. *Suppose that E is a Banach space and $\mathbf{M}^E := (E, \mathscr{B}(E))$ is a measurable space. Moreover let*

$$\mathcal{M}_{\mathbb{F}} := \left\{ \xi \in \mathcal{S}(E) \;\middle|\; \begin{array}{l} \xi_t \in \mathcal{L}(\mathbf{P}, \mathbf{M}^E) \ \forall (t \in \mathcal{T}), \\ \xi \textit{ is adapted to } \mathbb{F} \end{array} \right\}. \tag{IV.8}$$

(1) $\xi \in \mathcal{M}_{\mathbb{F}}$ is called an E valued martingale with respect to $\mathbb{F}$ if for all $s \le t \in \mathcal{T}$

$$\mathbb{E}[\xi_t|\mathcal{F}_s] = \xi_s, \ \mathbb{P} - a.s.$$

(2) $\xi \in \mathcal{S}(E)$ *is called square integrable if* $\xi_t \in \mathcal{L}^2(\mathbf{P}, \mathbf{M}^E)$ $\forall (t \in \mathcal{T})$.

Let us now outline a couple of useful theorems.

Theorem IV.23. *Suppose that* $\xi \in \mathcal{N}_W$. *Then we can define a stochastic process* $I \in \mathcal{S}(X)$ *in the following way*

$$I_t := \int_0^t \xi(s) dW(s), \ \forall (t \in \mathcal{T}).$$

Moreover

(A) I *is a square integrable* X *valued martingale with respect to* $\mathbb{F}$ *and trajectories of* I *are almost surely continuous.*

(B) For all $t \in \mathcal{T}$

(1) $\mathbb{E}\left[\int_0^t \xi(s) dW(s)\right] = 0,$

(2) $\mathbb{E}\left[\left\| \int_0^t \xi(s) dW(s) \right\|_X^2\right] = \int_0^t \mathbb{E}\left[\|\xi(s)\|_H^2\right] ds.$

Theorem IV.24. *Let* $p > 1$ *and let* E *be a separable Banach space. If* M *is a right-continuous* E *valued martingale then*

$$\begin{aligned}\mathbb{E}\left[\sup_{t \in \mathcal{T}} \|M(t)\|_E^p\right]^{\frac{1}{p}} &\leq \frac{p}{p-1} \sup_{t \in \mathcal{T}} \mathbb{E}\left[\|M(t)\|_E^p\right]^{\frac{1}{p}} \\ &\leq \frac{p}{p-1} \mathbb{E}\left[\|M(T)\|_E^p\right]^{\frac{1}{p}}.\end{aligned}$$

Theorem IV.25. *Let* Y *be a separable Hilbert space with the norm denoted by* $\|\cdot\|_Y$ *and define the following spaces:*

$$H' := \left\{ A \in L(\mathcal{H}, Y) \;\middle|\; \begin{array}{l} \|A\|_{H'} := \left(\sum_{n \in \mathbb{N}} \|A(\mathfrak{e}_n)\|_Y^2 \right)^{\frac{1}{2}} < \infty, \\ \mathfrak{e} := \{\mathfrak{e}_n\}_{n \in \mathbb{N}} \text{ is an orthonormal basis of } \mathcal{H} \end{array} \right\}, \tag{IV.9}$$

$$\overline{\mathcal{N}}_W := \left\{ \xi \in \mathcal{S}(H') \;\middle|\; \begin{array}{l} \mathbb{P}\left(\int_0^T \|\xi(s)\|_{H'}^2 ds < \infty \right) = 1, \\ \xi \text{ is progressively measurable} \end{array} \right\}. \tag{IV.10}$$

Moreover suppose in addition that $\xi \in \mathcal{N}_W$ and let $L \in L(X,Y)$ be a bounded linear operator from X to Y. Then $L \circ \xi \in \overline{\mathcal{N}}_W$ and

$$L\bigg(\int_0^T \xi(t)dW(t)\bigg) = \int_0^T L(\xi(t))dW(t),\ \mathbb{P}-a.s.$$

Theorem IV.26 (Kolmogorov Test)**.**

Suppose that E is a Banach space and $\xi \in \mathcal{S}(E)$. If there exist constants $C, \epsilon, \delta \in \mathbb{R}_+$ for all $s, t \in \mathcal{T}$ such that

$$\mathbb{E}\bigg[\|\xi_t - \xi_s\|_E^\delta\bigg] \leq C|t-s|^{1+\epsilon}$$

then ξ has a continuous modification (see Definition II.15).

Theorem IV.27 (BDG Type Inequality)**.**

Suppose that $p \in \mathbb{R}_2$ and $\xi \in \mathcal{N}_W$. Then

$$\mathbb{E}\bigg[\sup_{t\in\mathcal{T}}\bigg\|\int_0^t \xi(s)dW(s)\bigg\|_X^p\bigg]^{\frac{1}{p}} \leq p\bigg(\frac{p}{2(p-1)}\bigg)^{\frac{1}{2}}\bigg[\int_0^T \bigg(\mathbb{E}\bigg[\|\xi(s)\|_H^p\bigg]\bigg)^{\frac{2}{p}} ds\bigg]^{\frac{1}{2}}.$$

IV.3 Martingales and Wiener Process in $\mathbb{R}$

In this section we continue to work with a real valued Wiener process W defined on $\mathbf{MP}$ and assume that our filtration $\mathbb{F} := \{\mathcal{F}_t\}_{t\in\mathcal{T}}$ is suitably chosen so that the following properties are satisfied:

(1) For all $t \in \mathcal{T}$, $W(t)$ is $\mathcal{F}_t$ measurable,

(2) For all $s \leq t \in \mathcal{T}$, $W(t) - W(s)$ is independent of $\mathcal{F}_s$.

Unless stated otherwise, information in this subsection is based on [29]. We now would like to fix in place the following notation:

$$S_1 := \left\{ K \subset \mathcal{T} \times \Omega \;\middle|\; K = (s,t] \times A \text{ where } s < t \in \mathcal{T} \wedge A \in \mathcal{F}_s \right\},$$

$$S_2 := \left\{ K \subset \mathcal{T} \times \Omega \;\middle|\; K = \{0\} \times A \text{ where } A \in \mathcal{F}_0 \right\},$$

$$\mathcal{P} := \sigma(S_1 \cup S_2),$$

$$\mathbb{L} := \left\{ \xi \in \mathcal{S}(\mathbf{M}^{\mathbb{R}}) \;\middle|\; \begin{array}{l} \text{trajectories of } \xi \text{ are almost surely left continuous,} \\ \xi \text{ is adapted to } \mathbb{F} \end{array} \right\},$$

$$\overline{\mathbf{MP}} := (\overline{\Omega}, \mathcal{P}).$$

We note that $\mathcal{P}$ above is the smallest $\sigma-$algebra with respect to which all elements of $\mathbb{L}$ are measurable.

Definition IV.28. *For all $p \in \mathbb{R}_1$ we introduce the following spaces of stochastic processes.*

$$L^p_{ad} := \{\xi \in \mathcal{L}^p(\mathbf{MP}, \mathbf{M}^{\mathbb{R}}) \mid \xi \textit{ is adapted to } \mathbb{F}.\}$$

and a space

$$\mathcal{M}_{\mathbb{F}} := \left\{ \xi \in \mathcal{S}(\mathbf{M}^{\mathbb{R}}) \;\middle|\; \begin{array}{l} \xi_t \in \mathcal{L}(\mathbf{P}, \mathbf{M}^{\mathbb{R}}) \ \forall t \in \mathcal{T}, \\ \xi \textit{ is adapted to } \mathbb{F} \end{array} \right\}$$

Definition IV.29.

(1) $\xi \in \mathcal{M}_{\mathbb{F}}$ *is called a martingale with respect to* $\mathbb{F}$ *if for all* $s \leq t \in \mathcal{T}$

$$\mathbb{E}[\xi_t|\mathcal{F}_s] = \xi_s, \ \mathbb{P} - a.s.$$

(2) $\xi \in \mathcal{S}(\mathbf{M}^{\mathbb{R}})$ *is called square integrable if* $\xi_t \in \mathcal{L}^2(\mathbf{P}, \mathbf{M}^{\mathbb{R}}), \ \forall (t \in \mathcal{T})$.

(3) $\xi \in \mathcal{S}(\mathbf{M}^{\mathbb{R}})$ *is called predictable if* $\xi \in \mathcal{M}(\overline{\mathbf{MP}}, \mathbf{M}^{\mathbb{R}})$.

Theorem IV.30. *Let* ξ *be a right continuous, square integrable martingale with left-hand limits. Then there is a unique decomposition*

$$\xi_t^2 = L_t + A_t, \ \forall (t \in \mathcal{T})$$

where L *is a right continuous martingale with left-hand limits and* A *is a predictable, right continuous, and increasing process such that* $A(0) = 0$ *and* $A_t \in \mathcal{L}(\mathbf{P}, \mathbf{M}^{\mathbb{R}}), \ \forall (t \in \mathcal{T})$.

Remark IV.31. *Process* A *found by Theorem IV.30 will be called a quadratic variation of* ξ *(or a Meyer process) in this document and the following abbreviation will be used*

$$\langle \xi \rangle_t := A_t, \ \forall (t \in \mathcal{T}).$$

Moreover, one can show that

$$\langle W \rangle_t = t, \ \forall (t \in \mathcal{T}).$$

Theorem IV.32. *Let* $\xi \in L^2_{ad}$ *and define a stochastic process* X *in the following way*

$$X_t := \int_0^t \xi(s) dW(s), \ \forall (t \in \mathcal{T}).$$

Then

(A) X *is a martingale with respect to* $\mathbb{F}$ *and trajectories of* X *are almost surely continuous.*

(B) For all $t \in \mathcal{T}$ *the following statements hold:*

(1) $\mathbb{E}\Big[\int_0^t \xi(s)dW(s)\Big] = 0,$

(2) $\mathbb{E}\Big[\Big|\int_0^t \xi(s)dW(s)\Big|^2\Big] = \int_0^t \mathbb{E}\Big[|\xi(s)|^2\Big]ds,$

(3) $\langle X\rangle_t = \int_0^t |\xi(s)|^2 d\langle W\rangle_s.$

Following theorem is a useful result from [41].

Theorem IV.33 (Burkholder, Davis and Gundy Inequality)**.**

Let X be a continuous martingale. Then for all $t \in \mathcal{T}$ and all $p \in (0,\infty)$

$$\mathbb{E}\Big[\sup\Big\{|X_s|^p \;\Big|\; 0 \le s \le t\Big\}\Big] \le \mathbb{E}\Big[\Big(\langle X\rangle_t\Big)^{\frac{p}{2}}\Big].$$

Definition IV.34. *Suppose that $f \in L^2_{ad}$, $g \in L^1_{ad}$ and let ξ_0 be a $\mathcal{F}_0$ measurable random variable. An Itô process is a real valued stochastic process ξ satisfying*

$$\xi_t = \xi_0 + \int_0^t g(s)ds + \int_0^t f(s)dW(s), \;\forall(t \in \mathcal{T}). \tag{IV.11}$$

Theorem IV.35 (Itô Lemma)**.**

Let ξ be an Itô process satisfying equation (IV.11) above and suppose that $\theta : \mathbb{R}^2 \to \mathbb{R}$ is a continuous function such that all $\frac{\partial\theta}{\partial t}$, $\frac{\partial\theta}{\partial x}$ and $\frac{\partial^2\theta}{\partial x^2}$ are continuous functions from $\mathbb{R}^2$ to $\mathbb{R}$. Then $\theta \circ \xi$ is an Itô process satisfying

$$\theta(t,\xi_t) = \theta(0,\xi_0) + \int_0^t \mathcal{K}(s,\xi_s)ds + \int_0^t \frac{\partial\theta}{\partial x}(s,\xi_s)f(s)dW(s), \;\forall(t \in \mathcal{T})$$

where

$$\mathcal{K}(t,\xi_t) \coloneqq \frac{\partial\theta}{\partial t}(t,\xi_t) + \frac{\partial\theta}{\partial x}(t,\xi_t)g(t) + \frac{1}{2}\frac{\partial^2\theta}{\partial x^2}(t,\xi_t)f^2(t), \;\forall(t \in \mathcal{T}).$$

IV.4 Deterministic Ovsjannikov Equation

Unless stated otherwise, information in this subsection is based on [10, 11, 46, 47] and also on our work in Section II.

In this subsection we would like to address the problem of finding a unique solution f satisfying the following integral equation

$$f(t) = x_{\underline{\mathfrak{a}}} + \int_0^t F(f(s))ds, \quad \forall(t \in \mathcal{T}) \tag{IV.12}$$

using the method of Ovsjannikov. To this end let us begin by fixing a suitable scale of Banach spaces $\mathbb{X} := \{\mathbb{X}_{\mathfrak{a}}\}_{\mathfrak{a}\in\mathcal{A}}$, assuming that $x_{\underline{\mathfrak{a}}} \in X_{\underline{\mathfrak{a}}}$ and letting $F \in \mathcal{O}(\mathbb{X}, q)$ be an Ovsjannikov map on $\mathbb{X}$. The main result of this appendix, that is existence and uniqueness of f, is summarised in the Theorem IV.43 bellow.

We will now show how the proof of Theorem IV.43 can be obtained. We start by introducing a family $\mathbb{Y} := \{\mathbb{Y}_{\mathfrak{a}}\}_{\mathfrak{a}\in\mathcal{A}}$ where $\mathbb{Y}_{\mathfrak{a}}$ is the classical space of continuous $\mathbb{X}_{\mathfrak{a}}$ valued maps. That is for all $\mathfrak{a} \in \mathcal{A}$ we define

$$\mathbb{Y}_{\mathfrak{a}} := \mathcal{C}(\mathcal{T}, \mathbb{X}_{\mathfrak{a}}).$$

Remark IV.36. *It is important to understand that calculations in this subsection remain valid if we choose to proceed with the following definition*

$$\mathbb{Y}_{\mathfrak{a}} := \mathcal{B}(\mathcal{T}, \mathbb{X}_{\mathfrak{a}}), \quad \forall(\mathfrak{a} \in \mathcal{A}).$$

That is $\mathbb{Y}_{\mathfrak{a}}$ *is the space of bounded* $\mathbb{X}_{\mathfrak{a}}$ *valued maps.*

Now, for all $\alpha < \beta \in \mathcal{A}$ and $f \in \mathbb{Y}_{\alpha}$ the following simple statements are true:

(1) $\mathbb{Y}$ is a family of Banach spaces,

(2) $\mathbb{Y}_{\alpha} \prec \mathbb{Y}_{\beta}$, (IV.13)

(3) $||f||_{\mathbb{Y}_{\beta}} \leq ||f||_{\mathbb{Y}_{\alpha}}$.

Therefore, from the list (IV.13) above we can conclude, using the Definition II.2, that $\mathbb{Y}$ is a scale. Now continuing our work we define a map $\mathcal{J} : \overset{\uparrow}{\mathbb{Y}} \to \mathbb{Y}_{\overline{\mathfrak{a}}}$ by letting for all $f \in \overset{\uparrow}{\mathbb{Y}}$

$$\mathcal{J}(f)(t) := x_{\underline{\mathfrak{a}}} + \int_0^t F(f(s))ds, \quad \forall (t \in \mathcal{T}). \tag{IV.14}$$

The following result can now be proved.

Theorem IV.37. *$\mathcal{J} \in \mathcal{O}(\mathbb{Y}, q)$.*

Proof. Fix $\alpha < \beta \in \mathcal{A}$, $f, g \in \mathbb{Y}_\alpha$ and $t \in \mathcal{T}$. We now check that the integral map $\mathcal{J}$ satisfies the Definition II.4. We begin by using the general definition of a Bochner integral and the fact that $F \in \mathcal{O}(\mathbb{X}, q)$ to conclude that $\mathcal{J}|_{\mathbb{Y}_\alpha} : \mathbb{Y}_\alpha \to \mathbb{Y}_\beta$. Moreover we see that

$$\begin{aligned} ||\mathcal{J}(f)(t) - \mathcal{J}(g)(t)||_{\mathbb{X}_\beta} &\leq \int_0^t ||F(f(s)) - F(g(s))||_{\mathbb{X}_\beta} ds \\ &\leq \frac{L}{(\beta-\alpha)^q} \int_0^t ||f(s) - g(s)||_{\mathbb{X}_\alpha} ds \\ &\leq \frac{L}{(\beta-\alpha)^q} \int_0^t ||f - g||_{\mathbb{Y}_\alpha} ds. \end{aligned} \tag{IV.15}$$

Therefore we see that

$$\begin{aligned} ||\mathcal{J}(f) - \mathcal{J}(g)||_{\mathbb{Y}_\beta} &\leq \frac{L}{(\beta-\alpha)^q} \int_0^T ||f - g||_{\mathbb{Y}_\alpha} ds \\ &\leq \frac{LT}{(\beta-\alpha)^q} ||f - g||_{\mathbb{Y}_\alpha} \end{aligned}$$

hence the proof is complete. □

We now would like to define something called an iterated or a composite map. That is for all $n \in \mathbb{N}$ we define

$$\mathcal{J}^n := \overbrace{\mathcal{J} \circ \mathcal{J} \circ \cdots \circ \mathcal{J}}^{n \text{ times}}$$

and let $\mathcal{J}^0$ be the identity map from $\mathbb{Y}_{\underline{\mathfrak{a}}}$ to $\mathbb{Y}_{\underline{\mathfrak{a}}}$. Our next result shows that for all $n \in \mathbb{N}_0$ the composite map $\mathcal{J}^n$ is well defined.

Theorem IV.38. *For all* $n \in \mathbb{N}_0$

$$\mathcal{J}^n : \mathbb{Y}_{\underline{\mathfrak{a}}} \to \downarrow\mathbb{Y}. \tag{IV.16}$$

Proof. We prove this statement by induction. For $n = 0$ the statement (IV.16) is trivially true because $\mathbb{Y}_{\underline{\mathfrak{a}}} \subset \downarrow\mathbb{Y}$. Now suppose that induction hypothesis holds for some $n \geq 0$. Fix arbitrary $\mathfrak{a} \in (\underline{\mathfrak{a}}, \overline{\mathfrak{a}})$ and $p \in (\underline{\mathfrak{a}}, \mathfrak{a})$. Observe that induction hypothesis implies that $\mathcal{J}^n : \mathbb{Y}_{\underline{\mathfrak{a}}} \to \mathbb{Y}_p$. However because $\mathcal{J} \in \mathcal{O}(\mathbb{Y}, q)$ we know that $\mathcal{J}|_{\mathbb{Y}_p} \colon \mathbb{Y}_p \to \mathbb{Y}_{\mathfrak{a}}$ hence by composition $\mathcal{J} \circ \mathcal{J}^n$ it follows that $\mathcal{J}^{n+1} : \mathbb{Y}_{\underline{\mathfrak{a}}} \to \mathbb{Y}_{\mathfrak{a}}$ and since $\mathfrak{a} \in (\underline{\mathfrak{a}}, \overline{\mathfrak{a}})$ is arbitrary the proof is complete. □

Remark IV.39. *Observe that Theorem II.40 shows that if* $f \in \mathbb{Y}_{\underline{\mathfrak{a}}}$ *then the sequence* $\{\mathcal{J}^n(f)\}_{n=0}^{\infty}$ *belongs to* $\mathbb{Y}_{\mathfrak{a}}$ *for all* $\mathfrak{a} \in (\underline{\mathfrak{a}}, \overline{\mathfrak{a}})$.

Let us now, for a moment, consider some fixed $t_0 \in \mathcal{T}$, $\alpha < \beta \in (\underline{\mathfrak{a}}, \overline{\mathfrak{a}})$ and $f \in \mathbb{Y}_{\underline{\mathfrak{a}}}$. Moreover let us consider an arbitrary $n \in \mathbb{N}$ and a partition $\{\psi_i\}_{i=0}^n$ of $[\alpha, \beta]$ into n intervals of equal length. That is $\psi_0 = \alpha$, $\psi_n = \beta$ and $\psi_{i+1} - \psi_i = \frac{b-a}{n}$ for all $0 \leq i \leq n-1$. Letting

$$K_n^{n+1}(t) = \mathcal{J}^n(f)(t) - \mathcal{J}^{n+1}(f)(t), \quad \forall (t \in [0, t_0])$$

we see from Theorem IV.37 and IV.38 that

$$\begin{aligned}
||K_n^{n+1}(t_0)||_{\mathbb{X}_{\psi_n}} &\leq \frac{L}{(\psi_n - \psi_{n-1})^q} \int_0^{t_0} ||K_{n-1}^n(t_1)||_{\mathbb{X}_{\psi_{n-1}}} dt_1 \\
&\leq \frac{L}{(\psi_n - \psi_{n-1})^q} \frac{L}{(\psi_{n-1} - \psi_{n-2})^q} \int_0^{t_0} \int_0^{t_1} ||K_{n-2}^{n-1}(t_2)||_{\mathbb{X}_{\psi_{n-2}}} dt_2 dt_1 \\
&\leq L^n \left(\frac{\beta - \alpha}{n} \right)^{-qn} \int_0^{t_0} \int_0^{t_1} \cdots \int_0^{t_{n-1}} ||K_0^1(t_n)||_{\mathbb{X}_{\psi_0}} dt_n dt_{n-1} \cdots dt_1 \qquad \text{(IV.17)} \\
&\leq \frac{L^n}{(\beta - \alpha)^{qn}} n^{qn} ||K_0^1||_{\mathbb{Y}_{\psi_0}} \int_0^{t_0} \int_0^{t_1} \cdots \int_0^{t_{n-1}} dt_n dt_{n-1} \cdots dt_1 \\
&\leq \frac{L^n t_0^n}{(\beta - \alpha)^{qn}} \frac{n^{qn}}{n!} ||K_0^1||_{\mathbb{Y}_{\psi_0}}.
\end{aligned}$$

Hence, defining recursively a map $\mathcal{H}^n : \mathcal{C}(\mathcal{T},\mathbb{R}) \to \mathcal{C}(\mathcal{T},\mathbb{R})$ for all $n \in \mathbb{N}_0$ via formula

$$\mathcal{H}^n(t,f) := \begin{cases} f(t) & n = 0, \\ \int_0^t f(s)ds & n = 1, \\ \int_0^t \mathcal{H}^{n-1}(s,f)ds & n > 1. \end{cases} \tag{IV.18}$$

we see from inequalities (IV.17) that the following result can be formulated and proved.

Theorem IV.40. *Suppose $\alpha < \beta \in (\underline{\mathfrak{a}}, \overline{\mathfrak{a}})$ and $f, g \in \mathbb{Y}_{\underline{\mathfrak{a}}}$. Then for all $n \in \mathbb{N}$*

$$||\mathcal{I}^n(f) - \mathcal{I}^{n+1}(g)||_{\mathbb{Y}_\beta} \leq \frac{L^n T^n}{(\beta-\alpha)^{qn}} \frac{n^{qn}}{n!} ||f - \mathcal{I}(g)||_{\mathbb{Y}_\alpha}. \tag{IV.19}$$

Proof. Fixing $t \in \mathcal{T}$ we prove by induction that

$$||\mathcal{I}^n(f)(t) - \mathcal{I}^{n+1}(g)(t)||_{\mathbb{X}_\beta} \leq \frac{L^n}{(\beta-\alpha)^{qn}} n^{qn} \mathcal{H}^n(t, ||f - \mathcal{I}(g)||_{\mathbb{X}_\alpha})$$

from where inequality (IV.19) follows directly. Clearly case $n = 1$ follows immediately from the Theorem IV.37. Precisely speaking inequality (IV.15) shows that the induction hypothesis holds for $n = 1$. Now, suppose that the induction hypothesis holds for some $n \geq 1$. Choosing $\psi \in (\alpha, \beta)$ such that $\beta - \psi = \frac{\beta-\alpha}{n+1}$ we see, using Theorem IV.37, that

$$||\mathcal{I}^{n+1}(f)(t) - \mathcal{I}^{n+2}(g)(t)||_{\mathbb{X}_\beta} \leq \frac{L}{(\beta-\psi)} \int_0^t ||\mathcal{I}^n(f)(s) - \mathcal{I}^{n+1}(g)(s)||_{\mathbb{X}_\psi} ds.$$

Hence letting

$$\mathbf{A} := ||f - \mathcal{I}(g)||_{\mathbb{X}_\alpha}$$

and applying the induction hypothesis we get

$$\begin{aligned} ||\mathcal{I}^{n+1}(f)(t) - \mathcal{I}^{n+2}(g)(t)||_{\mathbb{X}_\beta} &\leq \frac{L}{(\beta-\psi)^q} \frac{L^n}{(\psi-\alpha)^{qn}} n^{qn} \int_0^t \mathcal{H}^n(s,\mathbf{A})ds \\ &\leq \frac{L^{n+1}}{(\beta-\psi)^q(\psi-\alpha)^{qn}} n^{qn} \mathcal{H}^{n+1}(t,\mathbf{A}) \end{aligned}$$

expanding further we see that

$$
\begin{aligned}
||\mathcal{I}^{n+1}(f)(t) - \mathcal{I}^{n+2}(g)(t)||_{\mathbb{X}_\beta} &\leq L^{n+1}\left(\frac{\beta-\alpha}{n+1}\right)^{-q}\left(\frac{n(\beta-\alpha)}{n+1}\right)^{-qn} n^{qn}\mathcal{H}^{n+1}(t,\mathbf{A}) \\
&\leq \frac{L^{n+1}}{(\beta-\alpha)^{q(n+1)}}\frac{(n+1)^{q(n+1)}}{n^{qn}} n^{qn}\mathcal{H}^{n+1}(t,\mathbf{A}) \\
&\leq \frac{L^{n+1}}{(\beta-\alpha)^{q(n+1)}}(n+1)^{q(n+1)}\mathcal{H}^{n+1}(t,\mathbf{A})
\end{aligned}
$$

Hence

$$
||\mathcal{I}^{n+1}(f)(t) - \mathcal{I}^{n+2}(g)(t)||_{\mathbb{X}_\beta} \leq \frac{L^{n+1}}{(\beta-\alpha)^{q(n+1)}}(n+1)^{q(n+1)}\mathcal{H}^{n+1}(t, ||f-\mathcal{I}(g)||_{\mathbb{X}_\alpha})
$$

and the proof is complete. □

> **Remark IV.41.** *It is clear from the definition of the composite map $\mathcal{I}^n$ that the Theorem IV.40 is trivially true for $n = 0$. Moreover it is essential that $\alpha \in (\underline{\mathfrak{a}}, \overline{\mathfrak{a}})$ because it is possible that $\mathcal{I}(f)$ does not belong to $\mathbb{Y}_{\underline{\mathfrak{a}}}$.*

Theorem IV.40 puts us now in a position to prove the following result.

Theorem IV.42. *Suppose that $q < 1$ and $F \in \mathcal{O}(\mathbb{X}, q)$. Then there exists a unique element $\phi \in \downarrow\mathbb{Y}$ such that $\mathcal{I}(\phi) = \phi$. Moreover if $\mathfrak{a} \in (\underline{\mathfrak{a}}, \overline{\mathfrak{a}})$ and $f \in \mathbb{Y}_{\underline{\mathfrak{a}}}$ then*

$$
\overbrace{\lim_{n\to\infty} \mathcal{I}^n(f)}^{in\ \mathbb{Y}_{\mathfrak{a}}} = \phi.
$$

Proof. Fix $f \in \mathbb{Y}_{\underline{\mathfrak{a}}}$ and $\mathfrak{a} \in (\underline{\mathfrak{a}}, \overline{\mathfrak{a}})$. Fix also an arbitrary $\gamma \in (\underline{\mathfrak{a}}, \mathfrak{a})$ and using theorem II.42 observe that for all $m \geq n \in \mathbb{N}$ we have

$$
\begin{aligned}
||\mathcal{I}^n(f) - \mathcal{I}^m(f)||_{\mathbb{Y}_{\mathfrak{a}}} &\leq \sum_{k=n}^{m-1} ||\mathcal{I}^k(f) - \mathcal{I}^{k+1}(f)||_{\mathbb{Y}_\gamma} \\
&\leq \sum_{k=n}^{m-1} \frac{L^k T^k}{(\mathfrak{a}-\gamma)^{qk}}\frac{n^{qk}}{k!}\ ||f-\mathcal{I}(f)||_{\mathbb{Y}_\gamma} \\
&\leq \sum_{k=n}^{\infty} \frac{L^k T^k}{(\mathfrak{a}-\gamma)^{qk}}\frac{n^{qk}}{k!}\ ||f-\mathcal{I}(f)||_{\mathbb{Y}_\gamma}. \qquad \text{(IV.20)}
\end{aligned}
$$

According to Theorem II.35 the right hand side of inequality (IV.20) above is a remainder of a convergent series. Therefore we conclude that sequence $\{\mathcal{I}^n(f)\}_{n\in\mathbb{N}}$ is Cauchy in $\mathbb{Y}_{\mathfrak{a}}$. Since $\mathfrak{a}$ is arbitrary, let us now consider $\alpha < \beta \in (\underline{\mathfrak{a}}, \overline{\mathfrak{a}})$ and

$$\overbrace{\lim_{n\to\infty} \mathcal{I}^n(f)}^{\text{in } \mathbb{Y}_\alpha} = \phi_\alpha$$

$$\overbrace{\lim_{n\to\infty} \mathcal{I}^n(f)}^{\text{in } \mathbb{Y}_\beta} = \phi_\beta.$$

Because $\mathbb{Y}$ is a scale, in particular $\mathbb{Y}_\alpha \prec \mathbb{Y}_\beta$, we see that

$$\|\phi_\beta - \phi_\alpha\|_{\mathbb{Y}_\beta} \leq \|\phi_\beta - \mathcal{I}^n(f)\|_{\mathbb{Y}_\beta} + \|\mathcal{I}^n(f) - \phi_\alpha\|_{\mathbb{Y}_\beta}$$

$$\leq \|\phi_\beta - \mathcal{I}^n(f)\|_{\mathbb{Y}_\beta} + \|\mathcal{I}^n(f) - \phi_\alpha\|_{\mathbb{Y}_\alpha}$$

which shows that $\phi_\beta = \phi_\alpha$. Therefore defining

$$\phi_\alpha =: \phi := \phi_\beta$$

we see that $\phi \in_{\downarrow} \mathbb{Y}$ and

$$\overbrace{\lim_{n\to\infty} \mathcal{I}^n(f)}^{\text{in } \mathbb{Y}_{\mathfrak{a}}} = \phi.$$

Now, from Theorem IV.37 it follows that $\mathcal{I}$ is a continuous map from $\mathbb{Y}_\gamma$ to $\mathbb{Y}_{\mathfrak{a}}$. Hence we see that

$$\mathcal{I}^{n+1}(f) \to \phi \text{ as } n \to \infty$$

$$\mathcal{I}^{n+1}(f) = \mathcal{I}(\mathcal{I}^n(f)) \to \mathcal{I}(\phi) \text{ as } n \to \infty$$

which shows that $\mathcal{I}(\phi) = \phi$. Finally suppose that there exists $\psi \in_{\downarrow} \mathbb{Y}$ such that $\psi \neq \phi$ and $\mathcal{I}(\psi) = \psi$. In this case it is clear that

$$||\mathcal{I}^n(\phi) - \mathcal{I}^{n+1}(\psi)||_{\mathbb{Y}_{\mathfrak{a}}} = ||\phi - \psi||_{\mathbb{Y}_{\mathfrak{a}}}.$$

However from Theorem IV.40 we can infer that

$$\begin{aligned} ||\mathcal{I}^n(\phi) - \mathcal{I}^{n+1}(\psi)||_{\mathbb{Y}_\mathfrak{a}} &\leq \frac{L^n T^n}{(\mathfrak{a}-\gamma)^{qn}} \frac{n^{qn}}{n!} ||\phi - \mathcal{I}(\psi)||_{\mathbb{Y}_\gamma} \\ &= \frac{L^n T^n}{(\mathfrak{a}-\gamma)^{qn}} \frac{n^{qn}}{n!} ||\phi - \psi||_{\mathbb{Y}_\gamma}. \end{aligned} \tag{IV.21}$$

Since, by Theorem II.35, the right hand side of inequality (IV.21) tends to zero we conclude that $||\phi - \psi||_{\mathbb{Y}_\mathfrak{a}} = 0$. Therefore ϕ is unique and the proof is complete. □

We now formulate and prove the main result of this appendix.

Theorem IV.43. *Suppose $x_{\underline{\mathfrak{a}}} \in X_{\underline{\mathfrak{a}}}$, $q < 1$ and $F \in \mathcal{O}(\mathbb{X}, q)$ are fixed. Then there exist a unique map $f \in \downarrow\mathbb{Y}$ such that*

$$f(t) = x_{\underline{\mathfrak{a}}} + \int_0^t F(f(s))ds, \quad \forall (t \in \mathcal{T}).$$

Moreover if $\mathfrak{a} \in (\underline{\mathfrak{a}}, \overline{\mathfrak{a}})$ and $g \in \mathbb{Y}_{\underline{\mathfrak{a}}}$ then

$$\overbrace{\lim_{n\to\infty} \mathcal{I}^n(g)}^{in\ \mathbb{Y}_\mathfrak{a}} = f.$$

Proof. This result follows directly from Theorem IV.42 above by letting $f := \phi$. □

Remark IV.44. *Current method can also be used to prove Theorem IV.43 when $q = 1$ by introducing a suitable upper bound on T.*

The final result of this appendix is a useful norm estimate. To prove this final result we now make two preliminary observations. First, suppose that $\alpha < \beta \in \mathcal{A}$ and $x \in \mathbb{X}_\alpha$. Then we can see that

$$\begin{aligned} \|F(x)\|_{\mathbb{X}_\beta} &= \|F(x) + F(0) - F(0)\|_{\mathbb{X}_\beta} \\ &\leq \|F(x) - F(0)\|_{X_\beta} + \|F(0)\|_{\mathbb{X}_\beta} \\ &\leq \frac{L}{(\beta-\alpha)^q} \|x\|_{\mathbb{X}_\alpha} + \|F(0)\|_{\mathbb{X}_\beta} \\ &\leq \frac{L}{(\beta-\alpha)^q} \left(P + \|x\|_{\mathbb{X}_\alpha} \right) \end{aligned}$$

where

$$P := \frac{\|F(0)\|_{\mathbb{X}_{\underline{\mathfrak{a}}}}(\overline{\mathfrak{a}}-\underline{\mathfrak{a}})^q}{L}. \tag{IV.22}$$

Second, suppose $\mathfrak{a} \in (\underline{\mathfrak{a}}, \overline{\mathfrak{a}})$ and $x_{\underline{\mathfrak{a}}} \in \mathbb{X}_{\underline{\mathfrak{a}}}$. Moreover consider a partition $\{\psi_i\}_{i=0}^{n+1}$ of $[\underline{\mathfrak{a}}, \mathfrak{a}]$ into $n+1$ intervals of equal length. That is $\psi_0 = \underline{\mathfrak{a}}$, $\psi_{n+1} = \mathfrak{a}$ and $\psi_{i+1} - \psi_i = \frac{\mathfrak{a}-\underline{\mathfrak{a}}}{n-1}$ for all $0 \le i \le n$. Now, from Theorem IV.40 we see that for all $n \in \mathbb{N}_0$ we have

$$\begin{aligned}
||\mathcal{T}^n(x_{\underline{\mathfrak{a}}})(t) - \mathcal{T}^{n+1}(x_{\underline{\mathfrak{a}}})(t)||_{\mathbb{X}_{\mathfrak{a}}} &\le \frac{L^n}{(\mathfrak{a}-\psi_1)^{qn}} n^{qn} \mathcal{H}^n(t, ||x_{\underline{\mathfrak{a}}} - \mathcal{T}(x_{\underline{\mathfrak{a}}})||_{\mathbb{X}_{\psi_1}}) \\
&\le \frac{L^n}{(\mathfrak{a}-\psi_1)^{qn}} n^{qn} \mathcal{H}^{n+1}(t, \|F(x_{\underline{\mathfrak{a}}})\|_{\mathbb{X}_{\psi_1}}) \\
&\le \frac{L^n}{(\mathfrak{a}-\psi_1)^{qn}} \frac{L}{(\psi_1 - \underline{\mathfrak{a}})} n^{qn} \mathcal{H}^{n+1}(t, P + \|x_{\underline{\mathfrak{a}}}\|_{\mathbb{X}_{\underline{\mathfrak{a}}}}) \qquad \text{(IV.23)} \\
&\le \frac{L^n T^{n+1}}{(\mathfrak{a}-\psi_1)^{qn}} \frac{L}{(\psi_1 - \underline{\mathfrak{a}})} \frac{n^{qn}}{(n+1)!} \left(P + \|x_{\underline{\mathfrak{a}}}\|_{\mathbb{X}_{\underline{\mathfrak{a}}}} \right) \\
&\le \frac{L^{n+1} T^{n+1}}{(\mathfrak{a}-\underline{\mathfrak{a}})^{q(n+1)}} \frac{(n+1)^{q(n+1)}}{(n+1)!} \left(P + \|x_{\underline{\mathfrak{a}}}\|_{\mathbb{X}_{\underline{\mathfrak{a}}}} \right).
\end{aligned}$$

We now obtain the norm estimate.

Theorem IV.45. *Let f be defined by Theorem IV.43 and suppose that $\mathfrak{a} \in (\underline{\mathfrak{a}}, \overline{\mathfrak{a}})$. Then*

$$||f(t)||_{\mathbb{X}_{\mathfrak{a}}} \le \sum_{n=0}^{\infty} \frac{L^n T^n}{(\mathfrak{a}-\underline{\mathfrak{a}})^{qn}} \frac{n^{qn}}{n!} \left(P + \|x_{\underline{\mathfrak{a}}}\|_{\mathbb{X}_{\underline{\mathfrak{a}}}} \right), \quad \forall (t \in \mathcal{T}).$$

Proof. From Theorem IV.42 it is clear that for all $t \in \mathcal{T}$ we have the following equality

$$\lim_{n\to\infty} \|\mathcal{T}^n(x_{\underline{\mathfrak{a}}})(t)\|_{\mathbb{X}_{\mathfrak{a}}} = \|f(t)\|_{\mathbb{X}_{\mathfrak{a}}}.$$

Hence we now use estimate (IV.23) to see that for all $n \in \mathbb{N}$ and all $t \in \mathcal{T}$ we have

$$\begin{aligned}
\|\mathcal{T}^n(x_{\underline{\mathfrak{a}}})(t)\|_{\mathbb{X}_{\mathfrak{a}}} - \|\mathcal{T}^0(x_{\underline{\mathfrak{a}}})(t)\|_{\mathbb{X}_{\mathfrak{a}}} &= \sum_{k=1}^{n} \|\mathcal{T}^k(x_{\underline{\mathfrak{a}}})(t)\|_{\mathbb{X}_{\mathfrak{a}}} - \|\mathcal{T}^{k-1}(x_{\underline{\mathfrak{a}}})(t)\|_{\mathbb{X}_{\mathfrak{a}}} \\
&\le \sum_{k=1}^{n} \|\mathcal{T}^{k-1}(x_{\underline{\mathfrak{a}}})(t) - \mathcal{T}^k(x_{\underline{\mathfrak{a}}})(t)\|_{\mathbb{X}_{\mathfrak{a}}}
\end{aligned}$$

Hence it follows that

$$\|\mathcal{I}^n(x_{\underline{\mathfrak{a}}})(t)\|_{\mathbb{X}_{\mathfrak{a}}} - \|\mathcal{I}^0(x_{\underline{\mathfrak{a}}})(t)\|_{\mathbb{X}_{\mathfrak{a}}} = \sum_{k=1}^{n} \frac{L^k T^k}{(\mathfrak{a}-\underline{\mathfrak{a}})^{qn}} \frac{k^{qk}}{k!} \left(P + \|x_{\underline{\mathfrak{a}}}\|_{\mathbb{X}_{\underline{\mathfrak{a}}}}\right).$$

Therefore for all $n \in \mathbb{N}$ and all $t \in \mathcal{T}$ we have

$$\begin{aligned}
\|\mathcal{I}^n(x_{\underline{\mathfrak{a}}})(t)\|_{\mathbb{X}_{\mathfrak{a}}} &\leq \|\mathcal{I}^0(x_{\underline{\mathfrak{a}}})(t)\|_{\mathbb{X}_{\mathfrak{a}}} + \sum_{k=1}^{n} \frac{L^k T^k}{(\mathfrak{a}-\underline{\mathfrak{a}})^{qn}} \frac{k^{qk}}{k!} \left(P + \|x_{\underline{\mathfrak{a}}}\|_{\mathbb{X}_{\underline{\mathfrak{a}}}}\right) \\
&\leq P + \|x_{\underline{\mathfrak{a}}}\|_{\mathbb{X}_{\underline{\mathfrak{a}}}} + \sum_{k=1}^{n} \frac{L^k T^k}{(\mathfrak{a}-\underline{\mathfrak{a}})^{qn}} \frac{k^{qk}}{k!} \left(P + \|x_{\underline{\mathfrak{a}}}\|_{\mathbb{X}_{\underline{\mathfrak{a}}}}\right) \\
&\leq \left(1 + \sum_{k=1}^{n} \frac{L^k T^k}{(\mathfrak{a}-\underline{\mathfrak{a}})^{qn}} \frac{k^{qk}}{k!}\right) \left(P + \|x_{\underline{\mathfrak{a}}}\|_{\mathbb{X}_{\underline{\mathfrak{a}}}}\right) \\
&\leq \sum_{k=0}^{n} \frac{L^k T^k}{(\mathfrak{a}-\underline{\mathfrak{a}})^{qn}} \frac{k^{qk}}{k!} \left(P + \|x_{\underline{\mathfrak{a}}}\|_{\mathbb{X}_{\underline{\mathfrak{a}}}}\right). \qquad \text{(IV.24)}
\end{aligned}$$

Finally taking the limit on both sides of inequality (IV.24) we see that for all $t \in \mathcal{T}$ we have

$$||f(t)||_{\mathbb{X}_{\mathfrak{a}}} \leq \sum_{n=0}^{\infty} \frac{L^n T^n}{(\mathfrak{a}-\underline{\mathfrak{a}})^{qn}} \frac{n^{qn}}{n!} \left(P + \|x_{\underline{\mathfrak{a}}}\|_{\mathbb{X}_{\underline{\mathfrak{a}}}}\right)$$

hence the proof is complete. □

Remark IV.46. *It is clear from the definition (IV.22) that if F is a linear map then $P \equiv 0$ hence in this case from Theorem IV.45 we see that for all $\mathfrak{a} \in (\underline{\mathfrak{a}}, \overline{\mathfrak{a}})$.*

$$||f(t)||_{\mathbb{X}_{\mathfrak{a}}} \leq \sum_{n=0}^{\infty} \frac{L^n T^n}{(\mathfrak{a}-\underline{\mathfrak{a}})^{qn}} \frac{n^{qn}}{n!} \|x_{\underline{\mathfrak{a}}}\|_{\mathbb{X}_{\underline{\mathfrak{a}}}}.$$

Remark IV.47. *It is also clear from Theorem IV.45 that*

$$||f||_{\mathbb{Y}_{\mathfrak{a}}} \leq \sum_{n=0}^{\infty} \frac{L^n T^n}{(\mathfrak{a}-\underline{\mathfrak{a}})^{qn}} \frac{n^{qn}}{n!} \left(P + \|x_{\underline{\mathfrak{a}}}\|_{X_{\underline{\mathfrak{a}}}}\right).$$

IV.5 Additional Estimates For Section II

IV.5.1 Continuous Dependence on the Initial Data

In this subsection let us assume that $\mathfrak{q} \in [0, \frac{1}{4\mathfrak{p}})$.

Theorem IV.48. *Suppose that $\alpha < \beta \in \mathcal{A}$ and $\zeta^1_{\underline{\mathfrak{a}}}, \zeta^2_{\underline{\mathfrak{a}}} \in \mathbb{X}_{\underline{\mathfrak{a}}}$. Moreover suppose that ξ is the unique strong solution of the equation (II.19) corresponding to the initial data $\zeta^1_{\underline{\mathfrak{a}}}$ and η is the unique strong solution of the equation (II.19) corresponding to the initial data $\zeta^2_{\underline{\mathfrak{a}}}$. Then*

$$||\xi - \eta||_{\mathbb{Y}^{2\mathfrak{p}}_{\beta}} \leq \|\zeta^1_{\underline{\mathfrak{a}}} - \zeta^2_{\underline{\mathfrak{a}}}\|_{\mathbb{X}_{\alpha}} \sum_{n=0}^{\infty} \frac{\bar{L}^n 2^n T^n}{(\beta-\alpha)^{\mathfrak{q}n}} \frac{n^{2\mathfrak{q}n}}{\sqrt[2\mathfrak{p}]{n!}}. \tag{IV.25}$$

Proof. Proof of this theorem rests on similar techniques that we employed before hence we shall omit some details here. We begin this proof by looking back at the Theorem II.45 from where we see that for all $t \in \mathfrak{T}$ we have

$$\lim_{n\to\infty} \mathbb{E}\Bigg[||\mathfrak{I}^n(\zeta^1_{\underline{\mathfrak{a}}})(t) - \mathfrak{I}^n(\zeta^2_{\underline{\mathfrak{a}}})(t)||^{2\mathfrak{p}}_{\mathbb{X}_\beta}\Bigg] = \mathbb{E}\Bigg[||\xi(t) - \eta(t)||^{2\mathfrak{p}}_{\mathbb{X}_\beta}\Bigg]. \tag{IV.26}$$

Following **<u>Observation III</u>** from subsection II.5.1 we now for each $n \in \mathbb{N}$ let $\phi^n := \{\phi^n_i\}^n_{i=0}$ be a partition of $[\alpha, \beta]$ into n intervals of equal length such that $\phi^n_0 = \beta$ and $\phi^n_n = \alpha$ Moreover for each $n \in \mathbb{N}$ let $\phi_{n-1} := \phi^n_{n-1}$ and $\phi := \{\phi_i\}^{\infty}_{i=0}$. It is clear that $\phi_n \downarrow \alpha$ as $n \to \infty$. On top of this a simple proof by induction shows that

$$\prod_{i=1}^{n} \left(\frac{i}{i+1} - \frac{i-1}{i}\right) = \prod_{i=1}^{n} \frac{1}{i(i+1)} \geq \frac{1}{n^{2n}}, \quad \forall (n \geq 2). \tag{IV.27}$$

Therefore we observe that

$$\begin{aligned}
\frac{1}{\phi_0 - \phi_1}\frac{1}{\phi_1 - \phi_2}\frac{1}{\phi_2 - \phi_3} &= \frac{1}{(\beta-\alpha)^3 \prod_{i=1}^3 \left(\frac{i}{i+1} - \frac{i-1}{i}\right)} \\
&\leq \frac{3^{2\cdot 3}}{(\beta-\alpha)^3}
\end{aligned}$$

and so one can prove by induction that for all $n \in \mathbb{N}$ we have

$$\frac{1}{\phi_0 - \phi_1} \cdots \frac{1}{\phi_{n-1} - \phi_n} \leq \frac{2n^{2n}}{(\beta-\alpha)^n}. \tag{IV.28}$$

Now by introducing the following notation

$$K_\beta^n(t) \coloneqq \mathbb{E}\left[||\mathcal{I}^n(\zeta_{\underline{\mathfrak{a}}}^1)(t) - \mathcal{I}^n(\zeta_{\underline{\mathfrak{a}}}^2)(t)||_{\mathbb{X}_\beta}^{2\mathfrak{p}}\right], \quad \forall (t \in \mathcal{T}) \tag{IV.29}$$

and invoking Theorem II.38 we see that inequality (II.28) implies that for all $t \in \mathcal{T}$ we have

$$K_\beta^n(t) \leq \|\zeta_{\underline{\mathfrak{a}}}^1 - \zeta_{\underline{\mathfrak{a}}}^2\|_{\mathbb{X}_\alpha}^{2\mathfrak{p}} + \mathbf{N}\left(\frac{L}{(\beta-\alpha)^{\mathfrak{q}}}\right)^{2\mathfrak{p}} \int_0^t K_\alpha^{n-1}(s)ds.$$

Hence using sequence ϕ we see that

$$\begin{aligned} K_{\phi_0}^n(t) \leq \|\zeta_{\underline{\mathfrak{a}}}^1 - \zeta_{\underline{\mathfrak{a}}}^2\|_{\mathbb{X}_\alpha}^{2\mathfrak{p}} + \|\zeta_{\underline{\mathfrak{a}}}^1 - \zeta_{\underline{\mathfrak{a}}}^2\|_{\mathbb{X}_\alpha}^{2\mathfrak{p}} T\mathbf{N}\left(\frac{L}{(\phi_0-\phi_1)^{\mathfrak{q}}}\right)^{2\mathfrak{p}} + \\ + \mathbf{N}^2\left(\frac{L}{(\phi_0-\phi_1)^{\mathfrak{q}}}\right)^{2\mathfrak{p}}\left(\frac{L}{(\phi_1-\phi_2)^{\mathfrak{q}}}\right)^{2\mathfrak{p}} \int_0^t \int_0^s K_{\phi_2}^{n-2}(\tau)d\tau ds + \cdots \\ \cdots + \|\zeta_{\underline{\mathfrak{a}}}^1 - \zeta_{\underline{\mathfrak{a}}}^2\|_{\mathbb{X}_{\phi_n}}^{2\mathfrak{p}} \mathbf{N}^n\left(\frac{L}{(\psi_0-\psi_1)^{\mathfrak{q}}} \cdots \frac{L}{(\psi_{n-1}-\psi_n)^{\mathfrak{q}}}\right)^{2\mathfrak{p}} \frac{T^n}{n!}. \end{aligned}$$

Therefore similar arguments as in Theorem II.49 and inequality (IV.28) above shows that

$$\mathbb{E}\left[||\mathcal{I}^n(\zeta_{\underline{\mathfrak{a}}}^1)(t) - \mathcal{I}^n(\zeta_{\underline{\mathfrak{a}}}^2)(t)||_{\mathbb{X}_\beta}^{2\mathfrak{p}}\right] \leq \|\zeta_{\underline{\mathfrak{a}}}^1 - \zeta_{\underline{\mathfrak{a}}}^2\|_{\mathbb{X}_\alpha}^{2\mathfrak{p}} \left(\sum_{k=0}^n \frac{\bar{L}^k 2^k T^k}{(\beta-\alpha)^{\mathfrak{q}k}} \frac{k^{2\mathfrak{q}k}}{\sqrt[2\mathfrak{p}]{k!}}\right)^{2\mathfrak{p}}.$$

Finally using Theorem IV.16 and II.35 we conclude that series on the left hand side of inequality above converges. Therefore our limit estimate (IV.26) shows that

$$||\xi - \eta||_{\mathbb{Y}_\beta^{2\mathfrak{p}}} \leq \|\zeta_{\underline{\mathfrak{a}}}^1 - \zeta_{\underline{\mathfrak{a}}}^2\|_{\mathbb{X}_\alpha} \sum_{k=0}^\infty \frac{\bar{L}^k 2^k T^k}{(\beta-\alpha)^{\mathfrak{q}k}} \frac{k^{2\mathfrak{q}k}}{\sqrt[2\mathfrak{p}]{k!}} \tag{IV.30}$$

hence the proof is complete. □

IV.5.2 Calculation for Theorem II.38

Using BDG Type Inequality (Theorem IV.27) we see that

$$\mathbb{E}\left[\left\| \int_0^t \bar{\Phi}(s)dW(s)\right\|_{\mathbb{X}_\beta}^{2\mathfrak{p}}\right] \leq (2\mathfrak{p})^{2\mathfrak{p}}\left(\frac{2\mathfrak{p}}{2(2\mathfrak{p}-1)}\right)^{\frac{2\mathfrak{p}}{2}}\left[\int_0^t \left(\mathbb{E}\left[\|\bar{\Phi}(s)\|_{\mathbb{H}_\beta}^{2\mathfrak{p}}\right]\right)^{\frac{2}{2\mathfrak{p}}} ds\right]^{\frac{2\mathfrak{p}}{2}} \tag{IV.31}$$

$$\leq \left(\frac{4\bar{\mathfrak{p}}^3}{(2\bar{\mathfrak{p}}-1)}\right)^{\mathfrak{p}}\left[\int_0^t \left(\mathbb{E}\left[\|\bar{\Phi}(s)\|_{\mathbb{H}_\beta}^{2\mathfrak{p}}\right]\right)^{\frac{1}{\mathfrak{p}}} ds\right]^{\mathfrak{p}} \tag{IV.32}$$

$$\leq \left(\frac{2\bar{\mathfrak{p}}^3}{(\bar{\mathfrak{p}}-1)}\right)^{\mathfrak{p}}\left[\int_0^t \left(\mathbb{E}\left[\|\bar{\Phi}(s)\|_{\mathbb{H}_\beta}^{2\mathfrak{p}}\right]\right)^{\frac{1}{\mathfrak{p}}} ds\right]^{\mathfrak{p}}, \tag{IV.33}$$

where $\bar{\mathfrak{p}} > \mathfrak{p}$. Now using Jensen Inequality (Theorem IV.21) we see that

$$\mathbb{E}\left[\left\| \int_0^t \bar{\Phi}(s)dW(s)\right\|_{\mathbb{X}_\beta}^{2\mathfrak{p}}\right] \leq \left(\frac{2\bar{\mathfrak{p}}^3}{(\bar{\mathfrak{p}}-1)}\right)^{\mathfrak{p}} t^{1-\frac{1}{\mathfrak{p}}}\left[\int_0^t \mathbb{E}\left[\|\bar{\Phi}(s)\|_{\mathbb{H}_\beta}^{2\mathfrak{p}}\right] ds\right]^{\frac{\mathfrak{p}}{\mathfrak{p}}} \tag{IV.34}$$

$$\leq \left(\frac{2\bar{\mathfrak{p}}^3}{\bar{\mathfrak{p}}-1}\right)^{\mathfrak{p}} T^{\frac{\mathfrak{p}-1}{\mathfrak{p}}} \int_0^t \mathbb{E}\left[\|\bar{\Phi}(s)\|_{\mathbb{H}_\beta}^{2\mathfrak{p}}\right] ds. \tag{IV.35}$$

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