On Supergravity on a 3-Torus, Automorphic Scalar Fields in 2-dimensional de Sitter space and Harmonics on Complex Spheres

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Abstract

In the first part of this thesis we study linearization stability conditions in quantum supergravity on a flat 3-torus. Solutions to linearized supergravity on this background space-time can only be extended to solutions of the non-linear theory if they satisfy additional quadratic constraints, called the linearization stability conditions. This situation is well known in linearized gravity. The novel feature in the case of supergravity is the appearance of fermionic linearization stability constraints, in addition to the kind of bosonic constraints which arise already for linearized gravity. We show how to incorporate the fermionic and bosonic linearization stability constraints in the quantum theory and construct a physical space of states by group-averaging.

Unlike higher dimensional de Sitter spaces, two dimensional de Sitter space is not simply connected. This allows for the existence of fields which pick up non-trivial phases when making a full rotation of the spatial sections. In the second part of this thesis we study the quantum theory of automorphic complex scalar fields in two dimensional de Sitter space, extending the work of Epstein and Moschella. We define de Sitter invariant vacuum states when corresponding unitary irreducible representations of the universal covering group of $\mathrm{SL}(2,\mathbb{R})$ exist. By calculating the two-point functions we show that these states can only be Hadamard if the field is periodic. We also define a class of de Sitter non-invariant Hadamard states for the automorphic theories.

In the final part of this thesis we study harmonics on complex spheres. Using Mackey's tensor product theorem, the harmonics on complex spheres can be used to decompose tensor products of principal series representations of the Lorentz group.

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Introduction

This thesis is composed of three parts, which can be treated independently. The first part concerns the linearization stability of supergravity when perturbed around a flat 3-toroidal background. The second part deals with free complex scalar fields on two-dimensional de Sitter space, when an additional twisted periodicity condition is imposed on the complex scalar fields. In the final part, the decomposition of tensor products of principal series representations of the Lorentz group is studied using functions defined on 2 complex-dimensional spheres.

In the first part of the thesis, we show that linearized supergravity on a 3-torus suffers not only from the bosonic linearization instability conditions that the total energy and momentum of the linearized perturbations have to vanish, which are conditions that already arise in linearized gravity, but also from a fermionic linearization stability condition which requires that the total supercharge of the linearized system has to vanish. If these conditions are not satisfied, the solutions to the linearized system can not be extended to solutions of the full non-linear supergravity theory. The main result of this part is the construction of a space of states for the quantum version of the linearized theory which satisfy the linearization stability conditions. To illustrate the construction of this space, we also provide an example of a particular state in this state.

In the second part of this thesis, we study automorphic complex scalar fields on two-dimensional de Sitter space. Two-dimensional de Sitter space is not simply-connected, so this provides a simple arena to study some effects of non-trivial topology in curved spacetimes. The automorphic scalar fields are not invariant when making a full rotation of the circular spatial sections, instead they pick up a phase factor. We study the canonical quantisation of these free fields. The main results of this section are the construction of de Sitter invariant states whenever a corresponding representation of the universal covering group of $\mathrm{SO}_0(2,1)$ exists, and we show that these invariant states can never be Hadamard if the field is not periodic. Thus only the periodic Bunch-Davies state is both de Sitter invariant and Hadamard. We also exhibit de Sitter non-invariant Hadamard states for the automorphic fields.

In the third part of the thesis, the decomposition of tensor products of principal series irreducible unitary representations of the Lorentz group into irreducible components is studied. We follow the well known approach of using Mackey's tensor product theorem to show that the study of the the tensor product is equivalent to studying a single induced representation. The main part of this chapter is showing how this induced representation can be understood as functions on a 2 complex-dimensional sphere. These functions are then studied and we recover the known result for the decomposition of the tensor product.

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Declaration

I declare that this thesis is a presentation of original work and I am the sole author. This work has not previously been presented for an award at this, or any other, University. All sources are acknowledged as References.

Part I is based on a paper published in collaboration with Atsushi Higuchi [1]. Part II is based on work done in collaboration with Atsushi Higuchi and David Serrano Blanco [2]. Part III is based on work done in collaboration with Atsushi Higuchi.

Part I

Supergravity on a 3-Torus: Linearization Stability Conditions with a Supergroup

1 Introduction

In this chapter we examine linearization stability conditions that arise when one linearizes supergravity in four dimensions around a flat background, whose spatial slices are 3-toruses. In general, to gain insight into complicated non-linear equations, one considers the linear equations of motion obeyed by small perturbations around known solutions. However, it is not always guaranteed that the solutions of the linearized equations of motions actually arise as approximations of solutions to the exact equations of motion. When this is not guaranteed, we say that the system is linearization unstable, and further conditions, known as linearization stability conditions need to be imposed to ensure that a solution of the linearized system extends to a solution of the non-linear system.

Classical field theory systems which display linearization instabilities are Maxwell Electrodynamics [3, 4] when the background has compact Cauchy surfaces, and classical general relativity [3, 5, 6, 7, 8, 9, 10, 11, 12, 13] provided that the background spacetime has compact Cauchy surfaces and admits Killing symmetries. In Maxwell Electrodynamics, Gauss's law says that the divergence of the electric field $\vec{E}$ is related to the charge density ρ by

$$\vec{\nabla} \cdot \vec{E} = \rho. \tag{1}$$

Thus integrating we find that

$$Q = \int \mathrm{d}^3\vec{x}\ \rho = \int \mathrm{d}^3\vec{x}\ \vec{\nabla} \cdot \vec{E} = 0, \tag{2}$$

because the divergence integral can be converted into a surface integral over the boundary of the Cauchy surfaces, but if the background has compact Cauchy surfaces there are no boundaries and the integral must vanish. However, in the linearized theory the charged matter decouples from the electromagnetic field, and therefore the total charge is not constrained in the linearized theory. Thus $Q = 0$ must be imposed as a linearization stability constraint. Similarly, in general relativity, it is the charges Q_ξ that generate the Killing symmetries along the Killing vector fields ξ^μ which must vanish, and these conditions must be imposed as linearization stability conditions on the linearized theory.

In the quantum version of the linearized system, the linearization stability conditions $Q = 0$ are imposed as restrictions on the physical states of the theory [12, 13]. That is, a state is said to be a physical state $|\text{phys}\rangle$ if

$$Q|\text{phys}\rangle = 0. \tag{3}$$

Thus the linearization stability conditions require that the physical states of the theory be invariant under the symmetries generated by the conserved charges. Naively taken however, this condition would be very restrictive. For example, four dimensional de Sitter space has compact Cauchy surfaces, and is invariant under the identity component of SO(4, 1). However, it is known that when quantising the gravitational perturbations around de Sitter space, only the vacuum state is invariant under the Killing symmetires, thus the space of physical states would appear to be very restricted [4, 14].

A way to deal with this problem is by a procedure known as group-averaging. In this approach, one starts with a non-invariant state $|\psi\rangle$ and averages over the symmetry group to define an invariant state $|\Psi\rangle$ by

$$|\Psi\rangle = \int_G \mathrm{d}g\; U(g)|\psi\rangle, \tag{4}$$

where G is the symmetry group, and $U(g)$ is the unitary operator which implements the symmetry described by $g \in G$ on the space of states. We have here assumed that G is unimodular, that is to say we can find a measure dg on the group which is invariant under both left- and right-translations [15]. All the groups considered in this part will be of this type. Then $|\Psi\rangle$ is by construction invariant under the action of the symmetry group, however if we calculate the inner product between invariant states

$$\begin{aligned} \langle \Psi_1 \,|\, \Psi_2 \rangle &= \int_G \mathrm{d}g\mathrm{d}h\; \langle \psi_1 | U^{-1}(g) U(h) | \psi_2 \rangle \\ &= \left[\int_G \mathrm{d}g\; 1 \right] \int \mathrm{d}g'\; \langle \psi_1 | U(g') | \psi_2 \rangle \\ &= V_G \int \mathrm{d}g'\; \langle \psi_1 | U(g') | \psi_2 \rangle, \end{aligned} \tag{5}$$

where V_G is the volume of the group G and we used the unimodularity of G to change the integration over h to $g' = g^{-1}h$. In particular, if the volume of the background symmetry group is infinite, as is the case for SO(4, 1), then these invariant states are not normalisable. This can be fixed by 'dividing' by the group volume, and redefining the inner product between invariant states as

$$(\Psi_1 \,|\, \Psi_2) = \int_G \mathrm{d}g\; \langle \psi_1 | U(g) | \psi_2 \rangle. \tag{6}$$

In this way one can hope to define a finite, positive definite, inner product between the invariant states. Indeed, this procedure can be carried out to obtain a Hilbert space of invariant states for linearized gravity in de Sitter space [4, 14], it can also be carried out for other free fields in de Sitter space [16, 17]. Group-averaging has also been studied in the context of constrained dynamical systems (see e.g. [18, 19, 20, 21]) and forms an important part of refined algebraic quantisation [22] and is well studied in Loop Quantum Gravity.

A comparatively simpler example of a gravitational system where the group averaging procedure can be explicitly carried out is by expanding the gravitational perturbations around a flat background metric, with line-element

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2. \tag{7}$$

The Cauchy surfaces are made compact by periodically identifying the spatial coordinates, with periods L_1, L_2 and L_3 respectively. This background metric is still invariant under the $\mathbb{R} \times U(1)^3$ group of space and time-translations. The quantised theory for this model [12, 23] has been studied using group-averaging to obtain a physical space of states. In this chapter we study four dimensional $\mathcal{N} = 1$ supergravity linearized around the same background. In this theory, in addition to the vanishing of the total energy and total momentum, one requires that the supercharge Q_α must also vanish. This arises because the supercharge can also be written as an integral over the boundary at infinity of Cauchy surfaces in asymptotically flat space times [24]. Thus if we work on the flat 3-torus, these integrals trivially vanish and we find $Q_\alpha = 0$. In this chapter we show how to incorporate this fermionic linearization stability constraint in the quantum theory.

1.1 Organisation of this Chapter

As a guide, the remainder of this chapter is organized as follows.

We begin first by building up classical field theory on a toroidal background. We first consider real and complex Klein-Gordon scalar field theory, constructing the Fourier expansion of the field and the bracket relations obeyed by the Fourier components, which become creation and annihilation operators on quantisation. We then consider the conserved charges of the system, including the total energy, total momentum and the charge for the complex scalar field. Finally, for a real massless scalar field, the torus allows spatially constant (or zero-momentum) solutions to the Klein-Gordon equation, and these are studied separately.

We then revisit the theory of a Majorana Spinor field, we again construct the Fourier expansion for this field on a toroidal background and find the classical (Dirac) bracket structure between the Fourier components. We then write down again the conserved Energy and Momentum in terms of the Fourier components and finish by considering the zero-momentum modes for the massless Majorana field.

Having constructed simple scalar and spinor fields, we then introduce supersymmetry by studying a simple massive non-interacting Wess-Zumino type model. We calculate the conserved supercharge of the system, and verify that the total energy, momentum and the supercharge verify the $\mathcal{N} = 1$ supersymmetry algebra relations.

After the detour into supersymmetry, we next introduce free massless vector field theory on the 3-torus. This is the first theory with a gauge-symmetry, which we deal with by explicitly fixing the gauge. Working in Coulomb gauge, we write down the Fourier expansion of the physical components and the classical bracket relations obeyed by the system. The zero-momentum modes are again separately analysed at the end.

Armed with a description of the electromagnetic field and complex scalars, we study a first example of linearization instability conditions and group-averaging by considering scalar quantum electrodynamics on a toroidal background. We show how the total electric charge of the system must vanish and impose this as a linearization stability condition on the quantised scalar field. We show how this constraint can be solved by averaging over the $U(1)$ group associated with the total electric charge.

Following on from this, we introduce spin-3/2 Rarita-Schwinger fields. This theory again has a gauge symmetry, which we explicitly fix. Particular attention is paid to the zero-momentum modes of this field, which are not as well studied as the modes with non-zero momentum. We calculate the classical brackets for zero-modes and show how these can be organised into independent Fermi oscillators (up to a sign).

We then move on to the spin-2 field theory describing linearized gravity. We write down the Fourier expansion for physical components of the spin-2 field, and also the total energy and momentum. The zero-momentum modes are then separately analysed. We then move on to recall how by studying the second order in perturbation theory, one finds that the total energy and total momentum in the linearized theory must vanish. We then recap how these constraints can be incorporated in the quantum theory by group averaging, following the method of [23].

Finally, we combine the spin-2 and spin-3/2 fields in a linearized version of four dimensional $\mathcal{N} = 1$ supergravity. A novel aspect of this theory is that the conserved supercharge Q also has to vanish. We show how this arises in the linearized theory by studying the second order in the perturbation theory. We then incorporate this constraint into the quantum theory, and show that this can be done by group-averaging over the supergroup of symmetries. To end the main content of the chapter, we illustrate the construction of the physical states in linearized supergravity on a 3-torus by constructing an explicit example of a physical state.

The main content of the chapter is supplemented by seven appendices. In appendix A, we collect the conventions we use throughout the thesis. Appendix B contains additional information about constrained Hamiltonian systems, in particular we recall how constrained systems can be dealt with in the Hamiltonian formalism by replacing Poisson brackets with Dirac brackets. Appendix C collects some information relating to Grassmann variables, which are used to study the classical theory of the spinor fields. In appendix D we collect some information regarding γ-matrices and prove a number of identities used in the main text. Appendix E is devoted to frame fields, which are required to couple gravity to fermions, as is done in supergravity. Appendix F calculates positive helicity vectors and spinors used in the construction of the spinor fields as well as the electromagnetic and gravitational field. In appendix G we study the quantum theory of the zero-momentum gravitino modes in more detail.

2 Classical Theory of Free Fields: Spin-0, Spin-1/2, Spin-1

Throughout this thesis, unless otherwise noted, we will assume that the background spacetime is four dimensional, with Minkowski metric given by

$$\mathrm{d}s^2 = \eta_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = -\mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2. \tag{8}$$

Further, we will assume that the spatial sections form 3-tori with lengths L_1, L_2 and L_3 in the x-, y- and z-directions respectively. We denote the total spatial volume by V and this is given by

$$V = L_1 L_2 L_3. \tag{9}$$

2.1 Scalar Field

We start with a free real scalar field ϕ. This theory is governed by an action

$$S[\phi] = \int \mathrm{d}^4x \; \frac{1}{2}(-\partial^\mu\phi\partial_\mu\phi - M^2\phi^2), \tag{10}$$

The equation of motion for this theory is obtained by extremizing the action with respect to the field. That is

$$\frac{\delta S}{\delta \phi} = \partial_\mu\partial^\mu\phi - M^2\phi = 0. \tag{11}$$

In the canonical setting, we are interested in the conjugate momentum π to ϕ, which is defined by

$$\pi(x) = \frac{\partial \mathcal{L}}{\partial\partial_0\phi(x)} = \frac{\partial}{\partial t}\phi(x). \tag{12}$$

The classical phase space coordinates for the scalar field are $\phi(\vec{x}, t)$ and $\pi(\vec{x}, t)$ taken at an arbitrary fixed equal time t (typically taken as $t = 0$). The canonical Poisson brackets between ϕ and π are given as

$$\{\phi(t, \vec{x}), \pi(t, \vec{y})\} = \delta^3(\vec{x} - \vec{y}), \tag{13}$$

where $\delta^3(\vec{x}-\vec{y})$ is the three dimensional Dirac-delta function defined so that for an arbitrary function $f(\vec{x})$

$$\int \mathrm{d}^3\vec{y}\; f(\vec{y})\delta(\vec{x} - \vec{y}) = f(\vec{x}). \tag{14}$$

To make progress, we expand ϕ as a Fourier series

$$\phi(x) = \frac{1}{\sqrt{V}} \sum_{\vec{k}} \phi(t, \vec{k}) e^{i\vec{k}\cdot\vec{x}}, \tag{15}$$

where V is the spatial volume of the torus and reality of ϕ requires that $\phi(t, \vec{k})^* = \phi(t, -\vec{k})$. Then the Klein-Gordon equation becomes

$$\left(\frac{d^2}{dt^2} + M^2 + \vec{k}^2\right)\phi(t, \vec{k}) = 0. \tag{16}$$

Thus, denoting $E_{\vec{k}} = +\sqrt{\vec{k}^2 + M^2}$, we find that the general solution is a superposition of positive and negative frequency solutions

$$\phi(t, \vec{k}) = A(\vec{k})e^{-iE_{\vec{k}}t} + B(\vec{k})e^{iE_{\vec{k}}t}. \tag{17}$$

The names positive and negative energy or frequency come from considering a Heisenberg type equation, applying $i\partial_t$ to these solutions gives positive or negative eigenvalues. Using the reality of $\phi(x)$, we can then write

$$\phi(t, \vec{x}) = \frac{1}{\sqrt{V}} \sum_{\vec{k}} \frac{1}{\sqrt{2E_{\vec{k}}}} \left(a(\vec{k})e^{ik\cdot x} + a^\dagger(\vec{k})e^{-ik\cdot x}\right), \tag{18}$$

where $k^\mu = (E_{\vec{k}}, \vec{k})$ and we have defined

$$A(\vec{k}) = \frac{1}{\sqrt{2E_{\vec{k}}}} a(\vec{k}), \qquad B(\vec{k}) = \frac{1}{\sqrt{2E_{\vec{k}}}} a^\dagger(-\vec{k}). \tag{19}$$

Similarly, we can expand the conjugate momentum as

$$\pi(t, \vec{x}) = \frac{1}{\sqrt{V}} \sum_{\vec{k}} (-i) \sqrt{\frac{E_{\vec{k}}}{2}} \left(a(\vec{k}) e^{ik\cdot x} - a^\dagger(\vec{k}) e^{-ik\cdot x} \right). \tag{20}$$

Isolating the coefficient $a(\vec{k})$ as

$$a(\vec{k}) = \frac{\sqrt{E_{\vec{k}}}}{2V} e^{iE_{\vec{k}} t} \int \mathrm{d}^3\vec{x} e^{-i\vec{k}\cdot\vec{x}} \left(\phi(t, \vec{x}) - \frac{i}{E_{\vec{k}}} \pi(t, \vec{x}) \right), \tag{21}$$

we can calculate the Poisson brackets between $a(\vec{k})$ and $a^\dagger(\vec{k})$, where we find

$$\{a(\vec{k}), a^\dagger(\vec{p})\} = -i\delta_D(\vec{k} - \vec{p}), \tag{22}$$

with all other Poisson brackets vanishing. Here $\delta_D(\vec{p})$ is a discrete momentum-space delta function, that is for any function $g(\vec{k})$ in momentum space,

$$\sum_{\vec{p}} g(\vec{p}) \delta_D(\vec{k} - \vec{p}) = g(\vec{k}). \tag{23}$$

2.1.1 Conserved Charges

The real Klein-Gordon scalar field theory has some continuous symmetries, arising from the invariance of the action under space-time transformations. As a consequence of Noether's theorem there are associated conserved charges. For invariance under space-time translations these are the total energy H and total momentum $\vec{P}$ of the system. Suppose that we make the infinitesimal space-time translation

$$x \mapsto x - \epsilon \tag{24}$$

Under this translation, the fields transform as $\phi(x) \mapsto \phi'(x) = \phi(x + \epsilon)$, so that the infinitesimal transformation of the field is

$$\delta_\epsilon \phi(x) = \phi'(x) - \phi(x) = \epsilon^\mu \partial_\mu \phi(x). \tag{25}$$

As the action is invariant under this transformation when the equations of motion are obeyed, we must have

$$\frac{\delta S}{\delta \phi} \delta_\epsilon \phi = \partial_\mu J^\mu, \tag{26}$$

We find that J^μ is linear in ϵ, so writing $J^\mu = T^{\mu\nu}\epsilon_\nu$, we find that the energy-momentum tensor is given by

$$T^{\mu\nu} = \partial^\mu \phi \partial^\nu \phi - \eta^{\mu\nu} \mathcal{L}. \tag{27}$$

The associated charges are then obtained in the usual manner as

$$P^\mu = \int \mathrm{d}^3\vec{x}\; T^{0\mu}. \tag{28}$$

A straightforward, if slightly tedious, calculation then yields

$$P^0 = H = \sum_{\vec{k}} E_{\vec{k}} a^\dagger(\vec{k})a(\vec{k}), \tag{29}$$

$$\vec{P} = \sum_{\vec{k}} \vec{k} a^\dagger(\vec{k})a(\vec{k}). \tag{30}$$

2.1.2 Charged Scalar Field

We can also consider complex scalar fields Φ, which are governed by an action

$$S[\Phi] = \int \mathrm{d}^4x \; \left(-\partial^\mu\Phi^\dagger\partial_\mu\Phi - M^2\Phi^\dagger\Phi\right). \tag{31}$$

Treating Φ and $\Phi^\dagger$ as independent, the phase space consists of $\{\Phi, \Pi, \Phi^\dagger, \Pi^\dagger\}$, with the conjugate momentum

$$\Pi(x) = \frac{\partial}{\partial t}\Phi^\dagger(x), \tag{32}$$

and the only non-zero canonical equal-time Poisson brackets being

$$\{\Phi(t,\vec{x}), \Pi(t,\vec{y})\} = \delta^3(\vec{x}-\vec{y}) = \{\Phi^\dagger(t,\vec{x}), \Pi^\dagger(t,\vec{y})\}. \tag{33}$$

The Fourier series can be written down as before, except that there is now no reality condition, which therefore yields

$$\Phi(x) = \frac{1}{\sqrt{V}} \sum_{\vec{k}} \frac{1}{\sqrt{2E_{\vec{k}}}} \left(a(\vec{k})e^{ik\cdot x} + b^\dagger(\vec{k})e^{-ik\cdot x}\right), \tag{34}$$

where in this case we have as non-zero Poisson brackets

$$\{a(\vec{k}), a^\dagger(\vec{p})\} = -i\delta_D(\vec{k}-\vec{p}) = \{b(\vec{k}), b^\dagger(\vec{p})\}. \tag{35}$$

In this case the total energy and momentum are given by

$$H = \sum_{\vec{k}} E_{\vec{k}} \left(a^\dagger(\vec{k})a(\vec{k}) + b^\dagger(\vec{k})b(\vec{k})\right), \tag{36}$$

$$\vec{P} = \sum_{\vec{k}} \vec{k} \left(a^\dagger(\vec{k})a(\vec{k}) + b^\dagger(\vec{k})b(\vec{k})\right). \tag{37}$$

A novel feature of this model is that there is an internal $U(1)$ global symmetry, given by

$$\Phi \mapsto \Phi'(x) = \exp(-ie\theta)\Phi(x), \tag{38}$$

where θ is a constant real number. The associated infinitesimal transformation is

$$\delta_\theta\Phi(x) = -ie\theta\Phi(x). \tag{39}$$

There is again an associated charge, which can be found as before, we write

$$\frac{\delta S}{\delta \Phi}\delta\Phi + \delta\Phi^\dagger \frac{\delta S}{\delta \Phi^\dagger} = -\partial_\mu J^\mu_\theta. \tag{40}$$

Then we can calculate that a suitable current is

$$J^\mu_\theta = -ie\theta(\Phi^\dagger \partial^\mu \Phi - \partial^\mu \Phi^\dagger \Phi). \tag{41}$$

Removing θ and integrating, we find the conserved charge Q

$$\begin{aligned} Q &= +ie \int \mathrm{d}^3\vec{x} \, \left(\Phi^\dagger \partial_t \Phi - \partial_t \Phi^\dagger \Phi\right), \\ &= e \sum_{\vec{k}} (a^\dagger(\vec{k})a(\vec{k}) - b^\dagger(\vec{k})b(\vec{k})). \end{aligned} \tag{42}$$

when coupled to an electric field, this has the interpretation of electric charge.

2.1.3 Massless Scalar Field

There is a novel subtlety when working on the torus if we take the massless limit $M^2 \to 0$. In this case the zero-modes $\vec{k} = 0$ need to be treated more carefully as in this case $E_{\vec{k}} \to 0$ as well. The zero-momentum $\vec{k} = 0$ modes correspond to spatially constant modes. Writing $\phi(t, \vec{x}) = \frac{1}{\sqrt{V}}\phi_0(t)$ in the scalar action (10), we find that these modes are described by a Lagrangian

$$L = \frac{1}{2}\left(\frac{\partial \phi_0}{\partial t}\right)^2. \tag{43}$$

This corresponds to the Lagrangian of a free particle, which can be analysed in the usual manner.

2.2 Majorana Spinor Field

We next consider spinor fields. We will exclusively deal with Majorana spinor fields, which should be viewed as the spinor equivalent of real fields. Let ψ_α be a four component spinor field. We will eventually want this to describe a theory of spin-1/2 particles, so by the spin-statistic theorem these will be fermions obeying Fermi-Dirac statistics. As a consequence the spinor field is composed of anticommuting variables

$$\psi_\alpha \psi_\beta = -\psi_\beta \psi_\alpha. \tag{44}$$

We refer to Appendix C for more on details on the classical mechanics of anti-commuting variables. To work with spinors, we introduce the γ-matrices γ^μ which obey the commutation relations

$$[\gamma^\mu, \gamma^\nu]_+ = 2\eta^{\mu\nu}. \tag{45}$$

See Appendix A and Appendix D for more properties of the γ-matrices. We say that the spinor field ψ is a Majorana spinor if it obeys

$$(\psi^\dagger)_\alpha = (iC\gamma^0\psi)_\alpha, \tag{46}$$

where the charge conjugation matrix C obeys $C^T = -C$ and

$$\gamma^{\mu T} = -C\gamma^\mu C^{-1}. \tag{47}$$

If we use the Majorana representation (422) for the γ-matrices, we may take $C = i\gamma^0$ and the Majorana condition reads

$$\psi_\alpha^\dagger = \psi_\alpha. \tag{48}$$

A suitable action for a theory of free Majorana spinors is

$$S[\psi] = -\int \mathrm{d}^4x\ \frac{1}{2}\bar{\psi}\left(\gamma^\mu\partial_\mu - m\right)\psi, \tag{49}$$

where $\bar{\psi} = \psi^T C$. Then the equation of motion obeyed by ψ is found to be

$$\frac{\delta S}{\delta \psi} = -C(\gamma^\mu\partial_\mu - m)\psi = 0, \tag{50}$$

multiplying by C^{-1} we recover the Dirac equation. We note here that the derivative is to be considered as a left-derivative, see Appendix C equation (459), so that

$$\delta S = \int \mathrm{d}^4x\ \delta\psi^T \frac{\delta S}{\delta\psi}. \tag{51}$$

Next, we calculate the conjugate momentum to ψ, we find here that

$$\Pi_\alpha = \frac{\partial \mathcal{L}}{\partial\partial_0\psi_\alpha} = -\frac{i}{2}\psi_\alpha. \tag{52}$$

We note here that this definition is not immediately consistent with the canonical Poisson brackets

$$\begin{aligned} &\{\psi(t,\vec{x}), \psi(t,\vec{y})\} = 0, \\ &\{\psi(t,\vec{x}), \Pi(t,\vec{y})\} = -\delta^3(\vec{x}-\vec{y}), \\ &\{\Pi(t,\vec{x}), \Pi(t,\vec{y})\} = 0. \end{aligned} \tag{53}$$

To proceed, we need to implement (52) as a constraint on the system, which can be done in the canonical formalism by using Dirac brackets. We refer to Appendix B for a review of constrained Hamiltonian systems. Denote the constraint as

$$\phi_\alpha = \Pi_\alpha + \frac{i}{2}\psi_\alpha, \tag{54}$$

this is a second-class constraint because

$$C_{\alpha\beta}(\vec{x},\vec{y}) = \{\phi_\alpha(t,\vec{x}), \phi_\beta(t,\vec{y})\} = -i\delta_{\alpha\beta}\delta^3(\vec{x}-\vec{y}). \tag{55}$$

Thus, the fundamental Dirac bracket we should impose is

$$\begin{aligned}\{\psi_\alpha(t,\vec{x}),\psi_\beta(t,\vec{y})\}_D &= \{\psi_\alpha(t,\vec{x}),\psi_\beta(t,\vec{y})\}_P \\ &\quad - \int \mathrm{d}^3\vec{w}\mathrm{d}^3\vec{z}\ \{\psi_\alpha(t,\vec{x}),\phi_\gamma(t,\vec{w})\}_P C^{-1}_{\gamma\delta}(\vec{w},\vec{z})\{\phi_\delta(t,\vec{z}),\psi_\beta(t,\vec{y})\}_P \\ &= -i\delta_{\alpha\beta}\delta^3(\vec{x}-\vec{y}),\end{aligned} \tag{56}$$

where $C^{-1}_{\alpha\beta}(\vec{x},\vec{y})$ is the inverse of $C_{\alpha\beta}(\vec{x},\vec{y})$ in the sense that

$$\int \mathrm{d}^3\vec{w}\ C_{\alpha\beta}(\vec{x},\vec{w})C_{\beta\gamma}(\vec{w},\vec{y}) = \delta_{\alpha\gamma}\delta^3(\vec{x}-\vec{y}). \tag{57}$$

Next, we find again the Fourier expansions, so let us write

$$\psi_\alpha(t,\vec{x}) = \frac{1}{\sqrt{V}}\sum_{\vec{k}}\psi(t,\vec{k})e^{i\vec{k}\cdot\vec{x}}, \tag{58}$$

where again the Majorana condition requires us to impose $\psi(t,\vec{k})^* = \psi(t,-\vec{k})$. Then the Dirac equation implies that $\psi(t,\vec{k})$ has to obey

$$\left(\gamma^0\frac{\partial}{\partial t} + i\vec{\gamma}\cdot\vec{k} - m\right)\psi(t,\vec{k}) = 0. \tag{59}$$

By noting that

$$(\gamma^\mu\partial_\mu + m)(\gamma^\mu\partial_\mu - m)\psi(x) = (\partial^\mu\partial_\mu - m^2)\psi(x) = 0, \tag{60}$$

we again look for positive and negative energy solutions, proportional to $\exp(\mp iE_{\vec{k}}t)$ respectively. In fact, there are two linearly independent solutions for the positive and negative frequency solutions, so that we may write

$$\psi_\alpha(t,\vec{k}) = \sum_{s=1,2}\left[A^s_\alpha(\vec{k})e^{-iE_{\vec{k}}t} + B^s_\alpha(\vec{k})e^{+iE_{\vec{k}}t}\right], \tag{61}$$

where we use $s = 1, 2$ to count the two linearly independent solutions. Taking into account the reality condition, we thus get the general expansion

$$\psi_\alpha(t,\vec{x}) = \frac{1}{\sqrt{V}}\sum_{\vec{k}}\sum_{s=1,2}\frac{1}{\sqrt{2E_{\vec{k}}}}\left[a_s(\vec{k})u^s(\vec{k})e^{ik\cdot x} + a^\dagger_s(\vec{k})v^s(\vec{k})e^{-ik\cdot x}\right], \tag{62}$$

where $u^s(\vec{k})$ are a normalised basis for the positive frequency solutions, normalized such that $u^{s*}_\alpha(\vec{k})u^r_\alpha(\vec{k}) = 2|\vec{k}|$. In the Majorana representation for the γ-matrices, we have $v^s(\vec{k}) = u^{s*}(\vec{k})$. With these conditions, it is possible to find that the classical Dirac bracket obeyed by the $a_s(\vec{k})$ are

$$\{a_s(\vec{k}), a^\dagger_r(\vec{p})\} = -i\delta_{sr}\delta_D(\vec{k}-\vec{p}). \tag{63}$$

To get a feel for these solutions, let us take $\vec{k} = k\vec{e}_1$, where $\vec{e}_1$ is a unit vector in the

x-direction. Then the positive frequency solution $u(\vec{k})$ obeys

$$\left(E_{\vec{k}}\gamma^0 - k\gamma^1\right) u(\vec{k}) = imu(\vec{k}) \tag{64}$$

That is, $u(\vec{k})$ is an eigenvector of the 4×4 matrix, written as 2×2 blocks,

$$\begin{pmatrix} -k & E_{\vec{k}} \\ -E_{\vec{k}} & k \end{pmatrix} \tag{65}$$

with eigenvalue $+im$, thus we find

$$u^s(k\vec{e}_1) = \begin{pmatrix} \sqrt{k - im}\xi^s \\ \sqrt{k + im}\xi^s \end{pmatrix}, \tag{66}$$

where ξ^s is an orthonormal basis for two dimensional space. In particular, if we make the choices

$$\xi^{\pm}(k\vec{e}_1) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ \mp i \end{pmatrix}, \tag{67}$$

then we show in Appendix E that in the massless limit the $u^s(k)$ are eigenspinors of the helicity operator with eigenvalues $\pm\frac{1}{2}$ respectively

The theory also has a conserved total energy and momentum, which we can calculate as

$$H = \sum_{\vec{k}} \sum_{s=1,2} E_{\vec{k}} a_s^\dagger(\vec{k}) a_s(\vec{k}), \tag{68}$$

$$\vec{P} = \sum_{\vec{k}} \sum_{s=1,2} \vec{k} a_s^\dagger(\vec{k}) a_s(\vec{k}). \tag{69}$$

2.2.1 Zero Modes for Massless Majorana Spinor

Consider now the massless case $m = 0$. On the torus we then again expect zero momentum modes. To analyse the zero modes, let us write

$$\psi(t, \vec{x}) = \frac{1}{\sqrt{V}}\psi(t), \tag{70}$$

then the action for the zero-momentum modes becomes

$$S = \int \mathrm{d}t \, \left[-\frac{1}{2}\psi^T C\gamma^0 \partial_0 \psi \right] \tag{71}$$

Then we can read off the canonical conjugate momentum π_α as

$$\pi_\alpha = \frac{1}{2}(\psi^T C\gamma^0)_\alpha. \tag{72}$$

In the Majorana representation simplifies to $\pi_\alpha = -\frac{i}{2}\psi_\alpha$. This again needs to be imposed as a second class constraint on the system. Note that the canonical Hamiltonian for this system vanishes, because

$$H_C = \dot{\psi}_\alpha \pi_\alpha - L = 0 \tag{73}$$

A quick calculation shows that the Dirac bracket between the ψ_α which we need to impose is

$$\{\psi_\alpha, \psi_\beta\}_D = -i\delta_{\alpha\beta}. \tag{74}$$

2.3 An Example of Global Supersymmetry

Let us consider an example of a theory which displays a global supersymmetry. This free massive Wess-Zumino [25] type model consists of two real scalar fields ϕ_1 and ϕ_2 as well as a Majorana spinor ψ, all of which have the same mass M. There are no interaction terms between the fields, and thus it is governed by the action

$$\begin{aligned} S[\phi_1, \phi_2, \psi] = \int \mathrm{d}^4x \, \Big[\frac{1}{2}\left(-\partial^\mu\phi_1\partial_\mu\phi_1 - M^2\phi_1^2 - \partial^\mu\phi_2\partial_\mu\phi_2 - M^2\phi_2^2\right) \\ - \frac{1}{2}\bar{\psi}(\gamma^\mu\partial_\mu - M)\psi\Big]. \end{aligned} \tag{75}$$

This action is invariant under the supersymmetry transformation

$$\begin{aligned} \phi_1 &\mapsto \phi_1 + \delta_\varepsilon\phi_1 = \phi_1 + \frac{1}{2}\bar{\varepsilon}\psi, \\ \phi_2 &\mapsto \phi_2 + \delta_\varepsilon\phi_2 = \phi_2 + \frac{i}{2}\bar{\varepsilon}\gamma_5\psi, \\ \bar{\psi} &\mapsto \bar{\psi} + \delta_\varepsilon\bar{\psi} = \bar{\psi} + \frac{1}{2}\bar{\varepsilon}(-\gamma^\mu\partial_\mu + M)\phi_1 + \frac{i}{2}\bar{\varepsilon}\gamma_5(-\gamma^\mu\partial_\mu + M)\phi_2, \end{aligned} \tag{76}$$

where ε is a Majorana spinor and we have defined $\gamma_5 = -i\gamma^0\gamma^1\gamma^2\gamma^3$. By Noether's theorem, associated with this transformation there is a conserved spinor supercurrent $\mathcal{J}^\mu$ and a conserved spinor supercharge Q. The conserved current obeys

$$\delta_\varepsilon\phi_1\frac{\delta S}{\delta\phi_1} + \delta_\varepsilon\phi_2\frac{\delta S}{\delta\phi_2} + \delta_\varepsilon\bar{\psi}\frac{\delta S}{\delta\bar{\psi}} = \partial_\mu(\bar{\varepsilon}\mathcal{J}^\mu). \tag{77}$$

One quickly finds that a suitable super current is therefore

$$\mathcal{J}^\mu = \frac{1}{2}\left[\gamma^\nu\partial_\nu(\phi_1 - i\gamma_5\phi_2) - M(\phi_1 + i\gamma_5\phi_2)\right]\gamma^\mu\psi, \tag{78}$$

so that the conserved supercharge is

$$\begin{aligned} Q &= \int \mathrm{d}^3\vec{x}\ \mathcal{J}^0 \\ &= \int \mathrm{d}^3\vec{x}\ \frac{1}{2}\left(\left[\gamma^\nu\partial_\nu(\phi_1 - i\gamma_5\phi_2) - M(\phi_1 + i\gamma_5\phi_2)\right]\gamma^0\psi\right). \end{aligned} \tag{79}$$

If we label the Fourier coefficients of the free scalar fields ϕ_i by $a_i(\vec{k})$ and $a_i^\dagger(\vec{k})$ for $i = 1, 2$ and for the Majorana spinor we use $b_s(\vec{k})$ and $b_s^\dagger(\vec{k})$, then we can calculate that

$$\begin{aligned} Q = &\frac{1}{2}\sum_{\vec{k}}\sum_s\left[-iu^s(\vec{k})a_1(\vec{k})b_s^\dagger(\vec{k}) + iv^s(\vec{k})a_1^\dagger(\vec{k})b_s(\vec{k})\right] \\ &+ \frac{1}{2}i\gamma_5\sum_{\vec{k}}\sum_s\left[-iu^s(\vec{k})a_2(\vec{k})b_s^\dagger(\vec{k}) + iv^s(\vec{k})a_2^\dagger(\vec{k})b_s(\vec{k})\right]. \end{aligned} \tag{80}$$

Similarly, we can write down the conserved total energy H and total momentum $\vec{P}$ by summing the independent contributions from each field as

$$P^\mu = \sum_{\vec{k}} k^\mu \left[a_1^\dagger(\vec{k})a_1(\vec{k}) + a_2^\dagger(\vec{k})a_2(\vec{k}) + \sum_s b_s^\dagger(\vec{k})b_s(\vec{k}) \right], \tag{81}$$

where $k^\mu = (E(\vec{k}), \vec{k})$ and $P^\mu = (P^0, \vec{P})$. It is then not too complicated to see that if we calculate the classical brackets we obtain

$$\{Q_\alpha, P^\mu\} = 0. \tag{82}$$

Similarly, it is possible to calculate that

$$\{Q_\alpha, Q_\beta\} = -\frac{i}{2}(\gamma_\mu \gamma^0)_{\alpha\beta} P^\mu, \tag{83}$$

in a way this says that the square of supersymmetry transformations is a space-time translation, so we should not view supersymmetry as an internal symmetry, but instead as a space-time symmetry.

As an aside, we note that this Wess-Zumino model can be made interacting with, for example, Yukawa couplings and quartic interactions between the scalars, while still preserving the supersymmetry of the theory. The form of the classical brackets between the Q and P are unchanged.

2.4 Massless Vector Field

Let us now consider electromagnetic theory, which is the theory of a free massless vector field A_μ. A suitable action for this theory is provided by

$$S[A_\mu] = \int \mathrm{d}^4\vec{x} \, \left[-\frac{1}{4} F_{\mu\nu}F^{\mu\nu} \right], \tag{84}$$

where the electromagnetic field strength tensor is defined by

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{85}$$

Varying the action with respect to A_μ yields the equations of motion

$$\frac{\delta S}{\delta A_\mu} = \partial_\nu F^{\mu\nu} = 0. \tag{86}$$

To see how this describes an electromagnetic field theory, we define the electric and magnetic fields $\vec{E}$ and $\vec{B}$ by

$$\begin{aligned} \vec{E} &= -\frac{\partial \vec{A}}{\partial t} - \vec{\nabla} A^0, \\ \vec{B} &= \vec{\nabla} \times \vec{A}. \end{aligned} \tag{87}$$

Then the equation of motion yields two of Maxwell's equations

$$\vec{\nabla} \cdot \vec{E} = 0, \tag{88}$$

$$\frac{\partial \vec{E}}{\partial t} - \vec{\nabla} \times \vec{B} = 0. \tag{89}$$

The other two Maxwell equations

$$\vec{\nabla} \cdot \vec{B} = 0, \tag{90}$$

$$\vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0, \tag{91}$$

are already inbuilt in this formalism by the definition of $\vec{E}$ and $\vec{B}$ in terms of A_μ.

A novel feature of this theory is gauge invariance, that is to say the formalism we have developed so far carries some redundancy which we need to deal with. In this case the redundancy can be seen in that if we modify A_μ by a total derivative the field strength tensor and hence equations of motions are unaffected, that is we can always make the mapping

$$A_\mu \mapsto A'_\mu = A_\mu + \partial_\mu \theta, \tag{92}$$

without affecting the physics. One way of dealing with this redundancy is by fixing the gauge, that is imposing extra conditions on the field A_μ which completely eliminate this freedom.

Let us begin by first considering the non-zero momentum modes. That is, we do not allow the field components to be spatially constant, as these again need to be treated seperately. For these modes we can choose to work in Coulomb gauge, which imposes the condition

$$\vec{\nabla} \cdot \vec{A} = 0. \tag{93}$$

This fixes the gauge completely, up to the zero-momentum sector which we will deal with separately. A further consequence of the Coulomb gauge condition is that A_0 is identically zero. This follows by considering the $\mu = 0$ component of the field equation (86), which in the Coulomb gauge becomes

$$\nabla^2 A_0 = 0. \tag{94}$$

Switching to momentum space, we quickly see that this has only solutions spatially constant solutions, which are treated separately. Thus for the non-zero momentum modes we can take $A_0 = 0$. The remaining equation of motion obeyed by $\vec{A}$ is then the wave equation

$$\partial_\mu \partial^\mu \vec{A} = 0. \tag{95}$$

After removing A_0 from the formalism, the remaining action for $\vec{A}$ is

$$S[\vec{A}] = \int \mathrm{d}^4 x \, \left[\frac{1}{2} \left(\frac{\partial A^i}{\partial t} \right)^2 - \frac{1}{2} \partial_j A^i \partial_j A^i + \frac{1}{2} \partial_i A^j \partial_j A^i \right]. \tag{96}$$

Then, the canonical momentum conjugate density Π^i to A^i is given by

$$\Pi^i = \frac{\partial A^i}{\partial t}, \tag{97}$$

and the canonical Poisson bracket relations for this theory are

$$\{A^i(t,\vec{x}), A^j(t,\vec{y})\}_P = 0, \tag{98}$$
$$\{A^i(t,\vec{x}), \Pi^j(t,\vec{y})\}_P = \delta^{ij}\delta^3(\vec{x}-\vec{y}), \tag{99}$$
$$\{\Pi^i(t,\vec{x}), \Pi^j(t,\vec{y})\}_P = 0. \tag{100}$$

so that the canonical Hamiltonian is given by

$$\begin{aligned} H_C &= \int \mathrm{d}^3\vec{x}\ \Pi^i \frac{\partial A^i}{\partial t} - L \\ &= \int \mathrm{d}^3\vec{x}\ \left[\frac{1}{2}\Pi^i\Pi^i + \frac{1}{2}\partial_j A^i \partial_j A^i - \frac{1}{2}\partial_i A^j \partial_j A^i\right]. \end{aligned} \tag{101}$$

On this theory we still need to impose the Coulomb condition constraint

$$\phi_1(t,\vec{x}) = \vec{\nabla}\cdot\vec{A}(t,\vec{x}) = 0. \tag{102}$$

Clearly this constraint can not be consistent with the canonical Poisson bracket structure, so we will need to find the correct Dirac bracket structure for the theory. First we need to check whether the constraint is consistent with the time evolution generated by the canonical Hamiltonian. Demanding that the Poisson bracket of H_C and ϕ_1 vanishes when the constraints are obeyed leads us to an additional constraint

$$\phi_2 = \vec{\nabla}\cdot\vec{\Pi} = 0. \tag{103}$$

These are all the constraints of the theory. To get at the Dirac brackets we calculate the matrix of Poisson brackets between the constraints

$$\begin{aligned} C_{ab}(\vec{x},\vec{y}) &= \{\phi_a(t,\vec{x}), \phi_b(t,\vec{y})\}_P \\ &= \begin{pmatrix} 0 & -\vec{\nabla}^2_{\vec{x}} \\ \vec{\nabla}^2_{\vec{x}} & 0 \end{pmatrix} \delta(\vec{x}-\vec{y}). \end{aligned} \tag{104}$$

Then we note the only bracket which needs to be modified is between A^i and Π^i, where we calculate

$$\begin{aligned} \{A^i(t,\vec{x}), \Pi^j(t,\vec{y})\}_D &= \{A^i(t,\vec{x}), \Pi^j(t,\vec{y})\}_P \\ &\quad - \int \mathrm{d}^3\vec{z}\mathrm{d}^3\vec{w}\{A^i(t,\vec{x}), \Pi^k(t,\vec{z})\}_P C_{21}(\vec{z},\vec{w})^{-1}\{A^l(t,\vec{w}), \Pi^j(t,\vec{y})\}_P \\ &= \left(\delta^{ij} - \frac{\partial^i_{\vec{x}}\partial^j_{\vec{x}}}{\vec{\nabla}^2_{\vec{x}}}\right)\delta^3(\vec{x}-\vec{y}). \end{aligned} \tag{105}$$

To better understand the content of this, let us move over to momentum space. Acting on $e^{i\vec{k}\cdot\vec{x}}$, we find that

$$\left(\delta^{ij} - \frac{\partial^i_{\vec{x}}\partial^j_{\vec{x}}}{\vec{\nabla}^2_{\vec{x}}}\right) e^{i\vec{k}\cdot\vec{x}} = \left(\delta^{ij} - \frac{k^i k^j}{\vec{k}^2}\right) e^{i\vec{k}\cdot\vec{x}}, \tag{106}$$

so the effect of this factor is to remove any component in the direction of $\vec{k}$, while any component perpendicular to $\vec{k}$ is left unaffected. Using Dirac brackets for the theory, we are allowed to set the constraints zero everywhere. In particular, we see that the action is the same as that of a massless scalar field for each component A^i, but the bracket structure is different, meanwhile imposing the condition on the Hamiltonian we can recover the Hamiltonian we know from electromagnetism

$$H = \int \mathrm{d}^3\vec{x}\ \frac{1}{2}\vec{\Pi}\cdot\vec{\Pi} + \frac{1}{2}\partial_j A^i \partial_j A^i = \int \mathrm{d}^3\vec{x}\ \frac{1}{2}\left(\vec{E}^2 + \vec{B}^2\right). \tag{107}$$

Now, let us find the Fourier expansion of the field operator

$$A^i(t,\vec{x}) = \frac{1}{\sqrt{V}} \sum_{\vec{k}\neq 0} A^i(t,\vec{k}) e^{i\vec{k}\cdot\vec{x}}, \tag{108}$$

where, as usual, reality means $A^i(t,-\vec{k}) = A^i(t,\vec{k})^*$. The Coulomb gauge condition requires that

$$\vec{k}\cdot\vec{A}(t,\vec{k}) = 0, \tag{109}$$

thus if we introduce two polarisation vectors $\vec{\epsilon}^{\,s}(\vec{k})$, $s = 1,2$, such that

$$\vec{k}\cdot\vec{\epsilon}(\vec{k}) = 0, \qquad \vec{\epsilon}^{\,s}(\vec{k})\cdot\vec{\epsilon}^{\,r}(\vec{k}) = \delta^{sr}, \tag{110}$$

then the Fourier expansion for the vector field can be written

$$\vec{A}(t,\vec{x}) = \frac{1}{\sqrt{V}} \sum_{k\neq 0} \sum_{s=1,2} \frac{1}{\sqrt{2|k|}} \left(a_s(\vec{k}) \epsilon^s(\vec{k}) e^{ik\cdot x} + a_s^\dagger(\vec{k}) \epsilon^s(\vec{k})^* e^{-ik\cdot x} \right), \tag{111}$$

where $k^\mu = (|k|, \vec{k})$. In terms of the a_s, the Dirac brackets of the theory become

$$\{a_s(\vec{k}), a_r^\dagger(\vec{p})\}_D = -i\delta_{sr}\delta_D(\vec{k}-\vec{p}), \tag{112}$$

with all other brackets vanishing.

Let us quickly give an example of suitable polarisation vectors $\vec{\epsilon}^{\,s}(\vec{k})$. Let $\vec{e}^{(3)}(\vec{k}) = \vec{k}/|\vec{k}|$ and then introduce unit vectors $\vec{e}^{(1)}(\vec{k})$ and $\vec{e}^{(2)}(\vec{k})$ so that these three unit vectors form a right-handed orthonormal basis for $\mathbb{R}^3$. Then we introduce circular polarisation vectors

$$\vec{\epsilon}^{\,\pm}(\vec{k}) = \frac{1}{\sqrt{2}} \left(\vec{e}^{(1)}(\vec{k}) \pm i\vec{e}^{(2)}(\vec{k}) \right). \tag{113}$$

In Appendix F we show that these vectors are eigenvectors of the helicity operator with eigenvalues ± 1.

2.4.1 Zero Momentum Modes

We have so far neglected the modes with zero momentum. For these modes we can note that the gauge transformation can not affect the spatial components, so these components are all physical, while we can still gauge away A_0. Thus, we set

$$A^0 = 0, \qquad \vec{A}(t, \vec{x}) = \frac{1}{\sqrt{V}}\vec{A}(t), \tag{114}$$

then the action for the physical zero modes is

$$S[\vec{A}] = \int \mathrm{d}t \; \frac{1}{2}\frac{\partial \vec{A}}{\partial t} \cdot \frac{\partial \vec{A}}{\partial t}, \tag{115}$$

which we can recognize as the action for three independent classical particles.

3 Example of Linearization Stability Conditions: Scalar QED on a Torus

To better understand the derivation of the linearization stability conditions, let us consider the simpler example of scalar electrodynamics on a flat torus. The dynamical fields in this theory are a complex scalar Φ and the electromagnetic field A_μ. The theory is governed by a minimal coupling action

$$S[A, \Phi] = \int \mathrm{d}^4x \; \left(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - D^\mu\Phi^\dagger D_\mu\Phi - M^2\Phi^\dagger\Phi\right), \tag{116}$$

where we have defined the electromagnetic field strength tensor by

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \tag{117}$$

and the gauge-covariant derivative D_μ is defined as

$$D_\mu = \partial_\mu - ieA_\mu, \tag{118}$$

where e is the electric charge of the field, which is a constant. This theory has invariance under finite gauge transformations given by

$$\Phi(x) \mapsto \Phi'(x) = \exp(-ie\theta(x))\Phi(x) \tag{119}$$

$$A_\mu(x) \mapsto A'_\mu(x) = A_\mu(x) - \partial_\mu\theta(x), \tag{120}$$

where $\theta(x)$ is an arbitrary function. The corresponding infinitesimal transformations are then

$$\delta_\theta\Phi(x) = \Phi'(x) - \Phi(x) = -ie\theta(x)\Phi(x), \tag{121}$$

$$\delta_\theta A_\mu(x) = A'_\mu(x) - A_\mu(x) = -\partial_\mu\theta(x). \tag{122}$$

The equations of motion for this theory can be derived in the usual manner, by requiring that $\delta S = 0$ when we make arbitrary variations of $\delta\Phi$ and δA_μ, that is

$$\frac{\delta S}{\delta \Phi(x)} = 0, \qquad \frac{\delta S}{\delta A_\mu(x)} = 0. \tag{123}$$

The resulting exact equations of motion are

$$D^\mu D_\mu \Phi - M^2 \Phi = 0, \tag{124}$$

$$\partial_\mu F^{\mu\nu} - ie(\Phi^\dagger D^\nu \Phi - D^\nu \Phi^\dagger \Phi) = 0. \tag{125}$$

Now suppose we expand the fields in some small parameter α. We assume that the fields Φ and A_μ describe the small perturbations around the classical background solution $\phi = A_\mu = 0$, so that we can write

$$\phi = \Phi^{(1)} + \Phi^{(2)} + \dots, \tag{126}$$

$$A_\mu = A_\mu^{(1)} + A_\mu^{(2)} + \dots, \tag{127}$$

where for a field F, $F^{(i)}$ denotes the component which scales like α^i. If we expand the action to second order, then the linearized theory of $(\Phi^{(1)}, A_\mu^{(1)})$ is described by the action

$$S^{\mathrm{Lin}}[A^{(1)}, \phi^{(1)}] = \int \mathrm{d}^4 x \, \left[-\frac{1}{4} F_{\mu\nu}^{(1)} F^{(1)\mu\nu} - \partial_\mu \Phi^{(1)\dagger} \partial^\mu \Phi^{(1)} - M^2 \Phi^{(1)\dagger} \Phi^{(1)} \right], \tag{128}$$

which is the action for the free electromagnetic field and the action for a free complex scalar field, with no interaction between them. As we have seen, the free complex scalar field Φ has a conserved charge, arising from the internal phase rotations,

$$Q = ie \int \mathrm{d}^3 \vec{x} \left[\Phi^{(1)\dagger} \frac{\partial \Phi^{(1)}}{\partial t} - \frac{\partial \Phi^{(1)\dagger}}{\partial t} \Phi^{(1)} \right]. \tag{129}$$

We now wish to show that on the torus, or more generally any compact space, the second order theory imposes that the charge Q is not just conserved but in fact must vanish identically. To see this, note that the exact equations of motion must vanish order by order, and consider the order α^2 part of the equation of motion for A_μ, which reads

$$\partial_\mu \left(\partial^\mu A^{(2)\nu} - \partial^\nu A^{(2)\mu} \right) - ie \left(\Phi^{(1)\dagger} \partial^\nu \Phi^{(1)} - \partial^\nu \Phi^{(1)\dagger} \Phi^{(1)} \right) = 0. \tag{130}$$

Taking the $\nu = 0$ component reduces this to

$$\vec{\nabla} \cdot \left(\vec{\nabla} A^{(2)0} + \frac{\partial \vec{A}^{(2)}}{\partial t} \right) = -ie \left(\Phi^{(1)\dagger} \frac{\partial \Phi^{(1)}}{\partial t} - \frac{\partial \Phi^{(1)\dagger}}{\partial t} \Phi^{(1)} \right). \tag{131}$$

In particular, note that the second-order fields enter this equation only as total spatial derivatives. It follows that if we integrate over the whole torus, where such integrals must vanish by Gauss's theorem as there is no boundary, then we must get zero. It follows that the total charge, Q must vanish, so

$$Q = 0. \tag{132}$$

This is the classical linearization stability condition. We can only hope to extend a solution of the linearized equations of motion to a full solution of the non-linear system if the system obeys this additional condition.

Next, let us examine how this linearization stability is implemented on the quantised version of the linearized theory. To quantise the complex scalar field, we promote the Fourier coefficients $a(\vec{k})$, $a^\dagger(\vec{k})$, $b(\vec{k})$ and $b^\dagger(\vec{k})$ to operators, which obey the commutation relations

$$\begin{aligned} [a(\vec{k}), a^\dagger(\vec{p})]_- &= \delta_D(\vec{k}-\vec{p}), \\ [b(\vec{k}), b^\dagger(\vec{p})]_- &= \delta_D(\vec{k}-\vec{p}), \end{aligned} \tag{133}$$

with all others vanishing. These commutation relations are obtained in canonical quantisation by multiplying the classical brackets by i and taking them to be commutators instead of classical brackets. We also need to provide a Hilbert space of states for these operators to act on. We define this to be the Fock space built on a vacuum state $|0\rangle$ which is annihilated by all $a(\vec{k})$ and $b(\vec{k})$, that is

$$a(\vec{k})|0\rangle = b(\vec{k})|0\rangle = 0, \qquad \text{for all } \vec{k}. \tag{134}$$

Other states are then created by acting with $a^\dagger(\vec{k})$ and $b^\dagger(\vec{k})$. For example, the state

$$\left|a, \vec{k}\right\rangle = a^\dagger(\vec{k})|0\rangle, \tag{135}$$

describes a single a-type excitation, with charge $Q = e$, energy $H = E(\vec{k}) = \sqrt{\vec{k}^2 + M^2}$ and momentum $\vec{P} = \vec{k}$. Similarly, applying $b^\dagger$ creates b-type excitation, which have the same energy and momentum as their a-type counterparts, but are oppositely charged $Q = -e$.

In the quantum theory, the linearization stability constraint $Q = 0$ should be imposed as a first class constraint, constraining the physical states $|\text{phys}\rangle$ by

$$Q|\text{phys}\rangle = 0. \tag{136}$$

By the definition of the total charge in terms of the creation and annihilation operators,

$$Q = e\sum_{\vec{k}} \left[a^\dagger(\vec{k})a(\vec{k}) - b^\dagger(\vec{k})b(\vec{k})\right], \tag{137}$$

it is clear that the physical states are those with with equal numbers of a- and b-type excitations. But let us see how this fact comes about if we apply the group averaging procedure over the generating $U(1)$ symmetry group.

In the group averaging procedure we begin with a general, possibly unphysical, state $|\text{state}\rangle$, and then define an associated physical state $|\text{phys}\rangle$ by

$$|\text{phys}\rangle = \int_0^{\frac{2\pi}{e}} \mathrm{d}\theta \; e^{i\theta Q}|\text{state}\rangle, \tag{138}$$

where the integral runs over the symmetry group which leads to the conserved charge Q, in our case of the complex scalar field the symmetry is the internal phase rotation

$$\Phi^{(1)} \mapsto \Phi^{(1)\prime} = \exp(-ie\theta)\Phi^{(1)}, \tag{139}$$

so the parameter $\theta \in [0, \frac{2\pi}{e})$. Now, suppose that we take $|\text{state}\rangle$ to be a state with $n_a(\vec{k})$ particles of type a with momentum $\vec{k}$ for each $\vec{k}$ and similarly $m_b(\vec{k})$ particles of type b with momentum $\vec{k}$, so that

$$\begin{aligned} |\text{state}\rangle &= \left|\{n_a(\vec{k}), m_b(\vec{k})\}\right\rangle \\ &= \prod_{\vec{k}} \left[\frac{1}{\sqrt{n_a(\vec{k})!m_b(\vec{k})!}} (a^\dagger(\vec{k})^{n_a(\vec{k})} b^\dagger(\vec{k})^{m_b(\vec{k})} \right] |0\rangle \end{aligned} \tag{140}$$

These kinds of states form a basis for the Hilbert space of the complex scalar field. On these states, the total charge acts as

$$Q\left|\{n_a(\vec{k}), m_b(\vec{k})\}\right\rangle = e\sum_{\vec{k}} \left(n_a(\vec{k}) - m_a(\vec{k})\right). \tag{141}$$

In particular, we can then evaluate the integral defining the physical state, as Q/e is an integer, we find that

$$\begin{aligned} |\text{phys}\rangle &= \frac{2\pi}{e}\delta(\sum_{\vec{k}} \left(n_a(\vec{k}) - m_a(\vec{k})\right))\left|\{n_a(\vec{k}), m_b(\vec{k})\}\right\rangle \\ &= \frac{2\pi}{e}\begin{cases} 0, & \text{if } \sum_{\vec{k}} \left(n_a(\vec{k}) - m_a(\vec{k})\right) \neq 0, \\ \left|\{n_a(\vec{k}), m_b(\vec{k})\}\right\rangle, & \text{if } \sum_{\vec{k}} \left(n_a(\vec{k}) - m_a(\vec{k})\right) = 0. \end{cases} \end{aligned} \tag{142}$$

Thus, for a general state composed of superpositions of these eigenstates, we see that the group averaging prescription indeed projects onto the physical space of states with total charge $Q = 0$.

4 Classical Theory of Free-Fields: Spin-3/2 and Spin-2

4.1 Rarita-Schwinger Fields

We next want to consider free massless fields of spin-3/2. These are described by a Majorana vector-spinor field $\Psi_{\mu\alpha}$. A suitable theory is described by the massless Rarita-Schwinger action

$$S[\Psi_\mu] = \int \mathrm{d}^4x \left[-\frac{1}{2}\bar{\Psi}_\mu \gamma^{\mu\nu\rho}\partial_\nu \Psi_\rho\right], \tag{143}$$

where $\gamma^{\mu\nu\rho}$ is the anti-symmetric element formed out of three γ-matrices defined in (470) of Appendix D. This action is invariant under a gauge transformation of the form

$$\Psi_{\mu\alpha} \mapsto \Psi'_{\mu\alpha} = \Psi_{\mu\alpha} + \partial_\mu \epsilon_\alpha, \tag{144}$$

where ϵ_α is a Majorana spinor function. Varying the action with respect to Ψ_μ yields the equation of motion in the form

$$\gamma^{\mu\nu\rho}\partial_\nu\Psi_\rho = 0. \tag{145}$$

To get at the physical degrees of freedom, we will completely fix the gauge. We will again at first neglect the modes with zero momentum and deal with these separately later. For the non-zero momentum modes, we impose the gauge-fixing condition

$$\vec{\gamma}\cdot\vec{\Psi} = 0. \tag{146}$$

This condition completely determines ϵ_α. Now, let us analyse the consequences of this condition. If we take the 0-component of the field equation, we have

$$\gamma^{0ij}\partial_i\Psi_j = -\gamma^0\vec{\nabla}\cdot\vec{\Psi} = 0, \tag{147}$$

where we used $\gamma^{0ij} = \gamma^0(\gamma^i\gamma^j - \delta^{ij})$, which follows from the definition (470) and using the Clifford algebra relations between the γ-matrices (466). As γ^0 is invertible, we learn that $\vec{\Psi}$ is divergence free. Next take the i-component of the field equation, which reads

$$\gamma^{ijk}\partial_j\Psi_k + \gamma^{i0j}\partial_0\Psi_j + \gamma^{ij0}\partial_j\Psi_0 = 0. \tag{148}$$

This can be simplified by noting again $\gamma^{0ij} = \gamma^0(\gamma^i\gamma^j - \delta^{ij})$ and

$$\gamma^{ijk} = \frac{1}{2}(\gamma^{ij}\gamma^k + \gamma^i\gamma^j\gamma^k + 2\delta^{ik}\gamma^j - 2\delta^{kj}\gamma^i - 2\delta^{ij}\gamma^k). \tag{149}$$

Using these we arrive at

$$\gamma^\mu\partial_\mu\Psi^i + \gamma^0(\gamma^i\gamma^j - \delta^{ij})\partial_j\Psi_0 = 0. \tag{150}$$

If γ^i is applied to this, we find

$$\vec{\gamma}\cdot\vec{\nabla}\Psi_0 = 0, \tag{151}$$

which leads us to conclude that in the non-zero momentum sector

$$\Psi_0 = 0, \tag{152}$$

while the equation of motion obeyed by the remaining components is the massless Dirac equation

$$\gamma\cdot\partial\vec{\Psi} = 0. \tag{153}$$

As the action is linear in derivatives of Ψ_μ and because of the gauge conditions we need to impose the system describing the Rarita-Schwinger field is singular. Thus the Poisson bracket structure will not be adequate to describe the theory and we need to calculate the Dirac brackets of the theory. This can be quite an involved task, however we can already anticipate the result. The gauge conditions tell us that the physical degrees of freedom are only those with helicity $\pm\frac{3}{2}$, thus we expect that the Fourier expansion for the non-zero momentum sector is of the form

$$\vec{\Psi}(t,\vec{x}) = \frac{1}{\sqrt{V}} \sum_{\vec{k}\neq 0} \sum_{s=\pm} \frac{1}{\sqrt{2|\vec{k}|}} \left[\vec{\epsilon}^{\,s}(\vec{k}) u^s(\vec{k}) b_s(\vec{k}) e^{ik\cdot x} + \vec{\epsilon}^{\,s}(\vec{k})^* v^s(\vec{k}) b_s^\dagger(\vec{k}) e^{-ik\cdot x} \right], \tag{154}$$

with the Dirac brackets between $b_s(\vec{k})$ given by

$$\{b_s(\vec{k}), b_r^\dagger(\vec{p})\}_D = -i\delta_{sr}\delta_D(\vec{k}-\vec{p}), \tag{155}$$

with all other brackets vanishing. Indeed this turns out to be the case [26].

Under a translation $x \mapsto x - \epsilon$, the Rarita-Schwinger field changes by $\delta_\epsilon \Psi_\mu = \epsilon^\nu \partial_\nu \Psi_\mu$, then we can calculate the energy momentum tensor by

$$\delta_\epsilon \Psi_\mu^T \frac{\delta S}{\delta \Psi_\mu} = -\partial_\mu T^\mu{}_\nu \epsilon^\nu. \tag{156}$$

We have

$$\begin{aligned} \delta_\epsilon \Psi_\mu^T \frac{\delta S}{\delta \Psi_\mu} &= -\epsilon^\lambda \partial_\lambda \Psi_\mu^T C \gamma^{\mu\nu\rho} \partial_\nu \Psi_\rho \\ &= \frac{1}{2}\partial_\nu \left(\epsilon^\lambda \Psi_\rho^T C \gamma^{\rho\nu\mu} \partial_\lambda \Psi_\mu \right) - \frac{1}{2}\partial_\lambda \left(\epsilon^\lambda \Psi_\rho^T C \gamma^{\rho\nu\mu} \partial_\nu \Psi_\mu \right), \end{aligned} \tag{157}$$

so that we read off

$$T^{\mu\nu} = -\frac{1}{2}\Psi_\rho^T C \gamma^{\rho\mu\lambda} \partial^\nu \Psi_\lambda + \eta^{\mu\nu} \Psi_\rho^T C \gamma^{\rho\lambda\tau} \partial_\lambda \Psi_\tau. \tag{158}$$

Note that when the equations of motion are obeyed the second term vanishes. Then we can calculate the conserved charges

$$\begin{aligned} H &= \int \mathrm{d}^3\vec{x}\; T^{00} = \int \mathrm{d}^3\vec{x} - \frac{1}{2}\Psi_i^T C \gamma^0 \gamma^{ij} \frac{\partial \Psi_j}{\partial t} \\ &= \sum_{\vec{k}\neq 0} \sum_{s=\pm} |\vec{k}| b_s^\dagger(\vec{k}) b_s(\vec{k}). \end{aligned} \tag{159}$$

Similarly,

$$\begin{aligned} \vec{P} &= \int \mathrm{d}^3\vec{x}\; T^{0i} = \int \mathrm{d}^3\vec{x} - \frac{1}{2}\Psi_i^T C \gamma^0 \gamma^{ij} \vec{\nabla} \Psi_j \\ &= \sum_{\vec{k}\neq 0} \sum_{s=\pm} \vec{k} b_s^\dagger(\vec{k}) b_s(\vec{k}). \end{aligned} \tag{160}$$

4.1.1 Zero Momentum Modes

The zero momentum modes need to be analysed separately again. Once again, the gauge freedom in this case can only change $\Psi_0(t)$, which we can set to zero. For the remaining components, let

$$\Psi_i = \frac{1}{\sqrt{V}} \psi_i(t), \tag{161}$$

then the action is

$$S[\psi_i] = \int \mathrm{d}t\; \frac{1}{2} \psi_i \gamma^0 \gamma^{ij} \frac{\partial \psi_j}{\partial t}. \tag{162}$$

Then the associated conjugate momentum is

$$\pi_{i\alpha} = -\frac{1}{2}\psi_j^T C\gamma^0\gamma^{ji}, \tag{163}$$

where as usual the momentum conjugate is defined as a left derivative with respect to $\dot{\psi}_{i\alpha}$. As this is a first order system, we need to impose this as a primary constraint on the system, thus let

$$\phi^i = \pi^i + \frac{1}{2}\psi_j^T C\gamma^0\gamma^{ji}. \tag{164}$$

As for the Majorana spinor zero-modes, the canonical Hamiltonian vanishes, so the primary Hamiltonian is simply

$$H_P = \phi^i_\alpha \lambda^i_\alpha, \tag{165}$$

where λ^i_α are constants. We wish to impose that the constraints are (weakly) conserved under the time evolution generated by H_P. Calculating the Poisson brackets between the constraints gives

$$\begin{aligned}\{\phi^i_\alpha, \phi^j_\beta\}_P &= \left\{\pi^i_\alpha + \frac{1}{2}(\psi_k^T C\gamma^0\gamma^{ki})_\alpha, \pi^j_\beta + \frac{1}{2}(\psi_l^T C\gamma^0\gamma^{kj})_\beta\right\}_P \\ &= -\frac{1}{2}(C\gamma^0\gamma^{ij})_{\alpha\beta} - \frac{1}{2}(C\gamma^0\gamma^{ij})_{\beta\alpha} \\ &= -(C\gamma^0\gamma^{ij})_{\alpha\beta}.\end{aligned} \tag{166}$$

where we made use of the canonical Poisson brackets for the system

$$\{\pi^i, \pi^j\}_P = 0, \quad \{\psi_{i\alpha}, \pi^j_\beta\}_P = -\delta^i_j\delta_{\alpha\beta}, \quad \{\psi_i, \psi_j\}_P = 0. \tag{167}$$

It follows that setting the time evolution to vanish requires $\lambda^i_\alpha = 0$ because

$$\{\phi^i_\alpha, H_P\}_P = -(C\gamma^0\gamma^{ij})_{\alpha\beta}\lambda^j_\beta, \tag{168}$$

and we used that if we let

$$D^{ij}_{\alpha\beta} = \frac{1}{2}\left[(\gamma^{ij} - \delta^{ij})\gamma^0 C^{-1}\right]_{\alpha\beta}, \tag{169}$$

then

$$D^{ij}_{\alpha\beta}\{\phi^j_\beta, \phi^k_\gamma\}_P = \delta^{ik}\delta_{\alpha\gamma}. \tag{170}$$

We note that all the constraints fall into the second class, so we calculate the Dirac bracket between ψ and itself to obtain

$$\begin{aligned}\{\psi_{i\alpha}, \psi_{j\beta}\}_D &= -\{\psi_{i\alpha}, \phi^k_\gamma\}_P D^{kl}_{\gamma\delta}\{\phi^l_\delta, \psi_{j\beta}\}_P \\ &= -\frac{1}{2}\left[(\gamma_{ij} - \delta_{ij})\gamma^0 C^{-1}\right]_{\alpha\beta}.\end{aligned} \tag{171}$$

To understand these relations a bit better, let us work again in our explicit Majorana representation, such that $C = i\gamma^0$ and write

$$\psi_{i\alpha} = \frac{1}{\sqrt{6}}(\gamma_i\eta)_\alpha + \sum_{A=1}^{2} T^A_{ij}(\gamma^j\eta^A)_\alpha, \tag{172}$$

where η and η^A, $A = 1, 2$ are anti-commuting spinors and we have introduced the matrices

$$T^1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad T^2 = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}. \tag{173}$$

We can isolate the η_α by contracting ψ_i with γ^i as $T^A_{ij}\gamma^i\gamma^j = 0$, which yields

$$\gamma^i \psi_i = \sqrt{\frac{3}{2}}\eta, \tag{174}$$

so that

$$\{\eta_\alpha, \eta_\beta\}_D = +i\delta_{\alpha\beta} \tag{175}$$

A short calculation shows that

$$T^A_{ij}\gamma^j T^B_{ik}\gamma^k = \delta^{AB}, \tag{176}$$

so that

$$T^A_{ij}\gamma^i\psi_{j\alpha} = \eta^A_\alpha. \tag{177}$$

It follows that

$$\{\eta_\alpha, \eta^A_\beta\}_D = 0, \qquad \{\eta^A_\alpha, \eta^B_\beta\}_D = -i\delta^{AB}\delta_{\alpha\beta}. \tag{178}$$

The Dirac bracket relations between the η^A are the same as for the zero modes of the Majorana spinor field, however the η_α come with opposite sign.

4.2 Linearized Gravity

The next theory we want study is that of a free massless spin-2 field. Such fields are necessarily associated with gravity, and it is possible to obtain a free theory if we consider small perturbations in the metric from a flat background. The vacuum Einstein equations can be derived from the Einstein-Hilbert action

$$S[g] = \frac{1}{2}\int \mathrm{d}^4x\sqrt{-g}R, \tag{179}$$

where $g_{\mu\nu}$ is the metric tensor and R is the Ricci scalar, the trace of the Ricci tensor $R_{\mu\nu}$ defined in our conventions as

$$R_{\mu\nu} = \left(\partial_\rho \Gamma^\rho{}_{\mu\nu} - \partial_\nu \Gamma^\rho{}_{\mu\rho} + \Gamma^\tau{}_{\mu\nu}\Gamma^\rho{}_{\tau\rho} - \Gamma^\tau{}_{\mu\rho}\Gamma^\rho{}_{\nu\tau}\right), \tag{180}$$

and the Christoffel symbols $\Gamma^\mu{}_{\nu\rho}$ are

$$\Gamma^\mu{}_{\nu\rho} = \frac{1}{2}g^{\mu\sigma}\left(\partial_\rho g_{\nu\sigma} + \partial_\nu g_{\rho\sigma} - \partial_\sigma g_{\nu\rho}\right). \tag{181}$$

Varying the Einstein-Hilbert action with respect to the metric $g_{\mu\nu}$ yields the vacuum Einstein equations

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 0 \tag{182}$$

Taking the trace of this, we can simplify this as $R_{\mu\nu} = 0$ for the vacuum case.

We will be interested in considering the theory described by the small perturbations

$h_{\mu\nu}$ from a flat background metric $\eta_{\mu\nu}$. To obtain a linear equation of motion, we should expand the Lagrangian $\sqrt{-g}R$ to quadratic order in $h_{\mu\nu}$. We follow the argument by Landau and Lifschitz [27]. First integrate by parts to write

$$\begin{aligned} S &= \int \mathrm{d}^4x \; \frac{1}{2}\sqrt{-g}g^{\mu\nu}R_{\mu\nu} \\ &= \int \mathrm{d}^4x \; \frac{1}{2}G + \int \mathrm{d}^4x \; \frac{1}{2}\partial_\mu(\sqrt{-g}w^\mu), \end{aligned} \tag{183}$$

where

$$w^\mu = g^{\rho\nu}\Gamma^\mu{}_{\rho\nu} - g^{\mu\nu}\Gamma^\rho{}_{\rho\nu}, \tag{184}$$

$$\begin{aligned} G = &-\partial_\rho(\sqrt{-g}g^{\mu\nu})\Gamma^\rho{}_{\mu\nu} + \partial_\nu(\sqrt{-g}g^{\mu\nu})\Gamma^\rho{}_{\mu\rho} \\ &+ \sqrt{-g}g^{\mu\nu}\left(\Gamma^\tau{}_{\mu\nu}\Gamma^\rho{}_{\tau\rho} - \Gamma^\tau{}_{\mu\rho}\Gamma^\rho{}_{\nu\tau}\right). \end{aligned} \tag{185}$$

Next, we recall that the derivatives of $\sqrt{-g}$ and $g^{\mu\nu}$ can be written in terms of Γ as

$$\partial_\rho(\sqrt{-g}g^{\mu\nu}) = \sqrt{-g}\Gamma^\tau{}_{\rho\tau}g^{\mu\nu} - \sqrt{-g}\left(\Gamma^\mu{}_{\rho\tau}g^{\tau\nu} + \Gamma^\nu{}_{\rho\tau}g^{\mu\tau}\right). \tag{186}$$

Inserting this into the previous expression for G yields

$$G = \sqrt{-g}g^{\mu\nu}\left(\Gamma^\rho{}_{\tau\nu}\Gamma^\tau{}_{\rho\mu} - \Gamma^\rho{}_{\mu\nu}\Gamma^\tau{}_{\rho\tau}\right). \tag{187}$$

To expand this to quadratic order in $h_{\mu\nu}$, we only need the linear order expression of the Christoffel symbol, which is

$$\Gamma^{(1)\mu}{}_{\nu\rho}[h] = \frac{1}{2}\eta^{\mu\sigma}\left(\partial_\nu h_{\sigma\rho} + \partial_\rho h_{\nu\sigma} - \partial_\sigma h_{\nu\rho}\right). \tag{188}$$

Thus the part of the action quadratic in $h_{\mu\nu}$ is, on performing another integration by parts,

$$S[h] = \int \mathrm{d}^4x \; \left(\frac{1}{4}\partial^\rho h_{\mu\rho}\partial_\sigma h^{\mu\sigma} - \frac{1}{8}\partial_\rho h_{\mu\nu}\partial^\rho h^{\mu\nu} - \frac{1}{4}\partial^\rho h_{\rho\sigma}\partial^\sigma h + \frac{1}{8}\partial_\rho h\partial^\rho h\right), \tag{189}$$

where in this expression we can raise and lower indices using the metric η and we have defined $h = \eta^{\mu\nu}h_{\mu\nu}$. The field equation obeyed by $h_{\mu\nu}$ is ten

$$\partial_\rho\partial^\rho h_{\mu\nu} - \partial^\rho(\partial_\nu h_{\mu\rho} + \partial_\mu h_{\nu\rho}) + \partial_\mu\partial_\nu h = 0. \tag{190}$$

In the non-linear theory, the theory is invariant if the metric is changed by a Lie derivative of the metric along any vector field. This invariance is inherited as the gauge symmetry of the linearized theory, which allows us to change

$$h_{\mu\nu} \mapsto h'_{\mu\nu} = h_{\mu\nu} + \partial_\mu\xi_\nu + \partial_\nu\xi_\mu, \tag{191}$$

for any ξ_μ. The linearized equation of motion is invariant under this gauge transformation. To fix the gauge freedom in the sector of the theory with non-zero momentum, we proceed as in [28, 26]. We first impose the condition

$$\partial^i h_{i\mu} = 0, \tag{192}$$

which completely fixes the gauge freedom. Then examining the equations of motion shows that additionally we have

$$h_{ii} = h_{0i} = h_{00} = 0. \tag{193}$$

The remaining components of h_{ij} then obey the wave equation

$$\partial_\mu \partial^\mu h_{ij} = 0, \tag{194}$$

subject to the transverse $\partial^i h_{ij} = 0$ and traceless h_{ii} condition. These conditions allow only the helicity ± 2 components to remain, and the resulting Fourier expansion for h_{ij} is

$$h_{ij}(t, \vec{x}) = \sqrt{\frac{2}{V}} \sum_{k \neq 0} \sum_{s=\pm} \frac{1}{\sqrt{|\vec{k}|}} \left(H^s_{ij}(\vec{k}) a_s(\vec{k}) e^{ik\cdot x} + H^{s*}_{ij}(\vec{k}) a^\dagger_s(\vec{k}) e^{-ik\cdot x} \right), \tag{195}$$

where

$$H^s_{ij}(\vec{k}) = \epsilon^s_i(\vec{k}) \epsilon^s_j(\vec{k}). \tag{196}$$

Then the Poisson bracket relations for the $a_s(\vec{k})$ are

$$\{a_s(\vec{k}), a^\dagger_r(\vec{p})\} = -i \delta_{sr} \delta_D(\vec{k} - \vec{p}), \tag{197}$$

with all other brackets vanishing.

The theory is still invariant under space-time translations, so we can calculate the conserved energy and momentum. Under the translation $x \mapsto x - \epsilon$ the field $h_{\mu\nu}$ changes by

$$\delta_\epsilon h_{\mu\nu} = \epsilon^\rho \partial_\rho h_{\mu\nu}, \tag{198}$$

and the energy momentum tensor is defined through

$$\delta_\epsilon h_{\mu\nu} \frac{\delta S}{\delta h_{\mu\nu}} = +\partial_\mu T^\mu{}_\nu \, \epsilon^\nu. \tag{199}$$

We can calculate that

$$\begin{aligned} \partial_\epsilon h_{\mu\nu} \frac{\delta S}{\delta h_{\mu\nu}} &= \frac{1}{4} \epsilon^\tau \partial_\tau h_{\mu\nu} \left[\partial^\rho \partial_\rho h^{\mu\nu} - \partial_\rho \left(\partial^\mu h^{\nu\rho} + \partial^\nu h^{\mu\rho} \right) + \partial^\mu \partial^\nu h \right] \\ &+ \frac{1}{4} \epsilon^\tau \partial_\tau h \left[\partial^\rho \partial^\sigma h_{\rho\sigma} - \partial^\rho \partial_\rho h \right]. \end{aligned} \tag{200}$$

It follows that a suitable candidate for $T^\mu{}_\nu$ is

$$\begin{aligned} T^\mu{}_\nu &= \frac{1}{4} \left[\partial_\nu h_{\rho\sigma} \partial^\mu h^{\rho\sigma} + \partial_\nu h^{\mu\rho} \partial_\rho h + \partial_\nu h \partial_\rho h^{\mu\rho} - \partial_\nu h \partial^\mu h - 2 \partial_\nu h^{\rho\mu} \partial^\sigma h_{\sigma\rho} \right] \\ &+ \frac{1}{4} \delta^\mu_\nu \left[\partial^\rho h_{\tau\rho} \partial_\sigma h^{\tau\sigma} - \frac{1}{2} \partial_\rho h_{\sigma\tau} \partial^\rho h^{\sigma\tau} - \partial^\rho h_{\rho\sigma} \partial^\sigma h + \frac{1}{2} \partial_\rho h \partial^\rho h \right]. \end{aligned} \tag{201}$$

In particular, if we impose the gauge-fixing conditions $h_{0\mu} = 0 = h = \partial_i h_{ij}$ we find

$$T^{00} = \frac{1}{8}\left[\frac{\partial h_{ij}}{\partial t}\frac{\partial h_{ij}}{\partial t} + \vec{\nabla} h_{ij} \cdot \vec{\nabla} h_{ij}\right], \tag{202}$$

so that if we insert the Fourier expansion for h_{ij} we find

$$H = \int \mathrm{d}^3\vec{x}\; T^{00} = \sum_{k\neq 0}\sum_{s=\pm} |\vec{k}| a_s^\dagger(\vec{k}) a_s(\vec{k}). \tag{203}$$

Similarly we get that

$$T^{0i} = \frac{1}{4}\left[-\partial_i h_{jk}\frac{\partial h_{jk}}{\partial t}\right], \tag{204}$$

which gives the total momentum as

$$\vec{P} = \int \mathrm{d}^3\vec{x}\; T^{0i} = \sum_{k\neq 0}\sum_{s=\pm} \vec{k} a_s^\dagger(\vec{k}) a_s(\vec{k}). \tag{205}$$

4.2.1 Zero Momentum Modes

In the sector with zero momentum, the gauge freedom can only affect the components h_{00} and h_{0i}, but the purely spatial components h_{ij} are unaffected by a gauge transformation because under the transformation labelled by ξ_μ we have

$$\begin{aligned} h_{00}(t) &\mapsto h_{00}(t) + 2\frac{\partial}{\partial t}\xi_0, \\ h_{0i}(t) &\mapsto h_{0i}(t) + \frac{\partial}{\partial t}\xi_i, \\ h_{ij}(t) &\mapsto h_{ij}(t). \end{aligned} \tag{206}$$

We can thus choose ξ_μ to fix the gauge such that $h_{00} = h_{0i} = 0$. The action for the physical modes h_{ij} then becomes

$$S = \int \mathrm{d}t\; \left(\frac{1}{8}\frac{\partial h_{ij}}{\partial t}\frac{\partial h_{ij}}{\partial t} - \left(\frac{\partial h}{\partial t}\right)^2\right), \tag{207}$$

where $h = \eta^{\mu\nu}h_{\mu\nu} = \delta^{ij}h_{ij}$. To analyse this system, let us write

$$h_{ij}(t) = \sqrt{\frac{2}{3}}\delta_{ij}c(t) + 2\sum_{A=1}^{5} T^A_{ij} c_A(t), \tag{208}$$

where the five symmetric traceless tensors T^A satisfy the orthonormality condition

$$T^A_{ij}T^B_{ij} = \delta^{AB}. \tag{209}$$

We explicitly choose T^A so that T^1 and T^2 are as defined for the Rarita-Schwinger zero-modes, so we make the choice

$$\begin{aligned} T^1 &= \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & 0\end{pmatrix}, \qquad T^2 = \frac{1}{\sqrt{6}}\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & -2\end{pmatrix},\\ T^3 &= \frac{1}{\sqrt{2}}\begin{pmatrix}0 & 1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}, \qquad T^4 = \frac{1}{\sqrt{2}}\begin{pmatrix}0 & 0 & 1\\ 0 & 0 & 0\\ 1 & 0 & 0\end{pmatrix},\\ T^5 &= \frac{1}{\sqrt{2}}\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 1\\ 0 & 1 & 0\end{pmatrix}. \end{aligned} \tag{210}$$

In terms of the variables c and c^A the action can then be re-written as

$$S[c, c^A] = \int \mathrm{d}t \, \left[-\frac{1}{2}\left(\frac{\partial c}{\partial t}\right)^2 + \sum_{A=1}^{5} \frac{1}{2}\left(\frac{\partial c_A}{\partial t}\right)^2 \right]. \tag{211}$$

Notice that c contributes to the action with a negative kinetic term. Defining the canonical conjugates c_P and c_{PA} to c and c_A by

$$c_P = -\frac{\partial c}{\partial t}, \qquad c_{PA} = \frac{\partial c_A}{\partial t}, \tag{212}$$

the equal-time bracket structure for this system is given by

$$\{c, c_P\} = 1, \qquad \{c_A, c_{PB}\} = \delta_{AB}, \tag{213}$$

with all other brackets between the c, c_P, c_A and c_{PA} vanishing. Note that the Hamiltonian for the zero-modes is non-zero and given by

$$H = -\frac{1}{2}c_P^2 + \sum_{A=1}^{5} \frac{1}{2} c_{PA}^2. \tag{214}$$

Importantly, this is not positive definite. In (super)gravity on the torus, we will find that we need to impose $H = 0$ as a linearisation stability condition. The negative contributions from $-c_P^2$ to the Hamiltonian will allow for non-trivial solutions to the constraint $H = 0$.

5 Linearization Stability Conditions in Gravity on a 3-Torus

5.1 Linearization Stability Conditions in Classical Gravity

We now wish to exhibit the linearization instability conditions for classical gravity expanded around a flat toroidal background. To this end, suppose that we expand the metric in some small parameter, say α, as

$$g_{\mu\nu} = \eta_{\mu\nu} + h^{(1)}_{\mu\nu} + h^{(2)}_{\mu\nu} + \ldots, \tag{215}$$

such that $h^{(0)} = \eta_{\mu\nu}$ and the scaling in α of each term is given by

$$h^{(i)}_{\mu\nu} \sim \alpha^i. \tag{216}$$

Now consider the exact equations of motion

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} g^{\rho\sigma} R_{\rho\sigma} = 0, \tag{217}$$

and expand these order by order in α. The term first order in α gives the equations of motion for the linearized system, described by $h^{(1)}_{\mu\nu}$. Ordinarily we would expect the term second order in α to provide equations of motion for $h^{(2)}_{\mu\nu}$, which can be solved once $h^{(1)}_{\mu\nu}$ has been determined. We will show that the quadratic expressions for the energy and momentum can be written in terms of integrals of total spatial derivatives of $h^{(2)}_{\mu\nu}$, which necessarily vanish when integrated over the whole torus. Thus the second order theory constrains the linearized theory because only solutions which have vanishing energy and momentum can arise as a linearization of an exact solution to the equations of motion. The discussion of the 2nd order theory here follows Wald [29].

In the expansion of objects like $R_{\mu\nu}$ let $R^{(1)}_{\mu\nu}[h]$ denote the terms which are linear in h and $R^{(2)}_{\mu\nu}[h]$ denote the terms which are quadratic in h. Then the terms of order α in the equations of motion are:

$$R^{(1)}_{\mu\nu}[h^{(1)}] - \frac{1}{2} \eta_{\mu\nu} \eta^{\rho\sigma} R^{(1)}_{\rho\sigma}[h^{(1)}] = 0, \tag{218}$$

by taking the trace with $\eta^{\mu\nu}$ this can be simplified to

$$R^{(1)}_{\mu\nu}[h^{(1)}] = 0. \tag{219}$$

Next, at order α^2, the equations of motion are

$$R^{(1)}_{\mu\nu}[h^{(2)}] - \frac{1}{2} \eta_{\mu\nu} \eta^{\rho\sigma} R^{(1)}_{\rho\sigma}[h^{(2)}] + R^{(2)}_{\mu\nu}[h^{(1)}] - \frac{1}{2} \eta_{\mu\nu} \eta^{\rho\sigma} R^{(2)}_{\rho\sigma}[h^{(1)}] = 0, \tag{220}$$

where we have already used the first order equations of motion $R^{(1)}_{\mu\nu}[h^{(1)}] = 0$.

Now let us write down the expressions for $R^{(1)}_{\mu\nu}[h]$ and $R^{(2)}_{\mu\nu}[h]$. We recall that the lowest contribution to the Christoffel symbol is $\Gamma^{(1)\mu}{}_{\nu\rho}[h]$, so that the expressions for the Riemann tensors are

$$R^{(1)}_{\mu\nu}[h] = \partial_\rho \Gamma^{(1)\rho}{}_{\mu\nu}[h] - \partial_\nu \Gamma^{(1)\rho}{}_{\mu\rho}[h], \tag{221}$$

$$\begin{aligned} R^{(2)}_{\mu\nu}[h] &= \partial_\rho \Gamma^{(2)\rho}{}_{\mu\nu}[h] - \partial_\nu \Gamma^{(2)\rho}{}_{\mu\rho}[h] \\ &\quad + \Gamma^{(1)\tau}{}_{\mu\nu}[h] \Gamma^{(1)\rho}{}_{\tau\rho}[h] - \Gamma^{(1)\tau}{}_{\mu\rho}[h] \Gamma^{(1)\rho}{}_{\nu\tau}[h]. \end{aligned} \tag{222}$$

The expressions for $\Gamma^{(1)\mu}{}_{\nu\rho}[h]$ and $\Gamma^{(2)\mu}{}_{\nu\rho}[h]$ in terms of h are

$$\Gamma^{(1)\mu}{}_{\nu\rho}[h] = \frac{1}{2} \eta^{\mu\sigma} \left(\partial_\nu h_{\rho\sigma} + \partial_\rho h_{\nu\sigma} - \partial_\sigma h_{\nu\rho} \right), \tag{223}$$

$$\Gamma^{(2)\mu}{}_{\nu\rho}[h] = -\frac{1}{2} h^{\mu\sigma} \left(\partial_\nu h_{\rho\sigma} + \partial_\rho h_{\nu\sigma} - \partial_\sigma h_{\nu\rho} \right). \tag{224}$$

We are now finally ready to write down the expression for $R^{(1)}_{\mu\nu}[h]$, which is

$$R^{(1)}_{\mu\nu}[h] = \frac{1}{2}\left[\partial^\rho(\partial_\mu h_{\nu\rho} + \partial_\nu h_{\mu\rho}) - \partial^\rho\partial_\rho h_{\mu\nu} - \partial_\mu\partial_\nu h\right]. \tag{225}$$

In particular $R^{(1)}_{\mu\nu}[h^{(1)}] = 0$ recovers the equation of motion for the linearized equation which we studied previously. Meanwhile, the expression for $R^{(2)}_{\mu\nu}[h]$ is

$$\begin{aligned} R^{(2)}_{\mu\nu}[h] = &\frac{1}{2}h^{\rho\sigma}\partial_\mu\partial_\nu h_{\rho\sigma} - \frac{1}{2}h^{\rho\sigma}\partial_\rho(\partial_\mu h_{\nu\sigma} + \partial_\nu h_{\mu\sigma}) + \frac{1}{4}\partial_\mu h_{\rho\sigma}\partial_\nu h^{\rho\sigma} \\ &+ \frac{1}{2}\partial^\sigma h^\rho{}_\nu \partial_\sigma h_{\rho\mu} - \frac{1}{2}\partial^\sigma h^\rho{}_\nu\partial_\rho h_{\sigma\mu} + \frac{1}{2}\partial_\sigma\left(h^{\sigma\rho}\partial_\rho h_{\mu\nu}\right) \\ &- \frac{1}{4}\partial^\rho h \partial_\rho h_{\mu\nu} - \frac{1}{2}\left(\partial_\mu h_{\nu\rho} + \partial_\nu h_{\mu\rho}\right)\left(\partial_\sigma h^{\rho\sigma} - \frac{1}{2}\partial^\rho h\right). \end{aligned} \tag{226}$$

Now, let us consider

$$\begin{aligned} G^{(1)}_{\mu\nu}[h^{(2)}] &= R^{(1)}_{\mu\nu}[h^{(2)}] - \frac{1}{2}\eta_{\mu\nu}\eta^{\rho\sigma}R^{(1)}_{\rho\sigma}[h^{(2)}] \\ &= \frac{1}{2}\left[\partial^\rho(\partial_\mu h^{(2)}_{\nu\rho} + \partial_\nu h^{(2)}_{\mu\rho}) - \partial^\rho\partial_\rho h^{(2)}_{\mu\nu} - \partial_\mu\partial_\nu h^{(2)}\right] \\ &\quad - \frac{1}{2}\eta_{\mu\nu}\left[\partial^\rho\partial^\sigma h^{(2)}_{\rho\sigma} - \partial^\rho\partial_\rho h^{(2)}\right], \end{aligned} \tag{227}$$

where $h^{(2)} = \eta^{\mu\nu}h^{(2)}_{\mu\nu}$. First we note that

$$\partial^\mu G^{(1)}_{\mu\nu}[h^{(2)}] = 0, \tag{228}$$

a fact which also follows generally from the contracted Bianchi identity, thus

$$t_{\mu\nu} = -\left(R^{(2)}_{\mu\nu}[h^{(1)}] - \frac{1}{2}\eta_{\mu\nu}\eta^{\rho\sigma}R^{(2)}_{\rho\sigma}[h^{(1)}]\right), \tag{229}$$

obeys a conservation equation $\partial^\mu t_{\mu\nu} = 0$, so the charges associated to these conservation laws are constant. We will show that these charges are exactly the total energy H and momentum $\vec{P}$ which we have met before, and these charges actually are not just constant but vanishing. We consider the components $G^{(1)}_{00}[h^{(2)}]$ and $G^{(1)}_{0i}[h^{(2)}]$, which can be written as

$$G^{(1)}_{00}[h^{(2)}] = \frac{1}{2}\left[\partial_j\partial_k h^{(2)}_{jk} - \partial_j\partial_j h^{(2)}_{ii}\right], \tag{230}$$

$$G^{(2)}_{0i}[h^{(2)}] = \frac{1}{2}\left[\partial_j\partial_0 h^{(2)}_{ij} + \partial_j\partial_i h^{(2)}_{0j} - \partial_j\partial_j h^{(2)}_{0i} - \partial_i\partial_0 h^{(2)}_{jj}\right], \tag{231}$$

which we both recognize as total spatial derivatives, so that their integrals over the spatial 3-torus necessarily vanish, that is

$$\int \mathrm{d}^3\vec{x}\; G^{(1)}_{00}[h^{(2)}] = 0 = \int \mathrm{d}^3\vec{x}\; G^{(1)}_{0i}[h^{(2)}]. \tag{232}$$

However, we recall from the α^2 contribution to the Einstein equation that

$$G^{(1)}_{\mu\nu}[h^{(2)}] = -\left(R^{(2)}_{\mu\nu}[h^{(1)}] - \frac{1}{2}\eta_{\mu\nu}\eta^{\rho\sigma}R^{(2)}_{\rho\sigma}[h^{(1)}]\right), \tag{233}$$

thus the vanishing of the integrals of $G^{(1)}_{00}[h^{(2)}]$ and $G^{(1)}_{0i}[h^{(2)}]$ imposes non-trivial quadratic conditions on $h^{(1)}_{\mu\nu}$, which are the linearization stability conditions. One can show that these integrals are suitably gauge-invariant [29] under $h^{(1)}_{\mu\nu} \mapsto h^{(1)}_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu$. Choosing a gauge as previously by imposing

$$h^{(1)}_{0\mu} = 0 = h^{(1)} = \partial_j h^{(1)}_{ij}, \tag{234}$$

for the non-zero momentum modes we find

$$\begin{aligned}
-t^{(\vec{P})}_{00} &= R^{(2)}_{00}[h^{(1)}] - \frac{1}{2}\eta_{00}\eta^{\rho\sigma}R^{(2)}_{\rho\sigma}[h^{(1)}] \\
&= \frac{1}{4}h^{(1)}_{jk}\frac{\partial^2 h^{(1)}_{jk}}{\partial t^2} - \frac{1}{8}\frac{\partial h^{(1)}_{jk}}{\partial t}\frac{\partial h^{(1)}_{jk}}{\partial t} + \frac{1}{4}h^{(1)}_{jk}\partial_i\partial_i h^{(1)}_{jk} \\
&\quad + \frac{3}{8}\partial_i h^{(1)}_{jk}\partial_i h^{(1)}_{jk} - \frac{1}{4}\partial_i h^{(1)}_{jk}\partial_j h^{(1)}_{ik}, \\
&= -\frac{1}{8}\frac{\partial h^{(1)}_{jk}}{\partial t}\frac{\partial h^{(1)}_{jk}}{\partial t} - \frac{1}{8}\vec{\nabla} h^{(1)}_{jk}\cdot\vec{\nabla} h^{(1)}_{jk} \\
&\quad + \frac{1}{2}\partial_i\left(h^{(1)}_{jk}\partial_i h^{(1)}_{jk}\right) - \frac{1}{4}\partial_i\left(h^{(1)}_{jk}\partial_j h^{(1)}_{ik}\right) \\
&= -T^{(\vec{P})}_{00} + \frac{1}{2}\partial_i\left(h^{(1)}_{jk}\partial_i h^{(1)}_{jk}\right) - \frac{1}{4}\partial_i\left(h^{(1)}_{jk}\partial_j h^{(1)}_{ik}\right).
\end{aligned} \tag{235}$$

For the modes with zero-momentum, we can impose

$$h^{(1)}_{\mu 0} = 0, \tag{236}$$

and of course $\partial_k h^{(1)}_{ij} = 0$, so that

$$-t^{(\vec{P}=0)}_{00} = -\frac{1}{8}\frac{\partial h^{(0)}_{ij}}{\partial t}\frac{\partial h^{(0)}_{ij}}{\partial t} + \frac{1}{8}\frac{\partial h^{(0)}_{ij}}{\partial t}\frac{\partial h^{(0)}_{ij}}{\partial t} = -T^{(\vec{P}=0)}_{00}. \tag{237}$$

Then the total energy density $t_{00} = T_{00}$ up to total spatial derivatives. Thus integrating this we find that one of the linearization constraints is that the total energy H has to vanish,

$$H = \int \mathrm{d}^3\vec{x}\ t^{00} = -\frac{1}{2}c_P^2 + \frac{1}{2}\sum_{A=1}^{5} c_{PA}^2 + \sum_{k\neq 0}\sum_{s=\pm}|\vec{k}|a_s^\dagger(\vec{k})a_s(\vec{k}) = 0. \tag{238}$$

Similarly, considering the $0i$ component yields

$$-t_{0i} = R^{(2)}_{0i}[h^{(1)}] = -T_{0i} + \frac{1}{2}\partial_i\left(h^{(1)}_{jk}\frac{\partial h^{(1)}_{jk}}{\partial t}\right) - \frac{1}{2}\partial_j\left(h^{(1)}_{jk}\frac{\partial h^{(1)}_{ik}}{\partial t}\right). \tag{239}$$

Thus also the total momentum $\vec{P}$ has to vanish

$$\vec{P} = \int \mathrm{d}^3\vec{x}\ t^{0i} = \sum_{k\neq 0}\sum_{s=\pm}\vec{k}a_s^\dagger(\vec{k})a_s(\vec{k}) = 0. \tag{240}$$

These are the classical linearization stability conditions. If these conditions are not satisfied, then the corresponding solutions of the linearized equations of motion does not arise from a solution to the full Einstein equations. We next examine how these constraints are implemented in the corresponding quantum theory of linearized gravity.

5.2 Quantum Linearization Stability Conditions

Let us now consider the canonically quantised theory of linearized gravity on a 3-torus spatial background. This quantum theory and the associated linearization stability conditions have been previously studied by Moncrief [13]. We will satisfy the linearization stability conditions and construct a physical Hilbert space of states by group averaging [30].

To obtain the quantised theory, we promote the Fourier coefficients $a_s(\vec{k})$ and $a_s^\dagger(\vec{k})$ to operators which obey the commutation relations

$$[a_s(\vec{k}), a_r^\dagger(\vec{p})]_- = \delta_{sr}\delta_D(\vec{k}-\vec{p}), \tag{241}$$

with all other commutators vanishing. The Hilbert space these operators act on is the Fock space generated from the vacuum state $|0\rangle$ defined by

$$a_s(\vec{k})|0\rangle = 0, \qquad \text{for all } s, \text{and } \vec{k}. \tag{242}$$

A suitable basis for this is provided by $\Big|\{n_s(\vec{k})\}\Big\rangle$, which obeys

$$a_s^\dagger(\vec{k})a_s(\vec{k})\Big|\{n_s(\vec{k})\}\Big\rangle = n_s(\vec{k})\Big|\{n_s(\vec{k})\}\Big\rangle, \tag{243}$$

for all all s and $\vec{k}$. These normalised states can be written

$$\Big|\{n_s(\vec{k})\}\Big\rangle = \prod_{\vec{k},s}\left[\frac{1}{\sqrt{n_s(\vec{k})!}}(a_s^\dagger(\vec{k}))^{n_s(\vec{k})}\right]|0\rangle. \tag{244}$$

The excitations created by $a_s^\dagger(\vec{k})$ will also be called gravitons. We still have to consider the contributions from the zero-modes, whose operators c, c_A and their conjugates c_P and c_{PA} obey the commutation relations

$$[c, c_P]_- = i, \qquad [c_A, c_{PB}]_- = i\delta_{AB}. \tag{245}$$

We can realise these commutation relations acting on normalisable wavefunctions $\Psi(c, c_A)$, with

$$c \mapsto \text{multiply by } c, \qquad\qquad c_P \mapsto -i\frac{\partial}{\partial c}, \tag{246}$$

$$c_A \mapsto \text{multiply by } c_A, \qquad\qquad c_{PA} \mapsto -i\frac{\partial}{\partial c_A}. \tag{247}$$

If $|\Psi_1\rangle$ and $|\Psi_2\rangle$ are represented by $\Psi_1(c, c_a)$ and $\Psi_2(c, c_a)$ respectively, then the inner product between them is given by

$$\langle\Psi_1\,|\,\Psi_2\rangle = \int \mathrm{d}c\mathrm{d}^5c_A\ \Psi_1^*(c, c_A)\Psi_2(c, c_A). \tag{248}$$

The Hilbert space for the gravitons and the zero-modes is then formed by the tensor product of these two spaces. Suppose now that we consider a state proportional to a single basis vector $\left|\{n_s(\vec{k})\}\right\rangle$, that is

$$|\text{state}\rangle = \Psi(c, c_A) \otimes \left|\{n_s(\vec{k})\}\right\rangle. \tag{249}$$

On such a state, accounting for the graviton and zero-momentum modes, the total energy and total momentum operators H and $\vec{P}$ act as

$$\vec{P}|\text{state}\rangle = \sum_{\vec{k}\neq 0}\sum_{s=\pm} \vec{k} n_s(\vec{k})|\text{state}\rangle, \tag{250}$$

$$H|\text{state}\rangle = \frac{1}{2}\left(+\frac{\partial^2}{\partial c^2} - \sum_{A=1}^{5}\frac{\partial^2}{\partial c_A^2} + M^2\right)|\text{state}\rangle, \tag{251}$$

where the squared mass term M^2 is defined as

$$\frac{1}{2}M^2 = \sum_{\vec{k}\neq 0}\sum_{s=\pm} |\vec{k}| n_s(\vec{k}). \tag{252}$$

Note that in defining H, we deal with possible ordering ambiguities when going from the classical theory to quantum theory by normal ordering to tame the infinity occurring from the vacuum energy contributions.

Now we want to incorporate the linearization stability constraints $H = 0$ and $\vec{P} = 0$. The proposal by Moncrief [12, 13] is to impose these as first class constraints on the physical states. Thus a physical state $|\text{phys}\rangle$ is one which obeys

$$\vec{P}|\text{phys}\rangle = 0, \tag{253}$$

$$H|\text{phys}\rangle = 0. \tag{254}$$

We notice that the Hamiltonian constraint then becomes a $(5 + 1)$-dimensional Klein-Gordon equation for the wavefunction $\Psi(c, c_A)$. However, in general solutions to this equation will not be normalisable with respect to the original inner product for $\Psi(c, c_A)$. The approach we will pursue, which was carried out [30], is to first define a set of non-normalisable invariant states which obey the linearization stability constraints by averaging over symmetries generated by H and $\vec{P}$ and then to redefine the inner-product for these states to make it finite. Start with a state $|\text{state}\rangle$ as in (249), then define a physical state $|\text{phys})$ by

$$|\text{phys}) = \frac{1}{2V}\int \mathrm{d}\alpha^0 \int \mathrm{d}^3\vec{\alpha}\ \exp\left[-i\alpha^0 H + i\vec{\alpha}\cdot\vec{P}\right]|\text{state}\rangle. \tag{255}$$

Since the operators H and $\vec{P}$ commute we can separately deal with the integral over the Hamiltonian and the momentum. First we deal with the momentum. As we are dealing with a single eigenstate $\left|\{n_i(\vec{k})\}\right\rangle$ we have

$$\begin{aligned}
&\frac{1}{V}\int \mathrm{d}^3\vec{\alpha}\ \exp\left[i\vec{\alpha}\cdot\vec{P}\right]|\mathrm{state}\rangle \\
&= \frac{1}{V}\int \mathrm{d}^3\vec{\alpha}\exp\left[i\vec{\alpha}\cdot\left(\sum_{k\neq 0}\sum_{s=\pm}\vec{k}n_s(\vec{k})\right)\right]|\mathrm{state}\rangle \\
&= \delta_D\left(\sum_{k\neq 0}\sum_{s=\pm}\vec{k}n_s(\vec{k})\right)|\mathrm{state}\rangle \\
&= \begin{cases} 0 & \text{if } \vec{P}|\mathrm{state}\rangle \neq 0, \\ |\mathrm{state}\rangle & \text{if } \vec{P}|\mathrm{state}\rangle = 0, \end{cases}
\end{aligned} \tag{256}$$

so that this projects onto the zero total momentum sector of the theory. Thus, let us define

$$\left|\{n_s(\vec{k})\}\right) = \begin{cases} 0 & \text{if } \sum_{k\neq 0}\sum_{s=\pm}\vec{k}n_s(\vec{k}) \neq 0, \\ \left|\{n_s(\vec{k})\}\right\rangle & \text{if } \sum_{k\neq 0}\sum_{s=\pm}\vec{k}n_s(\vec{k}) = 0. \end{cases} \tag{257}$$

Then turn to the Hamiltonian integral. For this, if the zero-mode wavefunction is given by $\Psi(c, c_A)$, let us Fourier transform to write

$$\Psi(c, c_A) = \int \frac{\mathrm{d}p^0}{2\pi}\int \frac{\mathrm{d}^5\vec{p}}{(2\pi)^5}\ F(p^0,\vec{p})e^{-ip^0c+i\vec{p}\cdot\vec{c}}. \tag{258}$$

Then we find that

$$\begin{aligned}
&\frac{1}{2}\int \mathrm{d}\alpha^0\ \exp\left[-i\alpha^0 H\right]|\mathrm{state}\rangle \\
&= \frac{1}{2}\int \mathrm{d}\alpha^0\ \exp\left[-\frac{1}{2}i\alpha^0\left(+\frac{\partial^2}{\partial c^2} - \sum_{A=1}^{5}\frac{\partial^2}{\partial c_A^2} + M^2\right)\right] \\
&\quad\times \int \frac{\mathrm{d}p^0}{2\pi}\int \frac{\mathrm{d}^5\vec{p}}{(2\pi)^5}\ F(p^0,\vec{p})e^{-ip^0c+i\vec{p}\cdot\vec{c}} \otimes \left|\{n_s(\vec{k})\}\right\rangle \\
&= \frac{1}{2}\int \frac{\mathrm{d}p^0}{2\pi}\int \frac{\mathrm{d}^5\vec{p}}{(2\pi)^5}\int \mathrm{d}\alpha^0\ \exp\left[-\frac{1}{2}i\alpha^0\left(-(p^0)^2 + \vec{p}^2 + M^2\right)\right] \\
&\quad\times F(p^0,\vec{p})e^{-ip^0c+i\vec{p}\cdot\vec{c}} \otimes \left|\{n_s(\vec{k})\}\right\rangle \\
&= \int \mathrm{d}p^0 \int \frac{\mathrm{d}^5\vec{p}}{(2\pi)^5}\delta\left((p^0)^2 - \vec{p}^2 - M^2\right)F(p^0,\vec{p})e^{-ip^0c+i\vec{p}\cdot\vec{c}} \otimes \left|\{n_s(\vec{k})\}\right\rangle.
\end{aligned} \tag{259}$$

Now recall that if $f(x)$ has simple zeroes at at $x = x_*$, then

$$\delta(f(x)) = \sum_{x_*}\frac{1}{|f'(x_*)|}\delta(x - x_*). \tag{260}$$

So let $E(\vec{p}) = +\sqrt{\vec{p}^2 + M^2}$, then

$$\begin{aligned}
&\int \mathrm{d}p^0 \int \frac{\mathrm{d}^5\vec{p}}{(2\pi)^5}\delta\left((p^0)^2 - \vec{p}^2 - M^2\right)F(p^0,\vec{p})e^{-ip^0c+i\vec{p}\cdot\vec{c}} \otimes \left|\{n_s(\vec{k})\}\right\rangle \\
&= \int \frac{\mathrm{d}^5\vec{p}}{(2\pi)^5}\left(f^{(+)}(\vec{p})e^{-iE(\vec{p})c+i\vec{p}\cdot\vec{c}} + f^{(-)}(\vec{p})e^{iE(\vec{p})c+i\vec{p}\cdot\vec{c}}\right) \otimes \left|\{n_s(\vec{k})\}\right\rangle,
\end{aligned} \tag{261}$$

where we introduced the functions $f^{(\pm)}$ defined by

$$f^{(\pm)} = \frac{F(\pm E(\vec{p}), \vec{p})}{2E(\vec{p})}. \tag{262}$$

Putting everything together, we find that group-averaging gives us physical states of the form

$$|\text{phys}) = \int \frac{\mathrm{d}^5\vec{p}}{(2\pi)^5} \left(f^{(+)}(\vec{p}) e^{-iE(\vec{p})c + i\vec{p}\cdot\vec{c}} + f^{(-)}(\vec{p}) e^{iE(\vec{p})c + i\vec{p}\cdot\vec{c}} \right) \otimes \left| \{n_s(\vec{k})\} \right), \tag{263}$$

and in fact we can note that this is a general solution for states obeying the linearization stability conditions with "mass-squared" M^2.

Finally we need to calculate the inner product defined by the group averaging prescription. Let us take two physical states $|\text{phys1})$ and $|\text{phys2})$, which can be obtained from group averaging from

$$\begin{aligned} |\text{state1}\rangle &= \Psi_1(c, \vec{c}) \otimes \left| \{n_s(\vec{k})\} \right\rangle, \\ |\text{state2}\rangle &= \Psi_2(c, \vec{c}) \otimes \left| \{m_s(\vec{k})\} \right\rangle. \end{aligned} \tag{264}$$

Then the inner product between $|\text{phys1})$ and $|\text{phys2})$ is defined by

$$(\text{phys1}\,|\,\text{phys2}) = \langle \text{state1}| \frac{1}{2V} \int \mathrm{d}\alpha^0 \int \mathrm{d}^3\vec{\alpha}\ \exp\left[-i\alpha^0 H + i\vec{\alpha}\cdot\vec{P}\right] |\text{state2}\rangle. \tag{265}$$

Inserting the Fourier transforms of Ψ_1 and Ψ_2, this becomes

$$\begin{aligned} (\text{phys1}\,|\,\text{phys2}) = \int \frac{\mathrm{d}^5\vec{p}}{(2\pi)^5} 2E(\vec{p}) &\left[f_1^{(+)*}(\vec{p}) f_2^{(+)}(\vec{p}) + f_1^{(-)*}(\vec{p}) f_2^{(-)}(\vec{p}) \right] \\ &\times \left(\{n_s(\vec{k})\} \,|\, \{m_s(\vec{k})\} \right). \end{aligned} \tag{266}$$

6 Linearization Stability Conditions in Supergravity on a 3-Torus

Finally we wish to discuss supergravity. This should be thought of as the theory of local supersymmetry. We have seen in the Wess-Zumino model that generally we expect the anti-commutator of two supersymmetry transformations to give a space-time translation, therefore if we have local supersymmetry transformations we should in general also have local space-time translations. This necessitates a coupling to gravity, described by a spin-2 boson. The superpartner to the spin-2 particle needs to be a fermion, differing in spin by a half-integer, so the simplest candidate is the spin-$\frac{3}{2}$ field. Indeed, the simplest theory with local supersymmetry that we can consider is in 4-spacetime dimensions. This theory has a single supersymmetry, which couples the frame field e_a^μ to the field Ψ_μ. This theory is called $4D$, $\mathcal{N} = 1$ supergravity [28]. A suitable action for this theory is

$$S[e, \Psi] = \int \mathrm{d}^4x\ e \left[\frac{1}{2} R_{\mu\nu}{}^{ab} e_a^\mu e_b^\nu - \frac{1}{2} \bar{\Psi}_\mu \gamma^{\mu\nu\rho} D_\nu \Psi_\rho + X_{(4\Psi)} \right], \tag{267}$$

where e^a_μ is the frame field, which determines the spacetime metric $g_{\mu\nu}$ through

$$g_{\mu\nu} = e^a_\mu \eta_{ab} e^b_\nu, \tag{268}$$

and we define $e = \det(e^a_\mu) = \sqrt{-g}$. The term $X_{(4\Psi)}$ is quartic in Ψ and is required to have local supersymmetry (see [26] for the explicit form of $X_{(4\Psi)}$). The numerical γ-matrices are γ^a, and we define $\gamma^\mu = e^\mu_a \gamma^a$. The local Lorentz covariant derivative D_μ of the field Ψ_μ is given by

$$D_\mu \Psi_\nu = \partial_\mu \Psi_\nu + \frac{1}{4}\omega_{\mu ab}\gamma^{ab}\Psi_\nu. \tag{269}$$

We refer to the appendix E for further details of the frame field e^μ_a and spin-connection $\omega_{\mu ab}$.

6.1 Linearized Supergravity

At the linearized level, the $4D$, $\mathcal{N} = 1$ supergravity theory we consider is described by a free massless spin-2 field, the graviton, and a free massless spin-$\frac{3}{2}$ Majorana spinor field, the gravitino [26]. Thus the action is

$$\begin{aligned} S[h,\Psi] = \int \mathrm{d}^4x \, \Big[&\frac{1}{4}\partial^\rho h_{\mu\rho}\partial_\sigma h^{\mu\sigma} - \frac{1}{8}\partial_\rho h_{\mu\nu}\partial^\rho h^{\mu\nu} - \frac{1}{4}\partial^\rho h_{\rho\sigma}\partial^\sigma h \\ &+ \frac{1}{8}\partial_\rho h \partial^\rho h - \frac{1}{2}\bar{\Psi}_\mu \gamma^{\mu\nu\rho}\partial_\nu \Psi_\rho\Big]. \end{aligned} \tag{270}$$

At the linearized level, this is invariant under the global supersymmetry transformation

$$h_{\mu\nu} \mapsto h'_{\mu\nu} = h_{\mu\nu} + \frac{1}{2}\left(\bar{\varepsilon}\gamma_\mu \Psi_\nu + \bar{\varepsilon}\gamma_\nu \Psi_\mu\right), \tag{271}$$

$$\Psi_\mu \mapsto \Psi'_\mu = \Psi_\mu + \frac{1}{4}\partial_\rho h_{\mu\nu}\gamma^{\nu\rho}\varepsilon, \tag{272}$$

where ε is an arbitrary Majorana spinor field. To see that indeed the action is invariant under this transformation, if we vary only the gravitino part, we find

$$\begin{aligned} \delta\mathcal{L}_{3/2} = \delta\left(-\frac{1}{2}\bar{\Psi}_\mu\gamma^{\mu\nu\rho}\partial_\nu\Psi_\rho\right) &= -\frac{1}{2}\bar{\Psi}_\mu\gamma^{\mu\nu\rho}\partial_\nu\delta\Psi_\rho \\ &= -\frac{1}{2}\bar{\Psi}_\mu\gamma^{\mu\nu\rho}\partial_\nu\partial_\tau h_{\rho\sigma}\gamma^{\sigma\tau}\varepsilon, \end{aligned} \tag{273}$$

where in the first line we have ignored a total derivative term, which will not contribute to the variation of the action. To make progress, we recall the following identity obeyed by the γ-matrices in four space-time dimensions [28]

$$\begin{aligned} \gamma^{\mu\nu\rho}\gamma^{\sigma\tau} = \gamma^{\mu\nu\tau}\eta^{\rho\sigma} &+ \gamma^{\rho\mu\tau}\eta^{\nu\sigma} + \gamma^{\nu\rho\tau}\eta^{\mu\sigma} - \gamma^{\mu\nu\sigma}\eta^{\rho\tau} - \gamma^{\rho\mu\sigma}\eta^{\nu\tau} \\ &- \gamma^{\nu\rho\sigma}\eta^{\mu\tau} + \gamma^\mu(\eta^{\nu\tau}\eta^{\rho\sigma} - \eta^{\rho\tau}\eta^{\nu\sigma}) \\ &+ \gamma^\nu(\eta^{\rho\tau}\eta^{\mu\sigma} - \eta^{\rho\sigma}\eta^{\mu\tau}) + \gamma^\rho(\eta^{\mu\tau}\eta^{\nu\sigma} - \eta^{\mu\sigma}\eta^{\nu\tau}). \end{aligned} \tag{274}$$

Contracting this with $\bar{\Psi}_\mu \partial_\nu \partial_\tau h_{\rho\sigma}$ one finds only the terms proportional to a single γ^λ contribute, so that we find

$$\begin{aligned}\delta\mathcal{L}_{3/2} = &-\frac{1}{4}\bar{\Psi}_\mu\gamma^\mu(\partial^\rho\partial_\rho h - \partial^\rho\partial^\sigma h_{\rho\sigma})\varepsilon - \frac{1}{4}\bar{\Psi}_\mu\gamma^\nu(\partial_\nu\partial_\rho h^{\rho\mu} - \partial_\nu\partial^\mu h)\varepsilon \\ &- \frac{1}{4}\bar{\Psi}_\mu\gamma^\rho(\partial^\mu\partial^\sigma h_{\rho\sigma} - \partial^\sigma\partial_\sigma h_\rho{}^\mu)\varepsilon.\end{aligned} \tag{275}$$

Now, we note that

$$\bar{\Psi}_\mu\gamma^\rho\epsilon = -\bar{\varepsilon}\gamma^\rho\Psi_\mu, \tag{276}$$

to write

$$\begin{aligned}\delta\mathcal{L}_{3/2} = &\frac{1}{4}\bar{\varepsilon}\gamma^\mu(\partial^\rho\partial_\rho h - \partial^\rho\partial^\sigma h_{\rho\sigma})\Psi_\mu + \frac{1}{4}\bar{\varepsilon}\gamma^\nu(\partial_\nu\partial_\rho h^{\rho\mu} - \partial_\nu\partial^\mu h)\Psi_\mu \\ &+ \frac{1}{4}\bar{\varepsilon}\gamma^\rho(\partial^\mu\partial^\sigma h_{\rho\sigma} - \partial^\sigma\partial_\sigma h_\rho{}^\mu)\Psi_\mu.\end{aligned} \tag{277}$$

Next, let us vary the part involving the graviton, where we find

$$\begin{aligned}\delta\mathcal{L}_2 &= \delta\left(\frac{1}{4}\partial^\rho h_{\mu\rho}\partial_\sigma h^{\mu\sigma} - \frac{1}{8}\partial_\rho h_{\mu\nu}\partial^\rho h^{\mu\nu} - \frac{1}{4}\partial^\rho h_{\rho\sigma}\partial^\sigma h + \frac{1}{8}\partial_\rho h\partial^\rho h\right) \\ &= \frac{1}{2}\partial^\rho\delta h_{\mu\rho}\partial_\sigma h^{\mu\sigma} - \frac{1}{4}\partial_\rho\delta h_{\mu\nu}\partial^\rho h^{\mu\nu} \\ &\quad - \frac{1}{4}\partial^\rho\delta h_{\rho\sigma}\partial^\sigma h - \frac{1}{4}\partial^\rho h_{\rho\sigma}\partial^\sigma\delta h + \frac{1}{4}\partial_\rho\delta h\partial^\rho h.\end{aligned} \tag{278}$$

Integrating by parts to move all the derivatives off the variations, and ignoring the boundary term, yield

$$\begin{aligned}\delta\mathcal{L}_2 &= -\frac{1}{2}\delta h_{\mu\rho}\partial^\rho\partial_\sigma h^{\mu\sigma} + \frac{1}{4}\delta h_{\mu\nu}\partial_\rho\partial^\rho h^{\mu\nu} \\ &\quad + \frac{1}{4}\delta h_{\rho\sigma}\partial^\rho\partial^\sigma h + \frac{1}{4}\delta h\partial^\rho\partial^\sigma h_{\rho\sigma} - \frac{1}{4}\delta h\partial_\rho\partial^\rho h \\ &= -\frac{1}{4}\bar{\varepsilon}(\gamma_\mu\Psi_\rho + \gamma_\rho\Psi_\mu)\partial^\rho\partial_\sigma h^{\mu\sigma} + \frac{1}{4}\bar{\varepsilon}\gamma_\nu\Psi_\mu\partial^\rho\partial_\rho h^{\mu\nu} + \frac{1}{4}\bar{\varepsilon}\gamma_\rho\Psi_\sigma\partial^\rho\partial^\sigma h \\ &\quad + \frac{1}{4}\bar{\varepsilon}\gamma^\mu\Psi_\mu\partial^\rho\partial^\sigma h_{\rho\sigma} - \frac{1}{4}\bar{\varepsilon}\gamma^\mu\Psi_\mu\partial_\rho\partial^\rho h \\ &= -\frac{1}{4}\bar{\varepsilon}\gamma^\mu(\partial_\rho\partial^\rho h - \partial^\rho\partial^\sigma h_{\rho\sigma})\Psi_\mu - \frac{1}{4}\bar{\varepsilon}\gamma^\nu(\partial_\nu\partial_\sigma h^{\mu\sigma} - \partial_\nu\partial_\mu h)\Psi_\mu \\ &\quad - \frac{1}{4}\bar{\varepsilon}\gamma^\rho(\partial^\mu\partial^\sigma h_{\rho\sigma} - \partial_\sigma\partial^\sigma h_\rho{}^\mu)\Psi_\mu.\end{aligned} \tag{279}$$

In particular, it follows that

$$\delta S = \int \mathrm{d}^4x\ \left(\delta\mathcal{L}_2 + \delta\mathcal{L}_{3/2}\right) = 0, \tag{280}$$

so that indeed the linearized supergravity action is invariant under the global supersymmetry transformation. Next let us calculate the conserved supercurrent $\mathcal{J}^\mu$ associated with this global transformation. For this we write

$$\delta_\varepsilon h_{\mu\nu}\frac{\delta S}{\delta h_{\mu\nu}} + \delta_\varepsilon\Psi_\mu^T\frac{\delta S}{\delta\Psi_\mu} = \partial_\mu(\bar{\varepsilon}\mathcal{J}^\mu). \tag{281}$$

The graviton contribution to (281) expression is given by

$$\begin{aligned}\delta_\varepsilon h_{\mu\nu}\frac{\delta S}{\delta h_{\mu\nu}} &= \frac{1}{4}\bar{\varepsilon}\gamma_\mu\Psi_\nu\left(\partial^\rho\partial_\rho h^{\mu\nu} - \partial_\rho(\partial^\mu h^{\nu\rho} + \partial^\nu h^{\mu\rho}) + \partial^\mu\partial^\nu h\right) \\ &\quad + \frac{1}{4}\bar{\varepsilon}\gamma^\mu\Psi_\mu\left(\partial^\rho\partial^\sigma h_{\rho\sigma} - \partial^\rho\partial_\rho h\right).\end{aligned} \tag{282}$$

Next the gravitino contribution to the expression for the divergence of the supercurrent $\mathcal{J}^\mu$

$$\begin{aligned}\delta_\varepsilon\Psi_\mu^T\frac{\delta S}{\delta\Psi_\mu} &= -\frac{1}{4}\partial_\tau h_{\mu\sigma}(\gamma^{\sigma\tau}\varepsilon)^T C\gamma^{\mu\nu\rho}\partial_\nu\Psi_\rho \\ &= \partial_\mu\left[-\frac{1}{4}\partial_\tau h_{\rho\sigma}(\gamma^{\sigma\tau}\varepsilon)^T C\gamma^{\rho\mu\nu}\Psi_\nu\right] \\ &\quad + \frac{1}{4}\partial_\nu\partial_\tau h_{\mu\sigma}(\gamma^{\sigma\tau}\varepsilon)^T C\gamma^{\mu\nu\rho}\Psi_\rho.\end{aligned} \tag{283}$$

Now, we use that

$$\begin{aligned}(\gamma^{\sigma\tau}\varepsilon)^T C\gamma^{\rho\mu\nu}\Psi_\nu &= \gamma^{\sigma\tau}_{\alpha\beta}\varepsilon_\beta C_{\alpha\gamma}\gamma^{\rho\mu\nu}_{\gamma\delta}\Psi_{\nu\delta} \\ &= -C_{\gamma\alpha}\gamma^{\sigma\tau}_{\alpha\beta}\varepsilon_\beta\gamma^{\rho\mu\nu}_{\gamma\delta}\Psi_{\nu\delta} \\ &= -\gamma^{\tau\sigma}_{\alpha\gamma}C_{\alpha\beta}\varepsilon_\beta\gamma^{\rho\mu\nu}_{\gamma\delta}\Psi_{\nu\delta} \\ &= -\varepsilon_\beta C_{\beta\alpha}\gamma^{\sigma\tau}_{\alpha\gamma}\gamma^{\rho\mu\nu}_{\gamma\delta}\Psi_{\nu\delta} \\ &= -\bar{\varepsilon}\gamma^{\sigma\tau}\gamma^{\rho\mu\nu}\Psi_\nu.\end{aligned} \tag{284}$$

This allows us to then write

$$\begin{aligned}\delta_\varepsilon\Psi_\mu^T\frac{\delta S}{\delta\Psi_\mu} &= \partial_\mu\left[+\frac{1}{4}\partial_\tau h_{\rho\sigma}\bar{\varepsilon}\gamma^{\sigma\tau}\gamma^{\rho\mu\nu}\Psi_\nu\right] \\ &\quad - \frac{1}{4}\partial_\nu\partial_\tau h_{\mu\sigma}\bar{\varepsilon}\gamma^{\sigma\tau}\gamma^{\mu\nu\rho}\Psi_\rho\end{aligned} \tag{285}$$

Using a transposed version of the previous identity for $\gamma^{\mu\nu\rho}\gamma^{\sigma\tau}$, the second term above can be rewritten as

$$\begin{aligned}-\frac{1}{4}\partial_\nu\partial_\tau h_{\mu\sigma}\bar{\varepsilon}\gamma^{\sigma\tau}\gamma^{\mu\nu\rho}\Psi_\rho &= -\frac{1}{4}\bar{\varepsilon}\gamma^\mu\Psi_\mu\left(\partial^\rho\partial^\sigma h_{\rho\sigma} - \partial^\rho\partial_\rho h\right) \\ &\quad - \frac{1}{4}\bar{\varepsilon}\gamma_\mu\Psi_\nu\left(\partial^\rho\partial_\rho h^{\mu\nu} - \partial_\rho(\partial^\mu h^{\nu\rho} + \partial^\nu h^{\mu\rho}) + \partial^\mu\partial^\nu h\right) \\ &= -\delta_\varepsilon h_{\mu\nu}\frac{\delta S}{\delta h_{\mu\nu}}.\end{aligned} \tag{286}$$

Thus combining everything, we find that the expression for the conserved supercurrent $\mathcal{J}^\mu$ is

$$\mathcal{J}^\mu = \frac{1}{4}\partial_\tau h_{\rho\sigma}\gamma^{\tau\sigma}\gamma^{\mu\rho\nu}\Psi_\nu. \tag{287}$$

The associated conserved supercharge Q is then expressed as

$$Q = \int \mathrm{d}^3\vec{x}\ \mathcal{J}^0 = \int \mathrm{d}^3\vec{x}\ \frac{1}{4}\partial_\tau h_{i\sigma}\gamma^{\tau\sigma}\gamma^{0ij}\Psi_j \tag{288}$$

Let us now calculate the contributions to the supercharge from the zero- and non-zero-momentum sectors of the theory. Note that there are no cross term contributions to supercharge.

In the zero momentum sector, recall that we have $h_{\mu 0} = 0 = \Psi_0$, so that

$$\begin{aligned} Q^{(0)} &= \int \mathrm{d}^3\vec{x}\ \frac{1}{4}\frac{dh_{ik}}{dt}\gamma^{0k}\gamma^{0ij}\Psi_j \\ &= \int \mathrm{d}^3\vec{x}\ \left[\frac{1}{4}\frac{dh}{dt}\gamma^i\Psi_i - \frac{1}{4}\frac{dh_{ij}}{dt}\gamma^i\Psi_j\right], \end{aligned} \tag{289}$$

where we used the γ-matrix identities

$$\gamma^{0ij} = \gamma^0\gamma^{ij}, \tag{290}$$

$$\gamma^k\gamma^{ij} = \gamma^{kij} + \gamma^j\delta^{ik} - \gamma^i\delta^{jk}, \tag{291}$$

the second identity is proved in Appendix D. Inserting the expressions for h_{ij} and Ψ_j, which in the zero-mode sectors are

$$\sqrt{V}h_{ij} = \sqrt{\frac{2}{3}}\delta_{ij}c + 2\sum_{A=1}^{5} T_{ij}^A c_A, \tag{292}$$

$$\sqrt{V}\Psi_i = \frac{1}{\sqrt{6}}\gamma_i\eta + \sum_{A=1}^{2} T_{ij}^A\gamma^j\eta_A, \tag{293}$$

yields

$$Q^{(0)} = -\frac{1}{2}\left(c_P\eta + \sum_{A=1}^{5}\sum_{B=1}^{2} T_{ij}^A\gamma^j T_{ik}^B\gamma^k c_{PA}\eta^B\right), \tag{294}$$

where we recall $c_P = -\partial_0 c$ and $c_{PA} = \partial_0 c_A$.

For the non-zero-momentum modes, we recall the gauge-fixing conditions

$$h = h_{0\mu} = \partial^i h_{ij} = 0, \qquad \gamma^i\Psi_i = \partial^i\Psi_i = \Psi_0 = 0. \tag{295}$$

Then the expression for the contributions $\hat{Q}$ to the total supercharge becomes

$$\hat{Q} = \int \mathrm{d}^3\vec{x}\ \frac{1}{4}\left[\partial_0 h_{ik}\gamma^{0k}\gamma^{0ij}\Psi_j + \partial_l h_{ik}\gamma^{lk}\gamma^{0ij}\Psi_j\right], \tag{296}$$

this can be simplified a little if we note that in 4-dimensions

$$\gamma^{lk}\gamma^{ij} = \gamma^{lj}\delta^{ik} - \gamma^{kj}\delta^{li} - \gamma^{li}\delta^{jk} + \gamma^{ki}\delta^{lj} + \delta^{lj}\delta^{ik} - \delta^{kj}\delta^{il}, \tag{297}$$

which is proved in Appendix D. Making also a partial integration and using $\gamma^\mu\partial_\mu\Psi_j = 0$,

$$\hat{Q} = \int \mathrm{d}^3\vec{x}\ \frac{1}{4}\left[-\frac{\partial h_{ij}}{\partial t}\gamma^i\Psi_j + h_{ij}\gamma^i\frac{\partial\Psi_j}{\partial t}\right]. \tag{298}$$

Inserting the mode expansions for h_{ij} and Ψ_j, this can be written as

$$\hat{Q} = \frac{i}{2}\sum_{\vec{k}\neq 0}\sum_{s=\pm}\left[\vec{\epsilon}^{\,s}(\vec{k})\cdot\vec{\gamma}u^{s*}(\vec{k})a_s(\vec{k})b_s^\dagger(\vec{k}) - \vec{\epsilon}^{\,s*}(\vec{k})\cdot\vec{\gamma}u^s(\vec{k})a_s^\dagger(\vec{k})b_s(\vec{k})\right]. \tag{299}$$

The total supercharge for the system is then

$$Q = Q^{(0)} + \hat{Q}. \tag{300}$$

The system also has conserved charges arising from spacetime translations, as the linearized theory does not provide interaction terms between the graviton and gravitino, these are simply found by adding the energy and momentum of both fields. Thus we have

$$H = -\frac{1}{2}c_P^2 + \sum_{A=1}^{5} \frac{1}{2}c_{PA}^2 + \sum_{k\neq 0}\sum_{s=\pm} |\vec{k}| \left[a_s^\dagger(\vec{k})a_s(\vec{k}) + b_s^\dagger(\vec{k})b_s(\vec{k}) \right], \tag{301}$$

$$\vec{P} = \sum_{k\neq 0}\sum_{s=\pm} \vec{k} \left[a_s^\dagger(\vec{k})a_s(\vec{k}) + b_s^\dagger(\vec{k})b_s(\vec{k}) \right]. \tag{302}$$

These operators yield the supersymmetry algebra

$$\{Q, P^\mu\} = 0, \tag{303}$$

$$\{Q_\alpha, Q_\beta\} = -\frac{i}{2}(\gamma_\mu \gamma^0)P^\mu, \tag{304}$$

where $P^\mu = (H, \vec{P})$.

In the next section we show that for the supergravitational theory we need to impose $H = \vec{P} = Q = 0$ as linearization stability conditions.

6.2 Linearization Stability Conditions

In this section, we want to show how $Q = 0$ arises as a linearization stability condition when the second order contributions are taken into account. Similar to the gravitational theory, the vanishing of the total energy H and total momentum $\vec{P}$ must also be imposed on the solutions.

The equation of motion for the gravitino including first and second order corrections is [28]

$$\gamma^{\mu\nu\rho} D_\nu \Psi_\rho = 0, \tag{305}$$

where the derivative $D_\nu \Psi_\rho$ is defined by

$$D_\nu \Psi_\rho = \partial_\nu \Psi_\rho + \frac{1}{4}\omega_{\mu ab}\gamma^{ab}\Psi_\rho. \tag{306}$$

Now expand in powers of some small parameter as before, with

$$\Psi = \Psi^{(1)} + \Psi^{(2)} + \ldots, \tag{307}$$

$$\gamma^\mu = \gamma^{(0)\mu} + \gamma^{(1)\mu} + \ldots, \tag{308}$$

$$e_a^\mu = e_a^{\mu(0)} + e_a^{\mu(1)} + \ldots, \tag{309}$$

$$\omega_{\mu ab} = \omega_{\mu ab}^{(1)} + \ldots. \tag{310}$$

Expanding the gravitino equation of motion order-by-order, at first order the field equation is

$$\gamma^{(0)\mu\nu\rho}\partial_\nu \Psi_\rho^{(1)} = 0. \tag{311}$$

Expanding the $\mu = 0$ component of the field equation at second order

$$\gamma^{(0)0ij}\partial_i \Psi_j^{(2)} + \gamma^{(1)0ij}\partial_i \Psi_j^{(1)} + \frac{1}{4}\gamma^{(0)0ij}\omega_{iab}\gamma^{ab}\Psi_j^{(1)} = 0. \tag{312}$$

This can be re-written as

$$\partial_i \left(\gamma^{(0)0ij}\Psi_j^{(2)} + \gamma^{(1)0ij}\Psi_j^{(1)}\right) + \frac{1}{4}\gamma^{(0)0ij}\omega_{iab}^{(1)}\gamma^{ab}\Psi_j^{(1)} - \left(\partial_i\gamma^{(1)0ij}\right)\Psi_j^{(1)} = 0. \tag{313}$$

In particular, we note that the second-order perturbation of the field $\Psi_j^{(2)}$ appears only as a spatial divergence. Thus, if we integrate over the whole 3-torus this will vanish. Thus we get a quadratic constraint on the first-order perturbations in the theory, which is a linearization stability condition. We now want to show that this condition corresponds exactly to the vanishing of the supercharge Q.

The space-time γ-matrices are covariantly conserved,

$$\nabla_\mu \gamma^\nu = 0, \tag{314}$$

see (493) in appendix E. It follows from this that

$$\begin{aligned}\partial_i\gamma^{(1)0ij} &= -\frac{1}{4}\omega_{iab}^{(1)}\left[\gamma^{ab}, \gamma^{(0)0ij}\right] \\ &\quad - \Gamma^{(1)0}{}_{i\rho}\gamma^{(0)\rho ij} - \Gamma^{(1)i}{}_{ik}\gamma^{(0)0kj} - \Gamma^{(1)j}{}_{ik}\gamma^{(0)0ik} \\ &= -\frac{1}{4}\omega_{iab}^{(1)}\left[\gamma^{ab}, \gamma^{(0)0ij}\right] \\ &\quad - \Gamma^{(1)\mu}{}_{\mu i}\gamma^{(0)0ij},\end{aligned} \tag{315}$$

where the symmetry of $\Gamma^{(1)\rho}{}_{ik}$ and anti-symmetry of $\gamma^{(0)ijk}$ was used to eliminate terms. The first-order Christoffel symbols we require is

$$\Gamma^{(1)\mu}{}_{\mu i} = \frac{1}{2}\partial_i h^{(1)}, \tag{316}$$

Integrating by parts and using the first-order equation of motion then yields

$$\partial_i\gamma^{(1)0ij}\Psi_j^{(1)} = -\frac{1}{4}\omega_{iab}^{(1)}\left[\gamma^{ab}, \gamma^{(0)0ij}\right]\Psi_j^{(1)} - \partial_i\left[h^{(1)}\gamma^{(0)0ij}\Psi_j\right]. \tag{317}$$

We thus find that the following integral vanishes

$$\int \mathrm{d}^3\vec{x}\, \left[\frac{1}{4}\omega_{iab}^{(1)}\gamma^{ab}\gamma^{(0)0ij}\Psi_j^{(1)}\right] = 0. \tag{318}$$

The frame fields are only defined up to local Lorentz transformations. We can use this freedom to choose $e_a^{(0)\mu} = \delta_a^\mu$. We can go further and choose $e_a^{(1)\mu}$ so that the linearized spin-connection is given by

$$\omega_{iab}^{(1)} = \frac{1}{2}\left(\partial_\nu h_{\sigma i}^{(1)} - \partial_\sigma h_{\nu i}^{(1)}\right)\delta_a^\sigma\delta_b^\nu. \tag{319}$$

To see this, we make us of (490) from Appendix E,

$$\partial_\mu e_\nu^a + \omega_\mu{}^a{}_b e_\nu^b - \Gamma^\rho{}_{\mu\nu}e_\rho^a = 0. \tag{320}$$

Rearranging for the spin-connection and making the antisymmetry in a and b explicit yields

$$\omega_{\mu ab} = \frac{1}{2}\Gamma^{\rho}{}_{\mu\nu}(e_{a\rho}e_b^{\nu} - e_{b\rho}e_a^{\nu}) - \frac{1}{2}\left(e_b^{\nu}\partial_{\mu}e_{a\nu} - e_a^{\nu}\partial_{\mu}e_{b\nu}\right). \tag{321}$$

Then at the linear level we find

$$\omega_{iab}^{(1)} = \frac{1}{2}\left(\partial_{\nu}h_{\sigma i}^{(1)} - \partial_{\sigma}h_{\nu i}^{(1)}\right)\delta_a^{\sigma}\delta_b^{\nu} - \frac{1}{2}\left[\delta_b^{\nu}\partial_i e_{a\nu}^{(1)} - \delta_a^{\nu}\partial_i e_{b\nu}^{(1)}\right]. \tag{322}$$

The term inside the square brackets involves only the antisymmetric components of $e_{ab}^{(1)}$. However, these are precisely the terms which can be modified using an infinitesimal local Lorentz transformation, under which

$$e_{a\nu}^{(1)} \mapsto e_{a\nu}^{(1)} + \lambda_{ab}(x)\delta_{\nu}^{b}, \tag{323}$$

where $\lambda_{ab} = -\lambda_{ba}$. It follows that we can choose the second term to vanish by an appropriate choice of λ_{ab}.

Inserting (319) into the linearization stability condition (318), we find that

$$Q = \int d^3\vec{x}\ \left[\partial_{\tau}h_{i\sigma}^{(1)}\gamma^{(0)\tau\sigma}\gamma^{(0)0ij}\Psi_j^{(1)}\right] = 0. \tag{324}$$

Thus, the vanishing of the supercharge appears as a linearization stability condition of linearized supergravity on a toroidal background.

The total energy H and total momentum $\vec{P}$ of the graviton and gravitino system also has to vanish. This can be noted by examining the graviton equation of motion at second order. We will only sketch the argument, since it is very similar to the general relativity case. The equation of motion of the frame field is determined through

$$\frac{\delta S}{\delta e_{\mu}^{a}}e_{\nu}^{a} = 0. \tag{325}$$

If $S_2[e]$ contains the Riemann term, that is

$$S_2[e] = \int d^4x\ e\left[\frac{1}{2}R_{\mu\nu}{}^{ab}e_a^{\mu}e_b^{\nu}\right], \tag{326}$$

then

$$\frac{1}{e}\frac{\delta S_s}{\delta e_a^{\mu}}e_{a\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R, \tag{327}$$

so the equation of motion can be written

$$R_{\mu\nu}[g] - \frac{1}{2}g_{\mu\nu}R[g] = T_{\mu\nu}^{\Psi}[e,\Psi], \tag{328}$$

where we have defined the tensor $T_{\mu\nu}^{\Psi}$ by

$$T_{\mu\nu}^{\Psi} = -\frac{1}{e}\frac{\delta S_{3/2}}{\delta e_a^{\mu}}e_{a\nu}, \tag{329}$$

and here

$$S_{3/2}[e,\Psi] = \int d^4x\ e\left[-\frac{1}{2}\bar{\Psi}_{\mu}\gamma^{\mu\nu\rho}D_{\nu}\Psi_{\rho} + X_{(4\Psi)}\right]. \tag{330}$$

Notice that $T^{\Psi}_{\mu\nu}$ is at least quadratic in Ψ, so if we expand the equation of motion with

$$\begin{aligned} e^a_\mu &= \delta^a_\mu + e^{(1)a}_\mu + e^{(2)a}_\mu + \ldots, \\ g_{\mu\nu} &= \eta_{\mu\nu} + h^{(1)}_{\mu\nu} + h^{(2)}_{\mu\nu} + \ldots, \end{aligned} \tag{331}$$

and Ψ expanded as before we find that at first order we have

$$R^{(1)}_{\mu\nu}[h^{(1)}] - \frac{1}{2}\eta_{\mu\nu}\eta^{\rho\sigma}R^{(1)}_{\rho\sigma}[h^{(1)}] = 0, \tag{332}$$

so that again $h^{(1)}_{\mu\nu}$ obeys the linearized equations of motion. At second order we find

$$R^{(1)}_{\mu\nu}[h^{(2)}] - \frac{1}{2}\eta_{\mu\nu}\eta^{\rho\sigma}R^{(1)}_{\rho\sigma}[h^{(2)}] = t^{h^{(1)}}_{\mu\nu} + T^{\Psi^{(1)}}_{\mu\nu}, \tag{333}$$

where $T^{\Psi^{(1)}}_{\mu\nu}$ is quadratic in $\Psi^{(1)}$ and does not contain any other fields, and as before

$$t^{h^{(1)}}_{\mu\nu} = -\left(R^{(2)}_{\mu\nu}[h^{(1)}] - \frac{1}{2}\eta_{\mu\nu}\eta^{\rho\sigma}R^{(2)}_{\rho\sigma}[h^{(1)}]\right). \tag{334}$$

This is exactly the same set-up as for the linearization stability conditions in classical linearized gravity, except for the addition of an energy-momentum tensor for the gravitino. It follows that the total energy and total momentum of the linearized system have to vanish, taking into account the contributions from both the gravitons and gravitinos. Thus we have again

$$H = 0, \tag{335}$$

$$\vec{P} = 0. \tag{336}$$

Let us now impose these three linearization stability conditions in the quantised version of linearized supergravity on a 3-torus.

6.3 Imposing the Quantum Linearization Stability Conditions

Next we turn to the canonically quantised linearized supergravity theory. As before for the gravitational theory, we promote the Fourier coefficients $a_s(\vec{k})$, $a^\dagger_s(\vec{k})$, $b_s(\vec{k})$ and $b^\dagger_s(\vec{k})$ to operators. They obey the (anti)-commutation relations

$$[a_s(\vec{k}), a^\dagger_r(\vec{p})]_- = \delta_{sr}\delta_D(\vec{k}-\vec{p}), \tag{337}$$

$$[b_s(\vec{k}), b^\dagger_r(\vec{p})]_+ = \delta_{sr}\delta_D(\vec{k}-\vec{p}), \tag{338}$$

with all other (anti)-commutators vanishing. The a-type operators are related to the gravitons, while the b-type operators are related to the gravitinos. A suitable Hilbert space for these operators to act on is the Fock space generated from a vacuum state $|0\rangle$, which is defined to be annihilated by all annihilation type operators $a_s(\vec{k})$ and $b_s(\vec{k})$, that is

$$a_s(\vec{k})|0\rangle = b_s(\vec{k})|0\rangle = 0, \qquad \text{for all } s \text{and } \vec{k}. \tag{339}$$

A suitable basis for this Hilbert space is composed of states which contain $n_s^B(\vec{k})$ graviton particles generated by $a_s^\dagger(\vec{k})$ and $n_s^F(\vec{k})$ gravitino particles generated by $b_s^\dagger(\vec{k})$, where of course $n_s^F(\vec{k})$ must be 0 or 1 as the gravitino particles are fermions. Thus we have a basis

$$\left|\{n_s^B(\vec{k})\}\right\rangle \otimes \left|\{n_s^F(\vec{k})\}\right\rangle = \prod_{\vec{k},s}\left[\frac{1}{\sqrt{n_s(\vec{k})!}}(a_s^\dagger(\vec{k}))^{n_s^B(\vec{k})}(b_s^\dagger(\vec{k}))^{n_s^F(\vec{k})}\right]|0\rangle. \tag{340}$$

For the bosonic zero-modes, we promote c, c_P, c_A and c_{PA} to operators obeying the non-zero commutation relations

$$[c, c_P]_- = i, \qquad [c_A, c_{PB}]_- = 0, \tag{341}$$

exactly as before. We can again represent them as acting on wavefunctions $\Psi = \Psi(c, c_A)$ as we did in the gravitational case. We also have to promote the modes η_α and η_α^A to operators, which obey anti-commutation relations

$$[\eta_\alpha, \eta_\beta]_+ = -\delta_{\alpha\beta}, \tag{342}$$

$$[\eta_\alpha^A, \eta_\beta^A]_+ = \delta_{\alpha\beta}\delta^{AB}. \tag{343}$$

Let $\mathcal{H}_P$ denote the Fock space of the graviton and gravitino modes with momentum. Let $\mathcal{H}_B$ denote the space of the bosonic graviton zero mode space, and let $\mathcal{H}_F$ denote the (indefinite) fermionic gravitino zero mode space. The total space for our theory will be the tensor product of these spaces

$$\mathcal{H} = \mathcal{H}_F \otimes \mathcal{H}_B \otimes \mathcal{H}_P. \tag{344}$$

The gravitino zero-momentum modes only enter the linearization stability conditions through the supercharge Q, and the supercharge commutes with the total energy and the total momentum. We thus focus first on the bosonic linearization stability conditions $H = 0$ and $\vec{P} = 0$. These are imposed exactly as before for the gravitational case. Let $\mathcal{H}^{\text{phys},B}$ denote the space of states which obey the bosonic linearization stability conditions, that is

$$|\text{phys}, B\rangle \in \mathcal{H}^{\text{phys},B} \tag{345}$$

if

$$H|\text{phys}, B\rangle = \vec{P}|\text{phys}, B\rangle = 0. \tag{346}$$

Suppressing the fermionic zero-modes, the states are of the form

$$|\text{phys}, B\rangle = \Psi(c, c_A) \otimes \left|\{n_s^B(\vec{k})\}\right\rangle \otimes \left|\{n_s^F(\vec{k})\}\right\rangle, \tag{347}$$

subject to

$$\sum_{\vec{k}}\sum_{s=\pm}\vec{k}\left(n_s^B(\vec{k}) + n_s^F(\vec{k})\right) = 0, \tag{348}$$

and the wavefunction $\Psi(c, c_A)$ must obey the $(5+1)$-dimensional Klein-Gordon equation

$$\left(+\frac{\partial^2}{\partial c^2} - \sum_{A=1}^{5} \frac{\partial^2}{\partial c_A^2} + M^2\right) \Psi(c, c_A) = 0. \tag{349}$$

where we have defined again

$$\frac{1}{2}M^2 = \sum_{k\neq 0}\sum_{s=\pm} |\vec{k}| \left(n_s^B(\vec{k}) + n_s^F(\vec{k})\right). \tag{350}$$

The inner product between these states obtained by group averaging is precisely as in the gravitational case, but now also including the gravitino modes with momentum.

Next let us impose the fermionic linearization stability conditions on the states which already obey the bosonic linearization stability conditions. Let $\mathcal{H}^{\text{phys}}$ denote the space of states which obey all the stability conditions, that is

$$|\text{phys}\rangle \in \mathcal{H}^{\text{phys}} \tag{351}$$

if $|\text{phys}\rangle \in \mathcal{H}^{\text{phys},B}$ and

$$Q_\alpha|\text{phys}\rangle = 0, \qquad \alpha = 1, 2, 3, 4. \tag{352}$$

Note that if we already satisfy the bosonic constraints, then the supercharges Q_α anti-commute,

$$[Q_\alpha, Q_\beta]_+|\text{phys}, B\rangle = 0, \qquad \text{if } P^\mu|\text{phys}, B\rangle = 0. \tag{353}$$

Now split the supercharge again into its zero-momentum $Q^{(0)}$ and non-zero-momentum $\hat{Q}$ contributions. We recall that

$$Q^{(0)} = -\frac{1}{2}\left(c_P\eta + \sum_{A=1}^{5}\sum_{B=1}^{2} T_{ij}^A \gamma^j T_{ik}^B \gamma^k c_{PA}\eta^B\right). \tag{354}$$

Then calculating the anti-commutator between the zero-momentum contribution yields

$$[Q_\alpha^{(0)}, Q_\beta^{(0)}]_+ = \frac{1}{4}\left(-c_P^2 + \sum_{A=1}^{5} c_{AP}^2\right)\delta_{\alpha\beta}, \tag{355}$$

where we have to note that

$$\sum_{A=1}^{5}\sum_{B=1}^{2}\sum_{C=1}^{5} T_{ij}^A T_{ik}^B T_{lm}^B T_{ln}^C \gamma^j\gamma^k\gamma^m\gamma^n = \delta^{AC}, \tag{356}$$

which is verified by explicit calculation using the definition of the matrices T_{ij}^A in (210). If we're restricted to $\mathcal{H}^{\text{phys},B}$ we can then rewrite this as

$$[Q_\alpha^{(0)}, Q_\beta^{(0)}]_+ = -\frac{1}{4}M^2\delta_{\alpha\beta}, \tag{357}$$

and then it follows that

$$[\hat{Q}_\alpha, \hat{Q}_\beta]_+ = \frac{1}{4}M^2\delta_{\alpha\beta}. \tag{358}$$

We decompose $\mathcal{H}^{\text{phys},B}$ into eigenspaces of M^2 and work in the distinct eigenspaces. We need to consider two cases, first we consider the case $M^2 > 0$, and later we discuss the possibility $M^2 = 0$. Now, let us combine $Q^{(0)}$ and $\hat{Q}$ into creation- and annihilation-type operators, similar to the method in Appendix G for the gravitino zero-modes (see also [31]). Define

$$a_1 = \frac{\sqrt{2}}{M}(Q_1^{(0)} + iQ_2^{(0)}), \qquad a_2 = \frac{\sqrt{2}}{M}(Q_3^{(0)} + iQ_4^{(0)}) \tag{359}$$

$$b_1 = \frac{\sqrt{2}}{M}(\hat{Q}_1 + i\hat{Q}_2), \qquad b_2 = \frac{\sqrt{2}}{M}(\hat{Q}_3 + i\hat{Q}_4), \tag{360}$$

which obey the anti-commutation relations

$$\begin{aligned} [a_i, a_j^\dagger]_+ &= -\delta_{ij}, \\ [b_i, b_j^\dagger]_+ &= +\delta_{ij}. \end{aligned} \tag{361}$$

We now build the space of states from a positively normalised state $|0, B\rangle$ which obeys

$$a_i|0, B\rangle = 0, \qquad b_i|0, B\rangle = 0, \qquad i = 1, 2, \tag{362}$$

and $|0, B\rangle \in \mathcal{H}^{\mathrm{phys},B}$ and $M^2 > 0$. Built from this state we have a 16-dimensional Fock space obtained by applying the creation-operators $a_i^\dagger$ and $b_i^\dagger$. We denote a basis for this space by

$$|m_1 m_2 n_1 n_2\rangle = (a_1^\dagger)^{m_1}(a_2^\dagger)^{m_2}(b_1^\dagger)^{n_1}(b_2^\dagger)^{n_2}|0, B\rangle, \tag{363}$$

where each of m_1, m_2, n_1 and n_2 can either be 0 or +1. We now want to consider a state $|\mathrm{phys}\rangle$ which obeys additionally the fermionic linearization stability condition

$$Q_\alpha|\mathrm{phys}\rangle = 0, \tag{364}$$

which in the new notation can be written as

$$(a_i + b_i)|\mathrm{phys}\rangle = (a_i^\dagger + b_i^\dagger)|\mathrm{phys}\rangle = 0. \tag{365}$$

Now, in general we can write

$$|\mathrm{phys}\rangle = \sum_{m,n} c_{m_1 m_2 n_1 n_2}|m_1 m_2 n_1 n_2\rangle \tag{366}$$

Now, let us note that

$$(a_1 + b_1)|m_1 m_2 n_1 n_2\rangle = \begin{cases} 0 & \text{if } m_1 = 0, n_1 = 0, \\ (-1)^{m_2}|0 m_2 0 n_2\rangle & \text{if } m_1 = 0, n_1 = 1, \\ -|0 m_2 0 n_2\rangle & \text{if } m_1 = 1, n_1 = 0, \\ -|0 m_2 1 n_2\rangle + (-1)^{m_2}|1 m_2 0 n_2\rangle & \text{if } m_1 = 1, n_1 = 1. \end{cases} \tag{367}$$

Thus applying $(a_1 + b_1)$ to $|\mathrm{phys}\rangle$ we will get something proportional to $|0 m_2 1 n_2\rangle$ if $c_{1 m_2 1 n_2} \neq 0$, as terms proportional to $|0 m_2 1 n_2\rangle$ only arise from $|1 m_2 1 n_2\rangle$. Thus we must set $c_{1 m_2 1 n_2} = 0$. Similar reasoning with $(a_2 + b_2)$, $(a_1^\dagger + b_1^\dagger)$ and $(a_2^\dagger + b_2^\dagger)$ tells us that we need to only consider states of the form

$$|\text{phys}\rangle = A|1100\rangle + B|0110\rangle + C|1001\rangle + D|0011\rangle, \tag{368}$$

for some constants A, B, C and D. Note that these states are orthogonal to each other, and

$$\langle 1100\,|\,1100\rangle = \langle 0011\,|\,0011\rangle = -\langle 0110\,|\,0110\rangle = \langle 1001\,|\,1001\rangle = +1. \tag{369}$$

Then apply the linearization stability conditions. First apply $(a_1 + b_1)$,

$$(a_1 + b_1)|\text{phys}\rangle = -(A+B)|0100\rangle + (D-C)|0001\rangle = 0, \tag{370}$$

so that we learn $A = -B$ and $C = D$. Next, apply $(a_1^\dagger + b_1^\dagger)$, which gives

$$(a_1^\dagger + b_1^\dagger)|\text{phys}\rangle = B|1110\rangle + D|1011\rangle + A|1110\rangle - C|1011\rangle = 0, \tag{371}$$

so we learn nothing new from this condition. Then apply next $(a_2 + b_2)$, giving

$$(a_2 + b_2)|\text{phys}\rangle = A|1000\rangle - B|0010\rangle - C|1000\rangle - D|0010\rangle = 0, \tag{372}$$

from which we learn $A = D$ and $B = -C$. Finally applying $(a_2^\dagger + b_2^\dagger)$ yields again nothing new. Thus we must have,

$$|\text{phys}\rangle = A\left[|1100\rangle - |0110\rangle + |1001\rangle + |0011\rangle\right]. \tag{373}$$

This state now obeys all the linearization stability conditions, the bosonic as well as the fermionic ones, however we note that

$$\langle \text{phys}\,|\,\text{phys}\rangle = 0. \tag{374}$$

We thus still have to redefine the inner product on $\mathcal{H}^{\text{phys}}$ to be positive definite. To do this, we will proceed again by group averaging. We first show that averaging a general state in $\mathcal{H}^{\text{phys},B}$ over the supergroup generated by the supercharges Q_α necessarily leads to a state proportional to $|\text{phys}\rangle$. Then we apply the group-averaging method to define a new, positive definite, inner product.

In the group averaging procedure, we start with a general $|\text{state}\rangle$ and then a physical state is defined by

$$|\text{phys}\rangle = -\frac{1}{2V}\int \mathrm{d}^4\alpha \mathrm{d}^4\theta\;\exp\left(-i\alpha\cdot P - \bar{\theta}Q\right)|\text{state}\rangle, \tag{375}$$

where θ_α, for $\alpha = 1,2,3,4$ is an anti-commuting number, and $\alpha = (\alpha^0, \vec{\alpha})$ as before. We define $\mathrm{d}^4\theta = \mathrm{d}\theta_4\mathrm{d}\theta_3\mathrm{d}\theta_2\mathrm{d}\theta_1$, and the integration over θ_α is defined by [31]

$$\int \mathrm{d}\theta_\alpha\;\theta_\alpha = 0, \qquad \int \mathrm{d}\theta_\alpha = 0. \tag{376}$$

However, Q_α and P^μ commute, so we can first group-average over the constraint

$P^\mu = 0$, to obtain $|\text{phys}, B\rangle$ and then

$$|\text{phys}\rangle = -\int \mathrm{d}^4\theta \ \exp(-\bar{\theta}Q)|\text{phys}, B\rangle. \tag{377}$$

Acting on states which already obey $P^\mu = 0$, the supercharges Q_α anti-commute and it follows that

$$|\text{phys}\rangle = -Q_1Q_2Q_3Q_4|\text{phys}, B\rangle. \tag{378}$$

Now, we can write

$$(Q_1 - iQ_2)(Q_1 + iQ_2) = i(Q_1Q_2 - Q_2Q_1) = 2iQ_1Q_2, \tag{379}$$

if we use that Q_1 and Q_2 anti-commute. Thus

$$\begin{aligned} |\text{phys}\rangle &= \frac{1}{4}(Q_1 - iQ_2)(Q_1 + iQ_2)(Q_3 - iQ_4)(Q_3 + iQ_4)|\text{phys}, B\rangle \\ &= \frac{M^4}{16}(a_1^\dagger + b_1^\dagger)(a_1 + b_1)(a_2^\dagger + b_2^\dagger)(a_2 + b_2)|\text{phys}, B\rangle. \end{aligned} \tag{380}$$

Note that each of the brackets can be anti-commuted. In general we write $|\text{phys}, B\rangle$ as a linear combination of the $|m_1m_2n_1n_2\rangle$, however note that

$$\begin{aligned} (a_1 + b_1)|0m_20n_2\rangle &= 0, \\ (a_1^\dagger + b_1^\dagger)|1m_21n_2\rangle &= 0, \\ (a_2 + b_2)|m_10n_10\rangle &= 0, \\ (a_2^\dagger + b_2^\dagger)|m_11n_11\rangle &= 0. \end{aligned} \tag{381}$$

Thus we only still need to consider $|1100\rangle$, $|0110\rangle$, $|1001\rangle$ and $|0011\rangle$. We can calculate that

$$\begin{aligned} &(a_1^\dagger + b_1^\dagger)(a_1 + b_1)(a_2^\dagger + b_2^\dagger)(a_2 + b_2)|1100\rangle \\ &= (a_1^\dagger + b_1^\dagger)(a_1 + b_1)(a_2^\dagger + b_2^\dagger)|1000\rangle \\ &= (a_1^\dagger + b_1^\dagger)(a_1 + b_1)\left[-|1100\rangle - |1001\rangle\right] \\ &= (a_1^\dagger + b_1^\dagger)\left[|0100\rangle + |0001\rangle\right] \\ &= |1100\rangle - |0110\rangle + |1001\rangle + |0011\rangle, \end{aligned} \tag{382}$$

where the sign in the second term follows from the anti-commutation relations as

$$b_1^\dagger|0100\rangle = b_1^\dagger a_2^\dagger|0, B\rangle = -a_2^\dagger b_1^\dagger|0, B\rangle = -|0110\rangle. \tag{383}$$

Similarly,

$$\begin{aligned} &(a_1^\dagger + b_1^\dagger)(a_1 + b_1)(a_2^\dagger + b_2^\dagger)(a_2 + b_2)|0110\rangle \\ &= |1100\rangle - |0110\rangle + |1001\rangle + |0011\rangle, \\ &(a_1^\dagger + b_1^\dagger)(a_1 + b_1)(a_2^\dagger + b_2^\dagger)(a_2 + b_2)|1001\rangle \\ &= -\left[|1100\rangle - |0110\rangle + |1001\rangle + |0011\rangle\right], \\ &(a_1^\dagger + b_1^\dagger)(a_1 + b_1)(a_2^\dagger + b_2^\dagger)(a_2 + b_2)|1001\rangle \\ &= |1100\rangle - |0110\rangle + |1001\rangle + |0011\rangle. \end{aligned} \tag{384}$$

Thus we find again, by group-averaging, that

$$\begin{aligned} |\text{phys}\rangle &= -\int \mathrm{d}^4\theta\ \exp(-\bar{\theta}Q)|\text{phys}, B\rangle \\ &= A\left[|1100\rangle - |0110\rangle + |1001\rangle + |0011\rangle\right], \end{aligned} \tag{385}$$

for some constant A. However, as mentioned before, currently this state would have zero norm, so we ought to redefine the inner product. According to the group-averaging prescription, we define, if

$$|\text{phys}\rangle = -\frac{1}{2V}\int \mathrm{d}^4\alpha \mathrm{d}^4\theta\ \exp\left(-i\alpha\cdot P - \bar{\theta}Q\right)|\text{state}\rangle \tag{386}$$

$$\begin{aligned} (\text{phys1}\,|\,\text{phys2}) &= \langle\text{state1}|\left[-\frac{1}{2V}\int \mathrm{d}^4\alpha \mathrm{d}^4\theta\ \exp\left(-i\alpha\cdot P - \bar{\theta}Q\right)\right]|\text{state2}\rangle \\ &= \langle\text{phys}, B1|\left[-\int \mathrm{d}^4\theta \exp\left(-i\bar{\theta}Q\right)\right]|\text{phys}, B2\rangle \\ &= -\langle\text{phys}, B1|Q_1Q_2Q_3Q_4|\text{phys}, B2\rangle \\ &= \langle\text{phys}, B1\,|\,\text{phys2}\rangle. \end{aligned} \tag{387}$$

Now, if for $i = 1, 2$ we write

$$|\text{phys}, Bi\rangle = \kappa_i^{(1)}|1100\rangle + \kappa_i^{(2)}|0110\rangle + \kappa_i^{(3)}|1001\rangle + \kappa_i^{(4)}|0011\rangle + \dots, \tag{388}$$

where the ellipses denote linear combinations of states which vanish when the combination $Q_1Q_2Q_3Q_4$ are applied, then

$$|\text{phys}i\rangle = \frac{M^4}{16}\lambda_i\left[|1100\rangle - |0110\rangle + |1001\rangle + |0011\rangle\right] \tag{389}$$

where

$$\lambda_i = \kappa_i^{(1)} + \kappa_i^{(2)} - \kappa_i^{(3)} + \kappa_i^{(4)} \tag{390}$$

Then we find that the redefined inner product $(\text{phys1}\,|\,\text{phys2})$ between $|\text{phys1}\rangle$ and $|\text{phys2}\rangle$ is given by

$$(\text{phys1}\,|\,\text{phys2}) = \frac{M^4}{16}\lambda_1^*\lambda_2. \tag{391}$$

In this manner we obtain a new inner product, which now is positive definite, at least for $M^2 > 0$.

Let us quickly address the case $M^2 = 0$. In this case, after imposing the bosonic linearization stability conditions, we have no graviton or gravitino contributions with non-zero momentum. Thus the supercharge can be taken to be of the form

$$Q = Q^{(0)} = \frac{1}{2}\left(-c_P\eta + \sum_{A=1}^{5}\sum_{B=1}^{2} T^A_{ij}\gamma^j T^B_{ik}\gamma^k c_{PA}\eta^B\right) \tag{392}$$

Then consider graviton zero-mode states which are smeared against the plane waves

$$\Psi(c, c_A) = \exp\left[\mp i|\vec{p}|c + i\vec{p}\cdot\vec{c}\right]. \tag{393}$$

With respect to these plane waves we have

$$Q^{(0)} = \frac{p}{2}\left[\eta + R\right], \tag{394}$$

where the anti-commutation relations now read

$$[\eta_\alpha, \eta_\beta]_+ = -[R_\alpha, R_\beta]_+ = \delta_{\alpha\beta}. \tag{395}$$

Then we can proceed as for the $M^2 > 0$ case, but this time forming ladder operators out of η and R instead.

6.4 Example of a Physical State

To illustrate the construction of the space of physical states which obey the linearization instability conditions, let us now consider an explicit example. We will take a state of two particles, moving along the x-axis in opposing directions with $|\vec{k}| = k$, we will also assume that both particles have positive helicity. Then we have $M^2 = 4k$.

First we calculate the non-zero-momentum contribution to the total supercharge (300), which we denote

$$\hat{Q} = \frac{i}{2}\sum_{\vec{k}\neq 0}\sum_{s=\pm}\left[\vec{\epsilon}^{\,s}(\vec{k})\cdot\vec{\gamma}u^{s*}(\vec{k})a_s(\vec{k})b_s^\dagger(\vec{k}) - \vec{\epsilon}^{\,s*}(\vec{k})\cdot\vec{\gamma}u^s(\vec{k})a_s^\dagger(\vec{k})b_s(\vec{k})\right]. \tag{396}$$

For $\vec{\epsilon}^{\,+}(\pm k\vec{e}_1)$ we can take (from (113) with slight modification)

$$\vec{\epsilon}^{\,+}(k\vec{e}_1) = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\+i\end{pmatrix}, \qquad \vec{\epsilon}^{\,+}(-k\vec{e}_1) = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\-i\end{pmatrix} \tag{397}$$

and for the eigenspinors $u^+(\pm k\vec{e}_1)$ we take the massless limit of (66) to get

$$u^+(k\vec{e}_1) = \sqrt{\frac{k}{2}}\begin{pmatrix}1\\-i\\1\\-i\end{pmatrix}, \qquad u^+(-k\vec{e}_1) = \sqrt{\frac{k}{2}}\begin{pmatrix}1\\i\\-1\\-i\end{pmatrix}. \tag{398}$$

Thus, we calculate

$$\vec{\epsilon}^{\,+}(k\vec{e}_1)\cdot\vec{\gamma}u^{+*}(k\vec{e}_1) = \frac{\sqrt{k}}{2}\begin{pmatrix}0 & 0 & i & 1\\0 & 0 & 1 & -i\\i & 1 & 0 & 0\\1 & -i & 0 & 0\end{pmatrix}\begin{pmatrix}1\\i\\1\\i\end{pmatrix} = \sqrt{k}\begin{pmatrix}i\\1\\i\\1\end{pmatrix}. \tag{399}$$

Similarly,

$$\vec{\epsilon}^{+}(-k\vec{e}_1)\cdot\vec{\gamma}u^{+*}(-k\vec{e}_1) = \frac{\sqrt{k}}{2}\begin{pmatrix} 0 & 0 & -i & 1 \\ 0 & 0 & 1 & i \\ -i & 1 & 0 & 0 \\ 1 & i & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ -i \\ -1 \\ i \end{pmatrix} = \sqrt{k}\begin{pmatrix} i \\ -1 \\ -i \\ 1 \end{pmatrix}. \tag{400}$$

Introduce the notation $a_{(\pm)} = a_+(\pm k\vec{e}_1)$ and similarly for $b_{(\pm)}$, the contribution to the supercharge (396) from $\pm k\vec{e}_1$ is given by

$$\hat{Q} = \frac{\sqrt{k}}{2}\begin{pmatrix} -a_{(+)}b^\dagger_{(+)} - a^\dagger_{(+)}b_{(+)} - a_{(-)}b^\dagger_{(-)} - a^\dagger_{(-)}b_{(-)} \\ i(a_{(+)}b^\dagger_{(+)} - a^\dagger_{(+)}b_{(+)}) - i(a_{(-)}b^\dagger_{(-)} - a^\dagger_{(-)}b_{(-)}) \\ -a_{(+)}b^\dagger_{(+)} - a^\dagger_{(+)}b_{(+)} + a_{(-)}b^\dagger_{(-)} + a^\dagger_{(-)}b_{(-)} \\ i(a_{(+)}b^\dagger_{(+)} - a^\dagger_{(+)}b_{(+)}) + i(a_{(-)}b^\dagger_{(-)} - a^\dagger_{(-)}b_{(-)}) \end{pmatrix}. \tag{401}$$

As a quick sanity check, we can note that $\hat{Q}^\dagger = \hat{Q}$. We now combine the components of the supercharge, including the zero-momentum components $Q^{(0)}$ into ladder operators. As in (359) we define

$$a_1 = \frac{1}{\sqrt{2k}}(Q^{(0)}_1 + iQ^{(0)}_2), \qquad a_2 = \frac{1}{\sqrt{2k}}(Q^{(0)}_3 + iQ^{(0)}_4), \tag{402}$$

where we used $M^2 = 4k$. Similarly following (360), define

$$\begin{aligned} b_1 &= \frac{1}{\sqrt{2k}}(\hat{Q}_1 + i\hat{Q}_2) = \frac{1}{\sqrt{2}}(-a_{(+)}b^\dagger_{(+)} - a^\dagger_{(-)}b_{(-)}) \\ b_2 &= \frac{1}{\sqrt{2k}}(\hat{Q}_3 + i\hat{Q}_4) = \frac{1}{\sqrt{2}}(-a_{(+)}b^\dagger_{(+)} + a^\dagger_{(-)}b_{(-)}) \end{aligned} \tag{403}$$

For $i = 1, 2$ the non-zero anti-commutation obeyed by these are

$$[a_i, a^\dagger_j]_+ = -\delta_{ij}, \qquad [b_i, b^\dagger_j]_+ = \delta_{ij}. \tag{404}$$

Now let us define a suitable state $|\text{phys}, B\rangle$ obeying the bosonic linearization stability conditions

$$H|\text{phys}, B\rangle = \vec{P}|\text{phys}, B\rangle = 0. \tag{405}$$

We will take the non-zero-momentum part to be a positive helicity graviton and a positive helicity gravitino, with momenta $\pm k\vec{e}_1$ respectively, this thus satisfies the constraint that the momentum vanishes. For the graviton zero-modes, consider a Gaussian

$$\Psi(c, c_A) = Ne^{-c^2}e^{-\vec{c}^2}. \tag{406}$$

In momentum space, we have

$$\Psi(c, c_A) = N\pi^3 \int \frac{\mathrm{d}p^0}{2\pi}\int \frac{\mathrm{d}^5\vec{p}}{(2\pi)^5}\; \exp\left[-\frac{1}{4}((p^0)^2 + \vec{p}^2)\right]\exp\left[-ip^0c + i\vec{p}\cdot\vec{c}\right]. \tag{407}$$

Then, applying the bosonic group averaging procedure to this state, we find that

$$|\mathrm{phys}, B\rangle = \int \frac{\mathrm{d}^5\vec{p}}{(2\pi)^5} \left[f^{(+)}(\vec{p})e^{-iE(\vec{p})c+i\vec{p}\cdot\vec{c}} + f^{(-)}(\vec{p})e^{iE(\vec{p})c+i\vec{p}\cdot\vec{c}}\right] \otimes a^{\dagger}_{(+)}b^{\dagger}_{(-)}|0\rangle, \tag{408}$$

where $E(\vec{p}) = \sqrt{\vec{p}^2 + M^2}$ and

$$f^{(\pm)}(\vec{p}) = \frac{\pi^3 N e^{-M^2/4}e^{-\vec{p}^2/2}}{2\sqrt{\vec{p}^2+M^2}}. \tag{409}$$

The group-averaged norm for this state is then

$$\begin{aligned} \langle \mathrm{phys}, B\,|\,\mathrm{phys}, B\rangle &= \int \frac{\mathrm{d}^5\vec{p}}{(2\pi)^5} 2E(\vec{p}) \left[|f^{(+)}(\vec{p})|^2 + |f^{(-)}(\vec{p})|^2\right] \\ &\quad \times \langle 0|b_{(-)}a_{(+)}a^{\dagger}_{(+)}b^{\dagger}_{(-)}|0\rangle \\ &= \frac{\pi^3|N|^2 e^{-M^2/2}}{2^6}\int_0^{\infty} \mathrm{d}p\ \frac{p^4 e^{-p^2}}{\sqrt{p^2+M^2}} < \infty. \end{aligned} \tag{410}$$

We choose N such that this is normalised to unity.

Now it is time to add the gravitino zero-modes, η_α and η^A_α. Let $|0F\rangle$ be the state defined in the gravitino-zero mode appendix, which is annihilated by each of d_i and d^A_i. Define $\bar{a}_1$ and $\bar{a}_2$ to be the restrictions of a_1 and a_2 (formed from the graviton and gravitino zero-mode) to the eigenspaces of the graviton zero-modes with $c_P = p^0$ and $c_{PA} = p_A$. Then define,

$$\left|p^0, \vec{p}\right\rangle = N_{p^0\vec{p}}\bar{a}_1\bar{a}_2 d^{\dagger}_2 d^{\dagger}_1|0F\rangle, \tag{411}$$

where the normalisation constant ensures that

$$\langle p^0, \vec{p}\,|\,p^0, \vec{p}\rangle = +1. \tag{412}$$

Then the state

$$\begin{aligned} |\mathrm{phys}, B\rangle = \int \frac{\mathrm{d}^5\vec{p}}{(2\pi)^5} \Big[& f^{(+)}(\vec{p})e^{-iE(\vec{p})c+i\vec{p}\cdot\vec{c}} \otimes |E(\vec{p}), \vec{p}\rangle \\ &+ f^{(-)}(\vec{p})e^{iE(\vec{p})c+i\vec{p}\cdot\vec{c}} \otimes |-E(\vec{p}), \vec{p}\rangle\Big] \otimes\ a^{\dagger}_{(+)}b^{\dagger}_{(-)}|0\rangle \end{aligned} \tag{413}$$

obeys the bosonic linearization stability conditions, as well as

$$a_i|\mathrm{phys}, B\rangle = 0, \qquad i = 1, 2 \tag{414}$$

Notice also that this state obeys

$$b^{\dagger}_i|\mathrm{phys}, B\rangle = 0, \qquad i = 1, 2 \tag{415}$$

thus in the previous notation, we are considering the state $|0011\rangle$. Now, to satisfy the fermionic linearization constraint, we define

$$|\mathrm{phys}\rangle = \frac{M^4}{16}(a^{\dagger}_1 + b^{\dagger}_1)(a_1 + b_1)(a^{\dagger}_2 + b^{\dagger}_2)(a_2 + b_2)|\mathrm{phys}, B\rangle \tag{416}$$

this yields

$$|\text{phys}\rangle = \frac{M^4}{16}\Big(a_1^\dagger a_2^\dagger a_{(-)}^\dagger b_{(+)}^\dagger + \frac{1}{\sqrt{2}}(-a_1^\dagger + a_2^\dagger)a_{(+)}^\dagger a_{(-)}^\dagger \\ - \frac{1}{\sqrt{2}}(a_1^\dagger + a_2^\dagger)b_{(+)}^\dagger b_{(-)}^\dagger + a_{(+)}^\dagger b_{(-)}^\dagger\Big)|\Psi\rangle, \tag{417}$$

where we introduced $|\Psi\rangle$ by

$$|\text{phys}, B\rangle = a_{(+)}^\dagger b_{(-)}^\dagger |\Psi\rangle. \tag{418}$$

Then the norm of this state is, using the group-averaging inner product,

$$(\text{phys}\,|\,\text{phys}) = \langle \text{phys}, B\,|\,\text{phys}\rangle = \frac{M^4}{16}, \tag{419}$$

as we would expect from the non-invariant state $|\text{phys}, B\rangle = |0011\rangle$ that we started from.

7 Conclusion

In this chapter we have noted that there are fermionic linearization stability conditions as well as bosonic ones in 4-dimensional $\mathcal{N} = 1$ supergravity on a 3-torus background. Then we showed that states satisfying both fermionic and bosonic quantum linearization stability conditions can be constructed by group-averaging over the supergroup of global supersymmetry and spacetime translation symmetry.

States satisfying the bosonic quantum linearization stability conditions were seen to have infinite norm in the original Hilbert space. This infinity results from the infinite volume of the symmetry group generated by the LSCs. Roughly speaking, this infinite volume is factored out in the group-averaging inner product. It is interesting that the inner product of states satisfying all quantum linearization stability conditions have zero norm in the Hilbert space of states satisfying only the bosonic ones. The finite group-averaging inner product is obtained by factoring out zero in this case.

It would also be interesting to investigate whether there are analogues of LSCs in String Theory. A preliminary investigation in this direction [32] did not find such analogues in Bosonic String Theory, but since String Theory contains General Relativity, we believe there should be analogues of linearization stability conditions in (Super)String Theory on any background spacetime with compact Cauchy surfaces.

A A Note on Conventions

In this appendix we try to collect the main conventions we follow for easy reference. We try to follow the conventions of Freedman and van Proeyen [28], except we do not raise and lower spinor indices. For indices, we try to follow the following conventions:

- $\mu, \nu, \ldots = 0, 1, 2, 3$ are spacetime indices,
- $i, j, \ldots = 1, 2, 3$ are space indices,
- $\alpha, \beta, \ldots = 1, 2, 3, 4$ are spinor indices,
- $s, r, \ldots = 1, 2$ or $+, -$ are helicity indices,

- $a, b, \ldots = 0, 1, 2, 3$ are local frame indices.

Unless otherwise indicated, repeated indices are implicitly summed over. We use a mostly-positive metric convention $(-, +, +, +)$, so that the spacetime element for the flat-Minkowski space is

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}\vec{x} \cdot \mathrm{d}\vec{x}. \tag{420}$$

Throughout we work on a background spacetime whose spatial sections are flat 3-tori, with lengths L_1, L_2 and L_3 in the x-, y- and z-directions respectively. The spatial volume is $V = L_1 L_2 L_3$.

We use γ-matrices which obey

$$[\gamma^\mu, \gamma^\nu]_+ = 2\eta^{\mu\nu}, \tag{421}$$

when we need a particular representation of the γ-matrices, we will use a Majorana representation

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \gamma^1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \gamma^2 = \begin{pmatrix} 0 & \sigma^1 \\ \sigma^1 & 0 \end{pmatrix}, \quad \gamma^3 = \begin{pmatrix} 0 & \sigma^3 \\ \sigma^3 & 0 \end{pmatrix}, \tag{422}$$

where the Pauli-matrices σ^i are

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{423}$$

If A and B are two operators, we denote the commutator and anti-commutator of them by

$$[A, B]_- = AB - BA, \qquad [A, B]_+ = AB + BA. \tag{424}$$

B Constrained Hamiltonian Systems

Throughout, we want to study physical systems using the canonical formalism. In this formalism, we imagine that the system is described by some coordinates q and associated momenta p. Together the q and p, taken at an an arbitrary fixed equal time t, form coordinates for the phase space of the system. The momenta and coordinates combine into the Hamiltonian H,

$$H = H(p, q), \tag{425}$$

which determines the time-evolution of the system through Hamilton's equations

$$\begin{aligned} \frac{d}{dt} q &= \{q, H\}, \\ \frac{d}{dt} p &= \{p, H\}, \end{aligned} \tag{426}$$

where $\{A, B\}$ denotes the bracket between A and B. The brackets are typically determined from the canonical (equal-time) Poisson brackets

$$\begin{aligned}
\{q(t), q(t)\}_P &= 0, \\
\{q(t), p(t)\}_P &= 1, \\
\{p(t), p(t)\}_P &= 0,
\end{aligned} \tag{427}$$

as well as the anti-symmetry rule $\{A, B\} = -\{B, A\}$, Leibniz rule $\{A, BC\} = B\{A, C\} + \{A, B\}C$ and the Jacobi identity

$$\{A, \{B, C\}\} + \{B, \{C, A\}\} + \{C, \{A, B\}\} = 0. \tag{428}$$

Typically, the Poisson bracket defined by

$$\{A, B\}_P = \frac{\partial A}{\partial q}\frac{\partial B}{\partial p} - \frac{\partial A}{\partial p}\frac{\partial B}{\partial q} \tag{429}$$

satisfies these properties, and we can verify that this indeed gives rise to the canonical Poisson brackets. The notation here is suggestive of a single q and p, but this is schematic for more general situations where there are multiple coordinates or even continuous coordinates. In those cases there is a summation (and/or integration) over the variables.

As a standard example, a particle mass m moving freely in two dimensions has phase space coordinates $\{q_1, q_2, p_1, p_2\}$ and the Hamiltonian for the system is

$$H = \frac{1}{2m}(p_1^2 + p_2^2). \tag{430}$$

The associated equations of motion are then the well known

$$\frac{d}{dt}p = 0, \qquad m\frac{d}{dt}q = p. \tag{431}$$

This formalism is sufficient for describing unconstrained systems, however it is possible that the system has some constraints on it, which relate the momenta and coordinates. In this case, the formalism we have set up needs to be modified. To motivate the need for the modifications, consider again a mass m particle moving in two dimensions, but now constrain it to move on a circle radius 1, so impose

$$\phi(q_1, q_2) = q_1^2 + q_2^2 - 1 = 0. \tag{432}$$

One way to solve this system is to solve the constraint automatically by introducing a set of canonical variables (θ, p_θ), related to $\{q_1, q_2, p_1, p_2\}$ by

$$\begin{aligned}
q_1 &= \cos\theta, & q_2 &= \sin\theta, \\
p_1 &= -\sin\theta p_\theta, & p_2 &= \cos\theta p_\theta.
\end{aligned} \tag{433}$$

In terms of the new variables, the Hamiltonian is

$$H = \frac{1}{2m}p_\theta^2, \tag{434}$$

so imposing the Poisson brackets $\{\theta, p_\theta\} = 1$ and all others vanishing, we find the equations of motion

$$\frac{d}{dt}p_\theta = 0, \qquad m\frac{d}{dt}\theta = p_\theta. \tag{435}$$

If we calculate the brackets between the variables $\{q_1, q_2, p_1, p_2\}$ using the fundamental bracket $\{\theta, p_\theta\} = 1$, we find

$$\begin{aligned} \{q_i, q_j\} &= 0, \\ \{q_i, p_j\} &= \delta_{ij} - q_i q_j, \\ \{p_i, p_j\} &= -q_i p_j + p_j q_i. \end{aligned} \tag{436}$$

If we adopt these brackets, when the constraint is obeyed the function $\phi(q_1, q_2)$ has vanishing bracket with all the canonical coordinates, so it is consistent to set it to zero before the brackets are taken.

In general, the method for dealing with such constraints in the Hamiltonian formalism goes back to Dirac [33], and involves replacing the Poisson bracket with a more general Dirac bracket. We now give a quick account of this method, based on Dirac [33], Das [34] and Henneaux and Teitelboim [31].

Suppose we start with a Lagrangian $L = L(q^i, \dot{q}^i)$, which determines the equations of motion of the system through extremising of the action

$$S = \int \mathrm{d}t \; L(q^i, \dot{q}^i). \tag{437}$$

In moving over to the Hamiltonian formalism, we define a canonical conjugate momentum p_i by

$$p_i = \frac{\partial L}{\partial \dot{q}^i}, \tag{438}$$

and then define the Hamiltonian by

$$H_C = p_i \dot{q}^i - L. \tag{439}$$

Typically, we assume that we have as many independent momentum variables as we have velocity variables, and so we can replace the velocities in favour of the momenta. However, it may occur that the relations defining the momenta are not independent, then we get a number of primary constraints

$$\phi_m(p, q) = 0, \qquad m = 1, 2, \ldots, M \tag{440}$$

which arise from the definition of the momenta. Then the Hamiltonian we have defined is not the most general one we could consider. Adding any number of the primary constraints to the Hamiltonian should make no difference when the primary constraints are obeyed. We thus consider the primary Hamiltonian

$$H_P = H_C + \lambda_m \phi_m(p, q), \tag{441}$$

where the λ_m are undetermined Lagrange multipliers. Using standard Poisson brackets, the evolution of some phase-space function $g(p, q)$ is given by

$$\dot{g}(p, q) = \{g(p, q), H_P\}_P = \{g(p, q), H_C\}_P + \lambda_m\{g(p, q), \phi_m(p, q)\}_P. \tag{442}$$

We first ask that the constraints $\phi_m(p, q)$ are preserved under time-evolution. For this, we need to be careful, we must only impose the constraint $\phi(q, p) = 0$ after Poisson brackets are taken. To distinguish this, suppose we have two quantities A and B, we say that A and B are weakly equal, written

$$A \approx B, \tag{443}$$

if A and B are equal if the constraints are obeyed, $\phi(p, q) = 0$. If they are equal even if the constraints are not obeyed, we say they are strongly equal and write $A = B$. We thus ask that

$$\dot{\phi}_m(p, q) \approx 0. \tag{444}$$

In general there are then a few possibilities. Either $\dot{\phi}_m(p, q) \approx 0$ is already true by the existing constraints, $\dot{\phi}_m(p, q) \approx 0$ could determine some of the Lagrange multipliers λ_m or we find

$$\dot{\phi}(p, q) \approx \chi(p, q) \approx 0, \tag{445}$$

where $\chi(p, q)$ is a function of the p, q which does not vanish if the other constraints are obeyed. We must then incorporate $\chi(p, q)$ as an additional, secondary, constraint into the theory. We then repeat the procedure with the secondary constraints, until the the procedure terminates and we have determined all the constraints, primary and secondary. Let the secondary constraints $\chi(p, q)$ be denotes by

$$\phi_m(p, q) = 0, \qquad m = M + 1, \ldots N. \tag{446}$$

We now further classify the constraints. We say that a constraint is first class if it has weakly vanishing Poisson bracket with all constraints. Thus $\phi_i(p, q)$ is first class if

$$\{\phi_i(p, q), \phi_m(p, q)\}_P \approx 0 \qquad m = 1, 2, \ldots N. \tag{447}$$

If a constraint is not first class, then we say it is second class and has a non-vanishing Poisson bracket with at least one other constraint. In general, there must always be an even number of second class constraints [33].

Typically, first class constraints are related to gauge transformations, certainly in all cases we consider this is true. One way to deal with the first class constraints is to impose additional gauge-fixing constraints, one for each first class constraint, which turn the first class constraints into second class constraints. We will thus assume that all the constraints are second class.

Thus let $\phi_m(p, q)$ denote now the collection of all the second class constraints. In general, the constraints will not be compatible with the canonical Poisson brackets between the dynamical variables. This is fixed by introducing so-called Dirac brackets. Define the matrix C_{mn} of Poisson brackets between the constraints by

$$C_{mn} = \{\phi_m, \phi_n\}_P. \tag{448}$$

This matrix turns out to always be invertible, and can then be used to define the Dirac bracket between two functions f and g by

$$\{f, g\}_D = \{f, g\}_P - \{f, \phi_m\}_P C^{-1mn} \{\phi_n, g\}_P. \tag{449}$$

The nice property about Dirac brackets is that the constraints can be strongly set to vanish, because

$$\{f, \phi_m\}_D = 0, \tag{450}$$

for any second-class constraint ϕ_m and any phase space function $f(p, q)$. Thus when working with Dirac brackets, we may take the constraints to vanish even before taking brackets. Furthermore, it is possible to show that the algebraic properties obeyed by the Dirac brackets are the same as those for normal Poisson brackets.

Returning to the example of a particle constrained to a ring, this can be described by the Lagrangian

$$L(q_i, \dot{q}_i, F) = \frac{1}{2}\left[m(\dot{q}_1^2 + \dot{q}_2^2) - F(q_1^2 + q_2^2 - 1)\right]. \tag{451}$$

However, when calculating the momentum conjugate for the dynamical variable F, we find

$$p_F = \frac{\partial L}{\partial \dot{F}} = 0. \tag{452}$$

Thus, this must be imposed as a primary constraint on the theory. After carrying out the constraint analysis [34], one finds that the Dirac brackets between q_i and p_i are given by

$$\begin{aligned}
\{q_i, q_j\}_D &= 0, \\
\{q_i, p_j\}_D &= \delta_{ij} - q_i q_j, \\
\{p_i, p_j\}_D &= -q_i p_j + p_j q_i,
\end{aligned} \tag{453}$$

and the Hamiltonian for the system, after the constraints are strongly incorporated, is

$$H = \frac{1}{2m}(p_1^2 + p_2^2). \tag{454}$$

This exactly reproduces the results for the particle constrained to the ring we found previously by examining the system in angular coordinates.

C Grassmann Variables

In this appendix we collect some information on Grassmann, or anti-commuting, variables as needed for the "classical" treatment of spinor fields and supersymmetry. The treatment here is based on Rogers [35] and Henneaux and Teitelboim [31]. Grassmann numbers form a vector space, with a product (such as matrix multiplication), multiplication by (complex) scalars and a set of generating elements ξ^A which anti-commute, that is

$$\xi^A \xi^B + \xi^B \xi^A = 0. \tag{455}$$

For example, the Grassmann algebra with two generators can be realised as 4×4 matrices, with

$$\xi^1 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \xi^2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{456}$$

In general, we will realise bosonic dynamical variables q^i as even elements of a Grassman algebra, while fermionic dynamical variables θ^α are odd elements of a Grassman algebra, that is

$$\begin{aligned} q^i(t) &= q^i_0(t) + q^i_{AB}(t)\xi^A\xi^B + \dots, \\ \theta^\alpha(t) &= \theta^\alpha_A(t)\xi^A + \theta^\alpha_{ABC}(t)\xi^A\xi^B\xi^C + \dots. \end{aligned} \tag{457}$$

Now we can consider functions f, which map from a set of dynamical variables (q^i, θ^α) to a new point $f(q^i, \theta^\alpha)$. We will only consider so called superfunctions f, which are defined by the fact that they depend only on combinations of q^i and θ^α, and do not involve the Grassmann generators ξ^A explicitly. Then a general superfunction f admits an expansion of the form

$$f(q,\theta) = f_0(q) + f_\alpha(q)\theta^\alpha + f_{\alpha\beta}(q)\theta^\alpha\theta^\beta + \dots, \tag{458}$$

where we require $f_{\alpha\beta}(q) = -f_{\beta\alpha}(q)$. Given a superfunction, the left-derivative is defined by

$$\delta f = \delta\theta^\alpha \frac{\partial^L f}{\partial\theta^\alpha}, \tag{459}$$

that is, when we vary θ, we place the variation on the left. We will always take derivatives with respect to odd dynamical variables to be left-derivatives.

Suppose that A and B are dynamical variables and α is a complex scalar, we can define complex conjugation by

$$(AB)^* = B^*A^*, \qquad (A^*)^* = A, \qquad (\alpha A)^* = \alpha^* A^*. \tag{460}$$

A variable is real if $A^* = A$ and imaginary if $A^* = -A$. We will typically take the dynamical variables θ^α and q^i to be real variables.

Now, let us do classical mechanics with both even and odd variables. Assume that the equations of motion for the system are obtained by extremising an action of the form

$$S[q,\theta] = \int \mathrm{d}t\; L(q,\dot{q},\theta,\dot{\theta}), \tag{461}$$

where L is assumed to be a real, Grassman even function, and dots denote differentiation with respect to t. Canonical momenta are then defined

$$p_i = \frac{\partial L}{\partial \dot{q}^i}, \qquad \pi_\alpha = \frac{\partial^L L}{\partial\dot{\theta}^\alpha}. \tag{462}$$

Then the Hamiltonian is defined

$$H = \dot{q}^i p_i + \dot{\theta}^\alpha \pi_\alpha - L. \tag{463}$$

In this formalism, Hamilton's equations of motion can then be written

$$\dot{p}_i = -\frac{\partial H}{\partial q^i}, \quad \dot{q}^i = \frac{\partial H}{\partial p_i}, \quad \dot{\pi}_\alpha = -\frac{\partial^L H}{\partial \theta^\alpha}, \quad \dot{\theta}^\alpha = -\frac{\partial^L H}{\partial \pi_\alpha}. \tag{464}$$

This can be written in terms of Poisson brackets in the usual manner if we introduce the non-vanishing Poisson brackets between the dynamical variables

$$\{q^i, p_j\}_P = \delta^i_j, \qquad \{\theta^\alpha, \pi_\beta\}_P = -\delta^\alpha_\beta, \tag{465}$$

and extend the algebraic properties of Poisson brackets to include odd-variables.

D A Note on γ-Matrices

In this appendix we collect some information on γ-matrices, and prove some identities used in the main text, the treatment of the γ-matrices is based of Freedman and Van Proeyen [28]. The four γ-matrices we use are the four 4×4 matrices γ^0, γ^1, γ^2 and γ^3, which are defined to obey the anti-commutation relations

$$[\gamma^\mu, \gamma^\nu]_+ = \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\eta^{\mu\nu}, \tag{466}$$

where $\eta^{\mu\nu} = \mathrm{diag}(-1, +1, +1, +1)$. When necessary to indicate the components of a γ-matrix, we will use the indices $\alpha, \beta, \ldots = 1, 2, 3, 4$, so that $\gamma^\mu_{\alpha\beta}$ denotes the $\alpha\beta$ component of the matrix γ^μ.

An explicit realisation of the γ-matrices is provided by the following really real or Majorana representation

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \gamma^1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \gamma^2 = \begin{pmatrix} 0 & \sigma^1 \\ \sigma^1 & 0 \end{pmatrix}, \quad \gamma^3 = \begin{pmatrix} 0 & \sigma^3 \\ \sigma^3 & 0 \end{pmatrix}, \tag{467}$$

where the Pauli-matrices σ^i are

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{468}$$

By taking sums and matrix products the γ-matrices generate a Clifford algebra. Due to the defining relation, we can see that the antisymmetric products are sufficient to span the entire algebra. Let us define

$$\gamma^{\mu\nu} = \frac{1}{2}(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu), \tag{469}$$

$$\gamma^{\mu\nu\rho} = \frac{1}{3!}(\gamma^\mu\gamma^\nu\gamma^\rho + \gamma^\nu\gamma^\rho\gamma^\mu + \gamma^\rho\gamma^\mu\gamma^\nu - \gamma^\mu\gamma^\rho\gamma^\nu - \gamma^\nu\gamma^\mu\gamma^\rho - \gamma^\rho\gamma^\nu\gamma^\mu). \tag{470}$$

For the top element $\gamma^{\mu\nu\rho\sigma}$ it is more convenient to write

$$\gamma^{\mu\nu\rho\sigma} = -\epsilon^{\mu\nu\rho\sigma}\gamma^0\gamma^1\gamma^2\gamma^3, \tag{471}$$

where $\epsilon_{\mu\nu\rho\sigma}$ is the anti-symmetric Levi-Civita symbol, defined by

$$\epsilon_{\mu\nu\rho\sigma} = \begin{cases} +1 & \text{if } \mu\nu\rho\sigma \text{ even permutation of 0123,} \\ -1 & \text{if } \mu\nu\rho\sigma \text{ odd permutation of 0123,} \\ 0 & \text{ otherwise.} \end{cases} \tag{472}$$

In particular $\epsilon^{0123} = -1$.

We now want to prove some γ-matrix identities. First

$$\gamma^k \gamma^{ij} = \gamma^{kij} + \gamma^j \delta^{ik} - \gamma^i \delta^{jk} \tag{473}$$

Note that both sides of the result vanish if $i = j$. Thus suppose that $i \neq j$, then there are two different cases we need to consider, first if $k = i$, and if $k \neq i$ and $k \neq j$. In the first case $\gamma^k \gamma^i = \delta^{ik} = +1$, so that the result is γ^j. In the second case $\gamma^k \gamma^{ij} = \gamma^k \gamma^i \gamma^j = \gamma^{kij}$ because k, i and j are all distinct. Then the result follows by antisymmetry in i and j.

Next we prove

$$\begin{aligned} \gamma^{\mu\nu\rho}\gamma^{\sigma\tau} = {} & \gamma^{\mu\nu\tau}\eta^{\rho\sigma} + \gamma^{\rho\mu\tau}\eta^{\nu\sigma} + \gamma^{\nu\rho\tau}\eta^{\mu\sigma} - \gamma^{\mu\nu\sigma}\eta^{\rho\tau} - \gamma^{\rho\mu\sigma}\eta^{\nu\tau} \\ & - \gamma^{\nu\rho\sigma}\eta^{\mu\tau} + \gamma^{\mu}(\eta^{\nu\tau}\eta^{\rho\sigma} - \eta^{\rho\tau}\eta^{\nu\sigma}) \\ & + \gamma^{\nu}(\eta^{\rho\tau}\eta^{\mu\sigma} - \eta^{\rho\sigma}\eta^{\mu\tau}) + \gamma^{\rho}(\eta^{\mu\tau}\eta^{\nu\sigma} - \eta^{\mu\sigma}\eta^{\nu\tau}). \end{aligned} \tag{474}$$

To prove this, let us first note that μ, ν, ρ, σ and τ can not all be distinct. Furthermore, by antisymmetry no two indices on the same γ-matrix can be the same. It thus suffices to consider two cases, first when μ, ν, ρ and τ are all distinct and $\rho = \sigma$, second when μ, ν, ρ are all distinct and $\rho = \sigma$ and $\nu = \tau$. In the first case we have

$$\gamma^{\mu\nu\rho}\gamma^{\sigma\tau} = \gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma^\tau = \gamma^\mu\gamma^\nu\gamma^\tau\eta^{\rho\sigma} = \eta^{\rho\sigma}\gamma^{\mu\nu\tau}, \tag{475}$$

where we used that $\rho = \sigma$ and μ, ν, τ and σ are distinct. In the second case we have

$$\gamma^{\mu\nu\rho}\gamma^{\sigma\tau} = \gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma^\tau = \gamma^\mu\gamma^\nu\gamma^\tau\eta^{\rho\sigma} = \eta^{\rho\sigma}\eta^{\nu\tau}\gamma^{\mu}, \tag{476}$$

where μ, ν and ρ are all distinct and $\nu = \tau$ and $\mu = \sigma$. The other terms in the result then follow by using the anti-symmetry of the γ-matrices.

Finally we wish to consider

$$\gamma^{lk}\gamma^{ij} \tag{477}$$

Actually, this can be worked out from the previous identity by noting $\gamma^0\gamma^{lk} = \gamma^{0lk}$. Then

$$\gamma^{0lk}\gamma^{ij} = \gamma^{0lj}\delta^{ik} + \gamma^{k0i}\delta^{lj} - \gamma^{0li}\delta^{jk} - \gamma^{k0j}\delta^{li} + \gamma^0(\delta^{lj}\delta^{ik} - \delta^{kj}\delta^{il}). \tag{478}$$

Then multiplying by γ^0 yields the result

$$\gamma^{lk}\gamma^{ij} = \gamma^{lj}\delta^{ik} - \gamma^{kj}\delta^{li} - \gamma^{li}\delta^{jk} + \gamma^{ki}\delta^{lj} + \delta^{lj}\delta^{ik} - \delta^{kj}\delta^{il}. \tag{479}$$

E A Note on Frame Fields

In this appendix we collect some useful facts about the frame fields e^a_μ, and associated objects such as the spin-connection $\omega_{\mu ab}$. As before, we follow the treatment of Freedman and Van Proeyen [28]

The frame fields e^a_μ are related to the space-time metric $g_{\mu\nu}$ through

$$g_{\mu\nu} = e^a_\mu \eta_{ab} e^b_\nu, \tag{480}$$

clearly this equation only defines the frame fields up to local Lorentz transformations. The frame, or local Lorentz, indices a, b, ... can be raised and lowered with the Minkowski metric η_{ab}, while the space-time indices μ and ν are raised and lowered with the space-time metric $g_{\mu\nu}$. The frame field e^a_μ transforms as a vector under local Lorentz transformations and as a 1-form under space-time transformations. In particular, in the language of 1-forms we can write

$$e^a = e^a_\mu \mathrm{d}x^\mu. \tag{481}$$

Throughout we will always take the spin-connection $\omega_{\mu ab}$ to be torsion free, that is we define the spin-connection 1-form

$$\omega_{ab} = \omega_{\mu ab}\mathrm{d}x^\mu \tag{482}$$

through the first structure equation

$$\mathrm{d}e^a + \omega^a{}_b \wedge e^b = 0. \tag{483}$$

It is possible to consider also connections ω_{ab} with torsion, which have the torsion 2-form T^a on the right hand side of the first structure equation (483). Such connections appear naturally when considering Supergravity in the first order formalism, see [28]. Working in components, we can use the first structure equation to evaluate $\omega_{\mu ab}$ in terms of e^a_μ as

$$\begin{aligned}\omega_\mu{}^{ab} = \frac{1}{2}e^{a\nu}\left(\partial_\mu e^b_\nu - \partial_\nu e^b_\mu\right) - \frac{1}{2}e^{b\nu}\left(\partial_\mu e^a_\nu - \partial_\nu e^a_\mu\right)\\ - \frac{1}{2}\left(e^{a\nu}e^{b\rho} - e^{b\nu}e^{a\rho}\right)e_{c\mu}\partial_\nu e^c_\rho.\end{aligned} \tag{484}$$

The point of the spin-connection is to define derivatives which are covariant with respect to local Lorentz transformations of the frame. For example, if V^a is a local Lorentz vector, then $\partial_\mu V^a$ will not be a local Lorentz vector. We fix this by defining the Lorentz covariant derivative D_μ of a vector field by

$$D_\mu V^a = \partial_\mu V^a + \omega_\mu{}^a{}_b V^b. \tag{485}$$

Other tensor covariant derivatives are defined similarly, and importantly the Minkowski metric η_{ab} has vanishing covariant derivative $D_\mu \eta_{ab} = 0$. For a spinor such as the Majorana field ψ, we define the Lorentz covariant derivative by

$$D_\mu \Psi = \left(\partial_\mu + \frac{1}{4}\omega_{\mu ab}\gamma^{ab}\right)\Psi \tag{486}$$

Given the local Lorentz covariant derivatives, let V^ρ be a space-time vector, then define

$$\begin{aligned}\nabla_\mu V^\rho &= e^\rho_a D_\mu V^a \\ &= \partial_\mu V^\rho + e^\rho_a \left(\partial_\mu e^a_\nu + \omega_\mu{}^a{}_b e^b_\nu\right) V^\nu.\end{aligned} \tag{487}$$

We thus define an object $\Gamma^\rho{}_{\mu\nu}$ by

$$\Gamma^\rho{}_{\mu\nu} = e^\rho_a(\partial_\mu e^a_\nu + \omega_\mu{}^a{}_b e^b_\nu). \tag{488}$$

Inserting the definition of $\omega_{\mu ab}$ in terms of the frame field, we find that as the notation suggests $\Gamma^\rho{}_{\mu\nu}$ is indeed the Christoffel symbol

$$\Gamma^\rho{}_{\mu\nu} = \frac{1}{2}g^{\rho\sigma}\left(\partial_\mu g_{\sigma\nu} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu}\right). \tag{489}$$

Furthermore, the defining relation for $\Gamma^\rho{}_{\mu\nu}$ also tells us that

$$\nabla_\mu e^a_\nu = \partial_\mu e^a_\nu + \omega_\mu{}^a{}_b e^b_\nu - \Gamma^\rho{}_{\mu\nu} e^a_\rho = 0. \tag{490}$$

Then consider the space-time covariant derivative of the space-time γ-matrix γ_ν. This has three types of indices, two spinor indices (row and column) and a space-time index. Thus

$$\nabla_\mu \gamma_\nu = \partial_\mu \gamma_\nu + \frac{1}{4}\omega_{\mu ab}\left[\gamma^{ab}, \gamma^\nu\right]_- - \Gamma^\rho{}_{\mu\nu}\gamma_\rho \tag{491}$$

To evaluate this, write $\gamma_\nu = \gamma^c e_{c\nu}$ and use

$$[\gamma^{ab}, \gamma^c]_- = 2\gamma^a\eta^{bc} - 2\gamma^b\eta^{ac} \tag{492}$$

to note that

$$\nabla_\mu\gamma_\nu = \gamma^a\nabla_\mu e_{a\nu} = 0, \tag{493}$$

whence the space-time γ-matrices are covariantly conserved.

Finally, define the Riemann tensor $R_{\mu\nu}{}^{ab}$ in terms of the spin-connection by

$$R_{\mu\nu}{}^{ab} = \partial_\mu\omega_\mu{}^{ab} - \partial_\nu\omega_\nu{}^{ab} + \omega_\mu{}^a{}_c\omega_\nu{}^{cb} - \omega_\nu{}^a{}_c\omega_\mu{}^{cb}. \tag{494}$$

Then the curvature 2-form defined by

$$\rho^{ab} = \frac{1}{2}R_{\mu\nu}{}^{ab}\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu, \tag{495}$$

obeys the second structure equation

$$\mathrm{d}\omega^{ab} + \omega^a{}_c \wedge \omega^{cb} = \rho^{ab}. \tag{496}$$

F A Note on Helicities

In this appendix we verify the claims about the helicities of $\vec{\epsilon}^{\pm}(\vec{k})$ and $u^{\pm}(\vec{k})$. Helicity is defined as the projection of the spin $\vec{S}$ along the direction $\vec{k}$ of the momentum of a particle or field.

Under a rotation $R(\vec{\theta})$ the vector $\vec{A}$ transforms as

$$A_i \mapsto A'_i = R(\vec{\theta})_{ij} A_j, \tag{497}$$

where a general rotation is given by

$$R(\theta) = \exp\left[-i\vec{\theta}\cdot\vec{S}\right], \tag{498}$$

and the generators $\vec{S}$ are

$$S_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad S_2 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}, \quad S_3 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{499}$$

In particular, if we consider a rotation about the x-axis, so that $\vec{\theta} = (\theta, 0, 0)$, we find

$$R_1(\theta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix} \tag{500}$$

Notice that the generators S_1, S_2 and S_3 obey the commutation relations

$$[S_i, S_j] = i\epsilon_{ijk} S_k \tag{501}$$

Now, suppose that we choose $\vec{k} = k\vec{e}_1$, so that the momentum is purely in the x-direction. Then the helicity h is given by

$$h = \frac{\vec{S}\cdot\vec{k}}{|\vec{k}|} = S_1. \tag{502}$$

We can note that S_1 has eigenvalues $+1$, 0 and -1, with normalised eigenvectors $\vec{e}_1$, $\vec{\epsilon}^{\pm}(k\vec{e}_1)$ corresponding to the eigenvalues 0 and ± 1 respectively, where

$$\epsilon^{\pm}(k\vec{e}_1) = \frac{1}{\sqrt{2}}(\vec{e}_2 \pm i\vec{e}_3). \tag{503}$$

We quickly check that indeed

$$S_1\epsilon^{\pm}(k\vec{e}_1) = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ \pm i \end{pmatrix} = \pm\frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ \pm i \end{pmatrix} = \pm\epsilon^{\pm}(k\vec{e}_1). \tag{504}$$

Meanwhile, under a rotation $R(\vec{\theta})$, the spinor ψ_α transforms as

$$\psi_\alpha \mapsto \psi'_\alpha = D(\vec{\theta})_{\alpha\beta}\psi_\beta, \tag{505}$$

where the general $D(\vec{\theta})$ is given by

$$D(\vec{\theta}) = \exp\left[-i\vec{\theta}\cdot\vec{S}\right], \tag{506}$$

and this time the generators are given by

$$S_i = \frac{i}{8}\epsilon_{ijk}[\gamma^j, \gamma^k]. \tag{507}$$

Working explicitly in the Majorana representation for the γ-matrices,

$$S_1 = -\frac{1}{2}\begin{pmatrix} \sigma^2 & 0 \\ 0 & \sigma^2 \end{pmatrix}, \quad S_2 = \frac{i}{2}\begin{pmatrix} 0 & -\sigma^3 \\ \sigma^3 & 0 \end{pmatrix}, \quad S_3 = \frac{i}{2}\begin{pmatrix} 0 & \sigma^1 \\ -\sigma^1 & 0 \end{pmatrix}. \tag{508}$$

A quick calculation verifies that again

$$[S_i, S_j] = i\epsilon_{ijk}S_k. \tag{509}$$

Choosing again $\vec{k} = k\vec{e}_1$, so that the helicity operator is S_1. This has eigenvalues $\pm\frac{1}{2}$, both with multiplicity 2. Writing out S_1 in its 4×4 form,

$$S_1 = \frac{1}{2}\begin{pmatrix} 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \end{pmatrix}, \tag{510}$$

we see that a suitable basis normalised of eigenvectors is

$$v_1^{\pm} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ \mp i \\ 0 \\ 0 \end{pmatrix}, \quad v_2^{\pm} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 0 \\ 1 \\ \mp i \end{pmatrix}, \tag{511}$$

where $v_i^{\pm}$ have eigenvalue $\pm\frac{1}{2}$. In particular, it follows that

$$u^{\pm}(k\vec{e}_1) = \begin{pmatrix} \sqrt{k}\xi^{\pm} \\ \sqrt{k}\xi^{\pm} \end{pmatrix}, \qquad \xi^{\pm} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ \mp i \end{pmatrix}, \tag{512}$$

are of helicity $\pm\frac{1}{2}$ and also solve the massless the Dirac equation.

G More on the Quantised Zero-Momentum Gravitino Modes

In this appendix we consider the quantised zero-momentum sector for the gravitino. The approach follows [31]. In this sector we have self-adjoint anti-commuting operators η_α and η_α^A for $A = 1, 2$ and $\alpha = 1, 2, 3, 4$. They obey the anti-commutation relations

$$[\eta_\alpha, \eta_\beta]_+ = -\delta_{\alpha\beta}, \tag{513}$$

$$[\eta_\alpha^A, \eta_\beta^A]_+ = \delta_{\alpha\beta}\delta^{AB}, \tag{514}$$

with all other anti-commutators vanishing. We wish to find a suitable space on which these operators can act. To this end, we can combine them into creation- and annihilation-type by defining

$$
\begin{aligned}
d_1 &= \frac{1}{\sqrt{2}}(\eta_1 + i\eta_2), \\
d_2 &= \frac{1}{\sqrt{2}}(\eta_3 + i\eta_4),
\end{aligned} \tag{515}
$$

and for $A = 1, 2$

$$
\begin{aligned}
d_1^A &= \frac{1}{\sqrt{2}}(\eta_1^A + i\eta_2^A), \\
d_2^A &= \frac{1}{\sqrt{2}}(\eta_3^A + i\eta_4^A),
\end{aligned} \tag{516}
$$

These operators then obey the anti-commutation relations

$$[d_i, d_j^\dagger]_+ = -\delta_{ij}, \tag{517}$$

$$[d_i^A, d_j^{A\dagger}]_+ = \delta_{ij}\delta^{AB}. \tag{518}$$

where $i, j = 1, 2$. We thus have have 6 pairs of operators which obey fermionic creation- and annihilation-type anti-commutation relations. We can thus have these operators acting on a $2^6 = 64$ dimensional space, by defining a "vacuum" state $|0\rangle$ annihilated by each d_i and d_i^A, that is

$$d_i|0\rangle = d_i^A|0\rangle = 0, \tag{519}$$

and we choose this state to be (positively) normalised, $\langle 0 | 0 \rangle = +1$. Other states are then built by acting with $d_i^\dagger$ and $d_i^{A\dagger}$. Note that not all states in this space will have positive norm. For example

$$\|d_i^\dagger|0\rangle\|^2 = \langle 0|d_i d_i^\dagger|0\rangle = -1. \tag{520}$$

Part II

Automorphic Scalar Fields in two-dimensional de Sitter Space

8 Introduction

Lower dimensional models are frequently useful in understanding novel effects in quantum field theory as the relative simplicity of these situations may allow for the existence of exact solutions. The hope is that such situations provide useful toy models and that certain behaviours can be generalised to more realistic situations.

Unlike higher dimensional de Sitter spaces, the two-dimensional de Sitter space dS_2, is not simply connected. Correspondingly, the behaviour of the fields under complete traversals of non-contractible loops must be specified. This topological non-triviality allows for unexpected behaviour. For Dirac spinor fields in two-dimensional de Sitter space, Epstein and Moschella [36] found that an anti-periodic Neveu-Schwarz boundary condition is more natural than a periodic Ramond boundary condition. On conformally mapping the spinors from a Lorentzian cylinder to two-dimensional de Sitter space, only the anti-periodic spinor fields possess a form of invariance under all de Sitter transformations. Furthermore, for free periodic and anti-periodic real scalar fields on two-dimensional de Sitter [37, 38], Epstein and Moschella showed that behaviour of the anti-periodic scalar fields is quite different from the periodic scalars. For masses corresponding to the complementary series in the periodic case, the anti-periodic fields never admit de Sitter invariant two-point functions. This can be understood from the representation theory of $\mathrm{SL}(2,\mathbb{R})$, where there are no unitary irreducible representations corresponding to this mass range. Epstein and Moschella also showed that for the anti-periodic case there does not exist a natural analogue of the Bunch-Davies vacuum state [39, 40, 41] for any value of the mass, and correspondingly one loses the associated Gibbons-Hawking thermal state [42].

More generally, the non-trivial fundamental group $\pi_1(\mathrm{dS}_2) = \mathbb{Z}$ of the two-dimensional de Sitter space allows for the existence of automorphic scalar fields [43, 44, 45, 46, 47]. The automorphic scalar fields are generically complex scalar fields, which transform under a unitary representation of $\pi_1(\mathrm{dS}_2)$ on traversal of the non-contractible loop. Working in global coordinates on the de Sitter manifold, this can be expressed as the scalar $\Phi(t,\phi)$ having the following periodicity condition imposed

$$\Phi(t,\phi+2\pi) = e^{2\pi i\beta}\Phi(t,\phi), \tag{521}$$

where β is a real number. These fields can naturally be viewed as single-valued fields on the universal covering space $\widetilde{\mathrm{dS}}_2$ of two-dimensional de Sitter space, and transform under representations of the $\widetilde{\mathrm{SL}}(2,\mathbb{R})$, the universal covering group of the de Sitter symmetry group $\mathrm{SO}_0(2,1)$. In this part, we study properties of the automorphic scalar field and analyse the implications of a quantum field theory built upon it that different values of the phase parameter β yield. In particular we investigate compatibility between de Sitter invariance and the Hadamard condition for the resulting states associated to the automorphic field. We find that in general only the periodic $\beta = 0$ fields have a de Sitter invariant Hadamard vacuum state; we do this by constructing the two-point functions for the de Sitter invariant states and finding that they do not have the correct singularity structure to be locally Hadamard.

8.1 Organisation of this Chapter

The remainder of this chapter is organized as follows.

We begin with a review of two-dimensional de Sitter space. We introduce the coordinate systems used and review the causal and geodesic structure of the spacetime. Finally, we write down the Killing vectors of the spacetime, and recall that they form an $\mathrm{sl}(2,\mathbb{R})$ algebra.

We then review the canonical quantisation of automorphic scalar fields in two-dimensional de Sitter space. We decompose the field into mode functions which automatically satisfy the automorphic condition, and then split the space of solutions into a positive and a negative norm space with respect to the Klein-Gordon inner product on two-dimensional de Sitter space. This then allows us to define annihilation- and creation-type operators and build up a Fock space for the theory in a usual manner.

Following on from this, we then start to impose additional restrictions on the mode functions (or equivalently on the states). We first investigate when the Fock vacuum state is de Sitter invariant, which requires the mode functions to form a basis for a unitary irreducible representation of the symmetry group. Having found the symmetric states, we then additionally want to check if they are in a sense "physically reasonable", which we do by asking that the states obey the Hadamard condition.

Finally, we define a class of de Sitter non-invariant states which obey the Hadamard condition for all possible automorphic fields in two-dimensional de Sitter space.

The main content of the chapter is supplemented by three appendices. In the first appendix we review the unitary irreducible representations of the $\mathrm{sl}(2,\mathbb{R})$ algebra. In the second appendix we consider certain automorphic sums, which are encountered when considering the de Sitter non-invariant Hadamard states. In the final appendix we prove a formula related to Legendre equations used in the main part of the text.

9 Geometry of Two-Dimensional de Sitter Space

Two dimensional de Sitter space can be simply realised as a hyperboloid embedded within a three-dimensional Minkowski space. Let X^0, X^1 and X^2 be coordinates for the Minkowski space, with metric

$$\mathrm{d}s^2 = -(\mathrm{d}X^0)^2 + (\mathrm{d}X^1)^2 + (\mathrm{d}X^2)^2. \tag{522}$$

Then the de Sitter hyperboloid is defined by the equation

$$-(X^0)^2 + (X^1)^2 + (X^2)^2 = 1. \tag{523}$$

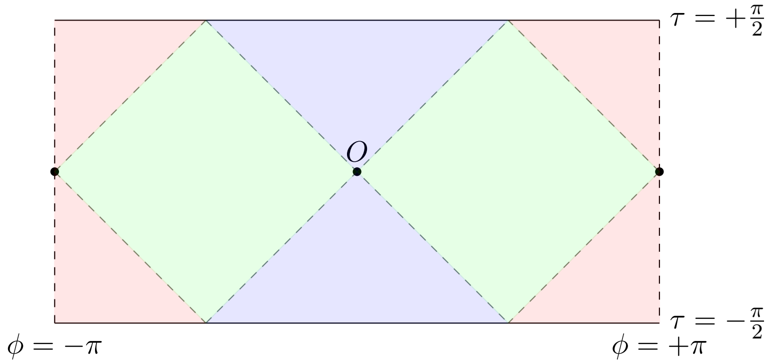

Figure 1: Carter-Penrose diagram for two-dimensional de Sitter space. The green area can be connected to the origin O by a space-like geodesic. The blue area can be connected to the origin by a time-like geodesic. The red shaded area can not be connected to the origin by a geodesic.

A suitable set of coordinates are the global coordinates (t, ϕ) defined by

$$\begin{aligned} X^0 &= \sinh t, \\ X^1 &= \cosh t \cos \phi, \\ X^2 &= \cosh t \sin \phi, \end{aligned} \tag{524}$$

which cover the entire de Sitter manifold, with $t \in (-\infty, \infty)$ and $\phi \sim \phi + 2\pi$. In the global coordinates the metric for de Sitter space is given by

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \cosh^2 t \mathrm{d}\phi^2 \tag{525}$$

The spatial slices in these coordinate systems are constant X^0 or t slices, which are circles with a contracting, then expanding radius $r = \cosh t$.

As a two-dimensional metric, this is conformally flat, which can be seen by going to conformal coordinates (τ, ϕ) with $\tau \in (-\frac{\pi}{2}, \frac{\pi}{2})$ by

$$\sinh t = \tan \tau. \tag{526}$$

Then the metric in conformal coordinates takes the form

$$\mathrm{d}s^2 = \sec^2 \tau(-\mathrm{d}\tau^2 + \mathrm{d}\phi^2). \tag{527}$$

From this, we can read off the causal structure of two-dimensional de Sitter space, as seen in the Carter-Penrose diagram in Figure 1.

The embedding space picture also makes it convenient to talk about the geodesic structure of de Sitter space. As is the case for geodesics on spheres, the geodesics of de Sitter space are the intersections of the hyperboloid and planes through the origin [48]. Thus, following Synge, we can easily determine the geodesics as follows. Let $X^A = P^A$, with $A = 0, 1, 2$ be a point on the de Sitter hyperboloid and suppose that the geodesic goes along the direction Q^A from P^A. To remain on the hyperboloid, these are constrained by

$$\eta_{AB}P^AP^B = 1, \qquad \eta_{AB}P^AQ^B = 0, \tag{528}$$

where $\eta_{AB} = \text{diag}(-,+,+)$ is the metric on the Minkowski space. Now the plane containing P^A, Q^A and also the origin in Minkowski space can then be given parametrically by

$$X^A(p,q) = pP^A + qQ^A, \tag{529}$$

where $p, q \in (-\infty, \infty)$. Then the intersection between this plane and the hyperboloid is determined by

$$\eta_{AB}X^A(p,q)X^B(p,q) = p^2 + q^2\eta_{AB}Q^AQ^B = 1. \tag{530}$$

Now the value of $\eta_{AB}Q^AQ^B$ depends on whether the geodesic is null, spacelike or timelike. If the geodesic is null, then $\eta_{AB}Q^AQ^B = 0$ and $p = 1$, so the geodesics are given by the straight lines

$$X^A(q) = P^A + qQ^A. \tag{531}$$

For a space-like geodesic, using proper-length l as the parameter along the geodesic, we have $\eta_{AB}Q^AQ^B = +1$, so that

$$p^2 + q^2 = 1, \tag{532}$$

and therefore the geodesics are given by

$$X^A(l) = P^A \cos l + Q^A \sin l. \tag{533}$$

In particular, note that all the space-like geodesics emanating from P^A will intersect again after a geodesic distance $l = \pi$ at the anti-podal point $-P^A$.

For the time-like geodesics we use proper-time τ as the parameter along the geodesic, so that $\eta_{AB}Q^AQ^B = -1$. The geodesic is thus given by

$$X^A(\tau) = P^A \cosh\tau + Q^A \sinh\tau. \tag{534}$$

Let x and y be two points in two-dimensional de Sitter space and let $X^A(x)$ and $X^A(y)$ be the corresponding embedding space coordinates for these points. Then the hyperbolic distance $Z(x,y)$ between these points is defined as

$$Z(x,y) = \eta_{AB}X^A(x)X^B(y). \tag{535}$$

If x and y can be connected by a geodesic then we note that

$$Z(x,y) = \cos\mu(x,y) = \begin{cases} \cos l(x,y) & \text{if space-like,} \\ \cosh\tau(x,y) & \text{if time-like,} \\ 1 & \text{if null.} \end{cases} \tag{536}$$

Thus $\mu(x,y)$ is the geodesic distance between x and y if they are space-like separated and proportional to the proper time between x and y if they are time-like separated. Conversely, any two-point with $Z(x,y) > -1$ can be connected by geodesics. Meanwhile, the point x can not be connected by a geodesic to the interior of the light-cone of the anti-podal point $\bar{x}$, because here $Z < -1$. When $Z = -1$ the points are connected to the anti-podal point by a null geodesic and only the antipodal point $\bar{x}$ itself can be connected to x by a geodesic, as can be seen from (533). For points x and y which are not connected by a geodesic, we use the relation

$$Z(x, y) = \eta_{AB} X^A(x) X^B(y) = \cos\mu(x, y) \tag{537}$$

to define what we mean by $\cos\mu(x, y)$. Note that this leaves some ambiguity in the definition of $\mu(x, y)$, but we will deal only with $\cos\mu(x, y)$ which does not suffer from this ambiguity.

The symmetries which leave the hyperboloid invariant are those symmetries of the embedding Minkowski space which leave the origin invariant. Thus the symmetry group is the group SO(2,1) of Lorentz transformations in $(2+1)$-dimensions. As a Lorentz transformation, the SO(2,1) element Λ acts on the embedding space coordinates by

$$X^A \mapsto X'^A = \Lambda^A{}_B X^B. \tag{538}$$

The symmetry algebra SO(2,1) is generated by a single rotation L and two boosts B_1 and B_2. Acting on functions of the global coordinates (t, ϕ), these generators are

$$\begin{aligned} L &= X^1 \frac{\partial}{\partial X^2} - X^2 \frac{\partial}{\partial X^1} = \frac{\partial}{\partial \phi}, \\ B_1 &= X^1 \frac{\partial}{\partial X^0} + X^0 \frac{\partial}{\partial X^1} = \cos\phi \frac{\partial}{\partial t} - \tanh t \sin\phi \frac{\partial}{\partial \phi}, \\ B_2 &= X^2 \frac{\partial}{\partial X^0} + X^0 \frac{\partial}{\partial X^2} = \sin\phi \frac{\partial}{\partial t} + \tanh t \cos\phi \frac{\partial}{\partial \phi}. \end{aligned} \tag{539}$$

The algebra obeyed by these generators is

$$[L, B_1] = -B_2, \quad [L, B_2] = B_1, \quad [B_1, B_2] = L. \tag{540}$$

This algebra admits a quadratic Casimir element Q given by

$$Q = -L^2 + B_1^2 + B_2^2. \tag{541}$$

In terms of the generators in global coordinates this is

$$Q = \frac{\partial^2}{\partial t^2} + \tanh t \frac{\partial}{\partial t} - \frac{1}{\cosh^2 t} \frac{\partial^2}{\partial \phi^2}, \tag{542}$$

which we can recognize as being (minus) the box-operator of two-dimensional de Sitter space

$$\Box = \frac{1}{\sqrt{-g}} \partial_\mu \left(\sqrt{-g} g^{\mu\nu} \partial_\nu \right) = -Q. \tag{543}$$

This algebra can also be realised by the $\mathrm{sl}(2, \mathbb{R})$ matrices

$$L = \frac{1}{2} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad B_1 = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad B_2 = \frac{1}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{544}$$

which is expected because $SL(2,\mathbb{R})$ is a double covering of $SO_0(2,1)$, which is the component of $SO(2,1)$ connected to the identity.

10 Canonical Quantisation in two-dimensional de Sitter Space

Let $\Phi(x)$ be a classical, generically complex, Klein-Gordon scalar field on two-dimensional de Sitter space. This field obeys the Klein-Gordon equation

$$(\Box - M^2)\Phi(x) = 0, \tag{545}$$

where M^2 is a real number. Generically, we could also have a term proportional to the Ricci scalar R, however this is a constant for de Sitter space, and thus for compactness has been absorbed into the mass parameter M. Further, suppose that the field Φ is automorphic in the sense that it picks up a phase when making a full traversal of the spatial sections. In global coordinates, this can be implemented as

$$\Phi(t, \phi + 2\pi) = e^{2\pi i\beta}\Phi(t, \phi), \tag{546}$$

where the phase $\beta \in [0, \frac{1}{2}]$. Generically this requires the field Φ to be complex, however there are two special values $\beta = 0$ and $\beta = \frac{1}{2}$, corresponding to periodic and anti-periodic boundary condition, which also allow for real fields. These real fields have previously been studied by Epstein and Moschella [37, 38], who realised them as single-valued fields living on the double cover of two-dimensional de Sitter space.

To quantize this system, we will follow the canonical approach, as in Birrell and Davis [49], Parker and Toms [50] or Wald [51]. As part of the canonical quantisation, we promote $\Phi(x)$ to an operator-valued quantum field and impose the curved-space versions of equal-time canonical commutation relations on the field $\Phi(x)$ and its canonical momentum conjugate $\Pi(x)$. We will only work with two-dimensional de Sitter space, which has a well defined causal structure. Therefore, working in global coordinates (t, ϕ) we may use t as a time coordinate, and the Cauchy surfaces of constant t act as suitable equal-time surfaces. To define the canonical momentum-conjugate $\Pi(x)$, we note that the Klein-Gordon equation of motion can be derived from the Lagrangian density

$$\begin{aligned} \mathcal{L} &= \sqrt{-g}\left(-g^{\mu\nu}\partial_\mu\Phi^\dagger(x)\partial_\nu\Phi(x) - M^2\Phi^\dagger(x)\Phi(x)\right) \\ &= \cosh t\left(\frac{\partial\Phi^\dagger}{\partial t}\frac{\partial\Phi}{\partial t} - \frac{1}{\cosh^2 t}\frac{\partial\Phi^\dagger}{\partial\phi}\frac{\partial\Phi}{\partial\phi} - M^2\Phi^\dagger\Phi\right), \end{aligned} \tag{547}$$

where we have used $\Phi^\dagger(x)$ to denote the complex conjugate field to $\Phi(x)$, anticipating the later quantisation. Then, defining the momentum conjugate as in flat space yields

$$\Pi(x) = \frac{\partial\mathcal{L}}{\partial\partial_0\Phi(x)} = \cosh t\frac{\partial\Phi^\dagger}{\partial t}(t, \phi). \tag{548}$$

Then the appropriate equal-time commutation relations to impose on the fields are

$$\big[\Phi(t,\phi),\Phi(t,\phi')\big]=0 \tag{549}$$

$$\big[\Phi(t,\phi),\Pi(t,\phi')\big]=i\delta_\beta(\phi-\phi') \tag{550}$$

$$\big[\Pi(t,\phi),\Pi(t,\phi')\big]=0, \tag{551}$$

where we have defined the δ-function $\delta_\beta(\phi-\phi')$ by

$$\delta_\beta(\phi-\phi')=\sum_{m\in\mathbb{Z}+\beta}\frac{1}{2\pi}e^{im(\phi-\phi')}=\delta_0(\phi-\phi')e^{i\beta(\phi-\phi')}, \tag{552}$$

where $\delta_0(\phi-\phi')$ is the usual periodic δ-function on a circle. This ensures that if $f(\phi)$ is a smooth automorphic function obeying $f(\phi+2\pi)=e^{2\pi i\beta}f(\phi)$, then one has

$$\int_0^{2\pi}\mathrm{d}\phi'\ \delta_\beta(\phi-\phi')f(\phi')=f(\phi). \tag{553}$$

Similar relations are imposed for the complex-conjugate field and its associated canonical conjugate momentum. As in flat space, to make progress it will be convenient to decompose the free field into independent modes. To this end, on the space of classical solutions of the Klein-Gordon equation we define a Klein-Gordon type inner product by

$$(\Phi_1,\Phi_2)(t)=i\int_0^{2\pi}\mathrm{d}\phi\cosh t\left[\Phi_1^\dagger(t,\phi)\frac{\partial\Phi_2}{\partial t}(t,\phi)-\frac{\partial\Phi_1^\dagger}{\partial t}(t,\phi)\Phi_2(t,\phi)\right]. \tag{554}$$

This product is conserved in the sense that

$$(\Phi_1,\Phi_2)(t_1)=(\Phi_1,\Phi_2)(t_2), \tag{555}$$

provided that Φ_1 and Φ_2 are both solutions to the classical Klein-Gordon equation. As in flat space, this product is not necessarily positive. Let $\mathcal{S}_\beta$ be the classical space of solutions all with the same automorphy condition labelled by the shared value β. We look to split $\mathcal{S}_\beta$ into a positive and a negative subspace, labelled by $\mathcal{S}_\beta^+$ and $\mathcal{S}_\beta^-$ respectively so that

$$\mathcal{S}_\beta=\mathcal{S}_\beta^+\oplus\mathcal{S}_\beta^-. \tag{556}$$

That is to say, if $\Psi_+\in\mathcal{S}_\beta^+$ and $\Psi_-\in\mathcal{S}_\beta^-$, then we have

$$(\Psi_+,\Psi_+)>0,\qquad(\Psi_+,\Psi_-)=0,\qquad(\Psi_-,\Psi_-)<0. \tag{557}$$

In canonical quantisation, the positive definite subspace $\mathcal{S}_\beta^+$ provides the one-particle subspace of the theory.

Let us now construct a the subspaces $\mathcal{S}_\beta^+$ and $\mathcal{S}_\beta^-$. We can satisfy the automorphy condition automatically if we look for modes of the form

$$\Phi(t,\phi)=\Phi_m(t)e^{im\phi}, \tag{558}$$

where $m \in \mathbb{Z} + \beta$. The equation satisfied by the functions $\Phi_m(t)$ is

$$\frac{1}{\cosh t}\frac{d}{dt}\left(\cosh t\frac{d\Phi_m}{dt}\right) + \left(\frac{m^2}{\cosh^2 t} + M^2\right)\Phi_m = 0. \tag{559}$$

This is a second order ordinary differential equation, and in particular it is real. Thus, if we assume $\Phi_m(t)$ and $\Phi_m^*(t)$ are linearly independent, then the space of solutions is spanned by $\Phi_m(t)$ and $\Phi_m^*(t)$. Now suppose that we define

$$F_m(t,\phi) = \Phi_m(t)e^{im\phi} \tag{560}$$

and assume that these form an orthonormal basis of the positive subspace, so that

$$(F_m, F_n) = \delta_{mn}. \tag{561}$$

Then a suitable orthonormal basis for $\mathcal{S}_\beta^-$ can be formed by

$$G_m^*(t,\phi) = \Phi_m^*(t)e^{im\phi}, \tag{562}$$

and from the definition of the Klein-Gordon inner product we can note that these functions obey

$$(G_m^*, G_n^*) = -\delta_{mn}, \tag{563}$$

$$(F_m, G_m^*) = 0. \tag{564}$$

The completeness of these functions means that an arbitrary solution Ψ of the Klein-Gordon equation can be expanded as

$$\Psi(t,\phi) = \sum_{m\in\mathbb{Z}+\beta}\left[(F_m,\Psi)F_m(t,\phi) - (G_m^*,\Psi)G_m^*(t,\phi)\right]. \tag{565}$$

Expanding out the Klein-Gordon products equation (565) can be rearranged to

$$\begin{aligned}\Psi(t,\phi) = i\int_0^{2\pi} \mathrm{d}\phi'\ \Psi(\phi',t)\cosh t\sum_m\left[-\partial_t F_m^*(\phi',t)F_m(\phi,t) + \partial_t G_m(\phi',t)G_m^*(\phi,t)\right]\\ +i\int_0^{2\pi}\mathrm{d}\phi'\ \partial_t\Psi(\phi',t)\cosh t\sum_m\left[F_m^*(\phi',t)F_m(\phi,t) - G_m(\phi',t)G_m^*(\phi,t)\right].\end{aligned} \tag{566}$$

As $\Psi(t,\phi)$ in (565) was assumed to be an arbitrary solution of the Klein-Gordon equation, we can choose $\Psi(t,\phi)$ and $\partial_t\Psi(\phi,t)$ independently at a given time t. Therefore we can read off that the completeness requires

$$\cosh t\sum_m\left[\partial_t F_m^*(\phi',t)F_m(\phi,t) - \partial_t G_m(\phi',t)G_m^*(\phi,t)\right] = i\delta_\beta(\phi-\phi'), \tag{567}$$

and

$$\sum_m\left[F_m^*(\phi',t)F_m(\phi,t) - G_m(\phi',t)G_m^*(\phi,t)\right] = 0. \tag{568}$$

Similarly, considering the expansion of $\partial_t \Psi(t,\phi)$ yields the further condition

$$\sum_m \left[\partial_t F_m^*(t,\phi')\partial_t F_m(t,\phi) - \partial_t G_m(t,\phi')\partial_t G_m^*(t,\phi)\right] = 0. \tag{569}$$

With this in mind, now we use the basis $\{F_m, G_m^*\}$ to expand the quantum field as

$$\Phi(t,\phi) = \sum_{m\in\mathbb{Z}+\beta} \left[a_m F_m(t,\phi) + b_m^\dagger G_m^\dagger(t,\phi)\right], \tag{570}$$

Comparing with the expansion of an arbitrary solution, we can read off that

$$a_m = (F_m, \Phi), \quad b_m^\dagger = -(G_m^*, \Phi). \tag{571}$$

Then we can calculate

$$\begin{aligned}
[a_m, a_n^\dagger] &= -[(F_m,\Phi),(F_n^*,\Phi^\dagger)] \\
&= \int_0^{2\pi} \mathrm{d}\phi \mathrm{d}\phi' \ \cosh^2 t \bigg(- \frac{\partial F_m^*}{\partial t}(t,\phi) F_n(t,\phi') \left[\Phi(t,\phi), \frac{\partial \Phi^\dagger}{\partial t}(t,\phi')\right] \\
&\qquad - F_m^*(t,\phi)\frac{\partial F_n}{\partial t}(t,\phi')\left[\frac{\partial \Phi^\dagger}{\partial t}(t,\phi), \Phi(t,\phi')\right]\bigg) \\
&= -i \int_0^{2\pi} \mathrm{d}\phi \ \cosh t \left(\frac{\partial F_m^*}{\partial t}(t,\phi) F_n(t,\phi) - F_m^*(t,\phi)\frac{\partial F_n}{\partial t}(t,\phi)\right) \\
&= (F_m, F_n) = \delta_{mn}
\end{aligned} \tag{572}$$

Similarly, we can show that

$$[b_m, b_n^\dagger] = \delta_{mn}, \tag{573}$$

and all other commutators vanish, that is the a_m and b_n obey independent ladder operator algebras.

Conversely, it is also possible to show that if we assume a_m and b_m obey independent ladder operator algebras, and also assume the form of the field operator in terms of the ladder operators then the associated fields obey the canonical commutation relations. For instance,

$$\begin{aligned}
&\left[\Phi(t,\phi), \frac{\partial \Phi^\dagger}{\partial t}(t,\phi')\right] \\
&= \sum_{m,n}\left(F_m(t,\phi)\frac{\partial F_n^*}{\partial t}(t,\phi')[a_m, a_n^\dagger] + G_m^*(t,\phi)\frac{\partial G_m}{\partial t}(t,\phi')[b_m^\dagger, b_n]\right) \\
&= \sum_{m\in\mathbb{Z}}\left(F_m(t,\phi)\frac{\partial F_m^*}{\partial t}(t,\phi') - G_m^*(t,\phi)\frac{\partial G_m}{\partial t}(t,\phi')\right) \\
&= \frac{i}{\cosh t}\delta_\beta(\phi-\phi').
\end{aligned} \tag{574}$$

Given this, the Fock space of states is defined as follows. Let $|0\rangle$ be the vacuum state defined by

$$a_m|0\rangle = b_m|0\rangle = 0, \qquad \text{for all } m \in \mathbb{Z}+\beta. \tag{575}$$

Other states are then created by acting on $|0\rangle$ with the creation operators $a_m^\dagger$ and $b_m^\dagger$.

It is important to note that the Fock vacuum state is not defined uniquely. The non-uniqueness can be traced back to the choice of basis $\{F_m, G_m^*\}$ for the classical space of solutions $\mathcal{S}_\beta$. For instance, suppose that we define

$$\begin{aligned} f_m(t,\phi) &= \alpha_m F_m(t,\phi) + \beta_m G_m^*(t,\phi), \\ g_m^*(t,\phi) &= \alpha_m^* G_m^*(t,\phi) + \beta_m^* F_m(t,\phi). \end{aligned} \tag{576}$$

Then we get a new orthonormal basis $\{f_m, g_m^*\}$ if we choose

$$|\alpha_m|^2 - |\beta_m|^2 = 1. \tag{577}$$

In this context, this kind of change of basis is usually known as a Bogoliubov transformation. With respect to this new basis, we can expand

$$\Phi(t,\phi) = \sum_{m\in\mathbb{Z}} \left(A_m f_m(t,\phi) + B_m^\dagger g_m^*(t,\phi) \right), \tag{578}$$

so we can relate the new ladder operators A_m and B_m to the old by

$$a_m = \alpha_m A_m + \beta_m^* B_m^\dagger, \qquad b_m^\dagger = \beta_m A_m + \alpha_m B_m^\dagger. \tag{579}$$

Now suppose that $|0'\rangle$ is the Fock vacuum state defined with respect to A_m and B_m. We calculate the expected number of b_m-excitations in this state by

$$\langle 0'|b_m^\dagger b_m|0'\rangle = |\beta_m|^2 \langle 0'|A_m A_m^\dagger|0'\rangle = |\beta_m|^2. \tag{580}$$

Thus if $\beta_m \neq 0$, $|0\rangle$ and $|0'\rangle$ are inequivalent vacuum states. In particular, if we think of b_m-excitations as "particles", not all observers will agree on their definition of particles or the number of particles in a given state. In general it is also possible to consider Bogoliubov transformations which mix different values of m, however we will restrict to considering only transformations for fixed m.

With these considerations in mind, let us return to the case of two-dimensional de Sitter space. The functions $\Phi_m(t)$ satisfy the equation

$$\frac{1}{\cosh t}\frac{d}{dt}\left(\cosh t \frac{d\Phi_m}{dt}\right) + \left(\frac{m^2}{\cosh^2 t} + M^2\right)\Phi_m = 0. \tag{581}$$

If we let $u = i\sinh t$, then this equation can be recast as

$$\frac{d}{du}\left((1-u^2)\frac{d}{du}\Phi_m\right) + \left(-M^2 - \frac{m^2}{1-u^2}\right)\Phi_m = 0, \tag{582}$$

so that if we make the identification

$$M^2 = -l(l+1), \tag{583}$$

then this can be recognized as an associated Legendre equation with labels m and l [52, Eq. 8.700]. We assume that the mass-square is always positive, so that $-1 < l < 0$ or $l \in -\frac{1}{2} + i\mathbb{R}$. Notice that we can further restrict the value of l by noting that M^2 is unchanged if we replace l by $-(l+1)$, so that we may take $-\frac{1}{2} < l < 0$. The solutions are Ferrer's functions [53], defined in terms of hypergeometric functions as

$$\mathsf{P}_l^{-m}(u) = \frac{1}{\Gamma(1+m)}\left(\frac{1-u}{1+u}\right)^{m/2} F\left(1+l, -l; 1+m; \frac{1-u}{2}\right). \tag{584}$$

These functions have a branch cut running from $u = +1$ to ∞ and from $u = -1$ to $-\infty$ along the real axis. Furthermore, we note that for the values of m and l we consider the identity

$$\mathsf{P}_l^{-m}(u)^* = \mathsf{P}_l^{-m}(u^*), \tag{585}$$

holds and that $\mathsf{P}_l^{-m}(i\sinh t)$ and $\mathsf{P}_l^{-m}(-i\sinh t)$ are linearly independent provided that $l + m \notin \mathbb{Z}$ [52]. For the values of l and m we consider in this thesis it will always be the case that $l + m \notin \mathbb{Z}$. Then, we set

$$\Phi_m(t) = \sqrt{\frac{\Gamma(m+l+1)\Gamma(m-l)}{4\pi}}\mathsf{P}_l^{-m}(i\sinh t), \tag{586}$$

then the associated modes

$$F_m(t,\phi) = \sqrt{\frac{\Gamma(m+l+1)\Gamma(m-l)}{4\pi}}\mathsf{P}_l^{-m}(i\sinh t)e^{im\phi}, \tag{587}$$

$$G_m^*(t,\phi) = \sqrt{\frac{\Gamma(m+l+1)\Gamma(m-l)}{4\pi}}\mathsf{P}_l^{-m}(-i\sinh t)e^{im\phi}. \tag{588}$$

We will further restrict the allowed value of l to

$$l \in -\frac{1}{2} + i\mathbb{R}, \quad \text{or} \quad -\frac{1}{2} < l < -|\beta|, \tag{589}$$

with $m \in \mathbb{Z} + \beta$. These restrictions ensure that $(m \pm l \pm 1)(m \mp l)$ are always positive, see Appendix H. It follows also then that $\Gamma(m+l+1)\Gamma(m-l) > 0$. The normalisation $(F_m, F_n) = -(G_m^*, G_n^*) = \delta_{mn}$ follows as

$$\begin{aligned}
&i\int_0^{2\pi} \mathrm{d}\phi\ e^{i(n-m)\phi}\cosh t \\
&\left[\mathsf{P}_l^{-m}(-i\sinh t)\frac{d}{dt}\mathsf{P}_l^{-m}(i\sinh t) - \frac{d}{dt}\mathsf{P}_l^{-m}(-i\sinh t)\mathsf{P}_l^{-m}(i\sinh t)\right] \\
&= 2\pi\delta_{mn}(1-u^2)\left[\mathsf{P}_l^{-m}(u)\frac{d}{du}\mathsf{P}_l^{-m}(-u) - \frac{d}{du}\mathsf{P}_l^{-m}(u)\mathsf{P}_l^{-m}(-u)\right] \\
&= \frac{4\pi}{\Gamma(m-l)\Gamma(m+l+1)}\delta_{mn},
\end{aligned} \tag{590}$$

where we made use of the Wronskian identity [54, Eq. 14.2.3]

$$\mathsf{P}_l^{-m}(u)\frac{d}{du}\mathsf{P}_l^{-m}(-u) - \frac{d}{du}\mathsf{P}_l^{-m}(u)\mathsf{P}_l^{-m}(-u) = \frac{2}{\Gamma(m-l)\Gamma(m+l+1)(1-u^2)}. \tag{591}$$

These modes satisfy the completeness relations. As we have said, we can also make a Bogoliubov transformation in each mode. Thus, absorbing an overall phase, we will consider mode functions of the form

$$f_m(t,\phi) = \cosh\alpha_m F_m(t,\phi) + e^{i\gamma_m}\sinh\alpha_m G_m^*(t,\phi), \tag{592}$$

$$g_m^*(t,\phi) = e^{-i\gamma_m}\sinh\alpha_m F_m(t,\phi) + \cosh\alpha_m G_m^*(t,\phi), \tag{593}$$

where γ_m and α_m are real numbers.

At this point, it is convenient to make contact with the quantisation conditions of Epstein and Moschella [37]. We recall that they consider real scalar fields, corresponding to the values $\beta = 0$ and $\beta = \frac{1}{2}$. For a real scalar field, we ought to be able to use $\{f_m, f_n^*\}$ as a basis, thus we need to choose α_m and γ_m such that

$$(f_m, f_n^*) = 0, \tag{594}$$

for all m and n when $\beta = 0$ or $\beta = \frac{1}{2}$. Taking the complex conjugate of f_m gives

$$f_m^*(t,\phi) = \cosh\alpha_m F_m^*(t,\phi) + e^{-i\gamma_m}\sinh\alpha_m G_m(t,\phi). \tag{595}$$

As F_m^* and G_m^* are both proportional to $e^{-im\phi}$ we should be able to express $f_m^*(t,\phi)$ in terms of $F_{-m}(t,\phi)$ and $G_{-m}^*(t,\phi)$. Indeed, if we use the connection formula [54, Eq. 14.2.7]

$$\frac{\sin(l-m)\pi}{\Gamma(m+l+1)}\mathsf{P}_l^m(u) = \frac{\sin l\pi}{\Gamma(l-m+1)}\mathsf{P}_l^{-m}(u) - \frac{\sin m\pi}{\Gamma(l-m+1)}\mathsf{P}_l^{-m}(-u), \tag{596}$$

we find that

$$\begin{aligned} f_m^*(t,\phi) &= \cosh\alpha_m F_m^*(t,\phi) + e^{-i\gamma_m}\sinh\alpha_m G_m(t,\phi) \\ &= -\frac{1}{\sqrt{\sin(m-l)\pi\sin(m+l)\pi}} \\ &\quad\times\Big[\left(\cosh\alpha_m\sin m\pi + e^{-i\gamma_m}\sinh\alpha_m\sin l\pi\right)F_{-m} \\ &\qquad + \left(\cosh\alpha_m\sin l\pi + e^{-i\gamma_m}\sinh\alpha_m\sin m\pi\right)G_{-m}^*\Big]. \end{aligned} \tag{597}$$

Requiring therefore the reality condition $(f_m, f_n^*) = 0$ imposes a non-trivial restriction when $n = -m$. Calculating (f_{-m}, f_m^*) then gives

$$\begin{aligned} 0 &= \cosh\alpha_{-m}\left(\cosh\alpha_m\sin m\pi + e^{-i\gamma_m}\sinh\alpha_m\sin l\pi\right) \\ &\quad - e^{-i\gamma_{-m}}\sinh\alpha_{-m}\left(\cosh\alpha_m\sin l\pi + e^{-i\gamma_m}\sinh\alpha_m\sin m\pi\right). \end{aligned} \tag{598}$$

If $m\in\mathbb{Z}$ we can rewrite this condition as,

$$e^{i\gamma_m}\cosh\alpha_{-m}\sinh\alpha_m - e^{i\gamma_{-m}}\cosh\alpha_m\sinh\alpha_{-m} = 0. \tag{599}$$

Meanwhile, if $m\in\frac{1}{2}+\mathbb{Z}$ we can instead rewrite theses conditions as

$$\cosh\alpha_m\cosh\alpha_{-m} - e^{i(\gamma_m+\gamma_{-m})}\sinh\alpha_m\sinh\alpha_{-m} = \delta_m\sin l\pi \tag{600}$$

$$e^{i\gamma_{-m}}\cosh\alpha_m\sinh\alpha_{-m} - e^{i\gamma_m}\cosh\alpha_{-m}\sinh\alpha_m = \delta_m\sin m\pi, \tag{601}$$

for some constants δ_m. These conditions are precisely the quantisation conditions of

Epstein and Moschella.

11 States and Two-Point Functions

11.1 Symmetries

In this section we look for vacuum states which are invariant under the $\mathrm{SO}_0(2,1)$ symmetry group. This will impose further restrictions on the mode functions. Suppose that Λ is a $\mathrm{SO}(2,1)$ transformation. Let this be implemented on the field operator by $U(\Lambda)$ and suppose that the associated action on the space of solutions is given by

$$U(\Lambda)F_m = \sum_{n\in\mathbb{Z}} \left(U(\Lambda)_{mn} F_n + \overline{U(\Lambda)}_{mn} G_n^* \right), \tag{602}$$

$$U(\Lambda)G_m^* = \sum_{n\in\mathbb{Z}} \left(V(\Lambda)_{mn} F_n + \overline{V(\Lambda)}_{mn} G_n^* \right). \tag{603}$$

Note that here we $\overline{U(\Lambda)}_{mn}$ is not the complex conjugate of $U(\Lambda)_{mn}$, and similarly for $\overline{V(\Lambda)}$ and $V(\Lambda)$. Then we have

$$\begin{aligned} U(\Lambda)\Phi &= \sum_m \left[(U(\Lambda)F_m)a_m + (U(\Lambda)G_m^*)b_m^\dagger \right] \\ &= \sum_{mn} \left[(U(\Lambda)_{mn}F_n + \overline{U(\Lambda)}_{mn}G_n^*)a_m + (V(\Lambda)_{mn}F_n + \overline{V(\Lambda)}_{mn}G_n^*)b_m^\dagger \right] \\ &= \sum_{nm} \left[(a_m U(\Lambda)_{mn} + b_m^\dagger V(\Lambda)_{mn})F_n + (a_m \overline{U(\Lambda)}_{mn} + b_m^\dagger \overline{V(\Lambda)}_{mn})G_n^* \right]. \end{aligned} \tag{604}$$

Thus, on the annihilation and creation operators we have

$$U(\Lambda)a_n = \sum_{m\in\mathbb{Z}} \left(a_m U(\Lambda)_{mn} + b_m^\dagger V(\Lambda)_{mn} \right), \tag{605}$$

$$U(\Lambda)b_n^\dagger = \sum_{m\in\mathbb{Z}} \left(a_m \overline{U(\Lambda)}_{mn} + b_m^\dagger \overline{V(\Lambda)}_{mn} \right). \tag{606}$$

However, as we have seen for the Bogoliubov transformations, in order for the vacuum state to be invariant, we do not want to allow mixing between the $\mathcal{S}^+$ and $\mathcal{S}^-$ subspaces and we also want the transformation to be unitary. Thus we require

$$V(\Lambda)_{mn} = 0, \qquad \overline{U(\Lambda)}_{mn} = 0, \tag{607}$$

and

$$\begin{aligned} \sum_{n\in\mathbb{Z}} U(\Lambda)_{mn} U(\Lambda)_{kn}^* &= \delta_{mk}, \\ \sum_{n\in\mathbb{Z}} \overline{V(\Lambda)}_{mn} \overline{V(\Lambda)}_{kn}^* &= \delta_{mk}. \end{aligned} \tag{608}$$

Now, we recall that the generators of the symmetry group can be given in global coordinates as

$$
\begin{aligned}
L &= \frac{\partial}{\partial \phi}, \\
B_1 &= \cos\phi \frac{\partial}{\partial t} - \tanh t \sin\phi \frac{\partial}{\partial \phi}, \\
B_2 &= \sin\phi \frac{\partial}{\partial t} + \tanh t \cos\phi \frac{\partial}{\partial \phi}.
\end{aligned} \tag{609}
$$

Then, if we act with $e^{2\pi L}$ on some function $F(\phi)$, we find that

$$
e^{2\pi L} F(\phi) = F(\phi + 2\pi), \tag{610}
$$

so that acting on the automorphic field $\Phi(t,\phi)$ shows that $e^{2\pi L}$ is a scalar as

$$
e^{2\pi L}\Phi(t,\phi) = \Phi(t,\phi+2\pi) = e^{2\pi i\beta}\Phi(t,\phi). \tag{611}
$$

Thus the group under whose representations the field transforms is $\widetilde{\mathrm{SL}(2,\mathbb{R})}$, labelled by β. In Appendix H we recall the classification of the irreducible unitary representations of this group. To further classify the irreducible representation, we also need the action of the quadratic Casimir operator $Q = -\Box$ in global coordinates. Thus, the eigenvalue equation

$$
Q\Phi(t,\phi) = l(l+1)\Phi(t,\phi) \tag{612}
$$

is the Klein-Gordon equation obeyed by the field $\Phi(t,\phi)$ with mass $M^2 = -l(l+1)$. With the restriction

$$
l \in -\frac{1}{2} + i\mathbb{R}^+, \quad \text{or} \quad -\frac{1}{2} < l < |\beta|, \tag{613}
$$

we are looking at irreducible principal and complementary series representations of $\widetilde{\mathrm{SL}(2,\mathbb{R})}$. Thus the question of whether the Fock vacuum state $|0\rangle$ is de Sitter invariant is equivalent to the question of whether the modes $\{f_m\}$ and $\{g_m^*\}$ form a basis for an irreducible representation. For this, we will check the action of L, B_1 and B_2 on the modes. Firstly, note that

$$
Lf_m = imf_m, \qquad Lg_m^* = img_m^*. \tag{614}
$$

Next, to find the action of B_1 and B_2 on the modes, we use the ladder operators $B_\pm = B_1 \pm iB_2$, which in globabl coordinates act on the fields as

$$
B_+ = e^{i\phi}\left(\frac{\partial}{\partial t} + i\tanh t \frac{\partial}{\partial \phi}\right), \tag{615}
$$

$$
B_- = e^{-i\phi}\left(\frac{\partial}{\partial t} - i\tanh t \frac{\partial}{\partial \phi}\right). \tag{616}
$$

To find the action of $B_\pm$, we first recall the recurrence relations obeyed by the associated Legendre functions [52]

$$
\left(\sqrt{1-u^2}\frac{d}{du} - \frac{mu}{\sqrt{1-u^2}}\right)\mathsf{P}_l^{-m}(u) = -\mathsf{P}_l^{-m+1}(u) \tag{617}
$$

$$
\left(\sqrt{1-u^2}\frac{d}{du} + \frac{mu}{\sqrt{1-u^2}}\right)\mathsf{P}_l^{-m}(u) = (l-m)(l+m+1)\mathsf{P}_l^{-m-1}(u), \tag{618}
$$

whence it follows that

$$\left(\frac{d}{dt} - m\tanh t\right)\Phi_m(t) = -i\sqrt{(m-l)(m+l+1)}\Phi_{m+1}(t), \tag{619}$$

$$\left(\frac{d}{dt} + m\tanh t\right)\Phi_m(t) = -i\sqrt{(m-l-1)(m+l)}\Phi_{m-1}(t). \tag{620}$$

In particular, letting again $u = i\sinh t$ it then follows that

$$\begin{aligned} B_+F_m(t,\phi) &= ie^{i(m+1)\phi}\sqrt{\frac{\Gamma(m+l+1)\Gamma(m-l)}{4\pi}}\left(\sqrt{1-u^2}\frac{d}{du} + \frac{mu}{\sqrt{1-u^2}}\right)\mathsf{P}_l^{-m}(u) \\ &= -i\sqrt{\frac{\Gamma(m+l+1)\Gamma(m-l)}{4\pi}}(m-l)(l+m+1)\mathsf{P}_l^{-m-1}(i\sinh t)e^{i(m+1)\phi} \\ &= -i\sqrt{(m-l)(m+l+1)}F_{m+1}(t,\phi). \end{aligned} \tag{621}$$

Similarly, we find also that

$$B_+G^*_m(t,\phi) = +i\sqrt{(m-1)(m+l+1)}G^*_{m+1}(t,\phi), \tag{622}$$

$$B_-F_m(t,\phi) = -i\sqrt{(m-l-1)(m+l)}F_{m-1}(t,\phi), \tag{623}$$

$$B_-G^*_m(t,\phi) = +i\sqrt{(m-l-1)(m+l)}G^*_{m-1}(t,\phi). \tag{624}$$

Thus $\{F_m\}$ and $\{(-1)^m G^*_m\}$ have the same transformation under the $\mathrm{so}(2,1)$ symmetry algebra generated by the Killing vectors, and that they transform as the basis vectors of a unitary irreducible representation of $\widetilde{\mathrm{SL}(2,\mathbb{R})}$. The general

$$f_m(t,\phi) = \cosh\alpha_m F_m(t,\phi) + e^{i\gamma_m}\sinh\alpha_m G^*_m(t,\phi), \tag{625}$$

$$g^*_m(t,\phi) = e^{-i\gamma_m}\sinh\alpha_m F_m(t,\phi) + \cosh\alpha_m G^*_m(t,\phi), \tag{626}$$

where γ_m and α_m are real numbers, and we can without loss of generality assume that $\alpha_m \geq 0$. Therefore it follows that these transform also as basis vectors of the same irreducible unitary representation precisely if we choose that

$$\alpha_m = \alpha, \qquad \gamma_m = \gamma + m\pi, \tag{627}$$

where α and γ are m-independent constant real numbers. It follows that if l and m are chosen so that there exists a corresponding $\widetilde{\mathrm{SL}(2,\mathbb{R})}$ representation, then this representation can be realised using the mode functions of automorphic scalar fields in two-dimensional de Sitter space. Notice that the vacuum state in each case is not unique, indeed the freedom in the choice of these states is precisely the same as for α-vacua [41, 55] of scalar fields in higher-dimensional de Sitter spaces.

To compare with the results of Epstein and Moschella [37], let us impose again additionally the reality conditions for the periodic, $\beta = 0$, and antiperiodic, $\beta = \frac{1}{2}$ cases. When $m \in \mathbb{Z}$, the reality condition requires

$$e^{i\gamma_m}\cosh\alpha_{-m}\sinh\alpha_m - e^{i\gamma_{-m}}\cosh\alpha_m\sinh\alpha_{-m} = 0, \tag{628}$$

while de Sitter invariance requires $\alpha_m = \alpha$ and $\gamma_m = m\pi + \gamma$ for constant α and γ.

Inserting these values for α_m and γ_m, we see that the reality condition imposes no further restrictions on α and γ.

For the anti-periodic case $m \in \mathbb{Z} + \frac{1}{2}$, and the reality conditions are

$$\begin{aligned} \cosh\alpha_m \cosh\alpha_{-m} - e^{i(\gamma_m+\gamma_{-m})} \sinh\alpha_{-m} \sinh\alpha_{-m} &= \delta_m \sin l\pi, \\ e^{i\gamma_{-m}} \cosh\alpha_m \sinh\alpha_{-m} - e^{i\gamma_m} \cosh\alpha_{-m} \sinh\alpha_m &= \delta_m \sin m\pi. \end{aligned} \tag{629}$$

Now let $l = -\frac{1}{2} + i\lambda$, as for the anti-periodic case there are no complementary series representations, then $\sin l\pi = -\cosh\lambda\pi$. Inserting the values for α_m and γ_m and eliminating δ_m yields

$$1 - e^{2i\gamma} \tanh^2\alpha = 2i \cosh\lambda\pi e^{i\gamma} \tanh\alpha. \tag{630}$$

This can be recognised as a quadratic in $ie^{i\gamma}\tanh\alpha$, with solutions

$$ie^{i\gamma} \tanh\alpha = e^{\pm\lambda\pi}. \tag{631}$$

As we can choose $\alpha > 0$ without loss of generality, the reality then requires us to choose $e^{i\gamma} = -i$. Then the condition can be rewritten as

$$\coth 2\alpha = \cosh\lambda\pi, \tag{632}$$

which agrees with the result of Epstein and Moschella [37].

11.2 Hadamard States

11.2.1 Adiabatic States

We have by now constructed a family of de Sitter invariant states, however symmetry is not the only kind of extra condition we may wish to impose on states. For example, it is not guaranteed *a priori* that all the states we have defined lead to theories that can be considered physically reasonable. In general, the definition of what we call particles is dependent on the choice of observer and associated vacuum state, as a result of the choice of basis we expand the field in. On short distance scales, the space-time appears locally flat, thus if we consider the high (angular)-momentum modes which probe the short distance scales we would on physical grounds not expect significant changes to the number of excitations corresponding to very large momenta. This is the basis of the adiabatic condition [49, 50, 56], which we can take as asking that the basis $\{f_m\}$ for $\mathcal{S}^+_\beta$ behaves as $m \to \infty$ like

$$f_m(t,\phi) \sim \frac{1}{\sqrt{4\pi|m|}} \exp\left[i(m\phi - |m| \arctan\sinh t)\right]. \tag{633}$$

This follows if we consider the Klein-Gordon equation (581) for modes modes proportional to $e^{im\phi}$. If m^2 is much larger than $\cosh t$, this is well approximated by

$$\cosh t \frac{d}{dt}\left(\cosh t \frac{d\Phi_m}{dt}\right) = -m^2\Phi_m. \tag{634}$$

If we introduce $y = y(t)$ by

$$\frac{d}{dy} = \cosh t \frac{d}{dt}, \tag{635}$$

which can be solved for $y = \arctan \sinh t$, the equation becomes

$$\frac{d^2\Phi_m}{dy^2} = -m^2\Phi_m. \tag{636}$$

Thus we find that a general solution has asymptotic behaviour as $m \to \infty$ of

$$f_m(t,\phi) \sim \frac{1}{\sqrt{4\pi|m|}} e^{im\phi} \left(A_m \exp[-i|m| \arctan \sinh t] + B_m \exp[+i|m| \arctan \sinh t] \right). \tag{637}$$

This can be thought of as a superposition of asymptotically positive and negative frequency solutions. The content of the adiabatic condition is then to say that physical states have modes such that $A_m \to 1$ and $B_m \to 0$ as $m \to \infty$. This ensures that the number of excitations in the high momentum modes does not change rapidly.

So we wish to investigate whether any of the symmetric states we have found previously are physical in the sense that they additionally obey the adiabatic condition. To this end, recall that

$$F_m(t,\phi) = \Phi_m(t) e^{im\phi}, \tag{638}$$

where

$$\Phi_m(t) = \sqrt{\frac{\Gamma(m+l+1)\Gamma(m-l)}{4\pi}} \mathsf{P}_l^{-m}(i \sinh t). \tag{639}$$

When m is large and positive, the behaviour of the associated Legendre functions is [54, Eq. 14.15.1]

$$\mathsf{P}_l^{-m}(\pm u) \sim \frac{1}{\Gamma(1+m)} \left(\frac{1 \mp u}{1 \pm u} \right)^{m/2}. \tag{640}$$

Recalling further that

$$\exp(-i \arctan y) = \left(\frac{1-iy}{1+iy} \right)^{1/2}, \tag{641}$$

implies that as $m \to +\infty$ we have

$$\mathsf{P}_l^{-m}(i \sinh t) \sim \frac{1}{\Gamma(1+m)} \exp(-im \arctan \sinh t). \tag{642}$$

We then wish to investigate the behaviour as $m \to +\infty$ of

$$\frac{\Gamma(m+l+1)\Gamma(m-l)}{\Gamma(1+m)^2}. \tag{643}$$

To this end, we recall the Stirling approximation [54, Eq. 5.11.3] which says that provided $|\arg z| \leq \pi - \epsilon$ for some positive ϵ, as $z \to \infty$ we have

$$\Gamma(z) = \sqrt{\frac{2\pi}{z}} z^z e^{-z} \left(1 + O\left(\frac{1}{z} \right) \right), \tag{644}$$

from which it follows that as $m \to +\infty$

$$\frac{\Gamma(m+l+1)\Gamma(m-l)}{\Gamma(1+m)^2} \sim \frac{1}{m}. \tag{645}$$

Thus we find that for large positive m

$$F_m(t,\phi) \sim \frac{1}{\sqrt{4\pi m}} \exp(im\phi - im \arctan \sinh t). \tag{646}$$

Similarly, we have

$$G_m^*(t,\phi) \sim \frac{1}{\sqrt{4\pi m}} \exp(im\phi + im \arctan \sinh t). \tag{647}$$

For the negative m, we recall the connection formula to note that as $m \to -\infty$

$$\begin{aligned} F_m(t,\phi) \sim & \frac{1}{\sqrt{4\pi|m|}} \frac{1}{\sqrt{\sin^2 l\pi - \sin^2 \beta}} \exp(im\phi) \\ & \times \left[\sin l\pi \exp(im \arctan \sinh t) + \sin m\pi \exp(-im \arctan \sinh t)\right]. \end{aligned} \tag{648}$$

We are now ready to check when it is possible to have physical and de Sitter symmetric vacuum states. Indeed, recall that for the de Sitter invariant states, we have in general modes of the form

$$f_m(t,\phi) = \cosh \alpha F_m(t,\phi) + (-1)^m e^{i\gamma} \sinh \alpha G_m^*(t,\phi), \tag{649}$$

for constant α and γ and $m \in \mathbb{Z} + \beta$. Thus, we can read off that in order to have the right behaviour as $m \to +\infty$ we must choose

$$\alpha = 0. \tag{650}$$

However, if we then examine the behaviour as $m \to -\infty$ we see that we must choose $\sin m\pi = 0$, which is to say that $\beta = 0$. Therefore only in the periodic theory ($\beta = 0$) is it possible to have a vacuum state that is both physically reasonable according to the adiabatic principle and also de Sitter invariant. Indeed, the resulting state is of the well known Bunch-Davies form.

11.2.2 De Sitter Invariant Hadamard States

Another, more formal, definition of a physically acceptable state is the Hadamard condition [51]. The Hadamard condition constrains the short-distance singular structure of the two-point function $\mathcal{W}(x,y) = \langle 0|\Phi(x)\Phi^\dagger(y)|0\rangle$, where x and y label two space-time points. In two dimensions, we say that the vacuum $|0\rangle$ is a Hadamard state if

$$\begin{aligned} \mathcal{W}(x,y) &= \langle 0|\Phi(x)\Phi^\dagger(y)|0\rangle \\ &= -\frac{1}{4\pi} V(x,y) \log\left[\frac{\mu(x,y)^2}{2} + i\epsilon \mathrm{sign}(x^0 - y^0)\right] + W(x,y), \end{aligned} \tag{651}$$

where ϵ is a positive infinitesimal, $\mu(x,y)$ denotes the geodesic-distance between x and y, and $W(x,y)$ and $V(x,y)$ are smooth functions with $V(x,y)$ state independent. The

Hadamard condition is physically motivated similarly to the adiabatic condition. The Hadamard condition requires that the singularity structure of the two-point function in a general spacetime matches as much as possible the singularity structure of the two-point function in a flat spacetime [51]. Generally, for globally hyperbolic spacetimes one expects there to exist a large class of Hadamard states [51, 57].

In this section, we will calculate the two-point functions for the de Sitter invariant vacuum states and show that unless $\beta = 0$ and $\alpha = 0$ the two-point function is always singular at two antipodal points. For non-automorphic fields, Radzikowski [58] has proved that a Hadamard state can not have other non-local singularities.

Suppose that we are working in a state that is invariant under connected component of the de Sitter group. In such a state the two-point function can be determined as a function of the geodesic distance [59]. Thus we may write $\mathcal{W} = \mathcal{W}(\mu)$, and this two-point function must still satisfy the Klein-Gordon equation

$$(\Box_x - M^2)\mathcal{W}(\mu(x,y)) = 0. \tag{652}$$

In particular, let $Z = \cos\mu(x,y)$, then we look for a function $F(Z)$ that solves

$$\Box_x F(Z) = (1-Z^2)\frac{d^2F}{dZ^2} - 2Z\frac{dF}{dZ}, \tag{653}$$

so that if we again set $M^2 = -l(l+1)$ we find that $\mathcal{W}(Z)$ obeys a Legendre equation

$$(1-Z^2)\frac{d^2\mathcal{W}}{dZ^2} - 2Z\frac{d\mathcal{W}}{dZ} + l(l+1)\mathcal{W} = 0. \tag{654}$$

Thus a general de Sitter invariant two-point function must take the form

$$\mathcal{W}(\mu) = A\mathsf{P}_l(-\cos\mu) + B\mathsf{P}_l(+\cos\mu). \tag{655}$$

For the automorphic fields, we then extend this two-point function by

$$\mathcal{W}(t,\phi+2\pi M; t',\phi'+2\pi N) = e^{2\pi i(M-N)\beta}\mathcal{W}(t,\phi;t',\phi'), \tag{656}$$

where M and N are integers. We are interested in the singular behaviour of these functions, so we note that when l is not an integer $\mathsf{P}_l(x)$ is singular with a branch cut from $x=-1$ to $-\infty$ and we have $\mathsf{P}_l(1) = 1$. The behaviour as $x \to -1$ can be determined using the following expression for $\mathsf{P}_l(x)$

$$\mathsf{P}_l(x) = \left[-\frac{\sin l\pi}{\pi}\log\frac{1-x}{1+x} + C_l\right]\mathsf{P}_l(-x) - \frac{2}{\pi}R_l(x), \tag{657}$$

where

$$R_l(x) = \lim_{m\to 0}\frac{\partial}{\partial m}F\left(l+1,-l;1-m;\frac{1+x}{2}\right), \tag{658}$$

and letting ψ denote the digamma function and γ Euler's constant

$$C_l = \frac{2\sin l\pi}{\pi}[\gamma + \psi(l+1)] + \cos l\pi. \tag{659}$$

We note that $R_l(x)$ is analytic at $x = -1$. Then as $x \to -1$ we have

$$\mathsf{P}_l(x) = \frac{\sin l\pi}{\pi} \log(1+x) + O(1). \tag{660}$$

The formula (657) is proved by differentiating the associated Legendre equation for $\Gamma(1-m)\mathsf{P}_l^m(x)$ and taking the limit as $m \to 0$. We relegate the details to Appendix J.

Recall that for the de Sitter invariant vacuum states, we can use the following positive modes

$$f_m(t,\phi) = \cosh\alpha F_m(t,\phi) + e^{im\pi}e^{i\gamma}\sinh\alpha G_m^*(t,\phi). \tag{661}$$

Then define

$$\begin{aligned}
\mathcal{W}_\beta^{(\alpha)}(t,\phi;0,0) &= \langle 0|\Phi(t,\phi)\Phi^\dagger(0,0)|0\rangle \\
&= \sum_{m\in\mathbb{Z}+\beta} f_m(t,\phi)f_m^*(0,0) \\
&= \sum_{m\in\mathbb{Z}+\beta} \Big[\cosh^2\alpha F_m(t,\phi)F_m^*(0,0) + \sinh^2\alpha G_m^*(t,\phi)G_m(0,0) \\
&+ \cosh\alpha\sinh\alpha \\
&\times \left(e^{im\pi}e^{i\gamma}G_m^*(t,\phi)F_m^*(0,0) + e^{-im\pi}e^{-i\gamma}F_m(t,\phi)G_m(0,0)\right)\Big].
\end{aligned} \tag{662}$$

Here we used the de Sitter invariance of the two-point function to set one of the points to be at the origin of our coordinate system. Next, if we define

$$\mathcal{W}_\beta^{(0)}(t,\phi) = \sum_{m\in\mathbb{Z}+\beta} F_m(t,\phi)F_m^*(0,0), \tag{663}$$

it is sufficient to study this two-point function because the general two-point function is

$$\begin{aligned}
\mathcal{W}_\beta^{(\alpha)}(t,\phi) &= \cosh^2\alpha \mathcal{W}_\beta^{(0)}(t,\phi) + \sinh^2\alpha \mathcal{W}_\beta^{(0)*}(t,-\phi) \\
&+ \cosh\alpha\sinh\alpha\left[e^{i\gamma}\mathcal{W}_\beta^{(0)*}(t,-\phi-\pi) + e^{-i\gamma}\mathcal{W}_\beta^{(0)}(t,\phi-\pi)\right],
\end{aligned} \tag{664}$$

where we use that $F_m(0,0)$ and $G_m^*(0,0)$ are real, $G_m^*(t,\phi) = F_m^*(t,-\phi)$ and $e^{-im\pi}F_m(t,\phi) = F_m(t,\phi-\pi)$. Therefore it is enough to consider the simpler function $\mathcal{W}_\beta^{(0)}(t,\phi)$.

We know in general that if (t,ϕ) can be connected to the origin by a spacelike geodesic that

$$\mathcal{W}_\beta^{(0)}(t,\phi) = A_\beta \mathsf{P}_l(-\cos\mu) + B_\beta \mathsf{P}_l(+\cos\mu), \tag{665}$$

and we will determine constants A_β and B_β by matching the logarithmic singularities with the mode-sum expressions.

For convenience, let us switch to conformal coordinates (τ,ϕ) with $\sinh t = \tan\tau$. In conformal coordinates, we have

$$\cos\mu(\tau,\phi;0,0) = \sec\tau\cos\phi. \tag{666}$$

We expect the singularities as $\tau \pm \phi$ approach 0 and $\pm\pi$. For example as $\tau - \phi = 0$ is approached from $\phi > 0$ we have

$$\cos\mu(\tau = \phi - \epsilon, \phi; 0, 0) = 1 - \epsilon \tan\phi + O(\epsilon^2), \tag{667}$$

where $\epsilon > 0$ is a small number. Using that $\epsilon = \phi - \tau$, we find that the singular part as we approach $\tau - \phi = 0$ from $0 < \phi < \pi$ is

$$W_\beta^{(0)} \sim \frac{A_\beta \sin l\pi}{\pi} \log(\phi - \tau). \tag{668}$$

Similarly, as $\tau - \phi = 0$ is approached from $\phi < 0$, we have

$$\cos\mu(\tau = \phi + \epsilon, \phi; 0, 0) = 1 + \epsilon \tan\phi + O(\epsilon^2). \tag{669}$$

In this case $\epsilon = \tau - \phi$ and thus in the region $-\pi < \phi < 0$ we have

$$\mathcal{W}_\beta^{(0)} \sim \frac{A_\beta \sin l\pi}{\pi} \log(\tau - \phi), \tag{670}$$

as $\tau - \phi$ approaches zero. Working out the other singularities from $\tau + \phi = 0$, $\tau + \phi = \pm\pi$ and $\tau - \phi = \pm\pi$ then yields the singularity structure as

$$\begin{aligned} \mathcal{W}_\beta^{(0)} \approx \frac{A_\beta \sin \pi l}{\pi} &\left[\log(\phi - \tau) + \log(\phi + \tau)\right] \\ + \frac{B_\beta \sin \pi l}{\pi} &\left[\log(\pi - \phi + \tau) + \log(\pi - \phi - \tau)\right], \end{aligned} \qquad 0 < \phi < \pi, \tag{671}$$

and

$$\begin{aligned} \mathcal{W}_\beta^{(0)} \approx \frac{A_\beta \sin \pi l}{\pi} &\left[\log(\tau - \phi) + \log(-\tau - \phi)\right] \\ + \frac{B_\beta \sin \pi l}{\pi} &\left[\log(\pi + \phi + \tau) + \log(\pi + \phi - \tau)\right], \end{aligned} \qquad -\pi < \phi < 0. \tag{672}$$

These expressions are valid provided that (τ, ϕ) and $(0, 0)$ are connectable by a spacelike geodesic.

Now match these singularities with the singularities extracted from the mode-sum expression for $\mathcal{W}_\beta^{(0)}$. As the simplest case, let us first deal with $\beta = 0$. In this case the mode-sum expression for $W_0^{(0)}(\tau, \phi)$ is

$$\mathcal{W}_0^{(0)}(\tau, \phi) = \sum_{m \in \mathbb{Z}} F_m(\tau, \phi) F_m^*(0, 0). \tag{673}$$

Separating the $m = 0$ and $m > 0$ and $m < 0$ modes yields

$$\begin{aligned} \mathcal{W}_0^{(0)}(\tau, \phi) = F_0(\tau, \phi) &F_0^*(0, 0)^* \\ &+ \sum_{m=1}^{\infty} \left[F_m(\tau, \phi) F_m(0, 0) + F_{-m}(\tau, \phi) F_{-m}^*(0, 0)\right]. \end{aligned} \tag{674}$$

This can be rewritten in terms of $F_m(\tau, \phi) = \Phi_m(\tau) e^{im\phi}$, where we recall that

$$\Phi_m(\tau) = \sqrt{\frac{\Gamma(m + l + 1)\Gamma(m - l)}{4\pi}} \mathsf{P}_l^{-m}(i \tan\tau). \tag{675}$$

When $m \in \mathbb{Z}$ we use (596) to note that

$$\frac{\sin(l-m)\pi}{\Gamma(m+l+1)}\mathsf{P}_l^m(i\tan\tau) = \frac{\sin l\pi}{\Gamma(l-m+1)}\mathsf{P}_l^{-m}(i\tan\tau). \tag{676}$$

Then we find that

$$\begin{aligned}\Phi_{-m}(\tau) &= \sqrt{\frac{\Gamma(l-m+1)\Gamma(-m-l)}{4\pi}}\mathsf{P}_l^m(i\tan\tau) \\ &= -\frac{\sin l\pi}{\pi}\sqrt{\frac{\Gamma(l-m+1)\Gamma(-m-l)}{4\pi}}\frac{\Gamma(m+l+1)}{\Gamma(l-m+1)}\frac{\pi}{\sin(m-l)\pi}\mathsf{P}_l^{-m}(i\tan\tau) \\ &= -\frac{\sin l\pi}{\pi}\sqrt{\Gamma(m-l)\Gamma(-m-l)\Gamma(l-m+1)\Gamma(l+m+1)}\Phi_m(\tau) \\ &= \Phi_m(\tau).\end{aligned} \tag{677}$$

where we noted that $|\sin l\pi| = -\sin l\pi$ and also used the reflection formula [54, Eq. 5.5.3]

$$\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin \pi x}. \tag{678}$$

Then the mode-sum expression (674) for $\mathcal{W}_0^{(0)}(\tau,\phi)$ can be rewritten as

$$\mathcal{W}_0^{(0)}(\tau,\phi) = \Phi_0(\tau)\Phi_0^*(0) + \sum_{m=1}^{\infty}\Phi_m(\tau)\Phi_m^*(0)[e^{im\phi}+e^{-im\phi}]. \tag{679}$$

To extract the singular part, we recall that as $m\to\infty$

$$\Phi_m(\tau,\phi) \sim \frac{1}{\sqrt{4\pi m}}e^{-im\tau}\left[1+O\left(\frac{1}{m}\right)\right], \tag{680}$$

and that we can drop sums of terms which behave like $m^{-1-\epsilon}$ for $\epsilon > 0$. Thus

$$\mathcal{W}_0^{(0)}(\tau,\phi) \approx \frac{1}{4\pi}\sum_{m=1}^{\infty}\frac{1}{m}\left[e^{-im(\tau-\phi)}+e^{-im(\tau+\phi)}\right]. \tag{681}$$

In order for these sums to converge, we should understand

$$\tau\pm\phi \mapsto \tau\pm\phi - i\epsilon \tag{682}$$

With this understood, the sum can be evaluated using the Taylor expansion

$$\log(1-x) = -\sum_{n=1}^{\infty}\frac{1}{n}x^n, \tag{683}$$

provided that $|x|<1$. It follows that the singular part of the two-point function $\mathcal{W}_0^{(0)}(\tau,\phi)$ is given by

$$\begin{aligned}\mathcal{W}_0^{(0)}(\tau,\phi) &\approx -\frac{1}{4\pi}\left[\log\left(1-e^{-i(\tau-\phi-i\epsilon)}\right)+\log\left(1-e^{-i(\tau+\phi-i\epsilon)}\right)\right] \\ &\approx \begin{cases} -\frac{1}{4\pi}\left[\log(\phi-\tau+i\epsilon)+\log(\phi+\tau-i\epsilon)\right], & \text{if } 0<\phi<\pi, \\ -\frac{1}{4\pi}\left[\log(-\phi+\tau-i\epsilon)+\log(-\phi-\tau+i\epsilon)\right], & \text{if } -\pi<\phi<0,\end{cases}\end{aligned} \tag{684}$$

provided that (τ, ϕ) is connected to the origin by a space-like geodesic. Suppressing again the $i\epsilon$ prescription, we can match the singularities with those obtained by the general expression, whence we find

$$A_\beta = -\frac{1}{4\sin\pi l}, \qquad B_\beta = 0, \tag{685}$$

so that

$$\mathcal{W}_0^{(0)}(\tau, \phi) = -\frac{1}{4\sin\pi l}\mathsf{P}_l(-\cos\mu(\tau, \phi; 0, 0)), \tag{686}$$

for space-like separated points. To extend the two-point function to points which are not connected to the origin by a spacelike geodesic, we analytically continue through the boundaries. For this, we choose the principal branches for the logarithm functions, in particular, this means that for $x > 0$ we continue to negative values by

$$\log(-x \pm i\epsilon) = \log|x| \pm i\pi. \tag{687}$$

Then extending across $\phi - \tau = 0$ from $\phi > 0$, we should replace the logarithmic singularities

$$-\frac{1}{4\pi}\log(\phi - \tau + i\epsilon) \mapsto -\frac{1}{4\pi}\left[\log(\tau - \phi) + i\pi\right]. \tag{688}$$

The non-singular terms added in this expression leads to additions of $P_l(\cos\mu)$, and it follows that extending from positive ϕ we should take in the future light-cone

$$\mathcal{W}_0^{(0)}(\tau, \phi) = -\frac{1}{4\sin l\pi}\tilde{\mathsf{P}}_l(-\cos\mu) - \frac{i}{4}\mathsf{P}_l(\cos\mu), \tag{689}$$

where we have defined for $x > 1$

$$\tilde{\mathsf{P}}_l(-x) = \left[-\frac{\sin l\pi}{\pi}\log\left(\frac{x+1}{x-1}\right) + C_l\right]\mathsf{P}_l(x) + R_l(-x). \tag{690}$$

We can also reach the future light-cone by extending from $\phi < 0$ across $\phi + \tau = 0$. Then we should replace

$$-\frac{1}{4\pi}\log(-\phi - \tau + i\epsilon) \mapsto -\frac{1}{4\pi}\left[\log(\phi + \tau) + i\pi\right]. \tag{691}$$

We find again that we should take

$$\mathcal{W}_0^{(0)}(\tau, \phi) = -\frac{1}{4\sin l\pi}\tilde{\mathsf{P}}_l(-\cos\mu) - \frac{i}{4}\mathsf{P}_l(\cos\mu) \tag{692}$$

in the future light cone. For the past light-cone, we can extend from $\phi > 0$ through $\tau + \phi = 0$, which leads us to replace

$$-\frac{1}{4\pi}\log(\phi + \tau - i\epsilon) \mapsto -\frac{1}{4\pi}\left[\log(-\phi - \tau) - i\pi\right]. \tag{693}$$

Thus in the past light-cone of the origin we should take

$$\mathcal{W}_0^{(0)}(\tau, \phi) = -\frac{1}{4\sin l\pi}\tilde{\mathsf{P}}_l(-\cos\mu) + \frac{i}{4}\mathsf{P}_l(\cos\mu), \tag{694}$$

this formula is also obtained by extending from $\phi < 0$. We still need to extend the two-point function to the interior of the future and past lightcones of the anti-podal point $(0, \pm\pi)$. As we have seen, these points can not be connected to the origin by a geodesic, thus we can not interpret μ as a geodesic distance, however we can still extend the definition of μ to these points. As a function μ, the two-point function $\mathcal{W}_0^{(0)}$ is not singular when passing through $\cos\mu = -1$, it follows that for these points we still have

$$\mathcal{W}_0^{(0)} = -\frac{1}{4\sin l\pi}\mathsf{P}_l(-\cos\mu). \tag{695}$$

These results can be summarised as

$$\mathcal{W}_0^{(0)}(\tau,\phi) = -\frac{1}{4\sin l\pi}\mathsf{P}_l(-\cos\mu + i\epsilon\tau). \tag{696}$$

Having seen the simpler case of $\beta = 0$, let us now return to general $\beta(\neq 0)$. The mode-sum expression for $\mathcal{W}_\beta^{(0)}$ is

$$\mathcal{W}_\beta^{(0)}(\tau,\phi) = \sum_{m\in\mathbb{Z}+\beta} F_m(\tau,\phi)F_m^*(0,0) \tag{697}$$

Seperate into modes with $m > 0$ and $m < 0$, so let $m = n + \beta$ and $-m' = -n - 1 + \beta$. Then we can rewrite

$$\mathcal{W}_\beta^{(0)}(\tau,\phi) = \sum_{n=0}^{\infty}\left[F_m(\tau,\phi)F_m^*(0,0) + F_{-m'}(\tau,\phi)F_{-m'}^*(0,0)\right], \tag{698}$$

Recall that we have

$$\begin{aligned} F_m(\tau,\phi) &= \Phi_m(\tau)e^{im\phi} \\ &= \sqrt{\frac{\Gamma(m+l+1)\Gamma(m-l)}{4\pi}}\mathsf{P}_l^{-m}(i\tan\tau)e^{im\phi}. \end{aligned} \tag{699}$$

We now want to find the general m version of (677), relating Φ_{-m} to Φ_m and Φ_m^*. This is done again using the connection formula (596), however this time with $m \in \mathbb{Z} + \beta$. Arguing as for (677) the result is

$$\Phi_{-m}(\tau) = \frac{1}{\sqrt{\sin^2 l\pi - \sin^2\beta\pi}}\left[\sin m\pi\Phi_m^*(\tau) - \sin l\pi\Phi_m(\tau)\right]. \tag{700}$$

Thus we can re-write the series expression for $\mathcal{W}_\beta^{(0)}(\tau,\phi)$ as

$$\begin{aligned} \mathcal{W}_\beta^{(0)}(\tau,\phi) = \sum_{n=0}^{\infty}\Phi_m(\tau)\Phi_m^*(0)e^{im\phi} &+ \frac{\sin^2 l\pi}{\sin^2 l\pi - \sin^2\beta\pi}\sum_{n=0}^{\infty}\Phi_{m'}(\tau)\Phi_{m'}(0)e^{-im'\phi} \\ &+ \frac{\sin^2\beta\pi}{\sin^2 l\pi - \sin^2\beta\pi}\sum_{n=0}^{\infty}\Phi_{m'}^*(\tau)\Phi_{m'}(0)e^{-im'\phi} \\ &+ e^{\pm i\beta\pi}\frac{\sin l\pi\sin\beta\pi}{\sin^2 l\pi - \sin^2\beta\pi}\sum_{n=0}^{\infty}\left[\Phi_{m'}^*(\tau)\Phi_{m'}^*(0) + \Phi_{m'}(\tau)\Phi_{m'}(0)\right]e^{-im'(\phi\mp\pi)}, \end{aligned} \tag{701}$$

noting that $-\sin m'\pi = e^{\pm i\beta\pi}\sin\beta\pi e^{\mp im'\pi}$. From (646) as $n\to\infty$ we have

$$\Phi_m(\tau) \sim \frac{1}{\sqrt{4\pi m}} e^{-im\tan^{-1}\tan\tau}\left(1+O\left(\frac{1}{m}\right)\right), \tag{702}$$

where we used again that in conformal coordinates $\sinh t = \tan\tau$. To get at the singular parts of these sums we proceed as before, dropping first sums of terms which scale like $m^{-1-\epsilon}$ for $\epsilon > 0$. Then the singular part of the first sum can be extracted from

$$\sum_{n=0}^{\infty}\Phi_m(\tau)\Phi_m^*(0)e^{im\phi} \approx \frac{1}{4\pi}\sum_{n=1}^{\infty}\frac{1}{n+\beta}e^{-i(n+\beta)(\tau-\phi)}. \tag{703}$$

Then as $n\to\infty$

$$\frac{1}{n+\beta} = \frac{1}{n}\left(1+O\left(\frac{1}{n}\right)\right). \tag{704}$$

To actually evaluate the sum, we should again understand $\tau-\phi\mapsto\tau-\phi-i\epsilon$ for $\epsilon>0$, and then the singular part is extracted as

$$\sum_{n=0}^{\infty}\Phi_m(\tau)\Phi_m^*(0)e^{im\phi} \approx -\frac{1}{4\pi}e^{-i\beta(\tau-\phi)}\log\left[1-e^{-i(\tau-\phi-i\epsilon)}\right]. \tag{705}$$

The other sums can be analysed similarly, leading to

$$\sum_{n=0}^{\infty}\Phi_{m'}(\tau)\Phi_{m'}(0)e^{-im'\phi} \approx -\frac{1}{4\pi}e^{i\beta(\tau+\phi)}\log\left[1-e^{-i(\tau+\phi-i\epsilon)}\right] \tag{706}$$

$$\sum_{n=0}^{\infty}\Phi_{m'}^*(\tau)\Phi_{m'}(0)e^{-im'\phi} \approx -\frac{1}{4\pi}e^{i\beta(\tau-\phi)}\log\left[1-e^{i(\tau-\phi+i\epsilon)}\right], \tag{707}$$

and the final sum

$$\begin{aligned}&\sum_{n=0}^{\infty}\left[\Phi_{m'}^*(\tau)\Phi_{m'}^*(0)+\Phi_{m'}(\tau)\Phi_{m'}(0)\right]\\ &\approx -\frac{1}{4\pi}\Big(e^{-i\beta(\tau-\phi\pm\pi)}\log\left[1-e^{-i(\tau-\phi\pm\pi+i\epsilon)}\right]\\ &\qquad + e^{i\beta(\tau+\phi\mp\pi)}\log\left[1-e^{-i(\tau+\phi\mp\pi-i\epsilon)}\right]\Big).\end{aligned} \tag{708}$$

Then, the logarithmic singularities of $\mathcal{W}_\beta^{(0)}(\tau,\phi)$ in the region connected to the origin by a spacelike geodesic with $0<\phi<\pi$ are

$$\begin{aligned}\mathcal{W}_\beta^{(0)}(\tau,\phi) \approx -\frac{1}{4\pi}\Big[&\log(\phi-\tau+i\epsilon)+\frac{\sin^2 l\pi}{\sin^2 l\pi-\sin^2\beta\pi}\log(\phi+\tau-i\epsilon)\\ &+\frac{\sin^2\beta\pi}{\sin^2 l\pi-\sin^2\beta\pi}\log(\phi-\tau-i\epsilon)+e^{+i\beta\pi}\frac{\sin l\pi\sin\beta\pi}{\sin^2 l\pi-\sin^2\beta\pi}\\ &\times\left[\log(\pi+\tau-\phi+i\epsilon)+\log(\pi-\tau-\phi+i\epsilon)\right]\Big].\end{aligned} \tag{709}$$

Ignoring for this region the $i\epsilon$ prescription, and comparing with the general logarithmic singularity structure of the two-point function we can determine

$$A_\beta = -\frac{1}{4}\frac{\sin l\pi}{\sin^2 l\pi-\sin^2\beta\pi}, \qquad B_\beta = -\frac{1}{4}e^{i\beta\pi}\frac{\sin\beta\pi}{\sin^2 l\pi-\sin^2\beta\pi}, \tag{710}$$

so that for $0 < \phi < \pi$ we have

$$\begin{aligned} \mathcal{W}^{(0)}_\beta(\tau,\phi) = &-\frac{1}{4}\frac{\sin l\pi}{\sin^2 l\pi - \sin^2\beta\pi}\mathsf{P}_l(-\cos\mu) \\ &-\frac{1}{4}e^{i\beta\pi}\frac{\sin\beta\pi}{\sin^2 l\pi - \sin^2\beta\pi}\mathsf{P}_l(\cos\mu). \end{aligned} \tag{711}$$

Similarly, for the points connected to the origin by a spacelike geodesic with $-\pi < \phi < 0$, the logarithmic singularity structure is

$$\begin{aligned} \mathcal{W}^{(0)}_\beta(\tau,\phi) \approx -\frac{1}{4\pi}\Big[&\log(\tau-\phi-i\epsilon) + \frac{\sin^2 l\pi}{\sin^2 l\pi - \sin^2\beta\pi}\log(-\phi-\tau+i\epsilon) \\ &+\frac{\sin^2\beta\pi}{\sin^2 l\pi-\sin^2\beta\pi}\log(\tau-\phi+i\epsilon) + e^{-i\beta\pi}\frac{\sin l\pi\sin\beta\pi}{\sin^2 l\pi - \sin^2\beta\pi} \\ &\times[\log(\pi-\tau+\phi-i\epsilon)+\log(\pi+\tau+\phi-i\epsilon)]\Big]. \end{aligned} \tag{712}$$

Thus, matching the singularity structure as before, we find that in this diamond

$$\begin{aligned} \mathcal{W}^{(0)}_\beta(\tau,\phi) = &-\frac{1}{4}\frac{\sin l\pi}{\sin^2 l\pi - \sin^2\beta\pi}\mathsf{P}_l(-\cos\mu) \\ &-\frac{1}{4}e^{-i\beta\pi}\frac{\sin\beta\pi}{\sin^2 l\pi - \sin^2\beta\pi}\mathsf{P}_l(\cos\mu). \end{aligned} \tag{713}$$

We now want to extend the two point function into the future and past light-cones of the origin, as well as the future and past of the anti-podal point to the origin. If we come from the spacelike region with $0 < \phi < \pi$, we need to replace the logarithmic singularity

$$\begin{aligned} &\log(\phi-\tau+i\epsilon) + \frac{\sin^2\beta\pi}{\sin^2 l\pi - \sin^2\beta\pi}\log(\phi-\tau+i\epsilon) \\ \mapsto &\frac{\sin^2 l\pi}{\sin^2 l\pi-\sin^2\beta\pi}\log(\tau-\phi) + i\pi\frac{\sin^2 l\pi - 2\sin^2\beta\pi}{\sin^2 l\pi - \sin^2\beta\pi}, \end{aligned} \tag{714}$$

in the expression for $\mathcal{W}^{(0)}_\beta$. Thus in the future light-cone of the origin, where $\tau > |\phi|$, the two-point function is given by

$$\begin{aligned} \mathcal{W}^{(0)}_\beta(\tau,\phi) = &-\frac{1}{4}\frac{\sin l\pi}{\sin^2 l\pi - \sin^2\beta\pi}\tilde{\mathsf{P}}_l(-\cos\mu) \\ &-\frac{1}{4}\left[\frac{\cos\beta\pi\sin\beta\pi}{\sin^2 l\pi - \sin^2\beta\pi} + i\right]\mathsf{P}_l(\cos\mu). \end{aligned} \tag{715}$$

In the past light-cone of the origin, where $\tau < -|\phi|$ the expression can be similarly found as

$$\begin{aligned} \mathcal{W}^{(0)}_\beta(\tau,\phi) = &-\frac{1}{4}\frac{\sin l\pi}{\sin^2 l\pi - \sin^2\beta\pi}\tilde{\mathsf{P}}_l(-\cos\mu) \\ &-\frac{1}{4}\left[\frac{\cos\beta\pi\sin\beta\pi}{\sin^2 l\pi - \sin^2\beta\pi} - i\right]\mathsf{P}_l(\cos\mu). \end{aligned} \tag{716}$$

Finally we need to find $\mathcal{W}^{(0)}_\beta(\tau,\phi)$ in the future and past light-cones of the point $(0,\pi)$. To get into the future light-cone of this point, we have to pass through $\pi-\phi-\tau = 0$, so we find that

$$\begin{aligned}\mathcal{W}_\beta^{(0)}(\tau,\phi) &= -\frac{1}{4}\frac{\sin l\pi}{\sin^2 l\pi - \sin^2\beta\pi}\mathsf{P}_l(-\cos\mu) \\ &\quad -\frac{1}{4}e^{i\beta\pi}\frac{\sin\beta\pi}{\sin^2 l\pi - \sin^2\beta\pi}\left[\tilde{\mathsf{P}}_l(\cos\mu) + i\sin l\pi\mathsf{P}_l(-\cos\mu)\right] \\ &= -\frac{1}{4}\frac{e^{i\beta\pi}}{\sin^2 l\pi - \sin^2\beta\pi}\left[\sin\beta\pi\tilde{\mathsf{P}}_l(\cos\mu) + \cos\beta\pi\sin l\pi\mathsf{P}_l(-\cos\mu)\right].\end{aligned} \tag{717}$$

The same expression can be found for $\mathcal{W}_\beta^{(0)}(\tau,\phi)$ in the past light-cone of $(0,\pi)$.

The expression of the two-point function $\mathcal{W}_\beta^{(0)}$ for all other values of ϕ can be found using the automorphic condition using shifts of ϕ by 2π.

We would now have all the ingredients to calculate the actual two-point function $\mathcal{W}_\beta^{(\alpha)}(\tau,\phi)$ for general α. What we notice is that if $\beta \neq 0$ or $\alpha \neq 0$, then we must necessarily have singularities not just when $\cos\mu = 1$ but also when $\cos\mu = -1$. Furthermore, the coefficient of $\mathsf{P}_l(-\cos\mu)$ when $\beta \neq 0$ differs from the same coefficient when $\beta = 0$ by a factor of $\sin^2 l\pi/(\sin^2 l\pi - \sin^2\beta\pi)$. Thus it follows when $\beta \neq 0$ these states can not be locally Hadamard as they do not have the same strength singularities at $\cos\mu = +1$. This verifies the claim that there are de Sitter invariant Hadamard states only in the periodic case, which occurs if we choose $\alpha = 0$

11.3 De Sitter Non-Invariant Hadarmard States

In the previous section we argued that the only de Sitter invariant vacuum Hadamard state occurs for a periodic field. While de Sitter invariance is a pleasant feature for a state to have, the Hadamard condition is essential for a state to be considered physically reasonable. In this section we present a set of modes which heuristically, by the adiabatic argument, should lead to a Hadamard vacuum state. We then consider the two-point function and argue that the associated vacuum state is indeed Hadamard.

Define the mode functions

$$F_m(t,\phi) = \Phi_{|m|}(t)e^{im\phi}, \tag{718}$$

$$G_m^*(t,\phi) = \Phi_{|m|}^*(t)e^{im\phi}, \tag{719}$$

for $m \in \mathbb{Z} + \beta$, where

$$\Phi_{|m|}(t) = \sqrt{\frac{\Gamma(|m|-l)\Gamma(|m|+l+1)}{4\pi}}\mathsf{P}_l^{-|m|}(i\sinh t). \tag{720}$$

Note that if $\beta = 0$, these modes just correspond to the invariant Bunch-Davies vacuum state again. As $m \to \pm\infty$ the modes behave as

$$F_m(t,\phi) \sim \frac{1}{\sqrt{4\pi|m|}}e^{i(m\phi - |m|\arctan\sinh t)}, \tag{721}$$

and similarly for G_m^*, as we expect for an adiabatic vacuum state. Thus it is reasonable to expect the associated vacuum states to be Hadamard. Let us consider the associated two-point function $\tilde{\mathcal{W}}_\beta$, which has the mode-sum form (for $\beta \neq 0$)

$$\begin{aligned}\tilde{\mathcal{W}}_\beta(\tau,\phi;\tau',\phi') &= \sum_{m\in\mathbb{Z}} \Phi_{|m|}(\tau)\Phi^*_{|m|}(\tau')e^{im(\phi-\phi')} \\ &= \sum_{n=0}^{\infty} \Phi_m(\tau)\Phi^*_m(\tau')e^{im(\phi-\phi')} \\ &\quad + \sum_{n=0}^{\infty} \Phi_{m'}(\tau)\Phi^*_{m'}(\tau')e^{-im'(\phi-\phi')},\end{aligned} \tag{722}$$

where we defined $m = n + \beta$ and $m' = n + 1 - \beta$, and we again switched to conformal coordinates (τ, ϕ). Notice that as we have given up de Sitter invariance, we can no longer place one of the points on the two-point function at the origin. Now, using the definition of the Ferrer's functions

$$\mathsf{P}_l^{-m}(i\tan\tau) = \frac{1}{\Gamma(1+m)}e^{-im\tau}F\left(1+l,-l;1+m;\frac{1-i\tan\tau}{2}\right), \tag{723}$$

we can rewrite the mode-sum expression for the two-point function as

$$\tilde{\mathcal{W}}_\beta(\tau,\phi;\tau',\phi') = \sum_{n=0}^{\infty} f(n+\beta;\tau,\tau')z_1^{n+\beta} + \sum_{n=0}^{\infty} f(n+1-\beta;\tau,\tau')z_2^{n+1-\beta}, \tag{724}$$

where we have introduced

$$z_1 = e^{-i(\tau-\tau'-\phi+\phi')}, \qquad z_2 = e^{-i(\tau-\tau'+\phi-\phi')}, \tag{725}$$

and

$$\begin{aligned}f(s;\tau,\tau') = &\frac{\Gamma(s-l)\Gamma(s+l+1)}{4\pi\Gamma(1+s)^2} \\ &\times F\left(1+l,-l;1+s;\frac{1-i\tan\tau}{2}\right)F\left(1+l,-l;1+s;\frac{1+i\tan\tau'}{2}\right).\end{aligned} \tag{726}$$

For $\beta = 0$, the mode-sum expression for the two-point function is

$$\mathcal{W}_0(\tau,\phi;\tau',\phi') = \sum_{n=0}^{\infty} f(n;\tau,\tau')z_1^n + \sum_{n=1}^{\infty} f(n;\tau,\tau')z_2^n. \tag{727}$$

In order for these series to converge, we should understand $|z_1| < 1$ and $|z_2| < 1$, so we introduce the $i\epsilon$ prescriptions

$$z_1 = e^{-i(\tau-\tau'-\phi+\phi'-i\epsilon)}, \qquad z_2 = e^{-i(\tau-\tau'+\phi-\phi'-i\epsilon)}. \tag{728}$$

To show that $\tilde{\mathcal{W}}_\beta(\tau,\phi;\tau',\phi')$ corresponds to a Hadamard state, we will subtract the known Hadamard state $\mathcal{W}_0$ and analyse the remainder, we will find that remainder is suitably analytic and conclude that $\tilde{\mathcal{W}}_\beta$ defines a Hadamard vacuum state.

The difference between $\tilde{\mathcal{W}}_\beta$ and $\mathcal{W}_0$ can be written

$$\begin{aligned}\Delta\tilde{\mathcal{W}}_\beta(\tau,\phi;\tau',\phi') &= \tilde{\mathcal{W}}_\beta(\tau,\phi;\tau',\phi') - \mathcal{W}_0(\tau,\phi;\tau',\phi') \\ &= \sum_{n=1}^{\infty}\left[f(n+\beta;\tau,\tau')z_1^{n+\beta} - f(n)z_1^n\right] \\ &\quad + \sum_{n=1}^{\infty}\left[f(n+\beta;\tau,\tau')z_2^{n+\beta} - f(n)z_2^n\right] \\ &\quad + f(\beta;\tau,\tau')z_1^\beta + f(1-\beta;\tau,\tau')z_2^{1-\beta} - f(0;\tau,\tau').\end{aligned} \tag{729}$$

In the Appendix I, we show that these series can be analytically continued for $|z_i| > 0$ and $|\arg(z_i)| < 2\pi$, provided that $|f(s;\tau,\tau')|$ grows at most polynomially for Re $(s) > 0$. In fact, as $s \to \infty$ with Re $s > 0$ we know from (645) that

$$\frac{\Gamma(s-l)(\Gamma(s+l+1)}{\Gamma(1+s)^2} \sim \frac{1}{s}. \tag{730}$$

Further, for large $|c|$ with Re $c > 0$ and Re$z = 0$ we have [54, Eq. 15.12.2]

$$F\left(a,b;c;\frac{1-z}{2}\right) = 1 + \left(\frac{1}{c}\right). \tag{731}$$

Thus we find that $f(s;\tau,\tau')$ in fact decays as $s \to \infty$. It follows then from the argument in Appendix I that $\Delta\tilde{\mathcal{W}}_\beta(\tau,\phi;\tau',\phi')$ is analytic provided that $|(\tau \pm \phi) - (\tau' \pm \phi')| < 2\pi$.

In particular, $\tilde{\mathcal{W}}_\beta(\tau,\phi;\tau',\phi')$ and $\mathcal{W}_0$ have the same light-cone singularity structure. The failure of the analyticity at $|(\tau \pm \phi) - (\tau' \pm \phi')| = 2\pi$ arises due to the difference in periodicity of the functions, as $\tilde{\mathcal{W}}_\beta$ picks up a phase of $2\pi\beta$, while $\mathcal{W}_0$ is periodic. Now $|\tau - \tau'| \leq \pi$, so $\tilde{\mathcal{W}}_\beta$ has the same singularity structure as $\mathcal{W}_0$ in a region containing $-\pi < \phi - \phi' < \pi$, for arbitrary τ and τ', which covers the two-dimensional de Sitter space. In this way we can interpret $\tilde{\mathcal{W}}_\beta((\tau,\phi;\tau',\phi')$ as defining a de Sitter non-invariant Hadamard state.

12 Conclusion

In this chapter we studied complex scalar fields on two-dimensional de Sitter space, which obeyed an additional periodicity condition when making a full spatial rotation. We reviewed how these fields are quantised in the canonical formalism. The symmetry group of these automorphic theories is $\widetilde{\mathrm{SL}(2,\mathbb{R})}$, and we showed that whenever there is a corresponding irreducible unitary representation of $\widetilde{\mathrm{SL}(2,\mathbb{R})}$, there is a two parameter (or one complex parameter) family of de Sitter invariant vacuum states for the complex field. For a real anti-periodic field, this two-parameter family of states gets restricted down to a single state, as found by Epstein and Moschella [37, 38].

We then showed that only the periodic Bunch-Davies type vacuum state is a de Sitter invariant Hadamard state, however we were still able to present a class of de Sitter non-invariant Hadamard state for all other values of the automorphic parameter β. In fact, these de Sitter non-invariant Hadamard states can be shown to exhibit an approximate Gibbons-Hawking effect [2], and in this sense the Gibbons-Hawking effect is not completely lost once the field is no longer periodic in contrast to claims of Epstein and Moschella [37].

H Irreducible Unitary Representations of $\mathrm{SO}_0(2,1)$ and its Universal Covering Group

In this appendix we describe the irreducible unitary representations of $\mathrm{SO}_0(2,1)$, the connected component of the Lorentz group in $(2+1)$-dimensions. The treatment is based on Kitaev [60].

We can realise the generators of the Lie algebra as differential operators acting on functions on two-dimensional de Sitter space. In global coordinates (t,ϕ) the generators can be written

$$L = \frac{\partial}{\partial \phi}, \tag{732}$$

$$B_1 = \cos\phi \frac{\partial}{\partial t} - \tanh t \sin\phi \frac{\partial}{\partial \phi}, \tag{733}$$

$$B_2 = \sin\phi \frac{\partial}{\partial t} + \tanh t \cos\phi \frac{\partial}{\partial \phi}. \tag{734}$$

The brackets obeyed by these generators are then

$$\begin{aligned} [L, B_1] &= -B_2, \\ [L, B_2] &= B_1, \\ [B_1, B_2] &= L. \end{aligned} \tag{735}$$

Note that if we define the operator Q by

$$\begin{aligned} Q &= -L^2 + B_1^2 + B_2^2 \\ &= \frac{1}{\cosh t}\frac{\partial}{\partial t}\left(\cosh t \frac{\partial}{\partial t}\right) - \frac{1}{\cosh^2 t}\frac{\partial^2}{\partial \phi^2}, \end{aligned} \tag{736}$$

then Q commutes with each generator L, B_1 and B_2. We will call Q the quadratic Casimir operator. Next, let us define

$$R(\lambda) = \exp\left[\lambda L\right] \tag{737}$$

This operator corresponds to a shift in ϕ by λ. For example,

$$\begin{aligned} R(\lambda)B_1(t,\phi)R(-\lambda) &= B_1(t,\phi)\cos\lambda - B_2(t,\phi)\sin\lambda \\ &= \cos(\phi+\lambda)\frac{\partial}{\partial t} - \tanh t \sin(\phi+\lambda)\frac{\partial}{\partial \phi} \\ &= B_1(t, \phi+\lambda), \end{aligned} \tag{738}$$

and similarly for $B_2(t,\phi)$. It follows that $R(2n\pi)$ for $n \in \mathbb{Z}$ also commutes with all elements of the algebra. In fact, the elements $R(2n\pi)$ form the center of the simply connected Lie group obtained by exponentiating the Lie algebra generated by $\{L, B_1, B_2\}$.

We now want to consider the irreducible unitary representations of this algebra. In a unitary representation of the algebra, the generators are to be represented by anti-Hermitian operators. It follows that the quadratic Casimir operator Q is Hermitian, while

$R(2\pi)$ is unitary. Thus the eigenvalues q of Q are real, while the eigenvalues r of $R(2\pi)$ must be on the unit circle. Let us write

$$q = l(l+1), \qquad r = e^{2\pi i\beta}. \tag{739}$$

Now let us work out the allowed values for l and β. For β, it is possible to restrict to $\beta \in (-\frac{1}{2}, \frac{1}{2}]$ and we should identify β with $\beta \pm 1$. Next, notice that we can always make the transformation

$$l \mapsto l' = -(l+1), \tag{740}$$

under which $q \mapsto q'$. It follows that we can restrict to $\text{Re}\, l \geq -\frac{1}{2}$. Now, let $l = A + iB$, then

$$q = l(l+1) = A(A+1) - B^2 + iB(2A+1). \tag{741}$$

Thus for q to be real, we must have $B = 0$ or $A = -\frac{1}{2}$. It follows that

$$l \in \Big[-\frac{1}{2}, \infty\Big), \quad \text{or} \quad l \in \left(-\frac{1}{2} - i\infty, -\frac{1}{2} + i\infty\right). \tag{742}$$

In each irreducible unitary representation, we choose to diagonalise Q and $R(2\pi)$, so we can label the representations by these eigenvalues. We can choose one further operator to diagonalise, and we will choose to diagonalise L. Thus let Ψ_m denote a basis of the representation such that

$$\begin{aligned} L\Psi_m &= i(m+\beta)\Psi_m, \\ R(2\pi)\Psi_m &= e^{2\pi i\beta}\Psi_m, \\ Q\Psi_m &= l(l+1)\Psi_m, \end{aligned} \tag{743}$$

where $m \in \mathbb{Z}$. Now, let us define ladder type operators $B_\pm$ by

$$B_+ = B_1 + iB_2, \qquad B_- = B_1 - iB_2. \tag{744}$$

These then obey the commutation relations

$$\begin{aligned} [L, B\pm] &= \pm iB_\pm, \\ [B_+, B_-] &= -2iL. \end{aligned} \tag{745}$$

It follows that $B_\pm$ raise the L eigenvalue by $\pm i$, as

$$LB_+\Psi_m = \left([L, B_+] + B_+L\right)\Psi_m = i(m+1+\beta)\Psi_m. \tag{746}$$

It follows that $B_+\Psi_m \propto \Psi_{m+1}$ unless $B_+\Psi_m = 0$, and similarly for B_-. To see when $B_\pm\Psi_m = 0$, let us work out the norms of these elements.

$$\|B_+\Psi_m\|^2 = (B_+\Psi_m, B_+\Psi_m) = -(\Psi_m, B_-B_+\Psi_m), \tag{747}$$

where we used $B_+^\dagger = -B_-$. However, $Q = -L^2 + B_-B_+ - iL$ so that

$$\begin{aligned}\|B_+\Psi_m\|^2 &= -(\Psi_m, (Q + L^2 + iL)\Psi_m) \\ &= -l(l+1) + (m+\beta)^2 + (m+\beta) \\ &= (m+\beta-l)(m+\beta+l+1),\end{aligned} \tag{748}$$

assuming we have normalised so that $\|\Psi_m\|^2 = 1$. Similarly, we can calculate

$$\|B_-\Psi_m\|^2 = (m+\beta+l)(m+\beta-l-1). \tag{749}$$

In an irreducible unitary representation, both $\|B_\pm\Psi_m\|$ need to be non-negative, this leads to the classification of the representations.

The simplest is the trivial representation, where we take $l = \beta = 0$. There is a single state in this representation with $m = 0$.

Next are the continuous series of representations. In these representations we always have $\|B_\pm\Psi_m\| > 0$. There are two types of continuous series representations. First are the principal series representations, for which we let $l = -\frac{1}{2} + i\nu$. Then

$$\|B_\pm\Psi_m\|^2 = \left|m + \beta \pm \frac{1}{2} + i\nu\right|^2 \tag{750}$$

Thus principal series exist for all values of β, and m takes all values in $\mathbb{Z}$. The complementary series representations take $l \in \mathbb{R}$. In order for $\|B_\pm\Psi_m\|^2 > 0$ we need $\beta - l$ and $\beta+l+1$ to lie between the same two consecutive integers, similarly, $\beta+l$ and $\beta-l-1$ also need to lie between the same two consecutive integers. For a given β, this occurs exactly when

$$l \in \left(-\frac{1}{2}, -|\beta|\right). \tag{751}$$

Thus for all values, except $\beta = \frac{1}{2}$ we have complementary series representations. These also have m taking all values in $\mathbb{Z}$.

The final possibility are the discrete series representations, in which one of the equations $B_\pm\Psi_m = 0$ holds. When $B_-\Psi_m = 0$, we have a lowest weight, while $B_+\Psi_m = 0$ gives a highest weight representation.

This exhausts the classification of the unitary irreducible representations of the algebra. We will choose the phases of the basis vector Ψ_m such that in a representation the operators $\{L, B_+, B_-\}$ act as

$$B_+\Psi_m = -i\sqrt{(m+\beta-l)(m+\beta+l+1)}\Psi_{m+1}, \tag{752}$$

$$L\Psi_m = i(m+\beta)\Psi_m, \tag{753}$$

$$B_-\Psi_m = -i\sqrt{(m+\beta+l)(m+\beta-l-1)}\Psi_{m-1}. \tag{754}$$

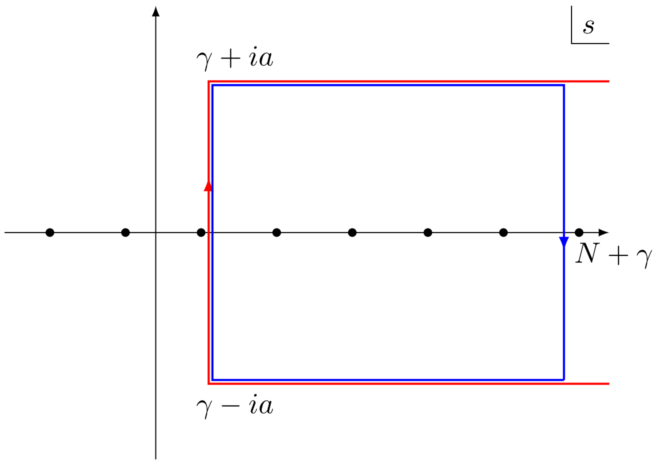

Figure 2: The integration contour C in red and C_N in blue in the complex s plane. The dots represent the poles of the integrand at $n+\beta$ for integer n.

I Integral Representation of the Difference of two Series

In this appendix we wish to consider sums of the form

$$S_\beta(z) = \sum_{n=1}^{\infty} f(n+\beta) z^{n+\beta}, \qquad |z| < 1. \tag{755}$$

We will assume that $-\frac{1}{2} < \beta \leq \frac{1}{2}$, $f(t)$ is analytic on the half-plane Re $t > 0$, and grows at most polynomially as $|t| \to \infty$. We first want to express $S_\beta(z)$ as the integral

$$S_\beta(z) = \frac{i}{2} \int_C \mathrm{d}s\ f(s) z^s \cot \pi(s-\beta), \tag{756}$$

where the contour C is composed of straight lines, starting at $\infty - ia$ connecting to $\gamma - ia$, $\gamma + ia$ and then ending at $\infty + ia$, where we choose $|\beta| < \gamma < 1 - |\beta|$ and a is a positive real constant. When $\beta = \frac{1}{2}$, we choose $\frac{1}{2} < \gamma < \frac{3}{2}$. The value of γ is chosen so that the pole at β is never within the contour and the first pole within the contour is at $1+\beta$. Then (756) is proved by making use of the residue theorem applied to

$$S_\beta^N(z) = \frac{i}{2} \int_{C_N} \mathrm{d}s\ f(s) z^s \cot \pi(s-\beta), \tag{757}$$

where the contour C_N now consists of a rectangle, traversed clockwise with vertices at $(N+\gamma, \pm ia)$ and $(\gamma, \pm ia)$. The integrand has simple poles at $s = n+\beta$, for $n = 1 \ldots N$ with residues $\pi^{-1} f(n+\beta) z^{n+\beta}$. Thus by the residue theorem

$$S_\beta^N(z) = \sum_{n=1}^{N} f(n+\beta) z^{n+\beta}, \tag{758}$$

so that as $N \to \infty$ we have $S_\beta^N \to S_\beta$. Now, consider the contribution to the integral $S_\beta^N(z)$ from the straight line contour running from $N+\gamma+ia$ to $N+\gamma-ia$, letting

$s = N + \gamma + it$ with t running from a to $-a$ we consider

$$\begin{aligned} I_N &= \left| \int_a^{-a} i\mathrm{d}t \; f(N+\gamma+it) z^{N+\gamma+it} \cot \pi(\gamma - \beta + it) \right| \\ &\leq \int_{-a}^{a} \mathrm{d}t \; |f(N+\gamma+it)| |z^{N+\gamma+it}| |\cot \pi(\gamma - \beta + it)|. \end{aligned} \tag{759}$$

However, $|z^{N+\gamma+it}| = e^{(N+\gamma)\log|z| - (\arg z)t}$ decays exponentially as $N \to \infty$, while we have assumed $f(s)$ grows at most polynomially. It follows that as $N \to \infty$ we must have $I_N \to 0$. Thus the result

$$S_\beta(z) = \frac{i}{2} \int_C \mathrm{d}s \; f(s) z^s \cot \pi(s - \beta) \tag{760}$$

follows.

Next, we want to consider the difference

$$S_\beta(z) - S_0(z) = \sum_{n=1}^{\infty} \left[f(n+\beta) z^{n+\beta} - f(z) z^n \right]. \tag{761}$$

We can clearly use the same integral contour for $S_0(z)$ and $S_\beta(z)$, so that for $|z| < 1$ we have

$$\begin{aligned} S_\beta(z) - S_0(z) &= \frac{i}{2} \int_C \mathrm{d}s \; f(s) z^s \left[\cot \pi(s-\beta) - \cot \pi s \right] \\ &= i \sin \beta \int_C \mathrm{d}s \; \frac{f(s) z^s}{\cos \pi\beta - \cos \pi(2s - \beta)}. \end{aligned} \tag{762}$$

Denote integrand here by

$$G(s) = \frac{f(s) z^s}{\cos \pi\beta - \cos \pi(2s-\beta)}. \tag{763}$$

First, consider the size of the numerator that the numerator

$$|f(s) z^s| = |f(s)| e^{\log|z| \mathrm{Re}(s) - \arg(z) \mathrm{Im}(s)} \leq |f(s)| e^{\log|z| \mathrm{Re}(s) + \arg(z) |\mathrm{Im}(s)|}. \tag{764}$$

Meanwhile, the denominator

$$\begin{aligned} &\cos \pi\beta - \cos \pi(2s - \beta) \\ &= \cos \pi\beta - \frac{1}{2} \left[e^{-2\pi \mathrm{Im}(s) + i(2\pi \mathrm{Re}(s) - \beta)} + e^{+2\pi \mathrm{Im}(s) - i(2\pi \mathrm{Re}(s) - \beta)} \right]. \end{aligned} \tag{765}$$

Then consider the contour C_{R+} running in straight lines $\gamma + i(a+R)$ to $\gamma + ia$ and then to $\gamma + R + ia$ and then closing back to $\gamma + i(a+R)$ in a quarter-circle. There are no poles within the closed contour, so

$$\oint_{C_{R+}} \mathrm{d}s \; G(s) = 0. \tag{766}$$

If we choose a large enough, then we can take

$$|\cos \pi\beta - \cos \pi(2s-\beta)| \geq \frac{1}{3} e^{2\pi |\mathrm{Im} s|}, \tag{767}$$

so that

$$|G(s)| \leq 3|f(s)| \exp\left[\log|z|\mathrm{Re}(s) - [2\pi - \arg(z)]\,\mathrm{Im}(s)\right]. \tag{768}$$

Thus $G(s) \to 0$ with exponential decay as either $\mathrm{Re}(s) \to \infty$ for $\mathrm{Im}(s) \geq a$ and $|z| < 1$ or as $\mathrm{Im}(s) \to \infty$ with $\mathrm{Re}(s) \geq \gamma$ and $|\arg(z)| < 2\pi$. In particular it follows that the contribution of the integral over the quarter circle of C_{R+} tends to zero as $R \to \infty$. Considering similarly a quarter circle in the lower half plane shows that it is possible to deform the contour C to the straight line contour with $\mathrm{Re}(s) = \gamma$, so that

$$S_\beta(z) - S_0(z) = i\sin\beta \int_{\gamma - i\infty}^{\gamma + i\infty} \mathrm{d}s \; \frac{f(s) z^s}{\cos\pi\beta - \cos\pi(2s - \beta)}. \tag{769}$$

We can note that this integral converges even for $|z| > 1$, provided we still take $|\arg(z)| < 2\pi$. Thus this integral representation provides an analytic continutation of $S_\beta(z) - S_0(z)$ to all $|z| > 0$ with $|\arg(z)| < 2\pi$.

J Derivation of Equation (657)

In this appendix we derive equation (657). The Legendre equation is

$$\left[(1 - x^2)\frac{d^2}{dx^2} - 2x\frac{d}{dx} + l(l+1)\right] F(x) = 0. \tag{770}$$

This has two linearly independent solutions $\mathsf{P}_l(x)$ and $\mathsf{Q}_l(x)$, which are real on $-1 < x < 1$. We define [53]

$$\mathsf{P}_l(x) = F\left(l + 1, -l; 1; \frac{1 - x}{2}\right), \tag{771}$$

the second Legendre function $\mathsf{Q}_l(x)$ is related to this by [54, Eq. 14.9.10]

$$\frac{2}{\pi}\sin l\pi \mathsf{Q}_l(x) = \cos l\pi \mathsf{P}_l(x) - \mathsf{P}_l(-x). \tag{772}$$

We further note that $\mathsf{P}_l(1) = 1$ and that as $x \to +1$ from below we have [54, Eq. 14.8.3]

$$\mathsf{Q}_l(x) = \frac{1}{2}\log\left(\frac{2}{1 - x}\right) - \gamma - \psi(l + 1) + O(1 - x), \tag{773}$$

where $\gamma \approx 0.5772$ is Euler's constant and $\psi(x)$ is the digamma function.

Consider the associated Legendre equation

$$\left[(1 - x^2)\frac{d^2}{dx^2} - 2x\frac{d}{dx} - \frac{m^2}{1 - x^2} + l(l+1)\right] \Gamma(1 - m)\mathsf{P}_l^m(x) = 0, \tag{774}$$

where we have multiplied by $\Gamma(1 - m)$ for convenience and $\mathsf{P}_l^m(x)$ is a Ferrer's function. Differentiating with respect to m and then letting $m \to 0$ yields

$$\left[(1 - x^2)\frac{d^2}{dx^2} - 2x\frac{d}{dx} + l(l+1)\right] f_l(x) = 0, \tag{775}$$

where

$$f_l(x) = \frac{1}{2}\mathsf{P}_l(x)\log\left(\frac{1 + x}{1 - x}\right) + R_l(-x), \tag{776}$$

and

$$R_l(-x) = \lim_{m\to 0} \frac{\partial}{\partial m} F\left(l+1, -l; 1-m; \frac{1-x}{2}\right). \tag{777}$$

As $f_l(x)$ solves the Legendre equation, we must be able to write

$$f_l(x) = A\mathsf{P}_l(x) + B\mathsf{Q}_l(x). \tag{778}$$

As $R_l(-1) = 0$ and $\mathsf{P}_l(1) = 1$, we find that

$$\lim_{x\to 1} f_l(x) - \mathsf{Q}_l(x) = \gamma + \psi(l+1). \tag{779}$$

It follows that we need to take $B = 1$ and $A = \gamma + \psi(l+1)$, which yields

$$\frac{1}{2}\mathsf{P}_l(x)\log\left(\frac{1+x}{1-x}\right) + R_l(-x) = (\gamma + \psi(l+1))\mathsf{P}_l(x) + \mathsf{Q}_l(x). \tag{780}$$

Using this to eliminate $\mathsf{Q}_l(x)$ in (772) and replacing x with $-x$ we obtain (657)

$$\mathsf{P}_l(x) = \left[-\frac{\sin l\pi}{\pi}\log\frac{1-x}{1+x} + C_l\right]\mathsf{P}_l(-x) - \frac{2}{\pi}R_l(x), \tag{781}$$

where

$$C_l = \frac{2\sin l\pi}{\pi}[\gamma + \psi(l+1)] + \cos l\pi. \tag{782}$$

Part III

Harmonics on Complex Spheres

13 Introduction

In this chapter we study the behaviour and properties of harmonics on complex spheres. We begin by motivating the study of these harmonics by showing how they arise when studying the tensor product of principal series $\mathrm{SO}(3,1)$ representations. Using Mackey's tensor product theorem it is possible to show that the tensor product of the principal series representations of either $\mathrm{SL}(2,\mathbb{R})$ or $\mathrm{SO}(3,1)$ is equivalent to another induced representation on two-dimensional real or complex de Sitter space. In the complex case, the study can then be related to harmonic functions on complex spheres.

This chapter is structured as follows. We begin by recalling some background information related to the structure theory of non-compact groups and induced representations. Next we realise the principal series representations of $\mathrm{SL}(2,\mathbb{R})$ as induced representations. We then carry on by recalling the methods employed by Repka [61] and Martin [62] for decomposing tensor products of $\mathrm{SL}(2,\mathbb{R})$ principal series tensor products. Throughout, we will not be interested in the precise proofs of the statements, but the ideas will be illustrated primarily by reference to the examples. After this background information on $\mathrm{SL}(2,\mathbb{R})$, we move on to deal with $\mathrm{SL}(2,\mathbb{C})$ or $\mathrm{SO}(3,1)$. We here also recall the structure theory and principal series representations. We then show how in the decomposition of the tensor product of the principal series representations one is led to consider harmonics on complex spheres. We then analyse these spherical harmonics first for a three-dimensional complex sphere, and verify that they correctly lead to the known decomposition of $\mathrm{SO}(3,1)$ principal series tensor products. We then generalise our study of the spherical harmonics to higher dimensions.

The main text of this chapter is supplemented by Appendix K, in which we review the classification of unitary irreducible representations of $\mathrm{SL}(2,\mathbb{C})$.

14 Background

In this section, we will deal with representations of non-compact Lie groups, primarily $\mathrm{SL}(2,\mathbb{R})$. To begin, we recall some aspects of the structure theory of these groups. We introduce parts of the general structure theory of non-compact semisimple Lie groups, but we will not worry about the technical details. We carry the notation and definitions of [63] and [15].

14.1 Structure Theory of Non-Compact Groups

The group $\mathrm{SL}(2,\mathbb{R})$, which we will use as our simplest example, consists of the 2×2 real matrices with unit modulus, that is

$$\mathrm{SL}(2,\mathbb{R}) = \left\{ g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \ : \ ad - bc = 1, \ a,b,c,d \in \mathbb{R} \right\}. \tag{783}$$

It follows that the associated Lie algebra, $\mathrm{sl}(2,\mathbb{R})$, can be realised as the real 2×2 matrices with vanishing trace, that is

$$\mathrm{sl}(2,\mathbb{R}) = \left\{ X = \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \ : \ a,b,c, \in \mathbb{R} \right\}. \tag{784}$$

With respect to the Cartan involution $\theta X = -X^\dagger$ on the algebra, we can decompose $\mathrm{sl}(2,\mathbb{R})$ into positive and negative eigenspaces, which we call $\mathfrak{k}$ and $\mathfrak{p}$ respectively. Explicitly

$$\mathfrak{k} = \{X \in \mathrm{sl}(2,\mathbb{R}) \ : \ \theta X = X\} = \left\{ \begin{pmatrix} 0 & -b \\ b & 0 \end{pmatrix} \ : \ b \in \mathbb{R} \right\}, \tag{785}$$

$$\mathfrak{p} = \{X \in \mathrm{sl}(2,\mathbb{R}) \ : \ \theta X = -X\} = \left\{ \begin{pmatrix} a & b \\ b & -a \end{pmatrix} \ : \ a,b \in \mathbb{R} \right\}. \tag{786}$$

Using the algebra identification $\mathrm{sl}(2,\mathbb{R}) \cong \mathrm{so}(2,1)$, we can think of the elements of $\mathfrak{k}$ as the generators of rotations, while the elements of $\mathfrak{p}$ are generators of boosts. Exponentiating $\mathfrak{k}$ leads to the maximal compact subgroup $K = \mathrm{SO}(2)$ of $\mathrm{SL}(2,\mathbb{R})$.

Next, we define $\mathfrak{a}$ to be the algebra generated by any boost in $\mathfrak{p}$. We choose

$$\mathfrak{a} = \left\{ \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix} \ : \ a \in \mathbb{R} \right\}. \tag{787}$$

In the general case, $\mathfrak{a}$ consists of a maximal abelian algebra from elements in $\mathfrak{p}$. This suggests that we should look to diagonalise the adjoint action with respect to this algebra, that is we look for elements X such that

$$[H,X] = \lambda(H)X, \tag{788}$$

for all elements $H \in \mathfrak{a}$, and some restricted roots λ. For $\mathrm{sl}(2,\mathbb{R})$, a quick calculation gives that there are two roots $\lambda = \pm 1$. We let the positive and negative root spaces be $\mathfrak{n}$ and $\bar{\mathfrak{n}}$ respectively, which for $\mathrm{sl}(2,\mathbb{R})$ are

$$\mathfrak{n} = \left\{ \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} \ : \ x \in \mathbb{R} \right\}, \bar{\mathfrak{n}} = \left\{ \begin{pmatrix} 0 & 0 \\ y & 0 \end{pmatrix} \ : \ y \in \mathbb{R} \right\}. \tag{789}$$

Exponentiating the subgroups A, N and $\bar{N}$ respectively, we find

$$A = \left\{ \begin{pmatrix} e^\alpha & 0 \\ 0 & e^{-\alpha} \end{pmatrix} \ : \ \alpha \in \mathbb{R} \right\}, N = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \ : \ x \in \mathbb{R} \right\}. \tag{790}$$

Finally, let M be the subgroup of K which under the adjoint action on the algebra leaves $\mathfrak{a}$ invariant. For $\mathrm{sl}(2,\mathbb{R})$, it is possible to see that

$$M = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \right\} \cong \mathbb{Z}_2. \tag{791}$$

Note that in general this group does not have to be connected.

14.1.1 Iwasawa and Bruhat Decompositions

We have now assembled all the pieces required to give some decompositions of $\mathrm{SL}(2,\mathbb{R})$.

The first decomposition we introduce is the Iwasawa or KNA decomposition, which decomposes a general element g into an element of K, N and A:

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} e^{\alpha} & 0 \\ 0 & e^{-\alpha} \end{pmatrix} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, \tag{792}$$

where $\theta \in [0, 2\pi)$, $\alpha, x \in \mathbb{R}$.

Secondly, we introduce the Bruhat or $\bar{N}NAM$ decomposition. Here the elements of $\mathrm{SL}(2,\mathbb{R})$ are decomposed as

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \pm \begin{pmatrix} 1 & 0 \\ y & 1 \end{pmatrix} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} e^{\alpha} & 0 \\ 0 & e^{-\alpha} \end{pmatrix}, \tag{793}$$

with $x, y, \alpha \in \mathbb{R}$. We note that if $a = 0$, then the element g does not admit a Bruhat decomposition, but still admits an Iwasawa decomposition.

14.2 Induced Representations

Next we introduce induced representations. The idea here is to start with a representation of a subgroup and to promote this to a representation of a larger group. We will not deal with the more general situation, but use a simplified construction based on the definition given by Berndt [15], which is sufficient for the cases considered.

We start with a group G and a closed subgroup H. We assume that G is unimodular, that is to say G admits a measure which is both left- and right-invariant. Now, we will assume that G admits a Mackey decomposition with respect to H. Given an element $g \in G$ let x denote the associated point in the coset space G/H. Then we assume that we can decompose the element g uniquely as

$$g = s(x)h(x), \qquad h(x) \in H, s(x) \in G. \tag{794}$$

Furthermore, we will assume that the coset space G/H admits an invariant measure $\mathrm{d}\mu(x)$. Now the representation π of G induced from the representation π_0 of H can be constructed as follows:

We realise the representations as functions $f(x)$ on the coset space G/H, with

$$\pi(g)f(x) = \pi_0(h(g^{-1}, x)^{-1})f(g^{-1} \cdot x), \tag{795}$$

where we define $h(g^{-1}, x)$ through

$$g^{-1}s(x) = s(g^{-1} \cdot x)h(g^{-1}, x). \tag{796}$$

That this forms a representation follows from the cocycle condition

$$h((g_1 g_2)^{-1}, x) = h(g_2^{-1}, g_1^{-1} \cdot x)h(g_1^{-1}, x). \tag{797}$$

Furthermore, we introduce an inner product for the representation π by

$$(f,g)_\pi = \int_{G/H} \mathrm{d}\mu(x)\ (f(x),g(x))_{\pi_0}. \tag{798}$$

14.2.1 Principal Series SL$(2,\mathbb{R})$ representations

We illustrate the above ideas by considering some representations of SL$(2,\mathbb{R})$. The principal series representations of SL$(2,\mathbb{R})$ can be realised as representations induced from the subgroup NAM. In particular, let

$$h = \pm \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} e^\alpha & 0 \\ 0 & e^{-\alpha} \end{pmatrix} \in NAM, \tag{799}$$

then a suitable representation of NAM is

$$\pi^s_\epsilon = \epsilon(\pm 1) e^{\alpha s}, \tag{800}$$

where $\epsilon(+1) = +1$ and $\epsilon(-1) = \pm 1$.

Then we form the representation of SL$(2,\mathbb{R})$ which this induces as $\Pi^s_\epsilon = \mathrm{ind}^{\mathrm{SL}(2,\mathbb{R})}_{MAN}\ \pi^s_\epsilon$. Using the Bruhat decomposition, we can realise the induced representation as acting on functions on $\bar{N} \cong \mathbb{R}$, thus we next calculate the action of SL$(2,\mathbb{R})$ on $\bar{N}$. Let

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \tag{801}$$

then we find

$$\begin{aligned} g^{-1}\begin{pmatrix} 1 & 0 \\ y & 1 \end{pmatrix} &= \begin{pmatrix} d-by & -b \\ -c+ay & a \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ \frac{ay-c}{d-by} & 1 \end{pmatrix} \begin{pmatrix} 1 & -b(d-by) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} d-by & 0 \\ 0 & \frac{1}{d-by} \end{pmatrix}. \end{aligned} \tag{802}$$

It follows that the induced representation can be realised on functions of $\mathbb{R}$ as

$$\Pi^s_\epsilon(g) f(y) = \epsilon\left(\frac{d-by}{|d-by|}\right) |d-by|^{-s} f\left(\frac{ay-c}{d-by}\right). \tag{803}$$

However, this representation is not yet unitary with respect to the L^2 product on $\mathbb{R}$. This measure on $\bar{N}$ is not invariant under the SL$(2,\mathbb{R})$ action, indeed

$$\mathrm{d}\left(\frac{ay-c}{d-by}\right) = \frac{\mathrm{d}y}{(d-by)^2}. \tag{804}$$

This leads us to the unitary principal series representations

$$\Psi^{is}_\epsilon(g) f(y) = \epsilon\left(\frac{d-by}{|d-by|}\right) |d-by|^{-1-is} f\left(\frac{ay-c}{d-by}\right), \tag{805}$$

because then

$$\|\Psi^{is}_\epsilon f\|^2 = \int_{-\infty}^{\infty} \mathrm{d}y \, \frac{1}{(d-by)^2} \left| f\left(\frac{ay-c}{d-by}\right)\right|^2 \\ = \int_{-\infty}^{\infty} \mathrm{d}z \, |f(z)|^2 = \|f\|^2 \tag{806}$$

We want to check that this indeed defines a principal series $\mathrm{SL}(2,\mathbb{R})$ representation. To this end, we calculate the actions of the algebra generators

$$L = \frac{1}{2}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad B_1 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad B_2 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{807}$$

For an algebra element X, the corresponding representation operator ψ^{is}_ϵ is calculated by

$$\psi^{is}_\epsilon(X) f(y) = \lim_{t\to 0} \frac{d}{dt}\left(\Psi^{is}_\epsilon\left[\exp(tX)\right] f(y)\right). \tag{808}$$

We will typically abuse the notation for the algebra representation operators and denote the representation operator by the same name as the generator. Then for the actions of L, B_1 and B_2 we calculate

$$L = -\frac{1}{2}(1+y^2)\frac{\partial}{\partial y} - \frac{1}{2}(1+is)y, \\ B_1 = y\frac{\partial}{\partial y} + \frac{1}{2}(1+is), \\ B_2 = -\frac{1}{2}(1-y^2)\frac{\partial}{\partial y} + \frac{1}{2}(1+is)y. \tag{809}$$

A quick calculation then reveals that on the representation we then find

$$Q = -(L)^2 + (B_1)^2 + (B_2)^2 = -\frac{1}{4}(s^2+1) = \left(-\frac{1}{2} + i\frac{s}{2}\right)\left(\frac{1}{2} + i\frac{s}{2}\right), \tag{810}$$

and we also have $R[2\pi] = \exp(2\pi L) = \epsilon(-1)$, which tells us that indeed Ψ^{is}_ϵ is an irreducible unitary principal series representation of $\mathrm{SL}(2,\mathbb{R})$, with $l = -\frac{1}{2} + i\frac{s}{2}$ in the notation used in appendix H. If $\epsilon(-1) = +1$, we get a representation of $\mathrm{SO}_0(2,1)$.

For later use, the same principal series representation can also be realised by inducing from $\bar{N}AM$ instead. Suppose that

$$h' = \pm\begin{pmatrix} 1 & 0 \\ y & 1 \end{pmatrix}\begin{pmatrix} e^{\alpha} & 0 \\ 0 & e^{-\alpha} \end{pmatrix}, \tag{811}$$

with the representation of $\bar{N}AM$ given as before by

$$\pi'^{s}_\epsilon = \epsilon(\pm 1) e^{\alpha s}. \tag{812}$$

Then we can form the unitary principal series representation as

$$\Psi'^{is}_\epsilon = \mathrm{ind}^{\mathrm{SL}(2,\mathbb{R})}_{\bar{N}AM}\, \pi'^{s}_\epsilon \tag{813}$$

with

$$\Psi'^{is}_{\epsilon}(g)f(x) = \epsilon\left(\frac{a-cx}{|a-cx|}\right)|a-cx|^{-1+is} f\left(\frac{dx-b}{a-cx}\right). \tag{814}$$

A quick calculation easily verifies that indeed $\Psi'^{is}_{\epsilon} \cong \Psi^{is}_{\epsilon}$.

14.3 Tensor Product of Principal Series $\mathrm{SL}(2,\mathbb{R})$ representations

We now want to calculate the decomposition into irreducibles of the product of two principal series representations of $\mathrm{SL}(2,\mathbb{R})$. Following the approach of Repka [61] and Martin [62], the starting point for this is Mackey's tensor product theorem. Before we state the theorem, we need a further definition.

Suppose that G is a group with two subgroups H_1 and H_2. Then the double coset $H_1 g H_2$ the element $g \in G$ is in is defined by

$$H_1 g H_2 = \{h_1 g h_2 \in G \ : \ h_1 \in H_1,\ h_2 \in H_2\}. \tag{815}$$

Importantly, the double cosets $H_1\backslash G/H_2$ partition the group. However different double cosets are allowed to contain differing numbers of elements of G. This is unlike ordinary cosets, which are required to all contain the same number of group elements.

To quickly demonstrate this, we give a discrete example: Consider the finite group $G = D_4 = \langle b, c : b^2 = c^4 = 1, bcb = c^{-1}\rangle$. Let $H_1 = H_2 = C_2 = \{1, b\}$. Then

$$H_1 1 H_2 = \{1, b\}, \qquad H_1 c H_2 = \{c, bc, bc^3, c^3\}, \qquad H_1 c^2 H_2 = \{c^2, bc^2\}. \tag{816}$$

14.3.1 Mackey's Tensor Product Theorem

We take a simplified version of Mackey's Tensor Product Theorem [64]. This version will be sufficient for the examples we consider in this thesis.

Suppose H_1 and H_2 are closed subgroups of a group G. Suppose further that there are only countably many $H_1\backslash G/H_2$ double cosets. Suppose that π_1 and π_2 are representations of H_1 and H_2 respectively. Let x and y be elements of G. Then denote

$$G_{x,y} = x^{-1}H_1 x \cap y^{-1} H_2 y. \tag{817}$$

Next, we define

$$\pi_{x,y}(g) = \pi_1(xgx^{-1}) \otimes \pi_2(ygy^{-1}), \tag{818}$$

$$\pi^{x,y} = \mathrm{ind}^G_{G_{x,y}} \pi_{x,y}. \tag{819}$$

Then the claim is that $\pi^{x,y}$, up to equivalence, is determined by the double coset d to which the element xy^{-1} belongs. Therefore we write $\pi^{x,y} = \pi^d$. Furthermore we have

$$\mathrm{ind}^G_{H_1} \pi_1 \otimes \mathrm{ind}^G_{H_2} \pi_2 \cong \bigoplus_d \pi^d, \tag{820}$$

where the sum runs over all cosets d which are not of measure zero.

14.3.2 Applying Mackey's Tensor Product Theorem to $\mathrm{SL}(2,\mathbb{R})$

We now apply Mackey's tensor product theorem to the $\mathrm{SL}(2,\mathbb{R})$ situation. We set $G = \mathrm{SL}(2,\mathbb{R})$

$$H_1 = \bar{N}AM, \qquad H_2 = NAM. \tag{821}$$

Now we need to calculate the $\bar{N}AM\backslash\mathrm{SL}(2,\mathbb{R})/NAM$ double cosets. Using the Iwasawa decomposition, we note that the $\mathrm{SL}(2,\mathbb{R})/NAM$ cosets are determined uniquely as

$$\mathrm{SL}(2,\mathbb{R})/NAM = \left\{ \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \ : \ \theta \in [0,\pi) \right\}. \tag{822}$$

Next, note that as long as $\theta \neq \frac{\pi}{2}$ we have

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \tan\theta & 1 \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ 0 & \sec\theta \end{pmatrix} \in \bar{N}NAM. \tag{823}$$

Thus there are two double cosets,

$$d_1 = \bar{N}AM \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} NAM, \qquad d_2 = \bar{N}AM \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} NAM. \tag{824}$$

Notice that in the Iwasawa decomposition, all the elements of the double coset d_2 have the same $K = \mathrm{SO}(2)$ angle $\theta = \frac{\pi}{2}$, thus with respect to the Haar measure on the group this is a set of measure zero. It follows that in Mackey's tensor product theorem only the double coset d_1 contributes. For the representations of NAM and $\bar{N}AM$ we choose, in the notation used previously,

$$\pi_1 = \pi'^{is_1}_{\epsilon_1}, \qquad \pi_2 = \pi^{is_2}_{\epsilon_2}. \tag{825}$$

As the only contributing double coset d_1 contains the identity element, we may choose $x = y = 1$. Now it is clear that

$$\bar{N}AM \cap NAM = AM, \tag{826}$$

Meanwhile,

$$\begin{aligned} \pi_{x,y}\left(\pm\begin{pmatrix} e^{\alpha} & 0 \\ 0 & e^{-\alpha} \end{pmatrix}\right) &= \pi_1\left(\pm\begin{pmatrix} e^{\alpha} & 0 \\ 0 & e^{-\alpha} \end{pmatrix}\right) \otimes \pi_2\left(\pm\begin{pmatrix} e^{\alpha} & 0 \\ 0 & e^{-\alpha} \end{pmatrix}\right) \\ &= \epsilon_1(\pm 1)\epsilon_2(\pm 1) e^{i\alpha(s_1+s_2)}. \\ &= \pi^{is_1+is_2}_{\epsilon_1\epsilon_2}\left(\pm\begin{pmatrix} e^{\alpha} & 0 \\ 0 & e^{-\alpha} \end{pmatrix}\right), \end{aligned} \tag{827}$$

where we regard $\pi^{is_1+is_2}_{\epsilon_1\epsilon_2}$ as a representation of AM.

The end result is that Mackey's tensor product tells us

$$\mathrm{ind}_{NAM}^{\mathrm{SL}(2,\mathbb{R})}\, \pi^{is_1}_{\epsilon_1} \otimes \mathrm{ind}_{NAM}^{\mathrm{SL}(2,\mathbb{R})}\, \pi^{is_2}_{\epsilon_2} \cong \mathrm{ind}_{AM}^{\mathrm{SL}(2,\mathbb{R})}\, \pi^{i(s_1+s_2)}_{\epsilon_1\epsilon_2}. \tag{828}$$

14.3.3 Representations Induced from AM

Mackey's tensor product theorem says that if we want to understand the tensor product of two principal series representations, we can alternatively analyse a representation induced from AM. Using the Iwasawa decomposition, the elements of $\mathrm{SL}(2,\mathbb{R})/MA$ can be labelled by a real number x and an angle $\theta \in [0,\pi)$ with

$$s(x,\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}. \tag{829}$$

If we work with the Bruhat $\bar{N}NAM$ decomposition, we instead label the cosets by y and z, labelling elements of $\bar{N}N$. These are related to x and θ by noting that an elements of $\bar{N}N$ can be written

$$\begin{pmatrix} 1 & 0 \\ y & 1 \end{pmatrix} \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} e^{-\beta} & e^{\beta} z \\ e^{-\beta} y & e^{\beta}(1+zy) \end{pmatrix} \begin{pmatrix} e^{\beta} & 0 \\ 0 & e^{-\beta} \end{pmatrix}, \tag{830}$$

thus, for $\theta \in (0, \frac{\pi}{2})$, the coset labelled by (x,θ) can also be labelled by

$$y = \tan\theta, \qquad z = x\cos^2\theta - \sin\theta\cos\theta, \tag{831}$$

meanwhile, if $\theta \in (\frac{\pi}{2}, \pi)$, the coset labelled by (x,θ) is instead labelled by

$$y = \tan\theta, \qquad z = -x\cos^2\theta + \sin\theta\cos\theta. \tag{832}$$

The elements with $\theta = \frac{\pi}{2}$ are not captured by the Bruhat decomposition. We now carry on working with the Bruhat decomposition. First we want to find the $\mathrm{SL}(2,\mathbb{R})$ action on $\bar{N}N$, let

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \tag{833}$$

then

$$g^{-1} \begin{pmatrix} 1 & 0 \\ y & 1 \end{pmatrix} \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} d - by & (d-by)z - b \\ -c + ay & (-c+ay)z + a \end{pmatrix}, \tag{834}$$

so that

$$y \mapsto y' = \frac{ay - c}{d - by}, \qquad z \mapsto z' = (d - by)(-b + (d - by)z). \tag{835}$$

Note that

$$\mathrm{d}y' \wedge \mathrm{d}z' = \mathrm{d}y \wedge \mathrm{d}z, \tag{836}$$

so that this space admits an invariant measure. Thus, on these elements, the induced representation, call it Π^{is}_{ϵ} is realised as

$$\Pi^{is}_{\epsilon}(g) f(y,z) = \epsilon\left(\frac{d - by}{|d - by|}\right) |d - by|^{-is} f\left(\frac{ay - c}{d - by}, (d - by)(-b + (d - by)z)\right) \tag{837}$$

Suppose that we Fourier transform in z by defining

$$\tilde{f}(y,\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathrm{d}z \; e^{-i\omega z} f(y,z). \tag{838}$$

This is clearly a unitary operation. Let $\tilde{\Pi}_\epsilon^{is}$ denote the Fourier transformed representation. Then note that

$$\begin{aligned}
&\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \mathrm{d}z \ \exp(-i\omega z)F((d-by)(-b+(d-by)z) \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \frac{\mathrm{d}z'}{(d-by)^2} \ \exp(-i\omega z)F(z') \\
&= \frac{1}{\sqrt{2\pi}}\exp\left(-i\omega\frac{b}{d-by}\right)\int_{-\infty}^{\infty}\frac{\mathrm{d}z'}{(d-by)^2}\exp\left(-i\frac{\omega}{(d-by)^2}z'\right)F(z') \\
&= \frac{1}{(d-by)^2}\exp\left(-i\omega\frac{b}{d-by}\right)\tilde{F}\left(\frac{\omega}{(d-by)^2}\right).
\end{aligned} \tag{839}$$

The end result is that

$$\begin{aligned}
&\tilde{\Pi}_\epsilon^{is}(g)\tilde{f}(y,\omega) \\
&= \epsilon\left(\frac{d-by}{|d-by|}\right)|d-by|^{-2-is}\exp\left(-i\omega\frac{b}{d-by}\right)\tilde{f}\left(\frac{ay-c}{d-by},\frac{\omega}{(d-by)^2}\right).
\end{aligned} \tag{840}$$

Suppose we then define

$$\tilde{f}(y,\omega) = \omega^{-is/2}\tilde{F}(y,\omega), \tag{841}$$

then it is immediately clear that

$$\tilde{\Pi}_\epsilon^{is}(g)\left[\omega^{-is/2}\tilde{F}(y,\omega)\right] = \omega^{-is/2}\tilde{\Pi}_\epsilon^{0}(g)\tilde{F}(y,\omega), \tag{842}$$

which proves that we have the unitary equivalence

$$\Pi_\epsilon^{is} \cong \Pi_\epsilon^0. \tag{843}$$

The upshot of this is that we only have to consider the, comparatively, simpler representation

$$\Pi_\epsilon^0 = \mathrm{ind}_{MA}^{\mathrm{SL}(2,\mathbb{R})}\,\pi_\epsilon^0. \tag{844}$$

14.3.4 Decomposing the Tensor Product using de Sitter Space

Using the Bruhat decomposition coordinates we can calculate an invariant metric for $\mathrm{SL}(2,\mathbb{R})$. In the Bruhat decomposition $g = g(y,z,\alpha)$ is given by

$$g(y,z,\alpha) = \begin{pmatrix}1 & 0\\ y & 1\end{pmatrix}\begin{pmatrix}1 & z\\ 0 & 1\end{pmatrix}\begin{pmatrix}e^\alpha & 0\\ 0 & e^{-\alpha}\end{pmatrix} \tag{845}$$

Then a metric on the group which is given by

$$\mathrm{d}s^2 \propto \mathrm{Tr}\left[g^{-1}(y,z,\alpha)\mathrm{d}g(y,z,\alpha)g^{-1}(y,z,\alpha)\mathrm{d}g(y,z,\alpha)\right], \tag{846}$$

which leads to

$$\mathrm{d}s^2 \propto (\mathrm{d}\alpha - z\mathrm{d}y)^2 + \mathrm{d}y(\mathrm{d}z - z^2\mathrm{d}y). \tag{847}$$

To find the metric on the quotient space $\mathrm{SL}(2,\mathbb{R})/MA$, we look for directions which are perpendicular to ∂_α, which we can see to be $z\partial_\alpha + \partial_y$, then because

$$(z\partial_\alpha + \partial_y)(\mathrm{d}\alpha - z\mathrm{d}y) = 0, \tag{848}$$

we should take for the invariant measure on $\mathrm{SL}(2,\mathbb{R})$ in these coordinates the metric

$$\mathrm{d}s^2 \propto \mathrm{d}y(\mathrm{d}z - z^2\mathrm{d}y). \tag{849}$$

Making the coordinate transformations

$$z = -\frac{1}{Z}, \qquad Y = 2y - Z, \tag{850}$$

we find that

$$\mathrm{d}s^2 = \frac{1}{Z^2}(-\mathrm{d}Y^2 + \mathrm{d}Z^2), \tag{851}$$

which we recognize as the Poincaré patches of de Sitter space. The regions $Z > 0$ and $Z < 0$ cover both the expanding and contracting patches.

It follows that if we take the tensor product of two principal series representations of $\mathrm{SO}(2,1)$, that is to say we let $\epsilon(-1) = +1$, then to decompose this tensor product into irreducible representations it is sufficient to look for solutions to the scalar field equation in de Sitter space. We look for solutions to the scalar field equation which are either square-integrable or δ-function integrable. We work in global coordinates (t, ϕ) in two-dimensional de Sitter space, where the metric takes the form

$$\mathrm{d}s^2 = \mathrm{d}t^2 - \cosh^2 t \; \mathrm{d}\phi^2 \tag{852}$$

Then the scalar field equation we wish to consider is

$$\Box\Phi(t,\phi) = \frac{1}{\cosh t}\frac{\partial}{\partial t}\left(\cosh t\frac{\partial\Phi}{\partial t}\right) - \frac{1}{\cosh^2 t}\frac{\partial^2\Phi}{\partial\phi^2} = -M^2\Phi(t,\phi). \tag{853}$$

Let $\Phi(t,\phi) = \Phi_m(t)e^{im\phi}$, where $m \in \mathbb{Z}$ so that the function is single-valued on de Sitter space, then

$$\frac{1}{\cosh t}\frac{d}{dt}\left(\cosh t\frac{d\Phi_m}{dt}\right) + \frac{m^2}{\cosh^2 t}\Phi_m = -M^2\Phi_m. \tag{854}$$

As before, we write $M^2 = -l(l+1)$ and restrict Re $l \geq -\frac{1}{2}$, then the equation obeyed by Φ_m is an associated Legendre equation in the variable $i\sinh t$, and the linearly independent solutions are

$$\Phi_m(t) = A\mathsf{P}_l^{-m}(i\sinh t) + B\mathsf{P}_l^{-m}(-i\sinh t), \tag{855}$$

where the $\mathsf{P}_l^{-m}(x)$ denote associated Legendre functions. Now, as $t \to \infty$, both these solutions behave as [54, Eq. 14.8.2]

$$\left|\mathsf{P}_l^{-m}(\pm i\sinh t)\right| \sim e^{|t|\mathrm{Re}\; l}, \tag{856}$$

provided that l is not an integer. While the measure behaves as

$$\sqrt{-g}\mathrm{d}^2x = \cosh t \mathrm{d}t\mathrm{d}\phi \sim e^{|t|}\mathrm{d}t\mathrm{d}\phi. \tag{857}$$

If follows that square-integrable solutions are not possible, while δ-function integrable solutions are possible only when

$$l = -\frac{1}{2} + i\lambda, \qquad \lambda \in \mathbb{R} \tag{858}$$

which is to say when l is such that these functions fall into the principal series of representations, as defined in Appendix H of Part II. Indeed it is possible to show that $\mathsf{P}_l^{-m}(\pm i \sinh t)$ both lead to δ-function integrable solutions for each m, and it follows that there are precisely two independent copies of the principal series for each $\lambda \in \mathbb{R}$.

There is one possibility we have so-far missed, which can lead to square-integrable solutions. This is when $l \in \mathbb{Z}$, in which case the associated Legendre functions reduce to associated Legendre polynomials. Now, notice that for each m the functions

$$\Psi_m(t, \phi) = e^{im\phi} \cosh^{-m} t \tag{859}$$

obey

$$\frac{1}{\cosh t} \frac{d}{dt} \left(\cosh t \frac{d}{dt} \Psi_m \right) = m(m-1)\Psi_m - m^2 \frac{1}{\cosh^2 t} \Psi_m, \tag{860}$$

that is the functions obey the associated Legendre equation with $l = m - 1$. For $m \geq 1$ these functions are square-integrable on the de Sitter space. Moreover, these functions are annihilated by the so$(2,1)$ lowering operator

$$L_- = e^{-i\phi} \left(\frac{\partial}{\partial t} - i \tanh t \frac{\partial}{\partial \phi} \right). \tag{861}$$

So these functions form the lowest weight vector in a discrete series representation of SO$(2,1)$ as described in Appendix H of Part II. We denote it as $T_{m,+}$. The other vectors can be found by applying the raising operator

$$L_+ = e^{i\phi} \left(\frac{\partial}{\partial t} + i \tanh t \frac{\partial}{\partial \phi} \right). \tag{862}$$

Similarly, the functions

$$\tilde{\Psi}_m(t, \phi) = e^{-im\phi} \cosh^{-m} t, \tag{863}$$

form highest weight vectors for the discrete series representations $T_{-m,+}$ of SO$(2,1)$.

Putting everything together, when we decompose the square-integrable solutions to the scalar field equation on de Sitter space, we find that in terms of the irreducible representations of SO$(2,1)$ we have two copies of every principal series representation and also one copy of every discrete series representation with $m \geq 1$. Thus the decomposition of our tensor product is

$$\pi_+^{is} \otimes \pi_+^{ir} \cong 2 \int_0^\infty \mathrm{d}\lambda \; \pi_+^{i\lambda} \bigoplus_{m \neq 0, m \in \mathbb{Z}} T_{m,+}, \tag{864}$$

which agrees with the well known results in Martin [62] and Repka [61].

15 Complex Spheres and SO(3, 1) Representations

With the background of SO(2, 1) representations set up, we now consider SO(3, 1) representations. We first set up the necessary pieces of the structure theory for the case of SO(3, 1).

15.1 Structure Theory of SO(3, 1)

We will make use of the identification of SL(2, ℂ) as the double covering group of SO(3, 1). Here we have

$$\mathrm{SL}(2,\mathbb{C}) = \left\{ g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \ : \ \alpha\delta - \beta\gamma = 1,\ \alpha,\beta,\gamma,\delta \in \mathbb{C} \right\}. \tag{865}$$

Then the associated Lie algebra is composed of the complex trace-free 2×2 matrices

$$\mathrm{sl}(2,\mathbb{C}) = \left\{ X = \begin{pmatrix} \alpha & \beta \\ \gamma & -\alpha \end{pmatrix} \ : \ \alpha,\beta,\gamma \in \mathbb{C} \right\}. \tag{866}$$

Then we can read off that $\mathfrak{k}$ consists of an su(2) subalgebra

$$\mathfrak{k} = \left\{ X = \begin{pmatrix} ia & \beta \\ -\beta^\dagger & -ia \end{pmatrix} \ : \ a \in \mathbb{R},\ \beta \in \mathbb{C} \right\} \tag{867}$$

while for the boost generators $\mathfrak{p}$ we have

$$\mathfrak{p} = \left\{ X = \begin{pmatrix} a & \beta \\ \beta^\dagger & -a \end{pmatrix} \ : \ a \in \mathbb{R},\ \beta \in \mathbb{C} \right\}. \tag{868}$$

For the privileged boost $\mathfrak{a}$, we again choose the diagonal element of $\mathfrak{p}$, so that

$$\mathfrak{a} = \left\{ \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix} \ : \ a \in \mathbb{R} \right\}, \qquad A = \left\{ \begin{pmatrix} e^a & 0 \\ 0 & e^{-a} \end{pmatrix} \ : \ a \in \mathbb{R} \right\}. \tag{869}$$

As $\mathfrak{a}$ has the same form for sl(2, ℂ) and sl(2, ℝ), we can quickly see that there are again two restricted roots $\lambda = \pm 2$, with positive and negative root spaces this time being two-dimensional with

$$\mathfrak{n} = \left\{ \begin{pmatrix} 0 & z \\ 0 & 0 \end{pmatrix} \ : \ z \in \mathbb{C} \right\},\ \bar{\mathfrak{n}} = \left\{ \begin{pmatrix} 0 & 0 \\ w & 0 \end{pmatrix} \ : \ w \in \mathbb{C} \right\}, \tag{870}$$

which exponentiate to

$$N = \left\{ \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} \ : \ z \in \mathbb{C} \right\},\ \bar{N} = \left\{ \begin{pmatrix} 1 & 0 \\ w & 1 \end{pmatrix} \ : \ w \in \mathbb{C} \right\}. \tag{871}$$

Finally we need to work out the group M of elements of K whose adjoint action commutes with $\mathfrak{a}$, this is easily calculated as

$$M = \left\{ m = \begin{pmatrix} e^{ib} & 0 \\ 0 & e^{-ib} \end{pmatrix} \;:\; b \in \mathbb{R} \right\} \cong \mathrm{SO}(2). \tag{872}$$

Notice that if we let $\Lambda = a + ib$, then the elements of MA are of the form

$$ma = am = \begin{pmatrix} e^{a} & 0 \\ 0 & e^{-a} \end{pmatrix} \begin{pmatrix} e^{ib} & 0 \\ 0 & e^{-ib} \end{pmatrix} = \begin{pmatrix} e^{\Lambda} & 0 \\ 0 & e^{-\Lambda} \end{pmatrix} \tag{873}$$

We are now in a position to give the Bruhat $\bar{N}NAM$ decomposition of $\mathrm{SL}(2,\mathbb{C})$, which we will primarily be using from now on. Here we decompose a general element of $\mathrm{SL}(2,\mathbb{C})$ as

$$g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ w & 1 \end{pmatrix} \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} \begin{pmatrix} e^{\Lambda} & 0 \\ 0 & e^{-\Lambda} \end{pmatrix}, \tag{874}$$

where $w, z, \Lambda \in \mathbb{C}$.

15.2 Principal Series Representations of $\mathrm{SL}(2,\mathbb{C})$

The principal series representations are again induced from NAM. Let h be an element of NAM given by

$$h(z,\Lambda) = \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} \begin{pmatrix} e^{\Lambda} & 0 \\ 0 & e^{-\Lambda} \end{pmatrix} \tag{875}$$

then for $k \in \mathbb{Z}$ and $s \in \mathbb{R}$ let $\pi^{k,s}$ be the representation of NAM defined by

$$\pi(h(z,\Lambda)) = e^{is\mathrm{Re}\ \Lambda} e^{ik\mathrm{Im}\ \Lambda}. \tag{876}$$

If we let a generic element g of $\mathrm{SL}(2,\mathbb{C})$ be given by

$$g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \tag{877}$$

then proceeding as for $\mathrm{SL}(2,\mathbb{R})$ the representation $\Pi^{k,s}$ of $\mathrm{SL}(2,\mathbb{C})$ which is induced by $\pi^{k,s}$ is given by

$$\Pi^{k,s}(g)f(w) = |\delta - \beta w|^{-is} \left(\frac{\delta - \beta w}{|\delta - \beta w|} \right)^{-k} f\left(\frac{\alpha w - \gamma}{\delta - \beta w} \right). \tag{878}$$

However, this representation is again not yet unitary with respect to the L^2 product on $\mathbb{C}$ because the measure $\mathrm{d}w\mathrm{d}\bar{w}$ on $\mathbb{C}$ is not invariant under the $\mathrm{SL}(2,\mathbb{C})$ action. Indeed if

$$w' = \frac{\alpha w - \gamma}{\delta - \beta w}, \tag{879}$$

then

$$\mathrm{d}w'\mathrm{d}\bar{w}' = \frac{\mathrm{d}w\mathrm{d}\bar{w}}{|\delta - \beta w|^4}. \tag{880}$$

Whence the unitary induced representation $\Psi^{k,s}$ of $\mathrm{SL}(2,\mathbb{C})$ is given by

$$\Psi^{k,s}(g)f(w) = |\delta - \beta w|^{-2-is}\left(\frac{\delta-\beta w}{|\delta-\beta w|}\right)^{-k} f\left(\frac{\alpha w - \gamma}{\delta - \beta w}\right). \tag{881}$$

We also note that not all the labels (k,s) lead to inequivalent irreducible unitary representations. Indeed we have the unitary equivalence [63]

$$\Psi^{k,s} \cong \Psi^{-k,-s}. \tag{882}$$

Moreover, we note that as for $\mathrm{SL}(2,\mathbb{R})$, it is irrelevant whether we induce from the lower or upper triangular subgroups. These lead to equivalent representations, that is

$$\mathrm{ind}_{MAN}^{\mathrm{SL}(2,\mathbb{C})}\,\pi^{k,s} \cong \mathrm{ind}_{MA\bar{N}}^{\mathrm{SL}(2,\mathbb{C})}\,\pi^{k,s}, \tag{883}$$

where in the right hand side, we view $\pi^{k,s}$ as a representation of $\bar{N}AM$ by

$$\pi^{k,s}\left(\begin{pmatrix}1 & 0\\ w & 1\end{pmatrix}\begin{pmatrix}e^{\Lambda} & 0\\ 0 & e^{-\Lambda}\end{pmatrix}\right) = e^{is\mathrm{Re}\,\Lambda}e^{ik\mathrm{Im}\,\Lambda}. \tag{884}$$

Calculating the infinitesimal action of the elements of a basis for $\mathfrak{k}$ and $\mathfrak{p}$ to find the quadratic Casimir operators acting on the representation, it is possible to show that the representation $\Psi^{k,s}$ has parameters $l_0 = \frac{k}{2}$ and $\rho = i\frac{s}{2}$ in the notation used in the classification of Appendix K. Only the representations with even k form principal series representations of $\mathrm{SO}(3,1)$.

15.3 Tensor Product of $\mathrm{SL}(2,\mathbb{C})$ representations

We have now set up the necessary structure theory and principal series representations to consider the tensor product of $\mathrm{SL}(2,\mathbb{C})$ representations. We follow essentially the same procedure as for $\mathrm{SL}(2,\mathbb{R})$ representations as much as possible. The first step is to apply Mackey's tensor product theorem to show that the tensor product of the principal series is equivalent to a representation induced from the diagonal group AM. We then show that the representation space of the representation induced from AM can be viewed as functions living on complex spheres. For $\mathrm{SO}(3,1)$ representations we then decompose the tensor product into irreducibles by finding the harmonics on the complex sphere which make up the representation.

As for $\mathrm{SL}(2,\mathbb{R})$, apply Mackey's tensor product theorem with

$$H_1 = \bar{N}AM, \qquad H_2 = NAM, \tag{885}$$

and $\pi_1 = \pi^{k_1,s_1}$ and $\pi_2 = \pi^{k_2,s_2}$. Then applying Mackey's tensor product theorem as for $\mathrm{SL}(2,\mathbb{R})$ show that [62]

$$\mathrm{ind}_{H_1}^{\mathrm{SL}(2,\mathbb{C})}\,\pi_1 \otimes \mathrm{ind}_{H_2}^{\mathrm{SL}(2,\mathbb{C})}\,\pi_2 \cong \Psi^{s_1,k_1}\otimes\Psi^{s_2,k_2} \cong \mathrm{ind}_{AM}^{\mathrm{SL}(2,\mathbb{C})}\,\pi^{k_1+k_2,s_1+s_2}, \tag{886}$$

where we regard $\pi^{k,s}$ as a representation of AM in the obvious manner for the final induced representation. Furthermore, for $\mathrm{SL}(2,\mathbb{C})$ it is again the case that the label s on the representation induced from AM does not matter, that is for all s we have

$$\operatorname{ind}_{AM}^{\mathrm{SL}(2,\mathbb{C})} \pi^{k,s} \cong \operatorname{ind}_{AM}^{\mathrm{SL}(2,\mathbb{C})} \pi^{k,0}, \tag{887}$$

this is proved in general by Martin [62], Theorem 2.

In particular, now if we recall that $\Psi^{k,s} \cong \Psi^{-k,-s}$, this tells us that if $k_1 + k_2$ is even, which is in particular the case if we have $\mathrm{SO}(3,1)$ representations, then

$$\Psi^{k_1,s_1} \otimes \Psi^{k_2,s_2} \cong \operatorname{ind}_{AM}^{\mathrm{SL}(2,\mathbb{C})} \pi^{k_1+k_2,0} \cong \operatorname{ind}_{AM}^{\mathrm{SL}(2,\mathbb{C})} \pi^{0,0}. \tag{888}$$

Thus, to decompose the tensor product of principal series Lorentz group representations, it is equivalently possible to decompose representation $\operatorname{ind}_{AM}^{\mathrm{SL}(2,\mathbb{C})} \pi^{0,0}$. We now analyse this representation, which is realised on square integrable functions on $\mathrm{SL}(2,\mathbb{C})/MA$. First, let us consider this quotient space using the Bruhat decomposition.

15.3.1 The Quotient Space $\mathrm{SL}(2,\mathbb{C})/MA$

As for $\mathrm{SL}(2,\mathbb{R})$, in the Bruhat decomposition we can write an element of $\mathrm{SL}(2,\mathbb{C})$ as

$$g(w,z,\Lambda) = \begin{pmatrix} 1 & 0 \\ w & 1 \end{pmatrix} \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} \begin{pmatrix} e^{\Lambda} & 0 \\ 0 & e^{-\Lambda} \end{pmatrix}, \tag{889}$$

with w, z and Λ complex numbers. The elements of h of MA are of the form

$$h(\Lambda) = \begin{pmatrix} e^{\Lambda} & 0 \\ 0 & e^{-\Lambda} \end{pmatrix}, \tag{890}$$

which suggests that we can identify the quotient space elements $\mathrm{SL}(2,\mathbb{C})/MA$ with $g(w,z,0)$. We can calculate an invariant complex metric on $\mathrm{SL}(2,\mathbb{C})$ by calculating

$$\mathrm{d}s^2_{\mathbb{C}} \propto \mathrm{Tr}\left[g(w,z,\Lambda)^{-1}\mathrm{d}g(w,z,\Lambda)g(w,z,\Lambda)^{-1}\mathrm{d}g(w,z,\Lambda)\right]. \tag{891}$$

We have

$$g(w,z,\Lambda)^{-1} = \begin{pmatrix} e^{-\Lambda}(1+zw) & -e^{-\Lambda}z \\ -e^{\Lambda}z & e^{\Lambda} \end{pmatrix}, \tag{892}$$

and

$$\mathrm{d}g(w,z,\Lambda) = \begin{pmatrix} e^{\lambda}\mathrm{d}\Lambda & e^{-\Lambda}\mathrm{d}z - ze^{-\Lambda}\mathrm{d}\Lambda \\ e^{\lambda}\mathrm{d}w + we^{\Lambda}\mathrm{d}\Lambda & -e^{-\Lambda}\mathrm{d}\Lambda(1+wz) + e^{-\Lambda}(z\mathrm{d}w + w\mathrm{d}z) \end{pmatrix}, \tag{893}$$

which yields to the complex metric

$$\mathrm{d}s^2_{\mathbb{C}} \propto (\mathrm{d}\Lambda - z\mathrm{d}w)^2 + \mathrm{d}w(\mathrm{d}z - z^2\mathrm{d}w). \tag{894}$$

From this, we can get a real metric by adding the complex conjugate part. In particular, we note that the $\mathrm{SL}(2,\mathbb{C})$ action on the group itself does not mix complex conjugates, so the real metric has the same invariance properties as the complex metric. Next, we can get from this a metric on the quotient space $\mathrm{SL}(2,\mathbb{C})/MA$ by taking out directions which are orthogonal to ∂_Λ. We have already seen in the real case how this leads to the metric on the quotient space

$$\mathrm{d}s^2 \propto \mathrm{d}w(\mathrm{d}z - z^2\mathrm{d}w). \tag{895}$$

We again make the coordinate transformation

$$z = -\frac{1}{Z}, \qquad W = 2w - Z, \tag{896}$$

to get to the Poincaré patch of (complex) de Sitter space metric

$$\mathrm{d}s^2 \propto \frac{1}{Z^2}(-\mathrm{d}W^2 + \mathrm{d}Z^2). \tag{897}$$

Thus we take as our real metric on the quotient space $\mathrm{SL}(2,\mathbb{C})$ the real part of the above metric

$$\mathrm{d}s^2 = \frac{1}{Z^2}(-\mathrm{d}W^2 + \mathrm{d}Z^2) + \text{c.c.}, \tag{898}$$

where c.c. denotes the complex conjugate of the previous part. This four dimensional manifold can be embedded in a six-dimensional flat space with metric

$$\mathrm{d}s^2 = \mathrm{d}Z_1^2 + \mathrm{d}Z_2^2 + \mathrm{d}Z_3^2 + \text{c.c.}, \tag{899}$$

subject to the constraint

$$Z_1^2 + Z_2^2 + Z_3^2 = 1, \tag{900}$$

where each $Z_i = X_i + iY_i$ is a complex coordinate. The connection between the two can be made for example by letting

$$\begin{aligned} -iZ_1 &= \frac{1}{2}\left(-\frac{1}{Z} + Z - \frac{W^2}{Z}\right), \\ Z_2 &= \frac{1}{2}\left(-\frac{1}{Z} - Z + \frac{W^2}{Z}\right), \\ Z_3 &= -\frac{W}{Z}. \end{aligned} \tag{901}$$

15.3.2 The Induced Representation on $\mathrm{SL}(2,\mathbb{C})/MA$

We have now seen how the quotient space $\mathrm{SL}(2,\mathbb{C})/MA$ can be realised as a complex sphere. Let us now consider the representation of $\mathrm{SL}(2,\mathbb{C})$ induced from $\pi^{0,0}$ on AM. We have already seen how this representation is equivalent to the tensor product of two $\mathrm{SO}(3,1)$ principal series representations. We note that this representation can be viewed as acting on square-integrable complex functions f depending on the complex variables w, z and perhaps their complex conjugates. Furthermore, recall that if

$$g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \tag{902}$$

then we have seen that the representation acts as

$$\left[\mathrm{ind}_{AM}^{\mathrm{SL}(2,\mathbb{C})}\,\pi^{0,0}\right](g)f(w,z) = f\left(\frac{\alpha w - \gamma}{\delta - \beta w}, (\delta - \beta w)(-\beta + (\delta - \beta w)z)\right). \tag{903}$$

We now want to find how the complex generators of $\mathrm{sl}(2,\mathbb{C})$ act in this representation. Thus we define a complex basis for the $\mathrm{sl}(2,\mathbb{C})$ algebra by $M_i = \frac{1}{2}\sigma_i$, where the σ_i are the Pauli matrices given by

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{904}$$

From this we are then able to calculate the action of the generators M_i in the representation. Indeed we find that

$$M_1 = \left(-\frac{1}{2} + \frac{w^2}{2}\right)\partial_w + \left(-\frac{1}{2} - wz\right)\partial_z, \tag{905}$$

$$M_2 = \left(-\frac{i}{2} - \frac{iw^2}{2}\right)\partial_w + \left(\frac{i}{2} + iwz\right)\partial_z, \tag{906}$$

$$M_3 = w\partial_w - z\partial_z. \tag{907}$$

Next, let us see how these act in the complex-sphere picture of our representation space. Start with M_3, in the (W, Z) coordinates of the complex de Sitter space we have

$$M_3 = Z\partial_Z + W\partial_W \tag{908}$$

Now in the (Z_1, Z_2, Z_3) coordinates on the complex sphere we have

$$\partial_W = -\frac{iW}{Z}\partial_{Z_1} + \frac{W}{Z}\partial_{Z_2} - \frac{1}{Z}\partial_{Z_3}, \tag{909}$$

and

$$\partial_Z = \frac{i}{2}\left(\frac{1}{Z^2} + 1 + \frac{W^2}{Z^2}\right)\partial_{Z_1} + \frac{1}{2}\left(\frac{1}{Z^2} - 1 - \frac{W^2}{Z^2}\right)\partial_{Z_2} + \frac{W}{Z^2}\partial_{Z_3}. \tag{910}$$

Which means that

$$M_3 = Z\partial_Z + W\partial_W = -iZ_2\partial_{Z_1} + iZ_1\partial_{Z_2}, \tag{911}$$

and similarly for M_1 and M_2. In particular, we can take as a real basis of generators for the representation

$$M_{ij} = Z_i\partial_{Z_j} - Z_j\partial_{Z_j}, \tag{912}$$

and their complex conjugates $\overline{M}_{ij}$. These obey the commutation relations

$$[M_{ij}, M_{kl}] = \delta_{jk}M_{il} - \delta_{ik}M_{jl} - \delta_{jl}M_{ik} + \delta_{il}M_{jk}, \tag{913}$$

and

$$[M_{ij}, \overline{M}_{kl}] = 0. \tag{914}$$

Thus we can recover the $\mathrm{so}(3,1)$ commutation relations if we define

$$M_{12} = \frac{1}{2}\left(L_3 + iB_3\right), \quad M_{23} = \frac{1}{2}\left(L_2 + iB_2\right), \quad M_{31} = \frac{1}{2}\left(L_1 + iB_1\right), \tag{915}$$

so that the commutation relations return the familiar so(3,1) commutation relations in the form

$$[L_i, L_j] = -\epsilon_{ijk} L_k, \quad [L_i, B_j] = -\epsilon_{ijk} B_k, \quad [B_i, B_j] = \epsilon_{ijk} L_k. \tag{916}$$

Now, if we rewrite the quadratic Casimir operator $M_{ij} M_{ij}$, in terms of L and B we recover the usual so(3,1) Casimir operators $\vec{L}^2 - \vec{B}^2$ and $\vec{L} \cdot \vec{B}$ as

$$\begin{aligned} 2M_{ij}M_{ij} &= L_1^2 + L_2^2 + L_3^2 - B_1^2 - B_2^2 - B_3^2 + 2i(L_1B_1 + L_2B_2 + L_3B_3) \\ &= \vec{L}^2 - \vec{B}^2 + 2i\vec{L} \cdot \vec{B}, \end{aligned} \tag{917}$$

If we define complex spherical coordinates (θ, ϕ) by

$$\begin{aligned} Z_1 &= \sin\theta \cos\phi, \\ Z_2 &= \sin\theta \sin\phi, \\ Z_3 &= \cos\theta, \end{aligned} \tag{918}$$

then the usual calculation yields

$$\frac{1}{2} M_{ij} M_{ij} = \frac{1}{\sin\theta} \frac{\partial}{\partial\theta} \left(\sin\theta \frac{\partial}{\partial\theta} \right) + \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\phi^2}, \tag{919}$$

which is the complex Laplacian $\Box$ on the complex sphere. Thus, in order to decompose the induced representation into SL$(2, \mathbb{C})$ irreducible representations we have to find the eigenfunctions of the Laplacian on the complex sphere.

15.4 Harmonics on the Complex Sphere

We are interested in finding complex eigenfunctions of the Laplacian on the complex sphere. We look for delta-function square-integrable, single-valued functions $f(\theta, \phi, \bar{\theta}, \bar{\phi})$ which obey

$$\frac{1}{2} M_{ij} M_{ij} f(\theta, \phi, \bar{\theta}, \bar{\phi}) = -\nu(\nu + 1) f(\theta, \phi, \bar{\theta}, \bar{\phi}), \tag{920}$$

$$\frac{1}{2} \overline{M}_{ij} \overline{M}_{ij} f(\theta, \phi, \bar{\theta}, \bar{\phi}) = -\bar{\nu}(\bar{\nu} + 1) f(\theta, \phi, \bar{\theta}, \bar{\phi}). \tag{921}$$

Then the SO(3,1) Casimir operators are

$$(\vec{L}^2 - \vec{B}^2) f = -2[\nu(\nu + 1) + \bar{\nu}(\bar{\nu} + 1)] f, \tag{922}$$

$$(\vec{L} \cdot \vec{B}) f = i[\nu(\nu + 1) - \bar{\nu}(\bar{\nu} + 1)] f. \tag{923}$$

Now, we look for a basis of functions in our unitary representation which diagonalise L_3 and B_3, with

$$L_3 f = (M_{12} + \overline{M}_{12}) f = imf, \quad B_3 f = -i(M_{12} - \overline{M}_{12}) f = i\lambda f, \tag{924}$$

where m and λ are real numbers. Thus as $M_{12} = \partial_\phi$, separating the variables in this way we are trying to find functions of the form

$$f(\theta, \phi, \bar{\theta}, \bar{\phi}) = F(\theta, \bar{\theta}) \exp\left(im\frac{\phi + \bar{\phi}}{2} - \lambda\frac{\phi - \bar{\phi}}{2} \right). \tag{925}$$

We first note that in order to get a function which is single valued around $Z_3 = 0$, we need to have 2π-periodicity in the real part of the complex spherical coordinate ϕ. We should therefore take $m \in \mathbb{Z}$.

If we switch back to $Z_3 = \cos\theta$, then the equations obeyed by $F(Z_3, \bar{Z}_3)$ are

$$\frac{d}{dZ_3}\left((1 - Z_3^2)\frac{d}{dZ_3}F \right) - \left(\frac{m + i\lambda}{2} \right)^2 \frac{1}{1 - Z_3^2}F = -\nu(\nu + 1)F, \tag{926}$$

$$\frac{d}{d\bar{Z}_3}\left((1 - \bar{Z}_3^2)\frac{d}{d\bar{Z}_3}F \right) - \left(\frac{m - i\lambda}{2} \right)^2 \frac{1}{1 - \bar{Z}_3^2}F = -\bar{\nu}(\bar{\nu} + 1)F, \tag{927}$$

which we recognize as associated Legendre equations. For brevity, define

$$\mu = \frac{m + i\lambda}{2}. \tag{928}$$

Two linearly independent solutions to the associated Legendre equation are $P_\nu^{-\mu}(x)$ and $\mathbf{Q}_\nu^\mu(x)$ defined by [53]

$$P_\nu^{-\mu}(x) = \left(\frac{x - 1}{x + 1} \right)^{\frac{\mu}{2}} \mathbf{F}\left(\nu + 1, -\nu; 1 + \mu; \frac{1 - x}{2} \right), \tag{929}$$

$$\mathbf{Q}_\nu^\mu(x) = 2^\nu\Gamma(\nu + 1)\left(\frac{x - 1}{x + 1} \right)^{\frac{\mu}{2}} (x - 1)^{-\nu-1}\mathbf{F}\left(\nu + 1, \nu - \mu + 1; 2\nu + 2; \frac{2}{1 - x} \right), \tag{930}$$

where we have defined

$$\mathbf{F}(a, b; c; z) = \frac{1}{\Gamma(c)}F(a, b; c; z), \tag{931}$$

and $F(a, b, ; c; z)$ is the hypergeometric function. These functions have branch point singularities at $x = \pm 1$ and $x = \infty$, the principal branch has a branch cut running along the real axis from $-\infty$ to $+1$. Note that as $x \to \infty$ [53]

$$P_\nu^{-\mu}(x) \sim \frac{\Gamma(\nu + \frac{1}{2})}{\sqrt{\pi}\Gamma(\nu + \mu + 1)}(2x)^\nu, \tag{932}$$

$$\mathbf{Q}_\nu^\mu(x) \sim \frac{\sqrt{\pi}}{\Gamma(\nu + \frac{3}{2})}(2x)^{-\nu-1}. \tag{933}$$

Similarly, as $x \to +1$ [53]

$$P_\nu^{-\mu}(x) \sim \frac{(x - 1)^{\mu/2}}{2^{\mu/2}\Gamma(\mu + 1)} \tag{934}$$

$$\mathbf{Q}_\nu^\mu(x) \sim \frac{2^{\mu/2-1}\Gamma(\mu)}{\Gamma(\nu + \mu + 1)}(x - 1)^{-(\mu/2)} \tag{935}$$

For later square-integrability, we are interested as functions which decay as $|Z_3| \to \infty$, and remain square-integrable around $Z_3 = \pm 1$, so we focus on

$$F_{\mu,\nu}(Z_3, \bar{Z}_3) = N_{\mu,\nu}\left[P_\nu^{-\mu}(Z_3)\mathbf{Q}_{\bar{\nu}}^{\bar{\mu}}(\bar{Z}_3) + B_{\mu,\nu}\mathbf{Q}_\nu^{\mu}(Z_3)P_{\bar{\nu}}^{-\bar{\mu}}(\bar{Z}_3)\right], \tag{936}$$

where $B_{\mu,\nu}$ is an undetermined constant and $N_{\mu,\nu)}$ is a normalisation constant. We wish to look for eigenfunctions $f(Z, \bar{Z})$ which are single-valued over the whole complex sphere. It suffices to consider the behaviour around two of the singular points of our functions, $Z_3 = \pm 1$. To do this, first investigate the exponential factors in

$$f_{\mu,\nu}(Z, \bar{Z}) = F_{\mu,\nu}(Z_3, \bar{Z}_3) \exp\left(im\frac{\phi + \bar{\phi}}{2} - \lambda\frac{\phi - \bar{\phi}}{2}\right). \tag{937}$$

In terms of Z_1, Z_2 and Z_3, the exponential factors can be rewritten as

$$e^{i\phi} = \frac{Z_1 + iZ_2}{\sqrt{1 - Z_3^2}}, \quad e^{i\bar{\phi}} = \frac{\bar{Z}_1 + i\bar{Z}_2}{\sqrt{1 - \bar{Z}_3^2}}, \tag{938}$$

so that

$$\begin{aligned} G_\mu(Z, \bar{Z}) &= \exp\left(im\frac{\phi + \bar{\phi}}{2} - \lambda\frac{\phi - \bar{\phi}}{2}\right) \\ &= \left(\frac{Z_1 + iZ_2}{\sqrt{1 - Z_3^2}}\right)^\mu \left(\frac{\bar{Z}_1 + i\bar{Z}_2}{\sqrt{1 - \bar{Z}_3^2}}\right)^{\bar{\mu}}. \end{aligned} \tag{939}$$

Suppose now we make a closed loop around $Z_3 = 1$, on the complex sphere $Z_1^2+Z_2^2+Z_3^2 = 1$. For concreteness, let $\epsilon(\psi) > 0$ be small and then parametrise the path in Z_3 by

$$Z_3 = 1 - \frac{\epsilon(\psi)}{2}e^{i\psi}, \tag{940}$$

where ψ is a real parameter and we assume further that $\epsilon(\psi)$ is 2π-periodic so that the path is closed. This restricts Z_1 and Z_2 such that

$$Z_1^2 + Z_2^2 = \epsilon(\psi)e^{i\psi}. \tag{941}$$

The possible closed paths for Z_1 and Z_2 can then be parameterised by a complex 2π-periodic function $g(\psi)$ as

$$Z_1 + iZ_2 = \sqrt{\epsilon(\psi)}g(\psi), \qquad Z_1 - iZ_2 = \sqrt{\epsilon(\psi)}\frac{e^{i\psi}}{g(\psi)}. \tag{942}$$

Then along this path we have

$$\begin{aligned} G_\mu(Z, \bar{Z}) &= \left(\frac{Z_1 + iZ_2}{\sqrt{1 - Z_3^2}}\right)^\mu \left(\frac{\bar{Z}_1 + i\bar{Z}_2}{\sqrt{1 - \bar{Z}_3^2}}\right)^{\bar{\mu}} \\ &= \left(g(\psi)e^{-i\psi/2}\right)^\mu \left(\frac{e^{-i\psi/2}}{\overline{g(\psi)}}\right)^{\bar{\mu}} \\ &= e^{-i\psi(\mu+\bar{\mu})/2}g(\psi)^\mu\overline{g(\psi)}^{-\bar{\mu}}. \end{aligned} \tag{943}$$

Recalling that $\mu = \frac{1}{2}(m + i\lambda)$ with $m \in \mathbb{Z}$ and writing $g(\psi)$ in polar form as

$$g(\psi) = r(\psi)e^{i\varphi(\psi)}, \tag{944}$$

where the 2π-periodicity requires that $\varphi(\psi+2\pi) = \varphi(\psi)+2k\pi$ with $k \in \mathbb{Z}$. Then it follows that

$$G_\mu(Z, \bar{Z}) = e^{-im\psi/2} r(\psi)^{i\lambda} e^{im\varphi(\psi)}. \tag{945}$$

In particular, it follows that as $\psi \mapsto \psi + 2\pi$, so that we fully encircle the point $Z_3 = +1$ in a positive sense,

$$G_\mu(Z, \bar{Z}) \mapsto (-1)^m G_\mu(Z, \bar{Z}). \tag{946}$$

It follows that for $f_{\mu,\nu}(Z, \bar{Z})$ to be single-valued around $Z_3 = 1$, we require that

$$F_{\mu,\nu}(Z_3, \bar{Z}_3) \mapsto (-1)^m F_{\mu,\nu}(Z_3, \bar{Z}_3), \tag{947}$$

as we encircle $Z_3 = +1$, but not $Z_3 = -1$, once in a positive sense. To this end, we note that if only $Z_3 = +1$ is encircled once in a positive sense [54, Eq. 14.24.3-4]

$$\begin{aligned} P_\nu^{-\mu}(Z_3) &\mapsto e^{\mu\pi i} P_\nu^{-\mu}(Z_3), \\ \mathbf{Q}_\nu^\mu(Z_3) &\mapsto e^{-\mu\pi i}\mathbf{Q}_\nu^\mu(Z_3) - \frac{i\pi}{\Gamma(\nu-\mu+1)} P_\nu^{-\mu}(Z_3). \end{aligned} \tag{948}$$

Noting that if Z_3 circles in a positive sense, $\bar{Z}_3$ circles in a negative sense, so that similarly we have

$$\begin{aligned} P_{\bar{\nu}}^{-\bar{\mu}}(\bar{Z}_3) &\mapsto e^{-\bar{\mu}\pi i} P_{\bar{\nu}}^{-\bar{\mu}}(\bar{Z}_3), \\ \mathbf{Q}_{\bar{\nu}}^{\bar{\mu}}(\bar{Z}_3) &\mapsto e^{\bar{\mu}\pi i}\mathbf{Q}_{\bar{\nu}}^{\bar{\mu}}(\bar{Z}_3) + \frac{i\pi}{\Gamma(\bar{\nu}-\bar{\mu}+1)} P_{\bar{\nu}}^{-\bar{\mu}}(\bar{Z}_3). \end{aligned} \tag{949}$$

Thus, we have

$$\begin{aligned} F_{\mu,\nu}(Z_3, \bar{Z}_3) \mapsto N_{\mu\nu}\Big[& e^{\mu\pi i} P_\nu^{-\mu}(Z_3) \left(e^{\bar{\mu}\pi i}\mathbf{Q}_{\bar{\nu}}^{\bar{\mu}}(\bar{Z}_3) + \frac{i\pi}{\Gamma(\bar{\nu}-\bar{\mu}+1)} P_{\bar{\nu}}^{-\bar{\mu}}(\bar{Z}_3) \right) \\ & + B_{\mu,\nu} e^{-\bar{\mu}\pi i} P_{\bar{\nu}}^{-\bar{\mu}}(\bar{Z}_3) \left(e^{-\mu\pi i}\mathbf{Q}_\nu^\mu(Z_3) - \frac{i\pi}{\Gamma(\nu-\mu+1)} P_\nu^{-\mu}(Z_3) \right) \Big] \\ = {} & (-1)^m F_{\mu,\nu}(Z_3, \bar{Z}_3) \\ & + i\pi N_{\mu\nu} \left[\frac{e^{\mu\pi i}}{\Gamma(\bar{\nu}-\bar{\mu}+1)} - B_{\mu,\nu} \frac{e^{-\bar{\mu}\pi i}}{\Gamma(\nu-\mu+1)} \right] P_\nu^{-\mu}(Z_3) P_{\bar{\nu}}^{-\bar{\mu}}(\bar{Z}_3). \end{aligned} \tag{950}$$

It follows that for the function to be single valued around $Z_3 = +1$, we must choose $B_{\mu,\nu}$ such that

$$\frac{e^{\mu\pi i}}{\Gamma(\bar{\nu}-\bar{\mu}+1)} = B_{\mu,\nu} \frac{e^{-\bar{\mu}\pi i}}{\Gamma(\nu-\mu+1)}. \tag{951}$$

Next, we investigate the behaviour around the singular point $Z_3 = -1$. As the function is single-valued around $Z_3 = +1$ now, we can consider a path that encircles both $Z_3 = +1$ and $Z_3 = -1$ once in a positive sense. Along such a path,

$$G_\mu(Z, \bar{Z}) \mapsto G_\mu(Z, \bar{Z}). \tag{952}$$

Along such a path, we have [54, Eq. 14.24.1-2]

$$\begin{aligned} P_\nu^{-\mu}(Z_3 e^{2\pi i}) &= e^{2\nu\pi i} P_\nu^{-\mu}(Z_3) + \frac{4i \sin \nu\pi}{\Gamma(\mu-\nu)} \mathbf{Q}_\nu^\mu(Z_3), \\ \mathbf{Q}_\nu^\mu(Z_3 e^{2\pi i}) &= e^{-2\nu\pi i} \mathbf{Q}_\nu^\mu(Z_3), \end{aligned} \tag{953}$$

and similarly,

$$\begin{aligned} P_{\bar\nu}^{-\bar\mu}(\bar Z_3 e^{-2\pi i}) &= e^{-2\bar\nu\pi i} P_{\bar\nu}^{-\bar\mu}(\bar Z_3) - \frac{4i \sin \bar\nu\pi}{\Gamma(\bar\mu-\bar\nu)} \mathbf{Q}_{\bar\nu}^{\bar\mu}(\bar Z_3), \\ \mathbf{Q}_{\bar\nu}^{\bar\mu}(\bar Z_3 e^{-2\pi i}) &= e^{2\bar\nu\pi i} \mathbf{Q}_{\bar\nu}^{\bar\mu}(\bar Z_3). \end{aligned} \tag{954}$$

Thus,

$$\begin{aligned} F_{\mu,\nu}(Z_3 e^{2\pi i}, \bar Z_3 e^{-2\pi i}) &= N_{\mu,\nu}\Big[e^{2\bar\nu\pi i} \left(e^{2\nu\pi i} P_\nu^{-\mu}(Z_3) + \frac{4i\sin\nu\pi}{\Gamma(\mu-\nu)} \mathbf{Q}_\nu^\mu(Z_3) \right) \mathbf{Q}_{\bar\nu}^{\bar\mu}(\bar Z_3) \\ &\quad + B_{\mu,\nu} e^{-2\nu\pi i} \mathbf{Q}_\nu^\mu(Z_3) \left(e^{-2\bar\nu\pi i} P_{\bar\nu}^{-\bar\mu}(\bar Z_3) - \frac{4i\sin\bar\nu\pi}{\Gamma(\bar\mu-\bar\nu)} \mathbf{Q}_{\bar\nu}^{\bar\mu}(\bar Z_3) \right) \Big] \\ &= N_{\mu,\nu}\Big[e^{2(\nu+\bar\nu)\pi i} P_\nu^{-\mu}(Z_3) \mathbf{Q}_{\bar\nu}^{\bar\mu}(\bar Z_3) + B_{\mu,\nu} e^{-2(\nu+\bar\nu)\pi i} \mathbf{Q}_\nu^\mu(Z_3) P_{\bar\nu}^{-\bar\mu}(\bar Z_3) \\ &\quad + 4i \mathbf{Q}_\nu^\mu(Z_3) \mathbf{Q}_{\bar\nu}^{\bar\mu}(\bar Z_3) \left(e^{2\bar\nu\pi i} \frac{\sin\nu\pi}{\Gamma(\mu-\nu)} - e^{-2\nu\pi i} \frac{\sin\bar\nu\pi}{\Gamma(\bar\mu-\bar\nu)} \right) \Big]. \end{aligned} \tag{955}$$

In particular, to get a single-valued function we need to take

$$e^{2(\nu+\bar\nu)\pi i} = 1, \tag{956}$$

that is,

$$\nu = \frac{n}{2} + i\kappa, \tag{957}$$

for $\kappa \in \mathbb{R}$ and $n \in \mathbb{Z}$, actually we may assume $n \geq -1$ as we have Re $\nu \geq -\frac{1}{2}$. Then the function $F_{\mu,\nu}(Z_3, \bar Z_3)$ is single valued going along this path if we additionally have

$$e^{2\bar\nu\pi i} \frac{\sin\nu\pi}{\Gamma(\mu-\nu)} - B_{\mu,\nu} e^{-2\nu\pi i} \frac{\sin\bar\nu\pi}{\Gamma(\bar\mu-\bar\nu)} = 0. \tag{958}$$

This yields no additional conditions. Indeed, inserting the value for $B_{\mu,\nu}$ we found previously into the above yields

$$\begin{aligned} & e^{-\mu\pi i} \frac{\sin\nu\pi}{\pi\Gamma(\mu-\nu)\Gamma(\nu-\mu+1)} - e^{\bar\mu\pi i} \frac{\sin\bar\nu\pi}{\pi\Gamma(\bar\mu-\bar\nu)\Gamma(\bar\nu-\bar\mu+1)} \\ &= e^{-\mu\pi i} \sin\nu\pi \sin(\mu-\nu)\pi - e^{\bar\mu\pi i} \sin\bar\nu\pi \sin(\bar\mu-\bar\nu)\pi \\ &= -\frac{1}{4}\left[e^{2i\mu\pi} - e^{2i\nu\pi} - e^{-i(2\mu-2\nu)} - e^{2i\bar\mu\pi} + e^{2i\bar\nu\pi} + e^{(2\bar\mu-\bar\nu)\pi i} \right] \\ &= 0, \end{aligned} \tag{959}$$

if we use that $e^{2\pi i\mu} = e^{-2\pi i\bar\mu}$ and $e^{2\pi i\nu} = e^{-2\pi i\bar\nu}$. Therefore, it follows that with this choice of $B_{\mu,\nu}$ and ν the functions $f_{\mu,\nu}(Z, \bar Z)$ are indeed single-valued as functions on the complex sphere.

Let us note that if we use [54]

$$\begin{aligned} P_\nu^{-\mu}(Z_3 e^{i\pi}) &= e^{i\pi\nu} P_\nu^{-\mu}(Z_3) + \frac{2}{\Gamma(\mu-\nu)} \mathbf{Q}_\nu^{\mu}(Z_3), \\ P_{\bar\nu}^{-\bar\mu}(\bar Z_3 e^{-i\pi}) &= e^{i\pi\nu} P_{\bar\nu}^{-\bar\mu}(\bar Z_3) + \frac{2}{\Gamma(\bar\mu-\bar\nu)} \mathbf{Q}_{\bar\nu}^{\bar\mu}(\bar Z_3), \end{aligned} \tag{960}$$

to eliminate $\mathbf{Q}_\nu^{\mu}(Z_3)$ and $\mathbf{Q}_{\bar\nu}^{\bar\mu}(\bar Z_3)$ we find that we can rewrite $F_{\mu,\nu}(Z_3,\bar Z_3)$ as

$$F_{\mu,\nu}(Z_3,\bar Z_3) = \tilde N_{\mu,\nu}\left[P_\nu^{-\mu}(Z_3) P_{\bar\nu}^{-\bar\mu}(\bar Z_3 e^{-i\pi}) + C_{\mu,\nu} P_\nu^{-\mu}(Z_3 e^{i\pi}) P_{\bar\nu}^{-\bar\mu}(\bar Z_3)\right], \tag{961}$$

where the new constants $C_{\mu,\nu}$ are related to the $B_{\mu,\nu}$ by

$$\begin{aligned} C_{\mu,\nu} &= \frac{\Gamma(\mu-\nu)}{\Gamma(\bar\mu-\bar\nu)} B_{\mu,\nu} \\ &= \frac{\sin\nu\pi}{\sin\bar\nu\pi} \\ &= \begin{cases} +1, & \text{if } n \text{ is odd}, \\ -1, & \text{if } n \text{ is even}. \end{cases} \\ &= -(-1)^n. \end{aligned} \tag{962}$$

Now let us use the value of ν to determine the values of the SO$(3,1)$ Casimir operators, which yields

$$\begin{aligned} (\vec B^2 - \vec L^2) &= 2[\nu(\nu+1) + \bar\nu(\bar\nu+1)] = n(n+2) - (2\kappa)^2, \\ \vec B\cdot\vec L &= i[\nu(\nu+1) - \bar\nu(\bar\nu+1)] = -(2\kappa)(n+1). \end{aligned} \tag{963}$$

Thus these fall into irreducible unitary SO$(3,1)$ representations with parameters $l_0 = (n+1)$ and $\rho = (2\kappa)$ in the notation of Appendix K. The functions with fixed ν form basis vectors for the representation $\Psi^{s,k}$ of SO$(3,1)$ with $s=\kappa$ and $k = 2(n+1)$. As we have argued previously, the representation formed by the harmonics on the complex sphere is equivalent to the tensor product of two principal series representations Ψ^{s_1,k_1} and Ψ^{s_2,k_2} of the Lorentz group. Thus we recover the classical result [62, 65] for the decomposition of Lorentz group tensor products

$$\Psi^{s_1,k_1} \otimes \Psi^{s_2,k_2} \cong \bigoplus_{k\geq 0} \int \mathrm{d}s\ \Psi^{s,k}, \tag{964}$$

where the sum extends only over the even k.

15.4.1 Normalisation of the Eigenfunctions

Next let us determine the normalisation constants $N_{\mu,\nu}$. We choose these to normalise the L^2 inner product of the eigenfunctions. To integrate over the complex-sphere, we work in coordinates $(Z_3,\bar Z_3,\phi,\bar\phi)$, with

$$Z_1 + iZ_2 = Z_3 e^{i\phi}. \tag{965}$$

In these coordinates, the metric on the complex sphere can be written

$$\mathrm{d}s^2 = \frac{\mathrm{d}Z_3^2}{1-Z_3^2} + (1-Z_3^2)\mathrm{d}\phi^2 + \text{c.c.} \tag{966}$$

and it follows that the associated volume form is $\mathrm{d}Z_3\mathrm{d}\bar{Z}_3\mathrm{d}\phi\mathrm{d}\bar{\phi}$. Therefore, the L^2 product between $f_{\mu,\nu}(Z_3,\bar{Z}_3,\phi,\bar{\phi})$ and $f_{\mu',\nu'}(Z_3,\bar{Z}_3,\phi,\bar{\phi})$ is given by

$$(f_{\mu',\nu'},f_{\mu,\nu}) = \int \mathrm{d}Z_3\mathrm{d}\bar{Z}_3\mathrm{d}\phi\mathrm{d}\bar{\phi}\ \overline{f_{\mu',\nu'}(Z_3,\bar{Z}_3,\phi,\bar{\phi})}f_{\mu,\nu}(Z_3,\bar{Z}_3,\phi,\bar{\phi}). \tag{967}$$

Let us first deal with the ϕ dependent parts. Here we need to calculate

$$\int \mathrm{d}\phi\mathrm{d}\bar{\phi}\ e^{i(\mu-\mu')\phi}e^{i(\bar{\mu}-\bar{\mu}')\bar{\phi}}. \tag{968}$$

Let $\phi=\psi+i\chi$, so that $\mathrm{d}\phi\wedge\mathrm{d}\bar{\phi}=-2i\mathrm{d}\psi\wedge\mathrm{d}\chi$ recall that $\mu=\frac{1}{2}(m+i\lambda)$, so that

$$e^{i\mu\phi}e^{i\bar{\mu}\phi} = e^{im\psi}e^{-i\lambda\chi}. \tag{969}$$

Then

$$2\int_0^{2\pi}\mathrm{d}\psi\ e^{i(m-m')\psi}\int_{-\infty}^{\infty}\mathrm{d}\chi\ e^{-i(\lambda-\lambda')\chi} = 8\pi^2\delta_{m,m'}\delta(\lambda-\lambda'), \tag{970}$$

so that, up to the normalisation, this factor sets $\mu=\mu'$. It then remains to calculate

$$I_{\mu,\nu,\nu'}(Z_3,\bar{Z}_3) = \int \mathrm{d}Z_3\mathrm{d}\bar{Z}_3\ F_{\mu,\nu}(Z_3,\bar{Z}_3)\overline{F_{\mu,\nu'}(Z_3,\bar{Z}_3)}. \tag{971}$$

For this we will use Sturm-Liouville theory, but first let us note that

$$\overline{F_{\mu,\nu}(Z_3,\bar{Z}_3)} = (-1)^{\nu+\bar{\nu}+1}F_{\mu,\nu}(Z_3,\bar{Z}_3), \tag{972}$$

where $\nu+\bar{\nu}=n\in\mathbb{Z}$. Next, we will consider the self-adjoint operator

$$\frac{1}{4i}X = \frac{d}{dZ_3}\left((1-Z_3^2)\frac{d}{dZ_3}\right) - \frac{d}{d\bar{Z}_3}\left((1-\bar{Z}_3^2)\frac{d}{d\bar{Z}_3}\right). \tag{973}$$

We know that when acting on $F_{\mu,\nu}$ the action of X is given by

$$\begin{aligned} XF_{\mu,\nu} &= -4i\left[\nu(\nu+1)-\bar{\nu}(\bar{\nu}+1)\right]F_{\mu,\nu} \\ &= 8\kappa(n+1), \end{aligned} \tag{974}$$

where we recalled that $\nu=\frac{n}{2}+i\kappa$. Working with to complex polar coordinates $Z_3=re^{i\theta}$, we then write

$$\mathrm{d}Z_3\wedge\mathrm{d}\bar{Z}_3 = -2ir\mathrm{d}r\wedge\mathrm{d}\theta \tag{975}$$

and

$$\frac{\partial}{\partial Z_3} = \frac{1}{2}e^{-i\theta}\left(\frac{\partial}{\partial r}-\frac{i}{r}\frac{\partial}{\partial\theta}\right). \tag{976}$$

Then we find that

$$\begin{aligned} X = {} & 2\sin 2\theta\frac{\partial^2}{\partial r^2} - \frac{2}{r^2}\sin 2\theta\frac{\partial^2}{\partial\theta^2} + \left[-4r+\frac{4}{r}\cos 2\theta\right]\frac{\partial^2}{\partial r\partial\theta} \\ & -\frac{2}{r}\sin 2\theta\frac{\partial}{\partial r} - \frac{4}{r^2}\left(\cos 2\theta + r^2\right)\frac{\partial}{\partial\theta}. \end{aligned} \tag{977}$$

Then it is possible to check that this is operator is indeed formally self-adjoint with respect to the L^2 inner product

$$(f,g) = 2\int \mathrm{d}r\mathrm{d}\theta\ r\overline{f(r,\theta)}g(r,\theta). \tag{978}$$

Furthermore, we note that X can be put into a Sturm-Liouville form

$$X = \nabla_\alpha(P^{\alpha\beta}\nabla_\beta) = \frac{1}{r}\partial_\alpha(rP^{\alpha\beta}\partial_\beta) \tag{979}$$

with

$$\begin{aligned} P^{rr} &= 2\sin 2\theta, \\ P^{r\theta} &= -2r + \frac{2}{r}\cos 2\theta, \\ P^{\theta\theta} &= -\frac{2}{r^2}\sin 2\theta. \end{aligned} \tag{980}$$

Then

$$\begin{aligned} &8[\kappa(n+1) - \kappa'(n'+1)]I_{\mu,\nu,\nu'} \\ &= 2\int \mathrm{d}r\mathrm{d}\theta\ r\left[(XF_{\mu,\nu})\overline{F_{\mu,\nu'}} - F_{\mu,\nu}(X\overline{F_{\mu,\nu'}})\right] \\ &= 2\int \mathrm{d}r\mathrm{d}\theta\ \left[\partial_\alpha(rP^{\alpha\beta}\partial_\beta F_{\mu,\nu})\overline{F_{\mu,\nu'}} - F_{\mu,\nu}\partial_\alpha(rP^{\alpha\beta}\partial_\beta\overline{F_{\mu,\nu'}})\right] \\ &= 2\lim_{r\to\infty}\int_0^{2\pi}\mathrm{d}\theta\ r\left[(P^{r\beta}\partial_\beta F_{\mu,\nu})\overline{F_{\mu,\nu'}} - F_{\mu,\nu}(P^{r\beta}\partial_\beta\overline{F_{\mu,\nu'}})\right]. \end{aligned} \tag{981}$$

Recall that

$$F_{\mu,\nu}(Z_3,\bar{Z}_3) = \tilde{N}_{\mu,\nu}\left[P_\nu^{-\mu}(Z_3)P_{\bar{\nu}}^{-\bar{\mu}}(\bar{Z}_3e^{-i\pi}) + (-1)^{\nu+\bar{\nu}+1}P_\nu^{-\mu}(Z_3e^{i\pi})P_{\bar{\nu}}^{-\bar{\mu}}(\bar{Z}_3)\right], \tag{982}$$

so that as $r\to\infty$, we have

$$\begin{aligned} F_{\mu,\nu} &\sim \tilde{N}_{\mu,\nu}A_{\mu,\nu}\left[e^{i\gamma_{\mu,\nu}}Z_3^\nu\bar{Z}_3^{-\bar{\nu}-1} + (-1)^{\nu+\bar{\nu}+1}\mathrm{c.c.}\right] \\ &= \tilde{N}_{\mu,\nu}A_{\mu,\nu}\left[r^{2i\kappa-1}e^{i\gamma_{\mu,\nu}+i\theta(n+1)} + (-1)^{n+1}r^{-2i\kappa-1}e^{-i\gamma_{\mu,\nu}-i\theta(n+1)}\right], \end{aligned} \tag{983}$$

where we have introduced $A_{\mu,\nu} > 0$ and $\gamma_{\mu,\nu}\in\mathbb{R}$ defined by

$$A_{\mu,\nu}e^{i\gamma_{\mu,\nu}} = \frac{2^{\nu-\bar{\nu}}\Gamma(\nu+\frac{1}{2})}{\Gamma(\nu+\mu+1)\Gamma(\bar{\mu}-\bar{\nu})\Gamma(\bar{\nu}+\frac{3}{2})}. \tag{984}$$

In particular, we note that as $r\to\infty$, we have $F_{\mu,\nu}$ scaling like r^{-1}. It thus follows that only the $-2r$ part of $P^{r\theta}$ will give a non-zero contribution as we take the limit $r\to\infty$.

Thus we have

$$\begin{aligned}
&8[\kappa(n+1)-\kappa'(n'+1)]I_{\mu,\nu,\nu'}\\
&=4\lim_{r\to\infty}\int_0^{2\pi}\mathrm{d}\theta\ r^2\left[F_{\mu,\nu}\left(\frac{\partial}{\partial\theta}\overline{F_{\mu,\nu'}}\right)-\left(\frac{\partial}{\partial\theta}F_{\mu,\nu}\right)\overline{F_{\mu,\nu'}}\right]\\
&=4\tilde{N}_{\mu,\nu}\tilde{N}_{\mu,\nu'}A_{\mu,\nu}A_{\mu,\nu'}\\
&\quad\lim_{r\to\infty}\int_0^{2\pi}\mathrm{d}\theta\Big\{i(n'-n)\Big[(-1)^{n'+1}r^{2i(\kappa+\kappa')}e^{i(\gamma_{\mu,\nu}+\gamma_{\mu,\nu'}+(n+n'+2)\theta)}\\
&\qquad-(-1)^{n+1}r^{-2i(\kappa+\kappa')}e^{-i(\gamma_{\mu,\nu}+\gamma_{\mu,\nu'}+(n+n'+2)\theta)}\Big]\\
&\qquad-i(n+n'+2)\Big[(-1)^{n+n'}r^{2i(\kappa-\kappa')}e^{i(\gamma_{\mu,\nu}-\gamma_{\mu,\nu'}+(n-n')\theta)}\\
&\qquad-r^{-2i(\kappa-\kappa')}e^{-i(\gamma_{\mu,\nu}-\gamma_{\mu,\nu'}+(n-n')\theta)}\Big]\Big\}\\
&=16\pi\tilde{N}_{\mu,\nu}\tilde{N}_{\mu,\nu'}A_{\mu,\nu}A_{\mu,\nu'}\lim_{r\to\infty}\sin\left[2(\kappa-\kappa')\log r+\gamma_{\mu,\nu}-\gamma_{\mu,\nu'}\right]\delta_{n,n'},
\end{aligned}\tag{985}$$

where we used the fact that $n+n'+2>0$ to set a term proportional to $\delta_{n,-n'-2}$ to zero. Now use that

$$\lim_{s\to\infty}\frac{\sin sx}{x}=\pi\delta(x),\tag{986}$$

with $\log r\to\infty$ as $r\to\infty$ to arrive at

$$\begin{aligned}
&8[\kappa(n+1)-\kappa'(n'+1)]I_{\mu,\nu,\nu'}\\
&=16\pi^2\tilde{N}^2_{\mu,\nu}A^2_{\mu,\nu}\delta(\kappa-\kappa')\delta(\kappa-\kappa')\delta_{n,n'}.
\end{aligned}\tag{987}$$

Hence we get the result

$$I_{\mu,\nu,\nu'}=2\pi^2\tilde{N}^2_{\mu,\nu}A^2_{\mu,\nu}\delta(\kappa-\kappa')\delta_{n,n'},\tag{988}$$

we then choose $\tilde{N}^2_{\mu,\nu}=(2\pi^2A^2_{\mu,\nu})^{-1}$ to normalise so that

$$(f_{\mu',\nu'},f_{\mu',\nu'})=\delta(\kappa-\kappa')\delta(\lambda-\lambda')\delta_{n,n'}\delta_{m,m'},\tag{989}$$

where

$$\nu=\frac{n}{2}+i\kappa,\qquad\mu=\frac{m+i\lambda}{2}.\tag{990}$$

15.5 Generalising to Higher Dimensional Complex Spheres

Let us note that these results can easily be generalised to higher dimensional complex spheres. That is, we can easily write down single-valued, delta-function normalisable harmonics, this is done inductively mirroring the way higher dimensional spherical harmonics are defined on real spheres. Let $Z_1,Z_2,\ldots,Z_{N+1}$ be complex coordinates, related by

$$Z_1^2+Z_2^2+\ldots+Z_{N+1}^2=1\tag{991}$$

Introduce complex spherical coordinates $(\theta_1, \theta_2, \ldots, \theta_N)$ in the usual manner

$$\begin{aligned} Z_1 &= \sin\theta_N \sin\theta_{N-1} \ldots \sin\theta_2 \sin\theta_1, \\ Z_2 &= \sin\theta_N \sin\theta_{N-1} \ldots \sin\theta_2 \cos\theta_1, \\ &\vdots \\ Z_N &= \sin\theta_N \cos\theta_{N-1}, \\ Z_{N+1} &= \cos\theta_N. \end{aligned} \tag{992}$$

In these coordinates, the complex Laplacians $\Box_N$ and $\bar{\Box}_N$ on the complex sphere are then

$$\Box_N = \frac{1}{\sin^{N-1}\theta_N} \frac{\partial}{\partial\theta_N} \left(\sin^{N-1}\theta_N \frac{\partial}{\partial\theta_N} \right) + \frac{1}{\sin^{N-1}\theta_N} \Box_{N-1}, \tag{993}$$

and

$$\bar{\Box}_N = \frac{1}{\sin^{N-1}\bar{\theta}_N} \frac{\partial}{\partial\bar{\theta}_N} \left(\sin^{N-1}\bar{\theta}_N \frac{\partial}{\partial\bar{\theta}_N} \right) + \frac{1}{\sin^{N-1}\bar{\theta}_N} \bar{\Box}_{N-1}. \tag{994}$$

Now look for functions $f_{L_N}(Z, \bar{Z})$ which obey

$$\begin{aligned} \Box_N f_{L_N}(Z, \bar{Z}) &= -L_N(L_N + N - 1) f_L(Z, \bar{Z}), \\ \bar{\Box}_N f_{L_N}(Z, \bar{Z}) &= -\bar{L}_N(\bar{L}_N + N - 1) f_L(Z, \bar{Z}). \end{aligned} \tag{995}$$

Using the shorthand Z to denote $(Z_1, \ldots, Z_N)$, let us write

$$f_{L_N} = F_{L_N, L_{N-1}}(Z_{N+1}, \bar{Z}_{N+1}) f_{L_{N-1}, \ldots, L_1} \left(\frac{Z}{\sqrt{1 - Z_{N+1}^2}}, \frac{\bar{Z}}{\sqrt{1 - \bar{Z}_{N+1}^2}} \right), \tag{996}$$

where for each $i = 1, 2, \ldots, N-1$ we have

$$L_i = \frac{n_i}{2} + i\kappa_i, \tag{997}$$

with Re $L_i \geq -\frac{i-1}{2}$, except Re L_1, which can take any integer or half-integer value. Furthermore

$$\begin{aligned} \Box_i f_{L_{N-1}, \ldots, L_1} &= -L_i(L_i + i - 1) f_{L_{N-1}, \ldots, L_1}, \\ \bar{\Box}_i f_{L_{N-1}, \ldots, L_1} &= -\bar{L}_i(\bar{L}_i + i - 1) f_{L_{N-1} \ldots, L_1}. \end{aligned} \tag{998}$$

Then, define

$$\begin{aligned} F_{L_N, L_{N-1}}(Z_{N+1}, \bar{Z}_{N+1}) = &\frac{1}{(1 - Z_{N+1}^2)^{(N-2)/2}} \frac{1}{(1 - \bar{Z}_{N+1}^2)^{(N-2)/2}} \\ &\times G_{L_N, L_{N-1}}(Z_{N+1}, \bar{Z}_{N+1}). \end{aligned} \tag{999}$$

Then $G_{L_N, L_{N-1}}$ obeys an associated Legendre equation in Z_{N+1} with parameters $L_{N-1} + \frac{1}{2}(N-2)$ and $L_N + \frac{1}{2}(N-2)$ [66, 67]. Furthermore $G_{L_N, L_{N-1}}$ also obeys an associated Legendre equation in $\bar{Z}_{N+1}$ with parameters $\bar{L}_{N-1} + \frac{1}{2}(N-2)$ and $\bar{L}_N + \frac{1}{2}(N-2)$. That is

$$0 = \frac{d}{dZ_{N+1}} \left[(1 - Z_{N+1}^2) \frac{d}{dZ_{N+1}} F_{L_N, L_{N-1}} \right] + \left[\left(L_N + \frac{N-2}{2} \right) \left(L_N + \frac{N}{2} \right) - \frac{\left(L_{N-1} + \frac{N-2}{2} \right)^2}{1 - Z_{N+1}^2} \right] F_{L_N, L_{N-1}}, \tag{1000}$$

and similarly in the conjugated variables. Thus the delta-function square integrable solutions are of the form

$$\begin{aligned} F_{L_N, L_{N-1}} = &\frac{\tilde{N}}{(1 - Z_{N+1}^2)^{(N-2)/2} (1 - \bar{Z}_{N+1}^2)^{(N-2)/2}} \\ &\times \Big[P_{L_N + (N-2)/2}^{-[L_{N-1} + (N-2)/2]} (Z_{N+1}) P_{\bar{L}_N + (N-2)/2}^{-[\bar{L}_{N-1} + (N-2)/2]} (\bar{Z}_{N+1} e^{-i\pi}) \\ &\quad + C P_{L_N + (N-2)/2}^{-[L_{N-1} + (N-2)/2]} (Z_{N+1} e^{i\pi}) P_{\bar{L}_N + (N-2)/2}^{-[\bar{L}_{N-1} + (N-2)/2]} (\bar{Z}_{N+1}) \Big], \end{aligned} \tag{1001}$$

where $\tilde{N}$ is again a normalisation constant and C is an undetermined constant. In order for the eigenfunction f_{L_N} to be single-valued on the complex sphere, arguing as in the previous section requires that

$$e^{2\pi i (L_N + \bar{L}_N)} = 1, \tag{1002}$$

which leads to

$$L_N = \frac{n_N}{2} + i\kappa_N, \tag{1003}$$

with $\kappa_N \in \mathbb{R}$ and $n_N \in \mathbb{Z}$, and we also have the restriction Re $n_N \geq -\frac{1}{2}(N-1)$, and then

$$C = \frac{\sin\left(L_N + \frac{N-2}{2} \right) \pi}{\sin\left(\bar{L}_N + \frac{N-2}{2} \right) \pi} = -(-1)^{n_N + N}. \tag{1004}$$

The integral determining the normalisation constants $\tilde{N}$ can be evaluated in the same manner as the previous section.

These N-dimensional complex spherical harmonics form a basis of functions for the induced representation

$$\pi = \mathrm{ind}_{\mathrm{SO}(N,\mathbb{C})}^{\mathrm{SO}(N+1,\mathbb{C})} \pi^0, \tag{1005}$$

where π^0 denotes the trivial representation of $\mathrm{SO}(N, \mathbb{C})$. Here $\mathrm{SO}(N, \mathbb{C})$ denotes the group of complex orthogonal matrices. The coset space formed from these two groups is the N-dimensional complex sphere. In the case $N = 2$, the accidental isomorphism $\mathrm{SO}(3, \mathbb{C}) \cong \mathrm{SO}(3, 1)$ and $\mathrm{SO}(2, \mathbb{C}) \cong MA$ allowed us to connect the induced representation π to the tensor product of principal series Lorentz group. This connection is lost in higher dimensions.

16 Conclusion

In this chapter, we studied the well known decomposition into irreducible representations of tensor products of principal series unitary irreducible representations of the Lorentz group. The principal series representations are realised as induced representations, and applying Mackey's tensor product theorem to the tensor product of the principal series representations, we argued that the resulting induced representation could be understood as describing harmonics on a complex sphere. We then analysed these harmonics, and showed that they exist as single-valued, delta-function square-integrable functions for parameter values which ensure that they form basis vectors for exactly the principal series representations which appear in the decomposition of tensor product, so that we recover the known result. We then generalised the treatment of the harmonics from 2 complex-dimensional spheres to N complex-dimensions, although these harmonics are not in general related to representations of Lorentz groups.

K Unitary Irreducible Representations of the Lorentz Group

In this appendix we recall the classification of the unitary irreducible representations of the Lorentz group SO(3, 1) by infinitesimal considerations. This classification is well known [68], the treatment here is based on Ohnuki [69].

The corresponding Lie algebra so(3, 1) is composed of 4×4 real matrices X which obey

$$X^T\eta + \eta X = 0, \tag{1006}$$

where η is the Minkowski matrix $\eta = \mathrm{diag}(-1, +1, +1, +1)$. If a, b, c, d, e and f are real numbers, then a general element is of the form

$$X = \begin{pmatrix} 0 & a & b & c \\ a & 0 & d & e \\ b & -d & 0 & f \\ c & -e & -f & 0 \end{pmatrix}. \tag{1007}$$

We introduce rotation generators L_1, L_2 and L_3 as

$$L_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad L_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \quad L_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{1008}$$

Similarly, we define the boost generators B_1, B_2 and B_3 by

$$B_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad B_3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}. \tag{1009}$$

In terms of these operators, the commutation relations can be written

$$[L_i, L_j] = \epsilon_{ijk}L_k, \tag{1010}$$

$$[L_i, B_j] = \epsilon_{ijk}B_k, \tag{1011}$$

$$[B_i, B_j] = -\epsilon_{ijk}L_k. \tag{1012}$$

Thus the rotations L_i form an so(3) subalgebra, and the boosts transform as a vector under the rotations. The irreducible representations of so(3) are labelled by l, which can be integer or half-integer. It is possible to choose a basis $|m\rangle$ for the representation labelled by fixed l such that

$$\vec{L}^2|m\rangle = -l(l+1)|m\rangle, \qquad L_3|m\rangle = -im|m\rangle, \tag{1013}$$

with m taking the values $-l, -l+1, \ldots, l-1, l$. Define the angular momentum ladder operators $L_\pm = L_1 \pm iL_2$, which respectively raise and lower the eigenvalue m by one. The basis can be chosen such that

$$\begin{aligned} L_+|m\rangle &= +\sqrt{(l-m)(l+m+1)}|m+1\rangle, \\ L_-|m\rangle &= -\sqrt{(l+m)(l-m+1)}|m-1\rangle. \end{aligned} \tag{1014}$$

Now consider an irreducible unitary representation of so(3,1). Importantly, if we restrict to an so(3) subalgebra, the representation of so(3,1) decomposes into irreducible representations of so(3), labelled by different values of l, and each of these values can appear at most once. Thus we can introduce a non-degenerate basis $|l, m\rangle$ for the so(3,1) representation, with each m taking values $-l, -l+1, \ldots, +l$. Under the action of L_3 and $L_\pm$ we have

$$\begin{aligned} L_+|l,m\rangle &= +\sqrt{(l-m)(l+m+1)}|l,m+1\rangle, \\ L_3|l,m\rangle &= -im|l,m\rangle, \\ L_-|l,m\rangle &= -\sqrt{(l+m)(l-m+1)}|l,m-1\rangle. \end{aligned} \tag{1015}$$

We now wish to consider the action of the boost generators B_1, B_2 and B_3. As the boost generators transform as a vector ($l = 1$) under rotations it follows from the addition of angular momentum that they can change $|l, m\rangle$ only to $|l', m\rangle$ with $|l - l'| \leq 1$. Define

$$B_\pm = B_1 \pm iB_2. \tag{1016}$$

Then we have the commutation relations

$$[L_3, B_3] = 0, \qquad [L_3, B_\pm] = \mp iB_\pm, \tag{1017}$$

which tell us that B_3 does not change the m values, while $B_\pm$ raise and lower the m value by 1 respectively. We are thus able to write

$$B_3|l,m\rangle = A(l+1;l,m)|l+1,m\rangle + A(l;l,m)|l,m\rangle + A(l-1;l,m)|l-1,m\rangle, \tag{1018}$$

$$\begin{aligned} B_+|l,m\rangle = {}& B(l+1;l,m)|l+1,m+1\rangle + B(l;l,m)|l,m+1\rangle \\ &+ B(l-1;l,m)|l-1,m+1\rangle, \end{aligned} \tag{1019}$$

$$\begin{aligned} B_-|l,m\rangle = {}& C(l+1;l,m)|l+1,m-1\rangle + C(l;l,m)|l,m-1\rangle \\ &+ C(l-1;l,m)|l-1,m-1\rangle. \end{aligned} \tag{1020}$$

Now, using the commutation relations $[L_+, B_+] = 0 = [L_-, B_-]$ we can determine that

$$\begin{aligned} B(\tilde{l}; l, m) &= B(\tilde{l}; l)\sqrt{\frac{\Gamma(\tilde{l}+m+2)\Gamma(l-m+1)}{\Gamma(l+m+1)\Gamma(\tilde{l}-m)}}, \\ C(\tilde{l}; l, m) &= C(\tilde{l}; l)\sqrt{\frac{\Gamma(\tilde{l}+m+1)\Gamma(\tilde{l}-m+2)}{\Gamma(l-m+1)\Gamma(\tilde{l}+m)}}. \end{aligned} \tag{1021}$$

Meanwhile, the commutation relations $[L_-, B_+] = 2iB_3 = -[L_+, B_-]$ lead to

$$\begin{aligned} 2iA(\tilde{l}; l, m) &= \frac{B(\tilde{l}; l, m)}{\sqrt{(\tilde{l}+m+1)(\tilde{l}-m)}}\left[l(l+1) - \tilde{l}(\tilde{l}+1) + 2m\right], \\ -2iA(\tilde{l}; l, m) &= \frac{C(\tilde{l}; l, m)}{\sqrt{(\tilde{l}-m+1)(\tilde{l}+m)}}\left[\tilde{l}(\tilde{l}+1) - l(l+1) + 2m\right]. \end{aligned} \tag{1022}$$

If we denote

$$A_0(l) = B(l; l), \quad A_+(l) = B(l+1; l), \quad A_-(l) = B(l-1; l), \tag{1023}$$

it follows then that

$$\begin{aligned} B_+|l, m\rangle &= \sqrt{(l+m+2)(l+m+1)}A_+(l)|l+1, m+1\rangle \\ &\quad + \sqrt{(l+m+1)(l-m)}A_0(l)|l, m+1\rangle \\ &\quad + \sqrt{(l-m)(l-m-1)}A_-(l)|l-1, m+1\rangle, \\ B_3|l, m\rangle &= +i\sqrt{(l+m+1)(l-m+1)}A_+(l)|l+1, m\rangle \\ &\quad - imA_0(l)|l, m\rangle \\ &\quad - i\sqrt{(l-m)(l+m)}A_-(l)|l-1, m\rangle, \\ B_-|l, m\rangle &= \sqrt{(l-m+2)(l-m+1)}A_+(l)|l+1, m-1\rangle \\ &\quad - \sqrt{(l+m)(l-m+1)}A_0(l)|l, m-1\rangle \\ &\quad + \sqrt{(l+m)(l+m-1)}A_-(l)|l-1, m-1\rangle. \end{aligned} \tag{1024}$$

For a unitary representation, we require that $B_3^\dagger = -B_3$ and $B_\pm^\dagger = -B_\mp$, which imposes that

$$\overline{A_0(l)} = A_0(l), \qquad A_+(l) = -\overline{A_-(l+1)}. \tag{1025}$$

We still need to impose the commutation relations

$$[B_\pm, B_3] = \mp iL_\pm, \qquad [B_+, B_-] = 2iL_3. \tag{1026}$$

These lead to the recurrence relations

$$0 = A_+(l)\left[(l+2)A_0(l+1) - lA_0(l)\right], \tag{1027}$$

$$1 = -(2l+3)A_+(l)A_-(l+1) - A_0(l)^2 + (2l-1)A_-(l)A_+(l-1), \tag{1028}$$

where the second equality only holds if $l > 0$, as we had to divide by m to get it, which is always possible unless $l = 0$.

Now, let us denote the minimum l value which occurs in the unitary irreducible representation by l_0. Furthermore, let us choose the phases of the basis vectors such that $A_-(l)$ is real. Then these recurrence relations can be solved as

$$\begin{aligned} A_0(l) &= \frac{i\rho l_0}{l(l+1)}, & l &> 0, \\ A_-(l) &= \frac{1}{l}\sqrt{\frac{(l^2-l_0^2)(l^2-\rho^2)}{4l^2-1}}, & l &\geq 1. \end{aligned} \tag{1029}$$

As A_0 is a real number, if $l_0 \neq 0$ we must have that $\rho \in i\mathbb{R}$. If $l_0 = 0$ we must still have

$$1 - \rho^2 \geq 0. \tag{1030}$$

If equality holds, so that $\rho = \pm 1$, then we get the one-dimensional trivial representation. If equality does not hold, we can have $\rho \in i\mathbb{R}$ or $-1 < \rho < 1$. The representations where $\rho \in i\mathbb{R}$ are called the principal series representations. The representations with $-1 < \rho < 1$ are the complementary series representations. The values of l which appear in these representations are $l_0, l_0 + 1, \ldots$, where l_0 is either an integer or half-integer.

Let us note that in $\mathrm{SO}(3,1)$, we have

$$R_3[2\pi] = \exp[2\pi L_3] = 1. \tag{1031}$$

Meanwhile in the representations we find

$$R_3[2\pi]|l,m\rangle = \exp[2\pi L_3]\,|l,m\rangle = e^{-2\pi i m}|l,m\rangle = e^{2\pi i l_0}|l,m\rangle. \tag{1032}$$

Thus only the representations with l_0 integer actually correspond to $\mathrm{SO}(3,1)$ representations. The half-integers lead to representations of the group $\mathrm{SL}(2,\mathbb{C})$ which forms the double cover of $\mathrm{SO}(3,1)$.

Finally, let us note that the $\mathrm{so}(3,1)$ algebra has two Casimir operators

$$Q_1 = \vec{L}^2 - \vec{B}^2, \qquad Q_2 = \vec{L}\cdot\vec{B}. \tag{1033}$$

In each irreducible unitary representation these are constant and we can calculate explicitly that

$$\vec{B}^2 - \vec{L}^2 = (l_0+1)(l_0-1) - \rho^2, \tag{1034}$$

$$\vec{L}\cdot\vec{B} = -i\rho l_0. \tag{1035}$$

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