Semigroup and monoid presentations

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Abstract

The aim of this thesis is two-fold. First we investigate the class of right ideal Howson semigroups inspired by a question posed by Steinberg. Right ideal Howson semigroups are defined by the finitary property that the intersection of any two finitely generated right ideals is also finitely generated. We obtain semigroup presentations for right ideal Howson semigroups which are universal in a certain sense. In addition, we provide examples of right ideal Howson semigroups with a specific focus on coherent monoids, varieties of bands and other finiteness conditions. Dual results hold for left ideal Howson semigroups. The second part of this thesis concerns finding semigroup presentations for semigroups of the form ST, where S and T are subsemigroups of some common semigroup U, such that for every $a \in T$ we have $aS \subseteq Sa$ and if $xa = yb$ then $x = y$ for every $x, y \in S$ and $a, b \in T$. Significantly, we obtain a semigroup presentation for the singular part of the partial endomorphism monoid of a free G-act of finite rank. This builds on the work of Al-Aadhami, Dolinka, East, Feng and Gould. We also use our methods to give presentations for almost-factorisable inverse semigroups.

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Introduction

In many ways, one may regard a semigroup as a generalisation of a group, as its name would suggest. That being said, it is rather surprising that the notion of a group largely predates the notion of a semigroup; an observation made persistently by the modern day semigroup community. In 2002, Schein remarked that "irreversible processes are much more common than reversible ones, one meets function and transformation semigroups much more often than groups (and much more often than one thinks he does)" [89]. To better understand the way in which semigroup theory developed as a theory, we start by summarising the origins of group theory.

It is often accredited to Cayley, the mathematician well-known amongst group theorists and of whom the terms 'Cayley table' and 'Cayley graph' are in honour, for first attempting to define a group abstractly in his landmark 1854 paper '*On the theory of groups as depending on the symbolic equation* $\theta^n = 1$' [11, 54]. Specifically, he remarked that a group is a "set of symbols, all of them different, and such that the product of any two of them (no matter what order), or the product of any one of them into itself, belongs to the set" [11, 54].

In the late 1800's it was of particular interest to mathematicians to intertwine the existing theories of finite groups with those of infinite groups. However, it became apparent to De Séguier that many of the existing formulations of finite groups did not give rise to infinite groups when infinitely many elements were being considered [54]. It is for that reason De Séguier talked of so called 'non-groups' and eventually the terms 'semigroup' and 'semigroupe' were used [54]. However, the usage of these terms differed considerably from how we use them today, and were closer, in a certain sense, to our modern definition of a group. After many years of adjustments made to the definitions, Hilton set up the modern definition of a semigroup of which we are so fond [53, 54].

Armed with an associative binary operation, semigroup theorists soon took to work uncovering the complex and beautiful mathematics that semigroup theory had to offer. In 1940 Rees developed the first structural result for semigroups: he provided a construction for a particular family of matrix semigroups (later known as Rees matrix semigroups) [56, 87]. Later, in 1954, Vagner and Preston famously showed that if S is an inverse semigroup then there exists some set X such that there is a faithful representation from S to the symmetric inverse monoid on X [56, 85]: this is an analogous result to Cayley's Theorem for groups [11]. The symmetric inverse monoid consists of all partial one-to-one maps of X under composition of partial maps. We will comment more on the development of the symmetric inverse monoid below. In 1965, Krohn and Rhodes showed that every finite semigroup can be decomposed into a special kind of wreath product [66, 36]. Their theory generalised the existing Jordan-Hölder decomposition for finite groups [83]. Semigroup theory has continued to flourish since and remains a very lively area of research within the mathematical community. In particular, the study of inverse semigroups is of special interest.

The origin of the theory of inverse semigroups lies deep within geometry, rather than merely as a natural generalisation of group theory. The primary focus of the Erlangen Programm, organised by Klein in the late 1800's, was to classify geometries using groups [54, 64]. The idea was that groups of automorphisms could be assigned to geometries in such a way that they could then be used as invariants. For example, affine geometry considers invariants of the affine group [89]. However, it was soon observed by Veblen and Whitehead that such an approach would not suffice for differential geometry (where the group of automorphisms is trivial) [54, 93]. Together, they attempted to generalise the idea of a group in their paper '*The foundations of differential geometry*' to something they called a 'pseudogroup', in the hopes that this would broaden the scope of the Erlangen Programm [54, 93]. These 'pseudogroups' involved partial bijective maps with a corresponding partial composition. However, the partial compositions in which Veblen and Whitehead were interested were not closed, posing several problems for the geometer duo.

This sparked interest from the mathematician Vagner, who understood that such partial maps could be identified with binary relations [54, 91]. Equipped with this understanding, it was not long until Vagner could extend the operation and called such a structure a 'generalised group' [54, 92]. The phraseology was later changed to 'inverse semigroup' by Preston, who had simultaneously and independently produced a formulation of such a structure [54, 84]. Thus, the theory of inverse semigroups emerged and it quickly attracted the attention of semigroup theorists and group theorists alike [56, 80, 74]. As above, both Vagner and Preston developed a representation of inverse semigroups by partial one-to-one mappings.

Inverse semigroups are, as the name would suggest, a special kind of semigroup involving inverses of elements in a sense weaker than that of group theory. It is not so surprising, given the nature of pseudogroups, that 'local' identities and 'local' inverses will be important. The emergence of the textbooks of Clifford and Preston [15], and then Howie [56], (in the English speaking world) inspired a rapid development of the subject; both within and away from inverse semigroups.

In this thesis, we focus on a particular venture within semigroup theory: that of finding semigroup presentations. At a very basic level, a presentation for a semigroup is a certain kind of description for a semigroup consisting of two parts: a generating set and a set of relations between the generators. Moreover, every identity that can possibly hold true, in terms of the generators in the semigroup, can be derived from the set of relations. Using a board game analogy, one can think of the pieces you are playing with and the rules of the game as the generating set and set of relations respectively. The benefit of such an analogy, which highlights the appeal of finding semigroup presentations, is clear: for a board game with potentially infinitely many moves, a finite 'simple-to-use' set of rules would be preferred. It is for this reason that obtaining semigroup presentations is of special interest for a range of semigroups, and likewise, groups and monoids.

In 1998, Lavers provided an explicit construction of a monoid presentation for a general product of monoids in terms of the monoid presentations for the constituent monoids [68]. Likewise, semigroup presentations for semidirect products, in which the first co-ordinate is a monoid element and the second co-ordinate is a semigroup element, were considered by Al-Aadhami, Dolinka, East, Feng and Gould [2].

The seemingly innocuous question of whether or not a subsemigroup of a finitely presented semigroup is finitely presented was answered, in special cases, by Campbell, Robertson, Ruškuc and Thomas in 1996 [9]. Among other interesting results, they showed that any finitely generated right ideal of a free semigroup is finitely presented. In 2009, for instance, Cain successfully proved that every finitely generated subsemigroup of the direct product of a virtually free group and a commutative group is finitely presented [7].

Word problems for groups and monoids for which the presentations contain a single relation (so-called *one-relator presentations*) have also received special attention. In 1932, Magnus showed that the word problem for one-relator group presentations was decidable [73]. This work was later extended by Adian who showed in 1966 that the word problem is also decidable for special one-relator monoid presentations, where the single relation is of the form $w = 1$ [1, 49]. More recently, Gray showed in 2020 that there is a one-relator inverse monoid presentation with undecidable word problem [49]. Semigroup presentations remain a highly attractive area of research for semigroup theorists.

The first aim of this thesis is to obtain semigroup presentations for a class of semigroups which we call *right ideal Howson semigroups*. An algebra exhibits the *Howson property* if the intersection of two finitely generated subalgebras is also finitely generated. This term is in honour of the author of [57], who showed that the intersection of finitely generated subgroups of free groups is finitely generated. There have been a number of investigations of the Howson property for other classes of algebras. In particular, the Howson property for inverse semigroups has been studied by several authors such as Jones and Trotter [59, 60], Lawson and Vdovina [71] and Silva and Soares [90]. By contrast to the situation for groups, free inverse semigroups have the Howson property if and only if they are free on a one-element set [60].

The aim of [10], of which Chapter 5 is based, is to change tack and to consider the Howson property for semigroups regarded as *semigroup acts* over themselves, so that the right subacts of a semigroup S are precisely its right ideals. We consider $\emptyset$ as being a right ideal with empty set of generators. We say that a semigroup S is *right ideal Howson* if the intersection of any two finitely generated right ideals of S is finitely generated. The notion of a *left ideal Howson* semigroup is dually defined. Notice that since intersection distributes over union, we have that a semigroup is right ideal Howson if and only if the intersection of principal right ideals is finitely generated. This is a fact we will continually call upon throughout Chapter 5. In this thesis we will explicitly refer to and give results for right ideal Howson semigroups; clearly, the dual results hold for left ideal Howson semigroups. Certainly for a commutative semigroup, the notions of right ideal Howson and left ideal Howson coincide; similar remarks apply to related definitions.

The property of being right ideal Howson is a *finiteness condition* for a semigroup; that is to say any finite semigroup is right ideal Howson. In Section 5.2 we show how it is connected to other finiteness conditions that have been studied recently, such as that of being right coherent [42, 43] or right Noetherian [79].

Semigroups that are right ideal Howson abound. We list some examples here, that may easily be verified by consulting any standard semigroup text such as [15, 56]: groups, inverse semigroups (which we visit in Lemma 5.2.6), completely (0-)simple semigroups, free semigroups and free monoids. We present many others subsequently in this thesis. The reader may note that any of the semigroups in the previous list display the extra condition that the intersection of principal right ideals is empty or principal. Monoids that satisfy this extra condition have been well-studied by Clifford, Cherubini and Petrich: the latter authors referring to this condition (for left ideals) as *Clifford's condition* [13]. Indeed, Clifford [14] showed that bisimple inverse monoids can be viewed as inverse hulls of right cancellative monoids satisfying Clifford's condition. This connection has been developed by a number of authors such as Lawson [70], McAlister [75] and Reilly [88]. The notion of being *finitely aligned* [25] is closely connected with that of being right ideal Howson, and coincides for many semigroups, including monoids. Indeed, it is noted in [25] that finitely aligned semigroups may be called *right Howson*. It is true that every finitely aligned semigroup is right ideal Howson. However, as we show in Remark 5.4.9, a right ideal Howson semigroup need not be finitely aligned. These discrepancies are essentially due to the way in which principal right ideals and adjoined identities are handled. We introduce the term right ideal Howson to distinguish from that of being finitely aligned, and to make clear we are talking of right ideals and not right congruences. Explicit connections between finitely aligned semigroups, higher rank graphs and constructions of C^*-algebras are given in [25].

The motivation for the second part of this thesis was initially to find a semigroup presentation for the *singular part of the partial endomorphism monoid of a free G-act of finite rank n* (where G is a group), which is denoted by $\mathcal{SP}\mathrm{End}F_n(G)$. Endomorphism monoids of free G-acts, and ultimately of independence algebras (as introduced by Gould in [41]), have been studied in a variety of contexts. Fountain and Lewin considered products of idempotent endomorphisms of an independence algebra of finite rank [35]. Dolinka, Gould and Yang investigated free idempotent-generated semigroups and endomorphism monoids of free G-acts [17]. This work was later built upon by Gould and Yang who investigated idempotent-generated semigroups and endomorphism monoids of independence algebras more generally [47]. Obtaining a presentation for $\mathcal{SP}\mathrm{End}F_n(G)$ builds on the work of Al-Aadhami, Dolinka, East, Feng and Gould: in which they provided a semigroup presentation for the wreath product $M^n \rtimes \mathcal{ST}_n$ where M is a monoid and $\mathcal{ST}_n$ denotes the singular part of the full transformation monoid [2]. Indeed, the singular part of the endomorphism monoid of a free G-act, denoted by $\mathcal{S}\,\mathrm{End}\,F_n(G)$ is isomorphic to $G^n \rtimes \mathcal{ST}_n$. Understandably, the degree of difficulty is increased in the case of $\mathcal{SP}\,\mathrm{End}\,F_n(G)$ given the interplay between 'singular' and 'partial' elements within the semigroup.

While obtaining a presentation $\mathcal{SP}\mathrm{End}F_n(G)$ is our primary focus, we develop a method by which we arrive at semigroup presentations for other classes of semigroups too, namely almost-factorisable inverse semigroups (as described by Lawson [69]). This extends and adds to the body of work on factorisable inverse monoids given by Easdown, East and FitzGerald [20, 28].

We organise this thesis in the following way. The first three chapters are purely prelimary: general background material for semigroup (and monoid) theory. In Chapter 1 we cover preliminary material on what is meant by a semigroup, a monoid, and a group. We include a number of examples and some very basic results. Next, in Chapter 2, we talk about regular semigroups, inverse semigroups and their generalisations (all of which form examples of the semigroups for which we are finding presentations). Alongside the definitions, we provide some very basic examples to illustrate the distinctions between them. We also prove some elementary observations for regular and inverse semigroups. We introduce the notion of a presentation for semigroups and monoids in Chapter 3 where we offer a number of examples and some important terminology. In Chapter 4 we provide the definitions for noetherianity and coherency for semigroups and monoid respectively. We conclude this chapter by offering some basic examples. In Chapter 5, we define what is meant by a right ideal Howson semigroup and give some important classes of examples. We then provide semigroup presentations for right ideal Howson where we deal with non-commutative and commutative presentations separately. We show how these presentations are, in some sense, universal for the class of right ideal Howson semigroups. In the last part of the thesis, we discuss the question of finding presentations for the class of semigroups that exhibit a certain uniqueness property. In Chapter 6, we look at semigroups which display left-uniqueness (in a particular sense) and and provide semigroup presentations for them. Finally, we look at semigroups exhibiting a certain right-uniqueness and show how to find semigroup presentations for this class of semigroups in a very special case in Chapter 7.

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To Vicky

The bicyclic limerick:

Which semigroup has an identity?
And with one-sided inverse aplent-ity?
Admits finite presentation
Without hesitation?
It never fails to amaze-zes me.

For this, you won't have to be psychic,
But the lecture is starting, so be quick!
This monoid supreme,
And for Vicky– a dream,
And, as we all know, is bicyclic.

Declaration

I declare that this thesis is a presentation of original work and I am the sole author. This work has not previously been presented for an award at this, or any other, University. All sources are acknowledged as References.

Chapter 5 on 'Right ideal Howson semigroups' is joint research with Prof. Victoria Gould [10], where the introduction for [10] can also be found within the introductory chapter to this thesis. We also use parts of the preliminary chapter from [10] on coherency and noetherianity in Chapter 4 of this thesis. Chapter 6 on 'Semigroups with uniqueness in the first co-ordinate' is part of ongoing reasearch in collaboration with East, Dolinka, Gould and Zenab.

Chapter 1

Semigroups, monoids and groups

In Section 1.1 we introduce the basic theory of semigroups that will be made use of throughout this thesis. We start by giving the definitions of a semigroup, monoid and a group as well as their corresponding substructures. In Section 1.2 we define binary relations focussing on equivalence relations (notably Green's equivalence relations) and congruences. We also discuss quotient semigroups. We introduce the notions of external and internal general products and highlight the relationship between these two structures in Section 1.3.

1.1 Basic definitions

Throughout Section 1.1, and indeed throughout this thesis, we assume a basic knowledge of sets (including direct products of sets), n-ary functions and n-ary operations for $n \in \mathbb{N}$. As a convention, for an n-ary operation on a non-empty set X, say $* : X^n \to X$, we will write $x_1 \ldots x_n$ in the place of $(x_1, \ldots, x_n)*$ where $*$ is understood. We may, however, omit the explicit mention of a binary operation entirely and refer to 'multiplication' on a set when the binary operation is unambiguous.

Definition 1.1.1. [56] A binary operation on a set S is *associative* if

$$(xy)z = x(yz) \tag{1.1}$$

for every $x, y, z \in S$.

Often we will say that a binary operation on S satisfies the *Associative Law* whenever Equation 1.1 holds. Examples of associative binary operations abound. For instance, composition of functions on a set is familiar to almost every mathematician.

Example 1.1.2. The usual notions of addition and multiplication defined on

$$\begin{aligned}
&\mathbb{N} = \{1, 2, \ldots\} \\
&\mathbb{Z} = \{\ldots, -1, 0, 1, \ldots\} \\
&\mathbb{Q} = \left\{\frac{a}{b} : a \in \mathbb{Z}, b \in \mathbb{Z} \setminus \{0\}\right\} \\
&\mathbb{R} \text{ and} \\
&\mathbb{C} = \{x + iy : x, y \in \mathbb{R}\}
\end{aligned}$$

are binary operations.

Of course, not every binary operation is associative. This is a fact witnessed in the next example.

Example 1.1.3 (Rock, paper, scissors!)**.** Let $S = \{r, p, s\}$ and consider the binary operation on S given by the rule that

$$rp = pr = p,\ ps = sp = s,\ sr = rs = r$$

and $xx = x$ for every $x \in S$. Then, for example, we see that

$$(rp)s = ps = s \text{ and } r = rs = r(ps).$$

It follows from $s \neq r$ that $(rp)s \neq r(ps)$ and so this is an example of a binary operation on S that is not associative.

Often it may be more suitable to represent a binary operation on a finite set S via a multiplication table. Specifically, if we wish to show that $xy = z$ in S, we have an entry z in the row labelled by x and the column labelled by y as seen in Figure 1.1. Such a representation of the binary operation on S is called a *Cayley table* for S.

$$\begin{array}{c|ccc}
 & \cdots & y & \cdots \\
\hline
\vdots & & \vdots & \\
x & \cdots & z & \cdots \\
\vdots & & \vdots &
\end{array}$$

Figure 1.1: A Cayley table showing that $xy = z$ in S.

Definition 1.1.4. [56] A *semigroup* $(S, *)$ is a non-empty set S together with an associative binary operation $*$ on S.

As a convention, we will write S to denote the semigroup $(S, *)$ where the associative binary operation $*$ on S is already understood. The intuition behind associativity here is that any expression of the form $x_1 \dots x_n$, for $x_1, \dots, x_n \in S$ and $n \in \mathbb{N}$, is unambiguous; by which we mean that there is unique way to interpret such an expression via addition of brackets. This is an immediate consequence of Definition 1.1.1. For a given semigroup S and element $x \in S$, we will write

$$x^n = \underbrace{x \dots x}_{n \text{ times}}$$

for any $n \in \mathbb{N}$. With this notation in mind, we note that

$$x^a x^b = x^{a+b} \text{ and } (x^a)^b = x^{ab}$$

for every $x \in S$ and $a, b \in \mathbb{N}$. These rules are known as the *index laws*. We proceed by letting S denote a semigroup throughout this thesis.

Definition 1.1.5. [56] The *order* (or *cardinality*) of a semigroup S is the number of elements in S and is denoted by $|S|$.

We now highlight some well-known examples of semigroups.

Example 1.1.6 (Trivial semigroup)**.** If $|S| = 1$, say $S = \{e\}$, then a binary operation on S can only be given by the rule that $ee = e$. It is straightforward to see that such a binary operation is associative since

$$(ee)e = e^2 = e \text{ and } e(ee) = e^2 = e.$$

This is called the *trivial* semigroup.

Example 1.1.7 (Left-zero semigroups)**.** Let S be a non-empty set where multiplication is given by $xy = x$ for every $x, y \in S$. Then we see that

$$(xy)z = xz = x \text{ and } x(yz) = xy = x$$

for every $x, y, z \in S$ and so S is a semigroup. Such a semigroup is known as a *left-zero* semigroup. A *right-zero* semigroup is defined dually.

Example 1.1.8 (Null semigroups)**.** Let S be a non-empty set and fix some $a \in S$. Define multiplication on S by the rule that $xy = a$ for every $x, y \in S$. Then we have that

$$(xy)z = az = a \text{ and } x(yz) = xa = a$$

for every $x, y, z \in S$ and so S is a semigroup. A semigroup S of this form with $|S| \geqslant 2$ is called a *null* semigroup.

Example 1.1.9 (Free semigroup on X)**.** Let X be a non-empty set and let

$$S = \{x_1 \dots x_n : x_1, \dots, x_n \in X, n \in \mathbb{N}\}$$

with multiplication on S given by the rule that

$$(x_1 \dots x_n)(y_1 \dots y_m) = x_1 \dots x_n y_1 \dots y_m$$

for any $n, m \in \mathbb{N}$ with $x_1, \dots, x_n, y_1, \dots, y_m \in X$. Two elements, say $x_1 \dots x_n$ and $y_1 \dots y_m \in S$, are equal if and only if $n = m$ and $x_i = y_i$ for every $1 \leqslant i \leqslant n$.

It is clear to see

$$\begin{aligned}\big((x_1 \dots x_n)(y_1 \dots y_m)\big)(z_1 \dots z_p) &= (x_1 \dots x_n y_1 \dots y_m)(z_1 \dots z_p)\\ &= x_1 \dots x_n y_1 \dots y_m z_1 \dots z_p\end{aligned}$$

and similarly

$$\begin{aligned}(x_1 \dots x_n)\big((y_1 \dots y_m)(z_1 \dots z_p)\big) &= (x_1 \dots x_n)(y_1 \dots y_m z_1 \dots z_p)\\ &= x_1 \dots x_n y_1 \dots y_m z_1 \dots z_p\end{aligned}$$

for every $x_1 \dots x_n, y_1 \dots y_m, z_1 \dots z_p \in S$. This semigroup is called the *free semigroup on* X and is denoted by X^+. Here, the term 'free' is used in relation to free objects in a categorical sense: free semigroups are free objects in the category of semigroups.

In the case where $xy = yx$ for every $x, y \in S$, we say that S is *commutative*. For instance, any null semigroup is commutative. Otherwise we say that S is *non-commutative*; for example, any left-zero semigroup S with $|S| \geqslant 2$ is non-commutative.

Definition 1.1.10. Let S and T be semigroups. If T is a non-empty subset of S and $xy \in T$ for every $x, y \in T$ then we say that T is a *subsemigroup* of S and we denote this by $T \leqslant S$. We also say that S is an *oversemigroup* of T.

Certainly, every semigroup can be thought of as a subsemigroup, as well as an oversemigroup, of itself. In the case where T is a subsemigroup of S but $T \neq S$, we say that T is a *proper* subsemigroup of S and this may be denoted by $T < S$.

Example 1.1.11. Consider any left-zero semigroup S as in Example 1.1.7 where $|S| > 1$. Then any non-empty subset T of S, with $|T| < |S|$, forms a proper subsemigroup of S.

Let $S_1, \dots, S_n$ be a collection of subsets of a semigroup S for some $n \in \mathbb{N}$.

We write

$$S_1 \dots S_n = \{x_1 \dots x_n : x_i \in S_i \text{ for all } 1 \leqslant i \leqslant n\}.$$

We say that $S_1 \dots S_n$ is a *semigroup product*. Let $X \subseteq S$. If every element of S can be written in the form $x_1 \dots x_n$ where $x_i \in X$ for every $1 \leqslant i \leqslant n$ then we write $S = \langle X \rangle$, and say X *generates* S. If S is a commutative semigroup then $S_1 \dots S_n \leqslant S$ for any $S_1, \dots, S_n \leqslant S$.

Definition 1.1.12. [56] We say that I is a *left ideal* of S if $I \subseteq S$ such that $SI \subseteq I$. A *right ideal* of S is defined dually. An *ideal* of S is both a left and a right ideal of S. A *principal left ideal* of S is of the form $xS \cup \{x\}$ for some $x \in S$. We define a *principal (right) ideal* dually.

In Definition 1.1.12, it should be noted that we are using the convention that xS means $\{x\}S$. As with subsemigroups, every semigroup is an ideal of itself. In the case that I is a (left, right) ideal of S, but $I \neq S$, we say that I is *proper*. For example, the set of even natural numbers under multiplication is a proper ideal of the set of natural numbers. Note that, any ideal of a semigroup is a subsemigroup by definition, but the converse is false as seen in the following example.

Example 1.1.13. Let S be the semigroup of natural numbers under multiplication. Notice that $I = \{2^n : n \in \mathbb{N}\}$ forms a subsemigroup of S. However $sI \nsubseteq I$ for any $s \notin I$ and so I is not an ideal of S.

Definition 1.1.14. A left ideal is *finitely generated* if it is the finite union of principal left ideals. Dually for (right) ideals being finitely generated.

For instance, if S is the set of natural numbers under multiplication then the set of natural numbers divisible by 2 or 3 is a finitely generated ideal of S.

Definition 1.1.15. [56] An element $x \in S$ is called a *left-zero element* of S if

$$xy = x$$

for all $y \in S$; a *right-zero element* is defined dually. An element $x \in S$ is called a *zero element* of S if it is both a left zero and right zero element of S.

For instance, as seen in Example 1.1.7, every element of a left-zero semigroup S is a left-zero element of S. Likewise, in Example 1.1.8, the element $a \in S$ is a zero element of S.

Lemma 1.1.16. *Let S be a semigroup. If S contains a zero element then it is unique.*

Proof. Let S be a semigroup and let $x, y \in S$ be zero elements of S. This means that $xy = x$ (as x is a zero element) and $xy = y$ (as y is a zero element). Therefore we have shown that $x = y$. $\square$

Often, we will write 0 as the zero element of a semigroup. In the case where S and T both contain zero elements, we will write 0_S and 0_T to distinguish between the zero elements of S and T respectively.

Definition 1.1.17. [56] Let S be a semigroup and 0 be a formal symbol such that $0 \notin S$. Then we extend the multiplication in S to $S \cup \{0\}$ by setting

$$0x = x0 = 0 \text{ and } 00 = 0$$

for all $x \in S$. We write S^0, called S *with-zero-adjoined if necessary*, to mean the semigroup

$$S^0 = \begin{cases} S & \text{if } S \text{ contains a zero element;} \\ S \cup \{0\} & \text{otherwise.} \end{cases}$$

Notice that S^0 is a semigroup with a zero element.

Definition 1.1.18. [56] An element $x \in S$ is a *left-identity element* of S if

$$xy = y$$

for every $y \in S$, where a *right-identity element* of S is defined dually. An element $x \in S$ is called an *identity element* of S if it is both a left-identity and right-identity element of S.

As an example, one can choose to think of a left-zero semigroup as a semigroup in which every element is also a right-identity. In contrast, a null semigroup S contains a zero element and there is no identity element.

Lemma 1.1.19. *Let S be a semigroup. If S contains an identity element then it is unique.*

Proof. Let S be a semigroup and let $x, y \in S$ be identity elements of S. This means that $xy = x$ (as y is an identity element) and $xy = y$ (as x is an identity element). Therefore we have shown that $x = y$. $\square$

We will normally denote the identity element of a semigroup by 1. For any semigroup S with identity we define $x^0 = 1$ for any $x \in S$ (the index laws then hold for all $m, n \in \mathbb{N}^0$. As before, if S and T both contain identity elements then we will write 1_S and 1_T to separate the identity elements of S and T respectively.

Definition 1.1.20. [56] Let S be a semigroup and 1 be a formal symbol such that $1 \notin S$. Then we extend the multiplication in S to $S \cup \{1\}$ by setting

$$1x = x1 = x \text{ and } 11 = 1$$

for all $x \in S$. We write S^1, called S *with-identity-adjoined if necessary*, to mean the semigroup

$$S^1 = \begin{cases} S & \text{if } S \text{ contains an identity element;} \\ S \cup \{1\} & \text{otherwise.} \end{cases}$$

Notice that S^1 is a semigroup. Semigroups which contain identity elements have special properties and have long been of particular focus to semigroup theorists.

Definition 1.1.21. [56] A *monoid* is a semigroup S that contains an identity element.

Typically, we will denote an arbitrary monoid by M instead of S or S^1 (where appropriate) and we use this notation for the rest of the thesis. As with the case for semigroups, there is a corresponding notion of a monoid within a monoid: a *submonoid* N of a monoid M is a subsemigroup of M such that $1 \in N$. In what follows, we provide an interesting example of a monoid where, for convenience, we will set $\mathbb{N}^0 = \mathbb{N} \cup \{0\}$.

Example 1.1.22 (Bicyclic monoid)**.** Let B be the set of formal symbols

$$B = \{x^a y^b : a, b \in \mathbb{N}^0\}$$

with multiplication given by

$$(x^a y^b)(x^c y^d) = x^{a-b+t} y^{d-c+t} \text{ where } t = \max\{b, c\}$$

for every $a, b, c, d \in \mathbb{N}^0$. We verify that this is a semigroup and that it contains an identity element.

Let $a, b, c, d, h, k \in \mathbb{N}^0$, then

$$\begin{aligned}
\big((x^a y^b)(x^c y^d)\big)(x^h y^k) &= (x^{a-b+t} y^{d-c+t})(x^h y^k) \text{ where } t = \max\{b, c\} \\
&= x^{a-b+c-d+u} y^{k-h+u} \text{ where } u = \max\{d-c+t, h\} \\
(x^a y^b)\big((x^c y^d)(x^h y^k)\big) &= (x^a y^b)(x^{c-d+s} y^{k-h+s}) \text{ where } s = \max\{d, h\} \\
&= x^{a-b+v} y^{k-h+d-c+v} \text{ where } v = \max\{b, c-d+s\}.
\end{aligned}$$

From this point, it will suffice to show that $c - d + u = v$ in order to show that B forms a semigroup. Using the fact that max is associative

$$\begin{aligned}
c - d + u &= c - d + \max\big\{d - c + \max\{b, c\}, h\big\} \\
&= \max\big\{ \max\{b, c\}, c - d + h\big\} \\
&= \max\{b, c, c - d + h\} \\
&= \max\big\{b, \max\{c, c - d + h\}\big\} \\
&= \max\big\{b, c - d + \max\{d, h\}\big\} \\
&= v
\end{aligned}$$

and so B is a semigroup. To show that B has an identity element, consider $x^0y^0 \in B$. For any $a, b \in \mathbb{N}^0$

$$\begin{aligned}(x^0y^0)(x^ay^b) &= x^ty^{b-a+t} \text{ where } t = \max\{0, a\} \\ &= x^ay^b \\ (x^ay^b)(x^0y^0) &= x^{a-b+u}y^u \text{ where } u = \max\{b, 0\} \\ &= x^ay^b\end{aligned}$$

and so B is a monoid, called the *bicyclic monoid*.

The bicyclic monoid is given by a monoid presentation on two generators, x and y, where $yx = 1$ (but where we cannot deduce $xy = 1$). It may be realised by taking y to be the successor function on $\mathbb{N}^0$ and x to be the operation that sends 0 to 0 and otherwise subtracts 1. We discuss presentations in Chapter 3.

Example 1.1.23 (Power sets)**.** Let X be any set (possibly empty). The *power set* of X, denoted $\mathcal{P}(X)$ and given by

$$\mathcal{P}(X) = \{A : A \subseteq X\}$$

forms a monoid under set union where the identity element is $\emptyset$. Equally $\mathcal{P}(M)$ forms a monoid under subset multiplication with identity element $\{1\}$.

Definition 1.1.24. [56] An element $x \in M$ is a *left inverse* of $y \in M$ if

$$xy = 1$$

where dually, $y \in M$ is a *right inverse* of $x \in M$ is defined. An element $x \in M$ is an *inverse* of $y \in M$ if it is both a left inverse and right inverse of $y \in M$.

In the special case where $x \in M$ is a inverse of itself, that is $x = x^{-1}$, we say that x is *self-inverse*. Certainly, the identity element of any monoid is a simple example of a self-inverse element.

Definition 1.1.25. [56] A *group* is a monoid such that each element has an inverse.

Alternatively, we may define groups in the following way.

Definition 1.1.26. [56] A *group* is a semigroup S such that

$$Sx = xS = S$$

for every $x \in S$.

Proposition 1.1.27. *The definitions of a group given in Definition 1.1.25 and Definition 1.1.26 are equivalent.*

Proof. Suppose we begin by assuming G is a group in the sense of Definition 1.1.25. As $xG \subseteq G$ is clear for every $x \in G$, we continue by showing that $Gx \subseteq xG$ and $G \subseteq Gx$ for every $x \in G$. For every $x \in G$ there exists a unique inverse element, say $x^{-1} \in G$ such that $xx^{-1} = x^{-1}x = 1$. Thus, for any $x, y \in G$, we have that $yx = xx^{-1}yx$ and so $Gx \subseteq xG$. Similarly, for every $x, y \in G$, we have that $y = yx^{-1}x$ and so $G \subseteq Gx$. Therefore we have shown that

$$G \subseteq Gx \subseteq xG \subseteq G$$

for every $x \in G$. Hence, G is a semigroup in which $Gx = xG = G$ for every $x \in G$. This is exactly Definition 1.1.26.

For the reverse implication, we suppose for a semigroup S that

$$Sx = xS = S$$

holds for every $x \in S$. Fix $x \in S$. It follows immediately that there exists some $a, b \in S$ such that $ax = xb = x$. In turn, for every $y \in S$ there exists $c, d \in S$ such that $cx = xd = y$. Together, this implies that

$$ay = a(xd) = (ax)d = xd = y \text{ and } yb = (cx)b = c(xb) = cx = y$$

which give us that a and b are a left-identity and right-identity element of S respectively. As such, we have that $a = b = 1$ is the unique identity element of S. Lastly, for every $h \in S$, there exists some $k, \ell \in S$ such that $kh = h\ell = 1$. It is clear from here to see that

$$k = k1 = k(h\ell) = 1\ell = \ell$$

and so $k = \ell = h^{-1}$ is the unique inverse element of $h \in S$. Thus S is a group as defined in Definition 1.1.25. $\square$

A *subgroup* H of a group G is a submonoid of G such that for every $x \in H$ we have $x^{-1} \in H$. Moreover, a subgroup H in which for every $x \in H$ and $y \in G$ we have $yxy^{-1} \in H$ is called *normal* (where y^{-1} is the inverse element of y). We proceed throughout this thesis by adopting the convention that G will denote an arbitrary group.

Definition 1.1.28. [56] A *semigroup homomorphism* is a map $\phi : S \to T$ such that

$$(x\phi)(y\phi) = (xy)\phi$$

for every $x, y \in S$. A *monoid homomorphism* $\phi : M \to N$ between monoids M and N is a semigroup homomorphism with the additional property that $1_M\phi = 1_N$.

For the sake of brevity, we will just write homomorphism where semigroup (or monoid) is understood.

Definition 1.1.29. [56] An *isomorphism* is a bijective homomorphism.

If there exists a isomorphism between S and T then we say S and T are *isomorphic* and write $S \simeq T$. A homomorphism from S to itself is called an *endomorphism* and similarly an isomorphism from S to itself is known as an *automorphism*. We denote by $\operatorname{End} S$ and $\operatorname{Aut} S$ the monoid of endomorphisms and the group of automorphisms of S (under composition of functions) respectively. It is worth emphasising here that $\operatorname{End} S$ and $\operatorname{Aut} S$ form a monoid and a group respectively.

Definition 1.1.30. [56] Let $\phi : S \to T$ be a homomorphism. The *kernel of*

ϕ is denoted $\ker\phi$ and given by

$$\ker\phi = \big\{(x, y) \in S \times S : x\phi = y\phi\big\}$$

and similarly, the *image of* ϕ, written $\operatorname{im}\phi$, is given by

$$\operatorname{im}\phi = \{x\phi : x \in S\}.$$

For instance, if ϕ is a isomorphism then

$$\ker\phi = \big\{(x, x) : x \in S\big\} \text{ and } \operatorname{im}\phi = T.$$

A point worth emphasising in Definition 1.1.30 is that $\ker\phi$ is in fact a binary relation on S, not a subset of S. We are now going to discuss binary relations in Section 1.2.

1.2 Equivalence relations, congruences and quotient semigroups

Throughout this section we let X be a non-empty set and use R for a binary relation on X. For clarity, we will write R^S whenever we wish to emphasise that R is a relation on a set X. With regards to notation, we will use $(x, y) \in R$ and $x\,R\,y$ interchangeably (or say x and y are R-related).

Definition 1.2.1. [56] An *equivalence relation* R on X is a binary relation on X satisfying the following conditions:

(ER1) $(x, x) \in R$;

(ER2) if $(x, y) \in R$ then $(y, x) \in R$;

(ER3) if $(x, y), (y, z) \in R$ then $(x, z) \in R$;

for every $x, y, z \in X$. We will refer to a binary relation as being *reflexive* if (ER1) holds; *symmetric* if (ER2) holds and *transitive* if (ER3) holds.

For an equivalence relation R on a set X, we will write $[x]_R$ to mean the *equivalence class* of x with respect to R where

$$[x]_R = \{y \in X : (x, y) \in R\}.$$

Where R is understood, we will write $[x]$ instead of $[x]_R$. Notice that any equivalence relation R on a set X leads to a *partition* of the set. Specifically, for any X and R, we have $[x] \neq \emptyset$,

$$\bigcup_{x \in X} [x] = X \text{ and } [x] \cap [y] = \begin{cases} [x] & \text{if } x \in [y]; \\ \emptyset & \text{otherwise} \end{cases}$$

for every $x, y \in X$.

Of course, not every binary relation on a set will for an equivalence relation, as seen in the next example.

Example 1.2.2 (Partial orders)**.** Consider the set of natural numbers $\mathbb{N}$ and the binary relation R where $(n, m) \in R$ if and only if $n \leqslant m$. It is clear that $n \leqslant n$ for every $n \in \mathbb{N}$. It is also easy to see that if $n \leqslant m$ and $m \leqslant p$ then $n \leqslant p$ for every $n, m, p \in \mathbb{N}$. However, it is not true that R is symmetric since $n \leqslant m$ and $m \leqslant n$ only when $n = m$. Therefore R is not an equivalence relation. Such a binary relation R that satisfies the following conditions:

(PO1) $(x, x) \in R$;

(PO2) if $(x, y), (y, x) \in R$ then $x = y$;

(PO3) if $(x, y), (y, z) \in R$ then $(x, z) \in R$

for every $x, y, z \in R$ is known as a *partial order*. We refer to a binary relation satisfying (PO2) as being *anti-symmetric*.

We illustrate some important examples of equivalence relations which, as we will see, are used in abundance throughout this thesis.

Example 1.2.3 (Green's $\mathscr{L}$-relation)**.** [56] Let $\mathscr{L}$ be the relation on a semigroup S given by the rule that

$$(x, y) \in \mathscr{L} \iff S^1x = S^1y$$

for every $x, y \in S$. Clearly $\mathscr{L}$ satisfies (ER1) since $S^1x = S^1x$ for every $x \in S$ trivially. Likewise, if $(x, y) \in \mathscr{L}$, for $x, y \in S$, then $S^1x = S^1y$ and so too it is straightforward to see that $(y, x) \in \mathscr{L}$. Finally, if $(x, y), (y, z) \in \mathscr{L}$ for $x, y, z \in S$, then we have $S^1x = S^1y$ and $S^1y = S^1z$. Altogether this gives $S^1x = S^1z$ and so $(x, z) \in \mathscr{L}$. Hence $\mathscr{L}$ is an equivalence relation on S known as *Green's $\mathscr{L}$-relation*.

As it happens, Green's $\mathscr{L}$-relation is one of a family of equivalence relations introduced by Green. The following, along with $\mathscr{L}$, form a complete set of *Green's equivalence relations* for any semigroup S:

$$\begin{aligned} x\mathscr{R}y &\iff xS^1 = yS^1 \\ x\mathscr{H}y &\iff x\mathscr{L}y \text{ and } x\mathscr{R}y \\ x\mathscr{D}y &\iff x\mathscr{L}z \text{ and } z\mathscr{R}y \text{ for some } z \in S^1 \\ x\mathscr{J}y &\iff S^1xS^1 = S^1yS^1 \end{aligned}$$

for every $x, y \in S$ and for some $z \in S$. The bright-eyed and bushy-tailed reader may make the additional observation that, for instance, if $(x, y) \in \mathscr{L}$ then $(xz, yz) \in \mathscr{L}$ for every $z \in S$. This is a important property that we define next.

Definition 1.2.4. [56] An equivalence relation R on S is *left-compatible* if $(x, y) \in R$ implies that $(zx, zy) \in R$ for every $z \in S$. A *right-compatible* equivalence relation is defined dually. An equivalence relation on S is *compatible* if it is both left-compatible and right-compatible.

If an equivalence relation is left-compatible, we call it a *left congruence*; a *right congruence* is defined dually. With Green's equivalence relations in mind, $\mathscr{R}$ is a left congruence and $\mathscr{L}$ is a right congruence. An equivalence relation is called a *congruence* if it is both a left and right congruence. It is easily seen that an equivalence relation R is a congruence if and only if for all $(a,b),(c,d) \in R$ we have $(ac,bd) \in R$. Hence the terms 'compatible equivalence relation' and 'congruence' are synonymous. We highlight some special congruences below.

Example 1.2.5 (Universal congruence)**.** Let $\omega = \omega_S$ be the equivalence relation on S given by the rule that $(x,y) \in \omega$ for all $x,y \in S$. It follows immediately that $(zx,zy) \in \omega$ and similarly $(xz,yz) \in \omega$ for all $x,y,z \in S$, and so ω defines a congruence on S. This is known as the *universal congruence* on S.

Example 1.2.6 (Identity congruence)**.** Let $\Delta = \Delta_S$ be the equivalence relation on S defined by

$$\Delta = \big\{(x,x) : x \in S\big\}.$$

Clearly, if $(x,y) \in \Delta$ then $(zx,zy) \in \Delta$ for any $z \in S$, since $x = y$. Dually, one can argue that $(xz,yz) \in \Delta$ and so Δ defines a congruence on S called the *identity congruence* on S.

Definition 1.2.7. [56] Let R be a congruence on S. The *quotient semigroup*, denoted by S/R, is the semigroup with elements

$$S/R = \big\{[x] : x \in S\big\}$$

and multiplication given by

$$[x][y] = [xy]$$

for every $x,y \in S$.

It is clear to see that this multiplication is well-defined for if $[x] = [a]$ and $[y] = [b]$ then $(x,a),(y,b) \in R$. As R is a congruence this gives $(xa,yb) \in R$ and so $[xy] = [yb]$. In this way, for every semigroup S we have S/ω is trivial and $S \simeq S/\iota$. We illustrate a couple of (more interesting) examples of quotient semigroup in what follows.

Example 1.2.8 (Rees quotient semigroups)**.** [56] Let I be any ideal of a semigroup S and let

$$R = (I \times I) \cup \Delta_{S \setminus I}$$

be a binary relation on S. If $x \in I$ then $(x, x) \in I \times I$ and if $x \in S \setminus I$ then $(x, x) \in \Delta_{S \setminus I}$. Hence R satisfies (ER1). If $(x, y) \in I \times I$ then $x, y \in I$ and so $(y, x) \in I \times I$. On the other hand, if $(x, y) \in \Delta_{S \setminus I}$ then $x = y$ and so $(y, x) = (x, x) \in \Delta_{S \setminus I}$. Therefore R satisfies (ER2). Lastly, to verify that R satisfies (ER3) we consider a couple of cases. If $(x, y), (y, z) \in I \times I$ then $x, y, z \in I$ and so $(x, z) \in I$. Alternatively, if $(x, y), (y, z) \in \Delta_{S \setminus I}$ then $x = y = z \in S \setminus I$ and so $(x, z) = (x, x) \in \Delta_{S \setminus I}$. Thus R satisfies (ER3) and so is an equivalence relation on S. On the other hand, if $(x, y) \in I \times I$ and $(y, z) \in \Delta_{S \setminus I}$, then we see $y = z$ so that $(x, z) \in R$; similarly for the final case.

To show that R is a congruence on S, we show that it is left-compatible, where the proof of right-compatibility is entirely dual. Suppose that $(x, y) \in I \times I$ and $a \in S$, then clearly $ax, ay \in I$ as I is an ideal. It follows that $(ax, ay) \in I \times I$. Conversely, if $(x, y) \in \Delta_{S \setminus I}$ and $a \in S$ then $ax = ay$ as $x = y$. This gives us that $(ax, ay) = (ax, ax) \in \Delta_S$. Hence R is a congruence on S. The resulting quotient semigroup S/R is known as a *Rees quotient semigroup* where R is called a *Rees congruence* on S.

In the Rees quotient S/I, effectively we identify all the elements of I and set them to be a zero.

Example 1.2.9. Let S and T be semigroups and let $\phi : S \to T$ be a homomorphism. It is clear that $\ker \phi$ is an equivalence relation on T. That $\ker \phi$ is a (left, right) congruence follows from the fact that ϕ is a homomorphism.

We conclude this section by proving some important results regarding congruences on semigroups.

Lemma 1.2.10. *Let S be a semigroup and let R be a congruence on S. Then there exists a homomorphism $\phi : S \to S/R$ given by*

$$x\phi = [x]$$

for every $x \in S$.

Proof. Let S be a semigroup and R be a congruence on S. Let $\phi : S \to S/R$ be a map given by $x\phi = [x]$. Then

$$(xy)\phi = [xy] = [x][y] = (x\phi)(y\phi)$$

for every $x, y \in S$. □

Theorem 1.2.11 (The Fundamental Theorem of Homomorphisms for Semigroups)**.** *Let* S, T *be semigroups and let* $\phi : S \to T$ *be a homomorphism. Then* $\ker\phi$ *is a congruence on* S*,* $\operatorname{im}\phi$ *is a subsemigroup of* T *and*

$$S/\ker\phi \cong \operatorname{im}\phi.$$

1.3 General products

General products of semigroups generalise the familiar notion of direct products of semigroups. We begin by introducing what is meant when one semigroup *acts* on another. This will be crucial in defining general products.

Definition 1.3.1. Let S and T be semigroups. Then we say that T *acts on the left of* S *(by* $\bullet$*)* if there exists a map $T \times S \to S$ given by $(a, x) \mapsto a \bullet x$ such that

$$a \bullet (b \bullet x) = (ab) \bullet x$$

for every $a, b \in T$ and $x \in S$. Dually we have the notion of when S *acts on the right of* T *(by* $*$*)*.

Equivalently, we may say that T *has a left action on* S or dually S *has a right action on* T. Wherever a left action, say $\bullet$, and right action, say $*$, are understood, we will simply write ${}^{a}x$ and a^{x} instead of $a \bullet x$ and $a * x$ respectively. If T acts on the left of S such that

$${}^{a}x\,{}^{b}x = {}^{ab}x$$

for every $a, b \in T$ and $x \in S$, we say that T *acts on the left of* S *by homomorphisms.* Dually we define when S *acts on the right of* T *by homomorphisms.*

Definition 1.3.2. Let T act on the left of S and S act on the right of T such that the following conditions are satisfied:

(GP1) ${}^{a}(xy) = {}^{a}x \, {}^{a^x}y$;

(GP2) $(ab)^x = a^{{}^{b}x} \, b^x$;

for all $x, y \in S$ and $a, b \in T$. An *external general product* of S and T, denoted $S \bowtie T$, is the semigroup with elements

$$S \bowtie T = \big\{(x, a) : x \in S, a \in T\big\}$$

and multiplication given by

$$(x, a)(y, b) = \big(x \, {}^{a}y, a^y \, b\big).$$

In the special cases where ${}^{a}x = x$ ($a^x = a$) for every $x \in S$ and $a \in T$, the resulting external general product is known as a *semidirect product* and is written as $S \ltimes T$ ($S \rtimes T$).

We verify below that $S \bowtie T$ forms a semigroup. Indeed, we see that

$$\begin{aligned}
\big((x,a)(y,b)\big)(z,c) &= \big(x \, {}^{a}y, a^y \, b\big)(z,c) \\
&= \big(x \, {}^{a}y \, {}^{(a^y \, b)}z, (a^y \, b)^z \, c\big) \\
&= \big(x \, {}^{a}y \, {}^{(a^y)}({}^{b}z), (a^y)^{({}^{b}z)} \, b^z \, c\big) \\
&= \big(x \, {}^{a}(y \, {}^{b}z), a^{(y \, {}^{b}z)} \, b^z \, c\big) \\
&= (x,a)\big(y \, {}^{b}z, b^z \, c\big) \\
&= (x,a)\big((y,b)(z,c)\big)
\end{aligned}$$

for every $x, y, z \in S$ and $a, b, c \in T$.

Definition 1.3.3. Let S and T be subsemigroups of a semigroup U. Then U is an *internal general product* of S and T if $U = ST$ and every element of U can be expressed in the form xa for a unique $x \in S$ and unique $a \in T$.

Note that general products are also referred to as *Zappa-Szép products*. We will simply refer to a general product whenever internal or external general product is understood. Interestingly, we demonstrate below how one cannot simply pass between the two definitions: they are separate notions.

Proposition 1.3.4. *Let U be an internal general product of subsemigroups S and T. Then $S \bowtie T$ exists.*

Proof. If $a \in T$ and $x \in S$ then it follows that $ax \in U$. Since U is an internal general product of S and T, there must exist some unique $y \in S$ and unique $b \in T$ such that $ax = yb$. We show that by setting ${}^a x = y$ and $a^x = b$ we define a left action of T on S and a right action of S on T respectively satisfying (GP1) and (GP2).

We see that

$$ {}^{ab}x(ab)^x = (ab)x = a(bx) = a({}^b x b^x) = (a\,{}^b x)b^x = {}^a({}^b x)a^{{}^b x}\, b^x $$

for all $a, b \in T$ and $x \in S$. Therefore

$$ {}^{ab}x = {}^a({}^b x) \text{ and } (ab)^x = a^{{}^b x}\, b^x $$

by uniqueness. Similarly, one can show that

$$ a^{xy} = (a^x)^y \text{ and } {}^a(xy) = {}^a x\, {}^{a^x} y $$

for every $a \in T$ and $x, y \in S$. Hence we may form the external general product $S \bowtie T$. $\square$

Proposition 1.3.5. *Let $S \bowtie T$ be an external general product of semigroups. Then $S^1 \bowtie T^1$ is an external general product and an internal general product of subsemigroups $S^1 \bowtie \{1_T\}$ and $\{1_S\} \bowtie T^1$.*

Proof. Suppose that $S \bowtie T$ is an external general product of semigroups. We can extend the left action of T on S to a left action of T^1 on S^1 by setting ${}^a 1_S = 1_S$ and ${}^{1_T}x = x$ for every $x \in S^1$ and $a \in T^1$. Dually, we can extend the right action of S on T to a right action of S^1 on T^1 by setting

$1_T^x = 1_T$ and $a^{1_T} = a$ for every $x \in S$ and $a \in T$. It is routine to verify that (GP1) and (GP2) hold under this rule. As such, we can form the external general product $S^1 \bowtie T^1$.

To see that $S^1 \bowtie \{1_T\}$ is a subsemigroup of $S^1 \bowtie T^1$ we notice that

$$(x, 1_T)(y, 1_T) = (x\,{}^{1_T}y, 1_T^y\,1_T) = (xy, 1_T^2) = (xy, 1_T)$$

for every $x, y \in S^1$. Dually one can see that $\{1_S\} \bowtie T^1$ is a subsemigroup of $S^1 \bowtie T^1$ dually. For any $x \in S^1$ and $a \in T^1$ we have

$$(x, a) = (x1_S, 1_Ta) = (x\,{}^{1_T}1_S, 1_T^{1_S}a) = (x, 1_T)(1_S, a).$$

Clearly this factorisation is unique. Hence $S^1 \bowtie T^1$ is an external general product and an internal general product of $S^1 \bowtie \{1_T\}$ and $\{1_S\} \bowtie T^1$ as required. $\square$

To summarise Proposition 1.3.4 and Proposition 1.3.5, there are clear differences in going from an internal general product to an external general product and vice versa. Starting from an internal general product of the form ST, one can form the external general product $S \bowtie T$. From an external general product $S \bowtie T$, one can form the external general product $S^1 \bowtie T^1$ so that

$$S^1 \bowtie T^1 = (S^1 \bowtie \{1_T\})(\{1_S\} \bowtie T)$$

where $S^1 \simeq S^1 \bowtie \{1_T\}$ and $T^1 \simeq \{1_S\} \bowtie T^1$. For if we have an external general product $S \bowtie T$ where S and T are semigroups, then unless S and T are monoids we cannot necessarily express this as an internal product, since S and T may not embed into $S \bowtie T$.

Chapter 2

Regular semigroups, inverse semigroups and their generalisations

In Section 2.1 we begin by introducing the notion of regular elements and regular semigroups. We then give the definition of an inverse semigroup along with a few examples, some basic concepts and important results, in Section 2.2. To conclude, we briefly look into generalisations of regular and inverse semigroups; namely abundant semigroups in Subsection 2.3.1, adequate semigroups in Subsection 2.3.2 and ample semigroups in Subsection 2.3.3.

2.1 Basic definitions

We will let $E(S)$ denote the *set of idempotents* of a semigroup S, given by

$$E(S) = \{x \in S : x^2 = x\}.$$

Where S is clear, we will write E instead of $E(S)$.

Definition 2.1.1. [56] An element $x \in S$ is *regular* if there exists some $y \in S$ such that $xyx = x$.

Examples of semigroups that contain regular elements are plentiful: for instance, every monoid contains the identity element, which is regular. We highlight another family of semigroups that exhibit regular elements in the next example.

Example 2.1.2 (Bands and semilattices)**.** Let S be a semigroup such that $E = S$. Then it is clear that

$$x^3 = x^2x = xx = x$$

for every $x \in S$ and so every element of S is regular. Such a semigroup is called a *band*. A commutative band is known as a *semilattice*.

If S is a semilattice then S is partially ordered under $x \leqslant y$ if and only if $xy = x$. Under this partial order, for any $x, y \in S$ we have that xy is the greatest lower bound of x and conversely, if Y is a partially ordered set in which every pair of elements a, b has a greatest lower bound, denoted $x \wedge y$, then Y becomes a semilattice under the operation of $\wedge$.

Definition 2.1.3. [56] A semigroup S is *regular* if every element $x \in S$ is regular.

Regular semigroups are a well-studied class of semigroups and examples abound: groups, semilattices, left-zero semigroups, full transformation monoids, the trivial semigroup, the power set of a semigroup under union or intersection, and the set of natural numbers under max. We highlight a less straightforward example.

Example 2.1.4 (Bicyclic monoids)**.** Let B be the bicyclic monoid and let $x^ay^b \in B$. It is easy to check that for any $a \in \mathbb{N}^0$ we have $y^ax^a = x^0y^0$ and then

$$x^ay^n(x^by^a)x^ay^b = x^ay^b.$$

Therefore the bicyclic monoid B is regular.

Certainly there are natural examples of semigroups lurking in the semigroup wilderness that are not regular: free semigroups, null semigroups and the set of natural numbers under addition or multiplication to name a few.

Proposition 2.1.5. *[56] If S is regular then for every $x \in S$ there exists $y \in S$ such that $xyx = x$ and $yxy = y$.*

Proof. Let S be a regular semigroup and $x \in S$. Then there exists some $y \in S$ such that $xyx = x$ by definition. Now we set $z \in S$ such that $z = yxy$. With this in mind, it follows that

$$xzx = xy(xyx) = xyx = x$$

and similarly

$$zxz = yxy(xyx)y = y(xyx)y = yxy = z$$

which completes the proof. $\square$

2.2 Inverse semigroups

We start by defining a (new) notion of an inverse element for semigroups.

Definition 2.2.1. [56] Let S be a semigroup and suppose for an element $x \in S$ there exists some $y \in S$ such that $xyx = x$ and $yxy = y$. Then we say that y is an *inverse* of x and dually x is an *inverse* of y.

For every $x \in S$, we let $V(x) \subseteq S$ be the subset defined by

$$V(x) = \{y \in S : xyx = x \text{ and } yxy = y\}.$$

Thus $V(x)$ is referred to as the set of inverses of the element $x \in S$. With this in mind, Definition 2.1.3 is equivalent to saying that a semigroup S is regular if and only if $|V(x)| > 0$ for every $x \in S$.

Definition 2.2.2. A semigroup S is *inverse* if every element $x \in S$ has a unique inverse $x^{-1} \in S$.

We remark that if $e \in E$ then $e^{-1} = e$. Note that every group is an inverse semigroup. To deal with the obvious difficulty posed by two separate notions of an inverse, we will use the following convention: x' will denote the inverse of an element x in an arbitrary semigroup whereas x^{-1} will denote the inverse of an element x in an inverse semigroup sense. In this way, we will see there will be no ambiguity with using x^{-1} to denote the inverse of a group element $x \in G$. We prove the following important property of inverse semigroups.

Proposition 2.2.3. *[56] Let S be a regular semigroup. Then S is inverse if and only if E forms a commutative subsemigroup of S.*

Proof. Suppose first that S is inverse. Clearly S is regular. Let $e, f \in E$; we set $x = (ef)^{-1}$. This implies that $x \in S$ satisfies

$$(ef)x(ef) = ef \text{ and } x(ef)x = x.$$

With this in mind, we see that

$$(fxe)^2 = f(xefx)e = fxe$$

so $fxe \in E$ and $fxe = (fxe)^{-1}$ by an earlier comment. Further, this gives

$$(fxe)ef(fxe) = f(xefx)e = fxe \text{ and } (ef)(fxe)(ef) = efxef = ef$$

so that $ef = (fxe)^{-1}$. Since S is inverse this implies that $fxe = ef \in E$. Dually one can argue that $fe \in E$. Lastly, we see

$$(ef)(fe)(ef) = (ef)^2 = ef \text{ and } (fe)(ef)(fe) = (fe)^2 = fe.$$

Therefore $ef = (fe)^{-1} = fe$.

For the converse, suppose that S is regular and the idempotents of E commute. It is easy to see that products of idempotents are then idempotent, so that E forms a semilattice. If $x \in S$ and $y \in S$ is such that $x = xyx$, then xy and yx are idempotent. Consequently, if y and z are inverses of x then one can show that $y = z$. For,

$$\begin{aligned} y &= yxy = y(xzx)y = (yx)(zx)y = (zx)(yx)y \\ &= z(xyx)y = zxy = \cdots = zxz = z. \end{aligned}$$

□

We illustrate Proposition 2.2.3 with the following counterexample.

Example 2.2.4. Let S be a left-zero semigroup with $|S| > 1$. We have $E = S$ which immediately gives us that S is regular. However, for any distinct elements $x \neq y \in S$, we see $xy = x$ and $yx = y$. Thus S is not inverse as the idempotents do not commute.

We make immediate use of Proposition 2.2.3 in the next result.

Proposition 2.2.5. *[56] If S is an inverse semigroup then every $\mathscr{L}$-class and $\mathscr{R}$-class contains a unique idempotent.*

Proof. Let S be an inverse semigroup. By definition, for any $x \in S$ there exists an $x^{-1} \in S$ such that $xx^{-1}x = x$ and $x^{-1}xx^{-1} = x^{-1}$. Certainly $(x, x^{-1}x) \in \mathscr{L}$ and $(x, xx^{-1}) \in \mathscr{R}$. We have

$$(xx^{-1})^2 = (xx^{-1}x)x^{-1} = xx^{-1} \text{ and } (x^{-1}x)^2 = x^{-1}(xx^{-1}x) = x^{-1}x$$

so that xx^{-1}, $x^{-1}x \in E$ by definition.

Suppose that $e, f \in E$ such that $(e, f) \in \mathscr{L}$. This implies $S^1e = S^1f$ and so there exists some $x, y \in S^1$ such that $xe = f$ and $yf = e$. Then

$$e = yf = yf^2 = ef = fe = xe^2 = xe = f$$

since idempotents commute in S. This gives us that $e = f$ is the unique idempotent in the $\mathscr{L}$-class of e. A dual argument shows that each $\mathscr{R}$-class contains a unique idempotent. □

We now give some well-known examples of inverse semigroups.

Example 2.2.6 (Symmetric inverse monoids)**.** [56] Let X be a non-empty set and let $\mathcal{I}_X$ be the monoid

$$\mathcal{I}_X = \{\alpha : X \to X \mid \alpha \text{ is an injection}\}.$$

The composition in $\mathcal{I}_X$ is given by

$$\operatorname{dom}(\alpha\beta) = (\operatorname{im}\alpha \cap \operatorname{dom}\beta)\alpha^{-1}$$

and for any $x \in \operatorname{dom}(\alpha\beta)$ we have

$$x(\alpha\beta) = (x\alpha)\beta.$$

For every $\alpha \in \mathcal{I}_X$, we define α^{-1} with $\operatorname{dom}\alpha^{-1} = \operatorname{im}\alpha$, $\operatorname{im}\alpha^{-1} = \operatorname{dom}\alpha$ and $x\alpha^{-1} = y$ where $y\alpha = x$ for every $x \in \operatorname{dom}\alpha^{-1}$. Clearly $V(\alpha) = \{\alpha^{-1}\}$ for every $\alpha \in \mathcal{I}_X$ with α^{-1} defined as described in the above. Such an inverse semigroup is referred to as the *symmetric inverse monoid on X*.

Example 2.2.7 (Brandt semigroups)**.** [15, 56] Let G be a group and let I be a non-empty set. Then we let $B^0 = (I \times G \times I) \cup \{0\}$ be a set of formal symbols with multiplication given by

$$(i, x, j)(k, y, \ell) = \begin{cases} (i, xy, \ell) & \text{if } j = k; \\ 0 & \text{otherwise} \end{cases}$$

and with

$$(i, x, j)0 = 0(i, x, j) = 00 = 0$$

for every $(i, x, j), (k, y, \ell) \in B^0$.

Then for every $(i, x, j) \in B$ we see that

$$\begin{aligned} \big((i, x, j)(j, x^{-1}, i)\big)(i, x, j) &= (i, xx^{-1}, i)(i, x, j) \\ &= (i, 1, i)(i, x, j) \\ &= (i, 1x, j) \\ &= (i, x, j). \end{aligned}$$

It follows that $(i, x, j)^{-1} = (j, x^{-1}, i)$. Finally, we always have that $0^{-1} = 0$ since $(00)0 = 00 = 0$. The inverse semigroup B^0 is known as a *Brandt semigroup.*

Groups and semilattices are also perfectly good examples of inverse semigroups. In addition, it is easy to see that the idempotents of the bicyclic monoid are $\{x^a y^a : a \in \mathbb{N}^0\}$ and they commute. Therefore the bicyclic monoid is another example of an inverse monoid.

2.3 Generalisations of regular and inverse semigroups

As we will see, many natural generalisations of regular and inverse semigroups exist.

2.3.1 Abundant semigroups

Before we proceed, it will be important to first recall Green's equivalence relations $\mathscr{L}$ and $\mathscr{R}$ defined on any semigroup as seen in Section 1.2. We begin with a motivating example of a relation, defined for any semigroup, closely related to that of a Green's equivalence relation.

Definition 2.3.1 (Generalised Green's equivalence relations)**.** [29] Let S be a semigroup. The equivalence relation $\mathscr{L}^*$ on S is given by the rule that $(x, y) \in \mathscr{L}^*$ if and only if for any $a, b \in S^1$ we have

$$xa = xb \Longleftrightarrow ya = yb.$$

We define $\mathscr{R}^*$ dually and

$$\mathscr{H}^* = \mathscr{L}^* \cap \mathscr{R}^*.$$

These equivalence relations form the *Generalised Green's equivalence relations.*

Alternatively, an equivalent definition of the relation $\mathscr{L}^*$ on a semigroup S is to say that $(x, y) \in \mathscr{L}^*$ in S if and only if $(x, y) \in \mathscr{L}$ in some oversemigroup of S. That $\mathscr{L}^*$ is a right congruence on S follows precisely from $\mathscr{L}$ being a right congruence on an oversemigroup of S. Dually, one can argue that $\mathscr{R}^*$ is a left congruence on S.

Definition 2.3.2. [34, 30] A semigroup S is *left abundant* if every $\mathscr{R}^*$-class contains an idempotent. The definition of *right abundant* is dual. We say that S is *abundant* if it is both left abundant and right abundant.

Any regular semigroup is automatically abundant. Of course, many familiar examples of semigroups are not abundant: any semigroup S with $E = \emptyset$, such as the free semigroup on a non-empty set X, is certainly not abundant. On the other hand, the free monoid X^* is abundant, with a single idempotent.

Example 2.3.3 (Monoid of $n \times n$ integer matrices)**.** For an example of an abundant, non-regular monoid with a plethora of idempotents, we cite the monoid of $n \times n$ integer matrices $M_n(\mathbb{Z})$ [32]. In the case of $M_2(\mathbb{Z})$, we have

$$\begin{bmatrix} 0 & -1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ -1 & 1 \end{bmatrix} \in E\big(M_2(\mathbb{Z})\big)$$

however

$$\begin{bmatrix} 0 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ -1 & 1 \end{bmatrix} \neq \begin{bmatrix} 0 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 0 & 1 \end{bmatrix}.$$

One can easily extend the 2×2 idempotent matrices above to $n \times n$ idempotents in $M_n(\mathbb{Z})$ which do not commute.

2.3.2 Adequate semigroups

It is straightforward to see that semilattices are abundant semigroups since they are regular. That being said, semilattices satisfy the additional condition that idempotent elements in a semilattice commute: by virtue of how they are defined. This special property is shared by other well-known classes of abundant semigroups and consequently given a separate name.

Definition 2.3.4. [29] We say S is *left adequate* if S is left abundant and the idempotents of S commute. A *right adequate* semigroup is defined dually. We say that S is *adequate* if it is both left and right adequate.

While it is clear from Definition 2.3.4 that every adequate semigroup is abundant, there are examples of semigroups that are abundant yet not adequate: for instance the free left adequate semigroup as described by Kambites [61]. Any regular semigroup in which idempotents do not commute is not adequate. It is worthwhile mentioning here that adequate semigroups are to abundant as inverse semigroups are to regular semigroups.

2.3.3 Ample semigroups

So-called ample semigroups can be defined in a couple of different ways and we begin this subsection by describing these.

Definition 2.3.5. [23, 32] A semigroup S is *left ample* if S is left adequate and

$$S^1x \cap S^1e = S^1xe$$

for every $x \in S$ and $e \in E$. Dually, we form the definition of a *right ample semigroup*. We say that S is *ample* if S is left ample and right ample.

Ample semigroups were initially referred to as *type A* semigroups: the exact etymology behind this terminology is unclear (we refer the reader to wiser minds and semigroup folklore).

Definition 2.3.6. [23, 32] Let S be a semigroup such that F is a commutative subsemigroup of idempotents of S and let $^+ : S \to F$ be a unary operation. Then S satisfies the *left ample condition with respect to* F if the following condition holds

$$xf = (xf)^+x$$

for every $x \in S$ and $f \in F$. The *right ample condition with respect* F is defined dually.

In the case where S exhibits the *left ample condition with respect to* $E(S)$, then S is left ample as in Definition 2.3.5 (dually for right ample). If M is any submonoid of a symmetric inverse monoid, closed under $\alpha \mapsto \alpha^+ = \alpha\alpha^{-1}$, then M is left ample. Ample semigroups are always adequate by the very nature of how ample semigroups are defined.

Example 2.3.7 (Inverse semigroups)**.** As inverse semigroups are regular we always have that $\mathscr{R} = \mathscr{R}^*$ on any inverse semigroup S. Using Proposition 2.2.5, every $\mathscr{R}$-class contains a unique idempotent and so we set $x^+ = xx^{-1}$ to be the unique idempotent in the $\mathscr{R}$-class of x.

For every $x \in S$ and $e \in E$ we see that

$$\begin{aligned} xe &= (xx^{-1}x)e = x(x^{-1}x)e = xe(x^{-1}x) = xe^2(x^{-1}x) \\ &= x(ee^{-1})(x^{-1}x) = (xe)(e^{-1}x^{-1})x = (xe)(xe)^{-1}x = (xe)^+x \end{aligned}$$

and so S satisfies the left ample condition.

Example 2.3.8 (Right cancellative monoids)**.** Let M be a monoid with the property that

$$mp = np \Longrightarrow m = n$$

for every $m, n, p \in M$. Such a monoid is called *right cancellative.* It is easy to see that in a right cancellative monoid M, the identity is the only idempotent and that every element is $\mathscr{R}^*$-related to the identity. We have $a^+ = 1$ for every $a \in M$ and it is then clear that M is left ample. Conversely, any one idempotent right ample monoid must be right cancellative.

Chapter 3

Semigroup presentations

We begin in Section 3.1 by explaining what is meant by a semigroup presentation, as well as presentations for groups and monoids. In Section 3.2 we cover Adian presentations and their important connection with cancellative properties. Last, in Section 3.3, we provide a definition of rewriting systems, confluence and normal forms.

3.1 Basic definitions

Recall the free semigroup X^+ on X as described in Example 1.1.9. One can form the *free monoid* X^* *on* X by adjoining an identity element (normally denoted by ϵ or 1) to X^+. With regards to X^+ and X^*, we refer to the set X as an *alphabet*; the elements of which we call *letters*. We call an element $w \in X^+$ a word and let $|w|_x$ be the number of times the letter x appears in the word w (counting repeats). The *context* of a word $w \in X^+$, denoted by $c(w)$, is the set

$$c(w) = \{x_1, \ldots, x_n : w = x_1 \ldots x_n \text{ where } x_1, \ldots, x_n \in X\}.$$

For any binary relation R on X^+, we let $R^\sharp$ be the congruence on X^+ given by the rule that $(x, y) \in R^\sharp$ if and only if $x = y$ or there exists some $n \in \mathbb{N}$

such that

$$x = a_1 x_1 b_1,\ a_1 y_1 b_1 = a_2 x_2 b_2,\ \ldots,\ a_n y_n b_n = y \tag{3.1}$$

where $a_i, b_i \in X^*$ and $(x_i, y_i) \in R \cup R^{-1}$ for every $1 \leqslant i \leqslant n$. We will refer to such a sequence 3.1 as an *R-sequence* where the pair $(a_i x_i b_i, a_i y_i b_i)$ is an *elementary R-transition* for every $1 \leqslant i \leqslant n$. We remark that $R^\sharp$ is the smallest congruence on X^+ containing R. We define the congruence $R^\sharp$ on X^* in precisely the same way.

Definition 3.1.1. Let X be a non-empty set and let R be a binary relation on X^+. A *semigroup presentation*, denoted by $\mathrm{Sgp}\langle X : R\rangle$, defines the quotient semigroup $X^+/R^\sharp$. If R is a binary relation on X^* then a *monoid presentation*, denoted by $\mathrm{Mon}\langle X : R\rangle$, defines the quotient monoid $X^*/R^\sharp$.

Throughout this thesis, we will identify the semigroup (monoid) presentations with the semigroups (monoids) which they define. Where it is clear that a given presentation is for a semigroup or monoid, we will write $\langle X : R\rangle$ instead of $\mathrm{Sgp}\langle X; R\rangle$ or $\mathrm{Mon}\langle X : R\rangle$ respectively. Where $X = \{x_i : i \in I\}$, for some index I, we will often write

$$\big\langle x_i\,(i \in I) : R\big\rangle$$

in the place of $\langle X : R\rangle$. In the special case where I is finite, say $I = \{1, \ldots, n\}$, we use $\langle x_1, \ldots, x_n : R\rangle$ to denote $\langle X : R\rangle$ instead. Similarly, for an index J, we will sometimes use

$$\big\langle X : u_j = v_j\,(j \in J)\big\rangle$$

instead of $\langle X : R\rangle$ where R is the set of pairs (u_j, v_j), $j \in J$. In this convention, we are using $u = v$ in the place of $(u, v) \in R$. If R is a finite set, say $J = \{1, \ldots, m\}$, then we write $\langle X : u_1 = v_1, \ldots, u_m = v_m\rangle$ instead. We may use both of these conventions simultaneously.

Definition 3.1.2. A semigroup S *has a presentation $\langle X : R\rangle$ (via ϕ)* if there exists a surjective homomorphism $\phi : X^+ \to S$ such that $\ker\phi = R^\sharp$.

A monoid M *has presentation* $\langle X : R\rangle$ *via* ϕ if there exists a surjective homomorphism $\psi : X^* \to M$ such that $\ker \phi = R^\sharp$.

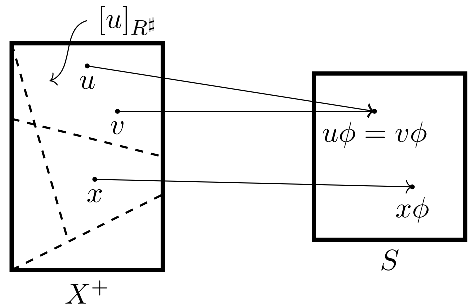

Figure 3.1: $R^\sharp$-related words in X^+ map to equal elements in S under ϕ.

Explicitly, this means that $u\phi = v\phi \in S$ if and only if $[u]_{R^\sharp} = [v]_{R^\sharp}$ for $u, v \in X^+$, that is, S is isomorphic to $\mathrm{Sgp}\langle X : R\rangle$ as in Definition 3.1.1. Certainly, every semigroup S has a presentation $\langle X : R\rangle$ via ϕ, where $X = S$, we take $\phi : X \to S$ as the identity map and

$$R = \big\{(x_1 \dots x_n, x) \in X^+ \times X : x_1 \dots x_n = x \in S\big\}.$$

A diagram illustrating Definition 3.1.2 is given in Figure 3.1.

Definition 3.1.3. A semigroup S is *finitely presented* if it has a presentation $\langle X : R\rangle$ such that X and R are finite.

In the case where $|x| = |y|$ for every $(x, y) \in R$, we say that the resulting presentation is *homogeneous*. The question as to whether or not a given semigroup can be finitely presented is still very much an open problem: special cases such as finite presentability of semidirect products of inverse semigroups [19] and finite presentability of HNN-extensions of inverse semigroups [18] have been studied. The following result concerning quotient semigroups follows from standard homomorphism theorems but we provide a proof for completeness.

Proposition 3.1.4. *Let X be a non-empty set and let P and Q be binary relations on X^+. Then $X^+/(P \cup Q)^\sharp \simeq (X^+/P^\sharp)/R^\sharp$ where R is the binary relation on $X^+/P^\sharp$ given by the rule that*

$$R = \left\{ \left([x]_{P^\sharp}, [y]_{P^\sharp} \right) : (x, y) \in Q \right\}.$$

Proof. Let X be a non-empty set and let P, Q and R be binary relations as above. We will use $[x]$ to denote the $P^\sharp$-class of $x \in X^+$ and $[[x]]$ to denote the $R^\sharp$-class of $[x]$. Define a map ϕ given by the rule

$$[[x]]\phi = [x]_{(P \cup Q)^\sharp}$$

for every $x \in X^+$.

Suppose that $x, y \in X^+$ such that $[[x]] = [[y]]$. This implies that either $[x] = [y]$ or there is a finite sequence of the form

$$[x] = [c_1][x_1][d_1],\ [c_1][y_1][d_1] = [c_2][x_2][d_2],\ \ldots,\ [c_m][y_m][d_m] = [y] \tag{3.2}$$

where $c_i, d_i \in X^+$ and $([x_i], [y_i]) \in R \cup R^{-1}$ for every $1 \leqslant i \leqslant m$. If $[x] = [y]$ then clearly $(x, y) \in (P \cup Q)^\sharp$ so we suppose that a sequence 3.2 exists instead. This means that there exists $r_0, \ldots, r_m \in \mathbb{N}$ such that

$$\begin{aligned} x =& z_1^{(0)},\ \ldots,\ z_{r_0}^{(0)} = c_1 x_1 d_1, \\ c_i y_i d_i =& z_1^{(i)},\ \ldots,\ z_{r_i}^{(i)} = c_{i+1}^{(i)} x_{i+1}^{(i)} d_{i+1}^{(i)}, \\ c_m y_m d_m =& z_1^{(m)},\ \ldots,\ z_{r_m}^{(m)} = y \end{aligned}$$

where $z_j^{(\ell)} = h_j^{(\ell)} p_j^{(\ell)} k_j^{(\ell)}$ and $z_{j+1}^{(\ell)} = h_j^{(\ell)} q_j^{(\ell)} k_j^{(\ell)}$ where $h_j^{(\ell)}, k_j^{(\ell)} \in X^+$ and $(p_j^{(\ell)}, q_j^{(\ell)}) \in P \cup P^{-1}$ for every $0 \leqslant \ell \leqslant m$ and $1 \leqslant j \leqslant r_\ell - 1$. As we have $(x_i, y_i) \in Q \cup Q^{-1}$ for every $1 \leqslant i \leqslant m$, it follows that $(x, y) \in (P \cup Q)^\sharp$. Therefore ϕ is a well-defined map as $[[x]]\phi = [[y]]\phi$.

Clearly ϕ is a surjective homomorphism. To show that ϕ is one-to-one, we see that if $[[x]]\phi = [[y]]\phi$ then either $x = y$ or there exists some $n \in \mathbb{N}$

such that

$$x = a_1 x_1 b_1,\ a_1 y_1 b_1 = a_2 x_2 b_2,\ \ldots,\ a_n y_n b_n = y \tag{3.3}$$

where $a_i, b_i \in X^*$ and $(x_i, y_i) \in (P \cup Q) \cup (P \cup Q)^{-1}$ for every $1 \leqslant i \leqslant n$. If $x = y$ then we are done, so we suppose instead that such a finite sequence exists. It follows directly from Equation 3.3 that

$$[x] = [a_1 x_1 b_1],\ [a_1 y_1 b_1] = [a_2 x_2 b_2],\ \ldots,\ [a_n y_n b_n] = [y].$$

If $(x_i, y_i) \in P \cup P^{-1}$ then $[x_i] = [y_i]$ and so $[[x_i]] = [[y_i]]$. On the other hand, if $(x_i, y_i) \in Q$ then $[x_i]$ is R-related to $[y_i]$ by definition. Dually in the case that $(x_i, y_i) \in Q^{-1}$. With this in mind, we have from Equation 3.3 that

$$[[x]] = [[a_1 x_1 b_1]] = [[a_1 y_1 b_1]] = [[a_2 x_2 b_2]] = \ldots = [[a_n y_n b_n]] = [[y]]$$

where $a_i, b_i \in X^*$ and

$$\big([x_i], [y_i]\big) \in R \cup R^{-1}$$

for every $1 \leqslant i \leqslant n$. This gives us that $(x, y) \in R^\sharp$ and so $[[x]] = [[y]]$. Hence ϕ is an isomorphism. □

We will write $\mathcal{C}X^+$ to denote the quotient semigroup $X^+/P^\sharp$ where

$$P = \big\{(xy, yx) : x, y \in X\big\}.$$

Definition 3.1.5. A *commutative semigroup presentation*, denoted $\langle \mathcal{C}X : R\rangle$, defines the quotient semigroup $\mathcal{C}X^+/R^\sharp$.

In light of Proposition 3.1.4, it is clear that $\langle \mathcal{C}X : R\rangle$ and $\langle X : P \cup Q\rangle$ define isomorphic quotient semigroups.

3.2 Adian presentations

We begin this section with a brief introduction to graphs. We introduce special kinds of graphs that can be used to understand cancellative properties of semigroups (from a given presentation).

Definition 3.2.1. A *(undirected) graph* (V, E) consists of a set of *vertices* V together with a set of *edges* E between the vertices.

We represent edges as a binary relation where $(a, b) \in E$ means there is an edge between the vertices a and b. Of course, in this way, we identify the pairs (a, b) and (b, a) in E. Often it will be more helpful to represent graphs visually. That is to say, a *graph drawing* of a graph (V, E) is a planar diagram obtained by drawing a line between distinct vertices x and y if and only if $(x, y) \in E$.

Definition 3.2.2. A *cycle* within a graph (V, E) is a finite sequence of distinct vertices $v_1, \ldots, v_n \in V$ with $n \geqslant 3$ such that

$$(v_1, v_2), (v_2, v_3), \ldots, (v_n, v_1) \in E.$$

We will denote such a cycle by $[v_1, \ldots, v_n]$. If a graph does not contain a cycle, it is said to be *cycle-free*.

Notice that in Definition 3.2.2, since $v_1, \ldots, v_n \in V$ are distinct, we are not regarding $[x]$ or $[x, y]$ as a cycle for any $(x, x), (x, y) \in E$. Given a cycle in (V, E) of the form $[v_1, \ldots, v_n]$, it follows $[v_i, \ldots, v_n, v_1, \ldots, v_{i-1}]$ is also a cycle in (V, E) for every $1 \leqslant i \leqslant n$.

Example 3.2.3. Let (V, E) be the graph given by $V = \{x, y, z, a, b\}$ and

$$E = \big\{(x, y), (x, z), (x, a), (y, z), (z, a), (a, a)\big\}.$$

Then (V, E) is not cycle-free since $[x, y, z]$, $[x, z, a]$ and $[x, y, z, a]$ are all examples of cycles. Conversely if

$$F = E \setminus \{(x, y), (x, z)\}$$

then (V, F) is a cycle-free graph. The graph drawing of (V, E) and (V, F) is given below in Figure 3.2.

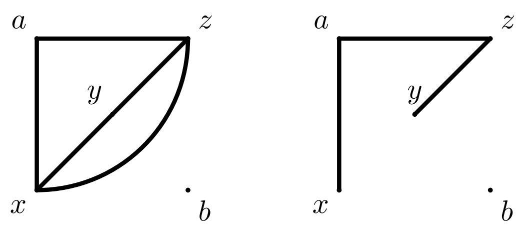

Figure 3.2: Graph drawing of (V, E) and (V, F) (from left to right).

Definition 3.2.4. [1, 3] Let $\mathrm{Sgp}\langle X : R\rangle$ be any fixed semigroup presentation. The *left Adian graph* of $\mathrm{Sgp}\langle X : R\rangle$ is the graph (X, E) where E is given by

$$E = \{(x, y) \in X \times X : (xa, yb) \in R \cup R^{-1} \text{ for some } a, b \in X^*\}.$$

Dually we can define a *right Adian graph* of some fixed semigroup presentation.

Suppose for a semigroup presentation $\langle X : R\rangle$ that $(x, y) \in R$ where $x, y \in X$. Then it follows immediately from Definition 3.2.4 that $(x, y) \in E \cap F$ where (X, E) and (X, F) represent the left and right Adian graphs of $\langle X : R\rangle$ respectively.

Theorem 3.2.5. *[1, 3, 58] Let S be a semigroup that has a presentation* $\mathrm{Sgp}\langle X : R\rangle$. *If the left Adian graph and right Adian graph of* $\mathrm{Sgp}\langle X : R\rangle$ *are both cycle-free then S can be embedded into a group.*

We highlight Theorem 3.2.5 in the following example.

Example 3.2.6. Consider the semigroup S that has semigroup presentation

$$\langle x, y, z : x^2 = yx, xz = zx, yz = yz^2.\rangle$$

The left and right Adian graphs are both cycle-free and so S can be embedded into a group. In contrast, if we were to adjoin the relation $yx = z^2$ into this presentation, then the left Adian graph of the resulting presentation would permit the cycle $[x, y, z]$. Thus we would not be able to deduce

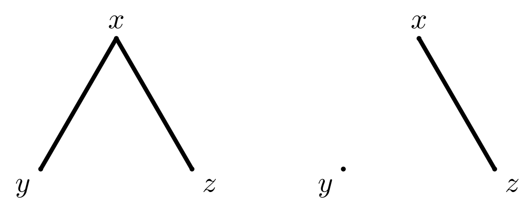

Figure 3.3: Graph drawing of a left and right Adian graph (from left to right) used in Example 3.2.6.

that the semigroup presentation defines a cancellative semigroup using Theorem 3.2.5. One can show (in this alternate case) that S cannot be embedded into a group since

$$[yx^2] = [z^2x] = [xz^2] = [x^3]$$

but $[x] \neq [y]$. The Adian graphs of this semigroup presentation are represented by graph drawings are shown above.

Adian's Theorem will be crucial in helping us prove some of the main results of Chapter 5.

3.3 Rewriting systems and normal forms

We conclude this chapter by providing a brief overview of rewriting systems and their applications with regards to semigroup presentations.

Definition 3.3.1. [50, 65] Let A be a non-empty set. A *rewriting system* R on A is a subset of $A \times A$ where the elements of R are called *rewriting rules*.

While rewriting rules are defined as ordered pairs of the form $(a, b) \in A \times A$, they are more commonly denoted by $a \to b$ for $a, b \in A$. We will proceed by adopting this convention.

Definition 3.3.2. [50, 65] Let X be a set and R be a rewriting system on X^+. A *single-step reduction relation* is the binary relation on X^+ denoted by $\xrightarrow{R}$ and defined by the rule that $x \xrightarrow{R} y$ for $x, y \in X^+$ if there exists $a, b, c, d \in X^*$ such that $a \to b \in R$, $x = cad$ and $y = cbd$.

Definition 3.3.3. [50, 65] Let X be a set and R be a rewriting system on X^+. A *reduction relation* on X^+ is the binary relation on X^+ denoted by $\xrightarrow{*,R}$ where $x \xrightarrow{*,R} y$ if $x = y$ or there exists $n \in \mathbb{N}$ such that

$$x = z_1 \xrightarrow{R} z_2 \xrightarrow{R} \cdots \xrightarrow{R} z_n = y$$

for some $z_i \in X^+$, $1 \leqslant i \leqslant n$.

The relation $\xrightarrow{*,R}$ is known as the *transitive and reflexive closure of* $\xrightarrow{R}$.

Definition 3.3.4. [50] A rewriting system R is *noetherian* if every sequence of the form

$$a_1 \xrightarrow{R} a_2 \xrightarrow{R} \cdots$$

is finite.

We give an example of a noetherian rewriting system on a free semigroup in the following example

Example 3.3.5. Let $X = \{x, y\}$ and R be a rewriting system on X^+ given by

$$R = \big\{xy \to x,\ yx \to x,\ x^2 \to x\big\}.$$

It is straightforward to see that R is a noetherian rewriting system since every rewriting rule in R is of the form $u \to v$ with $|v| < |u|$.

We give a name to an important property of rewriting systems.

Definition 3.3.6. [50] Let X be a set and R be a rewriting system on X^+. We say R is *locally confluent* if for every $x, y, z \in X^+$ with $x \xrightarrow{R} y$ and $x \xrightarrow{R} z$ there exists some $a \in X^+$ such that $y \xrightarrow{*,R} a$ and $z \xrightarrow{*,R} a$. We say R is *confluent* if for every $x, y, z \in X^+$ with $x \xrightarrow{*,R} y$ and $x \xrightarrow{*,R} z$ there exists some $a \in X^+$ such that $y \xrightarrow{*,R} a$ and $z \xrightarrow{*,R} a$.

It is immediate that any confluent rewriting system is locally confluent. It is known that one can obtain a partial converse to this statement in the case of noetherian rewriting systems.

Theorem 3.3.7. *[8] Let X be a set and R be a rewriting system on X^+. If R is noetherian then R is confluent if and only R is locally confluent.*

Given a noetherian and confluent rewriting system, we can identify those elements for which sequences of single-step reductions eventually terminate.

Definition 3.3.8. Let X be a set and R be a rewriting sytem on X^+. An element $x \in X^+$ is in *normal form* (or *irreducible*) if there exists no $y \in X^+$ such that $x \xrightarrow{*,R} y$.

Words in normal form become particularly useful when considering semigroup presentations. In particular, they can help us to identify when two words are equal in the free semigroup sense. The next result is known however we prove it for completeness.

Theorem 3.3.9. *Let X be a set and R be a rewriting system on X^+. If R is noetherian and confluent then the relation $\sim$ given by*

$$x \sim y \Longleftrightarrow x \xrightarrow{*,R} z \text{ and } y \xrightarrow{*,R} z \text{ for some } z \in X^+$$

for $x, y \in X^+$ is an equivalence relation. In addition, every $\sim$-class contains a unique word in normal form.

Proof. Let X be a set and let R be a noetherian and confluent rewriting system on X^+. By definition it is true that $x \xrightarrow{*,R} x$ for every $x \in X^+$ and so $\sim$ is reflexive. It is clear that $\sim$ is symmetric. If $x \sim y$ and $y \sim z$ then there exists some $a, b \in X^+$ such that

$$x \xrightarrow{*,R} a,\ y \xrightarrow{*,R} a,\ y \xrightarrow{*,R} b \text{ and } z \xrightarrow{*,R} b.$$

Since R is confluent, there exists some $c \in X^+$ such that $a \xrightarrow{*,R} c$ and $b \xrightarrow{*,R} c$. As $\xrightarrow{*,R}$ is transitive, this implies that $x \xrightarrow{*,R} c$ and $z \xrightarrow{*,R} c$. Therefore $x \sim z$ which means $\sim$ is transitive. Hence $\sim$ is an equivalence relation.

That every $\sim$-class of X^+ has a word of normal form follows from the fact that R is noetherian. $\square$

Let $\langle X : R \rangle$ be a semigroup presentation and suppose we consider R as a rewriting system on X^+. If one can show that R is noetherian and locally confluent then it follows that we can find a unique word in normal form in every $\sim$-class. Two words in normal form are equal in the free semigroup if and only if their corresponding $R^\sharp$-classes are equal. This is a technique we employ later in Chapter 5 to prove some cancellativity properties.

Chapter 4

Coherency and noetherianity

In Section 4.1, we explain what is meant by an S-act for a monoid S. Following that, we give the definition of a (left, right) coherent monoid as well as a weakly (left, right) coherent monoid in Section 4.2. We end Chapter 4 by recalling the definition of a (left, right) noetherian semigroup alongside providing some examples in Section 4.3. We also introduce weakly (left, right) noetherian semigroups in addition to decribing the relationship between coherent monoids and noetherian monoids.

4.1 S-acts

It will be useful to first recall Definition 1.3.1 in which we define what it means for a semigroup to act on the left of another semigroup.

Definition 4.1.1. Let S be a semigroup. A *left S-act* is a set A together with a map $S \times A \to A$ given by $(x, a) \mapsto x \bullet a$ satisfying

$$x \bullet (y \bullet a) = (xy) \bullet a$$

for every $x \in S$ and $a \in A$. If S is a monoid then we also require that

$$1 \bullet a = a$$

is satisfied for every $a \in A$. *Right S-acts* are defined dually.

Simply put, Definition 4.1.1 says that if S is a semigroup that acts on the left of a set A, then A is a left S-act. It is for this reason that we return to our previous notation where, if A is a left S-act, we will write ${}^{x}a$ instead of $x \cdot a$ for every $x \in S$ and $a \in A$ (dually for a right S-act). For any fixed semigroup S, we can always consider the map $S \times S \to S$ given by ${}^{x}y = xy$ for every $x, y \in S$ to obtain a left S-act. It follows that the set of left or right S-acts is always non-empty as we may regard S as a left and right S-act of itself. Note that we will regard $\emptyset$ as being a left and right S-act for any semigroup S. We highlight a more interesting example of a right S-act below.

Example 4.1.2. Let X be a non-empty set and let S be the semigroup of all functions from X to itself under composition. We consider the map $\mathcal{P}(X) \times S \to \mathcal{P}(X)$ given by $U^{\alpha} = U\alpha$ for every $U \in \mathcal{P}(X)$ and $\alpha \in S$. Clearly we have that

$$(U^{\alpha})^{\beta} = (U\alpha)^{\beta} = (U\alpha)\beta = U(\alpha\beta) = U^{\alpha\beta}$$

for every $U \in \mathcal{P}(X)$ and $\alpha, \beta \in S$. Therefore $\mathcal{P}(X)$ with the map $U^{\alpha} = U\alpha$ can be regarded as a right S-act.

We provide an example of a right M-act below to help build a broader picture.

Example 4.1.3. Let X be a non-empty set and consider the map given by $\mathbb{N} \times X^{*} \to \mathbb{N}$ given by $n^{x} = n + |x|$ for all $n \in \mathbb{N}$ and $x \in X^{*}$. It follows that

$$\begin{aligned}(n^{x})^{y} &= \big(n + |x|\big)^{y} \\ &= \big(n + |x|\big) + |y| \\ &= n + \big(|x| + |y|\big) \\ &= n + |xy| \\ &= n^{xy}\end{aligned}$$

for every $n \in \mathbb{N}$ and $x, y \in X^{*}$. Lastly, it is straightforward to verify that $n^{\epsilon} = n + |\epsilon| = n$ for every $n \in \mathbb{N}$. Therefore, $\mathbb{N}$ can be considered as a right

X^*-act by setting $n^x = n + |x|$ as above.

Definition 4.1.4. Let A be a left S-act. Then B is a *left S-subact* of A if $B \subseteq A$ is such that ${}^x b \in B$ for every $b \in B$ and $x \in S$.

Notice that if B is a left S-subact of A, then it is a left S-act in its own right by Definition 4.1.1. In the context of Example 4.1.3, it is certainly true that $\{n : m \leqslant n\}$ is a right X^*-subact of $\mathbb{N}$ for any fixed $m \in \mathbb{N}$.

Definition 4.1.5. Let A be a left S-act. A *congruence* on A is an equivalence relation θ on A such that for every $a, b \in A$ and $x \in S$ we have $(a, b) \in \theta$ implies $({}^x a, {}^x b) \in \theta$. A *congruence* on a right S-act is defined in dual way.

Given a congruence θ on a left S-act A, we may form the *quotient left S-act*

$$A/\theta = \big\{[a] : a \in A\big\}$$

with ${}^x[a] = [{}^x a]$ for every $a \in A$ and $x \in S$. We define a *quotient right S-act* dually.

Definition 4.1.6. Let A be a left S-act. A congruence θ on A is *finitely generated* if it is the smallest congruence containing a given finite set of elements of $A \times A$. A left S-act A is *finitely generated* if $A = \emptyset$ or there exists $n \in \mathbb{N}$ such that

$$A = (S^1 \bullet a) \cup \cdots \cup (S^1 \bullet a_n)$$

where $S^1 \bullet a_i = \{{}^x a_i : x \in S^1\}$ and $a_i \in A$ for every $1 \leqslant i \leqslant n$.

Of course, with Definition 4.1.6 in mind, it is routine to verify that Δ_A is always finitely generated as a left and right congruence. For any right S-act, say A, and for every $a \in A$ we define $\mathfrak{r}(a)$ on S by

$$\mathfrak{r}(a) = \big\{(x, y) \in S \times S : a^x = a^y\big\}.$$

One can show that $\mathfrak{r}(a)$ is a right congruence on S as follows.

Lemma 4.1.7. *Let A be a right S-act and let $a \in A$. Then $\mathfrak{r}(a)$ is a right congruence on S.*

Proof. It is straightforward to show that $\mathfrak{r}(a)$ is an equivalence relation on S. For any $(x, y) \in \mathfrak{r}(a)$ and $z \in S$, we have that

$$a^{xz} = (a^x)^z = (a^y)^z = a^{yz}$$

and so $(xz, yz) \in \mathfrak{r}(a)$ as required. $\square$

Dually, for any left S-act A and for every $a \in A$, we define $\mathfrak{l}(a)$ on S in a similar way.

4.2 Coherent monoids

We begin by introducing the notion of coherency for monoids as well as a related weaker notion.

Definition 4.2.1. [40, 95] A monoid is *weakly right coherent* if every finitely generated right ideal is finitely presented. A monoid is *right coherent* if every finitely generated right M-subact of every finitely presented right M-act is finitely presented. Dually for (weakly) left coherent monoids. A monoid is *(weakly) coherent* if it is both (weakly) left and (weakly) right coherent.

For example, Gould, Hartmann and Ruškuc were able to show that free monoids are coherent [43]. Moreover, Gould and Hartmann proved that free left ample monoids are right coherent [42]. It is clear that any finite monoid will also be coherent. We illustrate an example more explicitly below.

Example 4.2.2 (Finitely presented groups)**.** Let G be a group. Then we know that every principal right or principal left ideal of G is G itself by Definition 1.1.26. Thus every finitely generated left or finitely generated right ideal of G is also equal to G. Since G is finitely presented it is immediate that G is weakly coherent.

We are interested in these notions here due to the result below.

Theorem 4.2.3. *[40, Corollary 3.3, 3.4] A monoid M is weakly right coherent if and only if for any $a, b \in M$ the right congruence $\mathfrak{r}(a)$ is finitely generated, and the right ideal $aM \cap bM$ is finitely generated. A monoid M is right coherent if and only if for any finitely generated right congruence θ on M and any $[a], [b] \in M/\theta$ we have the right congruence $\mathfrak{r}([a])$ is finitely generated and the M-subact $[a]M \cap [b]M$ of M/θ is finitely generated.*

This theorem may be regarded as analogous to that of Chase for rings [12].

4.3 Noetherian semigroups

In a similar way to Section 4.2, we begin by introducing the notion of noetherian and weakly noetherian semigroups.

Definition 4.3.1. [79] A semigroup S is *weakly right noetherian* if every right ideal of S is finitely generated. A semigroup S is *right noetherian* if every right congruence on S is finitely generated and *(weakly) noetherian* if it is both (weakly) left and (weakly) right noetherian.

Any right noetherian monoid is right coherent, but weakly right noetherian monoids need not even be weakly right coherent [40]. Certainly it is true any group is weakly right noetherian but is not necessarily right noetherian. In addition, every finite semigroup and every semigroup where $\mathscr{R}$ is precisely the universal congruence ω, is trivially weakly right noetherian. We conclude this section, and chapter, by providing another example of a weakly right noetherian semigroup.

Example 4.3.2. Let S be a semigroup and with exactly two distinct $\mathscr{R}$-classes, say $[x]$ and $[y]$ for some $x, y \in S$. Let I be the intersection of two finitely generated right ideal of S. As I is a right ideal in its own right, I is either empty, or the union of $\mathscr{R}$-classes. That is to say, I is either empty, $[x]$, $[y]$ or $[x] \cup [y]$. In each case I is a finitely generated right ideal.

It was remarked by Miller that any semigroup with finitely-many $\mathscr{R}$-classes is weakly noetherian [78]. Miller also showed that if S is a weakly right noetherian semigroup and R is a congruence on S then the quotient semigroup S/R is also weakly right noetherian [78]. A partial converse occurs: if a quotient semigroup S/R is weakly noetherian and $R \subseteq \mathscr{R}$ then S is weakly noetherian [78].

Chapter 5

Right ideal Howson semigroups

We organise this chapter as follows. In Section 5.1 we recall essential terminology and fundamental results. In Section 5.2 we provide examples of right ideal Howson semigroups, with a particular focus on bands and coherent monoids. For each variety of bands, we give an explicit presentation of a right ideal Howson band belonging to the variety, and show that the lattice of varieties of bands splits into two with regard to Clifford's condition. We show that any semigroup given by a commutative presentation with finite set of relations is (right) coherent, and hence certainly (right) ideal Howson. In Section 5.3 we explore a number of closure results for the classes of right ideal Howson monoids and semigroups. We show that both the classes of right ideal Howson semigroups and right ideal Howson monoids are closed under free products. Right ideal Howson semigroups are not closed under direct products but, on the other hand, right ideal Howson monoids are closed under direct but not semidirect products. Finally, in Section 5.4, we consider a number of semigroup presentations, reflecting those given for bands in Section 5.2. We give presentations of right ideal Howson semigroups (which are also cancellative), commutative (right) ideal Howson semigroups and commutative cancellative (right) ideal Howson semigroups, all of which are universal in a given sense.

5.1 Basic definitions

We begin by defining the semigroups of which this chapter is concerned with.

Definition 5.1.1. A semigroup S is *right (left) ideal Howson* if the intersection of any two finitely generated right (left) ideals of S is finitely generated.

As intersection distributes over union, it is easy to see that the following lemma holds.

Lemma 5.1.2. *A semigroup is right (left) ideal Howson if and only if the intersection of principal right (left) ideals is finitely generated.*

Our next observations will be useful in what follows.

Lemma 5.1.3. *Let S be a semigroup such that for all $x, y \in S$ we have*

$$xS \cap yS = x_1S \cup \cdots \cup x_nS$$

where $x_i \in xS^1 \cap yS^1$ for all $1 \leqslant i \leqslant n$. Then S is right ideal Howson.

Proof. Let S be as given. Then, for any $x, y \in S$, we may write

$$xS^1 \cap yS^1 = \big(\{x\} \cap \{y\}\big) \cup \big(\{x\} \cap yS\big) \cup \big(xS \cap \{y\}\big) \cup (xS \cap yS).$$

Clearly, if there exists $z \in S^1$ such that $xz = y$ then $xS^1 \cap yS^1 = xS^1$ (the case where there exists $z \in S^1$ with $yz = x$ is entirely dual). However, if such a $z \in S^1$ does not exist (in either case) then we have by the above that $xS^1 \cap yS^1 = xS \cap yS$. Therefore, for such an $x, y \in S$, we have

$$xS^1 \cap yS^1 = x_1S \cup \cdots \cup x_nS \subseteq x_1S^1 \cup \cdots \cup x_nS^1 \subseteq xS^1 \cap yS^1$$

which gives us that $xS^1 \cap yS^1 = x_1S^1 \cup \cdots \cup x_nS^1$ as required. $\square$

Semigroups satisfying the hypothesis of Lemma 5.1.3 are called *finitely aligned* in [25]. However, as we show in Remark 5.4.9, a right ideal Howson semigroup need not be finitely aligned. For semigroups that are *right factorisable*, that is, semigroups S such that $xS = xS^1$ for any $x \in S$, the two notions coincide. This is the situation for monoids, inverse semigroups and bands, for example, but not for free semigroups.

The next observation we make follows quickly from the definition of a semigroup being right ideal Howson.

Lemma 5.1.4. *A semigroup S is right ideal Howson if and only if the monoid S^1 is right ideal Howson.*

We will say a right ideal I of S is *exactly n-generated* for some $n \in \mathbb{N}^0$, if there are n elements of S that generate I, but no $n-1$ elements will suffice. With this in mind, we note that $I = \emptyset$ if and only if I is exactly 0-generated.

Definition 5.1.5. A semigroup S satisfies (Rn) for $n \in \mathbb{N}^0$ if there exists some $x, y \in S$ such that $xS^1 \cap yS^1$ is exactly n-generated. The condition (Ln) is defined dually.

Example 5.1.6. Let X^+ be the free semigroup over X where $|X| > 1$. Then it follows that $xX^+ \cap yX^+ = \emptyset$ if and only if $x \neq y$, otherwise the intersection is principal (or exactly 1-generated). Therefore X^+ satisfies (Rn) if and only if $n = 1$.

From Lemma 5.1.4, one may derive another example, similar to that of Example 5.1.6, regarding free monoids.

5.2 Examples

Now we provide some natural examples of right ideal Howson semigroups. It will be helpful, in what follows, to recall the notions of coherency and noetherianity (as applied to monoids and semigroups respectively) as seen in Chapter 4.

5.2.1 Coherency and noetherianity

Any finite semigroup, (weakly) right noetherian semigroup or semigroup S such that S^1 is (weakly) right coherent is right ideal Howson: this is clear from the material covered in Chapter 4. There are many examples of right coherent monoids [45, 40]. Rédei's Theorem states that the free commutative monoid X^*, for any finite set X, is (right) noetherian [86]. Gould used this result to show that for any set X, the free commutative monoid $\mathcal{C}X^*$ is (right) coherent [40, Theorem 4.3]. Our next result is a significant extension of the latter fact. The proof makes essential use of the fact that for a commutative monoid, right congruences and congruences coincide.

Theorem 5.2.1. *Let M be a monoid given by a commutative presentation $\langle \mathcal{C}X : R\rangle$, where R is finite. Then M is (right) coherent.*

Proof. Let $\rho^\sharp$ be a finitely generated congruence on $M = \mathcal{C}X^*/R^\sharp$ where

$$\rho = \Big\{ \big([a]_R, [b]_R\big) : (a, b) \in \sigma \Big\}$$

for some finite subset σ of $\mathcal{C}X^* \times \mathcal{C}X^*$. We let $\nu = R \cup \sigma$ so that $\nu^\sharp$ is a finitely generated congruence on $\mathcal{C}X^*$. For convenience, we will let $[w]$ denote the $R^\sharp$-class of $w \in \mathcal{C}X^*$ and by $[[w]]$ the $\rho^\sharp$-class of $[w] \in M$. Taking care with the generators, it follows from the second isomorphism theorem [56, Theorem 1.5.4] that there exists a monoid isomorphism

$$\theta : M/\rho^\sharp \to \mathcal{C}X^*/\nu^\sharp : [[w]]\theta = [w]_\nu.$$

for every $w \in \mathcal{C}X^*$. We are considering the monoid M acting on the right of the M-act $M/\rho^\sharp$. To this end, note that for $[u] \in M$ and $[[w]] \in M/\rho^\sharp$ we have

$$[[w]][u] = [[w][u]] = [[wu]].$$

Similarly, $\mathcal{C}X^*$ acts on $\mathcal{C}X^*/\nu^\sharp$ by $[w]_\nu u = [wu]_\nu$.

Proposition 5.2.2. *Let $[[w]] \in M/\rho^\sharp$ and $[u], [v] \in M$, where $w, u, v \in \mathcal{C}X^*$. Then*

$$\big([u], [v]\big) \in \mathfrak{r}\big([[w]]\big) \Longleftrightarrow (u, v) \in \mathfrak{r}\big([w]_\nu\big).$$

Proof. Let $w, u, v \in \mathcal{C}X^*$. From the remarks above we have

$$\begin{aligned}
\big([u],[v]\big) \in \mathfrak{r}\big([[w]]\big) &\iff [[wu]] = [[wv]] \\
&\iff [wu]_\nu = [wv]_\nu \\
&\iff [w]_\nu u = [w]_\nu v \\
&\iff (u,v) \in \mathfrak{r}\big([w]_\nu\big)
\end{aligned}$$

and this concludes the proof. □

Let $w \in \mathcal{C}X^*$. Since ν is finite and $\mathcal{C}X^*$ is coherent, it follows that there exists a finite set of generators κ for $\mathfrak{r}([w]_\nu)$, say

$$\kappa = \big\{(u_i, v_i) : 1 \leqslant i \leqslant n\big\}$$

for some $n \in \mathbb{N}^0$ (where we will suppose for convenience that $\kappa = \kappa^{-1}$). We define

$$\eta = \left\{\big([u_i],[v_i]\big) : 1 \leqslant i \leqslant n\right\}$$

so that η is also finite and symmetric.

Proposition 5.2.3. *If $w \in \mathcal{C}X^*$ then, with the notation above, we have that*

$$\mathfrak{r}\big([[w]]\big) = \langle \eta \rangle.$$

Proof. That $\eta \subseteq \mathfrak{r}([[w]])$ follows from Proposition 5.2.2.

Suppose for the converse that $([u],[v]) \in \mathfrak{r}([[w]])$, so that $(u,v) \in \mathfrak{r}([w]_\nu)$, again by Proposition 5.2.2. Since κ generates $\mathfrak{r}([w]_\nu)$ there exists $\ell \in \mathbb{N}^0$ and a finite sequence of the form

$$u = z_0,\ z_1,\ \ldots,\ z_\ell = v$$

where $z_{i-1} = c_i p_i$ and $z_i = c_i q_i$ with $(p_i, q_i) \in \kappa$ and $c_i \in \mathcal{C}X^*$ for all $1 \leqslant i \leqslant \ell$. It follows that there is a sequence

$$[u]_R = [z_0]_R,\ [z_1]_R,\ \ldots,\ [z_\ell]_R = [v]_R$$

where $[z_{i-1}] = [c_i p_i] = [c_i][p_i]$ and $[z_i] = [c_i q_i] = [c_i][q_i]$ with $([p_i], [q_i]) \in \eta$ for all $1 \leqslant i \leqslant \ell$. Thus $\mathfrak{r}([[w]]) \subseteq \langle \eta \rangle$ and we therefore have equality. $\square$

Let $a, b \in \mathcal{C}X^*$ and let I and J denote the intersections

$$[[a]]M \cap [[b]]M \text{ and } [a]_\nu \mathcal{C}X^* \cap [b]_\nu \mathcal{C}X^*$$

respectively. We proceed to show that I is finitely generated.

Proposition 5.2.4. *If $w \in \mathcal{C}X^*$ then $[[w]] \in I \iff [w]_\nu \in J$.*

Proof. Suppose that $w, u, v \in \mathcal{C}X^*$. If $[[w]] = [[a]][u] = [[b]][v]$ then

$$\begin{aligned} [[a]][u] = [[b]][v] \quad &\iff [[au]] = [[bv]] \\ &\iff [au]_\nu = [bv]_\nu \\ &\iff [a]_\nu u = [b]_\nu v \end{aligned}$$

and so $[w]_\nu \in J$ as required. $\square$

Since ν is finitely generated, and $\mathcal{C}X^*$ is coherent, J is generated by a set

$$\left\{[ac]_\nu : c \in Y\right\}$$

for some finite subset Y of X^*. Proposition 5.2.4 then gives us that I is finitely generated by the set

$$\left\{[[ac]] : c \in Y\right\}$$

and this completes the proof of Theorem 5.2.1. $\square$

Using the fact that right coherent monoids are right ideal Howson and Theorem 5.2.1 we immediately deduce the following.

Corollary 5.2.5. *Let M be a monoid given by a commutative presentation $\langle X : R\rangle$, where R is finite. Then M is (right) ideal Howson.*

In particular, any finitely presented commutative semigroup or monoid is (right) ideal Howson. Changing tack, the following is well known.

Lemma 5.2.6. *Let S be an inverse semigroup. Then S is right ideal Howson, and S satisfies (Rn) if and only if $n = 1$.*

Proof. For any $a, b \in S$ we have $aS = aa^{-1}S$ and $bS = bb^{-1}S$. Since idempotents commute it follows that

$$aS \cap bS = aa^{-1}S \cap bb^{-1}S = aa^{-1}bb^{-1}S.$$

It follows from the definition that S is right ideal Howson. $\square$

5.2.2 Varieties of bands

As the free band on a finite set is finite, any finitely generated band is finite [56]. Hence, every finitely generated band is right and left ideal Howson.

Definition 5.2.7. [56] Let X be a countable set. For some $x, y \in X^+$, we say that a semigroup S satisfies an *identity* $x = y$ if for every choice of homomorphism $\theta : X^+ \to S$ we have $x\theta = y\theta$. A *semigroup variety* denotes a class of semigroups containing precisely all semigroups that satisfy a given collection of identities.

Throughout we let $\mathfrak{V}$ be a variety of bands. We recall the *variety of right regular bands* $\mathfrak{RR}$ (*variety of rectangular bands* $\mathfrak{RB}$) is determined by the identities $x^2 = x$ and $xy = yxy$ ($x^2 = x$ and $x = xyx$) (in addition to the identity guaranteeing associativity).

Theorem 5.2.8. *Let $\mathfrak{V}$ be a variety of bands. If $\mathfrak{V} \subseteq \mathfrak{RR}$ or $\mathfrak{V} \subseteq \mathfrak{RB}$, then every band $B \in \mathfrak{V}$ is right ideal Howson. Further, in this case B satisfies (Rn) if and only if $n = 1$, that is, B satisfies Clifford's condition.*

Proof. Suppose that $B \in \mathfrak{V}$. Since B is a band, we note that for any $a, b \in B$ we have $aB \cap bB \subseteq abB$. If $B \in \mathfrak{RR}$ then we have $ab \in aB$ and $ab = bab \in bB$, so that in this case $aB \cap bB = abB$ is principal. On the other hand, if $B \in \mathfrak{RB}$ then for all $a, b \in B$ we have $aB \cap bB$ is either principal if $(a, b) \in \mathscr{R}$ or empty else. $\square$

In order to prove the next therorem, we must draw upon Fennemore's result [27] concerning the defining identities of varieties of bands. This yields that if $\mathfrak{V}$ is a variety of bands not contained in $\mathfrak{RR}$ or $\mathfrak{RB}$, then $\mathfrak{V}$ is defined by an identity of the form $p = q$ where $c(p) = c(q)$ and the first letter of p and q are equal.

Theorem 5.2.9. *Let $\mathfrak{V}$ be a variety of bands not contained in $\mathfrak{RR}$ or $\mathfrak{RB}$. Then there exists a band $B_\infty \in \mathfrak{V}$ that is not right ideal Howson, and for each $n \in \mathbb{N}^0$ a band $B_n \in \mathfrak{V}$ that satisfies (Rn).*

Proof. Fix some $n \in \mathbb{N}^0$ and let X_n be the set of generators

$$X_n = \{a, b, u_i, v_i : 1 \leqslant i \leqslant n\}.$$

If $n = 0$ then we simply put $X_0 = \{a, b\}$. Define two subsets ρ_n and σ_n of $X_n^+ \times X_n^+$ as follows

$$\rho_n = \big\{(au_i, bv_i) : 1 \leqslant i \leqslant n\big\} \text{ and } \sigma_n = \big\{(w, w^2) : w \in X_n^+\big\}.$$

In this way, if $n = 0$ then $\rho_0 = \emptyset$.

Suppose V is the defining identity for $\mathfrak{V}$ and let B_n be the band with semigroup presentation $\langle X_n : \rho_n \cup \sigma_n \cup V\rangle$. From this point we will write X, ρ, σ and B instead of X_n, ρ_n, σ_n and B_n respectively. Moreover, for ease of notation we will let $\tau = \rho \cup \sigma \cup V$. It follows immediately the fact that B is finitely generated (as a band) that B is right ideal Howson.

Suppose that there exists some $s, t \in X^*$ such that $(as, bt) \in \tau^\sharp$. This implies that there exists a finite sequence of the form

$$as = z_0,\, z_1,\, \ldots,\, z_\ell = bt$$

where $z_{i-1} = c_i p_i d_i$ and $z_i = c_i q_i d_i$ with $(p_i, q_i) \in \tau \cup \tau^{-1}$ and $c_i, d_i \in X^*$ for all $1 \leqslant i \leqslant \ell$.

If (z_{i-1}, z_i) is an elementary $(\sigma \cup V)$-transition for all $1 \leqslant i \leqslant \ell$, then we immediately reach a contradiction since $(w, x) \in (\sigma \cup V)^\sharp$ implies that the first letter of w and x are equal. Therefore, we may assume that (z_{i-1}, z_i)

is an elementary ρ-transition for some $1 \leqslant i \leqslant \ell$. In order to avoid a similar contradiction, we may also assume that for at least one such $1 \leqslant i \leqslant \ell$, we have $c_i = \epsilon$. Thus, we have shown $z_i \in \{au_jd_i, bv_jd_i\}$ for some $1 \leqslant i \leqslant \ell$ and $1 \leqslant j \leqslant n$. It follows that

$$I \subseteq \bigcup_{1 \leqslant i \leqslant n} [au_i]B.$$

The reverse inclusion is clear from the form of the presentation.

To show that I is exactly n-generated, we note that if $(w, x) \in \tau^\sharp$ for some $w, x \in X^*$ and $u_i \in c(w)$ then $\{u_i, v_i\} \cap c(x) \neq \emptyset$. If $1 \leqslant i, j \leqslant n$ with $i \neq j$, then for no $t \in X^*$ do we have that $(au_i, au_jt) \in \tau^\sharp$. Thus $[au_i]B \not\subseteq [au_j]B$ and so I is exactly n-generated.

Concerning the latter part of the theorem, let

$$\begin{aligned} X_\infty &= \{a, b, u_i, v_i : i \in \mathbb{N}\}, \\ \rho_\infty &= \big\{(au_i, bv_i) : i \in \mathbb{N}\big\}, \\ \sigma_\infty &= \big\{(w, w^2) : w \in X_\infty^+\big\} \end{aligned}$$

and let B_∞ be the band with semigroup presentation $\langle X_\infty : \rho_\infty \cup \sigma_\infty \cup V\rangle$. It is easy to see that the intersection $[a]B \cap [b]B$ cannot be finitely generated and so B_∞ is not right ideal howson. $\square$

5.2.3 Other finiteness conditions

We end this section with a brief discussion of another finiteness condition for a monoid M, namely $\mathfrak{R}$ [44]. This condition arises from axiomatisability properties of classes of right M-acts and states that for any $a, b \in M$ the right M-subact of the direct product of the right M-act $M \times M$ given by

$$\mathfrak{R}(a, b) = \{(u, v) \in M \times M : au = bv\}$$

is finitely generated. It was shown that the property of satisfying $\mathfrak{R}$ is independent of being weakly right noetherian for a monoid [44].

Lemma 5.2.10. *If M is a monoid satisfying $\mathfrak{R}$ then M is right ideal Howson.*

Proof. Let $a, b \in M$. It is clear that if H is a finite set of generators for $\mathfrak{R}(a, b)$, then aK is a finite set of generators for $aM \cap bM$, where K is the set of first co-ordinates of elements of H. $\square$

On the other hand, there certainly exist examples of semilattices that do not satisfy $\mathfrak{R}$ [42]; since semilattices are inverse they are right and left ideal Howson. The next corollary comes from Lemma 5.2.10 and the results of [42]. We refer the reader to [46] for a description of restriction semigroups.

Corollary 5.2.11. *The free inverse monoid, the free ample (restriction) monoid, and the free left ample (restriction) monoid on any set is right and left ideal Howson.*

5.3 Closure results

In this section we explore a number of closure results regarding the class of right ideal Howson semigroups. Our first result is immediate, since any free semigroup X^+ is right ideal Howson.

Proposition 5.3.1. *The class of right ideal Howson semigroups is not closed under morphic image.*

We now consider free products of right ideal Howson semigroups.

5.3.1 Free products

We begin by defining a free product of semigroups and a free product of monoids.

Definition 5.3.2. [56] Let I be a non-empty set and let S_I be a semigroup (monoid) for every $i \in I$. Let

$$X = \bigcup_{i \in I} S_i$$

where we write the product of $x, y \in X^+$ (or X^*) as $x * y$. Consider the subsets $\mu \in X^+ \times X^+$ and $\nu \subseteq X^* \times X^*$ where

$$\mu = \{(x * y, xy) : x, y \in S_i, i \in I\}$$
$$\text{and } \nu = \mu \cup \{(1_{S_i}, 1_{S_j}) : i, j \in I\}.$$

The *free product of semigroups S_i ($i \in I$)* may be given by the quotient $X^+/\mu^\sharp$, and the *free product of monoids S_i ($i \in I$)* may be given by the quotient $X^*/\nu^\sharp$.

If $I = \{1, \ldots, n\}$ for some $n \in \mathbb{N}$, we may write $S_1 * \cdots * S_n$ for the semigroup or monoid free product, where the distinction will be clear from the context.

Proposition 5.3.3. *The class of right ideal Howson semigroups is closed under free products of semigroups.*

Proof. Let S be the semigroup free product of right ideal Howson semigroups S_i ($i \in I$). Let $[a], [b] \in S$ where $[a] = [a_1 * \cdots * a_n]$ and $[b] = [b_1 * \cdots * b_m]$ and consider the intersection $[a]S^1 \cap [b]S^1$. We show that this intersection is finitely generated. We may assume that n and m are *least* in the sense that there does not exist an $1 \leqslant k \leqslant n-1$ such that $a_k a_{k+1} \in S_i$ for some $i \in I$. It is clear that this intersection is either empty, $[a]S^1$, $[b]S^1$, or there exists some $[u], [v] \in S$ such that $[a * u] = [b * v]$. In the second and third cases the intersection is principal: we consider the final case.

Here we take

$$[u] = [u_1 * \cdots * u_k] \text{ and } [v] = [v_1 * \cdots * v_\ell]$$

for some least $k, \ell \in \mathbb{N}$, so that

$$[a * u] = [a_1 * \cdots * a_n * u_1 * \cdots * u_k]$$
$$\text{and } [b * v] = [b_1 * \cdots * b_m * v_1 * \cdots * v_\ell].$$

Let us assume $n \leq m$ and proceed with a case-by-case consideration.

(i) If $n < m$ then $a_i = b_i$ for all $1 \leqslant i \leqslant n-1$ and $b_n = a_n$ or $b_n = a_n u_1$. In either case, $[a]S^1 \cap [b]S^1 = [b]S^1$ is principal. Dually if $m < n$.

(ii) If $n = m$ then again $a_i = b_i$ for all $1 \leqslant i \leqslant n-1$. Suppose that $a_n, b_n \in S_j$; if $a_n = b_n$, $a_n = b_n w$ or $b_n = a_n w$ for some $w \in S_j^1$, then clearly $[a]S^1 \cap [b]S^1$ is principal. If this is not the case, then $a_n u_1 = b_n v_1$.

Now consider the intersection $a_n S_j^1 \cap b_n S_j^1$. Since S_j is right ideal Howson this intersection is finitely generated, say

$$a_n S_j^1 \cap b_n S_j^1 = \bigcup_{1 \leqslant i \leqslant r} a_n w_i S_j^1.$$

Clearly

$$\bigcup_{1 \leqslant i \leqslant r} [a_1 * \cdots * a_n w_i] S_j^1 \subseteq [a]S^1 \cap [b]S^1.$$

Conversely, suppose that $a_n u_1 = a_n w_i w$ for some $1 \leqslant i \leqslant r$ and $w \in S_j$. Then it follows that $[a * u] \in [a_1 * \cdots * a_n w_i]S^1$ for some $1 \leqslant i \leqslant r$. Thus

$$\bigcup_{1 \leqslant i \leqslant r} [a_1 * \cdots * a_n w_i] S^1 = [a]S^1 \cap [b]S^1$$

so that I is finitely generated as required.

In each case, the intersection is finitely generated and so this concludes the proof. □

The corresponding result holds for the free product of monoids.

Proposition 5.3.4. *The class of right ideal Howson monoids is closed under free products.*

Proof. The argument runs along the same lines as that of Proposition 5.3.3, but with added technicalities due to the extra relations in ν.

Let S be the (monoid) free product of right ideal Howson monoids S_i $(i \in I)$. Let $[a], [b] \in S$ and consider the intersection $[a]S \cap [b]S$. If $[a]$ is the identity of S, or an element that has a right inverse, then the result is clear. Thus we may suppose $[a] = [a_1 * \cdots * a_n]$ where $n \in \mathbb{N}$ is least and $a_j \in S_{j_k} \setminus \{1_{S_{j_k}}\}$ for $1 \leqslant j \leqslant n$. Now observe that if n' is greatest such that $a_{n'}$ is not right invertible, then $([a], [a_1 * \cdots * a_{n'}]) \in \mathscr{R}$. We may therefore assume from the outset that $a_n \in S_{n_k}$ is not right invertible. Similarly, we can assume $[b] = [b_1 * \cdots * b_m]$ where $m \in \mathbb{N}$ is least and $b_\ell \in S_{\ell_h} \setminus \{1_{S_{\ell_h}}\}$ for $1 \leqslant \ell \leqslant m$ and $b_m \in S_{m_h}$ is not right invertible.

If $[a]S \cap [b]S = \emptyset$, or if $[a] = [b * v]$ or $[b] = [a * u]$ for some $[u], [v] \in S$, then we are done. Suppose therefore that $[a*u] = [b*v]$ for some $[u], [v] \in S$. Given the fact that a_n, b_m are chosen to be not right invertible, the proof proceeds as in that of Proposition 5.3.3. $\square$

5.3.2 Direct and semidirect products

We begin with a negative result for direct products of semigroups.

Proposition 5.3.5. *The class of right ideal Howson semigroups is not closed under direct products.*

Proof. Consider the free monogenic semigroup $S = \langle a \rangle$. Clearly S is right ideal Howson since S is weakly (right) coherent (see Subsection 4.2). Indeed, for any $n, m \in \mathbb{N}$ we have that $a^n S^1 \cap a^m S^1 = a^k S^1$ where $k = \max\{n, m\}$. Let $T = S \times S$. One may then easily verify that the intersection $(a, a)T^1 \cap (a, a^2)T^1$ is generated by the set $\{(a^2, a^k) : k \geqslant 3\}$ and that for every $k, \ell \geqslant 3$ there does not exist some $h \geqslant 3$ such that $(a^2, a^k)(a^2, a^h) = (a^2, a^\ell)$. As every generating set for $(a, a)T^1 \cap (a, a^2)T^1$ must contain $\{(a^2, a^k) : k \geqslant 3\}$ it follows that T is not right ideal Howson. $\square$

On the other hand we have a positive result for right factorisable semigroups.

Proposition 5.3.6. *The class of right factorisable right ideal Howson semigroups is closed under direct products.*

Proof. Let S and T be right factorisable right ideal Howson semigroups.

Notice that for any $s \in S$ and $t \in T$ we have $sS = sS^1$, $tT = tT^1$ and also

$$(s,t)(S \times T)^1 = (s,t)(S \times T) = sS \times tT = sS^1 \times tT^1.$$

It then follows easily that $S \times T$ is right ideal Howson. $\square$

If S is free monogenic, then $S^1 \times S^1$ is right ideal Howson from Proposition 5.3.6, but Proposition 5.3.5 tells us that $S \times S$ is not right ideal Howson.

Corollary 5.3.7. *The class of right ideal Howson semigroups is not closed under taking subsemigroups.*

We now turn our attention to semidirect products. Let S and T be semigroups such that S acts on the left of T via morphisms. If the action of S on T is trivial, then $T \rtimes S$ is simply the direct product $T \times S$. Proposition 5.3.5 therefore tells us that the class of right ideal Howson semigroups is not closed under semidirect products. Considering now the case for monoids, S and T, where S acts as a monoid on T by monoid homomorphisms, the semidirect product $T \times S$ becomes a monoid. In view of Proposition 5.3.6 we know that the direct product of right ideal Howson monoids is right ideal Howson. By way of contrast we have the following.

Proposition 5.3.8. *The class of right ideal Howson monoids is not closed under semidirect product.*

Proof. Let $X = \{a, b, a_i : i \in \mathbb{N}\}$ and $A = \{a_i : i \in \mathbb{N}\}$. We consider the left action of X^* on $\mathcal{P}(X)$ where, for all $U \in \mathcal{P}(X)$, we have ${}^bU = \emptyset = {}^{a_i}U$ for all $i \in \mathbb{N}$ and

$${}^aU = \begin{cases} \emptyset & \text{if } U \cap A = \emptyset; \\ \{a, b\} \cup U & \text{otherwise.} \end{cases}$$

One may verify that this is a left action of X by monoid endomorphisms of the semilattice $\mathcal{P}(X)$ under union. The only case that needs thought is that of ${}^a(U \cup V)$ where $U \cap A \neq \emptyset$ but $V \cap A = \emptyset$ (or the dual). In this case

$${}^a(U \cup V) = \{a, b\} \cup U \cup V = \{a, b\} \cup U = {}^aU \cup \emptyset = {}^aU \cup {}^aV,$$

the second equality following since $V \subseteq \{a, b\}$. We may then extend the action of X to that of the monoid X^*. Let $S = \mathcal{P}(X) \rtimes X^*$ and consider

$$I = \big(\{a\}, a\big)S \cap \big(\{b\}, a\big)S.$$

Notice that if we set $Z_i = (\{a, b, a_i\}, a)$ for any $i \in \mathbb{N}$ we have

$$Z_i = \big(\{a, b, a_i\}, a\big) = \big(\{a\}, a\big)\big(\{a_i\}, \epsilon\big) = \big(\{b\}, a\big)\big(\{a_i\}, \epsilon\big) \in I.$$

Further, if $(Z, z) \in I$ then it is easy to see that $Z = \{a, b\} \cup (Z \cap A)$ where $Z \cap A \neq \emptyset$, and $z = az'$ for some $z' \in X^*$. In this way, if $a_i \in Z$ we have

$$(Z, z) = \big(\{a, b, a_i\}, a\big)(Z, z').$$

As there does not exist some $s \in S^1$ such that $Z_i s = Z_j$ for any $i \neq j$, it follows that I cannot be finitely generated. □

5.4 Semigroup presentations for right (left) ideal Howson semigroups

In Corollary 5.2.5 we show that any commutative semigroup presentation $\langle \mathcal{C}X : R\rangle$, where R is finite, gives rise to a (right) ideal Howson semigroup. In fact, one can show that both conditions (that of commutativity and the fact R is finite) are strictly necessary. We illustrate this by way of the examples below.

Example 5.4.1. Let S be given by the commutative semigroup presentation

$$\big\langle a, b, u_i, v_i, i \in \mathbb{N} : au_i = bv_i \, (i \in \mathbb{N})\big\rangle.$$

Then S is not ideal Howson.

Proof. This presentation is the commutative semigroup version of the band presentation of B_∞ given in Theorem 5.2.9. It is easy to see from the form of the presentation that

$$[a]S^1 \cap [b]S^1 = \bigcup_{i \in \mathbb{N}} [au_i]S^1$$

and is not finitely generated. $\square$

It is also easy to see that the (non-commutative) semigroup on the same presentation as that in Example 5.4.1 is not right ideal Howson; the presentation, however, is not finite. We now give an example of a finitely presented semigroup that is not right ideal Howson.

Example 5.4.2. Let S be given by the semigroup presentation

$$\langle a, b, c, d, p, q, u, v : auvc = bpqd, au = ua, ub = bp, uv = u^2v^2 \rangle.$$

Then S is not right ideal Howson.

Proof. Let $X = \{a, b, c, d, p, q, u, v\}$ and define subsets ρ and σ of $X^+ \times X^+$ by

$$\rho = \big\{(auvc, bpqd)\big\} \text{ and } \sigma = \big\{(au, ua), (ub, bp), (uv, u^2v^2)\big\}$$

and let $\tau = \rho \cup \sigma$. First, we note for any $t \in X^*$ that $[au^hvc] = [au^kvct]$ for $h, k \in \mathbb{N}$ if and only if $h = k$ and $t = \epsilon$. This is a fact witnessed by verifying, by induction on the length of a τ-sequence, that

$$[au^hvc] = \left\{ u^m au^{h+\ell-m} v^{\ell+1} c, u^r bp^s qd : \begin{array}{l} \ell, r, s \geqslant 0 \\ 0 \leqslant m \leqslant h + \ell \\ r + s = h \end{array} \right\}.$$

It follows that there does not exist some $[x], [y] \in S^1$ such that

$$[au^hvc][x] = [au^kvc] \text{ or } [au^kvc][y] = [au^hvc]$$

for any distinct $h, k \in \mathbb{N}$. Furthermore notice that $[au^hvc] \in I = [a]S^1 \cap [b]S^1$ for any $h \in \mathbb{N}$, since

$$au^hvc \, \sigma^\sharp \, u^{h-1}auvc \, \rho^\sharp \, u^{h-1}bpqd \, \sigma^\sharp \, bp^hqd.$$

To complete the proof, we must show that if $(aw, bx) \in \tau^\sharp$ for some $w, x \in X^*$ then $[aw] \in [au^h vc]S^1$ for some $h \geqslant 1$. Suppose therefore that $(aw, bx) \in \tau^\sharp$, so there exists a finite sequence of the form

$$aw = z_0,\, z_1,\, \ldots,\, z_n = bx$$

where $z_{i-1} = c_i s_i d_i$ and $z_i = c_i t_i d_i$ with $c_i, d_i \in X^*$ and $(s_i, t_i) \in \tau \cup \tau^{-1}$ for all $1 \leqslant i \leqslant n$. We claim that at least one elementary τ-transition must be an elementary ρ-transition of the form

$$(z_i, z_{i+1}) = (u^k auvcd_i, u^k bpqdd_i)$$

where $k \geqslant 0$ and $d_i \in X^*$. Suppose for contradiction that this is not the case; we argue that for each $0 \leqslant i \leqslant n$ we have $z_i = u^{k_i} az_i'$ for some $k_i \geqslant 0$ and $z_i' \in X^*$. Clearly this is true for $i = 0$. Suppose for induction that $z_i = u^{k_i} az_i'$ as given where $0 \leqslant i \leqslant n$. Avoiding the elementary ρ-transition of the form above, our possibilities for (z_i, z_{i+1}) are

$$(z_i, z_{i+1}) = \begin{cases} (u^{k_i} az_i', u^{k_i - 1} auz_i') & \text{if } k_i \geqslant 1; \\ (u^{k_i} az_i', u^{k_i+1} az_i'') & \text{if } z_i = uz_i''; \\ (u^{k_i} az_i', u^{k_i} az_{i+1}') & (z_i', z_{i+1}') \text{ an elementary } \tau\text{-transition.} \end{cases}$$

Thus if the first letter of z_n is not u, then it is certainly a, which forms a contradiction. Therefore for some $0 \leqslant i \leqslant n$ we must have $z_i = u^k auvcd_i$ where $k \geqslant 0$, and then

$$[ax] = [u^k auvcd_i] \in [au^h vc]S$$

where $h = k + 1 \in \mathbb{N}$. This completes the proof. $\square$

5.4.1 The case where S is non-commutative

We now turn to positive results, constructing semigroup presentations that are right (left, right and left) ideal Howson, which are, by construction, universal in a specific sense. The semigroups we construct in this subsection are also all cancellative. For $n, m \in \mathbb{N}^0$ we define an alphabet X_{nm} by

$$X_{nm} = \big\{a, b, u_i, v_i, p_j, q_j : 1 \leqslant i \leqslant n, 1 \leqslant j \leqslant m\big\}$$

and a relation ρ_{nm} on X^+_{nm} by

$$\rho_{nm} = \big\{(au_i, bv_i), (p_j a, q_j b) : 1 \leqslant i \leqslant n, 1 \leqslant j \leqslant m\big\}$$

Let $S_{\rho_{nm}}$ be the semigroup with presentation $\langle X_{nm} : \rho_{nm}\rangle$. If $n = 0$ then we simplify our ingredients considerably; we have

$$X_{0m} = \big\{a, b, p_j, q_j : 1 \leqslant j \leqslant m\big\} \text{ and } \rho_{0m} = \big\{(p_j a, q_j b) : 1 \leqslant j \leqslant m\big\}$$

and similarly if $m = 0$. If $m = n = 0$ then $X = \{a, b\}$ and $\rho_{00} = \emptyset$. In this case, $S_{\rho_{00}} = \{a, b\}^+$, being free, is certainly right and left ideal Howson, with intersections of right (left) ideals being empty or principal. Since $n, m \in \mathbb{N}^0$ are fixed, we simplify notation and denote X_{nm}, ρ_{nm} and $S_{\rho_{nm}}$ by X, ρ and S, respectively. Keeping this notation in mind, we consider a specific factorisation of elements of X^+.

Definition 5.4.3. Let $U(n)$ and $P(m)$ be the sets given by

$$U(n) = \{u_i, v_i : 1 \leqslant i \leqslant n\} \text{ and } P(m) = \{p_j, q_j : 1 \leqslant j \leqslant m\}$$

where we regard $U(n)$ and $P(m)$ to be empty if $n = 0$ and $m = 0$ respectively. Let

$$C(\rho) = \big\{\overline{p}c, c\overline{u}, \overline{p}c\overline{u} : c \in \{a, b\}, \overline{u} \in U(n), \overline{p} \in P(m)\big\}$$

and notice that $C(\rho)$ is closed under $\rho^\sharp$. For any $w \in X^+$ we may uniquely factorise w as

$$w = w_0 r_1 w_1 \dots w_{p-1} r_p w_p$$

for $p \in \mathbb{N}^0$ and subject to the following conditions for all $1 \leqslant i \leqslant p$ and $0 \leqslant j \leqslant p$:

(i) $r_i \in C(\rho)$;

(ii) $w_j \in X^*$ does not contain an element of $C(\rho)$ as a subword;

(iii) if the first letter of r_i is a or b then the last letter of w_{i-1} is not in $P(m)$;

(iv) if the last letter of r_i is a or b then the first letter of w_{i-1} is not in $U(n)$.

We call such a factorisation the *ρ-factorisation* of w with corresponding *ρ-length* equal to p.

Notice that, in Definition 5.4.3, we have $|r_i| \in \{2, 3\}$ for all $1 \leqslant i \leqslant p$.

Claim 5.4.4. *For $w, x \in X^+$ have $(w, x) \in \rho^\sharp$ if and only if w and x have ρ-factorisations*

$$w = w_0 r_1 w_1 \ldots w_{p-1} r_p w_p \text{ and } x = w_0 s_1 w_1 \ldots w_{p-1} s_p w_p$$

respectively, where $(r_i, s_i) \in \rho^\sharp$ for every $1 \leqslant i \leqslant p$.

Proof. If w and x are as given and $(r_i, s_i) \in \rho^\sharp$ for every $1 \leqslant i \leqslant p$, then clearly $(w, x) \in \rho^\sharp$.

For the converse, suppose that w is as given and we obtain the word y by applying a single relation of ρ on w. Then (from the definition of ρ-factorisation) we must have $y = w_0 s_1 w_1 \ldots w_{p-1} s_p w_p$ where, for all but one $1 \leqslant i \leqslant p$, we have $s_i = r_i$ and for a single $1 \leqslant j \leqslant p$ we have $r_j, s_j \in C(\rho)$ with $(r_j, s_j) \in \rho^\sharp$. Therefore y has the form required. The result then follows by induction on the length of a ρ-sequence starting from w and ending at x. □

Claim 5.4.5. *The semigroup S is right and left ideal Howson.*

Proof. We show that S is right ideal Howson, the argument for left ideals being dual.

Let $w, x \in X^+$ and put $I = [w]S^1 \cap [x]S^1$. Suppose that $I \neq \emptyset$, so that $(wh, xk) \in \rho^\sharp$ for some $h, k \in X^*$. By Claim 5.4.4 we have ρ-factorisations $wh = w_0 r_1 w_1 \ldots w_{p-1} r_p w_p$ and $xk = w_0 s_1 w_1 \ldots w_{p-1} s_p w_p$ for some $p \in \mathbb{N}^0$

where $(r_i, s_i) \in \rho^\sharp$ for all $1 \leqslant i \leqslant p$. Without loss of generality we may assume $|w| \leqslant |x|$ and then consider all possible cases for the ρ-factorisation of h.

(i) Suppose that $w = w_0 r_1 w_1 \ldots r_{i-1} s$ and $h = t r_i \ldots w_{p-1} r_p w_p$ where $w_i = st$ for some $0 \leqslant i \leqslant p$ and $s, t \in X^*$. Since $x = w_0 s_1 w_1 \ldots s_{i-1} s y$ for some $y \in X^*$, we have that

$$wy = w_0 r_1 w_1 \ldots r_{i-1} s y \,\rho^\sharp\, w_0 s_1 w_1 \ldots r s_{i-1} s y = x$$

and so $I = [x]S^1$ is principal.

(ii) Suppose now that $w = w_0 r_1 w_1 \ldots w_{i-1} s$ and $h = t w_i \ldots w_{p-1} r_p w_p$ where $r_i = st$ for some $1 \leqslant i \leqslant p$ and $s, t \in X^*$. We write $x = w_0 s_1 w_1 \ldots w_{i-1} s_i y$ for some $y \in X^*$ and see

$$\begin{aligned} wty &= w_0 r_1 w_1 \ldots w_{i-1} s t y \\ &= w_0 r_1 w_1 \ldots w_{i-1} r_i y \\ &\rho^\sharp\, w_0 s_1 w_1 \ldots w_{i-1} s_i y = x, \end{aligned}$$

so that $I = [x]S^1$ is principal.

(iii) Lastly, we suppose that w is as in case (ii) and $x = w_0 s_1 w_1 \ldots w_{i-1} u$ where $s_i = uv$ for some $1 \leqslant i \leqslant p$ and $u, v \in X^*$ with $|s| \leqslant |u|$. If $s = u$ then $[w] = [x]$ and we are done. Otherwise, let us define

$$R(\rho) = \big\{(p,q) : (sp, uq) \in \rho^\sharp \cap \big(C(\rho) \times C(\rho)\big)\big\}.$$

It follows $R(\rho)$ is finite since $C(\rho)$ is finite and certainly $(wp, xq) \in \rho^\sharp$ for all $(p,q) \in R(\rho)$. Since $(t, v) \in R(\rho)$ we have

$$I = \bigcup_{(p,q) \in R(\rho)} [wp]S^1$$

and so I is finitely generated.

Therefore, in each case for the ρ-factorisation of h, we have shown that the intersection is finitely generated and so S is right ideal Howson. $\square$

Claim 5.4.6. *The semigroup S satisfies (Rn) and (Lm). Moreover, the intersection of any two principal right (left) ideals of S requires at most n (m) generators.*

Proof. Again we only give the proof for right ideals. Let $w, x \in X^+$; we show the intersection $I = [w]S^1 \cap [x]S^1$ requires at most n generators. The only situation where I is not empty or principal is in the second situation of (iii). Here we consider all the possibilities for $(p, q) \in R(\rho)$; we can have $|r_i| = |s_i| = 2$ with $|s| = |u| = 1$ or $|s| = 1$ and $|u| = 2$, or $|r_i| = |s_i| = 3$ with $|s| = |u| = 1$ or $|s| = 1$ and $|u| = 2$, or 3, or $|s| = 2$ and $|u| = 2$ or 3. A case-by-case analysis for all the possibilities, paying special attention to the case where $m = 0$ or $n = 0$ or both, now gives the result. In particular, when $n \geqslant 1$ with $w = a$ and $x = b$ we have

$$I = [au_1]S^1 \cup \cdots \cup [au_n]S^1$$

which achieves the bound n as required. $\square$

We are now in a position to given the main result of this subsection.

Theorem 5.4.7. *Let $n, m \in \mathbb{N}^0$. The semigroup $S_{\rho_{nm}}$ is cancellative, right and left ideal Howson, and satisfies (Rn) and (Lm). Further, the intersection of any two principal right (left) ideals of S requires at most n (m) generators.*

Proof. From Adian's Theorem, it is immediate that there is an embedding of $S_{\rho_{nm}}$ in a group: cancellativity follows from this. The remainder of the result comes from Claims 5.4.4, 5.4.5 and 5.4.6. $\square$

We now show that our semigroup $S_{\rho_{nm}}$ is in a specific sense universal.

Proposition 5.4.8. *Let $n, m \in \mathbb{N}^0$. Suppose U is a semigroup containing elements α, β such that $\alpha U^1 \cap \beta U^1$ and $U^1\alpha \cap U^1\beta$ are each exactly n- and m-generated, respectively, by $\alpha\gamma_1 = \beta\delta_1, \ldots, \alpha\gamma_n = \beta\delta_n$ and $\mu_1\alpha =$*

$\pi_1\beta, \ldots, \mu_m\alpha = \pi_m\beta$, respectively. Then there is a homomorphism $\theta : S_{\rho_{nm}} \to U$ such that

$$[a]\theta = \alpha, [b]\theta = \beta, [u_i]\theta = \gamma_i, [v_i]\theta = \delta_i, [p_j]\theta = \mu_j \textit{ and } [q_j]\theta = \pi_j$$

for all $1 \leqslant i \leqslant n$ and $1 \leqslant j \leqslant m$.

Proof. Let $\psi : X_{nm}^+ \to U$ be given by determining its values on the elements of X_{nm} by

$$a\psi = \alpha,\ b\psi = \beta,\ u_i\psi = \gamma_i,\ v_i\psi = \delta_i,\ p_j\psi = \mu_j \text{ and } q_j\psi = \pi_j$$

for all $1 \leqslant i \leqslant n$ and $1 \leqslant j \leqslant m$. Clearly $\rho_{nm} \subseteq \ker\psi$ so that ψ induces a morphism $\theta : S_{\rho_{nm}} \to U$ as in the statement of the proposition. □

Remark 5.4.9. Let $n \geqslant 1$. The semigroup $S = S_{\rho_{mn}}$ is not finitely right aligned. To see this, suppose that

$$[a]S \cap [b]S = [w_1]S \cup \ldots \cup [w_\ell]S$$

for some $w_k \in X^+$ such that $[w_k] \in [a]S^1 \cap [b]S^1$ for all $1 \leqslant k \leqslant \ell$. We have $[au_1] = [bv_1] \in [a]S \cap [b]S$, so that $[au_1] = [bv_1] = [w_k][z]$ for some $k \in \underline{\ell}$ and $z \in X^+$. We must have that $w_k z = au_i$ or bv_i. If $w_k = a$ then we would not have $w_k \in [b]S^1$ and similarly if $w_k = b$ we reach a contradiction.

In fact, an easier approach to obtain a semigroup presentation that satisfies (Rn) and (Lm) for some fixed $n, m \in \mathbb{N}^0$, but producing a less tight result, runs as follows. For $n, m \in \mathbb{N}^0$ we define an alphabet Y_{nm} by

$$Y_{nm} = \{a, b, c, d, u_i, v_i, p_j, q_j : 1 \leqslant i \leqslant n, 1 \leqslant j \leqslant m\}$$

and a relation σ_{mn} on Y_{mn}^+ by

$$\sigma_{nm} = \{(au_i, bv_i), (p_jc, q_jd) : 1 \leqslant i \leqslant n, 1 \leqslant j \leqslant m\}$$

and, as a convention, we will let $T_{\sigma_{nm}}$ be the semigroup with presentation

$\langle Y_{nm} : \sigma_{nm}\rangle$. If $n = 0$ then (as before) we have

$$Y_{0m} = \big\{a, b, c, d, p_j, q_j : 1 \leqslant j \leqslant m\big\} \text{ and } \sigma_{0m} = \big\{(p_j c, q_j d) : 1 \leqslant j \leqslant m\big\}$$

and similarly when $m = 0$. If in fact $n = m = 0$ then we simply have $Y_{00} = \{a, b, c, d\}$ and $\sigma_{00} = \emptyset$. The proof for the following theorem is similar to that of Theorem 5.4.7 but rather simpler, since the complications of overlapping generators for $\sigma_{nm}^{\sharp}$ do not occur.

Theorem 5.4.10. *Let $n, m \in \mathbb{N}^0$. The semigroup $T_{\sigma_{nm}}$ is cancellative, right and left ideal Howson, and satisfies (Rn) and (Lm). Further, the intersection of any two principal right (left) ideals of $T_{\sigma_{nm}}$ requires at most n (m) generators.*

There is also a corresponding universal type result for $T_{\sigma_{nm}}$, analogous to that of Proposition 5.4.8. Namely, if U is a semigroup containing elements α, β, γ and δ such that $\alpha U^1 \cap \beta U^1$ and $U^1\gamma \cap U^1\delta$ are each exactly n- and m-generated respectively, then U^1 contains a morphic image of $T_{\sigma_{nm}}$ obtained as in Proposition 5.4.8.

5.4.2 The case where S is commutative

For any $n, m \in \mathbb{N}^0$, the semigroup $S_{\rho_{nm}}$ is not commutative. For instance, we have clearly have that $[ab] \neq [ba]$. In this section, we provide a commutative semigroup presentation that satisfies (Rn), which now coincides with (Ln), for any fixed $n \in \mathbb{N}$. Unlike the case for $S_{\rho_{nm}}$ we do not automatically have that the semigroups we construct are cancellative. However, we can construct natural quotients that satisfy (Rn) and are cancellative. For a fixed $n \in \mathbb{N}^0$ we define the alphabet X_n by

$$X_n = \{a, b, u_i, v_i : 1 \leqslant i \leqslant n\}$$

and the relations τ_n and υ_n on X_n^+ by

$$\begin{aligned} \tau_n &= \{(au_i, bv_i) : 1 \leqslant i \leqslant n\} \\ \text{and } \upsilon_n &= \{(au_i, bv_i), (u_iv_j, u_jv_i) : 1 \leqslant i, j \leqslant n, i \neq j\}. \end{aligned}$$

Similarly to the conventions in Subsection 5.4.1, we let S_{τ_n} and S_{υ_n} be the semigroups with commutative presentations $\langle \mathcal{C}X_n : \tau_n \rangle$ and $\langle \mathcal{C}X_n : \upsilon_n \rangle$ respectively. If $n = 0$ then we simply put $X_0 = \{a, b\}$ and

$$\tau_0 = \{(ab, ba), (ba, ab)\} = \upsilon_0.$$

Since $n \in \mathbb{N}^0$ is fixed, we will write X, τ, υ, S_τ and S_υ instead of X_n, τ_n, υ_n, S_{τ_n} and S_{υ_n} respectively. Also, note that we will continue to refer to *words* but we now mean elements of $\mathcal{C}X^+$ rather than X^+.

Theorem 5.4.11. *The semigroups S_{τ_n} and S_{υ_n} satisfy (R(n+1)).*

Proof. From Corollary 5.2.5, we have that both S_τ and S_υ are right and left ideal Howson. A straightforward argument, using the fact that both presentations are homogeneous, verifies that

$$[a]_\kappa S^1_\kappa \cap [b]_\kappa S^1_\kappa = [ab]_\kappa S^1_\kappa \cup \bigcup_{1 \leqslant i \leqslant n} [au_i]_\kappa S^1$$

for $\kappa = \tau$ or $\kappa = \upsilon$. In each case, no given generator is redundant. □

Notice that in S_τ for any $1 \leqslant i, j \leqslant n$ we have $[au_iv_j]_\tau = [au_jv_i]_\tau$ but if $i \neq j$ then $[u_iv_j]_\tau \neq [u_jv_i]_\tau$, so that if $n \geqslant 2$ the semigroup S_τ is not cancellative. However, we now set out to show that S_υ is cancellative. We begin by making some immediate observations about words in the same $\tau^\sharp$-class or $\upsilon^\sharp$-class.

Remark 5.4.12. For $w, x \in \mathcal{C}X^+$ with $(w, x) \in \tau^\sharp$ or $(w, x) \in \upsilon^\sharp$ we notice that

(i) $|w| = |x|$ since the presentations are homogeneous;

(ii) we may write $w = a^{p_0}b^{q_0}u_1^{p_1}v_1^{q_1}\dots u_n^{p_n}v_n^{q_n}$ for some $p_i, q_i \in \mathbb{N}^0$ for all $0 \leqslant i \leqslant n$.

With this in mind, for $w \in X^+$ we define $K(w)$ and $k_i(w)$ to be

$$\begin{aligned} K(w) &= |w|_b + \sum_{1 \leqslant i \leqslant n} |w|_{u_i}, \\ k_0(w) &= |w|_a + |w|_b \\ \text{and } k_i(w) &= |w|_{u_i} + |w|_{v_i} \end{aligned}$$

for all $1 \leqslant i \leqslant n$. Notice that for any $w, x \in \mathcal{C}X^+$ we have

$$K(wx) = K(w) + K(x) \text{ and } k_i(wx) = k_i(w) + k_i(x)$$

for any $0 \leqslant i \leqslant n$. We say that $w, x \in \mathcal{C}X^+$ are *balanced* if $k_i(w) = k_i(x)$ for all $0 \leqslant i \leqslant n$. The next claim is clear from the definition of τ and υ.

Claim 5.4.13. *If $w, x \in X^+$ are such that $(w, x) \in \tau^\sharp$ or $(w, x) \in \upsilon^\sharp$, then w and x are balanced and $K(w) = K(x)$.*

An easy argument gives:

Claim 5.4.14. *If $w, x \in X^+$ are such that $k_0(w), k_0(x) > 0$, then $(w, x) \in \tau^\sharp$ if and only if $(w, x) \in \upsilon^\sharp$.*

As τ and υ are homogeneous relations on $\mathcal{C}X^+$, the question follows of when $(w, x) \in \tau^\sharp$ or $(w, x) \in \upsilon^\sharp$ for two arbitrary words $w, x \in \mathcal{C}X^+$. To enable our proof of cancellativity, we establish explicitly the existence of a *normal form* in each $\tau^\sharp$-class and $\upsilon^\sharp$-class. First, we introduce a linear order on words in $\mathcal{C}X^+$. Let $w, x \in \mathcal{C}X^+$ where

$$w = a^{p_0}b^{q_0}u_1^{p_1}v_1^{q_1}\dots u_n^{p_n}v_n^{q_n} \text{ and } x = a^{r_0}b^{s_0}u_1^{r_1}v_1^{s_1}\dots u_n^{r_n}v_n^{s_n}.$$

We say that $w \leqslant x$ if and only if there exists some $0 \leqslant i \leqslant n$ such that $p_j = r_j$ for all $j < i$ and $p_i < r_i$ (or in fact $p_0 < r_0$ if indeed $i = 0$). One may verify the following.

Claim 5.4.15. *The relation $\leqslant$ is a partial order on $\mathcal{C}X^+$.*

In particular, $\leqslant$ is a partial order when restricted to any set of balanced words; certainly any $\tau^\sharp$-class or $\upsilon^\sharp$-class. Hence, let w^τ and w^υ be the unique words in $[w]_\tau$ and $[w]_\upsilon$ respectively that are *greatest* under $\leqslant$ (by which we mean if $w^\tau \leqslant w$ then $w^\tau = w$ and similarly for w^υ). We now draw upon some well-known results regarding rewriting systems, as seen in Section 3.3, to obtain w^τ and w^υ.

Claim 5.4.16. *The rewriting system on $[w]_\tau$ for all $w \in X^+$, given by the rewriting rules $bv_k \to au_k$ and $av_iu_j \to au_iv_j$ for all $1 \leqslant i,j,k \leqslant n$ with $i < j$, is confluent.*

Proof. If $w = ybv_k$ and $x = yau_k$ for some $y \in X^*$, then clearly $(w,x) \in \tau^\sharp$ and $w < x$. Correspondingly, if $w = yav_iu_j$ and $x = yau_iv_j$ for some $y \in X^*$ with $i < j$ then again $(w,x) \in \tau^\sharp$ and $w < x$. It follows that this is a noetherian rewriting system. It is routine to check that it is also locally confluent and thus confluent. □

Consequently, if $w \in \mathcal{C}X^+$, then applying the rewriting rules to w yields a unique reduced word $x \in [w]_\tau$, and x is independent of the choice of w. We know that $x \leqslant w^\tau$ and we deduce from $w^\tau \leqslant x^\tau = x$ that $x = w^\tau$. We say that w^τ is the word in *normal form* in $[w]_\tau$. An entirely similar argument can be made for a rewriting system consisting of rewriting rules $bv_k \to au_k$ and $v_iu_j \to u_iv_j$ (for all $i,j,k \in \mathbb{N}$ such that $i \leqslant j$) on elements of $[w]_\upsilon$ for any $w \in \mathcal{C}X^+$.

Claim 5.4.17. *If w^τ is in normal form in the $\tau^\sharp$-class of w, then it must be one of the following types:*

(NF1A) $w^\tau = a^{p_0}u_1^{p_1}u_2^{p_2}\dots u_{i-1}^{p_{i-1}}u_i^{p_i}v_i^{q_i}v_{i+1}^{q_{i+1}}\dots v_n^{q_n}$ *for* $p_0 > 0$ *and* $q_i > 0$ *or* $i-1 = n$;

(NF2) $w^\tau = a^{p_0}b^{q_0}u_1^{p_1}u_2^{p_2}\dots u_n^{p_n}$ *for* $q_0 > 0$;

(NF3) $w^\tau = u_1^{p_1}v_1^{q_1}\dots u_n^{p_n}v_n^{q_n}$.

Similarly, if w^υ is in normal form in the $\upsilon^\sharp$-class of w, then either it is type (NF1B) or (NF2) where

(NF1B) $w^\upsilon = a^{p_0} u_1^{p_1} u_2^{p_2} \dots u_{i-1}^{p_{i-1}} u_i^{p_i} v_i^{q_i} v_{i+1}^{q_{i+1}} \dots v_n^{q_n}$ *for* $q_i > 0$ *or* $i - 1 = n$.

To see this, one can verify that it is not possible to apply a rewriting rule (from the respective rewriting systems) to w^τ or w^υ as above. We now give a partial converse to Claim 5.4.13.

Claim 5.4.18. *Suppose $w, x \in X^+$ are in normal form and balanced with $k_0(w) = k_0(x) > 0$. Then:*

(i) if $|w|_a < |x|_a$ then $|w|_{u_\ell} \geqslant |x|_{u_\ell}$ for all $1 \leqslant \ell \leqslant n$ and $K(w) - K(x) > 0$;

(ii) if $|w|_a = |x|_a$ and $|w|_{u_k} > |x|_{u_k}$ for some $1 \leqslant k \leqslant n$, then $|w|_{u_\ell} \geqslant |x|_{u_\ell}$ for all $1 \leqslant \ell \leqslant n$ and $K(w) - K(x) > 0$.

Proof. Suppose that $w, x \in X^+$ and that $k_0(w) = k_0(x) > 0$ as in the statement above.

(i) If $|w|_a < |x|_a$, then as $k_0(x) > 0$, we must have that

$$w = a^{p_0} b^{q_0} u_1^{p_1} u_2^{p_2} \dots u_n^{p_n} \text{ for } q_0 > 0$$

is of type (NF2). Either

$$x = a^{r_0} b^{s_0} u_1^{p_1} u_2^{p_2} \dots u_n^{p_n} \text{ for } s_0 > 0$$

is also of type (NF2) or

$$x = a^{r_0} u_1^{p_1} u_2^{p_2} \dots u_{i-1}^{p_{i-1}} u_i^{r_i} v_i^{s_i} v_{i+1}^{p_{i+1}} \dots v_n^{p_n} \text{ for } r_0 > 0, s_i > 0 \text{ or } i - 1 = n$$

is of type (NF1A). In both cases we see that certainly $|w|_{u_\ell} \geqslant |x|_{u_\ell}$ for all $1 \leqslant \ell \leqslant n$. Since $|w|_a < |x|_a$ if and only if $|w|_b > |x|_b$ it is then clear that $K(w) - K(x) > 0$.

(ii) Suppose that $|w|_a = |x|_a$ and $|w|_{u_k} > |x|_{u_k}$ for some $1 \leqslant k \leqslant n$. Since $|w|_a = |x|_a$, either w and x are both of type (NF1A) or both of type

(F2). In the latter case, it is easy to see that $w = x$, contradicting the hypothesis. Thus

$$w = a^{p_0} u_1^{p_1} u_2^{p_2} \dots u_{i-1}^{p_{i-1}} u_i^{p_i} v_i^{q_i} v_{i+1}^{q_{i+1}} \dots v_n^{q_n} \text{ for } p_0 > 0, q_i > 0 \text{ or } i-1 = n$$

and

$$x = a^{p_0} u_1^{r_1} u_2^{r_2} \dots u_{j-1}^{r_{j-1}} u_j^{r_j} v_j^{s_j} v_{j+1}^{s_{j+1}} \dots v_n^{s_n} \text{ for } p_0 > 0, s_j > 0 \text{ or } j-1 = n$$

are of type (NF1A). We know that $|w|_{u_k} > |x|_{u_k}$ for some $1 \leqslant k \leqslant n$, and the only way this can occur is if $i > j$ or $i = j$ and $p_i > r_i$. In either case, $|w|_{u_\ell} \geqslant |x|_{u_\ell}$ for all $1 \leqslant \ell \leqslant n$. Again, it is clear that $K(w) - K(x) > 0$.

□

The next claim is now immediate.

Claim 5.4.19. *Suppose $w, x \in X^+$ are in normal form and balanced with $k_0(w) = k_0(x) > 0$. Then $w = x$, equivalently $(w, x) \in \tau^\sharp$, if and only if $K(w) = K(x)$.*

Theorem 5.4.20. *The semigroup S_υ is cancellative.*

Proof. Let $[w]_\upsilon, [x]_\upsilon, [h]_\upsilon \in S_\upsilon$ such that $[wh]_\upsilon = [xh]_\upsilon$. We may assume that w, x, h are in normal form. We have $(wh, xh) \in \upsilon^\sharp$ so that

$$k_i(w) + k_i(h) = k_i(wh) = k_i(xh) = k_i(x) + k_i(h),$$

giving $k_i(w) = k_i(x)$ for all $0 \leqslant i \leqslant n$ and, similarly, $K(w) = K(x)$. If $k_0(w) = k_0(x) > 0$, then by Claim 5.4.14 we have $(w, x) \in \tau^\sharp$ if and only if $(w, x) \in \upsilon^\sharp$, so that $w = x$ from Claim 5.4.19. Suppose therefore that $k_0(w) = k_0(x) = 0$; it follows that w, x are of the form (NF1B). Using the fact that w, x are balanced and $K(w) = K(x)$ a now familiar analysis again gives us that $w = x$. Certainly in each case we have $[w]_\upsilon = [x]_\upsilon$. □

Our semigroups S_τ and S_υ have universal properties corresponding to those for S_τ in Proposition 5.4.8.

Proposition 5.4.21. *Let S be a commutative (commutative and cancellative) semigroup such that S contains two principal right ideals αS^1 and βS^1 such that $\alpha S^1 \cap \beta S^1$ has exactly n generators $\alpha\gamma_1 = \beta\delta_1, \ldots, \alpha\gamma_n = \beta\delta_n$. Then there is a homomorphism $\theta : S_\tau \to S$ ($\theta : S_\upsilon \to S$) such that $[a]\theta = \alpha, [b]\theta = \beta, [u_i]\theta = \gamma_i$ and $[v_i]\theta = \delta_i$ for all $1 \leqslant i \leqslant n$.*

Chapter 6

Semigroup products with uniqueness in the first co-ordinate

We structure Chapter 6 in the following way. In Section 6.1, we begin by introducing some important definitions for this chapter including what is meant by a left-product (right-product) pair of semigroups. We concentrate further on such pairs that are so-called left-unique and right-unique. Then, in Section 6.2, we show how one may construct quotients of an external semidirect product of semigroups via certain 'left-unique' congruences. We change tack to some extent in Section 6.3 and consider internal semidirect products. We draw connections between the previous two sections in Section 6.4. In Section 6.5 we describe how to construct semigroup presentations for semigroups that arise as left-unique left product pairs. Finally, we give examples of classes of semigroups for which we can obtain semigroup presentations for in Section 6.6. Specifically, we show how $\mathcal{SP}\mathrm{End}F_n(G)$ is isomorphic to a semigroup product of a monoid M and semigroup S which arise as a left-unique left-product pair (and describe how to arrive at a semigroup presentation using this).

6.1 Basic definitions

We begin by giving a name to a special property exhibited by pairs of subsemigroups of a given semigroup.

Definition 6.1.1. Let U be a semigroup with subsemigroups S and T. Then we say S and T form a *left-product pair* (S, T) of U if $aS \subseteq Sa$ for every $a \in T$. We say (S, T) is a *right-product pair* if $xT \subseteq Tx$ for every $x \in S$. A *product pair* is both a left-product and right-product pair.

Of course, it is immediate from Definition 6.1.1 that (S, T) being a left-product pair of U implies that (T, S) is a right-product pair of U. This also means that (S, T) is a product pair of U if and only if (T, S) is a product pair of U. However, less can be said in general regarding the connection between (S, T) being a left-product pair of U implying that (S, T) is a right-product pair. We show in the following example that not every left-product pair of semigroups is a right-product pair.

Example 6.1.2 (Left zero semigroups)**.** Let U be a left zero semigroup and let S and T be subsemigroups of U such that $T \subset S$. Then

$$ax = aa = a$$

for every $a \in T$ and $x \in S$. Therefore (S, T) is a left-product pair of U. Fix some $y \in S \setminus T$. Then

$$ya = y \neq b = by$$

for every $a, b \in T$. This gives us that (S, T) is not a right-product pair of U.

Conversely, examples of product pairs abound. We provide some examples below.

Example 6.1.3. Let P be a monoid and let M be a subsemigroup of P. Then we see that

$$1m = m1 \text{ and } m1 = 1m$$

for every $m \in M$. This gives us that $(M, \{1\})$ is a left-product and right-product pair of P respectively. Hence $(M, \{1\})$ is a product pair of P.

Example 6.1.4 (Commutative semigroups)**.** Let U be a commutative semigroup with subsemigroups S and T. Then clearly we have that

$$ax = xa \text{ and } xa = ax$$

for every $a \in T$ and $x \in S$. This gives us that (S, T) is a left-product and right-product pair of U respectively. Therefore (S, T) is a product pair of U.

Example 6.1.5 (Normal subgroups)**.** Let G be a group and let N be a normal subgroup of G. Then it is clear that

$$gn = (gng^{-1})g \text{ and } ng = (ngn^{-1})n$$

for every $g \in G$ and $n \in N$. It follows $gN \subseteq Ng$ and $nG \subseteq Gn$ respectively for every $g \in G$ and $n \in N$. Therefore (N, G) is a product pair of G by definition.

Example 6.1.6 (Partial transformation monoids)**.** Consider $\mathcal{T}_n$, the full transformation monoid on $\{1, \ldots, n\}$, and the submonoid $\mathcal{M}_n$ of $\mathcal{PT}_n$ with elements

$$\mathcal{M}_n = \left\{ \mathrm{id}_A : A \subseteq \{1, \ldots, n\} \right\}$$

where $\mathrm{id}_A : A \to A$ is defined by the rule that

$$i\,\mathrm{id}_A = \begin{cases} i & \text{if } i \in A; \\ \text{undefined} & \text{otherwise.} \end{cases}$$

Let $\alpha \in \mathcal{T}_n$ and $A \subseteq \{1, \ldots, n\}$ so that $\mathrm{id}_A \in \mathcal{M}_n$. We set

$$B = (\operatorname{im} \alpha \cap A)\alpha^{-1}$$

so that $\mathrm{id}_B \in \mathcal{M}_n$. Then we see that

$$\begin{aligned} x \in \mathrm{dom}(\alpha\, \mathrm{id}_A) &\iff x \in (\mathrm{im}\, \alpha \cap \mathrm{dom}\, \mathrm{id}_A)\alpha^{-1} \\ &\iff x \in (\mathrm{im}\, \alpha \cap A)\alpha^{-1} \\ &\iff x \in (\mathrm{im}\, \alpha \cap A)\alpha^{-1} \cap \mathrm{dom}\, \alpha \\ &\iff x \in B \cap \mathrm{dom}\, \alpha \\ &\iff x \in \mathrm{im}\, \mathrm{id}_B \cap \mathrm{dom}\, \alpha \\ &\iff x \in (\mathrm{im}\, \mathrm{id}_B \cap \mathrm{dom}\, \alpha)(\mathrm{id}_B)^{-1} \\ &\iff x \in \mathrm{dom}(\mathrm{id}_B\, \alpha). \end{aligned}$$

Therefore we have $\alpha\mathcal{M}_n \subseteq \mathcal{M}_n\alpha$ for every $\alpha \in \mathcal{T}_n$ and so $(\mathcal{M}_n, \mathcal{T}_n)$ is a left-product pair of $\mathcal{PT}_n$.

The benefit of identifying such (left-, right-) product pairs of subsemigroups within some common oversemigroup is summarised in the next result.

Lemma 6.1.7. *Let U be a semigroup with subsemigroups S and T such that (S, T) is a left-product pair of U. Then ST is a subsemigroup of U.*

Proof. Let U be a semigroup with subsemigroups S and T such that (S, T) is a left-product pair of U. Then we see that

$$(xa)(yb) = x(ay)b = x(za)b = (xz)(ab)$$

for every $x, y \in S$, $a, b \in T$ and $z \in S$ such that $ay = za$. Therefore ST is a subsemigroup of U. $\square$

Note that the choice of U in Definition 6.1.1 is not unique; any oversemigroup of U will also contain (S, T) as a left-product pair. From this point, we will omit explicit mention of a semigroup U, as in Definition 6.1.1, whenever we suppose such a left-product (right-product) pair exists. Certainly, not every semigroup admits a left-product or right-product pair; considering any left zero or right zero semigroup respectively. We now concentrate on a special kind of (left-, right-) product pairs of semigroups that will become the main focus of this chapter.

Definition 6.1.8. Let (S,T) be a left-product pair. Then (S,T) is *left-unique* if

$$xa = yb \Longrightarrow x = y$$

for every $x, y \in S$ and $a, b \in T$. A left-product pair being *right-unique* is dually defined.

The next result follows in a straightforward way.

Lemma 6.1.9. *Let S and T be semigroups such that (S,T) forms a left-unique left-product pair. Then for every $a \in T$ and $x \in S$, there exists a unique $y \in S$ such that $ax = ya$.*

Proof. Let S and T be semigroups such that (S,T) forms a left-unique left-product pair. As (S,T) is a left-product pair, it follows that there exists a $y \in T$ such that $ax = ya$ for every $a \in T$ and $x \in S$. Using the fact that (S,T) is left-unique, if there also exists some $z \in S$ such that $ax = za$ then $ya = za$ and so $y = z$. $\square$

We conclude this section with an interesting observation regarding left-unique product pairs.

Theorem 6.1.10. *Let S and T be semigroups such that (S,T) is a right-unique product pair of U. Then $xa = ax$ for every $x \in S$ and $a \in T$.*

Proof. Let S and T be semigroups such that (S,T) is a left-unique product pair. For every $a \in T$ and $x \in S$ there exists $y \in S$ and $b \in T$ such that $xa = bx$ and $bx = yb$ (by virtue of (S,T) being a right-product and left-product pair, respectively). As (S,T) is right-unique, this implies that $a = b$ and so

$$xa = bx = ax$$

for every $x \in S$ and $a \in T$. $\square$

6.2 External setting

The aim of this section is to construct congruences on external semidirect products of semigroups such that elements of the semidirect product, equivalent under the congruences, have the same first coordinate. We start by defining an important property of equivalence relations.

Definition 6.2.1. Let S and T be semigroups such that $S \rtimes T$ exists. We say that a family $R = \{\rho_x : x \in S\}$ of equivalence relations on T is *right-aligned* if the following properties are satisfied:

(RA1) if $(a,b) \in \rho_x$ then $x\,{}^a y = x\,{}^b y$;

(RA2) if $(a,b) \in \rho_x$ then $(ca, cb) \in \rho_{y\,{}^c x}$;

(RA3) if $(a,b) \in \rho_x$ then $(ac, bc) \in \rho_{x\,{}^a y}$;

for every $x, y \in S$ and $a, b, c \in T$.

We highlight that this notion of right-aligned is unrelated to the notion of finitely aligned for semigroups in [25]. Next, we give a name to a certain kind of congruence on a semidirect product of semigroups that will be important throughout this chapter.

Definition 6.2.2. Let S and T be semigroups such that $S \rtimes T$ exists and such that θ is a congruence on $S \rtimes T$. We say θ is *left-unique* if

$$\big((x,a),(y,b)\big) \in \theta \Longrightarrow x = y$$

for every $x, y \in S$ and $a, b \in T$.

Using our notion of right-aligned equivalence relations, we proceed by constructing left-unique congruence on a semidirect product of semigroups.

Lemma 6.2.3. *Let S and T be semigroups such that $S \rtimes T$ exists. Suppose that there is a right-aligned family $R = \{\rho_x : x \in S\}$ of equivalence relations on T. Then the relation ρ, given by the rule*

$$\big((x,a),(y,b)\big) \in \rho \Longleftrightarrow x = y \text{ and } (a,b) \in \rho_x$$

defines a left-unique congruence on $S \rtimes T$.

Proof. To see that ρ defines an equivalence relation, we use the fact that ρ_x is an equivalence relation on T for every $x \in S$.

To see that ρ is a congruence on $S \rtimes T$, we suppose $((x,a),(y,b)) \in \rho$; this implies that $x = y$ and $(a,b) \in \rho_x$ by definition. Let $(z,c) \in S \rtimes T$ and consider the products

$$(z,c)(x,a) = (z\,{}^{c}x, ca) \text{ and } (z,c)(y,b) = (z\,{}^{c}y, cb).$$

It follows from $x = y$ that $z\,{}^{c}x = z\,{}^{c}y$ and we deduce from (RA2) that $(ca, cb) \in \rho_{z\,{}^{c}x}$. Hence ρ is a left congruence. To verify that ρ is also a right congruence, we consider the products

$$(x,a)(z,c) = (x\,{}^{a}z, ac) \text{ and } (y,b)(z,c) = (y\,{}^{b}z, bc).$$

That $x\,{}^{a}z = y\,{}^{b}z$ is derived from (RA1) and $x = y$. Hence we get that $(ac, bc) \in \rho_{x\,{}^{a}z}$ from (RA3). Therefore ρ is a (right) congruence on $S \rtimes T$. $\square$

Therefore, for any semigroups S and T such that $S \rtimes T$ exists, we have constructed a left-unique congruence on $S \rtimes T$ by utilising a right-aligned family of right congruences. Conversely, we show that the existence of a left-unique congruence on a semidirect product gives rise to a right-aligned family of equivalence relations.

Lemma 6.2.4. *Let S and T be semigroups such that $S \rtimes T$ exists. If ρ is a left-unique congruence on $S \rtimes T$ then there exists a right-aligned family of equivalence relations $R = \{\rho_x : x \in S\}$ on T where*

$$\rho_x = \Big\{(a,b) \in T \times T : \big((x,a),(x,b)\big) \in \rho\Big\}$$

for every $x \in S$.

Proof. Let S and T be semigroups and ρ be a left-unique congruence on $S \rtimes T$.

To prove that $R = \{\rho_x : x \in S\}$ is a right-aligned family of equivalence relations on T, we suppose first that $(a,b) \in \rho_x$ so that $((x,a),(x,b)) \in \rho$.

For any $y \in S$ and $c \in T$ we have the product

$$(x,a)(y,c) = (x\,{}^{a}y, ac) \text{ and } (x,b)(y,c) = (x\,{}^{b}y, bc).$$

By virtue of ρ being a (right) congruence, we have $((x\,{}^{a}y, ac), (x\,{}^{b}y, bc)) \in \rho$ and by virtue of ρ being left-unique we have $x\,{}^{a}y = x\,{}^{b}y$. Hence (RA1) is satisfied. In turn, we have

$$\big((x\,{}^{a}y, ac), (x\,{}^{a}y, bc)\big) = \big((x\,{}^{a}y, ac), (x\,{}^{b}y, bc)\big) \in \rho.$$

This gives us that $(ac, bc) \in \rho_{x\,{}^{a}y}$ which satisfies (RA3). Next we consider the product

$$(y,c)(x,a) = (y\,{}^{c}x, ca) \text{ and } (y,c)(x,b) = (y\,{}^{c}x, cb).$$

This gives us by definition that $(ca, cb) \in \rho_{y\,{}^{c}x}$ and so (RA2) is satisfied. We have shown that R defines a right-aligned family of equivalence relations on T as required. $\square$

6.2.1 The case that S and T are monoids

In the case where S and T are both monoids, the conditions required for a set of equivalence relations to be right-aligned are stronger and we examine them here.

Definition 6.2.5. Let S and T be monoids such that $S \bowtie T$ exists. We say that a family $R = \{\rho_x : x \in S\}$ of equivalence relations on T is *strongly right-aligned* if the following properties are satisfied:

(SRA1) if $(a, b) \in \rho_x$ then $x\,{}^{a}y = x\,{}^{b}y$;

(SRA2) if $(a, b) \in \rho_x$ then $(ca, cb) \in \rho_{{}^{c}x}$;

(SRA3) if $(a, b) \in \rho_x$ then $(ac, bc) \in \rho_{x\,{}^{a}y}$;

(SRA4) $\rho_x \subseteq \rho_{yx}$

for every $x, y \in S$ and $a, b, c \in T$.

Theorem 6.2.6. *Let S and T be monoids such that $S \rtimes T$ exists. Let $R = \{\rho_x : x \in S\}$ be a strongly right-aligned family of equivalence relations on T. Then R is a right-aligned family of equivalence relations on T.*

Proof. Suppose that S and T are monoids such that $S \rtimes T$ exists. Let $R = \{\rho_x : x \in S\}$ be a strong right-aligned family of equivalence relations on T. Clearly if (SRA1) holds then (RA1) holds as they are exactly the same. Similarly if (SRA3) holds then (RA3) holds. It remains to show that (RA2) is satisfied.

If $(a, b) \in \rho_x$ then $(ca, cb) \in \rho_{^c x}$ for every $c \in T$ by (SRA2). It follows from (SRA4) that $(ca, cb) \in \rho_{y\,^c x}$ for every $y \in S$. Therefore we have shown that (RA2) is satisfied. $\square$

6.3 Internal setting

We turn our attention to internal semidirect products. Specifically, we start with an internal semidirect product of semigroups ST and construct a congruence ρ such that $ST \simeq (S \rtimes T)/\rho$. We begin by building on Lemma 6.1.9 mentioned in Section 6.1.

Lemma 6.3.1. *Let S and T be semigroups such that (S, T) is a left-unique left-product pair. For every $a \in T$ and $x \in S$ with $ax = ya$, by setting $^a x = y$, we define a left action of T on S by homomorphisms.*

Proof. First, we verify that this defines a left action of T on S. For every $a, b \in T$ and $x \in S$, we see that

$$a(bx) = a(^b xb) = (a\,^b x)b = \big(^a(^b x)a\big)b \text{ and } (ab)x = {}^{ab}xab.$$

If follows from (S, T) being left-unique that $^a(^b x) = {}^{ab}x$ for every $a, b \in T$ and $x \in S$. Hence this defines a left action of T on S and we are left to check that this is a left action by homomorphisms. To see this, we notice

$$a(xy) = {}^a(xy)a \text{ and } (ax)y = (^a xa)y = {}^a x(ay) = {}^a x(\,^a ya) = (^a x\,^a y)a$$

which gives ${}^{a}(xy)a = ({}^{a}x\,{}^{a}y)a$. This implies, using the fact that (S,T) is left-unique, that ${}^{a}(xy) = {}^{a}x\,{}^{a}y$ for every $a \in T$ and $x, y \in S$. Therefore this defines a left action of T on S by homomorphisms as required. $\square$

Lemma 6.3.2. *Let S and T be semigroups such that (S,T) is a left-unique left-product pair. Let ρ_x be a relation on T defined as*

$$\rho_x = \big\{(a,b) \in T \times T : xa = xb\big\}$$

for every $x \in S$. Then $R = \{\rho_x : x \in S\}$ is a right-aligned family of right congruences on T.

Proof. To show that R is a right-aligned family of right congruences on T, we must show that each $\rho_x \in R$ satisfies (RA1), (RA2) and (RA3). For every $x \in S$, if $(a,b) \in \rho_x$ then $xa = xb$. It follows that

$$x\,{}^{a}y = (xa)y = (xb)y = x\,{}^{b}y$$

and so (RA1) is satisfied.

If $(a,b) \in \rho_x$ then $xa = xb$ and we see that

$$\begin{aligned}(y\,{}^{c}x)(ca) &= y({}^{c}xc)a = y(cx)a = yc(xa)\\ &= yc(xb) = y(cx)b = y({}^{c}xc)b = (y\,{}^{c}x)(cb)\end{aligned}$$

for every $y \in S$ and $c \in T$. This means that $(y\,{}^{c}x)(ca) = (y\,{}^{c}x)(cb)$ which gives $(ca, cb) \in \rho_{y\,{}^{c}x}$ by definition. Therefore (RA2) is satisfied.

Lastly, if $(a,b) \in \rho_x$ then $xa = xb$ and we see that

$$\begin{aligned}(x\,{}^{a}y)(ac) &= x({}^{a}ya)c = x(ay)c = (xa)yc\\ &= (xb)yc = x(by)c = x({}^{b}yb)c = (x\,{}^{b}y)(bc)\end{aligned}$$

for every $y \in S$ and $c \in T$. From (RA1) we know that $x\,{}^{a}y = x\,{}^{b}y$ and so $(x\,{}^{a}y)(ac) = (x\,{}^{a}y)(bc)$. This gives us that $(ac, bc) \in \rho_{x\,{}^{a}y}$ by definition and so (RA3) is satisfied. It is clear that ρ_x as defined above is a right congruence on T for every $x \in S$. $\square$

6.4 A main result

In this section, we show how one can amalgamate the results of Section 6.2.

Theorem 6.4.1. *Let S and T be subsemigroups such that (S,T) is a left-unique left-product pair. Then there exists a left action of T on S by homomorphisms given by ${}^a x = y$, where y is the unique element of S such that $ax = ya$, for all $x \in S$ and $a \in T$. Moreover, if*

$$\rho_x = \big\{(a,b) \in T \times T : xa = xb\big\}$$

then $R = \{\rho_x : x \in S\}$ is a right-aligned family of right congruences on T. If ρ is the relation on $S \rtimes T$ given by

$$\big((x,a),(y,b)\big) \in \rho \Longleftrightarrow x = y \textit{ and } (a,b) \in \rho_x$$

then ρ is a left-unique congruence on $S \rtimes T$ with $ST \simeq (S \rtimes T)/\rho$.

Proof. Let (S,T) be a left-unique left-product pair. By Lemma 6.3.1, T acts on the left of S by homomorphisms such that $ax = {}^a x a$ for all $x \in S$ and $a \in T$. Thus we can form the semidirect product $S \rtimes T$. It follows from Lemma 6.3.2 that $R = \{\rho_x : x \in S\}$ where

$$\rho_x = \big\{(a,b) \in T \times T : xa = xb\big\}$$

is a right-aligned family of right congruences on T. In addition, we showed in Lemma 6.2.3 that the relation ρ on $S \rtimes T$ given by

$$\big((x,a),(y,b)\big) \in \rho \Longleftrightarrow x = y \text{ and } (a,b) \in \rho_x$$

is a (left-unique) congruence. Hence, we can form the quotient semigroup $(S \rtimes T)/\rho$. It remains to show that the map $\phi : ST \to (S \rtimes T)/\rho$ given by

$$\phi : ST \to (S \rtimes T)/\rho : (xa)\phi = \big[(x,a)\big]$$

is an isomorphism.

First, we show ϕ is well-defined. If $x, y \in S$ and $a, b \in T$ such that $xa = yb$ then $x = y$ from the fact that (S, T) is left-unique. In turn this gives us that $(a, b) \in \rho_x$ and so $((x, a), (y, b)) \in \rho$. Therefore $(xa)\phi = (yb)\phi$. That ϕ is onto is clear from its definition. To show that ϕ is one-to-one and well-defined we see

$$\begin{aligned}(xa)\phi = (yb)\phi &\iff \big[(x, a)\big] = \big[(y, b)\big] \\ &\iff \big((x, a), (y, b)\big) \in \rho \\ &\iff x = y \text{ and } (a, b) \in \rho_x \\ &\iff xa = yb\end{aligned}$$

for every $x, y \in S$ and $a, b \in T$.

Lastly, to show that ϕ is a homomorphism, we have

$$\begin{aligned}\big((xa)(yb)\big)\phi &= \big((x\,{}^{a}y)(ab)\big)\phi \\ &= \big[(x\,{}^{a}y, ab)\big] \\ &= \big[(x, a)(y, b)\big] \\ &= \big((xa)\phi\big)\big((yb)\phi\big)\end{aligned}$$

for every $x, y \in S$ and $a, b \in T$. Therefore ϕ is an isomorphism and so $ST \simeq (S \rtimes T)/\rho$. $\square$

6.5 Semigroup presentations

We now turn our attention to obtaining semigroup presentations for semigroups of the form ST where (S, T) is a left-unique left-product pair, using results from Subsection 6.2 and Section 6.3. We start by recalling from Section 6.3 that, for a semigroup product of this kind, we may immediately form the semidirect product $S \rtimes T$ where ${}^{a}x \in S$ is defined to be the unique element satisfying $ax = {}^{a}xa$ for every $x \in S$ and $a \in T$.

Lemma 6.5.1. *Let $\Psi : S \rtimes T \to ST$ be given by*

$$(x, a)\Psi = xa$$

for every $x \in S$ and $a \in T$. Then Ψ is a surjective homomorphism.

Proof. For every $x, y \in S$ and $a, b \in T$ we have

$$\big((x, a)(y, b)\big)\Psi = (x\,{}^{a}y, ab)\Psi = x\,{}^{a}yab = xayb = \big((x, a)\Psi\big)\big((y, b)\Psi\big)$$

and so Ψ is a homomorphism. It is clear that Ψ is surjective since for every $xa \in ST$, with $x \in S$ and $a \in T$, we have $(x, a) \in S \rtimes T$ such that $(x, a)\Psi = xa$. $\square$

It is clear to verify that Lemma 6.5.1 follows directly from Theorem 6.4.1. We recall in this next result the congruence ρ on $S \rtimes T$ defined in Section 6.2.

Lemma 6.5.2. *Let S and T be semigroups such that (S, T) is a left-unique left-product pair. Then* $\ker \Psi = \rho$.

Proof. For every $x, y \in S$ and $a, b \in T$ we have

$$\begin{aligned}
\big((x, a), (y, b)\big) \in \ker \Psi &\iff (x, a)\Psi = (y, b)\Psi \\
&\iff xa = yb \\
&\iff x = y \text{ and } (a, b) \in \rho_x \\
&\iff \big((x, a), (y, b)\big) \in \rho
\end{aligned}$$

and so $\ker \Psi = \rho$. $\square$

We introduce some conventions and notation which will be useful throughout the remainder of this chapter. Let S and T be semigroups such that $S \rtimes T$ exists and has semigroup presentation $\langle X : R \rangle$ via ϕ. As ϕ is surjective, we let $w_{(x,a)}$ be a fixed word over X such that

$$w_{(x,a)}\phi = (x, a)$$

for every $x \in S$ and $a \in T$.

As a convention, for any formal symbol $\star$, we will write $u \sim_\star v$ and $u \approx_\star v$ instead of $(u,v) \in R_\star$ and $(u,v) \in R_\star^\sharp$ respectively.

Theorem 6.5.3. *Let S and T be semigroups such that (S,T) is a left-unique left-product pair and suppose that $S \rtimes T$ has presentation $\langle X : R \rangle$ via ϕ. Let $\Psi : S \rtimes T \to ST$ be given by $(x,a)\Psi = xa$ and suppose that*

$$\rho_x = \Big\{ \big((x,a),(x,b)\big) : xa = xb \Big\}.$$

If we define

$$R_\rho = \big\{ (w_{(x,a)}, w_{(x,b)}) : x \in S, (a,b) \in \rho_x \big\}$$

then ST has presentation $\langle X : R \cup R_\rho \rangle$ via $\phi\Psi$.

Proof. That the map $\phi\Psi$ is a surjective homomorphism follows from the fact that ϕ and Ψ are both surjective homomorphisms, using Lemma 6.5.1. For convenience we will set $\alpha = \phi\Psi$ for the remainder of this proof. We verify now that $(R \cup R_\rho)^\sharp = \ker \alpha$.

As $S \rtimes T$ has presentation $\langle X : R \rangle$ via ϕ, it is immediate that if $(u,v) \in R$ then $(u,v) \in \ker \phi$ by definition. This in turn gives us that $(u,v) \in \ker \alpha$ since $\ker \phi \subseteq \ker \alpha$.

On the other hand, let $(u,v) \in R_\rho$ be such that $u = w_{(x,a)}$ and $v = w_{(x,b)}$ with $(a,b) \in \rho_x$. From the definition of ρ_x, we have $xa = xb$ and so

$$\begin{aligned} u\alpha = (u\phi)\Psi = (w_{(x,a)}\phi)\Psi = (x,a)\Psi = xa = xb \\ = (x,b)\Psi = (w_{(x,b)}\phi)\Psi = (v\phi)\Psi = v\alpha. \end{aligned}$$

This gives us that $(u,v) \in \ker \alpha$. Altogether we have shown that $R \cup R_\rho \subseteq \ker \alpha$ and hence $(R \cup R_\rho)^\sharp \subseteq \ker \alpha$.

For the reverse inclusion, we suppose $(u,v) \in \ker \alpha$. We suppose that

$$u\phi = (x,a) = w_{(x,a)}\phi \text{ and } v\phi = (y,b) = w_{(y,b)}\phi$$

for $x, y \in S$ and $a, b \in T$. It implies from $u\alpha = v\alpha$ that $(x,a)\Psi = (y,b)\Psi$ and so $xa = yb$. Since (S,T) is left-unique, it follows that $x = y$ which means $(a, b) \in \rho_x$ by definition. Hence, we have

$$(w_{(x,a)}w_{(y,b)}) = (w_{(x,a)}, w_{(x,b)}) \in R_\rho.$$

Therefore

$$u \approx w_{(x,a)} \sim_\rho w_{(y,b)} \approx v$$

and so $\ker \alpha \subseteq (R \cup R_\rho)^\sharp$ as required. $\square$

For the next main result of this section, our intention is to reduce the size of the set of relations R_ρ by considering a certain special case. For each $x \in S$ we define a subset $\overline{\rho}_x \subseteq \rho_x$ such that $\overline{\rho}_x$ generates ρ_x as a right congruence.

Theorem 6.5.4. *Let S and T be semigroups such that (S, T) is a left-unique left-product pair and suppose that $S \rtimes T$ has presentation $\langle X : R \rangle$ via ϕ. Suppose that for every $a \in T$ there exists an $x \in S$ such that $a = xa$. Let $\Psi : S \rtimes T \to ST$ be given by $(x, a)\Psi = xa$ and suppose that*

$$\rho_x = \Big\{ \big((x, a), (x, b)\big) : xa = xb \Big\}$$

is generated as a right congruence by $\overline{\rho}_x \subseteq \rho_x$. If we define

$$R_{\overline{\rho}} = \big\{ (w_{(x,a)}, w_{(x,b)}) : x \in S, (a, b) \in \overline{\rho}_x \big\}$$

then ST has presentation $\langle X : R \cup R_{\overline{\rho}} \rangle$ via $\phi\Psi$.

Proof. With Theorem 6.5.3 in mind, given that $R_{\overline{\rho}} \subseteq R_\rho$, we only need to check that $R_\rho \subseteq (R \cup R_{\overline{\rho}})^\sharp$. Let $x \in S$ and $a, b \in T$ such that $(a, b) \in \rho_x$, so that $(w_{(x,a)}, w_{(x,b)}) \in R_\rho$. Given that $\overline{\rho}_x$ generates ρ_x as a right congruence, either $a = b$ or there exists a finite sequence of the form

$$a = a_1, \ldots, a_n = b \tag{6.1}$$

such that $a_i = c_i q_i$ and $a_{i+1} = d_i q_i$ for some $(c_i, d_i) \in \overline{\rho}_x \cup \overline{\rho}_x^{-1}$ and $q_i \in T^1$ for each $1 \leqslant i \leqslant n-1$. We emphasise here that $(w_{(x,c_i)}, w_{(x,d_i)}) \in R_{\overline{\rho}} \cup R_{\overline{\rho}}^{-1}$

by definition. If $a = b$ then clearly $(w_{(x,a)}, w_{(x,b)}) \in R_{\overline{\rho}}$ so we proceed by supposing that $a \neq b$ so that the sequence 6.1 exists. For the remainder of this proof, we fix some $1 \leqslant i \leqslant n$.

If $q_i = 1$, then $a_i = c_i$ and $a_{i+1} = d_i$ so that

$$w_{(x,a_i)} = w_{(x,c_i)} \sim_{\overline{\rho}} w_{(x,d_i)} = w_{(x,a_{i+1})}. \tag{6.2}$$

Therefore we have shown that $(w_{(x,a_i)}, w_{(x,a_{i+1})}) \in R_{\overline{\rho}}$ in this first case.

Now suppose that $q_i \in T$. By assumption, there exists $y \in S$ such that $q_i = yq_i$. We consider the product

$$(x, c_i)(y, q_i) = (x^{\,c_i}y, c_iq_i) = (x^{\,c_i}y, a_i).$$

By virtue of Ψ being a homomorphism from Lemma 6.5.1, we see that

$$x^{\,c_i}ya_i = (x^{\,c_i}y, a_i)\Psi = \big((x, c_i)\Psi\big)\big((y, q_i)\Psi\big) = xc_iyq_i = xc_iq_i = xa_i.$$

Hence, by the fact (S, T) is left-unique, this implies that $x^{\,c_i}y = x$. This immediately gives us that

$$(x, c_i)(y, q_i) = (x^{\,c_i}y, a_i) = (x, a_i). \tag{6.3}$$

A similar argument shows that

$$(x, d_i)(y, q_i) = (x^{\,d_i}y, a_{i+1}) = (x, a_{i+1}). \tag{6.4}$$

It follows from Equation 6.3 and Equation 6.4 that

$$w_{(x,c_i)}w_{(y,q_i)} \approx w_{(x,a_i)} \text{ and } w_{(x,d_i)}w_{(y,q_i)} \approx w_{(x,a_{i+1})}$$

respectively. In turn, we have that

$$w_{(x,a_i)} \approx w_{(x,c_i)}w_{(y,q_i)} \approx_{\overline{\rho}} w_{(x,d_i)}w_{(y,q_i)} \approx w_{(x,a_{i+1})}. \tag{6.5}$$

Altogether, Equation 6.2 and Equation 6.5 show that $(w_{(x,a_i)}, w_{(x,a_{i+1})}) \in$

$(R \cup R_{\overline{\rho}})^\sharp$ for every $1 \leqslant i \leqslant n$. Therefore we have $R_\rho \subseteq (R \cup R_{\overline{\rho}})^\sharp$ and so $(R \cup R_\rho)^\sharp = (R \cup R_{\overline{\rho}})^\sharp$. We already have shown from Theorem 6.5.3 that $(R \cup R_\rho)^\sharp = \ker(\phi\Psi)$ and so we are done. □

Note that the assumption in the previous claim can be simplified in certain circumstances, as seen in the following result.

Theorem 6.5.5. *Let S and T be semigroups such that (S, T) is a left-unique left-product pair. If S is such that every element of S has a left identity, then the following conditions are equivalent:*

- *for every $a \in T$ there exists $x \in S$ such that $a = xa$;*
- *$T \subseteq ST$.*

Proof. The forwards implication is straightforward to verify. For the reverse direction, suppose $T \subseteq ST$ and let $a \in T$. Since $T \subseteq ST$, it follows that $a = yb$ for some $y \in S$ and $b \in T$. By assumption, if $x \in S$ satisfies $y = xy$, then $a = yb = (xy)b = xa$. □

For instance, Theorem 6.5.5 includes the cases where S is regular or indeed where S is a monoid.

6.6 Applications

Now we consider a number of applications of the results obtained in Section 6.5. We demonstrate how a number of semigroups, of the form ST, arise as left-unique left-product pairs (S, T). As we will see, there are a whole host of important classes of semigroups that arise in this way.

6.6.1 Singular part of the partial endomorphism monoid of a free G-act of finite rank

We introduce the notion of a free G-act for a group G.

Definition 6.6.1. Let G be a group and fix $n \in \mathbb{N}$. Let $\{x_i : 1 \leqslant i \leqslant n\}$ be a set of formal symbols. The *free G-act of rank n* is the set

$$F_n(G) = \{gx_i : g \in G, 1 \leqslant i \leqslant n\}$$

where two elements $gx_i, hx_j \in F_n(G)$ are equal if and only if $g = h$ and $i = j$. We define ${}^{g}(hx_i) = (gh)x_i$ for every $g, h \in G$ and $1 \leqslant i \leqslant n$.

Definition 6.6.2. The *singular part of the partial endomorphism monoid* of a free G-act of rank n is a semigroup denoted by $\mathcal{SP}\mathrm{End}\, F_n(G)$ with elements

$$\mathcal{SP}\mathrm{End}\, F_n(G) = \big\{\alpha : F_n(G) \to F_n(G) \mid \operatorname{im}\alpha \subset F_n(G)\big\}$$

under composition of functions.

We recall the following semigroups:

$$\begin{aligned}
\mathcal{PT}_n &= \big\{\alpha : A \to B \mid A, B \subseteq \{1, \ldots, n\}\big\} \\
\mathcal{SPT}_n &= \big\{\alpha \in \mathcal{PT}_n : \operatorname{im}\alpha \subset \{1, \ldots, n\}\big\} \\
\mathcal{T}_n &= \big\{\alpha \in \mathcal{PT}_n : \operatorname{dom}\alpha = \{1, \ldots, n\}\big\} \\
\mathcal{ST}_n &= \big\{\alpha \in \mathcal{T}_n : \operatorname{im}\alpha \subset \{1, \ldots, n\}\big\}
\end{aligned}$$

For a fixed group G and $n \in \mathbb{N}$, we let $\mathcal{G}$ and $\mathcal{G}^0$ denote the direct product of n copies of G and the direct product of n copies of the (0-group) G^0 respectively.

Lemma 6.6.3. *Let G be a group and $n \in \mathbb{N}$. For every $(g_1, \ldots, g_n) \in \mathcal{G}^0$ and $\alpha \in \mathcal{PT}_n$, we define*

$$^{\alpha}(g_1, \ldots, g_n) = (g_{1\alpha}, \ldots, g_{n\alpha}) \tag{6.6}$$

where $g_{i\alpha} = 0$ if $i \notin \operatorname{dom}\alpha$. Then Equation 6.6 defines a left action of $\mathcal{PT}_n$ on $\mathcal{G}^0$.

Proof. If $\alpha, \beta \in \mathcal{PT}_n$ and $(g_1, \ldots, g_n) \in \mathcal{G}^0$ then

$$\begin{aligned}
{}^{\alpha}\big({}^{\beta}(g_1, \ldots, g_n)\big) &= {}^{\alpha}(g_{1\beta}, \ldots, g_{n\beta}) \\
&= {}^{\alpha}(h_1, \ldots, h_n) \text{ where } h_i = g_{i\beta} \\
&= (h_{1\alpha}, \ldots, h_{n\alpha}) \\
&= (g_{(1\alpha)\beta}, \ldots, g_{(n\alpha)\beta}) \\
&= (g_{1(\alpha\beta)}, \ldots, g_{n(\alpha\beta)}) \\
&= {}^{\alpha\beta}(g_1, \ldots, g_n).
\end{aligned}$$

It is routine to see that such an argument holds where $g_i = 0$. So Equation 6.6 defines a left action of $\mathcal{PT}_n$ on $\mathcal{G}^0$. $\square$

Lemma 6.6.4. *Let G be a group and $n \in \mathbb{N}$. Then Equation 6.6 defines a left action of $\mathcal{PT}_n$ on $\mathcal{G}^0$ by homomorphisms.*

Proof. We have already proved in Lemma 6.6.3 that Equation 6.6 defines a left action of $\mathcal{PT}_n$ on $\mathcal{G}^0$. To show that this is, in fact, a left action by homomorphisms, if $\alpha \in \mathcal{PT}_n$ and $(g_1, \ldots, g_n), (h_1, \ldots, h_n) \in \mathcal{G}^0$ then

$$\begin{aligned}
{}^{\alpha}\big((g_1, \ldots, g_n)(h_1, \ldots, h_n)\big) &= {}^{\alpha}(g_1h_1, \ldots, g_nh_n) \\
&= {}^{\alpha}(k_1, \ldots, k_n) \text{ where } k_i = g_ih_i \\
&= (k_{1\alpha}, \ldots, k_{n\alpha}) \\
&= (g_{1\alpha}h_{1\alpha}, \ldots, g_{n\alpha}h_{n\alpha}) \\
&= (g_{1\alpha}, \ldots, g_{n\alpha})(h_{1\alpha}, \ldots, h_{n\alpha}) \\
&= \big({}^{\alpha}(g_1, \ldots, g_n)\big)\big({}^{\alpha}(h_1, \ldots, h_n)\big).
\end{aligned}$$

One can verify that this argument holds where $g_i = 0$ or $h_i = 0$ for some $1 \leqslant i \leqslant n$. Therefore Equation 6.6 defines a left action of $\mathcal{PT}_n$ on $\mathcal{G}^0$ by homomorphisms. $\square$

Hence, we can form the semidirect product $\mathcal{G}^0 \rtimes \mathcal{PT}_n$ using Lemma 6.6.4. We now consider a special subsemigroup $\mathcal{P}_n$ of $\mathcal{G}^0 \rtimes \mathcal{PT}_n$.

Lemma 6.6.5. *Let G be a group and $n \in \mathbb{N}$. Then*

$$\mathcal{P}_n = \Big\{ \big((g_1, \dots, g_n), \alpha \big) \in \mathcal{G}^0 \times \mathcal{SPT}_n : g_i \neq 0 \Longleftrightarrow i \in \operatorname{dom} \alpha \Big\}$$

is a subsemigroup of $\mathcal{G}^0 \rtimes \mathcal{PT}_n$.

Proof. Let $\alpha, \beta \in \mathcal{SPT}_n$ and $(g_1, \dots, g_n), (h_1, \dots, h_n) \in \mathcal{G}^0$ such that

$$\big((g_1, \dots, g_n), \alpha \big), \big((h_1, \dots, h_n), \beta \big) \in \mathcal{P}_n.$$

Then the product

$$\begin{aligned} \big((g_1, \dots, g_n), \alpha \big) \big((h_1, \dots, h_n), \beta \big) &= \big((g_1, \dots, g_n)\, {}^{\alpha}(h_1, \dots, h_n), \alpha\beta \big) \\ &= \big((g_1, \dots, g_n)(h_{1\alpha}, \dots, h_{n\alpha}), \alpha\beta \big) \\ &= \big((g_1 h_{1\alpha}, \dots, g_n h_{n\alpha}), \alpha\beta \big). \end{aligned}$$

We are left to show that $g_i h_{i\alpha} = 0$ if and only if $i \notin \operatorname{dom}(\alpha\beta)$. To check this, we see that

$$g_i h_{i\alpha} = 0 \Longleftrightarrow g_i = 0 \text{ or } h_{i\alpha} = 0.$$

We deal with each of these cases separately. In the case that $g_i = 0$, this occurs if and only if $i \notin \operatorname{dom} \alpha$ and in turn this gives $i \notin \operatorname{dom}(\alpha\beta)$ as $\operatorname{dom}(\alpha\beta) \subseteq \operatorname{dom} \alpha$. On the other hand, if $h_{i\alpha} = 0$ then $i\alpha \notin \operatorname{dom} \beta$. This implies that $i \notin \operatorname{dom}(\alpha\beta)$.

For the reverse implication, suppose that $i \notin \operatorname{dom}(\alpha\beta)$. This implies that $i \notin (\operatorname{im} \alpha \cap \operatorname{dom} \beta)\alpha^{-1}$. If $i \notin \operatorname{dom} \alpha$ then $g_i = 0$ and if $i\alpha \notin \operatorname{dom} \beta$ then $h_{i\alpha} = 0$. In each case we have that $g_i h_{i\alpha} = 0$. □

For any $A \subseteq \{1, \dots, n\}$, we define

$$a_i = \begin{cases} 1 & \text{if } i \in A; \\ 0 & \text{otherwise.} \end{cases}$$

With this, we consider the subsemigroup $\mathcal{G} \rtimes \mathcal{ST}_n$ and the submonoid $\mathcal{E}_n$ of

$\mathcal{G}^0 \rtimes \mathcal{PT}_n$ where

$$\mathcal{E}_n = \Big\{ \big((a_1, \ldots, a_n), \mathrm{id}_A\big) : A \subseteq \{1, \ldots, n\} \Big\}$$

and use them in our next result.

Theorem 6.6.6. *Let G be a group and $n \in \mathbb{N}$. Then $(\mathcal{E}_n, \mathcal{G} \rtimes \mathcal{ST}_n)$ is a left-product pair of $\mathcal{G}^0 \rtimes \mathcal{PT}_n$.*

Proof. Let $A \subseteq \{1, \ldots, n\}$, $(g_1, \ldots, g_n) \in \mathcal{G}$ and $\alpha \in \mathcal{ST}_n$. Then we consider the product

$$\begin{aligned}
\big((g_1, \ldots, g_n), \alpha\big)\big((a_1, \ldots, a_n), \mathrm{id}_A\big) &= \big((g_1, \ldots, g_n)\,{}^{\alpha}(a_1, \ldots, a_n), \alpha\,\mathrm{id}_A\big) \\
&= \big((g_1, \ldots, g_n)(a_{1\alpha}, \ldots, a_{n\alpha}), \alpha\,\mathrm{id}_A\big) \\
&= \big((g_1 a_{1\alpha}, \ldots, g_n a_{n\alpha}), \alpha\,\mathrm{id}_A\big).
\end{aligned}$$

We set $B \subseteq \{1, \ldots, n\}$ to be

$$B = \big\{ i \in \{1, \ldots, n\} : i\alpha \in A \big\}$$

and recall our notation that

$$b_i = \begin{cases} 1 & \text{if } i \in B; \\ 0 & \text{otherwise.} \end{cases}$$

Then we see

$$\begin{aligned}
\big((b_1, \ldots, b_n), \mathrm{id}_B\big)\big((g_1, \ldots, g_n), \alpha\big) &= \big((b_1, \ldots, b_n)\,{}^{\mathrm{id}_B}(g_1, \ldots, g_n), \mathrm{id}_B\,\alpha\big) \\
&= \big((b_1, \ldots, b_n)(g_{1\,\mathrm{id}_B}, \ldots, g_{n\,\mathrm{id}_B}), \mathrm{id}_B\,\alpha\big) \\
&= \big((b_1 g_{1\,\mathrm{id}_B}, \ldots, b_n g_{n\,\mathrm{id}_B}), \mathrm{id}_B\,\alpha\big)
\end{aligned}$$

and so we are left to show that $\alpha\,\mathrm{id}_A = \mathrm{id}_B\,\alpha$ and $g_i a_{i\alpha} = b_i g_{i\,\mathrm{id}_B}$ for every $1 \leqslant i \leqslant n$. We consider the possibilities: $i\alpha \in A$ and $i\alpha \notin A$ for some fixed $1 \leqslant i \leqslant n$. If $i\alpha \in A$ then $i \in B$ by definition. It follows that $(i\alpha)\,\mathrm{id}_A = i\alpha = (i\,\mathrm{id}_B)\alpha$. In this case, we have $a_{i\alpha} = 1$ and $b_i = 1$ so that $g_i a_{i\alpha} = g_i = b_i g_{i\,\mathrm{id}_B}$. Conversely, if $i\alpha \notin A$ then $i \notin B$ by definition. This implies that $(i\alpha)\,\mathrm{id}_A$ and $(i\,\mathrm{id}_B)\alpha$ are both undefined. Here we have $a_{i\alpha} = 0$ and $b_i = 0$ so that $g_i a_{i\alpha} = 0 = b_i g_{i\,\mathrm{id}_B}$. $\square$

Theorem 6.6.7. *Let G be a group and $n \in \mathbb{N}$. Then $(\mathcal{E}_n, \mathcal{G} \rtimes \mathcal{ST}_n)$ is a left-unique left-product pair of $\mathcal{G}^0 \rtimes \mathcal{PT}_n$.*

Proof. We have already shown in Theorem 6.6.6 that $(\mathcal{E}_n, \mathcal{G} \rtimes \mathcal{ST}_n)$ is a left-product pair of $\mathcal{G}^0 \rtimes \mathcal{PT}_n$. Next, we show that $(\mathcal{E}_n, \mathcal{G} \rtimes \mathcal{ST}_n)$ is left-unique. Suppose $A, B \subseteq \{1, \ldots, n\}$ and $((g_1, \ldots, g_n), \alpha), ((h_1, \ldots, h_n), \beta) \in \mathcal{G} \rtimes \mathcal{ST}_n$ such that

$$\big((a_1, \ldots, a_n), \mathrm{id}_A\big)\big((g_1, \ldots, g_n), \alpha\big) = \big((b_1, \ldots, b_n), \mathrm{id}_B\big)\big((h_1, \ldots, h_n), \beta\big).$$

If this holds then

$$\big((a_1 g_{1\,\mathrm{id}_A}, \ldots, a_n g_{n\,\mathrm{id}_A}), \mathrm{id}_A\,\alpha\big) = \big((b_1 h_{1\,\mathrm{id}_B}, \ldots, b_n h_{n\,\mathrm{id}_B}), \mathrm{id}_B\,\beta\big).$$

It follows from $\mathrm{id}_A\,\alpha = \mathrm{id}_B\,\beta$ that $A = B$ and so $(\mathcal{E}_n, \mathcal{G} \rtimes \mathcal{ST}_n)$ is left-unique. $\square$

Theorem 6.6.8. *Let G be a group and $n \in \mathbb{N}$. Then $\mathcal{E}_n(\mathcal{G} \rtimes \mathcal{ST}_n) = \mathcal{P}_n$.*

Proof. Let $A \subseteq \{1, \ldots, n\}$, $(g_1, \ldots, g_n) \in \mathcal{G}$ and $\alpha \in \mathcal{ST}_n$. Consider the product

$$\begin{aligned}\big((a_1, \ldots, a_n), \mathrm{id}_A\big)\big((g_1, \ldots, g_n), \alpha\big) &= \big((a_1, \ldots, a_n)^{\,\mathrm{id}_A}(g_1, \ldots, g_n), \mathrm{id}_A\,\alpha\big) \\ &= \big((a_1, \ldots, a_n)(g_{1\,\mathrm{id}_A}, \ldots, g_{n\,\mathrm{id}_A}), \mathrm{id}_A\,\alpha\big) \\ &= \big((a_1 g_{1\,\mathrm{id}_A}, \ldots, a_n g_{n\,\mathrm{id}_A}), \mathrm{id}_A\,\alpha\big).\end{aligned}$$

We must show that $a_i g_{i\,\mathrm{id}_A} = 0$ if and only if $i \notin \mathrm{dom}(\mathrm{id}_A\,\alpha) = A$. We remark first that

$$a_i g_{i\,\mathrm{id}_A} = 0 \Longleftrightarrow a_i = 0 \text{ or } g_{i\,\mathrm{id}_A} = 0;$$

either of the conditions on the right hand side is equivalent to $i \notin A$.

We now show that every element of $\mathcal{P}_n$ can be written as a product of $\mathcal{E}_n$ and $\mathcal{G} \rtimes \mathcal{ST}_n$. Let $((g_1, \ldots, g_n), \alpha) \in \mathcal{P}_n$. Then we set $A = \operatorname{dom} \alpha$ and choose $\beta \in \mathcal{ST}_n$ such that $i\beta = i\alpha$ for every $i \in \operatorname{dom} \alpha$. Then

$$\begin{aligned}\big((a_1, \ldots, a_n), \operatorname{id}_A\big)\big((g_1, \ldots, g_n), \beta\big) &= \big((a_1, \ldots, a_n)\,{}^{\operatorname{id}_A}(g_1, \ldots, g_n), \operatorname{id}_A \beta\big) \\ &= \big((a_1, \ldots, a_n)(g_{1\operatorname{id}_A}, \ldots, g_{n\operatorname{id}_A}), \operatorname{id}_A \beta\big) \\ &= \big((a_1 g_{1\operatorname{id}_A}, \ldots, a_n g_{n\operatorname{id}_A}), \operatorname{id}_A \beta\big).\end{aligned}$$

Clearly we have $\operatorname{id}_A \beta = \alpha$. If $i \in \operatorname{dom} \alpha$ then $i \in A$. This gives us that $a_i = 1$ and $a_i g_{i\operatorname{id}_A} = g_i \neq 0$. If $i \notin \operatorname{dom} \alpha$ then $i \notin A$ and so $a_i = 0$ which gives $a_i g_{i\operatorname{id}_A} = 0$. $\square$

Since we have shown that:

- $(\mathcal{E}_n, \mathcal{G} \rtimes \mathcal{ST}_n)$ is a left-unique left-product pair of $\mathcal{G}^0 \rtimes \mathcal{PT}_n$ from Theorem 6.6.7;
- $\mathcal{E}_n(\mathcal{G} \rtimes \mathcal{ST}_n) = \mathcal{EGST}_n$ from Theorem 6.6.8

it follows that we may obtain a semigroup presentation for $\mathcal{EGST}_n$ using Theorem 6.5.3. Of course, this requires us to know a presentation for the semidirect product $\mathcal{E}_n \rtimes (\mathcal{G} \rtimes \mathcal{ST}_n)$ in advance. One can obtain a semigroup presentation for semidirect products of the form $M \rtimes S$ (where M is a monoid and S is a semigroup) in the case where a semigroup presentation of S is known [2]. As $\mathcal{E}_n$ is a monoid and $\mathcal{G} \rtimes \mathcal{ST}_n$ is a semigroup, this means finding a presentation for $\mathcal{E}_n \rtimes (\mathcal{G} \rtimes \mathcal{ST}_n)$ reduces to finding a presentation for $\mathcal{G} \rtimes \mathcal{ST}_n$. Again, since $\mathcal{G}$ is a monoid and $\mathcal{ST}_n$ is a semigroup, we can obtain a presentation for $\mathcal{G} \rtimes \mathcal{ST}_n$ by knowing a presentation for $\mathcal{ST}_n$. East provided a presentation for the singular part of the full transformation monoid in [21].

6.6.2 Almost-factorisable inverse semigroups

Almost-factorisable inverse semigroups, or *covering semigroups*, were first introduced by McAlister in 1967 [69]. As the term 'almost-factorisable' might suggest, there is a stronger notion of 'factorisable' that can be applied to inverse monoids. We start by defining such a notion.

Definition 6.6.9. [28, 69] Let M be an inverse monoid with $E = E(M)$ and G the group of units of M. Then M is *factorisable* if $M = EG$.

Certainly, any group is factorisable in this way. It is also easy to see that any band with an identity adjoined will be factorisable.

Example 6.6.10 (Full transformation monoids)**.** Let $n \in \mathbb{N}$ and consider $\mathcal{T}_n$ the full transformation monoid on $\{1, \ldots, n\}$. It is clear that $E = E(\mathcal{T}_n)$ and $\mathcal{S}_n$ are subsemigroups of $\mathcal{T}_n$ and $E\mathcal{S}_n \subseteq \mathcal{T}_n$.

For the reverse inclusion, let $\alpha \in \mathcal{T}_n$ and consider the element $\epsilon \in \mathcal{T}_X$ given by

$$i\epsilon = \min\big\{j \in \{1, \ldots, n\} : j\alpha = i\alpha\big\} = i_{\min}$$

for every $1 \leqslant i \leqslant n$. With this in mind, we let β be a fixed bijection such that

$$\operatorname{dom}\beta = \{1, \ldots, n\} \setminus \operatorname{im}\epsilon \text{ and } \operatorname{im}\beta = \{1, \ldots, n\} \setminus \operatorname{im}\alpha.$$

We define $\sigma \in \mathcal{T}_n$ given by

$$i\sigma = \begin{cases} i\alpha & \text{if } i \in \operatorname{im}\epsilon; \\ i\beta & \text{otherwise} \end{cases}$$

for every $1 \leqslant i \leqslant n$. If $i, j \in \operatorname{im}\epsilon$ then $i = i_{\min}$ and $j = j_{\min}$ by definition. If $i\alpha = j\alpha$ then $i_{\min} \leqslant j$ and dually $j_{\min} \leqslant i$. It follows that $i = j$ and so α is one-to-one when restricted to $\operatorname{im}\epsilon$.

To see that $\epsilon \in E$ we first set

$$i\epsilon^2 = (i\epsilon)\epsilon = \min\big\{j \in \{1, \ldots, n\} : j\alpha = (i\epsilon)\alpha\big\} = k.$$

It follows from $(i\epsilon)\epsilon, i\epsilon \in \operatorname{im}\epsilon$ and $(i\epsilon^2)\alpha = ((i\epsilon)\epsilon)\alpha = k\alpha = (i\epsilon)\alpha$ that $i\epsilon^2 = i\epsilon$ using the fact that α is one-to-one on $\operatorname{im}\epsilon$.

To prove that $\sigma \in \mathcal{S}_n$, we must be able to show that σ is a bijection. To start, we suppose that $i\sigma = j\sigma$ for some $1 \leqslant i \leqslant j \leqslant n$. If $i\sigma = i\alpha$ and $j\sigma = j\alpha$ then $i\alpha = j\alpha$ which gives $i = j$ as α is one-to-one on $\operatorname{im}\epsilon$.

If $i\sigma = i\alpha$ and $j\sigma = j\beta$ then we reach an immediate contradiction since $i\alpha \in \operatorname{im}\alpha$ but $j\beta \notin \operatorname{im}\alpha$. Dually if $i\sigma = i\beta$ and $j\sigma = j\alpha$. Lastly, if $i\sigma = i\beta$ and $j\sigma = j\beta$ then $i\beta = j\beta$ which gives $i = j$ since β is one-to-one. Thus σ is one-to-one.

To see that σ is onto, suppose that $1 \leqslant j \leqslant n$. If $j \in \operatorname{im}\alpha$ then there exists $1 \leqslant i \leqslant n$ such that $i\alpha = j$. It follows that $i_{\min}\sigma = i_{\min}\alpha = j$. If $j \notin \operatorname{im}\alpha$ then $j \in \operatorname{im}\beta$. Therefore there exists $i \in \operatorname{dom}\beta$ such that $i\beta = j$. Hence $i\sigma = i\beta = j$ and so we have shown that σ is onto. This means that $\beta \in \mathcal{S}_n$.

For any $1 \leqslant i \leqslant n$ we see that

$$i\alpha = i_{\min}\alpha = (i\epsilon)\alpha = (i\epsilon)\sigma$$

and so $\alpha = \epsilon\sigma$. Therefore we have shown that $\mathcal{T}_n \subseteq E\mathcal{S}_n$ and hence $\mathcal{T}_n = E\mathcal{S}_n$.

We draw on two results of Lawson that will be incredibly useful in obtaining almost-factorisable inverse semigroups from factorisable inverse monoids. It was proved that any inverse monoid is factorisable if and only if it is almost-factorisable [69]. Moreover Lawson proved that, for every almost-factorisable inverse semigroup S, there exists a factorisable inverse monoid $M = EG$ such that S is isomorphic to $EG \setminus G$. It follows from the next theorem that $EG \setminus G$ will always be a semigroup.

Theorem 6.6.11. *Let S be an almost-factorisable inverse semigroup isomorphic to $EG \setminus G$ for a factorisable inverse monoid $M = EG$. Then $(E \setminus \{1\}, G)$ is a left-unique left-product pair of M.*

Proof. To show that $(E \setminus \{1\}, G)$ is a left-product pair, we see $ge = (geg^{-1})g$ for every $e \in E \setminus \{1\}$ and $g \in G$. Note that $geg^{-1} \neq 1$, otherwise this would imply that $e = 1 \in E \setminus \{1\}$. Hence $g(E \setminus \{1\}) \subseteq (E \setminus \{1\})g$ for every $g \in G$.

To see that $(E \setminus \{1\}, G)$ is left-unique, suppose that $e, f \in E \setminus \{1\}$ and $g, h \in G$ such that $eg = fh$. This implies that $e = fhg^{-1}$ and $f = egh^{-1}$ and so $(e, f) \in \mathscr{R}$ by definition. From Proposition 2.2.5 we know that every

$\mathscr{L}$-class and every $\mathscr{R}$-class contains a unique idempotent. It follows that $e = f$ and so $(E \setminus \{1\}, G)$ is left-unique. $\square$

It is straightforward to see that $EG \setminus G = (E \setminus \{1\})G$ and so we may find presentations for almost-factorisable inverse semigroups using Theorem 6.5.3. However, this requires us to know of a presentation for $(E \setminus \{1\}) \rtimes G$ beforehand. Unlike in the case of Subsection 6.6.1, we cannot rely on an existing contruction of a presentation for $M \ltimes S$, where M is a monoid and S is a semigroup [2]. One can easily adapt this construction to reach a presentation for a semidirect product of the form $S \rtimes M$. Since $E \setminus \{1\}$ is a semigroup and G is a monoid, it follows that we can obtain semigroup presentations for $(E \setminus \{1\}) \rtimes G$ given we have a semigroup presentation for the semilattice $E \setminus \{1\}$. The question as to whether this semidirect product can be finitely presented was answered by Ruškuc and Dombi [19]. Note that this approach differs considerably from the approach of finding presentations of factorisable inverse monoids used by Easdown, East and FitzGerald [20].

Chapter 7

Semigroup products with uniqueness in the second co-ordinate

In this chapter we focus on right-unique left-product pairs, as described in Chapter 6. However, for reasons we will see, the results which arise in this chapter differ considerably from those in the previous chapter. In Section 7.1, we construct quotients of external semidirect products via congruences that are 'right-unique' (which is simply a dual notion of a congruence being left-unique). We change tack and consider internal semidirect products instead in Section 7.2. Finally, in Section 7.3, we investigate semigroup presentations for a very special case.

7.1 External setting

In a similar fashion to Section 6.2, in this section we construct right-unique congruences on semidirect products of semigroups. To begin, we introduce a dual notion to that of being right-aligned for a family of equivalence relations.

Definition 7.1.1. Let S and T be semigroups such that $S \rtimes T$ exists. We say that a family $\Lambda = \{\lambda_a : a \in T\}$ of equivalence relations on S is *left-aligned* if the following properties are satisfied:

(LA1) if $(x, y) \in \lambda_a$ then $(z\,{}^b x, z\,{}^b y) \in \lambda_{ba}$;

(LA2) if $(x, y) \in \lambda_a$ then $(x\,{}^a z, y\,{}^a z) \in \lambda_{ab}$

for every $x, y, z \in S$ and $a, b \in T$.

We introduce a dual notion to that of a left-unique congruence on a semidirect product of semigroups.

Definition 7.1.2. Let S and T be semigroups such that $S \rtimes T$ exists and θ is a congruence on $S \rtimes T$. Then θ is *right-unique* if

$$\big((x, a), (y, b)\big) \in \theta \Longrightarrow a = b$$

for every $x, y \in S$ and $a, b \in T$.

Utilising a family of left-aligned congruences, we construct a right-unique congruence on a semidirect product of semigroups.

Theorem 7.1.3. *Let S and T be semigroups such that $S \rtimes T$ exists. Suppose that there is a left-aligned family $\Lambda = \{\lambda_a : a \in T\}$ of equivalence relations on S. Then the relation λ on $S \rtimes T$ defined by*

$$\big((x, a), (y, b)\big) \in \lambda \Longleftrightarrow (x, y) \in \lambda_a \text{ and } a = b$$

defines a right-unique congruence on $S \rtimes T$.

Proof. Let S and T be semigroups such that $S \rtimes T$ exists and that $\Lambda = \{\lambda_a : a \in T\}$ is a left-aligned family of equivalence relations on S. Suppose that $((x, a), (y, b)) \in \lambda$ so that $a = b$ and $(x, y) \in \lambda_a$. Then we consider the products

$$(z, c)(x, a) = (z\,{}^c x, ca) \text{ and } (z, c)(y, a) = (z\,{}^c y, ca)$$

for every $z \in S$ and $a \in T$. It follows from (LA1) that $(z\,{}^{c}x, z\,{}^{c}y) \in \lambda_{ca}$ and so λ is a left congruence on S. On the other hand, we see that

$$(x, a)(z, c) = (x\,{}^{a}z, ac) \text{ and } (y, a)(z, c) = (y\,{}^{a}z, ac).$$

By considering (RA2) it is evident that $(x\,{}^{a}z, y\,{}^{a}z) \in \lambda_{ac}$ and so λ is a right congruence. Therefore λ is a congruence on $S \rtimes T$. We see that λ is right-unique by definition. $\square$

Conversely, we show in the next result that given a right-unique congruence on a semidirect product of semigroups, one can obtain a left-aligned family of equivalence relations on S.

Theorem 7.1.4. *Let S and T be semigroups such that $S \rtimes T$ exists. Let λ be a right-unique congruence on $S \rtimes T$ and λ_a be defined as*

$$\lambda_a = \Big\{(x, y) \in S \times S : \big((x, a), (y, a)\big) \in \lambda\Big\}.$$

Then $\Lambda = \{\lambda_a : a \in T\}$ is a left-aligned family of equivalence relations on S.

Proof. Let S and T be semigroups such that $S \rtimes T$. Now we suppose that λ is a right-unique congruence on $S \rtimes T$. It is clear to see that λ_a is an equivalence relation, for every $a \in T$, since λ is an equivalence relation.

If $x, y \in S$ with $(x, y) \in \lambda_a$ then $((x, a), (y, a)) \in \lambda$. Since λ is a congruence, we have

$$\big((z, b)(x, a), (z, b)(y, a)\big) = \big((z\,{}^{b}x, ba), (z\,{}^{b}y, ba)\big) \in \lambda \tag{7.1}$$

and

$$\big((x, a)(z, b), (y, a)(z, b)\big) = \big((x\,{}^{a}z, ab), (y\,{}^{a}z, ab)\big) \in \lambda \tag{7.2}$$

for every $z \in S$ and $b \in T$. From Equation 7.1 and Equation 7.2 it follows that (LA1) and (LA2) hold respectively. $\square$

7.1.1 The case where S and T are monoids

The conditions of Theorem 7.1.3 simplify considerably in case where S and T are monoids, and T acts unitarily on the left of S.

Definition 7.1.5. Let S and T be semigroups such that $S \rtimes T$ exists. We say that a family $\Lambda = \{\lambda_a : a \in T\}$ of equivalence relations on S is *strongly left-aligned* if the following properties are satisfied:

(SLA1) $\lambda_a \subseteq \lambda_{ab}$;

(SLA2) if $(x, y) \in \lambda_a$ then $({}^b x, {}^b y) \in \lambda_{ba}$;

(SLA3) if $(x, y) \in \lambda_a$ then $(x\, {}^a z, y\, {}^a z) \in \lambda_a$;

for every $x, y, z \in S$ and $a, b \in T$.

With this definition in mind, we show that a strongly left-aligned family of equivalence relations is always left-aligned.

Theorem 7.1.6. *Let S and T be semigroups such that $S \rtimes T$ exists. Let $\Lambda = \{\lambda_a : a \in T\}$ be a strongly left-aligned family of left congruences on S. Then Λ is a left-aligned family of left congruences on S.*

Proof. Let $\lambda_a \in \Lambda$ where Λ is a strongly left-aligned family of left congruences on S. If $(x, y) \in \lambda_a$ then by (SLA2) it follows $({}^b x, {}^b y) \in \lambda_{ba}$ for every $b \in T$. Since λ_{ba} is a left congruence this gives $(z\, {}^b x, z\, {}^b y) \in \lambda_{ba}$ for every $z \in S$. Hence (LA1) is satisfied.

Similarly, if $(x, y) \in \lambda_a$ then by (SLA3) we have $(x\, {}^a z, y\, {}^a z) \in \lambda_a$ for very $z \in S$. Then, by using (SLA1), we have $(x\, {}^a z, y\, {}^a z) \in \lambda_{ab}$ for every $b \in T$. This shows that λ_a satisfies (LA2). $\square$

We provide a partial converse to the previous theorem in the case that S and T are both monoids.

Theorem 7.1.7. *Let S and T be monoids such that $S \rtimes T$ exists. Let Λ be a left-aligned family of left congruences on S. Then Λ is a strongly left-aligned family of left congruences on S.*

Proof. Let $\lambda_a \in \Lambda$ where Λ is a left-aligned family of left congruences on S. Using (LA2), we see that if $(x,y) \in \lambda_a$ then $(x\,{}^a1, y\,{}^a1) \in \lambda_{ab}$ and so $(x,y) \in \lambda_{ab}$ for every $b \in T$. Therefore (SLA1) is satisfied.

By using (LA1), we see that if $(x,y) \in \lambda_a$ then $(1\,{}^bx, 1\,{}^by) \in \lambda_{ba}$ which gives $({}^bx, {}^by) \in \lambda_{ba}$ for every $b \in S$. This gives us that (SLA2) is satisfied.

Lastly, by considering (LA2), if $(x,y) \in \lambda_a$ then $(x\,{}^az, y\,{}^az) \in \lambda_{a1}$ and thus $(x\,{}^az, y\,{}^az) \in \lambda_a$. This satisfies (SLA3). $\square$

7.2 Internal setting

We briefly consider internal semidirect products. However, the connection with external semidirect products is not as tight as in Chapter 6, as we later explain. As in Section 6.3 it is the stronger set of conditions that allow us to realise $(S \rtimes T)/\lambda$, where S and T are semigroups and λ is a right-unique congruence, as an internal product.

Remark 7.2.1. *Suppose we begin with a left action of T on S by morphisms. We can extend such an action to an action by morphisms of T^1 on S^1 by setting*

$$ {}^{1_T}s = s \text{ and } {}^{1_T}1_S = 1_S $$

for every $s \in S$.

If $\Lambda = \{\lambda_a : a \in T\}$ is a family of left congruences on S satisfying (LA1) and (LA2), then one can naturally extend each λ_a to

$$ \lambda_a^1 = \lambda_a \cup \big\{(1_S, 1_S)\big\} \text{ with } \lambda_{1_T}^1 = \Delta_{S^1}. $$

For every $a \in T^1$, we have λ_a^1 is a left congruence on S^1 such that $\Lambda^1 = \{\lambda_a^1 : a \in T^1\}$ is a strongly left-aligned family of left congruences on S^1.

With Remark 7.2.1 in mind, we let $\Lambda^1 = \{\lambda_a^1 : a \in T^1\}$ and use Λ^1 to define a congruence θ^1 on $S^1 \rtimes T^1$. By considering θ^1, we may form the quotient $(S^1 \rtimes T^1)/\theta^1$ of the semidirect product $S^1 \rtimes T^1$. As before we have that $(S \rtimes T)/\theta$ naturally embeds into $(S^1 \rtimes T^1)/\theta^1$.

Theorem 7.2.2. *Let S and T be semigroups such that $S \rtimes T$ exists. Let λ be a right-unique congruence on $S \rtimes T$. If we define*

$$\overline{S} = \left\{ \left[(x, 1_T)\right]_{\theta^1} : x \in S \right\} \text{ and } \overline{T} = \left\{ [(1_S, a)]_{\theta^1} : a \in T \right\}$$

then $\overline{S} \cong S$ and $\overline{T} \cong T$. Moreover $(S^1 \rtimes T^1)/\theta^1$ is the union of the four disjoint semigroups

$$(S^1 \rtimes T^1)/\theta^1 = \left\{ \left[(1_S, 1_T)\right] \right\} \cup \overline{S} \cup \overline{T} \cup (S \rtimes T)/\theta.$$

such that $(\overline{S}, \overline{T})$ is a right-unique left-product pair.

Finding a converse argument to Theorem 7.2.2, in a similar fashion to Theorem 6.4.1, is not as straightforward as one may expect. The problem lies within the fact that one cannot obtain a left action of T on S by morphisms as previously achieved in Section 6.3.

7.3 A special case

We give a very special case of a right-unique left-product pair for which a semigroup presentation may be obtained.

Lemma 7.3.1. *Let S and T be semigroups such that (S, T) is a right-unique left-product pair. We define*

$$^{a}U = \bigcup_{x \in U} \{y \in S : ax = ya\} \tag{7.3}$$

for every $a \in T$ and $U \in \mathcal{P}(S)$. Then the following are satisfied:

(PA1) $^{a}U\, ^{a}V \subseteq {}^{a}(UV)$;

(PA2) $^{a}(^{b}U) \subseteq {}^{ab}U$;

for every $a, b \in T$ and $U, V \in \mathcal{P}(S)$.

Proof. Let $a \in T$ and $U, V \in \mathcal{P}(S)$. If $x \in {}^aU\,{}^aV$ then $x = yz$ where $y \in {}^aU$ and $z \in {}^aV$. Hence $ah = ya$ and $ak = za$ for some $h \in U$ and $k \in V$. It follows that $hk \in UV$ is such that

$$a(hk) = (ya)k = y(ak) = y(za) = (yz)a = xa$$

and so $x \in {}^a(UV)$ by definition. Hence we have shown that ${}^aU\,{}^aV \subseteq {}^a(UV)$ holds and so (PA1) is satisfied.

Now we let $a, b \in T$, $U \in \mathcal{P}(S)$ and $x \in {}^a({}^bU)$. Then $ay = xa$ for some $y \in {}^bU$, so $bz = yb$ for some $z \in U$. Since $z \in U$, we see that

$$(ab)z = a(bz) = a(yb) = (ay)b = (xa)b = x(ab)$$

and so $x \in {}^{ab}U$. This concludes the proof as we have shown ${}^a({}^bU) \subseteq {}^{ab}U$ which means (PA2) is satisfied. □

We remark that, using the notation from Lemma 7.3.1, we have ${}^aU = \emptyset$ if and only if $U = \emptyset$ since (S, T) is a left-product pair. We proceed by proving some important results which will be crucial in forming some of the main theorems of this chapter.

Lemma 7.3.2. *For every $a \in T$ and $U \in \mathcal{P}(S)$ we have that ${}^aUa = aU$.*

Proof. Let $a \in T$ and $U \in \mathcal{P}(S)$. Then we see that

$$\begin{aligned} x \in {}^aUa &\Longleftrightarrow x \in {}^a\{u\}a \text{ for some } u \in U \\ &\Longleftrightarrow x = va \text{ for some } v \in {}^a\{u\} \\ &\Longleftrightarrow x = va \text{ where } au = va \text{ for some } u \in U \\ &\Longleftrightarrow x = au \text{ for some } u \in U \\ &\Longleftrightarrow x \in aU. \end{aligned}$$

Therefore ${}^aU = Ua$ as required. □

Lemma 7.3.3. *Let $a \in T$, I be some non-empty index and let $U_i \in \mathcal{P}(S) \setminus \emptyset$*

for every $i \in I$. *Then*

$$ {}^{a}\left(\bigcup_{i \in I} U_i\right) = \bigcup_{i \in I} {}^{a}U_i. $$

Proof. For ease of notation, we begin by setting V and W to be

$$ V = \bigcup_{i \in I} U_i \text{ and } W = \bigcup_{i \in I} {}^{a}U_i. $$

With this in mind, we see that

$$ \begin{aligned} x \in {}^{a}V &\iff ay = xa \text{ for some } y \in V \\ &\iff ay = xa \text{ where } y \in U_i \text{ for some } i \in I \\ &\iff x \in {}^{a}U_i \text{ for some } i \in I \\ &\iff x \in W. \end{aligned} $$

Therefore we have shown that ${}^{a}V = W$ as required. □

It is worthwhile noting, in the following definition, that the converses of (PA1) and (PA2) do not necessarily hold.

Definition 7.3.4. Let S and T be semigroups such that (S, T) is a right-unique left-product pair. Then we say that (S, T) is *complete* if the following conditions are satisfied:

(CA1) ${}^{a}\{xy\} \subseteq {}^{a}\{x\}\,{}^{a}\{y\}$;

(CA2) there exists $y \in {}^{b}\{x\}$ such that ${}^{ab}\{x\} \subseteq {}^{a}\{y\}$

for every $a, b \in T$ and $x, y \in S$.

As we will see, it will be useful in what follows to adopt the following notation:

(PA3) ${}^{a}(UV) \subseteq {}^{a}U\,{}^{a}V$;

(PA4) ${}^{ab}U \subseteq {}^{a}({}^{b}U)$;

for every $a, b \in T$ and $U, V \in \mathcal{P}(S)$.

Lemma 7.3.5. *Let S and T be semigroups such that (S, T) is a right-unique left-product pair. Then (CA1) holds if and only if (PA3) holds. Additionally, (CA2) holds if and only if (PA4) holds.*

Proof. Suppose that (CA1) holds and let $a \in T$ and $U, V \in \mathcal{P}(S)$. If $x \in {}^{a}(UV)$ then

$$\begin{aligned} x \in {}^{a}(UV) &\iff x \in {}^{a}\{uv\} \text{ for some } u \in U \text{ and } v \in V \\ &\implies x \in {}^{a}\{u\}\,{}^{a}\{v\} \text{ for some } u \in U \text{ and } v \in V \\ &\iff x \in {}^{a}U\,{}^{a}V. \end{aligned}$$

Therefore we have shown ${}^{a}(UV) \subseteq {}^{a}U\,{}^{a}V$ and so (CA1) implies (PA3). To see that (PA3) implies (CA1) we set $U = \{x\}$ and $V = \{y\}$. Altogether we have shown that (CA1) holds if and only if (PA3) holds.

On the other hand, suppose that (CA2) and $a, b \in T$ and $U \in \mathcal{P}(S)$. If $x \in {}^{ab}U$ then

$$\begin{aligned} x \in {}^{ab}U &\iff x \in {}^{ab}\{u\} \text{ for some } u \in U \\ &\implies x \in {}^{a}\{v\} \text{ for some } v \in {}^{b}\{u\} \\ &\iff x \in {}^{a}\{v\} \text{ for some } v \in {}^{b}U \\ &\iff x \in {}^{a}({}^{b}U). \end{aligned}$$

Therefore we have shown that ${}^{ab}U \subseteq {}^{a}({}^{b}U)$ and so (CA2) implies (PA4). If we suppose that (PA4) holds with $a, b \in T$ then ${}^{ab}\{x\} \subseteq {}^{a}({}^{b}\{x\})$ by setting $U = \{x\}$ for any $x \in S$. Hence we have

$$\begin{aligned} z \in {}^{ab}\{x\} &\implies z \in {}^{a}\big({}^{b}\{x\}\big) \\ &\iff z \in {}^{a}\{y\} \text{ for some } y \in {}^{b}\{x\} \end{aligned}$$

and so there exists some $y \in {}^{b}\{x\}$ such that ${}^{ab}\{x\} \subseteq {}^{a}\{y\}$. Thus (PA4) implies (CA2) so that (CA2) holds if and only if (PA4) holds. $\square$

Equation 7.3 defines a left action of T on $\mathcal{P}(S)$ if conditions (PA2) and (PA4) hold for (S, T). In addition, Equation 7.3 defines a left action of T on

$\mathcal{P}(S)$ by morphisms if (PA1) and (PA3) are satisfied for (S,T). The following observation follows immediately from a combination of Definition 1.3.1, Definition 7.3.4, Lemma 7.3.1 and Lemma 7.3.5.

Lemma 7.3.6. *Let S and T be semigroups such that (S,T) is a right-unique left-product pair. Then (S,T) is complete if and only if Equation 7.3 defines a left action of T on $\mathcal{P}(S)$ by homomorphisms.*

With this in mind, for any semigroups S and T such that (S,T) forms a complete right-unique left-product pair we can form the semidirect product $\mathcal{P}(S) \rtimes T$.

Lemma 7.3.7. *Let S and T be semigroups such that (S,T) is a complete right-unique left-product pair and define*

$$\lambda_a = \big\{(U,V) \in \mathcal{P}(S) \times \mathcal{P}(S) : Ua = Va\big\}$$

for every $a \in T$. Then $\Lambda = \{\lambda_a : a \in T\}$ is a left-aligned family of left congruences on $\mathcal{P}(S)$.

Proof. If $a \in T$ then it is routine to see that λ_a is an equivalence relation on $\mathcal{P}(S)$. If $(U,V) \in \lambda_a$ and $W \in \mathcal{P}(S)$ then it is also clear $(WU, WV) \in \lambda_a$ so that λ_a is a left congruence. We continue the proof by verifying that (SLA1), (SLA2) and (SLA3) are satisfied.

Suppose $U, V \in \mathcal{P}(S)$ and $a \in T$ such that $(U,V) \in \lambda_a$. Then clearly (SLA1) holds since $(Ua)b = (Va)b$ for every $b \in T$. Further, if $(U,V) \in \lambda_a$ then using Lemma 7.3.2 we see that

$$\begin{aligned} {}^{b}U(ba) &= ({}^{b}Ub)a = (bU)a = b(Ua) \\ &= b(Va) = (bV)a = ({}^{b}Vb)a = {}^{b}V(ba) \end{aligned}$$

and so (SLA2) holds. Lastly, with $U, V, W \in \mathcal{P}(S)$ and $a \in T$ such that $(U,V) \in \lambda_a$. Then $Ua = Va$ and it follows from Lemma 7.3.2 that

$$\begin{aligned} (U\,{}^{a}W)a &= U({}^{a}Wa) = U(aW) = (Ua)W \\ &= (Va)W = V(aW) = V({}^{a}Wa) = (V\,{}^{a}W)a \end{aligned}$$

so (SLA3) holds. $\square$

Lemma 7.3.8. *Let S and T be semigroups such that (S,T) is a complete right-unique left-product pair. Then the relation*

$$\big((U,a),(V,b)\big) \in \lambda \iff (U,V) \in \lambda_a \text{ and } a = b$$

is a right-unique congruence on $\mathcal{P}(S) \rtimes T$.

Proof. To verify that λ is an equivalence relation on $\mathcal{P}(S) \rtimes T$ is straightforward. Therefore we proceed to show that λ is a left and a right congruence on $\mathcal{P}(S) \rtimes T$.

Let $U, V \in \mathcal{P}(S)$ and $a, b \in T$ be such that (U,a) and (V,b) are λ-related. This means that $(U,V) \in \lambda_a$ and $a = b$ by definition. If $W \in \mathcal{P}(S)$ and $c \in T$ then

$$(U,a)(W,c) = (U^aW, ac) \text{ and } (V,b)(W,c) = (V^bW, bc) = (V^aW, ac).$$

That $(U^aW, V^aW) \in \lambda_{ac}$ follows from (SLA1) and (SLA3) and so λ is a right congruence. On the other hand, we have

$$(W,c)(U,a) = (W^cU, ca) \text{ and } (W,c)(V,b) = (W^cV, cb) = (W^cV, ca).$$

That $(W^cU, W^cV) \in \lambda_{ca}$ follows from (SLA2) and the fact that λ_{ca} is a left congruence. Therefore λ is a (right and left) congruence.

Clearly λ is right-unique as a congruence by definition. $\square$

Lemma 7.3.9. *Let S and T be semigroups such that (S,T) is a complete right-unique left-product pair. For every $U, V \in \mathcal{P}(S)$ and for every $a, b \in T$ we have $(Ua)(Vb) = U^aV(ab)$.*

Proof. If $U, V \in \mathcal{P}(S)$ and $a, b \in T$ then using Lemma 7.3.2 we get that

$$(Ua)(Vb) = U(aV)b = U({}^aVa)b = (U\,{}^aV)(ab)$$

and so we are done. $\square$

Theorem 7.3.10. *Let S and T be semigroups such that (S,T) is a complete right-unique left-product pair. Then*

$$\big(\mathcal{P}(S) \rtimes T\big)/\lambda \cong \mathcal{P}(S)T$$

Proof. In this proof, we will write P instead of $\mathcal{P}(S)\setminus\emptyset$. With this in mind, we define a map $\theta : (P \rtimes T)/\lambda \to PT$ by

$$\big[(U,a)\big]\theta = Ua$$

for every $U \in P$ and $a \in T$. It is clear to see that θ is well-defined. That is, if $[(U,a)] = [(V,b)]$ then $((U,a),(V,b)) \in \lambda$ which gives $(U,V) \in \lambda_a$ and so $Ua = Va$. Therefore

$$\big[(U,a)\big]\theta = Ua = Va = \big[(V,a)\big]\theta = \big[(V,b)\big]\theta.$$

We now verify that θ defines an isomorphism. Firstly, θ is a homomorphism since

$$\begin{aligned}\Big(\big[(U,a)\big]\big[(V,b)\big]\Big)\theta &= \big[(U,a)(V,b)\big]\theta \\ &= \big[(U^aV,ab)\big]\theta \\ &= U^aVab \\ &= (Ua)(Vb) \\ &= \Big(\big[(U,a)\big]\theta\Big)\Big(\big[(V,b)\big]\theta\Big)\end{aligned}$$

To verify that θ is one-to-one, we see that if $(U,a)\theta = (V,b)\theta$ then $Ua = Vb$. In turn this gives us $a = b$ since (S,T) is right-unique. Thus $(U,V) \in \lambda_a$ and $a = b$ which gives us that (U,a) and (V,b) are λ-related. Lastly, it is clear to see that θ is onto; given any $U \in P$ and $a \in T$, we have that

$$\big[(U,a)\big]\theta = Ua$$

by definition. $\square$

With Theorem 7.3.10 in mind, we make an immediate observation.

Theorem 7.3.11. *Let S and T be semigroups such that (S,T) is a complete right-unique left-product pair. The subset*

$$P_1 = \Big\{(U,a) \in \big(\mathcal{P}(S) \setminus \emptyset\big) \rtimes T : |Ua| = 1\Big\}$$

forms a subsemigroup of $P \rtimes T$ where $P_1/\lambda \cong ST$.

Proof. Let S and T be semigroups such that (S,T) is a complete right-unique left-product pair. Let $(U,a),(V,b) \in P_1$ so that $|Ua| = |Vb| = 1$. Then we see that

$$1 = |Ua||Vb| = \big|(Ua)(Vb)\big| = \big|U(aV)b\big| = \big|U(\,^{a}Va)b\big| = \big|(U\,^{a}V)(ab)\big|$$

and so $(U\,^{a}V, ab) \in P_1$.

It is clear that θ, as described in Theorem 7.3.10, restricted to the subsemigroup P_1 of $P \rtimes T$, is an isomorphism where $\operatorname{im}\theta = ST$. $\square$

7.3.1 Semigroup presentations

For the very special case of complete right-unique left-product pairs discussed in Section 7.3, in Subsection 7.3.1 we offer semigroup presentations for semigroup products obtained from those pairs. It will be important to recall the notation used in Section 7.3 for this subsection. We begin by introducing a map $\Psi : P_1 \to ST$ given by

$$(U,a)\Psi = xa$$

where $Ua = \{xa\}$ for every $U \in \mathcal{P}(S)$ and $a \in T$. It is clear to see that Ψ is a surjective map, although it may not necessarily be one-to-one.

Lemma 7.3.12. *Let S and T be semigroups such that (S,T) is a complete right-unique left-product pair. Then* $\ker\Psi = \lambda$.

Proof. Let $U, V \in \mathcal{P}(S)$ and $a, b \in T$ such that $Ua = xa$ and $Vb = yb$ (where we identify elements of ST with singleton subsets). Then

$$\begin{aligned}
\big((U,a),(V,b)\big) \in \ker\Psi &\iff (U,a)\Psi = (V,b)\Psi \\
&\iff xa = yb \\
&\iff Ua = Vb \\
&\iff (U,V) \in \lambda_a \text{ and } a = b \\
&\iff \big((U,a),(V,b)\big) \in \lambda
\end{aligned}$$

and so $\ker\Psi = \lambda$ as required. □

Suppose that P_1 has the semigroup presentation $\langle X : R\rangle$ via ϕ. That is to say there exists surjective homomorphism $\phi : X^+ \to P_1$ such that $\ker\phi = R^\sharp$. For each $a \in T$ and $U \in \mathcal{P}(S)$, we let $w_{(U,a)}$ be a fixed word over X^+ such that $w_{(U,a)}\phi = (U,a)$. We also recall our notation from Section 6.5 where $\sim_* = R_*$ and $\approx_* = R_*^\sharp$ for any formal symbol $*$.

Theorem 7.3.13. *Let S and T be semigroups such that (S,T) is a complete right-unique left-product pair. Let $\Psi : P_1 \to ST$ be given by $(U,a)\Psi = xa$, where we identify elements of ST with singleton subsets, and suppose that*

$$\lambda_a = \Big\{\big((U,a),(V,a))\big) : Ua = Va\Big\}.$$

for every $a \in T$. If we define

$$R_\lambda = \big\{(w_{(U,a)}, w_{(V,a)}) : (U,V) \in \lambda_a \text{ and } a \in T\big\}$$

then ST has presentation $\langle X : R \cup R_\lambda\rangle$ via $\phi\Psi$.

Proof. For convenience we set $\alpha = \phi\Psi$. That α is surjective follows immediately from the fact that ϕ and Ψ are surjective in their own right. Thus we continue by proving that $\ker\alpha = (R \cup R_\lambda)^\sharp$ in order to complete the proof.

Suppose that $(p,q) \in \ker\alpha$ for some $p,q \in X^+$, say

$$p(\phi\Psi) = xa = (w_{(U,a)}\phi)\Psi \text{ and } q(\phi\Psi) = yb = (w_{(V,b)}\phi)\Psi$$

where $Ua = \{xa\}$ and $Vb = \{yb\}$ for some $x, y \in S$ and $a, b \in T$. Then it is clear that

$$xa = (p\phi)\Psi = (q\phi)\Psi = yb$$

and since (S, T) is right-unique, this implies that $a = b$. In turn, this gives us $Ua = Va$ and so $(U, V) \in \lambda_a$ which means $(w_{(U,a)}, w_{(V,b)}) \in R_\lambda$. It then follows that

$$p \sim w_{(U,a)} \sim_\lambda w_{(V,b)} \sim q$$

and so $(p, q) \in (R \cup R_\lambda)^\sharp$. Thus $\ker \alpha \subseteq (R \cup R_\lambda)^\sharp$ as required.

For the reverse inclusion we notice the following observations. Firstly, since $\ker \phi \subseteq \ker(\phi\Psi)$ we have that $R \subseteq \ker(\phi\Psi)$. Secondly, we have $R_\lambda \subseteq \ker \phi\Psi$ since

$$(w_{(U,a)}\phi)\Psi = (U, a)\Psi = xa = ya = (V, a)\Psi = (w_{(V,a)}\phi)\Psi$$

for every $U, V \in \mathcal{P}(S)$ and $a \in T$ such that $(U, V) \in \lambda_a$ with $Ua = \{xa\}$ and $Va = \{ya\}$. This implies that $(R \cup R_\lambda)^\sharp \subseteq \ker(\phi\Psi)$ and so $\ker \phi\Psi = (R \cup R_\lambda)^\sharp$. □

The next main result seeks to reduce the number of generators in the presentation provided in Theorem 7.3.13. We achieve this by supposing a generating set for λ_a for every $a \in T$. That is, for every $a \in T$, we let $\overline{\lambda}_a \subseteq \lambda_a$ such that $\overline{\lambda}_a$ generates λ_a as a left congruence.

Lemma 7.3.14. *Let S and T be semigroups such that (S, T) is a right-unique left-product pair. Suppose there exists some $a \in T$ such that $xa = x$ for every $x \in S$. Then $ab = b$ for every $b \in T$.*

Proof. Let $a \in T$ such that $xa = x$ for every $x \in S$. Then

$$x(ab) = (xa)b = xb$$

for every $x \in S$ and $b \in T$. Since (S, T) is right-unique we deduce that $ab = b$ for every $b \in T$. □

Theorem 7.3.15. *Let S and T be semigroups such that (S, T) is a complete* *right-unique left-product pair and suppose that* $P_1 \rtimes T$ *has presentation* $\langle X : R\rangle$ *via* ϕ*. Suppose that there exists* $b \in T$ *such that* $xb = x$ *for every* $x \in S$*. Let* $\Psi : P_1 \to ST$ *be given by* $(U, a)\Psi = xa$*, where we identify elements of* ST *with singleton subsets, and suppose that*

$$\lambda_a = \Big\{ \big((U, a), (V, a)\big) : Ua = Va \Big\}$$

is generated as a left congruence by $\overline{\lambda}_a$*. If we define*

$$R_{\overline{\lambda}} = \big\{ (w_{(U,a)}, w_{(V,b)}) : (U, V) \in \overline{\lambda}_a, a \in T \big\}$$

then ST *has presentation* $\langle X : R \cup R_{\overline{\lambda}}\rangle$ *via* $\phi\Psi$*.*

Proof. It follows in precisely the same way as in Theorem 7.3.13 that $\phi\Psi$ is a surjective homomorphism. We proceed by setting $\alpha = \phi\Psi$ for convenience. We must now show that $\ker \alpha = (R \cup R_{\overline{\lambda}})^\sharp$.

First, we know $R \subseteq \ker \phi$ and so $R \subseteq \ker \alpha$ since $\ker \phi \subseteq \ker \alpha$. Similarly, since $R_{\overline{\lambda}} \subseteq R_\lambda$ we have that $R_{\overline{\lambda}} \subseteq \ker \alpha$ as we have already shown that $R_\lambda \subseteq \ker \alpha$ from Theorem 7.3.13. Thus $(R \cup R_{\overline{\lambda}})^\sharp \subseteq \ker \alpha$.

For the reverse inclusion, we show that $\ker \alpha \subseteq (R \cup R_{\overline{\lambda}})^\sharp$. In order to do so, we will rely heavily on the fact that $\ker \alpha = (R \cup R_\lambda)^\sharp$ from Theorem 7.3.13. That is, if we can show that $R \cup R_\lambda \subseteq (R \cup R_{\overline{\lambda}})^\sharp$ then $(R \cup R_\lambda)^\sharp \subseteq (R \cup R_{\overline{\lambda}})^\sharp$ and we are done. As $R \subseteq (R \cup R_{\overline{\lambda}})^\sharp$ is clear, we now show that $R_\lambda \subseteq (R \cup R_{\overline{\lambda}})^\sharp$.

Let $(w_{(U,a)}, w_{(V,a)}) \in R_\lambda$ so that $(U, V) \in \lambda_a$ and $Ua = Va$. Since $\overline{\lambda}_a$ generates λ_a as a left congruence, either $U = V$ or there exists a sequence of the form

$$U = Z_1, Z_2, \ldots, Z_n = V$$

where $Z_i = C_i W_i$ and $Z_{i+1} = C_i W_{i+1}$ with $(W_i, W_{i+1}) \in \overline{\lambda}_a \cup (\overline{\lambda}_a)^{-1}$ and $C_i \in \mathcal{P}(S)^1$ for all $1 \leqslant i \leqslant n-1$. If $U = V$ then certainly we have that $(U, a) = (V, a)$ so that trivially $w_{(U,a)} \approx_{\overline{\lambda}} w_{(V,a)}$.

On the other hand, suppose that $U \neq V$ and fix some $1 \leqslant i \leqslant n$. If $C_i = 1_{\mathcal{P}(S)}$ then $Z_i = W_i$ and $Z_{i+1} = W_{i+1}$. Since $(W_i, W_{i+1}) \in \overline{\lambda}_a \cup \overline{\lambda}_a^{-1}$ it is then clear that $Z_i \approx_{\overline{\lambda}} Z_{i+1}$. If $C_i \in \mathcal{P}(S)$ then, by the assumption, there exists some $b \in T$ such that $xb = x$ for all $x \in S$. With this in mind we see that

$$bU = {}^bUb = {}^bU$$

for all $U \in \mathcal{P}(S)$. Now we consider the product

$$\begin{aligned}(C_i, b)(W_i, a) &= (C_i\,{}^bW_i, ba)\\ &= \big(C_i(bW_i), ba\big)\\ &= \big((C_ib)W_i, ba\big)\\ &= (C_iW_i, ba)\\ &= (Z_i, ba)\end{aligned}$$

where $C_ib = C_i$ follows from the fact that $xb = x$ for every $x \in S$. Likewise we see that $(C_i, b)(W_{i+1}, a) = (Z_{i+1}, ba)$ in the same manner as above. Using Lemma 7.3.14 we know that $ba = a$ for every $a \in T$. Therefore

$$\begin{aligned}w_{(Z_i,a)} = w_{(Z_i,ba)} &\approx w_{(C_i,b)}w_{(W_i,a)}\\ &\approx_{\overline{\lambda}} w_{(C_i,b)}w_{(W_{i+1},a)}\\ &\approx w_{(Z_{i+1},ba)} = w_{(Z_{i+1},a)}.\end{aligned}$$

This gives us that $(w_{(Z_i,a)}, w_{(Z_{i+1},a)}) \in (R \cup R_{\overline{\lambda}})^\sharp$ for every $1 \leqslant i \leqslant n$ and so we deduce that $(w_{(U,a)}, w_{(V,a)}) \in (R \cup R_{\overline{\lambda}})^\sharp$. Hence $R_\lambda \subseteq (R \cup R_{\overline{\lambda}})^\sharp$ as required. A further inductive argument will show us that $(R \cup R_\lambda)^\sharp \subseteq (R \cup \overline{R}_\lambda)^\sharp$ as required. $\square$

7.4 Applications

In this section, we discuss applications of right-unique left-product pairs.

7.4.1 Left restriction semigroups

We introduce a class of semigroup which remains highly popular amongst semigroup theorists (especially those from York).

Definition 7.4.1. [48] A unary semigroup S is *left restriction* with the unary operation denoted by $x \mapsto x^+$, if

(LAM1) $x^+x = x$;

(LAM2) $x^+y^+ = y^+x^+$;

(LAM3) $(x^+y)^+ = x^+y^+$;

(LAM4) $(xy)^+x = xy^+$

for all $x, y \in S$. Dually, we define when a unary semigroup is *right restriction* with the unary operation given by $x \mapsto x^*$. A semigroup is said to be *restriction* if it is both left and right restriction such that

$$(x^+)^* = x^+ \text{ and } (x^*)^+ = x^*.$$

Many natural examples of (left, right) restriction semigroups exist. Every inverse semigroup S can be regarded as a restriction semigroup by setting $x^+ = xx^{-1}$ and $x^* = x^{-1}x$ respectively for every $x \in S$. We provide an explicit example in what follows.

Example 7.4.2 (Partial transformation monoids)**.** Let X be a non-empty set and let $\mathcal{PT}_X$ denote the set

$$\mathcal{PT}_X = \{\alpha : A \to B \mid A, B \subseteq X\}.$$

The composition in $\mathcal{PT}_X$ is given by

$$\mathrm{dom}(\alpha\beta) = (\mathrm{im}\,\alpha \cap \mathrm{dom}\,\beta)\alpha^{-1}$$

and $x(\alpha\beta) = (x\alpha)\beta$ for all $x \in \mathrm{dom}(\alpha\beta)$. For every $A \subseteq X$, we let $\mathrm{id}_A \in \mathcal{PT}_X$ be given by

$$x\,\mathrm{id}_A = \begin{cases} x & \text{if } x \in A; \\ \text{undefined} & \text{otherwise.} \end{cases}$$

In the case where $A = \operatorname{dom}\alpha$ for some $\alpha \in \mathcal{PT}_X$ we will use $\mathrm{id}_{\overline{\alpha}}$ and $\mathrm{id}_{\operatorname{dom}\alpha}$ interchangeably. It is clear to see that $\mathrm{id}_A \in E$ for every $A \subseteq X$ and that $\{\mathrm{id}_A : A \subseteq X\}$ forms a commutative subsemigroup of $\mathcal{PT}_X$. To this end, we define a unary operation $^+ : \mathcal{PT}_X \to E$ by $\alpha^+ = \mathrm{id}_{\overline{\alpha}}$ and for every $\alpha \in \mathcal{PT}_X$. It is routine to verify (LAM1) and (LAM2) are satisfied under this unary operation.

Let $\alpha, \beta \in \mathcal{PT}_X$ and $x \in X$. That $(\alpha^+\beta)^+ = \alpha^+\beta^+$ then follows from

$$\begin{aligned}
x \in \operatorname{dom}(\alpha^+\beta)^+ &\iff x \in \operatorname{dom}(\alpha^+\beta)\\
&\iff x \in (\operatorname{im}\alpha^+ \cap \operatorname{dom}\beta)(\alpha^+)^{-1}\\
&\iff x\alpha^+ \in \operatorname{im}\alpha^+ \cap \operatorname{dom}\beta\\
&\iff x\alpha^+ \in \operatorname{im}\alpha^+ \cap \operatorname{dom}\beta^+\\
&\iff x \in (\operatorname{im}\alpha^+ \cap \operatorname{dom}\beta^+)(\alpha^+)^{-1}\\
&\iff x \in \operatorname{dom}(\alpha^+\beta^+).
\end{aligned}$$

So this unary operation satisfies (LAM3). We deduce that $(\alpha\beta)^+\alpha = \alpha\beta^+$ by observing

$$\begin{aligned}
x \in \operatorname{dom}\big((\alpha\beta)^+\alpha\big) &\iff x \in \big(\operatorname{im}(\alpha\beta)^+ \cap \operatorname{dom}\alpha\big)\big((\alpha\beta)^+\big)^{-1}\\
&\iff x(\alpha\beta)^+ \in \operatorname{im}(\alpha\beta)^+ \cap \operatorname{dom}\alpha\\
&\iff x(\alpha\beta)^+ \in \operatorname{dom}(\alpha\beta) \cap \operatorname{dom}\alpha\\
&\iff x(\alpha\beta)^+ \in (\operatorname{im}\alpha \cap \operatorname{dom}\beta)\alpha^{-1} \cap \operatorname{dom}\alpha\\
&\iff x(\alpha\beta)^+ \in (\operatorname{im}\alpha \cap \operatorname{dom}\beta)\alpha^{-1}\\
&\iff x(\alpha\beta)^+ \in (\operatorname{im}\alpha \cap \operatorname{dom}\beta^+)\alpha^{-1}\\
&\iff x(\alpha\beta)^+ \in \operatorname{dom}(\alpha\beta^+)\\
&\iff x \in \operatorname{dom}(\alpha\beta^+).
\end{aligned}$$

One can then verify that $x(\alpha\beta)^+\alpha = x\alpha\beta^+$ for all $x \in \operatorname{dom}(\alpha\beta^+)$. Hence the unary operation satisfies (LAM4). Thus S is left restriction.

It was shown by Branco, Gomes and Gould that the free left restriction monoid F is generated by a set of idempotents E and a free monoid over a certain set X [4]. They showed that if $e, f \in E$ and $x, y \in X^*$ such that $ex = fy$ then, by Theorem 5.1 of [4], we have $x = y$. That (E, X^*) is a left-product pair follows from (LAM4). Therefore (E, X^*) is a right-unique left-product pair of the free left restriction monoid $F = EX^*$.

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