Representation Theory for the Group $\mathrm{SL}_2(\mathbb{R})$

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Dedication

This thesis work is dedicated to

my mother, my father, my husband, my sister and my wonderful children

I am truly thankful for having you in my life

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Abstract

This thesis is concerned with the representation theory for the special linear Lie group $\mathrm{SL}_2(\mathbb{R})$. After the introduction and background material in the second chapter, we present the representation theory of the affine group, which is a subgroup of $\mathrm{SL}_2(\mathbb{R})$. The classification of the irreducible unitary representation of $\mathrm{SL}_2(\mathbb{R})$ was achieved by Bargmann [4]. Here, a new study of the $\mathrm{SL}_2(\mathbb{R})$ unitary representation is provided by using an inducing procedure and Gelfand method [13, VII]. Also, we investigate the representation of the group $\mathrm{SL}_2(\mathbb{R})$ on holomorphic spaces of function on the unit disc.

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Chapter 1

Introduction

The special linear group $\mathrm{SL}_2(\mathbb{R})$ is the group of 2×2 matrices with real entries and a determinant equal to one. It is an interesting and important example of a locally compact real Lie group of three dimension. In 1947, Bargmann classified the irreducible unitary representation of $\mathrm{SL}_2(\mathbb{R})$ [4]. His approach has been presented in different sources [20, 33, 40]. The main tool of Bargmann's classification is to work on the Lie algebra $\mathfrak{sl}_2(\mathbb{R})$. Gelfand studied the $\mathrm{SL}_2(\mathbb{R})$ representations on the Lie group instead of the Lie algebra $\mathfrak{sl}_2(\mathbb{R})$ [13, VII]. In this thesis, I use the induced representation technique (in the sense of Mackey [34]) and the Gelfand method [13, VII] to review the classification of the irreducible unitary representations on the Lie group $\mathrm{SL}_2(\mathbb{R})$.

The affine group is a subgroup of $\mathrm{SL}_2(\mathbb{R})$, and it is often used to build wavelets. To study the induced representation of the group $\mathrm{SL}_2(\mathbb{R})$ I start by considering the unitary representations of the affine group, which are due to Gelfand and Naimark[14] .

Another purpose of this thesis is to describe the $\mathrm{SL}_2(\mathbb{R})$ representations on spaces of holomorphic functions on the unit disc. The group $\mathrm{SL}_2(\mathbb{R})$ is more convenient for complex analysis in the upper half-plane. However, the group $\mathrm{SU}(1,1)$ of 2×2 matrices, with complex entries and a determinant equal to one, is well suited in unit disc $\mathbb{D}$. The Cayley transform (5.1) defines an isomorphism of the group $\mathrm{SL}_2(\mathbb{R})$ with the group $\mathrm{SU}(1,1)$. Lang [33, chapter IX] studied the holomorphic discrete series for the group $\mathrm{SL}_2(\mathbb{R})$ on the Bergman space in the upper half-plane and on the unit disc. Here, I study the mock discrete series for the group $\mathrm{SL}_2(\mathbb{R})$ and describe the $\mathfrak{sl}_2(\mathbb{R})$ Lie algebra vector module of the Dirichlet space on the unit disc.

1.1 The Structure of the Thesis

This thesis includes the following:

- In Chapter 2, we present a brief review of the background, including basic definitions and the results of group theory, Lie groups, Lie algebra and representations theory.

- In Chapter 3, we consider the construction of induced representations of the affine group. In particular, we describe three forms of the affine group representations: the left regular representation, the co-adjoint representation and the quasi-regular representation. Also, we discuss the covariant transform and its inverse, the contravariant transform. We find the intertwining operator between all three affine representations through the covariant transform. These are given in the following:

  (i) the Poisson integral between the quasi-regular representation on $H_2(\mathbb{R})$ and the left regular representation on $L_2(\mathrm{Aff}, d\nu)$.

  (ii) the Laplace transform between the co-adjoint representation on $L_2(\mathbb{R}_+)$ and the left regular representation on $L_2(\mathrm{Aff}, d\nu)$.

  (iii) the Fourier transform between the quasi-regular representation on $H_2(\mathbb{R})$ and the co-adjoint representation on $L_2(\mathbb{R}_+)$.

- In Chapter 4, we consider the $\mathrm{SL}_2(\mathbb{R})$ group and its infinite-dimensional representations. In particular, we present the irreducible unitary representations of the $\mathrm{SL}_2(\mathbb{R})$ determined originally by Bargmann [4].

In addition, we provide a classification of the $\mathrm{SL}_2(\mathbb{R})$ group representations by following the Gelfand method [13, Chapter VII]. To begin with, we induced a reducible representation of $\mathrm{SL}_2(\mathbb{R})$ in two subsections:

(i) we construct an induced representation on the vector space of smooth function W on the homogeneous space $X = \mathbb{R}^2 - \{(0,0)\}$. We get the following representation:

$$[U(g)f](w) = f(g^{-1} \cdot w),$$

where $w = (u, v) \in \mathbb{R}^2 - \{(0,0)\}$, and $f \in W$. The inverse is needed to satisfy the representation condition:

$$U(g_1 g_2) = U(g_1)U(g_2).$$

(ii) we use the technique of induction in stages to construct an induced representation on the homogeneous space $X = \mathbb{P}(\mathbb{R})$, which is the real projective line. We obtain the representation

$$[T_s(g)f](x) = |d - bx|^s \mathrm{sgn}^\epsilon(d - bx) f\left(\frac{ax - c}{d - bx}\right), \tag{1.1}$$

where $x \in \mathbb{P}(\mathbb{R})$ and $f \in W$, which is a vector space of smooth functions on $\mathbb{P}(\mathbb{R})$.

To study the unitarity of the representations (1.1), we start by finding the conditions needed for a bilinear functional to be invariant on a normed space. Then, we investigate the invariance of an inner product on the Hilbert space.

- In chapter 5, we study the representation of the $\mathrm{SL}_2(\mathbb{R})$ group on analytic spaces of function in the unit disc. This is done by studying the representation of the $\mathrm{SU}(1,1)$ group, since this is more convenient in the unit disc. The first section provides basic information about the $\mathrm{SU}(1,1)$ group. In the second section, we induce a representation of the $\mathrm{SU}(1,1)$ group from the subgroup K. In the third section, we present the actions of the ladder operators for the representation on the unit disc. Then, we study the $\mathrm{SU}(1,1)$ representation on the Dirichlet space.

Finally, we consider the mock discrete series representation in the real line, on the unit circle and on the group $\mathrm{SL}_2(\mathbb{R})$.

- In Chapter 6, we discuss further work.

Chapter 2

Preliminaries

In this chapter, we present some basic notions of group theory and representation theory that are needed for our study. Mainly, we use references [22] and [10].

2.1 Groups

Definition 2.1.1. [22] An abstract group (or simply group) is a non-empty set G for which there is a law of group multiplication, that is to say, a mapping $G \times G \to G$ with the properties:

(i) associativity: $g_1(g_2g_3) = (g_1g_2)g_3$;

(ii) the existence of an identity: there exists an element $e \in G$ such that $eg = g$ for all $g \in G$;

(iii) the existence of an inverse: for every $g \in G$ there exists an element $g^{-1} \in G$ such that $gg^{-1} = g^{-1}g = e$.

A group G is called commutative or abelian if in addition to the above properties we have

(iv) commutativity : $g_1g_2 = g_2g_1$ for all $g_1, g_2 \in G$.

Example 2.1.2. The following are abstract groups:

1. The set $\mathbb{R}$ of real numbers forms a commutative group under the operation of addition.

2. For $n \in \mathbb{Z}^+$, the set of all $n \times n$ invertible matrices with real entries forms a group with the operation of matrix multiplication. This group is non-commutative for $n \geq 2$.

Definition 2.1.3. [27] A subgroup of a group G is a subset H of G such that the restriction of multiplication from G to H makes H a group itself.

2.2 Homogeneous Spaces

Let G be a locally compact group and M be a locally compact Hausdorff space. A left action of G on M is a continuous map $(g, x) \mapsto gx$ from $G \times M$ to M such that:

1. $x \mapsto gx$ homeomorphism of M for each $g \in G$ and $x \in M$,

2. $g_1(g_2x) = (g_1g_2)x$ for all $g_1, g_2 \in G$ and $x \in M$.

A space M equipped with an action of is called a G-space. A G-space is called transitive if for every $x_1, x_2 \in M$ there exists $g \in G$ such that $gx_1 = x_2$. The homogeneous space is a transitive G-space.

In this section, we describe the construction of homogeneous spaces [22, §2.2]. Let us define the space of cosets $X = G/H$ by the equivalence relation: $g_1 \sim g_2$ if there exists $h \in H$ such that $g_1 = g_2h$. The space $X = G/H$ is a homogeneous space under the left G-action:

$$g : g_1H \to (gg_1)H. \tag{2.1}$$

It is more convenient to have parameterisations of $X = G/H$ and express the above action through those parameters. Suppose that we have chosen a representative in each equivalence class. In other words, we have a mapping $\mathsf{s} : X \to G$ such that it is a right inverse to the natural projection $\mathsf{p} : G \to G/H$, that is, $\mathsf{p}(\mathsf{s}(x)) = x$ for all $x \in X$.

The set G can be identified by the product $G \sim G/H \times H$, that is, $g = \mathsf{s}(\mathsf{p}(g))h$ for some $h \in H$ depending on g. From the definition of the maps s and p, the point $\mathsf{s}(\mathsf{p}(g))$ belongs to the same class of the point as g; that is, $\mathsf{s}(\mathsf{p}(g)) \sim g$. Then, any $g \in G$ has a unique decompostion of the form

$$g = \mathsf{s}(x)h, \tag{2.2}$$

where $x = \mathsf{p}(g)$ and $h \in H$. We define a map r associated with s through the identities:

$$h = \mathsf{r}(g) := \mathsf{s}(x)^{-1}g, \quad \text{where} \quad x = \mathsf{p}(g). \tag{2.3}$$

Then X is a left homogeneous space with the G-action defined in terms of s and p, as follows:

$$g : x \to g \cdot x = \mathsf{p}(g * \mathsf{s}(x)), \tag{2.4}$$

where $*$ is the multiplication on G and $\cdot$ is the action of G on X from the left. This is illustrated by the following commutative diagram:

$$\begin{array}{ccc} G & \xrightarrow{g*} & G \\ {\scriptstyle \mathsf{s}}\uparrow & & \downarrow{\scriptstyle \mathsf{p}} \\ X & \xrightarrow{g\cdot} & X \end{array}$$

2.3 Lie Groups and Lie Algebras

Definition 2.3.1. [17] A Lie group is a smooth manifold G that is also a group such that the group product $G \times G \to G$, and the inverse map $G \to G$ are smooth.

Example 2.3.2.

1. The $GL(n, \mathbb{R})$ group of $n \times n$ invertible matrices with real entries and matrix multiplication as the group law. This group is also called the matrix Lie group.

2. The $SL_2(\mathbb{R})$ group is a set of 2×2 matrices $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with real entries $a, b, c, d \in \mathbb{R}$ and a determinant equal to 1. The group law coincides with the matrix multiplication. The identity is the unit matrix, and the inverse is $g^{-1} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.

3. The Heisenberg group $\mathbb{H}^1$ is a set of triple real numbers (s, x, y) with the group multiplication:

$$(s, x, y) * (s', x', y') = (s + s' + \frac{1}{2}(x'y - xy'), x + x', y + y').$$

The identity is $(0, 0, 0)$ and $(s, x, y)^{-1} = (-s, -x, -y)$.

Definition 2.3.3. A Lie algebra $\mathfrak{g}$ is a vector space over a field K, with a bilinear map $[.,.] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ that satisfies the following conditions: for all X, Y and $Z \in \mathfrak{g}$, we have

(i) bilinearity: $[aX + bY, Z] = a[X, Z] + b[Y, Z]$ and for all scalars $a, b \in K$;

(ii) antisymmetry: $[X, Y] = -[Y, X]$ (or, equivalently, $[X, X] = 0$); and

(iii) the Jacobi identity: $[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0$.

The map $[.,.]$ is called a **commutator.**

An important class of Lie algebras is formed by matrix Lie algebras [23], which are subspaces of $M_n(K)$ (a Lie algebra of $n \times n$ matrices over the field K of real or complex numbers) and are closed with respect to the ordinary matrix commutator

$$[X, Y] = XY - YX. \tag{2.5}$$

Moreover, Ado's Theorem states that any Lie algebra is isomorphic to a matrix Lie algebra [23]. For every matrix Lie group G there is an associated matrix Lie algebra $\mathfrak{g}$. An important relation between them is the exponential map:

$$\begin{aligned} \exp &: \mathfrak{g} \to G; \\ &: X \mapsto \exp X = e^X. \end{aligned} \tag{2.6}$$

The expansion map for a matrix Lie group is defined by the following Taylor series:

$$\exp(X) = \sum_{k=0}^{\infty} \frac{(X)^k}{k!}. \tag{2.7}$$

2.4 Representations of Groups

Definition 2.4.1. [5, §1.1] Let G be a group with identity element e_G , and let V be a vector space. A representation π of G in V is a homomorphism of G into $GL(V)$ (the group of invertible, linear mappings that carry V to itself), that is

$$\pi : G \to GL(V), \quad g \mapsto \pi(g).$$

The representation operator $\pi(g) : V \to V, \quad g \in G$ satisfies the following properties:

$$\pi(g_1 g_2) = \pi(g_1)\pi(g_2), \quad \pi(e_G) = \mathbb{I}.$$

The repreaentation π is called linear if V is a linear space and the mappings $\pi(g)$ are linear operators. The space V is called the representation space of π.

Let π be a representation of a Lie group G on a Hilbert space $\mathcal{H}$. A strong continuity of π means that for any vector $u \in \mathcal{H}$ and for any convergent sequence $(g_j) \to g \in G$, we have [40, p.9]

$$\|\pi(g_j)u - \pi(g)u\| \to 0.$$

Definition 2.4.2. [40, p. 9] A representation π of a Lie group G on a Hilbert space $\mathcal{H}$ is called a unitary representation if the operator $\pi(g)$ is unitary, that is

$$\pi(g)^* = \pi(g)^{-1} = \pi(g^{-1}), \quad g \in G.$$

There is a natural equivalence relation on the set of all representations of a group, which is defined by an intertwining property.

Definition 2.4.3. [10] Let π_1 and π_2 be unitary representations of a Lie group G in spaces $\mathcal{H}_{\pi_1}$ and $\mathcal{H}_{\pi_2}$, respectively. An operator $U : \mathcal{H}_{\pi_1} \to \mathcal{H}_{\pi_2}$ is called an intertwining operator between π_1 and π_2 if for every $g \in G$, we have

$$U\pi_1(g) = \pi_2(g)U.$$

The set of all intertwining operators is denoted by $\mathcal{C}(\pi_1, \pi_2)$. The representations π_1 and π_2 are unitarily equivalent if $\mathcal{C}(\pi_1, \pi_2)$ contains a unitary operator U so that $\pi_1(g) = U\pi_2(g)U^{-1}$. We shall write $\mathcal{C}(\pi)$ for $\mathcal{C}(\pi, \pi)$, which is the space of the bounded operators on $\mathcal{H}_\pi$ that commute with $\pi(g)$.

Definition 2.4.4. [33] Let π be a representation of a Lie group G on the vector space V. Define the subspace V^∞ to consist of functions $f \in V$ such that the map $g \mapsto \pi(g)f$ is infinitely differentiable for any $g \in G$. Then, the derived representation generated by an element X of the corresponding Lie algebra $\mathfrak{g}$ is the representation $d\pi(X)$ of $\mathfrak{g}$ given as follows:

$$d\pi(X)f := \frac{d}{dt}\pi(\exp tX)f\Big|_{t=0}, \quad \text{where} \quad f \in V^\infty. \tag{2.8}$$

2.5 Decomposition of Representations

One of the main problems of the theory of representations is the problem of decomposing representations of a group G into the simplest possible components. In the following, we will provide some relevant notation.

Definition 2.5.1. [22] Let π be a linear representation of a Lie group G in a Hibert space $\mathcal{H}$. A linear subspace $L \subset \mathcal{H}$ is an invariant subspace for π if for any $x \in L$ and $g \in G$ the vector $\pi(g)x$ again belongs to L.

There are two trivial invariant subspaces, the null subspace and the entire space. All other invariant subspaces are non-trivial. Let π be a representation of a Lie group G on a Hilbert space $\mathcal{H}$. If there are only two trivial invariant subspaces, then π is an irreducible representation. Otherwise, we have a reducible representation.

Definition 2.5.2. [22] A representation on H is called decomposable if there are two non-trivial invariant subspaces H_1 and H_2 of H such that $H = H_1 \oplus H_2$.

Any unitary representation is either irreducible or decomposable. The irreducibility of representation is often established by Schur's lemma.

Lemma 2.5.3. (Schur's lemma)[10, lemma 3.5] Let G be a group and $\mathcal{C}(\pi)$ be the set of all intertwining operators.

- A unitary representation π of G is irreducible if and only if $\mathcal{C}(\pi)$ contains only scalar multiples of the identity.
- Suppose π_1 and π_2 are irreducible unitary representations of G. If π_1 and π_2 are equivalent,then $\mathcal{C}(\pi_1, \pi_2)$; is one-dimensional otherwise, $\mathcal{C}(\pi_1, \pi_2) = 0$.

Definition 2.5.4. [5, §5.1] A character χ of an Abelian locally compact group G is a continuous function $\chi : G \to \mathbb{C}$, which satisfies

$$|\chi(g)|= 1, \quad \chi(g_1 g_2) = \chi(g_1)\chi(g_2),$$

and for all $g_1, g_2 \in G$. That is, a character χ is a one-dimensional continuous irreducible unitary representation of G.

2.6 Induced Representations

In this section, we describe the construction of induced representations [10, 21, 22]. Let G be a group H be a closed subgroup of G; then $X = G/H$ is the left coset space. For a character $\chi : H \to \mathbb{T}$, where $\chi(h_1 h_2) = \chi(h_1)\chi(h_2)$ and $|\chi(h)|= 1$, let V_χ be the vector space of functions $F : G \to \mathbb{C}$ having the property:

$$F(gh) = \overline{\chi(h)}F(g), \quad \forall g \in G, h \in H. \tag{2.9}$$

The space V_χ is invariant under the left action of G, that is

$$\Lambda(g) : V_\chi \to V_\chi, \quad [\Lambda(g)F](g') = F(g^{-1}g'), \quad g, g' \in G. \tag{2.10}$$

The restriction of the left action of G on the space V_χ is called the induced representation.

An equivalent realisation of the above induced representation can be defined on the homogeneous space $X = G/H$. Let $\mathsf{s} : X \to G$, be a section map that is a right inverse of the natural projection map $\mathsf{p}: G \to X$, that is

$$\mathsf{p} \circ \mathsf{s} = \mathbb{I}_X.$$

Then the left action of G on the homogeneous space X is given by:

$$g \cdot x = \mathsf{p}(g\mathsf{s}(x)),$$

where $g \in G$ and $x \in X$. Any element $g \in G$ can be uniquely decomposed as $g = \mathsf{s}(\mathsf{p}(g))\mathsf{r}(g)$ where the map $\mathsf{r}: G \to H$ is given by $\mathsf{r}(g) = \mathsf{s}(\mathsf{p}(g))^{-1}g$.

Now, for a character χ of the subgroup H, introduce the lifting map $\mathcal{L}_\chi : W(X) \to V_\chi$, as follows:

$$[\mathcal{L}_\chi f](g) = \overline{\chi(\mathsf{r}(g))} f(\mathsf{p}(g)), \quad f \in W(X),$$

where $W(X) := \{f : X \to \mathbb{C}\}$ is the vector space of all complex functions on the homogeneous space $X = G/H$. Let the pulling map $\mathcal{P} : V_\chi \to W(X)$, given by:

$$[\mathcal{P}F](x) = F(\mathsf{s}(x)).$$

Proposition 2.6.1. Let the lifting map $\mathcal{L}_\chi$ and the pulling map $\mathcal{P}$ be as defined above. Then

1. $\mathcal{P} \circ \mathcal{L}_\chi = \mathbb{I}_{W(X)}$.
2. $\mathcal{L}_\chi \circ \mathcal{P} = \mathbb{I}_{V_\chi}$.

Proof. 1. Let $f \in W(X)$.

$$\begin{aligned} [(\mathcal{P}\mathcal{L}_\chi)f](x) &= [\mathcal{L}_\chi f](\mathsf{s}(x)) \\ &= \overline{\chi(\mathsf{r}(\mathsf{s}(x)))} f(\mathsf{p}(\mathsf{s}(x))) \\ &= \overline{\chi(\mathsf{s}(\mathsf{p}(s(x)))^{-1}s(x))} f(x) \\ &= \overline{\chi(\mathsf{s}(x)^{-1}s(x))} f(x) = f(x). \end{aligned}$$

2. Let $F \in V_\chi$.

$$\begin{aligned} [(\mathcal{L}_\chi\mathcal{P})F](g) &= \overline{\chi(\mathsf{r}(g))}(\mathcal{P}F)(\mathsf{p}(g)) \\ &= \overline{\chi(\mathsf{r}(g))} F(\mathsf{s}(\mathsf{p}(g)) \\ &= F(\mathsf{s}(\mathsf{p}(g)\mathsf{r}(g)) = F(g). \end{aligned}$$

∎

Next, the operator $\pi_\chi(g)$ on $W(X)$ is given as follows:

$$\pi_\chi(g) := \mathcal{P} \circ \Lambda(g) \circ \mathcal{L}_\chi. \tag{2.11}$$

This can be represented by the following commutative diagram:

$$\begin{array}{ccc} V_\chi & \xrightarrow{\Lambda(g)} & V_\chi \\ {\scriptstyle \mathcal{L}_\chi}\uparrow & & \downarrow{\scriptstyle \mathcal{P}} \\ W(X) & \xrightarrow{\rho_\chi(g)} & W(X) \end{array}$$

Figure 2.1: Induced representation from a character of a subgroup

Thus, the representation π_χ acts on $W(X)$ via the following explicit formula:

$$[\pi_\chi(g)f](x) = \bar{\chi}(\mathsf{r}(g^{-1} * \mathsf{s}(x)))f(g^{-1} \cdot x). \tag{2.12}$$

Theorem 2.6.2. The mapping $\pi_\chi : G \to GL(W(X))$, given by

$$g \mapsto \pi_\chi(g) = \mathcal{P} \circ \Lambda(g) \circ \mathcal{L}_\chi,$$

is a representation.

Proof. To show that π_χ is a representation, it is enough to prove that

$$\pi_\chi(g_1g_2) = \pi_\chi(g_1)\pi_\chi(g_2) \quad \text{and} \quad \pi_\chi(e_G) = \mathbb{I}$$

for all $g_1, g_2 \in G$, and e_G is the identity element of G. By using Proposition (2.6.1), we have

$$\begin{aligned} \pi_\chi(g_1g_2) &= \mathcal{P} \circ \Lambda(g_1g_2) \circ \mathcal{L}_\chi \\ &= \mathcal{P} \circ \Lambda(g_1) \circ \Lambda(g_2) \circ \mathcal{L}_\chi \\ &= \mathcal{P} \circ \Lambda(g_1) \circ \mathbb{I}_{V_\chi} \circ \Lambda(g_2) \circ \mathcal{L}_\chi \\ &= [\mathcal{P} \circ \Lambda(g_1) \circ \mathcal{L}_\chi] \circ [\mathcal{P} \circ \Lambda(g_2) \circ \mathcal{L}_\chi] \\ &= \pi_\chi(g_1) \circ \pi_\chi(g_2). \end{aligned}$$

In addition,

$$\begin{aligned}\pi_\chi(e_G) &= \mathcal{P} \circ \Lambda(e_G) \circ \mathcal{L}_\chi \\ &= \mathcal{P} \circ \mathbb{I}_{V_\chi} \circ \mathcal{L}_\chi \\ &= \mathcal{P} \circ \mathcal{L}_\chi = \mathbb{I}_{W(X)}.\end{aligned}$$

Hence, $\pi_\chi(g)$ is a representation. □

Chapter 3

Representations of the Affine Group

The affine group, denoted by Aff, is a non-commutative, locally compact Lie group of smallest dimensionality. The main purpose of this chapter is to study the intertwining operators between the affine group representations. We start by providing some important facts about the affine group. Then, we present the induced representations of the affine group following the study by Elmabrok [8]. Our main references are [8, 10, 21].

3.1 The Affine Group

The affine group is the set Aff $:= \{(a, b) : a > 0, b \in \mathbb{R}\}$, with the group law $*$ defined by

$$(a, b) * (a', b') = (aa', ab' + b), \tag{3.1}$$

where $(a, b), (a', b') \in$ Aff. The identity element is $e = (1, 0)$ and the inverse element is $(a, b)^{-1} = (a^{-1}, -ba^{-1})$.

For any $(a, b) \in$ Aff, we can define a transformation of the real line $A_{a,b} : \mathbb{R} \to \mathbb{R}$, by the following:

$$A_{a,b}(x) = (a, b) \cdot x = ax + b, \quad x \in \mathbb{R}.$$

$A_{a,b}$ is called an affine transformation of $\mathbb{R}$. This action is consistent with the group law (3.1). Hence, the affine group is called the $ax + b$ group as well.

The affine group is isomorphic to the group of all upper triangular 2×2 real matrices of the form:

$$g = \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix},$$

with matrix multiplication as the group law. The identity and the inverse matrices elements, respectively, are given as follows:

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad g^{-1} = \begin{pmatrix} a^{-1} & -ba^{-1} \\ 0 & 1 \end{pmatrix}.$$

Equivalently, any element (a, b) of the affine group can be identified as a real matrix of determinant one by the following relation:

$$(a, b) \Leftrightarrow \frac{1}{\sqrt{a}} \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}.$$

Hence, the Aff group is a subgroup of the $SL_2(\mathbb{R})$ group.

We can decompose the affine group Aff as a semi-direct product. That is Aff $=$ $A \ltimes N$, where N is the normal closed subgroup given by $\{(1, b) : b \in \mathbb{R}\}$ and the subgroup $A = \{(a, 0) : a > 0\}$. The subgroup N can be identified with $\mathbb{R}$ via the correspondence $(1, b) \leftrightarrow b$ and the subgroup A identify with $\mathbb{R}_+$ where $(a, 0) \leftrightarrow a$, [21].

In a geometric point of view, the affine group as a set can be identified with the upper half-plane

$$\mathbb{C}_+ = \{g = b + ia, \quad b \in \mathbb{R}, a > 0\}.$$

3.2 Affine Algebra

The Lie algebra of the affine group is denoted by $\mathfrak{aff}$. It consists of all real 2×2 matrices with second row 0. The following two elements form a basis of $\mathfrak{aff}$,

$$X_A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \text{and} \quad X_N = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. \tag{3.2}$$

Also, X_A and X_N generate one-parameter subgroups of the affine group, respectivley, given as follows:

$$a(t) = \begin{pmatrix} \mathrm{e}^t & 0 \\ 0 & 1 \end{pmatrix}, \quad \text{and} \quad n(t) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}. \tag{3.3}$$

3.3 Haar Measure

Definition 3.3.1. [10, §2.2] Let G be a locally compact group. A left (respectively, right) Haar measure μ on G is a nonzero Radon measure which is invariant under left (respectively, right) translation, that is, $\mu(gE) = \mu(E)$ $\Big($respectively, $\mu(Eg) = \mu(E)\Big)$ for every Borel set $E \subset G$ and $g \in G$.

The affine group has a left Haar measure , given as follows:

$$d\nu(a, b) = a^{-2} da db. \tag{3.4}$$

The measure $d\nu$, is left invariant under the natural action of Aff on itself. That is

$$\begin{aligned} d\nu((a', b') * (a, b)) &= d\nu(a'a, a'b + b') \\ &= (a'a)^{-2} d(a'a) d(a'b + b') \\ &= (a'a)^{-2} (a')^2 da db \\ &= d\nu(a, b). \end{aligned}$$

Moreover, the right Haar measure of the Aff group is given as follows:

$$d\mu(a, b) = a^{-1} da db, \tag{3.5}$$

where

$$d\mu((a, b) * (a', b')) = d\mu(a, b).$$

The affine group is non-unimodular group since

$$d\nu(a,b) = a^{-2}dadb = a^{-1}d\mu(a,b),$$

and $\triangle(a,b) = a^{-1}$ is the modular function of the group [21].

The invariant measure on the subgroup A is the Haar measure $\frac{da}{a}$, and on the subgroup N is the Lebesgue measure db.

3.4 Induced Representation of the Affine Group

The construction of a unitary induced representations is follows as in section §2.6. The affine group has three non-conjugated subgroups $\{e\}$, N and A. Hence, we can obtain the following unitary induced representations:

- the left regular representation on the group Aff itself which induced by a character of the subgroup $H = \{e\}$.
- the co-adjoint representation on the half real lines which induced by a character of the subgroup N.
- the quasi-regular representation on the real line which induced by a character of the subgroup A.

3.4.1 The Left Regular Representation

Let $H = \{e\}$, be the trivial subgroup of the affine group. The homogeneous space is $X = \text{Aff}/H \sim \text{Aff}$.

Let $L_2(\text{Aff}, d\nu)$, be the Hilbert space of all square integrable complex-valued functions on Aff with respect to the left Haar measure $d\nu$. The norm of any function $F \in L_2(\text{Aff}, d\nu)$ is given as follows:

$$\|F\|^2 = \int_{\text{Aff}} |F(a,b)|^2 \frac{dadb}{a^2} < \infty.$$

The unitary induced represenation of the affine group on $L_2(\text{Aff}, d\nu)$, is called the left regular representation and defined by the following unitary operator :

$$[\Lambda(a,b)F](u,v) := F((a,b)^{-1} * (u,v)) = F\left(\frac{u}{a}, \frac{v-b}{a}\right), \tag{3.6}$$

where $(u,v) \in \text{Aff}$.

3.4.2 The Co-adjoint Representation

For the normal subgroup $N = \{(1,b) : b \in \mathbb{R}\}$, the map

$$a \mapsto (a,0)N, \quad \text{where} \quad a \in \mathbb{R}_+,$$

identifies the homogeneous space $X = \text{Aff}/N$ with the the subgroup $A = \mathbb{R}_+$.

Let $\chi_\tau : N \to \mathbb{T}$ be the character of N, defined as follows:

$$\chi_\tau(1,b) = e^{2\pi i b\tau}, \qquad \tau \in \mathbb{R}. \tag{3.7}$$

This character induces a representation of the Aff group constructed in the Hilbert space $L_2^{\chi_\tau}(\text{Aff}, N)$. The space $L_2^{\chi_\tau}(\text{Aff}, N)$ consists of the complex functions F_τ : $\text{Aff} \to \mathbb{C}$, with the property

$$F_\tau(a,b) = \overline{\chi_\tau}(1, \frac{b}{a})F_\tau(a,0).$$

The invariant measure of the homogeneous space $X \sim \mathbb{R}_+$ is $\frac{da}{a}$, so the norm of F_τ is given by the following:

$$\|F_\tau\|_A^2 = \int_{\mathbb{R}_+} |F_\tau(a,0)|^2 \, \frac{da}{a}.$$

The space $L_2^{\chi_\tau}(\text{Aff}, N)$ is invariant under the left Aff-shifts (3.6). The restriction of the left Aff-shifts on the space $L_2^{\chi_\tau}(\text{Aff}, N)$ is the induced representation.

In accordance with the general conctruction in §2.6, we will obtain an equivalent form of this induced representation constructed in the left homogeneous space $X = \text{Aff}/N$ by using lifting and pulling maps.

First, because the affine group is a semi-direct product of subgroups N and A, there is a natural section map s for $\text{Aff}/N \cong \mathbb{R}_+$ in Aff. The map s is given as follows:

$$\mathsf{s} : \mathbb{R}_+ \to \text{Aff}, \quad \text{where} \quad \mathsf{s}(a) = (a,0). \tag{3.8}$$

This is the right inverse of the following natural projection map:

$$\mathsf{p} : \text{Aff} \to \mathbb{R}_+, \quad \text{where} \quad \mathsf{p}(a,b) = a, \tag{3.9}$$

such that $\mathsf{p}(\mathsf{s}(a)) = \mathsf{p}(a,0) = a$.

Then, the unique decomposition of any $(a,b) \in \text{Aff}$, takes the following form:

$$(a,b) = (a,0) * \left(1, \frac{b}{a}\right), \tag{3.10}$$

and the map $\mathsf{r} : \text{Aff} \to N$ is given by

$$\mathsf{r}(a,b) = \mathsf{s}(a)^{-1} * (a,b) = \left(1, \frac{b}{a}\right). \tag{3.11}$$

The space $X = \text{Aff}/N$ is a left homogeneous space under the Aff-action defined in terms of p and s as follows:

$$(a,b) : w \mapsto (a,b) \cdot w = \mathsf{p}((a,b) * \mathsf{s}(w)) = aw, \tag{3.12}$$

where $(a,b) \in \text{Aff}$, $w \in X$ and $\cdot$ is the action of Aff on X from the left.

Next, define the lifting map for the character χ_τ of the subgroup N, $\mathcal{L}_{\chi_\tau} : L_2(\mathbb{R}_+, \frac{da}{a}) \to L_2^{\chi_\tau}(\text{Aff}, N)$ by the following:

$$[\mathcal{L}_{\chi_\tau} f](a,b) := \overline{\chi_\tau}(\mathsf{r}(a,b)) f(\mathsf{p}(a,b)) = e^{-2\pi i \frac{b}{a}\tau} f(a),$$

where $f(a) = F_\tau(a,0)$ is a function on the subgroup A. Then, the pulling map is expressed as

$$\mathcal{P} : L_2^{\chi_\tau}(\text{Aff}, N) \to L_2(\mathbb{R}_+, \frac{da}{a}), \quad [\mathcal{P}F_\tau](a) = F_\tau(\mathsf{s}(a)) = f(a),$$

such that $\mathcal{P} \circ \mathcal{L}_{\chi_\tau} = I$, and $\mathcal{L}_{\chi_\tau} \circ \mathcal{P} = I$.

Therefore, the representation $\rho^+_{\chi_\tau} : L_2(\mathbb{R}_+, \frac{da}{a}) \to L_2(\mathbb{R}_+, \frac{da}{a})$, which is induced by the character χ_τ, is given as follows:

$$\rho^+_{\chi_\tau}(a,b) := \mathcal{P} \circ \Lambda(a,b) \circ \mathcal{L}_{\chi_\tau}.$$

To find the explicit formula of the representation $\rho^+_{\chi_\tau}(a,b)$, first apply the left action to the lifting map:

$$\begin{aligned}
\Lambda(a,b)[\mathcal{L}_{\chi_\tau}f](u,v) &= [\mathcal{L}_{\chi_\tau}f]\left((a,b)^{-1}*(u,v)\right)\\
&= [\mathcal{L}_{\chi_\tau}f]\left(\frac{u}{a},\frac{v-b}{a}\right)\\
&= \overline{\chi_\tau}\left(\mathsf{r}(\frac{u}{a},\frac{v-b}{a})\right)f\left(\mathsf{p}\left(\frac{u}{a},\frac{v-b}{a}\right)\right)\\
&= \overline{\chi_\tau}(1,\frac{v-b}{u})f(\frac{u}{a})\\
&= e^{-2\pi i(\frac{v-b}{u})\tau}f(\frac{u}{a})\\
&= e^{-2\pi i(\frac{-b}{u})\tau}e^{2\pi i(\frac{v}{u})t}f\left(\frac{u}{a}\right)\\
&= e^{2\pi i(\frac{b}{u})\tau}[\mathcal{L}_{\chi_\tau}f]\left(\frac{u}{a},\frac{v}{a}\right) = F^\tau_{(a,b)}(u,v).
\end{aligned} \tag{3.13}$$

Then, apply the pulling map to the function $F^\tau_{(a,b)}$:

$$\begin{aligned}
[\mathcal{P}F^\tau_{(a,b)}](u) &= F^\tau_{(a,b)}(\mathsf{s}(u))\\
&= F^\tau_{(a,b)}(u,0)\\
&= e^{2\pi i(\frac{b}{u})t}f\left(\frac{u}{a}\right).
\end{aligned} \tag{3.14}$$

Therefore, by (3.13) and (3.14), we obtain the following formula:

$$[\rho^+_{\chi_t}(a,b)f](u) = [\mathcal{P}\Lambda(a,b)\mathcal{L}_{\chi_t}](u) = e^{2\pi i\frac{b}{u}t}f\left(\frac{u}{a}\right), \tag{3.15}$$

where $f\in L_2(\mathbb{R}_+)$.

By changing the variable $t=u^{-1}$, $g(t)=t^{-\frac{1}{2}}f(t^{-1})$, we determine the following:

$$[\rho^+_{\chi_\tau}(a,b)g](t) = \sqrt{a}e^{2\pi ib\tau t}g(at), \tag{3.16}$$

where $g\in L_2(\mathbb{R}_+,da)$.

Proposition 3.4.1. [9] The co-adjoint representation $\rho^+_{\chi_\tau}$ (3.16) is irreducible for $\tau\neq 0$.

Proof. To show irreducibility, we will use Schur's lemma [10]. We will show that any operator T, permutable with the representation $\rho^+_{\chi_\tau}(a,b)$, is a multiple of the identity operator. The permutability of T with $\rho^+_{\chi_\tau}(a,b)$ implies its permutability with the derived representations:

$$[d\rho^+_{\chi_\tau}(X_A)g](u) = ug'(u) + \frac{1}{2}g(u), \tag{3.17}$$

$$[d\rho^+_{\chi_\tau}(X_N)g](u) = -2\pi i u g(u). \tag{3.18}$$

The latter differs from the operator of multiplication by u in a constant factor only. It is known that the only operators in the space of infinitely differentiable finite functions permutable with the operator of multiplication by u are operators of multiplication by a function [24, Lemma, p. 111] . Therefore, T has the form

$$Tg(u) = q(u)g(u).$$

To find $q(u)$, we use the permutability of T with

$$[\rho^+_{\chi_\tau}(a,0)g](u) = \sqrt{a}g(au).$$

The equality $T\rho^+_{\chi_\tau}(a,0) = \rho^+_{\chi_\tau}(a,0)T$, that means

$$q(u)\sqrt{a}g(au) = \sqrt{a}q(au)g(au), \qquad \text{for} \quad a \in \mathbb{R}_+.$$

Hence, $q(u) = q(au)$, and consequently, $q(u)$ is a constant. Finally, the representation $\rho^+_{\chi_\tau}$ is irreducible. □

Proposition 3.4.2. [9, Proposition 3.2.6] Two co-adjoint representations $\rho^+_{\chi_{\tau_1}}$ and $\rho^+_{\chi_{\tau_2}}$ of the affine group Aff are equivalent if and only if $\tau_1\tau_2 > 0$.

Proof. Let the numbers τ_1 and τ_2 be of the same sign (i.e. $\frac{\tau_1}{\tau_2} > 0$). From the identity

$$(a,0)^{-1} * (a',b') * (a,0) = \left(a', \frac{b'}{a}\right),$$

we have

$$[\rho^+_{\chi_{\tau_1}}(a,0)]^{-1} * \rho^+_{\chi_{\tau_1}}(a',b') * \rho^+_{\chi_{\tau_1}}(a,0) = \rho^+_{\chi_{\tau_1}}\left(a', \frac{b'}{a}\right).$$

Let $a = \frac{\tau_1}{\tau_2} > 0$, then

$$\left[\rho^+_{\chi_{\tau_1}}\left(\frac{\tau_1}{\tau_2}, 0\right)\right]^{-1} * \rho^+_{\chi_{\tau_1}}(a',b') * \rho^+_{\chi_{\tau_1}}\left(\frac{\tau_1}{\tau_2}, 0\right) = \rho^+_{\chi_{\tau_1}}\left(a', \frac{b'\tau_2}{\tau_1}\right).$$

Let $U = \rho^+_{\chi_{\tau_1}}\left(\frac{\tau_1}{\tau_2}, 0\right)$ be a unitary operator on $L_2(\mathbb{R}_+)$. We then obtain the following:

$$U^{-1}\rho^+_{\chi_{\tau_1}}(a',b')U = \rho^+_{\chi_{\tau_1}}\left(a', \frac{b'\tau_2}{\tau_1}\right).$$

By simple calculation, we determine that

$$\left[\rho^+_{\chi_{\tau_1}}\left(a', \frac{b'\tau_2}{\tau_1}\right) f\right](t) = [\rho^+_{\chi_{\tau_2}}(a', b')f](t),$$

which implies

$$U^{-1}\rho^+_{\chi_{\tau_1}}\left(a', \frac{b'\tau_2}{\tau_1}\right) U = \rho^+_{\chi_{\tau_2}}(a', b').$$

Hence,

$$\rho^+_{\chi_{\tau_1}} \cong \rho^+_{\chi_{\tau_2}}.$$

Next, let τ_1 and τ_2 be of different signs. We then have two cases:

In the first case, one number is positive and another is negative, so let $\tau_1 \in (0, \infty)$ and $\tau_2 \in (-\infty, 0)$. For χ_{τ_1}, fix the point $\tau_1 = 1$, and there is the following infinite-dimensional representation:

$$[\rho^+_{\chi_1}(a, b)g](t) = \sqrt{a}e^{2\pi ibt}g(at). \tag{3.19}$$

Also, for χ_{τ_2}, fix the point $\tau_2 = -1$, the infinite-dimensional representation is given by

$$[\rho^-_{\chi_{-1}}(a, b)g](t) = \sqrt{a}e^{-2\pi ibt}g(at). \tag{3.20}$$

We subsequently denote the representations $\rho^+_{\chi_1}$ and $\rho^-_{\chi_{-1}}$ as ρ^+_χ and ρ^-_χ, respectively. Assume that $\rho^+_\chi(a, b) \cong \rho^-_\chi(a, b)$. Then, there exists a unitary intertwining operator A:

$$A\rho^+_\chi(a, b) = \rho^-_\chi(a, b)A.$$

The Gårding space $\mathcal{G}(\rho^+) = \{\rho^+(f)t : t \in L_2(\mathbb{R}_+), f \in \mathcal{C}^\infty_0(\mathrm{Aff})\}$ [40] contains for any nonempty open interval, nonzero functions supported on that interval (because one can take a function supported on a smaller interval and mark it with a suitable $\mathcal{C}^\infty_0$ function on Aff that is supported on the small neighbourhood of the identity). Thus, for any $f \in \mathcal{G}(\rho^+)$, we have

$$A\left(\frac{1}{2\pi}d\rho^+_{\chi_1}(X_N) + iI\right) f = \left(\frac{1}{2\pi}d\rho^+_{\chi_{-1}}(X_N) + iI\right) Af$$

for the generator X_N of the subgroup N. Take $f \in \mathcal{G}(\rho^+)$, with the support $[\frac{1}{2}, \frac{3}{2}]$, then using (3.18), we have

$$\left\|A\left(\frac{1}{2\pi}d\rho_{\chi_1}^{+}(X_N)+iI\right)f\right\|^2 = \left\|\left(\frac{1}{2\pi}d\rho_{\chi_1}^{+}(X_N)+iI\right)f\right\|^2 = \int_{1/2}^{3/2}\left|\left[\frac{-2\pi}{2\pi}it+i\right]f(t)\right|^2\frac{dt}{t} \leq \frac{1}{4}\|f\|^2. \tag{3.21}$$

Conversely, we have

$$\left\|\left(\frac{1}{2\pi}d\rho_{\chi_1}^{-}(X_N)+iI\right)Af\right\|^2 = \int_0^{\infty}\left|\left(\frac{2\pi}{2\pi}it+i\right)[Af](t)\right|^2\frac{dt}{t} \geq \int_0^{\infty}|[Af](t)|^2\frac{dt}{t} = \|Af\|^2 = \|f\|^2. \tag{3.22}$$

In the second case, let $\tau_1 = 0$, and τ_2 is either positive or negative. Then, for $\tau_1 = 0$, the character $\chi_0(1,b) = 1$ is trivial and the operator $\rho_{\chi_0}^{+}(1,b)$ is the identity for all b, so the representation $\rho_{\chi_0}^{+}$ is not injective. However, the other representations of the Aff group are all injective since they are always represented by different unitary operators. Hence, they are not equivalent to $\rho_{\chi_0}^{+}$. □

Remark 3.4.3. Using the canonical embedding of $\mathbb{R}_{\pm}$ in $\mathbb{R}$, we can regard the spaces $L_2(\mathbb{R}_+)$ and $L_2(\mathbb{R}_-)$ as closed subspaces of $L_2(\mathbb{R})$.

For $f \in L_2(\mathbb{R}_+)$, we can define the map $\widetilde{f} : \mathbb{R} \to \mathbb{C}$ by the following:

$$\widetilde{f}(x) = \begin{cases} f(x), & x \geq 0 \\ 0, & x < 0 \end{cases}.$$

Then, $\widetilde{f} \in L_2(\mathbb{R})$, with $\|f\|_{L_2(\mathbb{R}_+)} = \|\widetilde{f}\|_{L_2(\mathbb{R})}$.

Similarly, for $f \in L_2(\mathbb{R}_-)$, define the map $\widetilde{g} : \mathbb{R} \to \mathbb{C}$ as follows:

$$\widetilde{f}(x) = \begin{cases} 0, & x > 0 \\ f(x), & x \leq 0 \end{cases}.$$

Thus, $\widetilde{f} \in L_2(\mathbb{R})$, with $\|f\|_{L_2(\mathbb{R}_-)} = \|\widetilde{f}\|_{L_2(\mathbb{R})}$. We conclude that

$$L_2(\mathbb{R}_{\pm}) = \{f \in L_2(\mathbb{R}) : \operatorname{supp}(f) \subseteq \mathbb{R}_{\pm}\}.$$

Lemma 3.4.4. The Hilbert space $L_2(\mathbb{R})$ contains two closed proper invariant subspaces $L_2(\mathbb{R}_+)$ and $L_2(\mathbb{R}_-)$, where

$$L_2(\mathbb{R}) = L_2(\mathbb{R}_+) \oplus L_2(\mathbb{R}_-).$$

Proof. We claim that $L_2(\mathbb{R}_+)^\perp = L_2(\mathbb{R}_-)$, as a subset of $L_2(\mathbb{R})$, where

$$L_2(\mathbb{R}_+)^\perp := \{g \in L_2(\mathbb{R}), \langle g, f\rangle = 0, \forall f \in L_2(\mathbb{R}_+)\}.$$

To this end, we shall show that $L_2(\mathbb{R}_+)^\perp \subset L_2(\mathbb{R}_-)$ and $L_2(\mathbb{R}_-) \subset L_2(\mathbb{R}_+)^\perp$.
Let $g \in L_2(\mathbb{R}_+)^\perp$, then

$$\langle g, f\rangle = \int_{\mathbb{R}} g(x)\overline{f(x)}dx = 0. \tag{3.23}$$

This implies that supp$(g) \subseteq (-\infty, 0)$. Moreover, if supp$(g) \nsubseteq (-\infty, 0)$, then there is $I =$ supp$(g) \cap [0, +\infty)$, such that $f = g\Big|_I \in L_2(\mathbb{R}_+)$. That is,

$$\int_{\mathbb{R}} g(x)\overline{f(x)}dx \geq \int_I g(x)\overline{g(x)}dx \geq 0,$$

which contradicts (3.23). Thus, $L_2(\mathbb{R}_+)^\perp \subset L_2(\mathbb{R}_-)$.

Conversely, if $g \in L_2(\mathbb{R}_-)$, we can consider g as a function in $L_2(\mathbb{R})$, where supp$(g) \subseteq (-\infty, 0)$. Then, for $f \in L_2(\mathbb{R}_+)$, we have

$$\int_{\mathbb{R}} g(x)\overline{f(x)}dx = \int_{\mathbb{R}_+} g(x)\overline{f(x)}dx + \int_{\mathbb{R}_-} g(x)\overline{f(x)}dx = 0.$$

Hence, $g \in L_2(\mathbb{R}_+)^\perp$, and since g was arbitrary, we obtain that $L_2(\mathbb{R}_-) \subset L_2(\mathbb{R}_+)^\perp$.

Finally, we have $L_2(\mathbb{R}_+) \subset L_2(\mathbb{R})$ and $L_2(\mathbb{R}_+)^\perp \subset L_2(\mathbb{R})$, by [37, Proposition 4.2], we obtain the following:

$$L_2(\mathbb{R}_+) \oplus L_2(\mathbb{R}_+)^\perp = L_2(\mathbb{R}_+) \oplus L_2(\mathbb{R}_-) = L_2(\mathbb{R}).$$

□

As a consequence of Lemma 3.4.4, the direct sum of the two irreducible representations ρ_χ^+ and ρ_χ^- is given as follows:

$$\rho_\chi(a, b) = \rho_\chi^+(a, b) \oplus \rho_\chi^-(a, b),$$

where

$$[\rho_\chi(a, b)g](t) = \sqrt{a}e^{2\pi ibt}g(at), \quad \text{for } g \in L_2(\mathbb{R}, da). \tag{3.24}$$

3.4.3 The Quasi-Regular Representation

For the subgroup $A = \{(a, 0), a > 0\}$, the following map:

$$b \mapsto (1, b)A, \quad \text{where} \quad b \in \mathbb{R},$$

identify the homogeneous space $X = \text{Aff}/A$ with the subgroup $N = \mathbb{R}$.
Let $\chi_\omega : A \to \mathbb{T}$ be a character of the subgroup A, defined by

$$\chi_\omega(a, 0) = a^{i\omega}, \qquad \omega \in \mathbb{R}. \tag{3.25}$$

This character induces a representation of the affine group constructed in the Hilbert space $L_2^{\chi_\omega}(\text{Aff})$. The space $L_2^{\chi_\omega}(\text{Aff})$ consists of the functions $F_\omega : \text{Aff} \to \mathbb{C}$, with the property

$$F_\omega(a, b) = \overline{\chi_\omega(a, 0)} F_\omega(1, b).$$

The invariant measure on the homogeneous space $X \sim \mathbb{R}$ is db. Then, the norm of the functions F_ω is given as follows:

$$\|F_\omega\|_N^2 = \int_{\mathbb{R}} |F_\omega(1, b)|^2 db.$$

The space $L_2^{\chi_\omega}(\text{Aff})$ is invariant under the left Aff-shifts (3.6). The restriction of the left Aff-shifts on the space $L_2^{\chi_\omega}(\text{Aff})$ is called the induced representation from the character χ_ω.

In the following, we will obtain an equivalent form of this induced representation constructed in the left homogeneous space $X = \text{Aff}/A$.
Let s be the section map from the homogeneous space $\text{Aff}/A = \mathbb{R}$ to the affine group, given by

$$\mathsf{s} : \mathbb{R} \to \text{Aff}, \quad \text{such that} \quad \mathsf{s}(b) = (1, b), \quad b \in \mathbb{R}. \tag{3.26}$$

The right inverse of s is the natural projection map, given as follows:

$$\mathsf{p} : \text{Aff} \to \mathbb{R}, \quad \text{where} \quad \mathsf{p}(a, b) = b. \tag{3.27}$$

Therefore, the unique decomposition of any $(a, b) \in \text{Aff}$, takes the following form:

$$(a, b) = (1, b) * (a, 0),$$

and the map $\mathsf{r} : \text{Aff} \to A$ is given by

$$\mathsf{r}(a,b) = \mathsf{s}(b)^{-1} * (a,b) = (a,0). \tag{3.28}$$

The space $X = \text{Aff}/A$ is a left homogeneous space under the Aff-action, defined in terms of p and s as follows:

$$(a,b) : x \mapsto (a,b) \cdot x = ax + b, \tag{3.29}$$

where $(a,b) \in \text{Aff}$, $x \in X$ and $\cdot$ is the action of Aff on X from the left.

Next, define the lifting map $\mathcal{L}_{\chi_\omega} : L_2(\mathbb{R}) \to L_2^{\chi_\omega}(\text{Aff})$, for the character χ_ω of the subgroup A by the following:

$$[\mathcal{L}_{\chi_\omega} f](a,b) := \overline{\chi_\omega(\mathsf{r}(a,b))} f(\mathsf{p}(a,b)) = a^{-i\omega} f(b),$$

where $f(b) = F_\omega(1,b)$ is a function on the subgroup A.

The pulling map is expressed as

$$\mathcal{P}' : L_2^{\chi_\omega}(\text{Aff}) \to L_2(\mathbb{R}),$$
$$[\mathcal{P}' F_\omega](b) = F_\omega(\mathsf{s}(b)) = f(b),$$

such that $\mathcal{P}' \circ \mathcal{L}_{\chi_\omega} = I$ and $\mathcal{L}_{\chi_\omega} \circ \mathcal{P}' = I$. Therefore, the character χ_ω induced the representation $\pi_{\chi_\omega} : L_2(\mathbb{R}) \to L_2(\mathbb{R})$, which is given as follows:

$$\pi_{\chi_\omega}(a,b) := \mathcal{P}' \circ \Lambda(a,b) \circ \mathcal{L}_{\chi_\omega}, \tag{3.30}$$

Furthermore, in order that the representation π_{χ_ω} to be unitary in $L_2(\mathbb{R})$ we will choose the exponent $-i\omega + \frac{1}{2}$ for the character (3.25). To find the formula of the representation $\pi_{\chi_\omega}(a,b)$, first, apply the left action to the lifting map:

$$\begin{aligned}
\Lambda(a,b)[\mathcal{L}_{\chi_\omega} f](u,v) &= [\mathcal{L}_{\chi_\omega} f]\left((a,b)^{-1} * (u,v)\right) \\
&= [\mathcal{L}_{\chi_\omega} f]\left(\frac{u}{a}, \frac{v-b}{a}\right) \\
&= \overline{\chi_\omega}\left(\mathsf{r}(\frac{u}{a}, \frac{v-b}{a})\right) f\left(\mathsf{p}\left(\frac{u}{a}, \frac{v-b}{a}\right)\right) \\
&= \overline{\chi_\omega}(\frac{u}{a}, 0) f(\frac{v-b}{a}) \\
&= \left(\frac{u}{a}\right)^{-i\omega + \frac{1}{2}} f(\frac{v-b}{a}) = F^\omega_{(a,b)}(u,v).
\end{aligned} \tag{3.31}$$

Then, apply the pulling map:

$$\begin{aligned}[\mathcal{P}' F^{\omega}_{(a,b)}](v) &= F^{\omega}_{(a,b)}(\mathsf{s}(v)) \\ &= F^{\omega}_{(a,b)}(1, v) \\ &= \left(\frac{1}{a}\right)^{-i\omega+\frac{1}{2}} f\left(\frac{v-b}{a}\right).\end{aligned} \tag{3.32}$$

Therefore, by (3.31) and (3.32) from (3.30), we obtain the explicit following formula:

$$[\pi_{\chi_\omega}(a,b)f](v) = \left(\frac{1}{a}\right)^{-i\omega+\frac{1}{2}} f\left(\frac{v-b}{a}\right), \tag{3.33}$$

where $f \in L_2(\mathbb{R})$.

Next, we will study the irreducibility of the representation π_{χ_ω}. Let $H_2(\mathbb{R})$ be the Hardy space defined by

$$H_2(\mathbb{R}) = \{f \in L_2(\mathbb{R}) : \text{supp}(\hat{f}) \subseteq [0, \infty)\},$$

where $\hat{f}$ is the Fourier transform for the function f and supp is the support of $\hat{f}$. Similarly, we defined the conjugate Hardy space $H_2^{\perp}(\mathbb{R})$, as

$$H_2^{\perp}(\mathbb{R}) = \{f \in L_2(\mathbb{R}) : \text{supp}(\hat{f}) \subseteq (-\infty, 0]\}.$$

Proposition 3.4.5. [43, Proposition 18.4] $H_2(\mathbb{R})$ and $H_2^{\perp}(\mathbb{R})$ are closed subspaces of $L_2(\mathbb{R})$.

Proposition 3.4.6. The Fourier transform $\mathcal{F} : H_2(\mathbb{R}) \to L_2(\mathbb{R}_+)$

$$[\mathcal{F}f](\lambda) = \hat{f}(\lambda) = \int_{-\infty}^{\infty} e^{-2\pi i x\lambda} f(x)dx, \quad \lambda \in \mathbb{R}, \tag{3.34}$$

intertwines the quasi-regular representation $\pi^+_{\chi_\omega} : H_2(\mathbb{R}) \to H_2(\mathbb{R})$ (3.33) and the co-adjoint representation $\rho^+_{\chi} : L_2(\mathbb{R}_+) \to L_2(\mathbb{R}_+)$ (3.19). That is:

$$\mathcal{F}\pi^+_{\chi_\omega}(a,b) = \rho^+_{\chi}(a,b)\mathcal{F}. \tag{3.35}$$

Proof. Let $f \in H_2(\mathbb{R})$ and $\pi^+_{\chi_\omega}(a,b)f \in H_2(\mathbb{R})$, be the quasi-regular representation given by (3.33). The Fourier transform of $\pi^+_{\chi_\omega}(a,b)f$, is expressed as follows:

$$\begin{aligned}
[\widehat{\pi^+_{\chi_\omega}(a,b)}f](\lambda) &= \int_{-\infty}^{\infty} e^{-2\pi i\lambda x}[\pi^+_{\chi_\omega}(a,b)f](x)dx \\
&= \int_{-\infty}^{\infty} e^{-2\pi i\lambda x} a^{i\omega-\frac{1}{2}} f\left(\frac{x-b}{a}\right)dx \\
&= a^{i\omega-\frac{1}{2}}\int_{-\infty}^{\infty} e^{-2\pi i\lambda(ay+b)}f(y)ady \\
&= a^{i\omega}a^{\frac{1}{2}}e^{-2\pi i\lambda b}\int_{-\infty}^{\infty} e^{-2\pi i\lambda ay}f(y)dy \\
&= a^{i\omega}a^{\frac{1}{2}}e^{-2\pi i\lambda b}\hat{f}(a\lambda) = a^{i\omega}[\rho^+_\chi(a,b)\hat{f}](\lambda).
\end{aligned}$$

□

By using the Szegö projection $P_{\mathbb{R}} : L_2(\mathbb{R}) \to H_2(\mathbb{R})$, the representation $\pi^+_{\chi_\omega} : H_2(\mathbb{R}) \to H_2(\mathbb{R})$ is given as follows:

$$\pi^+_{\chi_\omega}(a,b) = \mathcal{P}' \circ \Lambda(a,b) \circ \mathcal{L}_{\chi_\omega} \circ P_{\mathbb{R}}.$$

That is:

$$[\pi^+_{\chi_\omega}(a,b)f](v) = \left(\frac{1}{a}\right)^{-i\omega+\frac{1}{2}} f\left(\frac{v-b}{a}\right), \tag{3.36}$$

where $f \in H_2(\mathbb{R})$. Also, by using the complementary projection $P^\perp_{\mathbb{R}} : L_2(\mathbb{R}) \to H^\perp_2(\mathbb{R})$, the representation $\pi^-_{\chi_\omega} : H^\perp_2(\mathbb{R}) \to H^\perp_2(\mathbb{R})$, is given by the following:

$$\pi^-_{\chi_\omega}(a,b) = \mathcal{P}' \circ \Lambda(a,b) \circ \mathcal{L}_{\chi_\omega} \circ P^\perp_{\mathbb{R}}.$$

Thus,

$$[\pi^-_{\chi_\omega}(a,b)f](v) = \left(\frac{1}{a}\right)^{-i\omega+\frac{1}{2}} f\left(\frac{v-b}{a}\right), \tag{3.37}$$

where $f \in H^\perp_2(\mathbb{R})$.

Proposition 3.4.7. The quasi-regular representations $\pi^\pm_{\chi_\omega}$ are irreducible representations.

Proof. It is enough to prove it for $\pi^+_{\chi_\omega}$, then the same argument works for $\pi^-_{\chi_\omega}$. Let M be a nonzero closed invariant subspace of $H^+_2(\mathbb{R})$with respect to $\pi^+_{\chi_\omega}$. We then claim that $M = H^+_2(\mathbb{R})$. To this end, its enough to show that $M^\perp = \{0\}$.

Since $M \neq \{0\}$, we can pick $f \in M$ to be non zero function. Let $g \in M^\perp$, then we have $\langle g, \pi^+_\omega(a,b)f\rangle = 0$. By using Plancherel's theorem, we obtain

$$\begin{aligned} 0 =& \langle g, \pi^+_\omega(a,b)f\rangle \\ =& \langle \hat{g}, \widehat{\pi^+_\omega(a,b)}f\rangle \\ =& \int_{-\infty}^{\infty} \hat{g}(\lambda)\overline{\widehat{\pi^+_\omega(a,b)f}(\lambda)}d\lambda. \end{aligned}$$

From (3.35), we have

$$\begin{aligned} 0 =& \int_{-\infty}^{\infty} \hat{g}(\lambda)\overline{a^{i\omega}\rho^+_\chi(a,b)\hat{f}(\lambda)}d\lambda \\ =& a^{-i\omega+\frac{1}{2}} \int_{-\infty}^{\infty} e^{2\pi i b\lambda}\hat{g}(\lambda)\overline{\hat{f}(a\lambda)}d\lambda. \end{aligned}$$

Thus

$$\hat{g}(\lambda)\overline{\hat{f}(a\lambda)} = 0, \quad \text{for almost all} \quad \lambda \in \mathbb{R}.$$

Suppose that $\hat{g}(\lambda) \neq 0$ for all λ in a set S with positive measure. Then for all $\lambda \in S$, we obtain the following:

$$\hat{f}(a\lambda) = 0, \quad \forall a \in \mathbb{R}_+.$$

Thus $\hat{f} = 0$ and then $f = 0$; this is a contradiction. Hence, $g = 0$ for all $g \in M^\perp$. □

The Hilbert space $L_2(\mathbb{R})$ with respect to π_{χ_ω} contains precisely two closed proper invariant subspaces $H_2(\mathbb{R})$ and $H_2^\perp(\mathbb{R})$ such that

$$L_2(\mathbb{R}) = H_2(\mathbb{R}) \oplus H_2^\perp(\mathbb{R}).$$

Therefore, the representation π_{χ_ω} is decomposed into two irreducible representations. That is

$$\pi_{\chi_\omega}(a,b) = \pi^+_{\chi_\omega}(a,b) \oplus \pi^-_{\chi_\omega}(a,b).$$

3.5 Intertwining Operators

In this section, we study the intertwining operators between the induced representations of the affine group by using the covariant transform and the induced covariant transform. Figure 3.1 represents the intertwining operators.

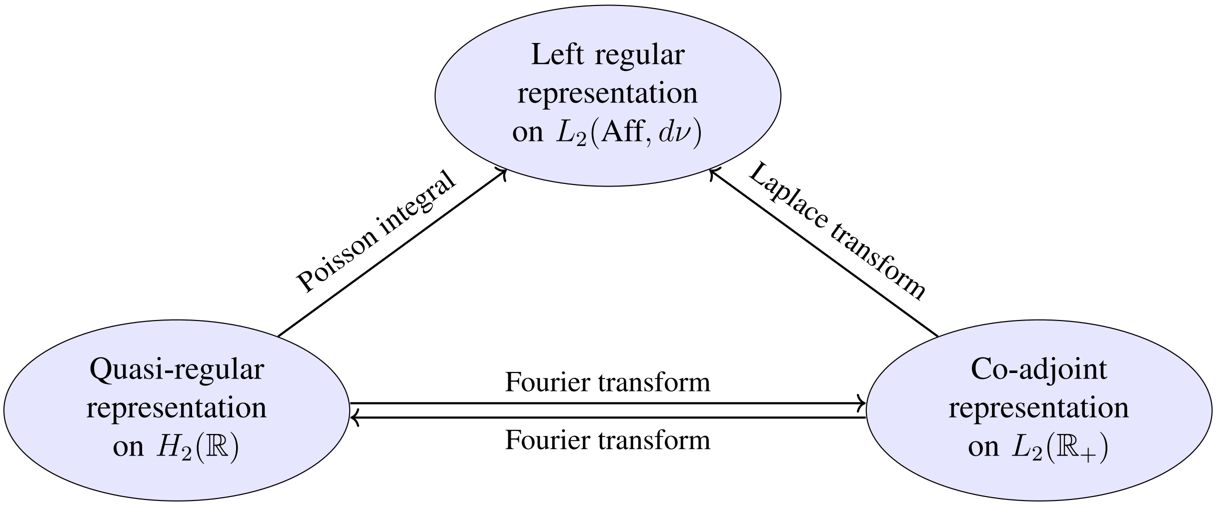

Figure 3.1: Intertwining operators between affine group representations

3.5.1 Covariant Transform

Definition 3.5.1. [29] Let ρ be a representation of a group G in a space V and F be an operator acting from V to a space U. We define a covariant transform $\mathcal{W}_F^\rho$ acting from V to the space $L(G, U)$ of U-valued functions on G by the formula:

$$\mathcal{W}_F^\rho : v \mapsto \hat{v}(g) = F(\rho(g^{-1})v), \quad v \in V, g \in G. \tag{3.38}$$

The operator F is called a fiducial operator.

Theorem 3.5.2. [29] The covariant transform (3.38) intertwines ρ and the left regular representation on $L(G, U)$:

$$\mathcal{W}\rho(g) = \Lambda(g)\mathcal{W}.$$

Example 3.5.3. [29] Let V be a Hilbert space with an inner product $\langle ., . \rangle$ and ρ be a unitary representation of a group G in the space V. Let $F : V \to \mathbb{C}$ be the functional $v \mapsto \langle v, v_0 \rangle$ defined by a vector $v_0 \in V$. The vector v_0 is called the mother wavelet. In the set-up, transformation (3.38) is the well-known expression for a wavelet transform

$$\mathcal{W} : v \mapsto \tilde{v}(g) = \langle \rho(g^{-1})v, v_0 \rangle = \langle v, \rho(g)v_0 \rangle, \quad v \in V, g \in G. \tag{3.39}$$

The family of the vectors $v_g = \rho(g)v_0$ is called wavelets or coherent states. The image of (3.39) consists of scalar valued functions on G.

Corollary 3.5.4. The Poisson integral for $a \in \mathbb{R}_+$ and $b \in \mathbb{R}$ is given by

$$[P\varphi](b,a) = \frac{1}{2}\int_{\mathbb{R}} \frac{a}{(x-b)^2+a^2}\varphi(x)dx. \tag{3.40}$$

It is the wavelet transform that intertwines the quasi-regular representations $\pi_\chi^\pm$ with the left regular representation $\Lambda(a,b)$ given by (3.6).

Proof. We will prove it for the representation π_χ^+ (3.36) and the result is valid for the representation π_χ^- (3.37). Let the fiducial operator $F : H_2(\mathbb{R}) \to \mathbb{C}$ be the functional $\varphi \mapsto \langle \varphi, \varphi_0\rangle$, and the mother wavelet be the conjugate Poisson kernel $\varphi_0 = \frac{-x}{\pi(1+x^2)}$. Then, $\bar{\varphi}_0 = \frac{1}{\pi(1+x^2)}$ is the Poisson kernel. Then, the wavelet transform $\mathcal{W}$ that intertwines the quasi-regular representation π_χ^+ with the left regular representation is given as follows:

$$\begin{aligned}
[\mathcal{W}\pi_\chi^+(a,b)\varphi](x) &= \langle \varphi, \pi_\chi^+(a,b)\varphi_0\rangle \\
&= \int_{\mathbb{R}} \varphi(x)\frac{1}{\sqrt{a}}\bar{\varphi}_0\left(\frac{x-b}{a}\right)dx \\
&= \frac{1}{\sqrt{a}\pi}\int_{\mathbb{R}} \varphi(x)\frac{1}{1+(\frac{x-b}{a})^2}dx \\
&= \frac{1}{\sqrt{a}\pi}\int_{\mathbb{R}} \varphi(x)\frac{a^2}{a^2+(x-b)^2}dx \\
&= \frac{\sqrt{a}}{\pi}\int_{\mathbb{R}} \varphi(x)\frac{a}{a^2+(x-b)^2}dx \\
&= \frac{2\sqrt{a}}{\pi}[P\varphi](b,a).
\end{aligned}$$

□

Corollary 3.5.5. The Laplace transform

$$F(a+ib) = \int_{\mathbb{R}_+} f(t)e^{-2\pi(a+ib)t}dt, \quad a+ib \in \mathbb{C},$$

is the wavelet transform that intertwines the co-adjoint representation of the affine group $\rho_\chi^\pm$ with the left regular representation $\Lambda(a,b)$ given by (3.6).

Proof. It is enough to prove it for the representation ρ_χ^+(3.19). The result works for

ρ_χ^-(3.20). Let the fiducial operator $F : L_2(\mathbb{R}_+) \to \mathbb{C}$ be the functional $f \mapsto \langle f, f_0 \rangle$, and the mother wavelet be $f_0(\lambda) = e^{2\pi\lambda}$. Then, the wavelet transform is given as follows:

$$\begin{aligned}
[\mathcal{W}_{f_0}\rho_\chi^+(a,b)f](\lambda) &= \langle f, \rho_\chi^+(a,b)f_0 \rangle \\
&= \int_{\mathbb{R}_+} f(\lambda)\overline{\rho_\chi^+(a,b)f_0(\lambda)}d\lambda \\
&= \int_{\mathbb{R}_+} f(\lambda)\sqrt{a}e^{-2\pi ib\lambda}\overline{f_0(a\lambda)}d\lambda \\
&= \sqrt{a}\int_{\mathbb{R}_+} f(\lambda)e^{-2i\pi b\lambda}e^{-2\pi a\lambda}d\lambda \\
&= \sqrt{a}\int_{\mathbb{R}_+} f(\lambda)e^{-2\pi(a+ib)\lambda}d\lambda = \sqrt{a}F(a+ib).
\end{aligned}$$

Hence, the intertwining operator between the co-adjoint representation and the left regular is the Laplace transform. □

Example 3.5.6. The representation of the affine group on $L_p(\mathbb{R})$ is given by

$$[\pi_p(a,b)f](x) = a^{\frac{-1}{p}} f\left(\frac{x-b}{a}\right). \tag{3.41}$$

Consider the operators $F_\pm : L_p(\mathbb{R}) \to \mathbb{C}$ defined by

$$F_\pm(f) = \frac{1}{\pi i}\int_{\mathbb{R}} \frac{f(x)}{i \pm x}dx.$$

In $L_2(\mathbb{R})$ we note that $F_+(f) = \langle f, c \rangle$, where $c(x) = \frac{1}{\pi i}\frac{1}{i+x}$. In $L_p(\mathbb{R})$ the covariant transform is given by

$$\widetilde{f}(a,b) = F(\pi_p((a,b)^{-1})f) = \frac{a^{\frac{1}{p}}}{\pi i}\int_{\mathbb{R}} \frac{f(x)}{x-(b+ia)}dx,$$

which is the Cauchy integral from $L_p(\mathbb{R})$ to the space of functions $\widetilde{f}(a,b)$ such that $a^{-\frac{1}{p}}\widetilde{f}(a,b)$ is in the Hardy space on the upper/lower half-plane $H_p(\mathbb{R}^2_\pm)$.

Proposition 3.5.7. [29] Let G be a Lie group and ρ be a representation of G in a space V. Let $[\mathcal{W}f](g) = F(\rho(g^{-1}f)$ be a covariant transform defined by a fiducial operator $F : V \to U$. Then the right shift $[\mathcal{W}f](gg')$ by g' is the covariant transform

$$[\mathcal{W}'f](g) = F'(\rho(g^{-1})f),$$

defined by the fiducial operator $F' = F \circ \rho(g^{-1})$. In other words, the covariant transform intertwines right shifts $R(g) : f(h) \to f(gh)$ on the group G with the associated action

$$\rho_B(g) : F \mapsto F \circ \rho(g^{-1}),$$

on fiducial operators

$$R(g) \circ \mathcal{W}_F = \mathcal{W}_{\rho_B(g)F}, \quad g \in G.$$

Corollary 3.5.8. [29] Let a fiducial operator F be a null solution for the operator $A = \sum_j a_j d\rho_B^{X_j}$, where $X_j \in \mathfrak{g}$ and a_j are constants. Then the covariant transform $[\mathcal{W}_F](g) = F(\rho(g^{-1})f)$ for any f satisfies

$$D(\mathcal{W}_F f) = 0 \quad where \quad D = \sum_j \bar{a}_j \mathfrak{L}^{X_j}.$$

Here, $\mathfrak{L}^{X_j}$ are the left invariant fields (Lie derivatives) on G corresponding to X_j.

Example 3.5.9. Consider the representation π (3.41) of the affine group with $p = 1$. Let $X_A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $X_N = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ be the basis of the Lie algebra $\mathfrak{g}$ of the affine group. They generate one-parameter subgroups of $\mathfrak{g}$

$$a(t) = \begin{pmatrix} e^t & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad n(t) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix},$$

then the derived representations are

$$[d\pi(X_A)f](x) = -f(x) - xf'(x),$$

$$[d\pi(X_N)f](x) = -f'(x).$$

The corresponding left invariant vector fields on the affine group are

$$\mathfrak{L}^{X_A} = a\partial_a, \quad \mathfrak{L}^{X_N} = a\partial_b.$$

The mother wavelet $\frac{1}{x+i}$ is a null solution of the operator

$$-d\pi(X_A) - id\pi(X_N) = I + (x+i)\frac{d}{dx}.$$

Therefore, the image of the covariant transform with the fiducial operator

$$F_+(f) = \frac{1}{\pi i}\int_{\mathbb{R}} \frac{f(x)}{i-x} dx,$$

consists of the null solutions to the operator

$$-\mathfrak{L}^{X_A} + i\mathfrak{L}^{X_N} = ia(\partial_b + i\partial_a),$$

that is essence of the Cauchy-Riemann operator $\partial_{\bar{z}} = \frac{\partial}{\partial x} + i\frac{\partial}{\partial y}$ in the upper half-plane.

3.5.2 Induced Covariant Transform

In this subsection, we study the covariant transform that produces functions on a homogeneous space rather than the entire group.

Definition 3.5.10. [30] Let H be a closed subgroup of the group G and $v_0 \in V$ such that

$$\rho(h)v_0 = \chi(h)v_0, \tag{3.42}$$

for some character χ of H where $h \in H$ and ρ is a unitary representation of the group G in the space V. For any continuous section $s : G/H \to G$ the map $v \mapsto \tilde{v}(x) = \tilde{v}(s(x))$, intertwines ρ with the representation ρ_χ in a certain function space on the homogeneous space G/H induced by the character χ of H. We call the map

$$\mathcal{W}_{v_0} : v \mapsto \tilde{v}(x) = \langle v, \rho(s(x))v_0\rangle, \quad \text{where} \quad x \in G/H \tag{3.43}$$

the induced wavelet transform.

Corollary 3.5.11. The induced wavelet transform that intertwines respectively the quasi-regular representations $\pi^+_{\chi_\omega}$(3.36) and $\pi^-_{\chi_\omega}$(3.37) with the co-adjoint representation ρ^+_χ (3.19) and ρ^-_χ (3.20) is the Fourier transform(3.34).

Proof. For simplicity it is enough to prove the corollary for the representation π^+_χ with $\omega = 0$. The same argument valid for π^-_χ. Let the mother wavelet be $\varphi_0(x) = e^{2\pi i x}$. It is clear that φ_0 satisfies the following condition:

$$\pi^+_\chi(1,b)\varphi_0 = \chi(1,b)\varphi_0,$$

where $\chi(1,b) = e^{2\pi i b}$ is the character of the subgroup N. Let $\mathsf{s} : \mathbb{R}_+ \to \mathrm{Aff}$ be the continuous section defined in (3.8). Then, for $f \in H_2(\mathbb{R})$ we calculate the induced wavelet transform as follows:

$$\begin{aligned}[\mathcal{W}_{\varphi_0} f](\lambda) &= \langle f, \pi_\chi^+(s(\lambda))\varphi_0\rangle \\ &= \langle f, \pi_\chi^+(\lambda, 0)\varphi_0\rangle \\ &= \int_{\mathbb{R}} f(x)\overline{\pi_\chi^+(\lambda, 0)\varphi_0(x)}dx \\ &= \frac{1}{\sqrt{\lambda}} \int_{\mathbb{R}} f(x)\overline{\varphi_0}\left(\frac{x}{\lambda}\right) dx \\ &= \frac{1}{\sqrt{\lambda}} \int_{\mathbb{R}} f(x)e^{-2\pi i \frac{x}{\lambda}}dx = \frac{1}{\sqrt{\lambda}}\hat{f}\left(\frac{1}{\lambda}\right), \quad \lambda \in \mathbb{R}_+.\end{aligned}$$

This is the Fourier transform. Next, for the co-adjoint representation ρ_χ^+, the mother wavelet $\psi_0(x) = 1$ satisfies the condition

$$\rho_\chi^+(a, 0)\psi_0 = \chi(a, 0)\psi_0,$$

where $\chi(a, 0) = a^{\frac{1}{2}}$ is the character of the subgroup A. Let s : $\mathbb{R} \to$ Aff be the continuous section defined in (3.26). Then, for $g \in L_2(\mathbb{R}_+)$ the induced wavelet transform $[\mathcal{W}_{\psi_0} g](\xi) = \langle g, \rho_\chi^+(s(\xi))\psi_0\rangle$ where $\xi \in \mathbb{R}$, is the Fourier transform. □

3.5.3 The Contravariant Transform

Define the left action Λ of a group G on a space of functions over G by

$$\Lambda(g) : f(h) \to f(g^{-1}h).$$

An object invariant under the left action Λ is called left invariant. In particular, let L and L' be two left invariant spaces of functions on G. We say that a pairing $\langle ., .\rangle$: $L \times L' \to \mathbb{C}$ is a left invariant if

$$\langle \Lambda(g)f, \Lambda(g)f'\rangle = \langle f, f'\rangle,$$

for all $f \in L$, $f' \in L'$.

Definition 3.5.12. [29] Let $\langle ., .\rangle$ be a left pairing on $L \times L'$ as above, let ρ be a representation of G in a spacw V, we define the function $w(g) = \rho(g)w_0$ for $w_0 \in V$ such that $w(g) \in L'$. The contravariant transform $\mathcal{M}^\rho_{w_0}$ is a map $L \to V$ defined by the pairing

$$\mathcal{M}^{\rho}_{w_0} : f \to \langle f, w \rangle, \quad \text{where} f \in L.$$

Definition 3.5.13. Let $\widetilde{H}^p(\mathbb{R}^2_+)$, $1 < p < \infty$, be the space of all holomorphic functions f which satisfy the following norm:

$$\|f\|_{\widetilde{H}^p} = \lim_{a\to 0} \frac{1}{a} \left(\int_{-\infty}^{\infty} |f(a,b)|^p db \right)^{\frac{1}{p}}.$$

Example 3.5.14. [29] Let G be the affine group with measure $d_\mu(a,b) = \frac{db}{a}$ and the representation π_p (3.41). The following invariant pairing on G is called Hardy pairing:

$$\langle f_1, f_2 \rangle = \lim_{a\to 0} \int_{-\infty}^{\infty} f_1(a,b) f_2(a,b) \frac{db}{a},$$

where $f_1 \in \widetilde{H}^p(\mathbb{R}^2_+)$ and $f_2 \in \widetilde{H}^q(\mathbb{R}^2_+)$ such that $\frac{1}{p} + \frac{1}{q} = 1$.
In this case, we can choose the function $v_0(x) = \frac{1}{i\pi} \frac{1}{x+i} \in L_p(\mathbb{R})$. Then, the contravariant transform is

$$\begin{aligned} [\mathcal{M}_{v_0} f](x) &= \langle f, \pi_p(a,b) v_0 \rangle \\ &= \lim_{a\to 0} \int_{-\infty}^{\infty} f(a,b) \frac{a^{\frac{-1}{p}+1}}{\pi i (x + ia - b)} db \\ &= \lim_{a\to 0} \frac{a^{\frac{-1}{p}+1}}{\pi i} \int_{-\infty}^{\infty} \frac{f(a,b) db}{b - (x + ia)}. \end{aligned} \tag{3.44}$$

The contravariant transform (3.44) is the boundary value of the the Cauchy integral as $a \to 0$.

Example 3.5.15. [29] Consider the affine group equipped with the following pairing which is the L_∞-version of the Hardy pairing:

$$\langle f_1, f_2 \rangle = \overline{\lim_{a\to 0}} \sup_{b\in\mathbb{R}} (f_1(a,b) f_2(a,b)), \tag{3.45}$$

where $\overline{\lim}$ is the upper limit and $f_1, f_2 \in L$ the space of all functions on the affine group, which the limit (3.45) is exists.

Define the following functions on $\mathbb{R}$:

$$v_0^+(t) = \begin{cases} 1, & \text{if } t = 0, \\ 0, & \text{if } t \neq 0. \end{cases} \tag{3.46}$$

The respective contravariant transforms are generated by the representation π_∞ (3.41):

$$[\mathcal{M}_{v_0^+} f](t) = \langle f(a,b), \pi_\infty(a,b) v_0^+(t) \rangle = \overline{\lim_{a\to 0}} f(a,t). \tag{3.47}$$

Chapter 4

Representations of the Group $\mathrm{SL}_2(\mathbb{R})$

The aim of this chapter is to explain the classification of the unitary $\mathrm{SL}_2(\mathbb{R})$ representations done by Gelfand [13]. First, basic information about the Lie group $\mathrm{SL}_2(\mathbb{R})$ is given. Then, we outline the classification of the unitary irreducible representation of the group $\mathrm{SL}_2(\mathbb{R})$ done by BargmannFi [4]. In section 4.3, we induce the $\mathrm{SL}_2(\mathbb{R})$ representation from the subgroup N. We get a representation constructed on a space of homogeneous functions in two variables. Then, we move in subsection 4.3.3 to induce the $\mathrm{SL}_2(\mathbb{R})$ representation in stages. Consequently, the representation of $\mathrm{SL}_2(\mathbb{R})$ acts on a space of functions of one variable. I devote the rest of the chapter to explaining the Gelfand classification of the $\mathrm{SL}_2(\mathbb{R})$ representations.

4.1 The Group $\mathrm{SL}_2(\mathbb{R})$

The Lie group $\mathrm{SL}_2(\mathbb{R})$ consists of 2×2 matrices with real entries and a determinant equal to one

$$\mathrm{SL}_2(\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : ad - bc = 1,\ a, b, c, d \in \mathbb{R} \right\}.$$

It acts on the upper half-plane by Möbius transformation

$$g \cdot z = \frac{az + b}{cz + d},$$

where $g \in \mathrm{SL}_2(\mathbb{R})$ and $z \in \{z \in \mathbb{C} : \mathrm{Im} z > 0\}$.

The group $\mathrm{SL}_2(\mathbb{R})$ contains the following three subgroups:

$$K = \left\{ \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} : \theta \in \mathbb{R} \right\}, \tag{4.1}$$

$$A = \left\{ \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix} : \alpha > 0 \right\}, \tag{4.2}$$

$$N = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} : x \in \mathbb{R} \right\}. \tag{4.3}$$

We have the Iwasawa decomposition $\mathrm{SL}_2(\mathbb{R}) = \mathrm{KAN}$. Therefore, every element $g \in \mathrm{SL}_2(\mathbb{R})$ has a unique representation as $g = kan$, where $k \in K$, $a \in A$ and $n \in N$. That is,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}. \tag{4.4}$$

The values of parameters in the above decomposition are as follows:

$$\alpha = \sqrt{a^2 + c^2}, \quad x = \frac{ab + cd}{a^2 + c^2}, \quad \theta = \arctan \frac{-c}{a}. \tag{4.5}$$

Consequently, $\cos\theta = \frac{a}{\sqrt{a^2+c^2}}$ and $\sin\theta = \frac{-c}{\sqrt{a^2+c^2}}$.

The Lie algebra $\mathfrak{sl}_2(\mathbb{R})$ is the set of all 2×2 real matrices of trace zero. It is a three-dimensional Lie algebra so we can choose a basis$\{Z, A, B\}$ of $\mathfrak{sl}_2(\mathbb{R})$ by setting

$$Z = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad A = \frac{1}{2} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \text{ and } B = \frac{1}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{4.6}$$

Note that

$$[Z, A] = 2B, \quad [Z, B] = -2A, \quad [A, B] = -\frac{1}{2} Z. \tag{4.7}$$

The exponential map of each matrix Z, A and B forms a one-dimensional subgroup of the group $SL_2(\mathbb{R})$ given as follows:

$$\exp(\theta Z) \in \left\{ \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} : \theta \in \mathbb{R} \right\}, \tag{4.8}$$

$$\exp(\theta A) \in \left\{ \begin{pmatrix} e^{\frac{-\theta}{2}} & 0 \\ 0 & e^{\frac{\theta}{2}} \end{pmatrix} : \theta \in \mathbb{R} \right\}, \tag{4.9}$$

$$\exp(\theta B) \in \left\{ \begin{pmatrix} \cosh\frac{\theta}{2} & \sinh\frac{\theta}{2} \\ \sinh\frac{\theta}{2} & \cosh\frac{\theta}{2} \end{pmatrix} : \theta \in \mathbb{R} \right\}. \tag{4.10}$$

4.2 Irreducible Unitary Representations of $\mathrm{SL}_2(\mathbb{R})$

The irreducible unitary strongly continuous representation of $\mathrm{SL}_2(\mathbb{R})$ was classified by Bargmann in 1947 [4], and his approach has been used in different sources , such as [33, 40]. Suppose that ρ is an irreducible unitary strongly continuous representation of $\mathrm{SL}_2(\mathbb{R})$ on a Hilbert space $\mathcal{H}$. The classification steps are as follows:

Step 1: Set the Gårding space[11] for ρ,

$$\mathcal{G}(\rho) = \{\rho(f)u : u \in \mathcal{H}, f \in C_0^\infty(G)\},$$

where $G = \mathrm{SL}_2(\mathbb{R})$. Denote the derived representations of the matrices Z, A, and B (4.6) by

$$d\rho(Z) = E, \; d\rho(A) = A_1, \text{ and } \; d\rho(B) = B_1.$$

From (4.7), we find that

$$[E, A_1] = 2B_1, \;\; [E, B_1] = -2A_1, \;\; \text{and} \;\; [A_1, B_1] = -\frac{1}{2}E. \tag{4.11}$$

Step 2: Consider the ladder operators

$$L_+ = A_1 - iB_1, \quad \text{and} \quad L_- = A_1 + iB_1. \tag{4.12}$$

Since ρ is unitary, then A_1 and B_1 are skew-symmetric. This implies that

$$L_+^* = -L_-.$$

From the commutator relation in (4.11), we have

$$[E, L_\pm] = \pm 2iL_\pm, \quad [L_+, L_-] = -iE. \tag{4.13}$$

Step 3: The Casimir operator given by $C := Z^2 - 4A^2 - 4B^2$, is an element of the centre of the universal enveloping algebra for the Lie algebra $\mathfrak{sl}_2(\mathbb{R})$. Therefore, by Schur's lemma [10], it acts as a scalar for the irreducible unitary representation ρ,

$$d\rho(C) = \lambda I. \tag{4.14}$$

Step 4: The decomposition into the irreducible subspace of the representation $\rho(K)$ on the Hilbert space $\mathcal{H}$ leads to the orthogonal sum, since K is a compact subgroup,

$$\mathcal{H} = \overline{\bigoplus_{k\in\mathbb{Z}} V_k}.$$

The unitary irreducible representation on the subgroup K is the character e^{iks}

$$\rho(\exp sZ) = \mathrm{e}^{iks} I \ \text{ on } V_k.$$

Thus,

$$\begin{aligned} E = d\rho(Z) &= \frac{d}{ds}\rho(\mathrm{e}^{sZ})|_{s=0} \\ &= \frac{d}{ds}\mathrm{e}^{iks}|_{s=0} \\ &= ik \ \text{ on } V_k. \end{aligned} \tag{4.15}$$

Moreover, for the Casimir operator $C := Z^2 - 4A^2 - 4B^2$, we have

$$\begin{aligned} d\rho(C) &:= d\rho(Z)^2 - 4d\rho(A)^2 - 4d\rho(B)^2 \\ &= E^2 - 4A_1^2 - 4B_1^2 \\ &= E^2 - 2(L_+L_- + L_-L_+). \end{aligned} \tag{4.16}$$

From(4.13), we have

$$4L_+L_- = E^2 - 2iE - \lambda,$$

$$4L_-L_+ = E^2 + 2iE - \lambda.$$

Then by (4.15),

$$-4L_+L_- = k^2 - 2k + \lambda, \tag{4.17}$$

$$-4L_-L_+ = k^2 + 2k + \lambda. \tag{4.18}$$

Since $L_+^* = -L_-$, then

$$\|L_-\|_{\mathcal{L}(V_k,V_{k-2})} = \frac{1}{2}[(k-1)^2 + \lambda - 1]^{\frac{1}{2}}, \tag{4.19}$$

$$\|L_+\|_{\mathcal{L}(V_k,V_{k+2})} = \frac{1}{2}[(k+1)^2 + \lambda - 1]^{\frac{1}{2}}. \tag{4.20}$$

Theorem 4.2.1. The ladder operators $L_\pm$ act as

$$L_\pm : V_k \to V_{k\pm 2}.$$

Proof. From the commutator relation (4.13), we have

$$[E, L_\pm] = \pm 2iL_\pm \Leftrightarrow EL_\pm = L_\pm E \pm 2iL_\pm.$$

Therefore, by (4.15), for $v \in V_k$,

$$E(L_\pm v) = L_\pm(Ev) \pm 2iL_\pm v = (k \pm 2)i(L_\pm v).$$

Step 5: We have the commutator relation $[L_+, L_-] = -iE$. Then, for each vector $v_k \in V_k$, where $k \in spec(1/i)E$, the collection of vectors

$$v_{k+2n} := (L_+)^n v_k,$$
$$v_{k-2n} := (L_-)^n v_k, \quad n \in \mathbb{Z}^+,$$

is invariant under the operators $L_+,\ L_-,\ E$. Therefore, V_k is a one-dimensional space.

Step 6: The ladder operators act on the vector spaces V_k where $k \in spec(1/i)E$. There are only four possibilities for the spectrum of the operator $(1/i)E$. First, if the ladder operators are two-sided infinite operators, given that the representation ρ is irreducible, the spectrum is either in the even or odd integer set. That is,

$$spec(1/i)E = \{.......-4, -2, 0, 2, 4......\},$$
$$spec(1/i)E = \{.....-5, -3, 1, -1, 3, 5, ...\}.$$

Second, if the ladder operators are one-sided infinite operators, then for $V_k \neq 0$, we have the following sets of spectrum:

- For $L_+ = 0$ on V_k, $\quad spec(1/i)E = \{.....n-4, n-2, n\}, \quad n \in \mathbb{Z}^+.$
- For $L_- = 0$ on V_k, $\quad spec(1/i)E = \{n, n+2, n+4, ...\}, \quad n \in \mathbb{Z}^+.$

Step 7: In each case above select a unit vector $v_k \in V_k$, $k \in spec(\frac{1}{i})E$. We have $L_+ v_k = \alpha_k v_{k+2}$. The absolute value of α_k is

$$|\alpha_k| = \frac{1}{2}[(k+1)^2 + \lambda - 1]^{\frac{1}{2}}. \tag{4.21}$$

The action of L_- on v_{k+2} is given as follows:

$$L_- v_{k+2} = \beta_k v_k, \quad \text{where} \quad \beta_k = -\overline{\alpha_k}.$$

Therefore, the type of the spectrum together with the value of $d\rho(C) = \lambda I$, fully determines the unitary irreducible representation of $\mathrm{SL}_2(\mathbb{R})$. This stated in the following theorem.

Theorem 4.2.2. [40] Any nontrivial irreducible unitary representation of $\mathrm{SL}_2(\mathbb{R})$ is unitary equivalent to one of the following types:

- Members of the holomorphic discrete series, denoted by ρ_n^+ such that

  $$d\rho_n^+(C) = 1 - (n-1)^2, \ n \in \mathbb{Z}^+, \tag{4.22}$$

  when $spec(1/i)E = \{n, n+2, ...\}$.

- Members of the anti-holomorphic discrete series, denoted by ρ_{-n}^- such that

  $$d\rho_{-n}^-(C) = 1 - (n-1)^2, \ n \in \mathbb{Z}^+, \tag{4.23}$$

  when $spec(1/i)E = \{....., n-4, n-2, n\}$.

- Mock discrete series ρ_1^+,ρ_{-1}^-, for $n = 1$.

- A member of the first principal series, denoted by ρ_{is}^e such that

  $$d\rho_{is}^e(C) = 1 + s^2, \ s \in \mathbb{R}, \tag{4.24}$$

  when $spec(1/i)E = \{......., -4, -2, 0, 2, 4, ...\}$.

- A member of the complementary series, denoted by ρ_s^e such that

  $$d\rho_s^e(C) = 1 - s^2, \ s \in (-1,1)\backslash\{0\}, \tag{4.25}$$

  when $spec(1/i)E = \{......., -4, -2, 0, 2, 4, ...\}$.

- A member of the second principal series, denoted by ρ^o_{is} such that

$$d\rho^o_{is}(C) = 1 + s^2,\ s \in \mathbb{R}\backslash\{0\}, \tag{4.26}$$

when $spec(1/i)E = \{..., -5, -3, -1, 1, 3, 5, ...\}$.

4.3 Induced Representation of the Group $\mathrm{SL}_2(\mathbb{R})$

In this section, we induce a representation of the group $\mathrm{SL}_2(\mathbb{R})$ from a trivial character of the subgroup N. We get a representation on a space of functions with two variables. Then, we can have this representation on a space of functions with one variable by using inducing in stages technique. That is, first induce a representation for the affine group from a trivial character of the subgroup N. We get an affine group representation that can be decomposed into a one-dimensional representation which is a complex character. Then, we induce a representation for the group $\mathrm{SL}_2(\mathbb{R})$ from a complex character of the affine group.

4.3.1 The $\mathrm{SL}_2(\mathbb{R})$ Induced Representation from the Subgroup N

Let $\chi_e : N \to \mathbb{T}$ be a trivial character of the subgroup N. The character χ_e induces a linear representation of $\mathrm{SL}_2(\mathbb{R})$. This induced representation is constructed in the vector space V, which consists of the functions $F_e : \mathrm{SL}_2(\mathbb{R}) \mapsto \mathbb{C}$ with the property

$$F_e\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \chi_e\begin{pmatrix} 1 & \frac{b}{a} \\ 0 & 1 \end{pmatrix} F\begin{pmatrix} a & 0 \\ c & a^{-1} \end{pmatrix}.$$

The space V is invariant under the left shift of the group $\mathrm{SL}_2(\mathbb{R})$. The restriction of the left shift on V is the left regular representation of the group $\mathrm{SL}_2(\mathbb{R})$, which is given by

$$[\Lambda(g)F_e](g') = F_e(g^{-1} * g'). \tag{4.27}$$

In the following, we obtain an equivalent induced representation constructed in the left homogeneous space $X = \mathrm{SL}_2(\mathbb{R})/\mathrm{N}$. The Iwasawa decomposition $\mathrm{SL}_2(\mathbb{R}) = \mathrm{KAN}$

implies that the homogeneous space $X = \mathrm{SL}_2(\mathbb{R})/\mathrm{N}$ topologically identifies to $KA \simeq \mathbb{T} \times \mathbb{R}_+ \simeq \mathbb{R}^2\backslash\{0\}$. Hence we can choose the section map to be given by

$$\begin{aligned} \mathsf{s} : X &\to \mathrm{SL}_2(\mathbb{R}), \\ : (u,v) &\mapsto \begin{pmatrix} u & 0 \\ v & u^{-1} \end{pmatrix}, \quad u > 0. \end{aligned}$$

The natural projection map will be

$$\begin{aligned} \mathsf{p} : \mathrm{SL}_2(\mathbb{R}) &\to \mathrm{X}, \\ : \begin{pmatrix} a & b \\ c & d \end{pmatrix} &\mapsto (a,c), \end{aligned}$$

such that s is the right inverse of p. Therefore, the unique decomposition of $g \in \mathrm{SL}_2(\mathbb{R})$ is of the form

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & 0 \\ c & a^{-1} \end{pmatrix} \begin{pmatrix} 1 & \frac{b}{a} \\ 0 & 1 \end{pmatrix}.$$

The map $\mathsf{r} : \mathrm{SL}_2(\mathbb{R}) \to \mathrm{N}$ is given by

$$\mathsf{r} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & \frac{b}{a} \\ 0 & 1 \end{pmatrix}. \tag{4.28}$$

The $\mathrm{SL}_2(\mathbb{R})$ action on the space $X = \mathrm{SL}_2(\mathbb{R})/\mathrm{N}$ can be expressed in terms of p and s as follows:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} : w \mapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} \cdot w = \mathsf{p}\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} \cdot \mathsf{s}(u,v)\right) = (du - bv, av - cu).$$

Let W be a vector space of function f on the homogeneous space X. The lifting map for the subgroup N and its character χ_e is given by:

$$\begin{aligned} [\mathcal{L}_{\chi_e} f] \begin{pmatrix} a & b \\ c & d \end{pmatrix} &:= \overline{\chi_e}\left(\mathsf{r}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) f\left(\mathsf{p}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) \\ &= f(a,c). \end{aligned} \tag{4.29}$$

Then, the pulling map $\mathcal{P} : V \to W$, which is the right inverse of the lifting map, is given by

$$[\mathcal{P}F](u,v) := F(\mathsf{s}(u,v)).$$

Therefore, the representation $U : W \to W$, which is induced by the character χ_e, is

$$U(g) = \mathcal{P} \circ \Lambda(g) \circ \mathcal{L}_{\chi_e}. \tag{4.30}$$

To calculate the explicit form of $U(g)$, take the left action of the lifting map

$$[\Lambda(g)\mathcal{L}_{\chi_e} f](g') = [\mathcal{L}_{\chi_e} f](g^{-1} * g') = f(da' - bc', ac' - a'c) = F_e(g'). \tag{4.31}$$

Then, apply pulling for the function F_e

$$\begin{aligned} [\mathcal{P}F_e](u, v) &= F_e(\mathsf{s}(u, v)) \\ &= F_e \begin{pmatrix} u & 0 \\ v & u^{-1} \end{pmatrix} \\ &= f(du - bv, av - cu). \end{aligned} \tag{4.32}$$

Hence, from(4.30), we obtain the following formula:

$$[U(g)f](u, v) = f(du - bv, av - cu), \tag{4.33}$$

where $(u, v) \in X$ and $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

4.3.2 Affine Group Representation Induced from a Trivial Character

For the trivial character χ_e, the induced representation of the subgroup N is $\rho^+_{\chi_e} : L_2(\mathbb{R}_+) \to L_2(\mathbb{R}_+)$ and is expressed as

$$[\rho^+_{\chi_e}(a, b)f](x) = \sqrt{a} f(ax), \quad f \in L_2(\mathbb{R}_+). \tag{4.34}$$

It is a reducible unitary representation. To decompose it into irreducible components, we will find the eigenfunction of the operator $\rho^+_{\chi_e}(a, b)f$ as follows:

$$[\rho^+_{\chi_e}(a, b)f](t) = \lambda_{a,b} f(t) \quad \Rightarrow \quad \sqrt{a} f(at) = \lambda_{a,b} f(t).$$

Let $f(t) = t^\alpha$, where $\alpha \in \mathbb{C}$. Then, we obtain

$$[\rho^+_{\chi_e}(a, b)](t^\alpha) = \sqrt{a}(at)^\alpha = a^{\alpha + \frac{1}{2}} t^\alpha.$$

Hence, the eigenfunction of $\rho^+_{\chi_e}(a,b)$ is t^α. Let the inverse Mellin transform (see A.1) be given by

$$[\mathsf{M}^{-1}\tilde{f}](t) = f(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} t^{-\frac{1}{2}+is}\tilde{f}(s)ds, \quad t \in \mathbb{R}_+, \tag{4.35}$$

where $\alpha = -\frac{1}{2} + is$. The function $\tilde{f}(s)$ is the Mellin transform $\tilde{f}(s) = [\mathsf{M}f](x) = \int_0^\infty x^s f(x)\frac{dx}{x}$. Therefore, we obtain

$$\begin{aligned}
[\rho^+_{\chi_e}(a,b)f](t) &= \sqrt{a}f(at) \\
&= \sqrt{a}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \tilde{f}(s)(at)^{-\frac{1}{2}+is}ds \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} a^{is}\tilde{f}(s)t^{-\frac{1}{2}+is}ds \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \chi_s(a,b)\tilde{f}(s)t^{-\frac{1}{2}+is}ds,
\end{aligned} \tag{4.36}$$

where $\chi_s(a,b) = a^{is}$ is a complex character of the affine group. Hence, the irreducible component of the representation $\rho^+_{\chi_e}$ (4.34) is the character χ_s.

4.3.3 Induction in Stages

Let P be the subgroup of $\mathrm{SL}_2(\mathbb{R})$, which is defined as follows:

$$P = \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} : a \in \mathbb{R}\backslash\{0\}, b \in \mathbb{R} \right\}.$$

There exists a homomorphism $T : P \to \mathrm{Aff}$ such that $T^{-1}(a,b)$ has two elements, one for $a > 0$ and the other for $a < 0$. The $SL_2(\mathbb{R})$ representations (4.33) can be obtained by induction in stages. That is

$$\mathrm{Ind}_P^{\mathrm{SL}_2(\mathbb{R})}[\mathrm{Ind}_N^P \chi_e] = \mathrm{Ind}_N^{\mathrm{SL}_2(\mathbb{R})}[\chi_e].$$

First induce the trivial character χ_e of the subgroup N to the affine group. We will obtain the co-adjoint representation $\rho^+_\chi : \mathsf{U} \to \mathsf{U}$, which is given as follows:

$$\left[\rho^+_{\chi_e}\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} g\right](t) = \sqrt{a}g(at).$$

The vector space U consists of all functions on the homogeneous space $\mathrm{Aff}/N =$

A. It is reducible, and from subsection 4.3.2 we can decompose it into irreducible component which is the following the character:

$$\chi_\alpha \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} = a^\alpha, \quad \alpha \in \mathbb{C}.$$

Therefore, for the subgroup $P = AN$, the character is given by

$$\chi_s \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} = |a|^s \mathrm{sgn}^\epsilon(a), \quad \epsilon = \{0, 1\}, s \in \mathbb{C}.$$

Next, the character χ_s induces a representation of the group $SL_2(\mathbb{R})$. This representation is constructed on the vector space V, which consist of the functions $F_s : \mathrm{SL}_2(\mathbb{R}) \to \mathbb{C}$ with the following property:

$$F_s \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \chi_s \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} F \begin{pmatrix} 1 & 0 \\ \frac{c}{a} & 1 \end{pmatrix}.$$

This vector space is invariant under the left shift of the group $\mathrm{SL}_2(\mathbb{R})$. The restriction of the left shift on this space is an induced representation.

An equivalent form of the induced representation can be constructed on the homogeneous space $X = \mathrm{SL}_2(\mathbb{R})/\mathrm{P}$. The space of the left cosets $X = \mathrm{SL}_2(\mathbb{R})/\mathrm{P}$ can be defined by the following equivalence relation: $g \sim g'$ if and only if there exists $x \in P$ such that $g = g'x$. Then, the equivalence class for all $g \in \mathrm{SL}_2(\mathbb{R})$ is given by the following:

$$[g] = \left[\begin{pmatrix} a & b \\ c & d \end{pmatrix} \right] = [c : a] = \begin{cases} [\frac{c}{a} : 1], & a \neq 0 \\ [1 : 0], & a = 0 \end{cases}.$$

Thus, we can identify the space $X = \mathrm{SL}_2(\mathbb{R})/\mathrm{P}$ by the real projective line $\mathbb{P}(\mathbb{R})$.

Next, let $\mathsf{s} : \mathbb{P}(\mathbb{R}) \to \mathrm{SL}_2(\mathbb{R})$ be the section map given by

$$\mathsf{s}(w) = \begin{pmatrix} 1 & 0 \\ w & 1 \end{pmatrix}. \tag{4.37}$$

The natural projection map will be

$$\begin{aligned} \mathsf{p} &: \mathrm{SL}_2(\mathbb{R}) \to \mathbb{P}(\mathbb{R}) \\ &: \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \frac{c}{a}, \end{aligned} \tag{4.38}$$

where $a \neq 0$, and $\mathsf{p} \circ \mathsf{s} = \mathbb{I}_{\mathbb{P}(\mathbb{R})}$. The unique decomposition of any $g \in \mathrm{SL}_2(\mathbb{R})$ defined by s is of the form

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \frac{c}{a} & 1 \end{pmatrix} \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}.$$

Hence, the map $\mathsf{r} : \mathrm{SL}_2(\mathbb{R}) \to \mathrm{P}$ is given by

$$\mathsf{r} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}. \tag{4.39}$$

The $\mathrm{SL}_2(\mathbb{R})$ action on the left homogeneous space $X = \mathrm{SL}_2(\mathbb{R})/\mathrm{P} \cong \mathbb{P}(\mathbb{R})$ is the Möbius transformation and we can express it in terms of p and s as follows:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} : w \mapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} \cdot w = \mathsf{p}\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} \cdot \mathsf{s}(x)\right) = \frac{ax - c}{d - bx}, \tag{4.40}$$

where $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R})$, $x \in \mathbb{P}(\mathbb{R})$ and $\cdot$ is the action of $\mathrm{SL}_2(\mathbb{R})$ on $\mathbb{P}(\mathbb{R})$ from the left.

Let W be the vector space of all functions on the homogeneous space $X = \mathbb{P}(\mathbb{R})$. The lifting map $\mathcal{L}_{\chi_s} : \mathsf{W} \to \mathsf{V}$ for the subgroup P and its character χ_s associates each function f on the projective line $\mathbb{P}(\mathbb{R})$ with a function F on the $\mathrm{SL}_2(\mathbb{R})$ group. That is

$$\begin{aligned} [\mathcal{L}_{\chi_s} f] \begin{pmatrix} a & b \\ c & d \end{pmatrix} &:= \overline{\chi_s}\left(\mathsf{r}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) f\left(\mathsf{p}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) \\ &= |a|^s \mathrm{sgn}^\epsilon(a) f\left(\frac{c}{a}\right), \end{aligned} \tag{4.41}$$

where $a \neq 0$. Then, the pulling map $\mathcal{P} : \mathsf{V} \to \mathsf{W}$, which is the right inverse of the lifting map, is given as follows:

$$[\mathcal{P}F](x) := F(\mathsf{s}(x)).$$

Therefore, the representation $T : \mathsf{W} \to \mathsf{W}$ that induced by the character χ_s is given as follows:

$$T(g) = \mathcal{P} \circ \Lambda(g) \circ \mathcal{L}_{\chi_s}. \tag{4.42}$$

The explicit formula of $T(g)$ is calculated as follows. First, take the left action of the lifting map

$$\begin{aligned}[\Lambda(g)\mathcal{L}_{\chi_s} f](g') &= [\mathcal{L}_{\chi_s} f](g^{-1}g') \\ &= |da' - bc'|^s \text{sgn}^\epsilon (da' - bc') f\left(\frac{ac' - ca'}{da' - bc'}\right) = F_s(g').\end{aligned} \tag{4.43}$$

Then, apply pulling to the function F_s

$$\begin{aligned}[\mathcal{P}F_s](x) &= F_s(\mathsf{s}(x)) \\ &= F_s \begin{pmatrix} 1 & 0 \\ w & 1 \end{pmatrix} \\ &= |d - bw|^s \text{sgn}^\epsilon (d - bx) f\left(\frac{ax - c}{d - bx}\right).\end{aligned} \tag{4.44}$$

Hence, by (4.43) and (4.44) from(4.42), we obtain the formula

$$[T_s(g)f](x) = |d - bx|^s \text{sgn}^\epsilon (d - bx) f\left(\frac{ax - c}{d - bx}\right), \tag{4.45}$$

where $f \in \mathsf{W}$ and $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

4.4 Gelfand Method to Classify the Group $\mathrm{SL}_2(\mathbb{R})$ Representation

In section 4.2, we present Bargmann's classification for the $\mathrm{SL}_2(\mathbb{R})$ representations which used the derived representation and find the vector modules of the representations on the Lie algebra $\mathfrak{sl}_2(\mathbb{R})$. In [13, chapter VII], the representations for the group $\mathrm{SL}_2(\mathbb{R})$ have been classified by working on the Lie group instead of the Lie algebra. The method is based on studying the invariance of bilinear functional on a normed space. Then, we move to study the invariance of the inner product on a Hilbert space.

The following sections explain the Gelfand method in details.

4.5 Invariant Bilinear Functionals

In section 4.3, the $\mathrm{SL}_2(\mathbb{R})$ representations are constructed on the vector space of functions W_t on the homogeneous space $X = \mathrm{SL}_2(\mathbb{R})/N = KAN/N$. The space X can be topologically identified as follows:

$$X = KA \simeq \mathbb{T} \times \mathbb{R}_+ \simeq \mathbb{R}^2 \backslash \{0\}.$$

Definition 4.5.1. Consider pairs of numbers $t = (s, \epsilon)$, where s is any complex number and $\epsilon = 0$ or 1. Then associate each such pair with the space W_t that consists of functions $f(x_1, x_2)$ with the following properties:

- Every function $f(x_1, x_2) \in W_t$ is homogeneous of degree $s-1$, and it has even parity if $\epsilon = 0$ and odd parity if $\epsilon = 1$. This means that for $a \neq 0$

$$f(ax_1, ax_1) = |a|^{s-1}\mathrm{sgn}^{\epsilon}(a) f(x_1, x_2).$$

- The function $f(x_1, x_2)$ is infinitely differentiable for every x_1 and x_2 except at the point $(0, 0)$.

In subsection 4.3.3, the $\mathrm{SL}_2(\mathbb{R})$ representations (4.45) have been constructed on the vector space of functions W_t on the real projective line $\mathbb{P}(\mathbb{R})$. We can realise the space W_t as the space of one variable by associating a function $f(x_1, x_2) \in W_t$ with a function $\varphi(x) \in \mathsf{W}_t$ as follows:

$$f(x_1, x_2) = |x_2|^{s-1}\mathrm{sgn}^{\epsilon}(x_2)\varphi\left(\frac{x_1}{x_2}\right). \tag{4.46}$$

Definition 4.5.2. From the relation (4.46), every function $\varphi(x) \in \mathsf{W}_t$ is given by $\varphi(x) = f(x, 1)$. Then, the function $\varphi(x)$ has the following properties:

- $\varphi(x)$ is infinitely differentiable.

- The function $\tilde{\varphi}(x) = f(1,x) = |x|^{s-1}\mathrm{sgn}^{\epsilon}(x)\varphi\left(\frac{1}{x}\right)$, is infinitely differentiable. Then, we obtain
$$\varphi(x) = |x|^{s-1}\mathrm{sgn}^{\epsilon}(x)\tilde{\varphi}\left(\frac{1}{x}\right) = |x|^{s-1}\mathrm{sgn}^{\epsilon}(x) f\left(1, \frac{1}{x}\right).$$
As $|x| \to \infty$, we have $\varphi(x) \sim |x|^{s-1}\mathrm{sgn}^{\epsilon}(x) f(1,0)$.

  This condition shows the behaviour of $\varphi(x)$ for large $|x|$. In particular, it implies that asymptotically as $|x| \to \infty$, the function $\varphi(x)$ goes as
$$\varphi(x) \sim C|x|^{s-1}\mathrm{sgn}^{\epsilon}(x).$$

In this section, we will study the case of the $\mathrm{SL}_2(\mathbb{R})$ representations (4.45) possessing an invariant bilinear functional. Associate the pairs of numbers $t_1 = (s_1, \epsilon_1)$ and $t_2 = (s_2, \epsilon_2)$ with the spaces W_{t_1} and W_{t_2},respectively. Then, consider the following two representations of $\mathrm{SL}_2(\mathbb{R})$:

$$[T_{s_1}(g)\varphi](x) = |d - bx|^{s_1 - 1}\mathrm{sgn}^{\epsilon_1}(d - bx)\varphi\left(\frac{ax - c}{d - bx}\right), \tag{4.47}$$
$$[T_{s_2}(g)\psi](x) = |d - bx|^{s_2 - 1}\mathrm{sgn}^{\epsilon_2}(d - bx)\psi\left(\frac{ax - c}{d - bx}\right), \tag{4.48}$$

acting on the spaces W_{t_1} and W_{t_2}, respectively.

A bilinear functional $(\cdot,\cdot) : \mathsf{W}_{t_1} \times \mathsf{W}_{t_2} \to \mathbb{R}$, is called invariant if

$$(T_{s_1}(g)\varphi, T_{s_2}(g)\psi) = (\varphi, \psi), \tag{4.49}$$

for all $g \in SL_2(\mathbb{R})$,$\varphi \in \mathsf{W}_{t_1}$ and $\psi \in \mathsf{W}_{t_2}$.

By the Iwasawa decomposition $\mathrm{SL}_2(\mathbb{R}) = \mathrm{KAN}$, every matrix $g \in \mathrm{SL}_2(\mathbb{R})$ can be written as a product of the following three matrices:

$$g_1 = \begin{pmatrix} 1 & 0 \\ x_0 & 1 \end{pmatrix} \in N, \quad g_2 = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix} \in A, \quad g_3 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in K. \tag{4.50}$$

Hence, the linear fractional transformation (4.40) can be obtained by combining the following three types of transformation:

- Translation: $x \to g_1^{-1} \cdot x = x - x_0$.

- Dilation: $x \to g_2^{-1} \cdot x = \alpha^2 x$.

- Inversion: $x \to g_3^{-1} \cdot x = \frac{-1}{x}$.

Therefore, in determining whether a bilinear functional is invariant, it is sufficient to consider the operators corresponding to the three matrices g_1, g_2 and g_3.

4.5.1 Invariance under Translation

For the matrix g_1, the representations (4.47) and (4.48) are given as follows :

$$[T_{s_1}(g_1)\varphi](x) = \varphi(x - x_0), \tag{4.51}$$

$$[T_{s_2}(g_1)\psi](x) = \psi(x - x_0). \tag{4.52}$$

We want to find a bilinear functional (φ, ψ) that satisfies the following condition :

$$(T_{s_1}(g_1)\varphi, T_{s_2}(g_2)\psi) = (\varphi, \psi).$$

We shall restrict our considerations to the infinitely differentiable functions with bounded support in the spaces W_{t_1} and W_{t_2}. Then, by the kernel theorem A.2.5 we can define an integral transform as follows:

$$L_k : \varphi \to L_k(\varphi) \quad \text{such that} \quad [L_k\varphi](x_2) = \int k(x_1, x_2)\varphi(x_1)dx_1.$$

Hence, we obtain

$$(L_k(\varphi), \psi) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} k(x_1, x_2)\varphi(x_1)\psi(x_2)dx_1dx_2,$$

where $x_1, x_2 \in \mathbb{R}$ and $k(x_1, x_2)$ is the kernel of the integral. We can consider

$$(\varphi, \psi) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} k(x_1, x_2)\varphi(x_1)\psi(x_2)dx_1dx_2. \tag{4.53}$$

Then, by using (4.51)and (4.52), we have

$$\begin{aligned}(T_{s_1}(g_1)\varphi, T_{s_2}(g_2)\psi) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} k(x_1 - x_0, x_2 - x_0)\varphi(x_1 - x_0)\psi(x_2 - x_0)dx_1dx_2 \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} k(x_1', x_2')\varphi(x_1')\psi(x_2')dx_1'dx_2' \\ &= (\varphi, \psi)\end{aligned}$$

where $x_1' = x_1 - x_0$, and $x_2' = x_2 - x_0$.

Therefore, the kernel is invariant under translation. We may associate $k(x_1, x_2)$ with a function of a single variable that is

$$k(x_1, x_2) = k(x_1 - x_2, 0) = k_0(x_1 - x_2).$$

Hence, every bilinear functional (φ, ψ) (4.53) invariant with respect to translation is of the form

$$(\varphi, \psi) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} k_0(x_1 - x_2)\varphi(x_1)\psi(x_2)dx_2dx_1. \tag{4.54}$$

4.5.2 Invariance under Dilation

Now, we wish to further that (φ, ψ) be invariant under the representations (4.47) and (4.48) for g_2. These operators are given as follows:

$$[T_{s_1}(g_2)\varphi](x) = |\alpha|^{-s_1+1}\text{sgn}^{\epsilon_1}(\alpha)\varphi(\alpha^2 x),$$

$$[T_{s_2}(g_2)]\psi(x) = |\alpha|^{-s_2+1}\text{sgn}^{\epsilon_2}(\alpha)\psi(\alpha^2 x).$$

The condition that (φ, ψ) be invariant under these operators may consequently be written as

$$(\varphi, \psi) = |\alpha|^{-s_1-s_2+2}\text{sgn}^{\epsilon_1+\epsilon_2}(\alpha)(\varphi(\alpha^2 x), \psi(\alpha^2 x)). \tag{4.55}$$

First, note that this requires that $\epsilon_1 = \epsilon_2$.

Let $x = x_1 - x_2$ in the integral (4.54). The bilinear functional will be given as follows:

$$(\varphi, \psi) = \int_{-\infty}^{\infty} k_0(x) \int_{-\infty}^{\infty} \varphi(x_1)\psi(x_1 - x)dx_1dx = (k_0, \omega) \tag{4.56}$$

where $\omega(x) = \int_{-\infty}^{\infty} \varphi(x_1)\psi(x_1 - x)dx_1$.

Next, substitute (k_0, ω) for (φ, ψ) in (4.55) considering that

$$\alpha^{-2}\omega(\alpha^2 x) = \int \varphi(\alpha^2 x_1)\psi(\alpha^2[x_1 - x])dx_1.$$

We get

$$(k_0, \omega) = |\alpha|^{-s_1-s_2}(k_0, \omega(\alpha^2 x)).$$

Let $\alpha > 0$ and replace α by $\alpha^{\frac{1}{2}}$; then, the above equation becomes

$$(k_0, \omega) = \alpha^{-\frac{1}{2}(s_1+s_2)}(k_0, \omega(\alpha x)),$$

which shows that k_0 is a homogeneous generalized function of degree $\lambda = -\frac{1}{2}(s_1 + s_2) - 1$.

Recall one of the basic properties of homogeneous generalized functions of a single variable [15]. For every complex number λ, there exists one even and one odd homogeneous generalized function of degree λ and every other homogeneous generalized function of this degree is a linear combination of these. Hence, $k_0(x)$ is given by one of the two following forms:

- If $\frac{1}{2}(s_1 + s_2) \neq 0, 1, 2...., n..$, where $n \in \mathbb{Z}$, then
$$k_0(x) = C_1|x|^{-\frac{1}{2}(s_1+s_2)-1} + C_2|x|^{-\frac{1}{2}(s_1+s_2)-1}\mathrm{sgn}x. \tag{4.57}$$
- If $\frac{1}{2}(s_1 + s_2) = 0, 1, 2, 3, ...n...$ is a non-negative integer, then
$$k_0(x) = C_1\delta^{\frac{1}{2}(s_1+s_2)}(x) + C_2x^{-\frac{1}{2}(s_1+s_2)-1}. \tag{4.58}$$
The function $\delta^{\frac{1}{2}(s_1+s_2)}(x)$ is the derivative of the delta function. It is defined by
$$\int \varphi(x_1)\delta^{\frac{1}{2}(s_1+s_2)}(x_1 - x_2) = \varphi^{\frac{1}{2}(s_1+s_2)}(x_2).$$

We established that an invariant bilinear functional (φ, ψ) can exist only if $\epsilon_1 = \epsilon_2$ for the representations (4.47) (4.48).

4.5.3 Invariance under Inversion

Let us now use the condition of invariance under inversion in addition to the invariance under translation and dilation . The operators $T_{s_1}(g)$ and $T_{s_2}(g)$ for the matrix g_3 are given as follows:

$$[T_{s_1}(g_3)\varphi](x) = |x|^{s_1-1}\mathrm{sgn}^{\epsilon}(x)\varphi\left(\frac{-1}{x}\right),$$
$$[T_{s_2}(g_3)\psi](x) = |x|^{s_2-1}\mathrm{sgn}^{\epsilon}(x)\psi\left(\frac{-1}{x}\right).$$

The invariant condition of bilinear functional (4.49) under $T_{s_1}(g_3)$ and $T_{s_2}(g_3)$ become

$$(T_{s_1}(g_3)\varphi, T_{s_2}(g_3)\psi) = (\varphi, \psi).$$

Then,by using (4.54) and changing the variable, we get

$$\int\int k_0(x_1-x_2)\varphi(x_1)\psi(x_2)dx_1dx_2 = \int\int k_0\left(\frac{x_1-x_2}{x_1x_2}\right)|x_1|^{-s_1-1}|x_2|^{-s_2-1}$$
$$\text{sgn}^{\epsilon}(x_1x_2)\varphi(x_1)\psi(x_2)dx_1dx_2. \quad (4.59)$$

To find the value of s_1 and s_2 for which (4.59) is valid, we will consider the different forms of $k_0(x)$, which are given by (4.57) and (4.58).

In the first case ,(4.57) $k_0(x)$ is invariant if C_1 or C_2 is zero. Hence, we get

$$k_0(x) = |x|^{-\frac{1}{2}(s_1+s_2)-1}\text{sgn}^{\nu}(x), \quad \nu = 0 \quad \text{or} \quad 1.$$

Then, we substitute $|x|^{-\frac{1}{2}(s_1+s_2)-1}\text{sgn}^{\nu}(x)$ for $k_0(x)$ in (4.59). We obtain that the bilinear functional is invariant if $s_1 = s_2 \neq 0, 1, 2.., n, ,$. In this case the invariant bilinear functional is given as follows:

$$(\varphi,\psi) = \int_{-\infty}^{\infty}|x_1-x_2|^{-s_1-1}\text{sgn}^{\epsilon}(x_1-x_2)\varphi(x_1)\psi(x_2)dx_1dx_2. \quad (4.60)$$

Similar, for (4.58) , $k_0(x)$ is invariant if C_1 or C_2 is zero. Then, we obtain

$$k_0(x) = \delta^{\frac{1}{2}(s_1+s_2)}(x), \quad \text{or} \quad k_0(x) = x^{-\frac{1}{2}(s_1+s_2)-1}.$$

We substitute $\delta^{\frac{1}{2}(s_1+s_2)}(x)$ for $k_0(x)$ in (4.59). We get the following invariant bilinear functionals:

- if $s_1 = s_2$ is an integer but the representation is not holomorphic, we have
$$(\varphi,\psi) = \int_{-\infty}^{\infty}\varphi^{s_1}(x)\psi(x)dx, \quad (4.61)$$
- if $s_1 = -s_2$, we have
$$(\varphi,\psi) = \int_{-\infty}^{\infty}\varphi(x_1)\psi(x_2)dx_1dx_2. \quad (4.62)$$

For $k_0(x) = x^{-\frac{1}{2}(s_1+s_2)-1}$, the invariant bilinear functional is given as follows:

$$(\varphi,\psi) = \int_{-\infty}^{\infty}(x_1-x_2)^{-s_1-1}\varphi(x_1)\psi(x_2)dx_1dx_2, \quad (4.63)$$

where $s_1 = s_2 \in \mathbb{Z}$ and the representation is holomorphic.

To conclude, the $SL_2(\mathbb{R})$ group representations T_{t_1} and T_{t_2} given by (4.47), (4.48) have an invariant bilinear functional if and only if $\epsilon_1 = \epsilon_2 = \{0, 1\}$ and either $s_1 = s_2$ or $s_1 = -s_2$, where $s_1, s_2 \in \mathbb{C}$.

4.6 Invariant Bilinear Functionals for Holomorphic Representations

In section 4.5, the bilinear functional (4.63) was invariant if $s_1 = s_2 = n \in \mathbb{Z}$. In this case, the representation operator is given by

$$[T_n(g)\varphi](x) = (d - bx)^{n-1}\varphi\left(\frac{ax - c}{d - bx}\right). \tag{4.64}$$

In this section, we illustrated the invariant subspaces of the $SL_2(\mathbb{R})$ representation T_n. The representation T_n is called holomorphic because it is constructed in a space of holomorphic functions. This is explained in the following text.

Let $\rho : H_2(\mathbb{R}) \to H_2(\mathbb{R})$ be the quasi-regular representation of the affine group given by

$$[\rho(a,b)f](x) = a^{\frac{-1}{2}} f\left(\frac{x-a}{b}\right).$$

Let the mother wavelet be $c(x) := \frac{1}{i\pi}\frac{1}{i\pm x}$, and let the operator $F_\pm : L_2(\mathbb{R}) \to \mathbb{C}$ be defined by

$$F_\pm(f) = \langle f, c\rangle = \frac{1}{\pi i}\int_{\mathbb{R}} \frac{f(x)}{i \pm x} dx.$$

Then, from the Definition 3.5.1, the covariant transform $\mathcal{W}_F^\rho : L_2(\mathbb{R}) \to H_2(\mathbb{R})$ becomes

$$[\mathcal{W}_{F_+}^\rho f](b + ai) = F_+(\rho(a,b)^{-1}f(t)) = \frac{1}{2\pi i}\int_{\mathbb{R}} \frac{f(t)}{t - (b + ai)} dt.$$

The image space for this covariant transform consists of the null solution of the Cauchy-Riemann equation $\partial_{\bar{z}}$ in the upper half-plane. This has been explained in example 3.5.9.

Also, for the affine group, consider the contravariant transform (see subsection 3.5.3) $\mathcal{M} : H_2(\mathbb{R}) \to L_2(\mathbb{R})$, which is given by

$$[\mathcal{M}f](t) = \lim_{a\to 0} f(a,t).$$

Therefore, the composition $\mathcal{M} \circ \mathcal{W}_{F_+}^\rho : H_2(\mathbb{R}) \to H_2(\mathbb{R})$ is given as follows:

$$[\mathcal{M} \circ \mathcal{W}_{F_+}^\rho f](t) = \lim_{a\to 0}\frac{1}{2\pi i}\int_{\mathbb{R}} \frac{f(t)}{t - (b + ia)} dt. \tag{4.65}$$

This shows that at $a = 0$, we get the boundary value of the Cauchy integral $[\mathcal{C}f](b+ia)$, and the vector space of functions $[\mathcal{C}f](b+i0)$ is the Hardy space on the real line.

Now, for nonnegative integer n, let D_n be the space with the invariant bilinear functional

$$(\varphi,\psi) = \int_{-\infty}^{\infty} (x_1 - x_2)^{-n-1}\varphi(x_1)\psi(x_2)dx_1dx_2. \tag{4.66}$$

To find the invariant subspaces of D_n, we choose the kernels $k_0(x) = (x - i0)^{-n-1}$ and $k_0(x) = (x + i0)^{-n-1}$. From (4.54), the functionals corresponding to them are

$$(\varphi,\psi)_+ = \int_{-\infty}^{\infty} (x_1 - x_2 - i0)^{-n-1}\varphi(x_1)\psi(x_2)dx_1dx_2, \tag{4.67}$$

$$(\varphi,\psi)_- = \int_{-\infty}^{\infty} (x_1 - x_2 + i0)^{-n-1}\varphi(x_1)\psi(x_2)dx_1dx_2, \tag{4.68}$$

where $\varphi(x)$ and $\psi(x) \in D_n$. From (4.65), we associate every $\varphi(x)$ with the following two bounded support functions:

$$\varphi_+(x) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{\varphi(x_1)}{x_1 - x - i0}dx_1, \tag{4.69}$$

$$\varphi_-(x) = -\frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{\varphi(x_1)}{x_1 - x + i0}dx_1. \tag{4.70}$$

These functions are in the Hardy space on the upper and lower half planes, respectively, and we have $\varphi(x) = \varphi_+(x) + \varphi_-(x)$.

Then, the bilinear functional on the upper and lower half planes, respectively, are given by the following:

$$(\varphi,\psi)_+ = \frac{2\pi i}{n}\int_{-\infty}^{\infty} \varphi_+^{(n)}(x)\psi(x)dx, \tag{4.71}$$

$$(\varphi,\psi)_- = \frac{2\pi i}{-n}\int_{-\infty}^{\infty} \varphi_-^{(n)}(x)\psi(x)dx. \tag{4.72}$$

The functions $\varphi_+^{(n)}(x)$ and $\varphi_-^{(n)}(x)$ are the nth derivative of $\varphi_+(x)$ and $\varphi_-(x)$, respectively, and are given as follows:

$$\varphi_+^{(n)}(x) = \frac{n}{2\pi i}\int_{-\infty}^{\infty} \frac{\varphi(x_1)}{(x_1 - x - i0)^{n+1}}dx_1, \tag{4.73}$$

$$\varphi_-^{(n)}(x) = \frac{-n}{2\pi i}\int_{-\infty}^{\infty} \frac{\varphi(x_1)}{(x_1 - x + i0)^{n+1}}dx_1. \tag{4.74}$$

Theorem 4.6.1. [13, p.410] The integrals $(\varphi,\psi)_+$ (4.71) and $(\varphi,\psi)_-$ (4.72) converge for arbitrary φ and $\psi \in D_n$, and hence, we define invariant bilinear functionals on all of D_n.

Let $D_n^- \subset D_n$ be a subspace of $\varphi(x)$ functions such that $(\varphi, \psi)_+ = 0$ for every $\psi \in D_n$. Equation (4.71) shows that D_n^- contains all $\varphi(x)$ functions such that $\varphi_+^{(n)}(x) = 0$. Hence, we obtain $\varphi^n(x) = \varphi_-^{(n)}(x)$ on the space D_n^-. Thus, $\varphi(x)$ is the boundary value of a holomorphic function in the lower half-plane.

Similarly, $(\varphi, \psi)_- = 0$ on a subspace $D_n^+ \subset D_n$ of the function $\varphi(x)$, which is the boundary value of a holomorphic function in the upper half-plane.

The intersection of D_n^+ and D_n^- is the finite dimensional subspace E_n of all polynomials of degree $n-1$ and less. To conclude, the space D_n of analytic representation contains three invariant subspaces: one finite dimensional and two infinite dimensional. In Lemma 4.7.4, we show that the quotient space D_n/E_n is the direct sum of the invariant subspaces D_n^+/E_n and D_n^-/E_n.

For $-n \in \mathbb{Z}_-$, let F_{-n} be the space where the invariant bilinear functional given by (4.66) is equal to zero . Hence, F_{-n} consists of functions $\varphi(x)$ that satisfy

$$\int_{-\infty}^{\infty} x^k \varphi(x) dx = 0, \quad k = 0, ..., -n-1. \tag{4.75}$$

Remark 4.6.2. For the homogeneous function $k_0(x) = x^{-n-1}$, let $k_1(x) = x^{-n-1} \ln|x|$ be an associated homogeneous function (see Definition A.2.6). That is

$$\begin{aligned} k_1(\alpha x) &= (\alpha x)^{-n-1} \ln|\alpha x| \\ &= \alpha^{-n-1} x^{-n-1} [\ln|\alpha| + \ln|x|] \\ &= \alpha^{-n-1} [x^{-n-1} \ln|x| + \ln|\alpha| x^{-n-1}] \\ &= \alpha^{-n-1} [k_1(x) + \ln|\alpha| k_0(x)]. \end{aligned}$$

The bilinear functional of $k_1(x) = x^{-n-1} \ln|x|$ is defined on the space F_{-n} and is given by

$$(\varphi, \psi)_1 = \int_{-\infty}^{\infty} (x_1 - x_2)^{-n-1} \ln|x_1 - x_2| \varphi(x_1) \psi(x_2) dx_1 dx_2. \tag{4.76}$$

By simple calculation, for g_2 (4.50) and T_n (4.64), we have

$$(T_n(g_2)\varphi, T_n(g_2)\psi))_1 = \left[(\varphi,\psi)_1 + \ln|\alpha^{-2}|(\varphi,\psi)\right], \tag{4.77}$$

where (φ, ψ) is given by (4.66). On the space F_{-n}, the invariant bilinear functional is $(\varphi, \psi) = 0$. Hence, we obtain

$$(T_n(g_2)\varphi, T_n(g_2)\psi)_1 = (\varphi,\psi)_1.$$

Therefore, the bilinear functional $(\varphi, \psi)_1$ is invariant under dilation on the space F_{-n}. Also, by direct calculation, $(\varphi, \psi)_1$ is invariant under inversion on F_{-n}, that is,

$$(T_n(g_3)\varphi, T_n(g_3)\psi)_1 = (\varphi,\psi)_1,$$

where g_3 is given in (4.50) and T_n is (4.64). Hence, $(\varphi, \psi)_1$ is an invariant bilinear functional on F_{-n}.

Next, for $k_1(x) = x^{-n-1}\ln|x|$, there exists the following kernels:

$$k_1^+(x) = \lim_{y\to+0} x^{-n-1}\ln|x-iy| = x^{-n-1}\ln|x-i0|, \tag{4.78}$$

$$k_1^-(x) = \lim_{y\to-0} x^{-n-1}\ln|x+iy| = x^{-n-1}\ln|x+i0|. \tag{4.79}$$

The functionals corresponding to $k_1^+(x)$ and $k_1^-(x)$ are

$$(\varphi,\psi)_1^+ = \int_{-\infty}^{\infty}(x_1-x_2)^{-n-1}\ln(x_1-x_2-i0)\varphi(x_1)\psi(x_2)dx_1dx_2,$$

$$(\varphi,\psi)_1^- = \int_{-\infty}^{\infty}(x_1-x_2)^{-n-1}\ln(x_1+x_2-i0)\varphi(x_1)\psi(x_2)dx_1dx_2.$$

Hence, F_{-n} is an invariant space and contains two invariant subspaces:

- The subspace F_{-n}^+ is the subspace of functions in F_{-n}, which are the boundary values of the function in the upper half plane, where $(\varphi, \psi)_1^- = 0$.
- The subspace F_{-n}^- is the subspace of functions in F_{-n}, which are the boundary values of function in the lower half plane, where $(\varphi, \psi)_1^+ = 0$.

Next, we want to show that the subspaces F_{-n}^+ and F_{-n}^- consist of the boundary values of holomorphic functions in the upper and lower half-planes, respectively.

For $\varphi(z)$, a holomorphic function in the upper half-plane, we have $\lim_{y_+\to 0}\varphi(z) = \varphi(x)$, where $z = x + iy$. Then, $\varphi(x)$ is the boundary value for $\varphi(z)$.

Let $\hat{\varphi}(\zeta) = \int_{-\infty}^{\infty} \varphi(x) e^{-2\pi i x \zeta}$ be the Fourier transform of $\varphi(x)$. Then, we obtain

$$\int_{-\infty}^{\infty} x^k \varphi(x) dx = (-2\pi i)^k \hat{\varphi}^k(0). \tag{4.80}$$

By Cauchy's integral theorem for the function $\varphi(z)$, which is holomorphic in the upper half-plane, we get $\hat{\varphi}(\zeta) = 0$, $\zeta > 0$. Hence, $\hat{\varphi}^k(0) = 0$, and (4.80) is equal to zero. This implies that $\varphi(x) \in F_{-n}^{+}$.

The same is noted for the boundary value of holomorphic function in the lower half-plane.

4.7 Equivalence of the $\mathrm{SL}_2(\mathbb{R})$ Representations

In this section, we study under which conditions the $\mathrm{SL}_2(\mathbb{R})$ representations T_{t_1} (4.47) and T_{t_2} (4.48) are equivalent.

Definition 4.7.1. For the representations T_{t_1} and T_{t_2}, an intertwining operator A is a continuous mapping of the space W_{t_1} onto the space W_{t_2}, that is,

$$AT_{t_1}(g) = T_{t_2}(g)A.$$

The representations T_{t_1} and T_{t_2} are equivalent if there exists an intertwining operator A which is one-to-one continuous mapping with the continuous inverse A^{-1} such that:

$$T_{t_1}(g) = AT_{t_2}(g)A^{-1}.$$

To obtain the conditions for the existence of an intertwining operator A, we establish a relation between the operator A and the bilinear functional (φ, ψ). Let W_{-t_2} be the space of the representation T_{-t_2} acting on. The space W_{-t_2} is associated with the pair of number $-t_2 = (-s_2, \epsilon_2)$. Then let $B(\psi, \varphi)$ be an invariant bilinear functional on the spaces W_{-t_2} and W_{t_1}. It is shown in section 4.5 that if $s_1 = -s_2$ then the invariant bilinear functional is given by the following:

$$B(\psi, \varphi) = \int_{-\infty}^{\infty} \psi(x)\varphi(x) dx. \tag{4.81}$$

Let $A : \mathsf{W}_{t_1} \to \mathsf{W}_{t_2}$ be a linear operator. Then, we associate with A the bilinear functional (φ, ψ) on the spaces W_{t_1} and W_{-t_2} as expressed by the following:

$$(\varphi, \psi) = B(\psi, A\varphi) = \int_{-\infty}^{\infty} \psi(x) A\varphi(x) dx, \tag{4.82}$$

where $\varphi \in \mathsf{W}_{t_1}$, $\psi \in \mathsf{W}_{-t_2}$.

Lemma 4.7.2. The linear operator $A : \mathsf{W}_{t_1} \to \mathsf{W}_{t_2}$ intertwines with the representations T_{t_1} and T_{t_2} if and only if $(\varphi, \psi) = B(\psi, A\varphi)$ invariant under T_{t_1} and T_{-t_2}.

Proof. From equation (4.7.1), we obtain the following:

$$B(T_{-t_2}(g)\psi, AT_{t_1}(g)\varphi) = B(T_{-t_2}(g)\psi, T_{t_2}(g)A\varphi),$$

where $\varphi \in \mathsf{W}_{t_1}$ and $\psi \in \mathsf{W}_{-t_2}$. The invariance of the bilinear functional $B(\psi, \varphi)$ implies that

$$B(T_{-t_2}(g)\psi, T_{t_2}(g)A\varphi) = B(\psi, A\varphi),$$

for all ψ and φ. Then, we have

$$B(T_{-t_2}(g)\psi, AT_{t_1}(g)\varphi) = B(\psi, A\varphi) = (\varphi, \psi).$$

Therefore, (φ, ψ) is invariant under $T_{t_1}(g)$ and T_{-t_2}. □

In section 4.5, we found the conditions under which the invariant bilinear functionals (φ, ψ) exist. By substituting $-s_2$ for s_2 in these conditions, we get that the $SL_2(\mathbb{R})$ representations T_{t_1} and T_{t_2} have an intertwining operator A, which maps W_{t_1} continuously into W_{t_2} if and only if $\epsilon_1 = \epsilon_2 = \{0, 1\}$ and either $s_1 = s_2$ or $s_1 = -s_2$, where $s_1, s_2 \in \mathbb{C}$.

To obtain the expression of such an operator A, first consider the case $s_1 = s_2$, the invariant bilinear functional is given by

$$(\varphi, \psi) = \lambda \int_{-\infty}^{\infty} \varphi(x)\psi(x) dx.$$

Comparing this with (4.82), we conclude that every operator A on W_{t_2} follows the condition that $AT_{t_1}(g) = T_{t_2}(g)A$ is a multiplier of the unit operator. This implies that $A = \lambda I$, where λ is constant. Therefore, by Schur's lemma, all the representations T_{t_1} and T_{t_2} except the holomorphic representation are irreducible.

Next, for the case $s_1 = -s_2$, we have two invariant bilinear functionals (4.60) and (4.61). For the functional given by (4.60), the operator A is expressed as follows:

$$A\varphi(x) = \lambda \int_{-\infty}^{\infty} |x_1 - x|^{-s_1-1} \mathrm{sgn}^{\epsilon}(x_1 - x)\varphi(x_1)dx_1.$$

For (4.61), the operator A is given as follows:

$$A\varphi(x) = \varphi^{(s)}(x).$$

Theorem 4.7.3. [13, p.416] Consider the representation operators $T_{t_1}(g)$ and $T_{t_2}(g)$ given by (4.47) and (4.48), respectively, possessing an intertwining operator A maps W_{t_1} continuously into W_{t_2}. Then, A is a one-to-one map, and $T_{t_1}(g)$,$T_{t_2}(g)$ are equivalent.

4.7.1 Equivalence of the Holomorphic Representation of $\mathrm{SL}_2(\mathbb{R})$

Consider the analytic representations T_n and T_{-n} given by (4.64) for $n \in \mathbb{Z}^+$. From section 4.6, the bilinear invariant functional is expressed as follows:

$$(\varphi, \psi) = \int_{-\infty}^{\infty} [\lambda_1 \varphi_+^{(n)}(x) + \lambda_2 \varphi_-^{(n)}(x)]\psi(x)dx,$$

where λ_1 and λ_2 are arbitrary constants. The functions ${\varphi_+}^{(n)}(x)$ and ${\varphi_-}^{(n)}(x)$ are given by (4.73) and (4.74),respectively. Hence, any operator intertwining with the holomorphic representations (4.64) is of the form

$$A'\varphi(x) = \frac{\lambda_1}{2\pi i} \int_{-\infty}^{\infty} \frac{\varphi^{(n)}(x_1)}{x_1 - x - i0} dx_1 - \frac{\lambda_2}{2\pi i} \int_{-\infty}^{\infty} \frac{\varphi^{(n)}(x_1)}{x_1 - x + i0} dx_1.$$

This shows that the holomorphic representations T_n and T_{-n} are inequivalent.

Let us illustrate the relations between the analytic representations. As mentioned in section 4.6 that for the analytic representations T_n and T_{-n} acting on D_n and D_{-n}, respectively, where $n \in \mathbb{Z}^+$, we have established the following:

- The space D_n contains three invariant subspaces:
  - E_n, the space of all polynomials of degree $n - 1$ and less,
  - D_n^+ ,the subspace of all functions $\varphi(x)$ that are boundary values of holomorphic functions on the upper half plane such that $A_-\varphi(x) = 0$, and

- D_n^-, the subspace of all functions $\varphi(x)$ that are boundary values of holomorphic functions on the lower half plane such that $A_+\varphi(x) = 0$. The intersection of D_n^+ and D_n^- is E_n, and their sum is the entire space D_n.

Here, A_+ and A_- maps D_n onto D_{-n} and are defined by

$$A_+\varphi(x) = \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\varphi^{(n)}(x_1)dx_1}{x_1 - x - i0}, \tag{4.83}$$

$$A_-\varphi(x) = -\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\varphi^{(n)}(x_1)dx_1}{x_1 - x + i0}. \tag{4.84}$$

- The space D_{-n} contains three subspaces:
  - F_n, the space of all $\varphi(x)$ such that

$$\int_{-\infty}^{\infty} x^k\varphi(x)dx = 0, \quad k = 0, \ldots - n - 1, \tag{4.85}$$

  - F_{-n}^+, the subspace of functions that are boundary values of holomorphic functions on the upper half plane, and
  - F_{-n}^-, the subspace of function that are the boundary values of holomorphic functions on the lower half plane.

Lemma 4.7.4. [13] The $SL_2(\mathbb{R})$ representations on the subspaces D_n/E_n and F_{-n} are reducible. Also, D_n/E_n and F_{-n} are direct sums of two invariant subspaces.

Proof. The quotient space D_n/E_n is the space of functions in D_n defined only up to the polynomial of degree $n-1$ and less. Consider the intertwining operators A_+ (4.83) and A_- (4.84) that maps the spaces D_n onto D_{-n}. The operators A_+ and A_- satisfy the following:

$$A_+T_n(g) = T_{-n}(g)A_+, \quad \text{and} \quad A_-T_n(g) = T_{-n}(g)A_-.$$

Every other intertwining operator for $T_n(g)$ and $T_{-n}(g)$ is a linear combination of A_+ and A_-.

Let $\varphi(x)$ be a function in the space D_n. In subsection 4.6, we show that

$$\varphi(x) = \varphi_+(x) + \varphi_-(x),$$

where the functions $\varphi_+(x)$ and $\varphi_-(x)$ are the boundary values of some holomorphic functions in the upper and lower half-planes, respectivly. That is $\varphi_+(x) \in D_n^+$ and $\varphi_-(x) \in D_n^-$. The above implies that space D_n/E_n is a direct sum of the form

$$D_n/E_n = D_n^+/E_n \oplus D_n^-/E_n.$$

Hence, the representation on the space D_n/E_n is reducible.

Next, let the subspaces F_{-n}^+ and F_{-n}^- be the images of the subspaces D_n^+ and D_n^- under the covariant transforms A_+ and A_-, respectively. The subspaces F_{-n}^+ and F_{-n}^- are invariant under T_{-n} and $F_{-n}^+ \cap F_{-n}^- = \{\phi\}$, respectivly. Thus, we have the direct sum

$$F_{-n} = F_{-n}^+ \oplus F_{-n}^-.$$

□

Remark 4.7.5. Since we have shown that the $SL_2(\mathbb{R})$ representations on the subspaces D_n^+/E_n and F_{-n}^+ are equivalent under the covariant transforms A_+, we can realise the representation in the upper half plane $\varphi(z)$. Then, the $SL_2(\mathbb{R})$ representation on $D_n^+/E_s \cong F_{-n}^+$ is given by

$$[T_n(g)\varphi](z) = (d - bz)^{n-1}\varphi\left(\frac{az - c}{d - bz}\right). \tag{4.86}$$

However, the subspace $D_n^+/E_n \cong F_{-n}^+$ does not consist of all analytic functions $\varphi(z)$ in the upper half-plane. The function $\varphi(z)$ must be infinitely differentiable together with $\tilde{\varphi}(z) = z^{n-1}\varphi(\frac{-1}{z})$ in the closed upper half-plane. The same is noted, for the $SL_2(\mathbb{R})$ representations on the subspaces $D_n^-/E_n \cong F_{-n}^-$.

Lemma 4.7.6. [13] The equivalence of the holomorphic representations T_n, T_{-n} in the following pairs of subspaces:

- E_n and D_{-n}/F_{-n}, where the intertwining operator is given by
$$A\varphi(x) = \int_{-\infty}^{\infty} (x_1 - x)^{n-1}\varphi(x_1)dx_1.$$
- D_n/E_n and F_{-n}, where A is the differential operator d^n/dx^n.

- D_n^+/E_n and F_{-n}^+ or D_n^-/E_n and F_{-n}^-, where the intertwining operator is A_+(4.83) or A_- (4.84).

4.8 Unitary Representations of the Group $\mathrm{SL}_2(\mathbb{R})$

Unitary representation is a representation on a Hilbert space with an invariant inner product. Hence we need to find the conditions under which it is possible to define an invariant inner product under the $\mathrm{SL}_2(\mathbb{R})$ representation. Recall that an inner product is a positive definite non-degenerate Hermitian functional. Hence, we start by studying the invariance of the Hermitian functional.

4.8.1 The Existence of an Invariant Hermitian Functional

Let W_t be the space of the representation T_t (4.47) associated with the pair of numbers $t = (s, \epsilon)$, $s \in \mathbb{C}$. Then, for $\bar{t} = (\bar{s}, \epsilon)$, we have the space $\mathsf{W}_{\bar{t}}$ of the representation $T_{\bar{t}}$, which is given as follows:

$$[T_{\bar{t}}(g)\psi](x) = |d - bx|^{\bar{s}-1}\mathrm{sgn}^{\epsilon}(d - bx)\psi\left(\frac{ax - c}{d - bx}\right). \qquad (4.87)$$

The Hermitian functional is defined as $\langle\varphi, \psi\rangle : \mathsf{W}_t \times \mathsf{W}_{\bar{t}} \to \mathbb{R}$.The goal of this subsection is to find the conditions under which this functional is invariant, that is

$$\langle\varphi, \psi\rangle = \langle T_t(g)\varphi, T_{\bar{t}}(g)\psi\rangle.$$

From section 4.5, the bilinear functional (φ, ψ) is invariant if and only if $s_1 = s_2$ or $s_1 = -s_2$. Let the number s_2 be the complex conjugate of s_1. Then, the bilinear functional (φ, ψ) will be converted to the Hermitian functional $\langle\varphi, \psi\rangle$. Therefore, the Hermitian functional $\langle\varphi, \psi\rangle$ is invariant if and only if $s = \bar{s}$ or $s = -\bar{s}$.

The expressions of the invariant Hermitian functional will be as follows:

- For $s = -\bar{s}$, i.e. s is pure imaginary, we have:

$$\langle\varphi, \psi\rangle = \int_{-\infty}^{\infty} \varphi(x)\overline{\psi}(x)dx. \qquad (4.88)$$

- For $s = \overline{s}$, i.e. if s is real, we have:

$$\langle \varphi, \psi \rangle = \int_{-\infty}^{\infty} |x_1 - x_2|^{-s_1 - 1} \text{sgn}^{\epsilon}(x_1 - x_2) \varphi(x_1) \overline{\psi}(x_2) dx_1 dx_2. \tag{4.89}$$

  Also, if s is a nonnegative integer and the representation is not holomorphic, the invariant Hermitian functional is

$$\langle \varphi, \psi \rangle = \int_{-\infty}^{\infty} \varphi^{(s)}(x) \overline{\psi}(x) dx.$$

- For the holomorphic representation (4.64) every invariant Hermitian functional is a linear combination of:

$$\langle \varphi, \psi \rangle_+ = \int_{-\infty}^{\infty} \varphi_+^{(n)}(x) \overline{\psi}(x) dx, \tag{4.90}$$

$$\langle \varphi, \psi \rangle_- = \int_{-\infty}^{\infty} \varphi_-^{(n)}(x) \overline{\psi}(x) dx. \tag{4.91}$$

  where $\varphi_+^{(n)}(x)$ and $\varphi_-^{(n)}(x)$ are given by (4.73) and (4.74), respectively.

4.8.2 Positive Definite Invariant Hermitian Functional

The invariant Hermitian bilinear functional given by (4.88), is positive definite for pure imaginary number s. The invariant Hermitian bilinear functional given by (4.89), is positive definite if $\epsilon = 0$ and $|s| < 1$ [13, p.427].

Next, for the holomorphic representation, every invariant Hermitian bilinear functional is a linear combination of (4.90) and (4.91). Consider $\langle \varphi, \psi \rangle_+ \neq 0$ as a Hermitian functional on the subspace D_n^+/E_n. We will show that $\langle \varphi, \psi \rangle_+$ is positive definite.

The Fourier transform of $\varphi^{(n)}(x)$ is given by $\mathcal{F}[\varphi^{(n)}(\zeta)] = (-i)^n \zeta^n \hat{\varphi}(\zeta)$, where $\hat{\varphi}(\zeta) = \int_{-\infty}^{\infty} \varphi(x) e^{i\zeta x} dx$.

Note that since $\varphi_+(x)$ is the boundary value of a holomorphic function on the upper half-plane, then the Fourier transform of $\varphi_+(x)$ is supported on $-\infty < \zeta < 0$. Then, the Plancherel theorem implies that

$$\langle \varphi, \psi \rangle_+ = i^{-n} \int_{-\infty}^{\infty} \varphi_+^{(n)}(x) \overline{\psi}(x) dx = \frac{1}{2\pi} \int_{-\infty}^{0} |\zeta|^n \hat{\varphi}(\zeta) \overline{\hat{\psi}}(\zeta) d\zeta.$$

Thus, $\langle \varphi, \psi \rangle_+$ is positive definite on D_n^+/E_n.

Similarly, the invariant Hermitian functional $\langle\varphi,\psi\rangle_-$ is positive definite on the subspace D_n^-/E_n since we have

$$\langle\varphi,\psi\rangle_- = i^{-n}\int_{-\infty}^{\infty}\varphi_-^{(n)}(x)\overline{\psi}(x)dx = \frac{1}{2\pi}\int_0^{\infty}\zeta^n\hat{\varphi}(\zeta)\overline{\hat{\psi}}(\zeta)d\zeta.$$

For the case that n is a negative integer, we have shown in the proof of Theorem 4.7.4 that the subspaces D_n^+/E_n and D_n^-/E_n map to the subspaces F_{-n}^+ and F_{-n}^- by the intertwining operator A_+ and A_-, respectively. Hence, the invariant Hermitian functionals on F_{-n}^+ and F_{-n}^- are positive definite.

Recall in Remark 4.7.5 that we can realise F_{-n}^+ as the space of holomorphic function in the upper half-plane. The representation in this case is defined by

$$[T_n(g)\varphi](z) = (d-bz)^{n-1}\varphi\left(\frac{az-c}{d-bz}\right), \quad \text{where} \quad z = x+iy. \tag{4.92}$$

The expression of the positive invariant Hermitian functionals for this model is of the form

$$\langle\varphi,\psi\rangle = \int_{\text{Im}z>0}\varphi(z)\overline{\psi}(z)\omega(z)dzd\bar{z},$$

where $\omega(z)$ is a positive function. To find the form of $\omega(z)$, we apply $T_n(g)$ (4.92) to $\varphi(z)$ and $\psi(z)$. Then, by direct calculation, the invariance condition is given by $\langle T_n(g)\varphi, T_n(g)\psi\rangle = \langle\varphi,\psi\rangle$, which is valid if and only if $\omega(z) = (\text{Im}z)^{-n-1} = y^{-n-1}$.

4.8.3 Representations of $\mathrm{SL}_2(\mathbb{R})$ on the Hilbert Space

We found in subsection 4.8.2 the condition under which there exists a positive definite Hermitian functional $\langle\varphi,\psi\rangle$ invariant under $T_t(g)$, that is

$$\langle T_t(g)\varphi, T_t(g)\psi\rangle = \langle\varphi,\psi\rangle.$$

We can consider such a Hermitian functional as an inner product in the space W_t. Then, if W_t is completed with respect to the norm

$$\|\varphi\|^2 = \langle\varphi,\varphi\rangle,$$

we obtain a Hilbert space $\mathcal{H}$.

The operators $T_t(g)$ on W_t can be extended uniquely to unitary operators on $\mathcal{H}$.

We denote these unitary operators, as before, by $T_t(g)$ such that they also satisfy the representation group property:

$$T_t(g_1g_2) = T_t(g_1)T_t(g_2).$$

Hence, these unitary operators form a representation of $\mathrm{SL}_2(\mathbb{R})$.

Lemma 4.8.1. [13]. For every representation T_t that possesses a positive definite Hermitian functional, a corresponding representation of $\mathrm{SL}_2(\mathbb{R})$ by unitary operators on the Hilbert space exists. In this correspondence, equivalent representations correspond to unitary equivalent representations and inequivalent representations correspond to inequivalent ones.

Next, we wish to classify the unitary representation of the $\mathrm{SL}_2(\mathbb{R})$ group.

- Representations of the principal (continuous) series:

  For $s = i\rho$ where $\rho \in \mathbb{R}$ and $\epsilon = 0$ or 1, the representations are defined by
  $$T_{i\rho}(g)\varphi(x) = |d - bx|^{i\rho-1}\mathrm{sgn}^{\epsilon}(d - bx)\varphi\left(\frac{ax - c}{d - bx}\right). \tag{4.93}$$
  From subsection 4.8.2 the inner product in this case is as follows:
  $$\langle\varphi, \varphi\rangle = \int_{-\infty}^{\infty} \varphi(x)\overline{\varphi}(x) < \infty.$$

- Representations of the complementary series:

  These representations are defined by a real parameter $s \neq 0$ in the interval $-1 < s < 1$. The inner product is given by
  $$\langle\varphi, \varphi\rangle = \int_{-\infty}^{\infty} |x_1 - x_2|^{s_1-1}\varphi(x_1)\overline{\varphi}(x_2)dx_1dx_2.$$
  The representation is defined by
  $$T_s(g)\varphi(x) = |d - bx|^{s-1}\varphi\left(\frac{ax - c}{d - bx}\right). \tag{4.94}$$

- Representations of the discrete series:

  For each integer number n, the inner product on the space of holomorphic functions in the upper half plane is given by
  $$\langle\varphi, \varphi\rangle = \int_{y>0}\int_{\mathbb{R}} |\varphi(x + iy)|^2 y^{-n-1}dxdy < \infty. \tag{4.95}$$

The representation is identified by

$$T_n(g)\varphi(z) = (d - bz)^{n-1}\varphi\left(\frac{az - c}{d - bz}\right), \quad n \in \mathbb{Z}. \tag{4.96}$$

Chapter 5

The $\mathrm{SL}_2(\mathbb{R})$ Representations on Spaces of Holomorphic Functions on the Unit Disc

We can realise the representations of the group $\mathrm{SL}_2(\mathbb{R})$ on the unit disc. This is due to an isomorphism between the group $\mathrm{SL}_2(\mathbb{R})$ and the group $\mathrm{SU}(1,1)$. This chapter is to describe the $\mathrm{SL}_2(\mathbb{R})$ representation on the Dirichlet space. Moreover, we study the mock discrete series representation on the real line, on the unit circle and on the group $\mathrm{SL}_2(\mathbb{R})$. To begin, we provide some basic information regarding the group $\mathrm{SU}(1,1)$. Then, we induce an $\mathrm{SU}(1,1)$ representation on the unit disc and describe the action of the ladder operators of this representation.

5.1 The Group $\mathrm{SU}(1,1)$

The Cayley transform of the upper-half plane to the unit disc $\mathbb{D}$ is defined by

$$w = \frac{z-i}{z+i}, \tag{5.1}$$

where $x \in \mathbb{D}$ and $z \in \{z \in \mathbb{C}, \mathrm{Im} z > 0\}$.

By the transformation (5.1) we can transfer the action of the group $\mathrm{SL}_2(\mathbb{R})$ from the upper half-plane to the action of the group $\mathrm{SU}(1,1)$ on the unit disc, where

$$\mathrm{SU}(1,1) = \left\{ \begin{pmatrix} \alpha & \overline{\beta} \\ \beta & \overline{\alpha} \end{pmatrix} : \alpha, \beta \in \mathbb{C}, |\alpha|^2 - |\beta|^2 = 1 \right\}.$$

Furthermore, the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R})$ can be an element of the group $\mathrm{SU}(1,1)$ by the following identity:

$$\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} = \begin{pmatrix} \alpha & \overline{\beta} \\ \beta & \overline{\alpha} \end{pmatrix}. \tag{5.2}$$

Next, any $g \in SU(1,1)$ has a unique decomposition of the form

$$\begin{aligned} \begin{pmatrix} \alpha & \overline{\beta} \\ \beta & \overline{\alpha} \end{pmatrix} &= |\alpha| \begin{pmatrix} 1 & \overline{\beta}\overline{\alpha}^{-1} \\ \beta\alpha^{-1} & 1 \end{pmatrix} \begin{pmatrix} \frac{\alpha}{|\alpha|} & 0 \\ 0 & \frac{\overline{\alpha}}{|\alpha|} \end{pmatrix} \\ &= \frac{1}{\sqrt{1-|u|^2}} \begin{pmatrix} 1 & u \\ \overline{u} & 1 \end{pmatrix} \begin{pmatrix} \mathrm{e}^{i\theta} & 0 \\ 0 & \mathrm{e}^{-i\theta} \end{pmatrix}, \end{aligned} \tag{5.3}$$

where $\theta = \arg \alpha$, $u = \overline{\beta}\overline{\alpha}^{-1}$ and $|u| < 1$ (since $|\alpha|^2 - |\beta|^2 = 1$). Let $u = r\mathrm{e}^{i\phi}$, then the identity (5.3) describes an element $g \in \mathrm{SU}(1,1)$ by a triplet of numbers (r, ϕ, θ) where $0 \leq r < 1$ and $-\pi < \phi, \theta \leq \pi$. The connection with the (α, β) coordinates is as follows:

$$\alpha = \frac{\mathrm{e}^{i\theta}}{\sqrt{1-|r|^2}}, \quad \beta = \frac{r\mathrm{e}^{i(\theta-\phi)}}{\sqrt{1-|r|^2}}, \tag{5.4}$$

$$r = \left|\frac{\beta}{\alpha}\right|, \quad \phi = -\arg\frac{\beta}{\alpha}, \quad \theta = \arg \alpha. \tag{5.5}$$

Moreover, the decomposition (5.3) can be rewritten with the same variables as

$$\begin{pmatrix} \alpha & \overline{\beta} \\ \beta & \overline{\alpha} \end{pmatrix} = \begin{pmatrix} \mathrm{e}^{i\frac{\phi}{2}} & 0 \\ 0 & \mathrm{e}^{-i\frac{\phi}{2}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{1-|r|^2}} & \frac{r}{\sqrt{1-|r|^2}} \\ \frac{r}{\sqrt{1-|r|^2}} & \frac{1}{\sqrt{1-|r|^2}} \end{pmatrix} \begin{pmatrix} \mathrm{e}^{i(\theta-\frac{\phi}{2})} & 0 \\ 0 & \mathrm{e}^{-i(\theta-\frac{\phi}{2})} \end{pmatrix} \tag{5.6}$$

The last presentation is a decomposition of the group $\mathrm{SU}(1,1)$ as the product KAK of its subgroups, which is called the Cartan decomposition.

The base of the Lie algebra $\mathfrak{sl}_2(\mathbb{R})$ was given by (4.6). By using the identity(5.2) for the base (4.6), we get the base for the Lie algebra $\mathfrak{su}(1,1)$ which consists of the following three matrices:

$$\tilde{Z} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \tilde{A} = \begin{pmatrix} 0 & -\frac{i}{2} \\ \frac{i}{2} & 0 \end{pmatrix} \text{ and } \tilde{B} = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}. \tag{5.7}$$

The matrices $\tilde{Z}$, $\tilde{A}$ and $\tilde{B}$ satisfy the commutation relation (4.7). Also, the exponential map of each matrix generates a one-dimensional subgroup of the $\mathrm{SU}(1,1)$ group, that is

$$\mathrm{e}^{\theta\tilde{Z}} = \begin{pmatrix} \mathrm{e}^{i\theta} & 0 \\ 0 & \mathrm{e}^{-i\theta} \end{pmatrix}, \tag{5.8}$$

$$\mathrm{e}^{\theta\tilde{A}} = \begin{pmatrix} \cosh\frac{\theta}{2} & -i\sinh\frac{\theta}{2} \\ i\sinh\frac{\theta}{2} & \cosh\frac{\theta}{2} \end{pmatrix}, \tag{5.9}$$

$$\mathrm{e}^{\theta\tilde{B}} = \begin{pmatrix} \cosh\frac{\theta}{2} & \sinh\frac{\theta}{2} \\ \sinh\frac{\theta}{2} & \cosh\frac{\theta}{2} \end{pmatrix}. \tag{5.10}$$

5.2 Induced Representation on the Unit Disc

In this section, we induce a representation of the group $\mathrm{SU}(1,1)$ from the subgroup K. The one-dimensional compact subgroup K is given as follows:

$$K = \left\{ \begin{pmatrix} \mathrm{e}^{i\theta} & 0 \\ 0 & \mathrm{e}^{-i\theta} \end{pmatrix}, \quad -\pi < \theta \leq \pi. \right\} \tag{5.11}$$

Using the decomposition (5.3) of any element $g \in \mathrm{SU}(1,1)$, we can identify the homogeneous space $X = \mathrm{SU}(1,1)/\mathrm{K}$ with the open unit disc $\mathbb{D}$. Let the section $\mathsf{s} : \mathbb{D} \to \mathrm{SU}(1,1)$ be defined as follows:

$$\mathsf{s} : u \mapsto \frac{1}{\sqrt{1-|u|^2}} \begin{pmatrix} 1 & u \\ \bar{u} & 1 \end{pmatrix}. \tag{5.12}$$

There is a natural projection map $\mathsf{p} : \mathrm{SU}(1,1) \to \mathbb{D}$, which assigns to an element of $\mathrm{SU}(1,1)$ its equivalence class in $\mathrm{SU}(1,1)/\mathrm{K}$. That is

$$\mathsf{p} : \begin{pmatrix} \alpha & \overline{\beta} \\ \beta & \overline{\alpha} \end{pmatrix} \mapsto \frac{\overline{\beta}}{\overline{\alpha}}. \tag{5.13}$$

Mapping $\mathsf{r} : \mathrm{SU}(1,1) \to \mathrm{K}$ associates f to the natural projection p, and the section s is

$$\mathsf{r} : \begin{pmatrix} \alpha & \overline{\beta} \\ \beta & \overline{\alpha} \end{pmatrix} \mapsto \begin{pmatrix} \frac{\alpha}{|\alpha|} & 0 \\ 0 & \frac{\overline{\alpha}}{|\alpha|} \end{pmatrix} \tag{5.14}$$

For the homogeneous space $\mathrm{SU}(1,1)/\mathrm{K}$ defines a left action denoted by "$\cdot$" as follows:

$$g : u \mapsto g \cdot u = \mathsf{p}(g * \mathsf{s}(u)), \tag{5.15}$$

where $*$ is the multiplication of the group $\mathrm{SU}(1,1)$.

The invariant measure $d\mu(u)$ on $\mathbb{D}$ comes from the decomposition $dg = d\mu(u)dk$, where dg and dk are Haar measure on $G = \mathrm{SU}(1,1)$ and K respectively. The measure $d\mu(u)$ is equal to

$$d\mu(u) = \frac{du \wedge d\bar{u}}{(1-|u|^2)^2}. \tag{5.16}$$

Let $\chi_n : \mathbb{T} \to \mathbb{C}$ be a character of the subgroup $K \simeq \mathbb{T}$ defined by:

$$\chi_n(w) = w^n, \quad n \in \mathbb{Z}. \tag{5.17}$$

This character induces a representation of $\mathrm{SU}(1,1)$ constructed in the Hilbert space $L_2^{\chi_n}(SU(1,1))$, consisting of the functions $F_n : SU(1,1) \to \mathbb{C}$ with the property

$$F_n\left[\begin{pmatrix} \alpha & \overline{\beta} \\ \beta & \overline{\alpha} \end{pmatrix}\right] = \chi_n\left(\frac{\alpha}{|\alpha|}\right) F\left(\frac{\overline{\beta}}{\overline{\alpha}}\right), \tag{5.18}$$

where $F \in L_2(\mathbb{D})$. Then, the norm of the function F_n is as follows:

$$\|F_n\|^2 = \int_{\mathbb{D}} |F(u)|^2 \frac{du \wedge d\bar{u}}{(1-|u|^2)^2}. \tag{5.19}$$

The space $L_2^{\chi_n}(\mathrm{SU}(1,1))$ is invariant under the left shift of the $\mathrm{SU}(1,1)$ group. The restriction of the left shift on $L_2^{\chi_n}(\mathrm{SU}(1,1))$ is the left regular representation of $\mathrm{SU}(1,1)$, given as follows:

$$[\Lambda(g)F_n](g') = F_n(g^{-1} * g'), \tag{5.20}$$

where $*$ is a matrix multiplication.

The lifting map $\mathcal{L}_{\chi_n} : L_2(\mathbb{D}) \to L_2^{\chi_n}(SU(1,1))$ for the subgroup K and its character χ_n is defined as follows:

$$\begin{aligned}[\mathcal{L}_{\chi_n} f] \begin{pmatrix} \alpha & \overline{\beta} \\ \beta & \overline{\alpha} \end{pmatrix} &= \chi_n \left(\mathsf{r} \begin{pmatrix} \alpha & \overline{\beta} \\ \beta & \overline{\alpha} \end{pmatrix} \right) f \left(\mathsf{p} \begin{pmatrix} \alpha & \overline{\beta} \\ \beta & \overline{\alpha} \end{pmatrix} \right) \\ &= \left(\frac{\bar{\alpha}}{|\alpha|} \right)^n f \left(\frac{\overline{\beta}}{\overline{\alpha}} \right). \end{aligned} \tag{5.21}$$

The pulling map is given by:

$$\mathcal{P} : L_2^{\chi_n}(SU(1,1)) \to L_2(\mathbb{D}),$$

$$\mathcal{P}(F(w, \bar{w})) = F(\mathsf{s}(w)),$$

such that $\mathcal{P} \circ \mathcal{L}_{\chi_n} = \mathbb{I}$ and $\mathcal{L}_{\chi_n} \circ \mathcal{P} = \mathbb{I}$.

Therefore, the representation $\pi_n : L_2(\mathbb{D}) \to L_2(\mathbb{D})$, which is induced by the character χ_n is given by:

$$\left[\pi_n \begin{pmatrix} \alpha & \overline{\beta} \\ \beta & \overline{\alpha} \end{pmatrix} \right] = \mathcal{P} \circ \Lambda \begin{pmatrix} \alpha & \overline{\beta} \\ \beta & \overline{\alpha} \end{pmatrix} \circ \mathcal{L}_{\chi_n}.$$

By simple calculation, we get:

$$\begin{aligned}[\pi_n(g) f](w, \bar{w}) &= \frac{(\overline{\alpha} - \overline{\beta} w)^n}{|\overline{\alpha} - \overline{\beta} w|^n} f \left(\frac{\overline{\alpha} w - \overline{\beta}}{\alpha - \beta w}, \frac{\alpha \overline{w} - \beta}{\overline{\alpha} - \overline{\beta} \overline{w}} \right) \\ &= \left(\frac{\overline{\alpha} - \overline{\beta} w}{\alpha - \beta \bar{w}} \right)^{\frac{n}{2}} f \left(\frac{\overline{\alpha} w - \overline{\beta}}{\alpha - \beta w}, \frac{\alpha \overline{w} - \beta}{\overline{\alpha} - \overline{\beta} \overline{w}} \right). \end{aligned} \tag{5.22}$$

Definition 5.2.1. For $n \in \mathbb{Z}$, an n-peeling is an isometry $\mathcal{P}_n : L_2(\mathbb{D}, dw) \to L_2(\mathbb{D}, (1-|w|^2)^{n-2} dw \wedge d\bar{w})$ defined as:

$$\mathcal{P}_n : f(w) \mapsto [\mathcal{P}_n f](w) = \frac{f(w)}{(1 - |w|^2)^{\frac{n}{2}}}, \quad w = u + iv. \tag{5.23}$$

The representation (5.22) is intertwined $\breve{\pi}_n \circ \mathcal{P}_n = \mathcal{P}_n \circ \pi_n$ by the n-peeling with the representation:

$$[\breve{\pi}_n(g) f](w) = (\overline{\alpha} - \overline{\beta} w)^{-n} f \left(\frac{\alpha w - \beta}{\overline{\alpha} - \overline{\beta} w} \right), \tag{5.24}$$

which is unitary in $L_2(\mathbb{D}, (1-|w|^2)^{n-2} dw \wedge d\bar{w})$. The demonstration of the intertwining properties is based on the following analogue of identity for the unit disk :

$$1-\left|\frac{\alpha w-\beta}{\overline{\alpha}-\overline{\beta}w}\right|=\frac{1-|w|^2}{|\overline{\alpha}-\overline{\beta}w|^2}.$$

By the identity (5.2), the matrix $\begin{pmatrix}\alpha & \overline{\beta}\\ \beta & \overline{\alpha}\end{pmatrix}\in \mathrm{SU}(1,1)$ transforms to $\begin{pmatrix}a & b\\ c & d\end{pmatrix}\in SL_2(\mathbb{R})$. Therefore, the representation $\breve{\rho}_n^K$ can be transformed to a holomorphic representation of the group $SL_2(\mathbb{R})$:

$$[\breve{\pi}_n(g)f](z)=(d-bz)^{-n}f\left(\frac{az-c}{d-bz}\right), \tag{5.25}$$

which is unitary on the upper half-plane where $z=x+iy\in\mathbb{R}^2_+$ with the measure $d\mu(g)=\frac{dxdy}{y^2}$.

5.3 Actions of Ladder Operators

In this section, we study the left and right actions of the ladder operators for the representation π_n given by (5.22). First, the derived representations are given as follows:

$$\begin{aligned}[Ef](w,\overline{w})&=\frac{d}{dt}\pi_n(\mathrm{e}^{t\tilde{Z}})f(w,\overline{w})|_{t=0}\\ &=[-inI-2iw\partial_w+2i\overline{w}\partial_{\bar{w}}]f(w,\overline{w}),\end{aligned} \tag{5.26}$$

$$\begin{aligned}[A_1f](w,\overline{w})&=\frac{d}{dt}\pi_n(\mathrm{e}^{t\tilde{A}})f(w,\overline{w})|_{t=0}\\ &=\left[\frac{ni}{4}(w+\overline{w})I+\frac{i}{2}(1+w^2)\partial_w-\frac{i}{2}(1+\overline{w}^2)\partial_{\bar{w}}\right]f(w,\overline{w}),\end{aligned} \tag{5.27}$$

$$\begin{aligned}[B_1f](w,\overline{w})&=\frac{d}{dt}\pi_n(\mathrm{e}^{t\tilde{B}})f(w,\overline{w})|_{t=0}\\ &=\left[\frac{n}{4}(w-\bar{w})I+\frac{1}{2}(w^2-1)\partial_w+\frac{1}{2}(\overline{w}^2-1)\partial_{\bar{w}}\right]f(w,\overline{w}).\end{aligned} \tag{5.28}$$

The ladder operators are defined as

$$\begin{aligned}L_+&=B_1-iA_1=\frac{n}{2}wI+w^2\partial_w-\partial_{\overline{w}},\\ L_-&=B_1+iA_1=\frac{-n}{2}\overline{w}I+\bar{w}^2\partial_{\overline{w}}-\partial_w,\end{aligned}$$

and satisfy the following relations:

$$[E, L_{\pm}] = \pm 2iL_{\pm}, \quad [L_+, L_-] = -iE. \tag{5.29}$$

The Casimir operator is given by

$$\begin{aligned} d\pi_n(C) &= E^2 - 2[L_+L_- + L_-L_+] \\ &= (w\overline{w} - 1)[n^2 I + 2nw\partial_w - 2n\bar{w}\partial_{\overline{w}} + 4(w\overline{w} - 1)\partial_w\partial_{\overline{w}}]. \end{aligned} \tag{5.30}$$

The Casimir operator in the polar coordinate $w = re^{i\theta}$ is as follows:

$$d\pi_n(C) = (r^2 - 1)(n^2 I - 2in\partial_\theta) - (r^2 - 1)^2(\partial_r^2 + r^{-1}\partial_r + r^{-2}\partial_\theta^2). \tag{5.31}$$

Lemma 5.3.1. The operator (5.26) has two eigenfunctions:

1. For $m \neq 2, 4, 6, 8......$,

$$\begin{aligned} f_{-\frac{m}{2},n}(w, \overline{w}) = w^{-\frac{m}{2}}(1 - w\bar{w})^{\frac{1\pm\sqrt{1-\mu}}{2}} F\Big(&\frac{1}{2}[1 + n - m \pm \sqrt{1-\mu}], \\ &\frac{1}{2}[1 - n \pm \sqrt{1-\mu}], 1 - \frac{m}{2}, w\overline{w}\Big), \end{aligned} \tag{5.32}$$

2. For $m \neq -2, -4, -6, -8......$,

$$\begin{aligned} \tilde{f}_{-\frac{m}{2},n}(w, \overline{w}) = \overline{w}^{\frac{m}{2}}(1 - w\overline{w})^{\frac{1\pm\sqrt{1-\mu}}{2}} F\Big(&\frac{1}{2}[1 - n + m \pm \sqrt{1-\mu}], \\ &\frac{1}{2}[1 + n \pm \sqrt{1-\mu}], 1 + \frac{m}{2}, w\overline{w}\Big), \end{aligned} \tag{5.33}$$

where F is a hypergeometric function.

Proof. To find the eigenfunction of the subgroup K, we will solve the following partial differential equation by using the method of characteristic

$$[Ef](w, \overline{w}) = [-inI - 2iw\partial_w + 2i\overline{w}\partial_{\bar{w}}]f(w, \bar{w}) = 0.$$

We can write the characteristics for this equation as follows:

$$\frac{df}{inf} = \frac{dw}{-2iw} = \frac{d\overline{w}}{2i\overline{w}}.$$

$$\frac{dw}{-2iw} = \frac{d\bar{w}}{2i\bar{w}} \Rightarrow \quad C_1 = w\overline{w}.$$

We need to obtain another integral curve which involves f. This is possible from the following equation:

$$\frac{df}{inf} = \frac{dw}{-2iw}, \quad \text{we get} \quad C_2 = w^{\frac{n}{2}} f.$$

Then, the general solution of (5.3) is of the form $C_2 = \phi(C_1)$, that is

$$f(w, \overline{w}) = w^{-\frac{n}{2}} \phi(w\bar{w}).$$

Now, for $m \in \mathbb{Z}$ the eigenfunction is given by

$$f_{-\frac{m}{2}}(w, \overline{w}) = w^{-\frac{m}{2}} \phi(w\overline{w}), \tag{5.34}$$

which satisfy

$$[Ef_m](w, \overline{w}) = i(m-n) f_m(w, \overline{w}). \tag{5.35}$$

Therefore, the eigenvalue of the operator E is $m - n$.

Next,let $w = re^{i\theta}$. Then the eigenfunction (5.34) will be given by

$$f_{-\frac{m}{2}}(r, \theta) = (re^{i\theta})^{-\frac{m}{2}} \phi(r^2).$$

The Casimir operator (5.31) is applied to $f_{-\frac{m}{2}}(r, \theta)$

$$[d\pi_n(C) f_{-\frac{m}{2},n}](r, \theta) = (re^{i\theta})^{-\frac{m}{2}} [(r^2-1)(n^2 - nm)\phi(r^2) \\ - 2(r^2-1)^2((-m+2)\phi'(r^2) + 2r^2\phi''(r^2))]. \tag{5.36}$$

To find the value of ϕ in (5.34), we need to solve the differential equation

$$[d\pi_n(C)f](r, \theta) = \mu f(r, \theta).$$

That is

$$[(r^2-1)(n^2-nm) - \mu]\phi(r^2) - 2(r^2-1)^2(-m+2)\phi'(r^2) \\ - 4(r^2-1)^2 r^2 \phi''(r^2) = 0. \tag{5.37}$$

Let $x = r^2$, then we get

$$[(x-1)(n^2-nm) - \mu]\phi(x) - 2(x-1)^2(-m+2)\phi'(x) \\ - 4(x-1)^2 x^2 \phi''(x) = 0. \tag{5.38}$$

Now, let

$$\phi(x) = x^{\alpha}(1-x)^{\beta}\psi(x),$$

and by substituting in (5.38), we get

$$\alpha = \frac{m}{2} \text{ or } 0, \quad \beta = \frac{1 \pm \sqrt{1-\mu}}{2}$$

. Hence, we have two solutions:

1. $\phi(x) = (1-x)^{\frac{1\pm\sqrt{1-\mu}}{2}}\psi(x)$,

2. $\phi(x) = x^{\frac{m}{2}}(1-x)^{\frac{1\pm\sqrt{1-\mu}}{2}}\psi(x)$.

By substituting the first solution in the differential equation(5.38), we get

$$x(1-x)\psi''(x) + \left(1 - \frac{m}{2} - (1 \pm \sqrt{1-\mu} + 1 - \frac{m}{2})x\right)\psi'(x) + \left[\frac{\mu}{2} + \left(\frac{-1 \mp \sqrt{1-\mu}}{2}\right)\left(1 - \frac{m}{2}\right) + \frac{1}{4}\left(n^2 - nm\right)\right]\psi(x) = 0. \quad (5.39)$$

This is a hypergeometric differential equation of the form

$$x(1-x)\psi''(x) + [c - (a+b+1)x]\psi'(x) - ab\psi(x) = 0.$$

By simple calculation, we get

$$a = \frac{1}{2}[1 + n - m \pm \sqrt{1-\mu}],$$
$$b = \frac{1}{2}[1 - n \pm \sqrt{1-\mu}],$$
$$c = 1 - \frac{m}{2}.$$

Then $\psi(x) = F(a,b,c,x)$, and the solution of (5.37) is

$$\phi(r^2) = (1-r^2)^{\frac{1\pm\sqrt{1-\mu}}{2}} F(a,b,c,r^2).$$

The hypergeometric function is given by

$$F(a,b,c,r^2) = 1 + \sum_{k=1}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{(r^2)^k}{k!}.$$

Finally, the eigenfunction is given by

$$f_{-\frac{m}{2},n}(w,\overline{w}) = w^{-\frac{m}{2}}(1-w\overline{w})^{\frac{1\pm\sqrt{1-\mu}}{2}} F(\frac{1}{2}[1+n-m\pm\sqrt{1-\mu}], \frac{1}{2}[1-n\pm\sqrt{1-\mu}], 1-\frac{m}{2}, w\overline{w}),$$

where $m \neq 2, 4, 6, 8.......$

Following the same calculation for the second solution, we get the eigenfunction

$$\tilde{f}_{-\frac{m}{2},n}(w,\overline{w}) = \overline{w}^{\frac{m}{2}}(1-w\bar{w})^{\frac{1\pm\sqrt{1-\mu}}{2}}F(\frac{1}{2}[1-n+m\pm\sqrt{1-\mu}],$$
$$\frac{1}{2}[1+n\pm\sqrt{1-\mu}],1+\frac{m}{2},w\overline{w}),$$

where $m \neq -2,-4,-6,-8.......$ □

The commutator relation $[E, L_{\pm}] = \pm 2iL_{\pm}$, implies that

$$[Ef_{-\frac{m}{2},n}(w,\bar{w})] = \mathrm{e}^{(m-n)i\theta} f_{-\frac{m}{2},n}(w,\bar{w}),$$

hence $E = (m-n)i$. Also, by using the relation $[L_+, L_-] = -iE$, and (5.30), we get the following identities:

$$4L_+L_- = E^2 - 2iE - d\pi_n(C),$$
$$4L_-L_+ = E^2 + 2iE - d\pi_n(C).$$

Then, for $d\pi_n(C) = \mu I$,

$$L_+L_- = -\frac{1}{4}[(m-n-1)^2 + \mu - 1], \tag{5.40}$$
$$L_-L_+ = -\frac{1}{4}[(m-n+1)^2 + \mu - 1]. \tag{5.41}$$

Now, since $L_+^* = -L_-$, we find that

$$\begin{aligned} \|L_-\| = \|L_-^*L_-\|^{\frac{1}{2}} &= \|-L_+L_-\|^{\frac{1}{2}} \\ &= \frac{1}{2}[(m-n-1)^2 + \mu - 1]^{\frac{1}{2}}. \end{aligned} \tag{5.42}$$

Similarly,

$$\|L_+\| = \frac{1}{2}[(m-n+1)^2 + \mu - 1]^{\frac{1}{2}}. \tag{5.43}$$

Let $1-\mu = (n-1)^2$, where n is an integer. The functions (5.32) are given by

$$f_{-\frac{m}{2},n}(w,\overline{w}) = w^{\frac{-m}{2}}(1-w\overline{w})^{\frac{n}{2}}. \tag{5.44}$$

Proposition 5.3.2. The functions $f_{-\frac{m}{2},n}(w,\overline{w}) = w^{\frac{-m}{2}}(1-w\overline{w})^{\frac{n}{2}}$, are L_2 summable for $n > 1$ and $m \leq 0$.

Proof. Let $w = r\mathrm{e}^{i\theta}$, then $f_{-\frac{m}{2},n}(r\mathrm{e}^{i\theta}, r\mathrm{e}^{-i\theta}) = (r\mathrm{e}^{i\theta})^{\frac{-m}{2}}(1-r^2)^{\frac{n}{2}}$. The measure is

$d\mu = \frac{rdr\wedge d\theta}{(1-r^2)^2}$. Then,

$$\begin{aligned}
\|f_{-\frac{m}{2},n}\|^2 &= \int_0^{2\pi}\int_0^1 |f_{-\frac{m}{2},n}(re^{i\theta}, re^{-i\theta})|^2 \frac{rdr\wedge d\theta}{(1-r^2)^2} \\
&= \int_0^{2\pi}\int_0^1 \left|(re^{i\theta})^{\frac{-m}{2}}(1-r^2)^{\frac{n}{2}}\right|^2 \frac{rdr\wedge d\theta}{(1-r^2)^2} \\
&= \int_0^{2\pi}\int_0^1 r^{-m}(1-r^2)^{n-2} rdrd\theta \\
&\leq \pi\int_0^1 (1-r^2)^{n-2}2rdr, \quad \text{for} \quad m \leq 0 \\
&= -\pi\frac{(1-r^2)^{n-1}}{n-1}\Big|_0^1 \\
&= \frac{\pi}{n-1} < \infty.
\end{aligned}$$

Hence, $f_{-\frac{m}{2},n}$ are L_2 summable if $n > 1$ and $m \leq 0$. □

By simple calculation, we get that,

$$[L_+ f_{-\frac{m}{2},n}](w,\overline{w}) = \left(n - \frac{m}{2}\right) f_{-\frac{m}{2}+1,n}(w,\overline{w}), \tag{5.45}$$

$$[L_- f_{-\frac{m}{2},n}](w,\overline{w}) = \frac{m}{2} f_{-\frac{m}{2}-1,n}(w,\overline{w}). \tag{5.46}$$

At $m = 0$, we have the function $f_{0,n}(w,\overline{w}) = (1-w\overline{w})^{\frac{n}{2}}$. Then $L_- f_{0,n}(w,\overline{w}) = 0$, which means that $f_{0,n}(w,\overline{w})$ is the vacuum of the operator L_-. This is represented by the following diagram:

$$0 \xleftarrow[L_-]{} f_{0,n} \underset{L_-}{\overset{L_+}{\rightleftarrows}} f_{1,n} \underset{L_-}{\overset{L_+}{\rightleftarrows}} f_{2,n} \underset{L_-}{\overset{L_+}{\rightleftarrows}} \cdots$$

Next, let $1-\mu = (n+1)^2$, then the functions (5.33) are given by

$$\tilde{f}_{-\frac{m}{2},n}(w,\overline{w}) = \overline{w}^{\frac{m}{2}}(1-w\overline{w})^{\frac{-n}{2}}, \tag{5.47}$$

which are L_2 summable for $n < -1$ and $m \geq 0$; that is

$$\|\tilde{f}_{-\frac{m}{2},n}\|^2 = \int_{\mathbb{D}} \left|\tilde{f}_{-\frac{m}{2},n}(w,\overline{w})\right|^2 \frac{dw\wedge d\overline{w}}{(1-|w|^2)} < \infty, \quad n < -1.$$

For the function (5.47), the ladder operators are given by:

$$[L_+\tilde{f}_{-\frac{m}{2},n}](w,\overline{w}) = \frac{-m}{2}\tilde{f}_{-(\frac{m}{2}+1),n}(w,\overline{w}), \tag{5.48}$$

$$[L_-\tilde{f}_{-\frac{m}{2},n}](w,\overline{w}) = \left(\frac{m}{2}-n\right)\tilde{f}_{-(\frac{m}{2}-1),n}(w,\overline{w}). \tag{5.49}$$

At $m = 0$, we get the function $\tilde{f}_{0,n}(w,\overline{w}) = (1-w\overline{w})^{\frac{-n}{2}}$. We can then see that $[L_+\tilde{f}_{0,n}](w,\overline{w}) = 0$, which means that $\tilde{f}_{0,n}$ is the vacuum of the operator L_+. This is represented by the following diagram:

$$\cdots \underset{L_-}{\overset{L_+}{\rightleftarrows}} \tilde{f}_{-2,n} \underset{L_-}{\overset{L_+}{\rightleftarrows}} \tilde{f}_{-1,n} \underset{L_-}{\overset{L_+}{\rightleftarrows}} \tilde{f}_{0,n} \xrightarrow{L_+} 0$$

Proposition 5.3.3. The Lie derivatives of the representation π_n are

$$\mathfrak{L}^{\tilde{Z}} = -\partial_\theta, \tag{5.50}$$

$$\mathfrak{L}^{\tilde{A}} = \frac{-r}{2}\sin(\phi-2\theta)\partial_\theta - \frac{1}{2}(1-u\bar{u})\left[e^{2i\theta}\partial_u + e^{-2i\theta}\partial_{\bar{u}}\right], \tag{5.51}$$

$$\mathfrak{L}^{\tilde{B}} = \frac{r}{2}\cos(\phi-2\theta)\partial_\theta - \frac{i}{2}(1-u\bar{u})\left[e^{2i\theta}\partial_u - e^{-2i\theta}\partial_{\bar{u}}\right]. \tag{5.52}$$

The right ladder operators are then represented by

$$\mathfrak{L}_+ = \mathfrak{L}^{\tilde{A}+i\tilde{B}} = e^{-2i\theta}\left[\frac{i}{2}u\partial_\theta - (1-u\bar{u})\partial_{\bar{u}}\right], \tag{5.53}$$

$$\mathfrak{L}_- = \mathfrak{L}^{\tilde{A}-i\tilde{B}} = -e^{2i\theta}\left[\frac{i}{2}\bar{u}\partial_\theta + (1-u\bar{u})\partial_u\right]. \tag{5.54}$$

Proof. The Lie derivative $\mathfrak{L}^X$ for an element X of the Lie algebra $\mathfrak{su}(1,1)$ is given by

$$[\mathfrak{L}^X F](g) = \frac{d}{dt}F(g\exp tX)|_{t=0}, \tag{5.55}$$

for any differentiable function F on $\mathrm{SU}(1,1)$ and $g = \begin{pmatrix} \alpha & \bar{\beta} \\ \beta & \bar{\alpha} \end{pmatrix}$.

We know that the space $L_2^{\chi_n}(\mathrm{SU}(1,1))$ consists of the functions $F_n : \mathrm{SU}(1,1) \to \mathbb{C}$ with the property

$$F_n\left[\begin{pmatrix} \alpha & \overline{\beta} \\ \beta & \overline{\alpha} \end{pmatrix}\right] = \chi_n\left(\frac{\alpha}{|\alpha|}\right) F\left(\frac{\overline{\beta}}{\overline{\alpha}}, \frac{\beta}{\alpha}\right),$$

where $F \in L_2(\mathbb{D})$.

Hence, for $v = \frac{\alpha}{|\alpha|} = \mathrm{e}^{i\theta}$ and $u = \frac{\overline{\beta}}{\bar{\alpha}} = r\mathrm{e}^{i\phi}$, we have

$$\begin{aligned}[\mathfrak{L}^X F_n](g) &= \frac{d}{dt} F_n(g \exp tX)\Big|_{t=0} \\ &= \frac{d}{dt}\chi_n(v(t))F\left(u(t), \bar{u}(t)\right)\Big|_{t=0} \\ &= \frac{\partial \chi_n}{\partial v}\frac{dv(t)}{dt}\Big|_{t=0} + \frac{\partial F}{\partial u}\frac{du(t)}{dt}\Big|_{t=0} + \frac{\partial F}{\partial \bar{u}}\frac{d\bar{u}(t)}{dt}\Big|_{t=0}.\end{aligned} \tag{5.56}$$

From section 5.1 we have $\tilde{Z}$, $\tilde{A}$ and $\tilde{B} \in \mathfrak{su}(1,1)$ given by (5.7). Then, the Lie derivatives corresponding to the subgroups $\exp t\tilde{Z}$,(5.8), $\exp t\tilde{A}$,(5.9) and $\exp t\tilde{B}$(5.10) are obtained through the differentiation of the right action of these subgroups as follows:

$$\begin{aligned}[\mathfrak{L}^{\tilde{Z}} F_n](g) &= \frac{d}{dt} F_n(g \exp t\tilde{Z})\Big|_{t=0} \\ &= \frac{d}{dt} F_n \begin{pmatrix} \alpha \mathrm{e}^{it} & \bar{\beta}\mathrm{e}^{-it} \\ \beta \mathrm{e}^{it} & \bar{\alpha}\mathrm{e}^{-it} \end{pmatrix}\Big|_{t=0} \\ &= \frac{d}{dt}\chi_n\left(\frac{\alpha \mathrm{e}^{it}}{|\alpha \mathrm{e}^{it}|}\right) F\left(\frac{\bar{\beta}\mathrm{e}^{-it}}{\bar{\alpha}\mathrm{e}^{-it}}, \frac{\beta \mathrm{e}^{-it}}{\alpha \mathrm{e}^{-it}}\right)\Big|_{t=0} \\ &= \frac{d}{dt}\chi_n(\mathrm{e}^{i(\theta+t)})F(u,\bar{u})\Big|_{t=0} \\ &= -\frac{\partial F}{\partial_\theta},\end{aligned}$$

where $\alpha = \frac{\mathrm{e}^{i\theta}}{\sqrt{1-|r|^2}}$ and $\beta = \frac{r\mathrm{e}^{i(\theta-\phi)}}{\sqrt{1-|r|^2}}$.

Similarly, it is easy to find that

$$\begin{aligned}[\mathfrak{L}^{\tilde{A}} F_n](g) &= \frac{d}{dt} F_n(g \exp t\tilde{A})\Big|_{t=0} \\ &= \frac{-r}{2}\sin(\phi - 2\theta)\frac{\partial F}{\partial_\theta} - \frac{1}{2}(1-u\bar{u})\left[\mathrm{e}^{2i\theta}\frac{\partial F}{\partial_u} + \mathrm{e}^{-2i\theta}\frac{\partial F}{\partial_{\bar{u}}}\right].\end{aligned}$$

$$\begin{aligned}[\mathfrak{L}^{\tilde{B}} F_n](g) &= \frac{d}{dt} F_n(g \exp t\tilde{B})\Big|_{t=0} \\ &= \frac{r}{2}\cos(\phi - 2\theta)\frac{\partial F}{\partial_\theta} - \frac{i}{2}(1-u\bar{u})\left[\mathrm{e}^{2i\theta}\frac{\partial F}{\partial_u} - \mathrm{e}^{-2i\theta}\frac{\partial F}{\partial_{\bar{u}}}\right].\end{aligned}$$

□

The function $f_{-\frac{m}{2},n}$ given by (5.44) is an eigenfunction with an eigenvalue in for the operator $\mathfrak{L}^{\tilde{Z}}$. That is, for

$$F_n(g) = \mathrm{e}^{int} f_{-\frac{m}{2},n}(w,\overline{w}),$$

we have

$$\mathfrak{L}^{\tilde{Z}} e^{int} f_{-\frac{m}{2},n}(w,\overline{w}) = ine^{int} f_{-\frac{m}{2},n}(w,\overline{w}). \tag{5.57}$$

Moreover, $\tilde{f}_{-\frac{m}{2},n}$ (5.47) is an eigenfunction with eigenvalue in for the operator $\mathfrak{L}^{\tilde{Z}}$.

Lemma 5.3.4. We have

$$\mathfrak{L}^{\tilde{A}\pm i\tilde{B}} : f_{-\frac{m}{2},n} \to f_{-\frac{m}{2},n\pm 2},$$

and

$$\mathfrak{L}^{\tilde{A}\pm i\tilde{B}} : \tilde{f}_{-\frac{m}{2},n} \to \tilde{f}_{-\frac{m}{2},n\pm 2}.$$

Proof. From the commutator relations $[\mathfrak{L}^{\tilde{Z}}, \mathfrak{L}^{\tilde{A}\pm i\tilde{B}}] = \pm 2i\mathfrak{L}^{\tilde{A}\pm i\tilde{B}}$, for the eigenfunction $f_{-\frac{m}{2},n}$ given by (5.44), we can see that

$$\begin{aligned}
[\mathfrak{L}^{\tilde{Z}} \mathfrak{L}^{\tilde{A}\pm i\tilde{B}}] e^{int} f_{-\frac{m}{2},n} &= \mathfrak{L}^{\tilde{A}\pm i\tilde{B}} (\mathfrak{L}^{\tilde{Z}} e^{int} f_{-\frac{m}{2},n}) \pm 2i\mathfrak{L}^{\tilde{A}\pm i\tilde{B}} e^{int} f_{-\frac{m}{2},n} \\
&= \mathfrak{L}^{\tilde{A}\pm i\tilde{B}} (nie^{int} f_{-\frac{m}{2},n}) \pm 2i\mathfrak{L}^{\tilde{A}\pm i\tilde{B}} e^{int} f_{-\frac{m}{2},n} \\
&= (n \pm 2) i\mathfrak{L}^{\tilde{A}\pm i\tilde{B}} e^{int} f_{-\frac{m}{2},n}.
\end{aligned} \tag{5.58}$$

Similarly,for the eigenfunction $\tilde{f}_{-\frac{m}{2},n}$ (5.47), we have

$$[\mathfrak{L}^{\tilde{Z}} \mathfrak{L}^{\tilde{A}\pm i\tilde{B}}] e^{int} \tilde{f}_{-\frac{m}{2},n} = (n \pm 2) i\mathfrak{L}^{\tilde{A}\pm i\tilde{B}} e^{int} \tilde{f}_{-\frac{m}{2},n}.$$

□

The vacuum $f_{0,n}(w,\overline{w}) = (1 - w\overline{w})^{\frac{n}{2}}$ is annihilated by the operator $\mathfrak{L}^{\tilde{A}+i\tilde{B}}$. That is, $[\mathfrak{L}^{\tilde{A}+i\tilde{B}} e^{int} f_{0,n}](w,\overline{w}) = 0$. Then, all the vectors $f_{j,n} = (L_+)^j f_{0,n}$ are vacuums of the operator $\mathfrak{L}^{\tilde{A}+i\tilde{B}}$ due to the commutation of the left and right actions:

$$\begin{aligned}
\mathfrak{L}^{\tilde{A}+i\tilde{B}} f_{j,n} &= \mathfrak{L}^{\tilde{A}+i\tilde{B}} (L_+)^j f_{0,n} \\
&= (L_+)^j \mathfrak{L}^{\tilde{A}+i\tilde{B}} f_{0,n} = 0.
\end{aligned} \tag{5.59}$$

For each vacuum $f_{0,n}$, the collection of vectors $f_{j,n} = (L_+)^j f_{0,n}$ forms an orthogonal basis of an irreducible component with the respective ladder operators (5.45) and (5.46). The left and the right actions for the eigenfunctions $f_{m,n}$ (5.44) jointly create the two-dimensional lattice structure that can be seen in the following diagram:

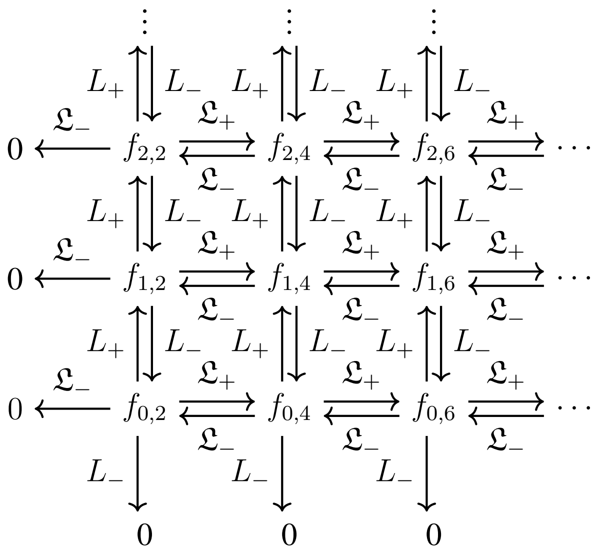

Figure 5.1: The left and the right actions of the ladder operators for $f_{\frac{-m}{2},n}$.

Furthermore, the function $\tilde{f}_{0,n}(w,\overline{w}) = (1-w\overline{w})^{\frac{-n}{2}}$ is a vacuum of the operator $\mathfrak{L}^{\tilde{A}-i\tilde{B}}$. That is, $[\mathfrak{L}^{\tilde{A}-i\tilde{B}}\mathrm{e}^{int}\tilde{f}_{0,n}](w,\overline{w}) = 0$. Then, all the vectors $\tilde{f}_{k,n} = (L_-)^k\tilde{f}_{0,n}$ are vacuums of the operator $\mathfrak{L}^{\tilde{A}-i\tilde{B}}$ due to the commutation of the left and right actions:

$$\begin{aligned} \mathfrak{L}^{\tilde{A}-i\tilde{B}}\tilde{f}_{k,n} &= \mathfrak{L}^{\tilde{A}-i\tilde{B}}(L_-)^k f_{0,n} \\ &= (L_-)^k\mathfrak{L}^{\tilde{A}-i\tilde{B}}\tilde{f}_{0,n} = 0. \end{aligned} \tag{5.60}$$

For each $\tilde{f}_{0,n}$, the collection of vectors $\tilde{f}_{k,n} = (L_-)^k f_{0,n}$ forms an orthogonal basis of an irreducible component with the respective ladder operators (5.48) and (5.49). The left and the right actions for the functions $\tilde{f}_{m,n}$ (5.47) jointly create the two-dimensional lattice structure that can be seen in the following diagram:

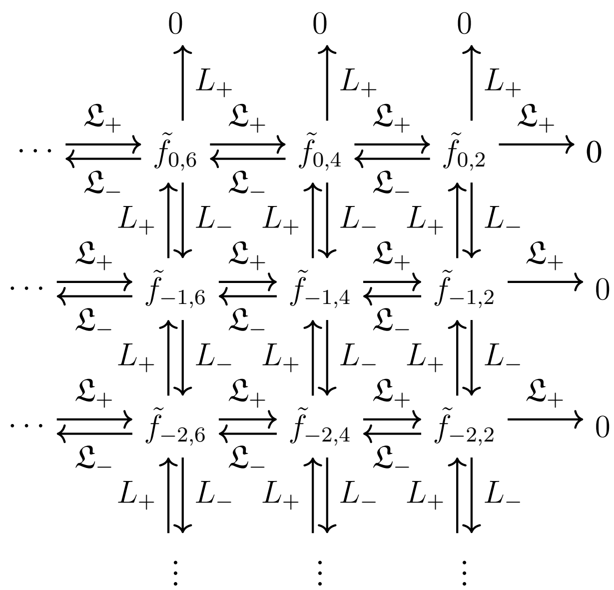

Figure 5.2: The left and the right actions of the ladder operators for $\tilde{f}_{\frac{-m}{2},n}$.

5.4 Representation on the Dirichlet Space

The Dirichlet space, the Hardy space and the Bergman space are the three classical spaces of holomorphic functions in the unit disc. In the present section, we find the $\mathfrak{su}(1,1)$ module (which is the space of the derived representation) on the Dirichlet space.

Definition 5.4.1. [7] The Dirichlet space $\mathcal{D}$ on the unit disc $\mathbb{D} = \{w : |w| < 1\}$ consists of the holomorphic functions $f(w)$ on $\mathbb{D}$ for which the following semi-norm is finite:

$$\mathcal{D}(f) := \left(\frac{1}{\pi} \int_{\mathbb{D}} |f'(w)|^2 dx dy \right)^{\frac{1}{2}}, \quad w = x + iy. \tag{5.61}$$

Definition 5.4.2. For $g = \begin{pmatrix} \alpha & \overline{\beta} \\ \beta & \overline{\alpha} \end{pmatrix} \in SU(1,1)$, the $\mathrm{SU}(1,1)$ representation on the Dirichlet space be defined by

$$[\breve{\pi}_0(g) f](w) = f\left(\frac{\overline{\alpha} w - \overline{\beta}}{\alpha - \beta z} \right). \tag{5.62}$$

The semi-norm $\mathcal{D}(f)$ is not a norm because $\mathcal{D}(f) = 0$ whenever f is a constant. Then, $\breve{\pi}_0$ is a non-unitary representation.

In the following, we present the $\mathfrak{su}(1,1)$ module for the representation $\breve{\pi}_0$ on the Dirichlet space. First, the representation $\breve{\pi}_0$ (5.62) is the $\mathrm{SU}(1,1)$ representation $\breve{\pi}_n$, for $n=0$. The representation $\breve{\pi}_n$ is given as follows:

$$[\breve{\pi}_n(g)f](w) = f\left(\frac{\overline{\alpha}w-\overline{\beta}}{\alpha-\beta w}\right)(\alpha-\beta w)^{-n}, \tag{5.63}$$

where $n \in \mathbb{Z}$. The derived representations for the basis $\{\tilde{Z}, \tilde{A}, \tilde{B}\}$ (5.7) are given in the following:

$$\begin{aligned} E = d\breve{\pi}_n^{\tilde{Z}} &= \frac{d}{dt}\breve{\pi}_n(\mathrm{e}^{t\tilde{Z}})f(w)|_{t=0} = [-inI - 2iw\partial_w]f(w),\\ A_1 = d\breve{\pi}_n^{\tilde{A}} &= \frac{d}{dt}\breve{\pi}_n(\mathrm{e}^{t\tilde{A}})f(w)|_{t=0} = \frac{i}{2}[nwI + (1+w^2)\partial_w]f(w),\\ B_1 = d\breve{\pi}_n^{\tilde{B}} &= \frac{d}{dt}\breve{\pi}_n(\mathrm{e}^{t\tilde{B}})f(w)|_{t=0} = \frac{1}{2}[nwI + (w^2-1)\partial_w]f(w). \end{aligned}$$

The commutator relations are

$$[E, A_1] = 2B_1,\ [E, B_1] = -2A_1,\ [A_1, B_1] = -\frac{1}{2}E.$$

The ladder operators are defined as

$$L_+ = A_1 + iB_1 = inwI + iw^2\partial_w, \quad L_- = A_1 - iB_1 = i\partial_w,$$

and

$$[E, L_+] = -2iL_+,\ [E, L_-] = 2iL_- \quad \text{and} \quad [L_+, L_-] = iE. \tag{5.64}$$

The Casimir operator is

$$d\breve{\pi}_n(C) = d\breve{\pi}_n(\tilde{Z}^2 - 4\tilde{A}^2 - 4\tilde{B}^2) = -n^2 + 2n. \tag{5.65}$$

The representation $\breve{\pi}_n$ on $L^2(\mathbb{D})$ is irreducible, and V_{n+2m} is the one-dimensional subspace generated by $w_{n,m}$ [33]. Indeed,

$$\breve{\pi}_n\left(\begin{pmatrix} \mathrm{e}^{i\theta} & 0 \\ 0 & \mathrm{e}^{-i\theta} \end{pmatrix}\right)(w_{n,m}) = \mathrm{e}^{-i\theta(n+2m)}w_{n,m}.$$

Hence, V_{n+2m} is an eigenspace of K with eigenvalue $\mathrm{e}^{-i\theta(n+2m)}$, which is the character of the subgroup K. Then

$$\breve{\pi}_n(\exp t\tilde{Z}) = \mathrm{e}^{-i(n+2m)t}I \quad \text{on} \quad V_{n+2m},$$

and the derived representation is given by

$$E = -i(n+2m)I \quad \text{on} \quad V_{n+2m}.$$

From the commutator relation (5.64), we have

$$\begin{aligned} E(L_+ w_{n,m}) &= L_+(E w_{n,m}) - 2iL_+ w_{n,m} \\ &= L_+(-i(n+2m)) - 2iL_+ = -i(n+2m+2)L_+, \\ E(L_- w_{n,m}) &= L_-(E w_{n,m}) + 2iL_- w_{n,m} \\ &= L_-(-i(n+2m)) + 2iL_+ = -i(n+2m-2)L_-. \end{aligned}$$

Therefore, the ladder operator $L_\pm$ acts as follows:

$$L_+ : V_{n+2m} \to V_{n+2m+2}, \quad L_- : V_{n+2m} \to V_{n+2m-2}.$$

$$\cdots \underset{L_-}{\overset{L_+}{\rightleftarrows}} V_{n+2m-2} \underset{L_-}{\overset{L_+}{\rightleftarrows}} V_{n+2m} \underset{L_-}{\overset{L_+}{\rightleftarrows}} V_{n+2m+2} \underset{L_-}{\overset{L_+}{\rightleftarrows}} \cdots$$

$V_{n+2m} = \{w_{n,m} : m = 0, 1, 2, 3.....\}$ is the lowest weight module and is given as follows:

$$\begin{aligned} E w_{n,m} &= -(n+2m)i w_{n,m}, \\ L_+ w_{n,m} &= A_1 w_{n,m} + iB_1 w_{n,m} = (n+m)i w_{n,m+1}, \quad m \in \mathbb{Z}_+ - \{0\}, \\ L_- w_{n,m} &= A_1 w_{n,m} - iB_1 w_{n,m} = mi w_{n,m-1}, \quad m \in \mathbb{Z}_+ - \{0\} \\ L_- w_{n,0} &= 0, \\ d\breve{\pi}_n(C) w &= (-n^2 + 2n) w, \quad w \in V_{n+2m}. \end{aligned}$$

The vector $w_{n,0}$ is called the lowest weight vector.

$$0 \underset{L_-}{\longleftarrow} w_{n,0} \underset{L_-}{\overset{L_+}{\rightleftarrows}} w_{n,1} \underset{L_-}{\overset{L_+}{\rightleftarrows}} w_{n,2} \underset{L_-}{\overset{L_+}{\rightleftarrows}} \cdots$$

$\bar{V}_{n+2m} = \{w_{n,m} : m = 0, 1, 2, 3.....\}$ is the highest weight module and is given as follows:

$$\begin{aligned}
Ew_{n,m} &= -(n-2m)iw_{n,m}, \\
L_{-}w_{n,m} &= A_1 w_{n,m} + iB_1 w_{n,m} = i(n+m)w_{n,m-1}, \quad m \in \mathbb{Z}_{+} - \{0\}, \\
L_{+}w_{n,m} &= A_1 w_{n,m} - iB_1 w_{n,m} = imw_{n,m+1}, \quad m \in \mathbb{Z}_{+} - \{0\} \\
L_{+}w_{n,0} &= 0, \\
d\breve{\pi}_n(C)w &= (-n^2+2n)w, \quad w \in V_{n-2m}.
\end{aligned}$$

The vector $w_{n,0}$ is called the highest weight vector.

$$\cdots \underset{L_-}{\overset{L_+}{\rightleftarrows}} w_{n,2} \underset{L_-}{\overset{L_+}{\rightleftarrows}} w_{n,1} \underset{L_-}{\overset{L_+}{\rightleftarrows}} w_{n,0} \xrightarrow[L_+]{} 0$$

The vector module V_{n+2m} is unitarizable if and only if $n > 0$, and $\bar{V}_{n+2m}$ is unitarizable if and only if $n < 0$ [20, p.96].

Next, for the Dirichlet space the $\mathfrak{su}(1,1)$ vector module is V_{0+2m}, that is given as follows:

$$\begin{aligned}
Ew_{0,m} &= -2imw_{0,m}, \\
L_{+}w_{0,m} &= imw_{0,m+1}, \quad m \in \mathbb{Z}_{+} - \{0\}, \\
L_{+}w_{0,0} &= 0, \\
L_{-}w_{0,m} &= imw_{0,m-1}, \quad m \in \mathbb{Z}_{+} - \{0\}, \\
L_{-}w_{0,0} &= 0, \\
d\breve{\pi}_0(C) &= 0.
\end{aligned}$$

$$0 \xleftarrow[L_-]{} [w_{0,0}] \xleftarrow[L_-]{} w_{0,1} \underset{L_-}{\overset{L_+}{\rightleftarrows}} w_{0,2} \underset{L_-}{\overset{L_+}{\rightleftarrows}} \cdots$$

In addition, $w_{0,0}$ is the highest weight vector for the vector module $\bar{V}_{0+2m}$ that is present in the following:

$$\cdots \underset{L_-}{\overset{L_+}{\rightleftarrows}} w_{0,2} \underset{L_-}{\overset{L_+}{\rightleftarrows}} w_{0,1} \xrightarrow{L_+} [w_{0,0}] \xrightarrow[L_+]{} 0$$

5.5 Mock Discrete Series Representation

The discrete series representations for the group $\mathrm{SL}_2(\mathbb{R})$ given by

$$\pi_n(g)\varphi(z) = \varphi\left(\frac{dz-b}{a-cz}\right)(a-cz)^{-n}, \quad n \in \mathbb{Z}. \tag{5.66}$$

is on the Bergman space where $n \geq 2$ [10, 18, 25]. Lang [33, IX] studies the discrete series on the group $\mathrm{SL}(\mathbb{R})$ in the upper half-plane and on the unit disc.

For $n = 1$, the $\mathrm{SL}_2(\mathbb{R})$ representation is called the mock discrete series. The representation space of the mock discrete series is the Hardy space [10, 18, 25]. The aim of this section is to consider the mock discrete series in the real line, on the unit circle and on the group $\mathrm{SL}_2(\mathbb{R})$.

5.5.1 Representation in the Real Line

In this subsection, we study the mock discrete representation on the space $L_2(\mathbb{R})$.

Theorem 5.5.1. Let $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R})$. We define

$$[\pi_1(g)f](x) = f(g^{-1}x)(a-cx)^{-1}, \tag{5.67}$$

on the space $L_2(\mathbb{R})$. Then π_1 is a unitary representation.

Proof. π_1 is a representation because $\pi_1(g_1g_2) = \pi_1(g_1)\pi_2(g_2)$ for all $g_1, g_2 \in SL_2(\mathbb{R})$.

To verify the unitary property for all $g \in SL_2(\mathbb{R})$, we have

$$\begin{aligned} \|\pi_1(g)f\|^2_{H_2(\mathbb{R})} &= \int_{\mathbb{R}} |\pi_1(g)f(x)|^2 dx \\ &= \int_{\mathbb{R}} |f(g^{-1}x)(a-cx)^{-1}|^2 dx \\ &= \int_{\mathbb{R}} |f(g^{-1}x)|^2 |a-cx|^{-2} dx \\ &= \int_{\mathbb{R}} |f(t)| dt \\ &= \|f\|^2_{L_2(\mathbb{R})}, \end{aligned}$$

where $t = g^{-1}x = \frac{dx-b}{a-cx}$ and $dt = (a-cx)^{-2}dx$. □

Lemma 5.5.2. Let m be a non-negative integer and let

$$\psi_m(x) = \sqrt{2}\left(\frac{x-i}{x+i}\right)^m (x+i)^{-1}. \tag{5.68}$$

Then $\psi_m \in L_2(\mathbb{R})$.

Proof. To prove this, we will show that $\|\psi_m(x)\|^2_{L_2(\mathbb{R})}$ is finite.

$$\begin{aligned} \|\psi_m(x)\|^2_{H_2(\mathbb{R})} &= \int_{\mathbb{R}} |\psi_m(x)|^2 dx \\ &= \int_{\mathbb{R}} \left|\sqrt{2}\left(\frac{x-i}{x+i}\right)^m (x+i)^{-1}\right|^2 dx \\ &= 2\int_{\mathbb{R}} \frac{1}{|x+i|^2} dx \\ &= 2\int_{-\infty}^{\infty} \frac{1}{1+x^2} dx \\ &= 2\pi. \end{aligned}$$

Hence, $\psi_m \in L_2(\mathbb{R})$. □

Theorem 5.5.3. The representation π_1 on $L_2(\mathbb{R})$ is irreducible. Let V_{1+2m} be the one-dimensional subspace generated by the function ψ_m. Then, V_{1+2m} is an eigenspace of K, and

$$H = \widehat{\bigoplus}_{n\geq 0} V_{1+2m},$$

is an orthogonal decomposition, with the lowest weight vector ψ_0 of weight 1.

Proof. To prove this theorem, we will change the representation under the isomorphism between the real line and the unit circle. This will be shown at the end of the next subsection. □

5.5.2 Representation on the Unit Circle

In this subsection, we review the mock discrete representation on the space $L_2(\mathbb{T})$, where $\mathbb{T}$ is the unit circle. That is, $\mathbb{T} = \{z \in \mathbb{C} : |z|= 1\} = \{e^{i\theta} : 0 \leq \theta \leq 2\pi\}$. The Cayley transformation between the unit circle and the real line is defined by

$$w = \frac{x-i}{x+i}, \tag{5.69}$$

where $w \in \mathbb{T}$ and $x \in \mathbb{R}$. The inverse mapping is:

$$x = -i\frac{w+1}{w-1}.$$

For a function f on the real line, let us define the map

$$T_1 f(w) = f\left(-i\frac{w+1}{w-1}\right)\left(\frac{-\sqrt{2}i}{w-1}\right). \tag{5.70}$$

Lemma 5.5.4. The map $T_1 : L^2(\mathbb{R}) \to L^2(\mathbb{T})$ is an isometry.

Proof. By changing the variable we get the following:

$$\begin{aligned}
\|T_1 f\|^2_{L^2(\mathbb{T})} &= \int_{\mathbb{T}} |T_1 f(e^{i\theta})|^2 d\theta \\
&= \int_{\mathbb{T}} |f\left(-i\frac{e^{i\theta}+1}{e^{i\theta}-1}\right)\left(\frac{-\sqrt{2}i}{e^{i\theta}-1}\right)|^2 d\theta \\
&= \int_{\mathbb{R}} |f(x)|^2 dx \\
&= \|f\|^2_{L^2(\mathbb{R})},
\end{aligned}$$

where $x = -i\frac{e^{i\theta}+1}{e^{i\theta}-1}$. □

Theorem 5.5.5. The functions $\{1, w, w^2, ...\}$ forms an orthonormal basis for $L^2(\mathbb{T})$.

Proof. From (5.68) and (5.70) we get the following functions on the unit circle:

$$T_1\psi_m(w) = w^m,$$

which satisfy the orthonormality property. That is,

$$\langle w^n, w^m \rangle = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{e}^{in\theta}\mathrm{e}^{-im\theta}d\theta = \begin{cases} 1, & m = n \\ 0, & m \neq n \end{cases}.$$

□

As we mentioned at the beginning, the action of the group $\mathrm{SL}_2(\mathbb{R})$ is transformed from the real line to the unit circle by (5.69). Also, the elements of $\mathrm{SL}_2(\mathbb{R})$ can be in $\mathrm{SU}(1,1)$ by the identity (5.2). Thus, our representation π_1 in (5.67) transforms to

$$\tilde{\pi_1}(g)f(w) = f\left(\frac{\overline{\alpha}w - \overline{\beta}}{\alpha - \beta w}\right)(\alpha - \beta w)^{-1}. \tag{5.71}$$

Similar to Theorem 5.5.3, $\tilde{H} = L^2(\mathbb{T})$ and $\tilde{V}_{1+2m}$ is the one-dimensional subspace generated by w^m and

$$\tilde{\pi_1}\left(\begin{pmatrix} \mathrm{e}^{i\theta} & 0 \\ 0 & \mathrm{e}^{-i\theta} \end{pmatrix}\right)(w^m) = \mathrm{e}^{-i\theta(1+2m)}w^m. \tag{5.72}$$

Hence, $\tilde{V}_{1+2m}$ is an eigenspace of K with eigenvalue $\mathrm{e}^{-i\theta(1+2m)}$.

Lemma 5.5.6. The elements $1, w, w^2, ...$ in $\tilde{H} = L^2(\mathbb{T})$ are analytic.

Proof. To prove that the elements $1, w, w^2, ...$ are analytic. we will show that the map $g \to \tilde{\pi_1}(g)w^m$ is analytic as follows:

$$\begin{aligned} \tilde{\pi_1}(g)w^m &= \left(\frac{\overline{\alpha}w - \overline{\beta}}{\alpha - \beta w}\right)^m (\alpha - \beta w)^{-1} \\ &= \left(\frac{\overline{\alpha}w - \overline{\beta}}{\alpha(1 - \frac{\beta}{\alpha}w)}\right)^m \left(\frac{1}{\alpha(1 - \frac{\beta}{\alpha})}\right). \end{aligned}$$

This power series converges since $|\frac{\beta}{\alpha}| < 1$. Thus, the map $\tilde{\pi_1}(g)w^m$ is analytic. □

5.5.3 Representation on the Group $\mathrm{SL}_2(\mathbb{R})$

From (5.2) we can see that

$$\begin{aligned} \alpha &= \frac{1}{2}((a - ic) + i(b - id)) = \frac{1}{2}(a + d - ic + ib), \\ \beta &= \frac{1}{2}(-ia + c + i(-ib + d)) = \frac{1}{2}(c + b - ia - id). \end{aligned}$$

α and β are functions of the group $\mathrm{SL}_2(\mathbb{R})$. So we can compute the following:

$$\alpha\left(\left(\begin{pmatrix} \mathrm{e}^t & 0 \\ 0 & \mathrm{e}^{-t} \end{pmatrix}\right)\right) = \frac{1}{2}(\mathrm{e}^t + \mathrm{e}^{-t}) = \cosh t, \tag{5.73}$$

$$\beta\left(\left(\begin{pmatrix} \mathrm{e}^t & 0 \\ 0 & \mathrm{e}^{-t} \end{pmatrix}\right)\right) = \frac{1}{2}(-i\mathrm{e}^t + i\mathrm{e}^{-t}) = -i\frac{\mathrm{e}^t - \mathrm{e}^{-t}}{2} = -i\sinh t. \tag{5.74}$$

By multiplying the following two matrices

$$\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} \text{ and } \begin{pmatrix} \alpha & \overline{\beta} \\ \beta & \overline{\alpha} \end{pmatrix},$$

we get

$$\alpha(k_\theta x) = \mathrm{e}^{i\theta}\alpha(x),\ \beta(k_\theta x) = \mathrm{e}^{-i\theta}\beta(x). \tag{5.75}$$

Lemma 5.5.7. Let $G = \mathrm{SL}_2(\mathbb{R})$, then the function α^{-1} is not in $L^2(G)$.

Proof. Using the following integral formula:

$$\begin{aligned}\int_G |\alpha|^{-2} dx &= \int_0^\infty (\cosh t)^{-2}\ \sinh 2t\ dt \\ &= \int_0^\infty (\cosh t)^{-2}\, 2\sinh t\ \cosh t\ dt \\ &= \int_0^\infty 2(\cosh t)^{-1}\ \sinh t\ dt.\end{aligned}$$

Let $u = \cosh t$ and $du = \sinh t\ dt$; then

$$2\int_0^\infty u^{-1}\ du = 2\ln u|_0^\infty = \infty,$$

as desired. □

Let H_2 be the closure of the linear span of the functions $\varphi_{1+2r} = \beta^r\alpha^{-(1+r)}$ for $r = 0, 1, 2, \ldots.$ The inner product of H_2 can be defined by pulling the function φ_{1+2r} to the unit circle,

$$\begin{aligned}\mathcal{P} &: H_2 \to L^2(\mathbb{T}) \\ &: \varphi_{1+2r}(\alpha, \beta) \to \varphi_{1+2r}(\mathrm{e}^{i\theta}, 0).\end{aligned}$$

Then

$$\begin{aligned}\langle \mathcal{P}\varphi_{1+2r}, \mathcal{P}\psi_{1+2r}\rangle &= \int_{\mathbb{T}} \mathcal{P}\varphi_{1+2r}(\alpha,\beta)\overline{\mathcal{P}\psi_{1+2r}(\alpha,\beta)}d\alpha d\beta \\ &= \int_{\mathbb{T}} \varphi_{1+2r}(\mathrm{e}^{i\theta},0)\overline{\psi_{1+2r}(\mathrm{e}^{i\theta},0)}d\theta.\end{aligned} \tag{5.76}$$

Let π be the representation of the left translation on $L^2(G)$:

$$\pi(y)f(x) = f(y^{-1}x). \tag{5.77}$$

Then, we obtain

$$\alpha_y = \alpha(y^{-1})\alpha + \overline{\beta}(y^{-1})\beta, \quad \beta_y = \beta(y^{-1})\alpha + \overline{\alpha}(y^{-1})\beta. \tag{5.78}$$

Theorem 5.5.8. The representation π given by (5.77) is unitary.

Proof. Let

$$g = \begin{pmatrix} \alpha & \overline{\beta} \\ \beta & \overline{\alpha} \end{pmatrix} \in SU(1,1) = G \text{ and } g' = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} \in \mathbb{T}.$$

Define the subgroup H

$$H = \left\{ \begin{pmatrix} \frac{|\alpha-i\overline{\beta}|}{\alpha-i\overline{\beta}}\alpha & \frac{|\alpha-i\overline{\beta}|}{\alpha-i\overline{\beta}}\overline{\beta} \\ \frac{|\alpha-i\overline{\beta}|}{\overline{\alpha}+i\beta}\beta & \frac{|\alpha-i\overline{\beta}|}{\overline{\alpha}+i\beta}\overline{\alpha} \end{pmatrix} : \alpha - i\beta \in \mathbb{R} \right\} \subset SU(1,1). \tag{5.79}$$

Then g can be written as

$$\begin{pmatrix} \alpha & \overline{\beta} \\ \beta & \overline{\alpha} \end{pmatrix} = \begin{pmatrix} \frac{\alpha-i\overline{\beta}}{|\alpha-i\overline{\beta}|} & 0 \\ 0 & \frac{\overline{\alpha}+i\beta}{|\alpha-i\overline{\beta}|} \end{pmatrix} \begin{pmatrix} \frac{|\alpha-i\overline{\beta}|}{\alpha-i\overline{\beta}}\alpha & \frac{|\alpha-i\overline{\beta}|}{\alpha-i\overline{\beta}}\overline{\beta} \\ \frac{|\alpha-i\overline{\beta}|}{\overline{\alpha}+i\beta}\beta & \frac{|\alpha-i\overline{\beta}|}{\overline{\alpha}+i\beta}\overline{\alpha} \end{pmatrix}, \tag{5.80}$$

where

$$\begin{pmatrix} \frac{\alpha-i\overline{\beta}}{|\alpha-i\overline{\beta}|} & 0 \\ 0 & \frac{\overline{\alpha}+i\beta}{|\alpha-i\overline{\beta}|} \end{pmatrix} \in G/H \simeq \mathbb{T}.$$

Recall that a representation space of π is the Hilbert space. To verify that π is unitary, we need to show that π is an isometry by using the property

$$F(gh) = F(g)F(h), \quad g \in G, h \in H. \tag{5.81}$$

$$\begin{aligned}\|\pi(g)(\mathcal{P}\varphi_{1+2r})\|^2_{L^2(\mathbb{T})} &= \int_{\mathbb{T}} \left|\mathcal{P}\varphi_{1+2r}(g^{-1}g')\right|^2 d\theta \\ &= \int_{\mathbb{T}} \left|\mathcal{P}\varphi_{1+2r}\begin{pmatrix} \overline{\alpha}e^{i\theta} & -\overline{\beta}e^{-i\theta} \\ -\beta e^{i\theta} & \alpha e^{-i\theta} \end{pmatrix}\right|^2 d\theta.\end{aligned}$$

Using the decomposition (5.80) and the property (5.81), we obtain

$$\begin{aligned}\|\pi(g)(\mathcal{P}\varphi_{1+2r})\|^2_{L^2(\mathbb{T})} &= \int_{\mathbb{T}} \left|\mathcal{P}\varphi_{1+2r}\begin{pmatrix}\frac{\overline{\alpha}e^{i\theta}+i\overline{\beta}e^{-i\theta}}{|\overline{\alpha}e^{i\theta}+i\overline{\beta}e^{-i\theta}|} & 0\\ 0 & \frac{\alpha e^{-i\theta}-i\beta e^{i\theta}}{|\overline{\alpha}e^{i\theta}+i\overline{\beta}e^{-i\theta}|}\end{pmatrix}\right|^2 \frac{1}{|\overline{\alpha}e^{i\theta}+i\overline{\beta}e^{-i\theta}|^2}d\theta\\ &= \int_{\mathbb{T}} \left|\mathcal{P}\varphi_{1+2r}\begin{pmatrix}e^{i\phi} & 0\\ 0 & e^{-i\phi}\end{pmatrix}\right|^2 d\phi\\ &= \|\mathcal{P}\varphi_{1+2r}\|^2_{L^2(\mathbb{T})},\end{aligned}$$

where

$$e^{i\phi} = \frac{\overline{\alpha}e^{i\theta}+i\overline{\beta}e^{-i\theta}}{|\overline{\alpha}e^{i\theta}+i\overline{\beta}e^{-i\theta}|} \text{ and } d\phi = \frac{1}{|\overline{\alpha}e^{i\theta}+i\overline{\beta}e^{-i\theta}|^2}.$$

□

Theorem 5.5.9. The functions φ_{1+2r} are eigenvectors of K with eigenvalue $\mathrm{e}^{-i(1+2r)\theta}$. H_2 is invariant under left translation by $\mathrm{SL}_2(\mathbb{R})$ it is irreducible and has the lowest weight vector equal to α^{-1}.

Proof. We can see from (5.75) that

$$\varphi_{1+2r}(k_\theta x) = \frac{\beta^r(k_\theta x)}{\alpha^{1+r}(k_\theta x)} = e^{-i(1+2r)\theta}\frac{\beta^r(x)}{\alpha^{1+r}(x)},$$

which proves the eigenvalue property. The function $\pi(y^{-1})\varphi_{1+2r}$ lies in the vector space generated by

$$\frac{1}{\alpha}\frac{1}{(1+\frac{\overline{\beta}(y)}{\alpha(y)}\frac{\beta}{\alpha})^{1+r}}\left(\frac{\beta}{\alpha}\right)^v,$$

since $|\frac{\beta}{\alpha}|< 1$ thus,

$$\left|\frac{\overline{\beta}(y)\beta}{\alpha(y)\alpha}\right| \leq \left|\frac{\beta(y)}{\alpha(y)}\right| < 1.$$

Therefore, the power series converges in H_2, and this proves the theorem. □

Chapter 6

Further Work

In section 3.4, we presented the affine group representations, which were induced from a complex valued character of the subgroups N and A. We can expand these ideas to double and dual numbers. The double number (also called a split complex number) is given by $\mathbb{O} = \{a + jb : j^2 = 1 \text{ and } a, b \in \mathbb{R}\}$. The triple $(\mathbb{O}, +, \times)$ is a commutative ring with identity [6, 27]. The dual number is given by $\mathbb{D} = \{a+\epsilon b : \epsilon^2 = 0 \text{ and } a, b \in \mathbb{R}\}$ and the triple $(\mathbb{D}, +, \times)$ is a commutative ring with identity [27, 44]. Therefore, we can have six different induced representations of the affine group.

Also, we can consider the representations of the group $\mathrm{SL}_2(\mathbb{R})$ induced from characters of its one-dimensional subgroups. The subgroup K requires only a complex valued character because it is a compact subgroup. For subgroups A and N, we can consider characters of complex, dual and double numbers.

In chapter 5 , we discussed the $\mathrm{SL}_2(\mathbb{R})$ representations on the Hardy, Bergman and Dirichlet spaces. We found that the vector module of the Lie algebra $\mathfrak{sl}_2(\mathbb{R})$ is unitary for the Hardy and Bergman spaces and non-unitary for the Dirichlet space. In [20, Theorem 1.1.13], all the possible vector modules of the Lie algebra $\mathfrak{sl}_2(\mathbb{R})$ have been presented. It is worth to studying the $\mathfrak{sl}_2(\mathbb{R})$ vector module on other spaces of holomorphic function, for instance the Poly-Bergman space [41].

Appendix A

A.1 Mellin Transform as a Unitary Operator on Hilbert Space

The Mellin transform on the Hilbert space $L_2(\mathbb{R}_+)$[3, §1.5] is a linear operator M : $L_2(\mathbb{R}_+) \to L_2(\mathbb{R})$, defined by:

$$[\mathsf{M}f](s) := \frac{1}{\sqrt{2\pi}} \int_0^\infty x^{-\frac{1}{2}-is} f(x)dx, \quad s \in \mathbb{R}. \tag{A.1}$$

This map is an isometry, that is, $\|\mathsf{M}f\|_{L_2(\mathbb{R}_+)} = \|f\|_{L_2(\mathbb{R}_+)}$. Moreover, it is a unitary operator and it has the Mellin inversion $\mathsf{M}^{-1} : L_2(\mathbb{R}) \to L_2(\mathbb{R}_+)$ given by:

$$[\mathsf{M}^{-1}\tilde{f}](x) = f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty x^{-\frac{1}{2}+is} \tilde{f}(s)ds, \quad x \in \mathbb{R}_+. \tag{A.2}$$

A.2 Generalised Functions

A generalised function (also called a distribution) is a generalisation of the classical notion of a function. In the following, we provide basic definitions. For more information, refer to [15, 38, 42].

Definition A.2.1. [38] Let Ω be an open subset of $\mathbb{R}^n$. The set of test functions $\mathcal{D}(\Omega)$ consists of all real functions $\varphi(x)$ defined in Ω vanishing outside a bounded subset of Ω that stays away from the boundary of Ω, such that all partial derivatives of all order of φ are continuous.

Remark A.2.2. [19] We can define the test function to be the elements of the space $C_0^\infty(\Omega)$.

Definition A.2.3. [38] The set of all continuous linear functional on $\mathcal{D}$ is denoted by $\mathcal{D}'$, and its elements are called generalised functions. By functional, we mean the real or complex valued function on $\mathcal{D}$ written (f, φ) where $\varphi \in \mathcal{D}$.

A generalized function f is a linear functional if it satisfies the identity:

$$a_1(f, \varphi_1) + a_2(f, \varphi_2) = (f, a_1\varphi_1 + a_2\varphi_2).$$

By continuous, we mean that if φ_1 is close enough to φ, then (f, φ_1) is close to (f, φ).

Remark A.2.4. [38] If f is a function such that the integral $\int f(x)\varphi(x)dx$ exists for every test function ϕ, then:

$$(f, \varphi) = \int f(x)\varphi(x)dx$$

defines a generalized function.

Theorem A.2.5. (The Kernel Theorem) [16, p.18]

Every bilinear functional (φ, ψ) on the space $\mathcal{D}$ of all infinitely differentiable functions that have bounded supports and which is continuous in each of the arguments φ and ψ has the form:

$$(\varphi, \psi) = (k, \varphi(x) \otimes \psi(y)),$$

where k is a continuous linear functional on the space $\mathcal{D}(X \times Y)$ of infinitely differentiable functions of two variables having bounded supports.

Definition A.2.6. A function $f(x)$ is called a homogeneous function of degree λ if:

$$f(\alpha x) = \alpha^\lambda f(x), \quad \alpha \neq 0.$$

A function $f_1(x)$ is called an associated homogeneous function of degree λ if:

$$f_1(\alpha x) = \alpha^\lambda [f_1(x) + \ln|\alpha| f_0(x)], \quad \alpha \neq 0.$$

$f_0(x)$ is a homogeneous function of of degree λ.

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