Pseudomonads, Relative Monads and Strongly Finitary Notions of Multicategory

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Submitted in accordance with the requirements
for the degree of Doctor of Philosophy

December 2021

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The Chapters based on work from jointly authored publications are the following, and the details of the relative publications, are the following.

- Chapter 1 and Chapter 2: all of the results in this chapter come from the paper *On the formal theory of pseudomonads and pseudodistributive laws*, authored by N. Gambino and G. Lobbia (see [GL21]). The contribution of all the authors is equally distributed.
- Chapter 3: all of the results in this chapter come from the preprint *Distributive laws for relative monads*, authored G. Lobbia (see [Lob20]).
- Chapter 4: all of the results in this chapter come from a joint work with J. Bourke and G. Lobbia. The contribution of all the authors is equally distributed.

alla mia famiglia

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Acknowledgements

First, I shall try to express my gratitude to my supervisor Nicola Gambino. Everything I have achieved during this PhD has been influenced by you. From the first meetings in Manchester until the last ones during the pandemic, you have always been there providing interesting discussions and friendly reassurance. Thank you for the suggestions, comments, corrections, support and for all the nice meetings.

There a lot of people who helped with the mathematics in this thesis. I will mention some of them and I apologise to the ones I have inevitably forgotten. Firstly, thanks to John Bourke, whom I worked with on all the results in Chapter 4. Starting a collaboration during a pandemic was not easy, but it has been a real pleasure. I am looking forward to joining you in Brno. I am very grateful to Steve Lack and Emily Riehl for the fruitful conversations about the content of Chapter 1 and Chapter 2. Finally, Chapter 3 would not be the same without the help of two more people: thanks to Martin Hyland for the interesting conversation about the definition of extension to Kleisli for relative monads and to Nathanael Arkor for all the comments, suggestions and interest in this work. Last but not least, I thank my examiners João Faria Martins and Marcelo Fiore for taking the time to read my thesis and for the helpful comments and suggestions.

The next debt of gratitude goes to the two maths families who welcomed me in these past few years. Thanks to the Logic Group in Leeds: I still remember when, at the end of my first day at the School of Maths, I was invited to a logic seminar which ended with one of the most beautiful phrases my ears will ever hear, “Shall we retire to the Fenton?”. You made me feel welcome from the very first day and provided an amazing environment to learn from and work with each other. Next, I would like to express my gratitude to ItaCa, the second maths family that adopted me. I feel incredibly lucky to be part of this amazing group and cannot wait for our future plans.

As many before me have said, writing a thesis in mathematics is not only about mathematics. There are a lot of people that, probably against their will, had a positive impact on this work. I will not try to name them all, but rather put them in some arbitrary categories (which will probably be ill-defined and not exhaustive, despite categories being the topic of this thesis).

Dear Leeds people: thanks for all the “Fenton at 5?”, the pub crawls, the seminars, (El) Pub(e), the nights ending in Call Lane or Stone Roses (or both), the walks on canals and all the weird (but fun) towns in Yorkshire, i LucroPhascistee (ciao Digos), the concerts, the pies, the ‘free’ coffee after lunch, the D&D sessions, Dino, the Sunday roasts and everything else my mind is not able to remember now. In spite of the pandemic, I had an amazing time in Leeds, and it is all because of you.

Dear friends in Italy and scattered around the world: whether you are people I have known all my life, schoolmates, Milanese friends who ended up in London, football friends, flatmates, friends I met in Germany or uni friends, I will never stop being pleasantly surprised about how we always manage to keep seeing each other and making fun memories (regardless of the distance, pandemic or, at this point, anything really).

Thanks to Casa Italia for all the laughs, interesting conversations and layers:

** sound of a washing machine in the background **
Home is wherever there are four glasses on the table.

Last but not least, grazie per l’ennesima volta alla mia famiglia. Non penso ve ne farete molto di questa dedica, ma spero potrà essere una nuova scusa per ridere insieme. Come direbbe la Mamma, siamo una famiglia strana, e ne sono incredibilmente felice.

I gratefully acknowledge the support of an EPSRC PhD Scholarship.

Abstract

In this thesis, we investigate two important notions of category theory: monads and multicategories.

First, we contribute to the formal theory of pseudomonads, *i.e.* the analogue for pseudomonads of the formal theory of monads. In particular, we solve a problem posed by Lack by proving that, for every Gray-category $\mathscr{K}$, there is a Gray-category $\mathbf{Psm}(\mathscr{K})$ of pseudomonads in $\mathscr{K}$. We then establish a triequivalence between $\mathbf{Psm}(\mathscr{K})$ and the Gray-category of pseudomonads introduced by Marmolejo and give a simpler version of his proof of the equivalence between pseudodistributive laws and liftings of pseudomonads to 2-categories of pseudoalgebras.

Secondly, we introduce the notion of a distributive law between a relative monad and a monad. We call this a relative distributive law and define it in any 2-category $\mathcal{K}$. In order to do that, we introduce the 2-category of relative monads in a 2-category $\mathcal{K}$. We relate our definition to the 2-category of monads in $\mathcal{K}$ defined by Street. Thanks to this view we prove two theorems regarding relative distributive laws and equivalent notions. We also describe what it means to have Eilenberg-Moore and Kleisli objects in this context and give examples in the 2-category of locally small categories.

Finally, we consider multicategories. It is known that monoidal categories have a finite definition, whereas multicategories have an infinite (albeit finitary) definition. Since monoidal categories correspond to representable multicategories, it goes without saying that representable multicategories should also admit a finite description. With this in mind, we give a new finite definition of a structure called a short multicategory, which has only multimaps of dimension at most four, and show that under certain representability conditions short multicategories correspond to various flavours of representable multicategories. This is done in both the classical and skew settings.

Contents

Introduction

Context and Motivation

Monads are one of the fundamental notions of category theory [Mac71, Chapter VI]. An example of the utility of monads is that they provide a homogeneous approach to the study of categories of sets equipped with algebraic structure, such as groups and monoids [BW85]. Furthermore, Beck's theorem on distributive laws between monads [Bec69] describes concisely the structure that is necessary and sufficient in order to combine two algebraic structures, so that the operations of one distribute over those of the other. For example, the monads for groups and for monoids can be combined via a distributive law to define the monad for rings. In this context, the formal theory of monads, originally introduced by Street [Str72] and later developed further by Lack and Street [LS02], has offered an elegant and mathematically efficient account of the theory of monads, starting from the observation that the notion of a monad can be defined within any 2-category (so that the usual notion is recovered by considering the 2-category of categories, functors and natural transformations). Among many other results, Lack and Street's work provides a characterisation of the existence of categories of Eilenberg-Moore algebras as a completeness property and, importantly for our purposes, a simple account of Beck's theorem on distributive laws.

In recent years, motivation from pure mathematics, *e.g.* in the theory of operads [FGHW08, FGHW17, GJ17, Gar08], and theoretical computer science, *e.g.* in the study of variable binding [FPT99, CW05, Cur12, TP06b], led to significant interest in pseudomonads [Lac00, Mar97, Mar04, Wal19], which are the counterparts of monads in 2-dimensional category theory, obtained by requiring the axioms for a monad to hold only up to coherent isomorphism rather than strictly [Bun74]. Here, one of the key issues has been the proof of a counterpart of Beck's theorem on distributive laws, which requires a satisfactory axiomatisation of the notion of a pseudostributive law [CHP03, Mar99, MW08, Tan04, TP06a], building on early work of Kelly [Kel74] on semi-strict distributive laws. This is a difficult question because such a notion necessarily involves complex coherence conditions.

Monads can also be generalised in another way. It is known that they were first introduced as endofunctors $S\colon \mathbb{C} \to \mathbb{C}$ with natural transformations $m\colon S^2 \to S$ and $s\colon 1_{\mathbb{C}} \to S$ acting as multiplication and unit. Then, Manes [Man76, Definition 3.2] introduced the equivalent notion of a Kleisli triple, which relies on a mapping of objects $S\colon \mathrm{Ob}(\mathbb{C}) \to \mathrm{Ob}(\mathbb{C})$, an extension operator sending any map $f\colon X \to SY$ to one of the type $f^{\dagger}\colon SX \to SY$ and a family of maps $s_X\colon X \to SX$. In recent years, monads with this description have been called *no–iteration monads* or *(left) extension systems*, and they have been studied in [Her20, MM07, MVM17, MW10]. This description of monads leads to a generalisation of them, known as *relative monads* [ACU15, Definition 2.1]. These are monad-like structure on a base functor $I\colon \mathbb{C}_0 \to \mathbb{C}$, i.e. for any $X \in \mathbb{C}_0$ an object $SX \in \mathbb{C}$, for any $X, Y \in \mathbb{C}_0$ an extension operator $(-)^{\dagger}_{S}\colon \mathbb{C}(IX, SY) \to \mathbb{C}(SX, SY)$ and a unit $s_X\colon IX \to SX$ satisfying unital and associativity laws. In [MRW02, Proposition 3.5] we find a characterization of distributive laws $d\colon ST \to TS$ in terms of S-algebras $\alpha\colon STS \to TS$ with some properties. This description is adapted in [MW10, Theorem 6.2] to extension systems. Then [Her20] provides the definition of a distributive law of a right extension system with respect to a left extension system (also called a *no-iteration distributive law* or a *distributive law in extensive form*), where a right extension system is the dual notion of an extension system. Finally, mixed distributive laws (between a monad and a comonad) have been studied in terms of extension systems in [MVM17].

Taking a step back, we can see how in the zoo of categorical structures, there are three closely related ones:

- monoidal categories [Mac63], which involve tensor products $A \otimes B$ and a unit I;
- closed categories [EK66], which involve an internal hom $[A, B]$ and unit I;
- multicategories [Lam69], which involve multimorphisms $A_1, \ldots, A_n \to B$ for all $n \in \mathbb{N}$.

It is well known that there are various correspondences between different flavours of these notions [Her00, Man09, BL18]. However, each of these concepts has various pros and cons.

- Monoidal categories are fairly straightforward to work in — for instance, it is easy to write down the definition of a monoid in a monoidal category. Another advantage is that while the definition is finite, they admit a coherence theorem — all diagrams commute [Mac63]. A disadvantage is that in practise, the tensor product is often constructed using colimits and so sometimes could be hard to describe explicitly.
- Closed categories have several advantages. Again they have a finite definition and a coherence theorem, though this is of a more complex nature [KM71, Sol97]. Another advantage is that the internal homs are often constructed using limits, and so easy to describe explicitly — see, for instance, the internal hom of vector spaces. The disadvantage is that the axiomatics of closed categories involve iterated contravariance, and this makes it quite hard to parse diagrams in a closed category.
- In a multicategory, the multimaps can often be described directly — see, for instance, multilinear maps of vector spaces — and this avoids the potentially

> complicated constructions of tensor products and internal homs using colimit and limits. A disadvantage is that the definition, is infinite (though finitary) in nature, and this sometimes makes it difficult to describe examples in full detail.

In this thesis, we contribute both to the formal theory of pseudomonads (Chapter 1 and Chapter 2) and the formal theory of relative monads (Chapter 3). In particular, we prove counterparts of Beck's theorem for distributive laws in both settings. Then, we study multicategories (Chapter 4) and try to address the disadvantage noticed above. In particular, we will provide an answer to the question: when is it possible to give a *finite* definition of multicategory? The plan of action will be to use the known equivalences between different flavours of monoidal category and multicategory. We believe our results will make it easier to construct examples of multicategorical structures in practice. For instance, we expect to get a shorter proof of Verity's multicategory of bicategories [Ver92, Definition 1.3.3].

Main Results

The aim of Chapter 1 and Chapter 2 is take some further steps in the development of the formal theory of pseudomonads. The main contribution of Chapter 1 is to answer the question raised in [Lac00] by showing that for every Gray-category $\mathscr{K}$, there is a Gray-category $\mathbf{Psm}(\mathscr{K})$ of pseudomonads, pseudomonad morphisms, pseudomonad transformations and pseudomonad modifications in $\mathscr{K}$.

Theorem 1.2.5. *Let $\mathscr{K}$ be a Gray-category. Then there is a Gray-category $\mathbf{Psm}(\mathscr{K})$, called the Gray-category of pseudomonads in $\mathscr{K}$, having pseudomonads in $\mathscr{K}$ as 0-cells, pseudomonad morphisms as 1-cells, pseudomonad transformations as 2-cells, and pseudomonad modifications as 3-cells.*

The main challenge in proving this Theorem was to show that the composition of pseudomonad morphisms is strictly associative (Lemma 1.2.8). Another result we prove is an analogue of a fundamental result of the formal theory of monads: in Proposition 2.2.4 we show that the objects of $\mathbf{Psm}(\mathbf{Psm}(\mathscr{K}))$ are exactly pseudodistributive laws in $\mathscr{K}$. Using this result, we give a new, simpler proof of Marmolejo's theorem regarding the equivalence between pseudodistributive laws and liftings of pseudomonads to 2-categories of pseudoalgebras, given as the proof of Theorem 2.2.5.

Theorem 2.2.5. *Let $\mathscr{K}$ be a Gray-category, (X, S) and (X, T) be pseudomonads in $\mathscr{K}$. A pseudodistributive law $d \colon ST \to TS$ is equivalent to a lifting of T to pseudo-S-algebras.*

The main result of Chapter 3 is a counterpart of Beck's equivalence for relative distributive laws, which is described in the theorem below. Here, we denote with $(X,\, I,\, T)$ a relative monad in $\mathcal{K}$ where $X, I \colon X_0 \to X$ are 1-cells in $\mathcal{K}$ and with $(S,\, S_0)$ two monads $S_0 \colon\; X_0 \to X_0$ and $S \colon X \to X$ which are *compatible* with I (Definition 3.5.1).

Theorem 3.6.19. *Let $\mathcal{K}$ be a 2-category, $(X,\, I,\, T)$ a relative monad in $\mathcal{K}$ and $(S,\, S_0)$ a compatible monad with I. The following are equivalent:*

(i) a relative distributive law of T over $(S,\, S_0)$;

(ii) a lifting $\hat{T} \colon S_0\text{-Alg}(-) \longrightarrow S\text{-Alg}(-)$ of T to the algebras of $(S,\, S_0)$;

(iii) a lifting $\tilde{S} \colon \mathrm{Mod}_T(-) \to \mathrm{Mod}_T(-)$ of S to the relative right modules of T.

The central issue with this Theorem was to find the right definitions of lifting to algebras and lifting to relative right modules.

Finally, the main contribution of Chapter 4 is to prove equivalences between different flavours of (skew) multicategories and our new notion of short (skew) multicategories, which involves multimaps of dimention at most four. We also show that these equivalences are compatible with the ones given in [Her00, BL18] for different flavours of multicategory and monoidal category. We start considering left representable short multicategories and we prove the following theorem. Here, we denote with $\mathbf{ShMult}_{lr}$ the category of left representable multicategories, with $\mathbf{Skew}_{ln}$ the category of left representable short multicategories and with $\mathbf{Skew}_{ln}$ the category of left normal monoidal categories.

Theorem 4.4.5. *The functor* $K\colon \mathbf{ShMult}_{lr} \to \mathbf{Skew}_{ln}$ *is an equivalence of categories, as is the forgetful functor* $U_{lr}\colon \mathbf{Mult}_{lr} \to \mathbf{ShMult}_{lr}$*. They also fit in a commutative triangle*

$$\begin{tikzcd}
\mathbf{Mult}_{lr} \arrow[dd, "T"'] \arrow[dr, "U_{lr}"] & \\
& \mathbf{ShMult}_{lr} \arrow[dl, "K"] \\
\mathbf{Skew}_{ln}. &
\end{tikzcd}$$

Then, we also consider the following cases:

- Theorem 4.4.6 provides an equivalence between representable multicategories and representable short multicategories.
- Theorem 4.4.7 and Theorem 4.4.8 show the equivalences in the closed left representable and closed representable case.
- Theorem 4.5.12 proves the left representable skew case.
- Finally, Theorem 4.5.15 is about the left representable closed skew case.

The crucial part of these theorems was to carefully check which conditions were enough to make the numerous diagrams in the proofs well-defined.

Outline

In Chapter 1 we start by recalling some useful background such as Gray-categories and pseudomonads (Section 1.1). We will assume familiarity with basic definitions of low dimensional category theory. For this notions, we refer the reader to [Lac10, KS74, JY21]. Then, in Section 1.2, we construct the 3-dimensional category $\mathbf{Psm}(\mathscr{K})$ of pseudomonads in a Gray-category $\mathscr{K}$. We conclude the chapter proving that $\mathbf{Psm}(\mathscr{K})$ is a Gray-category.

Then, in Chapter 2, we first introduce liftings to pseudoalgebras and then prove the equivalence between $\mathbf{Psm}(\mathscr{K})$ and $\mathbf{Lift}(\mathscr{K})$ (Section 2.1). At the end of the chapter, Section 2.2, we discuss pseudodistributive laws.

Next, we move to relative monads in Chapter 3. In Section 3.1 we introduce our notation and the definition of operator, which generalises the notion of a family of maps $\mathbb{C}(Fx, Gy) \to \mathbb{C}'(F'x, G'y)$ natural in x and y. Section 3.2 uses operators to generalise some results for relative monads in $\mathbf{Cat}$ to any 2-category $\mathcal{K}$. Then, in Section 3.3, we define explicitly the 2-category $\mathbf{Rel}(\mathcal{K})$ of relative monads in $\mathcal{K}$. We proceed in Section 3.4 giving the definition of algebras for a relative monad and use them to describe when a relative monad is induced by a relative adjunction. In Section 3.5 we define a relative distributive law and then prove the first Beck-type theorem (about distributive laws). Section 3.6 is devoted to the 2-isomorphism between $\mathbf{LiftR}(\mathcal{K})$ and $\mathbf{Rel}(\mathcal{K})$ and the second Beck-type theorem. We conclude the chapter with some examples.

Finally, we study strongly finitary notions of multicategory in Chapter 4. More precisely, in Section 4.1 we recall some important notions for multicategories. In Section 4.2 we introduce our new definition of *short multicategories* and related concepts such as representability and closedness. In Section 4.3 we give an overview on skew monoidal categories and skew multicategories. Then, Section 4.4 provides various equivalences between different flavour of short multicategories and skew monoidal categories. We conclude the chapter in Section 4.5 introducing short skew multicategories and describing analogues of the results in Section 4.4 appropriate to the skew setting.

1. On the Formal Theory of Pseudomonads

Introduction

The aim of this chapter is to give the right basis to prove a counterpart for pseudomonads of Beck's theorem on distributive laws. Given how the formal theory of monads offers a simple proof of this theorem, it seems natural to attack this problem by developing a formal theory of pseudomonads. In order to do this, however, one needs to face the challenge that, just as the formal theory of monads is formulated within 2-dimensional category theory [KS74], the formal theory of pseudomonads is formulated within 3-dimensional category theory [GPS95, Gur13], which is notoriously hard. In this setting, it is convenient to work with Gray-categories, *i.e.* semistrict tricategories [GPS95, Section 4.8], which are easier to handle than tricategories, but sufficiently general for many purposes, since every tricategory is triequivalent to a Gray-category [GPS95, Theorem 8.1].

In spite of significant advances in the creation of a formal theory of pseudomonads in the works cited above, there are still fundamental questions to be addressed. In particular, there is not yet a direct counterpart of the 2-category $\mathbf{Mnd}(\mathscr{K})$ of monads, monad morphisms and monad transformations in a 2-category $\mathscr{K}$, which is the the starting point of the formal theory of monads [Str72]. Filling this gap would involve the definition, for a Gray-category $\mathscr{K}$, of a 3-dimensional category $\mathbf{Psm}(\mathscr{K})$ having pseudomonads in $\mathscr{K}$ as 0-cells, pseudomonad morphisms as 1-cells, and appropriately defined pseudomonad transformations and pseudomonad modifications as 2-cells and 3-cells, respectively. This issue was raised by Lack in [Lac00, Section 6], who suspected that defining $\mathbf{Psm}(\mathscr{K})$ in this way would give rise only to a tricategory, not a Gray-category, and hence require lengthy verifications of the coherence conditions. For this reason, Lack preferred to define a Gray-category of pseudomonads in $\mathscr{K}$ using the description of pseudomonads in $\mathscr{K}$ as suitable lax functors and developing parts of the theory using enriched category theory.

Main results

The aim of this chapter is to take some further steps in the development of the formal theory of pseudomonads. In particular, our main contribution is the following:

- Theorem 1.2.5, which answers the question raised in [Lac00] by showing that for every Gray-category $\mathscr{K}$, there is a Gray-category $\mathbf{Psm}(\mathscr{K})$ of pseudomonads, pseudomonad morphisms, pseudomonad transformations and pseudomonad modifications in $\mathscr{K}$.

This result will be particularly useful in Chapter 2 where we will use it in various ways. For instance, since $\mathbf{Psm}(\mathscr{K})$ is a Gray-category, then we will be able to consider $\mathbf{Psm}(\,\mathbf{Psm}(\mathscr{K})\,)$ and study what are the objects of this Gray-category. Theorem 1.2.5 will also play a part in proving some results about pseudodistributive laws.

As the proof of our main result involves lengthy, subtle calculations with pasting diagrams, we tried to strike a reasonable compromise between rigour and conciseness by giving what we hope are the key diagrams of the proofs, and describing the additional steps in the text. When in doubt, we preferred to err on the side of rigour, since one of our initial goals was to answer the question raised in [Lac00] about whether $\mathbf{Psm}(\mathscr{K})$ is a Gray-category or not. For the convenience of the readers, some of the diagrams are confined to the Appendices A.

Outline

Section 1.1 provides background on Gray-categories, pseudomonads and 2-categories of pseudoalgebras.

Then, in Section 1.2 we define the Gray-category $\mathbf{Psm}(\mathscr{K})$.

1.1. Preliminaries

Gray-categories

We begin by reviewing the notion of a Gray-category and fixing some notation. A Gray-category can be defined very succinctly in terms of enriched category theory (see Remark 1.1.4). For our purposes, however, it is useful to give an explicit definition, which we recall from [Mar99, Section 2] in Definition 1.1.1 below. The explicit definition makes it easier to see that Gray-categories are special tricategories [GPS95, Proposition 3.1] in which the only non-strict operation is horizontal composition of 2-cells [GPS95, Section 5.2]. Throughout this paper, for a Gray-category $\mathscr{K}$, we use $X, Y, Z, \ldots$ to denote its 0-cells, $F\colon X \to Y$, $G\colon Y \to Z$, ...for its 1-cells, $f\colon F \to F'$, $g\colon G \to G'$ ... for 2-cells, and $\alpha\colon f \to f'$, $\beta\colon g \to g'$ ... for 3-cells.

When stating the definition of a Gray-category below, we make use of the notion of a cubical functor from [GPS95], which we unfold in Remark 1.1.2.

Definition 1.1.1. A Gray-category $\mathscr{K}$ consists of the the data in (G1)-(G4), subject to axioms (G5) and (G6), as given below.

(G1) A class of objects $\mathscr{K}_0$. We call the elements of $\mathscr{K}_0$ the 0-cells of $\mathscr{K}$.

(G2) For every $X\,, Y \in \mathscr{K}_0$, a 2-category $\mathscr{K}(X\,, Y)$. We refer to the n-cells of these 2-categories as the $(n+1)$-cells of $\mathscr{K}$.

(G3) For every X, Y, $Z \in \mathscr{K}_0$, a cubical functor

$$\mathscr{K}(Y,\, Z) \times \mathscr{K}(X,\, Y) \to \mathscr{K}(X,\, Z)\,,$$

whose action on $F\colon X \to Y$ and $G\colon Y \to Z$ is written $GF\colon X \to Z$, and whose action on $f\colon F \to F'$ and $g\colon G \to G'$ gives rise to an invertible 3-cell

$$\begin{tikzcd}
GF \arrow[r, "Gf"] \arrow[d, "gF"'] \arrow[dr, phantom, "\Downarrow g_f"] & GF' \arrow[d, "gF'"] \\
G'F \arrow[r, "G'f"'] & G'F'
\end{tikzcd}$$

called the *interchange maps* of $\mathscr{K}$.

(G4) For any $X \in \mathscr{K}_0$, a 1-cell $1_X\colon X \to X$. We call these the identity 1-cells of $\mathscr{K}$.

(G5) For every

$$\begin{tikzcd}
F \arrow[r, bend left, "f"] \arrow[r, bend right, "f'"'] \arrow[r, phantom, "\Downarrow\alpha"] & F' & G \arrow[r, bend left, "g"] \arrow[r, bend right, "g'"'] \arrow[r, phantom, "\Downarrow\beta"] & G' & K \arrow[r, bend left, "k"] \arrow[r, bend right, "k'"'] \arrow[r, phantom, "\Downarrow\gamma"] & K'
\end{tikzcd}$$

in $\mathscr{K}(X,\, Y)$, $\mathscr{K}(Y,\, Z)$ and $\mathscr{K}(Z,\, W)$, respectively,

$$(KG)F = K(GF)\,,$$

$$(KG)f = K(Gf)\,, \quad (Kg)F = K(gF)\,, \quad (kG)F = k(GF)\,,$$

$$(KG)\alpha = K(G\alpha)\,, \quad (K\beta)F = K(\beta F)\,, \quad (\gamma G)F = \gamma(GF)\,,$$

$$(Kg)_f = K(g_f)\,, \quad (kG)_f = k_{Gf}\,, \quad (k_g)F = k_{(gF)}\,.$$

(G6) For every X, the 2-functors

$$1_X(-)\colon \mathscr{K}(X,Y) \to \mathscr{K}(X,Y)\,, \quad (-)1_X\colon \mathscr{K}(X,Y) \to \mathscr{K}(X,Y)$$

defined by composition with $1_X\colon X \to X$, are identities.

Remark 1.1.2. Asserting that composition in a Gray-category $\mathscr{K}$ is a cubical functor means that the properties in (i)-(v) below hold, for every $F\,, F'\,, F''\colon X \to Y$, $G\,, G'\,, G''\colon Y \to Z$ and

$$\begin{tikzcd} F \arrow[r, bend left, "f"] \arrow[r, bend right, "f'"'] \arrow[r, phantom, "\Downarrow\phi"] & F' \arrow[r, "f''"] & F''\,, \end{tikzcd} \qquad \begin{tikzcd} G \arrow[r, bend left, "g"] \arrow[r, bend right, "g'"'] \arrow[r, phantom, "\Downarrow\psi"] & G' \arrow[r, "g''"] & G''\,. \end{tikzcd}$$

(i) Composition with 1-cells on either side,

$$(-)F\colon \mathscr{K}(Y,\,Z) \to \mathscr{K}(X,\,Z) \quad G(-) : \mathscr{K}(X,\,Y) \to \mathscr{K}(X,\,Z)\,,$$

is a strict 2-functor.

(ii) Composition with 2-cells,

$$(-)f : (-)F \to (-)F'\,, \quad g(-) : G(-) \to G'(-)\,,$$

is a pseudo-natural transformation.

(iii) Composition with 3-cells,

$$(-)\varphi\colon (-)f \to (-)f'\,, \quad \psi(-)\colon g(-) \to g'(-)\,,$$

is a modification.

(iv) The following coherence equations hold (which are equivalent to the ones making g_f the component of the pseudo-natural transformation $(-)f$ and $g(-)$):

$$\begin{tikzcd} GF \arrow[r, bend left, "Gf"] \arrow[r, bend right, "Gf'"'] \arrow[r, phantom, "\Downarrow G\varphi"] \arrow[d, bend right=40, "g'F"'] \arrow[d, bend left=40, "gF"] \arrow[d, phantom, "\underset{\gamma F}{\Leftarrow}"] & GF' \arrow[d, "gF'"] \arrow[dl, phantom, "\Swarrow g_{f'}"] \\ G'F \arrow[r, "G'f'"'] & G'F' \end{tikzcd} = \begin{tikzcd} GF \arrow[r, "G'f"] \arrow[d, "g'F"'] & GF' \arrow[d, bend right=40, "g'F'"'] \arrow[d, bend left=40, "gF'"] \arrow[d, phantom, "\overset{\gamma F'}{\Leftarrow}"] \arrow[dl, phantom, "g'_f\Swarrow"] \\ G'F \arrow[r, bend left, "Gf"] \arrow[r, bend right, "G'f'"'] \arrow[r, phantom, "\Downarrow G'\varphi"] & G'F' \end{tikzcd} \qquad (1.1.1)$$

$$\begin{tikzcd}
GF \arrow[r, "Gf"] \arrow[d, "gF"'] \arrow[dr, phantom, "\Downarrow g_f"] & GF' \arrow[d, "gF'"] \\
G'F \arrow[r, "G'f"'] \arrow[d, "g''F"'] \arrow[dr, phantom, "\Downarrow g''_f"] & G'F' \arrow[d, "g''F'"] \\
G''F' \arrow[r, "G''f"'] & G''F'
\end{tikzcd}
=
\begin{tikzcd}
GF \arrow[r, "Gf"] \arrow[d, "(g''g)F"'] \arrow[dr, phantom, "\Downarrow (g''g)_f"] & GF' \arrow[d, "(g''g)F'"] \\
G''F' \arrow[r, "G''f"'] & G''F'
\end{tikzcd}
\tag{1.1.2}$$

$$\begin{tikzcd}
GF \arrow[r, "Gf"] \arrow[d, "gF"'] \arrow[dr, phantom, "\Downarrow g_f"] & GF' \arrow[r, "Gf''"] \arrow[d, "gF'"'] \arrow[dr, phantom, "\Downarrow g_{f''}"] & GF'' \arrow[d, "gF''"'] \\
G'F \arrow[r, "G'f"'] & G'F' \arrow[r, "G'f''"'] & G'F''
\end{tikzcd}
=
\begin{tikzcd}
GF \arrow[r, "G(f''f)"] \arrow[d, "gF"'] \arrow[dr, phantom, "\Downarrow g_{(f''f)}"] & GF'' \arrow[d, "gF''"'] \\
G'F \arrow[r, "G'(f''f)"'] & G'F''.
\end{tikzcd}
\tag{1.1.3}$$

(v) The interchange map g_f is the identity 3-cell when either f or g is the identity.

Remark 1.1.3. When working with a Gray-category, we sometimes write $G \circ F$ instead of GF for cubical composition of 1-cells. For 2-cells, we write $g' \cdot g$ (or $g'\, g$) for the vertical composition and $g \circ f$ for cubical composition. For 3-cells, we write $\beta \circ \alpha$ for cubical composition, $\alpha' * \alpha$ for vertical composition in $\mathscr{K}(X, Y)$ and $\bar{\alpha} \cdot \alpha$ for horizontal composition in $\mathscr{K}(X, Y)$, where $\alpha' \colon f' \to f''$ and $\bar{\alpha} \in \mathscr{K}(X, Y)[F', F'']$.

Remark 1.1.4. We write **Gray** for the category of 2-categories and 2-functors. For 2-categories X and Y, let $[X, Y]$ be the 2-category of 2-functors from X to Y, pseudonatural transformations, and modifications [KS74]. This definition equips the category **Gray** with the structure of a closed category [EK66]. The closed structure of **Gray** is part of symmetric monoidal structure, whose tensor product is known as the *Gray tensor product* [GPS95, Section 4.8]. We will write $X \otimes Y$ for the Gray tensor product of 2-categories X and Y. A Gray-category can then be defined equivalently as a **Gray**-enriched category [GPS95, Section 5.1]. Since **Gray** is a monoidal closed category, it is enriched over itself. Therefore, it can be viewed as a Gray-category, as we will do from now on. More explicitly, **Gray** is the Gray-category having 2-categories as 0-cells, 2-functors as 1-cells, pseudonatural transformations as 2-cells, and modifications as 3-cells.

The notions of a *Gray-functor* and of a *Gray-natural transformation* are instances of the general notions of enriched functor and enriched natural transformation [Kel82, Section 1.2]. We will use the terminology of *Gray-modification* and *Gray-perturbation* to denote the strict counterparts of the corresponding tricategorical notions [GPS95, Section 3.3].

When working with the Yoneda embedding for Gray-categories, which is just an instance of the Yoneda embedding for enriched categories [Kel82, Section 2.4], we often identify an object $X \in \mathscr{K}$ with the representable Gray-functor $\mathscr{K}(-, X) \colon \mathscr{K}^{op} \to \mathbf{Gray}$ associated to it. Analogous conventions will be used also for the n-cells of $\mathscr{K}$, where $n = 1, 2, 3$. For further information on Gray-categories and tricategories, we invite the reader to refer to [GG09, GPS95, Gur13, Lac07].

Pseudomonads and their Pseudoalgebras

Let $\mathscr{K}$ be a Gray-category, to be considered fixed for the rest of this section. We recall the definition of a pseudomonad.

Definition 1.1.5. Let $X \in \mathscr{K}$. A *pseudomonad* on X in $\mathscr{K}$ consists of:

- a 1-cell $S : X \to X$ in $\mathscr{K}$;
- two 2-cells $m \colon S^2 \to S$ and $s : 1_X \to S$ in $\mathscr{K}$;
- three invertible 3-cells in $\mathscr{K}$ of the following form:

$$\begin{tikzcd}
S^3 \arrow[r, "Sm"] \arrow[d, "mS"'] \arrow[dr, phantom, "\Downarrow\mu"] & S^2 \arrow[d, "m"] \\
S^2 \arrow[r, "m"'] & S
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
S \arrow[r, "Ss"] \arrow[dr, bend right, "1_S"'] \arrow[dr, phantom, "\overset{\lambda}{\Leftarrow}"] & S^2 \arrow[d, "m"] & S \arrow[l, "sS"'] \arrow[dl, bend left, "1_S"] \arrow[dl, phantom, "\overset{\rho}{\Leftarrow}"] \\
 & S &
\end{tikzcd}$$

satisfying the coherence axioms in (1.1.4) and (1.1.5) below:

$$
\begin{tikzcd}
S^4 \arrow[rr, "S^2m"] \arrow[dr, "SmS"] \arrow[dd, "mS^2"'] & & S^3 \arrow[dr, "Sm"] \arrow[dl, phantom, "\Downarrow S\mu"] & \\
& S^3 \arrow[rr, "Sm"] \arrow[dd, "mS"] \arrow[dl, phantom, "\Downarrow \mu S"] \arrow[ddrr, phantom, "\Downarrow \mu"] & & S^2 \arrow[dd, "m"] \\
S^3 \arrow[dr, "mS"'] & & & \\
& S^2 \arrow[rr, "m"'] & & S
\end{tikzcd}
=
\begin{tikzcd}
S^4 \arrow[rr, "S^2m"] \arrow[dd, "mS^2"'] \arrow[ddrr, phantom, "\Downarrow m_m"] & & S^3 \arrow[dr, "Sm"] \arrow[dd, "mS"] & \\
& & & S^2 \arrow[dd, "m"] \arrow[dl, phantom, "\Downarrow \mu"] \\
S^3 \arrow[rr, "Sm"] \arrow[dr, "mS"'] & & S^2 \arrow[dr, "m"] \arrow[dl, phantom, "\mu \Downarrow"] & \\
& S^2 \arrow[rr, "m"] & & S
\end{tikzcd}
\tag{1.1.4}
$$

$$
\begin{tikzcd}
S^2 \arrow[drr, bend left, "1_{S^2}"] \arrow[dr, "SsS"] \arrow[ddr, bend right, "1_{S^2}"'] & & \\
& S^3 \arrow[r, "Sm"] \arrow[d, "Sm"] \arrow[u, phantom, "\Downarrow S\rho"] \arrow[l, phantom, "\Downarrow \lambda S"] \arrow[dr, phantom, "\Downarrow \mu"] & S^2 \arrow[d, "m"] \\
& S^2 \arrow[r, "m"'] & S
\end{tikzcd}
=
\begin{tikzcd}
S^2 \arrow[r, "m"] & S
\end{tikzcd}
\tag{1.1.5}
$$

For brevity, we will refer to an object $X \in \mathscr{K}$ and a pseudomonad $(S, m, s, \mu, \lambda, \rho)$ on X simply as a pseudomonad in $\mathscr{K}$ and write simply (X, S) to denote it.

Example 1.1.6. Many examples of pseudomonads can be found in [Tan04, Section 8.1], [CHP03, Section 2] and [FGHW08, Section 4.2]. Two of them are the pseudomonad of small categories with products and the pseudomonad of small symmetric monoidal categories [Tan04, Examples 8.1 and 8.3].

Note that the notion of a pseudomonad is self-dual, in the sense that a pseudomonad in $\mathscr{K}$ is the same thing as a pseudomonad in $\mathscr{K}^{op}$, where $\mathscr{K}^{op}$ is the Gray-category obtained from $\mathscr{K}$ by reversing the direction of the 1-cells, but not that of the 2-cells and 3-cells. As in the formal theory of monads, this is important to obtain results by duality.

Let (X, S) be a pseudomonad in $\mathscr{K}$. For $I \in \mathscr{K}$, there is a 2-category Ps-S-Alg(I) of I-indexed pseudo-S-algebras, pseudoalgebra morphisms, and pseudoalgebra 2-cells, whose definitions we recall below. An *I-indexed pseudoalgebra* for S consists of a 1-cell $A\colon I \to X$, called the underlying 1-cell of the pseudoalgebra, a 2-cell $a\colon SA \to A$, called the *structure map* of the pseudoalgebra, and invertible 3-cells

$$\begin{tikzcd}
S^2A \arrow[r, "Sa"] \arrow[d, "m_A"'] \arrow[dr, phantom, "\Downarrow \bar{a}"] & SA \arrow[d, "a"] \\
SA \arrow[r, "a"'] & A\,,
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r, "s_A"] \arrow[dr, "1_A"'] & SA \arrow[d, "a"] \arrow[dl, phantom, "\overset{\tilde{a}}{\Rrightarrow}"] \\
& A\,,
\end{tikzcd}$$

called the *associativity* and *unit* of the pseudoalgebra, satisfying the coherence axioms (1.1.6) and (1.1.7) stated below.

$$\begin{tikzcd}
S^3A \arrow[rr, "S^2a"] \arrow[dr, "Sm_A"] \arrow[dd, "m_{SA}"'] & & S^2A \arrow[dr, "Sa"] \arrow[dl, phantom, "\Downarrow S\bar{a}"] & \\
& S^2A \arrow[rr, "Sa"] \arrow[dd, "m_A"] \arrow[ddrr, phantom, "\Downarrow \bar{a}"] & & SA \arrow[dd, "a"] \\
S^2A \arrow[dr, "m_A"'] \arrow[ur, phantom, "\Downarrow \alpha_A"] & & & \\
& SA \arrow[rr, "a"'] & & A
\end{tikzcd}
=
\begin{tikzcd}
S^3A \arrow[rr, "S^2a"] \arrow[dd, "m_{SA}"'] \arrow[ddrr, phantom, "\Downarrow m_a"] & & S^2A \arrow[dr, "Sa"] \arrow[dd, "m_A"] & \\
& & & SA \arrow[dd, "a"] \arrow[dl, phantom, "\bar{a}\Downarrow"] \\
S^2A \arrow[rr, "Sa"] \arrow[dr, "m_A"'] \arrow[drr, phantom, "\Downarrow \bar{a}"] & & SA \arrow[dr, "a"] & \\
& SA \arrow[rr, "a"'] & & A\,.
\end{tikzcd}
\tag{1.1.6}$$

$$\begin{tikzcd}
SA \arrow[drr, bend left, "1_{SA}"] \arrow[dr, "Ss_A"] \arrow[ddr, bend right, "1_{SA}"'] & & \\
& S^2A \arrow[r, "Sa"] \arrow[d, "m_A"] \arrow[ur, phantom, "\Downarrow \tilde{a}"] \arrow[dl, phantom, "\Downarrow \lambda_A"] \arrow[dr, phantom, "\Downarrow \bar{a}"] & SA \arrow[d, "a"] \\
& SA \arrow[r, "a"'] & A
\end{tikzcd}
=
\begin{tikzcd}
SA \arrow[r, "a"] & A\,.
\end{tikzcd}
\tag{1.1.7}$$

As usual, we refer to a pseudoalgebra by the name of its underlying 1-cell, leaving the rest of its data implicit. Similar conventions will be implicitly assumed for other kinds of structures.

Proposition 1.1.7. *[Mar97, Lemma 9.1] Let (X, S) be a pseudomonad in $\mathscr{K}$, $I \in \mathscr{K}$ and A an I-indexed pseudoalgebra for S. Then, the coherence condition*

$$\begin{tikzcd}
SA \arrow[rr, "a"] \arrow[d, "s_{SA}"'] \arrow[drr, phantom, "\Downarrow s_a"] & & A \arrow[d, "s_A"] \arrow[ddr, bend left, "1_{SA}"] & \\
S^2A \arrow[rr, "Sa"] \arrow[dr, "m_A"'] \arrow[drr, phantom, "\Downarrow \bar{a}"] & & SA \arrow[dr, "a"'] \arrow[r, phantom, "\tilde{a}\Downarrow"] & \\
& SA \arrow[rr, "a"'] & & A
\end{tikzcd}
=
\begin{tikzcd}
SA \arrow[d, "s_{SA}"'] \arrow[ddr, bend left, "1_{SA}"] & & \\
S^2A \arrow[dr, "m_A"'] \arrow[r, phantom, "\Downarrow \rho_A"] & & \\
& SA \arrow[r, "a"'] & A
\end{tikzcd}
\tag{1.1.8}$$

is derivable.

Given pseudoalgebras A and B, a *pseudoalgebra morphism* $f : A \to B$ consists of a 2-cell $f : A \to B$ and an invertible 3-cell

$$\begin{tikzcd}
SA \arrow[r, "Sf"] \arrow[d, "a"'] \arrow[dr, phantom, "\Downarrow \bar{f}"] & SB \arrow[d, "b"] \\
A \arrow[r, "f"'] & B
\end{tikzcd}$$

satisfying the coherence conditions (1.1.9) and (1.1.10) stated below.

$$\begin{tikzcd}
S^2A \arrow[rr, "S^2f"] \arrow[dr, "Sa"] \arrow[dd, "m_A"'] \arrow[drr, phantom, "\Downarrow S\bar{f}"] & & S^2B \arrow[dr, "Sb"] & \\
& SA \arrow[rr, "Sf"] \arrow[dd, "a"] \arrow[ddrr, phantom, "\Downarrow \bar{f}"] \arrow[l, phantom, "\Downarrow \bar{a}"] & & SB \arrow[dd, "b"] \\
SA \arrow[dr, "a"'] & & & \\
& A \arrow[rr, "f"'] & & B
\end{tikzcd}
=
\begin{tikzcd}
S^2A \arrow[rr, "S^2f"] \arrow[dd, "m_A"'] \arrow[ddrr, phantom, "m_f \Downarrow"] & & SB \arrow[dd, "m_B"'] \arrow[dr, "Sb"] & \\
& & & SB \arrow[dd, "b"] \arrow[dl, phantom, "\Downarrow \bar{b}"] \\
SA \arrow[rr, "Sf"] \arrow[dr, "a"'] \arrow[drr, phantom, "\Downarrow \bar{f}"] & & SB \arrow[dr, "b"] & \\
& A \arrow[rr, "f"'] & & B\,.
\end{tikzcd}
\tag{1.1.9}$$

$$\begin{tikzcd}
A \arrow[d, "s_A"'] \arrow[ddr, bend left, "1_A"] & & \\
SA \arrow[dr, "a"'] \arrow[r, phantom, "\tilde{a}\Downarrow"] & & \\
& A \arrow[r, "f"'] & B
\end{tikzcd}
=
\begin{tikzcd}
A \arrow[rr, "f"] \arrow[d, "s_A"'] \arrow[drr, phantom, "\Downarrow s_f"] & & B \arrow[d, "s_B"] \arrow[ddr, bend left, "1_B"] & \\
SA \arrow[rr, "Sf"] \arrow[dr, "a"'] \arrow[drr, phantom, "\Downarrow \bar{f}"] & & SB \arrow[dr, "b"'] \arrow[r, phantom, "\tilde{b}\Downarrow"] & \\
& A \arrow[rr, "f"'] & & B\,.
\end{tikzcd}
\tag{1.1.10}$$

Given pseudoalgebra morphisms $f : A \to B$ and $g : A \to B$, a *pseudoalgebra* 2-*cell* consists of a 3-cell $\alpha : f \to g$ satisfying the coherence condition (1.1.11).

$$\begin{tikzcd}
SA \arrow[r, bend left, "Sf"] \arrow[r, bend right, "Sg"'] \arrow[r, phantom, "\Downarrow S\alpha"] \arrow[d, "a"'] \arrow[dr, phantom, "\Downarrow \bar{g}"] & SB \arrow[d, "b"] & & SA \arrow[r, bend left, "Sf"] \arrow[d, "a"'] \arrow[dr, phantom, "\Downarrow \bar{f}"] & SB \arrow[d, "b"] \\
A \arrow[r, bend right, "g"'] & B & = & A \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow \alpha"] & B\,.
\end{tikzcd} \tag{1.1.11}$$

There is a forgetful 2-functor $U_I : \text{Ps-}S\text{-Alg}(I) \to \mathscr{K}(I, X)$, defined by mapping a pseudo-S-algebra to its underlying 1-cell, which has a left pseudoadjoint, defined by mapping a 1-cell $A : I \to X$ to the free pseudoalgebra on it, given by the composite 1-cell $SA : I \to X$. Attentive readers will have observed that the directions of the structural 3-cells μ and λ for a pseudomonad as in Definition 1.1.5 match those of the 3-cells necessary to make SA into a pseudoalgebra.

The function mapping an object $I \in \mathscr{K}$ to the 2-category Ps-S-Alg(I) extends to a Gray-functor Ps-S-Alg : $\mathscr{K}^{op} \to$ **Gray**. We also have a Gray-transformation

$$U : \text{Ps-}S\text{-Alg} \to X\,, \tag{1.1.12}$$

with components given by the forgetful 2-functors $U_I : \text{Ps-}S\text{-Alg}(I) \to \mathscr{K}(I, X)$, for $I \in \mathscr{K}$. Note the use of our convention on the Yoneda lemma in (1.1.12). Note that the structure of pseudo-S-algebra on a 1-cell $A\colon I \to X$ can be viewed as a left S-action on A, associative and unital up to coherent isomorphism. For this reason, we sometimes refer to pseudoalgebras as *left pseudomodules*. This terminology is convenient when we discuss dualities in Section 2.2.

1.2. The Gray-category of Pseudomonads

The aim of this section is to introduce the 3-dimensional category **Psm**$(\mathscr{K})$ of pseudomonads in a Gray-category $\mathscr{K}$ and prove that it is a Gray-category. In order to do so, we review the notion of a pseudomonad morphism from [MW08] and introduce the notions of a pseudomonad transformation and modification. Again, we fix a Gray-category $\mathscr{K}$. When working with two pseudomonads (X, S) and (Y, T), we use m and s for the multiplication and unit of S, n and t for the multiplication and unit of T, but we use the same letters μ, λ, ρ for the structural 3-cells of both monads to simplify notation, as the context makes it always clear to which we are referring.

Definition 1.2.1. Let (X, S) and (Y, T) be pseudomonads in $\mathscr{K}$. A *pseudomonad morphism* $(F, \phi)\colon (X,\, S) \to (Y,\, T)$ consists of a 1-cell $F : X \to Y$, a 2-cell $\phi : TF \to FS$ and two invertible 3-cells

$$\begin{tikzcd}
T^2F \arrow[r, "T\phi"] \arrow[dd, "nF"'] & TFS \arrow[d, "\phi S"] \\
 & FS^2 \arrow[d, "Fm"] \\
TF \arrow[r, "\phi"'] & FS
\arrow[from=1-1, to=3-2, phantom, "\Downarrow \bar{\phi}"]
\end{tikzcd}
\qquad
\begin{tikzcd}
F \arrow[r, "tF"] \arrow[dr, bend right, "Fs"'] & TF \arrow[d, "\phi"] \\
 & FS.
\arrow[from=1-1, to=2-2, phantom, "\overset{\widetilde{\phi}}{\Longrightarrow}"]
\end{tikzcd}$$

These data are required to satisfy the coherence axioms in (1.2.1) and (1.2.2).

$$\begin{tikzcd}
T^3F \arrow[rr, "T^2\phi"] \arrow[dd, "nTF"'] \arrow[ddrr, "TnF"] & & T^2FS \arrow[dr, "T\phi S"] & & \\
& & \Downarrow T\bar{\phi} & TFS^2 \arrow[dr, "TFm"] & \\
T^2F \arrow[ddrr, "nF"'] & \Downarrow \mu F & T^2F \arrow[rr, "T\phi"'] \arrow[dd, "nF"'] & & TFS \arrow[d, "\phi S"] & = \\
& & & \Downarrow \bar{\phi} & FS^2 \arrow[d, "Fm"] & \\
& & TF \arrow[rr, "h"'] & & FS &
\end{tikzcd}$$

$$\begin{tikzcd}
T^3F \arrow[rr, "T^2\phi"] \arrow[dd, "nTF"'] & & T^2FS \arrow[dr, "T\phi S"] \arrow[dd, "nFS"] & & \\
& \Downarrow n_\phi & \Downarrow \bar{\phi}S & TFS^2 \arrow[d, "\phi S^2"'] \arrow[dr, "TFm"] & \\
T^2F \arrow[rr, "T\phi"] \arrow[ddrr, "nF"'] & & TFS \arrow[dr, "\phi S"'] & FS^3 \arrow[r, phantom, "\Downarrow \phi_m"] \arrow[d, "FmS"'] \arrow[dr, "FSm"] & TFS \arrow[d, "\phi S"] \\
& & \Downarrow \bar{\phi} & FS^2 \arrow[r, phantom, "\Downarrow F\mu"] \arrow[dr, "Fm"'] & FS^2 \arrow[d, "Fm"] \\
& & TF \arrow[rr, "\phi"'] & & FS
\end{tikzcd} \tag{1.2.1}$$

$$\begin{tikzcd}
TF \arrow[drr, bend left, "TFs"] \arrow[dr, "TtF"] \arrow[dddr, bend right, "1_{TF}"'] & \Downarrow T\tilde{\phi} & & & TF \arrow[drr, bend left, "TFs"] \arrow[d, "\phi"'] & & \\
& T^2F \arrow[r, "T\phi"] \arrow[dd, "nF"'] & TFS \arrow[d, "\phi S"] & = & FS \arrow[dr, "FSs"] \arrow[ddrr, bend right, "1_{FS}"'] & \Downarrow \phi_s & TFS \arrow[d, "\phi S"] \\
\Downarrow \lambda H & \Downarrow \bar{\phi} & FS^2 \arrow[d, "Fm"] & & & \Downarrow F\lambda & FS^2 \arrow[d, "Fm"] \\
& TF \arrow[r, "\phi"'] & FS & & & & FS
\end{tikzcd} \tag{1.2.2}$$

Proposition 1.2.2. *[MW08, Theorem 2.3] Let $(F, \phi) : (X, S) \to (Y, T)$ be a pseudomonad morphism. The coherence condition*

$$\begin{tikzcd}
TF \arrow[r, "\phi"] \arrow[d, "tTF"'] \arrow[dr, phantom, "\Downarrow t_\phi"] & FS \arrow[d, "tFS"'] \arrow[ddr, "FsS"] \arrow[dddrr, bend left, "1_{FS}"] \arrow[dr, phantom, "\Downarrow \tilde{\phi} S"] & & \\
T^2F \arrow[r, "T\phi"] \arrow[ddrr, "nF"'] & TFS \arrow[dr, "\phi S"] \arrow[d, phantom, "\Downarrow \bar{\phi}"] & \Downarrow F\rho_S & \\
 & & FS^2 \arrow[dr, "Fm"] & \\
 & & TF \arrow[r, "\phi"'] & FS
\end{tikzcd}
=
\begin{tikzcd}
TF \arrow[d, "tTF"'] \arrow[ddr, bend left, "1_{TF}"] & \\
T^2F \arrow[dr, "nF"'] \arrow[r, phantom, "\Downarrow \rho_T F"] & \\
 & TF \arrow[r, "\phi"'] & FS
\end{tikzcd}$$

is derivable.

Definition 1.2.3. Let $(F, \phi), (F', \phi') : (X, S) \to (Y, T)$ be pseudomonad morphisms. A *pseudomonad transformation* $(p, \bar{p}) : (F, \phi) \to (F', \phi')$ consists of a 2-cell $p : F \to F'$ and an invertible 3-cell

$$\begin{tikzcd}
TF \arrow[r, "Tp"] \arrow[d, "\phi"'] \arrow[dr, phantom, "\Downarrow \bar{p}"] & TF' \arrow[d, "\phi'"] \\
FS \arrow[r, "pS"'] & F'S
\end{tikzcd}$$

satisfying the coherence conditions in (1.2.3) and (1.2.4) below.

$$\begin{tikzcd}
T^2F \arrow[r, "T^2p"] \arrow[dd, "nF"'] \arrow[dr, "T\phi"] & T^2F' \arrow[dr, "T\phi'"] \arrow[d, phantom, "\Downarrow T\bar{p}"] & \\
 & TFS \arrow[r, "TpS"] \arrow[d, "\phi S"] \arrow[dr, phantom, "\Downarrow \bar{p}S"] & TF'S \arrow[d, "\phi' S"] \\
TF \arrow[dr, "\phi"'] \arrow[ur, phantom, "\Downarrow \bar{\phi}"] & FS^2 \arrow[r, "pS^2"] \arrow[d, "Fm"'] \arrow[dr, phantom, "\Downarrow p_m^{-1}"] & F'S^2 \arrow[d, "F'm"] \\
 & FS \arrow[r, "pS"'] & F'S
\end{tikzcd}
=
\begin{tikzcd}
T^2F \arrow[r, "T^2p"] \arrow[dd, "nF"'] \arrow[ddr, phantom, "\Downarrow n_p"] & T^2F' \arrow[dd, "nF'"] \arrow[dr, "T\phi'"] & \\
 & & TF'S \arrow[dd, "\phi' S"] \\
TF \arrow[r, "Tp"] \arrow[ddr, "\phi"'] & TF' \arrow[ddr, "\phi'"] \arrow[d, phantom, "\Downarrow \bar{p}"] \arrow[ur, phantom, "\Downarrow \bar{\phi}'"] & \\
 & & F'S^2 \arrow[d, "F'm"] \\
 & FS \arrow[r, "pS"'] & F'S
\end{tikzcd}
\tag{1.2.3}$$

$$\begin{tikzcd}
F \arrow[r, "p"] \arrow[d, "tF"'] \arrow[ddr, bend left, "Fs"] & F' \arrow[ddr, bend left, "F's"] & \\
TF \arrow[dr, "\phi"'] \arrow[r, phantom, "\Downarrow \tilde{\phi}"] & \arrow[r, phantom, "\Downarrow p_s^{-1}"] & {} \\
 & FS \arrow[r, "pS"'] & F'S
\end{tikzcd}
=
\begin{tikzcd}
F \arrow[r, "p"] \arrow[d, "tF"'] \arrow[dr, phantom, "\Downarrow t_p"] & F' \arrow[d, "tF'"] \arrow[ddr, bend left, "F's"] & \\
TF \arrow[r, "Tp"] \arrow[dr, "\phi"'] & TF' \arrow[dr, "\phi'"] \arrow[d, phantom, "\Downarrow \bar{p}"] \arrow[r, phantom, "\Downarrow \tilde{\phi}'"] & {} \\
 & FS \arrow[r, "pS"'] & F'S
\end{tikzcd}
\qquad (1.2.4)$$

Definition 1.2.4. Let $(p,\, \widetilde{p})$, $(p',\, \widetilde{p'}) : (F,\, \phi) \to (F',\, \phi')$ be pseudomonad transformations. A *pseudomonad modification* $\alpha : (p,\, \widetilde{p}) \to (p',\, \widetilde{p'})$ is a 3-cell $\alpha : p \to p'$ satisfying the coherence condition below.

$$\begin{tikzcd}
TF \arrow[r, bend left, "Tp"] \arrow[r, bend right, "Tp'"'] \arrow[r, phantom, "\Downarrow T\alpha"] \arrow[d, "\phi"'] & TF' \arrow[d, "\phi'"] \\
FS \arrow[r, bend right, "p'S"'] \arrow[ur, phantom, "\Downarrow \bar{p}'"] & F'S
\end{tikzcd}
=
\begin{tikzcd}
TF \arrow[r, bend left, "Tp"] \arrow[d, "\phi"'] & TF' \arrow[d, "\phi'"] \\
FS \arrow[r, bend left, "pS"] \arrow[r, bend right, "p'S"'] \arrow[r, phantom, "\Downarrow \alpha"] \arrow[ur, phantom, "\Downarrow \bar{p}"] & F'S
\end{tikzcd}
\qquad (1.2.5)$$

The following is our first main result, which solves the problem raised in [Lac00, Section 6].

Theorem 1.2.5. *Let $\mathscr{K}$ be a Gray-category. Then there is a Gray-category* **Psm**$(\mathscr{K})$*, called the Gray-category of pseudomonads in $\mathscr{K}$, having pseudomonads in $\mathscr{K}$ as 0-cells, pseudomonad morphisms as 1-cells, pseudomonad transformations as 2-cells, and pseudomonad modifications as 3-cells.*

The rest of this section is devoted to the proof of Theorem 1.2.5, which will be obtained by combining Lemmas 1.2.6, 1.2.8, 1.2.9 and 1.2.10 below. We begin by giving the definition of the hom-2-categories of **Psm**$(\mathscr{K})$.

Lemma 1.2.6. *Let $(X,\, S)$ and $(Y,\, T)$ be two pseudomonads in $\mathscr{K}$. Then there is a 2-category* **Psm**$(\mathscr{K})(\,(X,\, S),\, (Y,\, T)\,)$ *having pseudomonad morphisms from (X, S) to (Y, T) as 0-cells, pseudomonad transformations as 1-cells and pseudomonad modifications as 2-cells.*

Proof. First of all, for any pair of composable 1-cells $(p_0,\, \widetilde{p_0})\colon (F_0, \phi_0) \to (F_1, \phi_1)$ and $(p_1,\, \widetilde{p_1})\colon (F_1,\, \phi_1) \to (F_2,\, \phi_2)$ we define their composition as $(p_1 p_0,\, \widetilde{p_1 p_0})$ where $\widetilde{p_1 p_0}$ is defined as the pasting of

$$\begin{tikzcd}
TF_0 \arrow[r, "Tp_0"] \arrow[d, "\phi_0"'] \arrow[dr, phantom, "\Downarrow \widetilde{p_0}"] & TF_1 \arrow[r, "Tp_1"] \arrow[d, "\phi_1"] \arrow[dr, phantom, "\Downarrow \widetilde{p_1}"] & TF_2 \arrow[d, "\phi_2"] \\
F_0 S \arrow[r, "p_0 S"'] & F_1 S \arrow[r, "p_1 S"'] & F_2 S.
\end{tikzcd}$$

We want to show that composition is strictly associative. So let us consider three composable 1-cells

$$(F_0,\, \phi_0) \xrightarrow{(p_0,\widetilde{p_0})} (F_1,\, \phi_1) \xrightarrow{(p_1,\widetilde{p_1})} (F_2,\, \phi_2) \xrightarrow{(p_2,\widetilde{p_2})} (F_3,\, \phi_3)$$

By definition, the two possible composites are

$$\begin{aligned}
(p_2,\, \widetilde{p_2}) \cdot \big((p_1,\, \widetilde{p_1}) \cdot (p_0,\, \widetilde{p_0})\big) &= (p_2(p_1 p_0),\, \widetilde{p_2(p_1 p_0)})\,, \\
\big((p_2,\, \widetilde{p_2}) \cdot (p_1,\, \widetilde{p_1})\big) \cdot (p_0,\, \widetilde{p_0}) &= ((p_2 p_1) p_0,\, \widetilde{(p_2 p_1) p_0})\,.
\end{aligned}$$

We want to show that these are equal. Since $\mathscr{K}$ is a Gray-category, $p_2(p_1 p_0) = (p_2 p_1) p_0$. Moreover, $\widetilde{p_2(p_1 p_0)} = \widetilde{(p_2 p_1) p_0}$ since they are both the pasting of

$$\begin{tikzcd}
TF_0 \arrow[r, "Tp_0"] \arrow[d, "\phi_0"'] \arrow[dr, phantom, "\Downarrow \widetilde{p_0}"] & TF_1 \arrow[r, "Tp_1"] \arrow[d, "\phi_1"] \arrow[dr, phantom, "\Downarrow \widetilde{p_1}"] & TF_2 \arrow[r, "Tp_2"] \arrow[d, "\phi_2"] \arrow[dr, phantom, "\Downarrow \widetilde{p_2}"] & TF_3 \arrow[d, "\phi_3"] \\
F_0 S \arrow[r, "p_0 S"'] & F_1 S \arrow[r, "p_1 S"'] & F_2 S \arrow[r, "p_2 S"'] & F_3 S.
\end{tikzcd}$$

It remains to define the identity 1-cells of $\mathbf{Psm}(\mathscr{K})(\,(X,\, S),\, (Y,\, T)\,)$. For a pseudomonad morphism $(F, \phi)\colon (X, S) \to (Y, T)$, we define the identity on it to be

$$(1_F,\, 1_\phi) : (F,\, \phi) \to (F,\, \phi)\,.$$

This is allowed since $T1_F = 1_{TF}$ and $1_F S = 1_{FS}$. These can be shown to be a strict identities, using that $\mathscr{K}$ is a Gray-category and in particular Axiom (G6). $\square$

We proceed by defining the composition of 1-cells in $\mathbf{Psm}(\mathscr{K})$ and proving that is *strictly* associative, as required to have a Gray-category. Since a pseudomonad morphism is a tuple of the form $(F, \phi, \bar{\phi}, \tilde{\phi})$, where F is a 1-cell, ϕ is a 2-cell while $\bar{\phi}$ and $\tilde{\phi}$ are 3-cells, we will need to check equalities at three levels. The key level of the verification is that of 2-cells. Indeed, strict associativity at the level of 1-cells will follow easily from the strict associativity of composition of 1-cells in $\mathscr{K}$. The key issue are the equalities at the level of 2-cells, since 2-cells could be isomorphic (by means of an invertible 3-cell), but not equal. Instead, equalities of 3-cells will be quite straightforward. In fact, the required equations for 3-cells either hold strictly or they fail completely, since there are no 4-cells that could make these equations hold only up to isomorphism.

In the following, for a pseudomonad morphism $\underline{F} = (F,\, \phi,\, \bar{\phi},\, \widetilde{\phi})$, we define

$$\underline{F}^{-} := \bar{\phi}\,, \qquad \underline{F}^{\sim} := \widetilde{\phi}\,.$$

Let $(F,\, \phi) : (X,\, S) \to (Y,\, T)$ and $(G,\, \psi) : (Y,\, T) \to (Z,\, Q)$ be two pseudomonad morphisms. We define their composition as

$$(G,\, \psi, \underline{G}^{-}, \underline{G}^{\sim}) \circ (F,\, \phi, \underline{F}^{-}, \underline{F}^{\sim}) := \big(GF,\, G\phi \cdot \psi F\,, \underline{G} \circ \underline{F}^{-}, \underline{G} \circ \underline{F}^{\sim}\big) \tag{1.2.6}$$

where the invertible 3-cells are defined by the following pasting diagrams:

$$\begin{tikzcd}
 & Q^2GF \arrow[r, "Q\psi F"] \arrow[ddd, "m_Q GF"'] \arrow[dddr, phantom, "\bar{\psi}F \Downarrow"] & QGTF \arrow[r, "QG\phi"] \arrow[d, "\psi TF"'] \arrow[dr, phantom, "\psi_\phi \Downarrow"] & QGFS \arrow[d, "\psi FS"] \\
 & & GT^2F \arrow[r, "GT\phi"'] \arrow[dd, "GnF"'] \arrow[ddr, phantom, "G\bar{\phi} \Downarrow"] & GTFS \arrow[d, "G\phi S"] \\
\underline{G} \circ \underline{F}^{-} := & & & GFS^2 \arrow[d, "GFm"] \\
 & QGF \arrow[r, "\psi F"'] & GTF \arrow[r, "G\phi"'] & GFS
\end{tikzcd}$$

$$\begin{tikzcd}
& GF \arrow[rrr, "GFs"] \arrow[drr, "GtF"] \arrow[dr, "qGF"'] & & & GFS \\
\underline{G}\circ\underline{F}^{\sim} := & & & GTF \arrow[ur, "G\phi"'] & \\
& & QGF. \arrow[ur, "\psi F"'] & &
\arrow[from=2-4, to=1-4, phantom, "G\widetilde{\phi}\Downarrow"]
\arrow[from=2-3, to=3-3, phantom, "\widetilde{\psi}F\Downarrow"]
\end{tikzcd}$$

The proof that this definition gives a pseudomonad morphism is in Appendix A.2.

Remark 1.2.7. We did not consider any parenthesis in the diagrams above thanks to axiom (G5) for a Gray-category. Moreover since $Q(-)$ is a strict 2-functor we have $Q(G\phi \cdot \psi F) = QG\phi \cdot Q\psi F$ (and similarly for other compositions in the diagrams).

Lemma 1.2.8. *The composition of pseudomonad morphisms defined in* (1.2.6) *is strictly associative.*

Proof. From now on, let us consider three pseudomonad morphisms in $\mathscr{K}$:

$$(X,\, S) \xrightarrow{(F,\phi)} (Y,\, T) \xrightarrow{(G,\psi)} (Z,\, Q) \xrightarrow{(H,\xi)} (V,\, R).$$

In order to prove this statement we have to prove that the equation for associativity holds for the respective 1-, 2- and 3-cell components. For 1-cells, since $\mathscr{K}$ is a Gray-category, then $H(GF) = (HG)F$.

For 2-cells, the idea is to reduce both composites to $HG\phi \cdot H\psi F \cdot \xi GF$. On the one hand,

$$\begin{aligned}
H(G\phi\cdot\psi F)\cdot\xi GF &= [H(G\phi)\cdot H(\psi F)]\cdot\xi GF && \text{(because } H(-) \text{ is strict)}\\
&= [HG\phi\cdot H\psi F]\cdot\xi GF && \text{(by (G5))}\\
&= HG\phi\cdot H\psi F\cdot\xi GF && \text{(since } \mathscr{K}(X,\,V) \text{ is a 2-category).}
\end{aligned}$$

On the other hand,

$$HG\phi\cdot(H\psi\cdot\xi G)F = HG\phi\cdot[(H\psi)F\cdot(\xi G)F] \qquad \text{(because } (-)F \text{ is strict)}$$

$$= HG\phi \cdot [H\psi F \cdot \xi GF] \qquad \text{(by (G5))}$$

$$= HG\phi \cdot H\psi F \cdot \xi GF \qquad \text{(since } \mathscr{K}(X,\, V) \text{ is a 2-category).}$$

For 3-cells, to prove that $\big(\underline{H}(\underline{GF})\big)^{\sim} = \big((\underline{HG})\underline{F}\big)^{\sim}$ we just need to notice that, using the fact that $H(-)$ and $(-)F$ are strict 2-functors, both of them are pasting of:

$$\begin{tikzcd}
HGF \arrow[rrrr, "HGFs"] \arrow[rrrd, "HGtF"] \arrow[rrdd, "HqGF"] \arrow[rddd, "rHGF"'] & & & HG\widetilde{\phi}\Downarrow & HGFS \\
& & H\widetilde{\psi}F\Downarrow & HGTF \arrow[ur, "HG\phi"'] & \\
& \widetilde{\xi}GF\Downarrow & HQGF \arrow[ur, "H\psi F"'] & & \\
& RHGF. \arrow[ur, "\xi GF"'] & & &
\end{tikzcd}$$

We get the required equality by the pasting theorem for 2-categories [Pow90]. Finally, let us prove the equality on the other 3-cell component. By definition,

$$\begin{tikzcd}
& R^2HGF \arrow[rr] \arrow[dd] & & RHQGF \arrow[r] \arrow[d] \arrow[dr, phantom, "\xi_{(G\phi\cdot\psi F)}\Downarrow"] & RHGFS \arrow[d] \\
\underline{H}(\underline{GF})^{-} = & & \bar{\xi}GF\Downarrow & HQ^2GF \arrow[r] \arrow[d] \arrow[dr, phantom, "H(\underline{GF}^{-})\Downarrow"] & HQGFS \arrow[d] \\
& RHGF \arrow[rr] & & HQGF \arrow[r] & HGFS\,.
\end{tikzcd}$$

Using the definition of $\underline{GF}^{-}$and (1.1.3), the right-hand side pasting becomes

$$\begin{tikzcd}
R^2HGF \arrow[rr] \arrow[dddddd] & & RHQGF \arrow[rr] \arrow[dd] & & RHGTF \arrow[rr] \arrow[dd] & & RHGFS \arrow[dd] \\
& & & \Downarrow \xi_{\psi F} & & \Downarrow \xi_{G\phi} & \\
& & HQ^2GF \arrow[rr] \arrow[dddd] & & HGTFS \arrow[rr] \arrow[dd] & & HQGFS \arrow[dd] \\
& \Downarrow \bar{\xi}GF & & & & \Downarrow H(\psi_\phi) & \\
& & & H\bar{\psi}F \Downarrow & HGT^2F \arrow[rr] \arrow[dd] & & HGTFS \arrow[dd] \\
& & & & & \Downarrow HG\bar{\phi} & \\
RHGF \arrow[rr] & & HQGF \arrow[rr] & & HGTF \arrow[rr] & & HGFS\,.
\end{tikzcd}$$

Let us notice that, by (G5), $H(\psi_\phi) = H\psi_\phi$, $\xi_{\psi F} = (\xi_\psi)F$ and $\xi_{G\phi} = \xi G_\phi$. Moreover, using the definition of $\underline{HG}^{-}$and (1.1.2), the diagram above is equal to

$$\begin{tikzcd}
R^2HGF \arrow[rr] \arrow[dddd] & & RHGTF \arrow[rr] \arrow[dd] & & RHGFS \arrow[dd] \\
& & & \Downarrow (H\psi \cdot \xi G)_\phi & \\
& (\underline{HG}^{-})F \Downarrow & HGT^2FS \arrow[rr] \arrow[dd] & & HGTFS \arrow[dd] \\
& & & \Downarrow HG\bar{\phi} & \\
RHGF \arrow[rr] & & HGTF \arrow[rr] & & HGFS,
\end{tikzcd}$$

which is exactly the definition of $\underline{H}(\underline{GF})^{-}$. ☐

For brevity, we sometimes write $P_{\mathscr{K}}$ instead of $\mathbf{Psm}(\mathscr{K})$, so for any pair of pseudomonads (X, S) and (Y, T) the 2-category of pseudomonads morphisms from (X, S) to (Y, T) can be written as $P_{\mathscr{K}}\big((X, S), (Y, T)\big)$.

Lemma 1.2.9. *The definition of composition of pseudomonad morphisms extends to a cubical functor*

$$- \circ - : P_{\mathscr{K}}(\,(Y,\,T),\,(Z,\,Q)\,) \times P_{\mathscr{K}}(\,(X,\,S),\,(Y,\,T)\,) \longrightarrow P_{\mathscr{K}}(\,(X,\,S),\,(Z,\,Q)\,)$$

for $(X,S)\,,(Y,T)\,,(Z,Q) \in \mathbf{Psm}(\mathscr{K})$.

Proof. This is a just a long verification, but we spell it out in some detail. By the definition of a cubical functor, for $(F,\,\phi) : (X,\,S) \to (Y,\,T)$ and $(G,\,\psi) : (Y,\,T) \to (Z,\,Q)$ in $\mathbf{Psm}(\mathscr{K})$, we need to define strict 2-functors

$$F_\phi \colon P_{\mathscr{K}}(\,(Y,\,T),\,(Z,\,Q)\,) \to P_{\mathscr{K}}(\,(X,\,S),\,(Z,\,Q)\,)\,, \tag{1.2.7}$$

$$G_\psi \colon P_{\mathscr{K}}(\,(X,\,S),\,(Y,\,T)\,) \to P_{\mathscr{K}}(\,(X,\,S),\,(Z,\,Q)\,) \tag{1.2.8}$$

such that

$$F_\phi(\,(G,\,\psi)\,) = G_\psi(\,(F,\,\phi)\,) = (G,\,\psi) \circ (F,\,\phi)\,, \tag{1.2.9}$$

plus, for 2-cells $(p,\,\widetilde{p}) : (F,\,\phi) \to (F',\,\phi')$ and $(q,\,\widetilde{q}) : (G,\,\psi) \to (G',\,\psi')$, an invertible 3-cell in $\mathbf{Psm}(\mathscr{K})$

$$\begin{tikzcd}
(G,\,\psi) \circ (F,\,\phi) \arrow[r, "(G,\,\psi) \circ (p,\,\widetilde{p})"] \arrow[d, "(q,\,\widetilde{q}) \circ (F,\,\phi)"'] \arrow[dr, phantom, "\Downarrow \Sigma_{(p,\widetilde{p}),\,(q,\widetilde{q})}"] & (G,\,\psi) \circ (F',\,\phi') \arrow[d, "(q,\,\widetilde{q}) \circ (F',\,\phi')"] \\
(G',\,\psi') \circ (F,\,\phi) \arrow[r, "(G',\,\psi') \circ (p,\,\widetilde{p})"'] & (G',\,\psi') \circ (F',\,\phi')
\end{tikzcd} \tag{1.2.10}$$

satisfying axioms (1.1.1), (1.1.2) and (1.1.3).

We begin by defining F_ϕ in (1.2.7). Its action on objects is determined by (1.2.9). For its action on 1-cells, we send $(q,\,\widetilde{q})\colon (G,\psi) \to (G',\psi)$ to the pseudomonad modification $(qF,\widetilde{qF})\colon (GF, G\phi \cdot \psi F) \to (G'F, G'\phi \cdot \psi' F)$, where $\widetilde{qF}$ is defined as the following pasting:

$$\begin{tikzcd}
QGF \arrow[r, "QqF"] \arrow[d, "\psi F"'] \arrow[dr, phantom, "\Downarrow \widetilde{q}F"] & QG'F \arrow[d, "\psi' F"] \\
GTF \arrow[r, "qTF"'] \arrow[d, "G\phi"'] \arrow[dr, phantom, "\Downarrow q_{\phi}{}^{-1}"] & G'TF \arrow[d, "G'\phi"] \\
GFS \arrow[r, "qFS"'] & G'FS.
\end{tikzcd}$$

The action of F_ϕ on 3-cells $\beta : (q,\, \widetilde{q}) \to (q',\, \widetilde{q'})$ is defined by letting $\beta \circ (F,\, \phi) := \beta F$ in $\mathscr{K}$. The proof that this is a pseudomonad modification, and therefore a 3-cells in $\mathbf{Psm}(\mathscr{K})$, is in Appendix A.2.

We now show that F_ϕ is a 2-functor. For this, we use extensively the notation of Remark 1.1.3 to avoid writing some diagrams. To prove that composition is preserved strictly, we show that

$$F_\phi(\,(q',\, \widetilde{q'}) \cdot (q,\, \widetilde{q})\,) = F_\phi(q',\, \widetilde{q'}) \cdot F_\phi(q,\, \widetilde{q}) \tag{1.2.11}$$

for any

$$(G,\, \psi) \xrightarrow{(q,\widetilde{q})} (G',\, \psi') \xrightarrow{(q',\widetilde{q'})} (G',\, \psi')$$

in $P_{\mathscr{K}}(\,(Y,\, T),\, (Z,\, Q)\,)$. The composition $(q',\, \widetilde{q'}) \cdot (q,\, \widetilde{q})$ is defined as $(q'q,\, \widetilde{q'q})$ where $\widetilde{q'q}$ is defined as the pasting of

$$\begin{tikzcd}
QG \arrow[r] \arrow[d] \arrow[dr, phantom, "\Downarrow \widetilde{q}"] & QG' \arrow[r] \arrow[d] \arrow[dr, phantom, "\Downarrow \widetilde{q'}"] & QG' \arrow[d] \\
GT \arrow[r] & G'T \arrow[r] & G'T.
\end{tikzcd}$$

Using the equation (1.1.3) we can see that the 3-cells components of $F_\phi(\,(q', \widetilde{q'}) \cdot (q, \widetilde{q})\,)$ and $F_\phi(q',\, \widetilde{q'}) \cdot F_\phi(q,\, \widetilde{q})$ are two pasting of the diagram below:

$$\begin{tikzcd}
QG \arrow[r] \arrow[d] \arrow[rd, phantom, "\Downarrow \widetilde{q}"] & QG' \arrow[r] \arrow[d] \arrow[rd, phantom, "\Downarrow \widetilde{q'}"] & QG' \arrow[d] \\
GT \arrow[r] \arrow[d] \arrow[rd, phantom, "\Downarrow {q_\phi}^{-1}"] & G'T \arrow[r] \arrow[d] \arrow[rd, phantom, "\Downarrow {q'_\phi}^{-1}"] & G'T \arrow[d] \\
GF \arrow[r] & G'F \arrow[r] & G'F.
\end{tikzcd}$$

Moreover, $(q' \cdot q)F = q'F \cdot qF$ since $(-)F$ is a strict 2-functor (since $\mathscr{K}$ is a Gray category). Hence, the required equality in (1.2.11) holds. Let us also verify that F_ϕ preserves identities strictly. Recall from Lemma 1.2.6 that $1_{(G,\,\psi)} := (1_G,\, 1_\psi)$ in $P_{\mathscr{K}}(\,(Y,\,T),\,(Z,\,Q)\,)$. Therefore,

$$F_\phi(1_G,\, 1_\psi) = (1_G F,\, \widetilde{1_G F})$$

and moreover

$$\begin{aligned}
(1_G F,\, \widetilde{1_G F}) &= (1_{GF},\, (\,(1_G)_\phi \cdot 1_{\psi F}) * (1_{G\phi} \cdot 1_\psi F)\,) && \text{(by definition of } F_\phi) \\
&= (1_{GF},\, (1_{G\phi} \cdot 1_{\psi F}) * (1_{G\phi} \cdot 1_{\psi F})\,) && \text{(by Remark 1.1.2)} \\
&= (1_{GF},\, (1_{G\phi \cdot \psi F}) * (1_{G\phi \cdot \psi F})\,) && \text{(since } \cdot \text{ preserves identities)} \\
&= (1_{GF},\, 1_{G\phi \cdot \psi F}) && \text{(by (G2))} \\
&= 1_{(GF,\, G\phi \cdot \psi F)} \\
&= 1_{F_\phi(G,\,\psi)}\,,
\end{aligned}$$

as required.

We now define the 2-functor G_ϕ of (1.2.8). Again, its action on objects is determined by (1.2.9). On morphisms, it sends $(p, \widetilde{p})\colon (F, \phi) \to (F', \phi')$ to

$$(Gp, \widetilde{Gp})\colon (GF, G\phi \cdot \psi F) \to (GF', G\phi' \cdot \psi F')\,,$$

where $\widetilde{Gp}$ is defined as the following pasting:

$$\begin{tikzcd}
QGF \arrow[r, "QGp"] \arrow[d, "\psi F"'] & QGF' \arrow[d, "\psi F'"] \arrow[dl, phantom, "\Downarrow \psi_p"] \\
GTF \arrow[r, "GTp"'] \arrow[d, "G\phi"'] & GTF' \arrow[d, "G\phi'"] \arrow[dl, phantom, "\Downarrow G\widetilde{p}"] \\
GFS \arrow[r, "GpS"'] & GF'S \, .
\end{tikzcd}$$

On 3-cells $\alpha : (p, \widetilde{p}) \to (p', \widetilde{p'})$ we let $(G, \phi) \circ \alpha := G\alpha$, which is a 3-cell in $\mathbf{Psm}(\mathscr{K})$ by a similar argument to the one used for F_ϕ. The proof that this is a 2-functor is completely analogous to the one for F_ϕ and hence omitted.

To conclude the proof, we need to define the 3-cell $\Sigma_{(p,\widetilde{p}),(q,\widetilde{q})}$ in (1.2.10). We take this to be q_p, which is shown to be a pseudomonad modification in Appendix A.2. The required axioms for $\Sigma_{(p,\widetilde{p}),(q,\widetilde{q})}$, as in (1.1.1), (1.1.2) and (1.1.3), hold as they are instances of the ones for q_p for $\mathscr{K}$. □

Lemma 1.2.10. *The cubical functor providing composition in* $\mathbf{Psm}(\mathscr{K})$ *satisfies the coherence conditions of Axiom (G5).*

Proof. The first one is just Lemma 1.2.8. Since the definitions on 3-cells coincide with the ones in $\mathscr{K}$, all the equations regarding them hold directly. Therefore, we only need to prove the ones for 2-cells. Let us consider the following diagram in $\mathbf{Psm}(\mathscr{K})$:

$$\begin{tikzcd}
(X, S) \arrow[r, bend left, "{(F, \phi)}"] \arrow[r, bend right, "{(F', \phi')}"'] \arrow[r, phantom, "\Downarrow (p, \widetilde{p})"] & (Y, T) \arrow[r, bend left, "{(G, \psi)}"] \arrow[r, bend right, "{(G', \psi')}"'] \arrow[r, phantom, "\Downarrow (q, \widetilde{q})"] & (Z, Q) \arrow[r, bend left, "{(H, \xi)}"] \arrow[r, bend right, "{(H', \xi')}"'] \arrow[r, phantom, "\Downarrow (r, \widetilde{r})"] & (V, R)
\end{tikzcd}$$

We need to prove:

(i) $(H_\xi \circ G_\psi)\,(p, \widetilde{p}) = H_\xi(G_\psi\,(p, \widetilde{p})\,)$,

(ii) $(\, H_\xi\,(q,\, \widetilde{q})\,)F_\phi = H_\xi(\,(q,\, \widetilde{q})F_\phi\,),$

(iii) $(\,(r,\, \widetilde{r})\, G_\psi\,)F_\phi = (r,\, \widetilde{r})\,(G_\psi \circ F_\phi\,).$

At the 2-cells level we have $(HG)p = H(Gp)$ because $\mathscr{K}$ is a Gray-category. The same happens in (ii) and (iii) so we will just prove that the associated 3-cells are equal in each case. Let us start with part (i). On the one hand, by definition,

$$\widetilde{(HG)}p = \widetilde{H\widetilde{Gp}}\,,$$

and therefore

$$\begin{tikzcd}
RHGF \arrow[r] \arrow[d] \arrow[dr, phantom, "\Downarrow \xi_{Gp}"] & RHGF' \arrow[d] \\
HQGF \arrow[r] \arrow[d] \arrow[dr, phantom, "\Downarrow H\widetilde{Gp}"] & HQGF' \arrow[d] \\
HGFS \arrow[r] & HGF'S
\end{tikzcd}
=
\begin{tikzcd}
RHGF \arrow[r] \arrow[d] \arrow[dr, phantom, "\Downarrow \xi_{Gp}"] & RHGF' \arrow[d] \\
HQGF \arrow[r] \arrow[d] \arrow[dr, phantom, "\Downarrow H\psi_p"] & HQGF' \arrow[d] \\
HGTF \arrow[r] \arrow[d] \arrow[dr, phantom, "\Downarrow HG\widetilde{p}"] & HGTF' \arrow[d] \\
HGFS \arrow[r] & HGF'S.
\end{tikzcd}$$

On the other hand $H_\xi(G_\psi\,(p,\, \widetilde{p})\,) = (HG,\, H\psi \cdot \xi G)\,(p,\, \widetilde{p})$ so the associated 3-cell is, using (1.1.2),

$$\begin{tikzcd}
RHGF \arrow[r] \arrow[d] \arrow[dr, phantom, "\Downarrow (H\psi \cdot \xi G)_p"] & RHGF' \arrow[d] \\
HGTF \arrow[r] \arrow[d] \arrow[dr, phantom, "\Downarrow HG\widetilde{p}"] & HGTF' \arrow[d] \\
HGFS \arrow[r] & HGF'S
\end{tikzcd}
=
\begin{tikzcd}
RHGF \arrow[r] \arrow[d] \arrow[dr, phantom, "\Downarrow \xi G_p"] & RHGF' \arrow[d] \\
HQGF \arrow[r] \arrow[d] \arrow[dr, phantom, "\Downarrow H\psi_p"] & HQGF' \arrow[d] \\
HGTF \arrow[r] \arrow[d] \arrow[dr, phantom, "\Downarrow HG\widetilde{p}"] & HGTF' \arrow[d] \\
HGFS \arrow[r] & HGF'S.
\end{tikzcd}$$

But $\xi G_p = \xi_{Gp}$, since $\mathscr{K}$ is a Gray-category, and so the required equality holds.

For part (ii), by definition, the 3-cell component of $H_\xi\,(q,\,\widetilde{q})\,)F_\phi$ is:

$$\begin{tikzcd} RHGF \arrow[r] \arrow[d] \arrow[dr, phantom, "\Downarrow \widetilde{Hq}F"] & RHG'F \arrow[d] \\ HGTF \arrow[r] \arrow[d] \arrow[dr, phantom, "\Downarrow {Hq_\phi}^{-1}"] & HG'TF \arrow[d] \\ HGFS \arrow[r] & HG'FS \end{tikzcd} = \begin{tikzcd} RHGF \arrow[r] \arrow[d] \arrow[dr, phantom, "\Downarrow (\xi_q)F"] & RHGF' \arrow[d] \\ HQGF \arrow[r] \arrow[d] \arrow[dr, phantom, "\Downarrow H\widetilde{q}F"] & HQG'F \arrow[d] \\ HGTF \arrow[r] \arrow[d] \arrow[dr, phantom, "\Downarrow {Hq_\phi}^{-1}"] & HG'TF \arrow[d] \\ HGFS \arrow[r] & HG'FS \end{tikzcd} = \begin{tikzcd} RHGF \arrow[r] \arrow[d] \arrow[dr, phantom, "\Downarrow \xi_{qF}"] & RHGF' \arrow[d] \\ HQGF \arrow[r] \arrow[d] \arrow[dr, phantom, "\Downarrow H(\widetilde{qF})"] & HQG'F \arrow[d] \\ HGFS \arrow[r] & HG'FS. \end{tikzcd}$$

Finally, part (iii) is completely analogous to the first one using the inverse of 1.1.3 instead of 1.1.2. □

The combination of Lemmas 1.2.6, 1.2.8, 1.2.9 and 1.2.10 proves Theorem 1.2.5.

2. Pseudodistributive Laws

Introduction

In Chapter 1 we constructed the direct counterpart for pseudomonads of the 2-category **Mnd**$(\mathscr{K})$ of monads, monad morphisms and monad transformations in a 2-category $\mathscr{K}$, which is the the starting point of the formal theory of monads [Str72]. In this chapter, instead, we will consider another approach that was taken earlier by Marmolejo in [Mar99]. Marmolejo introduced, for a Gray-category $\mathscr{K}$, a Gray-category that we denote here **Lift**$(\mathscr{K})$ to avoid confusion, that has pseudomonads in $\mathscr{K}$ as 0-cells and liftings of 1-cells, 2-cells and 3-cells of $\mathscr{K}$ to 2-categories of pseudoalgebras as 1-cells, 2-cells and 3-cells, respectively. He then used **Lift**$(\mathscr{K})$ to introduce the notion of a lifting of a pseudomonad to pseudoalgebras and of a pseudodistributive law, proving the fundamental result that pseudodistributive laws are equivalent liftings of pseudomonads are equivalent to [Mar99, Theorems 6.2, 9.3 and 10.2], thus obtaining an analogue of Beck's result on distributive laws. Here, Marmolejo defined pseudodistributive laws explicitly, giving nine coherence conditions for them [Mar99]. Later, Marmolejo and Wood [MW08] showed not only that an additional tenth coherence condition, introduced by Tanaka [Tan04], can be derived from Marmolejo's conditions, but also that one of the original nine conditions introduced by Marmolejo is derivable from the others, thus reducing the number of coherence axioms for a pseudodistributive law to eight. Subsequent work on simplifying the axioms even further has been done by Walker [Wal21].

Main Results

The aim of this chapter is to continue the work done in Chapter 1. In particular, our main contributions are the following:

- Theorem 2.1.4, the analogue of a fundamental result of the formal theory of monads, asserting that $\mathbf{Psm}(\mathscr{K})$ is equivalent, in a suitable 3-categorical sense, to the Gray-category $\mathbf{Lift}(\mathscr{K})$;
- Proposition 2.2.4, recording that an object of $\mathbf{Psm}(\mathbf{Psm}(\mathscr{K}))$ is the same thing as a pseudodistributive law in $\mathscr{K}$;
- a new, simpler proof of Marmolejo's theorem of the equivalence between pseudodistributive laws and liftings of pseudomonads to 2-categories of pseudoalgebras, given as the proof of Theorem 2.2.5.

Theorem 1.2.5 supports the definition of a pseudodistributive law of [Mar99, MW08], since it allows us to show that a pseudodistributive law is the same thing as a pseudomonad in $\mathbf{Psm}(\mathscr{K})$ (Proposition 2.2.4), as one would expect by analogy with the situation in the formal theory of monads. Thanks to this observation, we provide an interpretation of the complex coherence conditions for a pseudodistributive law in terms of simpler ones, namely those for a pseudomonad morphism, a pseudomonad transformation and a pseudomonad modification (see Table 2.2 for details). This point of view allows us to give a principled presentation of the conditions for pseudodistributive laws of [Mar99, Tan04], included in Appendix A.1, which hopefully provides a useful reference for future work in this area. For the convenience of readers, we also describe how our formulation relates to the ones of Marmolejo and of Tanaka (see Table 2.1).

Theorem 2.1.4, which establishes the equivalence between $\mathbf{Psm}(\mathscr{K})$ and $\mathbf{Lift}(\mathscr{K})$, does not seem to be part of the literature (in part because its very statement requires the introduction of the 3-dimensional category $\mathbf{Psm}(\mathscr{K})$, which is defined here for the first time), but extends existing results. In particular, the equivalence between pseudomonad morphisms and liftings of morphisms to categories of pseudoalgebras is proved in [MW08]. Related results appear also in [Tan04], but with important differences. First, the work carried out therein is developed for the particular tricategory $\mathbf{2\text{-}Cat}_{\mathrm{psd}}$ of 2-categories, pseudofunctors, pseudonatural transformations and modifications, rather than for a general tricategory or Gray-category. While that is an important example (*cf.* Remark 2.2.7), restricting to a particular tricategory does not allow us to exploit the various dualities that are essential to derive results in the formal theory. Secondly, the results obtained therein focus on hom-2-categories of pseudomonad endomorphisms, *i.e.* of the form $\mathbf{Psm}(\mathscr{K})((X,S),(X,S))$, rather than on general hom-2-categories of pseudomonad morphisms.

Our proof of Theorem 2.2.5, which asserts the equivalence between pseudodistributive laws and liftings of pseudomonads to 2-categories of pseudoalgebras established in [Mar99], follows naturally combining Theorem 1.2.5 and Theorem 2.1.4. More specifically, combining our identification of pseudodistributive laws with pseudomonads in $\mathbf{Psm}(\mathscr{K})$ of Proposition 2.2.4 with the fact that a pseudomonad in $\mathbf{Lift}(\mathscr{K})$ is a lifting of a pseudomonad T to the 2-categories of pseudoalgebras of another pseudomonad S, we obtain the desired equivalence between a pseudodistributive law of S over T and a lifting of T to pseudo-S-algebras. This proof is simpler than that in [Mar99] since it takes a modular, more abstract, approach to the verification of the coherence conditions and avoids completely the notion of a composite of pseudomonads with compatible structure.

Outline

We introduce liftings to pseudoalgebras in Section 2.1 and then we prove the equivalence of $\mathbf{Psm}(\mathscr{K})$ and $\mathbf{Lift}(\mathscr{K})$.

We conclude in Section 2.2 by discussing pseudodistributive laws.

2.1. Liftings to Pseudoalgebras

We now recall from [Mar99, Section 7] and [Lac00, Section 6] the definition of the Gray-category **Lift**($\mathscr{K}$) of pseudomonads in $\mathscr{K}$ and liftings to pseudoalgebras. In [Mar99] this was written as **Psm**($\mathscr{K}$), but we prefer to use that notation for the Gray-category introduced in Section 1.2, since it seems the natural generalization of the 2-category of monads defined by Street in [Str72]. We will then show that **Lift**($\mathscr{K}$) is equivalent to **Psm**($\mathscr{K}$), which will be used in Section 2.2 for our results on pseudodistributive laws.

The 0-cells of **Lift**($\mathscr{K}$) are pseudomonads (X, S) in $\mathscr{K}$. For 0-cells (X, S) and (Y, T), a 1-cell $(F, \hat{F}) : (X, S) \to (Y, T)$ consists of a 1-cell $F : X \to Y$ in $\mathscr{K}$ and a Gray-transformation $\hat{F}$: Ps-S-Alg $\to$ Ps-T-Alg making the following diagram commute

$$\begin{tikzcd}
\text{Ps-}S\text{-Alg} \arrow[r, "\hat{F}"] \arrow[d, "U"'] & \text{Ps-}T\text{-Alg} \arrow[d, "U"] \\
X \arrow[r, "F"'] & Y
\end{tikzcd}$$

where, using implicitly the Yoneda lemma for Gray-categories, we write X and Y instead of $\mathscr{K}(-, X)$ and $\mathscr{K}(-, Y)$. We refer to $\hat{F}$ as a *lifting* of F to pseudoalgebras. Analogous terminology will be used for the 2- and 3-cells introduced below.

Lemma 2.1.1. *Let $(F, \phi) : (X, S) \to (Y, T)$ be a pseudomonad morphism. Then, there exists a lifting $\hat{F}$*: Ps-S-Alg $\to$ Ps-T-Alg *of $F \colon X \to Y$.*

Proof. Let us consider a fixed $I \in \mathscr{K}$. First, let us observe that if A is an I-indexed pseudo-S-algebra, then FA is naturally an I-indexed pseudo-T-algebra, with structure map given by the composite

$$TFA \xrightarrow{\phi_A} FSA \xrightarrow{Fa} FA$$

and associativity and unit 3-cells provided by the pasting diagrams

$$\begin{tikzcd}
T^2FA \arrow[r, "T\phi_A"] \arrow[dd, "n_{(FA)}"'] & TFSA \arrow[r, "TFa"] \arrow[d, "\phi_{SA}"] \arrow[dr, phantom, "\Downarrow \phi_a"] & TFA \arrow[d, "\phi_A"] \\
 & FS^2A \arrow[r, "FSa"'] \arrow[d, "Fm_A"] \arrow[dr, phantom, "\Downarrow F\bar{a}"] & FSA \arrow[d, "Fa"] \\
TFA \arrow[r, "\phi_A"'] & FSA \arrow[r, "Fa"'] & FA\,, \arrow[from=1-1, to=3-2, phantom, "\Downarrow \bar{\phi}_A"]
\end{tikzcd}
\qquad
\begin{tikzcd}
FA \arrow[r, "s_{FA}"] \arrow[dr, bend right, "Fs_A"'] \arrow[ddr, bend right=60, "1_{FA}"'] & TFA \arrow[d, "\phi_A"] \\
 & FSA \arrow[d, "Fa"] \\
 & FA\,. \arrow[from=1-1, to=2-2, phantom, shift left=3, "\overset{\tilde{\phi}_A}{\Longrightarrow}"] \arrow[from=2-1, to=3-2, phantom, shift left=3, "\overset{F\tilde{a}}{\Longrightarrow}"]
\end{tikzcd}$$

The coherence condition (1.1.6) for FA follows by an application of the coherence condition (1.2.1) for F and the coherence condition (1.1.6) for A. The coherence condition (1.1.7) for FA follows by applying the coherence condition (1.2.2) for F and the coherence condition (1.1.7) for A. Secondly, we observe that if $f : A \to B$ is a pseudo-S-algebra morphism, then $Ff : FA \to FB$ is naturally a pseudo-T-algebra morphism, as we have the following pasting diagram:

$$\begin{tikzcd}
TFA \arrow[r, "TFf"] \arrow[d, "\phi_A"'] \arrow[dr, phantom, "\Downarrow \phi_f"] & TFB \arrow[d, "\phi_B"] \\
FSA \arrow[r, "FSf"'] \arrow[d, "Fa"'] \arrow[dr, phantom, "\Downarrow F\bar{f}"] & FSB \arrow[d, "Fb"] \\
FA \arrow[r, "Ff"'] & FB\,.
\end{tikzcd}$$

The coherence conditions (1.1.9) and (1.1.10) follow immediately by the axioms for a Gray-category. Finally, if $\alpha : f \to g$ is a pseudo-S-algebra 2-cell, the required pseudo-T-algebra 2-cell is given by $F\alpha : Ff \to Fg$. We have thus defined the components of a Gray-natural transformation $\hat{F}$: Ps-S-Alg $\to$ Ps-T-Alg, which is clearly a lifting of $F : X \to Y$. ☐

Given 1-cells $(F, \hat{F}) : (X, S) \to (Y, T)$ and $(F', \hat{F}') : (X, S) \to (Y, T)$, a 2-cell $(p, \hat{p}) : (F, \hat{F}) \to (F', \hat{F}')$ in $\mathbf{Lift}(\mathscr{K})$ consists of a 2-cell $p : F \to F'$ in $\mathscr{K}$ and a Gray-modification $\hat{p} : \hat{F} \to \hat{F}'$ such that the following diagram commutes

$$\begin{tikzcd}
U\hat{F} \arrow[r, "U\hat{p}"] \arrow[d, equal] & U\hat{F}' \arrow[d, equal] \\
FU \arrow[r, "pU"'] & F'U.
\end{tikzcd}$$

The vertical arrows are the identities, which hold by the assumption that $\hat{F}$ and $\hat{F'}$ are liftings of F and F', respectively.

Lemma 2.1.2. *Let $(p, \tilde{p}) : (F, \phi) \to (F', \phi')$ be a pseudomonad transformation. Then, there exists a lifting $\hat{p} : \hat{F} \to \hat{F'}$ of $p : F \to F'$, where $\hat{F}$ and $\hat{F'}$ are the liftings of F and F' associated to the pseudomonad morphisms (F, ϕ) and (F', ϕ'), respectively, defined as in Lemma 2.1.1.*

Proof. Let $I \in \mathscr{K}$. We need to define a pseudonatural transformation $\hat{p} \colon \hat{F}_I \to \hat{F'}_I$. We define the component of $\hat{p}$ associated to an I-indexed pseudo-S-algebra A to be the I-indexed pseudo-T-algebra morphism given by $p_A : FA \to F'A$ and the 2-cell

$$\begin{tikzcd}
TFA \arrow[r, "Tp_A"] \arrow[d, "\phi_A"'] \arrow[dr, phantom, "\Downarrow \bar{p}_A"] & TF'A \arrow[d, "\phi'_A"] \\
FSA \arrow[r, "p_{SA}"'] \arrow[d, "Fa"'] \arrow[dr, phantom, "\Downarrow p_a^{-1}"] & F'SA \arrow[d, "F'a"] \\
FA \arrow[r, "p_A"'] & F'A\,.
\end{tikzcd}$$

To prove the condition (1.1.9) for the pseudoalgebra morphism p_A, we apply the axioms for a Gray-category and then condition (1.2.3) for the pseudomonad transformation p. To establish condition (1.1.10), it is sufficient to apply the coherence condition (1.2.4) for the pseudomonad transformation p, and then the axioms for a Gray-category. By definition, $\hat{p}$ is a lifting of p as required. □

Finally, for 2-cells $(p, \hat{p})$ and $(q, \hat{q})$, a 3-cell $\alpha : (p, \hat{p}) \to (q, \hat{q})$ consists of a 3-cell and $\alpha : p \to q$ and a Gray-perturbation $\hat{\alpha} : \hat{p} \to \hat{q}$ making the following diagram commute

$$\begin{tikzcd}
U\hat{p} \arrow[r, "U\hat{\alpha}"] \arrow[d, equal] & U\hat{q} \arrow[d, equal] \\
pU \arrow[r, "\alpha U"'] & qU\,.
\end{tikzcd}$$

As before, the vertical arrows are the identities that are part of the assumption that $\hat{p}$ and $\hat{q}$ are liftings of p and q, respectively. Composition and identities of **Lift**$(\mathscr{K})$ are defined in the evident way, using those of $\mathscr{K}$ and **Gray**.

Lemma 2.1.3. *Given a pseudomonad modification* $\alpha : (p, \tilde{p}) \to (q, \tilde{q})$ *we can define a lifting* $\hat{\alpha} : \hat{p} \to \hat{q}$ *of* α *as the Gray-perturbation whose components are, for a pseudo-*S*-algebra* A*, the 3-cells* $\alpha_A : p_A \to q_A$*.*

Proof. It suffices to check that, these 3-cells are a pseudo-T-algebra 2-cells. To prove this, apply the axioms for a Gray-category and the coherence axiom (1.2.5). □

We use these results to define a Gray-functor

$$\Phi : \mathbf{Psm}(\mathscr{K}) \to \mathbf{Lift}(\mathscr{K})\,.$$

On objects, Φ acts as the identity. For two pseudomonads (X, S) and (Y, T) in $\mathscr{K}$, the hom-2-functors

$$\Phi_{(X,S),\,(Y,T)} : \mathbf{Psm}(\mathscr{K})(\,(X,S),\ (Y,T)\,) \longrightarrow \mathbf{Lift}(\mathscr{K})(\,(X,S),\ (Y,T)\,)$$

are defined sending a pseudomonad morphism, pseudomonad transformation and pseudomonad modification to the associated liftings, using Lemmas 2.1.1, 2.1.2 and 2.1.3, respectively. Here, the Gray-functoriality of Φ is standard verification, which we omit for brevity, limiting ourselves to highlight that this includes checking that Φ preserves composition strictly, *i.e.* that the lifting associated to the composite of two pseudomonad morphisms is equal (rather than just equivalent by invertible 2-cells) to the composite of the liftings obtained from the pseudomonad morphisms. Theorem 2.1.4 states that the construction of $\mathbf{Psm}(\mathscr{K})$ given in Section 1.2 is equivalent to the one by Marmolejo in [Mar99].

Theorem 2.1.4. *The Gray-functor* $\Phi \colon \mathbf{Psm}(\mathscr{K}) \to \mathbf{Lift}(\mathscr{K})$ *is a triequivalence.*

Proof. Since Φ is clearly bijective on objects, it suffices to prove that locally it is a biequivalence. Let us begin by considering a lifting $\hat{F}$: Ps-S-Alg $\to$ Ps-T-Alg of a 1-cell $F : X \to Y$. By the definition of a lifting, the following diagram of

2-categories and 2-functors commutes:

$$\begin{tikzcd}
\text{Ps-}S\text{-Alg}(X) \arrow[r, "\hat{F}_X"] \arrow[d, "U_X"'] & \text{Ps-}T\text{-Alg}(X) \arrow[d, "U_X"] \\
\mathscr{K}(X,X) \arrow[r, "\mathscr{K}(X,F)"'] & \mathscr{K}(X,Y)\,.
\end{tikzcd} \tag{2.1.1}$$

Let us now observe that $S : X \to X$ can be regarded as an X-indexed pseudo-S-algebra, with structure map given by the 2-cell $m : S^2 \to S$. By the commutativity of the diagram (2.1.1), this pseudo-S-algebra is mapped by the 2-functor $\hat{F}_X$ into a pseudo-T-algebra with underlying 1-cell $FS : X \to Y$, with structure map a 2-cell of the form $\phi_0 : TFS \to FS$, and invertible 3-cells fitting in the diagrams

$$\begin{tikzcd}
T^2FS \arrow[r, "T\phi_0"] \arrow[d, "n_{FS}"'] \arrow[dr, phantom, "\Downarrow\,\bar{\phi}_0"] & TFS \arrow[d, "\phi_0"] \\
TFS \arrow[r, "h'"'] & FS
\end{tikzcd}
\qquad
\begin{tikzcd}
FS \arrow[r, "t_{FS}"] \arrow[dr, bend right, "1_{FS}"'] & TFS \arrow[d, "h'"] \arrow[dl, phantom, "\overset{\tilde{\phi}_0}{\Longrightarrow}"] \\
 & FS\,.
\end{tikzcd}$$

The desired pseudomonad morphism $(F,\phi) : (X,S) \to (Y,T)$ is then obtained by letting $\phi : TF \to FS$ be the composite

$$\begin{tikzcd}
TF \arrow[r, "TFs"] & TFS \arrow[r, "\phi_0"] & FS.
\end{tikzcd}$$

The appropriate 3-cells are provided by the following pasting diagrams

$$\begin{tikzcd}
T^2F \arrow[r, "T^2Fs"] \arrow[dd, "nF"'] \arrow[ddr, phantom, "\Downarrow\,n_{Fs}"] & T^2FS \arrow[r, "T\phi_0"] \arrow[dd, "nFS"'] \arrow[ddr, phantom, "\Downarrow\,\bar{\phi}_0"] & TFS \arrow[r, "TFsS"] \arrow[ddr, bend right, "\phi_0"'] & TFS^2 \arrow[d, "\phi_0 S"] \arrow[dl, phantom, "\gamma\Downarrow"] \\
 & & & FS^2 \arrow[d, "Fm"] \\
TF \arrow[r, "TFs"'] & TFS \arrow[rr, "\phi_0"'] & & FS
\end{tikzcd}
\qquad
\begin{tikzcd}
F \arrow[r, "tF"] \arrow[d, "Fs"'] \arrow[dr, phantom, "\overset{t_{Fs}}{\Longrightarrow}"] & TF \arrow[d, "TFs"] \\
FS \arrow[r, "tFS"] \arrow[dr, bend right, "1_{FS}"'] & TFS \arrow[d, "\phi_0"] \arrow[dl, phantom, "\overset{\tilde{\phi}_0}{\Longrightarrow}"] \\
 & FS\,,
\end{tikzcd}$$

where γ is the inverse to the 2-cell obtained from the following pasting of invertible

2-cells:

$$\begin{tikzcd}
TFS \arrow[d, "TFsS"'] \arrow[dr, bend left, "1_{TFS}"] \arrow[dr, phantom, "\Downarrow TF\rho"] & \\
TFS^2 \arrow[r, "TFm"] \arrow[d, "\phi_0 S"'] \arrow[dr, phantom, "\Downarrow F\alpha"] & TFS \arrow[d, "\phi_0"] \\
FS^2 \arrow[r, "Fm"'] & FS\,.
\end{tikzcd}$$

Let us now consider a lifting $(p, \hat{p}) : (F, \hat{F}) \to (F', \hat{F'})$ of a 2-cell $p : F \to F'$. We can define a pseudomonad transformation $p : (F, \phi) \to (F', \phi')$ by considering the following pasting diagram:

$$\begin{tikzcd}
TF \arrow[r, "Tp"] \arrow[d, "TFs"'] \arrow[dr, phantom, "\Downarrow Tp_u^{-1}"] & TF' \arrow[d, "TF's"] \\
TFS \arrow[r, "TpS"'] \arrow[d, "\phi_o"'] \arrow[dr, phantom, "\Downarrow \bar{p}S"] & TF'S \arrow[d, "\phi'_0"] \\
FS \arrow[r, "pS"'] & F'S\,,
\end{tikzcd}$$

in which the bottom 3-cell is part of the structure making $pS : FS \to F'S$ into a pseudoalgebra morphism. Finally, if $(\alpha, \hat{\alpha}) : (p, \hat{p}) \to (q, \hat{q})$ is a lifting of a 3-cell $\alpha : p \to q$, then $\alpha : p \to q$ is a pseudomonad modification. These definitions determine a 2-functor

$$\Psi_{(X,S),(Y,T)} : \mathbf{Lift}(\mathscr{K})\big((X,S),(Y,T)\big) \longrightarrow \mathbf{Psm}(\mathscr{K})\big((X,S),(Y,T)\big)$$

which provides the required quasi-inverse to $\Phi_{(X,S),(Y,T)}$. We omit the construction of the required invertible pseudonatural transformations $\eta : 1 \to \Psi\Phi$ and $\varepsilon : \Phi\Psi \to 1$, since this is not difficult. $\square$

2.2. Pseudodistributive Laws

Definition 2.2.1. Let (X, S) and (X, T) be pseudomonads in $\mathscr{K}$. A *pseudodistributive law* of T over S consists of a 2-cell $d : ST \to TS$ and invertible 3-cells

$$\begin{tikzcd}
S^2T \arrow[r, "Sd"] \arrow[dd, "mT"'] \arrow[ddr, phantom, "\Downarrow \bar{m}"] & STS \arrow[d, "dS"] \\
 & TS^2 \arrow[d, "Tm"] \\
ST \arrow[r, "d"'] & TS
\end{tikzcd}
\qquad
\begin{tikzcd}
T \arrow[r, "sT"] \arrow[dr, bend right, "Ts"'] \arrow[dr, phantom, "\overset{\bar{s}}{\Longrightarrow}"] & ST \arrow[d, "d"] \\
 & TS
\end{tikzcd}$$

$$\begin{tikzcd}
ST^2 \arrow[r, "Sn"] \arrow[d, "dT"'] \arrow[ddr, phantom, "\Downarrow \bar{n}"] & ST \arrow[dd, "d"] \\
TST \arrow[d, "Td"'] & \\
T^2S \arrow[r, "nS"'] & TS
\end{tikzcd}
\qquad
\begin{tikzcd}
 & ST \arrow[d, "d"] \\
S \arrow[ur, bend left, "St"] \arrow[r, "tS"'] \arrow[ur, phantom, "\Downarrow \bar{t}"] & TS
\end{tikzcd}$$

satisfying the coherence conditions (C1)-(C8) stated in Appendix A.1.

Remark 2.2.2. For the convenience of the reader, Table 2.1 describes the correspondence between the presentation of the coherence conditions for pseudodistributive laws here and in [Tan04, Mar99]. In the table, each row lists different formulations of the same axiom.

Remark 2.2.3. Our development in Section 2.1 allows us to give a clear explanation for the coherence conditions for pseudodistributive laws, summarised in Table 2.2.

The coherence axioms, (C9) and (C10) of Appendix A.1 have been shown to be derivable from the others in [MW08, Theorem 2.3 and Proposition 4.2]. Indeed, axiom (C9) is a particular case of a provable coherence condition for a pseudomonad morphism and follows from (C1) and (C2) (*cf.* Proposition 1.2.2). By duality, one can see that axiom (C10) is a particular case of a provable coherence condition for a pseudomonad op-morphism and follows from (C7) and (C8).

Appendix A.1	Marmolejo [Mar99]	Tanaka [Tan04]
(C1)	(coh 4)	(T6)
(C2)	(coh 2)	(T2)
(C3)	(coh 5)	(T9)
(C4)	(coh 3)	(T8)
(C5)	(coh 1)	(T1)
(C6)	(coh 6)	(T10)
(C7)	(coh 9)	(T7)
(C8)	(coh 7)	(T5)
(C9)	-	(T3)
(C10)	(coh 8)	(T4)

Table 2.1: Comparison of coherence conditions.

Axiom	Coherence condition
(C1) and (C2)	$(T, d) : (X, S) \to (X, S)$ is a pseudomonad morphism
(C3) and (C4)	$(n, \bar{n}) : (T, d)^2 \to (T, d)$ is a pseudomonad transformation
(C5) and (C6)	$(t, \bar{t}) : (X, 1_X) \to (T, d)$ is a pseudomonad transformation
(C7)	α is pseudomonad modification
(C8)	ρ is a pseudomonad modification
(C9)	λ is a pseudomonad modification

Table 2.2: Coherence axioms for pseudodistributive laws.

The explanation of the axioms for a pseudodistributive law in Remark 2.2.3 proves the following straightforward, but important, proposition.

Proposition 2.2.4. *The objects of* $\mathbf{Psm}(\mathbf{Psm}(\mathscr{K}))$ *are exactly pseudodistributive laws in* $\mathscr{K}$.

Proof. An object of $\mathbf{Psm}(\mathbf{Psm}(\mathscr{K}))$ consists of an object (X, S) of $\mathbf{Psm}(\mathscr{K})$, *i.e.* a pseudomonad in $\mathscr{K}$, together with a pseudomonad $(T, d)\colon (X, S) \to (X, S)$ on it in $\mathbf{Psm}(\mathscr{K})$, which is exactly a pseudodistributive law by Remark 2.2.3. □

We can now give a new simple proof of Marmolejo's fundamental result asserting the equivalence between a pseudodistributive law of a pseudomonad T over a pseudomonad S and a lifting of the pseudomonad T to the 2-categories of pseudoalgebras for S [Mar99].

Theorem 2.2.5. *Let $\mathscr{K}$ be a Gray-category, (X, S) and (X, T) be pseudomonads in $\mathscr{K}$. A pseudodistributive law $d \colon ST \to TS$ is equivalent to a lifting of T to pseudo-S-algebras.*

Proof. By Theorem 1.2.5, $\mathbf{Psm}(\mathscr{K})$ is a Gray-category and therefore we can consider the Gray-category $\mathbf{Psm}(\mathbf{Psm}(\mathscr{K}))$.

Next, observe that that $\mathbf{Psm}(-)$ preserves triequivalences between Gray-categories, i.e. given a triequivalence of Gray-categories $\Phi \colon \mathscr{K} \to \mathscr{K}'$, then it is possible to define a triequivalence $\mathbf{Psm}(\Phi) \colon \mathbf{Psm}(\mathscr{K}) \to \mathbf{Psm}(\mathscr{K}')$. The construction of $\mathbf{Psm}(\Phi)$ is evident, and the verification that it is a triequivalence is a long, but routine, calculation. For example, to prove essential surjectivity, we need to show that for every pseudomonad (X', T') in $\mathscr{K}'$, there is a pseudomonad (X, T) in $\mathscr{K}$ that is mapped by $\mathbf{Psm}(\Phi)$ to a pseudomonad that is biequivalent to (X', T') in $\mathbf{Psm}(\mathscr{K}')$. For this, one defines (X, T) using the essential surjectivity of Φ, carefully inserting coherence isomorphisms that are part of the given triequivalence where appropriate.

Applying this fact to the triequivalence of Theorem 2.1.4, we get a triequivalence:

$$\mathbf{Psm}(\mathbf{Psm}(\mathscr{K})) \simeq \mathbf{Psm}(\mathbf{Lift}(\mathscr{K}))\,.$$

An object on the left hand side is exactly a pseudodistributive law by Proposition 2.2.4. Similarly, an object on the right hand side consists exactly of a pseudomonad (X, S) in $\mathscr{K}$ and a pseudomonad $T \colon X \to X$ with a lifting $\hat{T} \colon \text{Ps-}S\text{-Alg} \to \text{Ps-}S\text{-Alg}$. □

We conclude the chapter by outlining how duality can be applied as in [Str72, Section 4] to obtain an equivalence between pseudodistributive laws and extensions to Kleisli objects (whenever they exist). Fix a Gray-category $\mathscr{K}$ and let (X, T) be a pseudomonad in it. By definition, a *right pseudo-T-module* in $\mathscr{K}$ is a left T-module in $\mathscr{K}^{op}$. We then have a Gray-functor

$$\mathrm{Mod}_T \colon \mathscr{K}^{op} \to \mathbf{Gray}\,. \tag{2.2.1}$$

Assuming the evident definition of a lifting to 2-categories of right pseudomodules, we have the following corollary of Theorem 2.2.5.

Corollary 2.2.6. *Let (X, S) and (X, T) be pseudomonads in $\mathscr{K}$. A pseudodistributive law $d : ST \to TS$ is equivalent to a lifting of S to right pseudo-T-modules.* ☐

The equivalence of Corollary 2.2.6 becomes more familiar under the assumption that $\mathscr{K}$ has Kleisli objects. Recall that a *Kleisli object* for a pseudomonad (X, T) in $\mathscr{K}$ is an 0-cell $X_T \in \mathscr{K}$ and a right pseudo-T-module $J_T \colon X \to X_T$, which is universal in the sense that the 2-functor

$$\mathscr{K}(X_T, I) \to \mathrm{Mod}_T(I)\,,$$

induced by composition with J_T, is an equivalence of 2-categories, thus making the Gray-functor in (2.2.1) representable. Now, a pseudodistributive law $d : ST \to TS$ is equivalent to a lifting of S to right pseudo-T-modules, as in

$$\begin{tikzcd}
\mathrm{Mod}_T \arrow[r, "\hat{S}"] \arrow[d, "U"'] & \mathrm{Mod}_T \arrow[d, "U"] \\
\mathscr{K}(-,\, X) \arrow[r, "S \circ -"'] & \mathscr{K}(-,\, X)\,.
\end{tikzcd}$$

This, in turn, is equivalent to

$$\begin{tikzcd}
X_T \arrow[r, "\hat{S}"] & X_T \\
X \arrow[u, "J_T"] \arrow[r, "S"'] & X \arrow[u, "J_T"']\,,
\end{tikzcd}$$

which describes an extension of S to the Kleisli object of T.

Remark 2.2.7. We conclude the chapter by briefly discussing the question of whether the Gray-category **Gray** has Kleisli objects. Given a 2-category X and pseudomonad $T\colon X \to X$, there are two reasonable options to be the Kleisli object for T, mirroring the one-dimensional situation. In both cases, the objects are the same objects as those of X, but they have different hom-categories of morphisms. The first option is to define the hom-category between two objects $x, y \in X$ to be $X(x, Ty)$. With this definition we only get a bicategory, not a 2-category, and so we step outside **Gray**. The second option (which we will call X_T), is to take the hom-category of morphisms between x and y to consist of pseudoalgebras morphisms from Tx to Ty (considered as free algebras). This is a 2-category and so one could try to show that it is a Kleisli object for **Gray**. In order to do this, one should prove that, for any 2-category I, there is an equivalence as in (2.2.1). However, the construction taking a I-indexed right pseudo-T-module to a 2-functor $X_T \to I$ is only a pseudofunctor and not a strict 2-functor, thus leading again outside **Gray**. The reason for this is that we need to use the pseudonaturality of the module action λ and other coherence isomorphisms. Because of this, it seems that **Gray** does not have Kleisli objects. We suspect that, once it is defined what it means for a tricategory to have Kleisli objects, it should be possible to show that the tricategory **2-Cat**$_{\mathrm{psd}}$ of 2-categories, pseudofunctors, pseudonatural transformations and modifications has Kleisli objects. The same should hold also for **Bicat**, the tricategory of bicategories, pseudofunctors, pseudonatural transformation and modifications. We leave the investigation of these problems to future research.

3. Distributive Laws for Relative Monads

The main aim of this chapter is to develop further the theory of distributive laws by presenting a possible definition of a distributive law between a relative monad T and a monad S, which we call *a relative distributive law* (Definition 3.5.2). In particular we prove a counterpart of Beck's equivalence for relative distributive laws (Theorem 3.6.19).

We take a 2-categorical approach to the subject, inspired by the formal theory of monads [LS02, Str72]. Let us briefly recall how distributive laws are treated there. First, for a 2-category $\mathcal{K}$, one introduces the 2-category $\mathbf{Mnd}(\mathcal{K})$ of monads, monad morphisms and monad 2-cells. Then, one defines the notions of an (indexed) left module and left module morphism, uses them to construct a 2-category $\mathbf{Lift}(\mathcal{K})$ of monads, liftings of maps to left modules and lifting of 2-cells to left modules, and proves that $\mathbf{Mnd}(\mathcal{K})$ and $\mathbf{Lift}(\mathcal{K})$ are 2-isomorphic. Once this is done, everything follows formally. First, one gets an equivalence between distributive laws (which are monads in $\mathbf{Mnd}(\mathcal{K})$) and liftings of monads to left modules (which are monads in $\mathbf{Lift}(\mathcal{K})$). Secondly, by duality, one obtains a 2-isomorphism $\mathbf{Mnd}(\mathcal{K}^{op})^{op}$ with $\mathbf{Lift}(\mathcal{K}^{op})^{op}$, which leads to the corresponding result on the equivalence between distributive laws and extensions of monads to right modules. Since representability of left and right modules corresponds to existence of Eilenberg-Moore and Kleisli objects, respectively, in that case one gets a version of Beck's theorem. Importantly, in the equivalence between distributive laws $d\colon ST \to TS$ and liftings of T to the category of Eilenberg-Moore algebras of S, one considers S as a part of an object of $\mathbf{Mnd}(\mathcal{K})$ and T as part of a monad morphism, while in the equivalence between distributive laws $d\colon ST \to TS$ and extensions of S to the Kleisli category of T, one considers T as part of an object of $\mathbf{Mnd}(\mathcal{K})$ and S as part of a monad morphism. The equivalence between all these notions is possible because of the aforementioned duality and because $\mathbf{Mnd}(\mathcal{K}^{op})^{op}$ has the same objects as $\mathbf{Mnd}(\mathcal{K})$, which are exactly distributive laws.

We will introduce a 2-category $\mathbf{Rel}(\mathcal{K})$ of relative monads, relative monad morphisms and relative monad transformation in $\mathcal{K}$. This 2-category generalises the one of no-iteration monads introduced in [Her20]. Importantly, $\mathbf{Rel}(\mathcal{K})$ is related more closely to $\mathbf{Mnd}(\mathcal{K}^{op})^{op}$ than to $\mathbf{Mnd}(\mathcal{K})$. Indeed, it contains $\mathbf{Mnd}(\mathcal{K}^{op})^{op}$ as a full sub-2-category (Proposition 3.3.5). This is motivated by the fact that relative monads are particularly suited to study Kleisli categories. Then, we extend some results of [ACU15, FGHW17] to our setting, proving them for any 2-category. Using this point of view, we introduce a notion of distributive law of a relative monad T on a monad (Definition 3.5.1), which we call *relative distributive law* (Section 3.5). We then show that it is equivalent to an object of $\mathbf{Mnd}(\mathbf{Rel}(\mathcal{K}))$.

The first difference we find between our work and the formal theory of monads is that the objects of $\mathbf{Rel}(\mathcal{K}^{op})$ are not the same as those of $\mathbf{Rel}(\mathcal{K})$. The issue is that the notion of operator (Definition 3.1.3), that is involved in the definition of a relative monad, does not dualise, i.e. an operator in $\mathcal{K}^{op}$ is not an operator in $\mathcal{K}$. For this reason, the duality available for monads fails and we need to consider separately left and right modules. In each case, we are able to prove some, but not all, counterparts of results valid in the classical case. Importantly, the combination of these results still allows us to obtain a version of Beck's theorem about distributive laws (Theorem 3.6.19).

Using left modules for a relative monad we are able to find a relative adjunction (Theorem 3.4.5). In particular, thanks to this result we can prove that if a 2-category has relative Eilenberg-Moore objects then any relative monad is induced by a relative adjunction (Theorem 3.4.6). On the other hand, we do not have a correspondence between relative monad morphisms and liftings of morphisms to left modules. Nevertheless, we get an equivalence between relative distributive laws and liftings of relative monads to left modules (Theorem 3.5.9).

Considering right modules we do not get a relative adjunction (see Remark 3.6.7). Instead, we use them to get an equivalence between relative monad morphisms and liftings of morphisms to right modules (Proposition 3.6.10). In particular, we can define a 2-category $\mathbf{LiftR}(\mathcal{K})$ of liftings to right modules and prove that it is 2-isomorphic to $\mathbf{Rel}(\mathcal{K})$. Thus, we get an equivalence between relative distributive laws and liftings of monads to right modules of a relative monad (Theorem 3.6.18), for which we use an argument similar to the one in [Str72].

A further motivation for this work is to provide a first step towards the definition of a notion of a pseudodistributive law between a relative pseudomonad [FGHW17] and a 2-monad [BKP89]. Part of a Beck-like theorem has already been translated in this setting [FGHW17, Theorem 6.3] without defining the notion of a relative pseudodistributive law. With this definition, it will be possible to interpret the results in [FGHW17, Section 7] with a *relative pseudodistributive law* of the presheaf relative pseudomonad on the 2-monad for free monoidal categories, or symmetric monoidal categories etc. Since this work, some steps in this direction have been made by Walker [Wal21].

3.1. Preliminary Definitions

Throughout this chapter, for a 2-category $\mathcal{K}$, we use letters X, Y, Z ... to denote 0-cells, $F\colon X \to Y$, $G\colon Y \to Z$ ... for 1-cells and $f\colon F \to G$, $\alpha\colon F \to F'$ ... for 2-cells. Regarding compositions, we will write $G \circ F$ or GF for composition of 1-cells. For 2-cells we denote with $\beta \circ \alpha\colon GF \to G'F'$ or juxtaposition for horizontal composition and $f' \cdot f\colon F \to H$ for vertical composition. We will denote with $(A, B)\colon O \to X; Y$ spans in $\mathcal{K}$ and with $(S, R)\colon Y; X \to Z$ cospans, i.e. diagrams as below.

$$\begin{tikzcd}
& O \arrow[dl, "A"'] \arrow[dr, "B"] & & X \arrow[dr, "R"'] & & Y \arrow[dl, "S"] \\
X & & Y & & Z &
\end{tikzcd}$$

When it will be clear from context, we will sometimes avoid saying explicitly which spans/cospans we are considering and might refer to them as $[S, R]$. For 2-categorical background we redirect the reader to [Gra74, Lac10].

Let us recall the definition of a relative monad [ACU15, Definition 2.1].

Definition 3.1.1. A *relative monad* on a functor $I : \mathbb{C}_0 \to \mathbb{C}$ consists of:

- an object mapping $S\colon \mathrm{Ob}(\mathbb{C}_0) \to \mathrm{Ob}(\mathbb{C})$;
- for any $x, y \in \mathbb{C}_0$, a map $(-)^*_S : \mathbb{C}(Ix, Sy) \to \mathbb{C}(Sx, Sy)$ (the *extension*);
- for any $x \in \mathbb{C}_0$, a map $s_x : Ix \to Sx$ (the *unit*);

satisfying the following axioms

- the *left unital law*, i.e. for any $k\colon Ix \to Sy$, $k = k^* \cdot s_x$;
- the *right unital law*, i.e. for any $x \in \mathbb{C}_0$, $s^*_x = 1_{Sx}$;
- the *associativity law*, i.e. for any $k : Ix \to Sy$ and $l : Iy \to Sz$, $(l^* {\cdot} k)^* = l^* {\cdot} k^*$.

Example 3.1.2. Many examples of relative monads can be found in [ACU15]. At the end of this chapter we will consider one of them, the relative monad of vector spaces [ACU15, Example 1.1].

A relative monad with $I = 1_{\mathbb{C}}$ is a no-iteration monad (also called extension system [MW10, Definition 2.3]). The notion of a no-iteration monad can be generalised to any 2-category $\mathcal{K}$ thanks to the definition of *pasting operators* [MW10, Definition 2.1], which is a mapping of 2-cells as shown below

$$\begin{tikzcd}
O \arrow[r, dotted, "B"] \arrow[dr, dotted, bend right, "A"'] \arrow[dr, phantom, "\overset{f}{\Longrightarrow}"] & Y \arrow[d, "S"] \arrow[rr, phantom, "\overset{(-)^{\#}}{\rightsquigarrow}"] & & O \arrow[r, dotted, "B"] \arrow[d, dotted, "A"'] \arrow[dr, phantom, "\overset{f^{\#}}{\Longrightarrow}"] & Y \arrow[d, "U"] \\
 & Z & & Z \arrow[r, "T"'] & Z',
\end{tikzcd}$$

indexed on spans $(A, B)\colon O \to Z; Y$ and satisfying two axioms. Similarly, the extension of a relative monad can be expressed as a mapping of the form

$$\begin{tikzcd}
\mathbf{1} \arrow[r, dotted, "y"] \arrow[d, dotted, "x"'] \arrow[dr, phantom, "\overset{k}{\Longrightarrow}"] & \mathbb{C}_0 \arrow[d, "S"] \arrow[rr, phantom, "\overset{(-)^*_S}{\rightsquigarrow}"] & & \mathbf{1} \arrow[r, dotted, "y"] \arrow[d, dotted, "x"'] \arrow[dr, phantom, "\overset{k^*_S}{\Longrightarrow}"] & \mathbb{C}_0 \arrow[d, "S"] \\
\mathbb{C}_0 \arrow[r, "I"'] & \mathbb{C} & & \mathbb{C}_0 \arrow[r, "S"'] & \mathbb{C},
\end{tikzcd}$$

where $\mathbf{1}$ is the terminal category and x, $y\colon \mathbf{1} \to \mathbb{C}_0$ are the constant functors to x and y respectively, satisfying two axioms. With this in mind, we can see how the next definition generalises pasting operators and gives us a way to define a relative monad in any 2-category.

Definition 3.1.3. Let

$$\begin{tikzcd}
X \arrow[dr, "F"'] & & Y \arrow[dl, "G"] & X \arrow[dr, "F'"'] & & Y \arrow[dl, "G'"] \\
 & Z & & & Z' &
\end{tikzcd}$$

be two cospans in a 2-category $\mathcal{K}$. An *operator* $(-)^{\#}\colon [F,\, G] \to [F',\, G']$ is a family of functions, for any span of arrows $(A,\, B)\colon O \to X; Y$

$$(-)^{\#}_{A,B} : \mathcal{K}[O, Z](FA,\, GB) \to \mathcal{K}[O, Z'](F'A,\, G'B)$$

$$\begin{tikzcd}
O \arrow[r, dotted, "B"] \arrow[d, dotted, "A"'] \arrow[dr, phantom, "\overset{f}{\Longrightarrow}"] & Y \arrow[d, "G"] \arrow[rr, phantom, "\overset{(-)^{\#}}{\rightsquigarrow}"] & & O \arrow[r, dotted, "B"] \arrow[d, dotted, "A"'] \arrow[dr, phantom, "\overset{f^{\#}}{\Longrightarrow}"] & Y \arrow[d, "G'"] \\
X \arrow[r, "F"'] & Z & & X \arrow[r, "F'"'] & Z',
\end{tikzcd}$$

satisfying the following axioms:

- *indexing naturality*, i.e. for any diagram
$$\begin{tikzcd}
O' \arrow[dr, "P"] & & \\
& O \arrow[r, dotted, "B"] \arrow[d, dotted, "A"'] \arrow[dr, phantom, "\overset{f}{\Longrightarrow}"] & Y \arrow[d, "G"] \\
& X \arrow[r, "F"'] & Z
\end{tikzcd}$$
, $(fP)^\# = f^\# P$;
- *left naturality*, i.e. for any diagram
$$\begin{tikzcd}
O \arrow[r, dotted, "B"] \arrow[d, dotted, bend right=50, "A'"'] \arrow[d, dotted, bend left=50, "A"] \arrow[d, phantom, "\overset{\alpha}{\Rightarrow}"] \arrow[dr, phantom, "\overset{f}{\Longrightarrow}"] & Y \arrow[d, "G"] \\
X \arrow[r, "F"'] & Z
\end{tikzcd}$$
, $(f \cdot F\alpha)^\# = f^\# \cdot F'\alpha$;
- *right naturality*, i.e. for any diagram
$$\begin{tikzcd}
O \arrow[r, dotted, bend left=40, "B'"] \arrow[r, dotted, bend right=40, "B"'] \arrow[r, phantom, "\Uparrow \beta"] \arrow[d, dotted, "A"'] \arrow[dr, phantom, "\underset{f}{\Longrightarrow}"] & Y \arrow[d, "G"] \\
X \arrow[r, "F"'] & Z
\end{tikzcd}$$
, $(G\beta \cdot f)^\# = G'\beta \cdot f^\#$.

The axioms of indexing, left and right naturality represent naturality in O, A and B respectively. When we consider $F = 1_X$ (so an operator $[1_X, G] \to [F', G']$) we get back the definition of *pasting operator* given in [MW10]. The conditions of whiskering and blistering of [MW10] correspond to indexing and left naturality, while right naturality is deducible from [MW10, Lemma 2.2] and the interchange law of $\mathcal{K}$. Pasting operators are also studied in [Her20], where both left and right pasting operators are introduced. Following the reasoning above we can see that right pasting operators are equivalent to operators in $\mathcal{K}^{op}$ with $F = 1_X$.

Example 3.1.4.

(i) Let us consider $\mathcal{K} = \mathbf{Cat}$. We will show that, in this particular 2-category, an operator is equivalent to a family of maps indexed by pairs of objects. Let $\mathbb{X}$, $\mathbb{Y}$, $\mathbb{Z}$ and $\mathbb{Z}'$ be categories and $(F, G)\colon \mathbb{X}; \mathbb{Y} \to \mathbb{Z}$ and $(F', G')\colon \mathbb{X}; \mathbb{Y} \to \mathbb{Z}'$

be two cospans in **Cat**. Let $(-)^{\#} \colon [F,\, G] \to [F',\, G']$ be an operator. Then, if we consider the span given by $(x,\, y) \colon \mathbf{1} \to \mathbb{X}; \mathbb{Y}$ with $x \in \mathbb{X}$ and $y \in \mathbb{Y}$, the operator $(-)^{\#}$ gives us family of maps

$$(-)^{\#}_{x,y} \colon \mathbb{Z}(Fx, Gy) \to \mathbb{Z}'(F'x, G'y)$$

$$\begin{tikzcd}
\mathbf{1} \arrow[r, dotted, "y"] \arrow[d, dotted, "x"'] \arrow[dr, phantom, "\overset{f}{\Longrightarrow}"] & \mathbb{Y} \arrow[d, "G"] & & \mathbf{1} \arrow[r, dotted, "y"] \arrow[d, dotted, "x"'] \arrow[dr, phantom, "\overset{f^{\#}}{\Longrightarrow}"] & \mathbb{Y} \arrow[d, "G'"] \\
\mathbb{X} \arrow[r, "F"'] & \mathbb{Z} & & \mathbb{X} \arrow[r, "F'"'] & \mathbb{Z}'
\arrow[from=1-2, to=2-4, phantom, "\overset{(-)^{\#}}{\rightsquigarrow}"]
\end{tikzcd}$$

Left and right naturality of $(-)^{\#}$ tell us that these maps are natural in x and y respectively. Conversely, if we have such a natural family of maps, then we can construct a pasting operator in the following way. For any span $(A,\, B) \colon \mathbb{O} \to \mathbb{X}; \mathbb{Y}$ and any natural transformation $f \colon FA \to GB$, we define the component of the natural transformation $f^{\#} \colon F'A \to G'B$ at $o \in \mathbb{O}$ as

$$(f)^{\#}_{Ao,Bo} \colon F'Ao \longrightarrow G'Bo$$

$$\begin{tikzcd}
\mathbf{1} \arrow[r, dotted, "Bo"] \arrow[d, dotted, "Ao"'] \arrow[dr, phantom, "\overset{f}{\Longrightarrow}"] & \mathbb{Y} \arrow[d, "G"] & & \mathbf{1} \arrow[r, dotted, "Bo"] \arrow[d, dotted, "Ao"'] \arrow[dr, phantom, "\overset{(f)^{\#}_{Ao,Bo}}{\Longrightarrow}"] & \mathbb{Y} \arrow[d, "G'"] \\
\mathbb{X} \arrow[r, "F"'] & \mathbb{Z} & & \mathbb{X} \arrow[r, "F'"'] & \mathbb{Z}'.
\arrow[from=1-2, to=2-4, phantom, "\overset{(-)^{\#}_{Ao,Bo}}{\rightsquigarrow}"]
\end{tikzcd}$$

Using naturality in a and b and naturality for f we can prove that $f^{\#}$ is also a natural transformation. Moreover this definition satisfies all the axiom of an operator: indexing naturality follows directly from the definition, left and right naturality follow from naturality in a and b respectively.

(ii) Let us look at the notion of operator when we set $\mathcal{K} = \mathcal{V}$-**Cat**, the 2-category of $\mathcal{V}$-categories with $\mathcal{V}$ a symmetric closed monoidal category. For background in enriched category theory we redirect the reader to [Kel82].

We have a description similar to the one in the previous example. Let $\mathbb{X}$, $\mathbb{Y}$, $\mathbb{Z}$ and $\mathbb{Z}'$ be $\mathcal{V}$-categories and $(F, G) \colon \mathbb{X}; \mathbb{Y} \to \mathbb{Z}$ and $(F', G') \colon \mathbb{X}; \mathbb{Y} \to \mathbb{Z}'$ be two cospans in $\mathcal{V}$-**Cat**. Also in this case an operator $(-)^{\#}\colon [F,\, G] \to [F',\, G']$ is equivalent to a family of functions, for any $a \in \mathbb{X}$ and $b \in \mathbb{Y}$,

$$(-)^{\#}_{a,b}\colon \overline{\mathbb{Z}}(Fa, Gb) \longrightarrow \overline{\mathbb{Z}'}(F'a, G'b)$$

natural in a and b, where $\overline{\mathbb{Z}}$ and $\overline{\mathbb{Z}}$ are the underlying categories of $\mathbb{Z}$ and $\mathbb{Z}'$, respectively. We get this characterisation setting $\mathbb{O} = \mathbb{I}$ the unit $\mathcal{V}$-category, which has one object and the monoidal unit $I \in \mathcal{V}$ as hom-object.

(iii) Let us consider another important example in $\mathcal{K} = \mathcal{V}$-**Cat** (we will use the same notation as above). If we have a natural family of maps in $\mathcal{V}$, for any $a \in \mathbb{X}$ and $b \in \mathbb{Y}$,

$$(-)^{\#}_{a,b}\colon \mathbb{Z}(Fa, Gb) \longrightarrow \mathbb{Z}'(F'a, G'b),$$

then we can construct a pasting operator in the following way. For any span $(A,\, B)\colon \mathbb{O} \to \mathbb{X}; \mathbb{Y}$ and any $\mathcal{V}$-natural transformation $f\colon RA \to SB$, then we define the $o \in \mathbb{O}$ component of $f^{\#}$ as

$$I \xrightarrow{f_o} \mathbb{Z}(FAo,\, GBo) \xrightarrow{(-)^{\#}_{Ao,Bo}} \mathbb{Z}'(F'Ao,\, G'Bo).$$

Using naturality in a and b and $\mathcal{V}$-naturality for f we can prove that $f^{\#}$ is also a $\mathcal{V}$-natural transformation. Moreover this definition satisties all the axiom of an operator: indexing naturality follows directly from the definition, left and right naturality follow from naturality in a and b respectively.

Remark 3.1.5. Let us state three properties that will be useful to prove that any relative adjunction induces a relative monad (Lemma 3.2.5).

- We can easily see that given two operators $[F_1,\, G_1] \to [F_2,\, G_2]$ and $[F_2,\, G_2] \to [F_3,\, G_3]$ we can construct a composition operator $[F_1,\, G_1] \to [F_3,\, G_3]$ composing component-wise.
- An example of an operator is, for any 1-cell $T\colon Z \to Z'$ in $\mathcal{K}$, $T(-)\colon [F,\, G] \to [TF,\, TG]$ defined as $Tf\colon TFA \to TGB$ for any $f\colon FA \to GB$. Indexing, left and right naturality in this case derive from the pasting theorem for 2-categories.

- Let $(-)^\sharp \colon [F,\,G] \to [F',\,G']$ be an operator between the cospans $(F,G)\colon X;Y \to Z$ and $(F',G')\colon X;Y \to Z'$. Then, given any 1-cells $F_0\colon X_0 \to X$ and $G_0\colon Y_0 \to Y$, we can construct a new operator

  $$(-)^\sharp \cdot (F_0, G_0)\colon [FF_0,\, GG_0] \to [F'F_0,\, G'G_0].$$

  For any span $(A,B)\colon O \to X_0;Y_0$ we define the action of $(-)^\sharp_{F_0,G_0}$ on a 2-cell $f\colon FF_0A \to GG_0B$ as

  $$[(f)^\sharp \cdot (F_0,G_0)]_{A,B} := (f)^\sharp_{F_0A,G_0B}.$$

  All three naturality axioms for $(-)^\sharp \cdot (F_0,G_0)$ hold since they are particular cases of the ones of $(-)^\sharp$.

We will use the following proposition in Section 3.2 to prove that any relative adjunction induces a relative monad (Lemma 3.2.5).

Proposition 3.1.6. *Let* $(-)^\#\colon [R,\,S] \to [T,\,U]$ *be an operator such that each* $(-)^\#_{A,B}$ *is an isomorphism. Then the family of functions sending any* $g \in \mathcal{K}[O,Z'](TA,\,UB)$ *to the unique* f *such that* $f^\#_{A,B} = g$ *forms an operator* $(-)^\flat\colon [T,\,U] \to [R,\,S]$.

Proof. Let us check all the axioms for $(-)^\flat$. For any $P\colon O' \to O$,

$$f^\flat P = (fP)^\flat \quad \iff \quad (f^\flat P)^\# = ((fP)^\flat)^\#.$$

The second equation is true using the fact that $(-)^\flat$ is locally the inverse of $(-)^\#$ and indexing naturality for $(-)^\#$. With the same reasoning we can prove that $(-)^\flat$ satisfies also the other axioms. □

3.2. Relative Monads in $\mathcal{K}$

Using Definition 3.1.3 we can define a relative monad in any 2-category $\mathcal{K}$ as follows.

Definition 3.2.1. A *relative monad* $(X,\, I,\, S)$ in $\mathcal{K}$ consists of a pair of objects $X,\, X_0 \in \mathcal{K}$ together with:

- two 1-cells $I, S \colon X_0 \to X$ (we say that S *is a relative monad on* I);
- an operator $(-)^{\dagger}_{S} \colon [I,\, S] \to [S,\, S]$ (the *extension operator*);
- a 2-cell $s \colon I \to S$ (the *unit*);

satisfying the following axioms:

- the *left unit law*, i.e. for any $A,\, B \colon O \to X_0$
$$\begin{tikzcd} IA \arrow[r, "sA"] \arrow[dr, bend right, "k"'] & SA \arrow[d, "k^{\dagger}"] \\ & SB; \end{tikzcd}$$
- the *right unit law*, i.e. $s^{\dagger} = 1_S$;
- the *associativity law*, i.e. for any 2-cells $k \colon IA \to SB$ and $l \colon IB \to SC$
$$\begin{tikzcd} SA \arrow[r, "k^{\dagger}"] \arrow[dr, bend right, "(l^{\dagger} \cdot k)^{\dagger}"'] & SB \arrow[d, "l^{\dagger}"] \\ & SC. \end{tikzcd}$$

Example 3.2.2.

(i) Let us consider relative monads in $\mathcal{K}$ with $X_0 = X$ and $I = 1_X$. Since operators with $I = 1_X$ are pasting operators, we get back exactly no-iteration monads in $\mathcal{K}$ [MW10, Theorem 2.4].

(ii) Thanks to part (i) of Example 3.1.4, setting $\mathcal{K} = \mathbf{Cat}$ gives back exactly the definition of relative monad given in [ACU15] and recalled in Definition 3.1.1 here.

(iii) Using part (ii) of Example 3.1.4 we can write more explicitly what is a relative monad $(\mathbb{X},\, I,\, S)$ in $\mathcal{V}$-**Cat**. Such an object consists of a pair of $\mathcal{V}$-categories $\mathbb{X}$ and $\mathbb{X}_0$ together with:

- two $\mathcal{V}$-functors $I, S\colon \mathbb{X}_0 \to \mathbb{X}$;
- a family of functions for any $a, b \in \mathbb{X}_0$, $(-)^{\dagger}_{S}\colon \overline{\mathbb{X}}(Ia,\, Sb) \to \overline{\mathbb{X}}(Sa,\, Sb)$ natural in a and b (with $U\mathbb{X}$ the underlying category of $\mathbb{X}$);
- a $\mathcal{V}$-natural transformation $s\colon I \to S$;

satisfying left/right unital law and associativity.

We notice that this is not what we expect to be an *enriched relative monad*, which would involve a natural family of maps in $\mathcal{V}$, for any $a \in \mathbb{X}$ and $b \in \mathbb{Y}$,

$$(-)^{\#}_{a,b}\colon \mathbb{Z}(Fa, Gb) \longrightarrow \mathbb{Z}'(F'a, Gb).$$

We get such a notion using the operators described in part (iii) of Example 3.1.4.

When we set $\mathcal{K} = \mathbf{Cat}$ we know that any relative monad is induced by a relative adjunction [ACU15]. It is natural to wonder if the same holds in any 2-category $\mathcal{K}$. In order to show this we start defining a relative adjunction in $\mathcal{K}$.

Definition 3.2.3. Let $I\colon C_0 \to C$ be a 1-cell in $\mathcal{K}$. A *relative adjunction in* $\mathcal{K}$ over I, denoted as $F\,{}_I\!\dashv G$ consists of an object D in $\mathcal{K}$ together with:

- two 1-cells $F\colon C_0 \to D$ and $G\colon D \to C$;
- a 2-cell $\iota\colon I \to GF$;

such that the operator $G(-)\iota\colon [F,\, 1_D] \to [I, G]$ induces isomorphisms, for any span $(A, B)\colon O \to C_0; D$,

$$\mathcal{K}[O,\, D](FA,\, B) \xrightarrow{\sim} \mathcal{K}[O,\, C](IA,\, GB).$$

Remark 3.2.4. By taking $I = 1_C$ we see that this definition generalises that of an adjunction in $\mathcal{K}$ [Lac10, Section 2.1]. Additionally, considering the case $\mathcal{K} = \mathbf{Cat}$ we see that it also generalises the definition of relative adjunction defined in [ACU15].

Lemma 3.2.5. *A relative adjunction $F_I \dashv G$ induces a relative monad (X, I, GF).*

Proof. By Proposition 3.1.6 the operator $G(-)\iota \colon [F, 1] \to [I, G]$ induces an operator $(-)^\flat \colon [I, G] \to [F, 1]$. Then, by Remark 3.1.5 we get an operator $(-)^\# := (-)^\flat \cdot (1, F) \colon [I, GF] \to [F, F]$. We define the extension operator $(-)^\dagger$ of $S =_{\mathrm{def}} GF$ as $(-)^\#$ composed after with $G(-)$ (by Remark 3.1.5 we get an operator of the required type). As unit we consider $s := \iota$.

The left unital law follows from the fact that $(-)^\#$ is the inverse of $G(-)\iota$, and therefore for any $f \colon IA \to SB$ we have $f = G(f^\#)\iota = f^\dagger\iota$.

Moreover we can deduce also the right unital law, as

$$\begin{aligned} (s)^\dagger &= G(\,(s)^\#\,) && \text{(by definition of } (-)^\dagger) \\ &= G(\,(G(1_F)\iota)^\#\,) && \text{(by definition of } s \text{ and indentity law in } \mathcal{K}) \\ &= G1_F = 1_{GF} = 1_S && \text{(since } (-)^\# \text{ inverse of } G(-)\iota). \end{aligned}$$

We have left to check the associativity law. Given $f \colon IA \to SB$ and $g \colon IB \to SC$, we have

$$\begin{aligned} (g^\dagger f)^\dagger &= G(\,(G(g^\#)f)^\#\,) && \text{(by definition of } (-)^\dagger) \\ &= G(\,(G(g^\#)G(f^\#)\iota)^\#\,) && \text{(by left unital law)} \\ &= G(\,(G(g^\# f^\#)\iota)^\#\,) && \text{(by strict functoriality of } G) \\ &= G(g^\# f^\#) = G(g^\#)G(f^\#) = g^\dagger f^\dagger && \text{(by left unital law).} \end{aligned}$$

□

3.3. The 2-category of Relative Monads

In this section, fixed a 2-category $\mathcal{K}$, we will give the definition of the 2-category $\mathbf{Rel}(\mathcal{K})$ of relative monads in $\mathcal{K}$.

Definition 3.3.1. Let (X, I, S) and (Y, J, T) be two relative monads in $\mathcal{K}$. A *relative monad morphism* $(F, F_0, \phi)\colon (X, I, S) \to (Y, J, T)$ consists of two 1-cells $F_0\colon X_0 \to Y_0$ and $F\colon X \to Y$ and a 2-cell $\phi\colon FS \to TF_0$ satisfying the following axioms:

- $FI = JF_0$;
- *unit law*, i.e. the following diagram commutes

$$\begin{tikzcd} & FS \arrow[d, "\phi"] \\ FI \arrow[ur, "Fs"] \arrow[r, "tF_0"'] & TF_0; \end{tikzcd}$$

- *extension law*, i.e. for any 1-cells $A, B\colon O \to X$ and 2-cell $k\colon IA \to SB$ the following diagram commutes

$$\begin{tikzcd} FSA \arrow[r, "\phi A"] \arrow[d, "F(k_S^\dagger)"'] & TF_0A \arrow[d, "(\phi B \cdot Fk)_T^\dagger"] \\ FSB \arrow[r, "\phi B"'] & TF_0B. \end{tikzcd}$$

Remark 3.3.2. A relative monad morphism $(F, F_0, \phi)\colon (X, I, S) \to (Y, J, T)$ between monads, i.e. when $X_0 = X$, $I = 1_X$, $Y_0 = Y$ and $J = 1_Y$, is the same as a monad morphism $(F, \phi)\colon (X, S) \to (Y, T)$ in $\mathbf{Mnd}(\mathcal{K}^{op})^{op}$.

Definition 3.3.3. Let $(F, F_0, \phi), (F', F'_0, \phi')\colon (X, I, S) \to (Y, J, T)$ be two relative monad morphisms. A *relative monad transformation* $(p, p_0)\colon (F, F_0, \phi) \to (F', F'_0, \phi')$ consists of two 2-cells $p\colon F \to F'$ and $p_0\colon F_0 \to F'_0$ such that:

- $Jp_0 = pI$;

- the following diagram commutes

$$\begin{tikzcd}
FS \arrow[r, "pS"] \arrow[d, "\phi"'] & F'S \arrow[d, "\phi'"] \\
TF_0 \arrow[r, "Tp_0"'] & TF'_0.
\end{tikzcd}$$

Remark 3.3.4. A relative monad transformation $(p,\, p_0)\colon (F,\, F_0,\, \phi) \to (F',\, F'_0,\, \phi')$ with $X_0 = X$, $I = 1_X$, $Y_0 = Y$ and $J = 1_Y$ is the same as a monad transformation of the form $p\colon (F,\, \phi) \to (F',\, \phi')$ in the sense of Street [Str72].

Proposition 3.3.5. *Let $\mathcal{K}$ be a 2-category. There is a 2-category* $\mathbf{Rel}(\mathcal{K})$ *of relative monads in $\mathcal{K}$ with relative monads, relative monad morphisms and relative monad transformations as 0-, 1- and 2-cells.* □

Using part (i) of Example 3.2.2 and Remarks 3.3.2 and 3.3.4 we get the following proposition, which shows how our definition of $\mathbf{Rel}(\mathcal{K})$ extends Street's definition of $\mathbf{Mnd}(\mathcal{K})$ [Str72], the 2-category of monads in a 2-category $\mathcal{K}$. Before stating the proposition we recall the definition of *full sub-2-category*.

Let $\mathcal{K}$ and $\mathcal{L}$ be two 2-categories. A 2-functor $J\colon \mathcal{K} \to \mathcal{L}$ exhibits $\mathcal{K}$ as a *full sub-2-category* of $\mathcal{L}$ if for all pair of objects $x,\, y \in \mathcal{K}$ the functor $J_{x,y}\colon \mathcal{K}(x,\, y) \to \mathcal{L}(Jx,\, Jy)$ is an equivalence of categories.

Proposition 3.3.6. $\mathbf{Mnd}(\mathcal{K}^{op})^{op}$ *is a full locally full sub-2-category of* $\mathbf{Rel}(\mathcal{K})$ *consisting of relative monads $(X,\, I,\, S)$ with $X_0 = X$ and $I = 1_X$.*

In Section 3.6 we will describe an equivalent way to define morphisms of relative monads using a generalised version of right modules. We will then build a 2-category equivalent to $\mathbf{Rel}(\mathcal{K})$.

3.4. Relative Algebras

We now introduce the notion of an Eilenberg-Moore object for a relative monad, which we will refer to as *relative EM object*. The approach used is the same as the one in [Str72]. The notion of *relative* algebra for a relative monad has been already introduced in [ACU15, MW10], here we complete it using our definition of operator (Definition 3.1.3). From now on we will consider a relative monad $(X,\, I,\, T) \in \mathbf{Rel}(\mathcal{K})$.

Definition 3.4.1. Let $K \in \mathcal{K}$. A *K-indexed relative EM-algebra* (or *relative left module*) consists of a 1-cell $M\colon K \to X$ together with an operator $(-)^m\colon [I,\, M] \to [T,\, M]$ satisfying the following axioms:

- *unit law*, i.e. for any span $(A,\, B) : O \to X_0; K$ and any 2-cell $h\colon IA \to MB$ the diagram below commutes

$$\begin{tikzcd} IA \arrow[r, "tA"] \arrow[dr, bend right, "h"'] & TA \arrow[d, "h^m"] \\ & MB; \end{tikzcd}$$

- *associativity law*, i.e. for any pair of spans $(A,\, B)\colon O \to X_0; K$ and $C\colon O \to X_0$, and any 2-cells $h\colon IA \to MB$ and $k\colon IC \to TA$, the diagram below commutes

$$\begin{tikzcd} TC \arrow[r, "k^\dagger"] \arrow[dr, bend right, "(h^m \cdot k)^m"'] & TA \arrow[d, "h^m"] \\ & MB. \end{tikzcd}$$

Definition 3.4.2. Let $K \in \mathcal{K}$ and let $(M,\, (-)^m)$ and $(N,\, (-)^n)$ be two K-indexed relative EM-algebras. A *K-indexed relative EM-algebra morphism* $f\colon (M,\, (-)^m) \to (N,\, (-)^n)$ consists of a 2-cell $f\colon M \to N$ such that for any 2-cell $h\colon IA \to MB$ (given any span $(A,\, B)\colon O \to X_0; K$):

$$\begin{tikzcd} TA \arrow[r, "h^m"] \arrow[dr, bend right, "(fB \cdot h)^n"'] & MB \arrow[d, "fB"] \\ & NB. \end{tikzcd}$$

Clearly there is a category $\mathrm{Mod}^T(K)$ of K-indexed relative algebras. Therefore we have an induced 2-functor

$$\mathrm{Mod}^T(-)\colon \mathcal{K}^{op} \to \mathbf{Cat}.$$

Definition 3.4.3. We say that a relative monad $(X, I, T) \in \mathbf{Rel}(\mathcal{K})$ has a *relative EM object* if $\mathrm{Mod}^T(-)\colon \mathcal{K}^{op} \to \mathbf{Cat}$ is representable. We will denote the representing object with $X^{I,T}$.

Example 3.4.4. Let T-Alg be the category of EM-algebras for a relative monad $(\mathbb{X}, I, T)$ in **Cat** defined in [ACU15]. We can see that this gives us a relative EM object for T. In order to prove it we just need to notice that, for any category $\mathbb{K}$, a $\mathbb{K}$-indexed relative algebra $M\colon \mathbb{K} \to \mathbb{X}$ is the same as endowing, for any $k \in \mathbb{K}$, each $Mk \in \mathbb{X}$ with a relative EM-algebra structure. Therefore each relative algebra M induces a functor $\bar{M}\colon \mathbb{K} \to T$-Alg. On the other hand if we start with a functor $\bar{M}$ then its composition with $U\colon T\text{-Alg} \to \mathbb{X}$ has a relative EM-algebra structure. These constructions are clearly inverses of each other and provide a natural isomorphism

$$\mathrm{Mod}^T(\mathbb{K}) \cong \mathbf{Cat}(\mathbb{K}, T\text{-Alg}).$$

For any $(X, I, T) \in \mathbf{Rel}(\mathcal{K})$, we can define $U\colon \mathrm{Mod}^T(-) \to \mathcal{K}(-, X)$ the forgetful natural transformation, sending a relative right T-module to its underlying 1-cell, and the *free relative algebra* transformation $F\colon \mathcal{K}(-, X_0) \to \mathrm{Mod}^T(-)$ defined as, for any indexing object $K \in \mathcal{K}$,

$$K \xrightarrow{M} X_0 \longmapsto (TM, (-)^{\dagger}_{*,M*'}).$$

Therefore we get a diagram in $\hat{\mathcal{K}}$ of the form

$$\begin{tikzcd}
& \mathrm{Mod}^T(-) \arrow[dr, "U"] \arrow[d, phantom, "\Uparrow t"] & \\
\mathcal{K}(-, X_0) \arrow[ur, "F"] \arrow[rr, "I_* := I \circ -"'] & {} & \mathcal{K}(-, X).
\end{tikzcd}$$

Lemma 3.4.5. *The natural transformations defined above form a relative adjunction* $F\ {}_{I_*}\dashv U$ *in* $\hat{\mathcal{K}}$. *Moreover, the relative monad induced by it is* $(\mathcal{K}(-, X), I_*, T_*)$.

Proof. In order to prove the first claim we need to find, for any $M \in \mathcal{K}(K, X_0)$ and $(N, (-)^n) \in \mathrm{Mod}^T(K)$, a natural bijection

$$\mathrm{Mod}^T(K)(FM, (N, (-)^n)) \cong \mathcal{K}[K, X](IM, N).$$

For any relative algebra map $\bar{f} : FM = (TM, (-)^\dagger) \to (N, (-)^n)$ we define the 2-cell $\bar{f}^\# : IM \to N$ as $\bar{f} \cdot tM$. On the other hand for any 2-cell $f : IM \to N$ we can define $f^\flat := f^n : TM \to N$ which is a relative algebras map because, for any $A, B : O \to X_0$ and $h : IA \to TMB$,

$$\begin{tikzcd}
TA \arrow[r, "h^\dagger"] \arrow[dr, bend right, "(f^nB \cdot h)^n"'] & TMB \arrow[d, "f^nB = (fB)^n"] \\
 & NB
\end{tikzcd}$$

which is true since $(-)^n$ is a relative algebra operator.

For any f we can see that $(f^\flat)^\# = f^n \cdot tM$ which is exactly equal to f by the unit law of $(-)^n$. Moreover, for any $\bar{f}$, we have

$$\begin{aligned}
(\bar{f}^\#)^\flat &= (\bar{f} \cdot tM)^n && \text{(by definitions)} \\
&= \bar{f} \cdot (tM)^\dagger && \text{(since } \bar{f} \text{ is in } \mathrm{Mod}^T(K)) \\
&= \bar{f} \cdot 1_{TM} = \bar{f} && \text{(by right unital law of } (-)^\dagger).
\end{aligned}$$

We can see that $UF = T_*$ and more generally the relative monad induced by $F\ {}_{I_*}\dashv U$ is the same as the one induced by (X, I, T) in $\hat{\mathcal{K}}$. □

Theorem 3.4.6. *If $\mathcal{K}$ has relative EM objects, then any relative monad is induced by a relative adjunction.*

Proof. The proof is just a matter of translating Lemma 3.4.5 using the Yoneda lemma, since the covariant Yoneda embedding reflects adjunctions [Gra74, Proposition I,6.4] and also relative adjunctions. □

3.5. Relative Distributive Laws

In this section, we will define the counterpart of distributive laws for a relative monad $I, T \colon X_0 \to X$ and a monad $S \colon X \to X$ which *restricts to* X_0. When I is an inclusion it is clear what we mean by this, but when I is any 1-cell we need to define a new notion. Therefore, with the following definition, we introduce the notion of *monad compatible* with a 1-cell $I : X_0 \to X$.

Definition 3.5.1. Let $I \colon X_0 \to X$ be 1-cell in a 2-category $\mathcal{K}$. A *monad compatible with* I consists of a pair of monads $(X_0,\, S_0)$ and $(X,\, S)$ in $\mathcal{K}$ such that $SI = IS_0$, $mI = Im_0$ and $sI = Is_0$. We will denote it with $(S,\, S_0)$.

To have a monad compatible with I is the same as lifting I to a morphism in $\mathbf{Mnd}(\mathcal{K})$ with corresponding 2-cell the identity, i.e. requiring $(I,\, 1) \colon (X_0,\, S_0) \to (X,\, S)$ to be a monad morphism.

Definition 3.5.2. Let $(X,\, I,\, T)$ be a relative monad in $\mathcal{K}$ and $(S,\, S_0)$ a monad compatible with I. A *relative distributive law* of T over $(S,\, S_0)$ consists of a 2-cell $d \colon ST \to TS_0$ in $\mathcal{K}$ satisfying the following axioms:

(D1)
$$\begin{tikzcd}
S^2T \arrow[r, "mT"] \arrow[d, "Sd"'] & ST \arrow[dd, "d"] \\
STS_0 \arrow[d, "dS_0"'] & \\
TS_0^2 \arrow[r, "Tm_0"'] & TS_0
\end{tikzcd}$$

(D2)
$$\begin{tikzcd}
& ST \arrow[dd, "d"] \\
T \arrow[ur, "sT"] \arrow[dr, "Ts_0"'] & \\
& TS_0
\end{tikzcd}$$

and for any object $O \in \mathcal{K}$, any pair of 1-cells A, $B \colon O \to X_0$ and any 2-cell $f \colon IA \to TB$

(D3)
$$\begin{tikzcd}
STA \arrow[r, "dA"] \arrow[d, "Sf_T^\dagger"'] & TS_0A \arrow[d, "(dB \cdot Sf)_T^\dagger"] \\
STB \arrow[r, "dB"'] & TS_0B
\end{tikzcd}$$

(D4)
$$\begin{tikzcd}
SI \arrow[r, "St"] \arrow[d, equal] & ST \arrow[d, "d"] \\
IS_0 \arrow[r, "tS_0"'] & TS_0
\end{tikzcd}$$

From now on, we will always consider a relative monad (X, I, T) and a monad (S, S_0) compatible with I.

Remark 3.5.3. We can see that, setting $I = 1_X$, we get back the definition of a distributive law between two monads T and S in $\mathcal{K}$ [MM07].

In the formal theory of monads [Str72] Street shows that a distributive law between two monads is an object of $\mathbf{Mnd}(\mathbf{Mnd}(\mathcal{K}))$, that is a monad in the 2-category of monads. The next Proposition proves a similar result for relative distributive laws.

Proposition 3.5.4. *A relative distributive law is the same thing as an object of* $\mathbf{Mnd}(\mathbf{Rel}(\mathcal{K}))$.

Proof. The table below provides a correspondence between axioms:

Axiom	In **Mnd**(**Rel**($\mathcal{K}$))
(D1) and $mI = Im_0$	$(m,\, m_0)$ is a 2-cell in **Rel**($\mathcal{K}$)
(D2) and $sI = Is_0$	$(s,\, s_0)$ is a 2-cell in **Rel**($\mathcal{K}$)
(D3), (D4) and $SI = IS_0$	$(S,\, S_0,\, d)$ is a 1-cell in **Rel**($\mathcal{K}$)

□

The aim of the last part of this section is to prove a Beck-like theorem for relative distributive laws, using liftings to the algebras of a monad $(S,\, S_0)$ compatible with I. First of all, we will explicitly define a lifting of a relative monad T to the algebras of $(S,\, S_0)$. Then we will show how we can go from relative distributive laws to liftings (Lemma 3.5.7) and vice versa (Theorem 3.5.8). Finally, we show that these constructions are inverses of each other.

Before proceeding with the definition of lifting to algebras, let us fix some notation. Given a monad $(S,\, S_0)$ compatible with $I\colon X_0 \to X$, we always get a natural transformation induced on indexed algebras, $I_*\colon S_0\text{-Alg}(-) \to S\text{-Alg}(-)$, defined, for any indexing object $K \in \mathcal{K}$ and any S_0-algebra $(M,\, m)$, as

$$(S_0M \xrightarrow{m} M) \longmapsto (SIM \xrightarrow{Im} IM),$$

where $Im\colon SIM \to IM$ is well-defined because $SI = IS_0$. Let us denote with U_0 and U the forgetful natural transformations from S_0-Alg$(-)$ and S-Alg$(-)$ into $\mathcal{K}(-, X_0)$ and $\mathcal{K}(-, X)$.

Definition 3.5.5. Let $(X,\, I,\, T)$ be a relative monad in $\mathcal{K}$. A *lifting of T to the algebras of* $(S,\, S_0)$ is a relative monad $(I_*,\, \hat{T})\colon S_0\text{-Alg}(-) \longrightarrow S\text{-Alg}(-)$, such that:

(i) the following diagram commutes

$$\begin{tikzcd}
S_0\text{-Alg}(-) \arrow[r, "\hat{T}"] \arrow[d, "U_0"'] & S\text{-Alg}(-) \arrow[d, "U"] \\
\mathcal{K}(-, X_0) \arrow[r, "T \circ -"'] & \mathcal{K}(-, X);
\end{tikzcd}$$

(ii) the extension operator $(-)^{\dagger}_{\widehat{T}}$ of $\widehat{T}$ is induced by the one of T, i.e. for any pair of K-indexed S_0-algebras $(M,\, m)$ and $(N,\, n)$ if the following diagram on the left commutes, then the one on the right commutes as well

$$\begin{tikzcd}
IS_0M \arrow[r, "Sfr"] \arrow[d, "Im"'] & STN \arrow[d, "\widehat{T}n"] & & STM \arrow[r, "Sf^{\dagger}"] \arrow[d, "\widehat{T}m"'] & STN \arrow[d, "\widehat{T}n"] \\
IM \arrow[r, "f"'] & TN & \Rightarrow & TM \arrow[r, "f^{\dagger}"'] & TN;
\end{tikzcd}$$

(iii) the unit $\hat{t}$ is induced by t, i.e. for any S_0-algebra $(M,\, m)$ the following diagram commutes

$$\begin{tikzcd}
IS_0M \arrow[r, "StM"] \arrow[d, "Im"'] & STM \arrow[d, "\widehat{T}m"] \\
IM \arrow[r, "tM"'] & TM.
\end{tikzcd}$$

Proposition 3.5.6. *Let $\widehat{T}$ be a lifting of T to the algebras of $(S,\, S_0)$ and let us denote with $\widehat{m_0M}$ the S-algebra structure on TS_0M given by $\widehat{T}$ applied to the free S_0-algebra $(S_0M,\, m_0M)$. Then, for any other S_0-algebra $(M,\, m)$ the S-algebra structure on TM given by $\widehat{T}$ is*

$$STM \xrightarrow{STs_0M} STS_0M \xrightarrow{\widehat{m_0M}} TS_0M \xrightarrow{Tm} TM.$$

Proof. We begin noticing that, by one of the algebra axioms, m is itself a S_0-algebra morphism between $(S_0M,\, m_0M)$ and $(M,\, m)$. Therefore the diagram below commutes, as it is the diagram making Tm a S-algebra morphism,

$$\begin{tikzcd}
STS_0M \arrow[r, "STm"] \arrow[d, "\widehat{m_0M}"'] & STM \arrow[d, "\widehat{T}m"] \\
TS_0M \arrow[r, "Tm"'] & TM.
\end{tikzcd}$$

Moreover, using the unit algebra axiom for m, we get the desired equality

$$\begin{tikzcd}
STM \arrow[r, "STs_0M"] \arrow[dr, bend right, "1_{STM} = S1_{TM}"'] & STS_0M \arrow[r, "\widehat{m_0M}"] \arrow[d, "STm"] & STM \arrow[d, "Tm"] \\
 & TS_0M \arrow[r, "\widehat{T}m"'] & TM.
\end{tikzcd}$$

$\square$

Lemma 3.5.7. *Let $d\colon ST \to TS_0$ be a relative distributive law of T over $(S,\, S_0)$. Then there is a lifting $\widehat{T}$ of T to the algebras of $(S,\, S_0)$ defined on K-indexed S_0-algebras $(M,\, m)$ as*

$$STM \xrightarrow{dM} TS_0M \xrightarrow{Tm} TM$$

and on morphisms of K-indexed S_0-algebras $f\colon (M,\, m) \to (N,\, n)$ as

$$Tf\colon (TM,\, Tm \cdot dM) \to (TN,\, Tn \cdot dN).$$

Proof. First of all, we need to verify that the definition above gives a K-indexed S-algebra structure. One can check this using (D1) for the compatibility axiom and (D2) for the unit. Now we have left to prove part (ii) and (iii) of the definition of a lifting. For the first one what we need to check is the following implication

$$\begin{tikzcd}
IS_0M \arrow[r, "Sf"] \arrow[dd, "Im"'] & STN \arrow[d, "dN"] & & STM \arrow[r, "Sf^\dagger"] \arrow[d, "dM"'] & STN \arrow[d, "dN"] \\
& TS_0N \arrow[d, "Tn"] & \Rightarrow & TS_0M \arrow[r, dotted, "(dN \cdot Sf)^\dagger"] \arrow[d, "Tm"'] & TS_0N \arrow[d, "Tn"] \\
IM \arrow[r, "f"'] & TN & & TM \arrow[r, "f^\dagger"'] & TN.
\end{tikzcd}$$

Using (D3) is enough to prove that the bottom square on the right commutes whenever the diagram on the left does.

$$\begin{aligned}
Tn \cdot (dN \cdot Sf)^\dagger &= (Tn \cdot dN \cdot Sf)^\dagger && \text{(by naturality of } (-)^\dagger) \\
&= (f \cdot Im)^\dagger && \text{(by diagram on the left)} \\
&= f^\dagger \cdot Tm && \text{(by naturality of } (-)^\dagger)
\end{aligned}$$

Similarly part (iii) follows from (D4) and the naturality of t. Indeed, for any K-indexed S_0-algebra $(M,\, m)$ the following diagram is commutative

$$\begin{tikzcd}
IS_0M \arrow[r, "StM"] \arrow[dd, "Im"'] \arrow[dr, "tS_0M"'] & STM \arrow[d, "dM"] \\
& TS_0M \arrow[d, "Tm"] \\
IM \arrow[r, "tM"'] & TM.
\end{tikzcd}$$

□

Theorem 3.5.8. *Let $\widehat{T}$ be a lifting of T to the algebras of $(S,\, S_0)$. Then d defined as*

$$ST \xrightarrow{STs_0} STS_0 \xrightarrow{\widehat{m_0}} TS_0$$

where $\widehat{m_0}$ is the X_0-indexed S-algebra structure of $\hat{T}(S_0,\, m_0)$, is a relative distributive law of T over $(S,\, S_0)$.

Proof. We need to prove that axioms (D1), (D2), (D3) and (D4) hold. In the following table we explain what will be used to prove each axiom.

Axiom	Axioms used in the proof
(D1)	S-algebra axiom for $\widehat{m_{0X}}$ and Proposition 3.5.6
(D2)	unit algebra axiom for $\widehat{m_{0X}}$
(D3)	(D4), (D1), part (ii) of Definition 3.5.5 and more
(D4)	part (iii) Definition 3.5.5

For (D1), (D2) and (D4) it is enough to write down what we get explicitly using the definition of d. The diagrams we get are the following

(D1)

$$\begin{tikzcd}
S^2T \arrow[r, "S^2Ts_0"] \arrow[dd, "mT"'] & S^2TS_0 \arrow[r, "S\widehat{m_0}"] \arrow[dd, "mTS_0"'] & STS_0 \arrow[r, "STs_0S_0"] \arrow[ddr, "\widehat{m_0}"'] & STS_0^2 \arrow[d, "\widehat{m_0S_0}"] \\
 & & & TS_0^2 \arrow[d, "Tm_0"] \\
ST \arrow[r, "STs_0"'] & STS_0 \arrow[rr, "\widehat{m_0}"'] & & TS_0
\end{tikzcd}$$

(D2)

$$\begin{tikzcd}
T \arrow[r, "sT"] \arrow[d, "Ts_0"'] & ST \arrow[d, "STs_0"] \\
TS_0 \arrow[r, "sTS_0"] \arrow[dr, bend right, "1_{TS_0}"'] & STS_0 \arrow[d, "\widehat{m_0}"] \\
 & TS_0
\end{tikzcd}$$

(D2)

$$\begin{tikzcd}
SI \arrow[r, "St"] \arrow[d, "SIs_0"] \arrow[dd, equal, bend right] & ST \arrow[d, "STs_0"] \\
IS_0^2 \arrow[r, "StS_0"] \arrow[d, "Im_0"] & STS_0 \arrow[d, "\widehat{m_0}"] \\
IS_0 \arrow[r, "tS_0"'] & TS_0.
\end{tikzcd}$$

Let us now look at (D3). For any 2-cell $\alpha\colon IA \to TB$ in $\mathcal{K}$ (for any pair of 1-cells $A, B\colon O \to X_0$), we need to prove that

$$\begin{tikzcd} STA \arrow[r, "S\alpha^\dagger"] \arrow[d, "dA"'] & STB \arrow[d, "dB"] \\ TS_0A \arrow[r, "(dB \cdot S\alpha)^\dagger"'] & TS_0B \end{tikzcd} \quad \text{i.e.} \quad \begin{tikzcd} STA \arrow[r, "S\alpha^\dagger"] \arrow[d, "STs_0A"'] \arrow[dr, phantom, "(*)"] & STB \arrow[d, "STs_0B"] \\ STS_0A \arrow[r, dotted, "S(dB \cdot S\alpha)^\dagger"] \arrow[d, "\widehat{m_0}A"'] \arrow[dr, phantom, "(**)"] & STS_0B \arrow[d, "\widehat{m_0}B"] \\ TA \arrow[r, "\alpha^\dagger"'] & TB. \end{tikzcd} \tag{3.5.1}$$

We will proceed proving that both squares $(*)$ and $(**)$ in diagram (3.5.1) are commutative. Diagram $(*)$ is the image through S of a diagram $(*')$

$$\begin{tikzcd} TA \arrow[r, "\alpha^\dagger"] \arrow[d, "Ts_0A"'] \arrow[dr, phantom, "(*')"] & TB \arrow[d, "Ts_0B"] \\ TS_0A \arrow[r, "(dB \cdot S\alpha)^\dagger"'] & TS_0B, \end{tikzcd}$$

so it suffices to prove the commutativity of $(*')$. By left naturality of $(-)^\dagger$ and the equality $(tS_0A)^\dagger = t^\dagger S_0A = 1_{S_0A}$

$$Ts_0A = (tS_0A \cdot Is_0A)^\dagger,$$

and therefore

$$\begin{aligned} (dB \cdot S\alpha)^\dagger \cdot Ts_0A &= ((dB \cdot S\alpha)^\dagger \cdot tS_0A \cdot Is_0A)^\dagger && \text{(by associativity of } (-)^\dagger) \\ &= (dB \cdot S\alpha \cdot Is_0A)^\dagger && \text{(by left unit of } (-)^\dagger) \\ &= (dB \cdot sTB \cdot \alpha)^\dagger && \text{(by naturality of } s \text{ and } Is_0 = sI) \\ &= (Ts_0B \cdot \alpha)^\dagger && \text{(by (D4))} \\ &= Ts_0B \cdot \alpha^\dagger && \text{(by associativity of } (-)^\dagger). \end{aligned}$$

For $(**)$ we need to use axiom (ii) of Definition 3.5.5. We can rewrite this axiom using Proposition 3.5.6 and the definition of d, in the following way:

$$\begin{tikzcd}
IS_0M \arrow[r, "Sf"] \arrow[dd, "Im"'] & STN \arrow[d, "dN"] & & STM \arrow[r, "Sf^{\dagger}"] \arrow[d, "dM"'] & STN \arrow[d, "dN"] \\
& TS_0N \arrow[d, "Tn"] & \Rightarrow & TS_0M \arrow[d, "Tm"'] & TS_0N \arrow[d, "Tn"] \\
IM \arrow[r, "f"'] & TN & & TM \arrow[r, "f^{\dagger}"'] & TN
\end{tikzcd}$$

for $(M,\, m)$ and $(N,\, n)$ two S_0-algebras and f a 2-cell in $\mathcal{K}$. If we consider the case with $(M,\, m) := (S_0A,\, m_0A)$, $(N,\, n) := (S_0B,\, m_0B)$ and $f := dB \cdot S\alpha$ we would get $(**)$ on the right (using again Proposition 3.5.6). So it is enough to prove that with these choices the diagram on the left is commutative, i.e.

$$\begin{tikzcd}
IS_0^2A \arrow[r, equal] \arrow[dd, "Im_0A"'] & S^2IA \arrow[r, "S^2\alpha"] \arrow[dd, "mIA"'] & S^2TB \arrow[r, "SdB"] \arrow[dd, "mTB"'] & STS_0B \arrow[d, "dS_0B"] \\
& & & TS_0^2B \arrow[d, "Tm_0B"] \\
IS_0A \arrow[r, equal] & SIA \arrow[r, "S\alpha"'] & STB \arrow[r, "dB"'] & TS_0B.
\end{tikzcd}$$

The square on the left commutes because $(S,\, S_0)$ is compatible with I, the one in the center commutes by naturality of m and the diagram on the right is (D1) applied to B. $\square$

In summary, Theorem 3.5.8 and Lemma 3.5.7 give us two constructions:

$$\begin{tikzcd}
\text{Rel. Distr. Laws} \arrow[r, bend left, "(-)^{\flat}"] & \text{Lift. to Alg.} \arrow[l, bend left, "(-)^{\#}"]
\end{tikzcd}$$

The following Theorem shows that these constructions are inverses of each other.

Theorem 3.5.9. *Let (X, I, T) be a relative monad in $\mathcal{K}$ and (S, S_0) a monad compatible with I. Relative distributive laws $d\colon ST \to TS_0$ of T over (S, S_0) are equivalent to liftings of T to the algebras of (S, S_0).*

Proof. Let start proving that $(d^\flat)^\# = d$:

$$\begin{aligned}
(d^\flat)^\# &= d^\flat(m_0) \cdot STs_0 && \text{(by definition of } (-)^\#\text{)}\\
&= Tm_0 \cdot dS_0 \cdot STs_0 && \text{(by definition of } (-)^\flat\text{)}\\
&= Tm_0 \cdot TS_0 s_0 \cdot d && \text{(by naturality of } d\text{)}\\
&= d && \text{(by right unit of } S_0\text{).}
\end{aligned}$$

On the other hand, using Proposition 3.5.6 we can see that $(\widehat{T^\#})^\flat = \widehat{T}$. □

This result gives us the first equivalence of the counterpart of Beck's Theorem for relative distributive laws. We will prove the second equivalence in Section 3.6 (Theorem 3.6.18), thus getting the entire counterpart (Theorem 3.6.19).

3.6. Relative Right Modules and Kleisli Objects

In the formal theory of monads [LS02, Str72], if we consider left modules (algebras) for a monad in $\mathcal{K}^{op}$ we get what are called *right modules* for a monad in $\mathcal{K}$. Using this duality, all the results for algebras can be translated easily to right modules. Unfortunately, when we consider relative monads it is not possible to take advantage of this duality. The issue is that the objects of $\mathbf{Rel}(\mathcal{K}^{op})$ are not relative monads. Indeed, we get two 1-cells, a unit 2-cell together with an extension operator in $\mathcal{K}^{op}$, which is not the same as an operator in $\mathcal{K}$. For this reason, we will need to define relative right modules explicitly.

In this section we will study right modules for relative monads, which we will call *relative right modules.* We will start giving the definition of the category $\mathrm{Mod}_T(K)$ of K-indexed relative right T-modules for an indexing object $K \in \mathcal{K}$ and a relative monad (X, I, T) in $\mathcal{K}$. This construction has to satisfy a couple of conditions.

First of all, we want that, whenever T is an actual monad in $\mathcal{K}$, then the notion of relative right module and the usual one of right module should coincide. Moreover, when we consider $\mathcal{K} = \mathbf{Cat}$, then the Kleisli category for a relative monad defined in [ACU15] should represent the 2-functor $\mathrm{Mod}_T(-)$ of relative right modules.

Once provided the appropriate setting, we will use $\mathrm{Mod}_T(-)$ to construct a 2-category $\mathbf{LiftR}(\mathcal{K})$ with objects relative monads in $\mathcal{K}$, 1-cells *lifting to relative right modules* and 2-cells *maps of liftings to relative right modules.* These concepts generalise the ones of lifting to right modules in the monad case. Then, we show that $\mathbf{LiftR}(\mathcal{K})$ is 2-isomorphic to $\mathbf{Rel}(\mathcal{K})$. Finally, thanks to this equivalence, we prove a Beck-like theorem stating that relative distributive laws are the same as liftings to relative right modules, with the appropriate definition of the latter.

The Category of Relative Right Modules

From now on we will consider a fixed relative monad $(X, I, T) \in \mathbf{Rel}(\mathcal{K})$ where $I \colon X_0 \to X$.

Definition 3.6.1. Let $K \in \mathcal{K}$. A *K-indexed relative right T-module* consists of a 1-cell $M \colon X_0 \to K$ together with an operator $(-)_m \colon [I, T] \to [M, M]$ satisfying the following axioms:

- *unit law*, i.e. $t_m = 1_M$;
- *associativity*, i.e. for any 2-cells $h \colon IA \to TB$ and $k \colon IB \to TC$ (given any three 1-cells $A, B, C \colon O \to X_0$)

$$\begin{tikzcd} MA \arrow[r, "h_m"] \arrow[dr, bend right, "(k^\dagger \cdot h)_m"'] & MB \arrow[d, "k_m"] \\ & MC. \end{tikzcd}$$

Definition 3.6.2. Let $(M, (-)_m)$ and $(N, (-)_n)$ be two K-indexed relative right T-modules. A *K-indexed relative right module morphism* $f\colon (M, (-)_m) \to (N, (-)_n)$ consists of a 2-cell $f\colon M \to N$ such that for any 2-cell $h\colon IA \to TB$ (given any pair of 1-cells $A, B\colon O \to X_0$):

$$\begin{tikzcd}
MA \arrow[r, "h_m"] \arrow[d, "fA"'] & MB \arrow[d, "fB"] \\
NA \arrow[r, "h_n"'] & NB.
\end{tikzcd}$$

Clearly the definitions above form a category $\mathrm{Mod}_T(K)$ of K-indexed relative right modules. Therefore we have an induced 2-functor

$$\mathrm{Mod}_T(-)\colon \mathcal{K} \to \mathbf{Cat}.$$

Remark 3.6.3. We briefly show that if $I = 1_X$, then $\mathrm{Mod}_T(K)$ is equal to the category of K-indexed right modules for a monad in the usual sense. Let us consider a relative right module for a monad (i.e. a relative monad with $I = 1_X$), we can prove that $\rho^M := (1_T)_m$ is a K-indexed right module structure. We need to prove the following axioms

$$\begin{tikzcd}
MT^2 \arrow[r, "\rho^M T"] \arrow[d, "Mn"'] & MT \arrow[d, "\rho^M"] \\
MT \arrow[r, "\rho^M"'] & M
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
M \arrow[r, "Mt"] \arrow[dr, equal, bend right] & MT \arrow[d, "\rho^M"] \\
 & M,
\end{tikzcd}$$

where $n := (1_T)^\dagger$. We can deduce them in the following way:

$$\begin{aligned}
\rho^M \cdot Mn &= (1_T)_m \cdot M(1_T)^\dagger && \text{(by definition)}\\
&= (1_T \cdot (1_T)^\dagger)_m = ((1_T)^\dagger \cdot 1_{T^2})_m && \text{(by left naturality of } (-)_m)\\
&= (1_T)_m \cdot (1_{T^2})_m && \text{(by associativity of } (-)_m)
\end{aligned}$$

$$
\begin{aligned}
&= (1_T)_m \cdot (1_T)_m T && \text{(by left naturality of } (-)_m\text{)}\\
&= \rho^M \cdot \rho^M T && \text{(by definitions)}
\end{aligned}
$$

$$
\begin{aligned}
\rho^M \cdot Mt &= (1_T)_m \cdot Mt && \text{(by definitions)}\\
&= (1_T \cdot t)_m = t_m && \text{(by left naturality of } (-)_m\text{)}\\
&= 1_M && \text{(by unit law for } (-)_m\text{).}
\end{aligned}
$$

On the other side, if we begin with a right module structure $\rho^M \colon MT \to M$, we can define a relative right module structure as

$$f \colon A \to TB \longmapsto f_m := \rho^M B \cdot Mf$$

Then by definition $t_m = \rho^M \cdot Mt$ which is equal to 1_M by the unit axiom for ρ^M. Therefore we have left to prove that, for any $f \colon A \to TB$ and $g \colon B \to TC$, $g_m \cdot f_m = (g^\dagger \cdot f)_m$ where $g^\dagger = nC \cdot Tg$. We can prove this by looking at the diagram

$$\begin{tikzcd}
MA \arrow[r, "Mf"] \arrow[dd, "Mf"'] & MTB \arrow[r, "\rho^M B"] \arrow[d, "MTg"] \arrow[ddl, equal] & MB \arrow[d, "Mg"] \\
 & MT^2C \arrow[r, "\rho^M C"'] \arrow[d, "MnC"'] & MTC \arrow[d, "\rho^M C"] \\
MTB \arrow[r, "Mg^\dagger"'] & MTC \arrow[r, "\rho^M C"'] & MC.
\end{tikzcd}$$

The commutativity follows from the naturality and multiplication axiom for ρ^M. Using indexing and left naturality for $(-)_m$ we can see that these two constructions provide a bijection between relative right modules for a monad and the usual notion of right modules.

Definition 3.6.4. We say that a relative monad $(X,\, I,\, T) \in \mathbf{Rel}(\mathcal{K})$ has a *relative Kleisli object* if $\mathrm{Mod}_T(-)$ is representable. We will denote a representing object with $X_{I,T}$.

Example 3.6.5. Let $\mathrm{Kl}(T)$ the Kleisli category for a relative monad $(\mathbb{X},\, I,\, T)$ in **Cat** [ACU15]. We can see that this gives us a relative Kleisli object for T. Let $(M,\, (-)_m)$ a $\mathcal{K}$-indexed relative right module. We can define a functor $\bar{M}\colon \mathrm{Kl}(T) \to \mathcal{K}$ the same as M on objects, and for any map $f\colon x \nrightarrow y \in \mathrm{Kl}(T)$, i.e. $f\colon Ix \to Ty \in \mathbb{X}$, $\bar{M}f := f_m\colon Mx \to My$. The unit law and associativity of $(-)_m$ ensure that $\bar{M}$ respects identities and composition respectively. Moreover, $\bar{M}$ defined in this way is such that $\bar{M}J_0 = M$. The equality on objects is trivially true, whilst for the action on maps we have

$$\begin{aligned} \bar{M}J_0 f &= (Tf \cdot t_x)_m && \text{(by definitions)} \\ &= Mf \cdot (t_x)_m && \text{(by right naturality of } (-)_m) \\ &= Mf \cdot 1_M = Mf && \text{(by unit law for } (-)_m). \end{aligned}$$

On the other hand, if we start with a functor $\bar{M}\colon \mathrm{Kl}(T) \to \mathcal{K}$ then we can define a $\mathcal{K}$-indexed relative right module in the following way. First of all we define $M\colon \mathbb{X}_0 \to \mathcal{K}$ as $\bar{M}J_0$. Then as operator structure we define, for any map $f\colon Ix \to Ty$, f_m as $\bar{M}f\colon Mx \to My$. In an analogous way as before, the functoriality of $\bar{M}$ proves the unit law and associativity for $(-)_m$ defined in this way.

In the formal theory of monads [Str72], the category of K-indexed right T-modules is shown to be equivalent to $\mathbf{Mnd}(\mathcal{K}^{op})^{op}[\,(X,\, T),\, (K,\, 1_K)\,]$, and therefore the Kleisli object construction gives a left adjoint of the inclusion $\mathcal{K} \to \mathbf{Mnd}(\mathcal{K}^{op})^{op}$ sending an object $K \in \mathcal{K}$ to the unital monad $(K,\, 1_K)$. The following proposition shows why this is not possible in our setting.

Proposition 3.6.6. *Let $(X,\, I,\, T)$ be a relative monad in $\mathcal{K}$ and $K \in \mathcal{K}$ an object. Then the category of relative monad morphisms $\mathbf{Rel}(\mathcal{K})[\,(X,\, I,\, T),\, (K,\, 1_K,\, 1_K)\,]$ is a subcategory of $\mathrm{Mod}_T(K)$.*

Proof. A relative monad map $(M,\, M_0,\, \rho) : (X,\, I,\, T) \to (K,\, 1_K,\, 1_K)$ consists of a pair of maps $M\colon X \to K$ and $M_0\colon X_0 \to K$ such that $MI = M_0$, and a 2-cell $\rho\colon MT \to M_0$ satisfying the axioms (for any 2-cell $f\colon IA \to TB$)

$$\text{(i)}\quad \begin{tikzcd} MI \arrow[r, "Mt"] \arrow[dr, equal, bend right] & MT \arrow[d, "\rho"] \\ & M_0 \end{tikzcd} \qquad \text{(ii)}\quad \begin{tikzcd} MSA \arrow[r, "\rho A"] \arrow[dd, "Mf^{\dagger}"'] & M_0A \arrow[d, "Mf"] \\ & MSB \arrow[d, "\rho B"] \\ MSB \arrow[r, "\rho B"'] & M_0B \end{tikzcd}\ .$$

We can endow M_0 with a relative right module structure defining its operator, for any 2-cell $f\colon IA \to TB$, as $f_m := \rho B \cdot Mf$. The unit law is guaranteed by diagram (i) above, while to prove associativity it is enough to precompose the two composite in diagram (ii) with Mf.

Moreover, let $(\alpha,\, \alpha_0)\colon (M,\, M_0,\, \rho) \to (M',\, M_0',\, \rho')$ be a relative monad transformation. Then the diagram

$$\begin{tikzcd} M_0A \arrow[r, equal] \arrow[d, "\alpha_0 A"'] & MIA \arrow[r, "Mf"] \arrow[d, "\alpha IA"'] & MSB \arrow[r, "\rho B"] \arrow[d, "\alpha SB"'] & M_0B \arrow[d, "\alpha_0 B"] \\ M_0'A \arrow[r, equal] & M'IA \arrow[r, "M'f"'] & M'SB \arrow[r, "\rho' B"'] & M_0'B \end{tikzcd}$$

is commutative using $\alpha_0 = \alpha I$, the naturality of α and the axiom for a relative monad transformation. Therefore α_0 is a relative right module morphism. □

An object of $\mathbf{Rel}(\mathcal{K})[\,(X,\, I,\, T),\, (K,\, 1_K,\, 1_K)\,]$ is similar to the usual notion of a right module, having a straightforward right action $\rho\colon MT \to M_0$ satisfying axioms similar to the one of a right module. So, one could wonder if this would be the appropriate definition for a relative right module over a relative monad. The main problem with this definition is that, in a general 2-category $\mathcal{K}$, is not possible to find a morphism from $\mathcal{K}(X, K)$ to $\mathbf{Rel}(\mathcal{K})[\,(X,\, I,\, T),\, (K,\, 1_K,\, 1_K)\,]$. We need a map of this kind to form a diagram like (3.6.1), which will be crucial in the next section, where we will define a lifting to relative right modules. Instead, for any $(X,\, I,\, T) \in \mathbf{Rel}(\mathcal{K})$ we can consider the diagram

$$\begin{tikzcd}
& \mathrm{Mod}_T(-) \arrow[dr, "J^*_{X_0}"] & \\
\mathcal{K}(X,-) \arrow[ur, "U^*_X"] \arrow[rr, "- \circ I"'] \arrow[rr, phantom, shift left=4, "\Uparrow t"] & & \mathcal{K}(X_0,-)
\end{tikzcd} \tag{3.6.1}$$

where $J^*_{X_0} \colon \mathrm{Mod}_T(-) \to \mathcal{K}(X_0,\, -)$ is a forgetful natural transformation, sending a K-indexed relative right T-module to its underlying 1-cell $M \colon X_0 \to K$, and $U^*_X \colon \mathcal{K}(X,\, -) \to \mathrm{Mod}_T(-)$ is a natural transformation defined as, for any $K \in \mathcal{K}$,

$$\begin{aligned}
(U^*_X)_K \colon \mathcal{K}(X,\, K) &\longrightarrow \mathrm{Mod}_T(K) \\
(M \colon X \to K) &\longmapsto (MT,\, M(-)^\dagger).
\end{aligned}$$

Remark 3.6.7. Whilst using relative algebras we were able to construct a relative adjunction (Lemma 3.4.5), the diagram in (3.6.1) does not always represent one. However, even if we would have such a relative adjuntion, then it would still not seem possible to prove a similar result to Theorem 3.4.6. Indeed, to define a relative adjunction in a 2-category we use operators, which are not self dual. Thus the contravariant Yoneda embedding $Y \colon \mathcal{K}^{op} \to [\mathcal{K},\, \mathbf{Cat}]$ does not preserve operators and so relative adjunctions, while the covariant Yoneda embedding $Y \colon \mathcal{K} \to [\mathcal{K}^{op},\, \mathbf{Cat}]$ does. Nevertheless, there are some 2-categories where we find a relative adjunction using Kleisli objects (for instance **Cat**, see [ACU15, Section 2.3]).

The 2-category **LiftR**($\mathcal{K}$)

Given a 2-category $\mathcal{K}$ we want to use the notion of relative right modules to define a 2-category **LiftR**($\mathcal{K}$) with the same objects as **Rel**($\mathcal{K}$) but 1- and 2-cells defined as *lifting to relative right modules*. Let us start fixing some notation. From now on we will consider two relative monads $(X,\, I,\, S)$ and $(Y,\, J,\, T)$ in $\mathcal{K}$. They induce the following diagrams:

$$\begin{tikzcd}
& \mathrm{Mod}_S(-) \arrow[dr, "J^*_{X_0}"] & \\
\mathcal{K}(X,-) \arrow[ur, "U^*_X"] \arrow[rr, "-\circ I"'] & \arrow[u, phantom, "\Uparrow s"] & \mathcal{K}(X_0,-),
\end{tikzcd}
\qquad
\begin{tikzcd}
& \mathrm{Mod}_T(-) \arrow[dr, "J^*_{Y_0}"] & \\
\mathcal{K}(Y,-) \arrow[ur, "U^*_Y"] \arrow[rr, "-\circ J"'] & \arrow[u, phantom, "\Uparrow t"] & \mathcal{K}(Y_0,-).
\end{tikzcd}$$

Definition 3.6.8. Let $(X,\, I,\, S)$ and $(Y,\, J,\, T)$ be two relative monads in $\mathcal{K}$. A *lifting to relative right modules* consists of two 1-cells $F\colon X \to Y$ and $F_0\colon X_0 \to Y_0$, a natural transformation $\tilde{F}\colon \mathrm{Mod}_T(-) \to \mathrm{Mod}_S(-)$ and a modification $\tilde{\phi}$ of the form

$$\begin{tikzcd}
\mathcal{K}(Y,-) \arrow[r, "-\circ F"] \arrow[d, "U^*_Y"'] \arrow[dr, phantom, "\Downarrow \tilde{\phi}"] & \mathcal{K}(X,-) \arrow[d, "U^*_X"] \\
\mathrm{Mod}_T(-) \arrow[r, "\tilde{F}"'] & \mathrm{Mod}_S(-)
\end{tikzcd}$$

satisfying the axioms:

(i) $FI = JF_0$ and the following diagram commutes

$$\begin{tikzcd}
\mathrm{Mod}_T(-) \arrow[r, "\tilde{F}"] \arrow[d, "J^*_{Y_0}"'] & \mathrm{Mod}_S(-) \arrow[d, "J^*_{X_0}"] \\
\mathcal{K}(Y_0,-) \arrow[r, "-\circ F_0"'] & \mathcal{K}(X_0,-).
\end{tikzcd}$$

(ii) The following pasting diagrams are equal

$$\begin{tikzcd}
& \mathrm{Mod}_T(-) \arrow[r, "\tilde{F}"] \arrow[dd, "J^*_{Y_0}"] & \mathrm{Mod}_S(-) \arrow[dd, "J^*_{X_0}"] & \\
\mathcal{K}(Y,-) \arrow[ur, "U^*_Y"] \arrow[dr, "-\circ J"'] \arrow[r, phantom, "\Uparrow t"] & {} & & = \\
& \mathcal{K}(Y_0,-) \arrow[r, "-\circ F_0"'] & \mathcal{K}(X_0,-) &
\end{tikzcd}$$

$$\begin{tikzcd}
 & & \mathrm{Mod}_T(-) \arrow[rr, "\tilde{F}"] & & \mathrm{Mod}_S(-) \arrow[ddr, "J^*_{X_0}"] & \\
= & \mathcal{K}(Y,-) \arrow[ur, "U^*_Y"] \arrow[rr, "-\circ F"'] \arrow[drr, "-\circ J"'] & & \mathcal{K}(X,-) \arrow[ur, "U^*_X"] \arrow[drr, "-\circ I"'] \arrow[ul, phantom, "\Nearrow \tilde{\phi}"] \arrow[rr, phantom, "\Uparrow s"] & & \\
 & & & \mathcal{K}(Y_0,-) \arrow[rr, "-\circ F_0"'] & & \mathcal{K}(X_0,-).
\end{tikzcd}$$

(iii) Let us denote with $(-)_{\tilde{F}m}$ and $(-)_{\tilde{F}T}$ the relative right module structure operators of $\tilde{F}(M,\,(-)_m)$ and $\tilde{F}(T,\,(-)^\dagger)$ respectively. Then, for any 2-cell $f\colon IA \to TB$ and any K-indexed relative right module $(M,\,(-)_m)$, the action of $\tilde{F}$ has to be

$$f_{\tilde{F}m} = (f_{\tilde{F}T} \cdot tF_0A)_m.$$

Since the last part of the definition above might seem a bit ad hoc, let us briefly explain where it comes from. The idea is that we want to write the action of $\tilde{F}$ on relative right module in terms of the *free* relative right modules. In the case of relative algebras, this property followed from the other axioms (Proposition 3.5.6). This will make sure that, in the definition of a lifting of a monad to relative right modules (Definition 3.6.15), all of the structure of the monad is lifted. When dealing with *right modules for a monad* this property follows from other axioms.

The following Lemma gives us a nice way to describe the action of $\tilde{F}$ in terms of a particular 2-cell ϕ. Thanks to this we can prove that a lifting to relative right modules is equivalent to a morphism of relative monads, as shown in Proposition 3.6.10.

Lemma 3.6.9. *Let* $(F, F_0, \tilde{F}, \tilde{\phi})$ *be a lifting to relative right modules. Let us denote by* $\phi \colon FS \to TF_0$ *the component of* $J^*_{X_0}\tilde{\phi}$ *relative to* 1_Y*.*

$$\begin{tikzcd}
\mathcal{K}(Y,Y) \arrow[r, "- \circ F"] \arrow[d, "U^*_Y"'] & \mathcal{K}(X,Y) \arrow[d, "U^*_X"] \arrow[dl, phantom, "\Downarrow \tilde{\phi}"] \\
\mathrm{Mod}_T(Y) \arrow[r, "\tilde{F}"'] \arrow[d, "J^*_{Y_0}"'] & \mathrm{Mod}_S(Y) \arrow[d, "J^*_{X_0}"] \\
\mathcal{K}(Y_0,Y) \arrow[r, "- \circ F_0"'] & \mathcal{K}(X_0,Y)
\end{tikzcd}
\qquad
\begin{tikzcd}
1_Y \arrow[rr, mapsto] \arrow[ddd, mapsto] & & F \arrow[dd, mapsto] \\
& & \\
& & FS \arrow[dl, Rightarrow, "\phi"] \\
T \arrow[r, mapsto] & TF_0 &
\end{tikzcd}$$

Then, for any 2-cell $f \colon IA \to SB$*,*

$$f_{\tilde{F}m} = (\phi B \cdot Ff)_m.$$

Proof. We know that $\phi \colon FS \to TF_0$ is a map in $\mathrm{Mod}_S(X)$, where the structure operators of FS and TF_0 are respectively $F(-)^\dagger_S$ and $(-)_{\tilde{F}T}$. Therefore, the following diagram is commutative

$$\begin{tikzcd}
FIA \arrow[r, equal] \arrow[d, "FsA"'] & JF_0A \arrow[d, "tF_0A"] \\
FSA \arrow[r, "\phi A"] \arrow[d, "Ff^\dagger_S"'] & TF_0A \arrow[d, "f_{\tilde{F}T}"] \\
FSB \arrow[r, "\phi B"'] & TF_0B.
\end{tikzcd}$$

Indeed, the top square commutes by part (ii) of Definition 3.6.8 and the bottom square because ϕ is a relative right module map. Thus, we can deduce

$$\begin{aligned}
f_{\tilde{F}m} &= (f_{\tilde{F}T} \cdot tF_0A)_m && \text{(by part (iii) of Definition 3.6.8)} \\
&= (\phi B \cdot Ff^\dagger_S \cdot FsA)_m && \text{(diagram above)} \\
&= (\phi B \cdot Ff)_m && \text{(by left unit law of } S\text{).}
\end{aligned}$$
□

Proposition 3.6.10. *A lifting to relative right modules is equivalent to a relative monad morphism between relative monads.*

Proof. We will start proving that given a lifting to relative right modules we get a relative monad morphism. Let $(X,\ I,\ S)$ and $(Y,\ J,\ T)$ be two relative monads in $\mathcal{K}$, $(\tilde{F}, F, F_0, \tilde{\phi})$ a lifting between them and denote with $\phi\colon FS \to TF$ the 2-cell given in Lemma 3.6.9. Axiom (ii) of Definition 3.6.8 is equivalent to the unit law for $(F,\ F_0,\ \phi)$ seen as a relative monad morphism. Moreover, the extension law for it follows from the fact that the components of $\tilde{\phi}$ are relative right modules morphisms and axioms (ii) and (iii) of Definition 3.6.8. More precisely, we have to prove that for any 2-cell $f\colon IA \to SB$ the following diagram is commutative

$$\begin{tikzcd}
FSA \arrow[r, "\phi A"] \arrow[d, "Ff^{\dagger}_{S}"'] & TF_0A \arrow[d, "(\phi B \cdot Sf)^{\dagger}_{T}"] \\
FSB \arrow[r, "\phi B"'] & TF_0B.
\end{tikzcd}$$

We know that $\phi\colon FS \to TF_0$ is a map in $\mathrm{Mod}_S(X)$, therefore for any $f\colon IA \to SB$ we have, using that the structure operators of FS and TF_0 are respectively $Ff^{\dagger}_{S}$ and $f_{\tilde{F}T}$,

$$\begin{tikzcd}
FSA \arrow[r, "\phi A"] \arrow[d, "Ff^{\dagger}_{S}"'] & TF_0A \arrow[d, "f_{\tilde{F}T}"] \\
FSB \arrow[r, "\phi B"'] & TF_0B.
\end{tikzcd}$$

Hence, we just need to prove that $f_{\tilde{F}T} = (\phi B \cdot Ff)^{\dagger}_{T}$, which is just a particular instance of Lemma 3.6.9. On the other hand if we start with a relative monad morphism $(F,\ F_0,\ \phi)$, we can define $\tilde{F}\colon \mathrm{Mod}_T(-) \to \mathrm{Mod}_S(-)$, for any $K \in \mathcal{K}$ and $(M,\ (-)_m) \in \mathrm{Mod}_T(K)$, as

$$\tilde{F}(M,\ (-)_m) := (MF_0,\ (\phi B \cdot F-)_m).$$

First of all we need to prove that $(\phi B \cdot F-)_m$ is a relative right module operator.

Unit Law:

$$\begin{aligned}(\phi B \cdot Fs)_m &= (tF_0)_m && \text{(by part (ii) of Definition 3.6.8)}\\ &= 1_{MF_0} && \text{(by unit law for } (-)_m).\end{aligned}$$

Associativity:

$$\begin{aligned}(k^\dagger \cdot h)_{\tilde{S}m} &= (\,\phi C \cdot Fk^\dagger_S \cdot Fh\,)_m && \text{(by definition)}\\ &= (\,(\phi C \cdot Fk)^\dagger_T \cdot \phi B \cdot Fh\,)_m && \text{(by Kleisli ext law for } \phi)\\ &= (\phi C \cdot Fk)_m \cdot (\phi B \cdot Fh)_m && \text{(by associativity of } (-)_m)\\ &= k_{\tilde{F}m} \cdot h_{\tilde{F}m} && \text{(by definition).}\end{aligned}$$

Moreover if $g\colon (M,\, (-)_m) \to (N,\, (-)_n)$ is a map in $\mathrm{Mod}_T(K)$, then applying the axiom for g to $\phi B \cdot Ff$ we get that $gF_0\colon \tilde{F}(M,\, (-)_m) \to \tilde{F}(N,\, (-)_n)$ is in $\mathrm{Mod}_S(K)$. Therefore $\tilde{F}\colon \mathrm{Mod}_T(-) \to \mathrm{Mod}_T(-)$ is well defined.

Then, we can define the component of $\tilde{\phi}$ at $K \in \mathcal{K}$ and $(M,\, (-)_m) \in \mathrm{Mod}_T(K)$ as $M\phi$. Looking at the definition of $\tilde{F}$ on relative right actions, we can see that the axiom for $M\phi$ to be a relative right module morphism is the same as the extension axiom for $(F,\, F_0,\, \phi)$.

Now we need to prove all the axioms of a lifting to relative right modules. The first one follows from definition, and part (ii) is equivalent to the unit one for a relative monad morphism. Finally, we can easily check that part (iii) of Definition 3.6.8 is satisfied, as $f_{\tilde{F}m} = (\phi B \cdot Ff)_m$ and $f_{\tilde{F}T} = (\phi B \cdot Sf)^\dagger$ by definition, and so

$$f_{\tilde{F}m} = (\phi B \cdot Ff)_m = (\,(\phi B \cdot Ff)^\dagger \cdot tF_0 A\,)_m = (f_{\tilde{F}T} \cdot tF_0 A)_m.$$

Lemma 3.6.9 guarantees that these constructions are inverses of each other. □

Definition 3.6.11. Let $(F,\, F_0,\, \tilde{F},\, \tilde{\phi})$ and $(F',\, F'_0,\, \tilde{F}',\, \tilde{\phi}')$ be two liftings to relative right modules from $(X,\, I,\, S)$ to $(Y,\, J,\, T)$, two relative monads in $\mathcal{K}$. A *map of liftings to relative right modules* $(p,\, p_0,\, \tilde{p})\colon (F,\, F_0,\, \tilde{F},\, \tilde{\phi}) \to (F',\, F'_0,\, \tilde{F}',\, \tilde{\phi}')$ consists of two 2-cells $p\colon F \to F'$ and $p_0\colon F_0 \to F'_0$ and a modification $\tilde{p}\colon \tilde{F} \to \tilde{F}'$ such that:

(i) $Jp_0 = pI$ and the following pasting diagrams are equal

$$\begin{tikzcd}[column sep=large, row sep=huge]
\mathrm{Mod}_T(-) \arrow[r, bend left, "\tilde{F}"] \arrow[r, bend right, "\tilde{F}'"'] \arrow[r, phantom, "\Downarrow \tilde{p}"] \arrow[d, "J^*_{Y_0}"'] & \mathrm{Mod}_S(-) \arrow[d, "J^*_{X_0}"] \\
\mathcal{K}(Y_0, -) \arrow[r, bend right, "\tilde{F}'"'] & \mathcal{K}(X_0, -)
\end{tikzcd}
=
\begin{tikzcd}[column sep=large, row sep=huge]
\mathrm{Mod}_T(-) \arrow[r, bend left, "\tilde{F}"] \arrow[d, "J^*_{Y_0}"'] & \mathrm{Mod}_S(-) \arrow[d, "J^*_{X_0}"] \\
\mathcal{K}(Y_0, -) \arrow[r, bend left, "- \circ F_0"] \arrow[r, bend right, "- \circ F_0'"'] \arrow[r, phantom, "\Downarrow - \circ p_0"] & \mathcal{K}(X_0, -);
\end{tikzcd}$$

(ii) the following pasting diagrams are equal

$$\begin{tikzcd}[column sep=large, row sep=huge]
\mathcal{K}(Y, -) \arrow[r, bend left, "- \circ F"] \arrow[r, bend right, "- \circ F'"'] \arrow[r, phantom, "\Downarrow - \circ p"] \arrow[d, "U^*_Y"'] \arrow[dr, phantom, "\Downarrow \tilde{\phi}'"] & \mathcal{K}(X, -) \arrow[d, "U^*_X"] \\
\mathrm{Mod}_T(-) \arrow[r, bend right, "\tilde{F}'"'] & \mathrm{Mod}_S(-)
\end{tikzcd}
=
\begin{tikzcd}[column sep=large, row sep=huge]
\mathcal{K}(Y, -) \arrow[r, bend left, "- \circ F"] \arrow[d, "U^*_Y"'] \arrow[dr, phantom, "\Downarrow \tilde{\phi}"] & \mathcal{K}(X, -) \arrow[d, "U^*_X"] \\
\mathrm{Mod}_T(-) \arrow[r, bend left, "\tilde{F}"] \arrow[r, bend right, "\tilde{F}'"'] \arrow[r, phantom, "\Downarrow \tilde{p}"] & \mathrm{Mod}_S(-).
\end{tikzcd}$$

Proposition 3.6.12. *Let* $(F, F_0, \tilde{F}, \tilde{\phi})$ *and* $(F', F_0', \tilde{F}', \tilde{\phi}')$ *be two liftings to relative right modules and* (F, F_0, ϕ) *and* (F', F_0', ϕ) *their corresponding relative monad morphisms (using Proposition 3.6.10). A map of liftings to relative right modules between them is equivalent to a relative monad transformation between the corresponding relative monad maps.*

Proof. Given $(p, p_0, \tilde{p})$ we can see that (p, p_0) is a relative monad transformation. This follows from $J^*_{X_0}\tilde{p} = (- \circ p_0)J^*_{Y_0}$ and part (ii) of Definition 3.6.11 applied to 1_Y. On the other hand, given a relative monad transformation $\tilde{p}$, to satisfy the first axiom of map of liftings to relative right modules, we need to choose $\tilde{p}$ as follows: for any $K \in \mathcal{K}$ and $(M, (-)_M) \in \mathrm{Mod}_T(K)$, then $\tilde{p}_{K,M}\colon \tilde{F}(M, (-)_m) \to \tilde{F}'(M, (-)_m)$ is defined as $Mp_0\colon (MF_0, (-)_{\tilde{F}m}) \to (MF'_0, (-)_{\tilde{F}'m})$. Thanks to Proposition 3.6.10 we know that for any 2-cells $f\colon IA \to SA$, $f_{\tilde{F}m} = (\phi B \cdot Ff)_m$ and $f_{\tilde{F}'m} = (\phi' B \cdot F'f)_m$. Therefore, to prove that $\tilde{p}_{K,M}$ is a map of relative right modules, it suffices to prove the following equality:

$$\begin{aligned}
Mp_0B \cdot (\phi B \cdot Ff)_m &= (Tp_0B\phi B \cdot Ff)_m && \text{(by right naturality of } (-)_m) \\
&= (\phi' B \cdot pSB \cdot Ff)_m && \text{(because } (p, p_0) \in \mathbf{Rel}(\mathcal{K})) \\
&= (\phi' B \cdot F'f \cdot pIA)_m && \text{(by naturality of } p) \\
&= (\phi' B \cdot F'f \cdot Jp_0A)_m && \text{(because } (p, p_0) \in \mathbf{Rel}(\mathcal{K})) \\
&= (\phi' B \cdot F'f)_m \cdot Mp_0A && \text{(by left naturality of } (-)_m).
\end{aligned}$$

These constructions are clearly inverses of each other. □

Proposition 3.6.13. *Let $\mathcal{K}$ be a 2-category. Then there exists a 2-category* $\mathbf{LiftR}(\mathcal{K})$ *of lifting to relative right modules with objects relative monads in $\mathcal{K}$, 1-cells liftings to relative right modules and 2-cells maps between them.*

Proof. The composition is given by composition in $\mathcal{K}$ and pasting the appropriate diagrams. The strictness of this operation follows from the strictness in $\mathcal{K}$ and the pasting Theorem for 2-categories [Pow90]. □

In the formal theory of monads [Str72] Street proved that the 2-category of monads in a 2-category is equivalent to the 2-category of liftings to indexed algebras. The following Theorem provides a similar result in the setting of relative monads, which will be useful to prove Theorem 3.6.18.

Theorem 3.6.14. *Let $\mathcal{K}$ be a 2-category. Then the 2-categories* $\mathbf{Rel}(\mathcal{K})$ *and* $\mathbf{LiftR}(\mathcal{K})$ *are 2-isomorphic.*

Proof. Let us define a 2-functor $\Gamma\colon \mathbf{Rel}(\mathcal{K}) \to \mathbf{LiftR}(\mathcal{K})$. On objects we take the identity, on 1-cells we use the correspondence seen in Proposition 3.6.10 and on 2-cells the one seen in Proposition 3.6.12. These propositions prove also that Γ is an isomorphism on hom-categories, and therefore a 2-isomorphism. □

Beck's Theorem for Relative Distributive Laws

Throughout this section we will consider a relative monad $(X,\, I,\, T) \in \mathbf{Rel}(\mathcal{K})$ and monads $(X,\, S)$, $(X_0,\, S_0) \in \mathbf{Mnd}(\mathcal{K})$ compatible with I. We will start by introducing the generalised notion of extensions to Kleisli categories, which we call *lifting to relative right modules*. Then we will prove that this concept is equivalent to relative distributive laws, providing a Beck-type equivalence.

Definition 3.6.15. We define a *lifting of S to the relative right modules of T* as a monad $\tilde{S}\colon \mathrm{Mod}_T(-) \to \mathrm{Mod}_T(-)$ satisfying the following properties.

(i) The pair $(\tilde{S},\, - \circ S_0)$ is a monad compatible with J_0^*.

(ii) The natural transformation $U^*\colon (\mathrm{Mod}_T(-),\, \tilde{S}) \to (\mathcal{K}(X,-),\, - \circ S)$ can be extended to a morphism in $\mathbf{Mnd}(\hat{\mathcal{K}}^{op})^{op}$. We will refer to the 2-cell making U^* a monad morphism as d^*.

(iii) The modification $t\colon (- \circ I,\, 1) \to (J_0^*,\, 1) \circ (U^*,\, d^*)$ is a 2-cell in $\mathbf{Mnd}(\hat{\mathcal{K}}^{op})^{op}$.

(iv) Let us denote with $(-)_{\tilde{S}m}$ and $(-)_{\tilde{S}T}$ the relative right module structure operators of $\tilde{S}(M,\, (-)_m)$ and $\tilde{S}(T,\, (-)^\dagger)$ respectively. Then, for any 2-cell $f\colon IA \to TB$ and any K-indexed relative right module $(M,\, (-)_m)$, the action of $\tilde{S}$ has to be

$$f_{\tilde{S}m} = (f_{\tilde{S}T} \cdot tS_0A)_m.$$

Example 3.6.16. Let us consider the case $\mathcal{K} = \mathbf{Cat}$. In this case, part (iv) of Definition 3.6.15 derives from the first three and properties of **Cat**. More precisely, we recall that f_m is equal to $\bar{M}f$, with $\bar{M}\colon \mathrm{Kl}(T) \to \mathbb{K}$ the functor associated to the relative right module M, and so $f_{\tilde{S}m} = \bar{M}\tilde{S}f$. On the other hand $f_{\tilde{S}T} = (\tilde{S}f)^\dagger$, therefore we get the equality in (iv).

Let us denote with $\mathbb{C}_0$ and $\mathbb{C}$ the categories on which we take the monads S_0 and S, and relative monad T. With this notation, we can rewrite the definition above in the following equivalent way. Let $(S,\, S_0)$ be a monad compatible with $I\colon \mathbb{C}_0 \to \mathbb{C}$

and $(\mathbb{C},\, I,\, T)$ a relative monad in **Cat**. We denote with $J_0 \colon \mathbb{C}_0 \to \mathrm{Kl}(T)$ and $U \colon \mathrm{Kl}(T) \to \mathbb{C}$ the functors forming the Kleisli relative adjunction (see [ACU15, Section 2.3]). We define an *extension of* S *to the Kleisli category of* T as a monad $\tilde{S} \colon \mathrm{Kl}(T) \to \mathrm{Kl}(T)$ such that:

(i) $\tilde{S}J_0 = J_0 S_0$ and $(J_0,\, 1)$ becomes a monad morphism, i.e. $\tilde{m}J_0 = J_0 m_0$ and $\tilde{s}J_0 = J_0 s_0$;

(ii) the functor $U \colon \mathrm{Kl}(T) \to \mathbb{C}$ is a monad morphism (with 2-cell d);

(iii) the unit $t \colon (I,\, 1) \to (U,\, d) \circ (J_0,\, 1)$ is a monad transformation.

Remark 3.6.17. More generally, if $\mathcal{K}$ has relative Kleisli objects $X_{I,T}$, then we can rewrite Definition 3.6.15 as a particular extension. First, let us notice that, if $\mathrm{Mod}_T(-)$ is represented by $X_{I,T}$, then there is a universal relative right module $(J_0,\, (-)_{j_0})$ with $J_0 \colon X_0 \to X_{I,T}$. Therefore, diagram 3.6.1 becomes equivalent (using Yoneda for 2-categories) to

$$\begin{tikzcd}
& X_{I,T} \arrow[dr, "U"] \arrow[d, phantom, "\Uparrow t"] & \\
X_0 \arrow[ur, "J_0"] \arrow[rr, "I"'] & & X,
\end{tikzcd}$$

with $UJ_0 = T$. Therefore, a lifting of a monad S to relative right modules of T becomes equivalent to an *extension of* S *to* $X_{I,T}$, i.e. a monad $\tilde{S} \colon X_{I,T} \to X_{I,T}$ such that

(i) the pair $(\tilde{S}, S_0)$ is a monad compatible with J_0;

(ii) the 1-cell $U \colon (X_{I,T},\, \tilde{S}) \to (X,\, S)$ is a monad morphism (with 2-cell d);

(iii) the 2-cell $t \colon (I,\, 1) \to (U,\, d) \circ (J_0,\, 1)$ is a monad transformation;

(iv) for any 2-cell $f \colon IA \to TB$ we have the following equality

$$\tilde{S}f_{j_0} = (U\tilde{S}f_{j_0} \cdot tS_0A)_{j_0}.$$

Theorem 3.6.18. *Let $(X,\, I,\, T)$ be a relative monad and $(S,\, S_0)$ a monad compatible with I, both in $\mathcal{K}$. Then, relative distributive laws $d\colon ST \to TS_0$ are equivalent to liftings of S to the relative right modules of T.*

Proof. We have already seen in Proposition 3.5.4 that relative distributive laws are the objects of $\mathbf{Mnd}(\mathbf{Rel}(\mathcal{K}))$. Now, using Theorem 3.6.14, we get that

$$\mathbf{Mnd}(\mathbf{Rel}(\mathcal{K})) \cong \mathbf{Mnd}(\mathbf{LiftR}(\mathcal{K})).$$

To get the conclusion, we just need to notice that an extension of S to the Kleisli category of T is just an object of $\mathbf{Mnd}(\mathbf{LiftR}(\mathcal{K}))$. ☐

Putting together Theorems 3.5.9 and 3.6.18 we get the following Theorem, which gives us the counterpart of Beck's Theorem for relative distributive laws.

Theorem 3.6.19. *Let $\mathcal{K}$ be a 2-category, $(X,\, I,\, T)$ a relative monad in $\mathcal{K}$ and $(S,\, S_0)$ a monad compatible with I. The following are equivalent:*

(i) *a relative distributive law of T over $(S,\, S_0)$;*

(ii) *a lifting $\hat{T}\colon S_0\text{-Alg}(-) \longrightarrow S\text{-Alg}(-)$ of T to the algebras of $(S,\, S_0)$;*

(iii) *a lifting $\tilde{S}\colon \mathrm{Mod}_T(-) \to \mathrm{Mod}_T(-)$ of S to the relative right modules of T.*

Let us consider the particular case $\mathcal{K} = \mathbf{Cat}$. We know that $\mathbf{Cat}$ has both relative EM objects and relative Kleisli objects (Examples 3.4.4 and 3.6.5). Using this property of $\mathbf{Cat}$ and Example 3.6.16 we can show that Theorem 3.6.19 can be rephrased in the following way.

Corollary 3.6.20. *Let $(\mathbb{C},\, I,\, T)$ be a relative monad in* **Cat** *and $(S,\, S_0)$ a monad compatible with I. The following are equivalent:*

(i) *a relative distributive law of T over $(S,\, S_0)$;*

(ii) *a lifting $\hat{T}\colon S_0\text{-Alg} \to S\text{-Alg}$ of T to the algebras of $(S,\, S_0)$;*

(iii) *an extension $\tilde{S}\colon Kl(T) \to Kl(T)$ of S to the Kleisli category of T.*

We conclude the chapter with a pair of examples of relative distributive laws in the 2-category of locally small categories **Cat**.

Example 3.6.21 (Power set and free monoids). We will consider a variation of a distributive law between two monads. Recall that there exists a distributive law between the power set monad P and the free monoid one S^M, given by

$$\begin{aligned} d_X \colon S^M P X &\longrightarrow P S^M X \\ I_1 ... I_n &\longmapsto \{a_1 ... a_n \mid a_i \in I_i\}. \end{aligned}$$

A problem arises if we want to impose some restrictions on the cardinality of sets. For example, given any infinite cardinal κ, let $\mathbf{Set}_{\leq\kappa}$ be the category of sets with cardinality less or equal to κ. Then, the restriction of P to $\mathbf{Set}_{\leq\kappa}$ it is not an endofunctor anymore. Nevertheless, we can recover its monad-like structure considering it as a relative monad on the inclusion $I \colon \mathbf{Set}_{\leq\kappa} \to \mathbf{Set}$. More precisely, we can take as unit $t_X(x) := \{x\}$ (for any $X \in \mathbf{Set}$ and $x \in X$) and as extension of $f \colon X \to PY$ the map

$$\begin{aligned} f^\dagger \colon PX &\longrightarrow PY \\ I &\longmapsto \textstyle\bigcup_{i\in I} f(i). \end{aligned}$$

Let us consider now the restriction of S^M to $\mathbf{Set}_{\leq\kappa}$. Let X be a set of cardinality at most k, then $S^M X$ has cardinality

$$|S^M X| = |\coprod_{n\in\mathbb{N}} |X|^n \,| \;\leq\; |\coprod_{n\in\mathbb{N}} \kappa^n \,| = |\coprod_{n\in\mathbb{N}} \kappa| = \kappa$$

and therefore we get $S^M_\kappa \colon \mathbf{Set}_{\leq\kappa} \to \mathbf{Set}_{\leq\kappa}$. In particular it means that (S^M, S^M_κ) is compatible with $I \colon \mathbf{Set}_{\leq\kappa} \to \mathbf{Set}$. With this point of view, we see that d becomes a relative distributive law of $T := P_\kappa \colon \mathbf{Set}_{\leq\kappa} \to \mathbf{Set}$ over (S^M, S^M_κ).

A similar situation arises when one works in set theories that do not have the power-set axiom, like Kripke-Platek set theory [Bar17] or Constructive Zermelo-Frankel set theory [Acz78]. There, the power set operation can be viewed as a relative monad over the inclusion of the category of sets into the category of classes.

Note also that the presheaf construction can then be viewed as a categorified version of the power-set monad [FGHW17]. See also [Hyl10].

Example 3.6.22 (Pointed vector spaces). In [ACU15, Example 1.1] is presented the relative monad V of vector spaces. In order to define it, let us fix a semiring R. For any set X, we will denote with $\delta_x \colon X \to R$ the map sending x to 1 and everything else to 0. Then, V is defined on the inclusion $I \colon \mathbf{Fin} \to \mathbf{Set}$ of finite cardinals into sets as follows:

- for any finite cardinal n, $Vn := \mathbf{Set}(In,\, R)$;
- the unit $v_n \colon In \to Vn$ is defined, for any $i \in In$, as $v_n(i) := \delta_i$;
- given $\alpha \colon In \to Vm$ we define its extension $\alpha^\dagger \colon Vn \to Vm$ as, for any $f \colon In \to R$,
$$\alpha^\dagger(f) := \sum_{i \in n} f(i) \cdot \alpha(i)(-) \colon Im \to R.$$

Let us consider the monad of pointed set, i.e. $SX := X+1$, the unit $s_X \colon X \to X+1$ is the canonical inclusion and the multiplication $m_X \colon X+2 \to X+1$ fixes any element of X and sends the two elements of 2 to the only one of 1. One can easily prove that the category of S-algebras is the category of pointed sets.

Clearly, defining S_f as the restriction of S to finite cardinals, we can see $(S,\, S_f)$ as a monad compatible with I. Moreover, there is a lifting of V to the algebras of $(S,\, S_f)$ defined as follows: for any finite pointed set $(n,\, i)$ we set $\hat{V}(n,\, i) := (\mathbf{Set}(In,\, R),\, \delta_i)$. Now we need to check that both the unit and the extension operator lift. First of all, we can see straight away that v_n is a map of pointed sets, since by definition it sends each i to the map δ_i. Then, if we consider a map of pointed sets $\alpha \colon (In,\, i) \to (Vm,\, \delta_j)$ we need to check that the extension $\alpha^\dagger \colon Vn \to Vm$ is still a map of pointed sets, i.e. the equality $\alpha^\dagger(\delta_i) = \delta_j$ holds.

For any $s \in m$

$$\begin{aligned}
[\alpha^\dagger(\delta_i)](s) &= \sum_{r \in n} \delta_i(r) \cdot [\alpha(r)](s) && \text{(by definition of } (-)^\dagger\text{)} \\
&= \alpha(i)(s) && \text{(by definition of } \delta_i\text{)} \\
&= \delta_j(s) && (\alpha \text{ map of pointed sets).}
\end{aligned}$$

Therefore, by Theorem 3.5.9, we have a relative distributive law of V over $(S,\, S_f)$. In particular, by Theorem 3.6.18, we have a monad $\tilde{S}$ induced on the Kleisli of V, i.e. vector spaces. What we get as algebras over this monad are *pointed vector spaces*.

4. Strongly Finitary Notions of Multicategory

Introduction

It is well known that there are various correspondences between different flavours of monoidal category and multicategory [Her00, Man09, BL18]. For instance, in his fundamental work [Her00] Hermida introduced representable multicategories and proved that they are equivalent to monoidal categories. Then, if we want to weaken the representable condition, for example considering only *left representable* multicategories, we end up in the world of *skew* monoidal categories. In [BL18] we can find the details of various equivalences between (skew) multicategories and skew monoidal categories. Since monoidal categories admit a definition involving finite data and finite axioms, it is natural to wonder if the same is possible for multicategories. Our goal in the present chapter is to describe a finite approach to the kinds of multicategory that arise in practise — these include representable and closed multicategories — with the goal of making examples of such notions easier to construct. We do this by introducing a structure called a *short multicategory*, which is not itself a multicategory, since it only has multimaps of dimension at most four. One of our main results shows that representable short multicategories are equivalent to representable multicategories, so providing a finite description of the latter. Moreover, we adapt all of these results to the setting of skew multicategories and skew monoidal categories described in [BL18].

Main Results

The main contribution of this work is to provide equivalences between different flavours of short (skew) multicategories and (skew) multicategories. In particular, we consider the following cases:

- Theorem 4.4.6 provides an equivalence between representable multicategories and representable short multicategories. We prove this as a consequence of the more general Theorem 4.4.5, which deals with left representable short multicategories.
- Theorem 4.4.7 and Theorem 4.4.8 show the equivalences in the closed left representable and closed representable case.
- Then, Theorem 4.5.12 proves the left representable skew case.
- Finally, Theorem 4.5.15 is about the left representable closed skew case.

We also show that these equivalences are compatible with the ones given in [Her00, BL18] for different flavours of multicategory and monoidal category.

Outline

In Section 4.1 we review the definition of a multicategory, before giving a slight reformulation of it better suited for our later use. We also recall some important notions for multicategories, such as representability and closedness.

In Section 4.2 we use the reformulation given in Section 4.1 to define *short multicategories*. We then define the notions of representability and closedness in the context of short multicategories.

In Section 4.3 we give an overview on skew monoidal categories and skew multicategories.

Section 4.4 provides various equivalences between different flavour of short multicategories and skew monoidal categories.

We conclude the chapter in Section 4.5 introducing short skew multicategories and describing analogues of the results in Section 4.4 appropriate to the skew setting.

4.1. Classical Multicategories

In this section we will recall the definitions of multicategories and morphisms between them. To begin with, a **multicategory** $\mathcal{C}$ consists of:

- a collection of objects;
- for each (possibly empty) list $a_1, ..., a_n$ of objects and object b, a set $\mathcal{C}_n(a_1, ..., a_n; b)$;
- for each object a an element $1_a \in \mathcal{C}_1(a; a)$.

The elements of the set $\mathcal{C}_n(a_1, ..., a_n; b)$ are called n-ary multimaps, with domain the list $a_1, \ldots, a_n$ and codomain b, whilst 1_a plays the role of the identity unary morphism. We sometimes write $\overline{a}$ for the list, and then $\mathcal{C}_n(\overline{a}; b)$ for the set of multimaps.

Substitution in a multicategory can be encoded in two ways. The best known one involves substitutions into all positions simultaneously. In this case, substitution is encoded by functions of the form

$$\begin{aligned} \mathcal{C}_n(b_1, ..., b_n; c) \times \prod_{i=1}^{n} \mathcal{C}_{k_i}(\overline{a_i}; b_i) &\longrightarrow \mathcal{C}_K(\overline{a_1}, .., \overline{a_n}; c) \\ (g, f_1, \ldots, f_n) &\longmapsto g \circ (f_1, \ldots, f_n) \end{aligned}$$

where $K = \sum_{i=1}^{n} k_i$. For such substitutions, there is a straightforward associativity axiom — see, for instance, Definition 2.1.1 of [Lei04] — and two identity axioms, which at $g \in \mathcal{C}_n(a_1, \ldots, a_n; b)$ are captured by the two equations

$$1_b \circ (g) = g = g \circ (1_{a_1}, ..., 1_{a_n}).$$

The original definition of multicategory, due to Lambek [Lam69], instead involved substitutions into a single position, and these are encoded by functions of the following form

$$- \circ_i - \colon \mathcal{C}_n(\overline{b}; c) \times \mathcal{C}_m(\overline{a}; b_i) \to \mathcal{C}_{n+m-1}(\overline{b}_{<i}, \overline{a}, \overline{b}_{>i}; c)$$
$$(g, f) \mapsto g \circ_i f$$

where $\overline{b}_{<i}$ and $\overline{b}_{>i}$ denote the sublists of $\overline{b}$ in indices less than and greater than i, respectively. To encode associativity of the $\circ_i$-type substitutions, one requires the following two collections of equations

$$\begin{aligned} h \circ_i (g \circ_j f) &= (h \circ_i g) \circ_{j+i-1} f \quad \text{for} \quad 1 \leq i \leq m, 1 \leq j \leq n \\ (h \circ_i g) \circ_{n+j-1} f &= (h \circ_j f) \circ_i g \quad \text{for} \quad 1 \leq i < j \leq m. \end{aligned}$$

Finally, there are the two identity axioms which at $g \in \mathcal{C}_n(a_1, \ldots, a_n)$ are captured by the two equations $1_b \circ_1 g = g = (\ldots((g \circ_1 1_{a_1}) \circ_2 1_{a_2}) \ldots \circ_n \circ 1_{a_n}$.

Given a multicategory with $\circ$-type substitutions, the corresponding $- \circ_i -$ are defined by

$$g \circ_i f = g \circ (1, \ldots, 1, f, 1, \ldots, 1)$$

where f is substituted in the i'th position. Given a multicategory with $\circ_i$-type substitutions, the corresponding $- \circ -$ are defined by

$$g \circ (f_1, \ldots, f_n) = (\ldots((g \circ_1 f_1) \circ_{k_1+1} f_2) \ldots \circ_{k_1+\ldots k_{n-1}+1} \circ f_n.$$

Each multicategory $\mathcal{C}$ has an underlying category $U\mathcal{C}$ with the same objects, and morphisms the unary ones, so that one can consider a multicategory $\mathcal{C}$ as a category $U\mathcal{C}$ equipped with additional structure. Thinking of a multicategory $\mathcal{C}$ as a category equipped with $\circ_i$-type substitution, we obtain the following reformulations, which will be our starting point in which follows. It is closely related to [BL18, Proposition 3.4].

Proposition 4.1.1. *A multicategory* $\mathcal{C}$ *is equivalently specified by:*

- *a category* $\mathbb{C}$*;*
- *for* $n \in \mathbb{N}$ *a functor* $\mathcal{C}_n(-;-)\colon (\mathbb{C}^n)^{op} \times \mathbb{C} \to \mathbf{Set}$ *such that, when* $n = 1$*, we have* $\mathcal{C}_1(-;-) = \mathbb{C}(-,-) : \mathbb{C}^{op} \times \mathbb{C} \to \mathbf{Set}$*;*
- *substitution functions*

  $$- \circ_i - \colon \mathcal{C}_n(\overline{b}; c) \times \mathcal{C}_m(\overline{a}; b_i) \to \mathcal{C}_{n+m-1}(\overline{b}_{<i}, \overline{a}, \overline{b}_{>i}; c)$$

  for $i \in \{1, \ldots, n\}$ *which are natural in each variable* $a_1, \ldots, a_m, b_1, \ldots, b_n, c$ *and satisfying the same associativity equations*

  $$h \circ_i (g \circ_j f) = (h \circ_i g) \circ_{j+i-1} f \quad \text{for} \quad 1 \leq i \leq m, 1 \leq j \leq n \tag{4.1.1}$$

  $$(h \circ_i g) \circ_{n+j-1} f = (h \circ_j f) \circ_i g \quad \text{for} \quad 1 \leq i < j \leq m \tag{4.1.2}$$

  as before. In this way, $U\mathcal{C} = \mathbb{C}$*.*

Proof. Given a structure as above, we can form a multicategory with objects those of $\mathbb{C}$, sets of multimaps $\mathcal{C}_n(\overline{a}; b)$, identities $1_a \in \mathbb{C}(a, a) = \mathcal{C}_1(a; a)$ and substitution functions $- \circ_i -$ as above. The only additional thing to note is that the two identity axioms for the multicategory are encoded by the fact that the functor $\mathbb{C}_n(-;-)\colon (\mathbb{C}^n)^{op} \times \mathbb{C} \to \mathbf{Set}$ preserves identities.

In the opposite direction, given a multicategory $\mathcal{C}$ with $\circ_i$-type operations, let $\mathbb{C} = U\mathcal{C}$ be its underlying category of unary morphisms. We must define a functor $\mathbb{C}_n(-;-)\colon (\mathbb{C}^n)^{op} \times \mathbb{C} \to \mathbf{Set}$ sending $(\overline{a}; b)$ to $\mathcal{C}_n(\overline{a}; b)$ on objects in such a way that the $- \circ_i -$ substitutions are natural in the sense described above, and such that $\mathcal{C}_1(-;-) = \mathbb{C}(-,-)$. In fact, the requirement that

$$- \circ_i - \colon \mathcal{C}_1(b; c) \times \mathcal{C}_n(\overline{a}; b) \to \mathcal{C}_n(\overline{a}; c)$$

be natural in b forces us to define $\mathcal{C}_n(\overline{a}; f) = f \circ_1 -$. Naturality of the $\circ_i$ also ensure natural of the associated $\circ$ operations, and in particular naturality of

$$- \circ - \colon \mathcal{C}_n(a_1, \ldots, a_n; b) \times \mathcal{C}_1(c_1, a_1) \times \ldots \times \mathcal{C}_1(c_n; a_n) \to \mathcal{C}_n(c_1, \ldots, c_n; b)$$

in $a_1, \dots, a_n$ forces us similarly to define $\mathcal{C}(f_1, \dots, f_n; b) = - \circ (f_1, \dots, f_n)$. With this definition of $\mathcal{C}(-;-)$ on morphisms, associativity of substitution implies that it is a functor and that the substitution maps are natural in each variable, and satisfy $\mathcal{C}_1(-;-) = \mathbb{C}(-,-)$.

These two constructions are inverse. □

Let us underline the fact that this proposition relies on the existence of identity morphisms. Instead, if we consider structures without identities these two presentations are not equivalent. For the specific case of operads (which are multicategories with one element) see for instance [Mar02, Part II, Section 1.3].

Naturally, there is a notion of morphism between multicategories, which we call here *multifunctor*. From now on, when talking about multicategories we will mean in the sense of Proposition 4.1.1.

Notation. From now on, when we will have a functor $F \colon \mathbb{C} \to \mathbb{D}$ and a list $\overline{a}$ of objects $a_1, ..., a_n$ in $\mathbb{C}$, then we will write $F\overline{a}$ for the list of objects $Fa_1, ..., Fa_n$ in $\mathbb{D}$.

Definition 4.1.2. Let $\mathcal{C}$ and $\mathcal{D}$ two multicategories. A *multifunctor* is a functor $F \colon \mathbb{C} \to \mathbb{D}$ together with natural families

$$F_n \colon \mathcal{C}_n(\overline{a}; b) \to \mathcal{D}_n(F\overline{a}; Fb)$$

for any $n \in \mathbb{N}$, such that when $n = 1$, then F_1 is the functor action. These families must commute with all substitution operators $\circ_i$.

Multicategories and multifunctors form a category **Mult**.

Representability

An important notion for multicategories is the one of representability. First, an **n-ary map classifier** for $\overline{a} = a_1, \dots, a_n$ consists of a representation of $\mathcal{C}_n(a_1, \dots, a_n; -) : \mathbb{C} \to \mathbf{Set}$ – in other words, a multimap

$$\theta_{\overline{a}} \colon a_1, \dots, a_n \to m(a_1, \dots, a_n)$$

for which the induced function $- \circ \theta_a : \mathcal{C}_1(m(a_1, \dots, a_n); b) \to \mathcal{C}_n(a_1, \dots, a_n; b)$ is a bijection for all b. We sometimes refer to such a multimap as a universal multimap and write $m\overline{a}$ for $m(a_1, ..., a_n)$. A n-ary map classifier is said to be *left universal* if, moreover, the induced function

$$- \circ_1 \theta_{\overline{a}} \colon \mathcal{C}_{1+r}(m\overline{a}, \overline{y}; d) \to \mathcal{C}_{n+r}(a_1, ..., a_n, \overline{y}; d)$$

is a bijection for any $\overline{y}$ of length r.

Definition 4.1.3 ([Her00, BL18])**.** Let $\mathcal{C}$ be a multicategory.

- $\mathcal{C}$ is said to be **weakly representable** when each of the functors $\mathcal{C}_n(\overline{a}; -) \colon \mathbb{C} \to \mathbf{Set}$ is representable, i.e. if it has all n-ary map classifiers $\theta_{\overline{a}}$.
- $\mathcal{C}$ is said to be **left representable** if it is weakly representable and all $\theta_{\overline{a}}$ are left universal.
- $\mathcal{C}$ is said to be **representable** if it is weakly representable and substitution with universal n-multimaps $\theta_{\overline{a}}$ induces bijections

  $$\mathcal{C}_{k+1}(\overline{x}, m\overline{a}, \overline{y}; b) \to \mathcal{C}_{k+n}(\overline{x}, \overline{a}, \overline{y}; b)$$

  for $\overline{x}$ and $\overline{y}$ tuples of appropriate length.

We will denote with $\mathbf{Mult}_{lr}$ and $\mathbf{Mult}_{rep}$ the full subcategories of $\mathbf{Mult}$ with objects, respectively, left representable multicategories and representable multicategories.

Closedness

Another important notion for multicategories is the one of closedness.

Definition 4.1.4. A multicategory $\mathcal{C}$ is said to be **closed** if for all pair of objects b and c there exists an object $[b, c]$ and binary map $e_{b,c}\colon [b, c], b \to c$ for which the induced function

$$e_{b,c} \circ_1 -\colon \mathcal{C}_n(\overline{x}; [b, c]) \to \mathcal{C}_{n+1}(\overline{x}, b; c)$$

is a bijection, for any tuple $\overline{x}$ of length n.

We will denote with $\mathbf{Mult}^{cl}_{lr}$ the full subcategory of **Mult** with objects left representable closed multicategories.

4.2. Short Multicategories

In this section, we will present a finite definition of certain multicategory-like structures, which we call short multicategories. Later on, under further assumptions, we will show that they are equivalent to known types of multicategory. We will take Proposition 4.1.1 as the grounds for our definition.

A **short multicategory** consists, to begin with, of a category $\mathbb{C}$ together with:

- for $n \leq 4$ a functor $\mathcal{C}_n(-; -) : (\mathbb{C}^n)^{op} \times \mathbb{C} \to \mathbf{Set}$ such that, when $n = 1$, we have $\mathcal{C}_1(-; -) = \mathbb{C}(-, -) : \mathbb{C}^{op} \times \mathbb{C} \to \mathbf{Set}$.

Remark 4.2.1. This says that for $n \leq 4$ we have sets $\mathcal{C}_n(x_1, \ldots, x_n; y)$ of n-ary multimaps (where the unary morphisms are those of $\mathbb{C}$) and n-ary multimaps can be precomposed and postcomposed by unary ones in a compatible manner. We sometimes refer to these compatibilities as *profunctoriality of n-ary multimaps.*

Furthermore, we require substitution functions

$$- \circ_i -\colon \mathcal{C}_n(\overline{b}; c) \times \mathcal{C}_m(\overline{a}; b_i) \to \mathcal{C}_{n+m-1}(b_{<i}, \overline{a}, b_{>i}; c)$$

for $i \in \{1, \ldots, n\}$ which are natural in each variable $a_1, \ldots, a_m, b_1, \ldots, b_n, c$, where:

- $n = 2, 3$, $m = 2$ (substitution of binary into binary and ternary);
- $n = 2$, $m = 3$ (substitution of ternary into binary);
- $n = 2, 3$, $m = 0$ (substitution of nullary into binary and ternary).

In the context of multimaps f, g and h of arity $2, n$ and p respectively, one can consider associativity equations of the form:

$$f \circ_i (g \circ_j h) = (f \circ_i g) \circ_{j+i-1} h \quad \text{for} \quad 1 \leq i \leq 2, 1 \leq j \leq n \tag{4.2.1}$$

$$(f \circ_1 g) \circ_{n+1} h = (f \circ_2 h) \circ_1 g \tag{4.2.2}$$

These are particular casse of the associativity equations (4.1.1,4.1.2) in Section 4.1. We require these equations in the following cases:

(a) $n = p = 2$;

(b) $n = 2$, $p = 0$;

(c) only for (4.2.2), $n = 0$, $p = 2$;

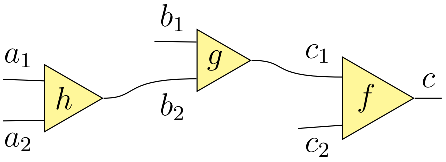
(d) only for (4.2.2), $n = p = 0$.

Let us explain these equations in a more digestible form, using diagrams.

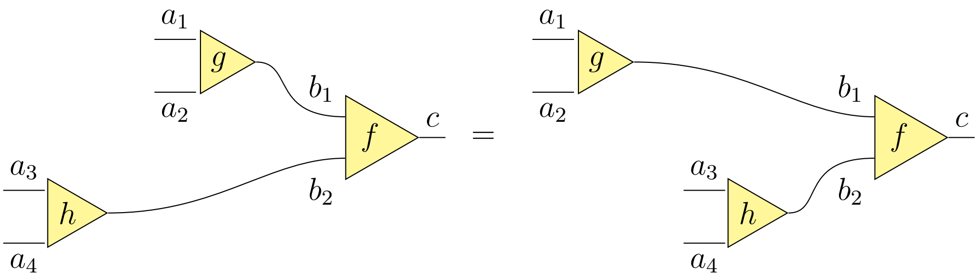
- (4.2.1.a) corresponds to the four equations

  $$(f \circ_i g) \circ_{i-j+1} h = f \circ_i (g \circ_j h)$$

  where $1 \leq i, j \leq 2$ with f, g and h binary. These amount to the fact that certain string diagrams are well-defined. For instance, if we set $i = 1$ and $j = 2$, we get that the two possible interpretations of the following string diagram are the same.



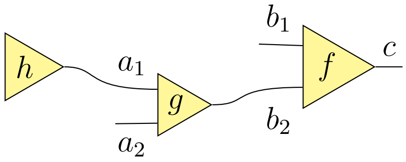
- (4.2.2.a) corresponds to the equation below.



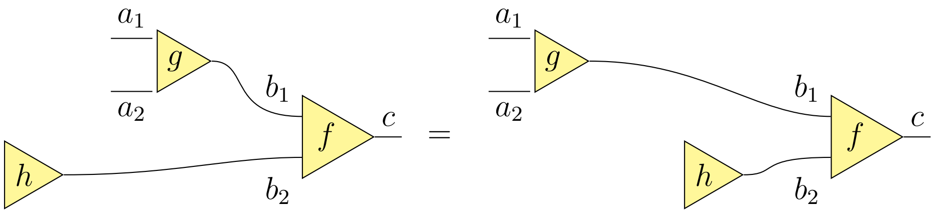
- (4.2.1.b) correspond to the four equations

$$(f \circ_i g) \circ_{i-j+1} h = f \circ_i (g \circ_j h)$$

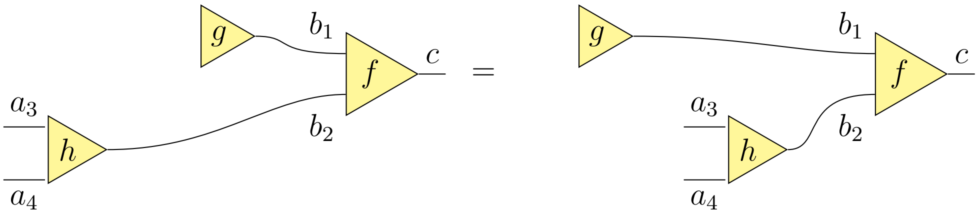
with f, g binary, h nullary and $1 \leq i, j \leq 2$. For instance, if we set $i = 2$ and $j = 1$ it says that the following string diagram is well-defined.



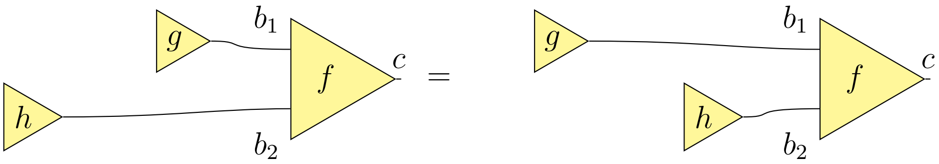
- (4.2.2.b) is the equation below.



- (4.2.2.c) is the equation below.



- (4.2.2.d) is the equation below.



Definition 4.2.2. Let $\mathcal{C}$ and $\mathcal{D}$ two short multicategories. A *morphism of short multicategories* is a functor $F\colon \mathbb{C} \to \mathbb{D}$ together with natural families

$$F_i\colon \mathcal{C}_i(\overline{a}; b) \to \mathcal{D}_i(F\overline{a}; Fb)$$

for any $0 \leq i \leq 4$, with $i = 1$ the functor action. These families must commute with all substitution operators $- \circ_i -$.

Short multicategories and their morphisms form a category **ShMult**. Naturally, there is a forgetful functor $U\colon$ **Mult** $\to$ **ShMult** which takes a multicategory $\mathcal{C}$ and *forgets* all the structure involving n-ary multimaps with $n \geq 4$. In particular, this functor forgets all substitutions which have as a result any n-ary multimaps with $n \geq 4$. For instance, it will not consider the substitution of ternary maps into ternary maps, since it gives out 5-ary multimaps.

Representability for Short Multicategories

We can define a **n-ary map classifier** for $\overline{a}$ also in **ShMult** as a representation of $\mathcal{C}_n(\overline{a}; -)\colon \mathbb{C} \to$ **Set**, i.e. a multimap

$$\theta_{\overline{a}}\colon a_1, ..., a_n \to m\overline{a}$$

for which the induced function $- \circ_1 \theta_{\overline{a}}\colon \mathcal{C}_1(m\overline{a}; b) \to \mathcal{C}_n(\overline{a}; b)$ is a bijection for all b.

Then, a binary map classifier is said to be **left universal** if, moreover, the induced function (where we write ab for $m(a, b)$)

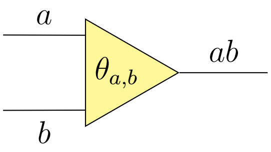
$$- \circ_1 \theta_{a,b}\colon \mathcal{C}_n(ab, \overline{x}; d) \to \mathcal{C}_{n+1}(a, b, \overline{x}; d)$$

is a bijection for $n = 2, 3$ and $\overline{x}$ a tuple of the appropriate length. Similarly a nullary map classifier $u \in \mathcal{C}_0(-; i)$ is said to be left universal if, moreover, the function (where we write i for $m(\,)$)

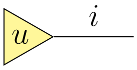
$$- \circ_1 u \colon \mathcal{C}_{1+n}(i, \overline{x}; d) \to \mathcal{C}_n(\overline{x}; d)$$

is a bijection for $n = 1, 2$ and $\overline{x}$ a tuple of the appropriate length. We remark that here we consider only $n = 1, 2$ and not $n = 3$ because in definition of short multicategory we have only substitution of nullary into binary and ternary.

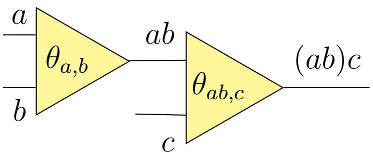
We will denote a binary multimap classifier as below



and the nullary map classifier by the following.

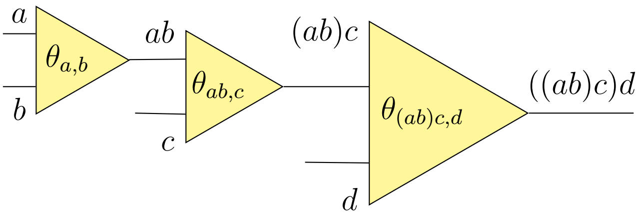


Proposition 4.2.3. *Suppose that binary map classifiers and nullary map classifiers exist and are left universal. Then the multimaps*



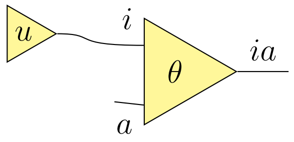
(4.2.3)

and



(4.2.4)

are 3-ary and 4-ary map classifiers and



(4.2.5)

is a unary map classifier.

Proof. Left universality implies that each component of the composite maps

$$\mathcal{C}_1((ab)c;d) \xrightarrow{-\circ\theta_{ab,c}} \mathcal{C}_2(ab,c;d) \xrightarrow{-\circ_1\theta_{a,b}} \mathcal{C}_3(a,b,c;d)$$

$$\mathcal{C}_1(((ab)c)d;e) \xrightarrow{-\circ\theta_{(ab)c,d}} \mathcal{C}_2((ab)c,d;e) \xrightarrow{-\circ_1\theta_{ab,c}} \mathcal{C}_3(ab,c,d;e) \xrightarrow{-\circ_1\theta_{a,b}} \mathcal{C}_4(a,b,c,d;e)$$

and

$$\mathcal{C}_1(ia;b) \xrightarrow{-\circ\theta_{i,a}} \mathcal{C}_2(i,a;b) \xrightarrow{-\circ_1 u} \mathcal{C}_1(a,b)$$

is a bijection; it follows that the composites are bijections, which says exactly that the three claimed multimaps are universal. ☐

Now, following the style of Definition 4.1.3, we will define the notion of representability for short multicategories. We will denote with $\mid \overline{x} \mid$ the length of a list $\overline{x}$.

Definition 4.2.4. Let $\mathcal{C}$ be a short multicategory.

- $\mathcal{C}$ is said to be **left representable** if it admits left universal nullary and binary map classifiers.
- $\mathcal{C}$ is said to be **representable** if it admits nullary and binary map classifiers such that the induced maps are bijections

$$- \circ_j u \colon \mathcal{C}_n(\overline{x}, i, \overline{y}; z) \to \mathcal{C}_{n-1}(\overline{x}, \overline{y}; z) \qquad \text{for } 1 \leq n \leq 3$$
$$- \circ_j \theta_{a,b} \colon \mathcal{C}_n(\overline{x}, ab, \overline{y}; z) \to \mathcal{C}_{n+1}(\overline{x}, a, b, \overline{y}; z) \qquad \text{for } 1 \leq n \leq 3$$

  where $0 \leq\mid \overline{x} \mid, \mid \overline{y} \mid\leq n-1$ and $j =\mid \overline{x} \mid +1$.

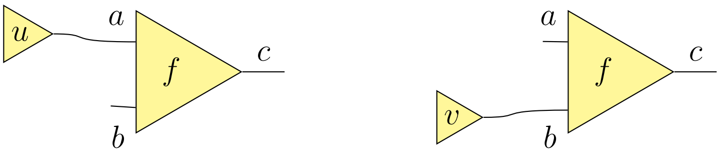
We will denote by $\mathbf{ShMult}_{lr}$ and $\mathbf{ShMult}_{rep}$ the full subcategories of **ShMult** with objects left representable/representable short multicategories. Naturally, the forgetful functor $U\colon \mathbf{Mult} \to \mathbf{ShMult}$ restricts to forgetful functors

$$U_{lr}\colon \mathbf{Mult}_{lr} \to \mathbf{ShMult}_{lr}$$
$$U_{rep}\colon \mathbf{Mult}_{rep} \to \mathbf{ShMult}_{rep}.$$

Notation. Let $\mathcal{C}$ be a short multicategory with a left universal binary classifier. Then we will use $(-)'\colon \mathcal{C}_n(\overline{a};b) \to \mathcal{C}_{n-1}(a_1a_2, a_3, ..., a_n; b)$ for the inverse of $-\circ_1 \theta a_1, a_2$: in other words, for any n-multimap f, f' is the unique $(n-1)$-multimap such that $f' \circ_1 \theta = f$.

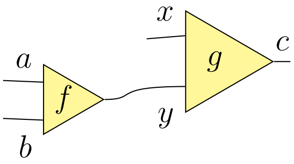
Lemma 4.2.5. *Let $\mathcal{C}$ and $\mathcal{D}$ be left representable short multicategories. A morphism $F\colon \mathcal{C} \to \mathcal{D}$ is uniquely specified by:*

- *A functor $F\colon \mathcal{C} \to \mathcal{D}$.*
- *Natural families $F_i\colon \mathcal{C}_i(\overline{a};b) \to \mathcal{D}_i(F\overline{a};Fb)$ for $i = 0, 2$ commuting with the substitutions*



(4.2.6)

and such that if we define, for any ternary map $h \in \mathcal{C}_3(\overline{a};b)$, $F_3h := F_2h' \circ_1 F_2\theta$, then F_3 also commutes with



(4.2.7)

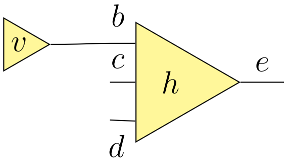
.

Proof. Let us check F commutes with $g \circ_1 f$ for $g\colon x, c \to d$ and $f\colon a, b \to x$ binary maps. By left representability, we know that $f = f' \circ \theta_{a,b}$.

$$F_3(g \circ_1 f) = F_2(g \circ_1 f') \circ_1 F_2(\theta_{a,b}) \qquad \text{(definition of } F_3\text{)}$$

$$
\begin{aligned}
&= [F_2(g) \circ_1 F_1(f')] \circ_1 F_2(\theta_{a,b}) && \text{(naturality of } F_2\text{)}\\
&= F_2(g) \circ_1 [F_1(f') \circ_1 F_2(\theta_{a,b})] && \text{(extranat. of sub. of bin. into bin. in } \mathcal{D}\text{)}\\
&= F_2(g) \circ_1 F_2(f' \circ \theta_{a,b}) && \text{(naturality of } F_2\text{)}\\
&= F_2(g) \circ_1 F_2(f) && \text{(definition of } f'\text{).}
\end{aligned}
$$

By definition F commutes with substitutions of binary maps into the second variable of other binary maps. Let us now consider the substitutions of nullary maps into ternary. Let $v \in \mathcal{C}_0(-;b)$ and $h \in \mathcal{C}_3(b,c,d;e)$.



By left representability, we know that $h = h' \circ_1 \theta_{b,c}$.

$$
\begin{aligned}
F_2(h \circ_1 v) &= F_2(\,(h' \circ_1 \theta_{b,c}) \circ_1 v) && \text{(by left representability)}\\
&= F_2(h' \circ_1 (\theta_{b,c} \circ_1 v)\,) && \text{(by (4.2.1.b) of } \mathcal{C}\text{)}\\
&= F_2(h') \circ_1 F_1(\theta_{b,c} \circ_1 v) && \text{(naturality of } F_2\text{)}\\
&= F_2(h') \circ_1 (\,F_2(\theta_{b,c}) \circ_1 F_0(v)\,) && \text{(by first one in (4.2.6))}\\
&= (\,F_2(h') \circ_1 F_2(\theta_{b,c})\,) \circ_1 F_0(v) && \text{(by (4.2.1.b) of } \mathcal{D}\text{)}\\
&= F_2(h' \circ_1 \theta_{b,c}) \circ_1 F_0(v) && \text{(by part above)}\\
&= F_2(h) \circ_1 F_0(v) && \text{(by left representability).}
\end{aligned}
$$

The proof for substitution of a nullary map in the third variable of a ternary one is analogous. For substitution of a nullary map in the second variable of a ternary one instead, we use the second part of (4.2.6).

Now we should prove that F also respects substitution of binary into ternary and viceversa. In order to prove this we need to first define F_4. Let $k \in \mathcal{C}_4(a,b,c,d;e)$. By left representability we have $k = k' \circ_1 \theta_{a,b}$, therefore we are forced to define

$$F_4(k) := F_3k' \circ_1 F_2\theta_{a,b} = (F_2k'' \circ_1 F_2\theta_{ab,c}) \circ_1 F_2\theta_{a,b}.$$

Let $h\colon c, d, e \to y$ be a ternary map and $f\colon a, b \to c$ a binary map. Then

$$\begin{aligned}
F_4(h \circ_1 f) &= F_3(\,(h \circ_1 f)'\,) \circ_1 F_2(\theta_{a,b}) && \text{(by definition of } F_4\text{)}\\
&= F_3(h \circ_1 f') \circ_1 F_2(\theta_{a,b}) && \text{(one checks that } (h \circ_1 f)' = h \circ_1 f'\text{)}\\
&= [F_3(h) \circ_1 F_1(f')] \circ_1 F_2(\theta_{a,b}) && \text{(naturality of } F_3\text{)}\\
&= F_3(h) \circ_1 [F_1(f') \circ_1 F_2(\theta_{a,b})] && \text{(properties of } \mathcal{D}\text{)}\\
&= F_3(h) \circ_1 F_2(f) && \text{(naturality of } F_2 \text{ and left representability).}
\end{aligned}$$

Using this, it is straightforward to prove that F preserves substitution of a ternary map into the first component of a binary one.

Next let $h\colon c, x, d \to y$ be a ternary map and $f\colon a, b \to x$ a binary map. We want to prove that F respects substitution of a binary map into the second variable of a ternary one. So, writing $h = h' \circ_1 \theta_{c,x}$,

$$\begin{aligned}
&F_4(h \circ_2 f)\\
&= F_3(\,(h \circ_2 f)'\,) \circ_1 F_2(\theta_{c,a}) && \text{(by definition of } F_4\text{)}\\
&= [\,F_2(\,(h \circ_2 f)''\,) \circ_1 F_2(\theta_{ca,b})\,] \circ_1 F_2(\theta_{c,a}) && \text{(by definition of } F_3\text{)}\\
&= [\,F_2(\,h' \circ_1 (cf' \circ \alpha_{c,a,b})\,) \circ_1 F_2(\theta_{ca,b})\,] \circ_1 F_2(\theta_{c,a}) && \text{(by } (h \circ_2 f)'' = h' \circ_1 (cf \circ \alpha)\,\text{)}\\
&= [\,(\,F_2(h') \circ_1 F_1(cf' \circ \alpha_{c,a,b})\,) \circ_1 F_2(\theta_{ca,b})\,] \circ_1 F_2(\theta_{c,a}) && \text{(profunct. binary maps)}\\
&= [\,F_2(h') \circ_1 (\,F_1(cf' \circ \alpha_{c,a,b}) \circ_1 F_2(\theta_{ca,b})\,)\,] \circ_1 F_2(\theta_{c,a}) && \text{(nat. sub. bin. into bin.)}\\
&= F_2(h') \circ_1 [\,(\,F_1(cf' \circ \alpha_{c,a,b}) \circ_1 F_2(\theta_{ca,b})\,) \circ_1 F_2(\theta_{c,a})\,] && \text{(by axiom (4.2.1.}b\text{) in } \mathcal{D}\text{)}\\
&= F_2(h') \circ_1 F_3(\,((cf' \circ \alpha_{c,a,b}) \circ_1 \theta_{ca,b}) \circ_1 \theta_{c,a}\,) && \text{(parts before and profunct.)}\\
&= F_2(h') \circ_1 F_3(\theta_{c,x} \circ_2 f) && \text{(routine checks)}\\
&= F_2(h') \circ_1 (\,F_2(\theta_{c,x}) \circ_2 F_2(f)\,) && \text{(parts before)}\\
&= (\,F_2(h') \circ_1 F_2(\theta_{c,x})\,) \circ_2 F_2(f) && \text{(by axiom (4.2.1.}b\text{) in } \mathcal{D}\text{)}\\
&= F_3(h' \circ_1 \theta_{c,x}) \circ_2 F_2(f) = F_3(h) \circ_2 F_2(f) && \text{(parts before and left represent.).}
\end{aligned}$$

Finally, substitution of a binary into the third variable of a ternary is similar to the previous cases, and we can use that to prove ternary into second input of a binary one. □

Closedness for Short Multicategories

We can adapt Definition 4.1.4 to short multicategories with the following.

Definition 4.2.6. A short multicategory is said to be **closed** if for all b, c there exists an object $[b, c]$ and binary map $e_{b,c} \colon [b, c], b \to c$ for which the induced function

$$e_{b,c} \circ_1 - \colon \mathcal{C}_n(\overline{x}; [b, c]) \to \mathcal{C}_{n+1}(\overline{x}, b; c)$$

is a bijection, for $n = 0, 1, 2, 3$.

In a closed short multicategory the assignment $(b, c) \mapsto [b, c]$, using the profunctoriality of binary maps, can be extended to a functor $\mathbb{C} \times \mathbb{C} \to \mathbb{C}$. We will denote with $\mathbf{ShMult}_{lr}^{cl}$ the full subcategory of **ShMult** with objects left representable closed short multicategories. Naturally, the forgetful functor $U \colon \mathbf{Mult} \to \mathbf{ShMult}$ restricts to a forgetful functor

$$U_{lr}^{cl} \colon \mathbf{Mult}_{lr}^{cl} \to \mathbf{ShMult}_{lr}^{cl}.$$

The next proposition gives a characterisation of closed short multicategories which are also left representable.

Proposition 4.2.7. *A closed short multicategory is left representable if and only if it has nullary map classifier and each $[b, -]$ has a left adjoint.*

Proof. If it is left representable and closed then the natural bijections

$$\mathcal{C}_1(ab; c) \cong \mathcal{C}_2(a, b; c) \cong \mathcal{C}_1(a; [b, c])$$

show that $-b \dashv [b, -]$. Conversely, if $[b, -]$ has left adjoint $-b$, then we have natural bijections

$$\mathbb{C}(ab, c) \cong \mathcal{C}_1(a; [b, c]) \cong \mathcal{C}_2(a, b; c)$$

and, by Yoneda, the composite is of the form $- \circ_1 \theta_{a,b}$ for a binary map classifier $\theta_{a,b} \colon a, b \to ab$ (which corresponds to 1_{ab}). It remains to show that this and the nullary map classifer are left universal. For the binary map classifier, we must show that $- \circ \theta_{a,b} \colon \mathcal{C}_{n+1}(ab, \overline{x}; c) \to \mathcal{C}_{n+2}(a, b, \overline{x}; c)$ is a bijection for all $\overline{x}$ of length 1 or 2, the case 0 being known. For an inductive style argument, suppose it is true for $\overline{x}$ of length $i \leq 1$. We should show that the bottom line below is a bijection

$$\begin{tikzcd}
\mathcal{C}_{i+1}(ab, \overline{x}; [y, c]) \arrow[r, "- \circ_1 \theta_{a,b}"] \arrow[d, "e_{y,c} \circ_1 -"'] & \mathcal{C}_{i+2}(a, b, \overline{x}; [y, c]) \arrow[d, "e_{y,c} \circ_1 -"] \\
\mathcal{C}_{i+2}(ab, \overline{x}, y; c) \arrow[r, "- \circ_1 \theta_{a,b}"'] & \mathcal{C}_{i+3}(a, b, \overline{x}, y; c])
\end{tikzcd}$$

but this follows from the fact that the square commutes, by associativity axiom (4.2.1.a), and the other three morphisms are bijections, by assumption. The case of the nullary map classifier is similar in form. □

4.3. Skew Notions

Now that we have introduced short multicategories, we shall review some important skew notions. In particular, we will recall the definitions of *skew monoidal category* [Szl12] and *skew multicategory* [BL18]. The first concept will be useful in Section 4.4 when we will consider various correspondences between different flavours of monoidal category and multicategory. Then, in Section 4.5 we will generalise the results of Section 4.4 to skew multicategories.

Skew Monoidal Categories

We start reviewing the definition of skew monoidal categories and morphisms between them. A **(left) skew monoidal category** $(\mathbb{C}, \otimes, i, \alpha, \lambda, \rho)$ [Szl12] is a category $\mathbb{C}$ together with a functor

$$\begin{aligned} \otimes \colon \mathbb{C} \times \mathbb{C} &\to \mathbb{C} \\ (a, b) &\mapsto ab, \end{aligned}$$

a unit object $i \in \mathbb{C}$, and natural families $\alpha_{a,b,c} \colon (ab)c \to a(bc)$ (the *associator*), $\lambda_a \colon ia \to a$ (the *left unit*) and $\rho_a \colon a \to ai$ (the *right unit*) satisfying five axioms which are neatly labelled by the five words

$$\begin{gathered} abcd \\ iab \ aib \ abi \\ ii \end{gathered}$$

of which the first refers to MacLane's pentagon axiom. More precisely, the axioms are the following

$$\begin{tikzcd}
 & & (ab)(cd) \arrow[drr, "\alpha_{a,b,cd}"] & & \\
((ab)c)d \arrow[urr, "\alpha_{ab,c,d}"] \arrow[dr, "\alpha_{a,b,c}d"'] & & & & a(b(cd)) \\
 & (a(bc))d \arrow[rr, "\alpha_{a,bc,d}"'] & & a((bc)d) \arrow[ur, "a\alpha_{b,c,d}"'] &
\end{tikzcd} \tag{4.3.1}$$

$$\begin{tikzcd}
(ia)b \arrow[r, "\alpha_{i,a,b}"] \arrow[dr, "\lambda_a b"'] & i(ab) \arrow[d, "\lambda_{ab}"] \\
 & ab
\end{tikzcd} \tag{4.3.2}$$

$$\begin{tikzcd}
ab \arrow[r, "\rho_{ab}"] \arrow[dr, "a\rho_b"'] & (ab)i \arrow[d, "\alpha_{a,b,i}"] \\
 & a(bi)
\end{tikzcd} \tag{4.3.3}$$

$$\begin{tikzcd}
ab \arrow[r, "\rho_a b"] \arrow[drr, "1_{ab}"'] & (ai)b \arrow[r, "\alpha_{a,i,b}"] & a(ib) \arrow[d, "a\lambda_b"] \\
 & & ab
\end{tikzcd} \tag{4.3.4}$$

$$\begin{tikzcd}
i \arrow[r, "\rho_i"] \arrow[dr, "1_i"'] & ii \arrow[d, "\lambda_i"] \\
 & i.
\end{tikzcd} \tag{4.3.5}$$

Example 4.3.1. Given a ring R, any R-bialgebroid defines a (right) skew monoidal structure on the category of right R-modules [Szl12, Section 3].

Definition 4.3.2. Let $(\mathbb{C}, \otimes, i, \alpha, \lambda, \rho)$ be a skew monoidal category.

- $\mathbb{C}$ is said to be **left normal** if λ is invertible.
- $\mathbb{C}$ is said to be **(left) closed** if the endofunctor $- \otimes b$ has a right adjoint $[b, -]$ for any $b \in \mathbb{C}$. We will sometime refer to *left closed* skew monoidal categories simply as *closed* skew monoidal.

Let $(\mathbb{C}, \otimes, i^C, \alpha^C, \lambda^C, \rho^C)$ and $(\mathbb{D}, \otimes, i^D, \alpha^D, \lambda^D, \rho^D)$ be two skew monoidal categories. A **lax monoidal functor** (F, f_0, f_2) [Szl12] consists of a functor $F\colon \mathbb{C} \to \mathbb{D}$, a map

$$f_0\colon i^D \to Fi^C$$

and a family of maps

$$f_2\colon FaFb \to F(ab)$$

natural in a and b and satisfying the following axioms:

$$\begin{tikzcd}
(FaFb)Fc \arrow[r, "f_2 \cdot Fc"] \arrow[d, "\alpha^D"'] & F(ab)Fc \arrow[r, "f_2"] & F(\,(ab)c\,) \arrow[d, "F\alpha^C"] \\
Fa(FbFc) \arrow[r, "Fa \cdot f_2"'] & FaF(bc) \arrow[r, "f_2"'] & F(\,a(bc)\,)
\end{tikzcd} \tag{4.3.6}$$

$$\begin{tikzcd}
iFa \arrow[r, "\lambda^D"] \arrow[d, "f_0 Fa"'] & Fa \\
FiFa \arrow[r, "f_2"'] & F(ia) \arrow[u, "F\lambda^C"']
\end{tikzcd} \tag{4.3.7}$$

$$\begin{tikzcd}
Fa \arrow[r, "F\rho^C"] \arrow[d, "\rho^D"'] & F(ai) \\
Fa.i \arrow[r, "Fa \cdot f_0"'] & FaFi. \arrow[u, "f_2"']
\end{tikzcd} \tag{4.3.8}$$

With an abuse of notation, we may write the associators, left/right unit maps as α, λ and ρ both in $\mathbb{C}$ and $\mathbb{D}$, omitting the superscript. Skew monoidal categories and lax monoidal functors form a category **Skew**. We will denote with $\mathbf{Skew}_{ln}$, $\mathbf{Skew}^{cl}$ and $\mathbf{Skew}^{cl}_{lr}$ the full subcategories with objects left normal/closed/left normal and closed skew monoidal categories.

Skew Multicategories

In this section we will recall the definition of skew multicategory and some other important notions, all of which can be found in [BL18].

Definition 4.3.3. [BL18, Definition 4.2] A skew multicategory consists of, a category $\mathbb{C}$ together with:

- for each $a \in \mathbb{C}$ a set $\mathcal{C}^l_0(-; a)$ of *nullary maps*;

- for each $n > 0$, each $a_1, ..., a_n \in \mathbb{C}$ and each $b \in \mathbb{C}$ a set $\mathcal{C}^t_n(\overline{a}; b)$ of *tight n-ary maps* natural in all components and such that, when $n = 1$, then $\mathcal{C}^t_1(a; b) = \mathbb{C}(a, b)$;
- for each $n > 0$, each $a_1, ..., a_n \in \mathbb{C}$ and each $b \in \mathbb{C}$ a set $\mathcal{C}^l_n(\overline{a}; b)$ of *loose n-ary maps* natural in all components;
- for each $n > 0$, each $a_1, ..., a_n \in \mathbb{C}$ and each $b \in \mathbb{C}$ a function

$$j_{\overline{a},b} \colon \mathcal{C}^t_n(\overline{a}; b) \to \mathcal{C}^l_n(\overline{a}; b).$$

On top of this there is further structure:

- substitution gives us multimaps $g(f_1, ..., f_n)$, which are tight just when g and f_1 are; these substitutions, moreover, commute with the comparisons viewing tight multimaps as loose.

Finally the usual associativity and unit axioms must be satisfied.

Example 4.3.4. In [BL18, Section 4.2] we can find some examples of skew multicategories. One of them is the skew multicategory of categories equipped with a choice of finite products. In this case loose multimaps are functors $\mathbb{C}_1 \times ... \times \mathbb{C}_n \to \mathbb{D}$ preserving products in each variable in the usual up to isomorphism sense, whereas tight multimaps preserve the given products *strictly in the first variable.*

Remark 4.3.5. We can identify multicategories as skew multicategories in which all multimorphisms are tight and loose, i.e. j is the indentity.

Skew multicategories have a notion of morphism between them, which we call here *skew multifunctor.* We recall that, given a functor $F \colon \mathbb{C} \to \mathbb{D}$, with $F\overline{a}$ we mean the list $Fa_1, ..., Fa_n$.

Definition 4.3.6. [BL18] Let $\mathcal{C}$ and $\mathcal{D}$ be skew multicategories. A **skew multifunctor** is a functor $F\colon \mathbb{C} \to \mathbb{D}$ together with natural families

$$\begin{aligned} F^t_n &\colon \mathcal{C}^t_n(\overline{a}; b) \to \mathcal{D}^t_n(F\overline{a}; Fb) && \text{for} && 1 \geq n \\ F^l_n &\colon \mathcal{C}^l_n(\overline{a}; b) \to \mathcal{D}^l_n(F\overline{a}; Fb) && \text{for} && 0 \geq n \end{aligned}$$

such that $F^t_1 \equiv F$. These families must commute with all substitution operators and j.

Skew multicategories and skew multifunctors form a category **SkMult**.

Left Representability and Closedness

A skew multicategory $\mathcal{C}$ is **weakly representable** [BL18, Section 4.4] if for each pair $x = t, l$ and $\overline{a} \in \mathbb{C}^n$ there exists an object $m^x\overline{a} \in \mathbb{C}$ and multimap

$$\theta^x_{\overline{a}} \in \mathcal{C}^x_n(\overline{a}; m^x\overline{a})$$

with the property that the induced function

$$- \circ_1 \theta^x_{\overline{a}} \colon \mathcal{C}^t_1(m^x\overline{a}; b) \to \mathcal{C}^x_n(\overline{a}; b)$$

is a bijection for all $b \in \mathbb{C}$. We call $\theta^t_{\overline{a}}$ a **tight n-ary map classifier** and $\theta^l_{\overline{a}}$ a **loose n-ary map classifier**. Moreover, we say that $\theta^x_{\overline{a}}$ is **left universal** if the induced function

$$- \circ_1 \theta^x_{\overline{a}} \colon \mathcal{C}^t_{1+r}(m^x\overline{a}, \overline{x}; b) \to \mathcal{C}^x_{n+r}(\overline{a}, \overline{x}; b)$$

for each $r \geq 0$, $\overline{x} \in \mathbb{C}^r$ and $b \in \mathbb{C}$.

Definition 4.3.7. [BL18, Definition 4.4] A skew multicategory $\mathcal{C}$ is said to be **left representable** if it is weakly representable and all universal multimaps $\theta^x_{\overline{a}}$ are left universal.

We will denote with $\textbf{SkMult}_{lr}$ the full subcategory of **SkMult** with objects left representable skew multicategories.

Definition 4.3.8. [BL18, Definition 4.6] A skew multicategory $\mathcal{C}$ is said to be **closed** if for all $b, c \in \mathbb{C}$ there exists an object $[b, c]$ and tight multimap $e_{b,c} \in \mathcal{C}^t_2([b, c], b; c)$ with the universal property that the induced function

$$e_{b,c} \circ_1 -\colon \mathcal{C}^x_n(\overline{a}; [b, c]) \to \mathcal{C}^x_{n+1}(\overline{a}, b; c)$$

is a bijection for all $a_1, ..., a_n \in \mathbb{C}$ and $x = t, l$.

We will denote with $\mathbf{SkMult}^{cl}_{lr}$ the full subcategory of **SkMult** with objects left representable closed skew multicategories.

We conclude this section explaining briefly how to construct an equivalence $T^{cl}_{lr}\colon \mathbf{Mult}^{cl}_{lr} \to \mathbf{Skew}^{cl}_{ln}$ between left representable closed multicategories and left normal skew monoidal closed categories. Even though this equivalence is not explicitly presented in [BL18], it follows directly from some of their results. In particular let us recall three.

Theorem 4.3.9. *[BL18, Theorem 6.1] There is a 2-equivalence between the 2-category* **Skew** *of skew monoidal categories and the 2-category of left representable skew multicategories.*

From this theorem they then deduce the following result.

Theorem 4.3.10. *[BL18, Theorem 6.3] There is a 2-equivalence between the 2-categories of left normal skew monoidal categories and of left representable multicategories.*

In the same way, one can prove that the existence of the equivalence $T^{cl}_{lr}\colon \mathbf{Mult}^{cl}_{lr} \to \mathbf{Skew}^{cl}_{ln}$ follows from the following result.

Theorem 4.3.11. *[BL18, Theorem 6.4] The 2-equivalence of Theorem 4.3.10 restricts to a 2-equivalence between the 2-category* $\mathbf{Skew}^{cl}_{ln}$ *of closed skew monoidal categories and the 2-category* $\mathbf{Mult}^{cl}_{lr}$ *of left representable closed skew multicategories.*

4.4. Short Multicategories vs Skew Monoidal Categories

In this section we will show that certain kinds of short multicategories are equivalent to certain kinds of multicategories. We will mostly consider kinds of representable multicategories because it will make proofs easier and many examples in the literature are of this kind (see for instance [Her00, Section 2.2]). The strategy will be to use known equivalences between different flavours of monoidal category and multicategory [Her00, BL18]. For example, the left representable case gives us the following picture

$$\begin{tikzcd}
\mathbf{Mult}_{lr} \arrow[dr, "U_{lr}"] \arrow[dd, "T"'] & \\
& \mathbf{ShMult}_{lr} \arrow[dl, dashed, "K"] \\
\mathbf{Skew}_{ln}. &
\end{tikzcd}$$

We will start showing how to construct the functor K and then prove it is an equivalence. The proof of the other cases will have the same structure.

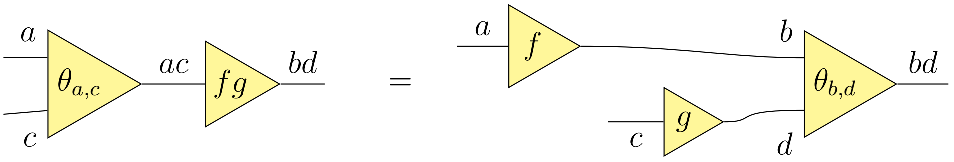
The Left Representable and Representable Cases

The first equivalence we will use is $T\colon \mathbf{Mult}_{lr} \to \mathbf{Skew}_{ln}$ between left representable multicategories and left normal skew monoidal categories [BL18, Theorem 6.3].

Notation. From now on, to increase readability of proofs, we will often mix algebraic parts and diagrams. For clarity, we shall explain what we mean with this. In a short multicategory any diagram has multiple interpretations, which are given by the order of the substitutions we apply. We presented some examples of this at the start of Section 4.2, when we explained how to interpret the associativity equations. For this reason, the formal proofs are always given by the algebraic expressions. However, the chain of equations can be quite long. We therefore add diagrams whenever the maps involved in the equations change and not only the bracketing.

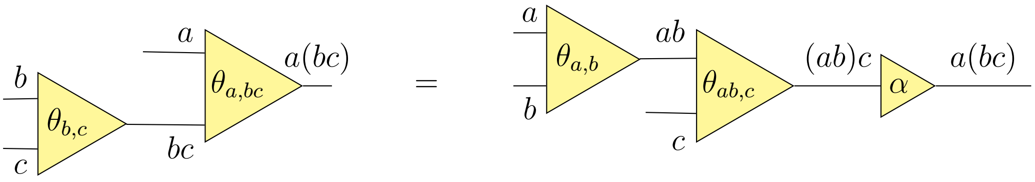
Lemma 4.4.1. *Given a left representable short multicategory* $\mathbb{C}$ *we can construct a left normal skew monoidal category* $K\mathbb{C}$ *in which:*

- *The tensor product* ab *of two objects* a *and* b *is the binary map classifier;*
- *The unit* i *is the nullary map classifier;*
- *Given* $f\colon a \to b$ *and* $g\colon c \to d$ *the tensor product* $fg\colon ac \to bd$ *is the unique morphism such that*



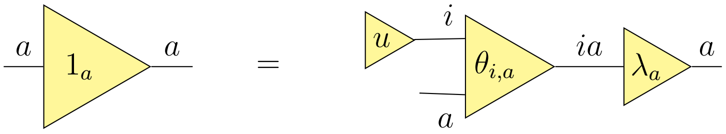
(4.4.1)

- *The associator* $\alpha\colon (ab)c \to a(bc)$ *is defined as the unique map such that*



(4.4.2)

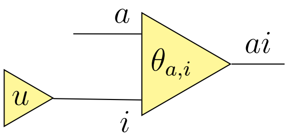
- *The left unit map* $\lambda\colon ia \to a$ *is defined as the unique map such that*



(4.4.3)

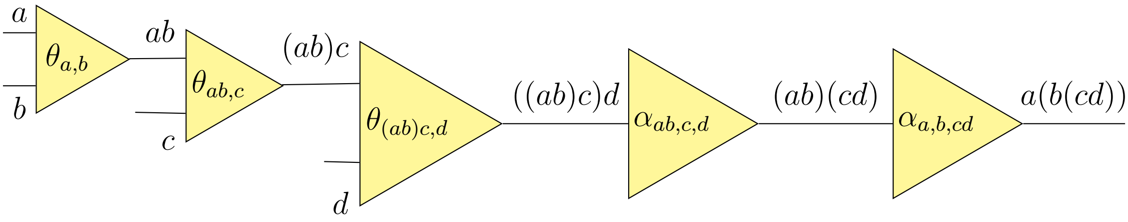
(which is invertible by left representability).

- *The right unit map* $\rho\colon a \to ai$ *is defined as*



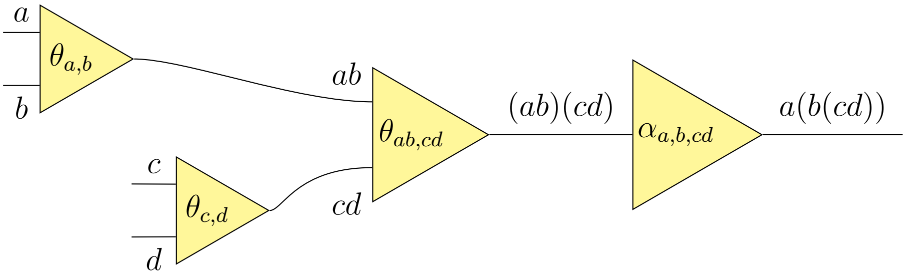
(4.4.4)

Proof. Functoriality of $\mathbb{C}^2 \to \mathbb{C} : (a,b) \mapsto ab$ follows from the universal property of the binary map classifier and profunctoriality of $\mathcal{C}_2(-;-)$. It remains to verify the five axioms for a skew monoidal category. We will start with the pentagon axiom (4.3.1). Since we need to prove an equality between two maps $((ab)c)d \to a(b(cd))$, by left representability, it is enough to prove that these maps become equal on precomposition with the left universal multimaps. Let us start with the top part, when we precompose with the universal 4-ary multimap of Proposition 4.2.3 we get

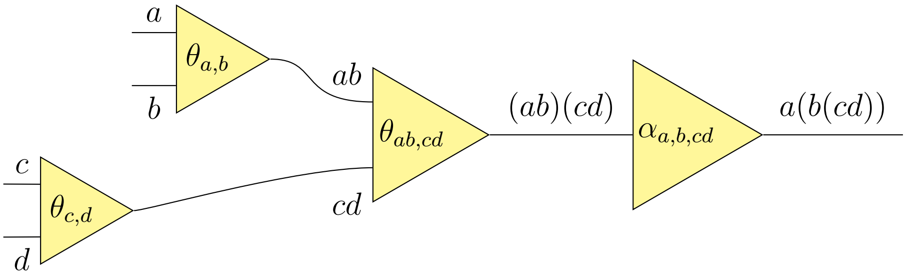


More precisely, we have

$$
\begin{aligned}
&(\alpha_{a,b,cd} \circ \alpha_{ab,c,d}) \circ [\,(\theta_{(ab)c,d} \circ_1 \theta_{ab,c}) \circ_1 \theta_{a,b}\,] \\
&= [\,(\alpha_{a,b,cd} \circ \alpha_{ab,c,d}) \circ (\theta_{(ab)c,d} \circ_1 \theta_{ab,c})\,] \circ_1 \theta_{a,b} && \text{(nat. sub. bin. into tern.)} \\
&= [\,\alpha_{a,b,cd} \circ [\alpha_{ab,c,d} \circ (\theta_{(ab)c,d} \circ_1 \theta_{ab,c})]\,] \circ_1 \theta_{a,b} && \text{(profunctoriality tern.)} \\
&= [\,\alpha_{a,b,cd} \circ (\theta_{ab,cd} \circ_2 \theta_{c,d})\,] \circ_1 \theta_{a,b} && \text{(definition of } \alpha)
\end{aligned}
$$

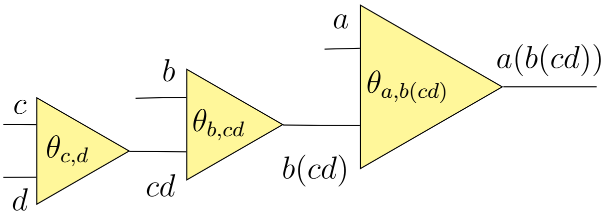


$$
\begin{aligned}
&= \alpha_{a,b,cd} \circ [\,(\theta_{ab,cd} \circ_2 \theta_{c,d}) \circ_1 \theta_{a,b}\,] && \text{(nat. sub. bin. into tern.)} \\
&= \alpha_{a,b,cd} \circ [\,(\theta_{ab,cd} \circ_1 \theta_{a,b}) \circ_3 \theta_{c,d}\,] && \text{(by axiom (4.2.2.}a\text{))}
\end{aligned}
$$



$$= [\, \alpha_{a,b,cd} \circ (\theta_{ab,cd} \circ_1 \theta_{a,b}) \,] \circ_3 \theta_{c,d} \qquad \text{(nat. sub. bin. into tern.)}$$

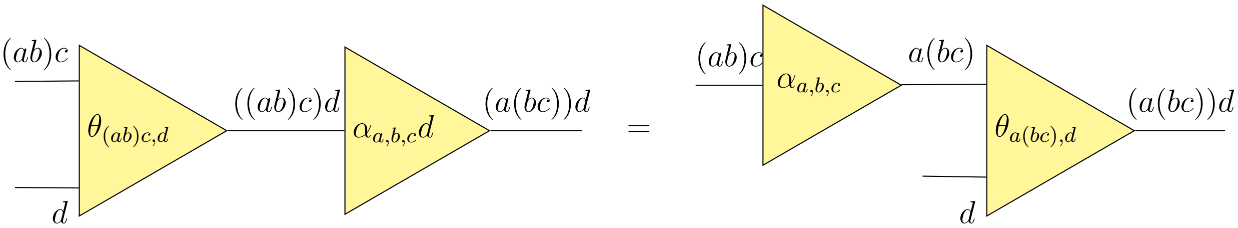
$$= (\theta_{a,b(cd)} \circ_2 \theta_{b,cd}) \circ_3 \theta_{c,d} \qquad \text{(by definition of } \alpha)$$



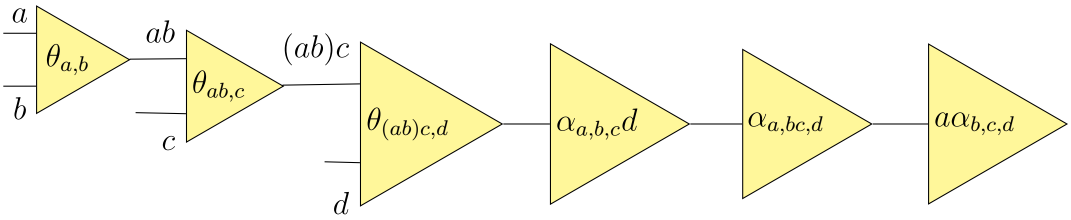
(4.4.5)

.

Now let us consider the bottom part of the pentagon axiom. Let us recall that $\alpha_{a,b,c} \cdot d$ is defined as the unique map such that



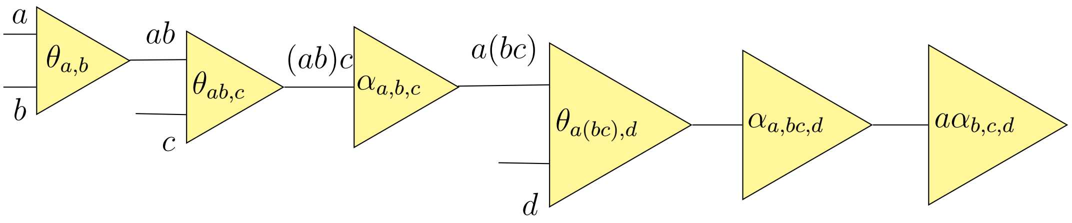
and $a \cdot \alpha_{b,c,d}$ is defined similarly. Then, the bottom part of the pentagon precomposed with left universal maps is equal to



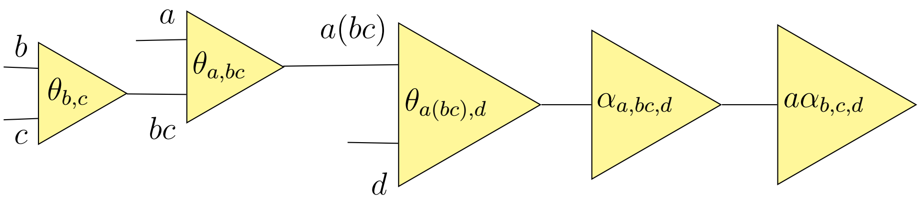
.

More precisely, we have

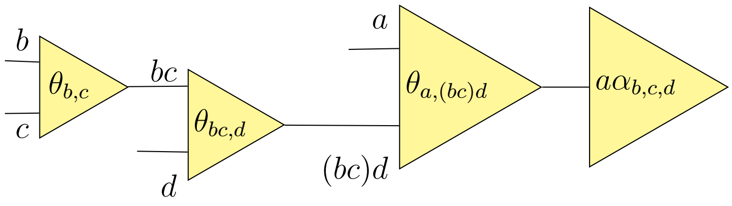
$$
\begin{aligned}
&[\,(a\cdot\alpha_{b,c,d}\circ\alpha_{a,bc,d})\circ\alpha_{a,b,c}\cdot d\,]\circ[\,(\theta_{(ab)c,d}\circ_1\theta_{ab,c})\circ_1\theta_{a,b}\,]\\
&=[\,[\,(a\cdot\alpha_{b,c,d}\circ\alpha_{a,bc,d})\circ\alpha_{a,b,c}\cdot d\,]\circ(\theta_{(ab)c,d}\circ_1\theta_{ab,c})\,]\circ_1\theta_{a,b} &&\text{(nat. sub. bin. into tern.)}\\
&=[\,(a\cdot\alpha_{b,c,d}\circ\alpha_{a,bc,d})\circ[\,\alpha_{a,b,c}\cdot d\;\circ(\theta_{(ab)c,d}\circ_1\theta_{ab,c})\,]\,]\circ_1\theta_{a,b} &&\text{(profunctoriality tern.)}\\
&=[\,(a\cdot\alpha_{b,c,d}\circ\alpha_{a,bc,d})\circ[\,(\alpha_{a,b,c}\cdot d\;\circ\theta_{(ab)c,d})\circ_1\theta_{ab,c}\,]\,]\circ_1\theta_{a,b} &&\text{(nat. sub. bin. into bin.)}\\
&=[\,(a\cdot\alpha_{b,c,d}\circ\alpha_{a,bc,d})\circ[\,(\theta_{a(bc),d}\circ_1\alpha_{a,b,c})\circ_1\theta_{ab,c}\,]\,]\circ_1\theta_{a,b} &&\text{(by definition of } \alpha\cdot d\text{)}
\end{aligned}
$$



$$
\begin{aligned}
&=[\,(a\cdot\alpha_{b,c,d}\circ\alpha_{a,bc,d})\circ[\,\theta_{a(bc),d}\circ_1(\alpha_{a,b,c}\circ\theta_{ab,c})\,]\,]\circ_1\theta_{a,b} &&\text{(extranat. sub. bin. into bin.)}\\
&=(a\cdot\alpha_{b,c,d}\circ\alpha_{a,bc,d})\circ[\,[\,\theta_{a(bc),d}\circ_1(\alpha_{a,b,c}\circ\theta_{ab,c})\,]\circ_1\theta_{a,b}\,] &&\text{(nat. sub. bin. into tern.)}\\
&=(a\cdot\alpha_{b,c,d}\circ\alpha_{a,bc,d})\circ[\,\theta_{a(bc),d}\circ_1[\,(\alpha_{a,b,c}\circ\theta_{ab,c})\circ_1\theta_{a,b}\,]\,] &&\text{(by axiom (4.2.1.}a\text{))}\\
&=(a\cdot\alpha_{b,c,d}\circ\alpha_{a,bc,d})\circ[\,\theta_{a(bc),d}\circ_1(\theta_{a,bc}\circ_2\theta_{b,c})\,] &&\text{(by definition of } \alpha\text{)}
\end{aligned}
$$

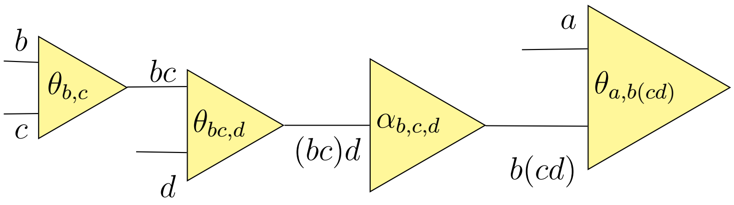


$$
\begin{aligned}
&=(a\cdot\alpha_{b,c,d}\circ\alpha_{a,bc,d})\circ[\,(\theta_{a(bc),d}\circ_1\theta_{a,bc})\circ_2\theta_{b,c}\,] &&\text{(by axiom (4.2.1.}a\text{))}\\
&=[\,(a\cdot\alpha_{b,c,d}\circ\alpha_{a,bc,d})\circ(\theta_{a(bc),d}\circ_1\theta_{a,bc})\,]\circ_2\theta_{b,c} &&\text{(by nat. sub. bin. into tern.)}\\
&=[\,a\cdot\alpha_{b,c,d}\circ[\,\alpha_{a,bc,d}\circ(\theta_{a(bc),d}\circ_1\theta_{a,bc})\,]\,]\circ_2\theta_{b,c} &&\text{(profunctoriality tern.)}\\
&=[\,a\cdot\alpha_{b,c,d}\circ(\theta_{a,(bc)d}\circ_2\theta_{bc,d})\,]\circ_2\theta_{b,c} &&\text{(by definition of } \alpha\text{)}
\end{aligned}
$$



$$= [\ (\ a \cdot \alpha_{b,c,d} \circ \theta_{a,(bc)d}\ ) \circ_2 \theta_{bc,d}\ ] \circ_2 \theta_{b,c} \qquad \text{(by nat. sub. bin. into bin.)}$$

$$= [\ (\ \theta_{a,b(cd)} \circ_2 \alpha_{b,c,d}\ ) \circ_2 \theta_{bc,d}\ ] \circ_2 \theta_{b,c} \qquad \text{(by definition of } a\alpha_{b,c,d})$$



$$= [\ \theta_{a,b(cd)} \circ_2 (\ \alpha_{b,c,d} \circ \theta_{bc,d}\ )\ ] \circ_2 \theta_{b,c} \qquad \text{(extranat. sub. bin. into bin.)}$$

$$= \theta_{a,b(cd)} \circ_2 [\ (\ \alpha_{b,c,d} \circ \theta_{bc,d}\ ) \circ_1 \theta_{b,c}\ ] \qquad \text{(by axiom (4.2.1.}a))$$

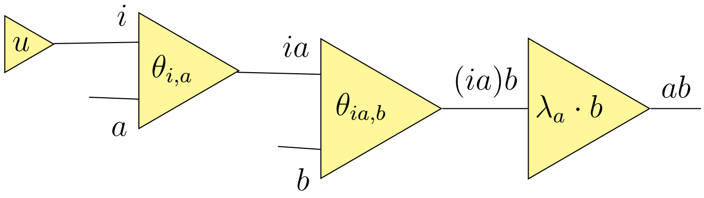
$$= \theta_{a,b(cd)} \circ_2 (\ \theta_{b,cd} \circ_2 \theta_{c,d}\ ) \qquad \text{(by definition of } \alpha)$$

$$= (\ \theta_{a,b(cd)} \circ_2 \theta_{b,cd}\ ) \circ_3 \theta_{c,d} \qquad \text{(by axiom (4.2.1.}a))$$

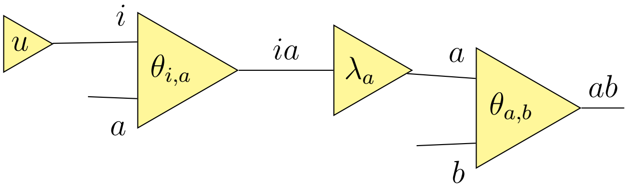
which is exactly (4.4.5). Similarly we use axioms (4.2.1.b) and (4.2.2.c) to prove the left unit axiom (4.3.2), which we recall below,

$$\begin{tikzcd}
(ia)b \arrow[r, "\alpha_{i,a,b}"] \arrow[dr, "\lambda_a b"'] & i(ab) \arrow[d, "\lambda_{ab}"] \\
 & ab.
\end{tikzcd}$$

Using left representability it is enough to prove the equality precomposing with the universal nullary map u and binary maps θ. So,

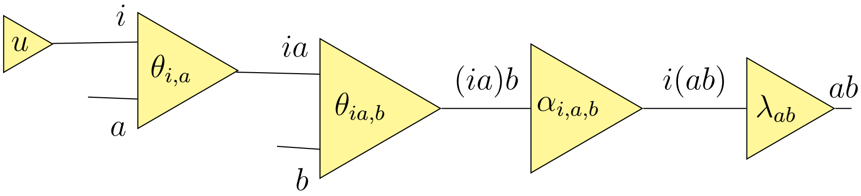


$$\begin{aligned}
&\lambda_a \cdot b \circ [\,(\,\theta_{ia,b} \circ_1 \theta_{i,a}\,) \circ_1 u\,] \\
&= [\,\lambda_a \cdot b \circ (\,\theta_{ia,b} \circ_1 \theta_{i,a}\,)\,] \circ_1 u && \text{(nat. sub. null. into bin.)} \\
&= [\,(\,\lambda_a \cdot b \circ \theta_{ia,b}\,) \circ_1 \theta_{i,a}\,] \circ_1 u && \text{(nat. sub. bin. into bin.)} \\
&= [\,(\,\theta_{a,b} \circ_1 \lambda_a\,) \circ_1 \theta_{i,a}\,] \circ_1 u && \text{(definition of } \lambda_a \cdot b)
\end{aligned}$$

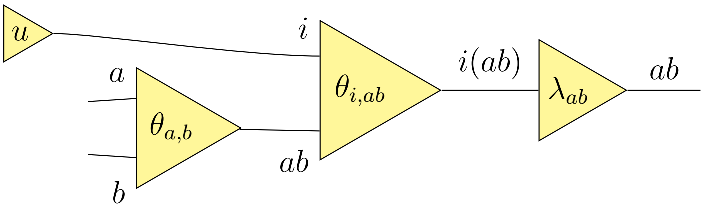


$$\begin{aligned}
&= [\,\theta_{a,b} \circ_1 (\,\lambda_a \circ_1 \theta_{i,a}\,)\,] \circ_1 u && \text{(extranat. sub. bin. into bin.)} \\
&= \theta_{a,b} \circ_1 [\,(\,\lambda_a \circ_1 \theta_{i,a}\,) \circ_1 u\,] && \text{(by axiom (4.2.1.}b)) \\
&= \theta_{a,b} \circ_1 1_a = \theta_{a,b} && \text{(by definition of } \lambda).
\end{aligned}$$

On the other hand,

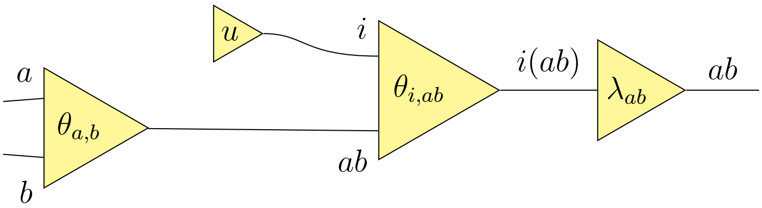


$$\begin{aligned}
&[\,[\,(\lambda_{ab} \circ \alpha_{i,a,b}) \circ \theta_{ia,b}\,] \circ_1 \theta_{i,a}\,] \circ_1 u \\
&= [\,[\,\lambda_{ab} \circ (\alpha_{i,a,b} \circ \theta_{ia,b})\,] \circ_1 \theta_{i,a}\,] \circ_1 u && \text{(profunctoriality bin.)} \\
&= [\,\lambda_{ab} \circ [\,(\alpha_{i,a,b} \circ \theta_{ia,b}) \circ_1 \theta_{i,a}\,]\,] \circ_1 u && \text{(nat. sub. bin. into bin.)} \\
&= [\,\lambda_{ab} \circ (\,\theta_{i,ab} \circ_2 \theta_{a,b}\,)\,] \circ_1 u && \text{(by definition of } \alpha)
\end{aligned}$$



$$= \lambda_{ab} \circ [\, (\, \theta_{i,ab} \circ_2 \theta_{a,b}\,) \circ_1 u \,] \qquad \text{(nat. sub. null. into bin.)}$$

$$= \lambda_{ab} \circ [\, (\, \theta_{i,ab} \circ_1 u \,) \circ \theta_{a,b} \,] = \qquad \text{(by axiom (4.2.2.c))}$$



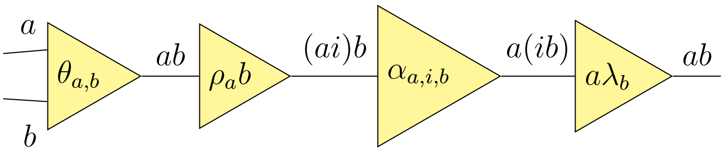
$$= [\, \lambda_{ab} \circ (\, \theta_{i,ab} \circ_1 u \,) \,] \circ \theta_{a,b} \qquad \text{(profunctoriality bin.)}$$

$$= 1_{a,b} \circ \theta_{a,b} = \theta_{a,b} \qquad \text{(definition of } \lambda).$$

Similarly, we use axioms (4.2.1.b) and (4.2.2.b) to prove the right unit axiom (4.3.3). Instead for the axiom (4.3.4)

$$\begin{tikzcd}
ab \arrow[r, "\rho_a b"] \arrow[drr, "1_{ab}"'] & (ai)b \arrow[r, "\alpha_{a,i,b}"] & a(ib) \arrow[d, "a\lambda_b"] \\
& & ab
\end{tikzcd}$$

we use axiom (4.2.1.b), more precisely:

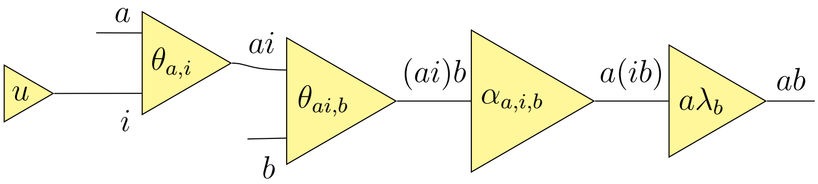


$$(a\lambda_b \circ \alpha_{a,i,b} \circ \rho_a b) \circ \theta_{a,b}$$

$$= (a\lambda_b \circ \alpha_{a,i,b}) \circ (\rho_a b \circ \theta_{a,b}) \qquad \text{(profunctoriality of binary maps)}$$

$$= (a\lambda_b \circ \alpha_{a,i,b}) \circ (\theta_{ai,b} \circ \rho_a) \qquad \text{(definition of } \rho b)$$

$$= (a\lambda_b \circ \alpha_{a,i,b}) \circ [\, \theta_{ai,b} \circ_1 (\theta_{a,i} \circ_2 u) \,] \qquad \text{(definition of } \rho)$$

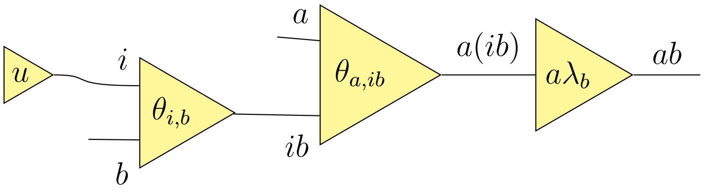


$$= (a\lambda_b \circ \alpha_{a,i,b}) \circ [\,(\theta_{ai,b} \circ_1 \theta_{a,i}) \circ_2 u\,] \qquad \text{(by axiom (4.2.1.}b\text{))}$$

$$= [\,(a\lambda_b \circ \alpha_{a,i,b}) \circ (\theta_{ai,b} \circ_1 \theta_{a,i})\,] \circ_2 u \qquad \text{(nat. sub. nullary into ternary)}$$

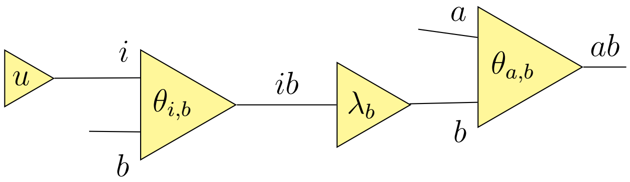
$$= [\,a\lambda_b \circ (\,\alpha_{a,i,b} \circ (\theta_{ai,b} \circ_1 \theta_{a,i})\,)\,] \circ_2 u \qquad \text{(profunctoriality ternary)}$$

$$= [\,a\lambda_b \circ (\theta_{a,ib} \circ_2 \theta_{i,b})\,] \circ_2 u \qquad \text{(definition of } \alpha\text{)}$$



$$= [\,(a\lambda_b \circ \theta_{a,ib}) \circ_2 \theta_{i,b}\,] \circ_2 u \qquad \text{(nat. sub. bin. into bin.)}$$

$$= [\,(\theta_{a,b} \circ_2 \lambda_b) \circ_2 \theta_{i,b}\,] \circ_2 u \qquad \text{(definition of } a\lambda\text{)}$$



$$= [\,\theta_{a,b} \circ_2 (\lambda_b \circ \theta_{i,b})\,] \circ_2 u \qquad \text{(extranat sub bin-into-bin)}$$

$$= \theta_{a,b} \circ_2 [\,(\lambda_b \circ \theta_{i,b}) \circ_2 u\,] \qquad \text{(by axiom (4.2.1.}b\text{))}$$

$$= \theta_{a,b} \circ_2 [\,\lambda_b \circ (\theta_{i,b} \circ_2 u)\,] \qquad \text{(nat. sub. null. into bin.)}$$

$$= \theta_{a,b} \circ_2 1_b = \theta_{a,b} \qquad \text{(definition of } \lambda\text{).}$$

Finally, we use axiom (4.2.2.d) to prove the axiom (4.3.5)

$$\begin{tikzcd} i \arrow[r, "\rho_i"] \arrow[dr, "1_i"'] & ii \arrow[d, "\lambda_i"] \\ & i. \end{tikzcd}$$

More precisely, we have to prove that $(\lambda_i \circ \rho_i) \circ u = u$ (by left representability).

$$\begin{aligned}(\lambda_i \circ \rho_i) \circ u &= \lambda_i \circ (\rho_i \circ u) && \text{(profunctoriality of nullary maps)}\\ &= \lambda_i \circ [\, (\theta_{i,i} \circ_2 u) \circ u \,] && \text{(definition of } \rho\text{)}\\ &= \lambda_i \circ [\, (\theta_{i,i} \circ_1 u) \circ u \,] && \text{(by axiom (4.2.2.}d\text{))}\\ &= [\, \lambda_i \circ (\theta_{i,i} \circ_1 u) \,] \circ u && \text{(profunctoriality of nullary maps)}\\ &= 1_i \circ u = u && \text{(definition of } \lambda\text{).} \ \square\end{aligned}$$

Before defining the functor $K\colon \mathbf{ShMult}_{lr} \to \mathbf{Skew}_{ln}$ on morphisms, we prove the following easy lemma.

Lemma 4.4.2. *Consider $\mathcal{C}, \mathcal{D} \in \mathbf{ShMult}_{lr}$ and a functor $F\colon \mathbb{C} \to \mathbb{D}$. There is a bijection between natural families*

$$F_{\overline{a},b}\colon \mathcal{C}_i(\overline{a}; b) \to \mathcal{D}_i(F\overline{a}; Fb)$$

and natural families

$$f_{\overline{a}}\colon m(F\overline{a}) \to F(m\overline{a})$$

where $m\overline{a}$ and $m(F\overline{a})$ are the n-ary map classifiers of the appropriate arity.

Proof. The bijection is governed by the following diagram

$$\begin{tikzcd} \mathcal{C}_i(\overline{a}; -) \arrow[r, "F_{\overline{a},-}"] & \mathcal{D}_i(F\overline{a}; F-) \\ \mathcal{C}_1(m\overline{a}, -) \arrow[u, "-\circ_1 \theta_{\overline{a}}"] \arrow[r, "F- \circ f_{\overline{a}}"'] & \mathcal{D}_1(m(F\overline{a}), F-) \arrow[u, "-\circ_1 \theta_{F\overline{a}}"'] \end{tikzcd} \tag{4.4.6}$$

in which the vertical arrows are natural bijections and the lower horizontal arrow corresponds to the upper one using the Yoneda lemma. □

Remark 4.4.3. Given a morphism $F\colon \mathcal{C} \to \mathcal{D} \in \mathbf{ShMult}_{lr}$ we obtain, applying the above lemma, natural families $f_2\colon FaFb \to F(ab)$ and $f_0\colon i \to Fi$ defining the *data* for a lax monoidal functor $KF\colon K\mathcal{C} \to K\mathcal{D}$. We will prove that is is a lax monoidal functor in Proposition 4.4.4.

Explicitly, $f_2\colon FaFb \to F(ab)$ is the unique morphism such that $f_2 \circ_1 \theta_{Fa,Fb} = F_2(\theta_{a,b})$ whilst f_0 is the unique morphism such that $f_0 \circ u = Fu$.

Notation. Let $\mathcal{C}$ be a short multicategory with a left universal nullary map classifier. Then we will use $(-)^* \colon \mathcal{C}_n(\overline{a};b) \to \mathcal{C}_{n+1}(i,\overline{a};b)$ for the inverse of $- \circ_1 u$: in other words, for any n-multimap f, then f^* is the unique $(n+1)$-multimap such that $f^* \circ_1 u = f$.

Proposition 4.4.4. *With the definition on objects given in Lemma 4.4.1 and on morphisms in Remark 4.4.3, we obtain a fully faithful functor* $K : \mathbf{ShMult}_{lr} \to \mathbf{Skew}_{ln}$.

Proof. By Lemma 4.2.5 a morphism of $\mathbf{ShMult}_{lr}(\mathcal{C},\mathcal{D})$ is uniquely specified by a functor $F : C \to D$ and natural families (F_2, F_0) satisfying three equations.

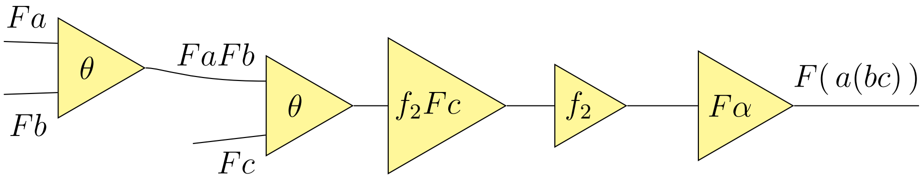
By Lemma 4.4.2, these natural families (F_2, F_0) bijectively correspond to natural families (f_2, f_0).

Therefore, if we can prove that (F_2, F_0) satisfy the equations of Lemma 4.2.5 if and only if (f_2, f_0) satisfy the equations for a lax monoidal functor, then we will have described a bijection $K_{\mathcal{C},\mathcal{D}} \colon \mathbf{ShMult}_{lr}(\mathcal{C},\mathcal{D}) \to \mathbf{Skew}_{ln}(K\mathcal{C}, K\mathcal{D})$.

Let us start assuming that (F_2, F_0) satisfy the equations of Lemma 4.2.5 and then prove that (f_2, f_0) satisfy the equations for a lax monoidal functor. The axiom (4.3.6)

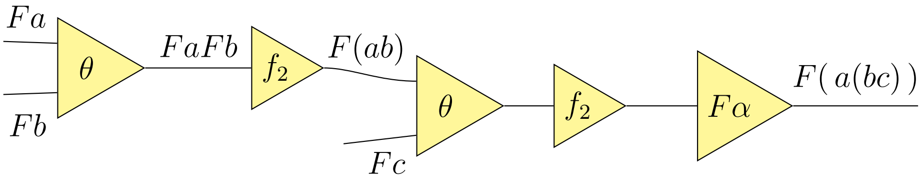
$$\begin{tikzcd}
(FaFb)Fc \arrow[r, "f_2 \cdot Fc"] \arrow[d, "\alpha"'] & F(ab)Fc \arrow[r, "f_2"] & F(\,(ab)c\,) \arrow[d, "F\alpha"] \\
Fa(FbFc) \arrow[r, "Fa \cdot f_2"'] & FaF(bc) \arrow[r, "f_2"'] & F(\,a(bc)\,)
\end{tikzcd}$$

is checked by precomposing with the universal ternary multimap. From now on, we will write only a diagramatic schema of these proofs, as the algebraic part is analogous to the previous ones. We start considering $F\alpha \circ f_2 \circ f_2 \cdot Fc$, which, after precomposing with the universal ternary map, corresponds to the following picture



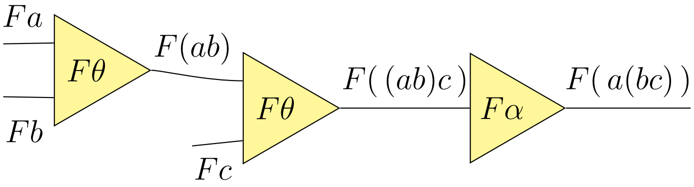
by definition of $f_2 \cdot Fc$

$=$



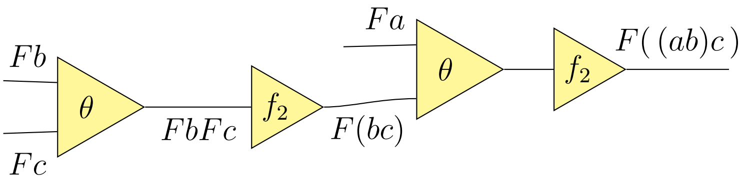
by definition of f_2

$=$



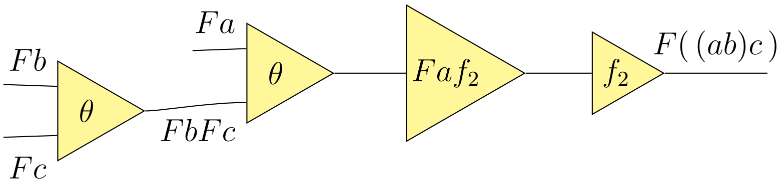
$$
\begin{aligned}
&= F\alpha \circ F_2\theta \circ_1 F_2\theta \\
&= F_2(\alpha \circ \theta) \circ_1 F_2\theta && \text{(by naturality of } F_2\text{)} \\
&= F_3(\,(\alpha \circ \theta_{a,b}) \circ_1 \theta_{ab,c}\,) && \text{(by Lemma 4.2.5)} \\
&= F_3(\,\theta_{b,c} \circ_2 \theta_{a,bc}\,) && \text{(by definition of } \alpha\text{)} \\
&= F_2\theta_{a,bc} \circ_2 F_2\theta_{b,c} && \text{(by Lemma 4.2.5)} \\
&= [\,f_2 \circ \theta_{Fa,F(bc)}\,] \circ_2 [\,f_2 \circ \theta_{Fb,Fc}\,] && \text{(by definition of } f_2\text{)}
\end{aligned}
$$

$=$



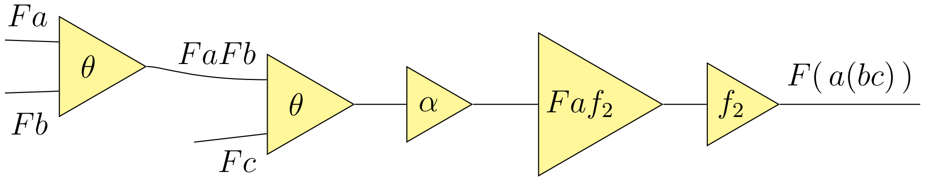
by definition of $Fa \cdot f_2$

$=$



by definition of α

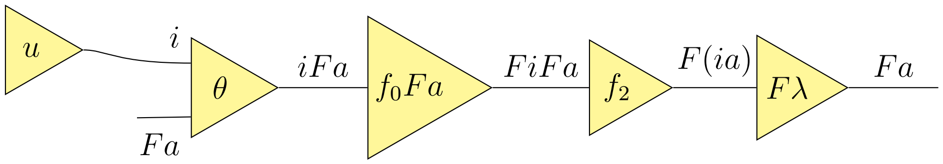
=



The last diagram corresponds to $f_2 \circ Fa \cdot f_2 \circ \alpha$, therefore the diagram (4.3.6) is commutative. Let us now prove the unit axioms, starting from (4.3.7)

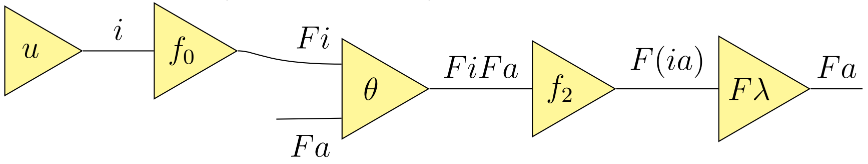
$$\begin{tikzcd}
iFa \arrow[r, "\lambda"] \arrow[d, "f_0 Fa"'] & Fa \\
FiFa \arrow[r, "f_2"'] & F(ia). \arrow[u, "F\lambda"']
\end{tikzcd}$$

We will prove this axioms showing that $F\lambda \cdot f_2 \cdot f_0 Fa$ satisfy the defining property (4.4.3) of λ.



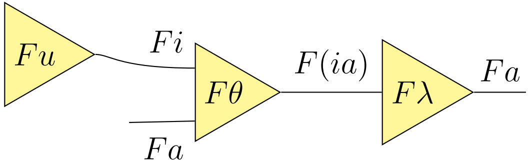
by definition of $f_0 \cdot Fa$

=



by definition of f_0 and f_2

=

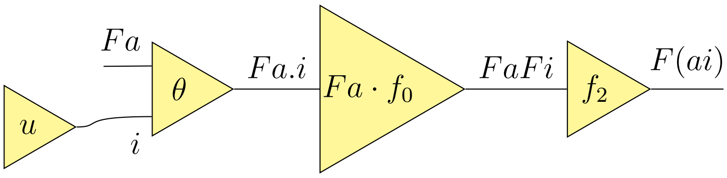


$$\begin{aligned}
&= F\lambda \circ F_2\theta \circ_1 F_0 u \\
&= F_2(\lambda \circ \theta) \circ_1 F_0 u && \text{(by naturality of } F_2\text{)} \\
&= F(\,(\lambda \circ \theta_{i,a}) \circ_1 u\,) && \text{(by Lemma 4.2.5)} \\
&= F(1_a) = 1_{Fa} && \text{(by defining property (4.4.3) of } \lambda\text{).}
\end{aligned}$$

Finally, we prove the right unit axiom (4.3.8)

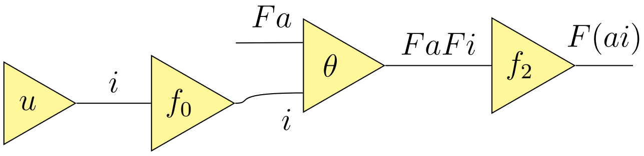
$$\begin{tikzcd}
Fa \arrow[r, "F\rho"] \arrow[d, "\rho"'] & F(ai) \\
Fa.i \arrow[r, "Fa\cdot f_0"'] & FaFi. \arrow[u, "f_2"']
\end{tikzcd}$$

We recall that the definition of ρ is given in (4.4.4).



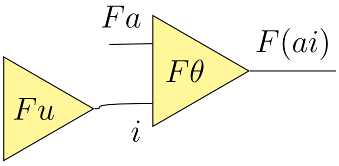
by definition of $Fa \cdot f_0$

$=$



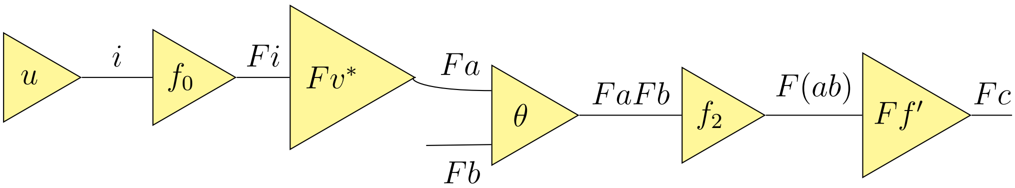
by definition of f_0 and f_2

$=$

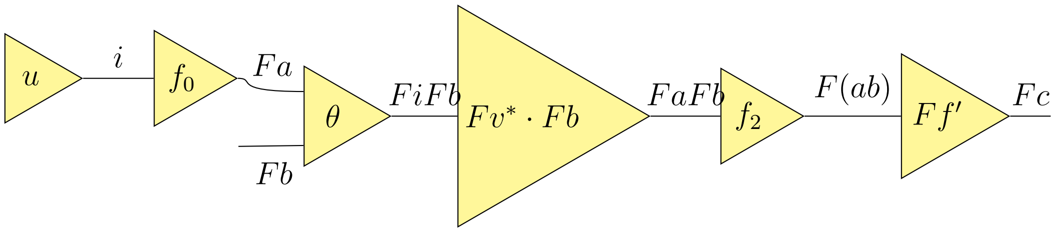


Then, by Lemma 4.2.5 $F_2\theta \circ_2 Fu = F(\theta \circ_2 u)$ which is exactly the definition of $F\rho$.

Finally, let us prove that if (f_2, f_0) satisfy the equations for a lax monoidal functor then (F_2, F_0) satisfy the equations of Lemma 4.2.5. Let us start with equations (4.2.6). Let $v \in \mathcal{C}_0(-;a)$ and $f \in \mathcal{C}_2(a,b;c)$, then $F_2f \circ_1 F_0v$ is defined as

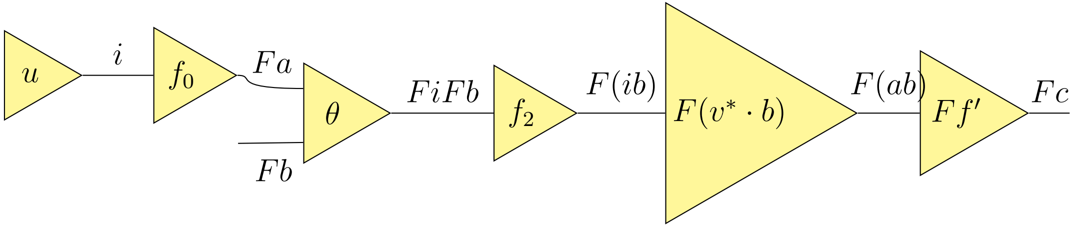


where v^* and f' correspond to v and f through left representability. Then, by definition of $Fv^* \cdot Fb$ we can rewrite the map above as



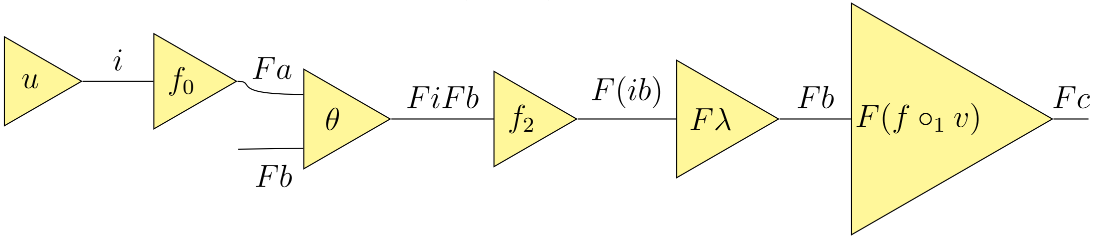
by naturality of f_2

=



since one can check that $f' \circ v^*b = (f \circ_1 v) \circ \lambda_b$ and functoriality of F

=



by definition of f_0Fb

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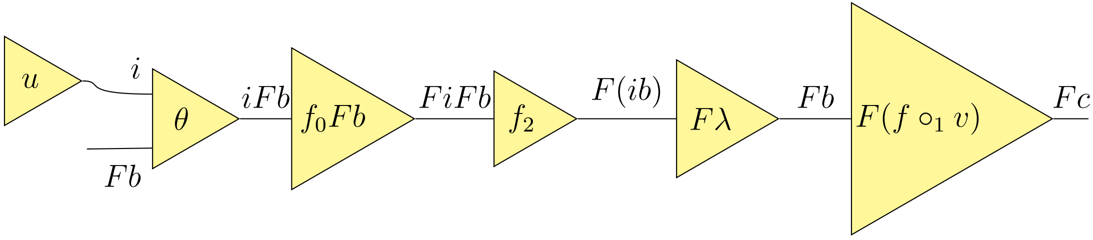
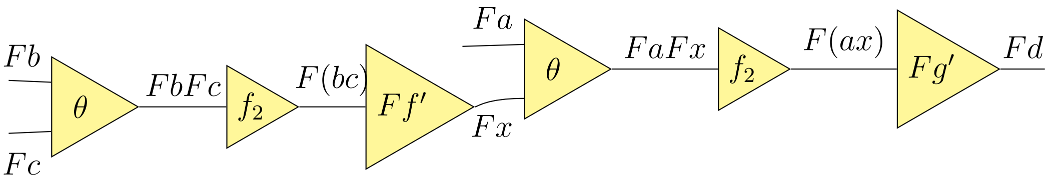
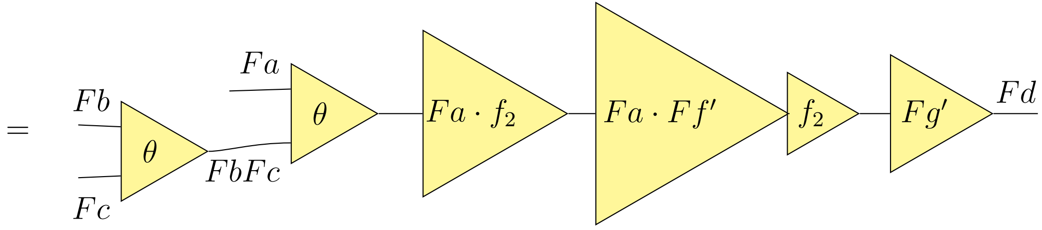


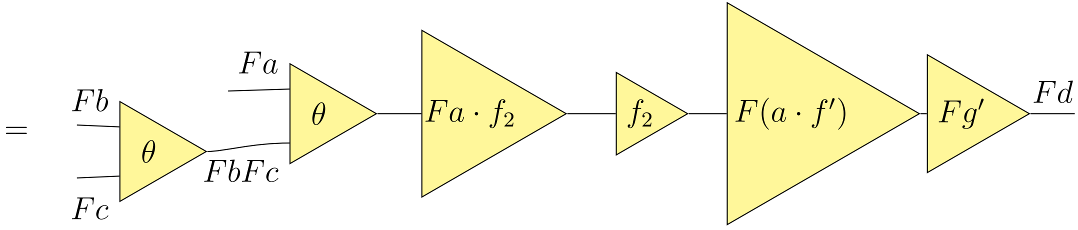
which is equal to $F(f \circ_1 v)$ by (4.3.7). Then, let us consider $s \in \mathcal{C}_0(-;b)$ and $f \in \mathcal{C}_2(a,b;c)$. Similarly, using (4.3.8) we can prove that $F(f \circ_2 s) = F_2 f \circ_2 F_0 v$. We have left to prove the equation (4.2.7). Let $f \in \mathcal{C}_2(b,c;x)$ and $g \in \mathcal{C}_2(a,x;d)$, then $F_2 g \circ_2 F_2 g$ is defined as



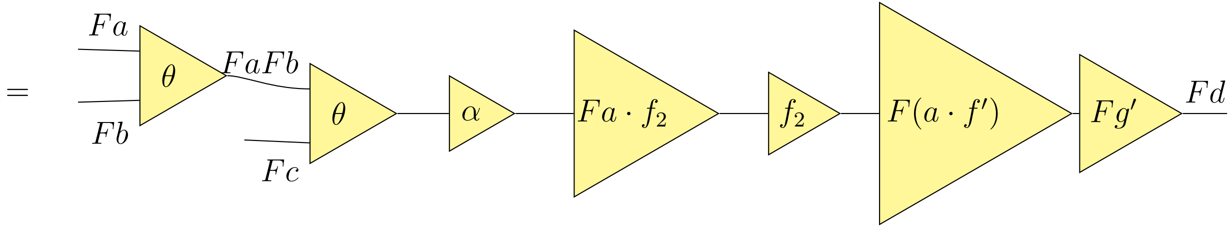
by definition of $Fa \cdot f_2$ and $Fa \cdot f'$



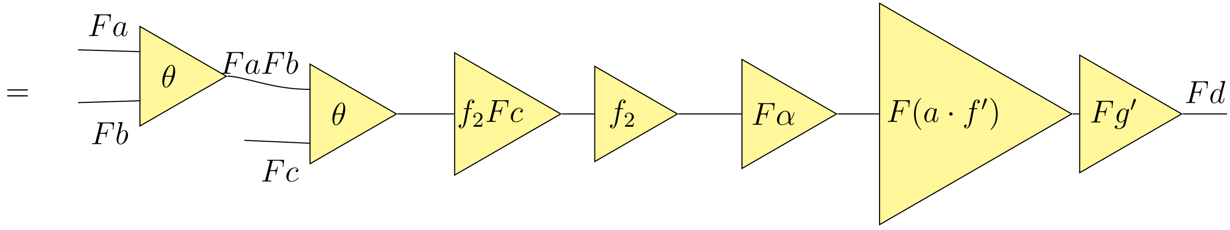
by naturality of f_2



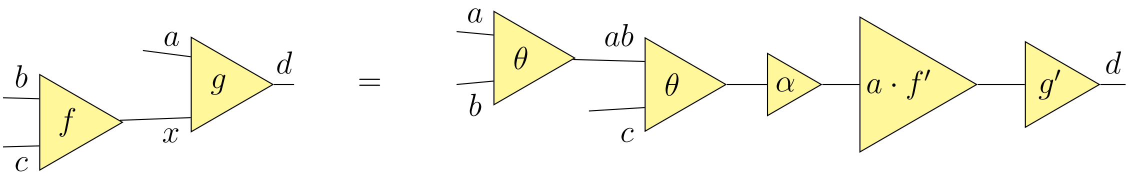
by definition of α



by diagram (4.3.6)



On the other hand, it is not hard to see that



$$\begin{aligned} F_3(g \circ_2 f) &= F_3(\, g' \circ a \cdot f' \circ \alpha \circ (\theta \circ_1 \theta)\,) \circ_1 F_2\theta && \text{(by equation above)} \\ &= Fg' \circ F(a \cdot f') \circ F\alpha \circ F_3(\theta \circ_1 \theta) && \text{(naturality of } F_3\text{).} \end{aligned}$$

We conclude noticing that $F_3(\theta \circ_1 \theta) = f_2 \circ f_2 Fc \circ (\theta \circ_1 \theta)$ by definition of F_3. In the end, Table 4.1 describes the correspondence between axioms.

(F_0, F_2)	(f_0, f_2)
(4.2.7)	Associator axiom
(4.2.6.a)	Left unit axiom
(4.2.6.b)	Right unit axiom

Table 4.1

Functoriality of K follows routinely from the definition of f_2 and f_0. ☐

Let us recall that there is a forgetful functor $U_{lr}\colon \mathbf{Mult}_{lr} \to \mathbf{ShMult}_{lr}$ and the authors of [BL18] construct an equivalence $T\colon \mathbf{Mult}_{lr} \to \mathbf{Skew}_{ln}$. Moreover, comparing the construction of K with that given in [BL18, Section 6.2], we see that the triangle

$$\begin{tikzcd}
\mathbf{Mult}_{lr} \arrow[dd, "T"'] \arrow[dr, "U_{lr}"] & \\
& \mathbf{ShMult}_{lr} \arrow[dl, "K"] \\
\mathbf{Skew}_{ln} &
\end{tikzcd}$$

is commutative.

Theorem 4.4.5. *The functor* $K\colon \mathbf{ShMult}_{lr} \to \mathbf{Skew}_{ln}$ *is an equivalence of categories, as is the forgetful functor* $U_{lr}\colon \mathbf{Mult}_{lr} \to \mathbf{ShMult}_{lr}$.

Proof. Let us show that K is an equivalence first. Since K is fully faithful by the preceding result, it remains to show that it is essentially surjective on objects. Since $T = KU$ and the equivalence T is essentially surjective, so is K, as required. Finally, since $T = KU$ and both T and K are equivalences, so is U_{lr}. ☐

Then, if we consider the forgetful functor $U_{rep}\colon \mathbf{Mult}_{rep} \to \mathbf{ShMult}_{rep}$ and the equivalence $T_{rep}\colon \mathbf{Mult}_{rep} \to \mathbf{Mon}$ given in [Her00], we get the following result.

Theorem 4.4.6. *The equivalence* $K\colon \mathbf{ShMult}_{lr} \to \mathbf{Skew}_{ln}$ *of Theorem 4.4.5 restricts to an equivalence* $K_{rep}\colon \mathbf{ShMult}_{rep} \to \mathbf{Mon}$ *between representable short multicategories and monoidal categories, which fits in the commutative triangle of equivalences*

$$\begin{tikzcd}
\mathbf{Mult}_{rep} \arrow[dr, "U_{rep}"] \arrow[dd, "T_{rep}"'] & \\
& \mathbf{ShMult}_{rep} \arrow[dl, "K_{rep}"] \\
\mathbf{Mon}. &
\end{tikzcd}$$

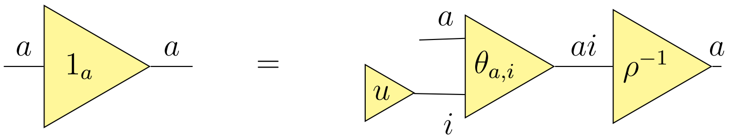
Proof. Let $\mathcal{C} \in \mathbf{ShMult}_{lr}$. If $\mathcal{C}$ is representable then $K\mathcal{C}$ has invertible left unit λ since it is skew left normal. Therefore, we have left to prove that α and ρ are isomorphisms as well. First, we can define the inverse of α through the chain of bijections

$$\begin{array}{ccc}
\mathcal{C}_3(a,b,c;(ab)c) \cong \mathcal{C}_2(a,bc;(ab)c) & \cong & \mathcal{C}_1(a(bc);(ab)c) \\
\theta_{ab,c} \circ_1 \theta_{a,b} & \longmapsto & \alpha^{-1}
\end{array}$$

Using the universal properties of $(ab)c$ and $a(bc)$ we can show that α and α^{-1} are inverses of each other. Then, ρ is defined as $\theta \circ_2 u$, see (4.4.4). We define ρ^{-1} as the map corresponding to 1_a through the following bijection

$$\mathcal{C}_1(ai;a) \cong \mathcal{C}_2(a,i;a) \cong \mathcal{C}_1(a;a),$$

i.e. ρ^{-1} is the unique map such that



(4.4.7)

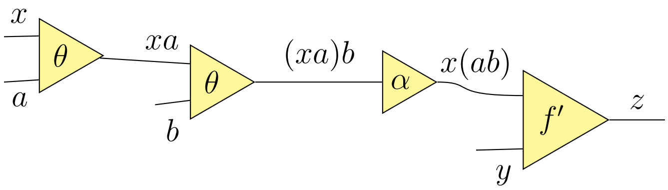
which means that $\rho^{-1}\rho = 1_a$. Using the universal property of ai we can also prove that $\rho\rho^{-1} = 1_{ai}$.

On the other side, if $K\mathcal{C}$ is monoidal, then α and ρ are invertible. Then

$$\mathcal{C}_3(x,ab,y;z) \cong \mathcal{C}_2(x(ab),y;z) \cong \mathcal{C}_2((xa)b,y;z) \cong \mathcal{C}_3(xa,b,y) \cong \mathcal{C}_4(x,a,b,y;z)$$

where the second isomorphism is given by pre-composition with $\alpha_{x,a,b}$ and the rest by left representability. We can see how this isomorphism sends a map $f\colon x, ab, y \to z$ to

$$[\,(f' \circ_1 \alpha) \circ_1 \theta_{xa,b}\,] \circ_1 \theta_{x,a} =$$



which can be proven to be equal to $f \circ_2 \theta_{a,b}$ using the definition of α and associativity equations in $\mathcal{C}$.

$$\begin{aligned}
&[\,(f' \circ_1 \alpha) \circ_1 \theta_{xa,b}\,] \circ_1 \theta_{x,a} \\
&= [\,f' \circ_1 (\alpha \circ \theta_{xa,b})\,] \circ_1 \theta_{x,a} && \text{(by extranat. sub. binary into binary)} \\
&= f' \circ_1 [\,(\alpha \circ \theta_{xa,b}) \circ_1 \theta_{x,a}\,] && \text{(by axiom (4.2.1.a))} \\
&= f' \circ_1 (\,\theta_{x,ab} \circ_2 \theta_{a,b}\,) && \text{(by definition of } \alpha) \\
&= (\,f' \circ_1 \theta_{x,ab}\,) \circ_2 \theta_{a,b} && \text{(by axiom (4.2.1.a))} \\
&= f \circ_2 \theta_{a,b} && \text{(by definition of } f').
\end{aligned}$$

Then, the isomorphisms

$$\mathcal{C}_2(x, ab; z) \cong \mathcal{C}_3(x, a, b; z) \qquad \text{and} \qquad \mathcal{C}_3(x, y, ab; z) \cong \mathcal{C}_4(x, y, a, b; z)$$

are constructed and shown to be induced by pre-composition with $\theta_{a,b}$ in a similar way. Finally, we show how u induces the required isomorphisms for a representable short multicategory. By left representability, the map

$$- \circ u\colon \mathcal{C}_n(i, \overline{a}; z) \to \mathcal{C}_{n-1}(\overline{a}; z)$$

is an isomorphism (for $\overline{a}$ of length $n-1$). Then, since ρ is invertible, we can define the following isomorphism

$$\mathcal{C}_3(a, i, b; z) \cong \mathcal{C}_2(ai, b; z) \cong \mathcal{C}_2(a, b; z)$$

where the last map is given by pre-composition with ρ in the first variable and the first one by left representability. Thus, a ternary map $k\colon a, i, b \to z$ is sent to $k' \circ_1 \rho$.

$$\begin{aligned} k' \circ_1 \rho_a &= k' \circ_1 (\theta_{a,i} \circ_2 u) && \text{(by definition of } \rho) \\ &= (k' \circ_1 \theta_{a,i}) \circ_2 u && \text{(by axiom (4.2.1.b))} \\ &= k \circ_2 u && \text{(by definition of } k'). \end{aligned}$$

Similarly, we can construct the isomorphism $\mathcal{C}_2(a, i; z) \cong \mathcal{C}_1(a; z)$ and prove that it is induced by pre-composition with u.

Since $\mathbf{ShMult}_{rep}$ and $\mathbf{Mon}$ are full subcategories of $\mathbf{ShMult}_{lr}$ and $\mathbf{Skew}_{ln}$, the fully faithfulness of K_{rep} follows from the one of K. Hence, U_{rep} is an equivalence as well. □

The Closed Left Representable Case

In this section we will consider the equivalence $T^{cl}_{lr}\colon \mathbf{Mult}^{cl}_{lr} \to \mathbf{Skew}^{cl}_{ln}$ between left representable closed multicategories and left normal skew monoidal closed categories. The existence of this equivalence follows from [BL18, Theorem 6.4] in the same way as [BL18, Theorem 6.3] follows from [BL18, Theorem 6.1].

Theorem 4.4.7. *The equivalence $K\colon \mathbf{ShMult}_{lr} \to \mathbf{Skew}_{ln}$ restricts to an equivalence $K^{cl}_{lr}\colon \mathbf{ShMult}^{cl}_{lr} \to \mathbf{Skew}^{cl}_{ln}$ between left representable closed short multicategories and left normal skew closed monoidal categories, which fits in the commutative triangle of equivalences*

$$\begin{tikzcd} \mathbf{Mult}^{cl}_{lr} \arrow[dd, "T^{cl}_{lr}"'] \arrow[dr, "U^{cl}_{lr}"] & \\ & \mathbf{ShMult}^{cl}_{lr} \arrow[dl, "K^{cl}_{lr}"] \\ \mathbf{Skew}^{cl}_{ln}. & \end{tikzcd}$$

Proof. Let $\mathcal{C}$ be in $\mathbf{ShMult}_{lr}$. If $\mathcal{C}$ is closed then we have natural isomorphisms

$$\mathbb{C}(ab, c) = \mathcal{C}_1(ab; c) \cong \mathcal{C}_2(a, b; c) \cong \mathcal{C}_1(a, [b, c]) = \mathbb{C}(a, [b, c])$$

so that $K\mathcal{C}$ is monoidal skew closed, as required. If $K\mathcal{C}$ is closed, then we have natural isomorphisms $\mathcal{C}_1(a; [b, c]) \cong \mathcal{C}_1(ab; c)$ for all a, b, c. By Yoneda, the composite

$$\mathcal{C}_1(a; [b, c]) \cong \mathcal{C}_1(ab; c) \cong \mathcal{C}_2(a, b; c)$$

is of the form $e_{b,c} \circ_1 -$ for a binary map $e_{b,c} \colon [b, c], b \to c$, and to show that $\mathcal{C}$ is closed we must prove that the function on the bottom row below is a bijection for tuples $\overline{a}$ of length 0 to 3.

$$\begin{tikzcd}
\mathcal{C}_1(m(\overline{a}); [b, c]) \arrow[r, "e_{b,c}\circ_1 -"] \arrow[d, "-\circ_1\theta_{\overline{a}}"'] & \mathcal{C}_2(m(\overline{a}), b; c) \arrow[d, "-\circ_1\theta_{\overline{a}}"] \\
\mathcal{C}_n(\overline{a}; [b, c]) \arrow[r, "e_{b,c}\circ_1 -"'] & \mathcal{C}_{n+1}(\overline{a}, b; c)
\end{tikzcd}$$

Since $\mathcal{C}$ underlies a left representable multicategory, there exists a left universal multimap $\theta_{\overline{a}} \colon \overline{a} \to m(\overline{a})$ and we have a commutative diagram as above in which the upper horizontal is invertible, as already established, and the two vertical functions by left universality, so that the lower horizontal is a bijection too. □

Putting together Theorem 4.4.6 and Theorem 4.4.7 we get the following result.

Theorem 4.4.8. *The equivalence* $K \colon \mathbf{ShMult}_{lr} \to \mathbf{Skew}_l$*, defined in Theorem 4.4.5, restricts to an equivalence* $K^{cl}_{rep} \colon \mathbf{ShMult}^{cl}_{rep} \to \mathbf{Mon}^{cl}$ *between short representable closed multicategories and closed monoidal categories, which fits in the commutative triangle of equivalences*

$$\begin{tikzcd}
\mathbf{Mult}^{cl}_{rep} \arrow[dr, "U^{cl}_{rep}"] \arrow[dd, "T^{cl}_{rep}"'] & \\
& \mathbf{ShMult}^{cl}_{rep} \arrow[dl, "K^{cl}_{rep}"] \\
\mathbf{Mon}^{cl}. &
\end{tikzcd}$$

4.5. Short Skew Multicategories

In this section we will adapt the definitions in Section 4.2 and the results in Section 4.4 to the skew setting. Once again, we will follow the style of Proposition 4.1.1.

A **short skew multicategory** consists, to begin with, of a category $\mathbb{C}$ together with:

- For $1 \leq n \leq 4$ a functor $\mathcal{C}^t_n(-;-) : (\mathbb{C}^n)^{op} \times \mathbb{C} \to \mathbf{Set}$ such that, when $n = 1$, we have $\mathcal{C}_1(-;-) = \mathbb{C}(-,-) : \mathbb{C}^{op} \times \mathbb{C} \to \mathbf{Set}$.
- For $n = 0, 1, 2$ an additional functor $\mathcal{C}^l_n(-;-) : (\mathbb{C}^n)^{op} \times \mathbb{C} \to \mathbf{Set}$ and dinatural transformation $j_n : \mathcal{C}^t_n(-;-) \to \mathcal{C}^l_n(-;-)$.

Remark 4.5.1. The l-typed multimaps, i.e. the elements of $\mathcal{C}^l_n(\overline{a}; b)$, are thought of as *loose*, whereas the t-typed multimaps, i.e. the elements of $\mathcal{C}^t_n(\overline{a}; b)$, are thought of as *tight*. The function j lets us view tights as loose. Nullary maps are thought of as loose. The tight unary maps are precisely the morphisms of $\mathbb{C}$. Both tight and loose n-ary multimaps f admit compatible precomposition and postcomposition by tight unary maps — that is, by the morphisms g of $\mathbb{C}$ — which we write as $f \circ g$ and $g \circ f$ respectively.

For $x, y \in \{t, l\}$, let us write

$$x \circ_i y = \begin{cases} t, & \text{if } x = y = t \text{ and } i = 1 \\ t, & \text{if } x = t \text{ and } i \neq 1 \\ l, & \text{otherwise} \end{cases}$$

Then in certain cases we require functions

$$- \circ_i - : \mathcal{C}^x_n(\overline{b}; c) \times \mathcal{C}^y_m(\overline{a}; b_i) \longrightarrow \mathcal{C}^{x \circ_i y}_{n+m-1}(b_{<i}, a, b_{>i}; c)$$

for $i \in \{1, \ldots, n\}$ which are natural in each variable $a_1, \ldots, a_m, b_1, \ldots, b_n, c$. Analogous to those from before, we require:

- $m = 2$, $n = 2, 3$ and $x = y = t$ (substitution of tight binary into tight binary/ternary);
- $m = 3$, $n = 2$ and $x = y = t$ (substitution of tight ternary into tight binary);
- $m = 0$, $n = 2, 3$ and $x = t$ (substitution of nullary into tight binary/ternary).

but also:

- $m = 0$, $n = 1$ and $x = y = l$ (substitution of nullary into loose unary);
- $m = 1$, $y = l$, $n = 2$ and $x = t$ (substitution of loose unary into tight binary);
- $m = 2$, $y = t$, $n = 1$ and $x = l$ (substitution of tight binary into loose unary).

In the context of a binary multimap f, and multimaps g and h of arity n and p respectively (all tight except nullary ones), one can consider associativity equations of the form:

$$f \circ_i (g \circ_j h) = (f \circ_i g) \circ h_{j+i-1} \quad \text{for} \quad 1 \leq i \leq m, 1 \leq j \leq n, \tag{4.5.1}$$

$$(f \circ_i g) \circ_j h = (f \circ_j h) \circ_{p+i} g \quad \text{for} \quad 1 \leq i \leq m, j < m - i. \tag{4.5.2}$$

We require these equations in the following cases:

(a) $n = p = 2$;

(b) $n = 2$, $p = 0$;

(c) only for (4.5.2), $n = 0$, $p = 2$;

(d) only for (4.5.2), $n = p = 0$.

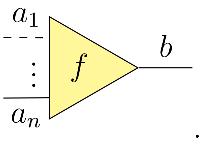
Remark 4.5.2. Let us unfold what naturality of j_n means. Let g be a tight binary map, p and q two tight unary maps and v a nullary map. Naturality of j_n means that the following equations hold:

$$\begin{array}{lll} g \circ_2 jp = g \circ_2 p & g \circ_1 jp = j(g \circ_1 p) & q \circ jp = j(q \circ p) \\ jp \circ g = j(p \circ g) & jp \circ v = p \circ v. & \end{array} \tag{4.5.3}$$

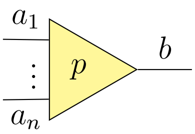
Moreover, if we consider the second and third naturality equation with $p = 1$. We get the following description for jg and jq:

$$jg = j(g \circ_1 1) = g \circ_1 j1$$
$$jq = j(q \circ_1 1) = q \circ_1 j1.$$

Notation. We will denote a tight n-ary multimaps as



and a loose n-ary multimaps as



Definition 4.5.3. Let $\mathcal{C}$ and $\mathcal{D}$ two short skew multicategories. A *morphism of short skew multicategories* is a functor $F\colon \mathcal{C} \to \mathcal{D}$ together with natural families

$$\begin{array}{lll} F_i^t\colon \mathcal{C}_i^t(\overline{a};b) \to \mathcal{D}_i^t(F\overline{a};Fb) & \text{for} & 1 \le i \le 4 \\ F_i^l\colon \mathcal{C}_i^l(\overline{a};b) \to \mathcal{D}_i^l(F\overline{a};Fb) & \text{for} & 0 \le i \le 2 \end{array}$$

such that $F_1^t \equiv F$ (with $F\overline{a}$ we mean the list $Fa_1, ..., Fa_n$). These families must commute with all substitution operators $\circ_i$ and j.

Short skew multicategories and their morphisms form a category **ShSkMult**. Naturally, there is a forgetful functor $U^s\colon$ **SkMult** $\to$ **ShSkMult**.

The Left Representable Case

A **tight binary map classifier** for a and b consists of a representation of $\mathbb{C}_2(a, b; -)\colon \mathbb{C} \to \mathbf{Set}$ – in other words, a tight binary map $\theta_{a,b}\colon a, b \to ab$ for which the induced function

$$- \circ \theta_{a,b} : \mathbb{C}^t_1(ab; c) \to \mathbb{C}^t_n(a, b; c)$$

is a bijection for all c. It is **left universal** if, moreover, the induced function

$$- \circ_1 \theta_{a,b}\colon \mathbb{C}^t_n(ab, \overline{x}; d) \to \mathbb{C}^t_{n+1}(a, b, \overline{x}; d)$$

is a bijection for $n = 2, 3$ and $\overline{x}$ a tuple of the appropriate length. A **nullary map classifier** is a representation of $\mathbb{C}^l_2(-; -)\colon \mathcal{C} \to \mathbf{Set}$ — thus, a certain nullary map $u \in \mathbb{C}^l_2(-; i)$. It is **left universal** if the induced function

$$- \circ_1 u : \mathbb{C}^t_{n+1}(i, \overline{x}; d) \to \mathbb{C}^l_n(\overline{x}; d)$$

is a bijection for each d and tuple $\overline{x}$ of length 1 and 2.

Definition 4.5.4. A short skew multicategory $\mathbb{C}$ is said to be **left representable** if it admits left universal nullary and tight binary map classifiers.

We will denote by $\mathbf{ShSkMult}_{lr}$ the full subcategory of **ShSkMult** with objects left representable short multicategories. Naturally, the forgetful functor $U^s\colon \mathbf{SkMult} \to \mathbf{ShSkMult}$ restricts to a forgetful functor $U^s_{lr}\colon \mathbf{SkMult}_{lr} \to \mathbf{ShSkMult}_{lr}$.

Notation. Let $\mathcal{C}$ be a short skew multicategory with a left universal tight binary and nullary classifier. Then we will use $(-)'\colon \mathcal{C}^t_n(\overline{a}; b) \to \mathcal{C}^t_{n-1}(a_1a_2, a_3, ..., a_n; b)$ for the inverse of $- \circ_1 \theta a_1, a_2$ and $(-)^*\colon \mathcal{C}^l_n(\overline{a}; b) \to \mathcal{C}^t_{n+1}(i, \overline{a}; b)$ for the inverse of $- \circ_1 u$. More precisely, for any tight n-multimap f, f' is the unique tight $(n-1)$-multimap such that $f' \circ_1 \theta = f$ and, for any loose n-multimap q, q^* the unique tight $(n+1)$-multimap such that $q^* \circ_1 u = q$.

We start proving Lemma 4.5.5 which will be useful in the proof of Lemma 4.5.6, which gives a characterisation of morphisms between left representable short skew multicategories.

Lemma 4.5.5. *Let $\mathcal{C}$ and $\mathcal{D}$ be left representable short skew multicategories and $F_0^l \colon \mathcal{C}_0^l(-;a) \to \mathcal{D}_0^l(-;Fa)$ and $F_2^t \colon \mathcal{C}_2^t(a,b;c) \to \mathcal{D}_2^t(Fa,Fb;Fc)$ two natural families. If we define, for any loose unary map q, $F_1^l q := F_2^t q^* \circ_1 F_0^l u$ (where u is the universal nullary map in $\mathcal{C}$ and q^* is the unique binary map such $q^* \circ_1 u = q$), then for any $v \in \mathcal{C}_0^l(-;a)$ and $f \in \mathcal{C}_2^t(a,b;c)$, we have*

$$F_1^l(f \circ_1 v) = F_2^t(f) \circ_1 F_0^l(v).$$

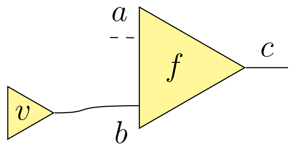
Proof. Let us consider $v \in \mathcal{C}_0^l(-;a)$ and $f \in \mathcal{C}_2^t(a,b;c)$. Then

$$\begin{aligned}
&F_1^l(f \circ_1 v) \\
&= F_2^t(\,(f \circ_1 v)^*\,) \circ_1 F_0^l(u) && \text{(by definition of } F_1^l\text{)} \\
&= F_2^t(\,f \circ_1 v^*\,) \circ_1 F_0^l(u) && \text{(because } (f \circ_1 v)^* = f \circ_1 v^*\text{)} \\
&= (\,F_2^t(f) \circ_1 F(v^*)\,) \circ_1 F_0^l(u) && \text{(by naturality of } F_2^t\text{)} \\
&= F_2^t(f) \circ_1 (\,F(v^*) \circ_1 F_0^l(u)\,) && \text{(by extranat. sub. nullary into tight binary)} \\
&= F_2^t(f) \circ_1 F_0^l(v^* \circ_1 u) = F_2^t(f) \circ_1 F_0^l(v) && \text{(by naturality of } F_0^l\text{).}
\end{aligned}$$

□

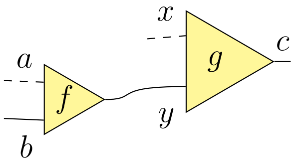
Lemma 4.5.6. *Let $\mathcal{C}$ and $\mathcal{D}$ be left representable short skew multicategories. A morphism $F \colon \mathcal{C} \to \mathcal{D}$ is uniquely specified by:*

- *A functor $F \colon \mathbb{C} \to \mathbb{D}$ (where $\mathbb{C}$ and $\mathbb{D}$ have as maps the tight unary maps of $\mathcal{C}$ and $\mathcal{D}$ respectively).*
- *Natural families $F_0^l \colon \mathcal{C}_0^l(-;a) \to \mathcal{D}_0^l(-;Fa)$ and $F_2^t \colon \mathcal{C}_2^t(a,b;c) \to \mathcal{D}_2^t(Fa,Fb;Fc)$ such that F commutes with*



(4.5.4)

and such that if we define, for any ternary tight map $h \in \mathcal{C}_3^t(\overline{a}; b)$*,* $F_3^t h := F_2^t h' \circ_1 F_2^t \theta$*, then* F *also commutes with*



(4.5.5)

and such that if we define, for any loose unary map q*,* $F_1^l q := F_2^t q^* \circ_1 F_0^l u$*, then, for any tight unary map* p*,*

$$F_1^l j(p) = jF_1^t p. \tag{4.5.6}$$

Proof. First of all, we need to define all of the natural families needed for a morphism in $\mathbf{ShSkMult}_{lr}$. We start with F_0^l, $F_1^t \equiv F$ and F_2^t given and we already defined F_1^l and F_3^t. So, we have left to define F_4^t and F_2^l:

- for $k \in \mathcal{C}_4^t(a, b, c, d; e)$ we define $F_4^t k := F_3^t k' \circ_1 F_2^t \theta$,
- for $r \in \mathcal{C}_2^l(a, b; c)$ we define $F_2^l r := F_3^t r^* \circ_1 F_0^l u$.

Then, we need to prove that these natural families commute with all substitutions, i.e.

(i) Tight binary into tight binary/ternary.

(ii) Tight ternary into tight binary.

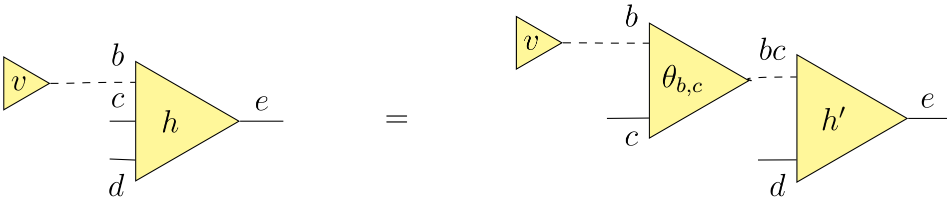
(iii) Nullary into tight binary/ternary.

(iv) Nullary into loose unary.

(v) Loose unary into tight binary.

(vi) Tight binary into loose unary.

Almost all of the first three can be all proved in an analogous way as in Lemma 4.2.5. For instance, to prove that F preserves $g \circ_1 f$ for f and g binary, we use (4.5.5) and naturality of F^t_2 (in tight maps).The only exceptions are the substitution of a nullary map into the first component of a binary/tight ternary. Lemma 4.5.5 proves the nullary into tight binary case. To prove the nullary into ternary case instead, we will assume substitution of loose unary in the first variable of a tight binary, which we will refer to as (v.1). Similarly, we will refer to substitution of loose unary in the second variable of a tight binary with (v.2).



We need to prove that $F^t_3(h) \circ_1 F^l_0(v) = F^l_2(h \circ_1 v)$.

$$
\begin{aligned}
& F^t_3(h) \circ_1 F^l_0(v) \\
&= (\, F^t_2(h') \circ_1 F^t_2(\theta_{b,c})\,) \circ_1 F^l_0(v) && \text{(by definition of } F^t_3) \\
&= F^t_2(h') \circ_1 (\, F^t_2(\theta_{b,c}) \circ_1 F^l_0(v)\,) && \text{(by axiom (4.5.1.b) in } \mathcal{D}) \\
&= F^t_2(h') \circ_1 F^l_1(\theta_{b,c} \circ_1 v) && \text{(by (4.5.4))} \\
&= F^l_2(\, h' \circ_1 (\theta_{b,c} \circ_1 v)\,) && \text{(by (v.1))} \\
&= F^l_2(\, (h' \circ_1 \theta_{b,c}) \circ_1 v\,) = F^l_2(\, h \circ_1 v\,) && \text{(by axiom (4.5.1.b) in } \mathcal{C}).
\end{aligned}
$$

(iv) Let $v \in \mathcal{C}^l_0(-;a)$ and $p \in \mathcal{C}^l_1(a;a')$, then

$$
\begin{aligned}
& F^l_1(p) \circ F^l_0(v) \\
&= (\, F^t_2(p^*) \circ_1 F^l_0(u)\,) \circ F^l_0(v) && \text{(by definition of } F^l_1) \\
&= (\, F^t_2(p^*) \circ_2 F^l_0(v)\,) \circ F^l_0(u) && \text{(by axiom (4.5.2.d) in } \mathcal{D}) \\
&= F^t_1(p^* \circ_2 v) \circ F^l_0(u) && \text{(by (4.5.4.b))} \\
&= F^l_0(\, (p^* \circ_2 v) \circ u\,) && \text{(by naturality of } F^l_0) \\
&= F^l_0(\, (p^* \circ_1 u) \circ v\,) = F^l_0(p \circ v) && \text{(by axiom (4.5.2.d) in } \mathcal{C}).
\end{aligned}
$$

(v.1) Let $p \in \mathcal{C}^l_1(a;a')$ and $f \in \mathcal{C}^t_2(a',b;c)$, then

$$\begin{aligned}
&F^l_2(f \circ_1 p)\\
&= F^t_3(\,(f \circ_1 p)^*\,) \circ_1 F^l_0(u) && \text{(by definition of } F^l_2\text{)}\\
&= F^t_3(\,f \circ_1 p^*\,) \circ_1 F^l_0(u) && \text{(since } (f \circ_1 p)^* = f \circ_1 p^*\text{)}\\
&= (\,F^t_2(f) \circ_1 F^t_2(p^*)\,) \circ_1 F^l_0(u) && \text{(by (i))}\\
&= F^t_2(f) \circ_1 (\,F^t_2(p^*) \circ_1 F^l_0(u)\,) && \text{(by axiom (4.5.1.b) in } \mathcal{C}\text{)}\\
&= F^t_2(f) \circ_1 (\,F^l_1(p^* \circ_1 u)\,) = F^t_2(f) \circ_1 F^l_1(p) && \text{(by axiom (4.5.4.a)).}
\end{aligned}$$

(v.2) Let $q \in \mathcal{C}^l_1(b;b')$ and $f \in \mathcal{C}^t_2(a',b;c)$, then

$$\begin{aligned}
&F^t_2(f) \circ_2 F^l_1(q)\\
&= F^t_2(f) \circ_2 (\,F^t_2(q^*) \circ_1 F^l_0(u)\,) && \text{(by definition of } F^l_1\text{)}\\
&= (\,F^t_2(f) \circ_2 F^t_2(q^*)\,) \circ_2 F^l_0(u) && \text{(by axiom (4.5.1.b))}\\
&= F^t_3(f \circ_2 q^*) \circ_2 F^l_0(u) && \text{(by (i))}\\
&= F^t_2(\,(f \circ_2 q^*) \circ_2 u\,) && \text{(by (iii))}\\
&= F^t_2(\,f \circ_2 (q^* \circ_1 u)\,) = F^t_2(\,f \circ_2 q) && \text{(by axiom (4.5.2.b) in } \mathcal{C}\text{).}
\end{aligned}$$

(vi) Let $f \in \mathcal{C}^t_2(a',b;c)$ and $p \in \mathcal{C}^l_1(c;c')$, then

$$\begin{aligned}
&F^l_2(p \circ f)\\
&= F^t_3(\,(p \circ f)^*\,) \circ_1 F^l_0(u) && \text{(by definition of } F^l_2\text{)}\\
&= F^t_3(\,p^* \circ_2 f\,) \circ_1 F^l_0(u) && \text{(since } (p \circ f)^* = p^* \circ_2 f\text{)}\\
&= (\,F^t_2(p^*) \circ_2 F^t_2(f)\,) \circ_1 F^l_0(u) && \text{(by (i))}\\
&= (\,F^t_2(p^*) \circ_1 F^l_0(u)\,) \circ F^t_2(f) = F^l_1(p) \circ F^t_2(f) && \text{(by axiom (4.5.2.b) in } \mathcal{C}\text{).}
\end{aligned}$$

Finally, let us prove that F commutes with j_1 and j_2. The assumption (4.5.6) is literally commutativity of F with j_1. So, let $f \in \mathcal{C}^t_2(a,b;c)$,

$$F^l_2 jf = F^l_2(\,f \circ_1 j1_a\,) = F^t_2 f \circ_1 F^l_1(j1_a) = F^t_2 f \circ_1 jF^t_1 1_a = F^t_2 f \circ_1 j1_{Fa} = j(F^t_2 f).$$

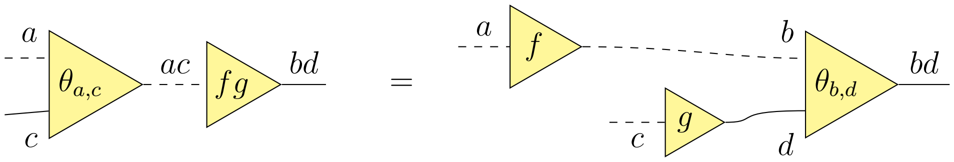
Here we have used naturality of j (in the style of Remark 4.5.2, assumption (4.5.6) and $F^t_1 1_a = 1_{Fa}$ (since F^t_1 is a functor). □

Remark 4.5.7. Let us briefly see what happens to this characterisation when we consider $j = 1$ both in $\mathcal{C}$ and $\mathcal{D}$, i.e. when they are left representable short *multicategories*. We can see that condition (4.5.6) implies that $F_1^t = F_1^l$, which, together with Lemma 4.5.5, gives back equation (4.2.6.a) in Lemma 4.2.5. Furthermore, conditions (4.5.4) and (4.5.5) correspond directly to (4.2.6.b) and (4.2.7), respectively. This shows how Lemma 4.5.6 corresponds to Lemma 4.2.5.

Now, let us recall that there is an equivalence $T^s\colon \mathbf{SkMult}_{lr} \to \mathbf{Skew}$ between left representable skew multicategories and skew monoidal categories [BL18, Theorem 6.1].

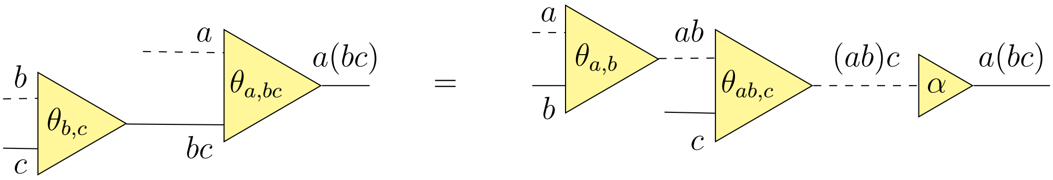
Lemma 4.5.8. *Given a left representable short skew multicategory $\mathcal{C}$ we can construct a skew monoidal category $K^s\mathcal{C}$ in which:*

- *The tensor product ab of two objects a and b is the tight binary map classifier;*
- *The unit i is the nullary map classifier;*
- *Given tight unary maps $f\colon a \to b$ and $g\colon c \to d$, the tensor product $fg\colon ac \to bd$ is the unique morphism such that*



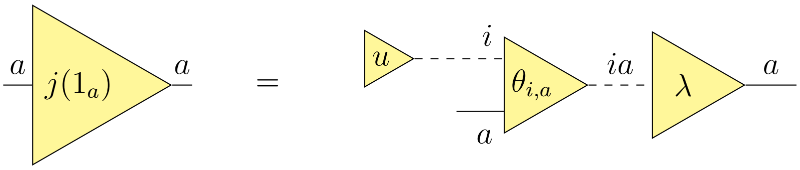
(4.5.7)

- *The associator $\alpha\colon (ab)c \to a(bc)$ is defined as the unique tight map such that*



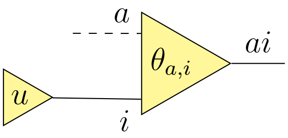
(4.5.8)

- *The left unit map* $\lambda\colon ia \to a$ *is defined as the unique tight unary map such that*



(4.5.9)

- *The right unit map* $\rho\colon a \to ai$ *is the tight unary map defined as*

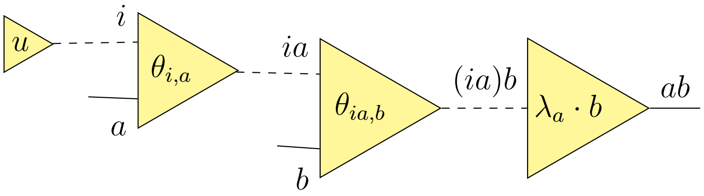


(4.5.10)

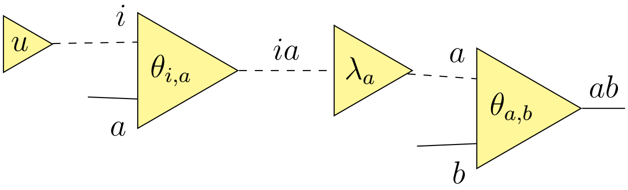
Proof. Functoriality of $\mathbb{C}^2 \to \mathbb{C} : (a,b) \mapsto ab$ follows from the universal property of the tight binary map classifier and profunctoriality of $\mathcal{C}_2^t(-;-)$. It remains to verify the 5 axioms for a skew monoidal category. Some of them have the same proof as short multicategories, we will give details only of the ones where we need to use new axioms. For instance, in the pentagon axiom (4.3.1) all the maps are tight, so the proof does not change (naturality and profunctoriality are defined using tight unary maps). Also the right unit axiom (4.3.3) has the same proof. We will prove the other three cases. Let us consider the axiom (4.3.2), which we recall below,

$$\begin{tikzcd}
(ia)b \arrow[r, "\alpha_{i,a,b}"] \arrow[dr, "\lambda_a b"'] & i(ab) \arrow[d, "\lambda_{ab}"] \\
& ab.
\end{tikzcd}$$

The strategy will be again to prove the equality pre-composing with the universal nullary map u and binary maps θ. So,

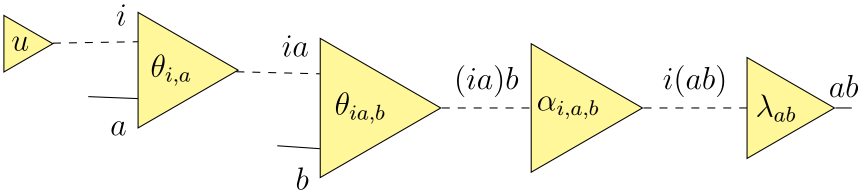


$$
\begin{aligned}
&\lambda_a \cdot b \circ [\,(\,\theta_{ia,b} \circ_1 \theta_{i,a}\,) \circ_1 u\,] \\
&= [\,\lambda_a \cdot b \circ (\,\theta_{ia,b} \circ_1 \theta_{i,a}\,)\,] \circ_1 u && \text{(nat., in tight unary, sub. null into tight bin.)} \\
&= [\,(\,\lambda_a \cdot b \circ \theta_{ia,b}\,) \circ_1 \theta_{i,a}\,] \circ_1 u && \text{(nat. sub. tight bin. into tight bin.)} \\
&= [\,(\,\theta_{a,b} \circ_1 \lambda_a\,) \circ_1 \theta_{i,a}\,] \circ_1 u && \text{(definition of } \lambda_a \cdot b\text{)}
\end{aligned}
$$

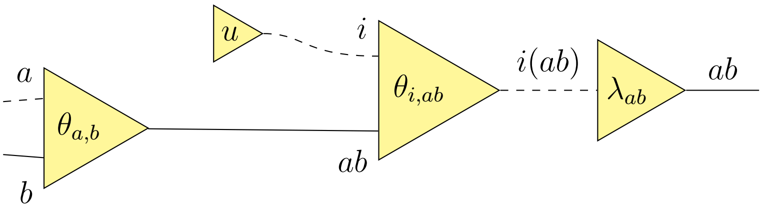


$$
\begin{aligned}
&= [\,\theta_{a,b} \circ_1 (\,\lambda_a \circ_1 \theta_{i,a}\,)\,] \circ_1 u && \text{(extranat. sub tight bin. into tight bin.)} \\
&= \theta_{a,b} \circ_1 [\,(\,\lambda_a \circ_1 \theta_{i,a}\,) \circ_1 u\,] && \text{(by axiom (4.5.1.}b\text{))} \\
&= \theta_{a,b} \circ_1 j1_a && \text{(by definition of } \lambda\text{)} \\
&= j(\theta_{a,b} \circ_1 1_a) = j\theta_{a,b} && \text{(by naturality of } j\text{).}
\end{aligned}
$$

Let us emphasise the last line, which is different from the normal short multicategories case. On the other hand, we start from



and we arrive to



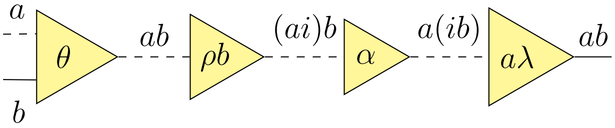
in the same way as the not-skew case. Then from here,

$$\begin{aligned}
&= [\, \lambda_{ab} \circ (\, \theta_{i,ab} \circ_1 u \,) \,] \circ \theta_{a,b} && \text{(nat. of sub. of tight bin. into loose unary)} \\
&= j1_{a,b} \circ \theta_{a,b} && \text{(definition of } \lambda\text{)} \\
&= j(1_{a,b} \circ \theta_{a,b}) = j\theta_{a,b} && \text{(by naturality of } j\text{).}
\end{aligned}$$

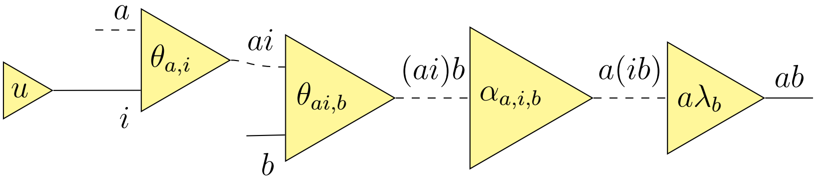
Again, we point out how this part changed from the not-skew case. In this case, in addition to naturality of j, we have to interpret the first line differently to the not-skew case. Here we view it as *naturality of substitution of tight binary into loose unary*, whereas with short multicategories we can see it as *profunctoriality of binary maps*. Then, we need to prove the axiom (4.3.4), which we recall below,

$$\begin{tikzcd}
ab \arrow[r, "\rho\cdot b"] \arrow[drr, equal, "1_{ab}"'] & (ai)b \arrow[r, "\alpha"] & a(ib) \arrow[d, "a\cdot\lambda"] \\
 & & ab.
\end{tikzcd}$$

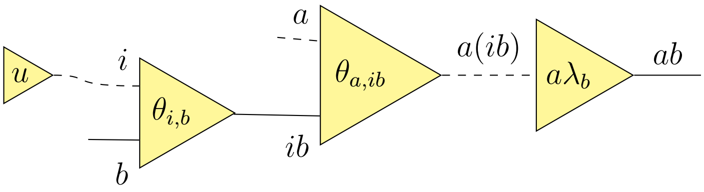
Pre-composing with the universal (tight) binary map we get



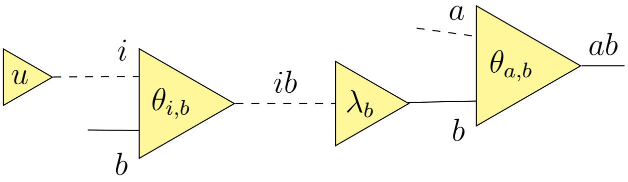
$$\begin{aligned}
&(\, a\lambda_b \circ \alpha_{a,i,b} \circ a\rho_a b \,) \circ \theta_{a,b} \\
&= (\, a\lambda_b \circ \alpha_{a,i,b} \,) \circ (\, a\rho_a b \circ \theta_{a,b} \,) && \text{(profunctoriality of tight bin.)} \\
&= (\, a\lambda_b \circ \alpha_{a,i,b} \,) \circ (\, \theta_{ai,b} \circ_1 \rho_a \,) && \text{(definition of } \rho_a \cdot b\text{)} \\
&= (\, a\lambda_b \circ \alpha_{a,i,b} \,) \circ [\, \theta_{ai,b} \circ_1 (\theta_{a,i} \circ_2 u) \,] && \text{(definition of } \rho_a\text{)}
\end{aligned}$$



$$
\begin{aligned}
&= (\,a\lambda_b \circ \alpha_{a,i,b}\,) \circ [\,(\theta_{ai,b} \circ_1 \theta_{a,i}) \circ_2 u\,] && \text{(by axiom (4.5.1.b))}\\
&= [\,(a\lambda_b \circ \alpha_{a,i,b}) \circ (\theta_{ai,b} \circ_1 \theta_{a,i})\,] \circ_2 u\,] && \text{(nat. sub. null. into tight tern.)}\\
&= [\,a\lambda_b \circ (\,\alpha_{a,i,b} \circ (\theta_{ai,b} \circ_1 \theta_{a,i})\,)\,] \circ_2 u\,] && \text{(profunctoriality tight tern)}\\
&= [\,a\lambda_b \circ (\theta_{a,ib} \circ_2 \theta_{i,b})\,] \circ_2 u\,] && \text{(definition of } \alpha)
\end{aligned}
$$



$$
\begin{aligned}
&= [\,(a\lambda_b \circ \theta_{a,ib}) \circ_2 \theta_{i,b}\,] \circ_2 u && \text{(nat. sub. tight bin. into tight bin.)}\\
&= [\,(\theta_{a,b} \circ_2 \lambda_b) \circ_2 \theta_{i,b}\,] \circ_2 u && \text{(definition of } a\lambda)
\end{aligned}
$$



$$
\begin{aligned}
&= [\,\theta_{a,b} \circ_2 (\lambda_b \circ \theta_{i,b})\,] \circ_2 u && \text{(extranat. sub. tight bin. into tight bin.)}\\
&= \theta_{a,b} \circ_2 [\,(\lambda_b \circ \theta_{i,b}) \circ_2 u\,] && \text{(by axiom (4.5.1.b))}\\
&= \theta_{a,b} \circ_2 [\,\lambda_b \circ (\theta_{i,b} \circ_2 u)\,] && \text{(nat. sub. null. into tight bin.)}\\
&= \theta_{a,b} \circ_2 j(1_b) && \text{(by definition of } \lambda)\\
&= \theta_{a,b} \circ_2 1_b = \theta_{a,b} && \text{(by naturality of } j)^*.
\end{aligned}
$$

Finally, we prove the axiom (4.3.5), which we recall below,

$$\begin{tikzcd}
i \arrow[r, "\rho_i"] \arrow[dr, "1_i"'] & ii \arrow[d, "\lambda_i"] \\
 & i.
\end{tikzcd}$$

More precisely, we have to prove that $(\lambda_i \circ \rho_i) \circ u = u$ (by left representability).

$$
\begin{aligned}
&(\lambda_i \circ \rho_i) \circ u \\
&= \lambda_i \circ (\rho_i \circ u) && \text{(profunctoriality of nullary maps)} \\
&= \lambda_i \circ [\,(\theta_{i,i} \circ_2 u) \circ u\,] && \text{(definition of } \rho) \\
&= \lambda_i \circ [\,(\theta_{i,i} \circ_1 u) \circ u\,] && \text{(by axiom (4.5.2.d))} \\
&= [\,\lambda_i \circ (\theta_{i,i} \circ_1 u)\,] \circ u && \text{(nat. of sub. of nullary into loose unary)}^* \\
&= j(1_i) \circ u && \text{(definition of } \lambda) \\
&= 1_i \circ u = u && \text{(by naturality of } j)^*.
\end{aligned}
$$

The main differences with respect to the not-skew case are underlined with a *. □

Before defining the functor $K^s \colon \mathbf{ShSkMult}_{lr} \to \mathbf{Skew}$, we prove the following easy lemma, which is the counterpart of Lemma 4.4.2.

Lemma 4.5.9. *Consider $\mathcal{C}, \mathcal{D} \in \mathbf{ShSkMult}_{lr}$ and a functor $F \colon C \to D$. There is a bijection between natural families, with $x \in \{t < l\}$,*

$$F^x_{\overline{a},b} \colon \mathcal{C}^x_i(\overline{a}; b) \to \mathcal{D}^x_i(F\overline{a}; Fb)$$

and natural families

$$f^x_{\overline{a}} \colon m^x(F\overline{a}) \to F(m^x\overline{a})$$

where $m^x\overline{a}$ and $m^x(F\overline{a})$ are the n-ary x-map classifiers of the appropriate arity.

Proof. The bijection is governed by the following diagram

$$
\begin{tikzcd}
\mathcal{C}^x_i(\overline{a}; -) \arrow[r, "F^x_{\overline{a},-}"] & \mathcal{D}^x_i(F\overline{a}; F-) \\
\mathcal{C}^t_1(m^x\overline{a}, -) \arrow[u, "-\circ_1\theta^x_{\overline{a}}"] \arrow[r, "F-\circ f^x_{\overline{a}}"'] & \mathcal{D}^t_1(m^x(F\overline{a}), F-) \arrow[u, "-\circ_1\theta^x_{F\overline{a}}"']
\end{tikzcd}
\tag{4.5.11}
$$

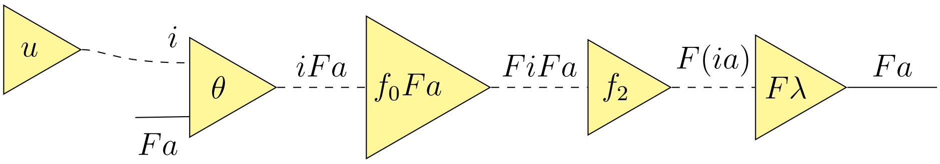
in which the vertical arrows are natural bijections and the lower horizontal arrow corresponds to the upper one using the Yoneda lemma. □

Remark 4.5.10. Given a morphism $F\colon \mathcal{C} \to \mathcal{D} \in \mathbf{ShSkMult}_{lr}$ we obtain, on applying the above lemma, natural families of tight maps $f_2\colon FaFb \to F(ab)$ and $f_0\colon i \to Fi$ defining the *data* for a lax monoidal functor $K^sF\colon K^s\mathcal{C} \to K^s\mathcal{D}$. That it is a lax monoidal functor follows directly from the following result.

Explicitly, $f_2\colon FaFb \to F(ab)$ is the unique morphism such that $f_2 \circ_1 \theta_{Fa,Fb} = F^t_2(\theta_{a,b})$ whilst f_0 is the unique morphism such that $f_0 \circ u = F^l_0 u$.

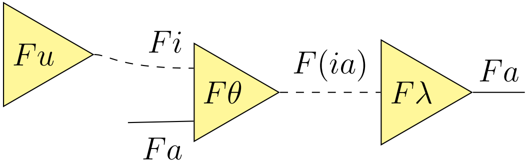
Proposition 4.5.11. *With the definition on objects given in Lemma 4.5.8 and on morphisms in Remark 4.5.10, we obtain a fully faithful functor* $K^s\colon \mathbf{ShSkMult}_{lr} \to \mathbf{Skew}$.

Proof. The proof is quite similar to the proof of Proposition 4.4.4. Using Lemma 4.5.6 and 4.5.9, it is enough to prove that (F^l_0, F^t_2) satisfy the equations (4.5.4, 4.5.5, 4.5.6) if and only if (f_0, f_2) satisfy the equations for a lax monoidal functor (4.3.6, 4.3.7, 4.3.8). Equation (4.5.5) corresponds to the associator axiom (4.3.6) for a lax monoidal functor. Since all maps involved are tight, this follows by the same proof in Proposition 4.4.4. In a similar way, we can prove how (4.5.4) corresponds to the right unit axiom (4.3.8) of a lax monoidal functor. The only part that changes significantly is the one regarding the left unit axiom (4.3.7). We need to change the proof because the substitution of a nullary map into the first variable of a tight binary gives a *loose unary* map. Let us start proving that if (F^l_0, F^t_2) satisfy (4.5.6), then (f_0, f_2) satisfy the left unit axiom (4.3.7). We will prove this axioms showing that $F\lambda \cdot f_2 \cdot f_0Fa$ satisfy the defining property (4.5.9) of λ.



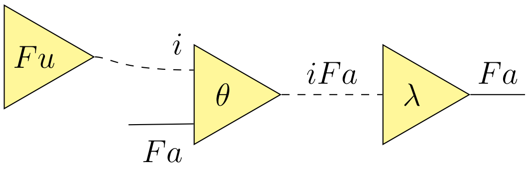
by definition of $f_0 \cdot Fa$, f_0 and f_2

$=$



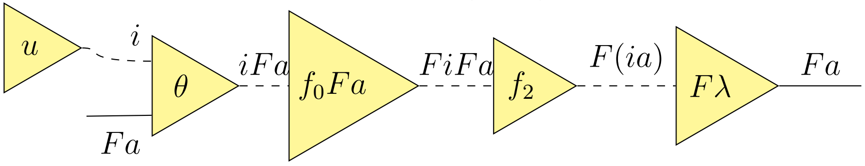
$$
\begin{aligned}
&(\, F\lambda \circ F_2^t\theta \,) \circ_1 F_0^l u \\
&= F_2^t(\lambda \circ \theta) \circ_1 F_0^l u && \text{(by naturality of } F_2^t\text{)} \\
&= F_1^l(\,(\lambda \circ \theta_{i,a}) \circ_1 u\,) && \text{(by Lemma 4.5.5)} \\
&= F_1^l(j1_a) && \text{(by defining property (4.5.9) of } \lambda\text{)} \\
&= j1_{Fa} && \text{(by assumption (4.5.6)).}
\end{aligned}
$$

On the other hand, let us assume (f_0, f_2) satisfy the left unit axiom (4.3.7). Then, by universal property of λ, $j1_{Fa}$ is equal to



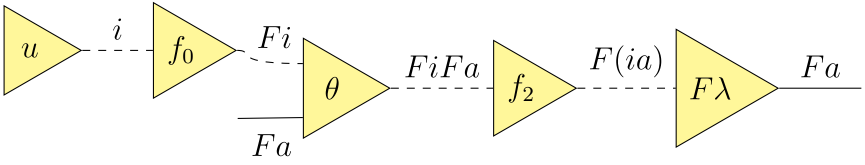
by left unit axiom (4.3.7)

=



by definition of f_0Fb

=



$$
\begin{aligned}
&= (\, F\lambda_a \circ F_2^t\theta_{i,a} \,) \circ_1 F_0^l u && \text{(by definition of } F_0^l \text{ and } F_2^t\text{)} \\
&= F_2^t(\, \lambda \circ \theta_{i,a} \,) \circ_1 F_0^l u && \text{(by naturality of } F_2^t\text{)} \\
&= F_1^l(\,(\lambda \circ \theta_{i,a}) \circ_1 u\,) && \text{(by Lemma 4.5.5)} \\
&= F_1^l(j1_a) && \text{(by defining property (4.5.9) of } \lambda\text{).}
\end{aligned}
$$

Therefore, we get a correspondence analogous to the one in Table 4.1, which is described in Table 4.2. □

(F^l_0, F^t_2)	(f_0, f_2)
(4.5.5)	Associator axiom (4.3.6)
(4.5.6)	Left unit axiom (4.3.7)
(4.5.4)	Right unit axiom (4.3.8)

Table 4.2

We recall that there is a forgetful functor $U^s_{lr} \colon \mathbf{SkMult}_{lr} \to \mathbf{ShSkMult}_{lr}$ and an equivalence $T^s \colon \mathbf{SkMult}_{lr} \to \mathbf{Skew}$ between left representable skew multicategories and skew monoidal categories. Moreover, comparing the construction of K^s with that given in [BL18, Section 6.2], we see that the triangle

$$\begin{tikzcd}
\mathbf{SkMult}_{lr} \arrow[dd, "T^s"'] \arrow[dr, "U^s_{lr}"] & \\
& \mathbf{ShSkMult}_{lr} \arrow[dl, "K^s"] \\
\mathbf{Skew}. &
\end{tikzcd}$$

is commutative.

Theorem 4.5.12. *The functor* $K^s \colon \mathbf{ShSkMult}_{lr} \to \mathbf{Skew}$ *is an equivalence of categories, as is the forgetful functor* $U^s \colon \mathbf{SkMult}_{lr} \to \mathbf{ShSkMult}_{lr}$*.*

Proof. Let us show that K^s is an equivalence first. Since K^s is fully faithful by Proposition 4.5.11, it remains to show that is essentially surjective on objects. Since $T^s = K^s U^s$ and the equivalence T^s is essentially surjective, so is K^s, as required. Finally, since $T^s = K^s U^s$ and both T^s and K^s are equivalences, so is U^s. □

The Closed Left Representable Case

Definition 4.5.13. A short skew multicategory is said to be **closed** if all $b, c \in \mathcal{C}$ there exists an object $[b, c]$ and a tight binary map $e_{b,c} \colon [b, c], b \to c$ for which the induced functions

$$\begin{aligned} e_{b,c} \circ_1 - \colon \mathcal{C}^t_n(\overline{x}; [b,c]) \to \mathcal{C}^t_{n+1}(\overline{x}, b; c), &\qquad \text{for } n = 1, 2, 3, \\ e_{b,c} \circ_1 - \colon \mathcal{C}^l_n(\overline{x}; [b,c]) \to \mathcal{C}^l_{n+1}(\overline{x}, b; c), &\qquad \text{for } n = 0, 1, \end{aligned}$$

are isomorphisms.

Once again, let us notice that the condition on n are determined by the definition of a short skew multicategory. For instance, when dealing with loose maps we only consider $n = 0, 1$ because we do not have ternary loose maps in a short skew multicategory.

We will denote with $\mathbf{ShSkMult}^{cl}_{lr}$ the fullsubcategory of **ShSkMult** with objects left representable closed short skew multicategories. Naturally, the forgetful functor $U^s_{lr} \colon \mathbf{SkMult}_{lr} \to \mathbf{ShSkMult}_{lr}$ restricts to a forgetful functor

$$U^{s,cl}_{lr} \colon \mathbf{SkMult}^{cl}_{lr} \to \mathbf{ShSkMult}^{cl}_{lr}.$$

Adapting Proposition 4.2.7, we get a characterisation of closed short skew multicategories which are also left representable.

Proposition 4.5.14. *A closed short skew multicategory is left representable if and only if it has a nullary map classifier and each $[b, -]$ has a left adjoint.*

Proof. If $\mathcal{C}$ is closed and left representable, then the natural bijections

$$\mathbb{C}(ab, c) = \mathcal{C}^t_1(ab; c) \cong \mathcal{C}^t_2(a, b; c) \cong \mathcal{C}^t_1(a; [b, c]) = \mathbb{C}(a, [b, c])$$

show that $-b \dashv [b, -]$. Conversely, if $[b, -]$ has a left adjoint. then we have natural isomorphisms

$$\mathcal{C}^t_1(ab; c) = \mathbb{C}(ab, c) \cong \mathbb{C}(a, [b, c]) = \mathcal{C}^t_1(a; [b, c]) \cong \mathcal{C}^t_2(a, b; c)$$

and, by Yoneda, the composite is of the form $- \circ_1 \theta_{a,b}$ for a tight binary map classifier $\theta_{a,b} \colon a, b \to ab$. It remains to show that this and the nullary map classifier are left universal. For the tight binary map classifier, we must show that $-\circ\, \theta_{a,b} \colon \mathcal{C}^t_{n+1}(ab, \overline{x}; c) \to \mathcal{C}^t_{n+2}(a, b, \overline{x}; c)$ is a bijection for all $\overline{x}$ of length 1 or 2, the case 0 being known. For an inductive style argument, suppose it is true for $\overline{x}$ of length $i \leq 1$. We should show that the bottom line below is a bijection

$$\begin{tikzcd}
\mathcal{C}^t_{i+1}(ab, \overline{x}; [y, c]) \arrow[r, "-\circ_1\theta_{a,b}"] \arrow[d, "e_{y,c}\circ_1 -"'] & \mathcal{C}^t_{i+2}(a, b, \overline{x}; [y, c]) \arrow[d, "e_{y,c}\circ_1 -"] \\
\mathcal{C}^t_{i+2}(ab, \overline{x}, y; c) \arrow[r, "-\circ_1\theta_{a,b}"'] & \mathcal{C}^t_{i+3}(a, b, \overline{x}, y; c])
\end{tikzcd}$$

but this follows from the fact that the square commutes, by associativity axiom (4.5.1.a), and the other three morphisms are bijections, by assumption. The case of the nullary map classifier is similar in form but uses associativity axiom (4.5.1.b). □

We recall that [BL18, Theorem 6.4] gives an equivalence $T^s_c \colon \mathbf{SkMult}^{cl}_{lr} \to \mathbf{Skew}^{cl}$ between left representable closed skew multicategories and skew closed monoidal categories.

Theorem 4.5.15. *The equivalence* $K^s \colon \mathbf{ShSkMult}_{lr} \to \mathbf{Skew}$ *restricts to an equivalence* $K^s_c \colon \mathbf{ShSkMult}^{cl}_{lr} \to \mathbf{Skew}^{cl}$ *between left representable closed short skew multicategories and skew closed monoidal categories, which fits in the commutative triangle of equivalences*

$$\begin{tikzcd}
\mathbf{SkMult}^{cl}_{lr} \arrow[dd, "T^s_c"'] \arrow[dr, "U^{s,cl}_{lr}"] & \\
& \mathbf{ShSkMult}^{cl}_{lr} \arrow[dl, "K^s_c"] \\
\mathbf{Skew}^{cl}. &
\end{tikzcd}$$

Proof. Let $\mathcal{C}$ be in $\mathbf{ShSkMult}_{lr}$. If $\mathcal{C}$ is closed then we have natural isomorphisms

$$\mathbb{C}(ab, c) = \mathcal{C}^t_1(ab; c) \cong \mathcal{C}^t_2(a, b; c) \cong \mathcal{C}^t_1(a, [b, c]) = \mathbb{C}(a, [b, c])$$

so that $K^s\mathcal{C}$ is monoidal skew closed, as required. If $K^s\mathcal{C}$ is closed, then we have natural isomorphisms $\mathcal{C}^t_1(a; [b, c]) \cong \mathcal{C}^t_1(ab; c)$ for all a, b, c. By Yoneda, the composite

$$\mathcal{C}^t_1(a; [b, c]) \cong \mathcal{C}^t_1(ab; c) \cong \mathcal{C}^t_2(a, b; c)$$

is of the form $e_{b,c} \circ_1 -$ for a tight binary map $e_{b,c} \colon [b, c], b \to c$, and to show that $\mathcal{C}$ is closed we must prove that

$$\begin{aligned} e_{b,c} \circ_1 - \colon \mathcal{C}^t_n(\overline{a}; [b, c]) &\to \mathcal{C}^t_{n+1}(\overline{a}, b; c), && \text{for } n = 2, 3, \\ e_{b,c} \circ_1 - \colon \mathcal{C}^l_n(\overline{a}; [b, c]) &\to \mathcal{C}^l_{n+1}(\overline{a}, b; c), && \text{for } n = 0, 1, \end{aligned}$$

are bijections. For the tight maps case we can consider the diagram

$$\begin{tikzcd} \mathcal{C}^t_1(m^t(\overline{a}); [b, c]) \arrow[r, "e_{b,c}\circ_1 -"] \arrow[d, "-\circ_1\theta_{\overline{a}}"'] & \mathcal{C}^t_2(m^t(\overline{a}), b; c) \arrow[d, "-\circ_1\theta_{\overline{a}}"] \\ \mathcal{C}^t_n(\overline{a}; [b, c]) \arrow[r, "e_{b,c}\circ_1 -"'] & \mathcal{C}^t_{n+1}(\overline{a}, b; c) \end{tikzcd}$$

where $\theta_{\overline{a}} \colon \overline{a} \to m^t(\overline{a})$ is the left universal tight n-multimap. More precisely,

$$\begin{aligned} &\text{for } n = 2, \quad m^t(a_1, a_2) := a_1a_2, && \text{and} \quad \theta_{\overline{a}} := \theta_{a_1,a_2}, \\ &\text{for } n = 3, \quad m^t(a_1, a_2, a_3) := (a_1a_2)a_3, && \text{and} \quad \theta_{\overline{a}} := \theta_{a_1a_2,a_3} \circ_1 \theta_{a_1,a_2}. \end{aligned}$$

The commutativity of the diagram with $n = 2$ follows from extranaturality of substitution of tight binary into tight ternary, whereas the one with $n = 3$ follows from the associativity axiom (4.5.1.a). Thus, since the two vertical functions are invertible by left representability and the upper horizontal by construction, the lower horizontal is invertible as well. Similarly, for the loose case we consider the diagram

$$\begin{tikzcd}
\mathcal{C}^t_1(m^l(\overline{a});[b,c]) \arrow[r, "e_{b,c}\circ_1 -"] \arrow[d, "-\circ_1\theta^l_{\overline{a}}"'] & \mathcal{C}^t_2(m^l(\overline{a}),b;c) \arrow[d, "-\circ_1\theta^l_{\overline{a}}"] \\
\mathcal{C}^l_n(\overline{a};[b,c]) \arrow[r, "e_{b,c}\circ_1 -"'] & \mathcal{C}^l_{n+1}(\overline{a},b;c)
\end{tikzcd}$$

where $\theta^l_{\overline{a}} \colon \overline{a} \to m(\overline{a})$ is the left universal loose n-multimap, i.e.

$$\text{for } n=0, \quad m^l(-) := i, \quad \text{and} \quad \theta^l_- := u,$$
$$\text{for } n=1, \quad m^l(a) := a, \quad \text{and} \quad \theta^l_a := \theta_{i,a} \circ_1 u.$$

This time, when $n = 0$, the commutativity of the diagram follows from extranaturality of substitution of nullary into tight binary, whereas when $n = 1$ follows from the associativity axiom (4.5.1.b). Then, since the other three maps are invertible, the lower horizontal map is an isomorphism. □

Conclusions

We conclude by summarising what we have done in this thesis and outlining some possible directions for future work.

In Chapter 1 we have answered a question raised in [Lac00] by showing that for every Gray-category $\mathscr{K}$, there is a Gray-category $\mathbf{Psm}(\mathscr{K})$ of pseudomonads, pseudomonad morphisms, pseudomonad transformations and pseudomonad modifications in $\mathscr{K}$ (Theorem 1.2.5). Then, in Chapter 2 we contributed to the formal theory of pseudomonads with Theorem 2.1.4, Proposition 2.2.4 and Theorem 2.2.5. A natural question that arises is whether the construction of $\mathbf{Psm}(-)$ extends easily to tricategories, i.e. if given a tricategory $\mathfrak{T}$ then $\mathbf{Psm}(\mathfrak{T})$ is a tricategory as well. The coherence theorem for tricategories could be helpful. We leave this to future work.

We also introduced a possible approach to the formal theory of relative monads in Chapter 3 and proved a counterpart of Beck's equivalence for relative distributive laws in Theorem 3.6.19. We believe that our definition of the 2-category of relative monads in any 2-category $\mathcal{K}$ will be useful in the future, as there have been already some works presented in seminars and conferences using this notion (see *The Nerve of a Relative Monad* by C. Walker at Masaryk University Algebra Seminar and *The formal theory of theories* by N. Arkor at CT 2021). Nonetheless, there are a lot of aspects that have not yet been investigated. For instance, one could study more in detail relative adjunctions in a 2-category. Another interesting direction to take is to put together Chapter 1, Chapter 2 and Chapter 3 to continue the study of relative pseudomonads started in [FGHW17].

Finally, in Chapter 4 we showed how we can construct different kinds of representable (skew) multicategories starting from a structure involving only multimaps of dimension at most 4. We needed 4-ary multimaps to encode the pentagon axiom in a skew monoidal category. For this reason, we believe that it would be impossible to prove such a result for fewer dimensions, even though we do not have a proof yet. It should be possible to prove similar results for the closed and biclosed setting and we intend to investigate this work soon. Another interesting case to consider is the one of braided and symmetric multicategories. We believe that using the results in [BL20, Appendix A] we will be able to get a finite description of braided and symmetric skew multicategories. More precisely, this would mean that to give a braided skew multicategory it will be enough to have four isomorphisms satisfying three equations.

A. Pseudomonads Diagrams

A.1. Coherence Conditions for Pseudodistributive Laws

We limit ourselves to drawing the boundaries of these diagrams and explain in text which 3-cells should be inserted in them, except for the 3-cells coming from the structure of a Gray-category of $\mathscr{K}$.

$$\begin{tikzcd}
S^3T \arrow[r, "S^2d"] \arrow[ddr, "SmT"] \arrow[ddd, swap, "mST"] & S^2TS \arrow[dr, "SdS"] & & & & \\
& & STS^2 \arrow[dr, "STm"] & & & \\
& S^2T \arrow[rr, "Sd"] \arrow[ddd, swap, "mT"] & & STS \arrow[dd, "dS"] & & = \\
S^2T \arrow[ddr, swap, "mT"] & & & & & \\
& & & TS^2 \arrow[d, "Tm"] & & \\
& ST \arrow[rr, swap, "d"] & & TS & &
\end{tikzcd}$$

$$\begin{tikzcd}
S^3T \arrow[r, "S^2d"] \arrow[ddd, "mST"'] & S^2TS \arrow[ddd, "mTS"'] \arrow[dr, "SdS"] & & \\
 & & S^2T \arrow[d, "dS^2"'] \arrow[dr, "STm"] & \\
 & & TS^3 \arrow[ddd, "TmS"'] \arrow[ddr, "TSm"] & STS \arrow[dd, "dS"] \\
S^2T \arrow[r, "Sd"] \arrow[dddrr, "mT"'] & STS \arrow[ddr, "dS"'] & & \\
 & & & TS^2 \arrow[dd, "Tm"] \\
 & & TS^2 \arrow[dr, "Tm"] & \\
 & & ST \arrow[r, "d"'] & TS
\end{tikzcd} \tag{C1}$$

In (C1), the left-hand side pasting is obtained using $S\bar{m}$, $\bar{m}$, and the associativity 3-cell of the pseudomonad S; the right-hand side pasting is obtained using the associativity 3-cell of the pseudomonad S and $\bar{m}$.

$$\begin{tikzcd}
ST \arrow[dr, "SsT"] \arrow[drr, bend left, "STs"] \arrow[dddr, bend right, "1_{ST}"'] & & \\
 & S^2T \arrow[r, "Sd"] \arrow[dd, "mT"] & STS \arrow[d, "dS"] \\
 & & TS^2 \arrow[d, "Tm"] \\
 & ST \arrow[r, "d"'] & TS
\end{tikzcd}
=
\begin{tikzcd}
ST \arrow[d, "d"'] \arrow[dr, bend left, "STs"] & \\
TS \arrow[dr, "TSs"] \arrow[ddr, bend right, "Id"'] & STS \arrow[d, "dS"] \\
 & TS^2 \arrow[d, "Tm"] \\
 & TS
\end{tikzcd} \tag{C2}$$

In (C2), the left-hand side pasting is obtained using $S\bar{s}$, $\bar{m}$, and the left unit 3-cell of the pseudomonad S; the right-hand side pasting is obtained using the left unit 3-cell of the pseudomonad S.

$$\begin{tikzcd}
S^2 \arrow[r, "S^2t"] \arrow[ddd, "m"'] \arrow[drr, bend right, "StS"'] \arrow[ddrr, bend right, "tS^2"'] & S^2T \arrow[dr, "Sd"] & & & S^2 \arrow[r, "S^2t"] \arrow[ddd, "m"'] & S^2T \arrow[dr, "Sd"] \arrow[ddd, "mT"'] & \\
 & & STS \arrow[d, "dS"] & = & & & STS \arrow[d, "dS"] \\
 & & TS^2 \arrow[dd, "Tm"] & & & & TS^2 \arrow[dd, "Tm"] \\
S \arrow[drr, bend right, "tS"'] & & & & S \arrow[r, "St"] \arrow[drr, bend right, "tS"'] & ST \arrow[dr, "d"] & \\
 & & TS & & & & TS
\end{tikzcd} \tag{C3}$$

For (C3), the left-hand side pasting is obtained using $S\bar{t}$, $\bar{t}S$; the right-hand side is obtained using $\bar{m}$ and $\bar{m}$.

$$\begin{tikzcd}
1_X \arrow[r, "t"] \arrow[d, "s"'] & T \arrow[d, "sT"] \arrow[ddr, bend left, "Ts"] & & & 1_X \arrow[r, "t"] \arrow[d, "s"'] & T \arrow[ddr, bend left, "Ts"] & \\
S \arrow[r, "St"] \arrow[drr, bend right, "tS"'] & ST \arrow[dr, "d"] & & = & S \arrow[drr, bend right, "tS"'] & & \\
 & & TS & & & & TS
\end{tikzcd} \tag{C4}$$

For (C4), the left-hand side pasting is obtained using $\bar{s}$ and $\bar{t}$; the right-hand side is obtained from pseudonaturality of t.

$$\begin{tikzcd}
S^2T^2 \arrow[rr, "S^2n"] \arrow[dr, "SdT"] \arrow[ddddd, "mT^2"'] & & S^2T \arrow[ddr, "Sd"] & \\
& STST \arrow[dr, "STd"] \arrow[dd, "dST"'] & & \\
& & ST^2S \arrow[r, "SnS"] \arrow[dd, "dTS"] & STS \arrow[dddd, "dS"] \\
& TS^2T \arrow[dr, "TSd"] \arrow[dddd, "TmT"'] & & \\
& & TSTS \arrow[dd, "TdS"] & \\
ST^2 \arrow[ddr, "dT"'] & & & \\
& & T^2S^2 \arrow[r, "nS^2"] \arrow[dd, "T^2m"] & TS^2 \arrow[dd, "Tm"] \\
& TST \arrow[dr, "Td"'] & & \\
& & T^2S \arrow[r, "nS"'] & TS
\end{tikzcd}
=
\begin{tikzcd}
S^2T^2 \arrow[rr, "S^2n"] \arrow[dd, "mT^2"'] & & S^2T \arrow[dd, "mT"'] \arrow[dr, "Sd"] & \\
& & & STS \arrow[dd, "dS"] \\
ST^2 \arrow[rr, "Sn"] \arrow[ddr, "dT"'] & & ST \arrow[dddr, "d"] & \\
& & & TS^2 \arrow[dd, "Tm"] \\
& TST \arrow[dr, "Td"'] & & \\
& & T^2S \arrow[r, "nS"'] & TS
\end{tikzcd}
\tag{C5}$$

For (C5), the left-hand side pasting is obtained using $S\bar{n}$, $\bar{n}S$ and $\bar{m}T$; the right-hand side pasting is obtained using $\bar{m}$ and $\bar{n}$.

$$
\begin{tikzcd}
T^2 \arrow[r, "n"] \arrow[d, "sT^2"'] & & T \arrow[d, "sT"] \arrow[dddr, bend left, "Ts"] & \\
ST^2 \arrow[rr, "Sn"] \arrow[dr, "dT"'] & & ST \arrow[ddr, "d"] & \\
& TST \arrow[dr, "Td"] & & \\
& & T^2S \arrow[r, "nS"] & TS
\end{tikzcd}
=
\begin{tikzcd}
T^2 \arrow[rr, "n"] \arrow[d, "sT^2"'] \arrow[ddr, bend left, "TsT"] \arrow[dddrr, bend left, "T^2s"] & & T \arrow[dddr, bend left, "Ts"] & \\
ST^2 \arrow[dr, "dT"'] & & & \\
& TST \arrow[dr, "Td"'] & & \\
& & T^2S \arrow[r, "nS"'] & TS
\end{tikzcd}
\tag{C6}
$$

In (C6), the left-hand side pasting is obtained using $\bar{s}$ and $\bar{n}$; the right-hand side pasting is obtained using $\bar{s}T$.

$$
\begin{tikzcd}
& ST^2 \arrow[dr, bend left, "Sn"] & \\
ST^3 \arrow[ur, bend left, "STn"] \arrow[dr, bend right, "SnT"'] \arrow[dd, "dT^2"'] & & ST \arrow[dddddd, "d"] \\
& S^2T \arrow[ur, bend right, "Sn"'] \arrow[dddd, "dT"] & \\
TST^2 \arrow[dd, "TdT"'] & & \\
& & \\
T^2ST \arrow[dr, bend right, "nST"'] \arrow[dd, "T^2d"'] & & \\
& TST \arrow[dd, "Td"] & \\
T^3S \arrow[dr, bend right, "nTS"'] & & TS \\
& T^2S \arrow[ur, bend right, "nS"'] &
\end{tikzcd}
=
\begin{tikzcd}
& ST^2 \arrow[dr, bend left, "Sn"] \arrow[dd, "dT"] & \\
ST^3 \arrow[ur, bend left, "STn"] \arrow[dd, "dT^2"'] & & ST \arrow[dddddd, "d"] \\
& TST \arrow[dddd, "Td"] & \\
TST^2 \arrow[ur, bend left, "TSn"] \arrow[dd, "TdT"'] & & \\
& & \\
T^2ST \arrow[dd, "T^2d"'] & & \\
& T^2S \arrow[dr, bend left, "nS"] & \\
T^3S \arrow[ur, bend left, "TnS"] \arrow[dr, bend right, "nTS"'] & & TS \\
& T^2S \arrow[ur, bend right, "nS"'] &
\end{tikzcd}
\tag{C7}
$$

For (C7), the left-hand side pasting is obtained using the associativity 3-cell of the pseudomonad T, $\bar{n}$ and $\bar{n}T$; the right-hand side pasting is obtained using $T\bar{n}$, $\bar{n}$ and the associativity 3-cell of the pseudomonad T.

$$\begin{tikzcd}
ST \arrow[rr, bend left, "1_{ST}"] \arrow[dr, "StT"] \arrow[dddd, "d"'] \arrow[ddr, "tST"] & & ST \arrow[dddd, "d"] \\
& ST^2 \arrow[ur, "Sn"] \arrow[d, "dT"] & \\
& TST \arrow[dd, "Td"] & \\
& & \\
TS \arrow[dr, "tTS"'] & & TS \\
& T^2S \arrow[ur, "nS"'] &
\end{tikzcd}
=
\begin{tikzcd}
ST \arrow[dddd, "d"'] & & \\
& & \\
& & \\
& & \\
TS \arrow[rr, bend left, "1_{TS}"] \arrow[dr, "tTS"'] & & TS \\
& T^2S \arrow[ur, "nS"'] &
\end{tikzcd}
\tag{C8}$$

For (C8), the left-hand side pasting is obtained using the right unit 3-cell of the pseudomonad T, $\bar{n}$, $\bar{t}S$; the right-hand side pasting is the right unit 3-cell of the pseudomonad T.

$$\begin{tikzcd}
ST \arrow[r, "d"] \arrow[d, "sST"'] & TS \arrow[d, "sTS"'] \arrow[ddr, "TsS"] \arrow[rrddd, bend left, "1_{TS}"] & & \\
S^2T \arrow[r, "Sd"] \arrow[ddrr, "mT"'] & STS \arrow[dr, "dS"] & & \\
& & TS^2 \arrow[dr, "Tm"] & \\
& & ST \arrow[r, "d"'] & TS
\end{tikzcd}
=
\begin{tikzcd}
ST \arrow[d, "sST"'] \arrow[ddr, bend left, "1_{ST}"] & & \\
S^2T \arrow[dr, "mT"'] & & \\
& ST \arrow[r, "d"'] & TS
\end{tikzcd}
\tag{C9}$$

For (C9), the left-hand side pasting is obtained using the right unit 3-cell of the pseudomonad S, $\bar{s}S$ and $\bar{m}$; the right-hand side pasting is obtained using the right unit 3-cell of the pseudomonad S.

$$
\begin{tikzcd}
& ST^2 \arrow[dr, bend left, "Sn"] & \\
ST \arrow[ur, bend left, "STt"] \arrow[rr, bend right, "1_{ST}"'] & & ST \arrow[dddd, "d"] \\
& & \\
& & \\
& & \\
& & TS
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
& ST^2 \arrow[dr, bend left, "Sn"] \arrow[dd, "dT"'] & \\
ST \arrow[ur, bend left, "STt"] \arrow[dddd, "d"'] & & ST \arrow[dddd, "d"] \\
& TST \arrow[dd, "Td"'] & \\
& & \\
& T^2S \arrow[dr, bend left, "nS"] & \\
TS \arrow[uuur, bend left, "TSt"] \arrow[ur, bend left, "TtS"'] \arrow[rr, bend right, "1_{TS}"'] & & TS
\end{tikzcd}
\tag{C10}
$$

For (C10), the left-hand side pasting uses the left unit 3-cell of the pseudomonad T. The right-hand side pasting is obtained using $\bar{n}$ and the left unit 3-cell of the pseudomonad T.

A.2. Some Technical Proofs

Coherence diagrams for $(GF,\, G\phi \cdot \psi F)$

We show only equation in (1.2.2). Using the coherence diagram (1.2.2) for $(G,\, \psi)$ and for $(F,\, \phi)$, it suffices to prove that the following two diagrams are equal:

$$\begin{tikzcd}
QGF \arrow[rrrd, bend left, "QGFs"] \arrow[rrd, "QGtF"] \arrow[dd, "\psi F"'] & & \Downarrow QG\widetilde{\phi} & \\
& \Downarrow \psi_{tF} & QGTF \arrow[r, "QG\phi"] \arrow[d, "\psi TF"'] \arrow[rd, phantom, "\psi_{\phi}\Downarrow"] & QGFS \arrow[d, "\psi FS"] \\
GTF \arrow[rr, "GTtF"] \arrow[rrdd, bend right, "1_{GTF}"'] & & GT^2F \arrow[r, "GT\phi"'] \arrow[dd, "GnF"'] \arrow[rdd, phantom, "G\bar{\phi}\Downarrow"] & GTFS \arrow[d, "G\phi S"] \\
& \Downarrow G\eta_T F & & GFS^2 \arrow[d, "GFm"] \\
& & GTF \arrow[r, "G\phi"'] & GFS
\end{tikzcd}$$

$$\begin{tikzcd}
QGF \arrow[rrd, bend left, "QGFs"] \arrow[dd, "\psi F"'] & & \\
& \Downarrow \psi F_s & QGFS \arrow[dd, "\psi FS"] \\
GTF \arrow[rd, "GTtF"'] \arrow[rrd, bend left, "GTFs"] \arrow[rddd, bend right, "1_GTF"'] & \Downarrow GT\widetilde{\phi} & \\
\Downarrow G\eta_T F & GT^2F \arrow[r, "GT\phi"'] \arrow[dd, "GnF"'] \arrow[rd, phantom, "\Downarrow G\bar{\phi}"] & GTFS \arrow[d, "G\phi S"] \\
& & GFS^2 \arrow[d, "GFm"] \\
& GTF \arrow[r, "G\psi"'] & GFS.
\end{tikzcd}$$

This equality holds using (1.1.1) and (1.1.2).

F_ϕ is well-defined

Given a pseudomonad transformation $(q, \bar{q})\colon (G, \psi) \to (G', \psi')$ in $P_{\mathscr{K}}(\,(Y, T), (Z, Q)\,)$ we want to show that $(qF, \overline{qF})$ is a pseudomonad transformation as well. We will show just equation (1.2.3), since (1.2.4) can be proved similarly. The required equality follows from equation (1.2.3) for q and the equation below, which can be proved using (1.1.1) twice.

$$\begin{tikzcd}
QGTF \arrow[rr, "QqTF"] \arrow[dr, "QG\phi"] \arrow[dd, "\psi TF"'] & & QG'TF \arrow[dr, "QG'\phi"] \arrow[dl, phantom, "\Rightarrow Qq_\phi{}^{-1}"] & \\
& QGFS \arrow[rr, "QqFS"] \arrow[dd, "\psi FS"] \arrow[ddrr, phantom, "\Downarrow \bar{q}FS"] & & QG'FS \arrow[dd, "\psi' FS"] \\
GT^2F \arrow[ur, phantom, "\psi_\phi \Downarrow"] \arrow[dr, "GT\phi"'] \arrow[ddd, "GnF"'] \arrow[ddr, phantom, "G\bar{\phi} \Downarrow"] & & & \\
& GTFS \arrow[rr, "qTFS"] \arrow[d, "G\phi S"] \arrow[drr, phantom, "\Downarrow q_{\phi S}{}^{-1}"] & & G'TFS \arrow[d, "G'\phi S"] \\
& GFS^2 \arrow[rr, "qFS^2"] \arrow[dd, "GFm"] \arrow[ddrr, phantom, "\Downarrow q_{Fm}{}^{-1}"] & & G'FS^2 \arrow[dd, "G'Fm"] \\
GTF \arrow[dr, "G\phi"'] & & & \\
& GFS \arrow[rr, "qFS"'] & & G'FS
\end{tikzcd}$$

$$=$$

$$\begin{tikzcd}
 & QG'TF \arrow[rr, "QqTF"] \arrow[dd, "\psi TF"'] \arrow[ddrr, phantom, "\bar{q}TF \Downarrow"] & & QG'TF \arrow[dr, "QG'\phi"] \arrow[dd, "\psi' TF"'] \arrow[ddr, phantom, "\psi'_{\phi} \Downarrow"] & \\
 & & & & QG'FS \arrow[dd, "\psi' FS"] \\
= & GT^2F \arrow[rr, "qT^2F"'] \arrow[dddd, "GnF"'] \arrow[ddddrr, phantom, "q_{nF} \Downarrow"] & & G'T^2F \arrow[dr, "G'T\phi"'] \arrow[dddd, "G'nF"'] \arrow[ddddr, phantom, "G'\bar{\phi} \Downarrow"] & \\
 & & & & G'TFS \arrow[d, "G'\phi S"] \\
 & & & & G'FS^2 \arrow[dd, "G'Fm"] \\
 & & & & \\
 & GTF \arrow[rr, "qTF"] \arrow[dr, "G\phi"'] \arrow[drr, phantom, "\Searrow q_{\phi}{}^{-1}"] & & G'TF \arrow[ddr, "G'\phi"] & \\
 & & GFS \arrow[rr, "qFS"'] & & G'FS
\end{tikzcd}$$

Let

$$(q, \bar{q}), (q', \bar{q'}) : (G, \psi) \to (G', \psi')$$

be pseudomonads transformations in $P_{\mathscr{K}}(\,(Y, T), (Z, Q)\,)$. Given a pseudomonad modification $\beta : (q, \bar{q}) \to (q', \bar{q'})$ we want to show that βF is a pseudomonad modification from $(qF, \bar{qF})$ to $(q'F, \bar{q'F})$. So we need to show the following equation

$$\begin{tikzcd}
QGF \arrow[rr, bend left, "QqF"] \arrow[rr, bend right, "Qq'F"'] \arrow[dd, "\psi F"'] & \Downarrow Q\beta F & QG'F \arrow[dd, "\psi' F"] & & QGF \arrow[rr, bend left, "QqF"] \arrow[dd, "\psi F"'] & & QG'F \arrow[dd, "\psi' F"] \\
& \Downarrow \bar{q}'F & & & & \Downarrow \bar{q}F & \\
GTF \arrow[rr, bend right, "q'TF"] \arrow[dd, "G\phi"'] & & G'TF \arrow[dd, "G'\phi"] & = & GTF \arrow[rr, bend left, "qTF"] \arrow[dd, "G\phi"'] & & G'TF \arrow[dd, "G'\phi"] \\
& \Downarrow {q'_\phi}^{-1} & & & & \Downarrow {q_\phi}^{-1} & \\
GFS \arrow[rr, bend right, "q'FS"'] & & G'FS & & GFS \arrow[rr, bend left, "qFS"] \arrow[rr, bend right, "q'FS"'] & \Downarrow \beta FS & G'FS.
\end{tikzcd}$$

This can be shown to hold using the coherence axiom for β and (1.1.3).

Coherence for q_p

Given $(p, \bar{p}) : (F,\, \phi) \to (F',\, \phi')$ and $(q, \bar{q}) : (G,\, \psi) \to (G',\, \psi')$ two 2-cells in $P_{\mathscr{K}}$ we want to prove that $q_p : G'p \cdot qF \to qF' \cdot Gp$ is a 3-cell in $P_{\mathscr{K}}$. First of all, it is useful to note that

$$\begin{tikzcd}
& & G'TF \arrow[drr, "G'Tp"] \arrow[dd, "G'\phi"] & & & & & & G'TF \arrow[drr, "G'Tp"] & & \\
GTF \arrow[urr, "qTF"] \arrow[dd, "G\phi"'] & \Downarrow {q_\phi}^{-1} & & \Downarrow G'\bar{p} & G'TF' \arrow[dd, "G'\phi'"] & & GTF \arrow[urr, "qTF"] \arrow[drr, "GTp"'] \arrow[dd, "G\phi"'] & & \Downarrow {q_{Tp}}^{-1} = {(qT)_p}^{-1} & & G'TF' \arrow[dd, "G'\phi'"] \\
& & G'FS \arrow[drr, "G'pS"] & & & = & & & GTF' \arrow[urr, "qTF'"'] \arrow[dd, "G\phi'"] & & \\
GFS \arrow[urr, "qFS"] \arrow[drr, "GpS"'] & & \Downarrow {q_{pS}}^{-1} & & G'F'S & & GFS \arrow[drr, "GpS"'] & \Downarrow G\bar{p} & & \Downarrow {q_{\phi'}}^{-1} & G'F'S. \\
& & GF'S \arrow[urr, "qF'S"'] & & & & & & GF'S \arrow[urr, "qF'S"'] & &
\end{tikzcd}$$

This equality is true since both diagrams are equal to the following one, using (1.1.3) for the one on the left-hand side and (1.1.1) for the one on the right-hand side.

$$\begin{tikzcd}
GTF \arrow[r, "qTF"] \arrow[d, "G(pS\cdot\phi)"'] \arrow[dr, phantom, "{q_{(pS\cdot\phi)}}^{-1} \Swarrow"] & G'TF \arrow[d, bend right=40, "G'(pS\cdot\phi)"'] \arrow[d, bend left=40, "G'(\phi'\cdot Tp)"] \arrow[d, phantom, "\overset{\Longleftarrow}{G'\bar{p}}"] \\
GF'S \arrow[r, "g'F"'] & G'F'S
\end{tikzcd}$$

The proof can be concluded using the invertibility of the 3-cells involved and (1.1.1).

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