



# Group field theory cosmology

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# Abstract

This thesis explores the cosmological sector of group field theory (GFT) which is a proposal for a theory of quantum gravity. We provide an overview of GFT and introduce the framework necessary to study cosmology in the context of GFT. We discuss two approaches to GFT cosmology in detail.

Firstly, we study the canonical approach where one implements the canonical quantisation program for a GFT coupled to one or more massless scalar fields. The main result is that we are able to derive equations for the volume which take the same functional form as the Friedmann equations from classical cosmology in certain limits.

Secondly, we employ a certain method of studying quantum systems with constraints in the context of GFT cosmology. We accomplish this by identifying a subset of one-body operators of the GFT with the classical observables. The resulting dynamics allow us to identify one of the operators with the extrinsic curvature of classical cosmology.



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# Chapter 1.

## Introduction

In the beginning of the twentieth century theoretical physics experienced two revolutions, both of which radically changed our understanding of reality. Firstly, the discovery of quantum theory was able to explain several phenomena that cannot be explained by classical physics alone—further broadening the gap between the world we experience and physical reality. Quantum theory is arguably contradictory to everyday experience and the debate about the correct interpretation is still ongoing. Secondly, the theory of relativity changed the way we think about space and time. The removal of the notion of a preferred reference frame culminates in the insight that gravity is a manifestation of the curvature of spacetime. Throughout the twentieth century advances in both these fields have progressed steadily. Early on it was realised that, ultimately, one should be able to find a theory which encompasses both of these essential theories. This marriage is referred to as *quantum gravity* and is the main focus of this thesis.

### 1.1. Quantum theory

We now give a brief overview of quantum theory. In 1900 Planck was able to derive a formula which correctly describes the spectrum of black body radiation [121]. The crucial idea was to postulate that the energy of an oscillator is quantised, i.e., that there is an amount of energy that cannot be subdivided further. This minimal amount of energy is related to a constant of nature,  $h \approx 6.6 \times 10^{-34} \text{ J Hz}^{-1}$ , which is now known as *Planck's constant*. In 1905 Einstein published a theoretical explanation of the photoelectric

effect which also relied on the idea that electromagnetic radiation is quantised. In the subsequent years models were developed from which one could derive this quantized nature of electromagnetic radiation. Notable contributions are Born, Heisenberg and Jordan's *matrix mechanics* [30] and Schrödinger's *wave function* formalism [134] both of which were later realised to be different representations of an underlying Hilbert space formulation.

Essentially, the Hilbert space formulation posits that a physical state of a system can be represented as a vector  $|\psi\rangle$  in some Hilbert space  $\mathcal{H}$ . Furthermore, the time evolution of such a state is fully captured by the Schrödinger equation,

$$i\hbar\frac{\partial}{\partial t}|\psi\rangle(t) = \hat{H}|\psi\rangle(t), \quad (1.1)$$

where  $\hbar = h/(2\pi)$  is the reduced Planck's constant<sup>1</sup> and  $\hat{H}$  is the Hamilton operator capturing the energy of the system. An important feature of the Schrödinger equation is that it is linear, i.e., if  $|\psi_1\rangle$  and  $|\psi_2\rangle$  are solutions, then so is  $c_1|\psi_1\rangle + c_2|\psi_2\rangle$ , where  $c_1$  and  $c_2$  are arbitrary complex numbers. This observation leads to the following unintuitive consequence. Assume that we have an apparatus which measures the value of an observable  $\hat{O}$  associated with a particle which can take two values,  $o_1$  and  $o_2$ . Furthermore, let  $|\psi_i\rangle$ ,  $i = 1, 2$  be eigenstates of  $\hat{O}$  such that  $\hat{O}|\psi_i\rangle = o_i|\psi_i\rangle$ ,  $i = 1, 2$ . If we were to perform repeat measurements of  $\hat{O}$  and always prepare the particle such that it is given by the state  $|\psi\rangle = c_1|\psi_1\rangle + c_2|\psi_2\rangle$  with  $|c_1|^2 + |c_2|^2 = 1$ , then we would measure  $o_1$  with a probability  $|c_1|^2$  and  $o_2$  with a probability  $|c_2|^2$ . This is known as *Born's rule*. The confusion arises if one insists that the particle should have a definite value of the observable  $\hat{O}$ , irrespective of whether or not one measures the value. This and similar problems in the interpretation of quantum mechanics arise if one separates the observer (oneself) from the system. The problem is resolved as follows: ultimately, both the observer and the system should be governed by the same physical laws and all of reality is described by a (complicated) state  $|\Psi_{\text{Universe}}\rangle$ . This point of view was proposed by Everett in 1957 which is now known as the *many-worlds interpretation* of quantum mechanics [56]. One of the issues of this interpretation is how to reconcile it with the probabilistic interpretation. Ideally, one should be able to start from the full theory and derive Born's rule. However, although there have been several attempts to solve this problem there is no consensus. A closely related concept is that of decoherence introduced in 1970 by Zeh [147]. Decoherence

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<sup>1</sup> Only in Chapters 1 and 6 do we write factors of  $\hbar$  explicitly. In the rest of the text we use units in which  $\hbar = 1$ .



reconciles the observed classical behaviour with the underlying quantum description by explaining how the interference between the different states (corresponding to distinct classical observations) is suppressed. The solution is that the quantum coherence is dissipated into the environment. Note that quantum decoherence does not solve the measurement problem as it provides no mechanism to explain why an observer observes a certain outcome rather than another one [3].

Another formulation of quantum theory is given by the *path integral approach* which was first formulated by Feynman in 1948 [57]. The main object of interest is the transition amplitude, i.e., it answers the question of how likely one is to end up in a final state when starting in some initial state. The radical insight is that that probability can be calculated by taking a weighted average over all intermediate states, where the weighting factor is the exponential of the action functional multiplied by the imaginary unit. For instance, if one wants to calculate the probability that one particle initially located at some point may be observed at some other point at some later time, then one would take the weighted average over all possible paths connecting the initial and final point. It is from this example that the formalism derives its name. More concretely, the example of the particle is captured by the equation

$$\langle q_f; t_f | q_i; t_i \rangle = \int_{\substack{q(t_i)=q_i \\ q(t_f)=q_f}} \mathcal{D}q \exp\left(\frac{i}{\hbar} \int_{t_i}^{t_f} dt L(q(t), q'(t))\right). \quad (1.2)$$

Although the path integral formulation is strikingly concise, it is technically quite challenging to implement in general. The main difficulty lying in rigorously defining what the path integral measure should be.

## 1.2. Relativity

This section provides an overview of relativity. In 1905 Einstein published his groundbreaking work [54] on what is now known as *special relativity*. Special relativity is based on two postulates. The first postulate states that physical laws should be the same in any inertial reference frame. This is the principle of relativity introduced by Galileo Galilei. The second postulate states that the speed of light is the same in every reference frame. This second postulate is the radical and unintuitive proposition made by Einstein. To

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appreciate why this is so unintuitive consider the following. From everyday experience we have come to understand that velocities are additive. That is, if a person were to run past me and throw a ball in their direction of motion then it is obvious to me that the ball will travel faster than if that same person had thrown the ball whilst standing. However, the second postulate states that if that person were to carry a torch and shine light in front of them then the light emitted from the torch would travel at the same speed as if that person were standing. Note that the situation is entirely symmetrical. Should I be the person shining the torch, the other person would measure the same speed of light irrespective whether they carry out their experiment while standing or running. The most profound consequence of special relativity is that it does away with the notion of a preferred time. Before special relativity, it was thought that there is some “universal time” that marches forward steadily and perpetually and that the physical laws describe the evolution with respect to that time. However, in special relativity the notion of time and in particular of simultaneity is dependent on the observer. The famous “twin paradox” illustrates this point: From the pair of twins Alice and Bob, Alice decides to travel through space on a fast spaceship whilst Bob remains on Earth. When Alice returns forty Earth years later she is a lot younger than her brother since from her perspective less time has elapsed. (Note that the reason this is called a paradox is because the question is why the situation is not symmetrical, i.e., why it cannot be Bob who is the younger upon Alice’s return. The resolution is that Alice does not remain in an inertial frame whilst travelling in the spaceship since she has to perform some acceleration to return to Earth.)

Shortly after Einstein’s 1905 paper on special relativity Hermann Minkowski realised that the underlying mathematical structure is that of a metric space, albeit with a twist. The metric most easily understandable by humans is that of Cartesian three-dimensional space. Say we choose Cartesian coordinates  $x$ ,  $y$  and  $z$ . If we want to calculate the distance between two points  $p_1 = (x_1, y_1, z_1)$  and  $p_2 = (x_2, y_2, z_2)$  then we may do this by using the metric  $d(p_1, p_2) = (x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2$ . The distance is always positive except for the special case in which the two points coincide,  $p_1 = p_2$ . Now special relativity suggests that space and time are deeply connected and it is reasonable to expect that one should add time as a fourth coordinate,  $t$ , and call the resulting four-dimensional space *spacetime*. Indeed, this is possible and it was Minkowski’s great insight that one should measure distances in four-dimensional spacetime with a new kind of metric [103]. Consider again a set of coordinates, now with an added time coordinate,  $t$ ,  $x$ ,  $y$  and  $z$ . Note that we use units in which the speed of light is equal to one,  $c = 1$ , and therefore all the coordinates

have the same dimension. If we now want to calculate the distance between two spacetime points  $p_1 = (t_1, x_1, y_1, z_1)$  and  $p_2 = (t_2, x_2, y_2, z_2)$  then the metric consistent with special relativity is given by  $d(p_1, p_2) = -(t_1 - t_2)^2 + (x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2$ . This metric is called *Minkowski metric* and the four-dimensional metric space equipped with this metric is referred to as *Minkowski space*. The crucial difference to the spatial metric is that the Minkowski metric is not positive definite, i.e., there may be two points  $p_1$  and  $p_2$  that have a distance of zero but do *not* coincide. All of special relativity can be derived by stating that spacetime is Minkowski space and that inertial reference frames are related by transformations which leave the metric invariant.

Einstein soon realised that it would be logical to extend the principle of relativity even further. Why should the physical laws be the same in two reference frames only if they are related by a special kind of transformations that leave the Minkowski metric invariant instead of more general transformations? After several years and acquiring mathematical knowledge previously deemed unnecessary for physics, Einstein completed his endeavour in 1915 and published his work on what is now known as *general relativity* [55]. The change of our understanding of space and time is comparably profound to that which occurred ten years earlier.

The key result of general relativity is that gravity is a manifestation of the curvature of spacetime and that the curvature of spacetime is influenced by the presence of matter. It is a well known result of classical mechanics that an object upon which no force acts travels along a straight line. In a Cartesian coordinate system straight lines are geodesics, i.e., the shortest paths connecting two given points. It turns out that gravity is the generalisation of this concept from classical mechanics to curved spacetimes. That is, a freely falling object travels on geodesics.

We now give a brief technical introduction to general relativity. General relativity can be viewed as the theory of the metric  $g$  defined on a manifold  $M$ . The theory can be defined via an action known as the *Einstein–Hilbert action*,

$$S_{\text{EH}}(g) = \frac{1}{2\kappa} \int_M d^4x \sqrt{-\det(g(x))} R(g(x)), \quad (1.3)$$

where the coupling constant is related to Newton's constant  $G$  by  $\kappa = 8\pi G$  and  $R$  is the Ricci curvature scalar given as a function of the metric. In the presence of matter the

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total action would be given by the sum

$$S(\phi, g) = S_{\text{EH}}(g) + S_{\text{Matter}}(\phi, g), \quad (1.4)$$

where  $\phi$  are the matter degrees of freedom. Let us briefly remark that from the point of view of effective field theory the gravitational action of a metric theory takes the form

$$S_{\text{grav}}(g) = \frac{1}{2\kappa} \int_M d^4x \sqrt{-\det(g(x))} (R(g(x)) - 2\Lambda + \dots), \quad (1.5)$$

where the ellipsis denotes higher-derivative terms which are invariant under diffeomorphisms. The constant  $\Lambda$  is the so-called *cosmological constant*. We emphasise that from the perspective of effective field theory one needs no justification for its addition. On the contrary, one actually would need to justify its exclusion. (Nevertheless we only consider “pure” Einstein gravity in the following, where  $\Lambda = 0$ .) The equation of motion obtained by varying the action with respect to the components of the metric,  $g_{\mu\nu}(x)$ , is the *Einstein equation*

$$R_{\mu\nu}(g(x)) - \frac{1}{2}R(g(x))g_{\mu\nu}(x) = \kappa T_{\mu\nu}(\phi(x), g(x)), \quad (1.6)$$

where  $R_{\mu\nu}$  are the components of Ricci curvature tensor and  $T_{\mu\nu}$  are the components of the energy momentum tensor of the matter specified in the action. The components of the energy momentum tensor can be calculated from (1.4),

$$T_{\mu\nu}(\phi(x), g(x)) = -\frac{2}{\sqrt{-\det(g(x))}} \frac{\delta S_{\text{Matter}}}{\delta g^{\mu\nu}(x)}(\phi, g). \quad (1.7)$$

Note that (1.6) is a non-linear partial differential equation for the metric which is already very difficult to solve for the case of vacuum  $T_{\mu\nu} = 0$ . The Einstein equation (1.6) can be solved analytically if one makes some simplifying assumptions. We will next discuss two solutions briefly.

The first example of a solution is given by the Schwarzschild metric which is named after Schwarzschild who published his result in 1916 [135]. The Schwarzschild metric is a static vacuum solution which is spherically symmetric. The metric is of particular interest for two reasons. Firstly, there is a region, called a *black hole* which cannot be exited once entered. Secondly, at the centre of the black hole there is a singularity. The notion of singularities in general relativity is a subtle one. A singularity is a point at which some curvature invariant of the metric diverges *and* which can be reached in a finite amount of time (as experienced by an observer). The appearance of the singularity at the centre of the black hole was for some time considered to possibly only occur due to the high degree

of symmetry in the solution. However, it was later shown that singularities generically appear due to gravitational collapse [85].

The second example is given by the Friedmann–Lemaître–Robertson–Walker (FLRW) metric named after Friedmann, Lemaître, Robertson and Walker [60, 93, 127, 142]. This solution describes the evolution of a maximally symmetric space which may contain homogeneously and isotropically distributed matter. The line element is given by

$$ds^2 = -dt^2 + a(t)^2(dx^2 + dy^2 + dz^2), \quad (1.8)$$

where  $a(t)$  is the *scale factor*. The Einstein equation (1.6) results in an ordinary differential equation for  $a(t)$ . The importance of the FLRW metric is that it describes the universe at large scales.

The current “standard model of cosmology” goes under the name “ $\Lambda$ CDM”, where  $\Lambda$  refers to the cosmological constant (cf. (1.5)) and CDM stands for cold dark matter, a postulated form of matter that interacts only gravitationally with the other forms of known matter. The FLRW metric is also generically singular; when one evolves the scale factor backwards in time, one arrives at a point in time where it is zero. This singularity is commonly referred to as the *big bang*.

### 1.3. Quantum gravity

Already in 1916 Einstein noted that quantum theory will need to be modified to accommodate the (then) new theory of gravitation [53]. The problem remains unsolved to this day, over a century later. In this section we give an overview of problem, key insights that allowed progress and some of the active areas of research trying to solve (part of) the problem. A textbook on quantum gravity in general is [91].

In [128] the early developments are separated into three major lines of research, namely the covariant, the canonical and the sum-over-histories lines of research. In more modern approaches the distinction between these different lines of research has become less prominent. Nevertheless, it proves useful to explain the main ideas of all three approaches.

The covariant line of research aims to apply the methods of particle physics to arrive at a theory of quantum gravity—or even more ambitiously at a theory of everything. From this perspective, gravity is viewed to be on an equal footing with the other forces. Gravity is described as a gauge field theory where in the case of gravity one is interested in a spin-2 field which is called graviton. Early success in this approach came in 1963 when Feynman was able to show that at tree level the calculations agree with general relativity [58]. In 1967 the Feynman rules for gravitons were completely worked out [45]. After these promising advances, the programme hit a road block in 1974 when 't Hooft and Veltman were able to show that gravitation coupled to matter is non-renormalisable [89]. This means that if one wanted to calculate, say, the probability of a certain physical phenomenon one would have to add an infinite number of parameters to the theory to obtain a finite result, making the theory non-predictive. There are two ideas how to circumvent this problem. Firstly, there is the idea that the non-renormalisable theory is missing degrees of freedom. Many different proposals concerning the nature of those missing degrees of freedom have been put forward such as supergravity, modified gravity and infinite derivative theories. Arguably, string theory is also a descendant of this line of thinking. Secondly, Weinberg proposed in 1978 [143] that whilst quantum gravity is not asymptotically free it might be *asymptotically safe*: the idea basically being that amongst all possible theories (values of coupling parameters) only those are physically realisable in which the coupling parameters approach a finite value at high energies (as opposed to going either to zero or diverging). That is, of the infinite number of parameters needed to calculate anything, only some of them need to be measured and all the others are fixed by this requirement.

The canonical line of research tries to identify the phase space variables relevant for general relativity and then perform a standard quantisation of that phase space. In 1959 Dirac completed the formulation of general relativity as a Hamiltonian system [47]. The main feature being that it is a totally constrained system, in the sense that the Hamiltonian is given as a linear combination of constraints. In 1961 Arnowitt, Deser and Misner greatly clarified the phase space structure by the introduction of new variables. This is now known as the Arnowitt–Deser–Misner (ADM) formalism [4]. The ADM metric takes the form

$$ds^2 = -N(t, \mathbf{x})^2 dt^2 + q_{ij}(t, \mathbf{x})(dx^i + N^i(t, \mathbf{x}))(dx^j + N^j(t, \mathbf{x})), \quad (1.9)$$

where  $N$  is the lapse function,  $\mathbf{N}$  is the shift vector and  $q$  is the three-metric defined

on a spatial slice. Note that it is assumed that spacetime can be foliated by a family of spacelike hypersurfaces. Using the methods of quantisation of constrained systems (see, e.g., [48]) DeWitt wrote down the *Wheeler–DeWitt equation* in 1967 [44],

$$\left( (2\kappa\hbar)^2 (q_{ik}q_{jl} - \frac{1}{2}q_{ij}q_{kl}) \frac{\delta}{\delta q_{ij}} \frac{\delta}{\delta q_{kl}} + \det(q)R(q) \right) \Psi(q) = 0, \quad (1.10)$$

where  $R(q)$  is the Ricci curvature scalar of the three-metric. Equation (1.10) should be viewed as a formal statement. In particular, the space in which the wave function  $\Psi(q)$  is a state is mathematically ill-defined. However, the equation has been studied extensively for the case in which the space of metrics (“superspace”) is greatly reduced by considering only metrics with a high degree of symmetry (“minisuperspace”). Studying such symmetry reduced models by solving the Wheeler–DeWitt equation is a central part of *quantum cosmology*. In 1986 Ashtekar introduced another set of variables (*Ashtekar variables*) which allow one to write general relativity as a Yang–Mills theory [5]. The use of these new variables culminated in the theory of *loop quantum gravity* (LQG)<sup>2</sup> where the kinematical Hilbert space of the theory is represented by *spin networks* which solve the diffeomorphism constraints of general relativity. Furthermore, LQG made great progress in the canonical approach by being able to identify quantum operators for geometric quantities such as area and volume. The striking result is that both area and volume have a discrete spectrum, suggesting that spacetime might be fundamentally discrete.

The sum-over-histories line of research aims to employ path integral quantisation to the gravitational field. That is, one is interested in calculating the partition function

$$Z = \int \mathcal{D}g e^{iS_{\text{EH}}(g)/\hbar} \quad (1.11)$$

non-perturbatively. As with the other lines of research one of the big problems is identifying the space of metrics. Formally one would like to integrate over the space of equivalence classes of metrics defined on four-dimensional spacetimes, where the equivalence relation is given by being related by a diffeomorphism. This space has a very complicated structure [81]. One idea in making this programme more feasible is to perform a Wick rotation which turns the partition function into

$$Z_E = \int \mathcal{D}g_E e^{-S_E(g_E)/\hbar}, \quad (1.12)$$

where the integral is now over Euclidean metrics (Riemannian metrics) on four-dimensional spaces (as opposed to spacetimes). This approach is known as *Euclidean quantum*

<sup>2</sup>An expanded discussion of LQG is given in Section 2.2.1.

*gravity* [65]. The Wick rotation is rather formal, however. There is no one-to-one correspondence between Riemannian and Lorentzian metrics and therefore one has changed the configuration space completely. Furthermore, even the Euclidean action,  $S_E$ , is not bounded from below, preventing one from generalising methods from statistical mechanics in a straightforward manner. Hartle and Hawking studied the *wave function of the universe* in this framework and put forward the proposal that one should sum over four-geometries which have the currently observed three-geometry as a boundary. Since the four-geometry should have no other boundaries, in particular not in the past, this is known as the *no boundary proposal* [86].

### 1.3.1. Discreteness in quantum gravity

As can be seen from the long history and plethora of ideas, the goal of formulating a theory of quantum gravity is an ambitious one. In order to circumvent the problems arising from the huge space of possible four-dimensional spacetimes it has proven fruitful to abandon the continuum and study discrete structures instead. Note that there are two philosophies concerning discrete structures in quantum gravity. The first point of view is that the discrete structure serves as a regulator, taming the ultraviolet divergences arising in the continuous theory. This is similar to lattice gauge theory, where one introduces an artificial lattice and then renormalises the theory by taking the limit where the lattice edge length goes to zero. The second point of view is that the discrete structure is actually realised in nature. That is, the apparent continuous spacetime is an emergent phenomenon of an underlying discrete structure.

In 1961 Regge introduced what is now known as *Regge calculus* [123]. Instead of a continuous spacetime, the underlying structure is a piecewise flat manifold which may be thought of as a graph where the edges are weighted by their geometric lengths. In piecewise flat manifolds curvature is located at loci with codimension 2. In 1968 Ponzano and Regge noticed a connection between an asymptotic expansion and the action of three-dimensional Regge calculus which lead to what is now known as the *Ponzano–Regge model* [17, 122].

One idea for studying the quantum theory of such triangulations is to use methods similar to those of statistical mechanics. The phase space is explored using Monte-Carlo



methods where an initially random triangulation is then “thermalised” by making arbitrary modifications which are then either rejected or accepted depending on the resulting change in the action. *Dynamical triangulation* is the theory resulting when one performs this procedure for Regge calculus. However, one finds that the resulting geometries are pathological in the sense that one never gets the desired dimension of spacetime, but rather finds that it is either too low or diverges<sup>3</sup>. However, considerably more encouraging results are obtained in *causal dynamical triangulation* where one enforces a topology  $\mathbb{R} \times \Sigma$  by hand. The main results are a second order phase transition and a de Sitter universe [97].

As mentioned above, one result of LQG is that space is discrete. One way of defining dynamics for LQG is by studying the transition amplitudes from one discrete three-geometry to another discrete three-geometry. This idea is formalised in the theory of *spin foams* [15, 117].

*Group field theory* (GFT) postulates that spacetime is constituted of building blocks which are sometimes referred to as “atoms of spacetime”. Just as matter is made up out of atoms, spacetime is thought to be made up out of something granular. (It is amusing to contemplate that the nomenclature might be similarly misguided as in the case of matter. That is, it is not unthinkable that the atoms of spacetime might themselves be composite objects.) GFT is technically interesting because it employs the techniques of quantum field theory but in quite a different manner to what has been referred to as “covariant lines of research”. Indeed, when viewed as a quantum field theory the “particles” are the postulated atoms of spacetime. In this sense, GFT is a quantum field theory *of* spacetime (as opposed to the usual notion of a quantum field theory on spacetime).

There are radical approaches that do away altogether with any preconceived notion of spacetime geometry. Rather, the objects of interest are pointlike events. Only through the causal relation between the points does spacetime emerge. One formulation of this idea is known as *causal sets*, a framework which famously predicted the order of magnitude of the cosmological constant [136]. Similar ideas form the basis of Wolfram’s new class of models where the proposal is that the universe is fundamentally described by a graph and a rule for updating that graph [145].

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<sup>3</sup>There are various ways to define dimensionality of discrete structures. One way is to observe the scaling behaviour of spheres with increasing radius. This is the Hausdorff dimension of the structure.

### 1.3.2. Observability of quantum gravity

Physics is usually considered to ultimately be an empirical science. That is, theories have to be confirmed or falsified by conducting experiments. This poses somewhat of a problem for theories of quantum gravity since quantum gravity effects are believed to become relevant only at very high energies. However, there is hope that nature provides natural laboratories which will soon provide opportunities to test and compare different theories of quantum gravity. We discuss the two such laboratories, namely those provided by cosmology and black holes. For a recent overview of quantum gravity phenomenology see, e.g., [1].

#### Cosmology

Standard FLRW cosmology describes a homogeneous and isotropic universe which is an excellent approximation to the universe we observe today. Hubble's observations confirmed that our universe is expanding and it was later realised that this expansion is accelerating. If one runs the evolution in time of the universe backwards one sooner or later reaches a regime which fails to satisfy the assumptions needed for an FLRW universe. Firstly, the distribution of matter will become less homogeneous and isotropic. Secondly, the energy density increases to a point at which quantum effects become important. Both these points can be addressed fairly well by considering a semiclassical model where the quantum field of the standard model of particle physics (and often also more speculative forms of matter) interact with each other on a classical background spacetime which allows for perturbations of the FLRW metric. However, if one evolves back in time *even further* the semiclassical description undeniably has to break down. There are two reasons for this. Firstly, the energy density will reach a scale at which quantum gravity effects become relevant. Secondly, the semiclassical theory does not resolve the big bang singularity.

The currently best measured feature which might provide signatures of quantum gravity is the cosmic microwave background (CMB) radiation. However, there are semiclassical models which agree well with observation. One of the hopeful prospects is that with increasing sensitivity of gravitational wave detectors one might be able to capture signals

dating back earlier in cosmological history than the CMB [35, 133].

The initial big bang singularity can be viewed as a breakdown of the (semi)classical theory. Indeed, a result shared by many different approaches is that the big bang singularity is replaced by a “big bounce” [32]. That is, before the universe entered the currently observed expanding phase it was contracting. In many models the bounce is symmetric but it need not be. Furthermore, there are models in which the bounces are cyclic<sup>4</sup> and models where the “pre-bounce” phase extend infinitely into the past<sup>5</sup>.

Most approaches to quantum gravity are technically challenging and it is hopeless to obtain results studying the full theory. Therefore, an appealing idea is that of symmetry reduction before quantisation. Needless to say, it is not obvious at all that truncating the theory *before* quantisation will result in anything resembling the quantisation of the full theory. Nevertheless, symmetry-reduced models have a long tradition in quantum gravity.

In general relativity the space of all metrics is called superspace. If one restricts this space to metrics of a high degree of freedom one studies minisuperspace. For instance, in the case of a flat FLRW metric with a massless scalar field, minisuperspace is parametrised by the scale factor,  $a$ , and the value of the massless scalar field,  $\chi$ . The resulting Wheeler-DeWitt equation of this model is

$$\left( \frac{\kappa \hbar^2}{6} a \frac{\partial}{\partial a} a \frac{\partial}{\partial a} - \frac{\partial^2}{\partial \chi^2} \right) \Psi(a, \chi) = 0. \quad (1.13)$$

In LQG the spatial metric is described by a graph. If one assumes that the graph is very regular, one arrives at a space which has a high degree of symmetry. Models inspired by this idea go under the umbrella term loop quantum cosmology (LQC), where the main result is that the big bang singularity is replaced by a big bounce [12]. The most studied

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<sup>4</sup>An interesting non-bouncing model featuring cyclicity is given by Penrose’s conformal cyclic cosmology [114].

<sup>5</sup>In the main text, the terms “future” and “past” refer to the value of the time coordinate. However, it is not a priori clear that a conscious entity has to experience time as flowing from values of low coordinate time to values of high coordinate time. It is an interesting thought experiment to consider observers living in what we would call the pre-bounce phase who would view our currently expanding universe as their respective pre-bounce phase, i.e., that they experience time “backwards” from our point of view.

system is that of a flat homogeneous isotropic universe coupled to a massless scalar field. To compare with classical physics one takes expectation values of the quantum operator corresponding to volume, the resulting effective Friedmann equation is of the form

$$\left(\frac{V'_\chi(\chi)}{V_\chi(\chi)}\right)^2 = \frac{3\kappa}{2} \left(1 - \frac{\rho(\chi)}{\rho_c}\right), \quad (1.14)$$

where the initial value depend on the choice of state,  $V_\chi$  is the volume as expressed as a function of the scalar field,  $\rho$  is the energy density of the scalar field and  $\rho_c$  is the critical energy density at which the bounce occurs ( $V'_\chi(\chi_{\text{bounce}}) = 0$  when  $\rho(\chi_{\text{bounce}}) = \rho_c$ ). Compared to the classical Friedmann equation in an FLRW universe coupled to a massless scalar field (cf. (A.42)) the second term on the right-hand side can be seen as a quantum correction arising from the quantum nature of spacetime itself. Note that the derivation of LQC from LQG remains an open problem [41].

In GFT results similar to that of LQC have been obtained [112]. In GFT the spacetime is assumed to be comprised of building blocks, e.g., simplices. Since we do not observe a discrete spacetime, the idea is that the continuous theory should be seen as an approximation to the underlying discrete structure constituted of a large number of building blocks. Drawing inspiration from condensed matter physics, the idea is then that the universe should be realised as a state with a large number of excitations, i.e., it should be described as a condensate state of GFT [75, 109]. The geometry of the universe is encoded in the shape of the building blocks. It is plausible that the observed isotropy of the universe can be explained as a consequence of the building blocks all having the same regular shape. With these considerations it was shown in [112] that the resulting effective Friedmann equations take the same functional form as (1.14).

## Black holes

Black holes have several mysterious properties which one tries to elucidate by quantum-gravitational considerations.

Semiclassical calculations have shown that black holes are not entirely black. Rather, they are believed to emit thermal radiation known as Hawking radiation [84]. The evaporation of a black hole leads to what is known as the “black hole information paradox” [66]: how

can unitary time evolution be reconciled with the existence of an event horizon? Note that this paradox comes about by treating the black hole classically and the matter as quantum fields. In a full theory of quantum gravity there would not be a paradox.

Another aspect which quantum gravity should clarify is the meaning of the singularity. Classically, when the gravitational pressure exceeds the pressure of matter (electroweak and strong), the matter collapses to a point and forms a black hole [38]. It is conceivable that at the Planck scale quantum gravity effects should become important and it might be that there is some repulsive force becoming relevant only at very short length scales, preventing the singularity from forming.

Since black holes emit Hawking radiation one may ascribe to them a temperature. Therefore, they also have an entropy. The Bekenstein–Hawking entropy of a black hole is given by

$$S_{\text{BH}} = \frac{k_{\text{B}}A}{4l_{\text{P}}^2}, \quad (1.15)$$

where  $k_{\text{B}}$  is Boltzmann’s constant,  $A$  is the surface area of the black hole and  $l_{\text{P}}$  is the Planck length. It is curious that the entropy should scale with the surface area rather than with the volume. It should be possible to arrive at (1.15) using the microcanonical ensemble of the microscopic theory of quantum gravity. Furthermore, it is natural to assume that there might be further quantum corrections to the semiclassical expression (1.15). This has been carried out in a string theory [98] and a loop quantum gravity [118] context, where in both cases one obtains the classical entropy to leading order. Perhaps one day measurements of the black hole entropy will have the final say in which theory of quantum gravity is correct.

## 1.4. Outline

The thesis is structured as follows. In Chapter 2 we introduce the formalism of GFT and show connections to other fields of research. Chapter 3 introduces the coupling of a GFT to one or more massless scalar fields which is the framework in which one can study the cosmological sector of GFT. We also list some of the different perspectives that have been explored in the study of GFT cosmology. Chapters 4 and 5 deal with the canonical approach to GFT cosmology for the coupling of multiple massless scalar

## *Chapter 1. Introduction*

fields and a single massless scalar field, respectively. In Chapter 6 we employ an effective approach for studying quantum systems with constraints to a simple GFT model where we identify the observables of interest with the one-body operators of the GFT. Finally, Chapter 7 summarises the thesis.

The novel contributions of this thesis can be found in the following chapters:

- Chapter 4: Canonical formalism for GFT with multiple scalar fields based on [77]
- Chapter 5: Canonical formalism for GFT with a single scalar field based on [76]
- Chapter 6: Effective one-body approach to GFT based on [72]

## Chapter 2.

# Group field theory

This chapter provides a technical introduction to the formalism of GFTs. We discuss two contexts in which GFTs arise, namely spin foams and tensor models.

### 2.1. Definition

In this section we provide a brief overview of the basic definition of GFT. Reviews of GFT can be found in [59, 92].

A GFT is a scalar field theory where the domain of the scalar field  $\varphi$  is a product Lie group  $G^M = G \times \cdots \times G$ , where  $G$  is a Lie group and there are  $M$  factors in the Cartesian product. We refer to the scalar field as the *group field* and note that it can be either real or complex, i.e.,

$$\varphi: G^M \rightarrow \mathbb{K}, \quad g_I \mapsto \varphi(g_I), \quad \mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}, \quad (2.1)$$

where the arguments of the group field are written in a compact notation which when expanded reads

$$\varphi(g_I) = \varphi(g_1, \dots, g_M). \quad (2.2)$$

The defining feature of a GFT is that one requires that the group field be invariant under a right-diagonal action of the Lie group  $G$ , i.e., for any element  $h$  of  $G$

$$\varphi(g_1 h, \dots, g_M h) = \varphi(g_1, \dots, g_M). \quad (2.3)$$

This invariance effectively means that the actual domain of the group field is given by  $G^M/G$ . The specific theory is given by an action functional (assuming a complex group field)

$$S: (\varphi, \bar{\varphi}) \mapsto S(\varphi, \bar{\varphi}) \in \mathbb{R}. \quad (2.4)$$

The classical equations of motion can then be obtained by requiring that the action is extremised, i.e.,

$$\frac{\delta S}{\delta \varphi(g_I)}(\varphi, \bar{\varphi}) = \frac{\delta S}{\delta \bar{\varphi}(g_I)}(\varphi, \bar{\varphi}) = 0. \quad (2.5)$$

The action of a model is usually split into a kinetic and an interaction part,

$$S(\varphi, \bar{\varphi}) = K(\varphi, \bar{\varphi}) + V(\varphi, \bar{\varphi}), \quad (2.6)$$

where the kinetic term is given by

$$K(\varphi, \bar{\varphi}) = \int d^M g d^M g' \bar{\varphi}(g_I) \mathcal{K}(g_I, g'_I) \varphi(g'_I), \quad (2.7)$$

where the integral measure is the normalised Haar measure,

$$d^M g = \prod_{a=1}^M dg_a, \quad \int_G dg_a = 1. \quad (2.8)$$

An example for an interaction term is given by

$$V(\varphi, \bar{\varphi}) = \int d^M g_1 \cdots d^M g_N \mathcal{V}(g_{1,I}, \dots, g_{N,I}) \varphi(g_{1,I}) \cdots \varphi(g_{N,I}) + \text{c. c.} \quad (2.9)$$

The simplest kinetic kernel would be the identity kernel,

$$\mathcal{K}(g_I, g'_I) = I(g_I, g'_I), \quad (2.10)$$

where the identity kernel,  $I$ , is defined such that  $\int d^M g' I(g_I, g'_I) f(g'_I) = f(g_I)$ . However, studies on GFT renormalisation indicate that the kinetic kernel should also include the Laplace–Beltrami operator [36],

$$\mathcal{K}(g_I, g'_I) = (m^2 + \Delta) I(g_I, g'_I), \quad (2.11)$$

where  $\Delta$  is the Laplace–Beltrami operator on the product group  $G^M$ ,

$$\Delta = \sum_{a=1}^M \Delta_a. \quad (2.12)$$



### 2.1.1. Mode expansion

For a compact Lie group  $G$  it follows from the Peter–Weyl theorem that one can write the group field as a sum over a countable set for a suitable choice of basis vectors on the space of square-integrable functions on  $G$ . Schematically, this means that one can write

$$\varphi(g_I) = \sum_J \varphi_J D_J(g_I), \quad (2.13)$$

where the  $\varphi_J$  are complex numbers. Inserting the mode expansion into the action gives

$$K(\varphi, \bar{\varphi}) = \sum_{JJ'} \bar{\varphi}_J \mathcal{K}^{JJ'} \varphi_{J'}, \quad (2.14a)$$

$$V(\varphi, \bar{\varphi}) = \sum_{J_1, \dots, J_N} \mathcal{V}^{J_1 \dots J_N} \varphi_{J_1} \dots \varphi_{J_N}, \quad (2.14b)$$

where we defined

$$\mathcal{K}^{JJ'} = \int d^M g d^M g' \overline{D_J(g_I)} \mathcal{K}(g_I, g'_I) D_{J'}(g'_I), \quad (2.15a)$$

$$\mathcal{V}^{J_1 \dots J_N} = \int d^M g_1 \dots d^M g_N \mathcal{V}(g_{1,I}, \dots, g_{N,I}) D_{J_1}(g_{1,I}) \dots D_{J_N}(g_{N,I}). \quad (2.15b)$$

One case of particular interest is where the Lie group is the special unitary group in two dimensions, i.e.,  $G = SU(2)$ .<sup>1</sup> In that case a suitable basis for the Lie group is given by the Wigner  $D$ -matrices,

$$\varphi(g_I) = \sum_{\underline{m}, \underline{n}, \underline{j}} \varphi_{(\underline{j})}^{\underline{mn}} \prod_{a=1}^M D_{m_a n_a}^{(j_a)}(g_a), \quad (2.16)$$

where we use a compact multi-index notation which has the expanded form

$$\varphi_{(\underline{j})}^{\underline{mn}} = \varphi_{(j_1, \dots, j_M)}^{m_1, \dots, m_M, n_1, \dots, n_M} \quad (2.17)$$

We use conventions in which the Wigner  $D$ -matrices have an inner product,

$$\int dg \overline{D_{mn}^{(j)}(g)} D_{m'n'}^{(j')}(g) = \frac{1}{2j+1} \delta^{jj'} \delta_{mm'} \delta_{nn'}. \quad (2.18)$$

However, the decomposition (2.16) does not obey the symmetry (2.3). By assuming that (2.3) holds one can write (2.16) in the form

$$\varphi(g_I) = \sum_{\underline{m}, \underline{n}, \underline{j}, \underline{\iota}} \varphi_{(\underline{j}, \underline{\iota})}^{\underline{m}} \mathcal{I}_{(\underline{j}, \underline{\iota})}^{\underline{n}} \prod_{a=1}^M D_{m_a n_a}^{(j_a)}(g_a), \quad (2.19)$$

<sup>1</sup>This gauge group  $SU(2)$  can be interpreted as spatial rotations in three-dimensional space.

where  $\mathcal{I}$  is an intertwining map (*intertwiner*) from the product space to the invariant subspace,  $\iota$  labels the invariant subspaces and the coefficients are related by

$$\varphi_{(\underline{j}, \iota)}^{\underline{m}} = \sum_{\underline{l}} \varphi_{(\underline{j})}^{\underline{m} \underline{l}} \mathcal{I}_{\underline{l}}^{(\underline{j}, \iota)}. \quad (2.20)$$

To see that (2.19) is true one may firstly use (2.3) to write

$$\varphi(g_I) = \int dh \varphi(g_I h), \quad (2.21)$$

use the fact that the Wigner  $D$ -matrices are a Lie group homomorphism, i.e.,  $D_{mn}^{(j)}(gh) = \sum_l D_{ml}^{(j)}(g) D_{ln}^{(j)}(h)$  and the definition of the intertwiners,

$$\int dh \prod_{a=1}^M D_{m_a n_a}^{(j_a)}(h) = \sum_{\iota} \mathcal{I}_{\underline{m}}^{(\underline{j}, \iota)} \mathcal{I}_{\underline{n}}^{(\underline{j}, \iota)}. \quad (2.22)$$

The specific form of the intertwiners depends on the number of copies of  $SU(2)$  in the Cartesian product  $G^M$ . (An overview of the recoupling theory of  $SU(2)$  can be found in [101].) In the case that the GFT field is real the coefficients in the expansion must satisfy [78, 105]

$$\overline{\varphi_{(\underline{j}, \iota)}^{\underline{m}}} = (-1)^{\sum_{a=1}^M (j_a - m_a)} \varphi_{(\underline{j}, \iota)}^{-\underline{m}}, \quad (2.23)$$

where  $-\underline{m} = (-m_1, \dots, -m_M)$ .

In the following we mostly assume the general case of a compact Lie group  $G$  and state explicitly when specialising to a particular group.

### 2.1.2. Quantum theory

So far the discussion of GFT has been entirely classical. We now turn to discuss a quantum formulation for GFT. In particular we discuss the operator and path integral formulations. Recall that we use units in which  $\hbar = 1$  (cf. the footnote on Page 2).

Firstly, we discuss the operator formulation of GFT. The framework is that of second quantisation of a non-relativistic field theory. We promote the scalar field  $\varphi$  and its complex conjugate  $\bar{\varphi}$  to operators  $\hat{\varphi}$  and  $\hat{\varphi}^\dagger$ , respectively. Furthermore we postulate that the operators obey bosonic statistics,

$$[\hat{\varphi}_J, \hat{\varphi}_{J'}^\dagger] = \delta_{JJ'}. \quad (2.24)$$

In the group representation an identity needs to satisfy (2.3) and we write symbolically

$$[\hat{\varphi}(g_I), \hat{\varphi}^\dagger(g'_I)] = \mathbb{I}(g_I, g'_I). \quad (2.25)$$

In the case of a compact group we have  $\mathbb{I}(g_I, g'_I) = \int dh \prod_{a=1}^M \delta(g'_a h g_a^{-1})$ , where  $\delta(\cdot)$  is the Dirac delta function on the group manifold  $G$ .

Given the bosonic statistics (2.24) it is natural to interpret  $\hat{\varphi}^\dagger$  and  $\hat{\varphi}$  as creation and annihilation operators, respectively. Furthermore, this perspective allows one to construct a one-quantum Hilbert space as follow. We define the Fock vacuum,

$$\hat{\varphi}_J |\emptyset\rangle = 0. \quad (2.26)$$

As will become clear in the sequel, the Fock vacuum can be interpreted as a “no geometry” state, i.e., a state which does not represent any space at all. Then the one-quantum Hilbert space  $\mathcal{H}$  has states

$$|\psi\rangle = \sum_J \psi_J \hat{\varphi}_J^\dagger |\emptyset\rangle. \quad (2.27)$$

From this one-quantum Hilbert space one can then construct the Fock space,

$$\mathcal{F} = \bigoplus_{n=0}^{\infty} S\mathcal{H}^{\otimes n}, \quad (2.28)$$

where  $S$  denotes symmetrisation.

In the usual setting of non-relativistic quantum field theory, dynamics would be governed by a Hamiltonian operator. We have defined the GFT to be defined via an action functional (2.4). The most direct way would be to promote the Euler–Lagrange equation to an operator equation,

$$\frac{\delta S}{\delta \varphi}(\hat{\varphi}, \hat{\varphi}^\dagger) = 0, \quad (2.29)$$

where one needs to choose some ordering of operators and a similar equation can be obtained by varying the action with respect to the second argument. Equation (2.29) is rather formal. One possibility is to view the operator obtained by the functional derivative of the action as a quantum constraint operator. The equations of motion are then implemented by requiring that physical states are annihilated by this constraint operator. This is the formalism of Dirac quantisation extended to the field theoretic setting of GFT. In Chapter 6 we use this quantum constraint as the starting point for an effective formulation.

The second quantum formalism used in the context of GFT is that of path integrals. The path integral for a GFT is defined by the partition function

$$Z = \int \mathcal{D}\varphi \mathcal{D}\bar{\varphi} e^{iS(\varphi, \bar{\varphi})}. \quad (2.30)$$

In the path integral setting the dynamical equations are given by the Schwinger–Dyson equations [99]. Since total derivatives under the integral vanish [148], one has for any function  $O$  of the fields

$$0 = \int \mathcal{D}\varphi \mathcal{D}\bar{\varphi} \frac{\delta}{\delta\varphi} \left( O(\varphi, \bar{\varphi}) e^{iS(\varphi, \bar{\varphi})} \right) = \left\langle \frac{\delta O}{\delta\varphi}(\varphi, \bar{\varphi}) + iO(\varphi, \bar{\varphi}) \frac{\delta S}{\delta\varphi}(\varphi, \bar{\varphi}) \right\rangle \quad (2.31)$$

and similarly for total derivatives with respect to  $\bar{\varphi}$ . These are the Schwinger–Dyson equations which give an infinite number of relations, e.g., by inserting for  $O(\varphi, \bar{\varphi})$  all the possible monomials. Although the path integral formalism is independent of the operator formalism, it is interesting that (2.31) can readily be converted into an equation involving operators acting on the GFT Fock space by replacing the fields by their corresponding operators and choosing some ordering prescription,

$$\left\langle \frac{\delta O}{\delta\varphi}(\hat{\varphi}, \hat{\varphi}^\dagger) + iO(\hat{\varphi}, \hat{\varphi}^\dagger) \frac{\delta S}{\delta\varphi}(\hat{\varphi}, \hat{\varphi}^\dagger) \right\rangle = 0. \quad (2.32)$$

This provides us with a different formulation of dynamics within the operator formalism. Note that by choosing  $O = 1$  one obtains the expectation value of the quantum Euler–Lagrange equation (2.29),

$$\left\langle \frac{\delta S}{\delta\varphi}(\hat{\varphi}, \hat{\varphi}^\dagger) \right\rangle = 0. \quad (2.33)$$

Although our main focus lies on the Hilbert space formulation of GFT, the relation to other theories of quantum gravity is most transparent in the path integral formulation. In the case that the GFT action has terms of an interacting group field theory, i.e., an action functional containing terms of higher multi-linearity in the group field than bilinear, the partition function may be expanded in terms of Feynman graphs in the same way as done in ordinary quantum field theory,

$$Z = \sum_{\Gamma} \frac{1}{|\text{sym}(\Gamma)|} \mathcal{A}(\Gamma), \quad (2.34)$$

where  $\Gamma$  is a Feynman graph,  $\mathcal{A}(\Gamma)$  is the associated Feynman amplitude and the symmetry factor is the cardinality of the symmetry group  $\text{sym}(\Gamma)$  which leaves the Feynman graph  $\Gamma$  invariant.

We now discuss some of the motivations for studying GFTs.

## 2.2. Relation to discrete geometry

The context in which GFTs arises is that of discrete geometry. Given a smooth manifold of dimension  $d$  one can approximate it by discretising it by polyhedra of dimension  $d$ . The most common discretisation is that of a triangulation, where the polyhedra are given by  $d$ -simplices.<sup>2</sup> Note that even though the discrete pieces are individually flat, their specific gluing does capture any intrinsic curvature of the manifold which can be calculated by computing the deficit angle at  $(d - 2)$ -dimensional intersections of polyhedra. (In  $d = 2$  the deficit angle is calculated at 0-dimensional vertices and in  $d = 3$  the deficit angle is calculated at 1-dimensional edges.)

Given a discretisation of a manifold one can construct a graph dual to that discretisation. Each polyhedron is assigned a vertex and vertices assigned to adjacent polyhedra are connected by a link. Thus the valency of the vertices is determined by the number of faces of the  $d$ -dimensional polyhedra in the discretisation and each link corresponds to a  $(d - 1)$ -dimensional face. In Fig. 2.1 some examples of polyhedra and their dual are shown. In Fig. 2.2 an example of a triangulation of a 2-dimensional manifold is shown.

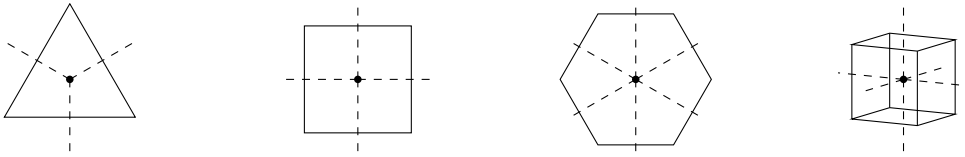


Figure 2.1.: Different polyhedra and their dual graphs. The polyhedron is shown as a solid line. The corresponding dual graph is drawn as a dashed line. From left to right: A triangle, a square, a hexagon, a cube.

After this short discussion of discrete geometry we return to the discussion of GFTs. So far we have been rather general in our discussion of the GFT action (2.4). Recall that we are considering group fields  $\varphi: G^M \rightarrow \mathbb{K}$ , where  $G$  is a Lie group. We are now going to specialise to the case of *simplicial GFTs*, where the action is the sum of a kinetic term, bilinear in the group field, and an interaction term,  $(M + 1)$ -linear in the group field. The claim is then that a suitable choice of action allows one to formulate a quantum-

<sup>2</sup>Clearly other “ $n$ -gulations” are possible. Indeed, one can even have discretisations with differently shaped polyhedra; An example for this is the  $C_{60}$  fullerene which can be viewed as a discretisation of the two-sphere comprised of pentagons and hexagons.

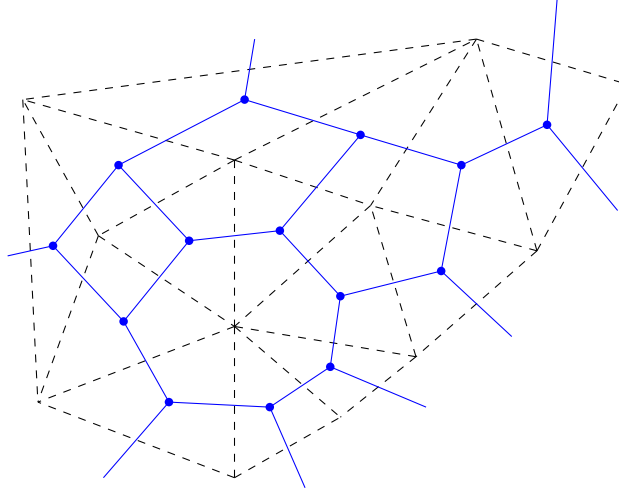


Figure 2.2.: An example of a discretisation and its dual graph. The dashed lines represent the discretisation which is a triangulation of a two-dimensional space. The blue solid graph is the graph dual to the discretisation.

geometric theory of an  $M$ -dimensional space. Heuristically, the group field corresponds to an  $(M - 1)$ -simplex and the interaction term glues together  $M + 1$  simplices to form an  $M$ -simplex. (A 2-dimensional triangle is comprised of three 1-dimensional edges. A 3-dimensional tetrahedron is comprised of four 2-dimensional triangles.) This is illustrated abstractly in Fig. 2.3, where the nodes correspond to  $(M - 1)$ -dimensional simplices and the links represent gluing along edges. More concretely, in the second-quantised framework discussed in Section 2.1.2 we constructed the one-quantum Hilbert space by acting with the creation operator  $\hat{\varphi}^\dagger$  on the no-geometry state  $|\emptyset\rangle$ . For example, a group field carrying three representation labels  $J_1$ ,  $J_2$  and  $J_3$  can then be viewed as creating a triangle the geometry of which is governed by the representation labels,

$$\hat{\varphi}_{J_1 J_2 J_3}^\dagger |\emptyset\rangle = \left| \begin{array}{c} J_1 \\ \diagup \quad \diagdown \\ \bullet \\ \diagdown \quad \diagup \\ J_3 \quad J_2 \end{array} \right\rangle. \quad (2.35)$$

Therefore, one may view the one-quantum states as *atoms of space* [106]. By repeatedly acting with creation operators and suitably contracting the labels, one may construct any discretisation compatible with the shape of building blocks provided.

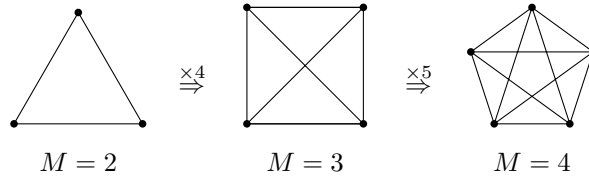


Figure 2.3.: The construction of  $M$ -simplices from  $(M - 1)$ -simplices in various dimensions. Each node represents a simplex of one dimension lower and the links represent gluing.

### 2.2.1. Loop quantum gravity and spin foams

One field from which GFT can be motivated is that of spin foams. Spin foams themselves are part of the LQG program which we now briefly discuss. According to [15] LQG is a “a very conservative approach to quantum gravity”. The reason for this claim being that, in essence, the goal is to perform a textbook quantisation of general relativity without adding any additional structure a priori. For detailed information on LQG there are textbooks [25, 61, 129, 139] and reviews [6, 10].

According to [15] the new ideas in LQG are

1. insistence on background-independence
2. use of loop representation of general relativity.

The first point is in stark contrast to the usual (perturbative) formulations of quantum field theories which require one to specify a background metric even if one considers the metric to be a quantum field itself. The second point is rather involved and introduces some ambiguity in the theory.

The usual starting point for a canonical treatment of general relativity is the ADM formulation in which the configuration space variable is given by the spatial metric  $q_{ab}$  and its momenta  $\pi^{ab}$  are related to the extrinsic curvature  $K^{ab}$ . The ADM Hamiltonian takes the form

$$H_{\text{ADM}} = \int d^3\mathbf{x} (NC - 2N_a C^a), \quad (2.36)$$

where  $N$  and  $N^a$  are the lapse and shift functions of the ADM metric (1.9),  $C$  is the

Hamiltonian constraint and  $C^a$  is the vector constraint. Explicitly, the Hamiltonian constraint is given by

$$C = \frac{1}{\det(q)} \left( \pi_{ab} \pi^{ab} - \frac{1}{2} \pi^2 \right) + \det(q) R(q) \quad (2.37)$$

and the vector constraint has components

$$C^a = D_b \pi^{ba}, \quad (2.38)$$

where  $D_a$  is the covariant derivative with respect to connection compatible with the spatial metric.

The classical theory LQG is based on is given by Holst gravity which has the same classical equations of motion as Einstein gravity. The Holst action the based on the first-order formulation in terms of tetrads  $e^I = e^I_\mu dx^\mu$  which satisfy

$$g_{\mu\nu} = \eta_{IJ} e^I_\mu e^J_\nu, \quad (2.39)$$

where  $\eta_{IJ}$  is the Minkowski metric. The curvature two-form of the spin connection is

$$R(\omega)^I{}_J = d\omega^I{}_J + \omega^I{}_K \wedge \omega^K{}_J \quad (2.40)$$

The Holst action is given by

$$S_{\text{Holst}}(e, \omega) = \frac{1}{2} \int \epsilon_{IJKL} e^I \wedge e^J \wedge \left( *R(\omega)_{IJ} + \frac{1}{\gamma} R(\omega)_{IJ} \right), \quad (2.41)$$

where  $*R(\omega)_{IJ} = \frac{1}{2} \epsilon_{IJKL} R(\omega)^{KL}$  is the Hodge dual and  $\gamma$  is a constant. The next step is the introduction of Ashtekar–Barbero variables,

$$A_a^i = \Gamma_a^i + \gamma K_a^i, \quad (2.42)$$

where  $\gamma$  is the Barbero–Immirzi parameter<sup>3</sup> (here taken to be the same as the coefficient in the action),  $\Gamma^i = \frac{1}{2} \epsilon^i{}_{jk} \omega^{kj}$  is the spatial component of the spin connection and  $K^i = \omega^{0i}$  is related to the extrinsic curvature. The constraints take the particularly nice form if one introduces the electric field and field strength tensor,

$$E_i^a = \det(e) e_i^a, \quad (2.43)$$

$$F(A)_{ab}^i = \partial_a A_b^i - \partial_b A_a^i + \epsilon^i{}_{jk} A_a^j A_b^k. \quad (2.44)$$

---

<sup>3</sup>The physical meaning of the Barbero–Immirzi parameter is a contested issue. Starting from the classical Holst action there are gauge choices leading to a classical theory independent of the Barbero–Immirzi parameter which would imply its absence in the quantum theory as well. For an overview see, e.g., [20, 52] and references therein.



In the canonical theory it turns out that the Ashtekar–Barbero variable and the electric field are conjugated variables,

$$\{A_a^i(\mathbf{x}), E_j^b(\mathbf{y})\} = \delta_j^i \delta_a^b \delta^3(\mathbf{x} - \mathbf{y}). \quad (2.45)$$

If one chooses the specific value<sup>4</sup> of the Barbero–Immirzi parameter  $\gamma = \pm i$  and the time gauge,  $e_a^0 = 0$ , one finds that the Hamiltonian is a sum of the constraints

$$C = \epsilon_i^{jk} F_{ab}^i E_j^a E_k^b, \quad (2.46a)$$

$$C_a = F_{ab}^i E_i^b, \quad (2.46b)$$

$$G_i = D_a E_i^a, \quad (2.46c)$$

where  $D_a$  is the covariant derivative of the one-form  $A$ . Note that the choice  $\gamma = \pm i$  corresponds to the Ashtekar–Barbero variables being the (anti-)self-dual part of the spin connection as can be seen from (2.42).

Note that in the above description the internal indices  $(i, j, k, \dots)$  are  $so(3)$  indices since the spatial part of the Minkowski metric is invariant under spatial rotations. However, since the Lie algebras  $so(3)$  and  $su(2)$  are isomorphic, one often instead interprets the indices to be  $su(2)$  indices. In particular, one then introduces the  $su(2)$ -valued one-form

$$A(\mathbf{x}) = A_a^i(\mathbf{x}) \sigma_i dx^a, \quad (2.47)$$

where the  $\sigma_i$  are the Pauli matrices.

Dirac formulated a way to quantise constrained dynamical systems [48]. It was the initial goal of the LQG programme to carry out this procedure for the gravitational theory. In Dirac quantisation, given a constraint  $\hat{C}$  the requirement on physical states is that they are in the kernel of  $\hat{C}$ , i.e.,

$$\hat{C}|\Psi\rangle = 0 \quad (2.48)$$

for physical states  $|\Psi\rangle$ .

The main result of the canonical LQG program is the implementation of this quantisation condition for the vector and Gauss constraints. The Hilbert space of states annihilated by those constraints is referred to as the *kinematical Hilbert space* and is the space of spin network states. A spin network state is a graph where the edges carry representation

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<sup>4</sup>A careful analysis for general  $\gamma$  in the time gauge can be found in [62].

labels and the vertices are labelled by intertwiners (compatible with the representation labels of the ingoing and outgoing edges). These spin networks can be interpreted as a dual graph of some discretisation of a spatial manifold. LQG defines operators which allow one to assign to each edge of the spin network an area and to each vertex a volume. This makes sense since when viewed as a dual graph, edges are dual to surfaces and vertices are dual to three-dimensional polyhedra (assuming we are considering  $(3 + 1)$ -dimensional general relativity). Another remarkable result of LQG is that the spectra of both the area and volume operators have a gap relative to the no-geometry state (the “vacuum” of the theory). That is, if there is some space, then it has to have a minimal size. The geometric operators in the quantum theory are defined as the quantisations of the corresponding integrated quantities in the classical theory [8, 9].

The results stated above indicate that there is some discreteness to the states of LQG. Nevertheless, the theory which was used a starting point, namely general relativity, is a continuous theory and it should be possible to establish a connection between the discrete structure and the continuous theory. This issue can be viewed from different angles. One point of view is that one of the defining properties of general relativity is its diffeomorphism invariance and that this must be encoded in the discrete structures [49]. Another point of view is that it should be possible to go from the discrete theory to the continuum via a refinement limit. By studying the theory at different coarse grainings one can establish a flow similar to the renormalisation group flow in a background-dependent theory [51].<sup>5</sup>

The above discussion was concerning the kinematical Hilbert space where the vector and Gauss constraints are satisfied. Implementation of the Hamiltonian constraint which would result in dynamics is technically challenging and there is no consensus what the correct way of proceeding should be. Spin foams are another proposal to implement dynamics that steps outside the framework of canonical quantisation and draws inspiration from the path integral approach of quantum mechanics and is sometimes referred to as covariant formulation of LQG. A textbook focussing on spin foams is [130] and reviews can be found in [15, 117].

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<sup>5</sup>The issue of lattice refinement has also been studied in the context of LQC. There the idea is that an expanding universe should correspond to vertices being added to the spin network states underlying the spatial geometry. See [132] for an overview.

The basic idea of spin foams is that in analogy to quantum field theory one can calculate a transition amplitude from an initial to a final state. In the case of spin foams the initial and final states are given by spin networks. On a technical level a spin foam is a decorated 2-complex where faces carry representation labels and edges carry intertwiner labels. Thus, any slice of a spin foam gives a spin network.

Schematically, what is calculated is a transition amplitude [59]

$$\langle s_f | s_i \rangle = \mathcal{A}(\Gamma) = \sum_{j_f, \iota_e} \prod_f \mathcal{A}_f(j_f) \prod_e \mathcal{A}_e(\iota_e) \prod_v \mathcal{A}_v(j_f, \iota_e), \quad (2.49)$$

where  $s_i$  and  $s_f$  are the initial and final spin networks, respectively, and  $\Gamma$  is the 2-complex with boundary  $\partial\Gamma = s_i \cup s_f$  and with faces  $f$  labelled by  $j_f$ , edges  $e$  labelled by  $\iota_e$  and vertices  $v$ . Assuming that one wants to model a four-dimensional spacetime, the spin networks would represent spatial three-dimensional manifolds and the spin foam can be viewed as the four-dimensional spacetime itself. The restriction to a single spin foam is problematic for two reasons. Firstly, four-dimensional general relativity has local degrees of freedom and there is no reason to believe that the result should be independent of the discretisation chosen.<sup>6</sup> Secondly, quantum mechanics, in particular the path integral approach, claims that one needs to take a weighted average over *all* possible intermediate states. Both problems can be addressed by taking the sum over all possible 2-complexes interpolating between the initial and final state, e.g.,

$$\langle s_f | s_i \rangle = \sum_{\Gamma} w_{\Gamma} \mathcal{A}(\Gamma). \quad (2.50)$$

This is exactly the type of expansion one finds in a Feynman expansion in GFT (2.34). Indeed, it was shown in [124] that for any spin foam model one can find a suitable GFT model which results in the desired spin foam amplitude  $\mathcal{A}(\Gamma)$ .

The above discussion shows how GFT is related to LQG and spin foams. One useful perspective is that GFT is the second quantisation of LQG [107, 108, 110]. Fock states in GFT can be identified with spin network states. This is comparable to ordinary quantum field theory where the excitations of the field correspond to particles. The transition amplitude from one state to another is then in both cases given by the overlap which can be computed by means of a sum over Feynman diagrams.

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<sup>6</sup>Note that even though only one 2-complex is chosen, one still sums over all possible labellings of the 2-complex. Therefore one does actually perform a sum over geometries, albeit not the most general one possible.

### 2.2.2. Tensor models

Group field theories can also be seen as a generalisation of tensor models which we will now briefly discuss.

The main idea of tensor models is to define a zero-dimensional quantum field theory of rank  $d$  tensors whose Feynman diagrams can be viewed as discrete manifolds.<sup>7</sup> The dimension of the tensors is denoted by  $N$ . One of the main focuses of tensor model theory is to find an expansion of the theory in terms of the dimension  $N$ . The idea is then that the large  $N$  limit corresponds to a continuum limit. Ideally, the choice of action of the tensor model is also in agreement with the discretised action obtained from a metric theory of gravity. Both of these points can be seen explicitly in the two-dimensional case discussed below.

We now discuss the simplest non-trivial case,  $d = 2$ , which are also known as matrix models. A review of matrix models can be found in [46].

Matrix models are a theory of quantum gravity in two dimensions. Indeed, matrix models can be viewed as corresponding to a direct discretisation of a metric theory of gravity. In two dimensions the (Riemannian) gravitational action is given by

$$\begin{aligned} S_{\text{grav}}^{(2d)}((M, g)) &= \frac{1}{2\kappa} \int_M d^2x \sqrt{g(x)} (-R(g(x)) + 2\Lambda) \\ &= -\frac{2\pi}{\kappa} \chi((M, g)) + \frac{\Lambda}{\kappa} A((M, g)), \end{aligned} \quad (2.51)$$

where  $(M, g)$  is a metric manifold  $M$  with metric  $g$ ,  $A((M, g)) = \int_M d^2x \sqrt{g(x)}$  is the total area,  $\chi((M, g)) = \frac{1}{4\pi} \int_M d^2x \sqrt{g(x)} R(g(x)) = 2 - 2h$  is the Euler characteristic which in two dimensions is related to the genus  $h$  of the manifold and  $\kappa$  and  $\Lambda$  are coupling constants. Were we now to discretise this manifold by a triangulation  $T$ , the action could be written as

$$\tilde{S}_{\text{grav}}^{(2d)}(T) = -\frac{2\pi}{\kappa} \chi(T) + \frac{\Lambda}{\kappa} A(T), \quad (2.52)$$

where the Euler characteristic is given by  $\chi(T) = V(T) - E(T) + F(T)$ , where  $V(T)$ ,  $E(T)$  and  $F(T)$  count the number of vertices, edges and faces of the triangulation, respectively, and assuming that each triangle has an area  $a_\Delta$  the area is given by  $A(T) = a_\Delta F(T)$ .

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<sup>7</sup>The reason we call them field theories nevertheless is because their description uses the same techniques as quantum field theory.

The main idea is that the Feynman expansion of a suitably chosen matrix model can be interpreted as a sum over random triangulations of the gravitational path integral,

$$Z_{\text{grav}}^{(2d)}(\kappa, \Lambda) = \sum_M \int \mathcal{D}g e^{-S_{\text{grav}}^{(2d)}((M,g))} \simeq \sum_T \frac{1}{|\text{sym}(T)|} e^{-\tilde{S}_{\text{grav}}^{(2d)}(T)} = \tilde{Z}_{\text{grav}}^{(2d)}(\kappa, \Lambda), \quad (2.53)$$

where  $\text{sym}(T)$  is the automorphism group which leaves the triangulation invariant and the sum over  $M$  is a sum over compact two-manifolds which is just the sum over topologies characterised by the genus of the surface. The relation ‘ $\simeq$ ’ is used to denote passing over to a discretised theory, where the sum over all triangulations  $T$  captures both the topological and metric content of the metric manifold  $(M, g)$ .

A suitable matrix model for establishing a connection to the gravitational action is given by the partition function<sup>8,9</sup>

$$e^{Z_{\text{mat}}(g,N)} = \int dM e^{-S_{\text{mat}}(M)}, \quad (2.54)$$

with an action

$$S(M) = N \left( \frac{1}{2} \text{tr}(M^2) + g \text{tr}(M^3) \right), \quad (2.55)$$

where  $N$  is the dimension of the Hermitian matrix  $M$ ,  $g$  is a coupling constant and the integral measure on the space of Hermitian matrices is normalised such that  $\int dM = 1$ . In the Feynman expansion of  $Z_{\text{mat}}(g, N)$  each vertex comes with a factor of  $gN$ , each edge (propagator) comes with a factor of  $N^{-1}$  and each loop comes with a factor of  $N$  due to the resulting trace over the identity matrix. The partition function is therefore given by a sum over Feynman graphs  $\Gamma$ ,

$$Z_{\text{mat}}(g, N) = \sum_{\Gamma} \frac{1}{|\text{sym}(\Gamma)|} g^{V(\Gamma)} N^{\chi(\Gamma)}, \quad (2.56)$$

where  $\Gamma$  is a graph with  $V(\Gamma)$  vertices and  $\chi(\Gamma)$  is its Euler characteristic. By noting that  $\Gamma$  can be interpreted as the dual graph of a triangulation  $T$ , i.e.,  $\Gamma = T^*$  and since  $\chi(T) = \chi(T^*)$  one arrives at the following relation

$$Z_{\text{mat}} \left( e^{-\frac{a\Delta\Lambda}{\kappa}}, e^{\frac{2\pi}{\kappa}} \right) = \tilde{Z}_{\text{grav}}^{(2d)}(\kappa, \Lambda). \quad (2.57)$$

<sup>8</sup>Here  $Z$  is the partition function, where the sum is over connected Feynman diagrams. By taking the exponential one can get the sum over all Feynman diagrams.

<sup>9</sup>The symbol  $M$  now denotes matrices and should not be confused with the  $M$  previously used to denote manifolds.

At this stage one might wonder what has been gained by considering the formulation of discrete gravity in two dimensions as a matrix model, since the sum over all Feynman diagrams is as complicated as the sum over all triangulations. The key insight is that for the matrix model it is possible to perform an expansion in powers of  $N^{-1}$  and, furthermore, the limit  $N \rightarrow \infty$  gives the continuum limit.<sup>10</sup> The continuum limit is often referred to as the *double scaling limit* since when taking the limit  $N \rightarrow \infty$  one has to tune the coupling constant to a critical value,  $g \rightarrow g_c$ . Heuristically, taking the limit of  $N \rightarrow \infty$  restores the infinite number of degrees of freedom of gravity and the tuning of the coupling  $g$  corresponds to considering arbitrarily fine triangulations.

Taking inspiration from the success of the matrix model discussed above, it is natural to hope that rank  $d$  tensors should correspond to modelling  $d$ -dimensional space. Unfortunately, already in three dimensions this approach faces severe obstacles, closely related to the fact that one cannot classify higher dimensional manifolds in a simple manner. As pointed out above, the crucial technical step is to perform an expansion in powers of  $N^{-1}$ .

We now turn to the connection of tensor models to GFTs. From the mode-expanded form of the action (2.14) it is apparent that GFTs which permit a mode expansion can be viewed as multi-tensor models with an infinite number of tensors<sup>11</sup>. For instance, in the case of the group  $G = SU(2)$ , the kinetic term in the action can be written as

$$K(\varphi, \bar{\varphi}) = \sum_{j, \iota, \underline{m}, \underline{n}} \bar{\varphi}_{(j, \iota)}^{\underline{m}} \mathcal{K}_{\underline{m}\underline{n}}^{(j, \iota)} \varphi_{(j, \iota)}^{\underline{n}}. \quad (2.58)$$

The representation label  $(j, \iota)$  labels the different tensors and the indices  $\underline{m}$  and  $\underline{n}$  label the components of the tensors  $\bar{\varphi}_{(j, \iota)}$  and  $\varphi_{(j, \iota)}$ , respectively.

Recently it has been shown that one can perform an  $N^{-1}$  expansion in models that are referred to as *coloured* group field theories (and coloured tensor models) [83]. Coloured GFTs generate graphs which are dual to simplicial manifolds in their Feynman expansion. In a coloured GFT there is a family  $\{\varphi_a\}_{a=1}^{d+1}$  of group fields each labelled by a colour  $a$ . The interaction term features each colour once and therefore the interaction term is of

<sup>10</sup>This is closely related to the  $N^{-1}$  expansion in quantum chromodynamics which is also the field in which matrix models were first introduced [137].

<sup>11</sup>If one considers quantum groups instead of classical Lie groups, one can obtain models which only have a finite number of representations. A well-known example of this is the Turaev–Viro model [141].

order  $d + 1$  in the group fields which agrees with our previous considerations on simplicial GFTs. Note that although one introduces additional fields the interaction term is more restrictive since it features each field only once. The resulting colouring of the Feynman graphs provide a better handle on the combinatorics needed to classify and calculate the their contributions to the state sum, allowing for the aforementioned  $N^{-1}$  expansion. For an overview of tensorial theories in the context of quantum gravity see [125, 126].

### 2.2.3. Boulatov model

We close this chapter by giving an explicit example of a GFT which provides an illustration of how the GFT partition function can be seen as a sum over discrete geometries.

The *Boulatov model* is a group field theory which describes three-dimensional Euclidean gravity [31]. The group field is real,  $\varphi: G^3 \rightarrow \mathbb{R}$ , and satisfies the cyclic permutation symmetry,

$$\varphi(g_1, g_2, g_3) = \varphi(g_2, g_3, g_1) = \varphi(g_3, g_2, g_1). \quad (2.59)$$

The Boulatov model is defined via an action

$$\begin{aligned} S(\varphi) = & \frac{1}{2} \int d^3g \varphi(g_1, g_2, g_3) \varphi(g_1, g_2, g_3) \\ & - \frac{\lambda}{4!} \int d^6g \varphi(g_1, g_2, g_3) \varphi(g_1, g_4, g_5) \varphi(g_2, g_5, g_6) \varphi(g_3, g_6, g_4). \end{aligned} \quad (2.60)$$

In the case where the gauge group is  $G = SO(3)$ , the (Euclidean) partition function of the theory can then be given as a perturbative expansion

$$Z = \int \mathcal{D}\varphi e^{-S[\varphi]} = \sum_C \lambda^{N_T(C)} \sum_{\{j_l\} \in \text{Irrep}} \prod_{l \in C} (2j_l + 1) \sum_{T \in C} \begin{Bmatrix} j_{T_1} & j_{T_2} & j_{T_3} \\ j_{T_4} & j_{T_5} & j_{T_6} \end{Bmatrix}, \quad (2.61)$$

where the sum over  $C$  denotes a sum over simplicial complexes,  $N_T(C)$  is the number of tetrahedra within that simplicial complex, the links  $l$  of the simplicial complex are labelled by representation labels  $j_l$  and  $\{\cdot\}$  is the Wigner  $6j$ -symbol for the six representation labels labelling the edges of each tetrahedron. The choice of the gauge group  $SO(3)$  can be seen as corresponding to local rotations in three-dimensional space which one would expect as the local gauge freedom in a three-dimensional theory of (Euclidean) gravity. The summands in the sum over complexes  $C$  in (2.61) are given by the *Ponzano–Regge*

*state sum* (weighted by the factor  $\lambda^{N_T(C)}$ ). Since the Ponzano–Regge model is a theory of discrete three-dimensional gravity [122] (see also [17]). Since in the Ponzano–Regge model the discretisation is fixed, we see that the Boulatov model does indeed feature a sum over discretisations.

Note that this pattern where a GFT provides a sum over all possible discretisations (including different topologies) is generic and extends to higher dimensions than three. For instance, the amplitudes of the Barrett–Crane model [16] can be viewed as the weights in a sum of a GFT defined on the group  $G = SO(4)/SO(3)$  [43]. Indeed, this is closely related to the discussion in Section 2.2.1 where we pointed out that any spin foam amplitude can be seen as a weight in a perturbative expansion of a GFT.



## Chapter 3.

# Group field theory cosmology

This chapter introduces the cosmological sector of GFT. By coupling to matter degrees of freedom it is possible to derive an equation similar to the Friedmann equation of FLRW cosmology. Reviews of GFT cosmology are given in [79, 120].

### 3.1. Overview

Group field theory aims to describe the microscopic degrees of freedom that constitute spacetime which are assumed to be fundamentally discrete. A question one faces when studying such discrete structures is what the connection to continuum physics is. One possibility would be to try to reconstruct the geometry of the spacetime manifold by some kind of coarse graining procedure. Ideally, one would be able to recover a smooth manifold by taking an adequate limit of the discrete theory. The details of such a limit remain an open question (see e.g. [50]) and we will not pursue it further in what follows.

Another possibility of making contact with continuum physics is by considering certain classes of states. Since we observe that spacetime can be approximated as a smooth manifold at all scales currently observable the conclusion is that the atoms of spacetime must be minuscule and that the universe we live in is made up of a very large number of such building blocks. Indeed, drawing inspiration from condensed matter physics it is natural to propose that the coherent excitation of the microscopic building blocks which gives rise to a smooth macroscopic spacetime can be modelled as a condensate state.

Although it would be desirable to be able to derive the spacetime manifold and its metric from the microscopic theory, it is possible to sidestep the introduction of coordinates by adopting a relational point of view. From the relational point of view coordinates are not necessary to describe the dynamics of the quantities of interest. Rather, what is important are the correlations of the relevant observables. The prototypical example of this is a flat FLRW universe coupled to a massless scalar field, where the physical observables are the scale factor and the value of the scalar field (cf. Appendix A). In a coordinate-based formulation one would write the physical observables as a function of coordinate time. However, it is possible to invert, say, the function of the scalar field as a function of time and this is able to write the scale factor as a function of the value of the scalar field. This can be extended to less symmetrical settings by introducing more matter degrees of freedom which then serve as a reference frame [33]. This relational framework can be imported into GFT by introducing matter degrees of freedom as the argument of the group field.

In summary, there are two main ideas which allow us to study the cosmological sector of GFT:

1. Choice of special states.
2. Coupling of matter degrees of freedom to act as reference frame.

We will next discuss these two points in more detail in the following sections.

### 3.1.1. Condensate states

Since we expect the continuous spacetime we observe to be constituted of a large number of discrete building blocks, one possibility is to model spacetime as a condensate state of the underlying GFT. Early works on this perspective are [67, 74, 75].

The simplest type of condensate is defined as

$$|\sigma\rangle = \mathcal{N}(\sigma) \exp\left(\int d^M g \sigma(g_I) \hat{\varphi}^\dagger(g_I)\right) |\emptyset\rangle, \quad (3.1)$$

where  $\mathcal{N}(\sigma)$  is a normalising factor

$$\mathcal{N}(\sigma) = \exp\left(-\frac{1}{2} \int d^M g |\sigma(g_I)|^2\right). \quad (3.2)$$

Similarly to the mode expansion (2.13) of the group field in the case of a compact Lie group, we define a mode expansion of the condensate wave function as

$$\sigma(g_I) = \sum_{J \in \mathcal{J}} \sigma_J D_J(g_I), \quad (3.3)$$

where  $\mathcal{J}$  is the set of labels. The imposition of the symmetry requirements is then achieved by considering only a subset of labels,

$$\bar{\mathcal{J}} \subset \mathcal{J}. \quad (3.4)$$

The condensate wave function which respects the desired symmetries then only has modes in that restricted subset,

$$\sigma_{\text{symmetries}}(g_I) = \sum_{J \in \bar{\mathcal{J}}} \sigma_J D_J(g_I). \quad (3.5)$$

The details of this truncation depend on the specific model studied and are quite involved. The most studied model is the case in which the group field is defined on four copies of  $SU(2)$  where the building blocks are interpreted as tetrahedra. A detailed explanation of how homogeneity and isotropy are implemented for this model can be found in [111]. Whilst discussing the implementation of homogeneity is beyond the scope of this text, isotropy can be understood more easily. In the mode expansion, the  $su(2)$  representation labels of the group field then correspond to the areas of the faces of a tetrahedron. To impose isotropy one then imposes the criterion that all the labels be the same, i.e., that the tetrahedra are all equilateral.

### 3.1.2. Coupling to matter degrees of freedom

In discrete approaches to quantum gravity, coupling to matter degrees of freedom is usually achieved by decorating each vertex of the discrete graph with the value of the matter degrees of freedom. If one interprets the vertex as being dual to a volume of space,

then the interpretation is that the matter degree of freedom takes the value defined at the vertex in this corresponding region of space. GFT is no exception to this idea and we now turn to the coupling of scalar fields to a GFT.

In ordinary quantum field theory, a real scalar field is a function mapping points of the spacetime manifold  $\mathcal{M}$  to the real numbers,  $\chi: \mathcal{M} \rightarrow \mathbb{R}$ . For our purposes we are only interested in the range of the scalar field, namely the real numbers. A group field coupled to a scalar field is then defined as a function

$$\varphi: G^M \times \mathbb{R} \rightarrow \mathbb{K}, \quad (g_I, \chi) \mapsto \varphi(g_I, \chi), \quad \mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}. \quad (3.6)$$

Coupling to more than one scalar field can be achieved by adding the desired number of fields as arguments to the group field. For instance, if we want to couple  $D$  scalar fields to the group field we have for the group field

$$\varphi: G^M \times \mathbb{R}^D \rightarrow \mathbb{K}, \quad (g_I, \chi_\alpha) \mapsto \varphi(g_I, \chi^\alpha), \quad \mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}, \quad (3.7)$$

where  $\chi^\alpha = (\chi^0, \dots, \chi^d)$  and  $D = d + 1$ .

As explained above the main reason for coupling the group field to one or more massless scalar fields is to view the matter fields as a reference frame, similar to models in classical general relativity as, e.g., Brown–Kuchař dust. In the case of a single scalar field we interpret that scalar field as a relational clock. In the case of multiple scalar fields we will view the scalar field  $\chi^0$  as a relational clock and the remaining scalar fields  $\chi = (\chi^1, \dots, \chi^d)$  as spatial. With this interpretation it is natural to identify  $D$  with the dimension of spacetime. However, we choose to leave it general in the following.

### 3.1.3. Quantum theory

The addition of the scalar field degrees of freedom to the GFT require that we revisit the discussion of the quantum theory discussed in Section 2.1.2. In the case of GFT cosmology we only consider operator formulations of the quantum theory. When specifying the commutation relations for this extended GFT there is some ambiguity.

The covariant extension of (2.24) is given by the commutator

$$[\hat{\varphi}_J(\chi^\alpha), \hat{\varphi}_{J'}^\dagger(\chi'^\alpha)] = \delta_{JJ'} \delta^D(\chi^\alpha - \chi'^\alpha). \quad (3.8)$$

This commutator was discussed in [74, 75, 79]. However, it is often desirable to define relational operators via

$$\hat{O}(\chi^0) = \sum_J \int d^d \chi \hat{O}_J(\chi^\alpha). \quad (3.9)$$

This could for instance provide us with a volume operator defined at a particular instance of relational time. The commutator of two such relational operators is formally divergent quantity when using the commutator (3.8) which would need to be regularised [13, 70, 73]. One possible choice for regularisation is to instead impose only an equal-time commutation relation [2, 70],

$$[\hat{\varphi}_J(\chi^0, \boldsymbol{\chi}), \hat{\varphi}_{J'}^\dagger(\chi^0, \boldsymbol{\chi}')] = \delta_{JJ'} \delta^d(\boldsymbol{\chi} - \boldsymbol{\chi}'). \quad (3.10)$$

There is another possibility. In the case of a flat FLRW universe with one or more massless scalar fields all the degrees of freedom are usually parametrised as a function of coordinate time. However, the functional dependence of the scalar fields is invertible and therefore and can go to a deparametrised description which does not feature the unobservable coordinate time. (See also Appendix A.) Taking inspiration from this fact we can consider one of the massless scalar fields, say  $\chi^0$ , as functioning as a ‘‘clock’’ in a deparametrised setting of the GFT. One can then employ the canonical quantisation procedure with respect to that massless scalar field clock to obtain yet another commutation relation [76, 77, 144],

$$[\hat{\varphi}_J(\chi^0, \boldsymbol{\chi}), \pi_{J'}(\chi^0, \boldsymbol{\chi}')] = i \delta_{JJ'} \delta^d(\boldsymbol{\chi} - \boldsymbol{\chi}'), \quad (3.11)$$

where  $\pi_J(\chi^\alpha)$  is the conjugate momentum of the group field  $\varphi_J(\chi^\alpha)$ . Note that in the case of canonical quantisation it is no longer the field operators which furnish a representation of the bosonic algebra. However, it is always possible to identify operators which satisfy the bosonic commutation relations by taking linear combinations of the canonically conjugate operators.

#### 3.1.4. Observables

Group field theory cosmology aims to provide a quantum gravitational model that is applicable to studying cosmology. In classical FLRW cosmology the system is entirely

captured by the scale factor  $a$ . Instead of the scale factor one can equivalently use any power of it as the variable, in particular one can also view the volume of the universe as the dynamical variable.

From the perspective put forward when discussing the relation of GFT to discrete geometry in Section 2.2 each field excitation corresponds to a spatial building block. The labelling of this building block captures its shape and in particular also its volume. Therefore, one can calculate the volume of a given state by summing over the number of building blocks multiplied by their respective volume. As explained in Section 3.1.3 different quantisation schemes have been put forward in the context of GFT cosmology. In the following we aim to be scheme-agnostic by simply stating that the quantum theory is given by the commutation relation

$$[\hat{a}_J(\chi^\alpha), \hat{a}_{J'}^\dagger(\chi'^\alpha)] = \delta_{JJ'} f(\chi^\alpha, \chi'^\alpha), \quad (3.12)$$

where the function  $f$  is real such that  $\hat{a}_J(\chi^\alpha)$  and  $\hat{a}_{J'}^\dagger(\chi'^\alpha)$  furnish a representation of the bosonic algebra. With this creation and annihilation operators we define a “density operator”

$$\hat{N}_J(\chi^\alpha) = \hat{a}_J^\dagger(\chi^\alpha) \hat{a}_J(\chi^\alpha), \quad (3.13)$$

which counts the number of particles of a given labelling. By rescaling the partial density operator one can get the “partial volume density”

$$\hat{V}_J(\chi^\alpha) = v_J \hat{N}_J(\chi^\alpha), \quad (3.14)$$

where  $v_J$  is the volume of a building block with labels  $J$  and  $\chi^\alpha$ . The value of the coefficient  $v_J$  can be obtained, e.g., from the volume operator defined in LQG which assigns to each node of a spin network (labelled by spin representation and intertwiner labels) a corresponding spatial volume.

Depending on the quantisation scheme chosen there are then two views to define observables. In a scheme in which the commutation relations are equal time commutation relations, the relational observables are defined as

$$\hat{O}(\chi^0) = \sum_J \int d^d \chi \hat{O}_J(\chi^\alpha). \quad (3.15)$$

We adopt this perspective in Chapters 4 and 5 where we discuss the canonical quantisation scheme in great detail. In the more covariant scheme with commutation relation (3.8)

the observables are then given by the fully integrated quantities

$$\hat{O} = \sum_J \int d^D \chi \hat{O}_J(\chi^\alpha). \quad (3.16)$$

We adopt this perspective in Chapter 6 where we discuss the one-body effective approach to GFT cosmology.

### 3.2. Action for GFT cosmology

We now turn to the question of how the action functional of a GFT (cf. (2.6)) should be modified to accommodate the ingredients needed for the cosmological setting.

Our main focus lies on the kinetic term of the GFT action which now needs to be generalised to

$$K(\varphi, \bar{\varphi}) = \int d^M g d^M g' d^D \chi d^D \chi' \bar{\varphi}(g_I, \chi^\alpha) \mathcal{K}(g_I, g'_I, \chi^\alpha, \chi'^\alpha) \varphi(g'_I, \chi'^\alpha). \quad (3.17)$$

As a first simplification we assume that the group  $G$  is compact and that we therefore can perform a mode expansion of the group field with respect to the group argument. Furthermore, we restrict the modes to those compatible with the conditions for homogeneity and isotropy outlined in Section 3.1.1. With these considerations the kinetic term takes the form

$$K(\varphi, \bar{\varphi}) = \sum_{JJ'} \int d^D \chi d^D \chi' \bar{\varphi}_J(\chi^\alpha) \mathcal{K}^{JJ'}(\chi^\alpha, \chi'^\alpha) \varphi_{J'}(\chi'^\alpha). \quad (3.18)$$

If we assume that the kinetic kernel in the group representation is only a function of the Laplace–Beltrami operator we can choose a basis in the mode expansion such that the kinetic kernel is diagonal (“spin representation”). We will do so and arrive at a kinetic kernel of the form

$$K(\varphi, \bar{\varphi}) = \sum_J \int d^D \chi d^D \chi' \bar{\varphi}_J(\chi^\alpha) \mathcal{K}^J(\chi^\alpha, \chi'^\alpha) \varphi_J(\chi'^\alpha). \quad (3.19)$$

The arguments  $\chi^\alpha$  and  $\chi'^\alpha$  are to be interpreted as being the values of scalar fields. In ordinary quantum field theory, the Lagrangian density of such a collection of non-interacting scalar fields on a manifold with metric  $g$  is given by

$$\mathcal{L}_{\text{sf}} = -\frac{1}{2} \sqrt{-\det(g)} g^{\mu\nu} \partial_\mu \chi^\alpha \partial_\nu \chi^\beta \delta_{\alpha\beta}, \quad (3.20)$$

### Chapter 3. Group field theory cosmology

where  $\delta_{\alpha\beta}$  is the metric on the set of scalar fields and we use it to raise and lower indices of the scalar fields in the following. This Lagrangian density is invariant under the action of the Euclidean group  $E(D)$ , i.e., it is invariant under:

- Translations:  $\chi^\alpha \mapsto \chi^\alpha + a^\alpha$ ,  $a^\alpha \in \mathbb{R}^D$
- Rotations:  $\chi^\alpha \mapsto R^\alpha_\beta \chi^\beta$ ,  $R^\alpha_\beta \in SO(D)$
- Inversion:  $\chi^\alpha \mapsto -\chi^\alpha$

Since we want our GFT action to be compatible with this interpretation, we are led to the following form of the kinetic kernel,

$$\mathcal{K}^J(\chi^\alpha, \chi'^\alpha) = \mathcal{K}^J((\chi - \chi')^2), \quad (3.21)$$

where  $(\chi - \chi')^2 = \delta_{\alpha\beta}(\chi^\alpha - \chi'^\alpha)(\chi^\beta - \chi'^\beta)$ . Inserting this form into the kinetic term one gets

$$\begin{aligned} K(\varphi, \bar{\varphi}) &= \sum_J \int d^D \chi d^D \chi' \bar{\varphi}_J(\chi^\alpha) \mathcal{K}^J((\chi - \chi')^2) \varphi_J(\chi'^\alpha) \\ &= \sum_J \int d^D \chi d^D \chi' \bar{\varphi}_J(\chi^\alpha) \mathcal{K}^J((\chi')^2) \varphi_J(\chi^\alpha + \chi'^\alpha) \\ &= \sum_J \int d^D \chi d^D \chi' \bar{\varphi}_J(\chi^\alpha) \mathcal{K}^J((\chi')^2) \\ &\quad \times \sum_{n=0}^{\infty} \frac{1}{n!} \chi'^{\alpha_1} \dots \chi'^{\alpha_n} \frac{\partial^n}{\partial \chi^{\alpha_1} \dots \partial \chi^{\alpha_n}} \varphi_J(\chi^\alpha) \\ &= \sum_{n=0}^{\infty} \sum_J \int d^D \chi \bar{\varphi}_J(\chi^\alpha) \mathcal{K}_J^{(2n)} \Delta_\chi^n \varphi_J(\chi^\alpha), \end{aligned} \quad (3.22)$$

where we defined

$$\Delta_\chi = \sum_{\alpha=0}^d \left( \frac{\partial}{\partial \chi^\alpha} \right)^2, \quad (3.23)$$

$$\mathcal{K}_J^{(2n)} = \frac{1}{(2n)!} \int d^D \chi' ((\chi')^2)^n \mathcal{K}^J((\chi')^2). \quad (3.24)$$

The kinetic term is therefore an infinite sum of terms with an integral power of the Laplace operator  $\Delta_\chi$  acting on the group field. Similar to what is done in effective field theory, we truncate this infinite sum after the second term. In the interpretation of the arguments



as being the values of scalar fields, the regime we are therefore interested in is where the value of the scalar field is varying slowly. The resulting kinetic term therefore is

$$K(\varphi, \bar{\varphi}) = \sum_J \int d^D \chi \bar{\varphi}_J(\chi^\alpha) \left( \mathcal{K}_J^{(0)} + \mathcal{K}_J^{(2)} \Delta_\chi \right) \varphi_J(\chi^\alpha). \quad (3.25)$$

This truncation restricts our discussion to the case where the group field  $\varphi$  varies slowly with respect to the massless scalar fields which is justified from an effective field theory point of view [94].

For a general action of an interacting GFT as defined in (2.4) the classical equations of motion then are given by

$$\left( \mathcal{K}_J^{(0)} + \mathcal{K}_J^{(2)} \Delta_\chi \right) \varphi_J(\chi^\alpha) + \frac{\delta V}{\delta \bar{\varphi}_J(\chi^\alpha)}(\varphi, \bar{\varphi}) = 0, \quad (3.26a)$$

$$\left( \mathcal{K}_J^{(0)} + \mathcal{K}_J^{(2)} \Delta_\chi \right) \bar{\varphi}_J(\chi^\alpha) + \frac{\delta V}{\delta \varphi_J(\chi^\alpha)}(\varphi, \bar{\varphi}) = 0. \quad (3.26b)$$

### 3.2.1. Symmetries

In the case in which we are interested in the regime in which GFT interactions are negligible the action is given only by the kinetic term,

$$S(\varphi, \bar{\varphi}) = K(\varphi, \bar{\varphi}). \quad (3.27)$$

We assume that the kinetic term is given by (3.25).

Continuous symmetries of the action lead to divergenceless currents per Noether's theorem [104]. The perspective taken here is that the group field is parametrised by a family of scalar function  $\varphi_J$  on the manifold  $\mathbb{R}^D$  and the action is given in terms of

$$S(\varphi, \bar{\varphi}) = \int d^D \chi \mathcal{L}(\varphi, \partial_\alpha \varphi, \bar{\varphi}, \partial_\alpha \bar{\varphi}), \quad (3.28)$$

where here and in the following  $\partial_\alpha = \partial/\partial\chi^\alpha$ . Noether's theorem states that to each continuous transformation that transforms the scalar field  $\chi^\alpha$  and group field  $\varphi_J$  as

$$\chi^\alpha \mapsto \chi^\alpha + \delta\chi^\alpha, \quad \varphi_J \mapsto \varphi_J + \delta\varphi_J \quad (3.29)$$

there is a Noether current given by [82]

$$J^\alpha = \sum_J \left( \frac{\partial \mathcal{L}}{\partial(\partial_\alpha \varphi_J)} \delta \varphi_J + \frac{\partial \mathcal{L}}{\partial(\partial_\alpha \bar{\varphi}_J)} \delta \bar{\varphi}_J \right) - \left( \sum_J \left( \frac{\partial \mathcal{L}}{\partial(\partial_\alpha \varphi_J)} \partial_\beta \varphi_J + \frac{\partial \mathcal{L}}{\partial(\partial_\alpha \bar{\varphi}_J)} \partial_\beta \bar{\varphi}_J \right) - \delta_\beta^\alpha \mathcal{L} \right) \delta \chi^\beta. \quad (3.30)$$

The crucial property of the Noether current is that its divergence vanishes,

$$\partial_\alpha J^\alpha = 0. \quad (3.31)$$

We have chosen the kinetic term such that it is invariant under transformation of the Euclidean group  $E(D)$  acting on the scalar field  $\chi^\alpha$ . Since the modes of the group field,  $\varphi_J$ , are scalars on the space of scalar fields,  $\chi^\alpha$ , they transform trivially under translations and rotations. That is, one has  $\delta \varphi_J = 0$  in (3.30). Therefore, for both translations and rotations of the scalar fields  $\chi^\alpha$ , the Noether current is given by

$$J_\alpha = \sum_J (\partial_\alpha \bar{\varphi}_J \partial_\beta \varphi_J + \partial_\beta \bar{\varphi}_J \partial_\alpha \varphi_J + \delta_{\alpha\beta} \mathcal{L}) \delta \chi^\beta. \quad (3.32)$$

Note that all the symmetries we consider are actually valid for each mode ‘ $J$ ’ independently due to the fact that the action is a sum of actions for each  $J$  and we neglected the interaction term which would give a coupling between different modes. Therefore, the divergenceless current  $J_\alpha$  is in fact a sum of divergenceless currents  $J_{J,\alpha}$ ,

$$J_\alpha = \sum_J J_{J,\alpha}. \quad (3.33)$$

For infinitesimal translations the transformation of the scalar fields is given by

$$\delta \chi^\alpha = \epsilon^\alpha \quad (3.34)$$

and we define the “energy-momentum tensor”<sup>1</sup>  $\Theta_{\alpha\beta}$  via

$$(J_{\text{trans}})_\alpha = \Theta_{\alpha\beta} \epsilon^\beta. \quad (3.35)$$

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<sup>1</sup>The names of the conserved charges discussed in this section are meant as an analogy only since they take the same functional form as the charges defined in ordinary scalar field theory. The usual notions of energy and momentum are defined with respect to a spacetime metric. Since GFT is a theory of spacetime itself no clear interpretation can be given on the physical significance of the conserved charges present in the GFT formalism.

For infinitesimal rotations the transformation of the scalar fields is given by

$$\delta\chi^\alpha = \omega^\alpha{}_\beta \chi^\beta, \quad (3.36)$$

where the matrix  $\omega^\alpha{}_\beta$  is antisymmetric, and we define the ‘‘angular momentum tensor’’  $M_{\alpha\beta\gamma}$  via

$$(J_{\text{rot}})_\alpha = M_{\alpha\beta\gamma} \omega^{\beta\gamma} \quad (3.37)$$

which is related to the energy-momentum tensor by the relation

$$M_{\alpha\beta\gamma} = \Theta_{\alpha\beta} \chi_\gamma - \Theta_{\alpha\gamma} \chi_\beta. \quad (3.38)$$

As explained above, our choice of action was motivated by considering the action of a collection of massless scalar fields in ordinary quantum field theory on curved spacetime (cf. (3.20)). Clearly the continuous transformations considered here will also give rise to Noether currents in the ordinary quantum field theory, where now  $\chi^\alpha$  takes the role of the fields and the arguments are the spacetime coordinates  $x^\mu$ . The scalar field theory defined on a background manifold also permits the definition of conjugate momenta,

$$\pi_\chi^\alpha = \frac{\partial \mathcal{L}_{\text{sf}}}{\partial \partial_0 \chi_\alpha} = -\sqrt{-\det(g)} g^{0\mu} \partial_\mu \chi^\alpha. \quad (3.39)$$

For translations that shift only one of scalar fields (and leave the others invariant) the resulting Noether current is

$$(J_{\text{sf, trans}})_\alpha^\mu = -\sqrt{\det(g)} g^{\mu\nu} \partial_\nu \chi_\alpha. \quad (3.40)$$

By integrating over the spatial coordinates of the time-component of the current one arrives at the conserved Noether charge

$$(P_{\text{sf}})^\alpha = - \int d^d \mathbf{x} \sqrt{-\det(g)} g^{0\mu} \partial_\mu \chi^\alpha = \int d^d \mathbf{x} \pi_\chi^\alpha \quad (3.41)$$

which corresponds to the total conjugate momentum of the scalar field  $\chi^\alpha$ . Similarly, rotations of the scalar fields (3.36) give rise to the Noether current

$$(J_{\text{sf, rot}})_{\alpha\beta}^\mu = -\frac{1}{2} \sqrt{-\det(g)} g^{\mu\nu} (\partial_\nu \chi_\alpha \chi_\beta - \partial_\nu \chi_\beta \chi_\alpha). \quad (3.42)$$

The conserved Noether charge is given by

$$\begin{aligned} (M_{\text{sf}})^{\alpha\beta} &= -\frac{1}{2} \int d^d \mathbf{x} \sqrt{-\det(g)} g^{0\nu} (\partial_\nu \chi_\alpha \chi_\beta - \partial_\nu \chi_\beta \chi_\alpha) \\ &= \frac{1}{2} \int d^d \mathbf{x} (\partial_\nu \pi_\chi^\alpha \chi^\beta - \pi_\chi^\beta \chi^\alpha). \end{aligned} \quad (3.43)$$

In Chapters 4 and 5 the formalism of GFT cosmology is discussed in a canonical framework. In those chapters we will identify the conserved quantities of the GFT models with those of the analogous system of ordinary quantum field theory when interpreting our results.

In the case of a complex group field, the action we are considering also exhibits a  $U(1)$  symmetry acting on the group field  $\varphi_J$ ,

$$\varphi_J(\chi^\alpha) \mapsto e^{i\phi} \varphi_J(\chi^\alpha) \quad (3.44)$$

which for infinitesimal  $\phi$  gives a divergenceless current  $J_\alpha = \phi I_\alpha$ ,

$$I_\alpha = i \sum_J (\partial_\alpha \bar{\varphi}_J \varphi_J - \bar{\varphi}_J \partial_\alpha \varphi_J) . \quad (3.45)$$

### 3.3. Approaches to GFT cosmology

This section provides an overview of some of the different approaches put forward in the study of GFT cosmology.

#### 3.3.1. Mean-field theory approach

The mean-field perspective draws from non-relativistic quantum field theory and is interested in states that satisfy

$$\langle \hat{\varphi}_J(\chi) \rangle = \sigma_J(\chi) . \quad (3.46)$$

Note that for simplicity we restrict ourselves to the case of a single scalar field  $\chi$ , which is also the most studied system in this context. Furthermore, a normal-ordering prescription is applied which eliminates any quantum corrections arising from commuting the field operators.

The equations of motions are simply those of the classical theory, where the group field is

replaced by the condensate wave function,

$$\left(\mathcal{K}_J^{(0)} + \mathcal{K}_J^{(2)} \Delta_\chi\right) \sigma_J(\chi^\alpha) + \frac{\delta V}{\delta \bar{\sigma}_J(\chi^\alpha)}(\sigma, \bar{\sigma}) = 0, \quad (3.47a)$$

$$\left(\mathcal{K}_J^{(0)} + \mathcal{K}_J^{(2)} \Delta_\chi\right) \bar{\sigma}_J(\chi^\alpha) + \frac{\delta V}{\delta \sigma_J(\chi^\alpha)}(\sigma, \bar{\sigma}) = 0. \quad (3.47b)$$

These equations are analogue to the Gross–Pitaevskii equations of non-relativistic quantum field theory.

The main object of interest is the relational volume operator which is defined as

$$\hat{V}(\chi) = \int d^M g d^M g' \hat{\varphi}^\dagger(g, \chi) V(g, g') \hat{\varphi}(g', \chi), \quad (3.48)$$

which in terms of the partial volume densities (and specialising to the case of a single scalar field) is given by

$$\hat{V}(\chi) = \sum_J v_J \hat{\varphi}_J^\dagger(\chi) \varphi_J(\chi). \quad (3.49)$$

The expectation value of the volume operator is then a functional of the condensate wave function,

$$\langle \hat{V}(\chi) \rangle_\sigma = \sum_J v_J |\sigma(\chi)|^2. \quad (3.50)$$

By solving the equations of motion for the condensate wave function it is possible to derive a Friedmann-like equation [111]

$$\left(\frac{\langle \hat{V}'(\chi) \rangle_\sigma}{\langle \hat{V}(\chi) \rangle_\sigma}\right)^2 = \left(\frac{2 \sum_J v_J \rho_J(\chi) \text{sgn}(\rho_J'(\chi)) \sqrt{E_J - \frac{Q_J^2}{\rho_J(\chi)^2} + m_J^2 \rho_J(\chi)^2}}{\sum_J v_J \rho_J(\chi)^2}\right)^2, \quad (3.51)$$

where  $m_J^2 = \mathcal{K}_J^{(0)}/\mathcal{K}_J^{(2)}$ ,  $\rho_J$  is the absolute value of  $\sigma_J$  and  $E_J$  and  $Q_J$  are conserved quantities.

### 3.3.2. Canonical approach

One of the main ideas for coupling the group field to massless scalar fields was to have access to some variable with respect to which one might adopt a relational point of view which circumvents the introduction of spacetime coordinates. The canonical approach to

GFT cosmology takes this perspective to its logical conclusion and treats one of the scalar fields as a time variable. With a time variable at our disposal it is possible to perform a Legendre transformation from the Lagrangian theory to a Hamiltonian theory. The quantum theory is then obtained by the usual process of canonical quantisation where the fields are promoted to operators and the commutation relation is directly related to the canonical Poisson structure. In the case of multiple massless scalar fields a choice has to be made as to which of them should serve as a relational clock. It turns out that this choice breaks the Euclidean symmetry present in the original formulation. To be concrete, assume that we choose  $\chi^0$  to serve as the clock variable then we have the canonical equal time commutation relation,

$$[\hat{\varphi}_J(\chi^0, \boldsymbol{\chi}), \hat{\pi}_J(\chi^0, \boldsymbol{\chi}')] = i\delta_{JJ'}\delta^d(\boldsymbol{\chi} - \boldsymbol{\chi}'), \quad (3.52)$$

where  $\hat{\pi}_J(\chi^\alpha)$  is the operator of the conjugate momentum. Note that this canonical commutation relation is not that of a bosonic creation and annihilation operators. In the sequel we show in detail how to perform a transformation to creation and annihilation operators which do satisfy the bosonic commutation relations. As mentioned above, it is always with respect to the bosonic operators which the number operator and volume operator are defined.

The canonical approach establishes connections to quantum field theory (in the case of multiple scalar fields) and quantum mechanics (in the case of a single scalar field). After having singled out one of the scalar fields,  $\chi^0$ , one can perform a Legendre transformation resulting in the Hamiltonian  $\hat{H}$  of the theory. In the Heisenberg picture the equations of motion of the theory are then given by the Heisenberg equation

$$\hat{O}'(\chi^0) = -i[\hat{O}(\chi^0), \hat{H}]. \quad (3.53)$$

The canonical approach was first proposed in [144] and then further investigated in [76, 77]. The discussion of this approach is the subject of Chapters 4 and 5, where we discuss the case of multiple scalar fields and a single scalar field, respectively.

### 3.3.3. Frozen formalism approach

As discussed in the previous section it is possible to deparametrise the theory before quantising which allows to carry out the procedure of canonical quantisation giving the commutation relations (3.52). However, we already mentioned that the more covariant commutator would be given by (3.8). In [71] it was shown how these two commutation relations can be reconciled.

The starting point for that discussion is the world line action

$$S(\varphi, \bar{\varphi}, N) = \sum_J \int d\tau d^D\chi \left[ \frac{i}{2} \left( \bar{\varphi}_J(\tau, \chi^\alpha) \frac{\partial \varphi}{\partial \tau}(\tau, \chi^\alpha) - \frac{\partial \bar{\varphi}_J}{\partial \tau}(\tau, \chi^\alpha) \varphi(\tau, \chi^\alpha) \right) + N(\tau, \chi) \bar{\varphi}(\tau, \chi^\alpha) \left( \mathcal{K}_J^{(0)} + \mathcal{K}_J^{(2)} \Delta_\chi \right) \varphi(\tau, \chi^\alpha) \right]. \quad (3.54)$$

The equations of motion for  $\varphi$ ,  $\bar{\varphi}$  and  $N$  together imply that  $\varphi$  and  $\bar{\varphi}$  are independent of  $\tau$ . The action also tells us that the field and its complex conjugate are canonically conjugate. Therefore the commutation relation from this theory is given by the same expression as (3.8),

$$[\hat{\varphi}_J(\chi^\alpha), \hat{\varphi}_{J'}^\dagger(\chi'^\alpha)] = \delta_{JJ'} \delta^D(\chi^\alpha - \chi'^\alpha). \quad (3.55)$$

The key realisation of [71] is as follows. If one starts with the kinematical (Fock) Hilbert space generated by the conjugate group field operator, one can define a projection to the physical Hilbert space by using group averaging techniques. The resulting inner product is then exactly the same as that of the canonical theory. In this sense both theories can be seen to be equivalent.

### 3.3.4. Thermal approach

One of the ideas of GFT in general is that the emergence of a continuous spacetime might be understood as a coherent excitation of a large number of spatial building blocks, i.e., a condensate. Condensation is a concept from many-body physics and statistical mechanics and is described as a phase transition. It is therefore natural to employ the tools of statistical mechanics and in particular of thermal field theory.

In [13, 14] this approach to GFT cosmology has been investigated in some detail. The

key object of interest in their approach is the thermal volume Gibbs state

$$\hat{\rho}_\beta = \frac{1}{Z_\beta} e^{-\beta \hat{V}}, \quad (3.56)$$

where  $Z_\beta = \text{tr}(\hat{\rho}_\beta)$ . Expectation values of observables are then computed via the thermal expectation value

$$\langle \hat{O} \rangle_\beta = \text{tr}(\hat{\rho}_\beta \hat{O}). \quad (3.57)$$

As in the other approaches listed in this section one is able to derive an effective Friedmann equation which takes a form similar to (3.51).

### 3.3.5. Coherent peaked states approach

Coherent peaked states present an interesting modification to the usual condensate states and are discussed in detail in [99, 100]. The model studied there is a GFT coupled to a single scalar field. The main goal of this approach is to ameliorate some of the conceptual issues arising in previous definitions of relational observables. For instance, one might ask why one would not integrate over the scalar field degrees of freedom in the definition of the volume operator (3.48). Instead, the proposition is to define the operators as the fully integrated (averaged) quantities, e.g.

$$\hat{V} = \int d^M g d^M g' d\chi \hat{\varphi}^\dagger(g, \chi) V(g, g') \hat{\varphi}(g', \chi). \quad (3.58)$$

Relational time is then introduced by considering special types of condensate wave functions, namely the class of coherent peaked states. A coherent peaked state is defined as the product wave function

$$\sigma(g_I, \chi; \chi_0, \epsilon) = \eta(\chi - \chi_0; \epsilon) \tilde{\sigma}(g_I, \chi), \quad (3.59)$$

where the states now carry to additional labels,  $\chi_0$  and  $\epsilon$ ,  $\eta$  is the peaking function which is a Gaussian distribution centred at  $\chi_0$  and variance  $\epsilon$ , and  $\tilde{\sigma}$  is the condensate wave function defining perturbations away from the peaking function. Note that just making an ansatz such as this does not change the dynamics. Indeed, the idea is to consider only the “instantaneous” equation of motion,

$$\left\langle \frac{\delta \widehat{S}(\varphi, \bar{\varphi})}{\delta \varphi(g_I, \chi_0)} \right\rangle_{\chi_0, \epsilon} = 0, \quad (3.60)$$



where the subscript indicates that the expectation value is to be taken for coherent peaked states with parameters  $\chi_0$  and  $\epsilon$  and the argument of the functional derivative is crucially the same as the parameter with respect to which the expectation value is taken.

The resulting Friedmann equations take the same functional form as those of the mean field approach (3.51), although the parameters appearing throughout are modified by the choice of coherent peaked state parameters.

### 3.3.6. One-body effective approach

There is a formulation of quantum mechanics in which the central role is not taken by states, but rather by the expectation values and moments of certain operators. The main advantage of this is that it allows one to be state-agnostic to some degree. Clearly, if one were to specify all the infinite possible moments, one would describe the state completely. The simplification is achieved by truncating a semi-classical expansion in orders of  $\hbar$  which corresponds to only considering moments up to a certain order<sup>2</sup>. This approach has been studied extensively for simple quantum mechanical systems, both relativistic and non-relativistic and has been employed to study the problem of time [27, 28, 29].

In the context of GFT the first step is to identify the algebra of operators one is interested in. One drastic simplification can be achieved by considering only one-body operators, that is operators that have been fully integrated, e.g.

$$\hat{O} = \int d^M g d^M g' d^D \chi d^D \chi' \hat{\varphi}^\dagger(g_I, \chi^\alpha) O(g_I, g'_I, \chi, \chi') \hat{\varphi}(g'_I, \chi'^\alpha). \quad (3.61)$$

Note that this is very similar to how we defined the volume operator above. The main difference lies in the fact that before we still were able to specify arbitrary states. In the one-body effective approach only the expectation values and moments of the set of operators considered enter. As mentioned above, if one were to specify all the infinite number of moments one specifies the state completely.

The effective approach in the context of GFT was studied in [72] and is the subject of Chapter 6.

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<sup>2</sup>In many cases the order of the moment does scale with the order in  $\hbar$ . Generically this is *still* a restriction on the class of admissible states, though.



## Chapter 4.

# Canonical formulation: Multiple scalar fields

This chapter is based on [77].

In this chapter we discuss the canonical formalism applied to GFT models defined with multiple scalar fields. This extends the formalism which was introduced for the case of a single scalar field of [144] which is also the subject of Chapter 5. Although the case of a single scalar field can be viewed as a special case of the case of more than one scalar field, the technical simplification arising from considering only a single scalar field warrants a separate treatment.

In this chapter we will only be interested in the case where the group field is real.<sup>1</sup> Furthermore we assume that we are in a regime where the group field interactions are negligible and that the action is given only by the kinetic term (3.25),

$$S(\varphi) = \frac{1}{2} \sum_J \int d^D \chi \varphi_J(\chi^\alpha) \left( \mathcal{K}_J^{(0)} + \mathcal{K}_J^{(2)} \Delta_\chi \right) \varphi_J(\chi^\alpha), \quad (4.1)$$

where we rescaled the field to introduce the customary factor of 1/2. We expect the action (4.1) to be valid in the mesoscopic regime, where there are a large number of excitations but the interactions of the quanta can still be neglected. More generally, one would also need to include an interaction term in the action. Note that since we are interested in

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<sup>1</sup>The formalism can be readily extended to the case of a complex group field which would lead to the introduction of an additional pair of creation and annihilation operators.

*multiple* scalar fields we demand that  $D \geq 2$ , i.e., there must be at least two massless scalar fields present. The case  $D = 1$  is treated separately in Chapter 5. The equations of motion are given by (cf. (3.26))

$$\left(\mathcal{K}_J^{(0)} + \mathcal{K}_J^{(2)} \Delta_\chi\right) \varphi_J(\chi^\alpha) = 0. \quad (4.2)$$

It is customary to introduce a new symbol,

$$m_J^2 = -\frac{\mathcal{K}_J^{(0)}}{\mathcal{K}_J^{(2)}}, \quad (4.3)$$

which brings the equations of motions into the same form as the equations of motion in Euclidean scalar field theory,

$$(\Delta_\chi - m_J^2) \varphi_J(\chi^\alpha) = 0. \quad (4.4)$$

Note that this notation is slightly misleading as  $m_J^2$  is not positive definite. Furthermore each mode ‘ $J$ ’ can have a different sign. For our purposes the case of positive  $m_J^2$  is of the greatest interest since it leads to exponentially growing (or decaying) solutions which we will identify with a cosmological expansion. Note that for  $m_J^2 < 0$  the solutions are oscillatory in relational time and therefore their contribution is expected to be subdominant at sufficiently late relational times.

In the following we will split the tuple of  $D = d + 1$  scalar fields,  $\chi^\alpha = (\chi^0, \chi^1, \dots, \chi^d) = (\chi^0, \chi^i) = (\chi^0, \chi^i)$  into a ‘temporal’ scalar field  $\chi^0$  and the ‘spatial’ scalar fields  $\chi^i$  which we will also write in vector form as  $\boldsymbol{\chi}$ . As the terminology suggests, we would like to interpret the scalar fields as defining a reference frame where the temporal scalar field acts as a (relational) clock and the spatial scalar fields act as a (relational) spatial frame of reference. Previous work on using multiple massless scalar fields in the context of GFT cosmology can be found in [63, 69, 70, 73].

## 4.1. Hamiltonian

If one interprets the temporal scalar field  $\chi^0$  as a clock it is possible to derive a Hamiltonian from (4.1) by a Legendre transform. To this end it is helpful to view the action (4.1) as being given in terms of a Lagrangian,

$$S(\varphi) = \int d\chi^0 L(\varphi, \partial_\alpha \varphi), \quad (4.5)$$

where  $\partial_\alpha = \partial/\partial\chi^\alpha$ . The conjugate momentum of the group field is then given by

$$\pi_J(\chi^\alpha) = \frac{\delta S(\varphi)}{\delta(\partial_0\varphi_J(\chi^\alpha))} = -\mathcal{K}_J^{(2)} \frac{\partial\varphi_J(\chi^\alpha)}{\partial\chi^0}. \quad (4.6)$$

In terms of conjugate momenta the action can be written as (with arguments omitted)

$$S(\varphi) = \sum_J \int d^D\chi \left( \pi_J \frac{\partial\varphi_J}{\partial\chi^0} - \frac{\mathcal{K}_J^{(2)}}{2} \left( -\frac{1}{|\mathcal{K}_J^{(2)}|^2} \pi_J^2 + \varphi_J (-\Delta_\chi + m_J^2) \varphi_J \right) \right), \quad (4.7)$$

where  $\Delta_\chi = (\partial/\partial\chi)^2$  denotes the Laplacian acting on the space of spatial scalar fields. From this one can read off the Hamiltonian

$$H = \sum_J \int d^d\chi \frac{\mathcal{K}_J^{(2)}}{2} \left( -\frac{1}{|\mathcal{K}_J^{(2)}|^2} \pi_J(\chi^\alpha)^2 + \varphi_J(\chi^\alpha) (-\Delta_\chi + m_J^2) \varphi_J(\chi^\alpha) \right). \quad (4.8)$$

The equation of motion (4.4) is an elliptic partial differential equation on  $\mathbb{R}^D$ . Elliptic partial differential equations are well-posed when formulated as a boundary value problem. That is, by specifying the value of the group field on some subspace in  $\mathbb{R}^D$  that bounds a  $D$ -dimensional region it is possible to solve the partial differential equation for the value of the group field in the interior. However, we are interested in formulating dynamics as an initial value problem. It is well known that elliptic differential equations are unstable when formulated as a Cauchy problem [113, 140]. Nevertheless we will proceed by formulating the theory as an initial value problem. Indeed, the unstable solutions are of particular interest for our interpretation in terms of an exponentially expanding universe. As we will discuss in the sequel, the insistence on treating this system as an initial value problem breaks the  $E(D)$  symmetry of the Lagrangian formulation.

In order to specify the initial value formulation we view the space  $\mathbb{R}^D$  as being foliated by  $d$ -dimensional slices. We assume that the group field is regular on each of these spatial slices in the sense that it is square integrable with respect to the standard measure and therefore permits a decomposition into Fourier modes. This assumption breaks the  $E(D)$  symmetry of the original theory to a subgroup  $E(1) \times E(d)$ . Note that as soon as we have chosen *one* foliation any other foliation is impermissible, as the group field would then not be square integrable on these other slices. This conundrum is illustrated in Fig. 4.1 where the green (solid) leaves depict the foliation which features square integrable functions on each leaf and the red (dotted) leaves are representative of any other choice of foliation.

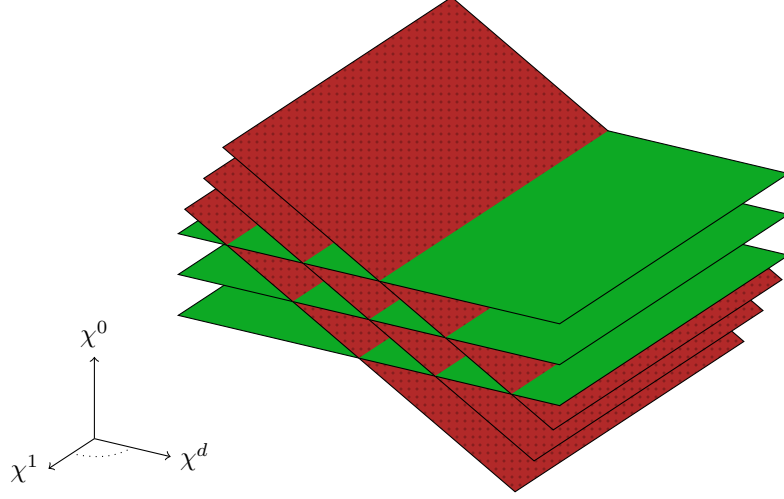


Figure 4.1.: Different foliations of  $\mathbb{R}^D$  into  $d$ -dimensional leaves. The green (solid) leaves show the foliation on which the group field is square integrable. The red (dotted) leaves show a foliation on which some of the group field modes would not be square integrable. The figure is taken from [77].

We assume from now on that the foliation we have chosen is such that the group field and its conjugate momentum are square integrable on the spatial slices. Their decomposition into Fourier modes is given by<sup>2</sup>

$$\varphi_J(\chi^\alpha) = \int \frac{d^d \mathbf{k}}{(2\pi)^d} e^{i\mathbf{k} \cdot \chi} \varphi_J(\chi^0, \mathbf{k}), \quad (4.9a)$$

$$\pi_J(\chi^\alpha) = \int \frac{d^d \mathbf{k}}{(2\pi)^d} e^{i\mathbf{k} \cdot \chi} \pi_J(\chi^0, \mathbf{k}). \quad (4.9b)$$

The Hamiltonian can then also be written in terms of the Fourier modes,

$$H = \sum_J \int \frac{d^d \mathbf{k}}{(2\pi)^d} H_J(\mathbf{k}). \quad (4.10)$$

The single-mode Hamiltonian densities are given explicitly by

$$H_J(\mathbf{k}) = \frac{\mathcal{K}_J^{(2)}}{2} \left( -\frac{1}{|\mathcal{K}_J^{(2)}|^2} \pi_J(\chi^0, -\mathbf{k}) \pi_J(\chi^0, \mathbf{k}) + \omega_J(\mathbf{k})^2 \varphi_J(\chi^0, -\mathbf{k}) \varphi_J(\chi^0, \mathbf{k}) \right), \quad (4.11)$$

<sup>2</sup>We adopt the convention where a function and its Fourier transform are distinguished by the names of their arguments. That is, a function  $f(\chi)$  has a Fourier transform denoted by  $f(\mathbf{k})$ . The mathematically inclined reader is encouraged to think of  $\chi$  and  $\mathbf{k}$  as not elements of  $\mathbb{R}^d$  but rather a “labelled”  $\mathbb{R}^d$ . The function  $f$  is then a piecewise function taking different values for different labels.

where we defined

$$\omega_J(\mathbf{k})^2 = \mathbf{k}^2 + m_J^2. \quad (4.12)$$

## 4.2. Quantisation

One of the big advantages of adopting the canonical framework is that quantisation can be carried out in the same standard framework of canonical quantisation. The classical phase space of the theory has a Poisson structure given by the Poisson brackets

$$\{\varphi_J(\chi^0, \boldsymbol{\chi}), \pi_{J'}(\chi^0, \boldsymbol{\chi}')\} = \delta^d(\boldsymbol{\chi} - \boldsymbol{\chi}')\delta_{JJ'}. \quad (4.13)$$

The quantisation of the theory is then achieved by promoting the field modes to operators and defining their canonical equal relational time commutation relation as

$$[\hat{\varphi}_J(\chi^0, \boldsymbol{\chi}), \hat{\pi}_{J'}(\chi^0, \boldsymbol{\chi}')] = i\delta_{JJ'}\delta^d(\boldsymbol{\chi} - \boldsymbol{\chi}')\hat{I}, \quad (4.14)$$

where  $\hat{I}$  is the identity operator. In the Fourier mode representation the commutation relation reads

$$[\hat{\varphi}_J(\chi^0, \mathbf{k}), \hat{\pi}_{J'}(\chi^0, \mathbf{k}')] = i\delta_{JJ'}(2\pi)^d\delta^d(\mathbf{k} - \mathbf{k}')\hat{I}. \quad (4.15)$$

As has been already alluded to in Section 3.1.3 we emphasise the appearance of *equal (relational) time* commutation relations. This needs to be contrasted to other (non-canonical) approaches where it is always assumed that the group field operators defined at different relational times  $\chi^0 \neq \chi'^0$  commute.

The next step is to write the group field operators in terms of creation and annihilation operators,

$$\hat{a}_J(\chi^0, \mathbf{k}) = A_J(\mathbf{k})\hat{\varphi}_J(\chi^0, \mathbf{k}) + \frac{i}{2A_J(\mathbf{k})}\hat{\pi}_J(\chi^0, \mathbf{k}), \quad (4.16a)$$

$$\hat{a}_J^\dagger(\chi^0, \mathbf{k}) = A_J(\mathbf{k})\hat{\varphi}_J(\chi^0, -\mathbf{k}) - \frac{i}{2A_J(\mathbf{k})}\hat{\pi}_J(\chi^0, -\mathbf{k}), \quad (4.16b)$$

where  $A_J(\mathbf{k})$  is an arbitrary real function and we made use of the fact that for a real group field

$$\hat{\varphi}_J^\dagger(\chi^0, \mathbf{k}) = \hat{\varphi}_J(\chi^0, -\mathbf{k}) \quad (4.17)$$

and similarly for  $\hat{\pi}_J(\chi^0, \mathbf{k})$ . The commutation relations for the operators  $\hat{a}$  and  $\hat{a}^\dagger$  are then those of bosonic creation and annihilation operators,

$$[\hat{a}_J(\chi^0, \mathbf{k}), \hat{a}_{J'}^\dagger(\chi^0, \mathbf{k}')] = \delta_{JJ'} (2\pi)^d \delta(\mathbf{k} - \mathbf{k}') \hat{I}. \quad (4.18)$$

Note that the corresponding Fock vacuum of this bosonic algebra satisfies

$$\hat{a}_J(\mathbf{k})|\oslash\rangle = 0. \quad (4.19)$$

The interpretation is now that the labelled nodes of a spin network are generated by acting with the creation operators  $\hat{a}_J^\dagger(\mathbf{k})$  on the Fock vacuum. In Chapter 2 a slightly different perspective was presented. In some sense the formalism of the present chapter is more satisfying as it is derived from the theory by a standard algorithm rather than merely postulated. Note furthermore the states constructed in this formalism can directly be interpreted as physical states of the GFT. This in contrast to other approaches where the Fock space is a “kinematical” Hilbert space and the physical states have to be obtained by a suitable “projection”. Below we will adopt the standard notation  $|\oslash\rangle = |0\rangle$  for the Fock vacuum.

The relations (4.16) can be inverted to obtain expressions for the group field and the conjugate momentum. Assuming for simplicity that  $A_J(\mathbf{k}) = A_J(-\mathbf{k})$ , the relations are

$$\hat{\varphi}_J(\chi^0, \mathbf{k}) = \frac{1}{2A_J(\mathbf{k})} \left( \hat{a}_J(\chi^0, \mathbf{k}) + \hat{a}_J^\dagger(\chi^0, \mathbf{k}) \right), \quad (4.20a)$$

$$\hat{\pi}_J(\chi^0, \mathbf{k}) = 2A_J(\mathbf{k}) \frac{1}{2i} \left( \hat{a}_J(\chi^0, \mathbf{k}) - \hat{a}_J^\dagger(\chi^0, \mathbf{k}) \right). \quad (4.20b)$$

Let us now turn to the question what a suitable choice of the function  $A_J(\mathbf{k})$  is.

If we assume that  $A_J(\mathbf{k}) = A_J(-\mathbf{k})$  the Hamiltonian takes the form,

$$\begin{aligned} \hat{H}_J(\mathbf{k}) = \frac{\mathcal{K}_J^{(2)}}{2} & \left( \left( \frac{A_J(\mathbf{k})^2}{|\mathcal{K}_J^{(2)}|^2} + \frac{\omega_J(\mathbf{k})^2}{4A(\mathbf{k})^2} \right) \left( \hat{a}_J(\mathbf{k})\hat{a}_J(-\mathbf{k}) + \hat{a}_J^\dagger(\mathbf{k})\hat{a}_J^\dagger(-\mathbf{k}) \right) \right. \\ & \left. - \left( \frac{A_J(\mathbf{k})^2}{|\mathcal{K}_J^{(2)}|^2} - \frac{\omega_J(\mathbf{k})^2}{4A(\mathbf{k})^2} \right) \left( \hat{a}_J(-\mathbf{k})\hat{a}_J^\dagger(-\mathbf{k}) + \hat{a}_J^\dagger(\mathbf{k})\hat{a}_J(\mathbf{k}) \right) \right), \end{aligned} \quad (4.21)$$

where we have written the expression in terms of the (time-independent) Schrödinger operators. This is justified because we will define time evolution *via* the Hamiltonian and therefore expect it to be time-independent. From this we see that in the case



that  $\omega_J(\mathbf{k}) \neq 0$  there is a natural choice for the function  $A_J(\mathbf{k})$  which simplifies (4.21) cancelling one of the two terms respectively,

$$A_J(\mathbf{k}) = \sqrt{\frac{|\omega_J(\mathbf{k})||\mathcal{K}_J^{(2)}|}{2}}. \quad (4.22)$$

For the case  $\omega_J(\mathbf{k}) = 0$  the natural choice for  $A_J(\mathbf{k})$  leading to a simple prefactor in (4.21) is given by

$$A_J(\mathbf{k}) = \sqrt{|\mathcal{K}_J^{(2)}|}. \quad (4.23)$$

Although this case is of marginal interest since in the  $\mathbf{k}$ -space this is a subspace with measure zero, we state the results for this case for completeness. With these choices the single-mode Hamiltonian is either of the type of the harmonic oscillator ( $\omega_J(\mathbf{k})^2 < 0$ ), of the squeezing type ( $\omega_J(\mathbf{k})^2 > 0$ ) known from quantum optics or of the special type ( $\omega_J(\mathbf{k}) = 0$ ). From the perspective of GFT cosmology the case  $\omega_J(\mathbf{k})^2 > 0$  is the interesting one, since it leads to exponentially growing (or shrinking) solutions. In contrast, the oscillatory solutions of the modes with  $\omega_J(\mathbf{k})^2 < 0$  would correspond to a static universe and modes with  $\omega_J(\mathbf{k}) = 0$  grow linearly.

An important point to highlight is that the distinction of which behaviour certain modes will exhibit does depend on the pair of labels  $(J, \mathbf{k})$ . That is, even in the case in which  $m_J^2 < 0$  the combined quantity  $\omega(\mathbf{k})^2 = m_J^2 + \mathbf{k}^2$  will be positive for large enough  $|\mathbf{k}|$ .

To summarise, there are three types of single-mode Hamiltonians. Those of the harmonic oscillator type ( $\omega_J(\mathbf{k})^2 < 0$ ),

$$\hat{H}_J(\mathbf{k}) = -\text{sgn}(\mathcal{K}_J^{(2)}) \frac{|\omega_J(\mathbf{k})|}{2} \left( \hat{a}_J(-\mathbf{k}) \hat{a}_J^\dagger(-\mathbf{k}) + \hat{a}_J^\dagger(\mathbf{k}) \hat{a}_J(\mathbf{k}) \right), \quad (4.24)$$

those of squeezing type ( $\omega_J(\mathbf{k})^2 > 0$ ),

$$\hat{H}_J(\mathbf{k}) = \text{sgn}(\mathcal{K}_J^{(2)}) \frac{\omega_J(\mathbf{k})}{2} \left( \hat{a}_J(\mathbf{k}) \hat{a}_J(-\mathbf{k}) + \hat{a}_J^\dagger(\mathbf{k}) \hat{a}_J^\dagger(-\mathbf{k}) \right), \quad (4.25)$$

and those of special type ( $\omega_J(\mathbf{k}) = 0$ ),

$$\hat{H}_J(\mathbf{k}) = \frac{1}{2} \text{sgn}(\mathcal{K}_J^{(2)}) \left( \hat{a}_J(\mathbf{k}) \hat{a}_J(-\mathbf{k}) + \hat{a}_J^\dagger(\mathbf{k}) \hat{a}_J^\dagger(-\mathbf{k}) - \hat{a}_J(-\mathbf{k}) \hat{a}_J^\dagger(-\mathbf{k}) - \hat{a}_J^\dagger(\mathbf{k}) \hat{a}_J(\mathbf{k}) \right). \quad (4.26)$$

The harmonic oscillator type Hamiltonian (4.24) is related to the number operator and counts the number of excitations of a specific mode. The squeezing type Hamiltonian

(4.25) has a term which creates a pair of excitations and a term destroying a pair of excitations. The special type Hamiltonian (4.26) is a sum of the other two types. In the next section we provide the dynamics generated for each type of Hamiltonian.

### 4.3. Equations of motion

In the Heisenberg picture the equations of motion are given by the Heisenberg equation,

$$\frac{\partial}{\partial \chi^0} \hat{a}_J(\chi^0, \mathbf{k}) = -i[\hat{a}_J(\chi^0, \mathbf{k}), \hat{H}], \quad (4.27a)$$

$$\frac{\partial}{\partial \chi^0} \hat{a}_J^\dagger(\chi^0, \mathbf{k}) = -i[\hat{a}_J^\dagger(\chi^0, \mathbf{k}), \hat{H}]. \quad (4.27b)$$

Depending on the type of mode, the equations of motion take different forms. For modes of the harmonic oscillator type ( $\omega_J(\mathbf{k})^2 < 0$ ) the solutions are given by

$$\hat{a}_J(\chi^0, \mathbf{k}) = \hat{a}_J(\mathbf{k}) e^{i \text{sgn}(\mathcal{K}_J^{(2)}) |\omega_J(\mathbf{k})| \chi^0}, \quad (4.28a)$$

$$\hat{a}_J^\dagger(\chi^0, \mathbf{k}) = \hat{a}_J^\dagger(\mathbf{k}) e^{-i \text{sgn}(\mathcal{K}_J^{(2)}) |\omega_J(\mathbf{k})| \chi^0}. \quad (4.28b)$$

For the modes of squeezing type ( $\omega_J(\mathbf{k})^2 > 0$ ) which we are most interested in the solutions to the equations of motion are given by

$$\hat{a}(\chi^0, \mathbf{k}) = \hat{a}(\mathbf{k}) \cosh(\omega_J(\mathbf{k}) \chi^0) - i \text{sgn}(\mathcal{K}_J^{(2)}) \hat{a}_J^\dagger(-\mathbf{k}) \sinh(\omega_J(\mathbf{k}) \chi^0), \quad (4.29a)$$

$$\hat{a}^\dagger(\chi^0, \mathbf{k}) = \hat{a}^\dagger(\mathbf{k}) \cosh(\omega_J(\mathbf{k}) \chi^0) + i \text{sgn}(\mathcal{K}_J^{(2)}) \hat{a}_J(-\mathbf{k}) \sinh(\omega_J(\mathbf{k}) \chi^0). \quad (4.29b)$$

Finally, the modes of special type ( $\omega_J(\mathbf{k}) = 0$ ) have solutions

$$\hat{a}(\chi^0, \mathbf{k}) = \hat{a}(\mathbf{k}) + i \left( \hat{a}(\mathbf{k}) - \hat{a}^\dagger(-\mathbf{k}) \right) \text{sgn}(\mathcal{K}_J^{(2)}) \chi^0, \quad (4.30a)$$

$$\hat{a}^\dagger(\chi^0, \mathbf{k}) = \hat{a}^\dagger(\mathbf{k}) - i \left( \hat{a}^\dagger(\mathbf{k}) - \hat{a}(-\mathbf{k}) \right) \text{sgn}(\mathcal{K}_J^{(2)}) \chi^0. \quad (4.30b)$$

From now on we will only consider modes of squeezing type.

## 4.4. Symmetries and conserved charges

As discussed in Section 3.2.1 the action considered here has several continuous symmetries which give rise to Noether currents which are divergenceless. Having now singled out one of the scalar fields to act as a relational clock,  $\chi^0$ , we can go one step further and define corresponding Noether charges which are conserved under relational time evolution,

$$Q = \int d^d \chi J^0(\chi^\alpha), \quad \frac{\partial}{\partial \chi^0} Q = 0. \quad (4.31)$$

We discuss now these conserved charges in the quantum theory of the canonical framework. We adopt a normal ordering prescription where creation operators are placed to the left of all annihilation operators which we denote by ‘ $\cdot \cdot \cdot$ ’.

Translations of the scalar field,  $\chi^\alpha \mapsto \chi^\alpha + \epsilon^\alpha$ , give rise to the energy-momentum tensor  $\Theta_{\alpha\beta}$  (cf. (3.35)). We define the resulting charge as the “momentum”

$$P_\alpha = \int d^d \chi \Theta_{0\alpha}(\chi^\alpha). \quad (4.32)$$

The temporal component is equal to minus the Hamiltonian,

$$\hat{P}_0 = -\hat{H} = \sum_J \int \frac{d^d \mathbf{k}}{(2\pi)^d} \text{sgn}(\mathcal{K}_J^{(2)}) \frac{\omega_J(\mathbf{k})}{2} \left( \hat{a}_J(\mathbf{k}) \hat{a}_J(-\mathbf{k}) + \hat{a}_J^\dagger(\mathbf{k}) \hat{a}_J^\dagger(-\mathbf{k}) \right), \quad (4.33)$$

where we have made the assumption that all the modes are of squeezing type. The spatial components are given by

$$\hat{\mathbf{P}} = \sum_J \hat{\mathbf{P}}_J = - \sum_J \int d^d \chi : \hat{\pi}_J(\chi^\alpha) \nabla \hat{\varphi}_J(\chi^\alpha) := \sum_J \int \frac{d^d \mathbf{k}}{(2\pi)^d} \mathbf{k} \hat{a}_J^\dagger(\mathbf{k}) \hat{a}_J(\mathbf{k}). \quad (4.34)$$

Since the time-dependence of the modes do not mix their  $J$  labels it is clear that each of the  $J$  modes is conserved separately. This can also be understood by noting that each  $J$  mode is invariant independently under the symmetry transformation. Therefore, any combination of modes is conserved. The conservation of spatial momentum can also be understood by noting that the squeezing Hamiltonian always produces two modes of the opposite wave vector  $\mathbf{k}$ .

Rotations of the scalar fields  $\chi^\alpha$  give rise to the angular momentum tensor  $M_{\alpha\beta\gamma}$  (cf. (3.37)). We define the conserved total angular momentum as

$$M_{\alpha\beta} = \int d^d \chi M_{0\alpha\beta}(\chi^\alpha). \quad (4.35)$$

The  $0i$ -component of the angular momentum operator is given by

$$\begin{aligned} \hat{M}_{0i} &= -\hat{P}_i \chi_0 \\ &- \frac{i}{2} \sum_J \text{sgn}(\mathcal{K}_J^{(2)}) \int \frac{d^d \mathbf{k}}{(2\pi)^d} \omega_J(\mathbf{k}) \left( \hat{a}_J(-\mathbf{k}) \frac{\partial}{\partial k^i} \hat{a}_J(\mathbf{k}) - \hat{a}_J^\dagger(-\mathbf{k}) \frac{\partial}{\partial k^i} \hat{a}_J^\dagger(\mathbf{k}) \right), \end{aligned} \quad (4.36)$$

where  $\hat{P}_i$  are the components of the spatial momentum given in (4.34). One can check by direct computation that the  $\hat{M}_{0i}$  are conserved by using the identities

$$\begin{aligned} &\sum_{JJ'} \int \frac{d^d \mathbf{k}}{(2\pi)^d} \frac{d^d \mathbf{k}'}{(2\pi)^d} [\hat{a}_J(-\mathbf{k}) \frac{\partial}{\partial k_i} \hat{a}_J(\mathbf{k}), \hat{a}_{J'}^\dagger(\mathbf{k}') \hat{a}_{J'}^\dagger(-\mathbf{k}')] \\ &= 2 \sum_J \int \frac{d^d \mathbf{k}}{(2\pi)^d} \left( \hat{a}_J^\dagger(\mathbf{k}) \frac{\partial}{\partial k_i} \hat{a}_J(\mathbf{k}) + \frac{\partial}{\partial k_i} \hat{a}_J(\mathbf{k}) \hat{a}_J^\dagger(\mathbf{k}) \right), \end{aligned} \quad (4.37a)$$

$$\begin{aligned} &\sum_{JJ'} \int \frac{d^d \mathbf{k}}{(2\pi)^d} \frac{d^d \mathbf{k}'}{(2\pi)^d} [\hat{a}_J^\dagger(-\mathbf{k}) \frac{\partial}{\partial k_i} \hat{a}_J^\dagger(\mathbf{k}), \hat{a}_{J'}(\mathbf{k}') \hat{a}_{J'}(-\mathbf{k}')] \\ &= 2 \sum_J \int \frac{d^d \mathbf{k}}{(2\pi)^d} \left( \frac{\partial}{\partial k_i} \hat{a}_J^\dagger(\mathbf{k}) \hat{a}_J(\mathbf{k}) + \hat{a}_J(\mathbf{k}) \frac{\partial}{\partial k_i} \hat{a}_J^\dagger(\mathbf{k}) \right) \end{aligned} \quad (4.37b)$$

and noting that the sum of the two contributions is a total derivative.

The  $ij$ -components of the angular momentum operator are

$$\hat{M}_{ij} = i \sum_J \int \frac{d^d \mathbf{k}}{(2\pi)^d} \hat{a}_J^\dagger(\mathbf{k}) \left( k_i \frac{\partial}{\partial k^j} - k_j \frac{\partial}{\partial k^i} \right) \hat{a}_J(\mathbf{k}). \quad (4.38)$$

Again, one can check by direct computation that the  $\hat{M}_{ij}$  are conserved by using the identities

$$\begin{aligned} &\sum_{JJ'} \int \frac{d^d \mathbf{k}}{(2\pi)^d} \frac{d^d \mathbf{k}'}{(2\pi)^d} [\hat{a}_J^\dagger(\mathbf{k}) k_i \frac{\partial}{\partial k_j} \hat{a}_J^\dagger(\mathbf{k}), \hat{a}_{J'}(\mathbf{k}') \hat{a}_{J'}(-\mathbf{k}')] \\ &= 2 \sum_J \int \frac{d^d \mathbf{k}}{(2\pi)^d} \delta_{ij} \hat{a}_J(\mathbf{k}) \hat{a}_J(-\mathbf{k}), \end{aligned} \quad (4.39a)$$

$$\begin{aligned} &\sum_{JJ'} \int \frac{d^d \mathbf{k}}{(2\pi)^d} \frac{d^d \mathbf{k}'}{(2\pi)^d} [\hat{a}_J^\dagger(\mathbf{k}) k_i \frac{\partial}{\partial k_j} \hat{a}_J^\dagger(\mathbf{k}), \hat{a}_{J'}^\dagger(\mathbf{k}') \hat{a}_{J'}^\dagger(-\mathbf{k}')] \\ &= -2 \sum_J \int \frac{d^d \mathbf{k}}{(2\pi)^d} \delta_{ij} \hat{a}_J^\dagger(\mathbf{k}) \hat{a}_J^\dagger(-\mathbf{k}). \end{aligned} \quad (4.39b)$$

One can check that the operators satisfy the commutation relations of the Euclidean

group  $E(D)$ ,

$$[\hat{M}_{\alpha\beta}, \hat{M}_{\gamma\delta}] = -i \left( \delta_{\alpha\gamma} \hat{M}_{\beta\delta} - \delta_{\alpha\delta} \hat{M}_{\beta\gamma} - \delta_{\beta\gamma} \hat{M}_{\alpha\delta} + \delta_{\beta\delta} \hat{M}_{\alpha\gamma} \right), \quad (4.40)$$

$$[\hat{M}_{\alpha\beta}, \hat{P}_\gamma] = -i \left( \delta_{\alpha\gamma} \hat{P}_\beta - \delta_{\beta\gamma} \hat{P}_\alpha \right). \quad (4.41)$$

Note that the expressions presented here for the modes of squeezing type highlight that the “temporal” and “spatial” direction in the space of scalar fields exhibit different behaviour. Let us reiterate that Euclidean  $E(D)$  symmetry is broken to a  $E(1) \times E(d)$  symmetry due to the unstable dynamics of the modes of squeezing type.

As mentioned in Section 3.2.1 translations of the scalar fields give rise to a conserved quantity in the usual quantum field theory where the scalar fields propagate on a curved spacetime (background) manifold. The total momentum of each scalar field is given by (cf. (3.41))

$$(P_{\text{sf}})^\alpha = \int d^d \mathbf{x} \pi_\chi^\alpha. \quad (4.42)$$

and the total angular momentum of each pair of scalar field is given by

$$(M_{\text{sf}})^{\alpha\beta} = \frac{1}{2} \int d^d \mathbf{x} \left( \pi_\chi^\alpha \chi^\beta - \pi_\chi^\beta \chi^\alpha \right), \quad (4.43)$$

where  $\pi_\chi^\alpha$  is the conjugate momentum of  $\chi_\alpha$ . We now propose the following identifications of the theory

$$P^\alpha \leftrightarrow (P_{\text{sf}})^\alpha, \quad M^{\alpha\beta} \leftrightarrow (M_{\text{sf}})^{\alpha\beta}. \quad (4.44)$$

The most important aspect of this identification is that the Hamiltonian  $\hat{H} = \hat{P}^0$  is interpreted as conjugate momentum of the massless scalar field *within the GFT formalism*. This ensures that the interpretation of the massless scalar field  $\chi^0$  as a clock is compatible with time evolution being generated by the Hamiltonian. This point has also been discussed in [2, 76] in the case of only a single scalar field.

## 4.5. Coherent states

In this section we present a class of coherent states which will serve as the initial states for time evolution.

We define coherent states to be given by

$$|\sigma\rangle = e^{-\|\sigma\|^2/2} \exp\left(\sum_J \int \frac{d^d \mathbf{k}}{(2\pi)^d} \sigma_J(\mathbf{k}) \hat{a}_J^\dagger(\mathbf{k})\right) |0\rangle, \quad (4.45)$$

where  $|0\rangle$  is the Fock vacuum and the norm  $\|\cdot\|$  which ensures normalisation is given by

$$\|\sigma\|^2 = \sum_J \int \frac{d^d \mathbf{k}}{(2\pi)^d} |\sigma_J(\mathbf{k})|^2. \quad (4.46)$$

These coherent states are completely characterised by specifying the family of functions  $\{\sigma_J(\mathbf{k})\}_J$ . In order to study the theory in the simplest cases we will make the further assumption that only *some* of the modes  $(J, \mathbf{k})$  are relevant. This can be achieved by choosing a suitable envelope function  $f_\epsilon(\mathbf{k})$  and write the complex function  $\sigma_J(\mathbf{k})$  as

$$\sigma_J(\mathbf{k}) = \sum_i f_\epsilon(\mathbf{k} - \mathbf{k}_i) \tau(\mathbf{k}_i), \quad (4.47)$$

where  $\tau(\mathbf{k}_i)$  is a complex number. The interpretation is that the mode  $J$  has only support on some  $\mathbf{k}$ , centred around the  $\mathbf{k}_i$ . The function  $f_\epsilon$  is required to satisfy

$$\lim_{\epsilon \rightarrow 0} \int \frac{d^d \mathbf{k}}{(2\pi)^d} |f_\epsilon(\mathbf{k})|^2 \phi(\mathbf{k}) = \phi(\mathbf{0}) \quad (4.48)$$

for some test function  $\phi(\mathbf{k})$ . In other words, we require the function  $f_\epsilon$  to converge to the delta distribution. The standard example for such an envelope is a Gaussian,

$$f_\epsilon(\mathbf{k}) = \left(\frac{4\pi}{\epsilon^2}\right)^{d/4} e^{-\frac{\mathbf{k}^2}{2\epsilon^2}}. \quad (4.49)$$

For the Gaussian envelope it is possible to perform an expansion in powers of  $\epsilon$ ,

$$\int \frac{d^d \mathbf{k}}{(2\pi)^d} |f_\epsilon(\mathbf{k})|^2 \phi(\mathbf{k}) = \phi(\mathbf{0}) + \frac{\epsilon^2}{4} (\nabla^2 \phi)(\mathbf{0}) + O(\epsilon^4). \quad (4.50)$$

Although the simple states which are peaked at only certain modes  $(J, \mathbf{k})$  are appealing, some cautionary remarks are in order. From the solutions to the equations of motion (4.29) we expect the time dependence to be proportional to  $e^{N\omega_J(\mathbf{k})\chi^0}$  where  $N$  is an integer. The first thing to note is that for any  $\mathbf{k}$  there is a  $\mathbf{k}'$  such that  $\omega_J(\mathbf{k}') > \omega_J(\mathbf{k})$  which would imply that the mode  $\mathbf{k}'$  grows faster than the  $\mathbf{k}$  mode. So unless one has very special fine tuned initial conditions where only exactly one mode is excited, the

relative significance of the modes will shift throughout time evolution. This can be seen by studying the Gaussian envelope (4.49) and solving the equation for the peak  $\mathbf{k}_*$ ,

$$\left. \frac{d}{d\mathbf{k}} \right|_{\mathbf{k}=\mathbf{k}_*} \left( |f_\epsilon(\mathbf{k} - \mathbf{k}_0)|^2 e^{N\omega_J(\mathbf{k})\chi^0} \right) = 0, \quad (4.51)$$

which to leading non-trivial order in  $\epsilon$  has a solution

$$\mathbf{k}_* = \mathbf{k}_0 \left( 1 + \frac{N\epsilon^2\chi^0}{2|\mathbf{k}_0|} \right), \quad (4.52)$$

which shows that the peak is moving linearly in time. For the example of a Gaussian envelope this result can be turned around and one can say that the approximation that the state is peaked at a particular value of  $\mathbf{k}_0$  is valid only for times  $\chi^0 < \chi_{\max}^0 \ll \frac{2|\mathbf{k}_0|}{N\epsilon^2}$ . In the following we will always assume that the peak  $\mathbf{k}_0$  of modes we consider remains unchanged by time evolution. This means that we consider the initial state to be peaked so sharply (small value of  $\epsilon$ ) such that the peak remains unchanged for a sufficiently long time.

## 4.6. Volume operator

The observable of main interest is the volume operator which we define to be given by

$$\hat{V}(\chi^0) = \sum_J \int \frac{d^d\mathbf{k}}{(2\pi)^d} \hat{V}_J(\chi^0, \mathbf{k}), \quad (4.53)$$

where the ‘‘partial volume densities’’ are given by

$$\hat{V}_J(\chi^0, \mathbf{k}) = v_J \hat{a}_J^\dagger(\chi^0, \mathbf{k}) \hat{a}_J(\chi^0, \mathbf{k}). \quad (4.54)$$

The meaning of the volume operator can be understood as a weighted sum over the number operators of each mode. The value of the weight can be obtained by comparing, e.g., with the volume operator defined in LQG as discussed in Section 3.1.4.

Inserting the solutions (4.29) to the equations of motion gives the explicit time dependence of the volume operator as

$$\begin{aligned} \hat{V}_J(\chi^0, \mathbf{k}) &= \frac{1}{2} \left( \hat{V}_J(\mathbf{k}) - \hat{V}_J(-\mathbf{k}) - v_J c_\infty \hat{I} \right) \\ &+ \frac{1}{2} \left( \hat{V}_J(\mathbf{k}) + \hat{V}_J(-\mathbf{k}) + v_J c_\infty \hat{I} \right) \cosh(2\omega_J(\mathbf{k})\chi^0) \\ &+ \hat{X}_J(\mathbf{k}) \sinh(2\omega_J(\mathbf{k})\chi^0), \end{aligned} \quad (4.55)$$

where we introduced the formally divergent quantity

$$c_\infty = \langle [\hat{a}_J(\mathbf{k}), \hat{a}_J^\dagger(\mathbf{k})] \rangle = (2\pi)^d \delta^d(\mathbf{k}) \Big|_{\mathbf{k}=0} \quad (4.56)$$

and defined the operator  $\hat{X}_J(\mathbf{k})$  as

$$\hat{X}_J(\mathbf{k}) = \text{sgn}(\mathcal{K}_J^{(2)}) \frac{i}{2} v_J \left( \hat{a}_J(\mathbf{k}) \hat{a}_J(-\mathbf{k}) - \hat{a}_J^\dagger(\mathbf{k}) \hat{a}_J^\dagger(-\mathbf{k}) \right). \quad (4.57)$$

The total volume is then given by

$$\begin{aligned} \hat{V}(\chi^0) = \sum_J \int \frac{d^d \mathbf{k}}{(2\pi)^d} \left[ -\frac{v_J c_\infty}{2} \hat{I} + \left( \hat{V}_J(\mathbf{k}) + \frac{v_J c_\infty}{2} \hat{I} \right) \cosh(2\omega_J(\mathbf{k})\chi^0) \right. \\ \left. + \hat{X}_J(\mathbf{k}) \sinh(2\omega_J(\mathbf{k})\chi^0) \right]. \end{aligned} \quad (4.58)$$

The appearance of the formally divergent terms proportional to  $c_\infty$  needs to be regularised. The simplest form of regularisation would be to simply set  $c_\infty = 0$  which would correspond to the adoption of a normal-ordering prescription. However, the same effect can be achieved by subtracting the vacuum contribution from the volume operator,

$$\hat{V}_{\text{reg}}(\chi^0) = \hat{V}(\chi^0) - \langle 0 | \hat{V}(\chi^0) | 0 \rangle \hat{I}. \quad (4.59)$$

In such a regularisation the regularised volume operator counts only the excitations relative to those already present in the vacuum state. Note that the appearance of divergent particle numbers also occurs in quantum optics and therefore should be no reason to reject this procedure [21].

Finally, note that there is a term in the volume operator (4.55) which is doubly divergent when no regularisation is in place,

$$\sum_J \frac{v_J c_\infty}{2} \int \frac{d^d \mathbf{k}}{(2\pi)^d} (\cosh(2\omega_J(\mathbf{k})\chi^0) - 1). \quad (4.60)$$

This term would then require further regularisation by imposing cutoffs on the sum and integral. In the following we will leave the formally divergent parameter  $c_\infty$  unregularised.



## 4.7. Effective Friedmann equations

As discussed in Appendix A the case of a homogeneous, isotropic universe with  $D = d + 1$  massless scalar fields has a Friedmann equation given by (cf. (A.52))

$$\left(\frac{V'_{\chi^0}(\chi^0)}{V_{\chi^0}(\chi^0)}\right)^2 = \frac{3\kappa}{2} \left(1 + \frac{\pi_\chi^2}{(\pi_\chi^0)^2}\right), \quad (4.61)$$

where  $V_{\chi^0}$  is the volume of the universe expressed as a function of the value of the massless scalar field  $\chi^0$  and  $\pi_\chi = (\pi_\chi^1, \dots, \pi_\chi^d)$  denotes the conjugate momenta of the spatial scalar fields.

A note on notation: we denote the relational volume, expressed as a function of the value of the massless scalar field, in the classical theory as  $V_{\chi^0}$  and the relational volume operator as  $\hat{V}$ . In the classical theory we adopt this notation to distinguish the function representing the relational volume from the more familiar function which expresses the volume as a function of coordinate time. In contrast, the volume operator of GFT was defined as a relational quantity, i.e., as a function of relational time. To summarise this remark, we note that the goal is to establish whether or not the following identification is justified:

$$\langle \hat{V}(\chi^0) \rangle \stackrel{?}{\longleftrightarrow} V_{\chi^0}(\chi^0). \quad (4.62)$$

In the case of the GFT considered here, we have the volume operator (4.55) at our disposal. Taking the (relational) time derivative of (4.55) gives

$$\begin{aligned} \partial_{\chi^0} \hat{V}_J(\chi^0, \mathbf{k}) &= 2\omega_J(\mathbf{k}) \left[ \frac{1}{2} \left( \hat{V}_J(\mathbf{k}) + \hat{V}_J(-\mathbf{k}) + v_J c_\infty \hat{I} \right) \sinh(2\omega_J(\mathbf{k})\chi^0) \right. \\ &\quad \left. + \hat{X}_J(\mathbf{k}) \cosh(2\omega_J(\mathbf{k})\chi^0) \right]. \end{aligned} \quad (4.63)$$

The expectation value of this can be rewritten as a function of the expectation value of the volume of the same mode,

$$\begin{aligned} \langle \partial_{\chi^0} \hat{V}_J(\chi^0, \mathbf{k}) \rangle &= \pm 2\omega_J(\mathbf{k}) \\ &\times \sqrt{\left( \langle \hat{V}_J(\chi^0, \mathbf{k}) \rangle - \langle \hat{V}_J(\mathbf{k}) \rangle \right) \left( \langle \hat{V}_J(\chi^0, \mathbf{k}) \rangle + \langle \hat{V}_J(-\mathbf{k}) \rangle + v_J c_\infty \right) + \langle \hat{X}_J(\mathbf{k}) \rangle^2}. \end{aligned} \quad (4.64)$$

This expression allows us to write the squared time derivative of the expectation value of

the volume operator as

$$\begin{aligned}
 \langle \hat{V}'(\chi^0) \rangle^2 &= \sum_{J,J'} \int \frac{d^d \mathbf{k}}{(2\pi)^d} \frac{d^d \mathbf{k}'}{(2\pi)^d} 4\omega_J(\mathbf{k})\omega_{J'}(\mathbf{k}') \\
 &\times \sqrt{\left( \langle \hat{V}_J(\chi^0, \mathbf{k}) \rangle - \langle \hat{V}_J(\mathbf{k}) \rangle \right) \left( \langle \hat{V}_J(\chi^0, \mathbf{k}) \rangle + \langle \hat{V}_J(-\mathbf{k}) \rangle + v_J c_\infty \right) + \langle \hat{X}_J(\mathbf{k}) \rangle^2} \\
 &\times \sqrt{\left( \langle \hat{V}_{J'}(\chi^0, \mathbf{k}') \rangle - \langle \hat{V}_{J'}(\mathbf{k}') \rangle \right) \left( \langle \hat{V}_{J'}(\chi^0, \mathbf{k}') \rangle + \langle \hat{V}_{J'}(-\mathbf{k}') \rangle + v_{J'} c_\infty \right) + \langle \hat{X}_{J'}(\mathbf{k}') \rangle^2}.
 \end{aligned} \tag{4.65}$$

Dividing both sides by  $\langle \hat{V}'(\chi^0) \rangle$  does not lead to any simplification. We now turn the question of possible physical interpretations of the above equation.

#### 4.7.1. Late time limit

A drastic simplification can be made by considering the late time limit of (4.65). Note that the only time-dependent quantity appearing on the right-hand side are the partial volume densities  $\hat{V}_J(\mathbf{k})$ . Indeed, as can be seen from the explicit expressions (4.29) the expectation value of the volume grows exponentially. From this it follows that for late times

$$\langle \hat{V}_J(\chi^0, \mathbf{k}) \rangle \gg \langle \hat{V}_J(\pm \mathbf{k}) \rangle, \quad \langle \hat{V}_J(\chi^0, \mathbf{k}) \rangle \gg \langle \hat{X}_J(\mathbf{k}) \rangle. \tag{4.66}$$

In the late time regime we therefore obtain the following asymptotic Friedmann equation,

$$\left( \frac{\langle \hat{V}'(\pm\infty) \rangle}{\langle \hat{V}(\pm\infty) \rangle} \right)^2 = \frac{\sum_{J,J'} \int \frac{d^d \mathbf{k}}{(2\pi)^d} \frac{d^d \mathbf{k}'}{(2\pi)^d} 4\omega_J(\mathbf{k})\omega_{J'}(\mathbf{k}') \langle \hat{V}_J(\pm\infty, \mathbf{k}) \rangle \langle \hat{V}_{J'}(\pm\infty, \mathbf{k}') \rangle}{\sum_{J,J'} \int \frac{d^d \mathbf{k}}{(2\pi)^d} \frac{d^d \mathbf{k}'}{(2\pi)^d} \langle \hat{V}_J(\pm\infty, \mathbf{k}) \rangle \langle \hat{V}_{J'}(\pm\infty, \mathbf{k}') \rangle}. \tag{4.67}$$

Note that the notation ‘ $\pm\infty$ ’ should not be taken literally. As has been already emphasised, we expect the validity of the simple non-interacting model to break down at some point. Rather the meaning of the notation is to signify times which are sufficiently late.

In order to further clarify the possibility of a cosmological interpretation, we restrict ourselves to the simple case in which only two modes  $(J_0, \pm \mathbf{k}_0)$  are excited. In that case (4.67) takes the form

$$\left( \frac{\langle \hat{V}'(\pm\infty) \rangle}{\langle \hat{V}(\pm\infty) \rangle} \right)^2 \approx 4\omega_{J_0}(\mathbf{k}_0)^2 = 4(m_{J_0}^2 + \mathbf{k}_0^2). \tag{4.68}$$

Note that the case of a single massless scalar field corresponds to the case in which  $\mathbf{k}_0 = \mathbf{0}$  or at least  $|\mathbf{k}_0| \ll |m_{J_0}|$ . As has been done in previous work, agreement with the classical theory of a flat FLRW universe with a massless scalar field can be obtained by defining  $m_{J_0}^2 = 3\kappa/8$  [2, 68, 76, 111, 112]. As discussed in Section 4.4 we propose to identify the conjugate momenta of the scalar field in the classical theory with the total momentum in the GFT formalism—the main justification stemming from the fact that they correspond to Noether charges generated by the same symmetry transformations, albeit in different theories. Symbolically, this identification amounts to

$$\langle \hat{H} \rangle \longleftrightarrow \pi_\chi^0, \quad \langle \hat{\mathbf{P}} \rangle \longleftrightarrow \boldsymbol{\pi}_\chi. \quad (4.69)$$

For the case in which only two modes  $(J_0, \pm\mathbf{k}_0)$  are relevant we expect the following relations

$$\langle \hat{H} \rangle^2 \approx \omega_{J_0}(\mathbf{k}_0)^2 \left\langle \hat{a}_{J_0}(\mathbf{k}_0) \hat{a}_{J_0}(-\mathbf{k}_0) + \hat{a}_{J_0}^\dagger(\mathbf{k}_0) \hat{a}_{J_0}^\dagger(-\mathbf{k}_0) \right\rangle^2, \quad (4.70)$$

$$\langle \hat{\mathbf{P}} \rangle^2 \approx \mathbf{k}_0^2 \left( \langle \hat{N}_{J_0}(\mathbf{k}_0) \rangle^2 - \langle \hat{N}_{J_0}(-\mathbf{k}_0) \rangle^2 \right). \quad (4.71)$$

Using these relations bring (4.68) into the suggestive form

$$\left( \frac{\langle \hat{V}'(\pm\infty) \rangle}{\langle \hat{V}(\pm\infty) \rangle} \right)^2 \approx 4m_{J_0}^2 \left( 1 + v^2 \frac{\langle \hat{\mathbf{P}} \rangle^2}{\langle \hat{H} \rangle^2} \right), \quad (4.72)$$

where we introduced the variable

$$v^2 = \frac{\mathbf{k}_0^2}{m_{J_0}^2} \frac{\langle \hat{H} \rangle^2}{\langle \hat{\mathbf{P}} \rangle^2}. \quad (4.73)$$

From this we see that  $v \approx 1$ , the condition for agreement with the classical theory, is achieved only for very special initial conditions. We emphasise that from the perspective of the underlying theory, the quantities  $\langle \hat{H} \rangle$  and  $\langle \hat{\mathbf{P}} \rangle^2$  are both conserved quantities and both can be seen as free parameters, encoding (some of) the initial conditions.

There is another possibility for understanding that generically  $v \neq 1$ . In the present discussion we have considered states where only two modes,  $(J_0, \mathbf{k}_0)$ , are relevant as good candidates to represent a homogeneous universe. However, this assumption might be misguided. Indeed, in previous work it was suggested that when interpreting the spatial scalar fields as a relational reference frame the case we describe here would correspond to a “periodically inhomogeneous” universe [63, 69]. From that point of view, a homogeneous universe would have to correspond to the case  $\mathbf{k} = \mathbf{0}$ . In [77, Appendix B] we discussed a

simple cosmological model of a spatially periodically inhomogeneous universe. However, there does not seem to be a simple connection between the simple case considered here and the classical theory.

In conclusion, the interpretation of the theory in terms of either a homogeneous or an inhomogeneous universe remains largely an open question.

## 4.8. Simple coherent states

To make things more concrete, we consider next the effective Friedmann equations for the choice of simple coherent states as the initial condition. In particular, we will discuss under what conditions we expect to find agreement with the classical theory.

In the case that only two modes  $(J_0, \pm \mathbf{k}_0)$  are relevant, the volume operator (4.55) takes a particularly simple form,

$$\langle \hat{V}(\chi^0) \rangle \approx \langle \hat{V} \rangle \cosh(2\omega_{J_0}(\mathbf{k}_0)\chi^0) + \langle \hat{X} \rangle \sinh(2\omega_{J_0}(\mathbf{k}_0)\chi^0), \quad (4.74)$$

where  $\hat{V} = \sum_J \int \frac{d^d \mathbf{k}}{(2\pi)^d} \hat{V}_J(\mathbf{k})$  is the total volume at initial times and we defined (cf. (4.57))  $\hat{X} = \sum_J \int \frac{d^d \mathbf{k}}{(2\pi)^d} \hat{X}_J(\mathbf{k})$ . The effective Friedmann equation (4.65) takes the form

$$\langle \hat{V}'(\chi^0) \rangle^2 = 4\omega_{J_0}(\mathbf{k}_0)^2 \left( \langle \hat{V}(\chi^0) \rangle^2 - \langle \hat{V} \rangle^2 + \langle \hat{X} \rangle^2 \right). \quad (4.75)$$

If one considers the even more specific state in which *only* a single mode,  $(J_0, \mathbf{k}_0)$  is relevant, both the expectation values of  $\hat{H}$  and of  $\hat{X}$  vanish. In that case the effective Friedmann equation takes the form

$$\frac{\langle \hat{V}'(\chi^0) \rangle_{\text{sm}}^2}{\langle \hat{V}(\chi^0) \rangle_{\text{sm}}^2} = 4\omega_{J_0}(\mathbf{k}_0)^2 \left( 1 - \frac{\langle \hat{V} \rangle_{\text{sm}}^2}{\langle \hat{V}(\chi^0) \rangle_{\text{sm}}^2} \right). \quad (4.76)$$

In this very simple case we are not able to provide an interpretation in terms of corrections to the classical equation which are similar to those encountered in LQC. This is due to the fact that for single-mode states the expectation value of  $\hat{H}$  vanishes and therefore we can not have terms proportional to the energy density as in LQC. As we will show below, in a slightly less restrictive case we are however able to provide an interpretation of the

Friedmann equation. An example of such a single mode state can be characterised by a coherent state (4.45) with coherent state function

$$(\sigma_{\text{sm}})_J(\mathbf{k}) = \delta_{JJ_0} f_\epsilon(\mathbf{k} - \mathbf{k}_0) \tau(\mathbf{k}_0). \quad (4.77)$$

The constant entering the effective Friedmann equation is then given by

$$\langle \hat{V} \rangle_{\text{sm}} = v_{J_0} |\tau(\mathbf{k}_0)|^2, \quad (4.78)$$

which is proportional to the expectation value of the number operator,  $\langle \hat{N} \rangle_{\text{sm}} = |\tau(\mathbf{k}_0)|^2$ .

We now turn the discussion of the case in which two modes,  $(J_0, \mathbf{k}_0)$ , are relevant in more detail. We refer to this type of states as *double mode* states. The effective Friedmann equation then takes the form

$$\frac{\langle \hat{V}'(\chi^0) \rangle_{\text{dm}}^2}{\langle \hat{V}(\chi^0) \rangle_{\text{dm}}^2} = 4\omega_{J_0}(\mathbf{k}_0)^2 \left( 1 - \frac{\langle \hat{V} \rangle_{\text{dm}}^2}{\langle \hat{V}(\chi^0) \rangle_{\text{dm}}^2} + \frac{\langle \hat{X} \rangle_{\text{dm}}^2}{\langle \hat{V}(\chi^0) \rangle_{\text{dm}}^2} \right). \quad (4.79)$$

Note that below we show that  $\langle \hat{V} \rangle_{\text{dm}}^2 - \langle \hat{X} \rangle_{\text{dm}}^2 \geq 0$  (cf. (4.89)) and therefore the right-hand side will generically be zero at some value of the relational time  $\chi^0$ . These double mode states can be realised as coherent states (4.45) by setting the coherent state function to

$$(\sigma_{\text{dm}})_J(\mathbf{k}) = \delta_{JJ_0} (f_\epsilon(\mathbf{k} - \mathbf{k}_0) \tau(\mathbf{k}_0) + f_\epsilon(\mathbf{k} + \mathbf{k}_0) \tau(-\mathbf{k}_0)). \quad (4.80)$$

The parameters in the effective Friedmann equation can readily be computed for the double mode coherent states,

$$\langle \hat{V} \rangle_{\text{dm}} = v_{J_0} (|\tau(\mathbf{k}_0)|^2 + |\tau(-\mathbf{k}_0)|^2), \quad (4.81)$$

$$\langle \hat{X} \rangle_{\text{dm}} = iv_{J_0} \text{sgn}(\mathcal{K}_{J_0}^{(2)}) (\tau(\mathbf{k}_0) \tau(-\mathbf{k}_0) - \bar{\tau}(\mathbf{k}_0) \bar{\tau}(-\mathbf{k}_0)). \quad (4.82)$$

Note that also in this case the expectation value of the volume operator is proportional to the expectation value of the number operator  $\langle \hat{V} \rangle_{\text{dm}} = v_{J_0} \langle \hat{N} \rangle_{\text{dm}}$ . The components of the total momentum are given by

$$\langle \hat{H} \rangle_{\text{dm}} = \text{sgn}(\mathcal{K}_{J_0}^{(2)}) \omega_{J_0}(\mathbf{k}_0) (\tau(\mathbf{k}_0) \tau(-\mathbf{k}_0) + \bar{\tau}(\mathbf{k}_0) \bar{\tau}(-\mathbf{k}_0)), \quad (4.83)$$

$$\langle \hat{\mathbf{P}} \rangle_{\text{dm}} = \mathbf{k}_0 (|\tau(\mathbf{k}_0)|^2 - |\tau(-\mathbf{k}_0)|^2). \quad (4.84)$$

We now revisit the question when these states result in a Friedmann equation for which we can provide an interpretation as modifications to the classical equations similar to the corrections found in LQC. In the case that  $\mathbf{k}_0^2 \ll m_{J_0}^2$  agreement with the classical

equation is obtained if we require that the expectation values of  $\hat{H}$  and  $\hat{\mathbf{P}}$  satisfy the following relation (cf. (4.73))

$$\frac{1}{\omega_{J_0}(\mathbf{k}_0)^2} \langle \hat{H} \rangle_{\text{dm}}^2 = \frac{1}{\mathbf{k}_0^2} \langle \hat{\mathbf{P}} \rangle_{\text{dm}}^2. \quad (4.85)$$

If we write the coherent state function in terms of their modulus and argument,

$$\tau(\mathbf{k}_0) = \rho_{\pm} e^{i\theta_{\pm}}, \quad (4.86)$$

then we get the condition that the moduli and arguments are related by

$$|\cos(\theta_+ + \theta_-)| = \frac{|\rho_+^2 - \rho_-^2|}{2\rho_+\rho_-} \quad (4.87)$$

which then further implies the necessary condition

$$(\sqrt{2} - 1)\rho_+ \leq \rho_- \leq (\sqrt{2} + 1)\rho_+. \quad (4.88)$$

From this we conclude that our proposed correspondence with the classical theory can only hold if the moduli are comparable size. This implies, that the initial number of quanta in both modes ( $J_0, \pm\mathbf{k}_0$ ) must be comparable in quantity.

We will now bring (4.79) into a different form which is more conducive for an interpretation. The first thing to note is that there is a relation between some of the expectation values,

$$\frac{v_{J_0}^2}{\omega_{J_0}(\mathbf{k}_0)^2} \langle \hat{H} \rangle_{\text{dm}}^2 + \frac{v_{J_0}^2}{\mathbf{k}_0^2} \langle \hat{\mathbf{P}} \rangle_{\text{dm}}^2 = \langle \hat{V} \rangle_{\text{dm}}^2 - \langle \hat{X} \rangle_{\text{dm}}^2. \quad (4.89)$$

With this relation (4.79) can be written as

$$\frac{\langle \hat{V}'(\chi^0) \rangle_{\text{dm}}^2}{\langle \hat{V}(\chi^0) \rangle_{\text{dm}}^2} = 4\omega_{J_0}(\mathbf{k}_0)^2 \left( 1 - \frac{v_{J_0}^2 \langle \hat{\mathbf{P}} \rangle_{\text{dm}}^2}{\mathbf{k}_0^2 \langle \hat{V}(\chi^0) \rangle_{\text{dm}}^2} - \frac{v_{J_0}^2 \langle \hat{H} \rangle_{\text{dm}}^2}{\omega_{J_0}(\mathbf{k}_0)^2 \langle \hat{V}(\chi^0) \rangle_{\text{dm}}^2} \right). \quad (4.90)$$

As already mentioned several times, we wish to interpret the Hamiltonian and spatial total momentum of the group field as corresponding to the conjugate momenta of the massless scalar field. In full analogy with this correspondence, we define the following energy densities,

$$\rho_{\chi^0}(\chi^0) = \frac{\langle \hat{H} \rangle_{\text{dm}}^2}{2\langle \hat{V}(\chi^0) \rangle_{\text{dm}}^2}, \quad \rho_{\mathbf{x}}(\chi^0) = \frac{\langle \hat{\mathbf{P}} \rangle_{\text{dm}}^2}{2\langle \hat{V}(\chi^0) \rangle_{\text{dm}}^2}. \quad (4.91)$$

Inserting these definitions into (4.90) gives the expression

$$\frac{\langle \hat{V}'(\chi^0) \rangle_{\text{dm}}^2}{\langle \hat{V}(\chi^0) \rangle_{\text{dm}}^2} = 4\omega_{J_0}(\mathbf{k}_0)^2 \left( 1 - \frac{2v_{J_0}^2}{\mathbf{k}_0^2} \rho_{\mathbf{x}}(\chi^0) - \frac{2v_{J_0}^2}{\omega_{J_0}(\mathbf{k}_0)^2} \rho_{\chi^0}(\chi^0) \right). \quad (4.92)$$

This form allows easier comparison with previous work in GFT cosmology and also to similar modifications to the Friedmann equations as derived from LQG and LQC [12, 138]. To complete the comparison, we introduce two further objects, namely the critical densities

$$\rho_{\chi,c}(J_0, \mathbf{k}_0) = \frac{\mathbf{k}_0^2}{2v_{J_0}^2}, \quad \rho_{\chi^0,c}(J_0, \mathbf{k}_0) = \frac{\omega_{J_0}(\mathbf{k}_0)^2}{2v_{J_0}^2}. \quad (4.93)$$

With the definition of the critical energy densities, the effective Friedmann equation takes the form

$$\frac{\langle \hat{V}'(\chi^0) \rangle_{\text{dm}}^2}{\langle \hat{V}(\chi^0) \rangle_{\text{dm}}^2} = 4\omega_{J_0}(\mathbf{k}_0)^2 \left( 1 - \frac{\rho_{\chi}(\chi^0)}{\rho_{\chi,c}(J_0, \mathbf{k}_0)} - \frac{\rho_{\chi^0}(\chi^0)}{\rho_{\chi^0,c}(J_0, \mathbf{k}_0)} \right). \quad (4.94)$$

This shows that in this model there is no cosmological singularity since the right-hand side can be zero. The functional form is very similar to the Friedmann-like equations arising from models in the context of LQG and LQC [12, 138]. Furthermore, the result obtained here is also very similar to results obtained previously in the context of GFT cosmology with a single scalar field [2, 76, 111]. In these previous works, the critical density also depended on the spin representation label. In the formalism of this section, the critical density is also labelled by the wave number  $\mathbf{k}_0$ . Note that the different values for the critical energy densities are once more a manifestation of the symmetry breaking induced by the singling out of a particular scalar field to act as the clock variable. Note that the critical density of the temporal scalar field reduces in the limit  $\mathbf{k}_0^2 \ll m_{J_0}^2$  to

$$\rho_{\chi^0,c}(J_0, \mathbf{k}_0)|_{\mathbf{k}_0 \rightarrow 0} = \frac{m_{J_0}^2}{2v_{J_0}^2} = \frac{3\pi}{2} \rho_{\text{P}} \left( \frac{v_{\text{P}}}{v_{J_0}} \right)^2, \quad (4.95)$$

which is compatible with the result of the work on GFT with a single scalar field.

An immediate consequence of the form of the effective Friedmann equation (4.94) is that the expectation value of the volume operator will undergo a bounce when

$$\frac{\rho_{\chi}(\chi^0)}{\rho_{\chi,c}(J_0, \mathbf{k}_0)} + \frac{\rho_{\chi^0}(\chi^0)}{\rho_{\chi^0,c}(J_0, \mathbf{k}_0)} = 1 \quad (4.96)$$

and therefore the expectation value of the volume operator remains positive through the relational time evolution. This last point implies that for generic states the big bang singularity is avoided in this model. Indeed, the only case in which there is no bounce is the case in which  $\rho_{\chi}(\chi^0) = \rho_{\chi^0}(\chi^0) = 0$ . In that case the effective Friedmann equation has the same functional form as the classical theory,

$$\langle \hat{V}'(\chi^0) \rangle^2 = 4\omega_{J_0}(\mathbf{k}_0)^2 \langle \hat{V}(\chi^0) \rangle_{\text{dm}}^2. \quad (4.97)$$

These results are independent of the number of massless scalar fields. The only requirement is that there are at least two massless scalar fields, i.e.,  $D \geq 2$ .

There is an interesting difference in the functional form that the effective Friedmann equations take when compared with the results for the case of only one massless scalar field ( $D = 1$ ) being present which was previously considered in [2, 76] and which is also the content of Chapter 5. In that case the right-hand side of the Friedmann equation (where the left-hand side is divided by the squared expectation value of the volume operator) also has a term which is proportional to  $\langle \hat{V}(\chi^0) \rangle^{-1}$ . We trace this discrepancy back to the fact that we had to regularise the volume operator by subtracting the vacuum contribution as discussed in Section 4.6. Indeed, had we not set  $c_\infty = 0$ , then the effective Friedmann equation for the double mode states would read

$$\frac{\langle \hat{V}'(\chi^0) \rangle_{\text{dm}}^2}{\langle \hat{V}(\chi^0) \rangle_{\text{dm}}^2} = 4\omega_{J_0}(\mathbf{k}_0)^2 \left( 1 + \frac{c_\infty v_{J_0}}{\langle \hat{V}(\chi^0) \rangle_{\text{dm}}} - \frac{\langle \hat{V} \rangle_{\text{dm}} (\langle \hat{V} \rangle_{\text{dm}} + c_\infty v_{J_0})}{\langle \hat{V}(\chi^0) \rangle_{\text{dm}}^2} + \frac{\langle \hat{X} \rangle_{\text{dm}}^2}{\langle \hat{V}(\chi^0) \rangle_{\text{dm}}^2} \right). \quad (4.98)$$

## 4.9. Comparison with mean-field theory of a complex group field

In this section we analyse the GFT with multiple scalar fields in the mean field formalism. The aspect we want to emphasise is that the breaking of covariance also arises in the mean field setting and is not an artefact of the canonical framework which we employed in this chapter. As has been stated before, the breaking of the covariance of the theory under the action of the Euclidean group  $E(D)$  to the smaller group  $E(1) \times E(d)$  is due to the insistence that there be a foliation of  $\mathbb{R}^D$  such that the group is square-integrable on the co-dimension 1 leaves. This point is also illustrated in Fig. 4.1.

We consider a complex group field  $\varphi: \mathbb{R}^D \rightarrow \mathbb{C}$ . The theory we defined before can be straightforwardly defined in terms of complex scalar fields. For definiteness, the action functional takes the following form which is analogous to (4.1)

$$S(\varphi, \bar{\varphi}) = \sum_J \int d^D \chi \bar{\varphi}_J(\chi^\alpha) \left( \mathcal{K}_J^{(0)} + \mathcal{K}_J^{(2)} \Delta_\chi \right) \varphi_J(\chi^\alpha), \quad (4.99)$$



#### 4.9. Comparison with mean-field theory of a complex group field

where the coefficients  $\mathcal{K}_J^{(n)}$  are real. The resulting equations of motion are

$$(\Delta_\chi - m_J^2) \varphi_J(\chi^\alpha) = 0, \quad (4.100a)$$

$$(\Delta_\chi - m_J^2) \bar{\varphi}_J(\chi^\alpha) = 0, \quad (4.100b)$$

where  $m_J^2 = -\mathcal{K}_J^{(0)}/\mathcal{K}_J^{(2)}$  as in (4.3). In the following we restrict ourselves to the case  $m_J^2 > 0$  which is the case for which there is an exponential expansion. As before, the breaking of the  $E(D)$  covariance occurs when we define the Fourier transform,

$$\varphi_J(\chi^\alpha) = \int \frac{d^d \mathbf{k}}{(2\pi)^d} e^{i\mathbf{k} \cdot \boldsymbol{\chi}} \varphi_J(\chi^0, \mathbf{k}) \quad (4.101)$$

which has equations of motion

$$\left( \frac{\partial^2}{\partial(\chi^0)^2} - \mathbf{k}^2 - m_J^2 \right) \varphi_J(\chi^0, \mathbf{k}) = 0. \quad (4.102)$$

The general solution to the equations of motion is

$$\varphi_J(\chi^0, \mathbf{k}) = \alpha_J(\mathbf{k}) e^{\sqrt{\mathbf{k}^2 + m_J^2} \chi^0} + \beta_J(\mathbf{k}) e^{-\sqrt{\mathbf{k}^2 + m_J^2} \chi^0}, \quad (4.103)$$

where  $\alpha_J(\mathbf{k})$  and  $\beta_J(\mathbf{k})$  are complex parameters. The volume in the mean-field theory is defined differently than in the canonical setting (cf. also the discussion in Chapter 3),

$$V(\chi^0) = \sum_J v_J \int d^d \boldsymbol{\chi} |\varphi_J(\chi^0, \boldsymbol{\chi})|^2 \quad (4.104)$$

which for the solution (4.103) reads

$$\begin{aligned} V(\chi^0) = \sum_J v_J \int \frac{d^d \mathbf{k}}{(2\pi)^d} & \left( |\alpha_J(\mathbf{k})|^2 e^{2\sqrt{\mathbf{k}^2 + m_J^2} \chi^0} \right. \\ & \left. + 2\text{Re}(\bar{\beta}_J(\mathbf{k}) \alpha_J(\mathbf{k})) + |\beta_J(\mathbf{k})|^2 e^{-2\sqrt{\mathbf{k}^2 + m_J^2} \chi^0} \right). \end{aligned} \quad (4.105)$$

For states sharply peaked at the modes  $(J_0, \pm \mathbf{k}_0)$  one obtains the effective Friedmann equation

$$\left( \frac{V'(\chi^0)}{V(\chi^0)} \right)^2 \approx 4 (m_{J_0}^2 + \mathbf{k}_0^2) = 4m_{J_0}^2 \left( 1 + \frac{\mathbf{k}_0^2}{m_{J_0}^2} \right) \quad (4.106)$$

which hearteningly agrees with (4.68). As before we would like to compare this with the classical expression (4.61). The first consequence is that if we relate the parameter  $m_{J_0}$  to the gravitational coupling constant as  $m_{J_0}^2 = 3\kappa/8$ , we have an agreement with the classical Friedmann equation. The next step is to establish a connection of the expression

$\mathbf{k}_0^2/m_{J_0}^2$  to the conserved charges of the theory. In order to obtain the conserved quantities of this theory, it is straightforward to adapt the procedure explained in Section 4.4. The total momentum is given by

$$\begin{aligned} \mathbf{P} &= \frac{1}{2} \sum_J \mathcal{K}_J^{(2)} \int d^d \boldsymbol{\chi} \left( \frac{\partial \bar{\varphi}_J(\chi^0, \boldsymbol{\chi})}{\partial \chi^0} \boldsymbol{\nabla} \varphi_J(\chi^0, \boldsymbol{\chi}) + \frac{\partial \varphi_J(\chi^0, \boldsymbol{\chi})}{\partial \chi^0} \boldsymbol{\nabla} \bar{\varphi}_J(\chi^0, \boldsymbol{\chi}) \right) \\ &= \sum_J \mathcal{K}_J^{(2)} \int \frac{d^d \mathbf{k}}{(2\pi)^d} 2\mathbf{k} \sqrt{\mathbf{k}^2 + m_J^2} \text{Im}(\bar{\beta}_J(\mathbf{k}) \alpha_J(\mathbf{k})) \end{aligned} \quad (4.107)$$

and the total energy is

$$\begin{aligned} E &= \frac{1}{2} \sum_J \mathcal{K}_J^{(2)} \int d^d \boldsymbol{\chi} \left( - \left| \frac{\partial \varphi_J(\chi^0, \boldsymbol{\chi})}{\partial \chi^0} \right|^2 + \left| \frac{\partial \varphi_J(\chi^0, \boldsymbol{\chi})}{\partial \boldsymbol{\chi}} \right|^2 + m_J^2 |\varphi_J(\chi^0, \boldsymbol{\chi})|^2 \right) \\ &= \sum_J \mathcal{K}_J^{(2)} \int \frac{d^d \mathbf{k}}{(2\pi)^d} 2(\mathbf{k}^2 + m_J^2) \text{Re}(\bar{\beta}_J(\mathbf{k}) \alpha_J(\mathbf{k})). \end{aligned} \quad (4.108)$$

As in the canonical framework it is clear that a priori there is no relation between these conserved quantities and therefore the term  $\mathbf{k}_0^2/m_{J_0}^2$  does not have a straightforward interpretation in terms of conserved quantities for generic initial conditions.

## Chapter 5.

# Canonical formulation: Single scalar field

This chapter is based on [76].

In Chapter 4 we studied a GFT model coupled to  $D$  massless scalar fields. This chapter deals with the case  $D = 1$ , i.e., the case in which there is only a single massless scalar field present, which is also the case that has been studied predominantly in other works on GFT cosmology. In Chapter 4 we have seen that the canonical framework applied to a GFT model with multiple scalar fields leads to a model which can be viewed as an ordinary quantum field theory defined on  $\mathbb{R}^D$ . Unsurprisingly, the case of a single scalar field,  $D = 1$ , then leads to a model which can be viewed as a quantum mechanical system. Due to the comparatively simple nature of this quantum mechanical model, it is possible to obtain some results which go beyond those we have presented for the case of multiple scalar fields.

## 5.1. General theory

As in Chapter 4 we will restrict ourselves to the case of a real group field. For the case of a single scalar field ( $D = 1$ ), the action (4.1) takes the form

$$S(\varphi) = \frac{1}{2} \sum_J \int d\chi \varphi_J(\chi) \left( \mathcal{K}_J^{(0)} + \mathcal{K}_J^{(2)} \partial_\chi^2 \right) \varphi_J(\chi). \quad (5.1)$$

Since there is only one scalar field, the Legendre transform with respect to the scalar field variables is unambiguous. The conjugate momentum is defined as (cf. (4.6))

$$\pi_J(\chi) = \frac{\delta S(\varphi)}{\delta \varphi'_J(\chi)} = -\mathcal{K}_J^{(2)} \varphi'_J(\chi). \quad (5.2)$$

The resulting Hamiltonian is then given by the expression (cf. (4.8))

$$H = \sum_J H_J = \sum_J \frac{\mathcal{K}_J^{(2)}}{2} \left( -\frac{1}{|\mathcal{K}_J^{(2)}|^2} \pi_J(\chi)^2 + m_J^2 \varphi_J(\chi)^2 \right), \quad (5.3)$$

where, as defined in (4.3),  $m_J^2 = -\mathcal{K}_J^{(0)}/\mathcal{K}_J^{(2)}$  and we emphasise again that  $m_J^2$  need not be positive.

The quantum theory can then be obtained by promoting the functions  $\varphi_J$  and  $\pi_J$  to operators which satisfy the equal time commutation relation

$$[\hat{\varphi}_J(\chi), \hat{\pi}_{J'}(\chi)] = i\delta_{JJ'} \hat{I}. \quad (5.4)$$

As we did for the case of multiple scalar field, we introduce a new set of operators via the relations (cf. (4.16))

$$\hat{a}_J(\chi) = A_J \hat{\varphi}_J(\chi) + \frac{i}{2A_J} \hat{\pi}_J(\chi), \quad (5.5a)$$

$$\hat{a}_J^\dagger(\chi) = A_J \hat{\varphi}_J(\chi) - \frac{i}{2A_J} \hat{\pi}_J(\chi), \quad (5.5b)$$

where  $A_J$  is an arbitrary real parameter. The operators  $\hat{a}_J$  and  $\hat{a}_J^\dagger$  satisfy the bosonic commutation relations (cf. (4.18))

$$[\hat{a}_J(\chi), \hat{a}_{J'}^\dagger(\chi)] = \delta_{JJ'} \hat{I}, \quad (5.6)$$

where  $\hat{I}$  is the identity operator. The interpretation is again, that acting with  $\hat{a}_J^\dagger$  creates a quantum of space with a geometry corresponding to the label  $J$  and likewise the operator

$\hat{a}_J$  annihilates such a quantum. In the following we will refer to the operators  $\hat{a}_J^\dagger$  and  $\hat{a}_J$  as creation and annihilation operators, respectively.

As was the case for multiple scalar fields, there is a natural value which the parameter  $A_J$  can take. For modes with  $m_J^2 \neq 0$  this natural choice is

$$A_J(\mathbf{k}) = \sqrt{\frac{|m_J| |\mathcal{K}_J^{(2)}|}{2}} \quad (5.7)$$

and for  $m_J^2 = 0$  it is

$$A_J = \sqrt{|\mathcal{K}_J^{(2)}|}. \quad (5.8)$$

The corresponding mode Hamiltonians  $\hat{H}_J$  are as follows: for the case  $m_J^2 > 0$  one gets the Hamiltonian of squeezing type

$$\hat{H}_J = \text{sgn}(\mathcal{K}_J^{(2)}) \frac{m_J}{2} \left( \hat{a}_J \hat{a}_J + \hat{a}_J^\dagger \hat{a}_J^\dagger \right), \quad (5.9)$$

for the case  $m_J^2 < 0$  one gets the Hamiltonian of harmonic oscillator type

$$\hat{H}_J = -\text{sgn}(\mathcal{K}_J^{(2)}) \frac{|m_J|}{2} \left( \hat{a}_J \hat{a}_J^\dagger + \hat{a}_J^\dagger \hat{a}_J \right), \quad (5.10)$$

and for the case  $m_J^2 = 0$  one gets the Hamiltonian of special type

$$\hat{H}_J = \frac{1}{2} \text{sgn}(\mathcal{K}_J^{(2)}) \left( \hat{a}_J \hat{a}_J + \hat{a}_J^\dagger \hat{a}_J^\dagger - \hat{a}_J \hat{a}_J^\dagger - \hat{a}_J^\dagger \hat{a}_J \right). \quad (5.11)$$

The relational volume operator is then defined as the weighted sum over all number operators,

$$\hat{V}(\chi) = \sum_J v_J \hat{N}_J, \quad (5.12)$$

where  $\hat{N}_J = \hat{a}_J^\dagger \hat{a}_J$  and  $v_J$  is the volume of a quantum of geometry with labels  $J$ . We now assume that the Hamiltonian is give by a sum over mode Hamiltonians  $\hat{H}_J$  which are all of the squeezing type (5.9). This makes sense since the time-dependence of the corresponding modes is exponential rather than oscillatory or linear as would be the case for the other cases. That is, for sufficiently late times it is reasonable to assume that all the modes which are not the squeezing type have become irrelevant. One can solve the equation of motion,

$$\hat{V}'(\chi) = -i[\hat{V}, \hat{H}](\chi), \quad (5.13)$$

analytically and the general expression is given by

$$\hat{V}(\chi) = \sum_J \left( -\frac{1}{2}v_J \hat{I} + \left( \hat{V}_J + \frac{v_J}{2} \hat{I} \right) \cosh(2m_J \chi) + \hat{X}_J \sinh(2m_J \chi) \right), \quad (5.14)$$

where we defined (cf. (4.57))

$$\hat{X}_J = \text{sgn}(\mathcal{K}_J^{(2)}) \frac{i}{2} v_J \left( \hat{a}_J \hat{a}_J - \hat{a}_J^\dagger \hat{a}_J^\dagger \right). \quad (5.15)$$

Note that (5.14) can be obtained from (4.55) by noting that in the case of a single scalar field the formally divergent quantity  $c_\infty$  defined in (4.56) is no longer divergent and in the case of a single scalar field we should set  $c_\infty = 1$  since for equal  $J$  (cf. (5.6)),

$$[\hat{a}_J, \hat{a}_J^\dagger] = \hat{I}. \quad (5.16)$$

The expectation value of the time derivative of the partial volume operators,  $\hat{V}_J$ , can be written in terms of the expectation value of the partial volume operators themselves,

$$\langle \hat{V}'_J(\chi) \rangle = \pm 2m_J \sqrt{\left( \langle \hat{V}_J(\chi) \rangle - \langle \hat{V}_J \rangle \right) \left( \langle \hat{V}_J(\chi) \rangle + \langle \hat{V}_J \rangle + v_J \right) + \langle \hat{X}_J \rangle^2}. \quad (5.17)$$

The resulting Friedmann equation is then given by

$$\begin{aligned} \langle \hat{V}'(\chi) \rangle^2 &= \sum_{J, J'} 4m_J m_{J'} \sqrt{\left( \langle \hat{V}_J(\chi) \rangle - \langle \hat{V}_J \rangle \right) \left( \langle \hat{V}_J(\chi) \rangle + \langle \hat{V}_J \rangle + v_J \right) + \langle \hat{X}_J \rangle^2} \\ &\quad \times \sqrt{\left( \langle \hat{V}_{J'}(\chi) \rangle - \langle \hat{V}_{J'} \rangle \right) \left( \langle \hat{V}_{J'}(\chi) \rangle + \langle \hat{V}_{J'} \rangle + v_{J'} \right) + \langle \hat{X}_{J'} \rangle^2}. \end{aligned} \quad (5.18)$$

Although this expression is a lot simpler than the analogue equation (4.65) of the previously studied case with multiple scalar fields, the equation does not allow for a straightforward cosmological interpretation. As before, the case in which only a single mode  $J_0$  is relevant simplifies things a lot. Assuming that only one mode  $J_0$  is relevant, the resulting effective Friedmann equation is given by

$$\left( \frac{\langle \hat{V}'(\chi) \rangle_{\text{sm}}}{\langle \hat{V}(\chi) \rangle_{\text{sm}}} \right)^2 = 4m_{J_0}^2 \left( 1 + \frac{v_{J_0}}{\langle \hat{V}(\chi) \rangle_{\text{sm}}} + \frac{\langle \hat{X} \rangle_{\text{sm}}^2 - \langle \hat{V} \rangle_{\text{sm}} \left( \langle \hat{V} \rangle_{\text{sm}} + v_{J_0} \right)}{\langle \hat{V}(\chi) \rangle_{\text{sm}}^2} \right). \quad (5.19)$$

In the case of a single scalar field the restriction to a single mode can be motivated as follows [68]. Since the solutions of the equations of motion scale are linear combinations of  $\exp(\pm m_J \chi^0)$ . In the case in which there is a maximal coefficient,  $|m_{J_0}| = \max_J \{|m_J|\}$ , the mode  $J_0$  would give the dominant contribution at sufficiently late times.

We now turn to the question of a possible interpretation of (5.19). Recall that the massless scalar field has an energy density

$$\rho_\chi(\chi) = \frac{\pi_\chi^2}{2V(\chi)^2}. \quad (5.20)$$

Furthermore, we put forward the interpretation that (the expectation value of) the Hamiltonian corresponds to the scalar field momentum,

$$\pi_\chi = \langle \hat{H} \rangle. \quad (5.21)$$

One can check by direct calculation that the expectation values of the operators  $\hat{X}_J$  satisfy

$$\langle \hat{X}_J \rangle^2 = v_J^2 \left( -\frac{\langle \hat{H}_J \rangle^2}{m_J^2} + \langle \hat{a}\hat{a} \rangle \langle \hat{a}^\dagger \hat{a}^\dagger \rangle \right). \quad (5.22)$$

Using this relation one can rewrite (5.19) to read

$$\begin{aligned} \left( \frac{\langle \hat{V}'(\chi) \rangle_{\text{sm}}}{\langle \hat{V}(\chi) \rangle_{\text{sm}}} \right)^2 &= 4m_{J_0}^2 \left( 1 + \frac{v_{J_0}}{\langle \hat{V}(\chi) \rangle_{\text{sm}}} \right. \\ &\left. - \frac{1}{\langle \hat{V}(\chi) \rangle_{\text{sm}}^2} \left( v_{J_0}^2 \frac{\langle \hat{H} \rangle_{\text{sm}}^2}{m_{J_0}^2} - v_{J_0}^2 \langle \hat{a}\hat{a} \rangle_{\text{sm}} \langle \hat{a}^\dagger \hat{a}^\dagger \rangle_{\text{sm}} + \langle \hat{V} \rangle_{\text{sm}} \left( \langle \hat{V} \rangle_{\text{sm}} + v_{J_0} \right) \right) \right). \end{aligned} \quad (5.23)$$

One can bring this equation into the even more suggestive form

$$\left( \frac{\langle \hat{V}'(\chi) \rangle_{\text{sm}}}{\langle \hat{V}(\chi) \rangle_{\text{sm}}} \right)^2 = 4m_{J_0}^2 \left( 1 - \frac{\rho_{\text{eff}}(\chi)}{\rho_c} \right) + \frac{4m_{J_0}^2 v_{J_0}}{\langle \hat{V}(\chi) \rangle_{\text{sm}}}, \quad (5.24)$$

where we defined the effective energy density

$$\rho_{\text{eff}}(\chi) = \rho_\chi(\chi) - \frac{2m_{J_0}^2 \langle \hat{a}\hat{a} \rangle_{\text{sm}} \langle \hat{a}^\dagger \hat{a}^\dagger \rangle_{\text{sm}}}{\langle \hat{V}(\chi) \rangle_{\text{sm}}^2} + \frac{2m_{J_0}^2 \langle \hat{V} \rangle_{\text{sm}} \left( \langle \hat{V} \rangle_{\text{sm}} + v_{J_0} \right)}{v_{J_0}^2 \langle \hat{V}(\chi) \rangle_{\text{sm}}^2} \quad (5.25)$$

and the critical energy density

$$\rho_c = \frac{m_{J_0}^2}{2v_{J_0}^2}. \quad (5.26)$$

The first thing to note is that we expect the limit  $\langle \hat{V}(\chi) \rangle_{\text{sm}} \rightarrow \infty$  to correspond to the classical limit. Agreement with the classical theory of a flat FLRW universe with a massless scalar field can then be obtained by identifying  $4m_{J_0}^2 = 3\kappa/2$ . With this identification we obtain for the critical energy density

$$\rho_c = \frac{3\kappa}{16v_{J_0}^2} = \frac{3\pi}{2} \rho_{\text{P}} \left( \frac{v_{\text{P}}}{v_{J_0}} \right)^2, \quad (5.27)$$

which is compatible with earlier results derived in the mean field formalism of GFT cosmology [111, 112]. The reason we brought the effective Friedmann equation into this specific form is to allow better comparison with previous results. For instance, we wish to compare our result for an effective Friedmann equation with that found in LQC. The reason we expect the results to be similar is that both GFT and LQC are descendants from the LQG research programme which is the primary motivation for considering discrete spacetimes. The effective Friedmann equation found in LQC takes the form [12, 22]

$$\left(\frac{V'_{\text{LQC},x}(\chi)}{V_{\text{LQC},x}(\chi)}\right)^2 = \frac{3\kappa}{2} \left(1 - \frac{\rho(\chi)}{\rho_{\text{LQC},c}}\right). \quad (5.28)$$

Compared to this form our result (5.24) has several extra terms which depend on the initial conditions.

## 5.2. Algebraic formulation

This section presents the derivation of the effective Friedmann equations from a different perspective. As we will see solving the equations of motion can be entirely circumvented for the special case in which only a single mode is relevant. Since we expect there to be a “fastest” growing mode it is reasonable to assume that the restriction to a single group field mode provides some perspective on the physical content of the theory.

The starting point for the algebraic discussion is the assumption that the Hamiltonian is given by the single mode Hamiltonian of squeezing type (5.9),

$$\hat{H} = -\frac{\omega}{2} \left(\hat{a}\hat{a} + \hat{a}^\dagger\hat{a}^\dagger\right). \quad (5.29)$$

More precisely, this expression corresponds to the mode Hamiltonian  $\hat{H}_{J_0}$  (5.9) with  $m_{J_0} = \omega$ ,  $\text{sgn}(\mathcal{K}_{J_0}) < 0$  and suppression of the mode labels of the operators. This simple form of squeezing Hamiltonian was also studied in the context of GFT in [2]. The crucial observation of [19] was that the squeezing type Hamiltonian and volume operator can be viewed as elements of a representation of the Lie algebra  $su(1, 1)$ . By using the Casimir of  $su(1, 1)$  it is possible to sidestep solving the equations of motion when deriving the effective Friedmann equation of the system. The analysis in [19] was carried out for a classical analogue system, where Lie algebra structure was realised via Poisson brackets. In this section we extend the method to the non-commutative quantum setting.



We next establish how the formalism developed so far is related to the Lie algebra  $su(1, 1)$ . In Appendix B we provide a brief overview of the representation theory of  $su(1, 1)$ . A representation of  $su(1, 1)$  in terms of bosonic creation and annihilation operators is given by the following identifications,

$$\hat{K}_0 = \frac{1}{4} (\hat{a}^\dagger \hat{a} + \hat{a} \hat{a}^\dagger), \quad \hat{K}_+ = \frac{1}{2} \hat{a}^2, \quad \hat{K}_- = \frac{1}{2} \hat{a}^{\dagger 2}. \quad (5.30)$$

These operators do indeed satisfy the commutation relation of  $su(1, 1)$ ,

$$[\hat{K}_0, \hat{K}_\pm] = \pm \hat{K}_\pm, \quad [\hat{K}_\mp, \hat{K}_\pm] = \pm 2\hat{K}_0 \quad (5.31)$$

as one can check by direct computation. In general, the algebra  $su(1, 1)$  has a Casimir element

$$\hat{C} = \hat{K}_0^2 - \frac{1}{2} (\hat{K}_+ \hat{K}_- + \hat{K}_- \hat{K}_+). \quad (5.32)$$

By inserting the defining relations (5.30) into the Casimir one finds that the bosonic Fock representation of  $su(1, 1)$  has a fixed Casimir,

$$\hat{C} = -\frac{3}{16}. \quad (5.33)$$

The Hamiltonian in terms of the  $su(1, 1)$  operators is given by

$$\hat{H} = -\omega (\hat{K}_+ + \hat{K}_-). \quad (5.34)$$

In the following we will consider this form of the Hamiltonian, (5.34), as the defining equation for the Hamiltonian of the quantum system. That is, we allow for representations other than the Fock representation. The representation we will consider is the ascending discrete series of  $su(1, 1)$ . This representation has states  $|k, m\rangle$  that satisfy the two eigenvalue equations,

$$\hat{C}|k, m\rangle = k(k-1)|k, m\rangle, \quad (5.35)$$

$$\hat{K}_0|k, m\rangle = (k+m)|k, m\rangle, \quad (5.36)$$

where  $k$  is a positive real number and  $m$  is a non-negative integer. The label  $k$  is called the *Bargmann index* of the representation. From (5.33) it follows that the bosonic representation corresponds to either  $k = 1/4$  or  $k = 3/4$ . The operator  $\hat{K}_0$  can be rewritten in the Fock representation as

$$\hat{K}_0 = \frac{1}{4} (2\hat{N} + \hat{I}), \quad (5.37)$$

where  $\hat{N} = \hat{a}^\dagger \hat{a}$  is the number operator and  $\hat{I}$  is the identity operator. The standard Fock representation satisfies  $\hat{N}|n\rangle = n|n\rangle$ . Considering the action of the operator  $\hat{K}_0$  on the Fock representation we observe that

$$\hat{K}_0|2n\rangle = \left(\frac{1}{4} + n\right)|2n\rangle, \quad (5.38a)$$

$$\hat{K}_0|2n+1\rangle = \left(\frac{3}{4} + n\right)|2n+1\rangle. \quad (5.38b)$$

Therefore we conclude that a Bargmann index  $k = 1/4$  corresponds to states where only modes of an even number of quanta are present and an index  $k = 3/4$  corresponds to the case where there are only modes with an odd number of quanta present. From the perspective of GFT cosmology therefore the index  $k = 1/4$  is of the greatest interest since it includes the Fock vacuum which we identify with the state which corresponds to no geometry.

### 5.2.1. Friedmann equation

In the simple setting we consider in this section, there is only one type of group field mode. Therefore the volume operator  $\hat{V}$  is simply a rescaling of the number operator  $\hat{N}$ ,

$$\hat{V} = v\hat{N}, \quad (5.39)$$

where  $v$  is the volume of a single quantum. As can be seen from Eq. (5.37) the operator  $\hat{K}_0$  is related to the volume operator  $\hat{V}$  by an affine transformation,

$$\hat{V} = v \left( 2\hat{K}_0 - \frac{1}{2}\hat{I} \right). \quad (5.40)$$

With this in mind, let us first derive a ‘‘Friedmann equation’’ for the expectation value of the operator  $\hat{K}_0$ . Explicitly, the aim is to obtain an expression

$$\langle \hat{K}'_0(\chi) \rangle^2 = f(\langle \hat{K}_0(\chi) \rangle), \quad (5.41)$$

where  $f$  is some functional where the only time-dependence arises through expectation values of the operator  $\hat{K}_0$ .

Note that we already solved the equation of motion for  $\hat{V}$  and the explicit solution is given in (5.14). Adapting that result to the set-up of this section and writing it in terms

of the  $su(1, 1)$  variables, the time dependence of  $\hat{K}_0$  is given by

$$\hat{K}_0(\chi) = \hat{K}_0 \cosh(2\omega\chi) + \frac{i}{2} \left( \hat{K}_+ - \hat{K}_- \right) \sinh(2\omega\chi). \quad (5.42)$$

However, the point of this section is to derive the effective Friedmann equation *without* solving the equations of motion explicitly. Therefore we will not make use of this explicit form in the following.

The equation of motion for the operator  $\hat{K}_0$  is given by

$$\hat{K}_0'(\chi) = -i[\hat{K}_0, \hat{H}](\chi) = i\omega \left( \hat{K}_+(\chi) - \hat{K}_-(\chi) \right). \quad (5.43)$$

Squaring this relation leads to the expression

$$\hat{K}_0'(\chi)^2 = -\omega^2 \left( \hat{K}_+(\chi)^2 + \hat{K}_-(\chi)^2 - \hat{K}_+(\chi)\hat{K}_-(\chi) - \hat{K}_-(\chi)\hat{K}_+(\chi) \right). \quad (5.44)$$

The crucial insight of [19] was that one can rewrite this as an expression where the only time-dependent operator is  $\hat{K}_0$  by using the two conserved quantities of the model, namely the Casimir and the Hamiltonian. In terms of these conserved quantities the above equation can be expressed as

$$\hat{K}_0'(\chi)^2 = 4\omega^2 \left( \hat{K}_0(\chi)^2 - \hat{C} - \frac{1}{4\omega^2} \hat{H}^2 \right). \quad (5.45)$$

As stated above, the goal is to obtain an equation where the left-hand side is given by  $\langle \hat{K}_0'(\chi) \rangle^2$ . However, if we take the expectation value of (5.45), the left-hand side is the expectation value of the operator squared,  $\langle \hat{K}_0'(\chi)^2 \rangle$ . To address this issue we introduce the following function,

$$G(\hat{A}, \hat{B}) = \frac{1}{2} \langle \hat{A}\hat{B} + \hat{B}\hat{A} \rangle - \langle \hat{A} \rangle \langle \hat{B} \rangle. \quad (5.46)$$

Note that this should *not* be interpreted as the (co)variance or uncertainty of the operators since we allow the arguments to be non-Hermitian operators and  $G(\hat{A}, \hat{B})$  therefore need not be real. Note that the function  $G$  is symmetric and  $\mathbb{C}$ -linear,

$$G(\hat{A}, \hat{B}) = G(\hat{B}, \hat{A}), \quad (5.47a)$$

$$G(\hat{A}, \hat{B} + \gamma\hat{C}) = G(\hat{A}, \hat{B}) + \gamma G(\hat{A}, \hat{C}), \quad (5.47b)$$

where  $\gamma$  is a complex number. With this function in hand, we take the expectation value of (5.45) and obtain

$$\langle \hat{K}_0'(\chi) \rangle^2 = 4\omega^2 \left( \langle \hat{K}_0(\chi) \rangle^2 - \langle \hat{C} \rangle - \frac{1}{4\omega^2} \langle \hat{H} \rangle^2 + X \right), \quad (5.48)$$

where the quantity  $X$  is defined as

$$X = G(\hat{K}_0(\chi), \hat{K}_0(\chi)) - \frac{1}{4\omega^2} \left( G(\hat{K}'_0(\chi), \hat{K}'_0(\chi)) + G(\hat{H}, \hat{H}) \right). \quad (5.49)$$

Note that the omission of a time-dependence on the left-hand side of the definition of  $X$  is not an oversight. We will now show that  $X$  is indeed time-independent. Using the expressions (5.34) and (5.43) we find that

$$G(\hat{K}'_0(\chi), \hat{K}'_0(\chi)) + G(\hat{H}, \hat{H}) = 4\omega^2 G(\hat{K}_+(\chi), \hat{K}_-(\chi)). \quad (5.50)$$

Note that since  $\hat{H}$  is time-independent one may replace the time-independent (Schrödinger) operators with the time-dependent (Heisenberg) operators evaluated at any time. Using this result the expression  $X$  takes the form

$$X = G(\hat{K}_0(\chi), \hat{K}_0(\chi)) - G(\hat{K}_+(\chi), \hat{K}_-(\chi)). \quad (5.51)$$

To show that this is time-independent note that the equation of motion for  $\hat{K}_\pm$  is given by

$$\hat{K}'_\pm(\chi) = -i[\hat{K}_\pm, \hat{H}](\chi) = \pm 2i\omega \hat{K}_0(\chi). \quad (5.52)$$

Using this and Eq. (5.43) we find

$$2G(\hat{K}_0(\chi), \hat{K}'_0(\chi)) - G(\hat{K}'_+(\chi), \hat{K}_-(\chi)) - G(\hat{K}_+(\chi), \hat{K}'_-(\chi)) = 0, \quad (5.53)$$

which confirms that  $X$  is time-independent and therefore given by

$$X = G(\hat{K}_0, \hat{K}_0) - G(\hat{K}_+, \hat{K}_-). \quad (5.54)$$

We remark that it is also possible to use the explicit form of the Casimir  $\hat{C}$  to get an equation which only depends on expectation values of the basic  $su(1,1)$  operators (as opposed to their squares as is implicit by the appearance of the function  $G(\cdot, \cdot)$ ). Explicitly, the expectation value of the Casimir (5.32) is given by

$$\langle \hat{C} \rangle = \langle \hat{K}_0 \rangle^2 - \langle \hat{K}_+ \rangle \langle \hat{K}_- \rangle + G(\hat{K}_0, \hat{K}_0) - G(\hat{K}_+, \hat{K}_-). \quad (5.55)$$

With this we see that the ‘‘Friedmann equation’’ (5.48) can also be written as

$$\langle \hat{K}'_0(\chi) \rangle^2 = 4\omega^2 \left( \langle \hat{K}_0(\chi) \rangle^2 - \langle \hat{K}_0 \rangle^2 - \frac{1}{4\omega^2} \langle \hat{H} \rangle^2 + \langle \hat{K}_+ \rangle \langle \hat{K}_- \rangle \right). \quad (5.56)$$

As explained above, the operator  $\hat{K}_0$  can be related to the volume operator  $\hat{V}$  of the Fock representation. The explicit relation is given by

$$\hat{K}_0 = \frac{1}{2v} \left( \hat{V} + \frac{v}{2} \right). \quad (5.57)$$

With this relation we can obtain the following effective Friedmann equation,

$$\langle \hat{V}'(\chi) \rangle^2 = 4\omega^2 \left( \langle \hat{V}(\chi) \rangle (\langle \hat{V}(\chi) \rangle + v) - \langle \hat{V} \rangle (\langle \hat{V} \rangle + v) - \frac{v^2}{\omega^2} \langle \hat{H} \rangle^2 + 4v^2 \langle \hat{K}_+ \rangle \langle \hat{K}_- \rangle \right). \quad (5.58)$$

Note that this is the same as the effective Friedmann equation we obtained from the explicit solution to the equations of motion in the idealised case where only one mode is relevant (cf. (5.23)).

### 5.3. Initial state

So far the discussion did not specialise to particular states. In this section we will show that the late time behaviour which we are most interested in for cosmological consideration depends on the choice of state. More precisely, we are interested in states that lead to solutions we consider to be semiclassical. As a measure of semiclassicality we introduce the relative uncertainty as a function of an observable  $\hat{O}$  and a state  $|\psi\rangle$ ,

$$r(\hat{O}, |\psi\rangle) = \frac{\langle \psi | \hat{O}^2 | \psi \rangle - \langle \psi | \hat{O} | \psi \rangle^2}{\langle \psi | \hat{O} | \psi \rangle^2}. \quad (5.59)$$

If the relative uncertainty is small we consider the state to be semiclassical for a given observable. We wish to identify those states which lead to a small relative uncertainty at late times for the volume operator and the Hamiltonian as those are the quantities for which we have an immediate classical interpretation. This measure of semiclassicality was discussed previously in the context of LQC in, e.g., [7] and in the context of GFT in [119].

The states we are going to consider are the following coherent states

- Fock coherent states
- Perelomov–Gilmore (PG) coherent states

- Barut–Girardello (BG) coherent states

Technical details concerning these states can be found in Appendix B. The Fock coherent states are the well-known coherent states of the harmonic oscillator. In terms of the bosonic creation and annihilation operators the Fock coherent states are given by

$$|\sigma\rangle = \exp(\sigma\hat{a}^\dagger - \bar{\sigma}\hat{a})|0\rangle, \quad (5.60)$$

where  $|0\rangle$  is the Fock vacuum. Note that as explained above the Fock coherent states are a superposition of two representations with different Bargmann index. (See (B.33) for an explicit expression.) The PG coherent states can be obtained by acting on the lowest state  $|k, 0\rangle$  with the squeezing operator

$$\hat{S}(\xi) = \exp(\xi\hat{K}_+ - \bar{\xi}\hat{K}_-). \quad (5.61)$$

We define the PG coherent states to be given by

$$|\zeta, k\rangle = \hat{S}\left(\frac{\zeta}{|\zeta|}\operatorname{artanh}(|\zeta|)\right)|k, 0\rangle. \quad (5.62)$$

The BG coherent states are eigenstates of the lowering operator  $\hat{K}_-$ . Their defining relation is given by the equation

$$\hat{K}_-|\mu, k\rangle = \mu|\mu, k\rangle. \quad (5.63)$$

We now turn to the question of what the relative uncertainties are for these classes of coherent states. Since we expect the semiclassicality to be a late time phenomenon we consider only the asymptotic volume operator. For Fock states the relative uncertainties are given by

$$r(\hat{H}, |\sigma\rangle) = \frac{2(1 + 2|\sigma|^2)}{(\sigma^2 + \bar{\sigma}^2)^2}, \quad (5.64a)$$

$$r(\hat{V}(\pm\infty), |\sigma\rangle) = \frac{2(1 \mp 2i(\sigma \pm i\bar{\sigma})^2)}{(1 \mp i(\sigma \pm i\bar{\sigma})^2)^2}. \quad (5.64b)$$

From these expressions it is clear that for large  $|\sigma|$  the relative uncertainties can be made arbitrarily small. Therefore Fock coherent states are compatible with our notion of semiclassicality.

For PG coherent states the relative uncertainties are given by

$$r(\hat{H}, |\zeta, k\rangle) = \frac{1}{2k} \frac{(1 + \zeta^2)(1 + \bar{\zeta}^2)}{(\zeta + \bar{\zeta})^2}, \quad (5.65a)$$

$$r(\hat{V}(\pm\infty), |\zeta, k\rangle) = \frac{1}{2k}. \quad (5.65b)$$

From this it follows that states which are compatible with our requirement of small uncertainties correspond to representations with large  $k$ . Whilst this is mathematically true, for the Fock representation we have  $k = 1/4$  (or  $k = 3/4$ ) and therefore the relative uncertainties are not small. We conclude that the PG coherent states do not satisfy our semiclassicality criterion.

For BG coherent states the relative uncertainties are given by

$$r(\hat{H}, |\mu, k\rangle) = \frac{2}{(\mu + \bar{\mu})^2} \left[ k + |\mu| \frac{I_{2k}(2|\mu|)}{I_{2k-1}(2|\mu|)} \right], \quad (5.66a)$$

$$\begin{aligned} r(\hat{V}(\pm\infty), |\mu, k\rangle) = & 2 \left[ -2|\mu|^2 I_{2k}(2|\mu|)^2 + (3 - 4k)|\mu| I_{2k}(2|\mu|) I_{2k-1}(2|\mu|) \right. \\ & \left. + (k \mp i(\mu - \bar{\mu}) + 2|\mu|^2) I_{2k-1}(2|\mu|)^2 \right] \\ & \times \left[ 2|\mu| I_{2k}(2|\mu|) + (2k \mp i(\mu - \bar{\mu})) I_{2k-1}(2|\mu|) \right]^{-2}. \end{aligned} \quad (5.66b)$$

These expressions are quite involved. One possible simplification is to consider the asymptotic expansion of the modified Bessel functions for large arguments. For large absolute value of the coherent state parameter,  $|\mu|$ , the relative uncertainties are given by

$$r(\hat{H}, |\mu, k\rangle) \stackrel{|\mu| \rightarrow \infty}{\sim} \frac{2|\mu|}{(\mu + \bar{\mu})^2}, \quad (5.67a)$$

$$r(\hat{V}(\pm\infty), |\mu, k\rangle) \stackrel{|\mu| \rightarrow \infty}{\sim} \frac{2}{2|\mu| \mp i(\mu - \bar{\mu})}. \quad (5.67b)$$

From these expressions it is apparent that BG coherent states with a coherent state parameter which has a sufficiently large absolute value are compatible with our notion of semiclassicality. However, since the analytic calculations are rather involved due to the appearance of the modified Bessel function we will not consider the BG coherent states in subsequent sections.

In the above expressions we see that for the Fock and BG coherent states the asymptotic value is generically not symmetric around the past, i.e., the asymptotic relative uncertainty

in the infinite past and the infinite future differ. To quantify this asymmetry we introduce an *asymmetry parameter*,

$$\eta(|\psi\rangle) = 1 - \min \left\{ \frac{r(\hat{V}(+\infty), |\psi\rangle)}{r(\hat{V}(-\infty), |\psi\rangle)}, \frac{r(\hat{V}(-\infty), |\psi\rangle)}{r(\hat{V}(+\infty), |\psi\rangle)} \right\}. \quad (5.68)$$

In Fig. 5.1 this asymmetry parameter is plotted as a function of the argument of the complex coherent state parameter for Fock and BG coherent states. (PG coherent states don't have any asymmetry.) Although we have chosen specific states in the figure, the situation is generic. Similar analyses concerning the asymmetry of the volume operator have been carried out within the context of LQC in [23, 24, 40]. The discussion presented here extends the results obtained in the mean field setting in [119].

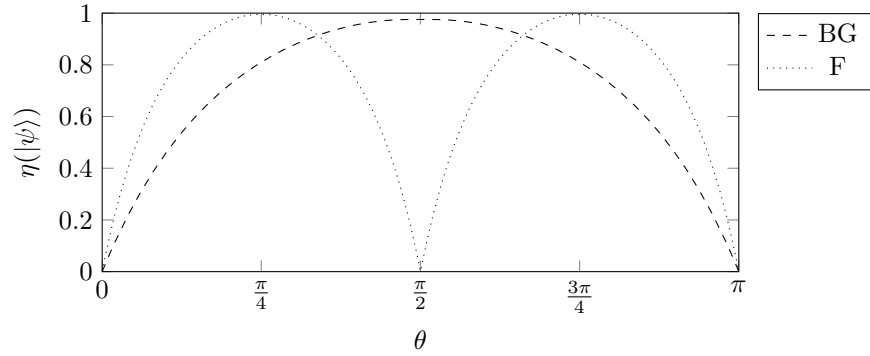


Figure 5.1.: The asymmetry parameter (5.68) shown as a function of the argument of the complex coherent state parameter. The Fock coherent states (F) plotted are given by  $|\sigma\rangle = |100\exp(i\theta)\rangle$  and the BG coherent states plotted are given by  $|\mu, k\rangle = |100\exp(i\theta), 1/4\rangle$ . This figure is taken from [76].

Up to now we only considered the asymptotic volume operators when discussing the relative uncertainty for different states. It is also interesting to investigate this relative uncertainty close to the bounce. Note that we have the explicit time-dependent expression for the volume operator at our disposal, cf. (5.42). In Fig. 5.2 we show the time dependence of the relative uncertainties for different values of the coherent state parameters close to the bounce. The plots illustrate the possibility of a time asymmetry in which the states on one side of the bounce have a small relative uncertainty, whilst having a large relative uncertainty on the other side of the bounce. Note that the complex parameter in the figure takes values only in the first quadrant of the complex plane. The behaviour for complex numbers in the rest of the complex plane is essentially the same as that shown. A notable difference is that for other choices it would be the past which has smaller relative



uncertainty than the future. (In the plots shown it is always the future with smaller relative uncertainty.)

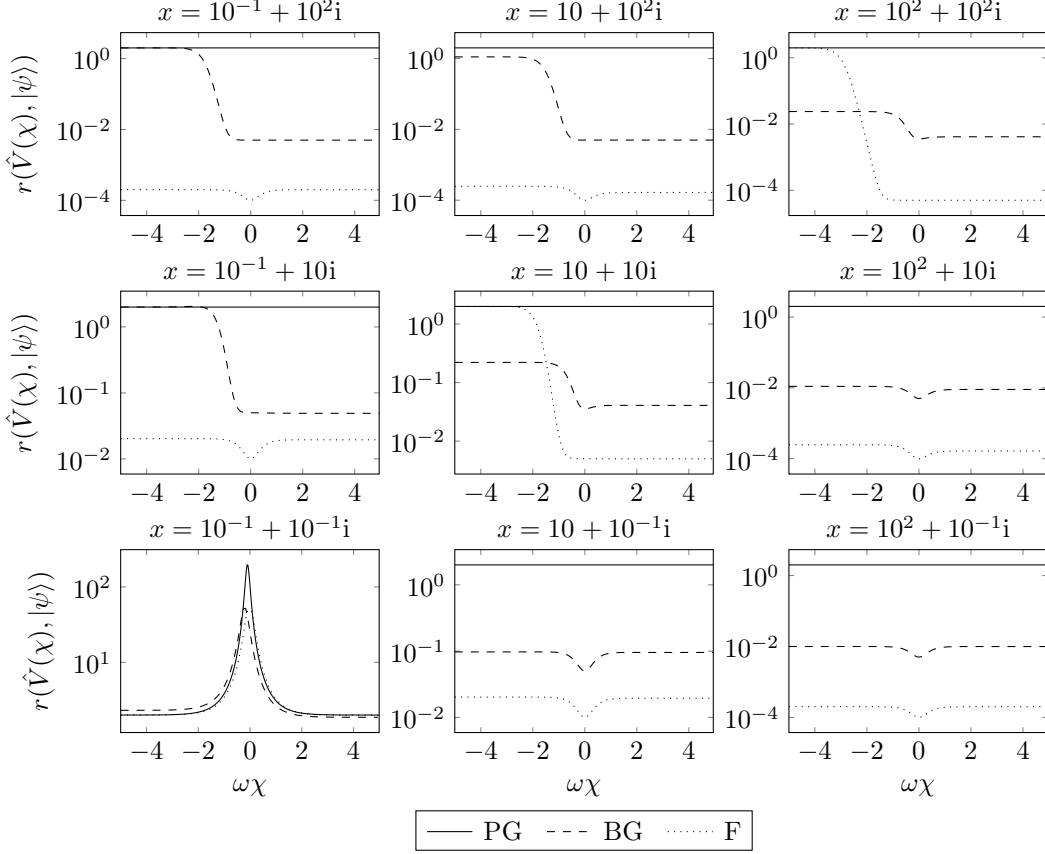


Figure 5.2.: The relative uncertainty (5.59) for different values of the coherent state parameters of the various classes of coherent states. The translation of the complex number  $x$  to the coherent state parameters is as follows: PG:  $|\zeta, k\rangle = |(x/|x|)\tanh(|x|), 1/4\rangle$  BG:  $|\mu, k\rangle = |x, 1/4\rangle$ , Fock:  $|\sigma\rangle \equiv |x\rangle$ . This figure is taken from [76].

The discussion above shows that of the classes of states considered here, the Fock coherent states and the BG coherent states are compatible with our desideratum that the relative uncertainties of the Hamiltonian and volume operator are small at late times. The PG coherent states do not satisfy this criterion.

## 5.4. Interactions

In this section we add an interaction term to the system discussed in Section 5.2. The cosmological solutions we have derived above all feature an exponentially growing volume which corresponds to an exponential growth in the expectation value of the number operators. As we have argued before, the free approximation should be thought of an approximation valid in the mesoscopic regime where there are a large number of quanta excited but not such a great number that one has to take interactions into consideration. By introducing an interaction term to the model we hope to make a step towards more general results in GFT cosmology. Related work on interacting GFT models has been carried out in [42] and [119]. In [42] interactions were added to the single-mode case by adding polynomial terms in the mean-field setting. One key insight from their work was that for a suitable interaction term and initial conditions their model leads to an inflationary universe. In [119] an interacting model is studied in the group representation (in contrast to the spin representation which we adopt). There it was shown that at early times an isotropic configuration is unstable. However, they also showed that at late times the evolution leads to an isotropisation.

The toy model we consider is given by the Hamiltonian

$$\hat{H} = -\omega \left( \hat{K}_+ + \hat{K}_- \right) + \lambda\omega \left( \hat{K}_+ + \hat{K}_- + 2\hat{K}_0 \right)^2. \quad (5.69)$$

In terms of the Fock creation and annihilation operator this can be written as

$$\hat{H} = -\omega \left( \hat{a}^2 + \hat{a}^{\dagger 2} \right) + \frac{\lambda\omega}{4} \left( \hat{a} + \hat{a}^\dagger \right)^4. \quad (5.70)$$

Recall that the creation and annihilation operators can be written in terms of the group field and its conjugate momentum (cf. (5.5)). The corresponding Hamiltonian would read

$$\hat{H} = \frac{1}{2|\mathcal{K}^{(2)}|} \hat{\pi}^2 - \frac{1}{2} |\mathcal{K}^{(0)}| \hat{\varphi}^2 + \lambda |\mathcal{K}^{(0)}|^{3/2} |\mathcal{K}^{(2)}|^{1/2} \hat{\varphi}^4, \quad (5.71)$$

where we suppressed the representation labels and made the identification  $\omega = m$ . From these formulas one can already discern the following. In the case  $\lambda > 0$  the interaction term has the opposite sign as the potential term coming from the free theory. The resulting potential is similar to that used in modelling the Higgs boson and solutions correspond to a cyclic universe. This agrees with the results of [42] where they found cyclic cosmologies

for a similar potential. In the case  $\lambda < 0$  the upside-down harmonic oscillator becomes an upside-down anharmonic oscillator leading to an even faster accelerated expansion.

In the following we present some cosmological implications of this interacting toy model. Unfortunately, the system is already complicated enough to prevent simply carrying over the analysis performed for the non-interacting model. As a first step we will derive the Friedmann equations in a classical analogue system. Then we will study the quantum system in a perturbative framework and finally we show some numerical, non-perturbative results for the quantum system.

#### 5.4.1. Classical analogue system

In this section we apply the algebraic method of Section 5.2 to a classical analogue system. More precisely, we now consider a Poisson manifold with variables  $\{K_0, K_+, K_-\}$  that have a Poisson structure specified by the Poisson brackets

$$\{K_0, K_\pm\} = \mp i K_\pm, \quad \{K_\mp, K_\pm\} = \mp i 2K_0. \quad (5.72)$$

The Hamiltonian and Casimir of the system are then simply given by (5.69) and (5.32) with the operators replaced by the corresponding phase space variables. In this section only we will consider the variable  $K_0$  to be related to the volume by the relation

$$K_0 = \frac{1}{2v} V. \quad (5.73)$$

We justify this by noting that in the quantum relation (5.57) the additional term was due to the non-commutativity of the quantum operators.

As we will see in a moment it is useful to relate the linear combinations  $K_+ + K_-$  and  $K_+ - K_-$  to the conserved quantities. Explicitly one finds

$$(K_+ - K_-)^2 = 4C - 4K_0^2 + (K_+ + K_-)^2, \quad (5.74a)$$

$$(K_+ + K_-) = \frac{1}{2\lambda} \left( 1 - 4\lambda K_0 \pm \sqrt{1 + 4\lambda \left( \frac{H}{\omega} - 2K_0 \right)} \right). \quad (5.74b)$$

Note that in (5.74a) we didn't take the square root as only the square will appear in later equations and in (5.74b) we will specify to the sign which is connected to the free theory in the limit  $\lambda \rightarrow 0$ , i.e., we choose the negative sign.

The equation of motion of  $K_0$  in the classical analogue system is

$$K_0'(\chi) = \{K_0, H\}(\chi) = i\omega(K_+ - K_-)(1 - 2\lambda(K_+ + K_- + 2K_0)). \quad (5.75)$$

Taking the square of this and using the relations (5.74) we get

$$K_0'(\chi)^2 = -\frac{\omega^2}{2\lambda^2} \left( 1 + 4\lambda \left( \frac{H}{\omega} - 2K_0(\chi) \right) \right) \times \left[ 1 - 8\lambda K_0(\chi) - (1 - 4\lambda K_0(\chi)) \sqrt{1 + 4\lambda \left( \frac{H}{\omega} - 2K_0(\chi) \right)} + 2\lambda \left( 4\lambda C + \frac{H}{\omega} \right) \right]. \quad (5.76)$$

This can be written as a Friedmann equation in terms of the volume via (5.73).

We now turn to the question of interpreting this result. In classical cosmology the right-hand side of the Friedmann equation is related to the energy density of the matter content of the universe. The Friedmann equation for the relational volume with  $n$  different types of matter when expressed as a function of the value of a massless scalar field is given by

$$\left( \frac{V_\chi'(\chi)}{V_\chi(\chi)} \right)^2 = \sum_{i=1}^n A_i V_\chi(\chi)^{1-w_i}, \quad (5.77)$$

where  $A_i$  are constants and  $w_i$  is the equation of state parameter of the matter species with label  $i$ . Usually the right-hand side of the Friedmann equation is considered to be related to the matter content (with the notable exceptions of the cosmological constant which is often call “dark energy” in this context and a curvature term). In our model the additional terms on the right-hand side of the Friedmann equation should be thought of as capturing quantum gravity effects<sup>1</sup>. Note that although we viewed the variables as coordinates of a Poisson manifold in this section, the model itself is derived from a bona fide quantum theory of gravity.

The appearance of a square root on the right-hand side of (5.76) makes the comparison to the classical equation (5.77) non-trivial. One possibility is to expand (5.76) as a series in  $\lambda$  around  $\lambda = 0$ . Writing this series in terms of the volume  $V$  (as opposed to  $K_0$ ) we

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<sup>1</sup>It is also possible to interpret these quantum gravity corrections as arising from a theory of modified gravity [37].

get

$$\begin{aligned}
V'(\chi)^2 = 4\omega^2 v^2 & \left\{ \frac{V(\chi)^2}{v^2} - \frac{H^2}{\omega^2} - 4C \left( 1 - 4\lambda \left( \frac{V(\chi)}{v} - \frac{H}{\omega} \right) \right) \right. \\
& - \sum_{n=1}^{\infty} \left[ 12\lambda^n \frac{(2n-2)!}{(n-1)!(n+2)!} \left( (2n-1)\frac{H}{\omega} + (3-n)\frac{V(\chi)}{v} \right) \right. \\
& \left. \left. \times \left( \frac{V(\chi)}{v} - \frac{H}{\omega} \right)^{n+1} \right] \right\}. \quad (5.78)
\end{aligned}$$

If one truncates this expansion at some order in  $\lambda$  it is important to note that the quantity which needs to be small for the approximation to be valid is actually the product  $\lambda V(\chi)$  which needs to be small. The leading order linear correction would correspond to an equation of state parameter  $w = 0$ , i.e., dust-like matter. Another interesting limit we can take is the limit where the volume tends to infinity. In this limit we find

$$\left( \frac{V'(\chi)}{V(\chi)} \right)^2 = 32\omega^2 \sqrt{-\lambda \frac{V(\chi)}{v}} + O(1). \quad (5.79)$$

This expression would correspond to an equation of state parameter  $w = 1/2$ . Note that the solution of this Friedmann equation (neglecting the subdominant terms) scales as  $V(\chi) \sim |\chi - \chi_0|$  for some constant  $\chi_0$ . Therefore, this solution would diverge at some finite value of  $\chi$ .

Another possibility of interpreting our result is to define an effective equation of state parameter as follows,

$$w_{\text{eff}}(\chi) = 1 - \frac{d \log((V'(\chi)/V(\chi))^2)}{d \log(V(\chi))}. \quad (5.80)$$

In Fig. 5.3 we show the Friedmann equation and the effective equation of state as a function of  $V$ . The plots illustrate the point that truncating any of these expressions at finite order in  $\lambda$  has a limited range of validity as explained above.

## 5.5. Quantum calculations

This section discusses the interacting toy model (5.69) as a quantum theory. Firstly we present some analytical results obtained using perturbation theory. Secondly we show some results obtained using numerical methods for the full theory.

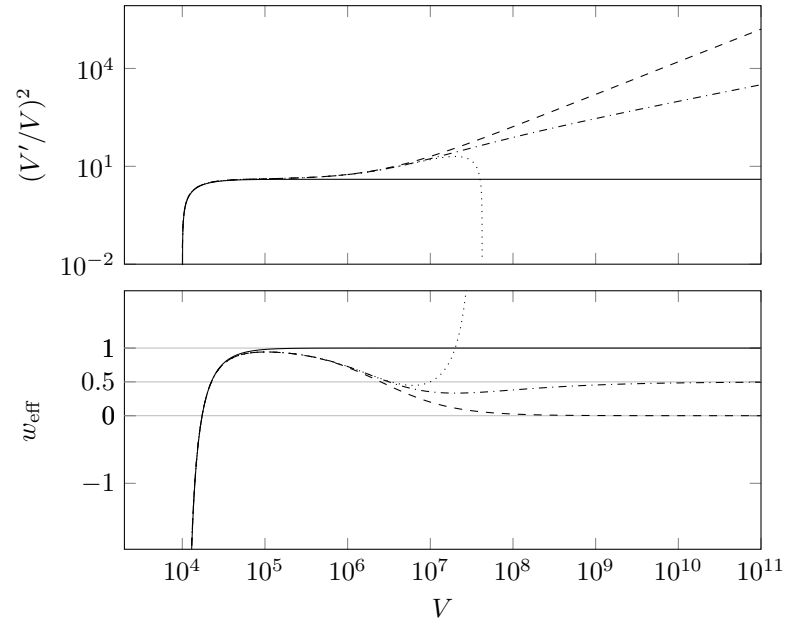


Figure 5.3.: The relative relational expansion rate squared (5.78) and the “effective equation of state parameter” (5.80) as functions of the volume  $V$ . The solid lines correspond to a truncation at zeroth order in  $\lambda$ . The dashed lines correspond to a truncation at first order in  $\lambda$ . The dotted lines correspond to a truncation at second order in  $\lambda$ . The dash-dotted lines correspond to the full non-perturbative case. The parameters are:  $v = 1$ ,  $\omega = 1$ ,  $\lambda = -10^{-7}$ ,  $H = -10040$ ,  $C = -3/16$ . (The choice of  $H$  corresponds to a Fock coherent state,  $\langle \sigma | \hat{H} | \sigma \rangle$ , for  $\sigma = 100$ .) This figure is taken from [76].

The dynamics generated by the Hamiltonian (5.69) are quite complicated and we employ methods of quantum perturbation theory to obtain analytical results. In this section we consider the number operator  $\hat{N}$  rather than the volume operator  $\hat{V}$  for notational simplicity. Note that the corresponding equations for the volume operator can be obtained by setting  $\hat{N} = \hat{V}/v$  everywhere. We expand the number operator as a series in  $\lambda$ ,

$$\hat{N}(\chi) = \sum_{n=0}^{\infty} \lambda^n \hat{N}_n. \quad (5.81)$$

We split the Hamiltonian (5.69) into a free part  $\hat{H}_0$  and an interaction term  $\hat{H}_1$ ,

$$\hat{H} = \hat{H}_0 + \lambda \hat{H}_1. \quad (5.82)$$

If we write the time evolution operator as a product,

$$\hat{U}(\chi) = \hat{U}_0(\chi) \hat{U}_1(\chi), \quad (5.83)$$

where  $\hat{U}_0(\chi^0) = \exp(-i\hat{H}_0\chi)$  we find that the interaction time evolution operator is given by

$$\hat{U}_1(\chi) = \text{T exp} \left( -i\lambda \int_0^\chi d\chi' \hat{U}_0^{-1}(\chi') \hat{H}_1 \hat{U}_0(\chi') \right), \quad (5.84)$$

where the time-ordered exponential is defined as

$$\text{T exp} \left( \int_0^\chi d\chi' \hat{f}(\chi') \right) = \sum_{n=0}^{\infty} \int_0^\chi d\chi_1 \cdots \int_0^{\chi_{n-1}} d\chi_n \hat{f}(\chi_1) \cdots \hat{f}(\chi_n). \quad (5.85)$$

The inverse operator  $\hat{U}^{-1}(\chi)$  requires an anti-time-ordered exponential. Note also that the expression  $\hat{U}_0^{-1}(\chi') \hat{H}_1 \hat{U}_0(\chi')$  amounts to replacing the operator  $\hat{K}_0$  in  $\hat{H}_1$  with the Heisenberg operator found in the free theory, (5.42). The operators  $\hat{K}_+$  and  $\hat{K}_-$  appear only as a sum,  $\hat{K}_+ + \hat{K}_-$ , which is proportional to the free Hamiltonian  $\hat{H}_0$  and therefore constant. We expand the interaction time evolution operator  $\hat{U}_1$  as a power series in  $\lambda$ ,

$$\hat{U}_1(\chi) = \sum_{n=0}^{\infty} \lambda^n \hat{U}_{1,n}(\chi). \quad (5.86)$$

In terms of the time evolution operator the general solution to the Heisenberg equation for any operator  $\hat{O}$  is given by

$$\hat{O}(\chi) = \hat{U}^{-1}(\chi) \hat{O} \hat{U}(\chi). \quad (5.87)$$

Inserting the perturbative expansions for  $\hat{O} = \sum_{n=0}^{\infty} \lambda^n \hat{O}_n$  and the time evolution operator and comparing order by order in  $\lambda$  we get

$$\hat{O}_n(\chi) = \sum_{m=0}^n \hat{U}_{1,m}^{-1}(\chi) \hat{U}_0^{-1}(\chi) \hat{O} \hat{U}_0(\chi) \hat{U}_{1,n-m}(\chi). \quad (5.88)$$

The Heisenberg equation can be written as the system of differential equations,

$$\hat{O}'_0(\chi) = -i[\hat{O}_0(\chi), \hat{H}_0], \quad (5.89)$$

$$\hat{O}'_n(\chi) = -i[\hat{O}_n(\chi), \hat{H}_0] - i[\hat{O}_{n-1}(\chi), \hat{H}_1], \quad n \geq 1. \quad (5.90)$$

With these equations in hand it is possible to carry out the calculation to any order in  $\lambda$  in principle. We will only do this to first order.

The first term in the expansion (5.81) is of course the same as in the free theory (cf. (5.42)),

$$\hat{N}_0(\chi) = -\frac{1}{2}\hat{I} + \left(\hat{N}_0 + \frac{1}{2}\hat{I}\right) \cosh(2\omega\chi) + i\left(\hat{K}_+ - \hat{K}_-\right) \sinh(2\omega\chi), \quad (5.91)$$

where we used the relation (5.37). Note that all the perturbative corrections of  $\hat{N}$  and  $\hat{K}_0$  are related by only a rescaling (as opposed to also having an additional shift),

$$\hat{N}_n(\chi) = 2(\hat{K}_0)_n(\chi), \quad n \geq 1. \quad (5.92)$$

The first order perturbative correction to the number operator is given by<sup>2</sup>

$$\begin{aligned} \hat{N}_1(\chi) = & (\hat{K}_+)^2 [3 - 2(2 + 3i\omega\chi)\cosh(2\omega\chi) + \cosh(4\omega\chi) + i\sinh(2\omega\chi)] + \text{h. c.} \\ & + \hat{K}_+(2\hat{N} + 3) [(i - 3\omega\chi)\sinh(2\omega\chi) - i\sinh(4\omega\chi)] + \text{h. c.} \\ & - \frac{1}{2}\sinh(2\omega\chi)^2 [3 + 4\hat{N}^2 + 8\hat{N} + 8\hat{K}_+\hat{K}_-]. \end{aligned} \quad (5.93)$$

The next step in deriving an effective Friedmann equation is to take the expectation value of  $\hat{N}(\chi)$  and express  $\langle \hat{N}'(\chi) \rangle$  as a function of  $\langle \hat{N}(\chi) \rangle$ . We were unable to carry this out in general. However, for certain states we managed to carry out the procedure. Firstly, for Fock coherent states we restrict ourselves to the case in which the coherent state parameter,  $\sigma = \sigma_1 + i\sigma_2$ , is either purely real or imaginary. For the case of a real coherent

<sup>2</sup>The analogous term in [76] has a wrong prefactor and is missing the term “ $i\hat{K}_+^2 \sinh(2\omega\chi) + \text{h. c.}$ ”.



state parameter, we find

$$\begin{aligned}
 \langle \hat{N}'(\chi) \rangle_{\text{F}}^2 &= 4\omega^2 \left( \langle \hat{N}(\chi) \rangle_{\text{F}}^2 + \langle \hat{N}(\chi) \rangle_{\text{F}} - \sigma_1^2(1 + \sigma_1^2) \right. \\
 &\quad + \frac{\lambda}{(1 + 2\sigma_1^2)^2} \left[ -4\langle \hat{N}(\chi) \rangle_{\text{F}}^3(3 + 12\sigma_1^2 + 4\sigma_1^4) \right. \\
 &\quad \quad - 6\langle \hat{N}(\chi) \rangle_{\text{F}}^2(3 + 15\sigma_1^2 + 12\sigma_1^4 + 4\sigma_1^6) \\
 &\quad \quad + 6\langle \hat{N}(\chi) \rangle_{\text{F}}(-1 - 5\sigma_1^2 + 4\sigma_1^6) \\
 &\quad \quad \left. \left. + 2\sigma_1^2(3 + 24\sigma_1^2 + 51\sigma_1^4 + 48\sigma_1^6 + 20\sigma_1^8) \right] + O(\lambda^2) \right). \tag{5.94}
 \end{aligned}$$

For imaginary coherent state parameter we find the following,

$$\begin{aligned}
 \langle \hat{N}'(\chi) \rangle_{\text{F}}^2 &= 4\omega^2 \left( \langle \hat{N}(\chi) \rangle_{\text{F}}^2 + \langle \hat{N}(\chi) \rangle_{\text{F}} - \sigma_2^2(1 + \sigma_2^2) \right. \\
 &\quad + \frac{\lambda}{(1 + 2\sigma_2^2)^2} \left[ -4\langle \hat{N}(\chi) \rangle_{\text{F}}^3(3 + 12\sigma_2^2 + 4\sigma_2^4) \right. \\
 &\quad \quad - 6\langle \hat{N}(\chi) \rangle_{\text{F}}^2(3 + 9\sigma_2^2 - 4\sigma_2^4 - 4\sigma_2^6) \\
 &\quad \quad + 6\langle \hat{N}(\chi) \rangle_{\text{F}}(-1 + \sigma_2^2 + 16\sigma_2^4 + 12\sigma_2^6) \\
 &\quad \quad \left. \left. + 2\sigma_2^2(3 + 6\sigma_2^2 - 15\sigma_2^4 - 24\sigma_2^6 - 4\sigma_2^8) \right] + O(\lambda^2) \right). \tag{5.95}
 \end{aligned}$$

Secondly, for PG states we find the following Friedmann equation

$$\begin{aligned}
 \langle \hat{N}'(\chi) \rangle_{\text{PG}}^2 &= 4\omega^2 \left\{ \left( \langle \hat{N}(\chi) \rangle_{\text{PG}} + \frac{1}{2} \right)^2 - 4k^2 - \frac{\langle \hat{H} \rangle_{\text{PG}}^2}{\omega^2} \right. \\
 &\quad + \lambda \frac{2k+1}{4k} \left[ -8\langle \hat{N}(\chi) \rangle_{\text{PG}}^3 + 12\langle \hat{N}(\chi) \rangle_{\text{PG}}^2 \left( \frac{\langle \hat{H} \rangle_{\text{PG}}}{\omega} - 1 \right) \right. \\
 &\quad \quad + 2\langle \hat{N}(\chi) \rangle_{\text{PG}} \left( 6\frac{\langle \hat{H} \rangle_{\text{PG}}}{\omega} + 16k^2 - 3 \right) \\
 &\quad \quad \left. \left. - \left( \frac{\langle \hat{H} \rangle_{\text{PG}}}{\omega} - 1 \right) \left( 2\frac{\langle \hat{H} \rangle_{\text{PG}}^2}{\omega^2} + \frac{\langle \hat{H} \rangle_{\text{PG}}}{\omega} + 16k^2 - 1 \right) \right] \right. \\
 &\quad \left. + O(\lambda^2) \right\}. \tag{5.96}
 \end{aligned}$$

It is very interesting that for PG coherent states this Friedmann equation can be written entirely in terms of the expectation values of the number operator and the Hamiltonian.

As already explained, the perturbative formulas have a rather limited range of applicability. More precisely, the perturbative formulas are useful as long as  $|\lambda\langle\hat{N}(\chi)\rangle| \ll 1$ . In order to investigate the non-perturbative regime we analysed the system numerically. Using the representation of the creation and annihilation operators as differential operators,

$$(\hat{a}\psi)(x) = \frac{1}{\sqrt{2}} \left( x + \frac{\partial}{\partial x} \right) \psi(x), \quad (\hat{a}^\dagger\psi)(x) = \frac{1}{\sqrt{2}} \left( x - \frac{\partial}{\partial x} \right) \psi(x), \quad (5.97)$$

we solved the time-dependent Schrödinger equation with initial states given by Fock coherent states. In Fig. 5.4 we show the Friedmann equation and the effective equation of state (5.80) for different truncations. In Fig. 5.5 we show the time dependence of the relative uncertainties for the interacting model. As expected, the relative uncertainties start growing as soon the interaction terms become relevant. This fact confirms that GFT interactions significantly can modify the late time behaviour relevant for cosmology.

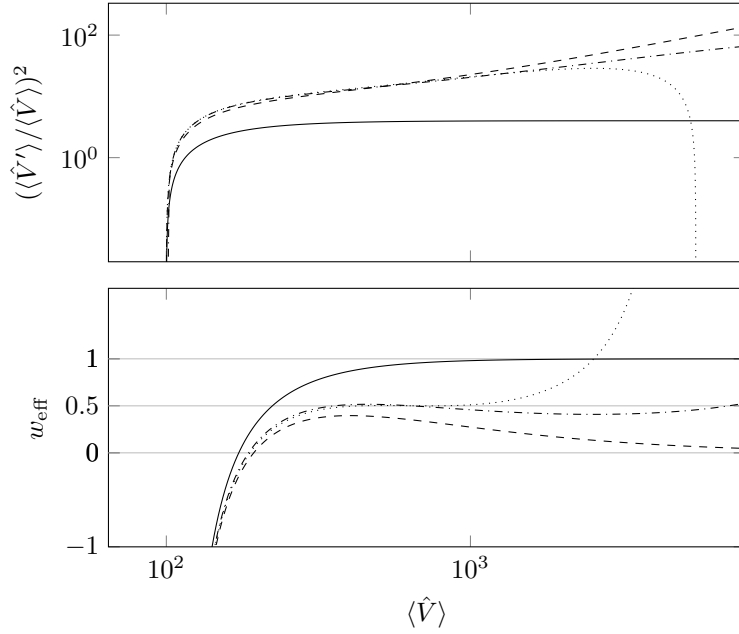


Figure 5.4.: Comparison of the perturbative analytical results with the non-perturbative numerical results. In the top diagram the value of the Friedmann equation is plotted as a function of the volume. The bottom diagram shows the effective equation of state parameter Eq. (5.80). The solid lines correspond to a truncation at zeroth order in  $\lambda$ . The dashed lines correspond to a truncation at first order in  $\lambda$ . The dotted lines correspond to a truncation at second order in  $\lambda$ . The dash-dotted lines correspond to the full non-perturbative case. The parameters are:  $v = 1$ ,  $\omega = 1$ ,  $\lambda = -10^{-3}$ ,  $\sigma = 10$ . This figure is taken from [76].

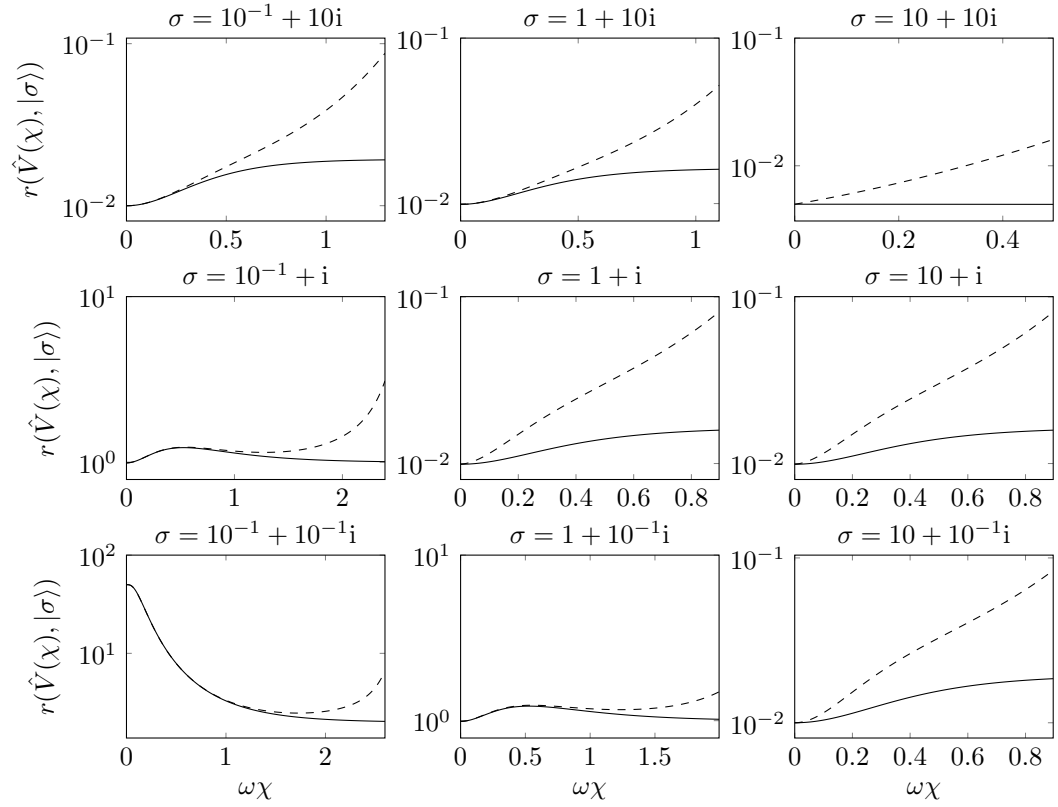


Figure 5.5.: The relative uncertainty (5.59) of the volume operator for different values of the coherent state parameter  $\sigma$  of Fock coherent states. The solid line is the free case ( $\lambda = 0$ ) and the dashed line corresponds to the interacting case with  $\lambda = -10^{-3}$ . This figure is taken from [76].

## Chapter 6.

# One-body effective approach

This chapter is based on [72].

One way of obtaining a quantum theory is to start at a classical theory and then quantise it by some method of quantisation. The well-understood method of canonical quantisation requires as its starting point a Hamiltonian formulation of a classical system. The Hamiltonian formulation itself is usually derived from the Lagrangian formulation defined by an action by carrying out a Legendre transformation. The Legendre transformation involves finding a map which allows one to express the velocities of the configuration variables as a function of the conjugate momenta. In the case that such a map does not exist (for a subset of velocities) one arrives at a Hamiltonian system with additional constraints that the momenta need to satisfy. One example of this is the covariant formulation of electrodynamics in vacuum in four dimensions where one finds as a constraint that the divergence of the electric field must vanish. For systems defined on a fixed spacetime manifold the quantisation of such systems is well understood [48, 87, 102]. However, if one starts with a theory that is background independent, such as general relativity, one is faced with the *problem of time* which has to do with the fact that there is no preferred notion of time in such a theory [90]. Mathematically one finds that the Hamiltonian obtained via a Legendre transformation is given as a sum of constraints and therefore vanishes on the constraint hypersurface. This can be understood by noting that in general relativity there is no preferred time parameter and therefore there can be no invariant Hamiltonian which generates time evolution.

One method of quantising a Hamiltonian system with constraints was first proposed by Dirac. The method of Dirac quantisation is formulated on a Hilbert space where the physical states are those which are annihilated by the constraint operator. Implementing the constraints on the Hilbert space level is however quite complicated in general. It therefore is desirable to sidestep the introduction of a Hilbert space entirely. We now turn to one proposal for how one might accomplish this. To this end we have to first introduce another perspective on the quantum theory: quantum mechanics can also be formulated as a Poisson-geometric theory where the quantum phase space is spanned by the expectation values and moments of the quantum variables<sup>1</sup>. A priori this quantum phase space is infinite dimensional as it is spanned by all the possible moments of the theory. A simplification arises by postulating that the moments capture the quantum behaviour of states and that higher moments correspond to higher orders in  $\hbar$ . Therefore one can arrive at a semiclassical theory by considering only a finite subspace of the quantum phase space. In [28] a way of implementing a quantum constraint on the quantum phase space in terms of effective constraints was put forward for non-relativistic quantum systems. The approach was further extended to relativistic quantum systems in [29] and invoked to study the problem of time in [26, 27].

One of the goals of this chapter is to extend this formalism to (group) field theory. In field theory the set of operators is a priori uncountably infinite. However, the formalism of effective constraints was developed for quantum systems with a finite number of operators. Our proposal is that we should pass to a description in terms of a finite set of observables by averaging over the microscopic degrees of freedom. This should be thought as a coarse graining procedure where we neglect the microscopic details and are only interested in the collective behaviour. In particular it is possible that different microscopic states correspond to the same macroscopic observable. Concretely we consider a set of one-body operators as the quantum degrees of freedom relevant for the physical system we try to model.

Having identified a finite set of observables, we apply this procedure to study the cosmological sector of GFT. As in other chapters of this thesis we consider a GFT coupled to a single scalar field. However, the geometric observable of key interest will turn out to be the extrinsic curvature (rather than the volume). The resulting dynamics we find is compatible with that of classical general relativity.

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<sup>1</sup>This set-up is also used in the geometrical formulation of quantum mechanics of [11].

## 6.1. Effective methods for quantum systems

In this section we provide an overview of the formalism and methods needed for the formalism of effective constraints.

As the quantum system of interest we consider the unital  $C^*$ -algebra generated by the variables  $\mathcal{A} = \{\hat{A}_i\}_{i=1}^N$  that satisfy the commutation relations

$$[\hat{A}_i, \hat{A}_j] = i\hbar f_{ij}^k \hat{A}_k, \quad (6.1)$$

where we assume that the structure constants  $f_{ij}^k$  are real. Note that this condition implies that the variable  $\hat{A}_i$  are invariant under the  $*$ -operation which we denote as  $\hat{A}_i^\dagger = \hat{A}_i$ . For definiteness we consider the case in which this quantum system is represented by operators acting on a Hilbert space. In particular, given a state  $|\psi\rangle$  of the Hilbert space we can define the expectation value  $\langle f(\hat{A}_i) \rangle = \langle \psi | f(\hat{A}_i) | \psi \rangle$  for any function  $f$  of the generators  $\hat{A}_i$ . Of particular interest are the expectation values of the operators themselves,  $\langle \hat{A}_i \rangle$ , which are of course related to the actually measured values of observables of a quantum system. To parametrise the expectation value of other polynomials of variables we introduce the notion of moments. A moment of order  $n$  is given by

$$\Delta(\hat{A}_{i_1}, \dots, \hat{A}_{i_n}) = \langle (\hat{A}_{i_1} - \langle \hat{A}_{i_1} \rangle) \cdots (\hat{A}_{i_n} - \langle \hat{A}_{i_n} \rangle) \rangle_{\text{Weyl}}, \quad (6.2)$$

where ‘‘Weyl’’ denotes totally symmetric ordering. For instance, a second order moment is

$$\Delta(\hat{A}_i, \hat{A}_j) = \frac{1}{2} \langle \hat{A}_i \hat{A}_j + \hat{A}_j \hat{A}_i \rangle - \langle \hat{A}_i \rangle \langle \hat{A}_j \rangle. \quad (6.3)$$

The moments defined in (6.2) are  $\mathbb{C}$ -multi-linear and totally symmetric. That is, for any complex number  $\alpha$  and any permutation  $\pi$  of the symmetric group  $S_n$  we have for moments of order  $n$

$$\Delta(\hat{A}_{i_1} + \alpha \hat{A}_{i'_1}, \dots, \hat{A}_{i_n}) = \Delta(\hat{A}_{i_1}, \dots, \hat{A}_{i_n}) + \alpha \Delta(\hat{A}_{i'_1}, \dots, \hat{A}_{i_n}), \quad (6.4a)$$

$$\Delta(\hat{A}_{i_1}, \dots, \hat{A}_{i_n}) = \Delta(\hat{A}_{\pi(i_1)}, \dots, \hat{A}_{\pi(i_n)}). \quad (6.4b)$$

The next idea is that the space spanned by the expectation values of the generators and moments can be endowed with a Poisson structure induced by the commutation relations

(6.1) resulting in a *quantum phase space*. The Poisson bracket on this quantum phase space is defined for any expectation values as

$$\{\langle f(\hat{A}_i) \rangle, \langle g(\hat{A}_i) \rangle\} = \frac{1}{i\hbar} \langle [f(\hat{A}_i), g(\hat{A}_i)] \rangle. \quad (6.5)$$

Note that linearity, antisymmetry and the Jacobi identity are automatically induced by the commutator. Extension to arbitrary functions on quantum phase space are achieved by imposition of the Leibniz rule,

$$\{f, gh\} = \{f, g\}h + g\{f, h\} \quad (6.6)$$

for any phase space functions  $f$ ,  $g$  and  $h$ . The Poisson bracket of an expectation value of one of the generators with a second order moment is given by

$$\begin{aligned} \{\langle \hat{A}_i \rangle, \Delta(\hat{A}_j, \hat{A}_k)\} &= \frac{1}{i\hbar} \left( \Delta([\hat{A}_i, \hat{A}_j], \hat{A}_k) + \Delta(\hat{A}_j, [\hat{A}_i, \hat{A}_k]) \right) \\ &= f_{ij}{}^l \Delta(\hat{A}_l, \hat{A}_k) + f_{ik}{}^l \Delta(\hat{A}_j, \hat{A}_l). \end{aligned} \quad (6.7)$$

The Poisson bracket of two second order moments is given by

$$\begin{aligned} &\{\Delta(\hat{A}_i, \hat{A}_j), \Delta(\hat{A}_k, \hat{A}_l)\} \\ &= \frac{1}{i\hbar} \left( \left( \langle [\hat{A}_i, \hat{A}_k] \rangle \Delta(\hat{A}_j, \hat{A}_l) + \Delta([\hat{A}_i, \hat{A}_k], \hat{A}_j, \hat{A}_l) \right. \right. \\ &\quad \left. \left. + \frac{1}{12} \langle [[[\hat{A}_i, \hat{A}_k], \hat{A}_j], \hat{A}_l] \rangle - \frac{1}{6} \langle [[[\hat{A}_i, \hat{A}_k], \hat{A}_l], \hat{A}_j] \rangle + (i \leftrightarrow j) \right) + (k \leftrightarrow l) \right) \\ &= \left( f_{ik}{}^m \langle \hat{A}_m \rangle \Delta(\hat{A}_j, \hat{A}_l) + f_{ik}{}^m \Delta(\hat{A}_m, \hat{A}_j, \hat{A}_l) \right. \\ &\quad \left. + \frac{(i\hbar)^2}{12} f_{ik}{}^m f_{mj}{}^n f_{nl}{}^o \langle \hat{A}_o \rangle - \frac{(i\hbar)^2}{6} f_{ik}{}^m f_{ml}{}^n f_{nj}{}^o \langle \hat{A}_o \rangle + (i \leftrightarrow j) \right) + (k \leftrightarrow l). \end{aligned} \quad (6.8)$$

From these explicit expressions it is already clear that the Poisson structure of the quantum phase space is quite complicated<sup>2</sup>. Furthermore, the full quantum phase space is infinite dimensional and therefore we expect that calculations can be carried out analytically in only the simplest cases.

However, if we content ourselves with a semiclassical theory which is valid only up to some order in  $\hbar$  it is possible to consider only a finite subset of phase space variables.

<sup>2</sup>We developed a Wolfram Mathematica package that provides functions that calculate the quantum phase space for an arbitrary system. The listing of the code can be found in Appendix C.



Note that this is a restriction on the states we consider permissible. In particular, we assume that states satisfy the semiclassicality condition

$$\Delta(\hat{A}_{i_1}, \dots, \hat{A}_{i_n}) = O(\hbar^{n/2}) \quad (6.9)$$

which is a non-trivial requirement. In the following we simply assume that the states we consider satisfy this property. At linear order in  $\hbar$  we therefore have for the truncated quantum phase space  $N$  expectational values  $\langle \hat{A}_i \rangle$  and  $N(N+1)/2$  second order moments  $\Delta(\hat{A}_i, \hat{A}_j)$ . The Poisson structure of this truncated quantum phase space will in general be degenerate. For instance, in the case  $N = 2$  there are  $2 + 3$  variables and the Poisson structure is therefore not invertible.

### 6.1.1. Effective constraints

The quantum phase space introduced above provides an alternative, equivalent formulation of quantum mechanics. The phase space is infinite-dimensional and the Poisson structure is quite complicated. The main advantage of using this approach is that it allows one to study constrained systems in a systematic way which we will now turn to.

In the Hilbert space setting, given a (quantum) constraint  $\hat{C}$  the method of Dirac quantisation prescribes that physical states are elements of the kernel of the constraints,

$$\hat{C}|\psi\rangle = 0 \quad (6.10)$$

for any physical state  $|\psi\rangle$ . Clearly, if (6.10) holds then also the expectation value of  $\hat{C}$  vanishes for physical states,

$$C = \langle \hat{C} \rangle = 0. \quad (6.11)$$

However, the converse is not true and the condition that the expectation value vanishes is not sufficient. The proposal of [28] to implement the constraint on the quantum phase space is to require that

$$C(\hat{f}) = \langle (\hat{f} - \langle \hat{f} \rangle) \hat{C} \rangle = 0 \quad (6.12)$$

for all polynomials  $\hat{f}$ . Note that there is no symmetrisation in the definition and the effective constraints  $C(\hat{f})$  are therefore complex in general. The constraints  $C$  and  $C(\hat{f})$

form a system of first class constraints as can be checked by direct computation. Indeed, one finds for the Poisson bracket of  $C$  with an effective constraint  $C(\hat{f})$

$$\{C, C(\hat{f})\} = \frac{1}{i\hbar} \langle [\hat{C}, (\hat{f} - \langle \hat{f} \rangle) \hat{C}] \rangle = \frac{1}{i\hbar} C([\hat{C}, \hat{f}]). \quad (6.13)$$

The Poisson bracket of two effective constraints  $C(\hat{f})$  and  $C(\hat{g})$  is given by

$$\begin{aligned} \{C(\hat{f}), C(\hat{g})\} = \frac{1}{i\hbar} \left( C([\hat{f}, \hat{g}] \hat{C}) + C(\hat{f}[\hat{C}, \hat{g}]) + C(\hat{g}[\hat{f}, \hat{C}]) \right. \\ \left. - C([\hat{f}, \hat{g}])C - C([\hat{C}, \hat{g}])\langle \hat{f} \rangle - C([\hat{f}, \hat{C}])\langle \hat{g} \rangle \right). \end{aligned} \quad (6.14)$$

This shows that the effective constraints are all first class constraints, i.e., the Poisson brackets amongst the effective constraints all vanish on the constraint-hypersurface. Note however, that the Poisson structure is in general degenerate and that the flows generated by the effective constraints are not independent.

### 6.1.2. Time in the effective approach

We now turn to the question of how the effective approach can be used to introduce a notion of time. This problem has been studied in detail in [26, 27].

We are going to work with a system with  $N$  quantum variables and we assume that there is one variable  $\hat{t}$  which forms a subalgebra with another variable  $\hat{p}_t$  and that they commute with the other variables. We make no restriction on the specific form of their commutation relation. Furthermore we restrict ourselves to the case of a quantum phase space truncated at second-order moments. That is we consider a quantum phase space with variables

$$\mathcal{V} = \{\langle \hat{a} \rangle\}_{\hat{a} \in \mathcal{A}} \cup \{\Delta(\hat{a}, \hat{b})\}_{\hat{a}, \hat{b} \in \mathcal{A}}. \quad (6.15)$$

There are  $N + 1$  effective constraints  $C$  and  $C(\hat{a})$ . Effective constraints arising from considering higher order polynomials are already captured by the first order monomials in the truncated phase space and therefore need not be considered. The effective constraints generate a flow on phase space,

$$v'(\tau) = \{v, c\}(\tau), \quad (6.16)$$

where  $v$  denotes any phase space variable and  $c$  is any of the effective constraints. We assume that  $N$  of the flows are independent. This is the case for the models studied in [26, 27, 28, 29]. The strategy is then to implement  $N - 1$  gauge fixing conditions and view the residual gauge flow as time evolution.

As we have stated above, we assume that there are two variables  $\hat{t}$  and  $\hat{p}_t$  that form a subalgebra that commutes with its complement. We furthermore assume that it is possible to eliminate  $\langle \hat{p}_t \rangle$  and  $\Delta(\hat{p}_t, \hat{a})$  for all  $\hat{a}$  in  $\mathcal{A}$  from the relations imposed by the effective constraints. The gauge fixing conditions are then defined as

$$G(\hat{a}) = \Delta(\hat{t}, \hat{a}) = 0, \quad \hat{a} \in \mathcal{A} \setminus \{\hat{p}_t\}. \quad (6.17)$$

This can be understood by noting that if one were to deparametrise the system before quantisation, then the parameter serving as a clock would necessarily be a real parameter and could not have fluctuations. One might also interpret this as the requirement that the clock should be as classical as possible. In this formalism neither the expectation values nor the moments need to be real in general. Indeed, it is quite generic that the moment  $\Delta(\hat{t}, \hat{p}_t)$  is purely imaginary. This is of course an artefact from the fact that we defined the effective constraints in such a way that they are complex relations in general.

After eliminating  $N + 1$  variables using the effective constraint and  $N - 1$  variables using the gauge fixing conditions, we are left with an  $N(N - 1)/2$ -dimensional reduced quantum phase space which we denote  $\tilde{\mathcal{V}}$ . Since we imposed  $N - 1$  gauge fixing conditions, there could in principle be two remaining gauge flows. If the variable  $\hat{t}$  is a suitable choice of clock, then only one of the remaining gauge flows will be non-trivial on the gauge-fixed constraint hypersurface. We assume that this is the case and refer to this residual constraint as the Hamiltonian constraint  $C_H$ . The Hamiltonian constraint generates a flow on the reduced phase space,

$$v'(\tau) = \{v, C_H\}(\tau), \quad v \in \tilde{\mathcal{V}}. \quad (6.18)$$

In the flow equation the parameter  $\tau$  is unphysical. We wish to express the phase space flow in a deparametrised manner. This is achieved by inverting the relation  $\langle \hat{t} \rangle(\tau)$  to arrive at a fully relational description,

$$v_{\text{rel}}(t) = (v \circ \langle \hat{t} \rangle^{-1})(t), \quad v \in \tilde{\mathcal{V}} \setminus \{\langle \hat{t} \rangle\}. \quad (6.19)$$

Note that it is of course possible that such a choice of clock can not be carried out globally. This is the case when the function  $\langle \hat{t} \rangle(\tau)$  is not a monotonic function. In [88] a method

of dealing with such systems was put forward which involves switching from one choice of clock to another.

### 6.1.3. Example: Non-relativistic free particle

As a simple example illustrating the above formalism we consider the non-relativistic free particle. The algebra we consider has variables,

$$\mathcal{A} = \{\hat{t}, \hat{p}_t, \hat{q}, \hat{p}_q\}, \quad (6.20)$$

with non-vanishing commutation relations

$$[\hat{t}, \hat{p}_t] = i\hbar, \quad [\hat{q}, \hat{p}_q] = i\hbar. \quad (6.21)$$

We consider here also only the quantum phase space truncated at second order in moments which is spanned by the 14 variables,

$$\begin{aligned} \mathcal{V} = \{ & \langle \hat{t} \rangle, \langle \hat{p}_t \rangle, \langle \hat{q} \rangle, \langle \hat{p}_q \rangle, \Delta(\hat{t}, \hat{t}), \Delta(\hat{t}, \hat{p}_t), \Delta(\hat{t}, \hat{q}), \Delta(\hat{t}, \hat{p}_q), \\ & \Delta(\hat{p}_t, \hat{p}_t), \Delta(\hat{p}_t, \hat{q}), \Delta(\hat{p}_t, \hat{p}_q), \Delta(\hat{q}, \hat{q}), \Delta(\hat{q}, \hat{p}_q), \Delta(\hat{p}_q, \hat{p}_q)\}. \end{aligned} \quad (6.22)$$

The quantum constraint for the non-relativistic free particle is given by

$$\hat{C} = \hat{p}_t + \frac{1}{2}\hat{p}^2. \quad (6.23)$$

The effective constraints defined in (6.11) and (6.12) are given by

$$C = \langle \hat{p}_t \rangle + \frac{1}{2}\Delta(\hat{p}_q, \hat{p}_q) + \frac{1}{2}\langle \hat{p}_q \rangle^2, \quad (6.24a)$$

$$C(\hat{t}) = \Delta(\hat{t}, \hat{p}_t) + \langle \hat{p}_q \rangle \Delta(\hat{t}, \hat{p}_q) + \frac{i\hbar}{2}, \quad (6.24b)$$

$$C(\hat{p}_t) = \Delta(\hat{p}_t, \hat{p}_t) + \langle \hat{p}_q \rangle \Delta(\hat{p}_t, \hat{p}_q), \quad (6.24c)$$

$$C(\hat{q}) = \Delta(\hat{p}_t, \hat{q}) + \langle \hat{p}_q \rangle \Delta(\hat{q}, \hat{p}_q) + \frac{i\hbar}{2}\langle \hat{p}_q \rangle, \quad (6.24d)$$

$$C(\hat{p}_q) = \Delta(\hat{p}_t, \hat{p}_q) + \langle \hat{p}_q \rangle \Delta(\hat{p}_q, \hat{p}_q). \quad (6.24e)$$

Setting these constraints to zero gives five equations which can be solved for the expectation value  $\langle \hat{p}_t \rangle$  and the four moments  $\Delta(\hat{t}, \hat{p}_t)$ ,  $\Delta(\hat{p}_t, \hat{p}_t)$ ,  $\Delta(\hat{p}_t, \hat{q})$  and  $\Delta(\hat{p}_t, \hat{p}_q)$ . Since we wish to interpret  $\hat{t}$  as the time variable, we choose the gauge fixing conditions (cf. (6.17))

$$G(\hat{a}) = \Delta(\hat{t}, \hat{a}) = 0, \quad \hat{a} \in \mathcal{A} \setminus \{\hat{p}_t\}. \quad (6.25)$$

Imposing both the effective constraints and the gauge fixing condition eliminates all the phase space variables involving the variables  $\hat{t}$  and  $\hat{p}_t$  except for the expectation value  $\hat{t}$ . The explicit form of the Poisson brackets of the effective constraints with the gauge fixing conditions is given for any  $\hat{a} \in \mathcal{A} \setminus \{\hat{p}_t\}$  by

$$\{C, G(\hat{a})\} \approx 0, \quad (6.26a)$$

$$\{C(\hat{t}), G(\hat{a})\} \approx 0, \quad (6.26b)$$

$$\{C(\hat{p}_t), G(\hat{t})\} \approx 2i\hbar, \quad (6.26c)$$

$$\{C(\hat{p}_t), G(\hat{q})\} \approx \frac{3i\hbar}{2} \langle \hat{p}_q \rangle + \langle \hat{p}_q \rangle \Delta(\hat{q}, \hat{p}_q), \quad (6.26d)$$

$$\{C(\hat{p}_t), G(\hat{p}_q)\} \approx \langle \hat{p}_q \rangle \Delta(\hat{p}_q, \hat{p}_q), \quad (6.26e)$$

$$\{C(\hat{q}), G(\hat{t})\} \approx 0, \quad (6.26f)$$

$$\{C(\hat{q}), G(\hat{q})\} \approx -\Delta(\hat{q}, \hat{q}), \quad (6.26g)$$

$$\{C(\hat{q}), G(\hat{p}_q)\} \approx -\frac{i\hbar}{2} - \Delta(\hat{q}, \hat{p}_q), \quad (6.26h)$$

$$\{C(\hat{p}_q), G(\hat{t})\} \approx 0, \quad (6.26i)$$

$$\{C(\hat{p}_q), G(\hat{q})\} \approx \frac{i\hbar}{2} - \Delta(\hat{q}, \hat{p}_q), \quad (6.26j)$$

$$\{C(\hat{p}_q), G(\hat{p}_q)\} \approx -\Delta(\hat{p}_q, \hat{p}_q), \quad (6.26k)$$

where ‘ $\approx$ ’ denotes restriction to the gauge-fixed constraint hypersurface. From these relations we see that both  $C$  and  $C(\hat{t})$  remain unfixed by the choice of gauge. However, one can check that the resulting gauge flow of  $C(\hat{t})$  is trivial on the subspace of remaining variables,  $\tilde{\mathcal{V}} = \{\langle \hat{t} \rangle, \langle \hat{q} \rangle, \langle \hat{p}_q \rangle, \Delta(\hat{q}, \hat{q}), \Delta(\hat{q}, \hat{p}_q), \Delta(\hat{p}_q, \hat{p}_q)\}$ . Therefore the Hamiltonian constraints of this system is given by

$$C_H = C. \quad (6.27)$$

The flow generated by the Hamiltonian constraint is defined by the set of differential equations (cf. (6.18))

$$\langle \hat{t} \rangle'(\tau) = 1, \quad (6.28a)$$

$$\langle \hat{q} \rangle'(\tau) = \langle \hat{p}_q \rangle(\tau), \quad (6.28b)$$

$$\langle \hat{p}_q \rangle'(\tau) = 0, \quad (6.28c)$$

$$\Delta(\hat{q}, \hat{q})'(\tau) = 2\Delta(\hat{q}, \hat{p}_q)(\tau), \quad (6.28d)$$

$$\Delta(\hat{q}, \hat{p}_q)'(\tau) = \Delta(\hat{p}_q, \hat{p}_q)(\tau), \quad (6.28e)$$

$$\Delta(\hat{p}_q, \hat{p}_q)'(\tau) = 0. \quad (6.28f)$$

This system of differential equations is solved by

$$\langle \hat{t} \rangle(\tau) = \langle \hat{t} \rangle(0) + \tau, \quad (6.29a)$$

$$\langle \hat{q} \rangle(\tau) = \langle \hat{q} \rangle(0) + \langle \hat{p}_q \rangle(0)\tau, \quad (6.29b)$$

$$\langle \hat{p}_q \rangle(\tau) = \langle \hat{p}_q \rangle(0), \quad (6.29c)$$

$$\Delta(\hat{q}, \hat{q})(\tau) = \Delta(\hat{q}, \hat{q})(0) + 2\Delta(\hat{q}, \hat{p}_q)(0)\tau + \Delta(\hat{p}_q, \hat{p}_q)(0)\tau^2, \quad (6.29d)$$

$$\Delta(\hat{q}, \hat{p}_q)(\tau) = \Delta(\hat{q}, \hat{p}_q)(0) + \Delta(\hat{p}_q, \hat{p}_q)(0)\tau, \quad (6.29e)$$

$$\Delta(\hat{p}_q, \hat{p}_q)(\tau) = \Delta(\hat{p}_q, \hat{p}_q)(0). \quad (6.29f)$$

It is easy to see that the relation (6.28a) can be inverted and one can arrive at a relational system where the other variables can be interpreted as function given in terms of the expectation value  $\langle \hat{t} \rangle$ .

It is interesting to compare the above expressions after deparametrisation with the analogue expressions obtained from solving the Schrödinger equation of the free non-relativistic free particle. Consider, for instance, the wave packet solution in the standard position basis

$$\psi(t, q) = \frac{1}{\pi^{1/4} \sqrt{\sigma} \sqrt{\sigma^{-2} + i\hbar t}} \exp\left(-\frac{q^2}{2(\sigma^{-2} + i\hbar t)}\right), \quad (6.30)$$

where  $\sigma$  is a real parameter. The wave function  $\psi$  is a solution of the Schrödinger equation,

$$\left(-i\hbar \frac{\partial}{\partial t} - \frac{\hbar^2}{2} \frac{\partial^2}{\partial q^2}\right) \psi(t, q) = 0. \quad (6.31)$$

In this context (6.29b) and (6.29c) above follow from Ehrenfest's theorem. The moments at a fixed moment of time are given by

$$\Delta(\hat{q}, \hat{q}) = \frac{1}{2\sigma^2} + \frac{\hbar^2 \sigma^2}{2} t^2, \quad (6.32a)$$

$$\Delta(\hat{q}, \hat{p}_q) = \frac{\hbar^2 \sigma^2}{2} t, \quad (6.32b)$$

$$\Delta(\hat{p}_q, \hat{p}_q) = \frac{\hbar^2 \sigma^2}{2}. \quad (6.32c)$$

It is remarkable that these expressions are compatible with (6.29d)–(6.29f) *without making any semi-classical approximation*.

## 6.2. Finite algebra in GFT

The goal of this chapter is to apply the method of effective constraints described in Section 6.1 to the GFT formalism. As a starting point for this endeavour we consider a GFT action<sup>3</sup>

$$S(\varphi, \bar{\varphi}) = \int d^M g d\chi \bar{\varphi}(g_I, \chi) \left( m^2 + \frac{\hbar^2}{M} \sum_{a=1}^M \Delta_{g_a} + \lambda \hbar^2 \Delta_\chi \right) \varphi(g_I, \chi). \quad (6.33)$$

The equations of motion can be obtained by varying the action with respect to the group field and its complex conjugate. We perform quantisation by promoting the group field  $\varphi$  to an operator  $\hat{\varphi}$  and its complex conjugate  $\bar{\varphi}$  to an operator  $\hat{\varphi}^\dagger$  with commutation relations (cf. (3.8))

$$[\hat{\varphi}(g_I, \chi), \hat{\varphi}^\dagger(g'_I, \chi')] = \mathbb{I}(g_I, g'_I) \delta(\chi - \chi'), \quad (6.34)$$

where  $\mathbb{I}(g_I, g'_I)$  is the identity kernel on the group manifold  $G^M$  (see also (2.25)). In the quantum theory the equations of motion are implemented by requiring that states  $|\psi\rangle$  solve the equation

$$\left( m^2 + \frac{\hbar^2}{M} \sum_{a=1}^M \Delta_{g_a} + \lambda \hbar^2 \Delta_\chi \right) \hat{\varphi}(g_I, \chi) |\psi\rangle = 0. \quad (6.35)$$

We will however adopt a weaker version of equation of motion obtained by multiplying with  $\hat{\varphi}^\dagger$  and integrating over the domain of the group field

$$\int d^M g d\chi \hat{\varphi}^\dagger(g_I, \chi) \left( m^2 + \frac{\hbar^2}{M} \sum_{a=1}^M \Delta_{g_a} + \lambda \hbar^2 \Delta_\chi \right) \hat{\varphi}(g_I, \chi) |\psi\rangle = 0. \quad (6.36)$$

The operator acting on the state is of the class of one-body operators. We interpret (6.36) as an equation

$$\hat{C}|\psi\rangle = 0, \quad (6.37)$$

where we define the constraint  $\hat{C}$  as

$$\hat{C} = \int d^M g d\chi \hat{\varphi}^\dagger(g_I, \chi) \left( m^2 + \frac{\hbar^2}{M} \sum_{a=1}^M \Delta_{g_a} + \lambda \hbar^2 \Delta_\chi \right) \hat{\varphi}(g_I, \chi). \quad (6.38)$$

The simplification of considering only the one-body version of the quantum equation of motion can be understood as follows. The goal of this chapter is to understand how the

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<sup>3</sup>This is the same as (3.25) with  $D = 1$ , explicit forms of  $\mathcal{K}^{(0)}$  and  $\mathcal{K}^{(2)}$  and written in the group representation.

effective approach of quantum mechanics can be used to study the cosmological sector of GFT. In cosmology we only consider a small number of degrees of freedom such as the scale factor of the FLRW universe. We argue that integrating over the microscopic degrees of freedom of the GFT corresponds to only considering the macroscopic behaviour resulting from the GFT. This is similar to a hydrodynamical approximation where one averages over the microscopic degrees of freedom to obtain a macroscopic description of a system. Whether or not this procedure imposes any restrictions on the cosmological models one can study in this framework is an issue left for future research.

Following the above discussion we consider as our quantum algebra the one-body operators of the GFT. However, a priori there is an infinite number of one-body operators and we need to restrict ourselves to a finite subset of those. From the one-body constraint (6.38) we are lead to the definitions

$$\hat{\Pi}_n = (i\hbar)^n \int d^M g d\chi \hat{\varphi}^\dagger(g_I, \chi) \partial_\chi^n \hat{\varphi}(g_I, \chi), \quad (6.39a)$$

$$\hat{\Lambda} = -\hbar^2 \sum_{a=1}^M \int d^M g d\chi \hat{\varphi}^\dagger(g_I, \chi) \Delta_{g_a} \hat{\varphi}(g_I, \chi). \quad (6.39b)$$

With these definitions the constraint (6.38) reads

$$\hat{C} = m^2 \hat{\Pi}_0 - \hat{\Lambda} - \lambda \hat{\Pi}_2. \quad (6.40)$$

Note that our definition of  $\hat{\Pi}_0$  corresponds to the usual definition of the number operator  $\hat{N}$  for many-body systems.

The one-body operators defined so far commute with each other and therefore we need to consider further operators to obtain non-trivial dynamics. For the matter sector we adopt the proposal of [99, 100] and define the scalar field operator as

$$\hat{X} = \int d^M g d\chi \hat{\varphi}^\dagger(g_I, \chi) \chi \hat{\varphi}(g_I, \chi). \quad (6.41)$$

This scalar field operator has non-trivial commutation relations with the operators  $\hat{\Pi}_n$ ,

$$[\hat{X}, \hat{\Pi}_n] = i\hbar n \hat{\Pi}_{n-1}. \quad (6.42)$$

Although this relation is only well-defined for  $n > 0$  we often will write it in this form also for the case  $n = 0$ . This is justified because in that case the non-existent operator  $\hat{\Pi}_{-1}$  is multiplied by zero.



We also wish to introduce an operator which has a non-trivial commutation relation with the operator  $\hat{\Lambda}$ . To achieve this we define an operator  $\hat{K}$  which satisfies the commutation relation

$$[\hat{\Lambda}, \hat{K}] = i\hbar\alpha\hat{K}. \quad (6.43)$$

Note that such an operator cannot be represented on a compact group manifold. To see this note that on a compact group manifold the operator  $\hat{\Lambda}$  is diagonal and has matrix elements  $\Lambda_{ij} = \Lambda_i\delta_{ij}$ , the matrix elements of  $\hat{K}$  would then have to satisfy

$$(\Lambda_i - \Lambda_j)K_{ij} = i\hbar\alpha K_{ij} \quad (6.44)$$

which implies that  $K_{ij} = 0$ . Since we wish to consider the commutation relation (6.43) we should think of the group manifold as being non-compact.

Furthermore it would be desirable to define the operator  $\hat{K}$  in the same way as we defined the other one-body operators, i.e., by explicitly stating the integral kernel in the definition of the one-body operator,

$$\hat{K} = \int d^M g d\chi d^M g' d\chi' \hat{\varphi}^\dagger(g_I, \chi) K(g_I, g'_I, \chi, \chi') \hat{\varphi}(g_I, \chi). \quad (6.45)$$

Even more desirable would be to have a geometric interpretation of such an operator provided by comparing it to an analogue operator acting on spin network states in LQG. Although it would be good to further investigate these questions, we will content ourselves with the ad hoc definition of (6.43).

The choice of (6.43) is not entirely arbitrary, however. Indeed with this choice of operator we are able to obtain a correspondence with a classical FLRW universe. To see this we employ the effective approach at the most semi-classical level, i.e., we consider the quantum phase space being spanned only by the expectation values of the operators. The effective constraint at this level is given by

$$C = \langle \hat{C} \rangle = m^2 \langle \hat{\Pi}_0 \rangle - \langle \hat{\Lambda} \rangle - \lambda \langle \hat{\Pi}_2 \rangle. \quad (6.46)$$

The only non-trivial flows generated by this constraint are given by (cf. (6.16))

$$\langle \hat{X} \rangle'(t) = -2\lambda \langle \hat{\Pi}_1 \rangle \quad (6.47a)$$

$$\langle \hat{K} \rangle'(t) = \alpha \langle \hat{K} \rangle. \quad (6.47b)$$

The other equations are all trivial,

$$\langle \hat{\Pi}_n \rangle'(t) = \langle \hat{\Lambda} \rangle'(t) = 0. \quad (6.48)$$

The non-constant solutions to the flow equations are then given by

$$\langle \hat{X} \rangle(t) = \langle \hat{X} \rangle(0) - 2\lambda \langle \hat{\Pi}_1 \rangle(0)t, \quad (6.49a)$$

$$\langle \hat{K} \rangle(t) = \langle \hat{K} \rangle(0)e^{\alpha t}. \quad (6.49b)$$

Note that both flows are invertible for all values of the flow parameter and could therefore be employed as a relational clock. For definiteness, we choose  $\langle \hat{X} \rangle$  as the clock variable. The relational evolution of  $\langle \hat{K} \rangle$  is then given by

$$(\langle \hat{K} \rangle \circ \langle \hat{X} \rangle^{-1})(\langle \hat{X} \rangle) = \langle \hat{K} \rangle(0) \exp\left(-\frac{\alpha}{2\lambda \langle \hat{\Pi}_1 \rangle(0)}(\langle \hat{X} \rangle - \langle \hat{X} \rangle(0))\right). \quad (6.50)$$

We wish to give a physical interpretation of the operator  $\hat{K}$ . To compare this to the classical FLRW universe with a massless scalar field, we state the relational dynamics obtained in Appendix A where we expressed the volume and its conjugate momentum as a function of the value of the massless scalar field (cf. (A.43)),

$$(V \circ \chi^{-1})(\chi) = V(0) \exp\left(\mp \sqrt{\frac{3}{2}} \kappa (\chi - \chi(0))\right), \quad (6.51a)$$

$$(p_V \circ \chi^{-1})(\chi) = p_V(0) \exp\left(\pm \sqrt{\frac{3}{2}} \kappa (\chi - \chi(0))\right). \quad (6.51b)$$

From this we see that depending on the initial conditions an interpretation of  $\langle \hat{K} \rangle$  as either the volume or the conjugate momentum would be possible since the exponents in (6.51) can have either sign and therefore both have the same functional form as (6.50). Since we have a notion of volume operator in GFT and its explicit form is known (cf. (3.48)), we conclude that the operator  $\hat{K}$  should be interpreted as the conjugate momentum of the volume which itself is related to the extrinsic curvature.

The above discussion shows that we can expect the operator  $\hat{K}$  to be useful for studying the cosmological sector of GFT. For completeness we consider the dynamics of the more general commutation relation

$$[\hat{\Lambda}, \hat{K}] = i\hbar \sum_{a \in \mathcal{V}} \alpha_a \hat{a}, \quad (6.52)$$

where we introduced the set of labels

$$\mathcal{V} = \{\Pi_0, \Pi_1, \Pi_2, X, K, \Lambda\}. \quad (6.53)$$

The differential equation specifying the flow of  $\langle \hat{K} \rangle$  is then given by

$$\langle \hat{K} \rangle'(t) = \sum_{a \in \mathcal{V}} \alpha_a \langle \hat{a} \rangle(t). \quad (6.54)$$

For  $\alpha_K \neq 0$  the solution is given by

$$\langle \hat{K} \rangle(t) = \left( \frac{x}{\alpha_K} - \frac{\alpha_X}{\alpha_K^2} 2\lambda \langle \hat{\Pi}_1 \rangle(0) \right) (e^{\alpha_K t} - 1) + \langle \hat{K} \rangle(0) e^{\alpha_K t} + \frac{\alpha_X}{\alpha_K} 2\lambda \langle \hat{\Pi}_1 \rangle(0) t \quad (6.55)$$

and in the case  $\alpha_K = 0$  by the expression

$$\langle \hat{K} \rangle(t) = \langle \hat{K} \rangle(0) + xt - \alpha_X \lambda \langle \hat{\Pi}_1 \rangle(0) t^2, \quad (6.56)$$

where in both cases we defined

$$x = \alpha_{\Pi_0} \langle \hat{\Pi}_0 \rangle(0) + \alpha_{\Pi_1} \langle \hat{\Pi}_1 \rangle(0) + \alpha_{\Pi_2} \langle \hat{\Pi}_2 \rangle(0) + \alpha_X \langle \hat{X} \rangle(0) + \alpha_\Lambda \langle \hat{\Lambda} \rangle(0). \quad (6.57)$$

In an FLRW universe with a massless scalar field all the relational variables are either of an exponential form or constant (cf. (A.43)). We wish to give an interpretation of  $\hat{K}$  in terms of those variables. The constant case could only be achieved for  $x = \alpha_X = 0$  which would require fine-tuning. The exponential form requires that  $\alpha_K \neq 0$ . The general case for  $\alpha_K \neq 0$  can be seen as a modification to the classical equations<sup>4</sup>. However, we require that the equation (6.55) agrees with the classical expression at low curvature ( $e^{\alpha_K t} \ll 1$ ) which implies that

$$x = \alpha_X = 0. \quad (6.58)$$

The vanishing of  $x$  can be achieved by fine-tuning the initial conditions or simply requiring that

$$\alpha_{\Pi_0} = \alpha_{\Pi_1} = \alpha_{\Pi_2} = \alpha_\Lambda = 0. \quad (6.59)$$

In the following we assume that this is the case and that the commutator is given by (6.43).

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<sup>4</sup>In LQC one also finds a modification to the classical expression. The extrinsic curvature in LQC is proportional to  $\arctan(\exp(\pm\sqrt{3\kappa/2\chi}))$  [12, 34]

### 6.3. Effective one-body relational cosmology

This section applies the effective formalism introduced in Section 6.1 to the algebra of one-body operators defined in Section 6.2.

Recall that the quantum algebra we are considering is given by

$$\mathcal{A} = \{\hat{X}, \hat{\Pi}_0, \hat{\Pi}_1, \hat{\Pi}_2, \hat{\Lambda}, \hat{K}\} \quad (6.60)$$

and that the non-trivial commutation relations are given by (cf. (6.42) and (6.43))

$$[\hat{X}, \hat{\Pi}_n] = i\hbar n \hat{\Pi}_{n-1}, \quad (6.61a)$$

$$[\hat{\Lambda}, \hat{K}] = i\hbar \alpha \hat{K}. \quad (6.61b)$$

The quantum constraint is defined as (cf. (6.40))

$$\hat{C} = m^2 \hat{\Pi}_0 - \hat{\Lambda} - \lambda \hat{\Pi}_2. \quad (6.62)$$

As in Section 6.1 we are interested in the lowest semi-classical order of the quantum phase space which we assume to correspond to a truncation at second order moments. Concretely, in our case this means that the effective quantum phase space we consider is spanned by 6 expectation values and 21 second order moments and therefore 27-dimensional. Explicitly, the variables of the phase space are given by

$$\begin{aligned} \mathcal{P} = \{ & \langle \hat{\Pi}_0 \rangle, \langle \hat{\Pi}_1 \rangle, \langle \hat{\Pi}_2 \rangle, \langle \hat{X} \rangle, \langle \hat{\Lambda} \rangle, \langle \hat{K} \rangle, \Delta(\hat{\Pi}_0, \hat{\Pi}_0), \Delta(\hat{\Pi}_0, \hat{\Pi}_1), \Delta(\hat{\Pi}_0, \hat{\Pi}_2), \\ & \Delta(\hat{X}, \hat{\Pi}_0), \Delta(\hat{\Pi}_0, \hat{\Lambda}), \Delta(\hat{\Pi}_0, \hat{K}), \Delta(\hat{\Pi}_1, \hat{\Pi}_1), \Delta(\hat{\Pi}_1, \hat{\Pi}_2), \Delta(\hat{X}, \hat{\Pi}_1), \\ & \Delta(\hat{\Pi}_1, \hat{\Lambda}), \Delta(\hat{\Pi}_1, \hat{K}), \Delta(\hat{\Pi}_2, \hat{\Pi}_2), \Delta(\hat{X}, \hat{\Pi}_2), \Delta(\hat{\Pi}_2, \hat{\Lambda}), \Delta(\hat{\Pi}_2, \hat{K}), \\ & \Delta(\hat{X}, \hat{X}), \Delta(\hat{X}, \hat{\Lambda}), \Delta(\hat{X}, \hat{K}), \Delta(\hat{\Lambda}, \hat{\Lambda}), \Delta(\hat{\Lambda}, \hat{K}), \Delta(\hat{K}, \hat{K}) \}. \end{aligned} \quad (6.63)$$

The Poisson structure of the effective phase space follows from the definition (6.5). For  $a \in \mathcal{V}$ , with  $\mathcal{V}$  being the set of labels defined in (6.53), the explicit expressions for the non-trivial Poisson brackets are given by<sup>5</sup>

$$\{\langle \hat{\Lambda} \rangle, \Delta(\hat{K}, \hat{a})\} = (1 + \delta_{aK}) \alpha \Delta(\hat{K}, \hat{a}), \quad (6.64a)$$

$$\{\langle \hat{K} \rangle, \Delta(\hat{\Lambda}, \hat{a})\} = -(1 + \delta_{a\Lambda}) \alpha \Delta(\hat{K}, \hat{a}), \quad (6.64b)$$

<sup>5</sup>In Appendix C.2 we show the Mathematica program used to perform the computations of this section.

### 6.3. Effective one-body relational cosmology

$$\{\langle \hat{X} \rangle, \Delta(\hat{\Pi}_n, \hat{a})\} = (1 + \delta_{a\Pi_n})n\Delta(\hat{\Pi}_{n-1}, \hat{a}), \quad a \neq \Pi_{m \neq n}, \quad (6.64c)$$

$$\{\langle \hat{X} \rangle, \Delta(\hat{\Pi}_n, \hat{\Pi}_m)\} = n\Delta(\hat{\Pi}_{n-1}, \hat{\Pi}_m) + m\Delta(\hat{\Pi}_n, \hat{\Pi}_{m-1}), \quad (6.64d)$$

$$\{\langle \hat{\Pi}_n \rangle, \Delta(\hat{X}, \hat{a})\} = -(1 + \delta_{aX})n\Delta(\hat{\Pi}_{n-1}, \hat{a}), \quad (6.64e)$$

$$\{\Delta(\hat{\Lambda}, \hat{X}), \Delta(\hat{K}, \hat{a})\} = (1 + \delta_{aK})\alpha\langle \hat{K} \rangle\Delta(\hat{X}, \hat{a}), \quad a \neq \Pi_n, \quad (6.64f)$$

$$\{\Delta(\hat{\Lambda}, \hat{X}), \Delta(\hat{\Pi}_n, \hat{a})\} = (1 + \delta_{a\Pi_n})n\langle \hat{\Pi}_{n-1} \rangle\Delta(\hat{\Lambda}, \hat{a}), \quad a \neq K, \Pi_{m \neq n}, \quad (6.64g)$$

$$\{\Delta(\hat{\Lambda}, \hat{X}), \Delta(\hat{K}, \hat{\Pi}_n)\} = n\langle \hat{\Pi}_{n-1} \rangle\Delta(\hat{\Lambda}, \hat{K}) + \alpha\langle \hat{K} \rangle\Delta(\hat{X}, \hat{\Pi}_n), \quad (6.64h)$$

$$\{\Delta(\hat{\Lambda}, \hat{X}), \Delta(\hat{\Pi}_n, \hat{\Pi}_m)\} = n\langle \hat{\Pi}_{n-1} \rangle\Delta(\hat{\Lambda}, \hat{\Pi}_m) + m\langle \hat{\Pi}_{m-1} \rangle\Delta(\hat{\Lambda}, \hat{\Pi}_n), \quad (6.64i)$$

$$\{\Delta(\hat{K}, \hat{X}), \Delta(\hat{\Lambda}, \hat{a})\} = -(1 + \delta_{a\Lambda})\alpha\langle \hat{K} \rangle\Delta(\hat{X}, \hat{a}), \quad a \neq \Pi_n, \quad (6.64j)$$

$$\{\Delta(\hat{K}, \hat{X}), \Delta(\hat{\Pi}_n, \hat{a})\} = (1 + \delta_{a\Pi_n})n\langle \hat{\Pi}_{n-1} \rangle\Delta(\hat{K}, \hat{a}), \quad a \neq \Lambda, \Pi_{m \neq n}, \quad (6.64k)$$

$$\{\Delta(\hat{K}, \hat{X}), \Delta(\hat{\Lambda}, \hat{\Pi}_n)\} = n\langle \hat{\Pi}_{n-1} \rangle\Delta(\hat{K}, \hat{\Lambda}) - \alpha\langle \hat{K} \rangle\Delta(\hat{X}, \hat{\Pi}_n), \quad (6.64l)$$

$$\{\Delta(\hat{K}, \hat{X}), \Delta(\hat{\Pi}_n, \hat{\Pi}_m)\} = n\langle \hat{\Pi}_{n-1} \rangle\Delta(\hat{K}, \hat{\Pi}_m) + m\langle \hat{\Pi}_{m-1} \rangle\Delta(\hat{K}, \hat{\Pi}_n), \quad (6.64m)$$

$$\{\Delta(\hat{\Lambda}, \hat{\Pi}_n), \Delta(\hat{K}, \hat{a})\} = (1 + \delta_{aK})\alpha\langle \hat{K} \rangle\Delta(\hat{\Pi}_n, \hat{a}), \quad a \neq X, \quad (6.64n)$$

$$\{\Delta(\hat{\Lambda}, \hat{\Pi}_n), \Delta(\hat{X}, \hat{a})\} = -(1 + \delta_{aX})n\langle \hat{\Pi}_{n-1} \rangle\Delta(\hat{\Lambda}, \hat{a}), \quad a \neq K, \quad (6.64o)$$

$$\{\Delta(\hat{\Lambda}, \hat{\Pi}_n), \Delta(\hat{K}, \hat{X})\} = -n\langle \hat{\Pi}_{n-1} \rangle\Delta(\hat{\Lambda}, \hat{a}) + \alpha\langle \hat{K} \rangle\Delta(\hat{\Pi}_n, \hat{a}), \quad (6.64p)$$

$$\{\Delta(\hat{K}, \hat{\Pi}_n), \Delta(\hat{\Lambda}, \hat{a})\} = -(1 + \delta_{a\Lambda})\alpha\langle \hat{K} \rangle\Delta(\hat{\Pi}_n, \hat{a}), \quad a \neq X, \quad (6.64q)$$

$$\{\Delta(\hat{K}, \hat{\Pi}_n), \Delta(\hat{X}, \hat{a})\} = -(1 + \delta_{aX})n\langle \hat{\Pi}_{n-1} \rangle\Delta(\hat{K}, \hat{a}), \quad a \neq \Lambda, \quad (6.64r)$$

$$\{\Delta(\hat{K}, \hat{\Pi}_n), \Delta(\hat{\Lambda}, \hat{X})\} = -n\langle \hat{\Pi}_{n-1} \rangle\Delta(\hat{K}, \hat{a}) - \alpha\langle \hat{K} \rangle\Delta(\hat{\Pi}_n, \hat{a}), \quad (6.64s)$$

$$\{\Delta(\hat{K}, \hat{K}), \Delta(\hat{\Lambda}, \hat{a})\} = -(1 + \delta_{a\Lambda})2\alpha\langle \hat{K} \rangle\Delta(\hat{K}, \hat{a}), \quad (6.64t)$$

$$\{\Delta(\hat{\Lambda}, \hat{\Lambda}), \Delta(\hat{K}, \hat{a})\} = (1 + \delta_{aK})2\alpha\langle \hat{K} \rangle\Delta(\hat{\Lambda}, \hat{a}), \quad (6.64u)$$

$$\{\Delta(\hat{\Lambda}, \hat{K}), \Delta(\hat{K}, \hat{a})\} = \alpha\langle \hat{K} \rangle\Delta(\hat{K}, \hat{a}), \quad a \neq \Lambda, \quad (6.64v)$$

$$\{\Delta(\hat{\Lambda}, \hat{K}), \Delta(\hat{\Lambda}, \hat{a})\} = -\alpha\langle \hat{K} \rangle\Delta(\hat{\Lambda}, \hat{a}), \quad a \neq K, \quad (6.64w)$$

$$\{\Delta(\hat{X}, \hat{X}), \Delta(\hat{\Pi}_n, \hat{a})\} = (1 + \delta_{a\Pi_n})2n\langle \hat{\Pi}_{n-1} \rangle\Delta(\hat{X}, \hat{a}), \quad a \neq \Pi_{m \neq n}, \quad (6.64x)$$

$$\{\Delta(\hat{X}, \hat{X}), \Delta(\hat{\Pi}_n, \hat{\Pi}_m)\} = 2\langle n\langle \hat{\Pi}_{n-1} \rangle\Delta(\hat{\Pi}_m, \hat{X}) + m\langle \hat{\Pi}_{m-1} \rangle\Delta(\hat{\Pi}_n, \hat{X}) \rangle, \quad (6.64y)$$

$$\{\Delta(\hat{X}, \hat{\Pi}_n), \Delta(\hat{\Pi}_m, \hat{a})\} = (1 + \delta_{a\Pi_m})m\langle \hat{\Pi}_{m-1} \rangle\Delta(\hat{\Pi}_n, \hat{a}), \quad a \neq X, \Pi_{l \neq m}, \quad (6.64z)$$

$$\{\Delta(\hat{X}, \hat{\Pi}_n), \Delta(\hat{\Pi}_m, \hat{\Pi}_l)\} = m\langle \hat{\Pi}_{m-1} \rangle\Delta(\hat{\Pi}_n, \hat{\Pi}_l) + l\langle \hat{\Pi}_{l-1} \rangle\Delta(\hat{\Pi}_n, \hat{\Pi}_m), \quad (6.64aa)$$

$$\{\Delta(\hat{X}, \hat{\Pi}_n), \Delta(\hat{X}, \hat{a})\} = -(1 + \delta_{aX})n\langle \hat{\Pi}_{n-1} \rangle\Delta(\hat{\Pi}_n, \hat{\Pi}_l), \quad a \neq \Pi_m, \quad (6.64ab)$$

$$\{\Delta(\hat{\Pi}_n, \hat{\Pi}_m), \Delta(\hat{X}, \hat{a})\} = -(1 + \delta_{aX})(n\langle \hat{\Pi}_{n-1} \rangle\Delta(\hat{\Pi}_m, \hat{a}) + m\langle \hat{\Pi}_{m-1} \rangle\Delta(\hat{\Pi}_n, \hat{a})). \quad (6.64ac)$$

From (6.11) and (6.12) we have the following set of effective constraints,

$$C = m^2 \langle \hat{\Pi}_0 \rangle - \langle \hat{\Lambda} \rangle - \lambda \langle \hat{\Pi}_2 \rangle, \quad (6.65a)$$

$$C(\hat{X}) = m^2 \Delta(\hat{X}, \hat{\Pi}_0) - \Delta(\hat{X}, \hat{\Lambda}) - \lambda \Delta(\hat{X}, \hat{\Pi}_2) - i\hbar \lambda \langle \hat{\Pi}_1 \rangle, \quad (6.65b)$$

$$C(\hat{\Pi}_0) = m^2 \Delta(\hat{\Pi}_0, \hat{\Pi}_0) - \Delta(\hat{\Pi}_0, \hat{\Lambda}) - \lambda \Delta(\hat{\Pi}_0, \hat{\Pi}_2), \quad (6.65c)$$

$$C(\hat{\Pi}_1) = m^2 \Delta(\hat{\Pi}_0, \hat{\Pi}_1) - \Delta(\hat{\Pi}_1, \hat{\Lambda}) - \lambda \Delta(\hat{\Pi}_1, \hat{\Pi}_2), \quad (6.65d)$$

$$C(\hat{\Pi}_2) = m^2 \Delta(\hat{\Pi}_0, \hat{\Pi}_2) - \Delta(\hat{\Pi}_2, \hat{\Lambda}) - \lambda \Delta(\hat{\Pi}_2, \hat{\Pi}_2), \quad (6.65e)$$

$$C(\hat{\Lambda}) = m^2 \Delta(\hat{\Pi}_0, \hat{\Lambda}) - \Delta(\hat{\Lambda}, \hat{\Lambda}) - \lambda \Delta(\hat{\Pi}_2, \hat{\Lambda}), \quad (6.65f)$$

$$C(\hat{K}) = m^2 \Delta(\hat{\Pi}_0, \hat{K}) - \Delta(\hat{\Lambda}, \hat{K}) + \frac{i\hbar}{2} \alpha \langle \hat{K} \rangle - \lambda \Delta(\hat{\Pi}_2, \hat{K}). \quad (6.65g)$$

Next we have to choose a gauge. Since we wish to employ the method of Section 6.1.2 we need a subalgebra which commutes with the rest of the algebra. In our case this means that we can not choose  $\hat{X}$  as a clock and we therefore choose the operator  $\hat{K}$  to act as a clock and eliminate its ‘‘conjugate’’  $\hat{\Lambda}$  via the effective constraints (6.65). The gauge fixing conditions therefore are given by

$$G(\hat{a}) = \Delta(\hat{K}, \hat{a}), \hat{a} \in \mathcal{A} \setminus \{\hat{\Lambda}\}. \quad (6.66)$$

As expected two of the constraints have a Poisson bracket which vanishes weakly, i.e., on the gauge-fixed constraint hypersurface. For any  $a \in \mathcal{V}$  the explicit expressions of the Poisson bracket of the gauge fixing condition and the effective constraints are given by

$$\{G(\hat{a}), C\} \approx 0, \quad (6.67a)$$

$$\{G(\hat{a}), C(\hat{\Pi}_n)\} \approx (1 - \delta_{aK}) \alpha \langle \hat{K} \rangle \left( \Delta(\hat{\Pi}_n, \hat{a}) - \delta_{aX} \frac{i\hbar}{2} n \langle \hat{\Pi}_{n-1} \rangle \right) \quad (6.67b)$$

$$\{G(\hat{a}), C(\hat{X})\} \approx (1 - \delta_{aK}) \alpha \langle \hat{K} \rangle \left( \Delta(\hat{X}, \hat{a}) + \delta_{a\Pi_n} \frac{i\hbar}{2} n \langle \hat{\Pi}_{n-1} \rangle \right), \quad (6.67c)$$

$$\{G(\hat{\Pi}_n), C(\hat{\Lambda})\} \approx \alpha \langle \hat{K} \rangle \left( m^2 \Delta(\hat{\Pi}_0, \hat{\Pi}_n) - \lambda \Delta(\hat{\Pi}_2, \hat{\Pi}_n) \right), \quad (6.67d)$$

$$\{G(\hat{X}), C(\hat{\Lambda})\} \approx \alpha \langle \hat{K} \rangle \left( m^2 \Delta(\hat{\Pi}_0, \hat{X}) - \lambda \Delta(\hat{\Pi}_2, \hat{X}) - 3i\hbar \lambda \langle \hat{\Pi}_1 \rangle \right), \quad (6.67e)$$

$$\{G(\hat{K}), C(\hat{\Lambda})\} \approx 2i\hbar \alpha^2 \langle \hat{K} \rangle^2, \quad (6.67f)$$

$$\{G(\hat{a}), C(\hat{K})\} \approx 0. \quad (6.67g)$$

In summary, the effective constraints (6.65) eliminate the expectation value  $\langle \hat{\Lambda} \rangle$  and the moments  $\Delta(\hat{\Lambda}, \hat{a})$  (for any  $\hat{a} \in \mathcal{A}$ ) and the gauge fixing conditions eliminate the moments

$\Delta(\hat{K}, \hat{a})$  (for  $\hat{a} \in \mathcal{A} \setminus \{\hat{\Lambda}\}$ ). Therefore, by imposing both the effective constraints (6.65) and the gauge fixing conditions (6.66) we arrive at the 15-dimensional reduced phase space spanned by the variables,

$$\begin{aligned} \tilde{\mathcal{P}} = \{ & \langle \hat{X} \rangle, \langle \hat{K} \rangle, \langle \hat{\Pi}_0 \rangle, \langle \hat{\Pi}_1 \rangle, \langle \hat{\Pi}_2 \rangle, \Delta(\hat{X}, \hat{X}), \Delta(\hat{X}, \hat{\Pi}_0), \Delta(\hat{X}, \hat{\Pi}_1), \Delta(\hat{X}, \hat{\Pi}_2), \\ & \Delta(\hat{\Pi}_0, \hat{\Pi}_0), \Delta(\hat{\Pi}_0, \hat{\Pi}_1), \Delta(\hat{\Pi}_0, \hat{\Pi}_2), \Delta(\hat{\Pi}_1, \hat{\Pi}_1), \Delta(\hat{\Pi}_1, \hat{\Pi}_2), \Delta(\hat{\Pi}_2, \hat{\Pi}_2) \}. \end{aligned} \quad (6.68)$$

On this reduced phase space the flow of  $C(\hat{K})$  is trivial and therefore the Hamiltonian constraint is given by

$$C_H = C. \quad (6.69)$$

The non-trivial equations of motion generated by the Hamiltonian constraint are

$$\langle \hat{X} \rangle'(t) = -2\lambda \langle \hat{\Pi}_1 \rangle(t), \quad (6.70a)$$

$$\langle \hat{K} \rangle'(t) = \alpha \langle \hat{K} \rangle(t), \quad (6.70b)$$

$$\Delta(\hat{X}, \hat{X})'(t) = -4\lambda \Delta(\hat{X}, \hat{\Pi}_1)(t), \quad (6.70c)$$

$$\Delta(\hat{X}, \hat{\Pi}_0)'(t) = -2\lambda \Delta(\hat{\Pi}_0, \hat{\Pi}_1)(t), \quad (6.70d)$$

$$\Delta(\hat{X}, \hat{\Pi}_1)'(t) = -2\lambda \Delta(\hat{\Pi}_1, \hat{\Pi}_1)(t), \quad (6.70e)$$

$$\Delta(\hat{X}, \hat{\Pi}_2)'(t) = -2\lambda \Delta(\hat{\Pi}_1, \hat{\Pi}_2)(t). \quad (6.70f)$$

From the equations of motion one can already see that the classical part corresponding to the expectation values decouples from the quantum corrections given by the moments. Perhaps this should not be that surprising as the same behaviour can be seen in the example studied in Section 6.1.3 for which we showed that the effective treatment captures the true quantum dynamics well. The non-constant solutions to the equations of motion are given by

$$\langle \hat{X} \rangle(t) = \langle \hat{X} \rangle(0) - 2\lambda \langle \hat{\Pi}_1 \rangle(0)t, \quad (6.71a)$$

$$\langle \hat{K} \rangle(t) = \langle \hat{K} \rangle(0)e^{\alpha t}, \quad (6.71b)$$

$$\Delta(\hat{X}, \hat{X})(t) = \Delta(\hat{X}, \hat{X})(0) - 4\lambda \Delta(\hat{X}, \hat{\Pi}_1)(0)t + 4\lambda^2 \Delta(\hat{\Pi}_1, \hat{\Pi}_1)(0)t^2, \quad (6.71c)$$

$$\Delta(\hat{X}, \hat{\Pi}_0)(t) = \Delta(\hat{X}, \hat{\Pi}_0)(0) - 2\lambda \Delta(\hat{\Pi}_0, \hat{\Pi}_1)(0)t, \quad (6.71d)$$

$$\Delta(\hat{X}, \hat{\Pi}_1)(t) = \Delta(\hat{X}, \hat{\Pi}_1)(0) - 2\lambda \Delta(\hat{\Pi}_1, \hat{\Pi}_1)(0)t, \quad (6.71e)$$

$$\Delta(\hat{X}, \hat{\Pi}_2)(t) = \Delta(\hat{X}, \hat{\Pi}_2)(0) - 2\lambda \Delta(\hat{\Pi}_1, \hat{\Pi}_2)(0)t. \quad (6.71f)$$

Note that although both  $\langle \hat{X} \rangle$  and  $\langle \hat{K} \rangle$  can be inverted and could be used as a clock, our choice of gauge suggests that we should deparametrise the system by interpreting the

variable  $\langle \hat{K} \rangle$  as the clock. Most importantly, the expectation value of the scalar field operator is then given by the relational expression

$$(\langle \hat{X} \rangle \circ \langle \hat{K} \rangle^{-1})(\langle \hat{K} \rangle) = \langle \hat{X} \rangle(0) - \frac{2\lambda \langle \hat{\Pi}_1 \rangle(0)}{\alpha} \log \left( \frac{\langle \hat{K} \rangle}{\langle \hat{K} \rangle(0)} \right). \quad (6.72)$$

This has the same functional form as in the classical FLRW universe with a massless scalar field where the value of the scalar field can be written as a function of the extrinsic curvature (cf. (6.51b))

$$(\chi \circ p_V^{-1})(p_V) = \chi(0) \pm \sqrt{\frac{2}{3\kappa}} \log \left( \frac{p_V}{p_V(0)} \right). \quad (6.73)$$

This shows that we recover the classical expression if we identify  $\langle \hat{X} \rangle$  times a constant with the scalar field  $\chi$  and  $\langle \hat{K} \rangle$  times a constant with the extrinsic curvature  $p_V$ . Note furthermore that the ratio of the coupling constants  $\lambda/\alpha$  is related to Newton's constant. We can eliminate the dependence on the initial condition  $\langle \hat{\Pi}_1 \rangle(0)$  by interpreting the scalar field  $\chi$  to be identified with the quotient  $\langle \hat{X} \rangle / \langle \hat{\Pi}_1 \rangle$ . This correspondence of GFT dynamics with the classical equations needs to be contrasted with other works in GFT cosmology such as [76, 111, 112] where the emergence of the classical theory was shown by studying the expectation value of the volume operator. In our setting the volume operator as defined in previous work would actually remain a constant since it commutes with all of the operators in the constraint (6.40). Note that the constancy of the volume operator can be traced back to the truncation of the action and algebra of observables. We would expect that less restrictive truncations would result in a non-trivial flow of the expectation value of the volume operator.

One notion of semi-classicality for states is that the relative uncertainty should be small. This is discussed, e.g., in Section 5.3 and [100]. In the model discussed here it would be the case that if we require that the relative uncertainties are small at small curvature then they also need to be small at large curvature. This can be seen by noting that in the limits  $|t| \rightarrow \infty$  the expectation value  $|\langle \hat{K} \rangle(t)|$  becomes either large or small depending on the signs of the coupling constants. In the limit of  $|t| \rightarrow \infty$  the relative uncertainty

$$\frac{\Delta(\hat{a}, \hat{a}')(t)}{\langle \hat{a} \rangle(t) \langle \hat{a}' \rangle(t)} \quad (6.74)$$

does however agree for both limits as can be seen by the explicit expressions (6.49). A related discussion on the smallness of fluctuation in the quantum regime in the context of LQC is given in [131].



## Chapter 7.

# Conclusions

In Chapter 1 we discussed the problem of quantum gravity and provided an overview of the early history. Since their inception in the beginning of the twentieth century both quantum theory and general relativity have been confirmed experimentally with ever increasing accuracy. Nevertheless it remains highly desirable to attain a quantum theory of gravity and even more so a theory which encompasses a description of both gravity and matter ultimately culminating in a theory which provides an explanation of all phenomena we observe in nature (i.e., a “theory of everything”). Quantum gravity effects are expected to become relevant at very high energy scales. Our current understanding of the universe as a whole suggests that the universe was once in a very hot and dense state and it seems natural that quantum gravity effects should become important in that regime. There exists a rather simple cosmological model which explains the current observations reasonably well. However, one needs to supply the model with initial conditions. Ideally, the full theory of quantum gravity would provide a process with which the initial conditions can be explained (or at least reduced to a smaller number of parameters). However one needs to be careful not to take such arguments too seriously. Statistical reasoning in a sample size of one can ultimately be completely misguided—it might just be that the universe we live in is the way it is for no underlying reason.

In Chapter 2 we introduced GFT which is a proposal for a theory of quantum gravity. GFT is a quantum field theory where the field is defined on a group manifold which is related to the gauge group of gravity. In certain interacting GFTs the Feynman graphs arising in a perturbative expansion can be seen as simplicial complexes that are dual to

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a discretisation of spacetime where the boundary is given by a discretisation of space. This fact provides a connection to other approaches to quantum gravity and random geometry such as spin foams and tensor models that offer a similar view on the structure of spacetime. One big improvement relative to the theory of spin foams is that GFT gives a prescription as to how different discrete structures should be summed over. In particular, the sum over discretisations also includes a sum over all topologies.

Chapter 3 explained the assumptions that provide the basis for studying the cosmological implications of GFT. The main idea is that just as classical cosmology is described by a small number of degrees of freedom, GFT cosmology should only require a small number of degrees of freedom to capture the cosmological dynamics. The main idea is that one can focus on special classes of condensate states which are characterised by a non-vanishing expectation value of the group field operator. This is similar to Bose–Einstein condensation in the theory of many-body systems. A further assumption which is made is that the domain of the group field is extended to also include additional arguments which are interpreted to be massless scalar fields. Finally one is interested in states which capture the notions of homogeneity and isotropy that are expected to hold at least in the classical limit. As discussed in the main text there are several different perspectives on how one should deal with such a system.

Chapters 4 and 5 discussed the canonical formalism of GFT cosmology. By choosing one of the scalar fields coupled to the GFT as a clock variable, one can perform the Legendre transformation from the Lagrangian theory to a Hamiltonian theory. The Hamiltonian theory can then be quantised by the standard method of canonical quantisation. In the case of multiple scalar fields, the assumption that the group field is square integrable for constant value of the clock scalar field breaks the covariance of the Lagrangian theory. In both cases we were able to arrive at equations which resemble the Friedmann equations of a classical FLRW universe in the presence of a massless scalar field. The agreement for the case of multiple scalar fields was however not generic and required a very special choice of initial conditions. In the case of a single massless scalar field we showed how one can arrive at the Friedmann-like equations without having to solve the equations of motion by using the underlying algebraic structure of the system. The majority of work on GFT cosmology neglects any type of GFT interactions. As a step to ameliorate the situation we studied a toy model in the case of a coupling a single scalar field with a simple interaction term. We studied the interacting system both as a classical and quantum

system. We showed that generically the assumption of negligible interactions breaks down at some point and that the dynamics can be dramatically changed by adding interactions. One particularly interesting possibility is that interactions lead to a cyclic cosmology where the universe is periodically expanding and contracting. The cosmologies studied in the canonical approach were the homogeneous and isotropic case. Accommodating more general geometries in the canonical framework is an interesting idea for further research. Another open problem is studying the effects of GFT interactions provided by a more realistic model than the toy model we studied.

In Chapter 6 we employed an effective approach to quantum theory to study GFT cosmology. In the effective approach one considers the so-called quantum phase space which is spanned by the expectation values and moments of some algebra of quantum variables. In general this point of view is equivalent to other quantum theories such as the Hilbert space formalism as the states in a Hilbert space can be specified entirely by its moments. The idea to arrive at a tractable system is to consider a certain semi-classical approximation. The assumption for the semi-classical approximation is that higher order moments correspond to higher orders in  $\hbar$  which is taken to be a measure of “quantumness”. By truncating the phase space at a certain order in moments, one is able to study a semi-classical theory up to a certain order in  $\hbar$ . Note that such a truncation limits the states which are described accurately by the theory. This truncated quantum phase space allows one to implement a quantum constraint in an effective manner by requiring that certain expectation values vanish. In order to apply this method to study GFT cosmology we identified a class of observables that we expect to capture the cosmological dynamics of the theory. This is similar to a hydrodynamical description where one averages over the microscopic degrees of freedom to arrive at a macroscopic description of a system. In contrast to other work on GFT cosmology the geometric operator we studied corresponds to the extrinsic curvature of an FLRW universe. An interesting property of dynamics we found is that the expectation values (corresponding to the classical sector) decouple from the moments (corresponding to the quantum corrections). However, since this behaviour also occurs in simple models such as the free non-relativistic particle this feature should not be interpreted as a shortcoming of the method. One of the most puzzling results is the time-independence of the expectation value of the volume operator. Further research is needed to better understand the relation of the effective one-body approach to the other approaches which find a non-constant expectation value of the volume operator. Another possible generalisation of the simple model we considered is to add an interaction

term to the GFT model used as the starting point.

One desideratum of a theory of quantum theory is to provide an explanation for the initial conditions needed for the standard cosmological model. In cosmological perturbation theory the deviations from perfect homogeneity and isotropy in the initial conditions are used to describe the formation of the currently observed large scale structures. In [70, 73] the feasibility of studying cosmological perturbations within a GFT setting was demonstrated in a mean-field setting. However, it would be good to extend the results to the other approaches of GFT cosmology. One particularly interesting idea is to identify operators which capture the relevant degrees of freedom within the full theory and study the system with the one-body effective approach. Another perhaps simpler step in this direction was to consider more general states which allow for some notion of anisotropy. Clearly the challenge also here lies in identifying the necessary degrees of freedom within the full theory.

The main result of GFT cosmology is the resolution of the big bang singularity. The big bang singularity is replaced by a bounce, i.e., one obtains as a result that the universe was in a contracting phase before it entered the currently observed expanding phase. In most approaches this is a result which is generic and does not require any fine-tuning of the initial conditions. Furthermore, agreement with the classical theory was also shown across approaches, albeit with the need to specify to specific initial states. These facts and the intersection of GFT with other approaches to quantum gravity and random geometry make a compelling case for further study of GFT in general and GFT cosmology in particular.

## Appendix A.

# Flat, homogeneous and isotropic universe with a massless scalar field

In this appendix we present the theory of a flat, homogeneous and isotropic universe with the matter content given by a massless scalar field, minimally coupled to the metric. Note that the scalar field  $\chi$  necessarily has the same symmetries as the metric tensor, i.e., it is only a function of coordinate time. More precisely, we consider a metric of the form

$$ds^2 = -N(t)^2 dt^2 + a(t)^2 \sum_{i=1}^d (dx^i)^2, \quad (\text{A.1})$$

where  $N$  is the lapse function and  $a$  is the scale factor.

### A.1. Lagrangian formulation

In this section we state the action of this symmetry-reduced setting and derive the equations of motion.

The Ricci curvature scalar of this metric is given by

$$R = \frac{2d}{N(t)^2} \left( \frac{a''(t)}{a(t)} + \frac{d-1}{2} \frac{a'(t)^2}{a(t)^2} - \frac{a'(t)}{a(t)} \frac{N'(t)}{N(t)} \right). \quad (\text{A.2})$$

Appendix A. Flat, homogeneous and isotropic universe with a massless scalar field

In the Einstein–Hilbert action, the curvature scalar is multiplied by the volume form,

$$\sqrt{-\det(g)}R = -d(d-1)\frac{a(t)^d}{N(t)}\frac{a'(t)^2}{a(t)^2} + 2d\frac{d}{dt}\left(\frac{a'(t)a(t)^{d-1}}{N(t)}\right). \quad (\text{A.3})$$

Since the last term is a total derivative<sup>1</sup>, the symmetry-reduced Einstein–Hilbert action is given by

$$S_{\text{grav}}(V, N) = -\frac{d-1}{d}\frac{1}{2\kappa}\int dt\frac{V(t)}{N(t)}\frac{V'(t)^2}{V(t)^2}, \quad (\text{A.4})$$

where  $\kappa$  is the gravitational coupling constant and we have written the action as a function of the volume,

$$V(t) = \int d^d\mathbf{x} a(t)^d. \quad (\text{A.5})$$

As stated above, the massless scalar field  $\chi$  has to have the same symmetries as the metric. Therefore the Lagrangian density of the massless scalar field is given by

$$\mathcal{L}_{\text{sf}}(\chi'(t), N(t)) = \frac{1}{2}\frac{1}{N(t)^2}\chi'(t)^2 \quad (\text{A.6})$$

and the action is given by

$$S_{\text{sf}}(\chi, V, N) = \frac{1}{2}\int dt\frac{V(t)}{N(t)}\chi'(t)^2. \quad (\text{A.7})$$

Combining the gravitational and matter actions we arrive at the total action,  $S = S_{\text{grav}} + S_{\text{sf}}$ ,

$$S(\chi, V, N) = \int dt\frac{V(t)}{N(t)}\left(-\frac{d-1}{d}\frac{1}{2\kappa}\left(\frac{V'(t)}{V(t)}\right)^2 + \frac{1}{2}\chi'(t)^2\right). \quad (\text{A.8})$$

By varying the action by  $\chi$ ,  $V$  and  $N$  respectively, one arrives at the following equations of motion,

$$\frac{d}{dt}\left(\frac{V(t)\chi'(t)}{N(t)}\right) = 0, \quad (\text{A.9a})$$

$$\frac{V''(t)}{V(t)} - \frac{V'(t)}{V(t)}\frac{N'(t)}{N(t)} = 0, \quad (\text{A.9b})$$

$$\left(\frac{V'(t)}{V(t)}\right)^2 = 2\kappa\frac{d}{d-1}\frac{1}{2}\chi'(t)^2. \quad (\text{A.9c})$$

---

<sup>1</sup>More precisely, one should add the Gibbons–Hawking–York boundary term to the action to cancel the boundary term arising from the total derivative [64, 146].

Note that the equations are not independent, for instance (A.9c) and (A.9a) together imply (A.9b). Equation (A.9a) is the Klein–Gordon equation, (A.9b) is the Raychaudhuri equation and (A.9c) is the Friedmann equation. By noting that the energy density of the massless scalar field is given by

$$\rho(t) = \frac{1}{2N(t)^2} \chi'(t)^2 \quad (\text{A.10})$$

one can bring the Friedmann equation (A.9c) into the more familiar form

$$\left( \frac{V'(t)}{N(t)V(t)} \right)^2 = 2\kappa \frac{d}{d-1} \rho(t). \quad (\text{A.11})$$

The action (A.8) has the following reparametrisation invariance,

$$S(\chi, V, N) = S(\chi \circ f, V \circ f, f' \cdot (N \circ f)), \quad (\text{A.12})$$

where  $f : \mathbb{R} \rightarrow \mathbb{R}$  is an arbitrary function<sup>2</sup> and ‘ $\cdot$ ’ denotes pointwise multiplication,  $(f \cdot g)(t) = f(t)g(t)$ . This reparametrisation invariance corresponds to the diffeomorphism invariance of general relativity.

## A.2. Hamiltonian formulation

In this section we review the Hamiltonian formulation of a flat FLRW universe with a massless scalar field.

The Hamiltonian formulation of this system is of particular interest since the Hamiltonian is given by a constraint as we will see momentarily. The conjugate momentum of the volume is given by

$$p_V(t) = \frac{\delta S}{\delta V'(t)}(\chi, V, N) = -\frac{1}{\kappa} \frac{d-1}{d} \frac{V'(t)}{N(t)V(t)}. \quad (\text{A.13})$$

The conjugate moment of the massless scalar field is given by

$$p_\chi(t) = \frac{\delta S}{\delta \chi'(t)}(\chi, V, N) = \frac{V(t)}{N(t)} \chi'(t). \quad (\text{A.14})$$

---

<sup>2</sup>In the case that the action functional is interpreted as a definite integral, the reparametrisation function  $f$  must leave the boundary invariant.

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Since the velocity of the lapse function doesn't feature in the action, one arrives at the primary constraint

$$p_N(t) = \frac{\delta S}{\delta N'(t)}(\chi, V, N) = 0. \quad (\text{A.15})$$

Carrying out the Legendre transformation, one finds that the Hamiltonian is given by

$$H(p_\chi, V, p_V, N) = -\frac{N}{2} \left( \kappa \frac{d}{d-1} V p_V^2 - \frac{1}{V} p_\chi^2 \right). \quad (\text{A.16})$$

The equations of motion in the Hamiltonian system are (with all arguments omitted)

$$\chi' = \{\chi, H\} = \frac{N}{V} p_\chi, \quad (\text{A.17a})$$

$$p_\chi' = \{p_\chi, H\} = 0, \quad (\text{A.17b})$$

$$V' = \{V, H\} = -\kappa \frac{d}{d-1} N V p_V, \quad (\text{A.17c})$$

$$p_V' = \{p_V, H\} = \frac{N}{2} \left( \kappa \frac{d}{d-1} p_V^2 + \frac{1}{V^2} p_\chi^2 \right), \quad (\text{A.17d})$$

$$N' = \{N, H\} = 0, \quad (\text{A.17e})$$

$$p_N' = \{p_N, H\} = \frac{1}{2} \left( \kappa \frac{d}{d-1} V p_V^2 - \frac{1}{V} p_\chi^2 \right) \quad (\text{A.17f})$$

Note that from the primary constraint (A.15),  $p_N(t) = 0$ , (A.17f) gives the secondary constraint

$$\kappa \frac{d}{d-1} V p_V^2 - \frac{1}{V} p_\chi^2 = 0. \quad (\text{A.18})$$

This then can be used to simplify the equation of motion (A.17d), e.g.,

$$p_V' = N \kappa \frac{d}{d-1} p_V^2 \quad (\text{A.19})$$

There are two things to note here. Firstly, (A.18) is the Friedmann equation (A.9c). Secondly, the Hamiltonian (A.16) is proportional to the constraint and therefore vanishes when the constraint is satisfied.

### A.2.1. Specific gauges

The Hamiltonian given in (A.16) can be seen as a constraint being multiplied by a Lagrange multiplier  $N$ . We have seen that different choices of  $N$  correspond to different



parametrisations of the system. We now solve the equations of motion for specific choices of gauge.

Firstly, we consider the proper time gauge, specified by the choice  $N(t) = 1$ . We denote the phase space function in the proper time gauge with a tilde. The Hamiltonian in proper time gauge is given by

$$\tilde{H}(\tilde{V}, \tilde{p}_V, \tilde{\chi}, \tilde{p}_\chi) = -\frac{1}{2} \left( \kappa \frac{d}{d-1} \tilde{V} \tilde{p}_V^2 - \frac{1}{\tilde{V}} \tilde{p}_\chi^2 \right). \quad (\text{A.20})$$

The equations of motion for this system are given by

$$\tilde{V}'(t) = -\kappa \frac{d}{d-1} \tilde{V}(t) \tilde{p}_V(t), \quad (\text{A.21a})$$

$$\tilde{p}_V'(t) = \kappa \frac{d}{d-1} \tilde{p}_V(t)^2, \quad (\text{A.21b})$$

$$\tilde{\chi}'(t) = \frac{\tilde{p}_\chi(t)}{\tilde{V}(t)}, \quad (\text{A.21c})$$

$$\tilde{p}_\chi'(t) = 0, \quad (\text{A.21d})$$

where we made use of the constraint. The system is solved by

$$\tilde{V}(t) = \tilde{V}(0) \left( 1 - \kappa \frac{d}{d-1} \tilde{p}_V(0)t \right), \quad (\text{A.22a})$$

$$\tilde{p}_V(t) = \frac{\tilde{p}_V(0)}{1 - \kappa \frac{d}{d-1} \tilde{p}_V(0)t}, \quad (\text{A.22b})$$

$$\tilde{\chi}(t) = \tilde{\chi}(0) - \frac{1}{\kappa} \frac{d-1}{d} \frac{\tilde{p}_\chi(0)}{\tilde{p}_V(0) \tilde{V}(0)} \log \left( 1 - \kappa \frac{d}{d-1} \tilde{p}_V(0)t \right), \quad (\text{A.22c})$$

$$\tilde{p}_\chi(t) = \tilde{p}_\chi(0). \quad (\text{A.22d})$$

Secondly, we consider the volume gauge, where the lapse function is equal to the volume,  $N(t) = V(t)$ . We notate the variables in the volume gauge with an overbar. The Hamiltonian in the volume gauge is given by

$$\bar{H}(\bar{V}, \bar{p}_V, \bar{\chi}, \bar{p}_\chi) = -\frac{1}{2} \left( \kappa \frac{d}{d-1} \bar{V}^2 \bar{p}_V^2 - \bar{p}_\chi^2 \right). \quad (\text{A.23})$$

The equations of motion in the volume gauge are given by

$$\bar{V}'(t) = -\kappa \frac{d}{d-1} \bar{V}(t)^2 \bar{p}_V(t), \quad (\text{A.24a})$$

$$\bar{p}'_V(t) = \kappa \frac{d}{d-1} \bar{V}(t) \bar{p}_V(t)^2, \quad (\text{A.24b})$$

$$\bar{\chi}'(t) = \bar{p}_\chi(t), \quad (\text{A.24c})$$

$$\bar{p}'_\chi(t) = 0. \quad (\text{A.24d})$$

This system of equations is solved by

$$\bar{V}(t) = \bar{V}(0) e^{-\kappa \frac{d}{d-1} \bar{p}_V(0) \bar{V}(0) t}, \quad (\text{A.25a})$$

$$\bar{p}_V(t) = \bar{p}_V(0) e^{\kappa \frac{d}{d-1} \bar{p}_V(0) \bar{V}(0) t}, \quad (\text{A.25b})$$

$$\bar{\chi}(t) = \bar{\chi}(0) + \bar{p}_\chi(0) t, \quad (\text{A.25c})$$

$$\bar{p}_\chi(t) = \bar{p}_\chi(0). \quad (\text{A.25d})$$

Note the different qualitative behaviour of the two gauges. In the volume gauge all the variables are well defined for all values of the parameter. In the proper time gauge, however, for the value of the parameter at which the volume would be zero, both the conjugate momentum of the volume,  $\tilde{p}_V$ , and the scalar field,  $\tilde{\chi}$  diverge. As has been emphasised before, different choices of gauge correspond to different parametrisations of the system and we now turn to the question of how one can address this ambiguity.

### A.3. Deparametrisation

In this section we discuss the distinction of parametrised and deparametrised formulations, both in general and with respect to the case of a flat FLRW metric with a massless scalar field.

In a physical model the variables used to describe a system are often not independent of each other. It is then possible to view this as a system with constraints. This point can already be appreciated in a simple classical mechanical system of a point particle with an action

$$S(q) = \int dt L(q(t), q'(t)). \quad (\text{A.26})$$

We assume that the relation

$$p_q = L^{(0,1)}(q, q') \quad (\text{A.27})$$

can be solved for the velocity  $q'$  and that therefore the Legendre transform gives the Hamiltonian

$$H(q, p_q) = q' p_q - L(q, q'), \quad (\text{A.28})$$

where  $q'$  is the solution of (A.27). The Hamiltonian equations of motion are then given by (with omitted arguments)

$$q' = \{q, H(q, p_q)\}, \quad (\text{A.29a})$$

$$p'_q = \{p_q, H(q, p_q)\}. \quad (\text{A.29b})$$

In the action (A.26)  $q$  is the position of the point particle and  $t$  is the parameter with respect to which the trajectories are parametrised. In classical mechanics this is of course the Newtonian time. However, from a relativistic point of view it might seem odd that if  $t$  represents time that it should be treated differently from the variable  $q$ . Indeed, one may instead consider the equivalent system

$$\tilde{S}(t, q) = \int d\tau \tilde{L}(t(\tau), t'(\tau), q(\tau), q'(\tau)), \quad (\text{A.30})$$

where the Lagrangian  $\tilde{L}$  is related to  $L$  via

$$\tilde{L}(t(\tau), t'(\tau), q(\tau), q'(\tau)) = t'(\tau) L\left(q(\tau), \frac{q'(\tau)}{t'(\tau)}\right) \quad (\text{A.31})$$

and the relation between the actions is given by

$$\tilde{S}(t, q) = S(q \circ t^{-1}). \quad (\text{A.32})$$

The conjugate momentum of  $t$  is given by

$$p_t = \frac{\partial \tilde{L}}{\partial t'}(t, t', q, q') = L\left(q, \frac{q'}{t'}\right) - \frac{q'}{t'} L^{(0,1)}\left(q, \frac{q'}{t'}\right) \quad (\text{A.33})$$

and the conjugate momentum of  $q$  is given by

$$p_q = \frac{\partial \tilde{L}}{\partial q'} = L^{(0,1)}\left(q, \frac{q'}{t'}\right). \quad (\text{A.34})$$

Appendix A. Flat, homogeneous and isotropic universe with a massless scalar field

The Legendre transformation reveals that the Hamiltonian is zero,

$$\tilde{H}(t, p_t, q, p_q) = t' p_t + q' p_q - \tilde{L}\left(q, \frac{q'}{t'}\right) = 0. \quad (\text{A.35})$$

Note that in (A.35) it is implicitly assumed that one can express the velocities  $t'$  and  $q'$  in terms of the momenta  $p_t$  and  $p_q$ . It turns out that in this case it is not possible to invert the relation as can be seen by noting that

$$\det\left(\frac{\partial^2 \tilde{L}(q, q'/t')}{\partial v^{i'} \partial v^{j'}}\right) = 0, \quad (\text{A.36})$$

where  $v^i = (t, q)$ . Since  $L$  is the Lagrangian of a classical system, one can invert (A.34) to write  $q'/t'$  as a function of  $p_q$ . Therefore (A.33) does not feature  $t'$  and cannot be inverted. Equation (A.33) is therefore a primary constraint.

The total Hamiltonian of the parametrised system is then given by the primary constraint, multiplied by an arbitrary function

$$\begin{aligned} H_{\text{T}}(t(\tau), p_t(\tau), q(\tau), p_q(\tau)) \\ &= N(\tau) \left( p_t(\tau) + \frac{q'(\tau)}{t'(\tau)} L^{(0,1)}\left(q(\tau), \frac{q'(\tau)}{t'(\tau)}\right) - L\left(q(\tau), \frac{q'(\tau)}{t'(\tau)}\right) \right) \\ &= N(\tau) (p_t(\tau) + H(q(\tau), p_q(\tau))), \end{aligned} \quad (\text{A.37})$$

where  $H$  is the Hamiltonian of the original system.

The equations of motion of the parametrised system are given by (with arguments omitted)

$$t' = \{t, H_{\text{T}}(t, p_t, q, p_q)\} = N, \quad (\text{A.38a})$$

$$p'_t = \{p_t, H_{\text{T}}(t, p_t, q, p_q)\} = 0, \quad (\text{A.38b})$$

$$q' = \{q, H_{\text{T}}(t, p_t, q, p_q)\} = N\{q, H(q, p_q)\}, \quad (\text{A.38c})$$

$$p'_q = \{p_q, H_{\text{T}}(t, p_t, q, p_q)\} = N\{p_q, H(q, p_q)\}. \quad (\text{A.38d})$$

Comparing these equations of motion with those of the deparametrised system given in (A.29) we see that for the choice  $N(\tau) = 1$  the equations of motion for  $q$  and  $p_q$  are the same and that the parameter is given by Newtonian time, i.e.  $t(\tau) = \tau$ . Furthermore, the case of general  $N$  can be viewed as a *reparametrisation* of the initial theory. This can be seen by inserting, e.g.,

$$q = \tilde{q} \circ f^{-1}, \quad p_q = \tilde{p}_q \circ f^{-1} \quad (\text{A.39})$$

into the Hamiltonian equations of motion (A.29) which gives the equations

$$\tilde{q}' = f' \{ \tilde{q}, H(\tilde{q}, \tilde{p}_q) \}, \quad (\text{A.40a})$$

$$\tilde{p}'_q = f' \{ \tilde{p}_q, H(\tilde{q}, \tilde{p}_q) \} \quad (\text{A.40b})$$

which for suitable choices of  $N$  and  $f$  are the same as (A.38).

We now return to the model of main concern in this appendix, namely the case of a flat FLRW universe with a massless scalar field. We have stressed above that the time variable is arbitrary in the sense that one may reparametrise the system at will. The conclusion of this is that the value of  $t$  is irrelevant. Indeed, we adopt a *relational* perspective. What counts is the relation of the variables  $V(t)$  and  $\chi(t)$  at any given time. In the case that  $\chi'(t) \neq 0$  for all  $t$ ,  $\chi(t)$  is a monotonic function and one may even go one step further and eliminate the time parameter  $t$  entirely by inverting the function to obtain a function  $\chi^{-1}: \mathbb{R} \rightarrow \mathbb{R}$ . We then define a new function,

$$V_\chi = V \circ \chi^{-1}. \quad (\text{A.41})$$

With this the Friedmann equation (A.9c) can be written in the compact form

$$\left( \frac{V'_\chi(\chi)}{V_\chi(\chi)} \right)^2 = \kappa \frac{d}{d-1}. \quad (\text{A.42})$$

In Appendix A.2.1 we solved the equations of motion in two different parametrisations. By inverting the solutions for  $\chi$ , we arrive at the following relational functions which are independent of any choice of parametrisation (and we chose the integration constants in such a way that the initial conditions agree in both parametrisations)

$$(V \circ \chi^{-1})(\chi) = V(0) \exp \left( \mp \sqrt{\kappa \frac{d}{d-1}} (\chi - \chi(0)) \right), \quad (\text{A.43a})$$

$$(p_V \circ \chi^{-1})(\chi) = p_V(0) \exp \left( \pm \sqrt{\kappa \frac{d}{d-1}} (\chi - \chi(0)) \right), \quad (\text{A.43b})$$

$$(p_\chi \circ \chi^{-1})(\chi) = p_\chi(0), \quad (\text{A.43c})$$

where we made use of the constraint

$$p_\chi(0)^2 = \kappa \frac{d}{d-1} V(0)^2 p_V(0)^2 \quad (\text{A.44})$$

and the choice of sign is determined by the initial conditions.

## A.4. Multiple scalar fields

The above considerations can be readily extended to the case of multiple scalar fields. In this section we collect the main results for that case.

The Lagrangian we are going to consider is given by

$$\mathcal{L}_{\text{sf}}(\chi'_\alpha(t), N(t)) = \frac{1}{2} \frac{1}{N(t)^2} \delta^{\alpha\beta} \chi'_\alpha(t) \chi'_\beta(t), \quad (\text{A.45})$$

where we assume that there are  $n + 1$  scalar fields,

$$\chi_\alpha = (\chi_0, \chi_1, \dots, \chi_n). \quad (\text{A.46})$$

The corresponding action is then given by the functional

$$S_{\text{sf}}(\chi_\alpha, V, N) = \frac{1}{2} \int dt \frac{V(t)}{N(t)} \delta^{\alpha\beta} \chi'_\alpha(t) \chi'_\beta(t). \quad (\text{A.47})$$

The conjugate momenta are given by

$$p_\chi^\alpha = p_{\chi_\alpha} = \frac{\delta S_{\text{sf}}}{\delta \chi'_\alpha(t)} = \frac{V(t)}{N(t)} \delta^{\alpha\beta} \chi'_\beta(t). \quad (\text{A.48})$$

The equation of motion of  $N$  is the Friedmann equation

$$\left( \frac{V'(t)}{V(t)} \right)^2 = 2\kappa \frac{d}{d-1} \frac{1}{2} \delta^{\alpha\beta} \chi'_\alpha(t) \chi'_\beta(t). \quad (\text{A.49})$$

Written in terms of the conjugate momenta of the scalar fields this reads

$$\left( \frac{V'(t)}{N(t)V(t)} \right)^2 = \kappa \frac{d}{d-1} \frac{1}{V(t)^2} \delta_{\alpha\beta} p_\chi^\alpha p_\chi^\beta, \quad (\text{A.50})$$

where we emphasise that the conjugate momenta are time independent.

If we want to adopt the relational point of view, we have to choose one of the scalar fields to serve as the relational clock variable. We will chose the scalar field  $\chi_0$  and define the relational volume as

$$V_{\chi_0} = V \circ (\chi_0)^{-1}. \quad (\text{A.51})$$

The resulting Friedmann equation is then given by

$$\left( \frac{V'_{\chi_0}(\chi_0)}{V_{\chi_0}(\chi_0)} \right)^2 = \kappa \frac{d}{d-1} \left( 1 + \frac{(p_\chi^1)^2 + \dots + (p_\chi^n)^2}{(p_\chi^0)^2} \right). \quad (\text{A.52})$$

## Appendix B.

# Representation theory of $su(1, 1)$

This appendix provides an overview of the representation theory of the Lie algebra  $su(1, 1)$  and coherent states.

The defining Lie bracket relations for  $su(1, 1)$  are given by

$$[\hat{K}_0, \hat{K}_1] = i\hat{K}_2, \quad [\hat{K}_1, \hat{K}_2] = -i\hat{K}_0, \quad [\hat{K}_2, \hat{K}_0] = i\hat{K}_1. \quad (\text{B.1})$$

As noted in [135] this can be written compactly as

$$[\hat{K}_i, \hat{K}_j] = i\varepsilon_{ijk}g^{kl}\hat{K}_l, \quad (\text{B.2})$$

where  $\varepsilon_{ijk}$  is the alternating tensor with  $\varepsilon_{012} = 1$  and  $g^{ij}$  are the components of the metric ( $g^{ij} = \text{diag}(-1, 1, 1)$ ). The Casimir of  $su(1, 1)$  is given by

$$\hat{C} = -g^{ij}\hat{K}_i\hat{K}_j = \hat{K}_0^2 - \hat{K}_1^2 - \hat{K}_2^2, \quad (\text{B.3})$$

where the choice of sign is conventional. An explicit representation of  $su(1, 1)$  can be given in terms of Pauli matrices

$$\hat{K}_0 \doteq \frac{1}{2}\hat{\sigma}_3, \quad \hat{K}_1 \doteq \frac{i}{2}\hat{\sigma}_1, \quad \hat{K}_2 \doteq \frac{i}{2}\hat{\sigma}_2, \quad (\text{B.4})$$

where the binary operator ‘ $\doteq$ ’ should be read as “is represented by” and the Lie bracket is of course given by the commutator. Note that this representation is non-unitary since the generators are not all Hermitian.

As in the case of  $su(2)$  it is possible to define raising and lowering operators

$$\hat{K}_\pm = \hat{K}_1 \pm i\hat{K}_2. \quad (\text{B.5})$$

Appendix B. Representation theory of  $su(1, 1)$

In terms of raising and lowering operator the algebra is given by the following Lie brackets,

$$[\hat{K}_0, \hat{K}_\pm] = \pm \hat{K}_\pm, \quad [\hat{K}_\mp, \hat{K}_\pm] = \pm 2\hat{K}_0. \quad (\text{B.6})$$

The Casimir is then given by

$$\hat{C} = \hat{K}_0^2 - \frac{1}{2} (\hat{K}_+ \hat{K}_- + \hat{K}_- \hat{K}_+). \quad (\text{B.7})$$

We now wish to sketch the different representations possible for  $su(1, 1)$ . In the following we will view the Lie algebra as operators acting on some Hilbert space. We characterise the representations by the eigenvalues of the operators  $\hat{C}$  and  $\hat{K}_0$ ,

$$\hat{C}|c_1, c_2\rangle = c_1|c_1, c_2\rangle, \quad (\text{B.8a})$$

$$\hat{K}_0|c_1, c_2\rangle = c_2|c_1, c_2\rangle. \quad (\text{B.8b})$$

Using the commutation relations one can verify the meaningfulness of calling  $\hat{K}_+$  and  $\hat{K}_-$  raising and lowering operators,

$$\hat{K}_\pm|c_1, c_2\rangle = c_\pm(c_1, c_2)|c_1, c_2 \pm 1\rangle. \quad (\text{B.9})$$

Using the Casimir and the fact that the adjoint of  $\hat{K}_+$  is  $\hat{K}_-$  we find

$$|c_\pm(c_1, c_2)|^2 = c_2(c_2 \pm 1) - c_1. \quad (\text{B.10})$$

Note that when constructing states by acting with the raising and lowering operators only the eigenvalue of  $\hat{K}_0$  changes. Assume that  $c_1$  is fixed and you start at some initial  $c_2$ . Then by acting with the raising and lowering operators you can reach states  $|c_1, c_2 \pm 1\rangle$ . Generically, one can act with the raising and lowering operators an infinite number of times. However, for special initial values it might be that the coefficient  $c_\pm$  vanishes after some iterations. In the case where the action of  $\hat{K}_-$  leads to such a vanishing coefficient we speak of an ascending series, in the case where there is a state annihilated by  $\hat{K}_+$  the family of states is called the descending series.

From now on we will be only considering the discrete ascending series. We will also adopt the standard states where the eigenvalues take a particular form

$$\hat{C}|k, m\rangle = k(k-1)|k, m\rangle, \quad (\text{B.11a})$$

$$\hat{K}_0|k, m\rangle = (k+m)|k, m\rangle, \quad (\text{B.11b})$$



where  $k$  is a positive real and  $m$  is a non-negative integer. Note that with this choice, the state  $m = 0$  is annihilated by  $\hat{K}_-$ ,

$$\hat{K}_-|k, 0\rangle = 0. \quad (\text{B.12})$$

The action of the raising and lowering operators has coefficients

$$\hat{K}_\pm|k, m\rangle = \sqrt{(k+m)(k+m-1) - k(k-1)}|k, m\rangle. \quad (\text{B.13})$$

The states  $|k, m\rangle$  can also be obtained by repeatedly acting with the raising operator on the lowest state  $|k, 0\rangle$ ,

$$|k, m\rangle = \sqrt{\frac{\Gamma(2k)}{\Gamma(2k+m)m!}}\hat{K}_+^m|k, 0\rangle. \quad (\text{B.14})$$

## B.1. Coherent states

Coherent states are classes of states which have desirable properties both from a mathematical and physical perspective. The most well known type of coherent states is those which describe the most semiclassical states of the quantum harmonic oscillator. In the case of the harmonic oscillator the coherent states are eigenstates of the annihilation operator and can also be generated by acting with a displacement operator on the ground state. Both these notions can be extended to  $su(1, 1)$  where, as we will see, those two notions no longer coincide.

A general discussion of coherent states can be found in the textbooks [39, 116].

### B.1.1. Perelomov–Gilmore coherent states

The first class we will discuss are the so-called Perelomov–Gilmore (PG) coherent states [80, 115, 116]. This class of coherent states can be generated by acting with an operator on the lowest state  $|k, 0\rangle$ .

The operator which generates the PG coherent states is given by

$$\hat{S}(\xi) = \exp(\xi\hat{K}_+ - \bar{\xi}\hat{K}_-). \quad (\text{B.15})$$

Appendix B. Representation theory of  $su(1, 1)$

This operator can be written in the ‘‘normal-ordered’’ form [116]

$$\hat{T}(\zeta) = e^{\zeta\hat{K}_+} e^{\eta\hat{K}_0} e^{-\bar{\zeta}\hat{K}_-}, \quad (\text{B.16})$$

where  $\zeta$  is defined as

$$\zeta = \frac{\xi}{|\xi|} \tanh(|\xi|), \quad \eta = \ln(1 - \tanh(|\xi|)^2) = \ln(1 - |\zeta|^2). \quad (\text{B.17})$$

Note that this relation can be verified rather straightforwardly in the representation as Pauli matrices (B.4). To summarise the two operators (B.15) and (B.16) are related by

$$\hat{T}(\zeta) = \hat{S} \left( \frac{\zeta}{|\zeta|} \text{artanh}(|\zeta|) \right). \quad (\text{B.18})$$

The product of two such  $\hat{T}$  operators can be explicitly given as

$$\hat{T}(\zeta_1)\hat{T}(\zeta_2) = \exp\left(\ln\left(\frac{1 + \zeta_1\bar{\zeta}_2}{1 + \bar{\zeta}_1\zeta_2}\right)\hat{K}_0\right)\hat{T}(\zeta_3), \quad \zeta_3 = \frac{\zeta_1 + \zeta_2}{1 + \zeta_1\bar{\zeta}_2} \quad (\text{B.19})$$

or with the opposite ordering

$$\hat{T}(\zeta_1)\hat{T}(\zeta_2) = \hat{T}(\zeta_3)\exp\left(\ln\left(\frac{1 + \zeta_1\bar{\zeta}_2}{1 + \bar{\zeta}_1\zeta_2}\right)\hat{K}_0\right), \quad \zeta_3 = \frac{\zeta_1 + \zeta_2}{1 + \bar{\zeta}_1\zeta_2}. \quad (\text{B.20})$$

We define the PG coherent states by acting with  $\hat{T}(\zeta)$  on the lowest state  $|k, 0\rangle$ ,

$$|\zeta, k\rangle = \hat{T}(\zeta)|k, 0\rangle = (1 - |\zeta|^2)^k \sum_{m=0}^{\infty} \sqrt{\frac{\Gamma(2k+m)}{\Gamma(2k)m!}} \zeta^m |k, m\rangle. \quad (\text{B.21})$$

Expectation values of ‘‘anti-normal-ordered’’ operators can be given in a closed form [95]

$$\begin{aligned} \langle \zeta, k | (\hat{K}_-)^p (\hat{K}_0)^q (\hat{K}_+)^r | \zeta, k \rangle &= \frac{(1 - |\zeta|^2)^{2k}}{\Gamma(2k)} \zeta^{p-r} \\ &\times \sum_{m=0}^{\infty} \frac{|\zeta|^{2m}}{m!} \frac{\Gamma(2k+m+p)\Gamma(m+p+1)}{\Gamma(m+p-r+1)} (k+m+p)^q. \end{aligned} \quad (\text{B.22})$$

The PG coherent states are eigenstates of the operator  $v^i \hat{K}_i$ , where [135]

$$(v_0, v_+, v_-) = \frac{1}{1 - |\zeta|^2} (1 + |\zeta|^2, 2\bar{\zeta}, 2\zeta) \quad (\text{B.23})$$

with eigenvalues  $k$ , i.e.

$$v^i \hat{K}_i |\zeta, k\rangle = g^{ij} v_j \hat{K}_j |\zeta, k\rangle = k |\zeta, k\rangle. \quad (\text{B.24})$$

As an aside let us remark that in [96] it was pointed out that the PG coherent states saturate the usual Robertson–Schrödinger uncertainty relation. A slight generalisation of their result is that for any generators  $\hat{A}$ ,  $\hat{B}$  the following inequality is saturated

$$\Delta(\hat{A}, \hat{A})\Delta(\hat{B}, \hat{B}) \geq \Delta(\hat{A}, \hat{B})^2 + \bar{\Delta}(\hat{A}, \hat{B})^2, \quad (\text{B.25})$$

where the moment  $\Delta(\cdot, \cdot)$  and “anti-moment”  $\bar{\Delta}(\cdot, \cdot)$  are defined as

$$\Delta(\hat{A}, \hat{B}) = \frac{1}{2} \left( \langle \hat{A}^\dagger \hat{B} + \hat{B}^\dagger \hat{A} \rangle - \langle \hat{A}^\dagger \rangle \langle \hat{B} \rangle - \langle \hat{B}^\dagger \rangle \langle \hat{A} \rangle \right), \quad (\text{B.26a})$$

$$\bar{\Delta}(\hat{A}, \hat{B}) = \frac{1}{2i} \left( \langle \hat{A}^\dagger \hat{B} - \hat{B}^\dagger \hat{A} \rangle - \langle \hat{A}^\dagger \rangle \langle \hat{B} \rangle + \langle \hat{B}^\dagger \rangle \langle \hat{A} \rangle \right). \quad (\text{B.26b})$$

Note that for Hermitian operators these definitions take the more familiar form,

$$\Delta(\hat{A}, \hat{B}) = \frac{1}{2} \langle \hat{A} \hat{B} + \hat{B} \hat{A} \rangle - \langle \hat{A} \rangle \langle \hat{B} \rangle, \quad (\text{B.27a})$$

$$\bar{\Delta}(\hat{A}, \hat{B}) = \frac{1}{2i} \langle [\hat{A}, \hat{B}] \rangle. \quad (\text{B.27b})$$

### B.1.2. Barut–Girardello coherent states

The second class of coherent states we discuss are the so-called Barut–Girardello (BG) coherent states [18]. These coherent states are defined as being the eigenstates of the lowering operator,

$$\hat{K}_- |\mu, k\rangle = \mu |\mu, k\rangle. \quad (\text{B.28})$$

This equation can be solved to obtain the states

$$|\mu, k\rangle = N(\mu, k) \sum_{m=0}^{\infty} \frac{\mu^m}{\sqrt{m! \Gamma(2k+m)}} |k, m\rangle, \quad (\text{B.29a})$$

$$N(\mu, k) = \sqrt{\frac{|\mu|^{2k-1}}{I_{2k-1}(2|\mu|)}}, \quad (\text{B.29b})$$

where  $I_\alpha(x)$  is the modified Bessel function of the first kind.

The expectation value of “normal-ordered” products of  $su(1, 1)$  elements is given by

$$\langle \mu, k | (\hat{K}_+)^p (\hat{K}_0)^q (\hat{K}_-)^r | \mu, k \rangle = \frac{|\mu|^{2k-1}}{I_{2k-1}(2|\mu|)} \bar{\mu}^p \mu^r \sum_{m=0}^{\infty} \frac{|\mu|^{2m}}{m! \Gamma(2k+m)} (k+m)^q. \quad (\text{B.30})$$

### B.1.3. Fock coherent states

As explained in the main text on Page 83, the  $su(1, 1)$  algebra can be realised by the bosonic creation and annihilation operators,

$$\hat{K}_0 = \frac{1}{4} (\hat{a}^\dagger \hat{a} + \hat{a} \hat{a}^\dagger), \quad \hat{K}_+ = \frac{1}{2} \hat{a}^2, \quad \hat{K}_- = \frac{1}{2} \hat{a}^{\dagger 2}. \quad (\text{B.31})$$

Note that this requires a Bargmann index of  $k = 1/4$  or  $k = 3/4$  corresponding to the case in which only modes with an even or odd number of quanta are excited, respectively.

Coherent states can then be defined by either relation,

$$|\sigma\rangle = \exp(\sigma \hat{a}^\dagger - \bar{\sigma} \hat{a}) |0\rangle, \quad (\text{B.32a})$$

$$\hat{a} |\sigma\rangle = \sigma |\sigma\rangle. \quad (\text{B.32b})$$

Explicitly the Fock coherent states are given by

$$\begin{aligned} |\sigma\rangle &= e^{-|\sigma|^2/2} \sum_{n=0}^{\infty} \frac{\sigma^n}{\sqrt{n!}} |n\rangle \\ &= e^{-|\sigma|^2/2} \left( \sum_{m=0}^{\infty} \frac{\sigma^{2m}}{\sqrt{(2m)!}} \left| \frac{1}{4}, m \right\rangle + \sum_{n=0}^{\infty} \frac{\sigma^{2m+1}}{\sqrt{(2m+1)!}} \left| \frac{3}{4}, m \right\rangle \right), \end{aligned} \quad (\text{B.33})$$

where  $|n\rangle$  are the eigenstates of the number operator and which shows that the Fock coherent states are a superposition of two distinct ascending series representations of  $su(1, 1)$ .

“Normal-ordered” expectation values for Fock coherent states are given by

$$\langle \sigma | (\hat{K}_+)^p (\hat{K}_0)^q (\hat{K}_-)^r | \sigma \rangle = \frac{1}{2^{p+q+r}} \bar{\sigma}^{2p} \sigma^{2r} \sum_{m=0}^q \binom{q}{m} \frac{1}{2^{q-m}} \langle \sigma | \hat{N}^m | \sigma \rangle, \quad (\text{B.34a})$$

$$\langle \sigma | \hat{N}^m | \sigma \rangle = \sum_{n=0}^m S(m, n) |\sigma|^{2n}, \quad (\text{B.34b})$$

where  $S(m, n)$  are the Stirling numbers of the second kind,

$$S(m, n) = \frac{1}{n!} \sum_{k=0}^n (-1)^k \binom{n}{k} (n-k)^m. \quad (\text{B.35})$$

## Appendix C.

# Mathematica package “EffectiveConstraints”

This appendix provides a listing of the Mathematica package `EffectiveConstraints` developed to carry out the computations of the formalism introduced in Section 6.1.

### C.1. Code listings

#### C.1.1. `init.wl`

```
Get["EffectiveConstraints`Usage`"]  
Get["EffectiveConstraints`QuantumAlgebra`"]  
Get["EffectiveConstraints`EffectiveAlgebra`"]  
Get["EffectiveConstraints`EffectiveConstraints`"]  
Get["EffectiveConstraints`Poisson`"]
```

#### C.1.2. `Usage.wl`

```
BeginPackage["EffectiveConstraints`"]
```

Appendix C. Mathematica package "EffectiveConstraints"

```
expVal::usage =  
"expVal[x] computes the expectation value of the operator x.";
```

```
moment::usage =  
"moment[x1, ..., xn] denotes the nth moment of the xs.";
```

```
toMoments::usage =  
"toMoments[expr, alg] converts all expectation values of more  
than one operator to the corresponding moments.";
```

```
toExpVal::usage =  
"toExpVal[expr] converts all moments present in expr to their  
Weyl ordered expansion in terms of expectation values.";
```

```
truncate::usage =  
"truncate[expr, ord] truncates expr to order ord in the  
moments. Effectively an expansion to order ord/2 in  $\hbar$ .";
```

```
effectivePhaseSpace::usage =  
"effectivePhaseSpace[ops, alg, ord] computes the effective  
phase space of ops up to order ord.
```

Output is of the form {effectiveVariables,  
effectivePoissonStructure} where

- \* effectiveVariables is a list of effective variables
- \* effectivePoissonStructure is a matrix of Poisson brackets
- \* of the effective  
variables";

```
effectiveConstraints::usage =  
"effectiveConstraints[c, ops, alg, ord] gives the effective  
constraints of the quantum constraint c up to order  
 $\hbar^{(ord / 2)}$ .
```

If the algebra is not fully specified, it is assumed that

```

commutators are of the order \[HBar].";

poissonBracket::usage =
"poissonBracket[x, y, vars, ps] computes the Poisson bracket of
x and y.
  * vars: List of variables
      Example: {q, p}
  * ps: Matrix corresponding to the Poisson structure of
  * 'vars'
      Example: {{0, 1}, {-1, 0}}";

diracBracket::usage =
"diracBracket[x, y, vars, ps, cs, delta] computes the Dirac
bracket of x and y, where cs are constraints and delta is the
inverse of the Poisson bracket matrix of the constraints.";

op::usage =
"op[x] represents the operator x.";

ncTimes::usage =
"ncTimes[x, y, ...] represents the noncommutative product of
operators. Only expressions wrapped in 'op' are considered to
be operators.";

ncPower::usage =
"ncPower[x, n] gives the noncommutative product of x n times
with itself.";

commutator::usage =
"commutator[x, y] gives the commutator of x and y.";

completeAlgebra::usage =
"completeAlgebra[ops, alg] gives the completion of a partial
algebra by antisymmetrizing and filling in the missing
commutators with trivial commutators.";

```

```
ordering::usage =  
"ordering[expr] brings the arguments of ncTimes into lexical  
order. ordering[expr, order] brings the arguments of ncTimes  
into the same same order as supplied by the optional argument  
order = {op[1], op[2], op[3], ...}.";
```

```
applyAlgebra::usage =  
"applyAlgebra[expr, alg] replaces commutators with their  
respective entry in alg.";
```

```
EndPackage[]
```

### C.1.3. QuantumAlgebra.wl

```
BeginPackage["EffectiveConstraints`"]
```

```
Get["EffectiveConstraints`Usage`"]
```

```
Begin["`Private`"]
```

```
Clear[op]
```

```
Clear[ncTimes]
```

```
ncTimes[] := 1
```

```
ncTimes[x___, ncTimes[y___], z___] := ncTimes[x, y, z]
```

```
ncTimes[x___, Plus[y1_, y2_], z___] :=  
ncTimes[x, y1, z] + ncTimes[x, Plus[y2], z]
```

```
ncTimes[x_op] := x
```



```

ncTimes[x_commutator] := x

ncTimes[x___, c_expVal, y___] := c ncTimes[x, y]
ncTimes[x___, c_expVal y__, z___] := c ncTimes[x, Times[y], z]

ncTimes[x___, c_, y___] /; FreeQ[c, op] := c ncTimes[x, y]
ncTimes[x___, c_ y__, z___] /; FreeQ[c, op] :=
c ncTimes[x, Times[y], z]

Clear[ncPower]

ncPower[x_, n_Integer] := ncTimes @@ Table[x, n]

Clear[commutator]

commutator[x_, x_] := 0

commutator[x_, _] /; NumericQ[x] := 0
commutator[_ , x_] /; NumericQ[x] := 0

commutator[HoldPattern[ncTimes[x_, y___]], z_] :=
ncTimes[commutator[x, z], y] + ncTimes[x, commutator[ncTimes[y], z]]

commutator[x_, HoldPattern[ncTimes[y_, z___]]] :=
ncTimes[commutator[x, y], z] + ncTimes[y, commutator[x, ncTimes[z]]]

commutator[c_ x_, y_] /; FreeQ[c, op] := c commutator[x, y]
commutator[x_, c_ y_] /; FreeQ[c, op] := c commutator[x, y]

commutator[x_ + y_, z_] := commutator[x, z] + commutator[y, z]
commutator[x_ , y_ + z_] := commutator[x, y] + commutator[x, z]

```

```

Clear[completeAlgebra]

completeAlgebra[ops_List, alg_] /; (
  MatchQ[ops, {__op}]
):=
Module[{
  rule,
  antisymmetrized,
  rest
},
  rule = RuleDelayed[
    commutator[a_, b_] -> c_,
    commutator[b, a] -> -c
  ];
  antisymmetrized = Flatten @ (
    {#, # /. rule}& /@ alg
  );
  rest = (commutator[#1, #2] -> 0)& @@@ Complement[
    DeleteCases[{x_, x_}] @ Tuples[ops, 2],
    Replace[
      antisymmetrized,
      (commutator[a_, b_] -> _) :> {a, b}, 1
    ]
  ];
  Join[antisymmetrized, rest]
]

```

```
Clear[commutatorReduce]
```

```
(* The reason of introducing this helper function is that
   defining such a rule for "commutator" would result in the
   need to wrap all left-hand sides containing a pattern with
   "commutator" in "HoldPattern". *)
```

```

commutatorReduce[expr_] :=
Module[{condition, localCommutator},
  condition[x_] := FreeQ[op][x] && FreeQ[commutator][x];
  localCommutator[x_, _] /; condition[x] = 0;
  localCommutator[_ , x_] /; condition[x] = 0;
  Composition[
    ReplaceAll[localCommutator -> commutator],
    ReplaceAll[commutator -> localCommutator]
  ][expr]
]

Clear[applyAlgebra]

applyAlgebra[expr_, alg_] :=
  FixedPoint[commutatorReduce @ ReplaceRepeated[#, alg] &, expr]
applyAlgebra[alg_] := applyAlgebra[#, alg]&

Clear[orderedPairs]

orderedPairs[l_List] :=
  {1[[#1]], 1[[#2]]} & @@@ Subsets[#, {2}] &@ Range[Length[l]]

Clear[pairToRule]

pairToRule[{a_, b_}] :=
  ncTimes[PatternSequence[x___, b, a, y___]] :>
  ncTimes[x, a, b, y] - ncTimes[x, commutator[a, b], y]

Clear[extractArgs]

extractArgs[expr_] :=

```

Appendix C. Mathematica package "EffectiveConstraints"

```
DeleteDuplicates @ Sort @ Flatten @ Last @ Reap[
  expr /. HoldPattern[x : ncTimes[args__]] :> (Sow[{args}]; x)
]

Clear[orderingAux]

orderingAux[expr_, order_:{}] :=
Module[{args, rules},
  args = If[
    order === {},
    extractArgs[expr],
    order
  ];
  rules = pairToRule /@ (orderedPairs @ args);
  ReplaceRepeated[expr, rules]
]

Clear[ordering]

ordering[expr_] := FixedPoint[orderingAux, expr]
ordering[expr_, order_] := FixedPoint[orderingAux[#, order]&, expr]

End[]

EndPackage[]
```

#### C.1.4. EffectiveAlgebra.wl

```
BeginPackage["EffectiveConstraints`"]

Get["EffectiveConstraints`Usage`"]
```

```

Begin["Private"]

\[HBar] = Global'\[HBar];

Clear[expVal]

SetAttributes[expVal, Listable]

expVal[c_?NumericQ] := c
expVal[c_] /; FreeQ[c, op] := c
expVal[c_ x__] /; FreeQ[c, op] := c expVal[Times[x]]
expVal[Plus[y1_, y2__]] := expVal[y1] + expVal[Plus[y2]]
expVal[Power[c_expVal, n_.] x_.] := c^n expVal[Times[x]]

Clear[moment]

SetAttributes[moment, Orderless]

moment[] := 1

moment[x___, c_, z___] /; (FreeQ[op][c] && FreeQ[commutator][c]) :=
c moment[x, z]
moment[x___, c_ y__, z___] /; (FreeQ[op][c] && FreeQ[commutator][c]) :=
c moment[x, Times[y], z]

moment[x___, y1_ + y2__, z___] :=
moment[x, y1, z] + moment[x, Plus[y2], z]

Clear[weylOrdering]

weylOrdering :=
  ReplaceAll[
    HoldPattern[expVal[ncTimes[xs__]]] :> weylExpVal[ncTimes[xs]]
  ]

```

]

Clear[weylExpVal]

```
weylExpVal[HoldPattern[ncTimes[x_]]] :=
Module[
  {
    permutations
  },
  permutations = Permutations[{x}];
  Times[
    1/Length[permutations],
    expVal[Plus @@ ncTimes @@@ Flatten /@ permutations]
  ]
]
```

Clear[toMoments]

```
toMoments[expr_] := toMoments[expr, {}]
toMoments[expr_, ops_, alg_] := toMoments[expr, alg]
toMoments[expr_, alg_] :=
Module[{repl},
  repl = HoldPattern[x : expVal[ncTimes[xs_]]] :> (
    x
    + moment[xs]
    - momentExpand[xs]
    (* The above adds and subtracts the same expression, but
       only expands the moment in terms of the expectation
       values for the subtracted part. *)
  );
  FixedPoint[
    Composition[
      Simplify,
```

```

    ordering,
    ReplaceAll[repl]
  ],
  ordering[expr]
] // applyAlgebra[#, alg]&
]

Clear[toExpVal]

toExpVal[expr_] :=
  ReplaceAll[HoldPattern[moment[xs_]] :> momentExpand[xs]] @ expr

Clear[momentExpand]

momentExpand[xs_] :=
  weylOrdering @ expVal[ncTimes @@ Table[x - expVal[x], {x, {xs}}]]

Clear[effPoissonBracket]

(* Basic properties of the Poisson bracket *)

effPoissonBracket[x_?NumericQ, y_, ___] := 0
effPoissonBracket[x_, y_?NumericQ, ___] := 0
effPoissonBracket[x_, x_, ___] := 0

effPoissonBracket[Plus[x_, y_], z_, ops_, alg_] :=
  Plus[
    effPoissonBracket[x, z, ops, alg],
    effPoissonBracket[Plus[y], z, ops, alg]
  ]
effPoissonBracket[x_, Plus[y_, z_], ops_, alg_] :=
  Plus[

```

Appendix C. Mathematica package "EffectiveConstraints"

```

    effPoissonBracket[x, y, ops, alg],
    effPoissonBracket[x, Plus[z], ops, alg]
]

effPoissonBracket[Power[x_, n_Integer], y_, ops_, alg_] :=
  n Power[x, n-1] effPoissonBracket[x, y, ops, alg]
effPoissonBracket[x_, Power[y_, n_Integer], ops_, alg_] :=
  n Power[y, n-1] effPoissonBracket[x, y, ops, alg]

effPoissonBracket[Times[x_, y_], z_, ops_, alg_] :=
  Plus[
    x effPoissonBracket[Times[y], z, ops, alg],
    Times[y] effPoissonBracket[x, z, ops, alg]
  ]
effPoissonBracket[x_, Times[y_, z_], ops_, alg_] :=
  Plus[
    y effPoissonBracket[x, Times[z], ops, alg],
    Times[z] effPoissonBracket[x, y, ops, alg]
  ]

HoldPattern[effPoissonBracket[expVal[x_], expVal[y_], ops_, alg_]] :=
  Composition[
    toMoments[#, alg]&,
    applyAlgebra[#, alg]&
  ][
    1 / (I \[HBar]) * expVal[commutator[x, y]]
  ]

effPoissonBracket[x_moment, y_, ops_, alg_] :=
  Composition[
    Simplify,
    toMoments[#, alg]&
  ][
    effPoissonBracket[toExpVal[x], y, ops, alg]
  ]

```



```

effPoissonBracket[x_, y_moment, ops_, alg_] :=
  - effPoissonBracket[y, x, ops, alg]

Clear[opsToVars]

opsToVars[ops_List] /; MatchQ[ops, {__op}] :=
  ops /. op -> Identity

Clear[unsortedTuples]

unsortedTuples[l_List, n_Integer] :=
  DeleteDuplicates[Sort /@ Tuples[l, n]]

Clear[truncate]

truncate[order_Integer] := truncate[#, order] &
truncate[expr_, order_Integer] :=
  Module[
    {repl, \[Lambda]},
    repl = ReplaceAll[{
      x:\[HBar] -> \[Lambda]^2 * x,
      moment -> (\[Lambda]^Length[{##}] * moment[##] &),
      commutator -> (\[Lambda]^2 * commutator[##] &)
    }];
    Normal@Series[repl@expr, {\[Lambda], 0, order}] /. \[Lambda] -> 1
  ]

Clear[effectiveVariables]

effectiveVariables[ops_List, ord_Integer] :=
  Join[

```

Appendix C. Mathematica package "EffectiveConstraints"

```
expVal /@ ops,  
moment @@@ Join @@ (  
  Table[unsortedTuples[ops, n], {n, 2, ord}]  
)  
]
```

```
Clear[effectivePoissonStructure]
```

```
effectivePoissonStructure[effVars_, ops_List, alg_List, ord_Integer] :=  
Module[{n},  
  n = Length[effVars];  
  truncate[ord] @ Simplify @ Normal @ SparseArray[  
    Flatten[#, 2]& @ Table[  
      With[{  
        val = effPoissonBracket[effVars[[ $i$ ]],  
        effVars[[ $j$ ]], ops, alg]  
      },  
      {{ $i$ ,  $j$ } -> val, { $j$ ,  $i$ } -> - val}  
    ],  
    {{ $i$ , 1, n-1}, { $j$ ,  $i$ , n}  
  ],  
  {n, n}  
]
```

```
Clear[effectivePhaseSpace]
```

```
effectivePhaseSpace[ops_List, alg_List, ord_Integer] :=  
Module[{effVars, effPS},  
  effVars = effectiveVariables[ops, ord];  
  effPS = effectivePoissonStructure[effVars, ops, alg, ord];  
  {effVars, effPS}  
]
```

```
End[]
```

```
EndPackage[]
```

### C.1.5. EffectiveConstraints.wl

```
BeginPackage["EffectiveConstraints`"]
```

```
Get["EffectiveConstraints`Usage`"]
```

```
Begin["`Private`"]
```

```
Clear[effectiveConstraints]
```

```
effectiveConstraints[c_, ops_, alg_, order_Integer] :=
```

```
  Composition[
    truncate[#, order]&,
    toMoments[#, ops, alg]&,
    Flatten
```

```
  ] [
```

```
    Table[
      computeEffectiveConstraints[c, ops, n],
      {n, 0, order - 1}
    ]
```

```
  ]
```

```
Clear[computeEffectiveConstraints]
```

```
computeEffectiveConstraints[c_, _, 0] := expVal[c]
```

```
computeEffectiveConstraints[c_, ops_, n_Integer] :=
```

```
  Module[{polyOrderings},
    polyOrderings = DeleteDuplicates[Sort /@ Tuples[ops, {n}]];
    Table[
```

Appendix C. Mathematica package "EffectiveConstraints"

```
expVal[
  ncTimes @@ Join[
    ReleaseHold[
      HoldPattern[# - expVal[#]]& /@ polyOrdering
    ],
    {c}
  ]
],
{polyOrdering, polyOrderings}
]
]

End[]

EndPackage[]
```

### C.1.6. Poisson.wl

```
BeginPackage["EffectiveConstraints`"]

Get["EffectiveConstraints`Usage`"]

Begin["Private`"]

poissonBracket[x_, y_, vars_, ps_] :=
  Grad[x, vars] . ps . Grad[y, vars]

diracBracket[x_, y_, vars_, ps_, cs_, delta_] :=
  poissonBracket[x, y, vars, ps] -
  Dot[
    poissonBracket[x, #, vars, ps]& /@ cs,
    delta,
    poissonBracket[#, y, vars, ps]& /@ cs
  ]
```

```
End[]
```

```
EndPackage[]
```

## C.2. One-body GFT calculation

The following listing shows how the calculation performed in Section 6.3 can be performed using the Mathematica package `EffectiveConstraints`.

```
<<"EffectiveConstraints"

vars = {
  \[DoubleStruckCapitalN],
  \[CapitalPi]1,
  \[CapitalPi]2,
  \[DoubleStruckCapitalX],
  \[CapitalLambda],
  \[CapitalKappa]
};
ops = op /@ vars;
algPartial = {
  commutator[
    op[\[DoubleStruckCapitalX]],
    op[\[CapitalPi]1]
  ] -> I \[HBar] op[\[DoubleStruckCapitalN]],
  commutator[
    op[\[DoubleStruckCapitalX]],
    op[\[CapitalPi]2]
  ] -> I \[HBar] 2 op[\[CapitalPi]1],
  commutator[
    op[\[CapitalLambda]],
```

Appendix C. Mathematica package "EffectiveConstraints"

```

    op[\[CapitalKappa]]
  ] -> I \[HBar] \[Alpha] op[\[CapitalKappa]]
};
alg = completeAlgebra[ops, algPartial];

(* Define the quantum constraint. *)
\[DoubleStruckCapitalC] = Sum[
  m^2 op[\[DoubleStruckCapitalN]],
  - op[\[CapitalLambda]],
  - \[Lambda] op[\[CapitalPi]^2]
];
(* Compute a list of effective constraints. *)
eff\[DoubleStruckCapitalC]s = effectiveConstraints[
  \[DoubleStruckCapitalC], ops, alg, 2
];

(* Compute the truncated quantum phase space. *)
{effVars, effPS} = effectivePhaseSpace[ops, alg, 2];

(* Define gauge fixing conditions. *)
\[DoubleStruckCapitalG]s = Map[
  moment[op[\[CapitalKappa]], #]&,
  Complement[ops, {op[\[CapitalLambda]}]}]
];
(* List of rules which impose the gauge fixing conditions. *)
repl\[DoubleStruckCapitalG]s = Map[
  Rule[#, 0]&,
  \[DoubleStruckCapitalG]s
];

(* Variables to eliminate via the effective constraints. *)

```

```

toEliminate = Join[
  {expVal[op[\[CapitalLambda]]]},
  Map[
    moment[op[\[CapitalLambda]], #]&,
    ops
  ]
];
(* List of rules which implement the effective constraints. *)
repl\[DoubleStruckCapitalC]s = First @ Solve[
  0 == # & /@ eff\[DoubleStruckCapitalC]s,
  toEliminate
];

(* The variables spanning the gauge-fixed constraint hypersurface.
* Overscript[\[ScriptCapitalP], ~] in the main text. *)
remainingVars = Complement[
  effVars,
  toEliminate,
  \[DoubleStruckCapitalG]s
];

(* Function which imposes both effective constraints and gauge
* fixing conditions. *)
toGaugeFixedConstraintHypersurface = Composition[
  ReplaceAll[repl\[DoubleStruckCapitalG]s],
  ReplaceAll[repl\[DoubleStruckCapitalC]s]
];

(* The matrix of Poisson brackets of the gauge fixing
* conditions \[DoubleStruckCapitalG]_a and the effective
* constraints \[DoubleStruckCapitalC],
* \[DoubleStruckCapitalC]_a on the gauge-fixed constraint

```

Appendix C. Mathematica package "EffectiveConstraints"

```

* hypersurface. *)
pbMatrix =
  toGaugeFixedConstraintHypersurface @
  Outer[
    poissonBracket[#1, #2, effVars, effPS]&,
    \[DoubleStruckCapitalG]s,
    eff\[DoubleStruckCapitalC]s
  ];

(* Determine which flow is nontrivial on the gauge-fixed constraint
* hypersurface. *)
unfixedConstraintsIndices = Position[Transpose @ pbMatrix, {0..}];
unfixedConstraints = Extract[
  eff\[DoubleStruckCapitalC]s, unfixedConstraintsIndices
];
flowNonTrivialQ[constraint_] := Not @ MatchQ[{0..}] @ Map[
  toGaugeFixedConstraintHypersurface @
  poissonBracket[constraint, #, effVars, effPS]&,
  remainingVars
];
{hamiltonianConstraint} = Select[unfixedConstraints, flowNonTrivialQ];

(* The set of differential equations on the reduced phase space.
* The time-dependent function are wrapped in 'u' and the initial
* values are wrapped in 'initial'. *)
toTimeDep = ReplaceAll[Rule[#, u[#][t]]& /@ remainingVars];
funcs = u /@ remainingVars;
eomsLHS = D[toTimeDep @ remainingVars, t];
eomsRHS = toTimeDep @ Map[
  poissonBracket[#, hamiltonianConstraint, effVars, effPS]&,
  remainingVars
];
eoms = Thread[Equal[eomsLHS, eomsRHS]];

```



## C.2. One-body GFT calculation

```
eomsInitial = Map[
  u[#][0] == initial[#]&,
  remainingVars
];

(* Solutions to the equations of motion. *)
solutions = First @ DSolve[
  Join[eoms, eomsInitial],
  funcs,
  t
];
```



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