A combinatorial approach to relative complete reducibility

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Abstract

For a reductive subgroup K of a reductive group G, the notion of relative complete reducibility gives an algebraic description of the closed K-orbits in G^n, where K acts by simultaneous conjugation. In this thesis we show that questions about reductive groups acting on arbitrary affine varieties can be translated to the setting of relative $\mathrm{GL}(V)$-complete reducibility. Furthermore, we present characterizations of relative $\mathrm{GL}(V)$-complete reducibility in terms of certain subsets of flags of V. These characterizations lead to combinatorial descriptions of closed orbits, which may assist in proving Tits' Centre Conjecture for convex subsets of spherical buildings.

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Introduction

The Centre Conjecture asserts that a convex subcomplex of a spherical building that is not completely reducible has a centre. The conjecture was motivated by the theory of algebraic groups [37], and a special case considered in Geometric Invariant Theory [28] was proven by Rousseau [32] and Kempf [22]. The conjecture regained interest following work of Serre [34] concerning complete reducibility in spherical buildings. The Centre Conjecture was proved for buildings of classical type and buildings of rank 2 by Mühlherr and Tits [27], and for the remaining buildings of exceptional type by Leeb and Ramos-Cuevas [23], [29]. The motivation for this thesis is to work towards a proof of a generalized Centre Conjecture for convex subsets of spherical buildings by studying G-complete reducibility.

The notion of G-complete reducibility for a reductive group G was developed by Serre [33] to extend standard results from the representation theory of algebraic groups. The notion has proven useful in the study of the structure of linear algebraic groups, and has applications in representation theory, geometric invariant theory, the aforementioned theory of buildings, and number theory [3–7, 9, 13, 18, 24, 25]. Serre's original definition was extended to non-connected groups by Bate, Martin and Röhrle [5] using a geometric approach inspired by Richardson's characterization of closed G-orbits in G^n, where G acts by simultaneous conjugation [30]. The notion of relative G-complete reducibility, introduced in [10], generalizes the study of closed G-orbits to the study of closed K-orbits for reductive subgroups K of G. In this thesis we study relative $\mathrm{GL}(V)$-complete reducibility, with the aim of providing a combinatorial description of what it means for a given K-orbit to be closed.

The first chapter of this thesis provides an overview of the theory of affine algebraic groups, introduces the notion of a spherical building, and summarises previous work related to the Centre Conjecture. In the second chapter we introduce G-complete reducibility and the generalized notion of relative G-complete reducibility, and demonstrate that questions about reductive groups acting on arbitrary affine varieties can be answered by studying relative $\mathrm{GL}(V)$-complete reducibility. The third chapter contains results related to $\mathrm{GL}(V)$-complete reducibility, including a new characterization in terms of certain flags of V. In the fourth chapter we introduce new partial orders for flags and obtain another characterization of relative $\mathrm{GL}(V)$-complete reducibility. The final chapter discusses potential further research, and explains how results in previous chapters extend to fields which are not algebraically closed.

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Declaration

This thesis contains material from [1], which was written in collaboration with Prof. Michael Bate, Dr Maike Gruchot, Dr Alastair Litterick, and Prof. Gerhard Röhrle. No claim of original work is made for Chapter 1, as well as Sections 2.1, 2.2, 2.3, and 5.2. Otherwise, I declare that this thesis is a presentation of original work and I am the sole author. This work has not previously been presented for an award at this, or any other, University. All sources are acknowledged as References.

Chapter 1

Preliminaries

In this chapter we provide an overview of the theory upon which this thesis is based. To introduce algebraic groups, we follow classic textbooks of Humphreys [21], Springer [36], and Borel [12]. We work in the context of affine algebraic groups rather than in the generalized setting of group schemes. We demonstrate various results and concepts in a general linear group, since this setting is the focus of future chapters. When discussing buildings, we introduce the original work of Tits [38] and refer to an article of Everitt [14]. The subsection which focuses on the Centre Conjecture summarises statements and arguments from a paper of Mühlherr and Tits [27].

1.1 Algebraic Groups

We begin by recalling some important definitions from algebraic geometry. For the first four chapters of this thesis we will work over an algebraically closed field k. Given a set X and a function $f : X \to k$, we denote evaluation at a point x by ϵ_x, so that $\epsilon_x(f) := f(x)$. An *affine variety* over k is a set X, together with a finitely-generated k-algebra $k[X]$ of k-valued functions $f : X \to k$, such that the evaluation map $x \to \epsilon_x$ gives a bijection $X \to \mathrm{Hom}_{k-alg}(k[X], k)$. There is a more general notion of an algebraic variety; since we only encounter non-affine varieties very briefly in this thesis (see Theorem 1.1.11) we suppress this and refer the reader to [17] for details.

Throughout this thesis affine varieties will be defined over k, and actions and representations are assumed to be rational maps. For affine varieties X and Y, a map $\phi : X \to Y$ is a *morphism* if $g \circ \phi \in k[X]$ for every $g \in k[Y]$, and we define the *comorphism* $\phi^\sharp : k[Y] \to k[X]$ to be the map such that $\phi^\sharp(g) = g \circ \phi$ for $g \in k[Y]$. Affine varieties can be equipped with a natural topology. For $S \subseteq k[X]$, define the subset $\mathcal{V}(S) = \{x \in X \mid f(x) = 0 \text{ for all } f \in S\} \subseteq X$; these sets form the closed sets of the *Zariski topology* on X. A topological space is said to be *irreducible* if it cannot be written as a union of two proper nonempty closed subsets.

Definition 1.1.1. An *affine algebraic group* is an affine variety G which is also a group such that the multiplication map $\mu : G \times G \to G$, where $\mu(x, y) = xy$, and the inverse map $\iota : G \to G$, where $\iota(x) = x^{-1}$, are morphisms of varieties.

Examples 1.1.2.

1. The affine line $k = \mathbb{A}^1$ equipped with the group laws $\mu(x, y) = x + y$, $\iota(x) = -x$ and the coordinate ring $k[T]$ is known as the *additive group* $\mathbb{G}_a$.

2. The affine open subset $k^* = \mathbb{A}^1 \setminus \{0\}$ equipped with the group laws $\mu(x, y) = xy$, $\iota(x) = x^{-1}$ and coordinate ring $k[T, T^{-1}]$ is known as the *multiplicative group* $\mathbb{G}_m$.

3. $\mathrm{GL}_n(k)$, the set of all $n \times n$ invertible matrices with entries in k, is an algebraic group when equipped with the familiar laws of matrix multiplication and inversion. The coordinate ring is generated by the coordinate functions T_{ij} and $\det(T_{ij})^{-1}$ for $1 \leq i, j \leq n$.

There are familiar definitions which we need to extend to the setting of algebraic groups.

Definition 1.1.3. Let G and G' be algebraic groups. A *homomorphism of algebraic groups* is a group homomorphism $\phi : G \to G'$ that is also a morphism of varieties. Notions of algebraic group isomorphisms and automorphisms follow naturally.

Definition 1.1.4. A *closed subgroup* of an algebraic group is a subgroup that is closed in the Zariski topology.

Remark 1.1.5. Algebraic groups can be defined over varieties which are not affine, but we restrict our attention to affine algebraic groups. These may be described as *linear algebraic groups*; when the underlying variety of an algebraic group is affine, there exists an isomorphism of G onto a closed subgroup of some $\mathrm{GL}_n(k)$ [36, Theorem 2.3.7(i)].

Proposition 1.1.6. [21, §7.4] *Let $\phi : G \to H$ be a homomorphism of algebraic groups. Then* $\mathrm{Ker}(\phi)$ *and* $\mathrm{Im}(\phi)$ *are closed subgroups of G and H, respectively.*

The determinant map $\det : \mathrm{GL}_n(k) \to \mathrm{GL}_1(k) = \mathbb{G}_m$ is an example of a homomorphism of algebraic groups. The kernel of this homomorphism is $\mathrm{SL}_n(k)$, which is an algebraic group in its own right. We do not need to show that $\mathrm{SL}_n(k)$ has the structure of an affine variety; any closed subgroup of an algebraic group is an algebraic group, and Proposition 1.1.6 tells us that it is a closed subgroup of $\mathrm{GL}_n(k)$. We can use this fact about closed subgroups, along with the fact that a direct product of algebraic groups (the usual direct product endowed with the Zariski topology) is an algebraic group, to obtain additional examples of algebraic groups.

Examples 1.1.7.

1. Affine m-space $\mathbb{A}^m$ may be viewed as the direct product of m copies of $\mathbb{G}_a$.

2. The group of diagonal $n \times n$ matrices $D_n(k)$ can be viewed as the direct product of n copies of $\mathbb{G}_m$, or as a closed subgroup of $\mathrm{GL}_n(k)$.

3. The groups of upper triangular matrices $B_n(k)$ and upper unitriangular matrices $U_n(k)$ are both closed subgroups of $\mathrm{GL}_n(k)$. Observe that U_2 is naturally isomorphic to the additive group $\mathbb{G}_a$.

We now introduce some important theory, demonstrating key ideas in the context of general linear groups.

Definition 1.1.8. There is a unique irreducible component G^0 of G containing the identity element which we will refer to as the *identity component*. We will say that an algebraic group is *connected* if $G = G^0$.

For a justification of the uniqueness of such an irreducible component, see [36, Proposition 2.2.1]; this proposition also proves that G^0 is the unique connected component of G containing the identity element, and that any closed subgroup of finite index in G contains G^0. The groups $\mathbb{G}_a$, $\mathbb{G}_m$ and $\mathrm{GL}_n(k)$ are all examples of connected algebraic groups.

Definition 1.1.9. The *radical* of G, denoted $R(G)$, is the unique maximal closed connected normal solvable subgroup of G. The *unipotent radical* of G, denoted $R_u(G)$, is the unique maximal closed connected normal unipotent subgroup of G. A connected group G is called *semisimple* if $R(G)$ is trivial. A group G is said to be *reductive* if $R_u(G)$ is trivial; note that we do not require that G is connected.

For a justification of the uniqueness of $R(G)$ and $R_u(G)$, see [36, 6.4.14]. Reductive algebraic groups are objects of interest throughout this thesis. For any algebraic group G, $R_u(G)$ consists of the unipotent elements of $R(G)$. When we study non-connected reductive groups, we will extend ideas from the study of connected reductive groups.

Definition 1.1.10. A linear algebraic group which is isomorphic to a product of copies of the multiplicative group $\mathbb{G}_m$ is known as a *torus*. A *maximal torus* of an algebraic group G is a subgroup T that is a torus which is not properly contained in any other subgroup of G that is also a torus.

The maximal tori of a given G are conjugate, see [36, Theorem 6.3.5]. A variety X is said to be *complete* if for every variety Y, the projection $q : X \times Y \to Y$ is a closed map. The following is one of the key theorems in the study of algebraic groups; see [21, Theorem 21.2] for a proof.

Theorem 1.1.11 (Borel's Fixed Point Theorem)**.** *Let G be a connected solvable linear algebraic group acting on a complete variety X. There exists an element $x \in X$ such that $gx = x$ for all $g \in G$.*

An important class of connected solvable subgroups are named after Borel.

Definition 1.1.12. A *Borel subgroup* of G is a maximal closed connected solvable subgroup of G.

Borel's Fixed Point Theorem can be used to show that the Borel subgroups of a given G are conjugate, see [36, Theorem 6.2.7]. This thesis repeatedly studies a class of subgroups which includes the Borel subgroups.

Definition 1.1.13. A subgroup P of G is called a *parabolic subgroup* if it contains a Borel subgroup.

The Borel subgroups are the minimal parabolic subgroups. A subgroup of G is parabolic if and only if the quotient variety G/P is complete; see [36, §6.2].

Example 1.1.14. Let $G = \mathrm{GL}_n(k)$. The radical $R(G)$ is the subgroup of diagonal matrices with equal nonzero entries. In this case we have that $R(G) = Z(G)$, where $Z(G)$ denotes the centre of G. The unipotent radical $R_u(G)$ is trivial, since the only unipotent scalar matrix is the identity. Hence G is reductive but not semisimple. For an example of a nontrivial unipotent radical, consider the subgroup B_n of upper triangular matrices; $R_u(B_n)$ is U_n, the subgroup of upper unitriangular matrices.

Any product of copies of $\mathbb{G}_m$ is isomorphic to some group of diagonal matrices; the subgroup of $n \times n$ diagonal matrices $D_n(k)$ is therefore a maximal torus in $G = \mathrm{GL}_n(k)$. In the case $G = \mathrm{GL}_3(k)$, the subgroup

$$T = \left\{ \begin{pmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & 1 \end{pmatrix} \in G \right\},$$

is an example of a torus (isomorphic to two copies of $\mathbb{G}_m$) which is not maximal.

The group of upper triangular matrices B_n is a Borel subgroup of $\mathrm{GL}_n(k)$, sometimes referred to as the standard Borel subgroup. In Chapters 3 and 4, one of our recurring examples will involve the study of parabolic subgroups of $\mathrm{GL}_4(k)$. We will see some parabolic subgroups which contain B_4, such as

$$\begin{pmatrix} * & * & * & * \\ 0 & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & 0 & * \end{pmatrix}, \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & * & * \end{pmatrix}, \text{ and } \begin{pmatrix} * & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \\ 0 & 0 & 0 & * \end{pmatrix}.$$

We will also see parabolics of the form

$$\begin{pmatrix} * & 0 & 0 & 0 \\ * & * & 0 & 0 \\ * & * & * & 0 \\ * & * & * & * \end{pmatrix}, \begin{pmatrix} * & * & 0 & 0 \\ * & * & 0 & 0 \\ * & * & * & * \\ * & * & * & * \end{pmatrix}, \text{ and } \begin{pmatrix} * & 0 & 0 & 0 \\ * & * & * & 0 \\ * & * & * & 0 \\ * & * & * & * \end{pmatrix},$$

which contain the Borel subgroup of lower triangular matrices. These triangular subgroups are conjugate, as one might expect. For a more interesting example of the conjugacy of Borel subgroups, we study the subgroup

$$H = \begin{pmatrix} * & 0 & * & * \\ * & * & * & * \\ 0 & 0 & * & 0 \\ 0 & 0 & * & * \end{pmatrix}.$$

Consider the matrices

$$x = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \text{ and } y = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

One can check that $x^{-1}Hx$ consists of upper triangular matrices and $y^{-1}Hy$ consists of lower triangular matrices. To derive these matrices, take V to be the natural module of column vectors of length 4 and let $e_1, \ldots, e_4$ be the natural basis of V. Choose another basis $v_1, \ldots, v_4$ for V compatible with the flag

$$\langle e_2 \rangle \subset \langle e_1, e_2 \rangle \subset \langle e_1, e_2, e_4 \rangle \subset V,$$

i.e. choose $v_1 \in \langle e_2 \rangle$, then choose $v_2 \in \langle e_1, e_2 \rangle \setminus \langle e_2 \rangle$. This flag was chosen as it contains a subspace of each intermediate dimension which is stabilized by H. We might as well choose $v_1 = e_2$, $v_2 = e_1$, $v_3 = e_4$, and $v_4 = e_3$. Then x is the change of basis matrix such that $xe_i = v_i$ and y is the change of basis matrix such that $ye_i = v_{5-i}$.

Remark 1.1.15. The fact that a Borel subgroup of $\mathrm{GL}(V)$ stabilizes such a flag of subspaces of V is a consequence of Theorem 1.1.11. The arguments above can be extended to $\mathrm{GL}_n(k)$ to prove a theorem of Kolchin, which can be used to prove that B_n is a Borel subgroup:

Theorem 1.1.16 (Lie-Kolchin Theorem)**.** [36, Theorem 6.3.1]. *Let H be a closed connected solvable subgroup of* $\mathrm{GL}_n(k)$*. Then H is conjugate to a subgroup of the group of upper triangular matrices B_n.*

Levi subgroups are another class of subgroups which appear throughout this thesis, and they are closely related to parabolic subgroups.

Definition 1.1.17. A *Levi subgroup* of G is a connected subgroup L of a parabolic subgroup P of G such that P is the semi-direct product of L and $R_u(P)$.

Levi subgroups are reductive, and a Levi subgroup of a Levi subgroup is a Levi subgroup; the class of subgroups is well-behaved with respect to this sort of descent, which will prove useful in Chapter 4.

Theorem 1.1.18. [36, Theorem 30.2]. *Any parabolic subgroup P of G has a Levi decomposition $P = L \ltimes R_u(P)$ and any two Levi subgroups are conjugate by an element of $R_u(P)$.*

We now introduce the notions of characters and cocharacters of algebraic groups. We will make extensive use of cocharacters in later chapters of this thesis.

Definition 1.1.19. Let G be a linear algebraic group. A homomorphism $\chi : G \to \mathbb{G}_m$ is called a *character* of G, and the set of characters of G is denoted by $X(G)$. A homomorphism $\lambda : \mathbb{G}_m \to G$ is called a *cocharacter* of G and the set of cocharacters of G is denoted by $Y(G)$.

Since the image of any nontrivial cocharacter of G is a one-dimensional torus, $Y(G)$ is the union of the sets $Y(T)$ as T runs over the maximal tori of G. Each of these sets $Y(T)$ carries the structure of an abelian group; we have $Y(T) \cong \mathbb{Z}^r$, where $r = \dim T$. Therefore, we use additive notation for cocharacters which evaluate in a common maximal torus T: if λ and μ are such cocharacters of G, we set $(n\lambda + m\mu)(a) := \lambda(a)^n \mu(a)^m$. Cocharacters are sometimes referred to as *one-parameter (multiplicative) subgroups*. We will see in Lemma 2.1.4 that parabolic subgroups of connected groups G can be characterized in terms of cocharacters of G.

Example 1.1.20. Let $G = \mathrm{GL}_n(k)$, and let $\mathbf{n} = (n_1, \dots, n_r)$ be an m-tuple such that $n_1 + \cdots + n_r = n$. Let $P_{\mathbf{n}}$ denote the subgroup of block upper triangular matrices with block sizes of $n_1, \dots, n_r$ down the main diagonal. This is a parabolic subgroup of G, and any parabolic of G will be (up to conjugation in G) of this form. Let $L_{\mathbf{n}}$ denote the subgroup of block diagonal matrices with block sizes of $n_1, \dots, n_r$ down the main diagonal. Then $L_{\mathbf{n}}$ is a Levi subgroup of $P_{\mathbf{n}}$. The following are examples of such subgroups in $\mathrm{GL}_4(k)$:

$$P_{(3,1)} = \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ 0 & 0 & 0 & * \end{pmatrix} \quad P_{(2,2)} = \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & * & * \end{pmatrix} \quad P_{(1,2,1)} = \begin{pmatrix} * & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \\ 0 & 0 & 0 & * \end{pmatrix}$$

$$L_{(3,1)} = \begin{pmatrix} * & * & * & 0 \\ * & * & * & 0 \\ * & * & * & 0 \\ 0 & 0 & 0 & * \end{pmatrix} \quad L_{(2,2)} = \begin{pmatrix} * & * & 0 & 0 \\ * & * & 0 & 0 \\ 0 & 0 & * & * \\ 0 & 0 & * & * \end{pmatrix} \quad L_{(1,2,1)} = \begin{pmatrix} * & 0 & 0 & 0 \\ 0 & * & * & 0 \\ 0 & * & * & 0 \\ 0 & 0 & 0 & * \end{pmatrix}$$

In the case $\mathbf{n} = (1, \dots, 1)$, $P_{\mathbf{n}}$ is the standard Borel subgroup of upper triangular matrices B_n, and $L_{\mathbf{n}}$ is the maximal torus of diagonal matrices D_n. In the case $\mathbf{n} = (n)$, we have $P_{\mathbf{n}} = L_{\mathbf{n}} = \mathrm{GL}_n(k)$. Once we have established the necessary correspondence, we will be able to read off the shape of parabolic subgroups by studying the diagonalisation of their corresponding cocharacters.

Earlier we saw that $\det : \mathrm{GL}_n(k) \to \mathrm{GL}_1(k) = \mathbb{G}_m$ is an example of a homomorphism of algebraic groups; this is therefore an example of a character of $\mathrm{GL}_n(k)$. Cocharacters of G will be of the form

$$\lambda(a) = \begin{pmatrix} a^{z_1} & & \\ & \ddots & \\ & & a^{z_n} \end{pmatrix},$$

where the powers z_i are integers. By conjugating if necessary, we can ensure that the powers of a decrease in size along the main diagonal.

Definition 1.1.21. Let T be a torus of G and let V be a G-module. For each $\chi \in X(T)$ we define the χ *weight space* of V to be

$$V_\chi = \{v \in V \mid t \cdot v = \chi(t)v \text{ for all } t \text{ in } T\}.$$

We say that χ is a *weight* of V if V_χ is non-zero.

The following makes use of the fact that commuting diagonalisable matrices are simultaneously diagonalisable [19, Theorem 1.3.12], and that a torus has to act diagonally on a vector space.

If V is a k-vector space and T is a torus over k acting linearly on V, then we can diagonalise the action. Hence there is a finite list of characters $\chi_i : T \to \mathbb{G}_m$ (with $1 \leq i \leq r$) such that $V = \bigoplus_{i=1}^r V_i$, where $V_i = \{v \in V \mid t \cdot v = \chi_i(t)v \text{ for all } t \in T\}$. We can further break up each V_i into one-dimensional subspaces because each $t \in T$ acts as a scalar on the whole of V_i.

The character group of a one-dimensional torus is isomorphic to the integers. So if we can write $T = \mathrm{Im}(\lambda)$ for a cocharacter $\lambda : \mathbb{G}_m \to G$, we can write $V = \bigoplus_{n\in\mathbb{Z}} V_n$ where $V_n = \{v \mid \lambda(a) \cdot v = a^n v\}$ and only finitely many V_n are non-zero. This relies on the fact that endomorphisms of the multiplicative group are of the form $a \mapsto a^n$ for some $n \in \mathbb{Z}$, see [36, Example 3.2.2].

Definition 1.1.22. We call $V_n = \{v \mid \lambda(a) \cdot v = a^n v\}$ a λ-*weight space* and say that λ *acts with weight* n on V_n.

Arguments on weight spaces will appear in later chapters of this thesis. We return to the context of $\mathrm{GL}_4(k)$ to provide some concrete examples.

Example 1.1.23. Let $G = \mathrm{GL}_4(k)$, and let $T = \mathrm{diag}(a^2, a^2, a, a)$ be a torus of G. Let V be the k-vector space spanned by the natural basis $e_1, \ldots, e_4$ (where e_i is the 4-dimensional column vector with 1 in the ith position and zeros elsewhere). Given the characters

$$\chi\begin{pmatrix} t^2 & & & \\ & t^2 & & \\ & & t & \\ & & & t \end{pmatrix} = t^2, \text{ and } \chi'\begin{pmatrix} t^2 & & & \\ & t^2 & & \\ & & t & \\ & & & t \end{pmatrix} = t,$$

one can check that $V_\chi = \langle e_1, e_2\rangle$ and $V_{\chi'} = \langle e_3, e_4\rangle$. Hence we may write $V = V_\chi \oplus V_{\chi'}$.

Let λ be the cocharacter defined by

$$\lambda(a) = \begin{pmatrix} a^2 & & & \\ & a^2 & & \\ & & a & \\ & & & a \end{pmatrix},$$

and observe that $T = \mathrm{Im}(\lambda)$. Since λ acts with weight 2 on $V_2 = \langle e_1, e_2\rangle$ and acts with weight 1 on $V_1 = \langle e_3, e_4\rangle$, we obtain the λ-weight space decomposition $V = V_2 \oplus V_1$.

We say that two cocharacters $\lambda, \mu \in Y(G)$ *commute* if there exists a common torus $T \subseteq G$, with $\mathrm{Im}(\lambda), \mathrm{Im}(\mu) \subseteq T$; this is equivalent to insisting that $\lambda(a)\mu(b) = \mu(b)\lambda(a)$ for all a, b. We will require the following fact about commuting cocharacters.

Proposition 1.1.24. *If $\lambda, \mu \in Y(G)$ commute, each λ-weight space is μ-stable and vice* *versa.*

Proof. If v is a vector with λ-weight n, that means that $\lambda(a) \cdot v = a^n v$ for $a \in k^*$. Then since λ and μ commute, for any $b \in k^*$ we have

$$\lambda(a) \cdot (\mu(b) \cdot v) = \mu(b) \cdot (\lambda(a) \cdot v) = \mu(b) \cdot (a^n v) = a^n(\mu(b) \cdot v),$$

so $\mu(b) \cdot v$ is a λ-weight vector with weight n. Hence the n-weight space for λ is μ-stable. $\square$

We will also make use of the following two lemmas concerning maximal tori.

Lemma 1.1.25. *Suppose λ is a cocharacter of an algebraic group G and T is a maximal torus of G. Then there exists $g \in G$ such that $g \cdot \lambda$, where $(g \cdot \lambda)(z) := g\lambda(z)g^{-1}$, is a cocharacter of T.*

Proof. We know that λ evaluates in some maximal torus, and that all maximal tori are conjugate [36, Theorem 6.3.5]. Hence λ can be conjugated into every other maximal torus. $\square$

Lemma 1.1.26. *Let P and Q be parabolic subgroups of a reductive group G. Then $P \cap Q$ contains a maximal torus of G.*

Proof. This is a consequence of the fact that the intersection of any two Borel subgroups of G contains a maximal torus; see [12, Corollary 14.13] $\square$

The theory in this section can be developed to introduce the Lie algebras of algebraic groups and obtain a classification of connected reductive groups using root data and Dynkin diagrams, see for example [20, §11]. The statements and proofs in this thesis will not require an understanding of this theory; we will be working in the convenient setting of a general linear group.

1.2 Buildings

Before providing a definition, we list three informal ways of thinking about buildings. One can work from the ground up and think of a building as a simplicial complex Δ with a highly structured decomposition as a union of subcomplexes called *apartments*. Each apartment is isomorphic to a complex attached to a fixed Coxeter group W. Alternatively, one can take a top-down approach and view a building as a chamber system Δ with a W-valued distance function $\delta : \Delta \times \Delta \to W$. These viewpoints are equivalent; chamber systems are examples of simplicial complexes where the chambers are maximal simplices. Alternatively, one can think of a geodesic metric space equipped with an atlas of embeddings $\Sigma \to \Delta$ for some model space Σ equipped with a W-action by isometries.

To present our definition of a building we require the following terminology, outlined in [38, §1]. Let Δ be a simplicial complex. Say that $A \in \Delta$ is a *face* of $B \in \Delta$ if $A \subseteq B$. The *rank* of an element $A \in \Delta$ is the number of minimal nonzero faces of A and elements of rank 1 are called *vertices*. The rank of a complex Δ is defined to be the supremum of the ranks of elements it contains. If $A \in \Delta$, the set of all elements of Δ which contain A together with the order relation induced by that of Δ is a complex called the *star* of A. For B in the star of A, the rank of B in the star of A is called the *codimension* of A in B, and is denoted by $\operatorname{codim}_B A$. A complex is called a *chamber complex* if every element is contained in a maximal element and if, given two maximal elements c and c', there exists a finite sequence

$$c = c_0,\ c_1, \dots,\ c_m = c'$$

such that

$$\operatorname{codim}_{c_{i-1}}(c_{i-1} \cap c_i) = \operatorname{codim}_{c_i}(c_{i-1} \cap c_i) \leq 1$$

for all $i = 1, \dots, m$. The maximal elements are then called *chambers*. A chamber complex is said to be *thin* if every element of codimension 1 is contained in exactly two chambers. A chamber complex is said to be *thick* if every element of codimension 1 is contained in at least three chambers.

Definition 1.2.1. [38, §3.1]. Let Δ be a complex, and let $\mathcal{A}$ be a set of subcomplexes of Δ. The pair $(\Delta, \mathcal{A})$ is called a *building* of which the elements of $\mathcal{A}$ are called *apartments* if the following conditions hold:

(B2) the elements of $\mathcal{A}$ are thin chamber complexes;

(B3) any two elements of Δ belong to an apartment;

(B4) if two apartments Σ and Σ' contain two elements $A, A' \in \Delta$, there exists an isomorphism of Σ onto Σ' which leaves invariant A, A', and all their faces.

Remark 1.2.2. Following [27], we view buildings as simplicial complexes and do not require buildings to be thick, which was an axiom (B1) in the original formulation. We reserve the term building for the structures described as 'weak buildings whose Weyl complexes are Coxeter complexes' in [38, §3.1].

Any representative of the isomorphism class of the elements of $\mathcal{A}$ will be called the *Weyl complex* of Δ. We call a building *spherical* when its Weyl complex is finite. The Weyl complex of a building is a Coxeter complex [38, Theorem 3.7]. An alternative top-down chamber system definition for a building can be found in [31].

Examples of buildings come from BN-pairs, also known as Tits systems. BN-pairs can be found in all reductive algebraic groups.

Definition 1.2.3. A *BN-pair* or *Tits system* for a group G consists of a pair of subgroups B and N of G which satisfy the following axioms:

(A1) G is generated by B and N.

(A2) $H = B \cap N$ is a normal subgroup of G.

(A3) The quotient $W = N/H$ is a Coxeter group with a generating set S.

(A4) For every $w \in W$ and $s \in S$, $sBwB \subset BwB \cup BswB$.

(A5) For every $s \in S$, we have $sBs \neq B$.

The group W is called the *Weyl group* of the BN-pair. The Weyl group of a Tits system is a Coxeter group; see [21, §29.4].

Theorem 1.2.4. [14, Theorem 5.1]. *Let G be a group with a BN-pair and let Δ be a chamber system over I with chambers the cosets G/B and adjacency defined by $a_1B \sim_i a_2B$ if and only if $a_1^{-1}a_2 \in B\langle s_i\rangle B$. Define a W-metric by $\delta(a_1B, a_2B) = g \in W$ if and only if $a_1^{-1}a_2 \in BgB$. Then (Δ, δ) is a thick building of type (W, S)*

Note that we could also identify the chambers of Δ with subgroups (conjugates of B) rather than cosets. We can get a BN-pair for every connected reductive algebraic group G over an algebraically closed field [21, §29.1], which results in a spherical building.

Example 1.2.5. [14, Example 5.1]. Recall that a *monomial matrix* is a matrix containing exactly one non-zero element in each row and column. A *permutation matrix* is a monomial matrix in which all non-zero elements are 1.

Let $G = \mathrm{GL}_n(k)$. Take $B \leq G$ to be the subgroup of upper triangular matrices, and take $N \leq G$ to be the subgroup of monomial matrices. Then $B \cap N = H \trianglelefteq G$ is the normal subgroup of diagonal matrices. Here, $W = N/H$ is the set of permutation matrices generated by $s_1, \ldots, s_{n-1}$, where s_i is the permutation matrix which is zero off the diagonal except for positions $(i, i+1)$ and $(i+1, i)$. In the case $n = 3$ these matrices would be:

$$s_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad s_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

Define lines $L_i = \{te_i \mid t \in k\}$ where e_i is the column vector with 1 in the ith position and zeros elsewhere. Then N permutes the set of lines $\{L_1, \ldots, L_n\}$ and this action can be used to show that W is isomorphic to the symmetric group S_n.

Note that the example for $\mathrm{GL}_n(k)$ is misleading; we cannot, in general, realise W as a subgroup of G. A discussion of the BN-pairs in simple algebraic groups over local fields can be found in [21, §35.4].

1.2.1 The Centre Conjecture

In a chamber complex Δ, a set L of chambers is called *convex* if every minimal gallery whose first and last chambers belong to L has all its terms in L. A chamber subcomplex Δ' of Δ is called *convex* if the set of all chambers in Δ' is convex. An intersection of convex chamber subcomplexes is convex; an arbitrary subcomplex of Δ is called *convex* if it is such an intersection. For spherical buildings, there is a notion of "opposition" for simplices: two simplices are opposite in Δ if they are opposite in some apartment (and therefore all apartments) containing them both. Roughly, the Centre Conjecture asserts that a convex subcomplex of a spherical building should be a "subbuilding", or should contain a natural "centre".

Theorem 1.2.6 (The Centre Conjecture)**.** *Let $\mathbf{\Delta}$ be a spherical building, and let $\tilde{\Delta}$ be a convex subcomplex of $\mathbf{\Delta}$. Then at least one of the following holds:*

(a) *for each simplex $A \in \tilde{\Delta}$ there is a simplex $B \in \tilde{\Delta}$ which is opposite to A in $\mathbf{\Delta}$;*

(b) *there exists a nontrivial simplex of $\tilde{\Delta}$ which is fixed by all automorphisms of $\mathbf{\Delta}$ which stabilize $\tilde{\Delta}$.*

Recall that the Weyl complexes of our buildings are Coxeter complexes. We say that a building is of type A_n (respectively B_n, D_n, etc.) if its Weyl Complex is a Coxeter complex of type A_n (respectively B_n, D_n, etc.). The Centre Conjecture for convex subcomplexes has been proven using type-based proofs in [27], [23], and [29]. We are interested in studying a strengthened version of the Centre Conjecture which considers the more general class of convex *subsets* of a spherical building. For convex subsets of dimension at most 2, the conjecture holds [2]. An approach towards proving a strengthened Centre Conjecture using Geometric Invariant Theory is discussed in [8]. We would like to develop an approach based on the ideas developed by Mühlherr and Tits in [27]. By identifying buildings with flag complexes of certain classes of incidence structures, the combinatorial properties of these incidence geometries can be used to prove the centre conjecture for convex subcomplexes. The rest of this subsection is dedicated to providing a detailed summary of their proof for buildings of type A_n.

The following definition and proposition are due to Serre [34]. The statements here are reproduced from [27].

Definition 1.2.7. [27, Definition 2.2]. Let $\mathbf{\Delta} = (\Delta, \subset)$ be a spherical building and let $\tilde{\Delta} \subset \Delta$ be a convex subcomplex of $\mathbf{\Delta}$. Then $\tilde{\Delta}$ is called *completely reducible* if for each simplex $A \in \tilde{\Delta}$ there exists a simplex $B \in \tilde{\Delta}$ which is opposite to $A \in \mathbf{\Delta}$.

Proposition 1.2.8. [27, Proposition 2.3]. *Let $\mathbf{\Delta} = (\Delta, \subset)$ be a spherical building and let $\tilde{\Delta} \subset \Delta$ be a convex subcomplex of $\mathbf{\Delta}$. Then the following are equivalent:*

(a) *$\tilde{\Delta}$ is completely reducible;*

(b) *for each vertex $X \in \tilde{\Delta}$ there exists a vertex $Y \in \tilde{\Delta}$ which is opposite to X in $\mathbf{\Delta}$.*

This proposition is used in the proof for buildings of type A_n, but is also used in the proofs for buildings of type $I_2(n)$, C_n, and D_n. Proving that an arbitrary vertex has an opposite is simpler, in practice, than proving that an arbitrary simplex has an opposite. We now describe the arguments used specifically for buildings of type A_n. The incidence structure corresponding to buildings of type A_n comes from projective geometry.

Definition 1.2.9. A point-line space consists of a set P and a set L of subsets of P; elements of P are *points* and elements of L are *lines.* A *projective space* is a linear point-line space $\mathcal{S} = (P, L)$ such that:

1. two distinct points are contained in exactly one line;
2. for any 5-tuple of pairwise distinct points a, b, c, p, q such that a, b, p and a, c, q are collinear on distinct lines, the lines containing b, c and p, q have a common

point.

An example of a projective space is given by the point-line space coming from a vector space, where points are one-dimensional subspaces and lines are two-dimensional subspaces. A *subspace* of a projective space $\mathcal{S} = (P, L)$ is a subset P' of P such that $|\ell \cap P'| \geq 2$ implies $\ell \subseteq P'$ for each line $\ell \in L$. Two subspaces U, V of a projective space $\mathcal{S} = (P, L)$ are said to be *complementary* if $U \cap V = \emptyset$ and $\langle U, V \rangle = P$, where $\langle U, V \rangle$ denotes the smallest subspace containing U and V.

Let $\mathcal{S} = (P, L)$ be a projective space, and let $\mathcal{V}(\mathcal{S})$ denote the set of all nontrivial subspaces of $\mathcal{S}$.

Definition 1.2.10. [27, Definition 4.2]. We call a subset Ω of $\mathcal{V}(\mathcal{S})$ *closed* if the following hold for all $U, V \in \Omega$:

(i) $\langle U, V \rangle \neq P$ implies $\langle U, V \rangle \in \Omega$.

(ii) $U \cap V \neq \emptyset$ implies $U \cap V \in \Omega$.

From this point on, $\Omega \subset \mathcal{V}(\mathcal{S})$ always denotes a closed set. Let P_Ω denote the set of all minimal elements in Ω. For an element $Z \in \Omega$, define $\Omega_Z := \{X \in \Omega \,|\, Z \neq X \subset Z\}$ and $P_\Omega(Z) := P_\Omega \cap \Omega_Z$. In words, Ω_Z is the set of subspaces in Ω which are properly contained in Z, and $P_\Omega(Z)$ is the set of minimal elements in Ω_Z.

Proposition 1.2.11. [27, Proposition 4.4]. *Suppose that $\mathcal{S}$ has finite rank k and that for each $X \in P_\Omega$ there exists a complement $Y \in \Omega$ of X in $\mathcal{S}$. Then the following hold for all $Z \in \Omega$:*

(a) *either $Z = \langle X \,|\, X \in P_\Omega(Z) \rangle$ or $Z \in P_\Omega$;*

(b) *there is $W \in \Omega$ which is a complement of Z in $\mathcal{S}$.*

Moreover, if $\Omega \neq \emptyset$, then $P = \langle X \mid X \in P_\Omega \rangle$.

There is a dual proposition for maximal elements of Ω. Let H_Ω denote the set of all maximal elements in Ω. Define $\Omega^Z := \{X \in \Omega \,|\, Z \neq X \supset Z\}$ and $H_\Omega(Z) := H_\Omega \cap \Omega^Z$. Then Ω^Z is the set of subspaces in Ω which properly contain Z, and $H_\Omega(Z)$ is the set of maximal elements in Ω^Z.

Proposition 1.2.12. [27, Proposition 4.5]. *Suppose that $\mathcal{S}$ has finite rank k and that for each $X \in H_\Omega$ there exists a complement $Y \in \Omega$ of X in $\mathcal{S}$. Then the following hold for all $Z \in \Omega$:*

(a) *either $Z = \bigcap_{X \in H_\Omega(Z)} X$ or $Z \in H_\Omega$;*

(b) *there is $W \in \Omega$ which is a complement of Z in $\mathcal{S}$.*

Moreover, if $\Omega \neq \emptyset$, then $\bigcap_{X \in H_\Omega} X = \emptyset$.

Although we do not need to reproduce it here, it is worth noting that the proof of Proposition 1.2.11 makes use of a fact about complementary subspaces of a projective space. When elements of Ω have $\mathcal{S}$-complements in Ω, one can check that assertion (a) of the centre conjecture holds; the following theorem is introduced to handle cases where such complements do not exist.

Theorem 1.2.13. [27, Theorem 4.6]. *Suppose that $\mathcal{S}$ has finite rank. Let C_p (respectively C_h) denote the set of all $X \in P_\Omega$ (respectively $X \in H_\Omega$) for which there is no $Y \in \Omega$ which is a complement of X in $\mathcal{S}$. Put $\mathcal{C}_p := \langle X \mid X \in C_p \rangle$ and $\mathcal{C}_h := \bigcap_{X \in C_h} X$. Then one of the following holds.*

(a) *For each $X \in \Omega$ there is $Y \in \Omega$ which is a complement of X in $\mathcal{S}$.*

(b) *$\mathcal{C}_p$ and $\mathcal{C}_h$ are incident elements which are both contained in Ω.*

Remark 1.2.14. We are particularly interested in mirroring this result in our own setting. We will summarise the ideas of this proof here, and will refer back to it in Chapter 4. First note that if C_p (respectively C_h) is empty then we can apply Proposition 1.2.11 (respectively Proposition 1.2.12) to conclude that we are in case (a) where every subspace in Ω has an $\mathcal{S}$ complement in Ω. We can therefore assume that both C_p and C_h are nonempty; it follows that $\mathcal{C}_p \neq \emptyset$ and $\mathcal{C}_h \neq P$. Take $X \in P_\Omega$ and $Y \in C_h$, and suppose that X is not contained in Y. Since X is in P_Ω, it follows that $X \cap Y = \emptyset$. Since Y is in H_Ω, it follows that $\langle X, Y \rangle = P$. Then X is a complement of Y in $\mathcal{S}$, contradicting our assumption that $Y \in C_h$. Hence every subspace $X \in P_\Omega$ is contained in every subspace $Y \in C_h$. Set $C := \langle X \mid X \in P_\Omega \rangle$. It follows that $\mathcal{C}_p \subset C \subset \mathcal{C}_h$, and that $\mathcal{C}_p \neq P$ and $\mathcal{C}_h \neq \emptyset$. Therefore $\mathcal{C}_p$ and $\mathcal{C}_h$ are incident elements in Ω.

This construction of a pair of incident elements is key. Theorem 1.2.15 will use the flag $\mathcal{C}_p \subset \mathcal{C}_h$ (which is fixed by inclusion-preserving and inclusion-reversing automorphisms of the flag complex alike) to prove that for buildings of type A_n, one of the conditions of the centre conjecture has to hold.

We require some additional terminology to discuss the remaining results of Section 4 of [27]. An *incidence structure* is a pair $\mathcal{G} = (V, \star)$ where V is a set and $\star$ is a reflexive and symmetric binary relation on V known as an *incidence relation*. A *flag* of $\mathcal{G} = (V, \star)$ is a subset of V whose elements are pairwise incident, and we denote the set of flags of $\mathcal{G}$ by $\operatorname{flag} \mathcal{G}$. We let $\operatorname{Flag} \mathcal{G} := (\operatorname{flag} \mathcal{G}, \subset)$ denote the *flag complex* of $\mathcal{G}$. For any subset Ω of V, let $\operatorname{flag} \Omega$ denote the set of all flags contained in Ω and let $\operatorname{Flag} \Omega$ denote the corresponding subcomplex of $\operatorname{Flag} \mathcal{G}$. It follows that the automorphism groups $\operatorname{Aut}(\mathcal{G})$ and $\operatorname{Aut}(\operatorname{Flag} \mathcal{G})$ are the same and that the stabilizer of $\Omega \subset V$ in $\operatorname{Aut}(\mathcal{G})$ corresponds to the stabilizer of $\operatorname{flag} \Omega$ in $\operatorname{Aut}(\operatorname{Flag} \mathcal{G})$.

Theorem 1.2.15. [27, Theorem 4.8]. *Let Ω be a closed subset of $\mathcal{V}(\mathcal{S})$. Then one of the following holds:*

(a) *for each $X \in \Omega$ there is $Y \in \Omega$ which is a complement of X in $\mathcal{S}$;*

(b) *the group $\operatorname{Stab}_{\operatorname{Aut}(\operatorname{Flag}(\mathcal{G}(\mathcal{S})))}(\operatorname{flag}(\Omega))$ fixes a non-trivial element in $\operatorname{flag}(\Omega)$.*

The final result required to prove the centre conjecture for buildings of type A_n is the following identification theorem.

Theorem 1.2.16. [27, Theorem 4.9]. *Let $\mathcal{S} = (P, L)$ be a projective space of finite rank*

n and let $\mathcal{G} := \mathcal{G}(\mathcal{S}) = (\mathcal{V}(\mathcal{S}), \star)$ be the associated incidence structure. Then we have the following:

(a) $\mathrm{Flag}(\mathcal{G})$ *is a building of type A_n.*

(b) *For $X, Y \in \mathcal{V}(\mathcal{S})$, the flags $\{X\}$ and $\{Y\}$ are opposite in* $\mathrm{Flag}(\mathcal{G})$ *if and only if Y is a complement of X in $\mathcal{S}$.*

(c) *Let Ω be a subset of $\mathcal{V}(\mathcal{S})$. Then* $\mathrm{flag}(\Omega)$ *is a convex subcomplex of* $\mathrm{Flag}(\mathcal{G})$ *if and only if Ω is closed.*

Conversely, if $\mathbf{\Delta} = (\Delta, \subset)$ is a building of type A_n, then there exists a projective space $\mathcal{S}'$ of rank n such that $\mathbf{\Delta}$ is isomorphic to $\mathrm{Flag}(\mathcal{G}(\mathcal{S}'))$.

To prove the centre conjecture for buildings of type A_n, the authors work with an isomorphic flag complex of a projective space $\mathcal{S}$. They describe a closed subset Ω of subspaces of $\mathcal{S}$ such that if each element of Ω has an $\mathcal{S}$ complement in Ω, each vertex in $\tilde{\Delta}$ has an opposite in $\tilde{\Delta}$; the complete reducibility of $\tilde{\Delta}$ follows by Proposition 1.2.8, and assertion (a) of the Centre Conjecture holds. If there is an element of Ω with no such complement, Theorem 1.2.15 guarantees the existence of a non-trivial element of $\mathrm{flag}\,\Omega$ fixed by certain stabilizers. This fixed flag corresponds to a nontrivial simplex in $\tilde{\Delta}$ satisfying assertion (b) of the Centre Conjecture.

1.2.2 The Strict Centre Conjecture

There is a version of the centre conjecture for convex subcomplexes that is stricter than the statement given in [27]. In applications of the centre conjecture, it is useful to have an extra condition on the nontrivial simplex described in the second assertion.

Theorem 1.2.17 (The Strict Centre Conjecture)**.** *Let $\mathbf{\Delta}$ be a spherical building, and let $\tilde{\Delta}$ be a convex subcomplex of $\mathbf{\Delta}$. Then at least one of the following holds:*

(a) *$\tilde{\Delta}$ is completely reducible;*

(b) *there exists a nontrivial simplex σ of $\tilde{\Delta}$ which is fixed by all automorphisms of $\mathbf{\Delta}$ which stabilize $\tilde{\Delta}$.* Additionally, σ has no opposite in $\tilde{\Delta}$.

With a little extra work, the proofs in [27, §4] can be shown to prove this stronger version. The following result and its proof uses terminology introduced in the previous subsection. Let $\mathcal{S} = (P, L)$ be a projective space and let Ω be a closed subset of $\mathcal{V}(\mathcal{S})$. Recall that C_p (respectively C_h) denotes the set of all $X \in P_\Omega$ (respectively all $X \in H_\Omega$) for which there is no $Y \in \Omega$ which is a complement of X in $\mathcal{S}$. Set $\mathcal{C}_p := \langle X \mid X \in C_p \rangle$ and $\mathcal{C}_h := \bigcap_{X \in C_h} X$.

Lemma 1.2.18. *Suppose that $\mathcal{C}_h$ is a proper subspace of $\mathcal{S}$. There are no elements of Ω that are complementary to $\mathcal{C}_h$ in $\mathcal{S}$.*

Proof. Since we are in a setting with finite rank, $\mathcal{C}_h$ is generated by a finite number of elements of C_h. We proceed by induction on the minimal number of elements of C_h required to generate $\mathcal{C}_h$. If $C_h = \{X\}$ is a singleton set, assuming that $\mathcal{C}_h = X$ has an $\mathcal{S}$-complement contained in Ω immediately leads to a contradiction; elements of C_h cannot have such complements.

Now let $W \in C_h$ and let $X \in \Omega$ be a subspace with no $\mathcal{S}$ complement in Ω. Suppose (for a contradiction) that there exists a subspace $Y \in \Omega$ that is complementary to $W \cap X$ in $\mathcal{S}$. Set $Y' = Y \cap X$. Note that $\langle X, Y \rangle = P$; since X has no $\mathcal{S}$ complement in Ω, we must have that $X \cap Y \neq \emptyset$. Then since Ω is a closed set containing X and Y, we have that $X \cap Y = Y' \in \Omega$. We know that $(W \cap X) \cap Y = \emptyset$, so the intersection $W \cap Y'$ must also be empty. Consider the subspace $\langle W, Y' \rangle$. If $\langle W, Y' \rangle = P$, then we would immediately have a contradiction, since Y' would be a complement to W in $\mathcal{S}$, and W has no such complement since it is an element of C_h. Assume that $\langle W, Y' \rangle \neq P$; then $\langle W, Y' \rangle \in \Omega$ since Ω is closed. But W is maximal in Ω and must be contained in $\langle W, Y' \rangle$, so $\langle W, Y' \rangle = W$. Hence the subspace Y' is contained in W, so our earlier result $W \cap Y' = \emptyset$ implies that $Y' = Y \cap X = \emptyset$. Since $\langle X, Y \rangle = P$, the subspace $Y \in \Omega$ is a complement to X in $\mathcal{S}$. This contradicts the assumption that X has no such complement. Hence there is no subspace in Ω that is complementary to $W \cap X$ in $\mathcal{S}$.

We have shown that intersecting an element of C_h with a subspace with no $\mathcal{S}$ complement in Ω results in another subspace with no $\mathcal{S}$ complement in Ω. Armed with this inductive step and the singleton set base case, we have proved that $\mathcal{C}_h = \bigcap_{X \in C_h} X$ has no $\mathcal{S}$ complement contained in Ω. $\square$

A similar series of arguments could be used to prove that there are no elements of Ω that are complementary to $\mathcal{C}_p$ in $\mathcal{S}$. Hence the flag $\mathcal{C}_p \subset \mathcal{C}_h$ and the corresponding nontrivial simplex used in the proof of the centre conjecture have no opposite.

Chapter 2

Complete reducibility

We begin this chapter by introducing the notion of G-complete reducibility, which generalizes the concept of complete reducibility from representation theory, and was first introduced by Jean-Pierre Serre [33], [34]. It was developed to extend standard results from the representation theory of algebraic groups by replacing representations $H \to \mathrm{GL}(V)$ with homomorphisms from H to an arbitrary reductive algebraic group G. Serre's definition was extended to non-connected groups by Bate, Martin and Röhrle [5] using a geometric approach inspired by Richardson, who had characterized the closed G-orbits in G^n in terms of the subgroup structure of G [30]. This approach proved useful and resulted in further joint work: see [6], [7], [9], and [3].

We then discuss the notion of relative G-complete reducibility, which was first introduced in [10]. This generalizes the study of closed G-orbits to the study of closed K-orbits for a reductive subgroup $K \leq G$. The final section of this chapter focuses on relative $\mathrm{GL}(V)$-complete reducibility and includes one of the main results of this thesis, Theorem 2.4.1. With the exception of subsection 2.4, none of the material presented in this chapter is original work.

2.1 Definitions and results

Definition 2.1.1. Let G be a connected reductive linear algebraic group over an algebraically closed field k. A closed subgroup H of G is said to be *G-completely reducible* (G-cr) provided that whenever H is contained in a parabolic subgroup P of G, it is contained in a Levi subgroup L of P.

We will require the following notion of the limit of a morphism, and a version of the Hilbert-Mumford theorem.

Definition 2.1.2. Let $\phi : \mathbb{G}_m \to X$ be a morphism of algebraic varieties. We say that $\lim\limits_{t\to 0} \phi(t)$ exists if there exists a morphism $\hat{\phi} : \mathbb{A}^1 \to X$ whose restriction to $\mathbb{G}_m$ is ϕ. If this limit exists, we set $\lim\limits_{t\to 0} \phi(t) = \hat{\phi}(0)$. If the morphism $\hat{\phi}$ exists it is unique, because k^* is dense in k.

We are primarily interested in when $\lim\limits_{t\to 0} \lambda(t) \cdot x$ exists for cocharacters λ. In these cases we may use the shorthand $\lim_\lambda x := \lim_{t\to 0} \lambda(t)\cdot x$.

Theorem 2.1.3 (Hilbert-Mumford Theorem)**.** *Let G be a reductive group acting on an affine variety X, and let $x \in X$. If $G\cdot x$ is not closed in X, then there exists a $\lambda \in Y(G)$ such that* $\lim_\lambda x$ *exists and does not belong to $G\cdot x$.*

A stronger version of the above due to Kempf is used in [5]; the original statement can be found in [22]. For fields which are not algebraically closed, a rational version of Theorem 2.1.3 is detailed in [4], which we will review in Chapter 5.

Note that when $G = \mathrm{GL}(V)$ for a finite dimensional k-vector space V, a subgroup H is G-completely reducible if and only if V is a completely reducible H-module. Recall that the Levi subgroups introduced in Example 1.1.20 took a block-diagonal form. For a closed subgroup $H \leq G$, we denote the centralizer and normalizer of H in G by $C_G(H)$ and $N_G(H)$, respectively. We will make use of the following characterization of parabolic subgroups and Levi subgroups of connected groups, see [30] and [36].

Lemma 2.1.4. [5, Lemma 2.4]. *Given a parabolic subgroup $P \leq G$ and any Levi subgroup $L \leq P$, there exists $\lambda \in Y(G)$ such that the following hold:*

(i) $P = P_\lambda := \{g \in G \mid \lim_{t\to 0} \lambda(t)g\lambda(t)^{-1} \textit{ exists } \}$.

(ii) $L = L_\lambda := C_G(\lambda(k^*))$.

(iii) *the map $c_\lambda : P_\lambda \to L_\lambda$ defined by*

$$c_\lambda(g) := \lim_{t\to 0} \lambda(t)g\lambda(t)^{-1}$$

is a surjective homomorphism of algebraic groups. Moreover, L_λ is the set of fixed points of c_λ and the kernel of c_λ is $R_u(P_\lambda)$.

Conversely, given any $\lambda \in Y(G)$, the subset P_λ defined as in part (i) is a parabolic subgroup of G, L_λ is a Levi subgroup of P_λ, and the map c_λ as defined in part (iii) has the described properties. Moreover, P_λ is a proper subgroup if and only if $\lambda(k^*) \subsetneq Z(G)$.

The following definition was introduced by Richardson; note that it does not depend on the choice of maximal torus S.

Definition 2.1.5. [30, Definition 16.1]. Let H be a closed subgroup of G and let S be a maximal torus of $C_G(H)$. Then H is said to be *strongly reductive in G* provided that H is not contained in any proper parabolic of $C_G(S)$.

Another characterization of strong reductivity can be obtained using Theorem 2.1.3 and the map c_λ from Lemma 2.1.4.

Lemma 2.1.6. [5, Lemma 2.17]. *Let H be a closed subgroup of G. Then H is strongly reductive in G if and only if for every cocharacter $\lambda \in Y(G)$ with $H \subseteq P_\lambda$, there exists $g \in G$ such that $c_\lambda(h) = ghg^{-1}$ for every $h \in H$.*

Richardson proved a connection between strong reductivity and topologically finitely generated subgroups.

Definition 2.1.7. A subgroup $H \leq G$ is said to be *topologically finitely generated* by elements $h_1, \dots, h_n \in G$ if H is the Zariski closure of the subgroup of G generated by those elements.

Remark 2.1.8. Not every G-completely reducible subgroup is topologically finitely generated, but Remark 2.9 and Lemma 2.10 of [5] justify restricting our attention to this case.

Theorem 2.1.9. [30, Theorem 16.4]. *If $H \leq G$ is topologically finitely generated by $h_1, \dots, h_n$, then H is strongly reductive in G if and only if the G-orbit of $(h_1, \dots, h_n)$ under the diagonal action of G on G^n by simultaneous conjugation is closed.*

When $n = 1$ this is the characterization of semisimple elements of G [35]. The following result shows that strong reductivity and complete reducibility are equivalent for subgroups of $\mathrm{GL}(V)$.

Lemma 2.1.10. [30, Lemma 16.2]. *Let H be a closed subgroup of* $\mathrm{GL}(V)$*. Then H is strongly reductive in* $\mathrm{GL}(V)$ *if and only if V is a semisimple H-module.*

The main result of [5] states that we can replace $\mathrm{GL}(V)$ in the above with an arbitrary reductive group G.

Theorem 2.1.11. [5, Theorem 3.1]. *Let H be a closed subgroup of a reductive group G. Then H is strongly reductive in G if and only if H is G-completely reducible.*

To understand an important corollary of the above, we require the following definition.

Definition 2.1.12. Let H be a subgroup which is not contained in any proper parabolic subgroup of G. Then H is trivially G-completely reducible and we say that H is *G-irreducible* (*G-ir*).

When $G = \mathrm{GL}(V)$, a subgroup H is G-irreducible if and only if V is an irreducible H-module. The following corollary reduces the study of G-cr subgroups of G to the study of L-ir subgroups of Levi subgroups L of G.

Corollary 2.1.13. [5, Corollary 3.5]. *Let H be a closed subgroup of G. Then the following are equivalent:*

(i) *H is strongly reductive in G;*

(ii) *H is G-completely reducible;*

(iii) *H is $C_G(S)$-irreducible, where S is a maximal torus of $C_G(H)$;*

(iv) *for every parabolic subgroup P of G which is minimal with respect to containing H, the subgroup is L-irreducible for some Levi subgroup L of P;*

(v) *there exists a parabolic subgroup P of G which is minimal with respect to containing H, such that H is L-irreducible for some Levi subgroup L of P.*

We are now able to apply results on strong reductivity and geometric invariant theory to study G-complete reducibility. The following is a useful corollary of Theorem 2.1.9

and Theorem 2.1.11.

Corollary 2.1.14. [5, Corollary 3.7]. *Let $x_1, \ldots, x_n \in G$ (for some $n \in \mathbb{N}$) and let H be the subgroup of G topologically generated by $x_1, \ldots, x_n$. Then H is G-completely reducible if and only if the orbit of $(x_1, \ldots, x_n)$ under the diagonal action of G on G^n by simultaneous conjugation is closed.*

This corollary provides us with the following result.

Proposition 2.1.15. [5, Proposition 3.12]. *Let H be a G-completely reducible subgroup of G. Then $C_G(H)^0$ is reductive. Moreover, let K be a closed subgroup of G satisfying $H^0 C_G(H)^0 \subseteq K \subseteq N_G(H)$. Then K^0 is reductive. In particular, $N_G(H)^0$ is reductive.*

The following collection of results provides two sets of criteria for a subgroup to be G-completely reducible, and two immediate consequences for normalizers and stabilizers.

Theorem 2.1.16. [5, Theorems 3.10 and 3.14, Corollaries 3.16 and 3.17].

Let H be a closed subgroup of G.

(i) *Let N be a closed normal subgroup of H. If H is G-completely reducible, then so is N. In particular, H^0 is G-completely reducible if H is.*

(ii) *Let H be G-completely reducible and suppose K is a closed subgroup of G satisfying $HC_G(H)^0 \subseteq K \subseteq N_G(H)$. Then K is G-completely reducible.*

(iii) *The subgroup H is G-completely reducible if and only if $N_G(H)$ is.*

(iv) *If H is G-completely reducible, then so is $C_G(H)$.*

Remarks 2.1.17.

(1) We can view (i) as a consequence of Tits' Center Conjecture [34, Proposition 2.11], and it answers a question of Serre posed in [33].

(2) Serre proves a partial converse to (i) with a characteristic restriction in [33]. Examples show that this restriction is necessary. Another partial converse is given by (ii) and further partial converses are discussed in [6].

(3) For $G = \mathrm{GL}(V)$, (i) is an instance of Clifford Theory and (iii) and (iv) are consequences of Clifford Theory and Wedderburn's Theorem.

(4) In general, the converse to (iv) is false; it does not hold for Borel subgroups. A partial converse is given in Corollary 3.18 of [5].

Further results of [5] provide connections between G-complete reducibility and regular subgroups, separability, and reductive pairs. We omit a discussion of these topics, and delay our discussion of G-complete reducibility over arbitrary fields until Chapter 5. We instead provide the extensions of definitions and results to non-connected G. We now only suppose that G is a linear algebraic group with G^0 reductive. We require analogues to earlier results, and will use terminology introduced in Lemma 2.1.4:

Definition 2.1.18. For a cocharacter $\lambda \in Y(G)$, we define the following subgroups:

$$P_\lambda := \{g \in G \mid \lim_{t \to 0} \lambda(t) g \lambda(t)^{-1} \text{ exists}\},$$
$$L_\lambda := C_G(\lambda(k^*)).$$

We call P_λ a *Richardson parabolic* (or *R-parabolic*) subgroup of G, and we call L_λ a *Richardson Levi* (or *R-Levi*) subgroup of P_λ. To justify calling such L_λ subgroups R-Levi subgroups, observe that $P_\lambda = C_G(\lambda(k^*)) \ltimes R_u(P_\lambda)$. For the remainder of this thesis we will use 'an R-Levi subgroup of G', to mean an R-Levi subgroup of some R-parabolic subgroup of G.

We saw that when G is connected, the R-parabolic and R-Levi subgroups are the parabolic and Levi subgroups. Any R-parabolic subgroup P of G is a parabolic subgroup of G [36, Lemma 6.2.4], but the converse is false, see [26, Remark 5.3]. We say that two R-parabolics of G are *opposite* if their intersection is an R-Levi subgroup of G. For non-connected groups, G-complete reducibility is defined as in the connected case, replacing parabolic and Levi subgroups with R-parabolic and R-Levi subgroups. The notions of G-irreducibility and strong reductivity can be extended similarly.

Definition 2.1.19. Let G be a reductive linear algebraic group over an algebraically closed field k. A subgroup H of G is said to be *G-completely reducible* (G-cr) provided that whenever H is contained in an R-parabolic subgroup P_λ for some $\lambda \in Y(G)$, there exists $\mu \in Y(G)$ such that $P_\lambda = P_\mu$ and H is contained in the R-Levi subgroup L_μ.

Alternatively, H is G-completely reducible if whenever H is contained in an R-parabolic subgroup P_λ for some $\lambda \in Y(G)$, there exists $u \in R_u(P_\lambda)$ such that $H \subseteq uL_\lambda u^{-1}$. This follows since all R-Levi subgroups of an R-parabolic subgroup P are $R_u(P)$-conjugate [5, Corollary 6.7]. The following results will be useful in later chapters.

Corollary 2.1.20. [5, Corollary 6.5]. *Let P be an R-parabolic subgroup of G and let T be a maximal torus of P. Then there exists $\lambda \in Y(T)$ such that $P = P_\lambda$. Moreover, L_λ is the unique R-Levi subgroup of P that contains T.*

Lemma 2.1.21. [5, Lemma 6.11]. *Let P be an R-parabolic subgroup of G with an R-Levi subgroup L. Then there exists a unique R-parabolic subgroup P^- of G such that $P \cap P^- = L$.*

Remark 2.1.22. These can be combined to show that for a given maximal torus T of an R-parabolic subgroup P, there exists a unique R-parabolic subgroup P^- such that $P \cap P^-$ is the unique R-Levi subgroup of P containing T.

2.2 Relative complete reducibility

We now extend our study of G-orbits to the study of K-orbits, where K is a reductive subgroup of G.

Definition 2.2.1. Let H and K be subgroups of a reductive algebraic group G, with K reductive. We say that H is *relatively G-completely reducible* (*relatively G-cr*) with respect to K if for every $\lambda \in Y(K)$ such that H is contained in P_λ, there exists $\mu \in Y(K)$ such that $P_\lambda = P_\mu$ and $H \subseteq L_\mu$.

Remarks 2.2.2.

(i) If $K = G$ this is the definition of G-complete reducibility.

(ii) A subgroup H is relatively G-cr with respect to K if and only if it is relatively G-cr with respect to K^0; we may assume without loss that K is connected. Every subgroup H is relatively G-cr with respect to K if K^0 is central in G [10, Remarks 3.2(i)].

(iii) If $H \subseteq K$ then H is relatively G-cr with respect to K if and only if H is K-cr. [10, Lemma 3.3(ii)]

(iv) In characteristic zero, $H \leq G$ is G-completely reducible if and only if H is reductive. There is no known analogous characterization of relative G-complete reducibility in this case [10, Remarks 3.2(iv)].

(v) The property that a G-cr subgroup is reductive [34, Prop 4.1] is not inherited by relatively G-cr subgroups. If $K \subseteq Z(G)$, all subgroups of G are relatively G-cr with respect to K; there may be non-reductive subgroups which are relatively G-cr with respect to K but not G-cr.

(vi) A G-cr subgroup is not necessarily relatively G-cr. Two examples are provided in [10, Remarks 3.2(v)]

We will sometimes need to consider parabolic and Levi subgroups of the reductive subgroup K. We denote these by $P_\lambda(K)$ and $L_\lambda(K)$, respectively. The main result of [10] is the following theorem:

Theorem 2.2.3. [10, Theorem 1.1]. *Let K be a reductive subgroup of G. Let H be the algebraic subgroup of G generated by elements $x_1, \ldots, x_n \in G$. Then $K \cdot (x_1, \ldots, x_n)$ is closed in G^n if and only if H is relatively G-completely reducible with respect to K.*

Due to the equivalence of strong reductivity and G-complete reducibility detailed in Theorem 2.1.11, the above can be viewed as a generalization of Theorem 2.1.9, which is the special case where $K = G$. This algebraic interpretation of closed orbits is somewhat surprising, but follows naturally after extending results from the study of G-complete reducibility to a relative setting. The following provides relative analogues of [5, Corollary 3.22] and Corollary 2.1.13.

Proposition 2.2.4. [10, Proposition 3.17] *Let H and K be subgroups of G and suppose K is reductive.*

(i) *Let S be a torus of $C_K(H)$ and let $L = C_G(S)$. Then H is relatively G-completely reducible with respect to K if and only if H is relatively L-completely reducible with respect to $K \cap L$.*

(ii) *The R-Levi subgroups L_μ of G for $\mu \in Y(K)$ that are minimal with respect to containing H are precisely the subgroups of the form $C_G(S)$ where S is a maximal* *torus of $C_K(H)$. If L is such an R-Levi subgroup of G, then H is relatively G-completely reducible with respect to K if and only if H is relatively L-irreducible with respect to $K \cap L$.*

There is a relative analogue of the first statement of Proposition 2.1.15.

Corollary 2.2.5. [10, Corollary 3.7]. *Let K be a reductive subgroup of G, and let H be a subgroup of G which is relatively G-completely reducible with respect to K. Then $C_K(H)$ is reductive.*

The following result generalizes the second statement of [10, Proposition 3.19] and can be applied to generalize Theorem 2.1.16(iii); recall that $N_G(H)$ and $HC_G(H)$ are automatically reductive if H is a G-completely reducible subgroup of G by Proposition 2.1.15.

Proposition 2.2.6. [10, Proposition 3.26]. *Suppose K is a reductive subgroup of G. Let H be a subgroup of G which is relatively G-completely reducible with respect to K, and suppose M is a reductive subgroup of G which contains H and is normalized by a maximal torus of $C_K(H)$. Then M is also relatively G-completely reducible with respect to K.*

Relative complete reducibility is an active area of research; we refer the reader to the recent papers [15] and [16].

2.3 Reducing to $G = \mathrm{GL}(V)$

Recall that in the case where $G = \mathrm{GL}(V)$ for a finite dimensional k-vector space V, a subgroup H is G-completely reducible if and only if V is a completely reducible H-module. This leads to a characterization of G-complete reducibility in terms of flags of subspaces of V; to provide the statement, we require some additional terminology. Recall that in Example 1.1.14 we worked with a Borel subgroup and a flag with subspaces of V of each intermediate dimension; flags of this type are called *complete* flags. If a flag is not complete we say that it is a *partial* flag. There is an obvious action of $\mathrm{GL}(V)$ on the set of all flags, and every Borel subgroup of $\mathrm{GL}(V)$ is the stabilizer of some complete flag. The parabolic subgroups of $\mathrm{GL}(V)$ are stabilizers of flags of subspaces in V. We say that two flags $A = A_1 \subset \cdots \subset A_n$ and $B = B_1 \subset \cdots \subset B_n$ are *opposite* or *complementary* if $A_i \oplus B_{n-i+1} = V$ for all i.

Proposition 2.3.1. *Let H be a subgroup of $G = \mathrm{GL}(V)$. Then H is G-completely reducible if and only if every H-stable flag of subspaces of V admits an H-stable complement.*

A similar characterization exists for relative $\mathrm{GL}(V)$-complete reducibility. For a reductive subgroup K of $\mathrm{GL}(V)$, write $\mathcal{P}_K$ for the set of parabolic subgroups $P_\lambda \leq \mathrm{GL}(V)$ with $\lambda \in Y(K)$. Let $\mathcal{F}_K$ denote the set of flags in V which correspond to the parabolic subgroups in $\mathcal{P}_K$ and let $\mathcal{F}_K^H$ denote the subset of flags in $\mathcal{F}_K$ stabilized by H.

Proposition 2.3.2. *Let H and K be subgroups of $G = GL(V)$, with K reductive. Then H is relatively G-completely reducible with respect to K if and only if each flag in $\mathcal{F}_K^H$*

has an opposite in $\mathcal{F}_K^H$.

This is a convenient characterization to work with and we will use it repeatedly throughout this thesis.

Example 2.3.3. Let $K = \mathrm{diag}(a, a, b) \leq G = \mathrm{GL}(V)$ where V is a 3-dimensional vector space with a natural basis e_1, e_2, e_3. Then we have that $\mathcal{F}_K = \{\langle e_1, e_2\rangle \subset V, \langle e_3\rangle \subset V\}$. See Example 3.2.3 for an extended example describing how the elements of $\mathcal{F}_K$ are determined. Consider the following subgroups of G:

$$H_1 = \begin{pmatrix} * & 0 & 0 \\ 0 & * & * \\ 0 & * & * \end{pmatrix}, \quad H_2 = \begin{pmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & * \end{pmatrix}, \quad H_3 = \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix}.$$

Since $\mathcal{F}_K^{H_1} = \{\emptyset\}$, H_1 is trivially relatively G-cr with respect to K. Since $\mathcal{F}_K^{H_2} = \mathcal{F}_K$ consists of two complementary flags, H_2 is relatively G-cr with respect to K. Finally, H_3 is *not* relatively G-cr with respect to K as $\mathcal{F}_K^{H_3}$ consists of the single flag $\langle e_1, e_2\rangle \subset V$.

Since every G embeds in some $\mathrm{GL}(V)$, the following result can be used to show that questions about relative G-complete reducibility can be reduced to questions about relative $\mathrm{GL}(V)$-complete reducibility.

Corollary 2.3.4. [10, Corollary 3.6]. *Let M be a reductive subgroup of G. Let H and K be subgroups of M and assume that K is reductive. Then H is relatively G-completely reducible with respect to K if and only if it is relatively M-completely reducible with respect to K.*

The extra combinatorial structure on flags makes it a lot easier to work in this setting, which motivated this thesis.

2.3.1 Extended example: an analogue for closed subsets

In this subsection we give an example of this set-up, mirroring [10, §5.1]. This example shows that in some special situations we can mimic constructions of [27] in this relative setting. In particular, there is a natural analogue of the closed subsets described in Definition 1.2.10.

Let V be a finite-dimensional k-vector space and set $G = \mathrm{GL}(V)$. Let U be a subspace of V, and pick a direct complement $\tilde{U}$. Let $K = \mathrm{GL}(U) \subseteq G$ be embedded in the obvious way, let H be a subgroup of G. For any pair A, B of H-submodules of V, let $\langle A, B\rangle$ denote the smallest H-submodule of V containing both A and B.

Definition 2.3.5. Call a subset Ω of the set of all H-submodules of V *closed* if the following hold for all $A, B \in \Omega$:

(i) $\langle A, B\rangle \neq V$ implies $\langle A, B\rangle \in \Omega$.

(ii) $A \cap B \neq \emptyset$ implies $A \cap B \in \Omega$.

For any subspace $W \subseteq V$, let $\sigma_H(W)$ be the H-submodule of V generated by the

$H-$submodules contained in W, and let $\iota_H(W)$ denote the smallest H-submodule of V containing W. Let V_U be the set of all proper nontrivial H-submodules of V contained in U, and let $V^{\tilde{U}}$ be the set of all proper nontrivial H-submodules of V containing $\tilde{U}$.

Lemma 2.3.6. *V_U is closed.*

Proof. Suppose $A, B \in V_U$. Since A and B are both contained in U, $\sigma_H(U)$ is a H-submodule containing A and B. Then $\langle A, B\rangle \subseteq \sigma_H(U) \subseteq U$, so $\langle A, B\rangle$ is a H-submodule contained in U. Both A and B are nontrivial, so $\langle A, B\rangle$ is a nontrivial H-submodule of V. Suppose now that $\langle A, B\rangle \neq V$. Then $\langle A, B\rangle$ is also a proper H-submodule of V, and so $\langle A, B\rangle \in V_U$.

Note that $A \cap B \subseteq A \subseteq U$. This guarantees that $A \cap B$ is contained in U, and that $A \cap B$ is a proper H-submodule, since A is. The only criterion for belonging to V_U that is not automatically satisfied is being nontrivial; if we assume that $A \cap B \neq \emptyset$, then we can conclude that $A \cap B$ belongs to V_U. $\square$

Lemma 2.3.7. *$V^{\tilde{U}}$ is closed.*

Proof. Suppose $A, B \in V^{\tilde{U}}$. We have that $\tilde{U} \subseteq A \subseteq \langle A, B\rangle$. As in the V_U case above, $\langle A, B\rangle$ is nontrivial since A and B are. If we suppose that $\langle A, B\rangle \neq V$, the H-submodule $\langle A, B\rangle$ will satisfy all the conditions for belonging to $V^{\tilde{U}}$.

Since $\tilde{U}$ is contained in both A and B, we have that $\tilde{U} \subseteq A \cap B$. Again, $A \cap B$ must be a proper H-submodule because A is. Supposing that $A \cap B \neq \emptyset$ is all that remains to guarantee that $A \cap B$ belongs to $V^{\tilde{U}}$. Hence, $V^{\tilde{U}}$ is closed. $\square$

With an extra assumption about relative G-complete reducibility, these closed sets lead to a pair of flag complexes with interesting complementary structures. Suppose now that H is relatively G-completely reducible with respect to K. By [10, Proposition 5.1], the following two conditions hold:

(1) every H-submodule of V contained in U has an H-complement in V containing $\tilde{U}$;

(2) every H-submodule of V containing $\tilde{U}$ has an H-complement in V contained in U.

The conditions above can be refined; [10, Proposition 5.3] states that H is relatively G-completely reducible with respect to K if and only if the following two conditions hold:

(3) $\sigma_H(U)$ is a completely reducible H-module;

(4) $V = \sigma_H(U) \oplus \iota_H(\tilde{U})$.

Consider a maximal flag of V_U given by

$$f = U_0 \subset U_1 \subset \cdots \subset \sigma_H(U),$$

where $U_i \in V_U$, and U_0 is minimal in U. Using the conditions above, we can find a maximal flag in $V^{\tilde{U}}$ that is complementary to f in V:

$$\tilde{f} = \iota_H(\tilde{U}) \subset \cdots \subset \tilde{U_1} \subset \tilde{U_0},$$

where $\tilde{U_i}$ denotes the H-complement to U_i that contains $\tilde{U}$. Observations like these form the basis of our attempts to mirror the techniques and results of Mühlherr and Tits in a relative general linear setting.

2.4 Main reduction theorem

We now introduce the main result of this chapter in order to explain our focus on the setting of general linear groups. Given an arbitrary reductive group G acting on an affine space X, the action can be linearised by embedding X into a suitable general linear group. Consequently, questions about reductive groups acting on arbitrary affine varieties can be translated to the setting of relative $\mathrm{GL}(V)$-complete reducibility. We aim to use the extra structure provided in this general linear setting to make some of these questions easier to answer.

Theorem 2.4.1. *Let G be a reductive group acting on an affine variety X. Then there exists a vector space W, an embedding $\phi : X \hookrightarrow \mathrm{GL}(W)$, and a homomorphism $\rho : G \to \mathrm{GL}(W)$ such that the following diagram commutes:*

$$\begin{tikzcd}
G \times X \arrow[rr, "g \cdot x"] \arrow[d, "(\rho(g), \phi(x))"] & & X \arrow[d, "\phi(g \cdot x)"] \\
\mathrm{GL}(W) \times \mathrm{GL}(W) \arrow[rr, "\rho(g)\phi(x)\rho(g)^{-1}"'] & & \mathrm{GL}(W)
\end{tikzcd}$$

Moreover, given any $x \in X$, the G-orbit of x is closed in X if and only if the subgroup generated by $\phi(x)$ is relatively $\mathrm{GL}(W)$-cr with respect to G.

Proof. We begin by showing that we can embed X in a vector space. To find a morphism from X into a vector space V, it will suffice to describe a surjective homomorphism from the coordinate ring of an affine m-space onto $k[X]$; it is straightforward to check that a surjective comorphism $\phi^\sharp : k[Y] \to k[X]$ corresponds to an injective morphism $\phi : X \to Y$ whose image is the closed subset of Y corresponding to the ideal $\mathrm{Ker}(\phi^\sharp) \subset k[Y]$.

Let E be a finite-dimensional G-stable subspace containing the generators of $k[X]$ - the existence of such a subspace is guaranteed by [36, Proposition 2.3.6]. Let $e_1, \ldots, e_m$ be a basis for E. We can define a k-algebra homomorphism from the coordinate ring of an affine m-space $k[Y_1, \ldots, Y_m]$ to $k[X]$ by mapping $Y_i \to e_i$ for all i. Describing a map of the generators Y_i in this way uniquely defines a k-algebra homomorphism

$k[Y_1,\ldots,Y_m] \to k[X]$. Each generator of $k[X]$ lies in E, so each generator is a linear combination of the e_i and therefore lies in the image of this homomorphism. Hence the homomorphism is surjective. Thus X embeds as a closed subvariety of the affine m-space V with coordinate ring $k[V] = k[Y_1,\ldots,Y_m]$.

We need to ensure that everything is compatible with the G-action. Since G acts linearly on $k[X]$ and E was chosen to be a G-stable subspace, G acts linearly on E. The action on E gives an action on $k[Y_1,\ldots,Y_m]$: for each i, and each $g \in G$, we know that

$$g \cdot e_j = \sum_{i=1}^{m} a_{ij}(g) e_i,$$

for some scalars $a_{ij}(g)$. We can define an action of G on $k[Y_1,\ldots,Y_m]$ by setting

$$g \cdot Y_j = \sum_{i=1}^{m} a_{ij}(g) Y_i,$$

for each j. With this description of how G acts on each generator, the action can be extended to the entire polynomial ring. From this action of G on the coordinate ring of V, we obtain an action of G on V. Given $f \in k[V]$, set $f(g \cdot v) = (g^{-1} \cdot f)v$.

At this point, we can replace G with its image in $\mathrm{GL}(V)$, and replace X with V. Now we set $W = k^{m+1}$ and define a homomorphism $\rho : G \to \mathrm{GL}(W)$ and an embedding $\psi : V \to \mathrm{GL}(W)$:

$$\rho(g) = \left(\begin{array}{c|c} A & 0 \\ \hline 0 & 1 \end{array}\right), \quad \psi(v) = \left(\begin{array}{c|c} I_m & v \\ \hline 0 & 1 \end{array}\right),$$

where A is the image of g in $\mathrm{GL}(V)$. We can check that acting by conjugation of $\rho(g)$ on the embedded image of v is equal to the embedded image of $g \cdot v$:

$$\begin{aligned} \rho(g)\psi(v)\rho(g)^{-1} &= \left(\begin{array}{c|c} A & 0 \\ \hline 0 & 1 \end{array}\right)\left(\begin{array}{c|c} I_m & v \\ \hline 0 & 1 \end{array}\right)\left(\begin{array}{c|c} A^{-1} & 0 \\ \hline 0 & 1 \end{array}\right) = \left(\begin{array}{c|c} A & Av \\ \hline 0 & 1 \end{array}\right)\left(\begin{array}{c|c} A^{-1} & 0 \\ \hline 0 & 1 \end{array}\right) \\ &= \left(\begin{array}{c|c} I_m & Av \\ \hline 0 & 1 \end{array}\right) = \psi(g \cdot v) \end{aligned}$$

Overall we obtain a closed embedding of X in $\mathrm{GL}(W)$ which preserves the necessary geometric properties. To prove the second statement, note that the G-orbit of x in X is closed if and only if the orbit of its image $v \in V$ is closed, since the image of X is closed in V. In turn, the orbit of v is closed in V if and only if the G-orbit of $\psi(v)$ is closed in $\mathrm{GL}(W)$, since $\psi(V)$ a closed subgroup of $\mathrm{GL}(W)$. Finally, the G-orbit of $\psi(v)$ is closed in $\mathrm{GL}(W)$ if and only if $\langle \psi(v) \rangle$ is relatively $\mathrm{GL}(W)$-cr with respect to G by Theorem 2.2.3. $\square$

Remark 2.4.2. One way to see how we obtain an action of G on V is to recognise that $k[Y_1,\ldots,Y_m]$ can be identified with the symmetric algebra on the dual space V^*, $k[V] = S(V^*)$. This symmetric algebra is the direct sum of all the symmetric powers of the dual space; once we have defined a linear action on V^*, it automatically extends to the whole of $S(V^*) = k[V]$. In coordinates, the argument says that the elements

$Y_1, \ldots, Y_m$ are the coordinate functions on V for some choice of basis. Then they are a basis for the dual space V^*, and we know that any action of G on the dual space gives rise to an action on V.

We conclude this chapter with a discussion of how the results in this thesis relate to the aforementioned subset version of the Centre Conjecture. Recall that we defined convex subcomplexes as intersections of convex chamber subcomplexes. The subset Centre Conjecture does not just look at subsets of the simplicial complex; we consider subsets which are not unions of simplices. This means that some parts of a simplex can lie in a subset whilst others do not. Such subsets arise naturally in a geometric invariant theory setting. The following paragraph describes a basic example which illustrates this idea.

Let $G = \mathrm{GL}_n$ act on k^n as usual, and let $v = e_1$ be the first standard basis vector. The diagonal cocharacters λ for which $\lim_\lambda v$ exists are precisely those for which the first entry is a^n with $n \geq 0$. All of the parabolics containing the diagonal torus T arise from such cocharacters, but we do not obtain every possible cocharacter. Then the set of cocharacters can tell things apart which the set of parabolics on its own cannot. The cocharacters $\lambda(a) = \mathrm{diag}(a, 1, \ldots, 1)$ and $\mu(a) = \mathrm{diag}(a^{-1}, a^{-2}, \ldots, a^{-2})$ give the same parabolic in GL_n, but $\lim_\lambda v$ exists and $\lim_\mu v$ does not.

If one is interested in this invariant theory setting, these sets of cocharacters need to be studied. They have a convex structure: if $\lim_\lambda v$ and $\lim_\mu v$ exist, then so does $\lim_{a\lambda+b\mu} v$ for positive a, b. The link between cocharacters and parabolics allows us to see these subsets as subsets of the associated building, but it is still not clear that they have a nice combinatorial structure; in the previous paragraph we observed that the simplicial structure on the set of parabolic subgroups is not necessarily compatible with the set of cocharacters for which the limit exists. Our aim, then, is to find a combinatorial structure that we can work with. Theorem 2.4.1 says that it will be enough to study relative $\mathrm{GL}(V)$-complete reducibility, and then we will be working with the possible combinatorial structures which arise in this setting.

The remaining results in this thesis provide combinatorial ways of stating that a given orbit is closed. The hope is that they will eventually assist in an argument which states that if a given orbit is not closed, then the associated set of cocharacters has a "centre".

Chapter 3

Relative $\mathrm{GL}(V)$-complete reducibility

This chapter is primarily based on a paper which was written in collaboration with Michael Bate, Maike Gruchot, Alastair Litterick, and Gerhard Röhrle [1]. Several results in the paper are formulated for arbitrary (possibly non-connected) reductive linear algebraic groups. Motivated by the previous discussion of Corollary 2.3.4, we translate these results and their proofs to the setting of $G = \mathrm{GL}(V)$. In doing so, we promote a corollary from the paper to Theorem 3.2.7, the main result of this chapter, which provides a characterization of relative $\mathrm{GL}(V)$-complete reducibility in terms of certain flags of V. A consequence of our Theorem 2.4.1 is that the following results can be interpreted as results which apply in a more general "geometric invariant theory" setting.

3.1 A result for arbitrary reductive G

For a reductive subgroup $K \leq G$, let $\mathcal{P}_K$ denote the set of R-parabolic subgroups $P_\lambda \leq G$ with $\lambda \in Y(K)$ and recall that we say that two R-parabolics of G are *opposite* if their intersection is an R-Levi subgroup of G. The following theorem is the main result of [1].

Theorem 3.1.1. [1, Theorem 1.2]. *Let $K \leq G$ be reductive algebraic groups with G connected, and let H be a subgroup of G. Then the following are equivalent:*

(i) *H is relatively G-completely reducible with respect to K.*

(ii) *Each maximal member of $\mathcal{P}_K$ containing H has an opposite in $\mathcal{P}_K$ which is maximal and contains H.*

(iii) *There is an R-Levi subgroup L_μ with $\mu \in Y(K)$, such that $H \leq L_\mu$ and H is relatively L_μ-irreducible with respect to $K \cap L_\mu$.*

Remarks 3.1.2.

(1) For $K = G = \mathrm{GL}(V)$, the conditions (i)–(iii) specialize to familiar representation-theoretic notions.

(2) Proposition 1.2.8 was a restatement of [34, Théorème 2.2], which states that when $K = G$ is a connected reductive group, a subgroup is G-completely reducible if and only if it lies in a Levi subgroup of each maximal parabolic subgroup containing it. The equivalence of conditions (i) and (ii) can be viewed as a generalization of this result to our relative setting.

The implications (i) $\Leftrightarrow$ (iii) $\Rightarrow$ (ii) all hold without the assumption that G is connected. The proof of the missing implication makes use of the fact that every element of $\mathcal{P}_K$ can be expressed as an intersection of maximal elements of $\mathcal{P}_K$ if G is connected. Some non-connected groups share this property (and the conclusion of Theorem 3.1.1 holds in these instances) but the following example demonstrates that it is not guaranteed.

Example 3.1.3. [1, Example 2.3]. Let T be a 1-dimensional torus, let $\langle x \rangle$ be cyclic of order 8 and let $G = K = T^8 \rtimes \langle x \rangle$, with x permuting the factors of T^8 in the obvious manner. Each R-parabolic subgroup of G contains $T^8 = G^0$, and every subgroup $G^0 \leq P \leq G$ arises as an R-parabolic subgroup P_λ, depending on whether the 1-dimensional torus $\lambda(k^*)$ is centralized by x, x^2, x^4 or none of these. The group G has a unique maximal proper R-parabolic subgroup $T^8 \rtimes \langle x^2 \rangle$. This has a subgroup $T^8 \rtimes \langle x^4 \rangle$ which is not the intersection of the maximal R-parabolics of G containing it.

We can address the problem of $G = T^8 \rtimes \langle x \rangle$ not being connected by working in a suitable $\mathrm{GL}_n(k)$ instead.

Example 3.1.4. We consider a similar set-up to that of Example 3.1.3, with 8 copies of a 1-dimensional torus and a cyclic group of order 8, but based inside $\mathrm{GL}_8(k)$. Let g be the matrix with ones above the main diagonal and in the bottom left, and zeroes everywhere else:

$$g = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

This is a permutation matrix and generates a subgroup of order 8. Elements centralised by g will be of the form $\mathrm{diag}(a, a, a, a, a, a, a, a)$, elements centralised by g^2 will be of the form $\mathrm{diag}(a, b, a, b, a, b, a, b)$, and elements centralised by g^4 will be of the form $\mathrm{diag}(a, b, c, d, a, b, c, d)$. Consider the parabolic subgroup P corresponding to the cocharacter $\lambda : t \to \mathrm{diag}(t^4, t^3, t^2, t, t^4, t^3, t^2, t)$:

$$P = \begin{pmatrix} * & * & * & * & * & * & * & * \\ 0 & * & * & * & 0 & * & * & * \\ 0 & 0 & * & * & 0 & 0 & * & * \\ 0 & 0 & 0 & * & 0 & 0 & 0 & * \\ * & * & * & * & * & * & * & * \\ 0 & * & * & * & 0 & * & * & * \\ 0 & 0 & * & * & 0 & 0 & * & * \\ 0 & 0 & 0 & * & 0 & 0 & 0 & * \end{pmatrix}.$$

The maximal parabolics containing P are as follows:

$$\begin{pmatrix} * & * & * & * & * & * & * & * \\ 0 & * & * & * & 0 & * & * & * \\ 0 & * & * & * & 0 & * & * & * \\ 0 & * & * & * & 0 & * & * & * \\ * & * & * & * & * & * & * & * \\ 0 & * & * & * & 0 & * & * & * \\ 0 & * & * & * & 0 & * & * & * \\ 0 & * & * & * & 0 & * & * & * \end{pmatrix} \begin{pmatrix} * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * \\ 0 & 0 & * & * & 0 & 0 & * & * \\ 0 & 0 & * & * & 0 & 0 & * & * \\ * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * \\ 0 & 0 & * & * & 0 & 0 & * & * \\ 0 & 0 & * & * & 0 & 0 & * & * \end{pmatrix} \begin{pmatrix} * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * \\ 0 & 0 & 0 & * & 0 & 0 & 0 & * \\ * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * \\ 0 & 0 & 0 & * & 0 & 0 & 0 & * \end{pmatrix}$$

Observe that P can be expressed as the intersection of these maximal parabolics. By replacing the G of Example 3.1.3 with GL_8, we obtain a combinatorial structure which is rich enough to see these details. By restricting our attention to relative $\mathrm{GL}(V)$-reducibility, we remove some technicalities whilst *generalizing* the set-up because of our Theorem 2.4.1.

3.2 Translation to $G = \mathrm{GL}(V)$

In this section we outline our alternative formulation of Theorem 3.1.1 where $G = \mathrm{GL}(V)$. We will require some additional terminology, and we take this opportunity to introduce some conventions we use when working with flags.

Recall that a *flag* is a sequence of subspaces of a finite-dimensional vector space V arranged by inclusion:

$$\{0\} \subset V_1 \subset \cdots \subset V_m \subset V. \tag{3.2.1}$$

We usually omit the zero subspace when describing a flag, and may omit V when there is no ambiguity about the ambient vector space. The *length* of a flag is the number of proper nontrivial subspaces that it contains; this must be finite, since the inclusions between subspaces are proper. Let f be the arbitrary flag introduced in 3.2.1 and set $d_i = \dim V_i$. We define the *signature* of f, denoted $\mathrm{sig}(f)$, to be the vector $(d_1, \ldots, d_m, \dim(V))$. We can describe complete and partial flags in terms of length or signature: a flag in an n-dimensional space is complete if and only if it has length $n-1$, and flags of length $n-1$ must have signature $(1, 2, \ldots, n)$. We say that a subspace U of V is a *subspace of a flag* f if $U = V_i$ for some i.

Remark 3.2.1. We will usually write flags "from left to right" and consider the smallest nontrivial subspace to be the "start" of the flag. When indexing the largest subspace by 1 will aid the clarity of a proof, we may write flags "from right to left" by writing, say, $f' = U_1 \supset \cdots \supset U_m$.

Recall that $\mathcal{F}_K$ denotes the set of flags in V which correspond to the parabolic subgroups in $\mathcal{P}_K$, the set of parabolic subgroups P_λ with $\lambda \in Y(K)$. Given that the parabolic subgroups in $\mathrm{GL}(V)$ form a poset under inclusion, we can apply a partial order to this set of flags. For two flags f and f' in $\mathcal{F}_K$, we set $f \preceq f'$ when $\mathrm{Stab}_G(f) \supseteq \mathrm{Stab}_G(f')$ and say that f is a *subflag* of f'.

Definition 3.2.2. Let $f \in \mathcal{F}_K$ and suppose that f is minimal with respect to this partial order, i.e. $f' \preceq f$ for $f' \in \mathcal{F}_K$ implies that $f' = f$. We say that f is a *basic* flag and let $\mathcal{B}_K$ denote the subset of basic flags in $\mathcal{F}_K$.

We say that a flag is H*-stable* if H stabilizes each subspace in the flag; that is, a flag f is H-stable if $H \subseteq \mathrm{Stab}_G(f)$. Let $\mathcal{B}_K^H$ denote the subset of H-stable flags in $\mathcal{B}_K$. To illustrate these ideas we introduce the following example, which will be a running example throughout the next two chapters.

Example 3.2.3. Let $G = \mathrm{GL}_4(k)$ and let $K = \mathrm{diag}(s, t, t^{-1}, s^{-1})$ with $s, t \in k^*$. Elements of $Y(K)$ will all send elements $a \in k^*$ to matrices of the form

$$\begin{pmatrix} a^x & 0 & 0 & 0 \\ 0 & a^y & 0 & 0 \\ 0 & 0 & a^{-y} & 0 \\ 0 & 0 & 0 & a^{-x} \end{pmatrix},$$

for some $x, y \in \mathbb{Z}$. For a given $\lambda_{x,y}$, we wish to find the corresponding parabolic subgroup $P_{x,y} := \{g \in G \mid \lim_{a\to 0} \lambda_{x,y}(a) g \lambda_{x,y}(a)^{-1} \text{ exists}\}$. Let $g = (g_{ij})$ be an arbitrary element of $\mathrm{GL}_4(k)$. To find these parabolic subgroups, we study the matrices of the form

$$\lambda_{x,y}(a) g \lambda_{x,y}(a)^{-1} = \begin{pmatrix} a^0 g_{11} & a^{x-y} g_{12} & a^{x+y} g_{13} & a^{2x} g_{14} \\ a^{y-x} g_{21} & a^0 g_{22} & a^{2y} g_{23} & a^{x+y} g_{24} \\ a^{-x-y} g_{31} & a^{-2y} g_{32} & a^0 g_{33} & a^{x-y} g_{34} \\ a^{-2x} g_{41} & a^{-x-y} g_{42} & a^{y-x} g_{43} & a^0 g_{44} \end{pmatrix}.$$

For the relevant limits to exist, we cannot have negative powers of a appearing in these matrices. If a negative power of a appears in the (i, j)th position of this matrix, we will require that $g_{ij} = 0$. With this in mind, it is easier to study the array

$$A_{x,y} = \begin{pmatrix} 0 & x-y & x+y & 2x \\ y-x & 0 & 2y & x+y \\ -x-y & -2y & 0 & x-y \\ -2x & -x-y & y-x & 0 \end{pmatrix},$$

which has as its (i, j)th entry the power of a appearing in front of g_{ij} in the matrix $\lambda_{x,y}(a) g \lambda_{x,y}(a)^{-1}$. Observe that $A_{x,y}$ is antisymmetric along the main diagonal (a property shared by other examples when K is a diagonal torus) and symmetric along the antidiagonal (a consequence of the symmetry of our particular K). Wherever a negative entry appears in $A_{x,y}$, we know a zero must appear for all elements of $P_{x,y}$. We can look at $\lambda_{1,1}$ for a specific example:

$$A_{1,1} = \begin{pmatrix} 0 & 0 & 2 & 2 \\ 0 & 0 & 2 & 2 \\ -2 & -2 & 0 & 0 \\ -2 & -2 & 0 & 0 \end{pmatrix} \longrightarrow P_{1,1} = \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & * & * \end{pmatrix}.$$

In this way we associate the cocharacter $\lambda_{1,1} \in Y(K)$ with the flag $\langle e_1, e_2 \rangle \subset k^4$. The set $\mathcal{F}_K$ contains the following flags of length one:

$$\langle e_1, e_2 \rangle \subset k^4, \quad \langle e_1, e_3 \rangle \subset k^4, \quad \langle e_2, e_4 \rangle \subset k^4, \quad \langle e_3, e_4 \rangle \subset k^4.$$

For the remainder of this thesis it will be useful to have a shorthand notation for flags. Our examples will involve a canonical basis $e_1, \ldots, e_n$ of k^n for some $n \leq 8$, and the spans of the e_i will be introduced independently. To represent the flag

$$f = \langle e_{i_1} \rangle \subset \langle e_{i_1}, e_{i_2}, e_{i_3} \rangle \subset k^n,$$

where we assume without loss that $i_2 < i_3$, we will use the shorthand $(i_1, i_2 i_3)$. We generalize this so that flags of length n are represented by vectors of length n; using this notation, the aforementioned flags of length one in $\mathcal{F}_K$ are denoted by (12), (13), (24), and (34). This notation was chosen to highlight the order in which the basis spans appear in proper nontrivial subspaces, and to indicate when spans of basis elements are introduced simultaneously. The following table lists every flag of $\mathcal{F}_K$:

Length	$\mathcal{F}_K$ flags
1	(12), (13), (24), (34)
2	(1, 23), (2, 14), (3, 14), (4, 23)
3	(1, 2, 3), (1, 3, 2), (2, 1, 4), (2, 4, 1) (3, 1, 4), (3, 4, 1), (4, 2, 3), (4, 3, 2)

One can check that this results in the following set of basic flags:

$$\mathcal{B}_K = \{(12),\ (13),\ (24),\ (34),\ (1, 23),\ (2, 14),\ (3, 14),\ (4, 23)\}.$$

Now let H be the following subgroup of G:

$$H = \begin{pmatrix} * & 0 & 0 & 0 \\ 0 & * & * & 0 \\ 0 & 0 & * & 0 \\ 0 & 0 & 0 & * \end{pmatrix}.$$

Then H stabilizes flags in which the span of e_3 is not introduced before the span of e_2.

This leads to the following sets of H-stable flags and H-stable basic flags:

$$F_K^H = \{(12),\ (24),\ (1,23),\ (2,14),\ (4,23),\ (1,2,3),\ (2,1,4),\ (2,4,1),\ (4,2,3)\}.$$
$$\mathcal{B}_K^H = \{(12),\ (24),\ (1,23),\ (2,14),\ (4,23)\}.$$

Proposition 3.2.4. *Let λ be a cocharacter and let $a_1 > \cdots > a_r$ be the distinct λ-weights on V, with corresponding weight spaces $X_1, \ldots, X_r$. Then the parabolic corresponding to λ is the stabilizer of the flag $U_1 \subset \cdots \subset U_r$, where $U_i = \sum_{j=1}^{i} X_i$ for each i.*

Proof. We may conjugate λ so that it is diagonal and of the form $t \mapsto \operatorname{diag}(t^{a_1}, \ldots, t^{a_r})$. The result follows from the form of the matrices such that $\lim_\lambda$ exists. □

In Example 3.2.3 we saw that the cocharacter $\lambda_{1,1} \in Y(K)$ corresponded to the flag $\langle e_1, e_2\rangle \subset k^4$; in this case, $\lambda_{1,1}$ acts with weight 1 on $\langle e_1, e_2\rangle$ and with weight -1 on $\langle e_3, e_4\rangle$. In the same example, the cocharacter $\lambda_{2,1} \in Y(K)$ acts with weights 2, 1, -1, and -2 on the subspaces $\langle e_1\rangle$, $\langle e_2\rangle$, $\langle e_3\rangle$, and $\langle e_4\rangle$, respectively. The cocharacter $\lambda_{2,1}$ therefore corresponds to the flag

$$\langle e_1\rangle \subset \langle e_1, e_2\rangle \subset \langle e_1, e_2, e_3\rangle \subset k^4.$$

In future arguments we use this fact to provide a weight space decomposition of V, indexing the weight spaces based on the order they "drop in" to a given flag.

When working with reductive subgroups K described by two parameters as in Example 3.2.3, we can graphically represent cocharacters and the flags stabilized by their associated parabolic subgroups. Our figures will be motivated by the following result which states that basic flags are represented by a single "ray" of cocharacters.

Lemma 3.2.5. *Suppose that λ, $\mu \in Y(K)$ commute and give rise to the same basic flag. Then λ and μ are positive multiples of each other.*

Proof. Let $f = U_1 \subset \ldots U_r \subset U_{r+1} = V$ be the basic flag such that $\operatorname{Stab}_G(f) = P_\lambda = P_\mu$. As discussed in Proposition 3.2.4, we can decompose V into λ-weight spaces $X_1, X_2, \ldots, X_{r+1}$ such that each U_i is the direct sum of the X_j with $j \le i$. For $1 \le i \le r+1$ let a_i and b_i denote, respectively, the weights of λ and μ on each X_i. The following chains of inequalities must hold:

$$a_1 > a_2 > \cdots > a_{r+1},$$
$$b_1 > b_2 > \cdots > b_{r+1}.$$

The cocharacters λ and μ both evaluate in a common maximal torus $T \subset K$ and we may denote their unique opposites (see Remark 2.1.22) in $Y(T)$ by $-\lambda$ and $-\mu$. Choose i' so that the quotient $\epsilon := \dfrac{a_i - a_{i+1}}{b_i - b_{i+1}}$ is minimal, where we run over all $1 \le i \le r$. Note that this quotient is always positive, due to the chains of inequalities above. Consider the cocharacter $\lambda + \epsilon(-\mu)$ and note that this acts with equal weight on $X_{i'}$ and $X_{i'+1}$ since $a_{i'} - \epsilon b_{i'} = a_{i'+1} - \epsilon b_{i'+1}$. We also have that $a_i - \epsilon b_i \ge a_{i+1} - \epsilon b_{i+1}$ for all i, due to our choice of ϵ. This cocharacter therefore corresponds to a proper subflag of f. Since f is a basic flag it must be the case that $\lambda + \epsilon(-\mu)$ corresponds to the empty flag and hence we conclude that $\lambda = \epsilon\mu$. $\square$

Example 3.2.6. We return to the setting of Example 3.2.3 where $G = \mathrm{GL}_4(k)$ and $K = \mathrm{diag}(s, t, t^{-1}, s^{-1})$ with $s, t \in k^*$. One can check that the cocharacters $\lambda_{3,3}$ and $\lambda_{1,1}$ commute and both correspond to the basic flag $\langle e_1, e_2\rangle \subset k^4$. Following the proof of Lemma 3.2.5 above, we can decompose $V = k^4$ into $\lambda_{3,3}$-weight spaces $X_1 = \langle e_1, e_2\rangle$ and $X_2 = \langle e_3, e_4\rangle$. Letting a_i and b_i denote the respective weights of $\lambda_{3,3}$ and $\lambda_{1,1}$, we get that $a_1 = 3 > a_2 = -3$ and $b_1 = 1 > b_2 = -1$. Then we choose $i' = 1$ so that

$$\epsilon = \frac{3-(-3)}{1-(-1)} = 3,$$

and conclude that $\lambda_{3,3} = 3\lambda_{1,1}$. We can plot our cocharacters $\lambda_{x,y}$ on a lattice as in

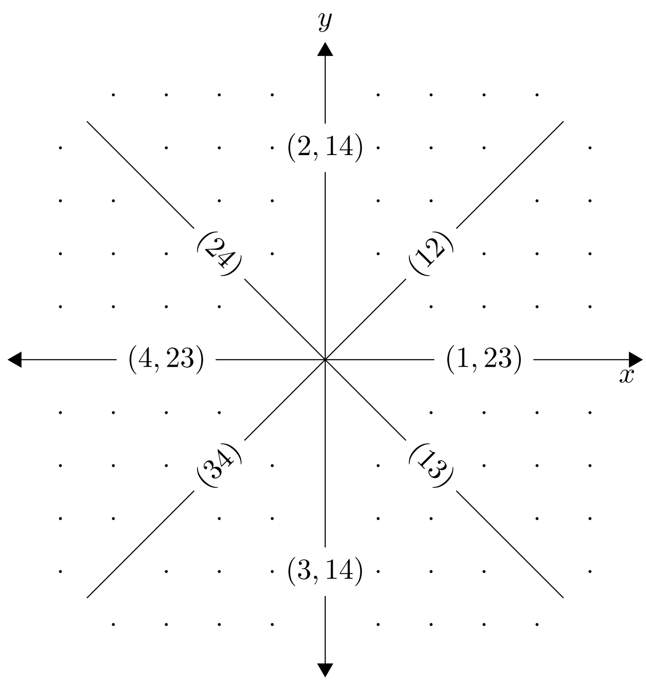


Figure 3.1: Cocharacter and stabilized flag diagram for $\mathcal{F}_K$ of Example 3.2.6

Figure 3.1. The rays of cocharacters which give rise to basic flags (such as $y = x > 0$) have been plotted and labelled accordingly. These basic rays are extremal points of cones of nonbasic flags. For example the cocharacter $\lambda_{2,1}$ is located in the cone outlined by $y = x > 0$ and $x > y = 0$, the cocharacter rays which correspond to basic flags (12) and $(1, 23)$ respectively. The parabolic associated to $\lambda_{2,1}$ stabilizes the flag $(1, 2, 3)$; observe that $\lambda_{2,1}$ can be expressed as a positive linear combination of cocharacters on these rays corresponding to subflags of $(1, 2, 3)$. Lemma 3.2.12 will prove that this is true for all cocharacters of nonbasic flags.

Now set $\lambda := \lambda_{(1,1)}$ and $\mu := \lambda_{(-2,2)}$. Then we have $P_\lambda = \mathrm{Stab}_G(\langle e_1, e_2\rangle \subset k^4)$ and $P_\mu = \mathrm{Stab}_G(\langle e_2, e_4\rangle \subset k^4)$. Observe the flags which are stabilized by the parabolic associated to the cocharacter $n\lambda + \mu$ as we vary the positive integer n:

$$\begin{aligned}
P_{\lambda+\mu} &= \mathrm{Stab}_G(\langle e_2\rangle \subset \langle e_2, e_4\rangle \subset \langle e_1, e_2, e_4\rangle \subset k^4) \\
P_{2\lambda+\mu} &= \mathrm{Stab}_G(\langle e_2\rangle \subset \langle e_1, e_2, e_4\rangle \subset k^4) \\
P_{3\lambda+\mu} &= \mathrm{Stab}_G(\langle e_2\rangle \subset \langle e_1, e_2\rangle \subset \langle e_1, e_2, e_4\rangle \subset k^4)
\end{aligned}$$

In general, taking a sufficiently large positive multiple of λ will result in basis vectors appearing in the flag in an order which first depends on the λ weighting. On basis vectors with the same λ weighting, the order is then decided by the weighting of μ. We can see here that for $n \geq 3$, the basis elements are introduced in the order e_2, e_1, e_4, e_3. This occurs since λ orders e_1 and e_2 before e_3 and e_4, then μ orders e_2 before e_1 and e_4 before e_3. Figure 3.2 demonstrates this graphically. For sufficiently large n, the cocharacter combination lies in a cone adjacent to the ray of cocharacters which are multiples of λ.

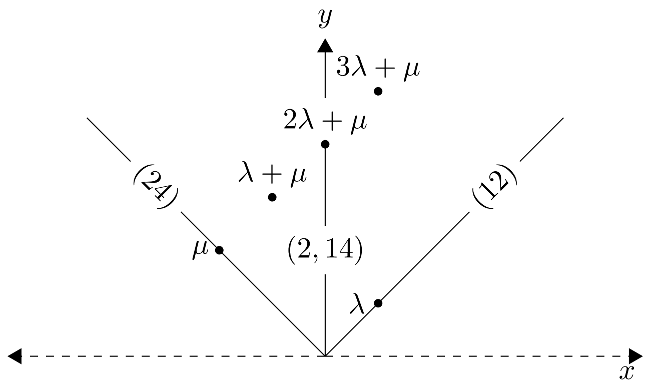


Figure 3.2: Cocharacter combination diagram for $\mathcal{F}_K$ of Example 3.2.6

Using our flag terminology, we can formulate Theorem 3.1.1 in the case where $G = \mathrm{GL}(V)$ as follows:

Theorem 3.2.7. [1, Corollary 1.5]. *Let H and K be closed subgroups of* $\mathrm{GL}(V)$ *with K reductive. Then the following are equivalent:*

(i) *H is relatively* $\mathrm{GL}(V)$*-completely reducible with respect to K.*

(ii) *Every flag in $\mathcal{B}_K^H$ has an opposite in $\mathcal{B}_K^H$*

(iii) *There is a maximal torus S of $C_K(H)$ such that H preserves the direct sum decomposition of V into simultaneous S-eigenspaces, and this decomposition gives a flag which is maximal among H-stable flags in $\mathcal{F}_K$.*

In the case $K = \mathrm{GL}(V)$, the maximal torus S in (iii) must be a product of the centres of the $\mathrm{GL}(V_i)$ where $V = V_1 \oplus \cdots \oplus V_r$ is a decomposition of V into irreducible H-modules; this gives the usual representation-theoretic characterization.

We begin our work towards a proof of Theorem 3.2.7 by collecting some useful results. The following lemma combines some of the results for parabolic subgroups provided in our preliminaries.

Lemma 3.2.8. *Suppose $f, g \in \mathcal{F}_K$. Then there exists a maximal torus S of K and $\lambda, \mu \in Y(S)$ such that $P_\lambda = \mathrm{Stab}_G(f)$, $P_\mu = \mathrm{Stab}_G(g)$.*

Proof. Choose any $\lambda, \mu \in Y(K)$ such that $\mathrm{Stab}_G(f) = P_\lambda$ and $\mathrm{Stab}_G(g) = P_\mu$. Then $P_\lambda(K)$ and $P_\mu(K)$ are parabolic subgroups of K, and hence contain a common maximal torus S by applying Lemma 1.1.26 to K. By Lemma 1.1.25, we may replace λ and μ with cocharacters of S. $\square$

A key ingredient in our proof is that a flag opposite a basic flag must be basic itself.

Lemma 3.2.9. *Let K be a reductive subgroup of $G = \mathrm{GL}(V)$. If $f \in \mathcal{B}_K$ and $g \in \mathcal{F}_K$ is opposite to f, then $g \in \mathcal{B}_K$.*

Proof. Let $f \in \mathcal{B}_K$ with an opposite $g \in \mathcal{F}_K$ and suppose that $g' \preceq g$ for some $g' \in \mathcal{B}_K$. By Lemma 3.2.8, there is a maximal torus T of K contained in $\mathrm{Stab}_G(f)$ and $\mathrm{Stab}_G(g)$, and there exists $\lambda \in Y(T)$ such that $P_\lambda = \mathrm{Stab}_G(f)$ and $P_{-\lambda} = \mathrm{Stab}_G(g)$. Since $T \subseteq P_{-\lambda} \subseteq \mathrm{Stab}_G(g')$, there exists $\mu \in Y(T)$ such that $P_\mu = \mathrm{Stab}_G(g')$. Consider the cocharacter $-\mu \in Y(T)$; since $P_{-\lambda} \subseteq P_\mu$, we must have that $P_\lambda \subseteq P_{-\mu}$. Let f' be the flag corresponding to $-\mu$, and observe that $f' \preceq f$. If g' were a proper subflag of g, then f' would be a proper subflag of f, since opposite flags have equal lengths. Since f is minimal in $\mathcal{F}_K$, we conclude that $g' = g$ and hence g is a basic flag. $\square$

We do not need the following lemma to prove Theorem 3.2.7, but will make use of some technology involved in its proof. In Chapter 4 we will utilise the full statement.

Lemma 3.2.10. *Let $f \in \mathcal{F}_K$ and let U be a subspace of f. Then there exists a basic flag f' such that $f' \preceq f$ and U is a subspace of f'.*

Proof. Let $f = (U_1 \subset \cdots \subset U_r \subseteq V) \in \mathcal{F}_K$ and let $U = U_i$ for some i. We argue by induction on the length r of f. If $r = 1$ then $f \in \mathcal{B}_K$ and there is nothing to prove. Suppose that $r > 1$ and that the statement is true for flags of length at most $r - 1$. We have seen that there is nothing to prove if $f \in \mathcal{B}_K$, so suppose that $f \notin \mathcal{B}_K$. There must exist a basic flag $f' \preceq f$. If U is a subspace of f' then we are done, so suppose that U is not a subspace of f'.

Let $\lambda, \mu \in Y(K)$ such that $P_\lambda = \mathrm{Stab}_G(f) \subseteq \mathrm{Stab}_G(f') = P_\mu$. There is a maximal torus T of G in P_λ such that $T \cap K$ is a maximal torus of K, and there is a Borel subgroup B of G such that $T \subseteq B \subseteq P_\lambda \subseteq P_\mu$. Hence we may assume $\lambda, \mu \in Y(T)$. Note that $Y(T)$ is isomorphic to a subgroup of $\mathbb{Z}^n$, where $n = \dim V$. Due to the conjugacy of parabolics, we may assume that for a n-tuple $(z_1, \ldots, z_n)$ in $\mathbb{Z}^n$ corresponding to a parabolic subgroup containing B, we have $z_1 \geq \cdots \geq z_n$. There exists a basis $v_1, \ldots, v_n$ of V such that $\mathrm{Stab}_G(\langle v_1 \rangle \subset \langle v_1, v_2 \rangle \subset \cdots \subset V) = B$. Let $(a_1, \ldots, a_n)$ and $(b_1, \ldots, b_n)$ be the n-tuples corresponding to P_λ and P_μ, respectively. Then we have

$$P_\lambda = \mathrm{Stab}_G(\langle v_1, \ldots, v_{\dim(U_1)} \rangle \subseteq \ldots \subseteq \langle v_1, \ldots, v_{\dim(U_r)} \rangle \subseteq V).$$

Hence

$$a_1 = \cdots = a_{\dim(U_1)} > a_{\dim(U_1)+1} = \cdots = a_{\dim(U_2)} > \cdots > a_{\dim(U_r)+1} = \cdots = a_n.$$

Choose i_0 so that the quotient $\dfrac{b_i - b_{i+1}}{a_i - a_{i+1}}$ is maximal, where we run over all i with $a_i \neq a_{i+1}$. Set $n_1 := b_{i_0} - b_{i_0+1}$ and $n_2 := a_{i_0} - a_{i_0+1}$. Let $(c_1, \ldots, c_n)$ be the n-tuple corresponding to the cocharacter $n_1\lambda - n_2\mu \in Y(T)$ and let $\tilde{f}$ be the corresponding flag. By construction, $c_i = n_1 a_i - n_2 b_i$ for all i. If $a_i > a_{i+1}$ and $b_i = b_{i+1}$, then $c_i > c_{i+1}$. If $a_i = a_{i+1}$ and $b_i = b_{i+1}$, then $c_i = c_{i+1}$. The case $a_i = a_{i+1}$ and $b_i > b_{i+1}$ does not occur, since $f' \preceq f$. If $a_i > a_{i+1}$ and $b_i > b_{i+1}$, then our choice of n_1 and n_2 ensures that we have

$$n_1(a_i - a_{i+1}) \geq n_2(b_i - b_{i+1}).$$

Hence $c_i \geq c_{i+1}$ in this case. So, whenever $a_i = a_{i+1}$ we have $c_i = c_{i+1}$ and whenever $a_i > a_{i+1}$ we have $c_i \geq c_{i+1}$. Thus $\tilde{f} \preceq f$.

Since U is a subspace of f but not of f', we have $a_{\dim(U)} > a_{\dim(U)+1}$ and $b_{\dim(U)} = b_{\dim(U)+1}$; this means that $c_{\dim(U)} > c_{\dim(U)+1}$ and so U is a subspace of $\tilde{f}$. Finally, observe that $a_{i_0} > a_{i_0+1}$ but $c_{i_0} = c_{i_0+1}$. This means that the length of $\tilde{f}$ is strictly smaller than r. By our induction hypothesis, there exists a basic flag $g \preceq \tilde{f}$ which contains U as a subspace. $\square$

Remark 3.2.11. The basic flag f' is not necessarily uniquely determined by the choice of U. For $K = \mathrm{diag}(s, t^3, t^2, t) \leq \mathrm{GL}_4(k)$, we have $(2, 1, 3) \in \mathcal{F}_K$ and $(12, 3), (2, 13) \in \mathcal{B}_K$. All of these flags contain the subspace $\langle e_1, e_2, e_3\rangle$, and we have that $(12, 3) \preceq (2, 1, 3)$ and $(2, 13) \preceq (2, 1, 3)$.

A key element in our proof of Theorem 3.2.7 is the idea of expressing arbitrary cocharacters in terms of "basic" cocharacters.

Lemma 3.2.12. *Let $\lambda \in Y(K)$ correspond to the flag $f \in \mathcal{F}_K$. Some positive integer multiple of λ can be expressed as a positive $\mathbb{Z}$-linear combination of cocharacters in $Y(K)$ which correspond to basic flags $f_i \preceq f$.*

It only makes sense to take linear combinations of cocharacters which commute; this is guaranteed in our set-up since the cocharacters map into the same torus.

Proof. We will argue inductively based on the length of the flag f. We immediately have a base case for our induction since a flag of minimal length must be basic. Suppose the result holds for flags of length at most l, and let the length of the flag f be $l + 1$. There is nothing to show if f is a basic flag, so assume $f \notin \mathcal{B}_K$. Then there exists a basic flag $f_0 \preceq f$; let λ_0 be its corresponding cocharacter.

Using the method outlined in the proof of Lemma 3.2.10, we can find positive integers a and b such that $\lambda_1 := a\lambda - b\lambda_0$ is a cocharacter corresponding to a flag $f_1 \preceq f$, with the length of f_1 being strictly shorter than the length of f. Since we can write $a\lambda = \lambda_1 + b\lambda_0$, and the cocharacters λ_0 and λ_1 correspond to flags with lengths at most l, the result holds for our flag of length $l + 1$. $\square$

Remark 3.2.13. As previously observed, the set of cocharacters $Y(T)$ for a fixed torus T is isomorphic as an abelian group to $\mathbb{Z}^r$, where $r = \dim T$. By tensoring with $\mathbb{Q}$, we can work with "rational cocharacters" and then this result can be reformulated to say that any $\lambda \in Y(K)$ can be expressed as a positive *rational* combination of basic cocharacters.

Example 3.2.14. Let $G = \mathrm{GL}_4(k)$ and let $K = \mathrm{diag}(w, x, y, z)$. Let $\lambda \in Y(K)$ be the cocharacter such that

$$\lambda(a) = \begin{pmatrix} a^3 & & & \\ & a^2 & & \\ & & a & \\ & & & 1 \end{pmatrix}, \quad P_\lambda = \begin{pmatrix} * & * & * & * \\ 0 & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & 0 & * \end{pmatrix}.$$

Then P_λ stabilizes the flag $f = \langle e_1 \rangle \subset \langle e_1, e_2 \rangle \subset \langle e_1, e_2, e_3 \rangle \subset k^4$. We can find an expression for λ in terms of basic cocharacters as follows:

Take $\lambda_1 \in Y(K)$ so that

$$\lambda_1(a) = \begin{pmatrix} a & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}, \quad P_{\lambda_1} = \begin{pmatrix} * & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \end{pmatrix}.$$

This gives rise to the basic flag $f_1 = \langle e_1 \rangle \subset k^4$; note that $f_1 \prec f$ in $\mathcal{F}_K$. Both P_λ and P_{λ_1} contain the standard Borel subgroup of upper triangular matrices, which contains the standard maximal torus of diagonal matrices. Two 4-tuples in $\mathbb{Z}^4$ corresponding to P_λ and P_{λ_1} are $\underline{a} = (3, 2, 1, 0)$ and $\underline{b} = (1, 0, 0, 0)$, respectively. To produce a shorter flag, we need to find the value of i_0 that maximises $(b_{i_0} - b_{i_0+1})(a_{i_0} - a_{i_0+1})^{-1}$; a quick check reveals that this is $i_0 = 1$. We therefore work with the integers $m := (b_1 - b_2) = 1$ and $n := (a_1 - a_2) = 1$, setting $\lambda_2 := m\lambda - n\lambda_1 = \lambda - \lambda_1$. We have that

$$\lambda_2(a) = \begin{pmatrix} a^2 & & & \\ & a^2 & & \\ & & a^1 & \\ & & & 1 \end{pmatrix}, \quad P_{\lambda_3} = \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ 0 & * & * & * \\ 0 & 0 & * & * \end{pmatrix}.$$

Then λ_2 corresponds to the flag $f_2 = \langle e_1, e_2 \rangle \subset \langle e_1, e_2, e_3 \rangle \subset k^4$, which is a proper subflag of f. At this point we can express λ as a combination of λ_1 and λ_2, which are both proper subflags of f. But λ_2 is not a basic flag; to express λ in terms of basic cocharacters, we need to repeat the process. Take $\lambda_3 \in Y(K)$ such that

$$\lambda_3(a) = \begin{pmatrix} a & & & \\ & a & & \\ & & 1 & \\ & & & 1 \end{pmatrix}, \quad P_{\lambda_3} = \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & * & * \end{pmatrix}.$$

We have that P_{λ_3} stabilizes the basic flag $f_3 = \langle e_1, e_2 \rangle \subset k^4$, and $f_3 \prec f_2 \prec f$. Two

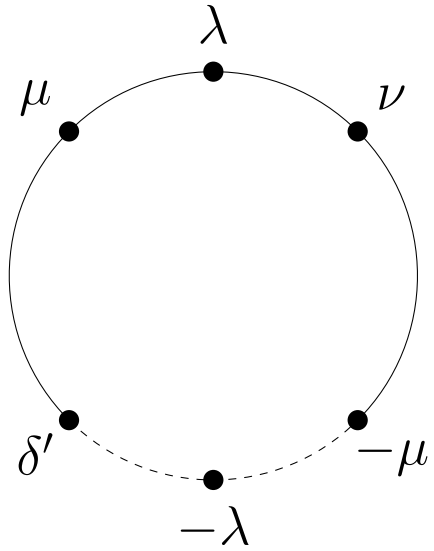


Figure 3.3: Cocharacters evaluating in a common maximal torus

4-tuples corresponding to P_{λ_2} and P_{λ_3} are, respectively, $\underline{c} = \underline{a} - \underline{b} = (2, 2, 1, 0)$ and $\underline{d} = (1, 1, 0, 0)$. These lead to integers that define the new cocharacter $\lambda_4 := \lambda_2 - \lambda_3$. The flag associated to λ_4 is the basic flag $f_4 = \langle e_1, e_2, e_3 \rangle \subset k^4$; note that $f_4 \prec f$. Now we have obtained an expression for λ in terms of basic cocharacters since we can write $\lambda = \lambda_2 + \lambda_1 = \lambda_4 + \lambda_3 + \lambda_1$.

We now have the results we need to present a proof of our main result. The implications (iii) $\Leftrightarrow$ (i) $\Rightarrow$ (ii) follow quickly,

Proof of Theorem 3.2.7. The equivalence of conditions (i) and (iii) follows from Proposition 2.2.4(ii): the centralizer of a maximal torus of $C_K(H)$ is a Levi subgroup L_μ, for some $\mu \in Y(K)$, which is minimal with respect to containing H. Then H is relatively $\mathrm{GL}(V)$-completely reducible with respect to K if and only if H is relatively L_μ-irreducible with respect to $K \cap L_\mu$.

Recall that H is relatively $\mathrm{GL}(V)$-completely reducible with respect to K if and only each flag in $\mathcal{F}_K^H$ has a complement in $\mathcal{F}_K^H$. Then the implication (i) $\implies$ (ii) follows immediately from Lemma 3.2.9. We prove the reverse implication by showing that (ii) implies that every flag in $\mathcal{F}_K^H$ has a complement in $\mathcal{F}_K^H$. Let $f \in \mathcal{F}_K$ be a flag with corresponding cocharacter $\lambda \in Y(K)$, and suppose that $H \subseteq P_\lambda$. We know from Lemma 3.2.12 that we can express λ as a linear combination of cocharacters λ_i which correspond to basic flags $f_i \preceq f$. We will argue inductively on the number of basic cocharacters required to express λ, lining up cocharacters in a maximal torus (see Figure 3.3) to show that λ has an opposite.

Assume that (ii) holds. The base case where f itself is a basic flag follows immediately. Suppose that $\lambda = \mu + \nu$, where μ corresponds to a basic flag and ν can be expressed with fewer basic cocharacters than λ. Suppose that the desired result holds for cocharacters expressed in fewer basic cocharacters than λ. We can arrange that $H \subseteq L_\mu = C_{\mathrm{GL}(V)}(\mathrm{Im}(\mu))$, so $\mu \in Y(C_K(H))$. Since μ and ν are in a common torus of K, we also have $\mu \in Y(L_\nu(K))$. Hence we have $\mu \in Y(C_K(H) \cap L_\nu(K)) = Y(C_{L_\nu(K)}(H))$. Observe that if $f_i \preceq f$, then $P_\lambda \subseteq P_{\lambda_i}$. Since we assumed that f is stabilized by H, we have $H \subseteq P_\lambda \subseteq P_{\lambda_i}$. Hence all the basic flags f_i are H-stable and so ν corresponds to a flag which is H-stable. Our induction hypothesis allows us to find an opposite δ to ν such that $H \subseteq L_\delta$, and so we will have $\delta \in Y(C_{P_\nu(K)}(H))$. Since $\mu \in Y(C_{P_\nu(K)}(H))$, we can conjugate δ by an element of $C_{P_\nu(K)}(H)$ to get δ' such that δ' and μ commute. Note that δ' will still be an opposite to ν, and that these both commute with μ. Hence δ' and ν both commute with $-\mu$, and these all evaluate in a common maximal torus T, say, of K. Since $\lambda = \mu + \nu$, λ also evaluates in T. The unique opposite to λ in $Y(T)$, which we may write as $-\lambda$, must be be $-\mu + \delta'$. The opposite to $\mu + \nu$ in this local picture must commute with H, so the flag corresponding to $-\lambda = -\mu + \delta'$ is stabilized by H. Thus we have shown the final implication $(ii) \implies (i)$. □

The final parts of the above proof are essentially an argument of Serre's written in cocharacter terminology, see the proof of [34, Théorème 2.2]. This "Levi circle" argument motivates Figure 3.3. Having put everything inside a common maximal torus, we are able to work in a two-dimensional vector space.

3.3 Further results

When $K = G = \mathrm{GL}(V)$ the set of basic flags $\mathcal{B}_K$ is the set of flags of length 1 in V. Henceforth, we will indicate when an arbitrary set F of flags in V contains only length one flags by writing $F \subseteq \mathcal{B}_{\mathrm{GL}(V)}$. Let $\mathcal{S}_K$ denote the set of subspaces of V which are subspaces of $\mathcal{F}_K$ flags. The following result is an immediate consequence of Lemma 3.2.10, and asserts that every element of $\mathcal{S}_K$ forms a K-flag of length one if and only if every basic K-flag has length one.

Corollary 3.3.1. [1, Corollary 4.2]. *Let K be a reductive subgroup of* $\mathrm{GL}(V)$. *Then the following are equivalent:*

(i) $\mathcal{S}_K = \{U \subseteq V \mid (U \subseteq V) \in \mathcal{F}_K\}$

(ii) $\mathcal{B}_K \subseteq \mathcal{B}_{\mathrm{GL}(V)}$

Example 3.3.2. Let $G = \mathrm{GL}_3(k)$ and $K = \mathrm{diag}(a, b, c) \leq G$. The following table lists the flags of $\mathcal{F}_K$:

Length	$\mathcal{F}_K$ flags
1	(1), (2), (3), (12), (13), (23)
2	$(1,2)$, $(1,3)$, $(2,1)$, $(2,3)$, $(3,1)$, $(3,2)$

This results in a set of basic flags $\mathcal{B}_K = \{(1), (2), (3), (12), (13), (23)\}$. Hence we can write $\mathcal{B}_K \subseteq \mathcal{B}_{\mathrm{GL}(V)}$.

The following result follows from Theorem 3.2.7 and Corollary 3.3.1, and mirrors the statement from representation theory that V is a completely reducible H-module if and only if every H-submodule has a complement.

Corollary 3.3.3. [1, Corollary 1.6]. *Let H and K be subgroups of* $\mathrm{GL}(V)$ *with K reductive. Suppose that $\mathcal{B}_K \subseteq \mathcal{B}_{\mathrm{GL}(V)}$; then the following are equivalent:*

(i) *H is relatively* $\mathrm{GL}(V)$*-completely reducible with respect to K.*

(ii) *For each H-stable $U \in \mathcal{S}_K$ there exists an H-stable $W \in \mathcal{S}_K$ such that $V = U \oplus W$ as an H-module.*

One can check that condition (i) of Corollary 3.3.1 is also satisfied when K is a maximal torus of G. Corollary 3.3.3 therefore applies in both of these cases. Corollary 3.3.3 can be viewed as a generalization of [10, Proposition 5.1]; recall that in Subsection 2.3.1 we saw that this proposition gives a representation-theoretic characterization of relative $\mathrm{GL}(V)$-complete reducibility in the case $K = \mathrm{GL}(U)$ for a subspace $U \subseteq V$. This is related to the condition of the corollary above in light of Corollary 3.3.1 and the following lemma.

Lemma 3.3.4. [1, Lemma 4.5]. *Let $G = \mathrm{GL}(V)$ and let $U \subseteq V$. Fix a complement $\widetilde{U}$ to U in V. Let $K = \mathrm{GL}(U) \leq G$, embedded via the decomposition $V = U \oplus \widetilde{U}$. Then $\mathcal{S}_K = \{W \subseteq V \mid (W \subseteq V) \in \mathcal{F}_K\}$.*

Proof. Let $f = (W_1 \subseteq \ldots \subseteq W_m \subseteq V) \in \mathcal{F}_K$. For each $1 \leq i \leq m$, we have $W_i \subseteq U$ or $\widetilde{U} \subseteq W_i$. Conversely, suppose that W is a subspace contained in U. We can find a complement W' to W containing $\widetilde{U}$. The cocharacter which acts with weight 1 on W and weight 0 on W' lies in $Y(K)$ and the corresponding flag is $(W \subseteq V) \in \mathcal{F}_K$. Similarly all flags $(W \subseteq V)$ with $\widetilde{U} \subseteq W$ lie in $\mathcal{F}_K$. Hence we have that

$$\mathcal{F}_K = \{f \in \mathcal{F}_{\mathrm{GL}(V)} \mid W_i \subseteq U \text{ or } \widetilde{U} \subseteq W_i \text{ for all subspaces } W_i \text{ of } f\},$$

and so $\mathcal{S}_K = \{W' \subseteq V \mid W' \subseteq U \text{ or } \widetilde{U} \subseteq W'\} = \{W \subseteq V \mid (W \subseteq V) \in \mathcal{F}_K\}$. □

The implication (i) ⇒ (ii) of Corollary 3.3.3 fails without the hypothesis on $\mathcal{B}_K$, as the following example demonstrates.

Example 3.3.5. [1, Example 4.3]. Let $G = \mathrm{GL}_4(k)$ and $K = \mathrm{diag}(s, t, t^{-1}, s^{-1})$ with $s, t \in k^*$. Let $e_1, \ldots, e_4$ be the standard basis of k^4 and let $U = \langle e_1, e_2, e_3 \rangle$. Suppose that H is the parabolic subgroup of G corresponding to the flag $(U \subseteq V)$:

$$H = \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ 0 & 0 & 0 & * \end{pmatrix}.$$

We saw in Example 3.2.3 that $\mathcal{B}_K$ contains flags of length two and that flags in $\mathcal{F}_K$ have signatures of $(1, 3, 4)$, $(2, 4)$, or $(1, 2, 3, 4)$. The group H is therefore not contained in P_λ for any nontrivial $\lambda \in Y(K)$. Hence H is trivially relatively G-cr with respect to K. Observe that $U \in \mathcal{S}_K$ and H stabilizes U. The complement to U in $\mathcal{S}_K$ is $W = \langle e_4 \rangle$ which is not stabilized by H.

If $\mathcal{B}_K \nsubseteq \mathcal{B}_{\mathrm{GL}(V)}$ then it is always possible to find a subgroup which shows that Corollary 3.3.3 fails.

Proposition 3.3.6. *Let $G = \mathrm{GL}(V)$. If $\mathcal{B}_K \nsubseteq \mathcal{B}_{\mathrm{GL}(V)}$ then there exists a subgroup $H \leq G$ such that H is relatively G-cr with respect to K and H stabilizes a subspace $U' \in \mathcal{S}_K$ but does not stabilize any complement to U'.*

Proof. By Corollary 3.3.1, $\mathcal{S}_K \neq \{U \subseteq V \mid (U \subseteq V) \in \mathcal{F}_K\}$. This means there exists a $U' \in \mathcal{S}_K$ such that $(U' \subseteq V) \notin \mathcal{F}_K$. Set $H := \mathrm{Stab}_G(U' \subseteq V)$. Then H is not contained in P_λ for any nontrivial $\lambda \in Y(K)$. Hence H is trivially relatively G-cr with respect to K. We have that H stabilizes $U' \in \mathcal{S}_K$ but does not stabilize any complement to U', since H is a maximal parabolic subgroup of G. □

The classical groups $\mathrm{Sp}(V)$ and $\mathrm{SO}(V)$ are natural candidates for the reductive subgroup K of $\mathrm{GL}(V)$. The following result characterizes relative $\mathrm{GL}(V)$-complete reducibility with respect to $\mathrm{Sp}(V)$ or $\mathrm{SO}(V)$ in terms of totally isotropic or totally singular subspaces.

Corollary 3.3.7. [1, Corollary 1.7]. *Let H be a subgroup of $\mathrm{GL}(V)$ and let $K = \mathrm{Sp}(V)$ (resp. $\mathrm{SO}(V)$). Then the following are equivalent:*

(i) *H is relatively $\mathrm{GL}(V)$-completely reducible with respect to K.*

(ii) *Whenever H stabilizes a totally isotropic (resp. totally singular) subspace U and its annihilator $U^\perp$, there exists a totally isotropic (resp. totally singular) subspace $W \subseteq V$ such that H stabilizes W and $W^\perp$, and $V = W \oplus U^\perp = U \oplus W^\perp$ as H-modules.*

The second condition requires that H stabilizes $U^\perp$ and $W^\perp$ because H is not necessarily a subgroup of K, and therefore may not leave the form on V invariant). In the setting of Corollary 3.3.7, flags in $\mathcal{F}_K$ have the form

$$U_1 \subset \ldots \subset U_r \subset U_r^\perp \subset \ldots \subset U_1^\perp,$$

and flags in $\mathcal{B}_K$ are of the form $U \subset U^\perp$ for a totally isotropic or totally singular subspace U. The result follows immediately from Theorem 3.2.7.

Corollary 3.3.7 studies cases where K acts irreducibly on V. The following result presents a characterization of relative $\mathrm{GL}(V)$-complete reducibility when V decomposes as a direct sum of K-modules, and follows immediately from [1, Corollary 4.7] and [10, Corollary 3.6].

Corollary 3.3.8. [1, Corollary 4.8]. *Let $G = \mathrm{GL}(V)$ and suppose that both H and K preserve a direct-sum decomposition $V = \bigoplus_{i=1}^n V_i$. Suppose also that $K = K_1 \times \cdots \times K_n$ where $K_i \leq \mathrm{GL}(V_i)$ for each i. Then H is relatively G-completely reducible with respect to K if and only if H is relatively G-completely reducible with respect to K_i for all i.*

These implications fail in general, as the following example demonstrates.

Example 3.3.9. [1, Example 4.9]. We return to our running example where $G = \mathrm{GL}_4(k)$ and $K = \mathrm{diag}(s, t, t^{-1}, s^{-1})$ with $s, t \in k^*$. Let $e_1, \ldots, e_4$ be the canonical basis of k^4 and set $V_1 = \langle e_1, e_2 \rangle$ and $V_2 = \langle e_3, e_4 \rangle$. Let K_i be the image of the projection from K to $\mathrm{GL}(V_i)$ for $i = 1, 2$.

Let H be the stabilizer of $U := \langle e_2, e_4 \rangle$ and recall that $(U \subset k^4) \in \mathcal{B}_K$. Then H is a maximal parabolic subgroup of G corresponding to a cocharacter of K, and is therefore not relatively G-cr with respect to K. However, H does not correspond to a cocharacter of K_1 or K_2, and is not contained in any parabolic subgroup of G corresponding to a character of K_1 or K_2. Hence H is trivially relatively G-completely reducible with respect to K_1 and K_2.

Now let H' be the stabilizer of $U' := \langle e_1 \rangle$ in G. Note that H' is a maximal parabolic subgroup of G and corresponds to a cocharacter of K_1. It is therefore not relatively G-completely reducible with respect to K_1. Since H' does not correspond to a cocharacter of K, it is trivially relatively G-completely reducible with respect to K.

The flags in $\mathcal{F}_K$ which are maximal with respect to the $\preceq$ partial order all have the same length.

Lemma 3.3.10. *Suppose that $f, g \in \mathcal{F}_K$ are maximal with respect to the partial ordering $\preceq$. Then f and g have equal length.*

Proof. By Lemma 3.2.8 there is a maximal torus S of K and commuting cocharacters $\lambda, \mu \in Y(S)$ such that $P_\lambda = \mathrm{Stab}_G(f)$ and $P_\mu = \mathrm{Stab}_G(g)$. By Proposition 1.1.24, the λ-weight spaces are μ-stable, and vice versa. Suppose that μ has more than one weight on a λ-weight space. Then for sufficiently large n, the cocharacter $n\lambda + \mu$ will correspond to a flag containing f as a subflag. This contradicts the maximality of f, and so we can conclude that μ has only one weight on every λ-weight space. A similar argument shows that λ has only one weight on every μ-weight space. $\square$

Most of the work is contained in Lemma 3.2.8, which allows us to find a suitable torus S to work with. Obtaining a torus that is compatible with two flags involves working with bases of $\mathrm{GL}_n(k)$, and using basis elements in a different orders corresponds to ordering weight spaces differently. Example 2.3.3 and Example 3.3.11 show that maximal flags need not be complete.

3.3.1 The structures of K and $\mathcal{F}_K$

We finish this chapter with an extended example which demonstrates how the structure of a reductive subgroup K relates to the structure of $\mathcal{F}_K$. We highlight some interesting features which will help the reader understand later arguments.

Example 3.3.11. Let $K = \mathrm{diag}(a, b, c, d) \leq \mathrm{GL}_4$ and $K' = \mathrm{diag}(a, b, c, d, a, b, c, d) \leq \mathrm{GL}_8$. Although K' sits inside a higher dimensional group, the structure of $F_{K'}$ is no more complex than that of $\mathcal{F}_K$. Both $\mathcal{F}_K$ and $\mathcal{F}_{K'}$ contain 74 flags; Figures 3.4 and 3.5 break these sets down by flag length and signature. Every flag in $\mathcal{F}_K$ has an analogue in $\mathcal{F}_{K'}$ with twice the signature. For example, the flag $(1, 23) \in \mathcal{F}_K$ corresponds to the flag $(15, 2367) \in \mathcal{F}_{K'}$. Although subspaces in F'_K flags contain the spans of eight basis vectors, the structure of K' restricts when these can appear; the spans of e_1 and e_5 must appear in $\mathcal{F}_{K'}$ flags simultaneously, and the other basis vectors are paired off similarly. This results in a combinatorial structure similar to that of $\mathcal{F}_K$. The convenient basis pairing allows us to see similarities in the representative matrices; the 8-dimensional parabolics of Example 3.1.4 appear as a square tiling of four of their corresponding 4-dimensional parabolics.

Signature	Flags
(1,4)	4
(2,4)	6
(3,4)	4

Signature	Flags
(1,2,4)	12
(1,3,4)	12
(2,3,4)	12

Signature	Flags
(1,2,3,4)	24

Figure 3.4: A breakdown of flags in $\mathcal{F}_K$ for $K = \text{diag}(a, b, c, d) \leq \text{GL}_4$

Signature	Flags
(2,8)	4
(4,8)	6
(6,8)	4

Signature	Flags
(2,4,8)	12
(2,6,8)	12
(4,6,8)	12

Signature	Flags
(2,4,6,8)	24

Figure 3.5: A breakdown of flags in $\mathcal{F}_{K'}$ for $K' = \text{diag}(a, b, c, d, a, b, c, d) \leq \text{GL}_8$

Now consider the subgroup $K'' = \text{diag}(a, b, c, d, a, a, a, a) \leq \text{GL}_8$. This subgroup results in a set $\mathcal{F}_{K''}$ which also contains 74 flags. The signatures of $\mathcal{F}_{K''}$ flags do not follow the pattern shared by the signatures of flags in F_K and $\mathcal{F}_{K'}$, but a correspondence still exists between the flags of these sets. Figure 3.6 provides a length and signature breakdown of the flags in $F_{K''}$. Like F_K and $F_{K'}$, the set $F_{K''}$ contains 14 flags of length one, 36 flags of length two, and 24 flags of length three. The spans of e_1, e_5, e_6, e_7, and e_8 appear simultaneously in $\mathcal{F}_{K''}$ flags; this results in a wider variety of signatures when compared with F_K.

Flags in $\mathcal{F}_K$ and $\mathcal{F}_{K'}$ are constructed using four weight spaces of equal dimension, whereas flags in $\mathcal{F}_{K''}$ are constructed using three one-dimensional weight spaces and a five-dimensional weight space. The correspondence between these sets exists because we are free to vary the weights on each space independently; in each set, a flag corresponds to one potential arrangement of the four given weight spaces. For this reason, the flag $(1, 23) \in \mathcal{F}_K$ corresponds to the flag $(15678, 23) \in F_{K''}$. Consider the 24 flags in $\mathcal{F}_K$ with signature $(1, 2, 3, 4)$; the signature of their corresponding flags in $\mathcal{F}_{K''}$ depends on where the span of e_1 appears in the $\mathcal{F}_K$ flag. Six of these $\mathcal{F}_K$ flags start with the span of e_1 and these correspond to the six flags in $\mathcal{F}_{K''}$ with signature $(5, 6, 7, 8)$. Another six of these $\mathcal{F}_K$ flags introduce the span of e_1 in their third subspace; these correspond to the flags in $\mathcal{F}_{K''}$ with signature $(1, 2, 7, 8)$. In summary, the structure of $\mathcal{F}_K$ depends on the relationships between the parameters of K; adding parameters or working in higher dimensions does not necessarily lead to higher complexity.

Signature	Flags
(1,8)	3
(5,8)	1
(2,8)	3
(6,8)	3
(3,8)	1
(7,8)	3

Signature	Flags
(1,2,8)	6
(1,6,8)	3
(5,6,8)	3
(1,3,8)	3
(1,7,8)	6
(5,7,8)	3
(2,3,8)	3
(2,7,8)	3
(6,7,8)	6

Signature	Flags
(1,2,3,8)	6
(1,2,7,8)	6
(1,6,7,8)	6
(5,6,7,8)	6

Figure 3.6: A breakdown of flags in $\mathcal{F}_{K''}$ for $K'' = \mathrm{diag}(a, b, c, d, a, a, a, a) \leq \mathrm{GL}_8$

Chapter 4

New partial orders on $\mathcal{F}_K$

In this chapter we introduce new partial orders for the set of K-flags $\mathcal{F}_K$, with a view to obtaining new subsets of minimal and maximal flags. We begin by introducing a partial order for basic flags, before extending it to the set of all flags, and later introduce a second "dual" partial order. We present various results which highlight connections between our new partial orders and previous work in this thesis. The main result of this chapter is Theorem 4.2.1, which provides a characterization of relative $\mathrm{GL}(V)$-complete reducibility in terms of a subset of K-flags which is minimal with respect to our new partial order. We include several examples to demonstrate interesting features and limitations of these new partial orders.

4.1 A partial order for $\mathcal{B}_K$

Recall that f is a subflag of g if every subspace in f appears in g. We introduce the following terminology to deal with a special class of subflags.

Definition 4.1.1. We say that a flag $a = A_1 \subset \cdots \subset A_r$ is a *truncation* of a flag $b = B_1 \subset \cdots \subset B_s$ if $r < s$ and $A_i = B_i$ for $1 \leq i \leq r$.

Note that if a and b are both basic flags, one cannot be a truncation of the other. We now introduce a binary relation on $\mathcal{B}_K$. Given two basic flags, we first check whether all the subspaces of one flag appear in the other. If this is not the case, we look for inclusion between the first pair of nonequal subspaces with equal index.

Definition 4.1.2. Take a pair of basic flags $a = A_1 \subset \cdots \subset A_r$ and $b = B_1 \subset \cdots \subset B_s$, with $r \leq s$. If $A_i = B_i$ for $1 \leq i \leq r$ we say that $a \leq b$. Otherwise, let j be the smallest integer such that $A_j \neq B_j$.

- If $A_j \subsetneq B_j$, we say that $a \leq b$.
- If $B_j \subsetneq A_j$, we say that $b \leq a$.
- If there is no inclusion between A_j and B_j, we say that a and b are incomparable.

Lemma 4.1.3. *The binary relation described in Definition 4.1.2 is a partial order on* $\mathcal{B}_K$.

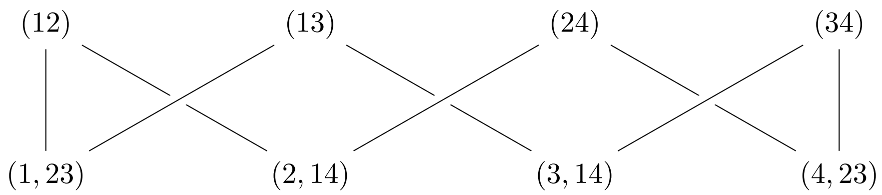


Figure 4.1: Hasse diagram of $\mathcal{B}_K$ for $K = \mathrm{diag}(s, t, t^{-1}, s^{-1}) \leq \mathrm{GL}_4(k)$

Proof. The relation is reflexive due to the conditions of the case where $A_i = B_i$ for $1 \leq i \leq r$. If we assume that $a \leq b$ and $b \leq a$, we must be in the case where the flags have equal length and $A_i = B_i$ for $1 \leq i \leq r = s$, hence the relation is antisymmetric. Let $c = C_1 \subset \cdots \subset C_t$ be a third basic flag and assume that $a \leq b$ and $b \leq c$. If $a = b$ or $b = c$ then $a \leq c$ follows immediately. If $b \not\leq a$ and $c \not\leq b$ then let j be the smallest integer such that $A_j \subsetneq B_j$, and let k be the smallest integer such that $B_k \subsetneq C_k$. Let $n = \min\{j, k\}$. Then $A_i = B_i = C_i$ for $i < n$, and $A_n \subsetneq C_n$. Hence $a \leq c$ and we conclude that the relation is transitive. $\square$

Remark 4.1.4. Knowing that $A_i = B_i$ for $1 \leq i \leq r$ is enough to conclude that $a = b$ since a cannot be a truncation of b. Later on we will be working with generic K-flags and we will not be able to rely on this property of basic flags.

We will study the subset of minimal basic flags in $\mathcal{B}_K$, henceforth denoted by $\mathcal{M}_K$.

Example 4.1.5. Let $G = \mathrm{GL}_4(k)$ and let K be the subgroup $\mathrm{diag}(s, t, t^{-1}, s^{-1})$. In Chapter 2.4 we saw that this results in the set of basic flags

$$\mathcal{B}_K = \{(12),\ (13),\ (24),\ (34),\ (1, 23),\ (2, 14),\ (3, 14),\ (4, 23)\}.$$

By applying our $\leq$ partial ordering to this set we obtain the Hasse diagram displayed in Figure 4.1 and the set of minimal basic flags $\mathcal{M}_K = \{(1, 23),\ (2, 14),\ (3, 14),\ (4, 23)\}$.

Remark 4.1.6. Our partial ordering does not result in a poset with meets and joins. For example, let $G = \mathrm{GL}_4(k)$ and let K be the subgroup $\mathrm{diag}(s^2, s, t, t^{-1})$. The Hasse diagram of Figure 4.2 demonstrates that there are pairs of elements with no greatest lower (or least upper) bound.

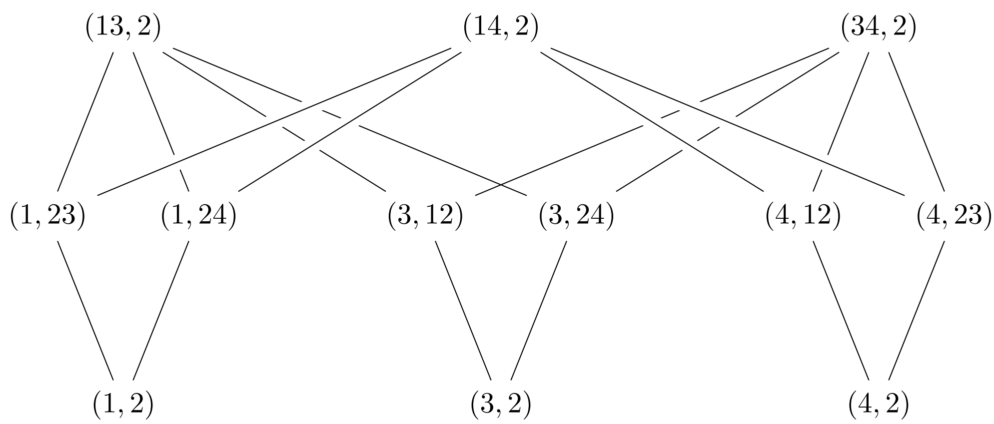


Figure 4.2: Hasse diagram of $\mathcal{B}_K$ for $K = \mathrm{diag}(s^2, s, t, t^{-1}) \leq \mathrm{GL}_4(k)$

The following result will be useful in future arguments concerning arbitrary flags.

Theorem 4.1.7. *Let $f = U_1 \subset \cdots \subset U_r$ and $g = W_1 \subset \cdots \subset W_s$ be flags in $\mathcal{F}_K$, and let $U_0 = \{0\}$. There exists a flag $h \in \mathcal{F}_K$ containing the subspaces $U_i + W_j$ such that $U_i \subseteq U_i + W_j \subseteq U_{i+1}$ for all $0 \leq i \leq r$ and all $1 \leq j \leq s$. In particular, h contains all $U_1 \cap W_j$ such that $U_1 \cap W_j \neq \{0\}$, and all $U_r + W_j$ such that $U_r + W_i \neq V$.*

The idea of the proof is that we can use the cocharacter of g to "break ties" between subspaces where the cocharacter of f has equal weight, as we observed in Example 3.2.6.

Proof. Let $f = U_1 \subset \cdots \subset U_r$ and let $g = W_1 \subset \cdots \subset W_s$. By Lemma 3.2.8 there is a maximal torus S of K and cocharacters $\lambda, \mu \in Y(S)$ such that $P_\lambda = \mathrm{Stab}_G(f)$ and $P_\mu = \mathrm{Stab}_G(g)$. We can decompose V into μ-weight spaces $X_1, X_2, \ldots, X_{s+1}$ so that each W_i is the direct sum of the X_j with $j \leq i$. Let the weight of μ on X_i be b_i for all i. We have, by choice, $b_1 > b_2 > \cdots > b_s > b_{s+1}$. Since λ and μ commute, μ stabilizes U_1. Hence U_1 decomposes into μ-weight spaces which must be of the form $U_1 \cap X_i$ when this intersection is nonempty. The subspace $U_1 \cap X_i$ has μ-weight b_i, and $U_1 \cap W_i$ is the direct sum of the $U_1 \cap X_j$ with $j \leq i$. Let a_i denote the weight of λ on U_i and choose a sufficiently large n so that $na_1 + b_1 > na_1 + b_2 > \cdots > na_1 + b_{s+1}$ are the greatest potential weights of $n\lambda + \mu$ on V and $na_r + b_1 > na_r + b_2 > \cdots > na_r + b_{s+1}$ are the least potential weights. The flag for $n\lambda + \mu$ will begin

$$U_1 \cap W_1 = U_1 \cap X_1 \subseteq (U_1 \cap X_1) \oplus (U_1 \cap X_2) = U_1 \cap W_2 \subseteq \ldots,$$

where we may have equality in some positions if the relevant weight does not appear inside U_1. A similar μ-weight space argument shows that the flag for $n\lambda + \mu$ will end

$$\ldots \subseteq U_r \oplus (\tilde{U_r} \cap X_1) = U_r + W_1 \subseteq \ldots \subseteq U_r + W_s \subseteq V,$$

where $\tilde{U_r}$ is a complement of U_r in V. Note that $\tilde{U_r}$ is the final λ-weight space in V and we may again have equality in some positions. Similar arguments prove the general result for subspaces between U_1 and U_r. $\square$

We saw flags in Example 3.2.6 which demonstrate the assertions of this result; we showed that the cocharacters corresponding to the flags (12) and (24) could be combined to give cocharacters corresponding to the flags $(2, 4, 1)$ and $(2, 1, 4)$.

The following results focus on the case where the set of H-stable minimal basic flags $\mathcal{M}_K^H$ consists of length 1 flags with opposites in $\mathcal{F}_K^H$. We know from Lemma 3.2.9 that these opposites are necessarily contained in $\mathcal{B}_K^H$.

Lemma 4.1.8. *If all $\mathcal{M}_K^H$ flags have length 1 and have opposites in $\mathcal{B}_K^H$, then all length 1 flags in $\mathcal{B}_K^H$ have opposites in $\mathcal{B}_K^H$.*

Proof. Suppose that all elements of $\mathcal{M}_K^H$ have length 1 and have opposites in $\mathcal{B}_K^H$. Let $a = A_1 \subset V \in \mathcal{B}_K^H$ be a flag of length 1 and suppose that $a \notin \mathcal{M}_K^H$. Suppose further that all H-stable flags $f < a$ of length 1 have opposites. There exists a flag $b = (B_1 \subset$

$V) \in \mathcal{M}_K^H$ such that $b < a$, which means that $B_1 \subsetneq A_1$. Since b is minimal, it has an opposite $c = (C_1 \subset V) \in \mathcal{M}_K^H$. This gives us an H-stable direct sum decomposition: $V = B_1 \oplus C_1$. Let λ_a, λ_b, and λ_c be cocharacters corresponding to the flags a, b, and c respectively. Consider the weights of these flags:

- Say λ_a acts with weight a_1 on $A_1 \subset V$, and with weight a_2 on the rest of V.
- Say λ_b acts with weight b_1 on $B_1 \subset V$, and with weight b_2 on the rest of V.
- As c is opposite b, we have that $\lambda_c = -\lambda_b$

Observe that $a_1 > a_2$ and $b_1 > b_2$. Now consider the linear combination $x\lambda_a + y\lambda_c$ where $x, y > 0$. This acts with weight $xa_1 - yb_1$ on B_1, with weight $xa_1 - yb_2$ on $A_1 \cap C_1$, and with weight $xa_2 - yb_2$ on $\tilde{A}_1$, where $\tilde{A}_1$ is the complement of A_1 in V. We would like this combination to result in an equal weighting for B_1 and $\tilde{A}_1$, so we wish to solve $xa_1 - yb_1 = xa_2 - yb_2$; this can be done by setting $x = b_1 - b_2$ and $y = a_1 - a_2$. Note that the following inequalities must hold for $x, y > 0$:

$$xa_1 - yb_2 > xa_2 - yb_2,$$

$$xa_1 - yb_2 > xa_1 - yb_1.$$

Hence the flag corresponding to the combination $x\lambda_a + y\lambda_c$ when $x = b_1 - b_2$ and $y = a_1 - a_2$ will be $f = A_1 \cap C_1 \subset V$, which is H-stable. Since $f < a$ and f has length 1, f has an opposite $g = G_1 \subset V \in \mathcal{B}_K^H$ by our earlier assumption. Consider the weights of these new flags:

- Say λ_f acts with weight f_1 on $A_1 \cap C_1 \subset V$, and with weight f_2 on the rest of V.
- As g is opposite f, we have that $\lambda_g = -\lambda_f$

The linear combination $\alpha\lambda_c + \beta\lambda_g$ acts with weight $-\alpha b_2 - \beta f_1$ on $A_1 \cap C_1$, with weight $-\alpha b_2 - \beta f_2$ on $C_1 \cap G_1$, and with weight $-\alpha b_1 - \beta f_2$ on B_1. We would like this combination to result in an equal weighting for $A_1 \cap C_1$ and B_1, so we wish to solve $-\alpha b_2 - \beta f_1 = -\alpha b_1 - \beta f_2$; this can be done by setting $\alpha = f_1 - f_2$ and $\beta = b_1 - b_2$. Note that the following inequalities must hold for $\alpha, \beta > 0$:

$$-\alpha b_2 - \beta f_2 > -\alpha b_1 - \beta f_2,$$
$$-\alpha b_2 - \beta f_2 > -\alpha b_2 - \beta f_1.$$

Hence the flag corresponding to the combination $\alpha\lambda_c + \beta\lambda_g$ will be $h = C_1 \cap G_1 \subset V$, which is also H stable. We have that $A_1 \oplus (C_1 \cap G_1) = V$, so we have a flag $h \in \mathcal{B}_K^H$ that is opposite a. Given our initial assumption that flags in $\mathcal{M}_K^H$ have opposites and have length 1, the result follows by induction. $\square$

Lemma 4.1.9. *If all $\mathcal{M}_K^H$ flags have length* 1 *and have opposites in $\mathcal{B}_K^H$, then all flags in $\mathcal{B}_K^H$ have length* 1.

Proof. Suppose all $\mathcal{M}_K^H$ flags have length 1 and have opposites in $\mathcal{B}_K^H$. Suppose there exists a flag $a = A_1 \subset A_2 \subset \cdots \subset V \in \mathcal{B}_K^H$ with length greater than one and assume that all flags $f < a$ have length 1. There exists a flag $b = B_1 \subset V \in \mathcal{M}_K^H$ such that $b < a$, which means that $B_1 \subsetneq A_1$. Since b is minimal, it has an opposite $c = (C_1 \subset V) \in \mathcal{M}_K^H$. Arguing similarly to Lemma 4.1.8, we can break A_2 down into three disjoint subspaces: B_1, $A_1 \cap C_1$, and A_2', where A_2' is the complement of A_1 in A_2. There is a basic flag $f \in \mathcal{B}_K^H$ containing $A_1 \cap C_1$; in fact $f = (A_1 \cap C_1 \subset V)$ since $f < a$, and flags preceding a must have length 1. By Lemma 4.1.8, f has an opposite $g \in \mathcal{B}_K^H$.

We will consider the linear combination $l\lambda_a + m\lambda_c + n\lambda_g$, which acts with the following weights:

$$\begin{aligned} &la_1 - mb_1 - nf_2 \text{ on } B_1 \\ &la_1 - mb_2 - nf_1 \text{ on } A_1 \cap C_1 \\ &la_2 - mb_2 - nf_2 \text{ on } A_2' \end{aligned}$$

We wish to construct a flag that has A_2 as its initial subspace; to do this, we will need to solve the following equations simultaneously:

$$la_1 - mb_1 - nf_2 = la_1 - mb_2 - nf_1$$

$$la_1 - mb_1 - nf_2 = la_2 - mb_2 - nf_2$$

Working with these equations yields the following:

$$m(b_1 - b_2) = n(f_1 - f_2)$$

$$l(a_1 - a_2) = m(b_1 - b_2)$$

We can therefore reach a solution by setting

$$m = (f_1 - f_2),\ n = (b_1 - b_2),\ \text{and } l = \frac{(f_1 - f_2)(b_1 - b_2)}{(a_1 - a_2)}.$$

The flag corresponding to $l\lambda_a + m\lambda_c + n\lambda_g$ has initial subspace A_2, and the rest of the flag is decided by λ_a, contradicting the fact that a is a basic flag. Hence there is no flag in $\mathcal{B}_K^H$ with length greater than one. $\square$

Combining Lemmas 4.1.8 and 4.1.9 and recalling that $\mathcal{M}_K^H \subseteq \mathcal{F}_K^H$ gives us the following result.

Lemma 4.1.10. *Every element of $\mathcal{M}_K^H$ has length 1 and an opposite in $\mathcal{B}_K^H$ if and only if every element of $\mathcal{B}_K^H$ has length 1 and an opposite in $\mathcal{B}_K^H$. Equivalently, every element of $\mathcal{M}_K^H$ has length 1 and an opposite in $\mathcal{B}_K^H$ if and only if H is relatively G-completely reducible with respect to K.*

The latter statement follows from Theorem 3.2.7. This condition of basic flags having length 1 is motivated by the case where $K = \mathrm{GL}(V)$. This result, then, can be viewed as an analogue of Proposition 1.2.11, and of the fact that a module is completely reducible if and only if every minimal (i.e., simple) submodule has a complement. We do not currently know if this result holds without the length 1 hypothesis on elements of $\mathcal{M}_K^H$, and so we try a different approach.

4.1.1 Extending the partial order

To extend our partial order on $\mathcal{B}_K$ to the entire set of K-flags, we begin by defining a similar binary relation.

Definition 4.1.11. Take a pair of flags $a = A_1 \subset \cdots \subset A_r$ and $b = B_1 \subset \cdots \subset B_s$ in $\mathcal{F}_K$, with $r \leq s$. If $A_i = B_i$ for $1 \leq i \leq r$ we say that $a \leq b$. Otherwise, let j be the smallest integer such that $A_j \neq B_j$.

- If $A_j \subsetneq B_j$, we say that $a \leq b$.
- If $B_j \subsetneq A_j$, we say that $b \leq a$.
- If there is no inclusion between A_j and B_j, we say that a and b are incomparable.

Lemma 4.1.12. *The binary relation described in Definition 4.1.11 is a partial order on $\mathcal{F}_K$.*

Proof. The relation is reflexive due to the conditions of the case where $A_i = B_i$ for $1 \leq i \leq r$. If we assume that $a \leq b$ and $b \leq a$, we must be in the case where the flags have equal length and $A_i = B_i$ for $1 \leq i \leq r = s$, hence the relation is antisymmetric. Let $c = C_1 \subset \cdots \subset C_t$ be a third flag and assume that $a \leq b$ and $b \leq c$. If $a = b$ or $b = c$ then $a \leq c$ follows immediately. If $b \nleq a$ and $c \nleq a$ then we are in one of the following four cases:

(1) There exist least integers w and x such that $A_w \subsetneq B_w$ and $B_x \subsetneq C_x$.

(2) There is a least integer y such that $A_y \subsetneq B_y$ and b is a truncation of c.

(3) a is a truncation of b and there is a least integer z such that $B_z \subsetneq C_z$.

(4) a is a truncation of b, and b is a truncation of c.

The transitivity proof from Lemma 4.1.3 gives us transitivity in the first case. In case (2) we have $A_i = B_i = C_i$ until $A_y \subsetneq B_y = C_y$, so $a \leq c$. In case (3) we have $A_i = B_i = C_i$ until $i = r < t$ or until $A_z = B_z \subsetneq C_z$; in either case we conclude that $a \leq c$. In case (4) we have $A_i = B_i = C_i$ for $1 \leq i \leq r < t$, and so $a \leq c$. $\square$

This partial order is similar to the lexicographic order; both compare sequences of elements by studying the first place where the sequences differ. We therefore refer to flags which are minimal with respect to this extended partial order as *minilex* flags.

Example 4.1.13. The Hasse diagram for $\mathcal{F}_K$ flags from our running example of $G = \mathrm{GL}_4(k)$ and $K = \mathrm{diag}(s, t, t^{-1}, s^{-1})$ is displayed in Figure 4.3, with the basic flags labelled in bold. The nonbasic flags of $\mathcal{F}_K$ are the minilex flags; they all precede their respective basic subflags which have signature $(1, 3, 4)$.

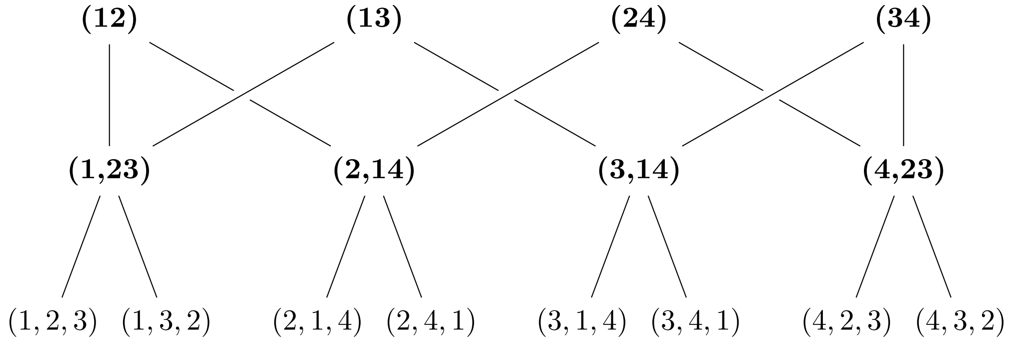


Figure 4.3: Hasse diagram of $\mathcal{F}_K$ for $K = \mathrm{diag}(s, t, t^{-1}, s^{-1}) \leq \mathrm{GL}_4(k)$

It is not always the case that nonbasic flags precede their basic subflags in our extended partial order, as the following example demonstrates.

Example 4.1.14. Let $G = \mathrm{GL}_4(k)$ and let K be the subgroup $\mathrm{diag}(s, t, sr^{-1}, sr^{-2})$. Then three of the minilex $\mathcal{F}_K$ flags are basic flags, as demonstrated in Figure 4.4 (where basic flags have again been labelled in bold).

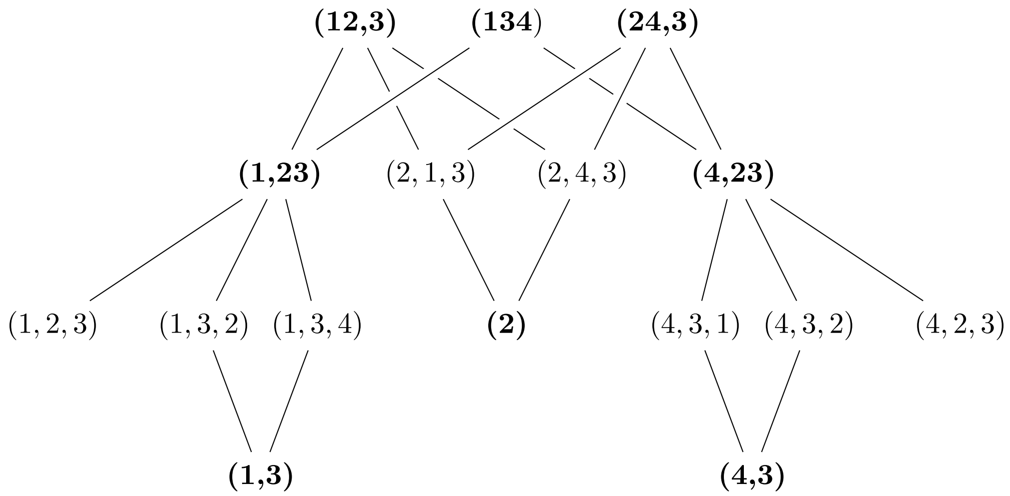


Figure 4.4: Hasse diagram of $\mathcal{F}_K$ for $K = \mathrm{diag}(s, t, sr^{-1}, sr^{-2}) \leq \mathrm{GL}_4(k)$

Remark 4.1.15. Extending the partial order to include all $\mathcal{F}_K$ flags still does not provide us with meets and joins. Recall the group $G = \mathrm{GL}_4(k)$ and the subgroup $K = \mathrm{diag}(s^2, s, t, t^{-1})$ from Remark 4.1.6. The partial order on $\mathcal{B}_K$ resulted in a subset of basic flags $S = \{(1, 23),\ (1, 24),\ (13, 2),\ (14, 2)\}$ with no notion of meets or joins; see the leftmost diagram of Figure 4.1.15. Extending to $\mathcal{F}_K$ reveals that no nonbasic flags are introduced which provide meets and joins. In fact, another problematic set $S' = \{(1, 2, 3),\ (1, 2, 4),\ (1, 23),\ (1, 24)\}$ is introduced, as demonstrated in the rightmost diagram of Figure 4.1.15.

The following result extends our study of opposed length 1 flags to minilex $\mathcal{F}_K$ flags.

Lemma 4.1.16. *Suppose all minilex $\mathcal{F}_K$ flags have length* 1 *and opposites in $\mathcal{F}_K$. Then all $\mathcal{B}_K$ flags have length* 1 *and opposites in $\mathcal{B}_K$.*

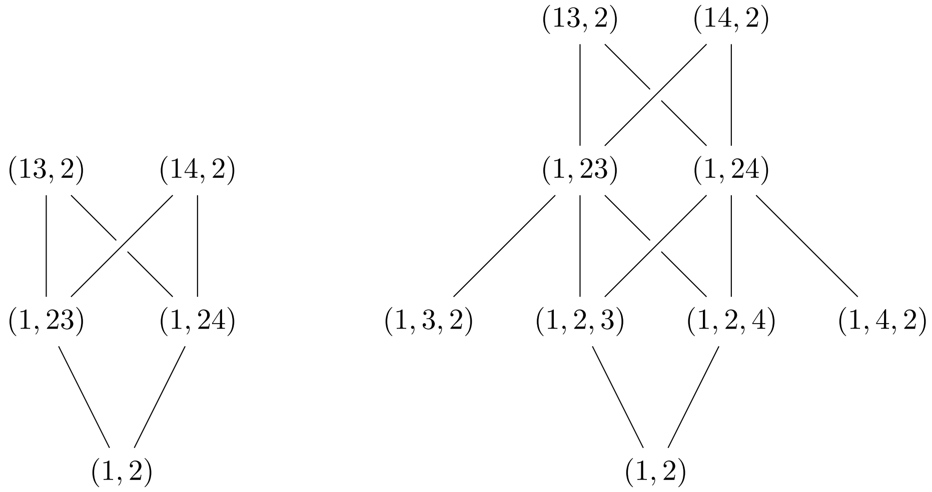


Figure 4.5: Hasse subdiagrams of $\mathcal{B}_K$ and $\mathcal{F}_K$ for $K = \mathrm{diag}(a^2, a, b, b^{-1}) \leq \mathrm{GL}_4(k)$

Proof. We prove that every minimal basic flag has an opposite in $\mathcal{B}_K$. Let $a \in \mathcal{M}_K$. There is a minilex flag $b \leq a$ with the same initial subspace as a; if its initial subspace was properly contained in the initial subspace of a, we could find a basic flag containing that subspace (by Lemma 3.2.10) which would contradict the minimality of a in $\mathcal{B}_K$. By assumption, b is a length 1 flag and is therefore a subflag of a unless it equals a. Since a is a basic flag, it must be the case that $b = a$ is a minilex flag of length 1. Hence all minimal basic flags are minilex flags of length 1 and have opposites in $\mathcal{F}_K$ by assumption. These opposites must be basic flags by Lemma 3.2.9. The conclusion for all basic flags follows by Lemma 4.1.10. $\square$

The following result arose from attempts to mirror the arguments of Kempf in [22] in our setting, with the goal of producing objects which are automorphism invariant. Fix a maximal torus T of K such that there exists an unopposed minilex $\mathcal{F}_K^H$ flag coming from some $\lambda \in Y(T)$. Let $U_1, \ldots, U_r$ be the initial subspaces of all such flags (listed without repeats) and let $\lambda_1, \ldots, \lambda_r$ be cocharacters of T giving rise to unopposed minilex flags $f_1, \ldots, f_r$ with initial subspaces $U_1, \ldots, U_r$. Then we have the following.

Lemma 4.1.17. *With notation as above, for each $1 \leq i \leq r$, let $W_i = \sum_{j=1}^{i} U_j$. Then:*

(i) *W_i is a proper subspace of V;*

(ii) *the sum for W_i is direct.*

In particular, $U_T := \bigoplus_{j=1}^{r} U_j$ is a proper subspace of V.

Proof. We prove the statement by induction. Observe that $W_1 = U_1$ trivially satisfies both properties. Suppose now that we have proved the statements for W_{i-1}.

Each intersection $U_j \cap U_k$ is trivial when $j \neq k$; this follows from Theorem 4.1.7 and the minimality of the initial subspaces of minilex flags. Then we can choose a basis $\mathcal{B}$ for V consisting of T-weight vectors such that each U_j has a basis $\mathcal{B}_j \subseteq \mathcal{B}$, and the subsets $\mathcal{B}_j$ are disjoint. Then $\bigcup_{j=1}^{i-1} \mathcal{B}_j$ is a basis for W_{i-1} and $\mathcal{B}_i$ is a basis for U_i. Hence $U_i \cap W_{i-1} = 0$ and and we can write $W_i = W_{i-1} \oplus U_i$, proving (ii).

To prove (i), suppose for contradiction that $W_i = V = U_1 \oplus \cdots \oplus U_i$. Let the λ_1-weight spaces of f_1 be $X_1, \ldots, X_k$ for some k, so that we can write

$$f_1 = X_1 \subset X_1 \oplus X_2 \subset \cdots \subset X_1 \oplus \cdots \oplus X_{k-1}.$$

By a previous argument to the above, the intersection of each U_j for $1 \leq j \leq i$ with the spaces in this flag is either 0 or U_j. Then for each $1 \leq j \leq i$ we have $U_j = X_{j'}$ for some $1 \leq j' \leq k$. Since we can write $V = U_1 \oplus \cdots \oplus U_i = X_1 \oplus \cdots \oplus X_k$, we must have that the $X_{j'}$ are the U_j, possibly reordered. Each U_j is stabilized by H, which means that each X_j' is stabilized by H. Then H stabilizes the flag

$$f_1' = X_k \subset X_k \oplus X_{k-1} \subset \cdots \subset X_2 \oplus \cdots \oplus X_k.$$

This is an H-stable opposite to f_1, providing the required contradiction. $\square$

Remark 4.1.18. Showing that intersections are trivial in the proof of (ii) relies on the fact that we are working in a fixed torus; this assumption allows us to fix a basis and work with subsets of that basis.

4.2 A second characterization of relative $\mathrm{GL}(V)$-complete reducibility

Like the subset of basic flags, the subset of minilex flags is a set of minimal flags which can be used to characterize relative G-complete reducibility. The following theorem is the main result of this chapter.

Theorem 4.2.1. *Let K and H be subgroups of $\mathrm{GL}(V)$, with K reductive. If each minilex H-stable K-flag has an H-stable opposite, then H is relatively G-completely reducible with respect to K.*

The proof depends on the hypothesis for minilex flags passing down to Levi subgroups. For the remainder of this section we will be assuming that each minilex H-stable K-flag has an H-stable opposite.

Lemma 4.2.2. *Suppose that $\lambda \in Y(K)$ is a cocharacter centralizing H and let $L_K = L_\lambda(K)$. Then a minilex H-stable L_K-flag is a minilex H-stable K-flag arising from a cocharacter of L_K. Moreover, every minilex H-stable K-flag is $C_K(H)$-conjugate to a minilex L_K-flag.*

Proof. Suppose $a = A_1 \subset \cdots \subset A_{r-1}$ is a minilex H-stable L_K-flag, and let $\alpha \in Y(L_K)$ be a cocharacter corresponding to a. Since $\alpha \in Y(L_K)$, we have that λ commutes with α and hence λ stabilizes the α-weight spaces in V by Lemma 1.1.24. We show that a is minilex amongst H-stable K-flags.

We begin by studying the λ-weights on the α-weight spaces. Denote the α-weight spaces by $X_1, X_2, \ldots, X_r$, labelled so that $A_j = \sum_{i=1}^{j} X_i$ for each $1 \leq j \leq r-1$. We claim that λ has only one weight on each X_i, except possibly X_r. To see this, suppose that λ has more than one weight on some X_i with $i < r$, and choose i minimal such that λ has multiple weights on X_i. Suppose the highest λ-weight space in X_i is W. For sufficiently large n, the flag corresponding to $n\alpha + \lambda \in Y(L_K)$ starts

$$A_1 \subset \cdots \subset A_{i-1} \subset A_{i-1} + W \subset \ldots,$$

which precedes a in our partial order because $A_{i-1} + W$ is a proper subspace of $A_{i-1} + X_i = A_i$. This new flag is H-stable, since the new cocharacter is a combination of α and λ. This contradicts the fact that a is minilex among H-stable L_K-flags, proving our claim; λ has only one weight on each X_i with $i < r$.

Now suppose that $b = B_1 \subset \cdots \subset B_{s-1}$ is a minilex H-stable K-flag such that $b \leq a$, and let $\beta \in Y(K)$ be a cocharacter giving rise to b which centralizes H. Such a cocharacter exists by our initial assumption on minilex flags in $\mathcal{F}_K^H$. If $b \neq a$, then $B_i = A_i$ until a point where $B_j \subsetneq A_j$, or b is a truncation of a. In either case, we work with the subflag $b' = B_1 \subset \cdots \subset B_j$ of b. Although b' may not be a K-flag, $Q = \mathrm{Stab}_K(b')$ is a parabolic subgroup of K because it contains $P_\beta(K)$. Hence, in particular, β evaluates in Q.

We claim that λ also evaluates in Q. Since λ stabilizes each α-weight space, λ stabilizes $B_1 = A_1, \ldots, B_{j-1} = A_{j-1}$. But $B_j \subseteq A_j$, and $A_j = A_{j-1} \oplus X_j = B_{j-1} \oplus X_j$, which implies that

$$B_j = B_{j-1} \oplus (B_j \cap X_j). \tag{4.2.1}$$

Since λ acts on X_j with a single weight, λ stabilizes every subspace of X_j, including $B_j \cap X_j$. Now (4.2.1) shows that λ stabilizes B_j also. Thus λ also evaluates in Q.

The previous two paragraphs imply that $\lambda, \beta \in Y(C_Q(H))$, so we may find some $C_Q(H)$-conjugate γ of β which commutes with λ. Conjugating in $C_Q(H)$ ensures that this γ still centralizes H, and because the element we conjugate with lies in Q, γ still corresponds to a flag which starts with the subflag b'. Then the γ-flag c is an H-stable L_K-flag such that $c \leq a$, so it must equal a, since a is minilex among L_K-flags. This implies that we had $a = b$ all along, proving that that minilex H-stable L_K-flags are minilex H-stable K-flags arising from cocharacters of L_K.

For the final part, suppose $d = D_1 \subset \cdots \subset D_{r-1}$ is a minilex H-stable K-flag. Then d comes from some cocharacter $\delta \in Y(K)$ which centralizes H. By conjugating in $C_K(H)$, we may line δ up with λ, to give a cocharacter $\delta' \in Y(L_K)$ corresponding to a flag $d' = D'_1 \subset \cdots \subset D'_{r-1}$. This must be a minilex L_K flag; if it was not, we could conjugate a flag preceding d' to obtain a flag preceding d in $\mathcal{F}_K^H$. □

We can now show that our hypothesis for minilex flags in $\mathcal{F}_K^H$ passes down to Levi subgroups of K.

Corollary 4.2.3. *With notation as above, each minilex H-stable L_K-flag has an H-stable opposite.*

Proof. Let a be a minilex H-stable L_K-flag. By Lemma 4.2.2, a is a minilex H-stable K-flag, and hence there is a cocharacter $\alpha \in Y(K)$ which gives rise to a and centralizes

H. Since a is an L_K-flag, λ evaluates in $\mathrm{Stab}_K(a)$. We may conjugate in $C_{\mathrm{Stab}_K(a)}(H)$ to line up α and λ without changing the flag a. Thus a has an H-stable L_K-opposite. $\square$

We could proceed inductively, applying this corollary until the process terminates. Alternatively, we can skip ahead as in the following lemma.

Lemma 4.2.4. *Let S be a maximal torus of $C_K(H)$ and let $L_K = C_K(S)$. Then each minilex H-stable L_K-flag has an H-stable opposite. Moreover, H is relatively G-cr with respect to K if and only if H is relatively G-cr with respect to L_K, and all minilex L_K-flags come from cocharacters which are central in L_K.*

Proof. We may find a $\lambda \in Y(S)$ such that $L_K = L_\lambda(K)$. Since $\lambda \in Y(S)$, we have that λ centralizes H. Hence Corollary 4.2.3 applies to show that minilex H-stable L_K-flags have H-stable opposites. The relative G-cr statement follows from [10, Proposition 3.17(i)]. For the final statement, note that a minilex L_K-flag has an H-stable opposite, and hence comes from a cocharacter which commutes with H. Such a cocharacter will evaluate in a maximal torus of $C_{L_K}(H)$, and the central torus S of L_K is the only such torus. $\square$

Armed with this result for Levi subgroups, we introduce a larger central torus and work with a new set of well-behaved flags. We obtain a necessary and sufficient condition for the relative G-complete reducibility of H with respect to K.

Lemma 4.2.5. *Suppose S is a maximal torus of $C_K(H)$ and let $L_K = C_K(S)$. Note that $C_G(S)$ is a Levi subgroup of $G = \mathrm{GL}(V)$ and let Z denote the centre of $C_G(S)$. Then H is relatively G-cr with respect to K if and only if H is relatively G-cr with respect to $L_K Z$.*

Proof. First off, note that $S \subseteq Z$ since Z is the centre of $C_G(S)$ and S is central. Also note that $C_G(S) = C_G(Z)$ and $L_K = C_K(S) = C_K(Z)$. Hence Z centralizes L_K, and hence Z is central in $L_K Z$. If T is any torus of $L_K Z$ that centralizes H, we may write $T = T_0 Z$ for some torus T_0 in L_K which centralizes H. But then $T_0 \subseteq S \subseteq Z$, because S is the unique maximal torus of $C_{L_K}(H)$. Hence Z is the unique maximal torus in $C_{L_K Z}(H)$.

Suppose H is relatively G-cr with respect to K and suppose $\lambda \in Y(L_K Z)$ is a cocharacter such that $H \subseteq P_\lambda$. Then we know that H is relatively G-cr with respect to L_K. Writing $\lambda = \lambda_0 + \lambda_1$ with $\lambda_0 \in Y(Z)$ and $\lambda_1 \in Y(L_K)$, we see that $H \subseteq P_{\lambda_1}$; because λ_0 centralizes H, we can find an $R_u(P_{\lambda_1}(L_K))$-conjugate $u \cdot \lambda_1$ of λ_1 which centralizes H. Because $u \in L_K$, u fixes $\lambda_0 \in Y(Z)$, and hence $u \cdot \lambda = u \cdot \lambda_0 + u \cdot \lambda_1 = \lambda_0 + u \cdot \lambda_1$ centralizes H. Furthermore, we see that $u \in P_\lambda$ because $u \in P_{\lambda_1} \cap L_{\lambda_0}$, and hence we are done – it suffices to show that H is fixed by some P_λ-conjugate to λ.

For the converse, suppose H is not relatively G-cr with respect to K. Then H is not relatively G-cr with respect to L_K, by Lemma 4.2.4, so we can find $\lambda \in Y(L_K)$ such that H is contained in P_λ and not contained in any Levi subgroup of P_λ coming from L_K. Note that S is contained in L_λ, and hence in P_λ. Suppose for a contradiction that H is in some Levi subgroup of P_λ coming from K. Then this Levi subgroup has the form L_μ, where μ commutes with H. The image of μ and S are both tori contained in $C_{P_\lambda(K)}(H)$, and S is maximal, hence we can conjugate μ in S by an element of $C_{P_\lambda(K)}(H)$. But this puts H is in a Levi subgroup coming from L_K, a contradiction.

Then there is a cocharacter $\lambda \in Y(K)$ which commutes with S, and such that H is contained in P_λ but in no Levi subgroup of P_λ coming from K. Since λ commutes with S, λ is a cocharacter of $L_K Z$. The Levi subgroups of P_λ coming from $L_K Z$ are the same as the Levi subgroups coming from L_K, because Z is central. Hence H is not contained in any Levi subgroup of P_λ coming from $L_K Z$, and so H is not relatively G-cr with respect to $L_K Z$. □

The following lemma is the final result that we need. Recall that we are still assuming that each minilex H-stable K-flag has an H-stable opposite.

Lemma 4.2.6. *With notation as above, H is relatively G-cr with respect to $L_K Z$.*

Proof. Suppose $a = A_1 \subset \cdots \subset A_{r-1}$ is an $L_K Z$-flag. We claim that a arises from a cocharacter of Z. Once this claim is established, the result follows immediately, because any cocharacter of Z centralizes H. Let a have representative cocharacter $\mu = \sigma + \beta$, with $\sigma \in Y(Z)$ and $\beta \in Y(L_K)$. Let β correspond to the L_K-flag b, and note that it is H-stable, since a is H-stable and σ centralizes H. We proceed with a sequence of refinements and reorderings of weight spaces making up b (and subsequent flags in the sequence), in order to show that in fact b comes from a cocharacter of Z.

First, we detail our process of refinement. If b does not begin with an H-stable minilex L_K flag (which holds in particular if b is not minilex itself), then choose a minilex flag $c \leq b$. The assumption that b does not begin with a minilex flag means that c is not a truncation of b. Since it is minilex, c comes from a cocharacter of S by Lemma 4.2.4. Say that $\gamma \in Y(S)$ gives rise to c. For a sufficiently large $n \in \mathbb{N}$ we have that $n\gamma + \beta$ gives rise to a refinement of c, a flag c_1 containing c as a subflag. Since c_1 arises from a combination of β and γ, it is a H-stable flag. Since c is minilex, c_1 must either equal c or be constructed by adding extra subspaces onto the end of c. In either case, c_1 begins with the minilex flag c, and hence $c_1 \leq b$ since c is not a truncation of b. Now we may write $\mu = (\sigma - n\gamma) + (n\gamma + \beta)$ with the first component in $Y(Z)$ and the second in $Y(L_K)$. Replacing σ with $\sigma - n\gamma$ and β with $n\gamma + \beta$, we can replace b with c_1 and see that we may assume that we have chosen σ and β so that the flag b corresponding to β begins with a minilex H-stable L_K-flag.

Next, we detail the process of rearrangement. Suppose b begins with a minilex H-stable flag c, arising from a cocharacter $\gamma \in Y(S)$, but b is not itself minilex. We may write b in the form

$$b = B_1 \subset \ldots B_{s-1} \subset B_s \subset \cdots \subset B_{t-1},$$

such that the minilex flag $c \leq b$ is the flag $c = B_1 \subset \cdots \subset B_{s-1}$. Let the β-weight spaces for b be $X_1, \ldots, X_t$, listed in decreasing order of weight so that $B_i = \sum_{j=1}^{i} X_j$ for each $1 \leq i \leq t-1$. Note that the γ-weight spaces are $X_1, \ldots, X_{s-1}$ and $\sum_{j=s}^{i} X_j$. By construction, the γ-weights on these spaces are also in strictly decreasing order of size. Finally, note that since the $X_1, \dots, X_s$ are distinct γ-weight spaces, they are sums of distinct Z-weight spaces: if γ-weights can tell these spaces apart then Z-weights can, because γ evaluates in Z. If we subtract a large multiple of γ from β, we obtain a cocharacter $\beta_1 = \beta - m\gamma$ with a corresponding flag b_1 which has the form

$$X_s \subset X_s \oplus X_{s+1} \subset \cdots \subset \sum_{j=s}^{t} X_j \subset \sum_{j=s-1}^{t} X_j \subset \cdots \subset \sum_{j=2}^{t} X_j.$$

This process has transferred the first s weight spaces from the flag b to the end of the flag b_1, in reverse order, by adding on a sufficiently large multiple of $-\gamma$. Note that, because $\gamma \in Y(Z)$, we may write $\sigma_1 = \sigma + m\gamma \in Y(Z)$ and we have $\mu = \sigma_1 + \beta_1$. That is, we may replace β with β_1 and σ with σ_1 without changing the flag a.

Now we can proceed with a sequence of refinements and reorderings. Given the flag a, refine it so that it begins with a minilex H-stable L_K-flag, and then reorder so that the corresponding Z-weight spaces at the start of this flag appear at the end. Then repeat with the new flag. If at any point we end up being able to replace a with a minilex H-stable L_K-flag, we stop. Otherwise, the process of refinement cannot continue forever, for dimension reasons. It will stop when all the subspaces of a are made up of combinations of Z-weight spaces; any space which is not a combination of Z-weight spaces would eventually reach the front of the flag after reordering. Thus we reach a point where the flag a arises from a cocharacter whose weight spaces are combinations of Z-weight spaces; but since Z is the centre of the Levi subgroup corresponding to the direct sum decomposition of V into the Z-weight spaces, this means that the corresponding cocharacter β lies in $Y(Z)$. Since a does not change throughout this process, we see that we had $\beta \in Y(Z)$ all along, and so a arises from a cocharacter of Z as claimed. □

Remark 4.2.7. Suppose we know that H is relatively G-cr with respect to K. If we pick any cocharacter $\lambda \in Y(K)$ such that $H \subseteq P_\lambda$, there is some K-conjugate of λ which centralizes H. Thus, every H-stable K-flag comes from a torus of $C_K(H)$. When we descend to L_K, every H-stable L_K-flag comes from a central cocharacter, which is what we have shown happens under an apparently weaker hypothesis.

Theorem 4.2.1 follows as a consequence of Lemma 4.2.5 and Lemma 4.2.6. We have now achieved one of our original goals: given a reductive group K acting on an affine variety X, and a point $x \in X$, we have found a "combinatorial" description of what it means for the K-orbit of x to be closed. Embed K and X inside a $\mathrm{GL}(V)$ as in Theorem 2.4.1 and let H be the subgroup corresponding to x. Then the set of cocharacters of K such that $\lim_\lambda x$ exists gives rise to the set F_K^H of H-stable K-flags; the K-orbit of x is closed if and only if every minilex flag in F_K^H has an opposite.

4.3 Further results

We have now seen two subsets of minimal flags (basic flags and minilex flags) which can be used to characterize relative $\mathrm{GL}(V)$-complete reducibility; a natural question to ask is whether the intersection of these subsets will provide us with a third characterization.

Our running example of $K = \text{diag}(s, t, t^{-1}, s^{-1}) \leq \text{GL}_4(k)$ presents an issue with this approach; the intersection of these subsets can be empty.

Example 4.3.1. Let $G = \text{GL}_4(k)$ and let be the subgroup $K = \text{diag}(s, t, t^{-1}, s^{-1})$. Recall that F_K consists of the following flags:

Length	$\mathcal{F}_K$ flags
1	(12), (13), (24), (34)
2	$(1,23)$, $(2,14)$, $(3,14)$, $(4,23)$
3	$(1,2,3)$, $(1,3,2)$, $(2,1,4)$, $(2,4,1)$ $(3,1,4)$, $(3,4,1)$, $(4,2,3)$, $(4,3,2)$

One can check (recalling the Hasse diagram of Figure 4.3) that a flag in $\mathcal{F}_K$ is basic if and only if it has length less than 3, and a flag in $\mathcal{F}_K$ is minilex if and only if it has length 3. Then there are no flags which are both minilex and basic.

This could be dealt with as a trivial or special case, but the following example (discovered with the aid of computer software) shows that the subset of minilex basic K-flags may only contain a single flag.

Example 4.3.2. Let $G' = \text{GL}_5(k)$ and let $K' = \text{diag}(s, t, t^{-1}, s^{-1}, u)$. Understanding the structure of F_K from Example 4.3.1 will help us to make sense of the structure of $\mathcal{F}_{K'}$, which contains 122 flags. Let $V = \langle e_1, e_2, e_3, e_4 \rangle$ and let $V' = \langle e_1, e_2, e_3, e_4, e_5 \rangle$. For a flag $f = (0 = U_0 \subset U_1 \subset \cdots \subset U_{l+1} = V)$ with length $l \geq 1$ in $\mathcal{F}_K$, there are $(2l + 3)$ corresponding flags in $\mathcal{F}_{K'}$. The additional subspace appearing in $\mathcal{F}_{K'}$ flags, the span of the canonical basis vector e_5, can appear at the following points in f:

- Between U_{i-1} and U_i for each $1 \leq i \leq l + 1$,
- at the same time as U_i for each $1 \leq i \leq l + 1$,
- or between $U_{l+1} = V$ and V'.

For example, $(12) \in \mathcal{F}_K$ corresponds to the $\mathcal{F}_{K'}$ flags $(5, 12), (12, 5)$, (125), (12), and $(12, 34)$. Note that our shorthand masks some complexity here; the flag $(12) \in \mathcal{F}_K$ represents $\langle e_1, e_2 \rangle \in k^4$, whereas $(12) \in F_{K'}$ represents $\langle e_1, e_2 \rangle \in k^5$. The 8 flags of length 3 in $\mathcal{F}_K$ correspond to 72 flags in $\mathcal{F}_{K'}$, the 4 flags of length 2 in $\mathcal{F}_K$ correspond to 28 flags in $\mathcal{F}_{K'}$, and the 4 flags of length 1 in $\mathcal{F}_K$ correspond to 20 flags in $\mathcal{F}_{K'}$. This accounts for 120 of the 122 flags in $F_{K'}$; no $\mathcal{F}_K$ flags correspond to the $\mathcal{F}_{K'}$ flags (1234) and (5). Most of the flags in $\mathcal{B}_{K'}$ correspond to flags in $\mathcal{B}_K$, as the following table illustrates.

$\mathcal{B}_K$ flag	Corresponding $\mathcal{B}_{K'}$ flags
(12)	(12), (125)
(13)	(13), (135)
(24)	(24), (245)
(34)	(34), (345)
(1,23)	(1,23), (1,235), (15,23)
(2,14)	(2,14), (2,145), (25,14)
(3,14)	(3,14), (3,145), (35,14)
(4,23)	(4,23), (4,235), (45,23)

The only elements of $\mathcal{B}_{K'}$ not listed above are the flags (1234) and (5). Then the set of minimal basic flags is $\mathcal{M}_{K'} = \{(1,23), (2,14), (3,14), (4,23), (5)\}$. If a basic flag is minimal in $\mathcal{F}_K$ then it must be minimal in $\mathcal{B}_{K'}$, so the set of minilex basics must be a subset of $\mathcal{M}_{K'}$. For each flag f of signature $(1,3,5)$ in $\mathcal{M}_{K'}$ there is a flag f' of signature $(1,2,3,5)$ in $\mathcal{F}_{K'}$ such that $f' \leq f$. Thus the only minilex basic flag in $\mathcal{F}_{K'}$ is (5).

The previous example demonstrates that even strong assumptions are unlikely to lead to positive results for arbitrary sets of minilex basic flags. There are, however, some interesting interactions between the partial orders we have introduced on $\mathcal{F}_K$. First we observe that flags and their subflags are always comparable.

Lemma 4.3.3. *Let $f, f' \in \mathcal{F}_K$ and suppose f' is a proper subflag of f. Then f' and f are comparable. Moreover, $f \leq f'$ unless f' is a truncation of f.*

Proof. Let $f = U_1 \subset \cdots \subset U_r$ and $f' = W_1 \subset \cdots \subset W_s$, with $s < r$ and suppose that f' is a proper subflag of f. Then either $W_i = U_i$ for all $1 \leq i \leq s$, or there exists a j such that $W_j \subsetneq U_j$ and $W_i = U_i$ for all $1 \leq i < j$. In either case, f' and f are comparable. In the first case, f' is a truncation of f and we have $f' \leq f$. Otherwise, we are in the second case where $f \leq f'$. $\square$

The following corollary for minilex basic flags follows immediately.

Corollary 4.3.4. *Suppose f' is a proper subflag of f which is minilex and basic. Then f' is a truncation of f.*

Flags which are maximal with respect to our new partial order must be minimal with respect to the $\preceq$ ordering on $\mathcal{F}_K$ discussed in Chapter 2.4.

Lemma 4.3.5. *If $f \in \mathcal{F}_K$ is maximal with respect to the partial order of Definition 4.1.11, then $f \in \mathcal{B}_K$.*

Proof. We show that every nonbasic flag is preceded by one of its basic subflags. Let $f = U_1 \subset \cdots \subset U_r$ be a nonbasic flag in $\mathcal{F}_K$. By Lemma 3.2.10, there exists a proper subflag $f' \in \mathcal{F}_K$ of f containing U_r. Then f' is a subflag of f which is not a truncation, so $f \leq f'$ by Lemma 4.3.3. $\square$

The following proposition demonstrates a connection between flags in $\mathcal{F}_K$ which are minimal with respect to exactly one of the partial orders we have introduced.

Proposition 4.3.6. *Let $f \in \mathcal{F}_K$ be a minilex nonbasic flag and suppose that all nonminilex basic $\mathcal{F}_K$ flags have opposites in $\mathcal{F}_K$. Then f has an opposite in $\mathcal{F}_K$.*

Proof. If $f \in \mathcal{F}_K$ is a minilex nonbasic flag, then $f \leq f_i$ for all basic flags $f_i \preceq f$. Then each f_i is a nonminilex basic flag and has an opposite in $\mathcal{B}_K$ by assumption. We showed in the proof of Theorem 3.2.7 that an opposite exists for an arbitrary flag in $\mathcal{F}_K$ if each of its basic subflags has an opposite. $\square$

4.3.1 A dual partial order

The binary relation introduced in Definition 4.1.11 has a natural dual on $\mathcal{F}_K$ which is also a partial order. Instead of beginning our subspace comparison at the start of the flags, we can begin by comparing subspaces at the end. From now on we shall denote the extended partial order of Definition 4.1.11 by $\leq_s$. We introduce a new binary relation $\leq_e$ as follows.

Definition 4.3.7. Take a pair of flags $a = A_1 \supset \cdots \supset A_r$ and $b = B_1 \supset \cdots \supset B_s$, with $r \leq s$. If $A_i = B_i$ for $1 \leq i \leq r$ we say that $b \leq_e a$. Otherwise, let j be the smallest integer such that $A_j \neq B_j$.

- If $A_j \subsetneq B_j$, we say that $a \leq_e b$.
- If $B_j \subsetneq A_j$, we say that $b \leq_e a$.
- If there is no inclusion between A_j and B_j, we say that a and b are incomparable.

Arguments dual to those of Lemma 4.1.12 show that this binary relation gives a partial order on $\mathcal{F}_K$. Observe that for proper subflags f_i which consist of the final n subspaces of a flag f (the dual notion of a truncation) we have $f \leq f_i$. This ensures that maximal subspaces (treated as flags of length 1) are maximal with respect to $\leq_e$. Flags in $\mathcal{F}_K$ which are maximal with respect to the extended partial order $\leq_e$ will be referred to as *maxilex* flags. Flags which are minilex are not necessarily maxilex, and vice versa.

Example 4.3.8. We return to the setting of Example 4.3.2, where we studied the subgroup $K' = \mathrm{diag}(s, t, t^{-1}, s^{-1}, u) \leq G' = \mathrm{GL}_5(k)$. The following table lists some flags in $\mathcal{F}_{K'}$ and indicates whether they are maximal or minimal with respect to either of our partial orders.

	$\leq_s$ minimal	$\leq_s$ maximal	$\leq_e$ minimal	$\leq_e$ maximal
$(1, 23)$				
$(1, 2, 3)$	✓			
(125)		✓		
(12)			✓	
$(15, 2, 3)$				✓
(5)	✓		✓	
(1234)		✓		✓
$(1, 2, 5, 3)$	✓			✓

Observe that a flag being $\leq_s$ minimal does not exclude it from being $\leq_e$ minimal or $\leq_e$ maximal, and being $\leq_e$ maximal does not exclude a flag from being $\leq_s$ maximal or $\leq_s$ minimal. Flags can satisfy one or two of these properties, or none at all. Roughly, the subspaces at the start of $\leq_s$ minimal flags and at the end of $\leq_e$ minimal flags are relatively low-dimensional. The subspaces at the start of $\leq_s$ maximal flags and at the end of $\leq_e$ maximal flags are relatively high-dimensional.

The following theorem demonstrates an important connection between minilex and maxilex flags.

Theorem 4.3.9. *Let K and H be subgroups of $\mathrm{GL}(V)$, with K reductive. An H-stable opposite of a minilex $\mathcal{F}_K^H$ flag is a maxilex $\mathcal{F}_K^H$ flag. Similarly, an H-stable opposite of a maxilex $\mathcal{F}_K^H$ flag is a minilex $\mathcal{F}_K^H$ flag.*

Proof. Let $f \in \mathcal{F}_K^H$ be a minilex flag. Suppose that $g \in \mathcal{F}_K^H$ is opposite f and that there is a flag $g' \in \mathcal{F}_K^H$ such that $g \leq_e g'$. We prove that $g' = g$. Let $f = A_1 \subset \cdots \subset A_r$, and let $g = B_r \subset \cdots \subset B_1$. Since $g \leq_e g'$, there is $1 \leq i \leq r+1$ such that g' ends with subspaces

$$B_{i-1} \subset \cdots \subset B_2 \subset B_1, \tag{4.3.1}$$

where if $i = 1$, the convention is that this is empty. Let $\lambda, \mu \in Y(K)$ be commuting cocharacters giving rise to f and g', respectively, and consider the cocharacter $n\lambda + \mu$ for large n. This gives a flag $f_1 \in \mathcal{F}_K^H$ which is a refinement of f. Since f is minilex, f is a truncation of f_1 by Lemma 4.3.3. By Proposition 1.1.24, the λ-weight spaces are all μ-stable, and vice versa, which implies that the first r distinct λ-weight spaces each have a single μ-weight.

Now consider the possible difference between g and g'. Suppose $i < r + 1$. Then either:

(1) g' is obtained from g by omitting subspaces, in which case g' is the flag displayed in Equation (4.3.1); or

(2) there is a proper inclusion $B_i \subsetneq B_i'$, where B_i' is the subspace preceding B_{i-1} in g'.

In case (1), since each B_k is a complement to A_k, and λ and μ commute, the flag for $-\mu$ is $A_1 \subset \cdots \subset A_{i-1}$. This is a flag in $\mathcal{F}_K^H$ which is a truncation of f, contradicting the fact that f is minilex.

Denote the λ-weight spaces by $X_1, \ldots, X_{r+1}$, in decreasing order of weight, so that each $A_j = \bigoplus_{k=1}^{j} X_k$. In case (2), μ has lowest weight spaces $X_1, X_2, \ldots, X_{i-1}$ in *increasing* order of weight; to see this, observe that $B_j = \bigoplus_{k=j+1}^{r+1} X_k$. Let Y_i denote the μ-weight space such that $B_{i-1} = B_i' \oplus Y_i$. We claim that $Y_i \subsetneq X_i$. To see this, note that since $B_i \subsetneq B_i'$ and $B_{i-1} = B_i' \oplus Y_i = B_i \oplus X_i$, we have $Y_i \cap B_i = 0$ and $\dim Y_i < \dim X_i$. Since λ and μ commute, Y_i is λ-stable, and hence is a sum of λ-weight spaces. The only λ-weight vectors in B_{i-1} which are not contained in B_i are those in X_i, so the fact that $B_i \cap Y_i = 0$ forces $Y_i \subset X_i$, and we have proved the claim. But $Y_i \subsetneq X_i$ implies that μ has at least two distinct weights on X_i, which is a contradiction to our earlier conclusion that μ has a single weight on each X_k with $1 \leq k \leq r$. Then both (1) and (2) lead to a contradiction, so we conclude that $i = r + 1$, and thus $g' = g$. A dual argument can be used to prove the second statement. $\square$

The subset of maxilex flags leads to a characterization of relative $\mathrm{GL}(V)$-complete reducibility which is dual to that of Theorem 4.2.1.

Theorem 4.3.10. *Let K and H be subgroups of $\mathrm{GL}(V)$, with K reductive. If each maxilex H-stable K-flag has an H-stable opposite, then H is relatively G-completely reducible with respect to K.*

Armed with Theorems 4.2.1 and 4.3.10, we can argue as follows. Given H and a K such that H is not relatively G-cr with respect to K, there will be minilex and maxilex H-stable K-flags which do not have H-stable opposites. Suppose $a = A_1 \subset \cdots \subset A_r$ is such a minilex flag and $b = B_1 \supset \cdots \supset B_s$ is such a maxilex flag. By Theorem 4.1.7, $A_1 \cap B_1$ and $A_1 + B_1$ appear as subspaces in a flag of $\mathcal{F}_K$ when the intersection is nontrivial and the sum is a proper subspace of V. The intersection $A_1 \cap B_1$ is trivial or all of A_1, since A_1 has no proper subspaces which occur in H-stable K-flags. Dually, $A_1 + B_1 = V$ or B_1, since no subspace of an H-stable K-flag can properly contain B_1. Together, these facts imply that A_1 and B_1 are complementary subspaces unless A_1 is contained in B_1. These statements mirror some of the arguments found in the proof of [27, Theorem 4.6], which we discussed in Remark 1.2.14. Unfortunately, we cannot obtain a completely analogous argument; complementary subspaces such as A_1 and B_1 can be found in flags which do not complement each other in $\mathcal{F}_K$, as the following example demonstrates.

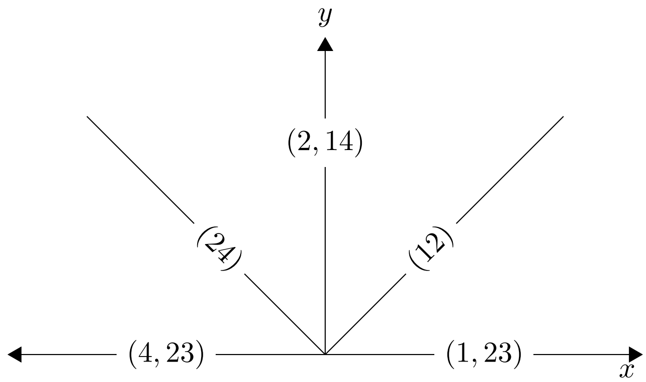
Example 4.3.11. Let $G = \mathrm{GL}_4(k)$, $K = \mathrm{diag}(s, t, t^{-1}, s^{-1})$, and let H be the subgroup of G given by

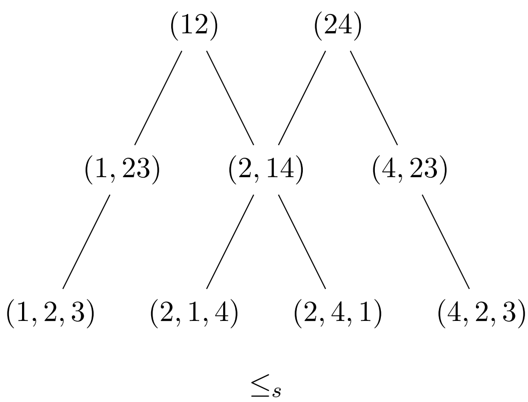
$$H = \begin{pmatrix} * & 0 & 0 & 0 \\ 0 & * & * & 0 \\ 0 & 0 & * & 0 \\ 0 & 0 & 0 & * \end{pmatrix}.$$

In Example 3.2.3 we saw that the H stable flags in $\mathcal{F}_K$ are the flags in which the span of e_3 is not introduced before the span of e_2:

$$\mathcal{F}_K^H = \{(1,23),(1,2,3),(12),(2,1,4),(2,14),(2,4,1),(24),(4,2,3),(4,23)\}.$$

Figure 4.3.1 provides a graphical representation of the cocharacters of $\mathcal{F}_K^H$ flags, and shows the Hasse diagrams for $\mathcal{F}_K^H$ under the $\leq_s$ and $\leq_e$ partial orderings. Observe that H is not $\mathrm{GL}(V)$-completely reducible with respect to K, as $(2, 14)$ has no opposite in $\mathcal{F}_K^H$. We have that $a = (1, 2, 3)$ is a minilex H-stable K-flag and $b = (4, 2, 3)$ is a maxilex H-stable K-flag, and both a and b have no opposite in $\mathcal{F}_K^H$. Note, however, that $A_1 = \langle e_1 \rangle$ and $B_3 = \langle e_2, e_3, e_4 \rangle$ are complementary subspaces.





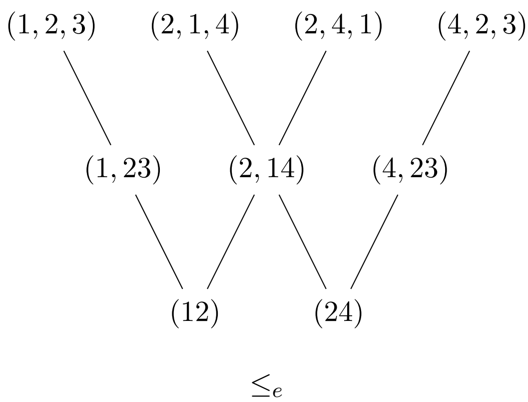


Figure 4.6: Cocharacter and Hasse diagrams for $\mathcal{F}_K^H$ of Example 4.3.11

Chapter 5

Further work and arbitrary fields

We conclude with a discussion of further work and an explanation of how results from previous chapters can be extended to arbitrary fields. Section 5.2 includes statements from the jointly written paper [1], and summarises known results from the theory of complete reducibility over arbitrary fields.

5.1 Strongly unopposed flags

Let H and K be subgroups of $G = \mathrm{GL}_n(k)$ with K reductive. In cases such as Example 4.3.11, unopposed minilex flags in $\mathcal{F}_K^H$ may have their initial subspace complemented by the final subspace of a maxilex flag. Some examples indicate that such flags may appear "on the edge" of $\mathcal{F}_K^H$, far from any potential centre. To work around these problematic flags, we introduce a stronger notion of opposition.

Definition 5.1.1. We say that a minilex flag in $\mathcal{F}_K^H$ is *strongly unopposed* if there is no flag in $\mathcal{F}_K^H$ containing a subspace complementary to its initial subspace. Dually, we say that a maxilex flag in $\mathcal{F}_K^H$ is strongly unopposed if there is no flag in $\mathcal{F}_K^H$ containing a subspace complementary to its final subspace.

Example 5.1.2. Recall the setting of Example 4.3.11: let $G = \mathrm{GL}_4(k)$, let $K = \mathrm{diag}(s, t, t^{-1}, s^{-1})$, and let H be the subgroup of G given by

$$H = \begin{pmatrix} * & 0 & 0 & 0 \\ 0 & * & * & 0 \\ 0 & 0 & * & 0 \\ 0 & 0 & 0 & * \end{pmatrix}.$$

We saw that this results in a set $\mathcal{F}_K^H$ with a pair of flags $(1, 2, 3)$ and $(4, 2, 3)$ which are unopposed but not strongly unopposed. Note, however, that there are minilex and maxilex flags which are strongly unopposed. In fact, the subset S_1 of strongly unopposed minilex flags in $\mathcal{F}_K^H$ is equal to the subset S_2 of strongly unopposed maxilex flags:

$$S_1 = \{(2, 1, 4),\ (2, 4, 1)\} = S_2$$

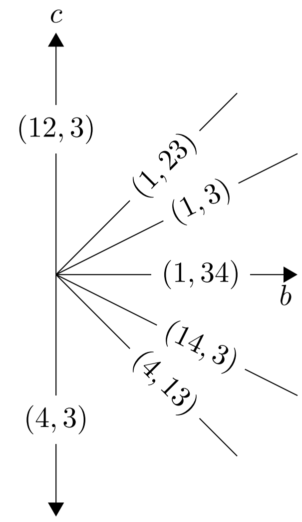


Figure 5.1: Cocharacter and stabilized flag diagram for $\mathcal{F}_K^H$ of Example 5.1.3

Recalling the discussion of Remark 1.2.14, let C_1 denote the initial subspaces of flags in S_1 and let C_2 denote the final subspaces of flags in S_2. Let $\mathcal{C}_1$ denote the sum of all subspaces in C_1, and let $\mathcal{C}_2$ denote the intersection of all subspaces in C_2. Then we have $\mathcal{C}_1 = \langle e_2 \rangle$, and $\mathcal{C}_2 = \langle e_1, e_2, e_4 \rangle$. In this example, $(\mathcal{C}_1 \subset \mathcal{C}_2) = (2, 14)$ is an element of $\mathcal{F}_K^H$ and is the "centre" we would expect: see the cocharacter diagram of Figure 4.3.1, and observe that $(2, 14)$ is fixed by the automorphism which swaps the spans of e_1 and e_4.

The following example introduces a set $\mathcal{F}_K^H$ where the subsets of minilex flags and maxilex flags are distinct.

Example 5.1.3. Let $K = \operatorname{diag}(b, b^{-1}, c^{-1}, c^{-2}) \leq G = \mathrm{GL}_4(k)$ and let H be the subgroup of G given by

$$H = \begin{pmatrix} * & * & 0 & 0 \\ 0 & * & 0 & 0 \\ 0 & 0 & * & 0 \\ 0 & 0 & 0 & * \end{pmatrix}.$$

Then the flags in $\mathcal{F}_K^H$ are the flags in which the span of e_2 is not introduced before the span of e_1; for a cocharacter $(b, -b, -c, -2c)$ to correspond to a flag in $\mathcal{F}_K^H$, we require $b \geq 0$. The following table lists the H-stable K-flags:

Length	$\mathcal{F}_K^H$ flags
2	$(1,23)$, $(1,3)$, $(1,34)$, $(14,3)$, $(4,13)$, $(12,3)$, $(4,3)$
3	$(1,2,3)$, $(1,3,2)$, $(1,3,4)$, $(1,4,3)$, $(4,1,3)$, $(4,3,1)$

Using the notation conventions of Example 5.1.2, we have $S_1 = \{(1,2,3), (1,3), (1,4,3)\}$, and $S_2 = \{(1,3,4), (14,3), (4,3,1)\}$. Then $\mathcal{C}_1 = \langle e_1 \rangle$, and $\mathcal{C}_2 = \langle e_1, e_3, e_4 \rangle$. We have that $(\mathcal{C}_1 \subset \mathcal{C}_2) = (1, 34)$ is an element of $\mathcal{F}_K^H$ and this again is the centre we would expect, given the cocharacter diagram of Figure 5.1.

As the following example demonstrates, there is no guarantee that C_1 and C_2 are both nonempty.

Example 5.1.4. Let $K = \mathrm{diag}(b, b^{-1}, c^{-1}, c^{-2}) \leq G = \mathrm{GL}_4(k)$ and let H be the subgroup of G given by

$$H = \begin{pmatrix} * & 0 & 0 & 0 \\ 0 & * & 0 & 0 \\ 0 & 0 & * & * \\ 0 & 0 & 0 & * \end{pmatrix}.$$

Then the flags in $\mathcal{F}_K^H$ are the flags in which the span of e_4 is not introduced before the span of e_3; for a cocharacter $(b, -b, -c, -2c)$ to correspond to a flag in $\mathcal{F}_K^H$, we require $c \geq 0$. The following table lists the H-stable K-flags:

Length	$\mathcal{F}_K^H$ flags
2	$(1,23)$, $(1,3)$, $(1,34)$, $(2,13)$, $(2,34)$, $(12,3)$, $(2,3)$
3	$(1,2,3)$, $(1,3,2)$, $(1,3,4)$, $(2,3,4)$, $(2,1,3)$, $(2,3,1)$

The set of minilex flags in $\mathcal{F}_K^H$ is $\{(1,3),\ (2,3),\ (1,2,3),\ (2,1,3)\}$; none of these flags are strongly unopposed. Using the notation conventions of Example 5.1.2, we have $S_1 = \emptyset$, and $S_2 = \{(1,3,2),\ (12,3),\ (2,3,1)\}$. Then $\mathcal{C}_1$ is undefined, and $\mathcal{C}_2 = \langle e_1, e_2, e_3 \rangle$. We want to work with the flag $(\mathcal{C}_2 \subset V) = (123)$, which is not a member of $\mathcal{F}_K^H$. We introduce an optimisation process to find the flag of $\mathcal{F}_K^H$ which is in some sense "closest" to the flag (123). The stabilizer of $\langle e_1, e_2, e_3 \rangle$ in G is given by

$$H' = \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ 0 & 0 & 0 & * \end{pmatrix},$$

and the associated cocharacter will be correspond to a 4-tuple (x, x, x, y) with $x > y$. Elements of $Y(K)$ correspond to 4-tuples $(b, -b, -c, -2c)$, and these 4-tuples can be treated as vectors. The cosine of the angle θ between these two is given by

$$\cos\theta = \frac{-c(x+2y)}{\sqrt{(x^2+4y^2)(2b^2+5c^2)}} = \frac{-c}{\sqrt{(2b^2+5c^2)}}\left(\frac{x+2y}{\sqrt{(x^2+4y^2)}}\right)$$

Recalling that $c \geq 0$ for flags in $\mathcal{F}_K^H$, choose x and y so that $x + 2y < 0$ and the cosine is positive. To minimize the angle between these vectors, we maximise the cosine. For a fixed nonzero value of c, setting $b = 0$ achieves this by minimizing the denominator above. Then the ray of cocharacters chosen by this procedure is the set of cocharacters of the form $(0, 0, -c, -2c)$, which corresponds to the flag $(12, 3)$ when $c > 0$. This flag is an element of $\mathcal{F}_K^H$ and is the centre we would expect; see the cocharacter diagram of Figure 5.2, and note that $(12, 3)$ is fixed by the automorphism which swaps the spans of e_1 and e_2.

Remark 5.1.5. As long as $x + 2y < 0$, we should be able to find a unique ray of cocharacters as above. Kempf proved that a nonzero linear functional is maximised on an open ray in [22, Lemma 2.3].

A natural question to ask is how far apart two flags can be if the initial subspace of one

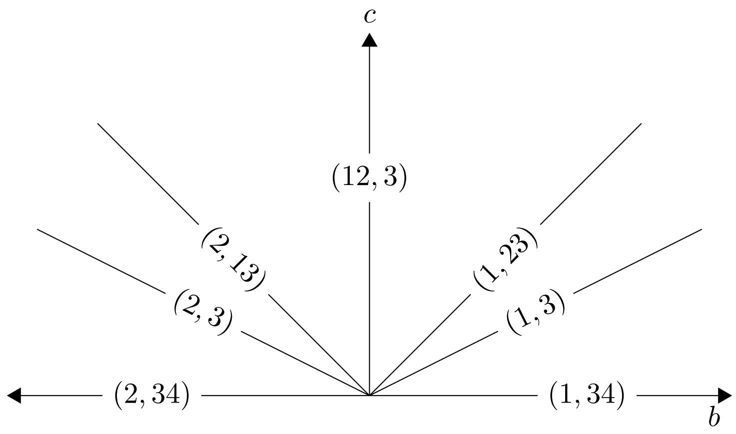


Figure 5.2: Cocharacter and stabilized flag diagram for $\mathcal{F}_K^H$ of Example 5.1.4

complements the final subspace of the other. We can create examples of such flags which are arbitrarily close. Let $G = \mathrm{GL}_n(k)$ and let K be the subgroup of diagonal matrices. Let $\mathbf{a}$ be the n-tuple where $a_1 = 2$ and $a_i = 1$ for all $2 \leq i \leq n$ and let $\mathbf{b}$ be the n-tuple where $b_1 = 0$ and $b_i = 1$ for all $2 \leq i \leq n$. The cosine of the angle θ between $\mathbf{a}$ and $\mathbf{b}$ is given by

$$\cos(\theta) = \frac{n-1}{\sqrt{4+n-1}\sqrt{n-1}} = \sqrt{\frac{n-1}{n+3}}.$$

Observe that the flags corresponding to $\mathbf{a}$ and $\mathbf{b}$ oppose each other in $\mathcal{F}_K$, and that the angle θ tends towards zero as n increases.

5.2 Arbitrary fields

We begin by recalling some definitions introduced in [21, §34]. Let k denote an arbitrary field. A variety X over $\bar{k}$ is said to be *defined over k* or *k-defined* if the ideal of all polynomials vanishing on X is generated by k-polynomials. A morphism of varieties is said to be *defined over k* if the coordinate functions are k-polynomials. We say that an algebraic group G is *defined over k* if G and its multiplication and inversion maps are defined over k. For a k-defined closed subgroup M of G, we write $Y_k(M)$ for the k-defined cocharacters of M. Let G denote a reductive k-defined algebraic group. If G is connected, a parabolic subgroup P of G is k-defined if and only if there exists $\lambda \in Y_k(G)$ such that $P = P_\lambda$, by [36, Lemma 15.1.2(ii)]. Note that this property is not shared by the parabolic subgroups of general non-connected groups; see [11, Remark 2.4] for an example.

5.2.1 Cocharacter-closure

Let G be a reductive group acting on an affine variety V over k. When working over a field that does not carry a natural topology, we can use the actions of cocharacters to define a topology on V, and specifically the G-orbits in V.

Definition 5.2.1. [4, Definition 1.2]. We say that a subset S of V is *cocharacter-closed* if for every $x \in S$ and every $\lambda \in Y_k(G)$ such that $x' = \lim_{a\to 0} \lambda(a) \cdot x$ exists, then $x' \in S$. We define the *cocharacter-closure of S*, denoted $\overline{S}^c$, to be the smallest subset of V such that $S \subseteq \overline{S}^c$ and $\overline{S}^c$ is cocharacter-closed.

Remarks 5.2.2.

1. The definition of the cocharacter-closure of a set makes sense because the intersection of cocharacter-closed subsets is cocharacter-closed.

2. We are most interested in the case $S = G \cdot x$ for some fixed x in a k-defined variety X. In this case, we only need to check $\lim_\lambda x$ for this fixed x. This is because $\lim_\lambda x$ exists if and only if $\lim_{g \cdot \lambda}(g \cdot x)$ exists and is G conjugate to x. This is a consequence of the fact that, for any a,

$$(g \cdot \lambda)(a) \cdot (g \cdot x) = (g\lambda(a)g^{-1}) \cdot (g \cdot x) = g \cdot (\lambda(a) \cdot x).$$

3. In the case $k = \bar{k}$, the Hilbert-Mumford Theorem implies that $G \cdot x$ is cocharacter-closed if and only if it is Zariski closed.

With the definition of cocharacter-closure, we can present the following rational version of the Hilbert-Mumford Theorem.

Theorem 5.2.3. [4, Theorem 1.3]. *Let V be an affine variety over k on which G acts, and let $v \in V$. Then there is a unique cocharacter-closed G-orbit $\mathcal{O}$ inside $\overline{G \cdot v}^c$. Moreover, there exists $\lambda \in Y_k(G)$ such that* $\lim_{a \to 0} \lambda(a) \cdot v$ *exists and lies in $\mathcal{O}$.*

Remark 5.2.4. A question concerning the behaviour of G-complete reducibility under separable field extensions is answered in [11]. It is unknown whether cocharacter-closedness of orbits behaves well with separable field extensions. The centre conjecture would solve this question and Theorem 2.4.1 shows that it would be enough to answer this question in the relative general linear setting described in this thesis.

Our reduction theorem, Theorem 2.4.1, goes through over a field using cocharacter-closed orbits instead of closed orbits. If the group G, the affine space X, and the action of G on X are all defined over k, then the polynomials used in the proof all have coefficients in k. Thus the embeddings constructed will be k-equivariant k-embeddings, and the homomorphism from G to $\mathrm{GL}(W)$ will be k-defined as well. The content of Chapter 4 requires no modifications to account for an arbitrary field, as our manipulations of flags and weight spaces never made use of the assumption that k was algebraically closed. An extended discussion of "Levi descent" over arbitrary fields can be found in [4, §5]. We conclude with a discussion of how the results of Chapter 3 can be extended to work over arbitrary fields.

5.2.2 Rational relative complete reducibility

Let k denote an arbitrary field, let G be a reductive k-defined group, and let K be a reductive k-defined subgroup of G. First, we recall the definition of relative G-complete reducibility over k from [10, Definition 4.1].

Definition 5.2.5. Let H be a subgroup of G. We say that H *is relatively G-completely reducible over k with respect to K* if for every $\lambda \in Y(K)$ such that P_λ is k-defined and

H is contained in P_λ, there exists $\mu \in Y(K)$ such that $P_\lambda = P_\mu$, H is contained in L_μ and L_μ is k-defined.

Remark 5.2.6. [1, Remark 5.2]. By [10, Lemma 4.8], a subgroup is relatively G-cr over k with respect to K if and only if for every $\lambda \in Y_k(K)$ such that $H \leq P_\lambda$, there exists $\mu \in Y_k(K)$ such that $P_\lambda = P_\mu$ and $H \leq L_\mu$. Hence it suffices to consider $\lambda \in Y_k(K)$ rather than all k-defined R-parabolics.

To state the rational analogue of Theorem 3.1.1, we need to introduce a rational analogue of $\mathcal{P}_K$, the set of R-parabolic subgroups P_λ with $\lambda \in Y(K)$. Let $\mathcal{P}_{K,k}$ denote the set of k-defined R-parabolic subgroups arising from cocharacters of K:

$$\mathcal{P}_{K,k} := \{P_\mu \mid \mu \in Y_k(K)\}.$$

Theorem 5.2.7. [1, Theorem 5.6] *Let $K \leq G$ be reductive k-defined algebraic groups with G connected, and let H be a subgroup of G. Then the following are equivalent:*

(i) *H is relatively G-completely reducible over k with respect to K.*

(ii) *Every maximal member of $\mathcal{P}_{K,k}$ containing H has an opposite in $\mathcal{P}_{K,k}$ which is maximal and contains H.*

(iii) *There is an R-Levi subgroup L_μ with $\mu \in Y_k(K)$, such that $H \leq L_\mu$ and H is relatively L_μ-irreducible over k with respect to $K \cap L_\mu$.*

The proof uses rational analogues of the results used in the proof of the algebraically closed case. There are a couple of subtleties concerning k-split tori and conjugation which are solved using relative root systems [12, §21] [36, §15, 16] and earlier results from the study of relative complete reducibility [10, Lemma 4.6] [4, Lemma 2.12].

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