Semigroups of straight I-quotients: a general approach

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Abstract

Let Q be an inverse semigroup. A subsemigroup S of Q is a *left I-order* in Q, and Q is a *semigroup of left I-quotients* of S, if every element in Q can be written as $a^{-1}b$, where $a, b \in S$ and a^{-1} is the inverse of a in the sense of inverse semigroup theory. If we insist on a and b being $\mathcal{R}$-related in Q, we say that S is *straight* in Q and Q is a *semigroup of straight left I-quotients* of S.

In Chapter 4, we give two equivalent sets of necessary and sufficient conditions for a semigroup to be a straight left I-order. The first set of conditions is in terms of two binary relations and an associated partial order and the proof relies on the meet structure of the $\mathcal{L}$-classes of inverse semigroups. The second set of conditions in terms of two binary relations and a ternary relation and the proof is purely algebraic.

We characterise right ample straight left I-orders that are embedded as a unary semigroup into their semigroups of straight left I-quotients. As a special case of this, we characterise two-sided ample left I-orders that are embedded into their semigroups of left I-quotients as (2,1,1)-algebras.

Straight left I-orders always intersect every $\mathcal{L}$-class of their semigroup of straight left I-quotients. We characterise straight left I-orders that intersect every $\mathcal{R}$-class of their semigroup of straight left I-quotients. We use this to prove that if a semigroup S has both a semigroup of straight left I-quotients, Q, and a semigroup of straight right I-quotients, P, then P and Q are isomorphic if and only if their $\mathcal{R}$ and $\mathcal{L}$ relations restricted to S are equal.

We characterise left I-orders whose semigroups of quotients have a chain of idempotents. As a special case of this, we characterise left I-orders in inverse ω-semigroups.

We determine when two semigroups of straight left I-quotients are isomorphic.

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Author's declaration

I declare that this thesis is a presentation of original work and I am the sole author. This work has not previously been presented for an award at this, or any other, university. All sources are acknowledged as references.

Contents

Chapter 1

Introduction

The theory of orders and quotients has its history in classical ring theory. Let R be a subring of a ring Q with multiplicative identity. Then Q is a *ring of left quotients* of R and R is an *left order* in Q if every $q \in Q$ can be written as $q = a^{-1}b$ for some $a, b \in R$, and if, in addition, every non-zero divisor in R has an inverse in Q. With this definition, we are able to describe the relationship between $\mathbb{Z}$ and $\mathbb{Q}$. Ore in 1931 [27] proved that a ring R has a ring of left quotients if and only if it is left Ore. By saying that a ring is left Ore, we mean that for any non-zero $a \in R$, $d \in \Delta$, we have $\Delta a \cap Rd \neq 0$, where Δ is the set of non-zero divisors of R.

Several definitions of a semigroup of quotients have been proposed and studied by a number of authors. The earliest definition is that of a group of left quotients, introduced by Dubreil in 1943 [4], building on Ore's work. A subsemigroup S of a group G is a *left order* in G and G is a *group of left quotients* of S if every $g \in G$ can be written as $g = a^{-1}b$ for some $a, b \in S$. Dubreil showed that a semigroup S has a group of left quotients if and only if S is right reversible and cancellative. By saying that a semigroup S is right reversible we mean for any $a, b \in S$, $Sa \cap Sb \neq \varnothing$.

Murata in 1950 [25] extended the notion of a group of left quotients to a semigroup of classical left quotients by letting the semigroup of quotients be a monoid, and considering inverses lying in the group of units. A subsemigroup S of a monoid M is a *classical left order* in M and M is a *semigroup of classical left quotients* of S if every $m \in M$ can be written as $m = a^{-1}b$ for some $a, b \in S$

where a^{-1} is the inverse of a in the group of units of M, and if, in addition, every cancellative element of S is in the group of units of M. Murata showed that a semigroup S has a monoid of classical left quotients if and only if S satisfies the left Ore-Asano condition. By saying that a semigroup S satisfies the left Ore-Asano condition, we mean that for any $a \in A$, $b \in S$, we have $Sa \cap Ab \neq \varnothing$, where A is the set of cancellative elements of S.

A different definition proposed by Fountain and Petrich in 1986 [9] was restricted to completely 0-simple semigroups of left quotients. Gould in 1986 [15] extended this concept to left orders in an arbitrary semigroup, which we will call left Fountain-Gould orders. A subsemigroup S of a semigroup Q is a *left Fountain-Gould order* in Q and Q is a *semigroup of left Fountain-Gould quotients* of S if every $q \in Q$ can be written as $q = a^{\#}b$ for some $a, b \in S$, where $a^{\#}$ is the inverse of a in some subgroup of Q, and if, in addition, every square-cancellable element of S lies in a subgroup of Q. We will define square-cancellable formally in Chapter 2, but for now it is enough to know that it is a necessary condition for an element to lie in a subgroup of an oversemigroup, in the same way that cancellativity is a necessary condition for an element to lie in the group of units of an oversemigroup. Additionally, S is a *straight left Fountain-Gould order* in Q, if every $q \in Q$ can be written as $q = a^{\#}b$ for some $a, b \in S$, such that $a \mathcal{R} b$ in Q. Gould in 2003 [17] gave necessary and sufficient conditions for a semigroup to be a straight left Fountain-Gould order.

The concept central to this thesis is that of a semigroup of left I-quotients, first defined by Ghroda and Gould in 2010 [14]. A subsemigroup S of an inverse semigroup Q is a *left I-order* in Q and Q is a *semigroup of left I-quotients* of S if every element in Q can be written as $a^{-1}b$ where $a, b \in S$ and a^{-1} is the inverse of a in the sense of inverse semigroup theory. Note that there is no additional condition guaranteeing that certain elements have inverses, as in the classical case and the Fountain-Gould case. The reason for this is that in an inverse semigroup, every element already has an inverse.

A subsemigroup S of an inverse semigroup Q is a *straight left I-order* in Q and Q is a *semigroup of straight left I-quotients* of S if every $q \in Q$ can be written as $q = a^{-1}b$ where $a, b \in S$ and $a \mathcal{R} b$ in Q.

The notion of semigroups of I-quotients has effectively been defined by a number of authors without using the above terminology. The first case of this is probably

Clifford in 1953 [2] where he showed that every right cancellative monoid S with the (LC) condition has a bisimple inverse monoid of left I-quotients. By saying that a semigroup S satisfies the (LC) condition we mean for any $a, b \in S$ there exists $c \in S$ such that $Sa \cap Sb = Sc$. Thus, (LC) is a stronger condition than right reversibility.

Left I-quotients have also appeared implicitly in work on inverse hulls of right cancellative semigroups developed in [26] and [23], and taken further in [1]. A related approach recently appeared in Exel and Steinberg's work on inverse hulls of 0-left cancellative semigroups [5]. All of these examples are left ample (or right ample), and so we can determine the structure of their inverse hulls using Theorem 3.7 of [10]. We will explore this more in Subsection 5.2.1.

Fountain and Kambites also utilise left I-quotients in Section 2 of [7], in which they use the fact that certain graph products are left I-orders in related inverse semigroups to show that that this relationship is in fact that of a semigroup and its inverse hull.

The main purpose of this thesis is to develop a comprehensive theory for semigroups of left I-quotients. The 'I' stands for 'Inverse semigroup'. Including this introduction, this thesis comprises eight chapters. In Chapter 2, we begin by introducing the standard semigroup theory used throughout the thesis along with the basics of inverse semigroup theory.

In Chapter 3, we give the formal definitions of left I-orders and of straight left I-orders, along with some preliminary properties of left I-orders. We also provide a number of examples of left I-orders. In the second section, we show many of the connections between the historical theory of semigroups of quotients and that of semigroups of I-quotients, including many results guaranteeing straightness. In the final section of this chapter, we determine when a homomorphism between straight left I-orders can be lifted to a homomorphism between the semigroups of straight left I-quotients. Consequently, we find necessary and sufficient conditions for a semigroup of straight left I-quotients to be unique.

In Chapter 4, we determine the conditions under which a semigroup S is a straight left I-order. We have two approaches which we cover in two separate sections. Section 4.2 characterises straight left I-orders using the meet structure of the $\mathcal{L}$-classes of inverse semigroups, and we give our conditions in terms of two binary relations and an associated partial order. Section 4.3 characterises straight left I-orders in a 'purely algebraic' way, and we give our conditions in terms of two binary relations and a ternary relation. The final section answers the much simpler question of whether a subsemigroup of a given inverse semigroup, Q, is straight left I-order in Q. Many of the results in the following chapters are characterisations of particular classes of semigroups of I-quotients and each of their proofs rely on at least one of the results from this chapter, simplifying them considerably in application.

In Chapter 5, we use the results in Chapter 4 to reprove some established results for semigroups of I-quotients. In the first section, we reprove a characterisation of straight left I-orders in primitive inverse semigroups from [12]. In the final section, we reprove the result that left ample semigroups are left I-orders in their inverse hull if and only if they have the (LC) condition from [10], and we apply this result to Exel and Steinberg's work on inverse hulls of 0-left cancellative semigroups [5].

In Chapter 6, we examine right ample left I-orders. In the first section, we characterise right ample straight left I-orders that are embedded into their semigroups of straight left I-quotients as (2,1)-algebras. In the final section, we characterise two-sided ample left I-orders that are embedded into their semigroups of left I-quotients as (2,1,1)-algebras.

In Chapter 7, we focus on straight left I-orders that intersect every $\mathcal{R}$-class of their semigroups of straight left I-quotients. In the first section, we characterise such straight left I-orders. In Section 7.2, we characterise left ample straight left I-orders that intersect every $\mathcal{R}$-class of their semigroup of straight left I-quotients. In the final section, we prove that if a semigroup S has both a semigroup of straight left I-quotients, Q, and a semigroup of straight right I-quotients, P, then P and Q are isomorphic if and only if their $\mathcal{R}$ and $\mathcal{L}$ relations restricted to S are equal.

In Chapter 8, we consider semigroups of left I-quotients with totally ordered idempotents. In the first section, we characterise left I-orders whose semigroups of left I-quotients have totally ordered idempotents. In the last section, we characterise left I-orders in inverse ω-semigroups, along with three special cases of inverse ω-semigroups: no kernel, simple and proper kernel.

Chapter 2

Preliminaries

In this chapter, we introduce the semigroup theory used throughout the thesis. All definitions and results are standard and can be found in [20] and [3] unless a reference is given.

2.1 General semigroup theory

A *semigroup* $S = (S, .)$ is a non-empty set S together with an associative binary operation on S. Unless stated otherwise, S denotes a semigroup throughout.

If S contains an element 1 such that $a1 = 1a = a$ for all $a \in S$, then 1 is called an *identity* and S is called a *monoid*. If S contains an element 0 such that $a0 = 0a = 0$ for all $a \in S$, then 0 is called a *zero* element of S. Note that identity and zero elements, if they exist, are unique.

Let M be a monoid. An element a of M is called a *unit* if there exists $b \in M$ such that $ab = ba = 1$.

We use S^1 to denote the semigroup S with identity adjoined if necessary. That is,

$$S^1 = \begin{cases} S & \text{if } S \text{ is a monoid,} \\ S \cup \{1\} & \text{otherwise,} \end{cases}$$

with the multiplication extended by defining $a1 = 1a = a$ for all $a \in S^1$.

A non-empty subset T of a semigroup S is a *subsemigroup* of S if T is a semigroup under the operation of S. In this case, we also say that S is an *oversemigroup* of T. If T is a group under the operation of S then it is called a *subgroup* of S. If S is a monoid, the units of S form a subgroup called *the group of units*.

An element $e \in S$ is an *idempotent* if $e^2 = e$. The set of all idempotents of S is denoted by $E(S)$. We define a partial order $\leqslant$ on $E(S)$ by

$$e \leqslant f \text{ if and only if } ef = fe = e$$

We call this the *natural partial order* on idempotents. We write $e < f$ to denote that $e \leqslant f$, but $e \neq f$.

A *band* is a semigroup where every element is idempotent. A *semilattice* is a commutative band.

Let A be a set with a partial order $\leqslant$, and let a and b be two elements of A. An element c of A is the *meet* (or greatest lower bound) of a and b if the following two conditions are satisfied:

(i) $c \leqslant a$ and $c \leqslant b$.

(ii) $h \leqslant a$ and $h \leqslant b$ implies that $h \leqslant c$.

We denote the fact that the meet of a and b exists and equals c by $a \wedge b = c$. A meet of a and b will not necessarily exist, but if it does exist then it is unique. If, for every $a, b \in A$, the meet of a and b exists, we say that A is a *meet semilattice* under $\leqslant$.

Proposition 1.3.2 of [20] demonstrates that semilattices are precisely meet semilattices. Given a semilattice $(S, \cdot)$, we can define a meet semilattice $(S, \leqslant)$ by taking $\leqslant$ to be the natural partial order on idempotents. Then $a \wedge b = a \cdot b$. Conversely, a meet semilattice $(S, \leqslant)$ is a semilattice under $\wedge$.

Let A and B be subsets of S. We write

$$AB = \{\, ab \mid a \in A, b \in B \,\}.$$

We write aB for $\{a\}B = \{ab \mid b \in B\}$.

A non-empty subset A of a semigroup S is called a *right ideal* if $AS \subseteq A$, a *left ideal* if $SA \subseteq A$ and an *ideal* if A is both a right ideal and a left ideal.

Lemma 2.1.1 (Principal Ideal Lemma for Idempotents)**.** *Let S be a semigroup and let $x \in S$ and $e \in E(S)$. Then $x \in Se$ if and only if $x = xe$.*

Proof. If $x \in Se$, then $x = se$ for some $s \in S$. Therefore, using the fact that e is an idempotent, we have

$$xe = se^2 = se = x.$$

Conversely, if $x = xe$, then $x \in Se$ since $x \in S$. □

An ideal M of a semigroup S is called *minimal* if it does not properly contain an ideal of S. A semigroup can have at most one minimal ideal. To see this, suppose that M and N are both minimal ideals of S. Then, since MN is an ideal contained in both M and N, we have that $M = MN = N$. Therefore a semigroup S either has no minimal ideals or a unique minimal ideal, which we call the *kernel* of S.

A binary relation ρ on S is *right compatible* if, for all $a, b, x \in S$,

$$(a, b) \in \rho \text{ implies that } (ax, bx) \in \rho.$$

Dually, ρ is *left compatible* if, for all $a, b, x \in S$,

$$(a, b) \in \rho \text{ implies that } (xa, xb) \in \rho.$$

Also, ρ is *compatible* if, for all $a, b, c, d \in S$,

$$(a, b) \in \rho \text{ and } (c, d) \in \rho \text{ implies that } (ac, bd) \in \rho.$$

A *right congruence* is a right compatible equivalence relation and a *left congruence* is a left compatible equivalence relation. A *congruence* is a compatible equivalence relation. Equivalently, a congruence is a relation that is both a right congruence and a left congruence.

Let ρ be a congruence on S. Denoting the equivalence class of $a \in S$ by $a\rho$, we can define a binary operation on the quotient set S/ρ in the following way:

$$(a\rho)(b\rho) = (ab)\rho.$$

With respect to this operation, S/ρ is a semigroup.

Let A be an ideal of S. Then for $a, b \in S$, we define $a \, \rho_A \, b$ to mean that either $a = b$ or that both a and b belong to A. We call ρ_A the *Rees congruence modulo* A. We shall write S/A to mean S/ρ_A, and we call S/A the *Rees factor semigroup of* S *modulo* A.

Let S and T be semigroups, T having a zero element, and let A be an ideal of S. The S is an *ideal extension of* A *by* T if the Rees factor semigroup S/A is isomorphic to T.

A preorder $\leqslant$ is a binary relation that is reflexive and transitive. Given a preorder on a set A, one may define an equivalence relation $\sim$ on A such that

$$a \sim b \text{ if and only if } a \leqslant b \text{ and } b \leqslant a.$$

It is then possible to define a partial order on $A/\sim$ by

$$[a] \leqslant [b] \text{ if and only if } a \leqslant b,$$

where $[a]$ and $[b]$ are the $\sim$-classes of a and b, respectively. We will call these the *associated equivalence relation* and the *associated partial order*, respectively.

The equivalence relation $\mathcal{R}$ on a semigroup S is defined by the rule that

$$a \, \mathcal{R} \, b \text{ if and only if } aS^1 = bS^1.$$

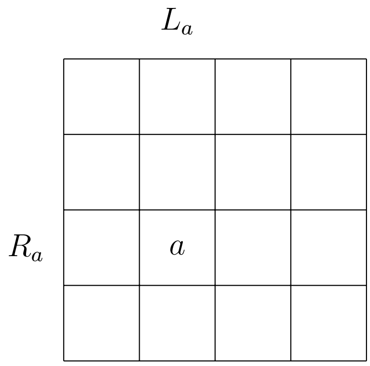
We might also write $\mathcal{R}^S$ if the semigroup S used to generate the relation is unclear. Dually, the equivalence relation $\mathcal{L}$ on a semigroup S is defined by the rule that

$$a \, \mathcal{L} \, b \text{ if and only if } S^1a = S^1b.$$

It is easy to see that $\mathcal{R}$ and $\mathcal{L}$ are a left congruence and a right congruence, respectively. We say that $a \, \mathcal{J} \, b$ if $S^1aS^1 = S^1bS^1$. The intersection of $\mathcal{R}$ and $\mathcal{L}$ is denoted by $\mathcal{H}$. It is a significant fact that $\mathcal{R}$ and $\mathcal{L}$ commute in the semigroup of binary relations on S, and consequently $\mathcal{D} = \mathcal{R} \circ \mathcal{L}$ is also an equivalence relation. We call these five equivalence relations *Green's relations*. The $\mathcal{R}$-class containing the element a will be written as R_a. Similarly for L_a, H_a, D_a and J_a.

It is convenient to visualise a $\mathcal{D}$-class as what Clifford and Preston [3] call an *eggbox picture*, in which each row represents an $\mathcal{R}$-class, each column represents

Figure 2.1: An eggbox picture



an $\mathcal{L}$-class, and each cell represents an $\mathcal{H}$-class. Note that this may well be an infinite eggbox.

We say that a semigroup S is *bisimple* if it consists of a single $\mathcal{D}$-class. We say that a semigroup S is *simple* if it consists of a single $\mathcal{J}$-class. Equivalently, we can define a simple semigroup as a semigroup with no proper ideals or a semigroup which is its own kernel. A semigroup S is *0-simple* if $S^2 \neq 0$ and $\{0\}$ and $S \setminus \{0\}$ are the only $\mathcal{J}$-classes.

Theorem 2.1.2 (Green's Theorem)**.** *If $a \in S$, then a lies in a subgroup of S if and only if $a \,\mathcal{H}\, a^2$.*

There is a preorder associated with Green's relation $\mathcal{R}$ which is defined by the rule that

$$a \leqslant_{\mathcal{R}} b \text{ if and only if } aS^1 \subseteq bS^1.$$

Dually, the preorder associated with Green's relation $\mathcal{L}$ is defined by the rule that

$$a \leqslant_{\mathcal{L}} b \text{ if and only if } S^1 a \subseteq S^1 b.$$

Note that these have associated equivalence relations, $\mathcal{R}$ and $\mathcal{L}$, respectively. We may write $a <_{\mathcal{L}} b$ to denote that $a \leqslant_{\mathcal{L}} b$, but that a and b are not $\mathcal{L}$-related.

Lemma 2.1.3. *Let S be a semigroup such that $E(S)$ is a semilattice and let $e, f \in E(S)$. Then $e \leqslant f$, $e \leqslant_{\mathcal{R}} f$ and $e \leqslant_{\mathcal{L}} f$ are all equivalent.*

Proof. We start by proving that $e \leqslant f$ and $e \leqslant_{\mathcal{L}} f$ are equivalent.

Let $e \leqslant f$. Then $e = ef$. Therefore $S^1 e = S^1 ef \subseteq S^1 f$, and so $e \leqslant_{\mathcal{L}} f$.

Conversely, let $e \leqslant_{\mathcal{L}} f$. Then $e \in S^1 f$. Since f is an idempotent, we have that $S^1 f = Sf$. We can therefore apply Lemma 2.1.1, to get that $e = ef$. Using the fact that idempotents commute, $fe = ef = e$, and so $e \leqslant f$.

The fact that $e \leqslant f$ and $e \leqslant_{\mathcal{R}} f$ are equivalent is dual, using the dual of Lemma 2.1.1. $\square$

There is another generalisation of Green's relation $\mathcal{R}$ which is defined by the rule that $a \, \mathcal{R}^* \, b$ if and only if the elements a, b of S are related by Green's relation $\mathcal{R}$ in some oversemigroup of S. According to Lemma 1.7 of [23], $a \, \mathcal{R}^* \, b$ is equivalent to the condition that $xa = ya$ if and only if $xb = yb$ for all $x, y \in S^1$. Given this, it is easy to see that $\mathcal{R}^*$ is transitive and then a left congruence on S. The relation $\mathcal{L}^*$ is defined dually. The $\mathcal{R}^*$-class containing the element a will be written as R^*_a. Similarly for L^*_a.

Lemma 2.1.4. *Let S be a semigroup such that $E(S)$ is a semilattice. Then there can be at most one idempotent in each $\mathcal{R}^*$-class ($\mathcal{L}^*$-class).*

Proof. Let $e, f \in E(S)$ such that $e \, \mathcal{R}^* \, f$. Then $e = 1e = ee$ implies that $f = 1f = ef$. Similarly, $f = ff$ implies that $e = fe$. Therefore, using the fact that idempotents commute, we have $e = fe = ef = f$.

The fact that there can be at most one idempotent in each $\mathcal{L}^*$-class is proved dually. $\square$

There are two further preorders associated with $\mathcal{R}^*$ and $\mathcal{L}^*$, namely $\leqslant_{\mathcal{R}^*}$ and $\leqslant_{\mathcal{L}^*}$. For $a, b \in S$, we say that $a \leqslant_{\mathcal{R}^*} b$ if and only if $a \leqslant_{\mathcal{R}} b$ in some oversemigroup of S. Note that this has associated equivalence relation $\mathcal{R}^*$. According to Lemma 2.2 of [17], $a \leqslant_{\mathcal{R}^*} b$ is equivalent to the condition that $xb = yb$ implies that $xa = ya$ for all $x, y \in S^1$. Clearly this is a left compatible preorder. The relation $\leqslant_{\mathcal{L}^*}$ is defined dually.

Lemma 2.1.5. *[18] For $e, f \in E(S)$,*

$$e \leqslant_{\mathcal{R}^*} f \text{ if and only if } e \leqslant_{\mathcal{R}} f$$

and

$$e \leqslant_{\mathcal{L}^*} f \text{ if and only if } e \leqslant_{\mathcal{L}} f.$$

An element $a \in S$ is *left cancellative* if $ab = ac$ implies $b = c$ for all $b, c \in S$. If every element in S is left cancellative, then S is called a *left cancellative semigroup*. Dually, an element $a \in S$ is *right cancellative* if $ba = ca$ implies $b = c$ for all $b, c \in S$. If every element in S is right cancellative, then S is called a *right cancellative semigroup*. An element $a \in S$ is called *cancellative* if it is both left cancellative and right cancellative. If every element in S is cancellative, then S is called a *cancellative semigroup*.

A semigroup S with zero is defined to be *0-right cancellative* if for all $a, b, c \in S$, $ab = ac \neq 0$ implies that $b = c$. Dually, a semigroup S with zero is defined to be *0-left cancellative* if for all $a, b, c \in S$, $ba = ca \neq 0$ implies that $b = c$.

An element $a \in S$ is *square-cancellable* in S if for all $x, y \in S^1$, $xa^2 = ya^2$ implies that $xa = ya$ and $a^2x = a^2y$ implies that $ax = ay$. This is clearly equivalent to $a\,\mathcal{H}^*\,a^2$, where $\mathcal{H}^* = \mathcal{R}^* \cap \mathcal{L}^*$.

Let S and T be semigroups. A function $\phi : S \to T$ is called a *homomorphism* of S to T, if for all $a, b \in S$, we have $(a\phi)(b\phi) = (ab)\phi$. If ϕ is injective, then ϕ is an *embedding* of S into T. Note that following the ring theoretic terminology, an embedding is an injective homomorphism and nothing more. If ϕ is surjective, then ϕ is an *epimorphism*. If ϕ is bijective, then ϕ is an *isomorphism*. We say that S and T are *isomorphic* if there is an isomorphism between S and T and we write $S \cong T$.

If a semigroup S has an additional unary operation, we call S a *unary semigroup*. If S and T are both unary semigroups and there exists an embedding $\phi : S \to T$ that preserves the unary operation, we say that S is embedded in T as a unary semigroup, or S is embedded in T as a (2,1)-algebra. A *bi-unary semigroup* is a semigroup equipped with two unary operations. If S and T are both bi-unary semigroups and there exists an embedding $\phi : S \to T$ that preserves both unary operations, we say that S is embedded in T as bi-unary semigroup, or S is embedded in T as a (2,1,1)-algebra.

A *transformation* on a set X is a function from X into itself. The set of all transformations on X is a semigroup under composition (from left to right). It is called the *full transformation semigroup* on X and is denoted by $\mathcal{T}_X$.

A *partial transformation* on a set X is a function α mapping a subset A of X into a subset B of X. The set of all partial transformations on X is a semigroup under the composition of partial mappings, that is,

$$\operatorname{dom}\alpha\beta = (\operatorname{im}\alpha \cap \operatorname{dom}\beta)\alpha^{-1} \text{ and } \forall\, x \in \operatorname{dom}\alpha\beta,\ x(\alpha\beta) = (x\alpha)\beta.$$

It is called the *partial transformation semigroup* on X and is denoted by $\mathcal{PT}_X$.

2.2 Inverse semigroups

An element a of a semigroup S is *regular* if there exists an element x in S such that $axa = a$. A semigroup S is *regular* if every element of S is regular.

An element $b \in S$ is an *inverse* of $a \in S$ if

$$a = aba \text{ and } b = bab.$$

We denote the set of inverses of a by $V(a)$. An *inverse semigroup* is a semigroup S such that $|V(a)| = 1$ for all $a \in S$. The unique element of $V(a)$ is denoted by a^{-1}. Equivalently, an inverse semigroup is a regular semigroup in which all the idempotents commute. It is worth noting that in an inverse semigroup S, $(a^{-1})^{-1} = a$ and $(ab)^{-1} = b^{-1}a^{-1}$ for all $a, b \in S$.

For an inverse semigroup, each $\mathcal{R}$-class and each $\mathcal{L}$-class contains exactly one idempotent, namely $aa^{-1} \in R_a$ and $a^{-1}a \in L_a$. We consequently obtain the following result.

Lemma 2.2.1. *Let a and b be elements of an inverse semigroup. Then $a \,\mathcal{R}\, b$ if and only if $aa^{-1} = bb^{-1}$, and $a \,\mathcal{L}\, b$ if and only if $a^{-1}a = b^{-1}b$.*

We can therefore immediately see that in an inverse semigroup,

$$x \,\mathcal{R}\, y \text{ if and only if } x^{-1} \,\mathcal{L}\, y^{-1}.$$

Lemma 2.2.2. *Let M be an inverse semigroup with an identity, and let a be in the group of units of M. Then the inverse of a in the group of units is equal to the inverse of a in the sense of inverse semigroup theory.*

Proof. Denote a' as the inverse of a in the group of units of M, and a^{-1} as the inverse of a in the sense of inverse semigroup theory. We see that

$$aa'a = a \text{ and } a'aa' = a',$$

so $a' \in V(a)$. Since M is an inverse semigroup, a^{-1} is the unique element in $V(a)$. Therefore $a' = a^{-1}$. □

The *symmetric inverse monoid* on a set X, denoted by $\mathcal{I}_X$, is the subsemigroup of $\mathcal{PT}_X$ consisting of the set of all one-to-one partial transformations of a set X. The symmetric inverse monoid is an inverse semigroup, where the inverse of α in the sense of inverse semigroup theory is the inverse of α as a map. That is, if $\alpha : \operatorname{dom}\alpha \to \operatorname{im}\alpha$, then $\alpha^{-1} : \operatorname{im}\alpha \to \operatorname{dom}\alpha$ such that

$$\alpha\alpha^{-1} = \iota_{\operatorname{dom}\alpha} \text{ and } \alpha^{-1}\alpha = \iota_{\operatorname{im}\alpha},$$

where ι_A is the identity map on A, for any $A \subseteq X$.

We give Lemmas 2.2.3 and 2.2.5 and their duals, as they are useful results that we will refer to throughout this thesis.

Lemma 2.2.3. *Let a and x be elements in an inverse semigroup. Then*

$$x \,\mathcal{R}\, xa \text{ if and only if } x = xaa^{-1}.$$

Proof. Let $x \,\mathcal{R}\, xa$. By Lemma 2.2.1, we know that this is equivalent to

$$xx^{-1} = (xa)(xa)^{-1} = xaa^{-1}x^{-1}$$

Therefore, using the fact that in an inverse semigroup idempotents commute,

$$x = xx^{-1}x = xaa^{-1}x^{-1}x = xx^{-1}xaa^{-1} = xaa^{-1}.$$

Conversely, let $x = xaa^{-1}$. Then

$$xx^{-1} = xaa^{-1}x^{-1} = (xa)(xa)^{-1}.$$

By Lemma 2.2.1, we know that this is equivalent to $x \,\mathcal{R}\, xa$. □

Lemma 2.2.4. *Let a and x be elements in an inverse semigroup. Then*

$$a \,\mathcal{L}\, xa \text{ if and only if } a = x^{-1}xa.$$

Lemma 2.2.5. *Let a and b be elements in an inverse semigroup. Then*

$$a \leqslant_{\mathcal{R}} b \text{ if and only if } bb^{-1}a = a.$$

$a \leqslant_{\mathcal{R}} b$ *if and only if* $bb^{-1}a = a$.

Proof. Let $a \leqslant_{\mathcal{R}} b$, i.e. $aS^1 \subseteq bS^1$. Then there exists an $x \in S^1$ such that $a = bx$. Then

$$bb^{-1}a = bb^{-1}bx = bx = a.$$

Conversely, let $bb^{-1}a = a$. Then

$$aS^{-1} = bb^{-1}aS^1 \subseteq bS^1.$$

□

Lemma 2.2.6. *Let a and b be elements in an inverse semigroup. Then $a \leqslant_{\mathcal{L}} b$ if and only if $ab^{-1}b = a$.*

Definition 2.2.7. Let S be a semigroup that embeds into an inverse semigroup Q. The *inverse hull* $\Sigma(S)$ of S is then the subsemigroup of Q generated by the elements of S and their inverses. Note that this is dependent on the embedding chosen, but this detail may be omitted in the case where S has a canonical embedding into an inverse semigroup.

2.2.1 The meet structure of inverse semigroups

Let Q be an inverse semigroup. As previously described, $\leqslant_{\mathcal{L}}$ is a preorder on Q defined by $a \leqslant_{\mathcal{L}} b$ if and only if $Q^1a \subseteq Q^1b$, with associated equivalence relation $\mathcal{L}$. The associated partial order on the $\mathcal{L}$-classes of Q is then:

$$L_a \leqslant_{\mathcal{L}} L_b \text{ if and only if } a \leqslant_{\mathcal{L}} b.$$

Using this partial order, we can consider the meet of two $\mathcal{L}$-classes. In a general semigroup, two $\mathcal{L}$-classes need not have a meet, but in an inverse semigroup the meet always exists.

Lemma 2.2.8. *Let Q be an inverse semigroup. Then $Q/\mathcal{L}$ is a meet semilattice under $\leqslant_{\mathcal{L}}$, with $L_a \wedge L_b = L_c$ if and only if $c^{-1}c = a^{-1}ab^{-1}b$.*

Proof. In an inverse semigroup, Q, every $\mathcal{L}$-class has a unique idempotent. Using Lemma 2.1.3, we see that for idempotents $e, f \in Q$,

$$L_e \leqslant_{\mathcal{L}} L_f \iff e \leqslant_{\mathcal{L}} f \iff e \leqslant f.$$

Therefore the poset $Q/\mathcal{L}$ is order isomorphic to the semilattice of idempotents under the natural partial order.

Proposition 1.3.2 of [20] gives us that the semilattice of idempotents is a meet semilattice with the meet of e and f equalling ef. Therefore $Q/\mathcal{L}$ is a meet semilattice with $L_e \wedge L_f = L_{ef}$. The result then immediately follows from the fact that $a \,\mathcal{L}\, a^{-1}a$. □

Lemma 2.2.9. *Let S be a semigroup such that $E(S)$ is a semilattice and let $e, f \in E(S)$. Then $L^*_e \wedge L^*_f = L^*_{ef}$.*

Proof. We know that $ef \leqslant_{\mathcal{L}^S} f$ and $ef = fe \leqslant_{\mathcal{L}^S} e$. Therefore $ef \leqslant_{\mathcal{L}^*} e, f$.

Now let $h \in S$ such that $h \leqslant_{\mathcal{L}^*} e, f$. By definition $h \leqslant_{\mathcal{L}^*} e$ implies that $h \leqslant_{\mathcal{L}^Q} e$ for some Q, an oversemigroup of S. Therefore, there exists $q \in Q^1$ such that $h = qe$ in Q. Similarly $h \leqslant_{\mathcal{L}^*} f$ implies that there exists P, an oversemigroup of S such that $h = pf$ in P, for some $p \in P^1$. Therefore, by calculating in P we obtain

$$hf = pf^2 = pf = h.$$

Consequently $h = hf = qef$ in Q, and so $h \leqslant_{\mathcal{L}^Q} ef$. Therefore $h \leqslant_{\mathcal{L}^*} ef$. □

2.3 Ample semigroups

Definition 2.3.1 (Left Ample Semigroup)**.** We define a semigroup S to be *left ample* if and only if every $\mathcal{R}^*$-c1ass contains an idempotent, $E(S)$ is a semilattice, and S satisfies the left ample condition which is:

$$(ae)^+a = ae \text{ for all } a \in S \text{ and } e \in E(S)$$

where, for $x \in S,\ x^+$ is the (unique) idempotent in the $\mathcal{R}^*$-class of x.

Note that in a left ample semigroup, $a\,\mathcal{R}^*\,b$ if and only if $a^+ = b^+$.

Definition 2.3.2 (Right Ample Semigroup)**.** We define a semigroup S to be *right ample* if and only if every $\mathcal{L}^*$-c1ass contains an idempotent, $E(S)$ is a semilattice, and S satisfies the right ample condition which is:

$$a(ea)^* = ea \text{ for all } a \in S \text{ and } e \in E(S)$$

where, for $x \in S,\ x^*$ is the (unique) idempotent in the $\mathcal{L}^*$-class of x.

We know that x^+ and x^* must be unique by Lemma 2.1.4. An *ample semigroup* is one which is both left and right ample.

Alternatively, we can define a semigroup S to be left ample or right ample by using the structure of $\mathcal{I}_X$. Let $\mathcal{I}_X$ be the symmetric inverse monoid on a set X. We can define three unary operations $^{-1}$, $^+$ and * as follows:

$$a^{-1} \text{ is the inverse of } a;\ a^+ = aa^{-1} \text{ and } a^* = a^{-1}a.$$

Let S be a subsemigroup of $\mathcal{I}_X$. If S is closed under $^{-1}$ then it is an inverse semigroup. If S is closed under $^+$ then S is a left ample semigroup. If S is closed under * then S is a right ample semigroup.

A unary semigroup is *left ample* if and only if it embeds as a unary semigroup in some $\mathcal{I}_X$, where the unary operation on $\mathcal{I}_X$ is $^+$. Left ample semigroups form a quasi-variety of unary semigroups. *Right ample semigroups* may be defined in a dual way as the subsemigroups of some $\mathcal{I}_X$ that are closed under *. Following [19], it is worth noting that an ample semigroup S may not be embeddable into an inverse semigroup in such a way that preserves both $^+$ and *.

We give some elementary properties of left ample semigroups. The duals of these properties apply to right ample semigroups.

Lemma 2.3.3. *Let S be a left ample semigroup. Then for all $a, b, x \in S$:*

(i) $a^+a = a$;

(ii) $(ab)^+ = (ab^+)^+$;

(iii) $(xa)^+x = xa^+$;

(iv) $x^+ = (xa)^+$ *if and only if* $x = xa^+$; *and*

(v) $(ab)^+ \leqslant a^+$.

Proof.

(i) We know that $a \mathcal{R}^* a^+$. Therefore, by the definition of $\mathcal{R}^*$, we have that $a^+a^+ = a^+$ implies that $a^+a = a$.

(ii) Using the fact that $\mathcal{R}^*$ is a left congruence, we see that $b \mathcal{R}^* b^+$ implies that $ab \mathcal{R}^* ab^+$.

(iii) Since a^+ is an idempotent, we can use the left ample condition to give us $xa^+ = (xa^+)^+x$. We can then apply (ii) to get the intended result.

(iv) Let $x^+ = (xa)^+$. Then, using (i) and (iii), we have

$$x = x^+x = (xa)^+x = xa^+.$$

Conversely, let $x = xa^+$. Then, applying $^+$ to both sides,

$$x^+ = (xa^+)^+ = (xa)^+,$$

using (ii) in the last equality.

(v) Since $(ab)^+a^+$ is an idempotent, we have that

$$(ab)^+a^+ = ((ab)^+a^+)^+ = ((ab)^+a)^+,$$

using (ii). We can then apply (iii) to obtain

$$((ab)^+a)^+ = (ab^+)^+ = (ab)^+,$$

using (ii) again. Putting these together and remember that idempotents commute, we have that $(ab)^+a^+ = a^+(ab)^+ = (ab)^+$, and so $(ab)^+ \leqslant a^+$.

□

Note the similarity between Lemma 2.3.3 (iv) and Lemma 2.2.3.

Lemma 2.3.4. *Let S be a left ample subsemigroup of an inverse semigroup Q. Then S is embedded as a unary semigroup into Q (i.e. embedded in such a way that $^+$ is preserved) if and only if $\mathcal{R}^Q \cap (S \times S) = \mathcal{R}^*$.*

Proof. Let S be embedded as a unary semigroup into Q. We know that $\mathcal{R}^Q \cap (S \times S) \subseteq \mathcal{R}^*$ is true by definition. Let $a, b \in S$ such that $a \,\mathcal{R}^*\, b$. Hence we have $a^+ = b^+$ in S and (by the preservation of $^+$) $a^+ = b^+$ in Q. Therefore $aa^{-1} = bb^{-1}$, giving us that $a \,\mathcal{R}^Q\, b$. Therefore $\mathcal{R}^* \subseteq \mathcal{R}^Q \cap (S \times S)$ as well.

Conversely, let $\mathcal{R}^Q \cap (S \times S) = \mathcal{R}^*$, and let $a \in S$. Since $a \,\mathcal{R}^*\, a^+$, we have that $a \,\mathcal{R}^Q\, a^+$. Since Q is inverse, there is a unique idempotent in each $\mathcal{R}^Q$-class. Therefore $a^+ = aa^{-1}$. □

Following [18], for any left ample semigroup S we can construct a (2,1)-embedding of S into the symmetric inverse semigroup $\mathcal{I}_S$ as follows. For each $a \in S$, we define $\rho_a \in \mathcal{I}_S$ by

$$\rho_a : Sa^+ \to Sa, \ \forall\, s \in Sa^+, \ s\rho_a = sa.$$

Then the map $\theta_S : S \to \mathcal{I}_S$ given by

$$a\theta_S = \rho_a,$$

is a (2,1)-embedding.

2.4 Semilattices of semigroups

We now describe the following well-known construction.

Definition 2.4.1. Let Y be a semilattice. A semigroup S is called a *semilattice Y of semigroups S_α, $\alpha \in Y$*, if S is the disjoint union $S = \bigcup_{\alpha \in Y} S_\alpha$, where $S_\alpha S_\beta \subseteq S_{\alpha\beta}$.

This definition gives us what we might call the "gross structure" of S. By this we mean that there are potentially many different ways for S to be a semilattice Y of semigroups S_α, $\alpha \in Y$. For $s_\alpha \in S_\alpha$ and $s_\beta \in S_\beta$, we know that $s_\alpha s_\beta \in S_{\alpha\beta}$, but there are no other restrictions except for the associativity of S. Its "fine structure" would then be how the products $s_\alpha s_\beta$ are located in $S_{\alpha\beta}$. We now introduce such a fine structure.

Definition 2.4.2. Let Y be a semilattice. To each $\alpha \in Y$ associate a semigroup S_α. For each pair $\alpha, \beta \in Y$, such that $\alpha \geqslant \beta$, let $\varphi_{\alpha,\beta} : S_\alpha \to S_\beta$ be a homomorphism such that the following conditions hold:

(i) $\varphi_{\alpha,\alpha} = \iota_{S_\alpha}$ for all $\alpha \in Y$, where ι_A is the identity map on a set A;

(ii) $\varphi_{\alpha,\beta}\varphi_{\beta,\gamma} = \varphi_{\alpha,\gamma}$ for all $\alpha, \beta, \gamma \in Y$ such that $\alpha \geqslant \beta \geqslant \gamma$.

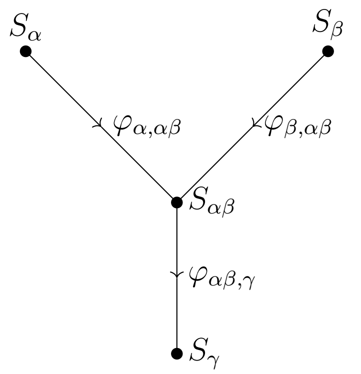
We define S as the disjoint union $S = \bigcup\limits_{\alpha \in Y} S_\alpha$ with multiplication

$$s_\alpha s_\beta = (s_\alpha \varphi_{\alpha,\alpha\beta})(s_\alpha \varphi_{\beta,\alpha\beta})$$

for $s_\alpha \in S_\alpha$ and $s_\beta \in S_\beta$.

With respect to this multiplication S is a semigroup called a *strong semilattice Y of semigroups S_α, $\alpha \in Y$.*

Figure 2.2: An example of a strong semilattice of semigroups



We could also consider strong semilattices of semigroups to be presheaves of semigroups over meet semilattices. A meet semilattice $(Y, \leqslant)$ can be regarded as a category with an arrow existing from α to β exactly when $\alpha \leqslant \beta$. Now consider the dual category $(Y, \leqslant)^{\mathrm{op}}$, in which the arrows are turned around. That is, there exists an arrow from α to β, $\phi_{\alpha,\beta}$, if and only if $\alpha \geqslant \beta$. Then the strong semilattices Y of semigroups are exactly the functors $F : (Y, \leqslant)^{\mathrm{op}} \to$ **Semigroups** to the category of semigroups, with $S_\alpha = F(\alpha)$ and $\varphi_{\alpha,\beta} = F(\phi_{\alpha,\beta})$.

Not every semilattice of semigroups is a strong semilattice of semigroups, but clearly every strong semilattice of semigroups is a semilattice of semigroups.

2.4.1 Clifford semigroups

In a semigroup S, an element c is defined to be *central* if $cs = sc$ for every $s \in S$. A *Clifford semigroup* is an inverse semigroup with central idempotents.

The next theorem from [20] gives an alternative description of Clifford semigroups.

Theorem 2.4.3 ([20, Theorem 4.2.1])**.** *Let S be a semigroup. Then the following statements are equivalent:*

(1) S is a Clifford semigroup;

(2) S is a semilattice of groups;

(3) S is a strong semilattice of groups;

(4) S is an inverse semigroup such that $xx^{-1} = x^{-1}x$ for all $x \in S$.

Let S be a Clifford semigroup. Since $xx^{-1} = x^{-1}x$ for all $x \in S$, we can use Lemma 2.2.1 to give us that $\mathcal{R} = \mathcal{L} = \mathcal{H}$ in S.

Lemma 2.4.4. *Let Y be a semilattice, and let S be a strong semilattice Y of groups G_α, $\alpha \in Y$. Let $g_\alpha \in G_\alpha$ and $h_\beta \in G_\beta$. Then the following statements are equivalent:*

(1) $\alpha \leqslant \beta$;

(2) $1_\beta\, g_\alpha = g_\alpha$, where 1_β is the identity of G_β;

(3) $g_\alpha\, 1_\beta = g_\alpha$, where 1_β is the identity of G_β;

(4) $g_\alpha \leqslant_{\mathcal{R}} h_\beta$*;*

(5) $g_\alpha \leqslant_{\mathcal{L}} h_\beta$*.*

Proof. (1) $\Rightarrow$ (2): Using the fact that S is a strong semilattice, we know that

$$1_\beta \, g_\alpha = (1_\beta \psi) g_\alpha,$$

for some homomorphism $\psi : G_\beta \to G_\alpha$. Since homomorphisms between groups preserve the identity, we know that $(1_\beta \psi) = 1_\alpha$. Therefore

$$1_\beta \, g_\alpha = 1_\alpha \, g_\alpha = g_\alpha.$$

(2) $\Rightarrow$ (4): By Lemma 2.2.5, $g_\alpha \leqslant_{\mathcal{R}} h_\beta$ if and only if $h_\beta h_\beta^{-1} g_\alpha = g_\alpha$. Since h_β is an element of the group G_β, we have that $h_\beta h_\beta^{-1} = 1_\beta$, where 1_β is the identity of G_β. Therefore $g_\alpha \leqslant_{\mathcal{R}} h_\beta$ if and only if $1_\beta \, g_\alpha = g_\alpha$.

(4) $\Rightarrow$ (1): Since $g_\alpha \, \mathcal{R} \, 1_\alpha$ and $h_\beta \, \mathcal{R} \, 1_\beta$, we know that $g_\alpha \leqslant_{\mathcal{R}} h_\beta$ implies that $1_\alpha \leqslant_{\mathcal{R}} 1_\beta$. Applying Lemma 2.1.3 gives $1_\alpha \leqslant 1_\beta$, and therefore $1_\alpha 1_\beta = 1_\alpha$. Since S is a semilattice, we know that $1_\alpha 1_\beta \in G_{\alpha\beta}$. Therefore $\alpha\beta = \alpha$, and so $\alpha \leqslant \beta$.

(1) $\Rightarrow$ (3) $\Rightarrow$ (5) is dual. □

Example 2.4.5. Let $G_\alpha = \{(\frac{a}{b})_\alpha \mid a, b \text{ positive odd integers}\}$ be the group of positive odd fractions, with normal fraction multiplication $(\frac{a}{b})_\alpha (\frac{c}{d})_\alpha = (\frac{ac}{bd})_\alpha$, and let $G_\beta = \{(\frac{a}{b})_\beta \mid a, b \text{ positive integers}\}$ be the group of positive rationals, with multiplication $(\frac{a}{b})_\beta (\frac{c}{d})_\beta = (\frac{ac}{bd})_\beta$.

We define $\varphi_{\alpha,\beta} : G_\alpha \to G_\beta$ by

$$\left(\tfrac{a}{b}\right)_\alpha \varphi_{\alpha,\beta} = \left(\tfrac{a}{b}\right)_\beta$$

Let $P = G_\alpha \,\dot\cup\, G_\beta$, where $\dot\cup$ denotes disjoint union. We extend the multiplication of G_α and G_β by

$$\left(\tfrac{a}{b}\right)_\alpha \left(\tfrac{c}{d}\right)_\beta = \left(\tfrac{c}{d}\right)_\beta \left(\tfrac{a}{b}\right)_\alpha = \left(\left(\tfrac{a}{b}\right)_\alpha \varphi_{\alpha,\beta}\right) \left(\tfrac{c}{d}\right)_\beta = \left(\tfrac{ac}{bd}\right)_\beta .$$

Then P is a strong semilattice Y of groups G_i, $i \in Y$, where $Y = \{\alpha, \beta\}$ with $\alpha \geqslant \beta$. Therefore P is a Clifford semigroup.

2.5 Primitive inverse semigroups

Remember that the natural partial order on idempotents is defined by $f \leqslant e$ if and only if $fe = ef = f$. An idempotent element e is called *primitive* if $e \neq 0$ and $f \leqslant e$ implies that either $e = f$ or $f = 0$. This concept is akin to atoms in lattices [20].

An inverse semigroup S with 0 such that $S \neq \{0\}$ is called a *primitive inverse semigroup* if all its nonzero idempotents are primitive.

Brandt semigroups are a special class of both primitive inverse semigroups and completely 0-simple semigroups. They are important for many reasons, one of which is that there is a one-to-one correspondence between Brandt semigroups and connected groupoids, as demonstrated by Proposition 6 of [22, Section 3.3]. The structure of completely 0-simple semigroups is given by the Rees Theorem [20, Theorem 3.2.3], of which the construction given in the following definition is a special case.

Definition 2.5.1 (Brandt semigroup)**.** Let G be a group and let I be a non-empty set. Then $B = B(G, I)$ is the set $(I \times G \times I) \cup \{0\}$ with multiplication

$$(i, a, j)(k, b, l) = \begin{cases} (i, ab, l) & \text{if } j = k, \\ 0 & \text{if } j \neq k. \end{cases}$$

We call such a semigroup a *Brandt semigroup.*

Lemma 2.5.2. *Let $B = B(G, I)$ be a Brandt semigroup. Then:*

(1) $(i, a, j)\,\mathcal{R}^B\,(k, b, l)$ if and only if $i = k$ and $a\,\mathcal{R}^T\,b$;

(2) $(i, a, j)\,\mathcal{L}^B\,(k, b, l)$ if and only if $j = l$ and $a\,\mathcal{L}^T\,b$;

(3) $(i, a, j)^{-1} = (j, a^{-1}, i)$.

Let $\{S_i : i \in \Lambda\}$ be a family of semigroups with zero, pairwise intersecting only at the zero element. Let $S = \bigcup\limits_{i \in \Lambda} S_i$ with multiplication

$$ab = \begin{cases} ab & \text{if } a, b \in S_i \text{ for some } i \in \Lambda, \\ 0 & \text{otherwise.} \end{cases}$$

We then call S the *0-direct union* of the S_i. In Theorem 5 of [22, Section 3.3], it is shown that every primitive inverse semigroup is a 0-direct union of Brandt semigroups, and vice versa.

Following Theorem 3.3.4 and Theorem 3.3.5 of [22], primitive inverse semigroups are exactly groupoids with zero adjoined and Brandt semigroups are exactly connected groupoids with zero adjoined.

2.6 Bruck-Reilly semigroups

We start this section by defining a very useful inverse semigroup. We define the *bicyclic monoid* as the set $\mathcal{B} = \mathbb{N}^0 \times \mathbb{N}^0$ along with the binary operation

$$(a, b)(c, d) = (a - b + \max\{b, c\}, d - c + \max\{b, c\}).$$

This is an inverse semigroup with $(a, b)^{-1} = (b, a)$. The idempotents of the bicyclic monoid are elements of the form (n, n) and under the natural ordering of idempotents these form a descending chain

$$(0, 0) \geqslant (1, 1) \geqslant (2, 2) \geqslant (3, 3) \geqslant \dots$$

Therefore the bicyclic monoid is an example of an inverse ω-semigroup defined in Definition 8.2.1.

We now define Bruck-Reilly semigroups, which are a generalisation of the bicyclic monoid.

Let T be a monoid with group of units H and let θ be a morphism from T into H. Then the *Bruck-Reilly semigroup* over T with respect to θ is $\mathbb{N}^0 \times T \times \mathbb{N}^0$ with multiplication

$$(m, a, n)(p, b, q) = (m - n + t, (a\theta^{t-n})(b\theta^{t-p}), q - p + t),$$

where $t = \max\{n, p\}$ and where θ^0 is interpreted as the identity map on T. We could equivalently write the multiplication as

$$(m, a, n)(p, b, q) = \begin{cases} (m - n + p, (a\theta^{p-n})b, q) & \text{if } n < p, \\ (m, ab, q) & \text{if } n = p, \\ (m, a(b\theta^{n-p}), q - p + n) & \text{if } n > p. \end{cases}$$

We denote the Bruck-Reilly semigroup over T with respect to θ by $BR(T, \theta)$. The proof that this multiplication is associative can be found in Section 5.6 of [20].

Bruck-Reilly semigroups are important due to several structure theorems by Reilly [29], Kochin [21] and Munn [24]. Reilly [29] showed that bisimple inverse ω-semigroups are isomorphic to Bruck-Reilly semigroups of the form $BR(G, \theta)$, where G is a group. Kochin [21] and Munn [24] extended this independently, showing that simple inverse ω-semigroups are isomorphic to Bruck-Reilly semigroups of the form $BR(T, \theta)$, where T is a finite chain of groups. Munn [24] also extended this to a structure theorem of arbitrary inverse ω-semigroups with kernel, which we will use heavily in Section 8.2.

We give some properties of Bruck-Reilly semigroups in the following Proposition.

Proposition 2.6.1 ([20, Proposition 5.6.6])**.** *Let T be a monoid and let $S = BR(T, \theta)$. Then:*

(1) $(m, a, n)\, \mathcal{R}^S\, (p, b, q)$ *if and only if* $m = p$ *and* $a\, \mathcal{R}^T\, b$*;*

(2) $(m, a, n)\, \mathcal{L}^S\, (p, b, q)$ *if and only if* $n = q$ *and* $a\, \mathcal{L}^T\, b$*;*

(3) *S is an inverse semigroup if and only if T is an inverse semigroup. If T is an inverse semigroup, then* $(m, a, n)^{-1} = (n, a^{-1}, m)$*.*

Chapter 3

Left I-orders

In this chapter, we will introduce the basic definitions of left I-orders and semigroups of left I-quotients and give some examples. We concentrate on straight left I-orders, the reason being that they are much easier to work with than arbitrary left I-orders. In many situations left I-orders are automatically straight, as we shall see in the following chapters.

We give some properties of straight left I-orders which we will use throughout this thesis, especially in the proofs of Theorem 3.3.7 and Theorem 4.2.1.

In Section 3.2, we show that left I-orders sit in the context of other types of left orders. Consequently, we can use an established result on Clifford semigroups of left Fountain-Gould quotients to obtain an analogous result for Clifford semigroups of left I-quotients.

In Section 3.3, we consider when a homomorphism between straight left I-orders can be lifted to a homomorphism between the semigroups of straight left I-quotients. Consequently, we determine necessary and sufficient conditions for two semigroups of straight left I-quotients of a given semigroup to be isomorphic.

All results are the joint work of myself and Professor Victoria Gould, unless indicated otherwise through the citation of a supporting reference.

3.1 Definitions and examples

We give the formal definition of a semigroup of left I-quotients, first introduced by Ghroda and Gould in [14].

Definition 3.1.1. Let S be a subsemigroup of an inverse semigroup Q. Then Q is a *semigroup of left I-quotients* of S, and S is a *left I-order* in Q, if every $q \in Q$ can be written as

$$q = a^{-1}b$$

for some $a, b \in S$, where a^{-1} denotes the inverse of a in the sense of inverse semigroup theory. *Right I-orders* and *semigroups of right I-quotients* are defined dually. If S is both a left I-order and a right I-order in Q, then we say that S is an I-order in Q, and Q is a semigroup of I-quotients of S.

We now define what it means for a left I-order to be straight. Straightness is not only a very useful property, but one that appears typical in left I-orders. Indeed, we cannot find a left I-order which is not straight. We conjecture that there are left I-orders which are not straight, but we have no counterexample to date.

Definition 3.1.2. A left I-order S in an inverse semigroup Q is *straight* in Q if every element in Q can be written as $a^{-1}b$ where $a, b \in S$ and $a \,\mathcal{R}^Q\, b$. We also say that Q is *straight* over S, and we call Q a *semigroup of straight left I-quotients* of S. *Semigroups of straight right I-quotients* are defined dually.

In this thesis we will see that straightness is a very important property for a left I-order to have. Most of the results in this thesis will be specifically about straight left I-orders. We will find in later results (for example, Lemma 3.3.4, Lemma 3.3.6 and Lemma 4.3.2) that, if S is straight, we can determine equalities and products in Q using equalities and relations between elements of S. This makes straight left I-orders easier to work with than general left I-orders. Because of this, it is of interest to determine when a left I-order is straight. In this regard, the following result is an important tool; it is an unpublished observation of Nassraddin Ghroda and Victoria Gould.

Lemma 3.1.3 (Ghroda, Gould)**.** *Let S be a left I-order in Q. Then S is straight in Q if and only if S intersects every $\mathcal{L}$-class of Q.*

Proof. Let S be straight in Q, and let $q = a^{-1}b \in Q$ such that $a, b \in S$ and $a\,\mathcal{R}^Q\,b$. Then

$$q^{-1}q = b^{-1}aa^{-1}b = b^{-1}bb^{-1}b = b^{-1}b,$$

and so $b \in S \cap L_q$.

Conversely, suppose $S \cap L_q \neq \varnothing$ for all $q \in Q$. Let $q \in Q$; we know that $q = a^{-1}b$, where $a, b \in S$. Then

$$q = a^{-1}aa^{-1}bb^{-1}b = a^{-1}fb,$$

where $f = aa^{-1}bb^{-1} \in E(Q)$. Since S intersects every $\mathcal{L}$-class, there exists $u \in S \cap L_f$, and so $f = u^{-1}u$. Hence

$$(ua)(ua)^{-1} = uaa^{-1}u^{-1} = ufaa^{-1}u^{-1} = ufu^{-1} = uu^{-1}.$$

Similarly $(ub)(ub)^{-1} = uu^{-1}$.

We can therefore write

$$q = a^{-1}fb = a^{-1}u^{-1}ub = (ua)^{-1}(ub),$$

where $ua\,\mathcal{R}^Q\,ub$. It follows that Q is straight over S. □

The rest of this section is devoted to illustrative examples. The first is a left I-order in an inverse semigroup with totally ordered idempotents. We will study these types of left I-orders in Chapter 8.

Example 3.1.4 (Bicyclic Monoid)**.** Let $\mathcal{B}$ be the bicyclic monoid and let S be the $\mathcal{R}$-class of the identity, $S = \{\,(0,n) \mid n \in \mathbb{N}^0\,\}$. We have that $\mathcal{B}$ is an inverse semigroup and for any $(a,b) \in \mathcal{B}$,

$$(a,b) = (a,0)(0,b) = (0,a)^{-1}(0,b)$$

so $\mathcal{B}$ is a semigroup of left I-quotients of S. Additionally, since $(0,a)\,\mathcal{R}^{\mathcal{B}}\,(0,b)$ for all $(0,a),(0,b) \in S$, we see that $\mathcal{B}$ is straight over S.

One question we can ask is if a given semigroup has at most one semigroup of left I-quotients. The answer to this is no. We will use a semilattice of groups to show this. In the case of semilattices of groups, semigroups of left I-quotients are exactly semigroups of left Fountain-Gould quotients, so this is not new.

Example 3.1.5 (Counterexample to Uniqueness)**.** Consider the semigroup $(\mathbb{N}, \cdot)$. It lies inside the group $(\mathbb{Q}^+, \cdot)$ and is a left I-order in $(\mathbb{Q}^+, \cdot)$, since for all $\frac{a}{b} \in \mathbb{Q}^+$, we have $\frac{a}{b} = b^{-1}a$ with $a, b \in \mathbb{N}$.

We will now show that the semigroup P from Example 2.4.5 is also a semigroup of left I-quotients of $(\mathbb{N}, \cdot)$. We see that P is an inverse semigroup with $(\frac{a}{b})_\alpha^{-1} = (\frac{b}{a})_\alpha$, and $(\frac{a}{b})_\beta^{-1} = (\frac{b}{a})_\beta$. We define an embedding $\phi : (\mathbb{N}, \cdot) \to P$ by

$$n\phi = \begin{cases} (\frac{n}{1})_\alpha & \text{if } n \text{ odd,} \\ (\frac{n}{1})_\beta & \text{if } n \text{ even.} \end{cases}$$

We see that P is a semigroup of left I-quotients of $(\mathbb{N}, \cdot)$, since

$$(\tfrac{a}{b})_\alpha = (b\phi)^{-1}(a\phi) \text{ and } (\tfrac{a}{b})_\beta = (2b\phi)^{-1}(2a\phi).$$

We know that P is not isomorphic to $(\mathbb{Q}^+, \cdot)$ because P has two idempotents whilst $(\mathbb{Q}^+, \cdot)$ only has one.

3.2 Connections with other types of semigroups of quotients

Up to this point, we have defined semigroups of quotients in terms of inverse semigroup theory. However, in generality, one could attempt to say that, for Q a semigroup and S a subsemigroup of Q, that Q is a *semigroup of left quotients* of S if every $q \in Q$ can be written as $q = a^{-1}b$ for some $a, b \in S$. However, this definition has no meaning without defining what we mean by a^{-1}. By using different interpretations of a^{-1}, we find different types of semigroups of left quotients.

In this section we look at two different types of semigroups of left quotients distinct from semigroups of left I-quotients. We look at the connections between them and semigroups of left I-quotients. We use one of these connections to obtain necessary and sufficient conditions for a semigroup to be a Clifford semigroup of I-quotients.

3.2.1 Semigroups of classical left quotients

The earliest type of semigroups of quotients are groups and use group inverses. This definition originates from work of Ore on rings of left quotients [27] and was formalised by Dubreil [4]. For a more comprehensive history, read [3, Section 1.10].

Definition 3.2.1. Let S be a subsemigroup of a group G. Then G is a *group of left quotients* of S, and S is a *left order* in G, if every $g \in G$ can be written as $g = a^{-1}b$ for some $a, b \in S$.

We know that every group is an inverse semigroup with the inverse elements in the sense of inverse semigroup theory being equal to the group inverses. Therefore it is easy to see that every group of left quotients is a semigroup of left I-quotients. Moreover, since the $\mathcal{R}$-relation in a group is the universal relation, every group of left quotients is a semigroup of straight left I-quotients.

The question of whether a semigroup has a group of left quotients was answered implicitly by Ore [27] and formalised by Dubreil [4]. We give the result below.

Definition 3.2.2. A semigroup S is *right reversible* or *left Ore* if for any $a, b \in S$, $Sa \cap Sb \neq \varnothing$. That is, there exists $u, v \in S$ with $ua = vb$.

Theorem 3.2.3 ([27], [4])**.** *A semigroup S has a group of left quotients if and only if S is cancellative and right reversible.*

We can generalise groups of quotients by letting the semigroup of quotients Q be a monoid and considering a^{-1} to be the inverse of a in the group of units. Note that this means a is a unit. We call these semigroups of classical left quotients to distinguish them from other types of semigroups of left quotients.

Definition 3.2.4 ([25])**.** Let S be a subsemigroup of a monoid M. Then M is a *monoid of classical left quotients* of S if

(i) for every cancellative $a \in S$, there exists an $a^{-1} \in M$;

(ii) every $m \in M$ can be written as $m = a^{-1}b$ for some $a, b \in S$.

This is a departure from Definition 3.2.1, since only certain elements need group inverses.

Proposition 3.2.5. *If M is a monoid of classical left quotients of S and M is an inverse semigroup, then M is a semigroup of straight left I-quotients of S.*

Proof. In this proof we will adopt the temporary convention that for $a \in M$, a' denotes the inverse of a in the group of units of M, and a^{-1} denotes the inverse of a in the sense of inverse semigroup theory.

Let $m \in M$. Since M is a monoid of classical left quotients of S, we know that we can write m as $m = a'b$ for some $a, b \in S$.

Since M is an inverse semigroup, we can apply Lemma 2.2.2 to obtain that $a^{-1} = a'$. Therefore, we can write m as $m = a^{-1}b$ for some $a, b \in S$. Hence M is a semigroup of left I-quotients of S.

We will now prove that M is straight over S by proving that S intersects every $\mathcal{L}$-class of M. Let $q \in M$. Since M is a monoid of classical left quotients of S, we know that we can write q as $q = a'b$ for some $a, b \in S$. We see that

$$aq = aa'b = 1b = b.$$

Therefore $q \mathcal{L} b$, and so $b \in L_q \cap S$. Therefore S intersects every $\mathcal{L}$-class of M. We apply Lemma 3.1.3 to obtain that M is straight over S. □

The question of whether a semigroup has a monoid of classical left quotients was answered by Murata [25]. We give the result below.

Definition 3.2.6. Let S be a semigroup. We say that S satisfies the *left Ore-Asano condition* if for every $b \in S$ and cancellative $a \in S$, there exists $b' \in S$ and cancellative $a' \in S$ such that $b'a = a'b$.

Theorem 3.2.7 ([25])**.** *A semigroup S has a monoid of classical left quotients if and only if S satisfies the left Ore-Asano condition.*

The proof of this theorem is similar to that of Theorem 3.2.3.

3.2.2 Semigroups of left Fountain-Gould quotients

A different definition of semigroups of quotients was proposed by Fountain and Petrich in 1986 [9], although this definition was restricted to completely 0-simple semigroups of quotients. Gould generalised this concept to all semigroups, and to the one-sided case later that same year [15]. In this definition elements are inverted by finding the inverse in a subgroup. In the literature, these are called semigroups of left quotients without an additional qualifier. However, in this thesis I will refer to them as semigroups of left Fountain-Gould quotients, in order to distinguish them from semigroups of I-quotients.

We use the convention that $a^{\#}$ denotes the inverse of a in some subgroup of Q. From knowledge of subgroups of semigroups, we know that if $a^{\#}$ exists, it is unique.

Definition 3.2.8. Let S be a subsemigroup of a semigroup Q. Then Q is a *semigroup of left Fountain-Gould quotients* of S, and S is a *left Fountain-Gould order* in Q if

(i) every square-cancellable element lies in a subgroup of Q;

(ii) every $q \in Q$ can be written as $q = a^{\#}b$ for some $a, b \in S$.

We give a connection between semigroups of Fountain-Gould quotients and semigroups of I-quotients.

Lemma 3.2.9. *Let S be a left Fountain-Gould order in Q with Q an inverse semigroup. Then S is a straight left I-order in Q.*

Proof. Let $a \in S$. If $a^{\#}$ exists, then clearly $a^{-1} = a^{\#}$. We have that every $q \in Q$ can be written as $q = a^{\#}b$ for some $a, b \in S$. Therefore every $q \in Q$ can be written as $q = a^{-1}b$ for some $a, b \in S$.

We will now prove that S is straight in Q by proving that S intersects every $\mathcal{L}$-class of Q. Let $q \in Q$. We know that $q \in Q$ can be written as $q = a^{\#}b$ for some $a, b \in S$. We see that

$$q = a^{\#}b \,\mathcal{L}^Q\, aa^{\#}b = a^{\#}ab \,\mathcal{L}^Q\, ab.$$

We know that $ab \in S$. Hence $b \in L_q \cap S$. Therefore S intersects every $\mathcal{L}$-class of Q. We apply Lemma 3.1.3 to obtain that S is straight in Q. □

We know that if an inverse semigroup is a semigroup of left Fountain-Gould quotients, then it is a semigroup of left I-quotients. However, there are semigroups of left I-quotients which are not semigroups of left Fountain-Gould quotients. We give an example taken from [10, Example 2.3].

Example 3.2.10. Let H be a left order in a group G, and let $B = B(G, I)$ be a Brandt semigroup over G with $|I| \geqslant 2$. Fixing $i \in I$, we define

$$S_i = \{(i, h, j) \mid h \in H,\ j \in I\} \cup \{0\}.$$

We claim that S_i is a left I-order in B, but not a left Fountain-Gould order in B. We start by acknowledging that S_i is a subsemigroup of B. To prove that S_i is a left I-order in B, we first notice that $0 = 0^{-1}0$. For non-zero $(j, g, k) \in B$, since we can write $g = a^{-1}b$ for $a, b \in H$, we have

$$(j, g, k) = (i, a, j)^{-1}(i, b, k).$$

To prove that S_i is not a left Fountain-Gould order in B, we first consider which elements are in subgroups. We know that 0 is in its own trivial subgroup. By Green's Theorem, we know that $(m, a, n) \in B$ lies in a subgroup of B if and only if $(m, a, n)\,\mathcal{H}\,(m, a, n)^2$, which is true if and only if $m = n$. Therefore non-zero elements of S_i that are in subgroups of B are elements of the form (i, a, i).

Assume S_i is a left Fountain-Gould order in B. Consider non-zero $(j, g, k) \in B$ such that $j \neq i$. By assumption, we can write this element as

$$(j, g, k) = s^{\#}t,$$

for $s, t \in S_i$. Since (j, g, k) is non-zero, we know that s and t are non-zero. Also, by definition, s is in a subgroup of B. Therefore we can write the above equation as

$$(j, g, k) = (i, a, i)^{\#}(i, b, n) = (i, a^{-1}, i)(i, b, n) = (i, a^{-1}b, n).$$

Since $j \neq i$, this is a contradiction.

Clifford semigroups of left quotients

Given the connection between left Fountain-Gould orders and left I-orders demonstrated in Lemma 3.2.9, we see that we can use established results for left Fountain-Gould orders to obtain results for left I-orders. In this subsection, we will use established conditions for a semigroup to be a left Fountain-Gould order in a Clifford semigroup to give conditions for a semigroup to be a left I-order in a Clifford semigroup.

The following theorem of Gould gives necessary and sufficient conditions for a semigroup to be a left Fountain-Gould order in a Clifford semigroup.

Theorem 3.2.11 ([16, Theorem 3.1])**.** *A semigroup S is a left Fountain-Gould order in a semilattice Y of groups G_α, $\alpha \in Y$, if and only if S is a semilattice Y of right reversible, cancellative semigroups S_α, $\alpha \in Y$.*

We use this theorem give the analogous result for Clifford semigroups of left I-quotients.

Corollary 3.2.12. *A semigroup S is a left I-order in a semilattice Y of groups G_α, $\alpha \in Y$ if and only if S is a semilattice Y of right reversible, cancellative semigroups S_α, $\alpha \in Y$.*

Proof. Let Q be a semilattice Y of groups G_α, $\alpha \in Y$. We know that every element of Q lies in a subgroup. Consequently, for every $a \in Q$, $a^\#$ exists and $a^{-1} = a^\#$. We can therefore see that a semigroup S is a left I-order in Q if and only if S is a left Fountain-Gould order in Q. By Theorem 3.2.11, we know that this is true if and only if S is a semilattice Y of right reversible, cancellative semigroups S_α, $\alpha \in Y$. □

3.3 Uniqueness and extension of homomorphisms

We have shown that a semigroup can have two non-isomorphic semigroups of straight left I-quotients. It is then natural to ask: if S has two semigroups of straight left I-quotients, Q and P, under what conditions are Q and P isomorphic?

To answer this we first consider a related question: when does a homomorphism from a straight left I-order lift to a homomorphism from its semigroup of left I-quotients? We will answer both of these questions in this section.

We begin by introducing the following notions.

Definition 3.3.1. Let S be a subsemigroup of Q and let $\phi : S \to P$ be a homomorphism of S into a semigroup P. If there is a homomorphism $\bar{\phi} : Q \to P$ such that $\bar{\phi}|_S = \phi$, then we say that ϕ lifts to Q. If ϕ lifts to an isomorphism, then we say that Q and P are isomorphic over S.

To achieve our goal, we must first examine when two quotients $a^{-1}b = c^{-1}d$ are equal, where $a, b, c, d \in S$ and S is a left I-order in Q with $a \,\mathcal{R}^Q\, b$ and $c \,\mathcal{R}^Q\, d$. This relation has already been determined by Ghroda and Gould [10].

Lemma 3.3.2 ([10, Lemma 2.7])**.** *Let S be a straight left I-order in Q. Let $a, b, c, d \in S$ with $a \,\mathcal{R}^Q\, b$ and $c \,\mathcal{R}^Q\, d$. Then $a^{-1}b = c^{-1}d$ in Q if and only if there exists $x, y \in S$ such that*

$$xa = yc,\ xb = yd,\ x \,\mathcal{R}^Q\, y,\ x^{-1} \,\mathcal{R}^Q\, a \text{ and } y^{-1} \,\mathcal{R}^Q\, c.$$

However, we need to be able to express the conditions in Lemma 3.3.2 entirely in terms of elements of S. We remind the reader that in an inverse semigroup Q, we have that $x \,\mathcal{R}^Q\, y$ if and only if $x^{-1} \,\mathcal{L}^Q\, y^{-1}$.

Lemma 3.3.3. *Let Q be an inverse semigroup and let $x, a \in Q$. Then*

$$x^{-1} \,\mathcal{R}^Q\, a \text{ if and only if } x \,\mathcal{R}^Q\, xa \,\mathcal{L}^Q\, a.$$

Proof. Let $x^{-1} \,\mathcal{R}^Q\, a$. Using the fact that $\mathcal{R}^Q$ is a left congruence, this implies

$$xa \,\mathcal{R}^Q\, xx^{-1} \,\mathcal{R}^Q\, x.$$

We know that $x^{-1} \,\mathcal{R}^Q\, a$ implies that $x \,\mathcal{L}^Q\, a^{-1}$. Therefore, using the fact that $\mathcal{L}^Q$ is a right congruence, we also have

$$xa \,\mathcal{L}^Q\, a^{-1}a \,\mathcal{L}^Q\, a.$$

Conversely, let $xa \in R_x \cap L_a$. By [20, Prop. 2.3.7], we have that $L_x \cap R_a$ contains an idempotent, e.

Then, as $x \,\mathcal{L}^Q\, e$, we have $x^{-1} \,\mathcal{R}^Q\, e^{-1} = e \,\mathcal{R}^Q\, a$. □

We can now rewrite Lemma 3.3.2 in terms of relations restricted to S. The next result is an adaptation of [10, Lemma 2.7].

xa		x
a		e

Lemma 3.3.4. *Let S be a straight left I-order in Q. Let $a, b, c, d \in S$ with $a \mathcal{R}^Q b$ and $c \mathcal{R}^Q d$. Then $a^{-1}b = c^{-1}d$ if and only if there exists $x, y \in S$ such that*

$$xa = yc, \ xb = yd, \ x \mathcal{R}^Q xa \mathcal{L}^Q a, \ \text{and} \ y \mathcal{R}^Q yc \mathcal{L}^Q c.$$

Note that since $xa = yc$, the conditions imply that $x \mathcal{R}^Q y$ and $a \mathcal{L}^Q c$.

This has internalised the condition $a^{-1}b = c^{-1}d$ to equalities on S and Green's relations in Q restricted to elements of S.

The next thing to address is multiplication on Q. Let $a^{-1}b$ and $c^{-1}d$ be elements of Q in standard form, meaning that $a, b, c, d \in S$ with $a \mathcal{R}^Q b$ and $c \mathcal{R}^Q d$. Since $b, c \in S$, we know that $bc^{-1} \in Q$. Therefore, since Q is a semigroup of straight left I-quotients of S, there exists $u, v \in S$ with $u \mathcal{R}^Q v$, such that

$$bc^{-1} = u^{-1}v$$

in Q. Therefore, multiplication on Q is given by

$$a^{-1}bc^{-1}d = (ua)^{-1}(vd),$$

where $bc^{-1} = u^{-1}v$ in Q. In the same way as we have internalised to S the condition that $a^{-1}b = c^{-1}d$, we need to be able to express $bc^{-1} = u^{-1}v$ solely in terms of elements of S. We start with a useful lemma of Ghroda and Gould.

Lemma 3.3.5 ([10, Lemma 2.6])**.** *Let b, c, u, v be elements of an inverse semigroup Q such that $u \mathcal{R}^Q v$. If $bc^{-1} = u^{-1}v$ then $ub = vc$.*

Proof. We have that

$$bc^{-1}cb^{-1} = (bc^{-1})(bc^{-1})^{-1} = (u^{-1}v)(u^{-1}v)^{-1} = u^{-1}vv^{-1}u = u^{-1}u,$$

as $u \mathcal{R}^Q v$. Therefore, using the fact that idempotents commute,

$$bc^{-1}c = bb^{-1}bc^{-1}c = bc^{-1}cb^{-1}b = u^{-1}ub.$$

We can left multiply this by u to obtain

$$ubc^{-1}c = ub. \tag{3.1}$$

We can also see that

$$vc = vv^{-1}vc = uu^{-1}vc = ubc^{-1}c. \tag{3.2}$$

We compare (3.1) and (3.2) to obtain our result. □

Lemma 3.3.6. *Let Q be an inverse semigroup and let $b, c, u, v \in Q$ such that $u \mathcal{R}^Q v$. Then $bc^{-1} = u^{-1}v$ in Q if and only if*

$$ub = vc,\ v \mathcal{R}^Q vc \text{ and } L^Q_b \wedge L^Q_c = L^Q_{ub}.$$

Proof. Let $bc^{-1} = u^{-1}v$. By Lemma 3.3.5, we have $ub = vc$. Since $u \mathcal{R}^Q v$, we know that $uu^{-1} = vv^{-1}$. Therefore, using $u^{-1}v = bc^{-1}$ and $ub = vc$, we have

$$v = vv^{-1}v = uu^{-1}v = ubc^{-1} = vcc^{-1}.$$

Therefore, by Lemma 2.2.3, we have $v \mathcal{R}^Q vc$. Finally, again using $bc^{-1} = u^{-1}v$ and $ub = vc$, we have

$$b^{-1}bc^{-1}c = b^{-1}u^{-1}vc = b^{-1}u^{-1}ub = (ub)^{-1}(ub).$$

Therefore, by Lemma 2.2.8, we have that $L^Q_b \wedge L^Q_c = L^Q_{ub}$.

Conversely, let

$$ub = vc,\ v \mathcal{R}^Q vc \text{ and } L^Q_b \wedge L^Q_c = L^Q_{ub}.$$

By Lemma 2.2.3, we know that $v \mathcal{R}^Q vc$ implies that $v = vcc^{-1}$. Using this along with $vc = ub$, we have

$$u^{-1}v = u^{-1}vcc^{-1} = u^{-1}ubc^{-1} = u^{-1}ubb^{-1}bc^{-1} = bb^{-1}u^{-1}ubc^{-1}, \tag{3.3}$$

using the fact that idempotents commute in the last equality. By Lemma 2.2.8, we know that $L_b^Q \wedge L_c^Q = L_{ub}^Q$ implies that $b^{-1}bc^{-1}c = (ub)^{-1}(ub)$. Therefore

$$bb^{-1}u^{-1}ubc^{-1} = b(ub)^{-1}(ub)c^{-1} = bb^{-1}bc^{-1}cc^{-1} = bc^{-1}. \tag{3.4}$$

Putting Equations (3.3) and (3.4) together, we obtain $u^{-1}v = bc^{-1}$. □

We now give necessary and sufficient conditions for a homomorphism from a straight left I-order to an inverse semigroup P to lift to a homomorphism from its semigroup of straight left I-quotients to P.

Theorem 3.3.7. *Let S be a straight left I-order in Q and let T be a subsemigroup of an inverse semigroup P. Suppose that $\phi : S \to T$ is a homomorphism. Then ϕ lifts to a (unique) homomorphism $\overline{\phi} : Q \to P$ if and only if for all $a, b, c \in S$:*

(i) $a\,\mathcal{R}^Q\,b$ implies that $a\phi\,\mathcal{R}^P\,b\phi$; and

(ii) $L_a^Q \wedge L_b^Q = L_c^Q$ implies that $L_{a\phi}^P \wedge L_{b\phi}^P = L_{c\phi}^P$.

If (i) and (ii) hold and $S\phi$ is a left I-order in P, then $\overline{\phi} : Q \to P$ is onto.

Proof. First, let ϕ lift to a homomorphism $\overline{\phi}$. Since homomorphisms preserve Green's relations, (i) holds. For (ii), let $L_a^Q \wedge L_b^Q = L_c^Q$. Since Q is inverse, Lemma 2.2.8 gives us that

$$a^{-1}ab^{-1}b = c^{-1}c$$

in Q. Since homomorphisms preserve inverses, we have that

$$(a\phi)^{-1}(a\phi)(b\phi)^{-1}(b\phi) = (c\phi)^{-1}(c\phi)$$

in P. Lemma 2.2.8 then gives us $L_{a\phi}^P \wedge L_{b\phi}^P = L_{c\phi}^P$. Therefore, we see that (ii) also holds.

Conversely, suppose (i) and (ii) hold. Note that by applying (ii) with $a = b$, we have that for all $a, c \in S$,

($*$) $a\,\mathcal{L}^Q\,c$ implies that $a\phi\,\mathcal{L}^P\,c\phi$.

We define $\overline{\phi} : Q \to P$ by

$$(a^{-1}b)\overline{\phi} = (a\phi)^{-1}b\phi$$

where $a, b \in S$ and $a \mathcal{R}^Q b$.

To show that $\bar{\phi}$ is well-defined, suppose that $a^{-1}b = c^{-1}d$ where $a, b, c, d \in S$, $a \mathcal{R}^Q b$ and $c \mathcal{R}^Q d$. Then by Lemma 3.3.4, there exists $x, y \in S$ such that

$$xa = yc,\ xb = yd,\ x \mathcal{R}^Q xa \mathcal{L}^Q a \text{ and } y \mathcal{R}^Q yc \mathcal{L}^Q c.$$

Therefore, using the fact that ϕ is a homomorphism along with (i) and $(*)$, we have that

$$x\phi a\phi = y\phi c\phi,\ x\phi b\phi = y\phi d\phi,\ x\phi \mathcal{R}^P x\phi a\phi \mathcal{L}^P a\phi \text{ and } y\phi \mathcal{R}^P y\phi c\phi \mathcal{L}^P c\phi.$$

Therefore, again by Lemma 3.3.4, we have $(a\phi)^{-1}b\phi = (c\phi)^{-1}d\phi$. Therefore $\bar{\phi}$ is well-defined.

To see that $\bar{\phi}$ lifts to ϕ, let $h \in S$. We can write $h = k^{-1}l$, where $k, l \in S$ with $k \mathcal{R}^Q l$. We see that

$$kh = kk^{-1}l = ll^{-1}l = l,$$

which also implies that

$$h = k^{-1}l = k^{-1}kh.$$

By Lemma 2.2.4, this implies that $h \mathcal{L}^Q kh$. Since ϕ is a homomorphism applying $(*)$ gives us that

$$k\phi h\phi = l\phi \text{ and } h\phi \mathcal{L}^P k\phi h\phi.$$

By Lemma 2.2.4, it follows that

$$h\phi = (k\phi)^{-1}k\phi h\phi = (k\phi)^{-1}l\phi = h\bar{\phi}.$$

To show that $\bar{\phi}$ is a morphism, let $a^{-1}b, c^{-1}d \in Q$ with $a \mathcal{R}^Q b$ and $c \mathcal{R}^Q d$. We know that $bc^{-1} = u^{-1}v$ for some $u, v \in S$ with $u \mathcal{R}^Q v$. By Lemma 3.3.6, this implies that

$$v \mathcal{R}^Q vc = ub \text{ and } L_b^Q \wedge L_c^Q = L_{ub}^Q.$$

Using (i) and (ii) this gives us

$$u\phi \mathcal{R}^P v\phi \mathcal{R}^P v\phi c\phi = u\phi b\phi \text{ and } L_{b\phi}^P \wedge L_{c\phi}^P = L_{u\phi b\phi}^P.$$

Since P is an inverse semigroup, we can apply Lemma 3.3.6 to obtain

$$b\phi(c\phi)^{-1} = (u\phi)^{-1}v\phi.$$

Multiplying in Q, we have

$$a^{-1}bc^{-1}d = a^{-1}u^{-1}vd = (ua)^{-1}vd,$$

with $ua\,\mathcal{R}^Q\,ub = vc\,\mathcal{R}^Q\,vd$. Therefore in P

$$\begin{aligned}((a^{-1}b)(c^{-1}d))\overline{\phi} &= ((ua)^{-1}vd)\overline{\phi}\\ &= ((ua)\phi)^{-1}(vd)\phi\\ &= (a\phi)^{-1}(u\phi)^{-1}v\phi d\phi\\ &= (a\phi)^{-1}b\phi(c\phi)^{-1}d\phi\\ &= (a^{-1}b)\overline{\phi}(c^{-1}d)\overline{\phi},\end{aligned}$$

so $\overline{\phi}$ is a morphism as required.

We have that $\overline{\phi}$ is unique, since homomorphisms must preserve inverses.

If (i) and (ii) hold and $S\phi$ is a left I-order in P, then for any $p \in P$ we have $p = (a\phi)^{-1}b\phi$ for some $a, b \in S$. Since $\overline{\phi}$ is a homomorphism, and homomorphisms between inverse semigroups preserve inverses, we have that

$$(a^{-1}b)\overline{\phi} = (a\overline{\phi})^{-1}b\overline{\phi} = (a\phi)^{-1}b\phi = p,$$

and hence $\overline{\phi}$ is onto. Note that we do not need $S\phi$ to be straight in P in order to do this. $\square$

We will now use Theorem 3.3.7 to prove that if S has two semigroups of straight left I-quotients, Q and P, then Q is isomorphic to P if and only if the restrictions of their $\mathcal{R}$ and $\mathcal{L}$-relations to S are equal. To do this we use the next result on the preorder associated with $\mathcal{L}$ in semigroups of straight left I-quotients.

Lemma 3.3.8. *Let S be a straight left I-order in Q and let $a, b \in Q$. Then*

$$a \leqslant_{\mathcal{L}^Q} b \text{ if and only if } a\,\mathcal{L}^Q\,db$$

for some $d \in S$.

Proof. Let $a \leqslant_{\mathcal{L}^Q} b$. By definition there exists $q \in Q$ such that $a = qb$. We can write q as $c^{-1}d$ with $c, d \in S$ and $c \,\mathcal{R}^Q\, d$. We can see that $a = c^{-1}db$ and $ca = db$, and so $a \,\mathcal{L}^Q\, db$. Conversely, if $a \,\mathcal{L}^Q\, db$, then since $db \leqslant_{\mathcal{L}^Q} b$, we obtain $a \leqslant_{\mathcal{L}^Q} b$. □

Theorem 3.3.9. *Let S be a straight left I-order in Q and let $\phi : S \to P$ be an embedding of S into an inverse semigroup P such that $S\phi$ is a straight left I-order in P. Then Q is isomorphic to P over S if and only if for any $a, b \in S$:*

(i) $a \,\mathcal{R}^Q\, b$ if and only if $a\phi \,\mathcal{R}^P\, b\phi$; and

(ii) $a \,\mathcal{L}^Q\, b$ if and only if $a\phi \,\mathcal{L}^P\, b\phi$.

Proof. If Q is isomorphic to P over S, then Green's relations are preserved.

Conversely suppose (i) and (ii) hold. Firstly we will use (ii) to show that for all $a, b, c \in S$:

($*$) $a \leqslant_{\mathcal{L}^Q} b$ if and only if $a\phi \leqslant_{\mathcal{L}^P} b\phi$; and

($**$) $L^Q_a \wedge L^Q_b = L^Q_c$ if and only if $L^P_{a\phi} \wedge L^P_{b\phi} = L^P_{c\phi}$.

We see that ($*$) is a direct consequence of Lemma 3.3.8.

For ($**$), let $a, b, c \in S$ such that

$$L^Q_a \wedge L^Q_b = L^Q_c.$$

Since P is an inverse semigroup, $P/\mathcal{L}^P$ is a meet semilattice by Lemma 2.2.8, so

$$L^P_{a\phi} \wedge L^P_{b\phi} = L^P_p$$

for some $p \in P$. Since $S\phi$ is a straight left I-order in P, Lemma 3.1.3 implies that $S\phi$ intersects every $\mathcal{L}$-class of P. Therefore there exists some $d \in S$ such that $p \,\mathcal{L}^P\, d\phi$. Therefore

$$L^P_{a\phi} \wedge L^P_{b\phi} = L^P_{d\phi}.$$

We know that $c \leqslant_{\mathcal{L}^Q} a, b$. We use ($*$) to obtain $c\phi \leqslant_{\mathcal{L}^P} a\phi, b\phi$. Therefore, by the definition of meet, this gives us $c\phi \leqslant_{\mathcal{L}^P} d\phi$.

Similarly, we apply $(*)$ to $d\phi \leqslant_{\mathcal{L}^P} a\phi, b\phi$ to obtain $d \leqslant_{\mathcal{L}^Q} a, b$. Therefore, by the definition of meet, $d \leqslant_{\mathcal{L}^Q} c$. Applying $(*)$ once again, we have that $d\phi \leqslant_{\mathcal{L}^P} c\phi$. Putting both together gives us $c\phi \, \mathcal{L}^P \, d\phi$. Therefore

$$L^P_{a\phi} \wedge L^P_{b\phi} = L^P_{c\phi}.$$

The converse is similar.

From Theorem 3.3.7, ϕ lifts to a homomorphism $\overline{\phi} : Q \to P$, where, for $a, b \in S$, $(a^{-1}b)\overline{\phi} = (a\phi)^{-1}b\phi$. Since every element of Q can be written as $a^{-1}b$, with $a, b \in S$, this wholly defines $\overline{\phi}$.

Since ϕ is an embedding, $\phi : S \to S\phi$ is an isomorphism. Therefore, $\phi^{-1} : S\phi \to S$ is also an isomorphism. By Theorem 3.3.7, ϕ^{-1} lifts to a homomorphism $\overline{\phi^{-1}} : P \to Q$, where for $a, b \in S$, $((a\phi)^{-1}b\phi)\overline{\phi^{-1}} = a^{-1}b$. Since $S\phi$ is a left I-order in P, this wholly defines $\overline{\phi^{-1}}$.

Clearly $\overline{\phi}$ and $\overline{\phi^{-1}}$ are mutually inverse, and so are isomorphisms. $\square$

There seems to be no simplification of Theorem 3.3.7 in general, along the lines of Theorem 3.3.9, the reason being that in order to obtain the preservation of the meet function, one must have two-sided preservation of the $\leqslant_{\mathcal{L}}$ function, which one cannot conclude from a homomorphism.

Chapter 4

The general case

In this chapter, we will determine the conditions under which a semigroup S is a straight left I-order. We adopt two approaches. The first makes use of the meet structure of the $\mathcal{L}$-classes of inverse semigroups, and we present our conditions in terms of two binary relations and an associated partial order. The second is 'purely algebraic' in that we give our conditions in terms of two binary relations and a ternary relation on S.

4.1 Preliminaries

Assume S has a semigroup of straight left I-quotients Q. We aim to identify properties of S inherited from Q with the eventual goal of reconstructing such a Q from these properties.

By definition, every element in Q can be written as $a^{-1}b$, where $a, b \in S$ and $a \,\mathcal{R}^Q\, b$. Therefore, we can reconstruct Q as ordered pairs of elements of S under an equivalence relation. That is, we have a bijective correspondence between Q and the set

$$\{(a, b) \mid a, b \in S,\ a \,\mathcal{R}^Q\, b\}/\sim\,,$$

where $(a, b) \sim (c, d)$ if and only if $a^{-1}b = c^{-1}d$ in Q. We have already determined this relation in terms of $\mathcal{R}^Q$ and $\mathcal{L}^Q$ in Lemma 3.3.4. The conditions given in Lemma 3.3.4 will determine our $\sim$.

The next thing to address is multiplication on Q. We note that for every $b, c \in S$,

$bc^{-1} \in Q$ and therefore, since Q is a semigroup of straight left I-quotients of S, there exists $u, v \in S$ with $u \mathcal{R}^Q v$, such that $bc^{-1} = u^{-1}v$ in Q. Therefore, multiplication on Q is given by $a^{-1}bc^{-1}d = (ua)^{-1}(vd)$, where $bc^{-1} = u^{-1}v$ in Q. In the same way that we internalised to S the condition that $a^{-1}b = c^{-1}d$ in Lemma 3.3.4, we need to find a method of expressing $bc^{-1} = u^{-1}v$ solely in terms of elements of S. In Section 4.2, we will use the meets of $\mathcal{L}$-classes applying Lemma 3.3.6. In Section 4.3, we will use a more algebraic approach, employing a ternary relation to express this relation.

4.2 The general case using an ordering on inverse semigroups

The aim of this section is to prove Theorem 4.2.1.

Before stating the result, we will first introduce the notation used in the Theorem 4.2.1 and throughout this section. We use $\mathcal{L}'$ to denote the equivalence relation associated with the preorder $\leqslant_l$. We use L'_a to denote the $\mathcal{L}'$-class of a. We use $\wedge$ to denote the meet on $\mathcal{L}'$-classes associated with the preorder $\leqslant_l$. For example $L'_a \wedge L'_b = L'_c$ denotes that the meet of the $\mathcal{L}'$-class of a and the $\mathcal{L}'$-class of b is the $\mathcal{L}'$-class of c. The relation $\mathcal{R}^*$ will always refer to S.

We use Greek letters in this theorem in order to lessen confusion when applying the listed properties.

Theorem 4.2.1. *Let S be a semigroup and let $\mathcal{R}'$ and $\leqslant_l$ be binary relations on S. Then S has a semigroup of straight left I-quotients, Q, such that $\mathcal{R}^Q \cap (S \times S) = \mathcal{R}'$ and $\leqslant_{\mathcal{L}^Q} \cap (S \times S) = \leqslant_l$ if and only if $\mathcal{R}'$ is a left compatible equivalence relation; $\leqslant_l$ is a preorder such that the $\mathcal{L}'$-classes form a meet semilattice under the associated partial order; and S satisfies Conditions (M1) - (M6).*

(M1) For all $\alpha, \beta \in S$, there exists $\gamma, \delta \in S$ such that

$$\gamma \, \mathcal{R}' \, \delta \, \mathcal{R}' \, \delta\beta = \gamma\alpha \;\; \textit{and} \;\; L'_\alpha \wedge L'_\beta = L'_{\gamma\alpha}.$$

(M2) Right multiplication distributes over meet, that is, for all $\alpha, \beta, \gamma, \delta \in S$,

$$L'_\alpha \wedge L'_\beta = L'_\gamma \text{ implies that } L'_{\alpha\delta} \wedge L'_{\beta\delta} = L'_{\gamma\delta}.$$

(M3) For all $\alpha, \beta \in S$, $\alpha\beta \leqslant_l \beta$.

(M4) $\mathcal{R}' \subseteq \mathcal{R}^*$.

(M5) Let $\alpha, \beta, \gamma, \delta \in S$ such that $\gamma \,\mathcal{R}'\, \gamma\alpha \,\mathcal{L}'\, \alpha$ and $\delta \,\mathcal{R}'\, \delta\beta \,\mathcal{L}'\, \beta$. Then $\gamma \,\mathcal{L}'\, \delta$ if and only if $\alpha \,\mathcal{R}'\, \beta$.

(M6) For all $\alpha, \beta, \gamma \in S$, $\alpha \,\mathcal{L}'\, \beta \,\mathcal{L}'\, \gamma\alpha = \gamma\beta$ implies that $\alpha = \beta$.

Before we prove this theorem, let us first discuss (M1) - (M6) and why they are natural properties for this context:

(M1) This is the equivalent to the Ore condition in Definition 3.2.2. It implies that for $\alpha, \beta \in S$, there exists $\gamma, \delta \in S$ such that $\alpha\beta^{-1} = \gamma^{-1}\delta$ with $\gamma \,\mathcal{R}^Q\, \delta$. We will refer to this condition as the Ore condition.

(M2) This property is true in any inverse semigroup and is a result of the fact that idempotents commute in inverse semigroups.

(M3) This property states that under the preorder $\leqslant_l$, final factors are larger than the product from which they are taken.

(M4) By definition, the restriction to S of $\mathcal{R}$ in any oversemigroup of S, should be contained in $\mathcal{R}^*$.

(M5) This property demonstrates the fact that in an inverse semigroup, $\alpha \,\mathcal{R}^Q\, \beta$ if and only if $\alpha^{-1} \,\mathcal{L}^Q\, \beta^{-1}$.

(M6) This is a cancellation property that occurs in an inverse semigroup.

Note that Properties (M2) - (M6) are true in all inverse semigroups, whilst Property (M1) is specific to straight left I-orders.

We start the proof of Theorem 4.2.1 by first proving the forward implication. We assume that S has a semigroup of straight left I-quotients, Q, and we put

$\mathcal{R}^Q \cap (S \times S) = \mathcal{R}'$, $\mathcal{L}^Q \cap (S \times S) = \mathcal{L}'$ and $\leqslant_{\mathcal{L}^Q} \cap (S \times S) = \leqslant_l$. From knowledge of Green's relations, we know that $\mathcal{R}'$ is a left congruence on S, and that $\leqslant_l$ is a preorder on S with the associated equivalence relation, $\mathcal{L}'$. Using Lemma 2.2.8, we know that $Q/\mathcal{L}^Q$ forms a meet semilattice under $\leqslant_{\mathcal{L}^Q}$. Since S intersects every $\mathcal{L}^Q$-class, this means that $S/\mathcal{L}'$ forms a meet semilattice under $\leqslant_l$. We now prove that Properties (M1) - (M6) hold.

(M1) Let $\alpha, \beta \in S$. Then $\alpha, \beta \in Q$ and so, by closure under taking of inverses and multiplication, $\alpha\beta^{-1} \in Q$. Since Q is a semigroup of straight left I-quotients of S, there exists $\gamma, \delta \in S$ such that $\alpha\beta^{-1} = \gamma^{-1}\delta$ with $\gamma \,\mathcal{R}'\, \delta$. Lemma 3.3.6 then gives the result.

(M2) Since Q is an inverse semigroup, we can use Lemma 2.2.8 to give us that $L'_\alpha \wedge L'_\beta = L'_\gamma$ is equivalent to $\alpha^{-1}\alpha\beta^{-1}\beta = \gamma^{-1}\gamma$. Therefore

$$\begin{aligned}(\alpha\delta)^{-1}(\alpha\delta)(\beta\delta)^{-1}(\beta\delta) &= \delta^{-1}\alpha^{-1}\alpha\delta\delta^{-1}\beta^{-1}\beta\delta \\ &= \delta^{-1}\alpha^{-1}\alpha\beta^{-1}\beta\delta \\ &= \delta^{-1}\gamma^{-1}\gamma\delta \\ &= (\gamma\delta)^{-1}(\gamma\delta).\end{aligned}$$

And so, using Lemma 2.2.8 again, $L'_{\alpha\delta} \wedge L'_{\beta\delta} = L'_{\gamma\delta}$.

(M3) This is true in any semigroup, since $Q^1\alpha\beta \subseteq Q^1\beta$.

(M4) Since $\mathcal{R}' = \mathcal{R}^Q \cap (S \times S)$ and Q is an oversemigroup of S, then, by definition, $\alpha \,\mathcal{R}'\, \beta$ implies that $\alpha \,\mathcal{R}^*\, \beta$.

(M5) By Lemma 3.3.3, in an inverse semigroup we have that $\gamma \,\mathcal{R}'\, \gamma\alpha \,\mathcal{L}'\, \alpha$ implies that $\gamma^{-1} \,\mathcal{R}^Q\, \alpha$, and similarly $\delta \,\mathcal{R}'\, \delta\beta \,\mathcal{L}'\, \beta$ implies that $\delta^{-1} \,\mathcal{R}^Q\, \beta$. Then $\alpha \,\mathcal{R}'\, \beta$ implies that $\gamma^{-1} \,\mathcal{R}^Q\, \alpha \,\mathcal{R}^Q\, \beta \,\mathcal{R}^Q\, \delta^{-1}$. We know that $\gamma^{-1} \,\mathcal{R}^Q\, \delta^{-1}$ implies that $\gamma \,\mathcal{L}^Q\, \delta$, and so $\gamma \,\mathcal{L}'\, \delta$. The converse is similar.

(M6) Since α and γ are elements in an inverse semigroup, $\alpha \,\mathcal{L}'\, \gamma\alpha$ if and only if $\alpha = \gamma^{-1}\gamma\alpha$ by Lemma 2.2.4. Similarly, $\beta \,\mathcal{L}'\, \gamma\beta$ if and only if $\beta = \gamma^{-1}\gamma\beta$. Therefore, $\gamma\alpha = \gamma\beta$ together with $\alpha \,\mathcal{L}'\, \gamma\alpha$ and $\beta \,\mathcal{L}'\, \gamma\beta$, implies that

$$\alpha = \gamma^{-1}\gamma\alpha = \gamma^{-1}\gamma\beta = \beta.$$

This proves the forward implication of Theorem 4.2.1.

We now prove the converse. This will consist of proving that the following construction, P, yields a semigroup of straight left I-quotients of S, with $\mathcal{R}' = \mathcal{R}^P \cap (S \times S)$ and $\leqslant_l = \leqslant_{\mathcal{L}^P} \cap (S \times S)$. For the convenience of the reader, we now set up the 'roadmap' for the proof.

Roadmap 4.2.2. Let S be a semigroup with $\mathcal{R}'$, $\leqslant_l$ and $\mathcal{L}'$ satisfying the conditions of Theorem 4.2.1. Note that by considering (M2) with $\alpha = \gamma$, we see that $\leqslant_l$ is right compatible. Therefore, since $\mathcal{L}'$ is an equivalence relation associated with a right compatible preorder, $\mathcal{L}'$ is a right congruence.

We begin by defining

$$\Sigma = \{(a,b) \in S \times S \mid a \,\mathcal{R}'\, b\}.$$

We then define an equivalence relation $\sim$ on Σ, by

$$(a,b) \sim (c,d)$$

if and only if there exists $x, y \in S$ such that

$$xa = yc,\ xb = yd,\ x \,\mathcal{R}'\, xa \,\mathcal{L}'\, a,\ y \,\mathcal{R}'\, yc \,\mathcal{L}'\, c.$$

Note that $x \,\mathcal{R}'\, y$ and $a \,\mathcal{L}'\, c$ as a consequence.

We show that this is an equivalence relation in Lemma 4.2.4. We use $[a,b]$ to denote the equivalence class of element (a,b) under $\sim$.

We then define $P = \Sigma/\!\sim$ and multiplication on P with the following rule:

$$[a,b][c,d] = [ua, vd],\ \text{where}\ u \,\mathcal{R}'\, v \,\mathcal{R}'\, vc = ub,\ L'_b \wedge L'_c = L'_{ub}\,.$$

Note that such a u and v exist in S by (M1).

We show that P is a semigroup in Lemma 4.2.5 and Lemma 4.2.6 and an inverse semigroup in Lemma 4.2.8 and Lemma 4.2.9.

We then show that S embeds into P, by defining $\phi : S \to P$ by $a\phi = [x, xa]$, where x is an element in S such that $x \,\mathcal{R}'\, xa \,\mathcal{L}'\, a$. The existence of such an x is a consequence of (M1) proved in Lemma 4.2.3. We will prove that this function is an embedding in Lemma 4.2.10.

We show that the restriction of $\mathcal{R}^P$ to $(S \times S)$ is $\mathcal{R}'$ in Lemma 4.2.11, and that the restriction of $\leqslant_{\mathcal{L}^P}$ to $(S \times S)$ is $\leqslant_l$ in Lemma 4.2.12. Lastly, we show that P is a semigroup of straight left I-quotients of $S\phi$ in Lemma 4.2.13.

Now that we have set up the 'roadmap', the rest of the section will be the 'road trip'. The properties in Theorem 4.2.1 will be used extensively, so the reader might prefer to have the list of properties in front of them whilst reading. For all of the following results in this section, S, $\mathcal{R}'$, $\leqslant_l$ and $\mathcal{L}'$ are as described in the conditions of Theorem 4.2.1, and Σ, $\sim$, P and ϕ are as described in Roadmap 4.2.2.

The following lemma will provide a few shortcuts in the proof.

Lemma 4.2.3.

(i) For all $a \in S$, there exists an $x \in S$ such that $x \,\mathcal{R}'\, xa \,\mathcal{L}'\, a$.

(ii) For all $a, b, x \in S$, $x \,\mathcal{R}'\, xa \,\mathcal{L}'\, a$ and $a \,\mathcal{R}'\, b$ implies that $x \,\mathcal{R}'\, xb \,\mathcal{L}'\, b$.

(iii) For all $x, a \in S$, we have $L'_{xa} \wedge L'_a = L'_{xa}$.

(iv) For all $a, b, x, y \in S$, $a \,\mathcal{R}'\, b$ and $xa \,\mathcal{L}'\, ya$ implies that $xb \,\mathcal{L}'\, yb$.

Proof.

(i) By applying (M1) with $\alpha = \beta = a$, there exists $x \in S$ such that $x \,\mathcal{R}'\, xa$ and $L'_{xa} = L'_a \wedge L'_a = L'_a$.

(ii) Let $a, b, x \in S$ such that $a \,\mathcal{R}'\, b$ and $x \,\mathcal{R}'\, xa \,\mathcal{L}'\, a$. Using the fact that $\mathcal{R}'$ is a left congruence, $b \,\mathcal{R}'\, a$ implies that $xb \,\mathcal{R}'\, xa \,\mathcal{R}'\, x$. By (i), there exists $y \in S$ such that $y \,\mathcal{R}'\, yb \,\mathcal{L}'\, b$. We can then use (M5), to see that $a \,\mathcal{R}'\, b$ implies that $x \,\mathcal{L}'\, y$. Therefore, using the fact that $\mathcal{L}'$ is a right congruence, $xb \,\mathcal{L}'\, yb \,\mathcal{L}'\, b$.

(iii) Let $x, a \in S$. By (M3), we know that $xa \leqslant_l a$, Therefore, by the definition of meet, we have that $L'_{xa} \wedge L'_a = L'_{xa}$.

(iv) Applying (M1) to xa and ya, there exists $w, z \in S$ such that

$$w \,\mathcal{R}'\, z \,\mathcal{R}'\, zya = wxa \;\text{ and }\; L'_{xa} \wedge L'_{ya} = L'_{wxa}.$$

Since $xa\,\mathcal{L}'\,ya$, this gives us

$$w\,\mathcal{R}'\,wxa\,\mathcal{L}'\,xa \text{ and } z\,\mathcal{R}'\,zya\,\mathcal{L}'\,ya.$$

Using the fact that $\mathcal{R}'$ is a left congruence, we have that $a\,\mathcal{R}'\,b$ implies that both $xa\,\mathcal{R}'\,xb$ and $ya\,\mathcal{R}'\,yb$. Therefore we can apply (ii) to both of the above equations to get

$$w\,\mathcal{R}'\,wxb\,\mathcal{L}'\,xb \text{ and } z\,\mathcal{R}'\,zyb\,\mathcal{L}'\,yb.$$

Also we can apply (M4) to $wxa = zya$ to get $wxb = zyb$. Therefore $xb\,\mathcal{L}'\,wxb = zyb\,\mathcal{L}'\,yb$.

□

Lemma 4.2.4. *The relation $\sim$ is an equivalence relation.*

Proof.

Reflexivity: Let $(a,b) \in \Sigma$. By definition, $(a,b) \sim (a,b)$ if there exists $x, y \in S$ such that $xa = ya$, $xb = yb$, $x\,\mathcal{R}'\,xa\,\mathcal{L}'\,a$, $y\,\mathcal{R}'\,ya\,\mathcal{L}'\,a$. By Lemma 4.2.3 (i), there is an $x \in S$ such that $x\,\mathcal{R}'\,xa\,\mathcal{L}'\,a$. Then take $y = x$ to get reflexivity.

Symmetry: Let $(a,b) \sim (c,d)$. By definition there must exist $x, y \in S$ such that

$$xa = yc,\ xb = yd,\ x\,\mathcal{R}'\,xa\,\mathcal{L}'\,a,\ y\,\mathcal{R}'\,yc\,\mathcal{L}'\,c,$$

By switching the roles of x and y, we can immediately see that $(c,d) \sim (a,b)$.

Transitivity: Let $(a,b) \sim (c,d)$. Therefore there exists $x, y \in S$ such that

$$xa = yc,\ xb = yd,\ x\,\mathcal{R}'\,xa\,\mathcal{L}'\,a,\ y\,\mathcal{R}'\,yc\,\mathcal{L}'\,c. \tag{4.1}$$

Suppose also that $(c,d) \sim (e,f)$. Then there exists $w, z \in S$ such that

$$wc = ze,\ wd = zf,\ w\,\mathcal{R}'\,wc\,\mathcal{L}'\,c,\ z\,\mathcal{R}'\,ze\,\mathcal{L}'\,e. \tag{4.2}$$

We need $(a,b) \sim (e,f)$. That is we need $X, Y \in S$ such that

$$Xa = Ye,\ Xb = Yf,\ X\,\mathcal{R}'\,Xa\,\mathcal{L}'\,a,\ Y\,\mathcal{R}'\,Ye\,\mathcal{L}'\,e. \tag{4.3}$$

We apply Property (M1) to y and w, to get that there exists $h, k \in S$ such that

$$h \,\mathcal{R}'\, k \,\mathcal{R}'\, kw = hy, \text{ and } L'_y \wedge L'_w = L'_{hy}. \tag{4.4}$$

We then take $X = hx$, $Y = kz$ giving us that

$$Xa = hxa = hyc = kwc = kze = Ye$$
$$Xb = hxb = hyd = kwd = kzf = Yf.$$

using (4.1), (4.4) and (4.2). Also since $\mathcal{R}'$ is a left congruence,

$$x \,\mathcal{R}'\, xa \implies X = hx \,\mathcal{R}'\, hxa = Xa$$
$$z \,\mathcal{R}'\, ze \implies Y = kz \,\mathcal{R}'\, kze = Ye.$$

Using Property (M2) we have that (4.4) implies that $L'_{yc} \wedge L'_{wc} = L'_{hyc}$. Hence, using $xa = yc$ from (4.1) and $wc \,\mathcal{L}'\, c$ from (4.2), we have $L'_{xa} \wedge L'_c = L'_{hxa}$. We can then use (4.1) to give us $a \,\mathcal{L}'\, xa = yc \,\mathcal{L}'\, c$, and so $L'_a \wedge L'_a = L'_{hxa}$. Therefore $Xa \,\mathcal{L}'\, a$.

The last relation needed can be obtained similarly or achieved quicker by noticing

$$e \,\mathcal{L}'\, c \,\mathcal{L}'\, a \,\mathcal{L}'\, Xa = Ye.$$

□

Lemma 4.2.5. *Multiplication in P is well-defined.*

Proof. Let $[a, b], [c, d] \in P$. From Roadmap 4.2.2, we have that

$$[a, b][c, d] = [ua, vd],$$

where $u, v \in S$ are the elements that exist by (M1) such that

$$u \,\mathcal{R}'\, v \,\mathcal{R}'\, vc = ub,\ L'_b \wedge L'_c = L'_{ub}.$$

We need to show that the product, $[a, b][c, d]$, depends neither upon the choice of representative for the equivalence class, nor the choice of u and v appearing in the rule for multiplication. We start with the choice of u and v.

Choice of u and v: Let

$$u\,\mathcal{R}'\,v\,\mathcal{R}'\,vc = ub,\ L'_b\ \wedge\ L'_c = L'_{ub}, \tag{4.5}$$

so that $[a,b][c,d] = [ua, vd]$. Also let

$$s\,\mathcal{R}'\,t\,\mathcal{R}'\,tc = sb,\ L'_b \wedge L'_c = L'_{sb}, \tag{4.6}$$

so that $[a,b][c,d] = [sa, td]$. We show that $(ua, vd) \sim (sa, td)$, which is true exactly if there exists $w, z \in S$ such that

$$wua = zsa,\ wvd = ztd,\ w\,\mathcal{R}'\,wua\,\mathcal{L}'\,ua,\ z\,\mathcal{R}'\,zsa\,\mathcal{L}'\,sa. \tag{4.7}$$

Applying Property (M1) to ua and sa, let w and z be elements such that

$$w\,\mathcal{R}'\,z\,\mathcal{R}'\,zsa = wua\ \text{ and }\ L'_{ua} \wedge L'_{sa} = L'_{wua}. \tag{4.8}$$

Using the fact that $a\,\mathcal{R}'\,b$, we see that

$$wua = zsa \overset{(M4)}{\Longrightarrow} wub = zsb.$$

Then, as $ub = vc$ and $tc = sb$ from (4.5) and (4.6), this gives us $wvc = ztc$. We then use $c\,\mathcal{R}'\,d$, to get

$$wvc = ztc \overset{(M4)}{\Longrightarrow} wvd = ztd.$$

From (4.5) and (4.6), we also see that

$$L'_{ub} = L'_b \wedge L'_c = L'_{sb} \implies ub\,\mathcal{L}'\,sb,$$

which, together with $a\,\mathcal{R}'\,b$, implies that $ua\,\mathcal{L}'\,sa$ by Lemma 4.2.3 (iv). Using the definition of $\wedge$, along with (4.7), we then have

$$L'_{ua} = L'_{sa} = L'_{ua} \wedge L'_{sa} = L'_{wua} = L'_{zsa}.$$

This gives us the required properties for $(ua, vd) \sim (sa, td)$.

First Variable: Let $(a,b) \sim (\tilde{a},\tilde{b})$. Therefore there exists $x, y \in S$ such that

$$xa = y\tilde{a},\ xb = y\tilde{b},\ x\,\mathcal{R}'\,xa\,\mathcal{L}'\,a,\ y\,\mathcal{R}'\,y\tilde{a}\,\mathcal{L}'\,\tilde{a}. \tag{4.9}$$

In order to show well-definedness in the first variable, we need that for all $[c,d] \in P$, $[a,b][c,d] = [\tilde{a},\tilde{b}][c,d]$. With that goal in mind, we apply (M1) to b and c, to get that there exists $u, v \in S$ such that

$$u\,\mathcal{R}'\,v\,\mathcal{R}'\,vc = ub \text{ and } L'_b \wedge L'_c = L'_{ub}. \tag{4.10}$$

Therefore $[a,b][c,d] = [ua, vd]$.

Our aim is to first find elements $\tilde{u}$ and $\tilde{v}$ which witness $[\tilde{a},\tilde{b}][c,d] = [\tilde{u}\tilde{a}, \tilde{v}d]$. We will then prove that $(ua, vd) \sim (\tilde{u}\tilde{a}, \tilde{v}d)$. Of course, we could use (M1) applied to $\tilde{b}$ and c, but for our purposes we need to be more careful.

Applying Property (M1) to u and x, we know that there exists $s, t \in S$ such that

$$s\,\mathcal{R}'\,t\,\mathcal{R}'\,tu = sx \text{ and } L'_x \wedge L'_u = L'_{sx}. \tag{4.11}$$

We take $\tilde{u} = sy$ and $\tilde{v} = tv$.

We want to prove that $[\tilde{a},\tilde{b}][c,d] = [\tilde{u}\tilde{a}, \tilde{v}d]$. To prove this, it is sufficient that $\tilde{u}\,\mathcal{R}'\,\tilde{v}\,\mathcal{R}'\,\tilde{v}c = \tilde{u}\tilde{b}$, and that $L'_{\tilde{b}} \wedge L'_c = L'_{\tilde{u}\tilde{b}}$. Rewriting this, we need to prove that

$$sy\,\mathcal{R}'\,tv\,\mathcal{R}'\,tvc = sy\tilde{b} \tag{4.12}$$

and

$$L'_{\tilde{b}} \wedge L'_c = L'_{sy\tilde{b}}. \tag{4.13}$$

We start by proving each relation in Equation (4.12) in turn:

We know that $y\,\mathcal{R}'\,x$ from (4.9) and $u\,\mathcal{R}'\,v$ from (4.10). Using the fact that $\mathcal{R}'$ is a left congruence, $y\,\mathcal{R}'\,x$ and $u\,\mathcal{R}'\,v$ imply that $sy\,\mathcal{R}'\,sx$ and $tu\,\mathcal{R}'\,tv$ respectively. Then, as $sx = tu$ from (4.11), this gives us that $sy\,\mathcal{R}'\,tv$. Using left compatibility of $\mathcal{R}'$ again, $v\,\mathcal{R}'\,vc$ from (4.10) implies that $tv\,\mathcal{R}'\,tvc$. Also, using $vc = ub$, $tu = sx$, $xb = y\tilde{b}$, we get

$$tvc = tub = sxb = sy\tilde{b}. \tag{4.14}$$

We now prove Equation (4.13). We can use (M2) to give us

$$L'_x \wedge L'_u = L'_{sx} \implies L'_{xb} \wedge L'_{ub} = L'_{sxb}.$$

Using Lemma 4.2.3 (ii), we have that $a \,\mathcal{R}'\, b$ imples that $x \,\mathcal{R}'\, xb \,\mathcal{L}'\, b$. Using $xb \,\mathcal{L}'\, b$ and $L'_b \wedge L'_c = L'_{ub}$, we have

$$L'_b \wedge (L'_b \wedge L'_c) = L'_{sxb}.$$

Therefore

$$L'_b \wedge L'_c = (L'_b \wedge L'_b) \wedge L'_c = L'_{sxb}. \tag{4.15}$$

Using Lemma 4.2.3 (ii) again, we have that $\tilde{a} \,\mathcal{R}'\, \tilde{b}$ and $y \,\mathcal{R}'\, y\tilde{a} \,\mathcal{L}'\, \tilde{a}$ by (4.9) implies that $y \,\mathcal{R}'\, y\tilde{b} \,\mathcal{L}'\, \tilde{b}$. Therefore, using (4.9), we have that $b \,\mathcal{L}'\, xb = y\tilde{b} \,\mathcal{L}'\, \tilde{b}$. Using this together with $xb = y\tilde{b}$ and (4.15), we have

$$L'_{\tilde{b}} \wedge L'_c = L'_{sy\tilde{b}},$$

which is (4.13). Therefore $[\tilde{a}, \tilde{b}][c, d] = [sy\tilde{a}, tvd]$.

Using $xa = y\tilde{a}$ from (4.9), this also means that $[\tilde{a}, \tilde{b}][c, d] = [sxa, tvd]$. Therefore, in order to have well-definedness in the first variable, one needs $(ua, vd) \sim (sxa, tvd)$. This is true exactly if there exists $w, z \in S$ such that

$$wua = zsxa,\ wvd = ztvd,\ w \,\mathcal{R}'\, wua \,\mathcal{L}'\, ua,\ z \,\mathcal{R}'\, zsxa \,\mathcal{L}'\, sxa.$$

Applying Property (M1) to ua and sxa, take w and z to be elements in S such that

$$w \,\mathcal{R}'\, z \,\mathcal{R}'\, zsxa = wua \text{ and } L'_{ua} \wedge L'_{sxa} = L'_{wua}. \tag{4.16}$$

Since $a \,\mathcal{R}'\, b$, we know that $wua = zsxa$ implies $wub = zsxb$ by (M4). We then use $sxb = tvc$ from (4.14) and $ub = vc$ from (4.10) to obtain $wvc = ztvc$. And therefore, using (M4) again, $c \,\mathcal{R}'\, d$ implies that $wvd = ztvd$.

Using Property (M2), $L'_x \wedge L'_u = L'_{sx}$ implies that $L'_{xa} \wedge L'_{ua} = L'_{sxa}$. We then use $xa \,\mathcal{L}'\, a$ from (4.9) and Lemma 4.2.3 (iii), to get

$$L'_{xa} \wedge L'_{ua} = L'_{sxa} \implies L'_a \wedge L'_{ua} = L'_{sxa} \implies L'_{ua} = L'_{sxa}.$$

We then compare this with $L'_{ua} \wedge L'_{sxa} = L'_{wua}$ from (4.16) to obtain $L'_{ua} = L'_{wua}$. Lastly, $sxa\,\mathcal{L}'\,ua\,\mathcal{L}'\,wua = zsxa$. Altogether, this proves that $(ua, vd) \sim (sxa, tvd) = (\tilde{u}\tilde{a}, \tilde{v}d)$.

Second Variable: Let $(c, d) \sim (\tilde{c}, \tilde{d})$. Therefore there exists $x, y \in S$ such that

$$xc = y\tilde{c},\ xd = y\tilde{d},\ x\,\mathcal{R}'\,xc\,\mathcal{L}'\,c,\ y\,\mathcal{R}'\,y\tilde{c}\,\mathcal{L}'\,\tilde{c}. \tag{4.17}$$

In order to show well-definedness in the second variable, we need that for all $[a, b] \in P$, $[a, b][c, d] = [a, b][\tilde{c}, \tilde{d}]$. With that goal in mind, given $[a, b] \in P$, we apply (M1) to b and c, to get that there exists $u, v \in S$ such that

$$u\,\mathcal{R}'\,v\,\mathcal{R}'\,vc = ub \text{ and } L'_b \wedge L'_c = L'_{ub}. \tag{4.18}$$

Therefore $[a, b][c, d] = [ua, vd]$.

Our aim is to find elements $\tilde{u}$ and $\tilde{v}$ which witness $[a, b][\tilde{c}, \tilde{d}] = [\tilde{u}a, \tilde{v}\tilde{d}]$. We will then prove that $(ua, vd) \sim (\tilde{u}a, \tilde{v}\tilde{d})$.

Applying Property (M1) to v and x, we know that there exists $p, q \in S$ such that

$$p\,\mathcal{R}'\,q\,\mathcal{R}'\,qx = pv \text{ and } L'_v \wedge L'_x = L'_{pv}. \tag{4.19}$$

We take $\tilde{u} = pu$ and $\tilde{v} = qy$.

We want to prove that $[a, b][\tilde{c}, \tilde{d}] = [\tilde{u}a, \tilde{v}\tilde{d}]$. To prove this, it is sufficient that

$$\tilde{u}\,\mathcal{R}'\,\tilde{v}\,\mathcal{R}'\,\tilde{v}\tilde{c} = \tilde{u}b \text{ and } L'_b \wedge L'_{\tilde{c}} = L'_{\tilde{u}b}. \tag{4.20}$$

Rewriting this, we need to prove that

$$pu\,\mathcal{R}'\,qy\,\mathcal{R}'\,qy\tilde{c} = pub \tag{4.21}$$

and

$$L'_b \wedge L'_{\tilde{c}} = L'_{pub}. \tag{4.22}$$

We start be proving each relation in Equation (4.21) in turn:

We know that $u\,\mathcal{R}'\,v$ from (4.18) and that $x\,\mathcal{R}'\,y$ from (4.17). Using the fact that $\mathcal{R}'$ is a left congruence, $u\,\mathcal{R}'\,v$ and $x\,\mathcal{R}'\,y$ imply that $pu\,\mathcal{R}'\,pv$ and $qx\,\mathcal{R}'\,qy$ respectively. Then, since $pv = qx$ from (4.19), this gives us that $pu\,\mathcal{R}'\,qy$. Using left compatibility of $\mathcal{R}'$ again, $u\,\mathcal{R}'\,ub$ from (4.18) implies that $pu\,\mathcal{R}'\,pub$. Also, using $ub = vc$ from (4.18), $pv = qx$ from (4.19), and $xc = y\tilde{c}$ from (4.17), we get

$$pub = pvc = qxc = qy\tilde{c}. \tag{4.23}$$

We now prove Equation (4.22). Using (M2) on (4.19), we have

$$L'_v \wedge L'_x = L'_{pv} \implies L'_{vc} \wedge L'_{xc} = L'_{pvc}.$$

We apply $L'_b \wedge L'_c = L'_{vc}$ from (4.18) and $xc\,\mathcal{L}'\,c$ from (4.17) to get

$$(L'_b \wedge L'_c) \wedge L'_c = L'_{pvc}.$$

Therefore

$$L'_b \wedge L'_c = L'_b \wedge (L'_c \wedge L'_c) = L'_{pvc}.$$

We apply $c\,\mathcal{L}'\,xc = y\tilde{c}\,\mathcal{L}'\,\tilde{c}$ from (4.17) and $ub = vc$ from (4.18) to get

$$L'_b \wedge L'_{\tilde{c}} = L'_{pub}.$$

This concludes the verification of Equations (4.21) and (4.22). Therefore $[a, b][\tilde{c}, \tilde{d}] = [pua, qy\tilde{d}] = [pua, qxd]$ using (4.17).

Therefore, in order to have well-definedness in the second variable, one needs $(ua, vd) \sim (pua, qxd)$. This is true exactly if there exists $w, z \in S$ such that

$$wua = zpua,\; wvd = zqxd,\; w\,\mathcal{R}'\,wua\,\mathcal{L}'\,ua,\; z\,\mathcal{R}'\,zpua\,\mathcal{L}'\,pua.$$

Applying Property (M1) to ua and pua, take w and z to be elements in S such that $w\,\mathcal{R}'\,z\,\mathcal{R}'\,zpua = wua$ and $L'_{ua} \wedge L'_{pua} = L'_{wua}$. We use $pub = qxc$ from (4.23) and $ub = vc$ from (4.18), along with $a\,\mathcal{R}'\,b$ and $c\,\mathcal{R}'\,d$ to obtain

$$wua = zpua \overset{(M4)}{\implies} wub = zpub \implies wvc = zqxc \overset{(M4)}{\implies} wvd = zqxd.$$

Using Property (M2), $L'_v \wedge L'_x = L'_{pv}$ implies that $L'_{vc} \wedge L'_{xc} = L'_{pvc}$. We then use $xc\,\mathcal{L}'\,c$ from (4.17) and Lemma 4.2.3 (iii), to get

$$L'_{vc} \wedge L'_{xc} = L'_{pvc} \implies L'_{vc} \wedge L'_c = L'_{pvc} \implies L'_{vc} = L'_{pvc} \implies L'_{ub} = L'_{pub}.$$

We apply Lemma 4.2.3 (iv) to $ub\,\mathcal{L}'\,pub$ and $a\,\mathcal{R}'\,b$ to obtain $ua\,\mathcal{L}'\,pua$. Therefore $L'_{wua} = L'_{ua} \wedge L'_{pua} = L'_{ua}$. Lastly $pua\,\mathcal{L}'\,ua\,\mathcal{L}'\,wua = zpua$, which gives us all the necessary conditions for $(ua, vd) \sim (pua, qxd) = (\tilde{u}a, \tilde{v}\tilde{d})$.

Note that by using well-definedness in the first variable and well-definedness in the second variable together, we can see that for $(a, b) \sim (\tilde{a}, \tilde{b})$ and $(c, d) \sim (\tilde{c}, \tilde{d})$, we get

$$(a, b)(c, d) \sim (\tilde{a}, \tilde{b})(c, d) \sim (\tilde{a}, \tilde{b})(\tilde{c}, \tilde{d}).$$

Therefore, by transitivity, this multiplication is well-defined. □

Lemma 4.2.6. *Multiplication in P is associative.*

Proof. Let $[a, b], [c, d], [e, f] \in P$.

Applying Property (M1) to b and c, we choose $u, v \in S$ satisfying

$$u\,\mathcal{R}'\,v\,\mathcal{R}'\,vc = ub \text{ and } L'_b \wedge L'_c = L'_{ub}. \tag{4.24}$$

This gives us that $[a, b][c, d] = [ua, vd]$. Similarly, we choose $p, q \in S$ satisfying

$$p\,\mathcal{R}'\,q\,\mathcal{R}'\,qe = pd \text{ and } L'_d \wedge L'_e = L'_{pd}. \tag{4.25}$$

Then $[c, d][e, f] = [pc, qf]$.

Applying Property (M1) to v and p, we know that there exists $i, j \in S$ such that

$$i\,\mathcal{R}'\,j\,\mathcal{R}'\,jp = iv \text{ and } L'_v \wedge L'_p = L'_{iv}. \tag{4.26}$$

We want to prove that

$$([a, b][c, d])[e, f] = [ua, vd][e, f] = [i(ua), (jq)f], \tag{4.27}$$

and that

$$[a, b]([c, d][e, f]) = [a, b][pc, qf] = [(iu)a, j(qf)]. \tag{4.28}$$

This would prove associativity.

In order to prove (4.27), we need

$$i\,\mathcal{R}'\,jq\,\mathcal{R}'\,jqe = ivd \tag{4.29}$$

and

$$L'_{vd} \wedge L'_e = L'_{ivd}. \tag{4.30}$$

We start by proving each relation in Equation (4.29) in turn:

Since $\mathcal{R}'$ is a left congruence, $q\,\mathcal{R}'\,p$ implies that $jq\,\mathcal{R}'\,jp$, which in turn is $\mathcal{R}'$-related to i. Using again the left compatibility of $\mathcal{R}'$, we see that $q\,\mathcal{R}'\,qe$ implies that $jq\,\mathcal{R}'\,jqe$. Also, using $qe = pd$ and $jp = iv$, we see that $jqe = jpd = ivd$.

We now prove Equation (4.30). We apply (M2) to $L'_v \wedge L'_p = L'_{iv}$ to give us that $L'_{vd} \wedge L'_{pd} = L'_{ivd}$. And so, using $L'_d \wedge L'_e = L'_{pd}$ and Lemma 4.2.3 (iii), we have

$$\begin{aligned} L'_{vd} \wedge L'_{pd} = L'_{ivd} &\implies L'_{vd} \wedge (L'_d \wedge L'_e) = L'_{ivd} \\ &\implies (L'_{vd} \wedge L'_d) \wedge L'_e = L'_{ivd} \\ &\implies L'_{vd} \wedge L'_e = L'_{ivd}. \end{aligned}$$

We now have proved both (4.29) and (4.30), which together gives us (4.27).

In order to prove (4.28), we need

$$iu\,\mathcal{R}'\,j\,\mathcal{R}'\,jpc = iub \tag{4.31}$$

and

$$L'_b \wedge L'_{pc} = L'_{iub}. \tag{4.32}$$

We start by proving each relation in Equation (4.31) in turn:

Since $\mathcal{R}'$ is a left congruence, $u\,\mathcal{R}'\,v$ implies that $iu\,\mathcal{R}'\,iv$, which in turn is $\mathcal{R}'$-related to j. Using again the left compatibility of $\mathcal{R}'$, we see that $c\,\mathcal{R}'\,d$ implies that $jpc\,\mathcal{R}'\,jpd$ and $p\,\mathcal{R}'\,pd$ implies that $jp\,\mathcal{R}'\,jpd$. Therefore $j\,\mathcal{R}'\,jp\,\mathcal{R}'\,jpd\,\mathcal{R}'\,jpc$. Also, using $ub = vc$ and $iv = jp$, we see that $iub = ivc = jpc$.

We now prove Equation (4.32). We apply (M2) to $L'_v \wedge L'_p = L'_{iv}$ to give us that $L'_{vc} \wedge L'_{pc} = L'_{ivc}$. And so, using $L'_b \wedge L'_c = L'_{vc}$ and Lemma 4.2.3 (iii), we have

$$\begin{aligned} L'_{vc} \wedge L'_{pc} = L'_{ivc} &\implies (L'_b \wedge L'_c) \wedge L'_{pc} = L'_{iub} \\ &\implies L'_b \wedge (L'_c \wedge L'_{pc}) = L'_{iub} \\ &\implies L'_b \wedge L'_{pc} = L'_{iub}. \end{aligned}$$

We have now proved both (4.31) and (4.32), which together gives us (4.28), finishing the proof. □

We have now proved that P is a semigroup. The following lemma provides a couple of useful shortcuts to help in the later parts of this proof.

Lemma 4.2.7. *These statements are true in P:*

(i) $[a, a] = [b, b]$ *if and only if* $a \, \mathcal{L}' \, b$*;*

(ii) $[a, b][b, a] = [a, a]$.

Proof.

(i) We know that $[a, a] = [b, b]$ if and only if there exists $w, z \in S$ such that

$$wa = zb, \;\; w \, \mathcal{R}' \, wa \, \mathcal{L}' \, a, \; z \, \mathcal{R}' \, zb \, \mathcal{L}' \, b. \tag{4.33}$$

Let $[a, a] = [b, b]$. Therefore there exists $w, z \in S$ satisfying (4.33). Hence $a \, \mathcal{L}' \, wa = zb \, \mathcal{L}' \, b$.

Conversely let $a \, \mathcal{L}' \, b$. Applying (M1) to a and b, there exists $w, z \in S$ such that $w \, \mathcal{R}' \, z \, \mathcal{R}' \, zb = wa$ and $L'_a \wedge L'_b = L'_{wa}$ Therefore, since $a \, \mathcal{L}' \, b$, we have $L'_{wa} = L'_a \wedge L'_b = L'_a$, and consequently $zb = wa \, \mathcal{L}' \, a \, \mathcal{L}' \, b$. Comparing with (4.33), we see that $[a, a] = [b, b]$.

(ii) By Lemma 4.2.3 (i), there exists $y \in S$ such that $y \, \mathcal{R}' \, yb \, \mathcal{L}' \, b$. By comparing with the definition of multiplication in Roadmap 4.2.2, we see that $[a, b][b, a] = [ya, ya]$. Since $a \, \mathcal{R}' \, b$, Lemma 4.2.3 (ii) gives us that $y \, \mathcal{R}' \, ya \, \mathcal{L}' \, a$. So by (i), $[a, b][b, a] = [a, a]$.

□

Lemma 4.2.8. *The semigroup P is regular.*

Proof. Let $[a, b] \in P$. By Lemma 4.2.7 (ii), $[a, b][b, a][a, b] = [a, a][a, b]$. By Lemma 4.2.3 (i), there exists $y \in S$ such that $y \, \mathcal{R}' \, ya \, \mathcal{L}' \, a$. Therefore, by our definition of multiplication, $[a, a][a, b] = [ya, yb]$.

We want to prove that $(a,b) \sim (ya,yb)$. That is there exists $w,z \in S$ such that

$$wa = zya,\ wb = zyb,\ w\,\mathcal{R}'\,wa\,\mathcal{L}'\,a,\ z\,\mathcal{R}'\,zya\,\mathcal{L}'\,ya.$$

Applying Property (M1) to a and ya, there exists $w,z \in S$ such that

$$w\,\mathcal{R}'\,z\,\mathcal{R}'\,zya = wa \text{ and } L'_a \wedge L'_{ya} = L'_{wa}.$$

We use (M4) to give us that $wa = zya$ implies $wb = zyb$. We can also use $a\,\mathcal{L}'\,ya$ to give us that $L'_{wa} = L'_a \wedge L'_{ya} = L'_a$. Therefore $a\,\mathcal{L}'\,ya\,\mathcal{L}'\,wa = zya$.

So we have that $[a,b][b,a][a,b] = [a,b]$. Therefore P is regular. $\square$

It is good to note that in the exactly same way $[b,a][a,b][b,a] = [b,a]$. Therefore $[b,a] \in V([a,b])$.

Lemma 4.2.9. *The semigroup P is an inverse semigroup, with $[a,b]^{-1} = [b,a]$.*

Proof. We start by identifying the idempotents of P.

Let $[a,b] \in P$ be an idempotent, i.e. let $[a,b][a,b] = [a,b]$. We know that $[a,b][a,b] = [ua,vb]$, where u and v are the elements that exists by (M1) such that

$$u\,\mathcal{R}'\,v\,\mathcal{R}'\,va = ub \text{ and } L'_a \wedge L'_b = L'_{ub}\,.$$

Consequently we know that $(a,b) \sim (ua,vb)$. Therefore there exists $w,z \in S$ such that

$$wa = zua,\ wb = zvb,\ w\,\mathcal{R}'\,wa\,\mathcal{L}'\,a,\ z\,\mathcal{R}'\,zua\,\mathcal{L}'\,ua.$$

By Lemma 4.2.3 (ii) we have that $a\,\mathcal{R}'\,b$ implies that $w\,\mathcal{R}'\,wb\,\mathcal{L}'\,b$. Also, by applying (M4) to both $wb = zvb$ and $wa = zua$ and using $va = ub$, we have

$$wa \overset{(M4)}{=} zva = zub \overset{(M4)}{=} wb.$$

Therefore $a\,\mathcal{L}'\,wa = wb\,\mathcal{L}'\,b$. We then apply Property (M6) to give us $a = b$. Therefore the idempotents of P are of the form $[a,a]$, where $a \in S$.

We now prove that idempotents commute. Let $[a,a],[b,b]$ be idempotents in P. Applying Property (M1) to a and b, we choose u and v such that

$[a, a][b, b] = [ua, vb]$, where

$$u\,\mathcal{R}'\,v\,\mathcal{R}'\,vb = ua \text{ and } L'_a \wedge L'_b = L'_{ua}.$$

By inspection we can see that v and u satisfy the necessary properties for $[b, b][a, a] = [vb, ua]$. And so we see that, since $ua = vb$, we have

$$[a, a][b, b] = [ua, vb] = [vb, ua] = [b, b][a, a].$$

Therefore the idempotents of P commute. Since P is also regular, this means that P is an inverse semigroup.

Moreover since $[b, a] \in V([a, b])$, we easily see that $[a, b]^{-1} = [b, a]$ for all $[a, b] \in P$. □

We now prove that S embeds into P. We do this by defining a function $\phi : S \to P$, by $a\phi = [x, xa]$, where x is the element such that $x\,\mathcal{R}'\,xa\,\mathcal{L}'\,a$, that exists by Lemma 4.2.3 (i). Note that $[x, xa] \in P$.

Lemma 4.2.10. *The function ϕ is an embedding.*

Proof.

Well-defined: Let $x\,\mathcal{R}'\,xa\,\mathcal{L}'\,a$ and let $y\,\mathcal{R}'\,ya\,\mathcal{L}'\,a$. By our definition, this means that $a\phi = [x, xa]$ and that $a\phi = [y, ya]$. Therefore, i order to prove that ϕ is well-defined, we need to prove that $(x, xa) \sim (y, ya)$. This is true exactly if there exists $w, z \in S$ such that

$$wx = zy,\ wxa = zya,\ w\,\mathcal{R}'\,wx\,\mathcal{L}'\,x,\ z\,\mathcal{R}'\,zy\,\mathcal{L}'\,y.$$

Applying Property (M1) to x and y, we take w and z to be elements in S such that $w\,\mathcal{R}'\,z\,\mathcal{R}'\,wx = zy$ and $L'_x \wedge L'_y = L'_{wx}$. Trivially $wxa = zya$. Using (M5), $x\,\mathcal{R}'\,xa\,\mathcal{L}'\,a$ and $y\,\mathcal{R}'\,ya\,\mathcal{L}'\,a$ implies that $x\,\mathcal{L}'y$. Therefore

$$L'_{wx} = L'_x \wedge L'_y = L'_x.$$

For the last necessary property, we notice $y\,\mathcal{L}'\,x\,\mathcal{L}'\,wx = zy$.

Homomorphism: Let $a, b \in S$, and let $x, y \in S$ such that $x\,\mathcal{R}'\,xa\,\mathcal{L}'\,a$ and

$y \mathcal{R}' yb \mathcal{L}' b$. Therefore, by definition, $a\phi = [x, xa]$ and $b\phi = [y, yb]$. Then

$$(a\phi)(b\phi) = [x, xa][y, yb] = [ux, vyb] = [ux, uxab],$$

where u and v are the elements that exist by (M1) such that

$$u \mathcal{R}' v \mathcal{R}' vy = uxa \text{ and } L'_{xa} \wedge L'_{y} = L'_{uxa}.$$

We want to prove that this is equal to $(ab)\phi$.

Using the fact that $\mathcal{R}'$ is a left conguence, we have that $yb \mathcal{R}' y$ implies that $vyb \mathcal{R}' vy$, and $xa \mathcal{R}' x$ implies that $uxa \mathcal{R}' ux$. Therefore

$$uxab = vyb \mathcal{R}' vy = uxa \mathcal{R}' ux.$$

We use $xa \mathcal{L}' a$ to obtain $L'_{a} \wedge L'_{y} = L'_{uxa}$. We can then apply Property (M2) to $L'_{a} \wedge L'_{y} = L'_{uxa}$ to give us that $L'_{ab} \wedge L'_{yb} = L'_{uxab}$. Using $yb \mathcal{L}' b$, this means that $L'_{ab} \wedge L'_{b} = L'_{uxab}$. We can then apply Lemma 4.2.3 (iii) to give us $ab \mathcal{L}' uxab$.

By the definition of ϕ, since $ux \mathcal{R}' uxab \mathcal{L}' ab$, this means that

$$(ab)\phi = [ux, uxab] = (a\phi)(b\phi).$$

Injective: Let $a, b \in S$ such that $a\phi = b\phi$. Therefore, choosing x and y such that $x \mathcal{R}' xa \mathcal{L}' a$ and $y \mathcal{R}' yb \mathcal{L}' b$, we have that $[x, xa] = [y, yb]$. This means there exists $w, z \in S$ such that

$$wx = zy,\ wxa = zyb,\ w \mathcal{R}' wx \mathcal{L}' x \text{ and } z \mathcal{R}' zy \mathcal{L}' y.$$

Therefore, using the fact that $\mathcal{L}'$ is a right congruence, we have that $x \mathcal{L}' wx$ implies that $xa \mathcal{L}' wxa$. Consequently, $a \mathcal{L}' xa \mathcal{L}' wxa$. Similarly, $y \mathcal{L}' zy$ implies that $yb \mathcal{L}' zyb$. And so, $b \mathcal{L}' yb \mathcal{L}' zyb = wxb$, using $zy = wx$ in the last equality. Therefore, we can apply Property (M6) giving us that $a \mathcal{L}' wxa = wxb \mathcal{L}' b$ implies that $a = b$. ☐

Lemma 4.2.11. *Let $a, b \in S$. Then $a \mathcal{R}' b$ if and only if $a\phi \mathcal{R}^P b\phi$.*

Proof. We have already proved that P is an inverse semigroup, so $a\phi \mathcal{R}^P b\phi$ if and only if $(a\phi)(a\phi)^{-1} = (b\phi)(b\phi)^{-1}$.

Let $x\,\mathcal{R}'\,xa\,\mathcal{L}'\,a$ and $y\,\mathcal{R}'\,yb\,\mathcal{L}'\,b$, so that $a\phi = [x, xa]$ and $b\phi = [y, yb]$. Then $(a\phi)(a\phi)^{-1} = [x, xa][xa, x] = [x, x]$, by Lemmas 4.2.9 and 4.2.7 (ii). Similarly $(b\phi)(b\phi)^{-1} = [y, yb][yb, b] = [y, y]$.

Therefore $a\phi\,\mathcal{R}^P\,b\phi$ if and only if $(x, x) \sim (y, y)$, which is true if and only if $x\,\mathcal{L}'\,y$, using Lemma 4.2.7 (i). We then use (M5) to give us that this is equivalent to $a\,\mathcal{R}'\,b$. □

Lemma 4.2.12. *Let $a, b \in S$. Then $a \leqslant_l b$ if and only if $a\phi \leqslant_{\mathcal{L}^P} b\phi$.*

Proof. We have already proved that P is an inverse semigroup, so $a\phi \leqslant_{\mathcal{L}^P} b\phi$ if and only if $a\phi = (a\phi)(b\phi)^{-1}(b\phi)$ by Lemma 2.2.6.

Let $x\,\mathcal{R}'\,xa\,\mathcal{L}'\,a$ and $y\,\mathcal{R}'\,yb\,\mathcal{L}'\,b$, so that $a\phi = [x, xa]$ and $b\phi = [y, yb]$. Using Lemma 4.2.7, we have $(b\phi)^{-1}(b\phi) = [yb, y][y, yb] = [yb, yb] = [b, b]$. Therefore

$$(a\phi)(b\phi)^{-1}(b\phi) = [x, xa][b, b] = [ux, vb],$$

where u and v are the elements that exist by (M1) such that

$$u\,\mathcal{R}'\,v\,\mathcal{R}'\,vb = uxa \text{ and } L'_{xa} \wedge L'_b = L'_{uxa}.$$

Note that since $xa\,\mathcal{L}\,a$, this means that $L'_a \wedge L'_b = L'_{uxa}$.

We use $vb = uxa$ to give us that $(a\phi)(b\phi)^{-1}(b\phi) = [ux, uxa]$. Therefore $a\phi \leqslant_{\mathcal{L}^P} b\phi$ if and only if $(x, xa) \sim (ux, uxa)$, which is true exactly if there exists $w, z \in S$ such that

$$wx = zux,\ wxa = zuxa,\ w\,\mathcal{R}'\,wx\,\mathcal{L}'\,x,\ z\,\mathcal{R}'\,zux\,\mathcal{L}'\,ux. \tag{4.34}$$

We know that $x\,\mathcal{R}'\,xa$, and so $ux\,\mathcal{R}'\,uxa$ as $\mathcal{R}'$ is a left congruence. Therefore we can use Lemma 4.2.3 (ii) to rewrite (4.34) to the equivalent expression (4.35). That is, $a\phi \leqslant_{\mathcal{L}^P} b\phi$ if and only if there exists $w, z \in S$ such that

$$wx = zux,\ wxa = zuxa,\ w\,\mathcal{R}'\,wxa\,\mathcal{L}'\,xa,\ z\,\mathcal{R}'\,zuxa\,\mathcal{L}'\,uxa. \tag{4.35}$$

Let $a\phi \leqslant_{\mathcal{L}^P} b\phi$, i.e. let w and z exist in S such that (4.35) is satisfied. We see that $uxa\,\mathcal{L}'\,zuxa = wxa\,\mathcal{L}'\,xa\,\mathcal{L}'\,a$. Therefore $L'_a \wedge L'_b = L'_{uxa} = L'_a$. By definition this means that $a \leqslant_l b$.

On the other hand, let $a \leqslant_l b$. By definition $L'_a \wedge L'_b = L'_a$. Therefore

$$L'_{uxa} = L'_a \wedge L'_b = L'_a = L'_{xa}.$$

Applying Property (M1) to xa and uxa, there exists $w, z \in S$ such that

$$w \,\mathcal{R}'\, z \,\mathcal{R}'\, zuxa = wxa \text{ and } L'_{uxa} \wedge L'_{xa} = L'_{wxa}.$$

Using $x \,\mathcal{R}'\, xa$, we know that $zuxa = wxa$ implies that $zux = wx$ by (M4). Using the fact that $uxa \,\mathcal{L}'\, xa$, we see that $L'_{wxa} = L'_{uxa} \wedge L'_{xa} = L'_{xa}$. Therefore $uxa \,\mathcal{L}'\, xa \,\mathcal{L}'\, wxa = zuxa$. This gives us (4.35), and so $a\phi \leqslant_{\mathcal{L}^P} b\phi$. □

Lemma 4.2.13. *The semigroup P is a semigroup of straight left I-quotients of $S\phi$.*

Proof. Let $[a, b] \in P$. Note that $a, b \in S$ with $a \,\mathcal{R}'\, b$.

Let $x \,\mathcal{R}'\, xa \,\mathcal{L}'\, a$ and $y \,\mathcal{R}'\, yb \,\mathcal{L}'\, b$, so that $a\phi = [x, xa]$ and $b\phi = [y, yb]$. By Lemma 4.2.11, $a\phi \,\mathcal{R}^P\, b\phi$. We have

$$(a\phi)^{-1}(b\phi) = [xa, x][y, yb] = [uxa, vyb],$$

where u and v are the elements that exist by (M1) such that

$$u \,\mathcal{R}'\, v \,\mathcal{R}'\, vy = ux \text{ and } L'_x \wedge L'_y = L'_{ux}.$$

We want to prove that $(a\phi)^{-1}(b\phi) = [a, b]$.

We see that $(a, b) \sim (uxa, vyb)$ exactly if there exists $w, z \in S$ such that

$$wa = zuxa,\ wb = zvyb,\ w \,\mathcal{R}'\, wa \,\mathcal{L}'\, a,\ z \,\mathcal{R}'\, zuxa \,\mathcal{L}'\, uxa.$$

Applying Property (M1) to a and uxa, we know that there exists $w, z \in S$ such that

$$w \,\mathcal{R}'\, z \,\mathcal{R}'\, zuxa = wa \text{ and } L'_a \wedge L'_{uxa} = L'_{wa}.$$

We see that $wa = zuxa$ implies $wb = zuxb$ by (M4), and therefore, since $ux = vy$ we have $wb = zvyb$. We use Property (M5) to get that $a \,\mathcal{R}'\, b$ implies $x \,\mathcal{L}' y$, and therefore $L'_{ux} = L'_x \wedge L'_y = L'_x$. We then use the fact that $\mathcal{L}'$ is a right congruence to give us that $ux \,\mathcal{L}'\, x$ implies $uxa \,\mathcal{L}'\, xa \,\mathcal{L}'\, a$. Therefore $L'_{wa} = L'_a \wedge L'_{uxa} = L'_a$, and so $zuxa = wa\,\mathcal{L}'\,a\,\mathcal{L}'\,uxa$. This gives us $[a, b] = (a\phi)^{-1}(b\phi)$, where $a\phi\,\mathcal{R}^P\,b\phi$. □

We have now finished the proof of Theorem 4.2.1.

4.3 The general case using a ternary relation on inverse semigroups

This is an alternative approach that does not use partial orders on $\mathcal{L}$-classes, but instead utilises a ternary relation that implicitly uses this natural order. In some cases it may be preferable to use this result instead of Theorem 4.2.1 because the meet structure of the $\mathcal{L}$-classes can be complicated to deal with directly. I will use the configuration explored in this section later to characterise straight left I-orders which intersect every $\mathcal{R}$-class of their semigroups of straight left I-quotients.

I now define the ternary relation $\mathcal{U}$ on an inverse semigroup Q.

Definition 4.3.1 ($\mathcal{U}$ relation)**.** Let Q be an inverse semigroup. Then

$$(b, c, u) \in \mathcal{U}^Q \iff u^{-1}\,\mathcal{R}^Q\,bc^{-1}.$$

Note that $(b, c, u) \in \mathcal{U}^Q$ if and only if $u^{-1}u = bc^{-1}cb^{-1}$ in Q.

As a motivation for introducing this relation, we will show how it can be used to construct a similar result to Lemma 3.3.6.

Lemma 4.3.2. *Let Q be an inverse semigroup and let $b, c, u, v \in Q$ such that $u\,\mathcal{R}^Q\,v$. Then $bc^{-1} = u^{-1}v$ in Q if and only if $ub = vc$ and $(b, c, u) \in \mathcal{U}^Q$.*

Proof. Let $bc^{-1} = u^{-1}v$. By Lemma 3.3.5, $ub = vc$. Also

$$u^{-1}u = u^{-1}vv^{-1}u = bc^{-1}cb^{-1}.$$

Therefore $(b, c, u) \in \mathcal{U}^Q$.

Conversely, let $ub = vc$ and $(b, c, u) \in \mathcal{U}^Q$. Since $(b, c, u) \in \mathcal{U}^Q$, we know that $u^{-1}u = bc^{-1}cb^{-1}$. By left multiplying by u, this gives us that $u = ubc^{-1}cb^{-1}$.

Therefore

$$ubb^{-1} = ubc^{-1}cb^{-1}bb^{-1} = ubc^{-1}cb^{-1} = u,$$

and so $u \mathcal{R} ub$ by Lemma 2.2.3. This means that $v \mathcal{R} u \mathcal{R} ub = vc$, and so $v = vcc^{-1}$ by Lemma 2.2.3. Therefore

$$u^{-1}v = u^{-1}vcc^{-1} = u^{-1}ubc^{-1} = bc^{-1},$$

where the final equality used $u^{-1}u \mathcal{R} bc^{-1}$, since $(b, c, u) \in \mathcal{U}^Q$. □

We will now prove the relationship between $\mathcal{U}$ and the meet of $\mathcal{L}$-classes.

Lemma 4.3.3. *Let Q be an inverse semigroup. Then $(b, c, u) \in \mathcal{U}^Q$ if and only if*

$$L_b^Q \wedge L_c^Q = L_{ub}^Q \text{ and } u \mathcal{R}^Q ub.$$

Proof. Let $(b, c, u) \in \mathcal{U}^Q$. Then $u^{-1}u = bc^{-1}cb^{-1}$. Then

$$(ub)^{-1}(ub) = b^{-1}u^{-1}ub = b^{-1}bc^{-1}cb^{-1}b = b^{-1}bc^{-1}c,$$

and so, $L_b^Q \wedge L_c^Q = L_{ub}^Q$ by Lemma 2.2.8.

We can left multiply $u^{-1}u = bc^{-1}cb^{-1}$ by u to obtain $u = ubc^{-1}cb^{-1}$. Therefore

$$(ub)(ub)^{-1} = ubb^{-1}u^{-1} = ubc^{-1}cb^{-1}bb^{-1}u^{-1} = ubc^{-1}cb^{-1}u^{-1} = uu^{-1},$$

so $u \mathcal{R}^Q ub$ as well.

Conversely, let $L_b^Q \wedge L_c^Q = L_{ub}^Q$ and let $u \mathcal{R}^Q ub$. By Lemma 2.2.3, $u \mathcal{R}^Q ub$ implies that $u = ubb^{-1}$, and so $u^{-1}u = u^{-1}ubb^{-1}$. Using $L_b^Q \wedge L_c^Q = L_{ub}^Q$, Lemma 2.2.8 gives us

$$\begin{aligned} b^{-1}u^{-1}ub = b^{-1}bc^{-1}c &\implies bb^{-1}u^{-1}ubb^{-1} = bb^{-1}bc^{-1}cb^{-1} \\ &\implies u^{-1}ubb^{-1} = bc^{-1}cb^{-1}. \end{aligned}$$

Putting these two together gives us $u^{-1}u = bc^{-1}cb^{-1}$. Therefore $(b, c, u) \in \mathcal{U}^Q$. □

We can say even more about the relationship between $\mathcal{U}^Q$ and the meet structure of $\mathcal{L}^Q$-classes in the case that Q is a semigroup of straight left I-quotients.

Lemma 4.3.4. *Let $a, b, c, d \in S$ and let S have a semigroup of straight left I-quotients Q. Then $L_a^Q \wedge L_b^Q = L_c^Q$ if and only if there exists $u \in S$ such that*

$$(a, b, u) \in \mathcal{U}^Q \text{ and } c \, \mathcal{L}^Q \, ua.$$

Proof. Let $L_a^Q \wedge L_b^Q = L_c^Q$. Then $c^{-1}c = a^{-1}ab^{-1}b$ by Lemma 2.2.8. Since Q is a semigroup of straight left I-quotients, there exists $u, v \in S$ with $u \, \mathcal{R}^Q \, v$ such that $ab^{-1} = u^{-1}v$. Lemma 4.3.2 gives us that $(a, b, u) \in \mathcal{U}^Q$ and $ua = vb$. Therefore

$$c^{-1}c = a^{-1}u^{-1}vb = (ua)^{-1}(ua).$$

Therefore $(a, b, u) \in \mathcal{U}^Q$ and $c \, \mathcal{L}^Q \, ua$.

Conversely, let there exist $u \in S$ such that $(a, b, u) \in \mathcal{U}^Q$ and $c \, \mathcal{L}^Q \, ua$. Therefore

$$c^{-1}c = a^{-1}u^{-1}ua = a^{-1}ab^{-1}ba^{-1}a = a^{-1}ab^{-1}b.$$

So by Lemma 2.2.8, $L_a^Q \wedge L_b^Q = L_c^Q$. $\square$

We will use this ternary relation to write another set of equivalent necessary and sufficient conditions for a semigroup to be a straight left I-order. We will use Theorem 4.2.1 in the proof of this result.

Theorem 4.3.5. *Let S be a semigroup and let $\mathcal{R}'$ and $\mathcal{L}'$ be binary relations on S and $\mathcal{U}'$ be a ternary relation on S. Then S has a semigroup of straight left I-quotients, Q, such that $\mathcal{R}^Q \cap (S \times S) = \mathcal{R}'$, $\mathcal{L}^Q \cap (S \times S) = \mathcal{L}'$, and $\mathcal{U}^Q \cap (S \times S \times S) = \mathcal{U}'$ if and only if $\mathcal{R}'$ is a left congruence, $\mathcal{L}'$ is a right congruence, and S satisfies Conditions (U1) - (U11).*

(U1) For all $\alpha, \beta \in S$, there exists $\gamma, \delta \in S$ such that $\gamma \, \mathcal{R}' \, \delta \, \mathcal{R}' \, \delta\beta = \gamma\alpha$, and $(\alpha, \beta, \gamma) \in \mathcal{U}'$.

(U2) For all $\alpha, \beta \in S$, $(\beta, \beta, \alpha) \in \mathcal{U}'$ if and only if $\alpha \, \mathcal{R}' \, \alpha\beta \, \mathcal{L}' \, \beta$.

(U3) For all $\alpha, \beta, \gamma, \delta \in S$, $(\alpha, \beta, \gamma) \in \mathcal{U}'$ and $(\delta, \epsilon, \beta) \in \mathcal{U}'$ implies that $(\alpha\delta, \epsilon, \gamma) \in \mathcal{U}'$.

(U4) For all $\alpha, \beta, \gamma, \delta \in S$, $(\alpha\beta, \gamma, \delta) \in \mathcal{U}'$ and $\alpha\beta \, \mathcal{L}' \, \beta$ implies that $(\beta, \gamma, \delta\alpha) \in \mathcal{U}'$.

(U5) For all $\alpha, \beta, \gamma, \delta \in S$, $(\alpha, \beta, \gamma) \in \mathcal{U}'$ and $\beta \, \mathcal{L}' \, \delta$ implies that $(\alpha, \delta, \gamma) \in \mathcal{U}'$.

(U6) Let $\alpha, \beta, \gamma, \delta \in S$ such that $(\alpha, \beta, \gamma) \in \mathcal{U}'$. Then $(\alpha, \beta, \delta) \in \mathcal{U}'$ if and only if $\gamma \, \mathcal{L}' \, \delta$.

(U7) For all $\alpha, \beta, \gamma, \delta \in S$, $\gamma\alpha = \delta\beta \, \mathcal{R}' \, \delta$ and $(\alpha, \beta, \gamma) \in \mathcal{U}'$ implies that $(\beta, \alpha, \delta) \in \mathcal{U}'$.

(U8) For all $\alpha, \beta, \gamma \in S$, $(\alpha\beta, \beta, \gamma) \in \mathcal{U}'$ implies that $(\alpha\beta, \alpha\beta, \gamma) \in \mathcal{U}'$.

(U9) $\mathcal{R}' \subseteq \mathcal{R}^*$.

(U10) Let $\alpha, \beta, \gamma, \delta \in S$ such that $\gamma \, \mathcal{R}' \, \gamma\alpha \, \mathcal{L}' \alpha$ and $\delta \, \mathcal{R}' \, \delta\beta \, \mathcal{L}' \beta$. Then $\gamma \, \mathcal{L}' \, \delta$ if and only if $\alpha \, \mathcal{R}' \, \beta$.

(U11) For all $\alpha, \beta, \gamma \in S$, $\alpha \, \mathcal{L}' \, \beta \, \mathcal{L}' \, \gamma\alpha = \gamma\beta$ implies that $\alpha = \beta$.

We start the proof of Theorem 4.3.5 by proving the forward implication. We assume that S has a semigroup of straight left I-quotients, Q, and we label $\mathcal{R}^Q \cap (S \times S) = \mathcal{R}'$, $\mathcal{L}^Q \cap (S \times S) = \mathcal{L}'$ and $\mathcal{U}^Q \cap (S \times S \times S) = \mathcal{U}'$. From knowledge of Green's relations, we know that $\mathcal{R}'$ will be a left congruence on S and $\mathcal{L}'$ will be a right congruence on S. Note that Properties (M1) - (M6) from Theorem 4.2.1 hold. We now prove that Properties (U1) - (U11) are satisfied.

(U1) Let $\alpha, \beta \in S$. Then $\alpha, \beta \in Q$ and so β^{-1} and hence $\alpha\beta^{-1} \in Q$. Since Q is a semigroup of straight left I-quotients of S, there exists $\gamma, \delta \in S$ such that $\alpha\beta^{-1} = \gamma^{-1}\delta$ with $\gamma \, \mathcal{R}' \, \delta$. Lemma 4.3.2 then gives us that $\delta\beta = \gamma\alpha$ and $(\alpha, \beta, \gamma) \in \mathcal{U}'$. Using Lemma 4.3.3, we see that $(\alpha, \beta, \gamma) \in \mathcal{U}'$ implies that $\gamma \, \mathcal{R}' \, \gamma\alpha$.

(U2) We have that $(\beta, \beta, \alpha) \in \mathcal{U}'$ if and only if $\alpha^{-1}\alpha = \beta\beta^{-1}\beta\beta^{-1} = \beta\beta^{-1}$, which is true exactly when $\alpha \, \mathcal{R}' \, \alpha\beta \, \mathcal{L}' \, \beta$ by Lemma 3.3.3.

(U3) Let $(\alpha, \beta, \gamma) \in \mathcal{U}'$ and $(\delta, \epsilon, \beta) \in \mathcal{U}'$. Therefore $\alpha\beta^{-1}\beta\alpha^{-1} = \gamma^{-1}\gamma$ and $\delta\epsilon^{-1}\epsilon\delta^{-1} = \beta^{-1}\beta$ in Q. Then

$$\gamma^{-1}\gamma = \alpha\beta^{-1}\beta\alpha^{-1} = \alpha\delta\epsilon^{-1}\epsilon\delta^{-1}\alpha^{-1} = (\alpha\delta)\epsilon^{-1}\epsilon(\alpha\delta)^{-1}.$$

Therefore $(\alpha\delta, \epsilon, \gamma) \in \mathcal{U}'$.

(U4) Let $(\alpha\beta, \gamma, \delta) \in \mathcal{U}'$ and $\alpha\beta \,\mathcal{L}'\, \beta$. Therefore $\delta^{-1}\delta = \alpha\beta\gamma^{-1}\gamma\beta^{-1}\alpha^{-1}$. By Lemma 2.2.4, $\beta \,\mathcal{L}'\, \alpha\beta$ implies that $\beta = \alpha^{-1}\alpha\beta$. Therefore

$$(\delta\alpha)^{-1}(\delta\alpha) = \alpha^{-1}\delta^{-1}\delta\alpha = \alpha^{-1}\alpha\beta\gamma^{-1}\gamma\beta^{-1}\alpha^{-1}\alpha = \beta\gamma^{-1}\gamma\beta^{-1},$$

and so $(\beta, \gamma, \delta\alpha) \in \mathcal{U}'$.

(U5) Let $(\alpha, \beta, \gamma) \in \mathcal{U}'$ and $\beta \,\mathcal{L}'\, \delta$. Then $\gamma^{-1}\gamma = \alpha\beta^{-1}\beta\alpha^{-1} = \alpha\delta^{-1}\delta\alpha^{-1}$, so $(\alpha, \delta, \gamma) \in \mathcal{U}'$.

(U6) Let $(\alpha, \beta, \gamma) \in \mathcal{U}'$. If $(\alpha, \beta, \delta) \in \mathcal{U}'$, then $\gamma^{-1}\gamma = \alpha\beta^{-1}\beta\alpha^{-1} = \delta^{-1}\delta$. Conversely if $\gamma \,\mathcal{L}'\, \delta$, then $\delta^{-1}\delta = \gamma^{-1}\gamma = \alpha\beta^{-1}\beta\alpha^{-1}$, and so $(\alpha, \beta, \delta) \in \mathcal{U}'$.

(U7) Let $\gamma\alpha = \delta\beta \,\mathcal{R}'\, \delta$ and $(\alpha, \beta, \gamma) \in \mathcal{U}'$. Then

$$\beta^{-1}\delta^{-1}\delta\beta = \alpha^{-1}\gamma^{-1}\gamma\alpha = \alpha^{-1}\alpha\beta^{-1}\beta\alpha^{-1}\alpha = \alpha^{-1}\alpha\beta^{-1}\beta.$$

Using $\delta\beta \,\mathcal{R}'\, \delta$, we know that

$$\delta^{-1}\delta = \beta\beta^{-1}\delta^{-1}\delta\beta\beta^{-1} = \beta\alpha^{-1}\alpha\beta^{-1}\beta\beta^{-1} = \beta\alpha^{-1}\alpha\beta^{-1}.$$

Therefore $(\beta, \alpha, \delta) \in \mathcal{U}'$.

(U8) Let $(\alpha\beta, \beta, \gamma) \in \mathcal{U}'$. Therefore

$$\gamma^{-1}\gamma = \alpha\beta\beta^{-1}\beta\beta^{-1}\alpha^{-1} = \alpha\beta\beta^{-1}\alpha^{-1}\alpha\beta\beta^{-1}\alpha^{-1}.$$

Therefore $(\alpha\beta, \alpha\beta, \gamma) \in \mathcal{U}'$.

(U9) (M4)

(U10) (M5)

(U11) (M6)

This proves the forward implication of Theorem 4.3.5.

We will prove the converse using Theorem 4.2.1. In order to do this, we need to find a suitable $\leqslant_l$ and $\wedge$.

We define $\leqslant_l$ as $a \leqslant_l b$ if and only if there exists $u \in S$ such that

$$(a, b, u) \in \mathcal{U}' \text{ and } ua \, \mathcal{L}' \, a. \tag{4.36}$$

We will prove that this is a preorder later. We will often use this definition in conjunction with Property (U1).

Lemma 4.3.6. *Let $a \leqslant_l b$. Then there exists $u, v \in S$ such that*

$$u \, \mathcal{R}' \, v \, \mathcal{R}' \, vb = ua, \ (a, b, u) \in \mathcal{U}' \ \textit{ and } \ ua \, \mathcal{L}' \, a.$$

Proof. Since $a \leqslant_l b$, we know that there exists $x \in S$ such that

$$(a, b, x) \in \mathcal{U}' \ \text{ and } \ xa \, \mathcal{L}' \, a.$$

Applying (U1) to a and b, there exists $u, v \in S$ such that

$$u \, \mathcal{R}' \, v \, \mathcal{R}' \, vb = ua \ \text{ and } \ (a, b, u) \in \mathcal{U}'.$$

By (U6), we have that $(a, b, x) \in \mathcal{U}'$ and $(a, b, u) \in \mathcal{U}'$ implies that $x \, \mathcal{L}' \, u$. Since $\mathcal{L}'$ is a right congruence, we have $xa \, \mathcal{L}' \, ua$, and therefore $ua \, \mathcal{L}' \, a$. □

We now need to find a suitable $\wedge$. We will define it first and prove some basic properties. We will prove that it is the meet of the $\mathcal{L}'$-classes with respect to $\leqslant_l$ later.

We define $\wedge$ as $L'_a \wedge L'_b = L'_c$ if and only if there exists $u \in S$ such that

$$(a, b, u) \in \mathcal{U}' \text{ and } ua \, \mathcal{L}' \, c. \tag{4.37}$$

The fact that this is a well-defined function on $\mathcal{L}'$-classes is not obvious and will be addressed in Lemma 4.3.9.

We will often use the definition of $\wedge$ in conjunction with (U1) as follows.

Lemma 4.3.7. *Let $L'_a \wedge L'_b = L'_c$. Then there exists $u, v \in S$ such that*

$$u \, \mathcal{R}' \, v \, \mathcal{R}' \, vb = ua, \ (a, b, u) \in \mathcal{U}' \ \textit{ and } ua \, \mathcal{L}' \, c.$$

Note that this gives an alternative stronger definition of $\wedge$.

Proof. Since $L'_a \wedge L'_b = L'_c$, by definition there exists $x \in S$ such that

$$(a, b, x) \in \mathcal{U}' \text{ and } xa \, \mathcal{L}' \, c.$$

Applying (U1) to a and b, there exists $u, v \in S$ such that

$$u \, \mathcal{R}' \, v \, \mathcal{R}' \, vb = ua \;\text{ and }\; (a, b, u) \in \mathcal{U}'.$$

By (U6), we have $(a, b, x) \in \mathcal{U}'$ and $(a, b, u) \in \mathcal{U}'$ together imply that $x \, \mathcal{L}' \, u$. Since $\mathcal{L}'$ is a right congruence, we have $xa \, \mathcal{L}' \, ua$, and therefore $ua \, \mathcal{L}' \, c$. □

In order to prove that $\wedge$ is the meet of the $\mathcal{L}'$-classes with respect to $\leqslant_l$ we will use the following lemma. We use $[a]$ to denote the $\sim$-class of a.

Lemma 4.3.8. *Let $\sim$ be an equivalence relation on S and let the $\sim$-classes of S form a semilattice under $\circ$. Additionally, let $\leqslant$ be the binary relation on S defined by $a \leqslant b$ if and only if $[a] \circ [b] = [a]$. Then $\leqslant$ is a preorder on S with the associated equivalence relation $\sim$. Moreover, $S / \sim$ is a meet semilattice under the partial order associated with $\leqslant$, and $\circ$ is the meet operation.*

Proof.

$\leqslant$ **reflexive**: Since $\sim$-classes of S form a semilattice under $\circ$, every element is an idempotent. Therefore, for all $a \in S$, we have $[a] \circ [a] = [a]$, and so $a \leqslant a$.

$\leqslant$ **transitive**: Let $a, b, c \in S$ such that $a \leqslant b$ and $b \leqslant c$. Therefore $[a] \circ [b] = [a]$ and $[b] \circ [c] = [b]$. Consequently

$$[a] \circ [c] = ([a] \circ [b]) \circ [c] = [a] \circ ([b] \circ [c]) = [a] \circ [b] = [a],$$

and so $a \leqslant c$.

Associated equivalence relation is $\sim$: Let $a, b \in S$ such that $a \leqslant b$ and $b \leqslant a$. Then, using the fact that $\circ$ is commutative,

$$[a] = [a] \circ [b] = [b] \circ [a] = [b],$$

and so $a \sim b$. The converse is clear.

$\circ$ **is the associated meet**: Let $[a] \circ [b] = [c]$. We see that

$$[a] \circ [c] = [a] \circ ([a] \circ [b]) = ([a] \circ [a]) \circ [b] = [a] \circ [b] = [c].$$

Therefore $c \preccurlyeq a$. Similarly $c \preccurlyeq b$.

Now let $h \in S$ such that $h \preccurlyeq a$ and $h \preccurlyeq b$. This means that $[h] \circ [a] = [h]$ and $[h] \circ [b] = [h]$. Then

$$[h] \circ [c] = [h] \circ ([a] \circ [b]) = ([h] \circ [a]) \circ [b] = [h] \circ [b] = [h].$$

And so $h \preccurlyeq c$. So $[c]$ is the meet of $[a]$ and $[b]$ with respect to $\preccurlyeq$.

$S / \sim$ **is a meet semilattice**: Let $a, b \in S$. Since the $\sim$-classes of S form a semigroup under $\circ$, there exists some $c \in S$ such that $[a] \circ [b] = [c]$. $\square$

Our aim is to apply Lemma 4.3.8 with $\sim \, = \, \mathcal{L}'$ and $\circ \, = \, \wedge$. It is obvious by comparing (4.36) and (4.37) that $a \preccurlyeq_l b$ if and only if $L'_a \wedge L'_b = L'_a$. Therefore our $\preccurlyeq$ in Lemma 4.3.8 will be $\preccurlyeq_l$. We now need to prove that the $\mathcal{L}'$-classes of S form a semilattice under $\wedge$.

Lemma 4.3.9. *The $\mathcal{L}'$-classes of S form a semilattice under $\wedge$.*

Proof. Commutativity will be proved first since it will make the rest of the proof much easier. Since we have not shown that $\wedge$ is a function yet, we can think of this as a result on the ternary relation $L'_a \wedge L'_b = L'_c$.

Commutativity: Let $L'_a \wedge L'_b = L'_c$. Then by Lemma 4.3.7, there exists $u, v \in S$ such that

$$u \,\mathcal{R}'\, v \,\mathcal{R}'\, vb = ua, \; (a, b, u) \in \mathcal{U}' \text{ and } ua \,\mathcal{L}'\, c.$$

By (U7), we see that $(b, a, v) \in \mathcal{U}'$. Combining this with $vb = ua \,\mathcal{L}'\, c$, this gives us that $L'_b \wedge L'_a = L'_c$.

Well-definedness: Let $L'_a \wedge L'_b = L'_c$. Then, by Lemma 4.3.7, there exists $u, v \in S$ such that

$$u \,\mathcal{R}'\, v \,\mathcal{R}'\, vb = ua, \; (a, b, u) \in \mathcal{U}' \text{ and } ua \,\mathcal{L}'\, c.$$

Firstly, let $b \,\mathcal{L}'\, \tilde{b}$. Then by (U5), we have $(a, \tilde{b}, u) \in \mathcal{U}'$. As we also have $ua \,\mathcal{L}'\, c$, it follows that $L'_a \wedge L'_{\tilde{b}} = L'_c$.

Secondly, let $c\,\mathcal{L}'\,\tilde{c}$. Then $(a,b,u) \in \mathcal{U}'$ and $ua\,\mathcal{L}'\,c\,\mathcal{L}'\,\tilde{c}$, so $L'_a \wedge L'_b = L'_{\tilde{c}}$.

Lastly, let $a\,\mathcal{L}'\,\tilde{a}$. Using commutativity and our previous observations on well-definedness, we have

$$L'_a \wedge L'_b = L'_c \implies L'_b \wedge L'_a = L'_c \implies L'_b \wedge L'_{\tilde{a}} = L'_c \implies L'_{\tilde{a}} \wedge L'_b = L'_c.$$

Associativity: Let $L'_a \wedge L'_b = L'_d$. Then, by Lemma 4.3.7, there exists $u, v \in S$ such that

$$u\,\mathcal{R}'\,v\,\mathcal{R}'\,vb = ua,\ (a,b,u) \in \mathcal{U}' \text{ and } ua\,\mathcal{L}'\,d. \tag{4.38}$$

Note that (U7) implies that $(b,a,v) \in \mathcal{U}'$ as well.

Also let $L'_b \wedge L'_c = L'_e$. Then, by Lemma 4.3.7, there exists $p, q \in S$ such that

$$p\,\mathcal{R}'\,q\,\mathcal{R}'\,qc = pb,\ (b,c,p) \in \mathcal{U}' \text{ and } pb\,\mathcal{L}'\,e. \tag{4.39}$$

Applying (U1) to v and p, there exists $i, j \in S$ such that

$$i\,\mathcal{R}'\,j\,\mathcal{R}'\,jp = iv \text{ and } (v,p,i) \in \mathcal{U}'. \tag{4.40}$$

Note that (U7) implies that $(p,v,j) \in \mathcal{U}'$ as well.

We will prove that

$$(L'_a \wedge L'_b) \wedge L'_c = L'_d \wedge L'_c = L'_{jqc} = L'_{iua} = L'_a \wedge L'_e = L'_a \wedge (L'_b \wedge L'_c). \tag{4.41}$$

We start by proving $jqc = iua$. This is true, since $qc = bc$ from (4.39), $jp = iv$ from (4.40), and $vb = ua$ from (4.38) together gives us $jqc = jpb = ivb = iua$.

We now prove that $L'_d \wedge L'_c = L'_{jqc}$. We see that

$$(v,p,i) \in \mathcal{U}', (b,c,p) \in \mathcal{U}' \overset{(U3)}{\implies} (vb,c,i) \in \mathcal{U}'. \tag{4.42}$$

Using $iv = jp$ from (4.40) and $pb = qc$ from (4.39), we see that $ivb = jpb = jqc$. In addition, we can use the fact that $\mathcal{R}'$ is a left congruence to get that $q\,\mathcal{R}'\,qc$ implies that $jq\,\mathcal{R}'\,jqc$. We can then apply (U7) to $ivb = jqc\,\mathcal{R}'\,jq$ and $(vb,c,i) \in \mathcal{U}'$ from (4.42) to obtain $(c,vb,jq) \in \mathcal{U}'$.

We know that $vb = ua\,\mathcal{L}'\,d$. Therefore we can apply (U5) to $(c,vb,jq) \in \mathcal{U}'$ to obtain $(c, d, jq) \in \mathcal{U}'$. This proves $L'_c \wedge L'_d = L'_{jqc}$.

Finally we prove $L'_a \wedge L'_e = L'_{iua}$. We have

$$(p, v, j) \in \mathcal{U}', (b, a, v) \in \mathcal{U}' \overset{(U3)}{\Longrightarrow} (pb, a, j) \in \mathcal{U}'.$$

Using $jp = iv$ and $vb = ua$, we see that $jpb = ivb = iua$. In addition, we can use the fact that $\mathcal{R}'$ is a left congruence to get that $u \,\mathcal{R}'\, ua$ implies that $iu \,\mathcal{R}'\, iua$. We can then apply (U7) to $jpb = iua \,\mathcal{R}'\, iu$ and $(pb, a, j) \in \mathcal{U}'$ from (4.3) to obtain $(a, pb, iu) \in \mathcal{U}'$.

We know that $pb \,\mathcal{L}'\, e$. Therefore we can apply (U5) to $(a, pb, iu) \in \mathcal{U}'$ to obtain $(a, e, iu) \in \mathcal{U}'$. This proves $L'_a \wedge L'_e = L'_{iua}$.

Therefore (4.41) is satisfied and so $\wedge$ is associative.

Idempotent: Let $L'_a \wedge L'_a = L'_c$. Then there exists $x \in S$ such that

$$(a, a, x) \in \mathcal{U}' \text{ and } xa \,\mathcal{L}'\, c.$$

Using (U2), we have $x \,\mathcal{R}'\, xa \,\mathcal{L}'\, a$. Therefore $L'_a = L'_{xa} = L'_c$, and so $L'_a \wedge L'_a = L'_a$ for all $a \in S$. □

Therefore we can apply Lemma 4.3.8 to get that $\leqslant_l$ is a preorder on S with the associated equivalence relation $\mathcal{L}'$. Moreover, $S/\mathcal{L}'$ is a meet semilattice under the associated partial order and $\wedge$ is the meet operation. Therefore, in order to prove that S, $\mathcal{R}'$ and $\leqslant_l$ satisfy the conditions of Theorem 4.2.1, all we need is (M1) - (M6).

This lemma will provide a shortcut in the proof of (M2).

Lemma 4.3.10. *For all $a \in S$, there exists an $x \in S$ such that $x \,\mathcal{R}'\, xa \,\mathcal{L}'\, a$.*

Proof. Applying Property (U1) with $\alpha = \beta = a$, there exists $x \in S$ such that $(a, a, x) \in \mathcal{U}'$. By Property (U2), this is equivalent to $x \,\mathcal{R}'\, xa \,\mathcal{L}'\, a$. □

(M1) Let $\alpha, \beta \in S$. By (U1) there exists $\gamma, \delta \in S$ such that $\gamma \,\mathcal{R}'\, \delta \,\mathcal{R}'\, \delta\beta = \gamma\alpha$, and $(\alpha, \beta, \gamma) \in \mathcal{U}'$. By definition, $L'_\alpha \wedge L'_\beta = L'_{\gamma\alpha}$.

(M2) Let $\alpha, \beta, \gamma, \delta \in S$ such that $L'_\alpha \wedge L'_\beta = L'_\gamma$. By Lemma 4.3.7, there exists

$u, v \in S$ such that

$$u\,\mathcal{R}'\,v\,\mathcal{R}'\,v\beta = u\alpha,\ (\alpha, \beta, u) \in \mathcal{U}' \ \text{ and } \ u\alpha\,\mathcal{L}'\,\gamma. \tag{4.43}$$

By Lemma 4.3.10, there exists $y \in S$ such that $y\,\mathcal{R}'\,y\delta\,\mathcal{L}'\,\delta$. Therefore $(\delta, \delta, y) \in \mathcal{U}'$ by (U2).

Applying (U1) to α and y, there exists $s, t \in S$ such that

$$s\,\mathcal{R}'\,t\,\mathcal{R}'\,ty = s\alpha \text{ and } (\alpha, y, s) \in \mathcal{U}'. \tag{4.44}$$

Applying (U1) to s and u, there exists $p, q \in S$ such that

$$p\,\mathcal{R}'\,q\,\mathcal{R}'\,qu = ps \text{ and } (s, u, p) \in \mathcal{U}'. \tag{4.45}$$

We can apply (U7) to $ps = qu\,\mathcal{R}'\,q$ and $(s, u, p) \in \mathcal{U}'$ to get $(u, s, q) \in \mathcal{U}'$.

Applying (U3) to $(\alpha, y, s) \in \mathcal{U}'$ and $(\delta, \delta, y) \in \mathcal{U}'$ implies that

$$(\alpha\delta, \delta, s) \in \mathcal{U}'. \tag{4.46}$$

Applying (U3) to $(u, s, q) \in \mathcal{U}'$ and $(\alpha\delta, \delta, s) \in \mathcal{U}'$ implies that

$$(u\alpha\delta, \delta, q) \in \mathcal{U}'. \tag{4.47}$$

We then apply (U8) to $(u\alpha\delta, \delta, q) \in \mathcal{U}'$ to give us $(u\alpha\delta, u\alpha\delta, q) \in \mathcal{U}'$. Therefore, by (U2)

$$q\,\mathcal{R}'\,qu\alpha\delta\,\mathcal{L}'\,u\alpha\delta. \tag{4.48}$$

Using the fact that $\mathcal{L}'$ is a right congruence, $u\alpha\,\mathcal{L}'\,\gamma$ implies that $u\alpha\delta\,\mathcal{L}'\,\gamma\delta$. Therefore, putting this together with (4.48) and $qu = ps$ from (4.45), we have

$$\gamma\delta\,\mathcal{L}'\,u\alpha\delta\,\mathcal{L}'\,qu\alpha\delta = ps\alpha\delta. \tag{4.49}$$

At the same time, using the fact that $\mathcal{R}'$ is a left congruence, $y\,\mathcal{R}'\,y\delta$ implies that $ty\,\mathcal{R}'\,ty\delta$. Since $ty = s\alpha$, this means that $s\alpha\,\mathcal{R}'\,s\alpha\delta$. Therefore, we can apply Property (U7) to $s(\alpha\delta) = (s\alpha)\delta\,\mathcal{R}'\,s\alpha$ and $(\alpha\delta, \delta, s) \in \mathcal{U}'$ from (4.46) to obtain $(\delta, \alpha\delta, s\alpha) \in \mathcal{U}'$.

Applying (U3) to $(s, u, p) \in \mathcal{U}'$ and $(\alpha, \beta, u) \in \mathcal{U}'$ implies that

$(s\alpha, \beta, p) \in \mathcal{U}'$. Since $\mathcal{R}'$ is a left congruence, $v\,\mathcal{R}'\,v\beta$ implies that $qv\,\mathcal{R}'\,qv\beta$. Also, using $ps = qu$ and $u\alpha = v\beta$, we have $ps\alpha = qu\alpha = qv\beta$. Therefore we can apply Property (U7) to $ps\alpha = qv\beta\,\mathcal{R}'\,qv$ and $(s\alpha, \beta, p) \in \mathcal{U}'$ to obtain $(\beta, s\alpha, qv) \in \mathcal{U}'$.

We then put the results from the two previous paragraphs together to obtain

$$(\beta, s\alpha, qv) \in \mathcal{U}', (\delta, \alpha\delta, s\alpha) \in \mathcal{U}' \overset{(U3)}{\Longrightarrow} (\beta\delta, \alpha\delta, qv) \in \mathcal{U}'.$$

We can apply (U8) to $(\alpha\delta, \delta, s) \in \mathcal{U}'$ from (4.46) to get $(\alpha\delta, \alpha\delta, s) \in \mathcal{U}'$. Therefore $s\,\mathcal{R}'\,s\alpha\delta\,\mathcal{L}'\,\alpha\delta$ by (U2). Since $\mathcal{R}'$ is a left congruence, this implies that $ps\,\mathcal{R}'\,ps\alpha\delta$. Also, using $v\beta = u\alpha$ and $qu = ps$, we have $qv\beta\delta = qu\alpha\delta = ps\alpha\delta$. Therefore we can apply Property (U7) to $qv\beta\delta = ps\alpha\delta\,\mathcal{R}'\,ps$ and $(\beta\delta, \alpha\delta, qv) \in \mathcal{U}'$ to obtain $(\alpha\delta, \beta\delta, ps) \in \mathcal{U}'$.

Therefore, we use this last relation and equation 4.49, to see that $ps \in S$ such that $(\alpha\delta, \beta\delta, ps) \in \mathcal{U}'$ and $\gamma\delta\,\mathcal{L}'\,ps\alpha\delta$. Therefore, by definition, we have $L'_{\alpha\delta} \wedge L'_{\beta\delta} = L'_{\gamma\delta}$.

(M3) Let $\alpha, \beta \in S$. Applying (U1) to $\alpha\beta$ and β, we have that there exists $u, v \in S$ such that $u\,\mathcal{R}'\,v\,\mathcal{R}'\,v\beta = u\alpha\beta$, and $(\alpha\beta, \beta, u) \in \mathcal{U}'$. Applying (U8), this gives us that $(\alpha\beta, \alpha\beta, u) \in \mathcal{U}'$. Applying (U2), this gives us $u\,\mathcal{R}'\,u\alpha\beta\,\mathcal{L}'\,\alpha\beta$. Since there exists $u \in S$ such that $(\alpha\beta, \beta, u) \in \mathcal{U}'$ and $u\alpha\beta\,\mathcal{L}'\,\alpha\beta$, this means that $\alpha\beta \leqslant_l \beta$.

(M4) (U9)

(M5) (U10)

(M6) (U11)

We have now shown that the relations $\mathcal{R}'$ and $\leqslant_l$ satisfy all of the conditions of Theorem 4.2.1. Applying Theorem 4.2.1 gives us that S has a semigroup of straight left I-quotients, Q, such that $\mathcal{R}^Q \cap (S \times S) = \mathcal{R}'$ and $\leqslant_{\mathcal{L}^Q} \cap (S \times S) = \leqslant_l$. Note that consequently $\mathcal{L}^Q \cap (S \times S) = \mathcal{L}'$ as this is the equivalence relation associated with $\leqslant_l$. Finally we need to check that $\mathcal{U}^Q \cap (S \times S \times S) = \mathcal{U}'$.

Lemma 4.3.11. *Let $b, c, u \in S$. Then $(b, c, u) \in \mathcal{U}'$ if and only if $(b, c, u) \in \mathcal{U}^Q$.*

Proof. Let $(b, c, u) \in \mathcal{U}'$. Referring to (4.37), this means that $L'_b \wedge L'_c = L'_{ub}$ by definition. Therefore, since $\leqslant_{\mathcal{L}^Q} \cap (S \times S) = \leqslant_l$, this also means that $L^Q_b \wedge L^Q_c = L^Q_{ub}$.

Applying (U1) to b and c, there exists $p, q \in S$ such that $p\,\mathcal{R}'\,q\,\mathcal{R}'\,qc = pb$ and $(b, c, p) \in \mathcal{U}'$. Applying (U6), we have that $(b, c, u) \in \mathcal{U}'$ and $(b, c, p) \in \mathcal{U}'$ implies that $u\,\mathcal{L}'\,p$.

Since Q is an inverse semigroup with $\mathcal{R}^Q \cap (S \times S) = \mathcal{R}'$ and $\mathcal{L}^Q \cap (S \times S) = \mathcal{L}'$, we have $u\,\mathcal{L}^Q\,p$ and $p\,\mathcal{R}^Q\,pb$. Since $\mathcal{R}^Q$ is left compatible, we can left multiply by p^{-1} to get $p^{-1}p\,\mathcal{R}^Q\,p^{-1}pb$. Since $u\,\mathcal{L}^Q\,p$, we know that $u^{-1}u = p^{-1}p$, and therefore $u^{-1}u\,\mathcal{R}^Q\,u^{-1}ub$. Left multiplying by u then gives us $u\,\mathcal{R}^Q\,ub$.

By Lemma 4.3.3, we can put these together to get that $L^Q_b \wedge L^Q_c = L^Q_{ub}$ and $u\,\mathcal{R}^Q\,ub$ implies that $(b, c, u) \in \mathcal{U}^Q$.

Conversely, let $(b, c, u) \in \mathcal{U}^Q$. We know that Q is an inverse semigroup. Therefore by Lemma 4.3.3, we know that $L^Q_b \wedge L^Q_c = L^Q_{ub}$ and $u\,\mathcal{R}^Q\,ub$. Therefore, since $\leqslant_{\mathcal{L}^Q} \cap (S \times S) = \leqslant_l$, we have $L'_b \wedge L'_c = L'_{ub}$.

Applying (U1) to b and c, there exists $p, q \in S$ such that $p\,\mathcal{R}'\,q\,\mathcal{R}'\,qc = pb$ and $(b, c, p) \in \mathcal{U}'$. By definition, $L'_b \wedge L'_c = L'_{pb}$. Therefore $L'_{ub} = L'_b \wedge L'_c = L'_{pb}$, that is $ub\,\mathcal{L}'\,pb$.

Since Q is an inverse semigroup with $\mathcal{R}^Q \cap (S \times S) = \mathcal{R}'$ and $\mathcal{L}^Q \cap (S \times S) = \mathcal{L}'$, we have $ub\,\mathcal{L}^Q\,pb$ and $p\,\mathcal{R}^Q\,pb$. From knowledge of Green's relations we know that $p\,\mathcal{R}^Q\,pb$ implies that $pp^{-1} = pbb^{-1}p^{-1}$, and therefore $p = pbb^{-1}p^{-1}p = pbb^{-1}$. Similarly, $u\,\mathcal{R}^Q\,ub$ implies that $u = ubb^{-1}$.

Since $\mathcal{L}^Q$ is right compatible, $ub\,\mathcal{L}^Q\,pb$ implies that $u = ubb^{-1}\,\mathcal{L}^Q\,pbb^{-1} = p$, and therefore $u\,\mathcal{L}'\,p$. We then apply (U6) to $(b, c, p) \in \mathcal{U}'$ to obtain $(b, c, u) \in \mathcal{U}'$. $\square$

We have now finished the proof of Theorem 4.3.5.

4.4 Straight left I-orders in given inverse semigroups

One may have noticed that in both Theorem 4.2.1 and Theorem 4.3.5, most of the properties required are true in all inverse semigroups, not just semigroups of straight left I-quotients. Consequently, if you are checking whether a given subsemigroup is a straight left I-order in a particular inverse semigroup, then this requires a much simpler result.

Corollary 4.4.1. *Let S be a subsemigroup of an inverse semigroup Q. Then S is a straight left I-order in Q if and only if for all $b, c \in S$ there exists $u, v \in S$ such that $u \,\mathcal{R}^Q\, v$ and $bc^{-1} = u^{-1}v$.*

Proof. Let $\mathcal{R}' = \mathcal{R}^Q \cap (S \times S)$ and let $\leqslant_l = \leqslant_{\mathcal{L}^Q} \cap (S \times S)$. Then $\mathcal{R}'$ will be a left congruence and $\leqslant_l$ will be a preorder, such that, denoting the associated equivalence relation by $\mathcal{L}'$, $S/\mathcal{L}'$ is a meet semilattice under the associated partial order. Properties (M2) - (M6) are true in all inverse semigroups, so are satisfied. Therefore the only property to check is the Ore condition, (M1), which by Lemma 3.3.6 is equivalent to the condition in the result. Conversely in a straight left I-order (M1) is always true. □

We will apply Corollary 4.4.1 on the semidirect product of a semilattice and a group.

Let Y be a semilattice and let G be a group acting by automorphisms on Y. That is, let $\cdot : G \times Y \to Y$ be a group action on Y, such that for every $g \in G$, the action by g is an automorphism on Y. Then we say that the semidirect product of Y and G is the semigroup made from pairs (α, g), where $\alpha \in Y$ and $g \in G$, with the multiplication

$$(\alpha, g)(\beta, h) = (\alpha(g \cdot \beta), gh).$$

We write this as $Y \rtimes G$. This is an inverse semigroup with

$$(\alpha, g)^{-1} = (g^{-1} \cdot \alpha, g^{-1})$$

and a semilattice of idempotents isomorphic to Y. Using Lemma 2.2.1, we also see that

$$(\alpha, g) \,\mathcal{R}\, (\beta, h) \text{ if and only if } \alpha = \beta$$

and

$$(\alpha, g) \,\mathcal{L}\, (\beta, h) \text{ if and only if } g^{-1} \cdot \alpha = h^{-1} \cdot \beta.$$

Corollary 4.4.2. *Let $Q = Y \rtimes G$, where Y is a semilattice and G is a group, and let S be a subsemigroup of Q. Then S is a straight left I-order in Q if and only if for all (α, g), $(\beta, h) \in S$, there exists (γ, f), $(\gamma, j) \in S$ such that $fg = jh$ and $\gamma = (f \cdot \alpha)(j \cdot \beta)$.*

Proof. By Corollary 4.4.1, S is a straight left I-order in Q if and only if for all $(\alpha, g), (\beta, h) \in S$ there exists $(\gamma, f), (\delta, j) \in S$ such that $(\gamma, f)\, \mathcal{R}^Q\, (\delta, j)$ and $(\alpha, g)(\beta, h)^{-1} = (\gamma, f)^{-1}(\delta, j)$.

Assume that S is a straight left I-order in Q, and consequently such a (γ, f) and (δ, j) exists. Since $(\gamma, f)\, \mathcal{R}^Q\, (\delta, j)$, we have that $\gamma = \delta$. We calculate

$$(\alpha, g)(\beta, h)^{-1} = (\alpha, g)(h^{-1} \cdot \beta, h^{-1}) = (\alpha(gh^{-1} \cdot \beta), gh^{-1}) \tag{4.50}$$

and

$$(\gamma, f)^{-1}(\gamma, j) = (f^{-1} \cdot \gamma, f^{-1})(\gamma, j) = (f^{-1} \cdot \gamma, f^{-1}j). \tag{4.51}$$

Comparing the last coordinate of (4.50) and (4.51), this gives us that $gh^{-1} = f^{-1}j$, which is equivalent to $fg = jh$ since G is a group. Comparing the first coordinate of (4.50) and (4.51), we have $f^{-1} \cdot \gamma = \alpha(f^{-1}j \cdot \beta)$. Acting on both sides by f we find that this implies that $\gamma = (f \cdot \alpha)(j \cdot \beta)$.

Conversely, assume that for all (α, g), $(\beta, h) \in S$, there exists (γ, f), $(\gamma, j) \in S$ such that $fg = jh$ and $\gamma = (f \cdot \alpha)(j \cdot \beta)$. Since G is a group, $fg = jh$ implies that $gh^{-1} = f^{-1}j$. Acting on $\gamma = (f \cdot \alpha)(j \cdot \beta)$ by f^{-1} we then obtain

$$f^{-1} \cdot \gamma = \alpha(f^{-1}j \cdot \beta) = \alpha(gh^{-1} \cdot \beta).$$

Therefore (4.50) and (4.51) are equal, and so $(\alpha, g)(\beta, h)^{-1} = (\gamma, f)^{-1}(\gamma, j)$ with $(\gamma, f)\, \mathcal{R}^Q\, (\gamma, j)$. Therefore S is a straight left I-order in Q by Corollary 4.4.1. $\square$

Chapter 5

Proving established results using my general theorem

Some characterisations of special cases of semigroups of I-quotients already exist in the work in [13], [12], [10] by Nassraddin Ghroda and Victoria Gould. In this chapter, we consider two of these established results and prove them both using the main result of the previous chapter, Theorem 4.2.1.

In Section 5.1, we reprove a characterisation of straight left I-orders in primitive inverse semigroups.

In Section 5.2, we reprove the fact that left ample semigroups are left I-orders in their inverse hull if and only if they have the (LC) condition. We apply this result to Exel and Steinberg's work on inverse hulls of 0-left cancellative semigroups [5], which is an example of when semigroups of I-quotients are used in the literature implicitly.

5.1 Primitive inverse semigroups of left I-quotients

In this section we reprove Theorem 3.1 of [12] using Theorem 4.2.1. This result provides a characterisation of left I-orders in primitive inverse semigroups. All such left I-orders are straight.

Recall that an inverse semigroup S with zero is a *primitive inverse semigroup* if all its nonzero idempotents are primitive, where a nonzero idempotent e of S is called *primitive* if $f \leqslant e$ implies that $f = 0$ or $e = f$.

A semigroup S with zero is *categorical* at 0 if for all $a, b, c \in S$, $ab \neq 0$ and $bc \neq 0$ implies $abc \neq 0$. We say that S is *0-cancellative* if for all $a, b, c \in S$, either one of $ab = ac \neq 0$ or $ba = ca \neq 0$ implies that $b = c$.

We will use the following facts about primitive inverse semigroups throughout this chapter.

Lemma 5.1.1. *Let Q be a primitive inverse semigroup. Then*

(i) Q is categorical at 0.

(ii) If $a, b \in Q \backslash \{0\}$, then $ab \neq 0$ if and only if $a^{-1}a = bb^{-1}$.

(iii) Q is 0-cancellative.

Proof. (i) [3, Lemma 7.61]

(ii) Let $a, b \in Q\backslash\{0\}$. We have that $a^{-1}a = bb^{-1}$ if and only if there exists an idempotent, e, contained in $L_a \cap R_b$. By [20, Prop. 2.3.7], this is true if and only if $ab \in R_a \cap L_b$.

a		ab
e		b

Let $ab \in R_a \cap L_b$. Since a is nonzero, zero cannot lie in R_a. Therefore $ab \neq 0$.

Conversely, let $ab \neq 0$. Since Q is a 0-direct union of Brandt semigroups, it follows that a and b are both nonzero elements of the same Brandt semigroup. Using Definition 2.5.1 and Lemma 2.5.2, we see that for a, b nonzero elements of a Brandt semigroup, $ab \neq 0$ implies that $ab \in R_a \cap L_b$.

(iii) From (ii), we can see that $ab = ac \neq 0$ implies that $a^{-1}a = bb^{-1}$ and $a^{-1}a = cc^{-1}$. Therefore $b = a^{-1}ab = a^{-1}ac = c$. Dually $ba = ca \neq 0$ implies that $b = c$.

□

We will need the following relation.

Definition 5.1.2 (λ relation)**.** Let S be a semigroup with 0. Then

$$a \lambda b \text{ if and only if } a = b = 0 \text{ or } Sa \cap Sb \neq \{0\}.$$

In the next proposition we identify some properties of a semigroup which has a primitive inverse semigroup of left I-quotients. Most of these properties are from Proposition 2.4 of [12].

We make the convention that if S is a left I-order in Q, then $\mathcal{R}$, $\mathcal{L}$ and $\leqslant_{\mathcal{L}}$ will be relations on Q, and $\mathcal{R}^*$ and λ will be relations on S.

Proposition 5.1.3. *Let S be a subsemigroup of a primitive inverse semigroup Q. If S is a left I-order in Q, then*

(1) S contains the 0 element of Q;

(2) S is a straight left I-order in Q;

(3) $Sa \neq 0$ for all non-zero $a \in S$;

(4) $\mathcal{L} \cap (S \times S) = \lambda$;

(5) $\mathcal{R} \cap (S \times S) = \mathcal{R}^*$;

(6) if $a, b \in S$, then $a \leqslant_{\mathcal{L}} b$ if and only if $a \lambda b$ or $a = 0$.

Proof. Let S be a left I-order in Q.

(1) By definition, since $0 \in Q$, we have that there exists $a, b \in S$ such that $a^{-1}b = 0$.

If a and b are in different Brandt semigroups, then $ab = 0$ is an element of S by closure. If a and b are both elements of the same Brandt semigroup, $B = B(G, I)$, let $a = (i, g, j)$ and $b = (k, h, l)$, where $g, h \in G$ and $i, j, k, l \in I$.

Since a^2 is an element of S, we have

$$a^2 = (i, g, j)(i, g, j)$$

and so either $0 \in S$ or $i = j$. Similarly, ab and b^2 are elements of S, and so we either have $0 \in S$ or $i = j = k = l$. In the latter case, we can then rewrite a and b as $a = (j, g, j)$ and $b = (j, h, j)$ and conclude

$$0 = a^{-1}b = (j, g^{-1}, j)(j, h, l) = (j, g^{-1}h, l) \neq 0,$$

which gives us a contradiction. Thus $0 \in S$.

(2) Suppose that $q \in Q$. If $q = 0$, then $q = 0^{-1}0$ and $0 \,\mathcal{R}\, 0$. If q is non-zero, then $q = a^{-1}b$, for some $a, b \in S$. Since $a^{-1}b \neq 0$, Lemma 5.1.1 (ii) gives us that $(a^{-1})^{-1}a^{-1} = aa^{-1} = bb^{-1}$, and so $a \,\mathcal{R}\, b$.

(3) Let $a = x^{-1}y \neq 0$ for some $x, y \in S$, where $x \,\mathcal{R}\, y$. Then $xa = y \neq 0$.

(4) If $a \,\lambda\, b$, either $a = b = 0$ and therefore $a \,\mathcal{L}\, b$, or $xa = yb \neq 0$ and therefore $x^{-1}x = aa^{-1}$ and $y^{-1}y = bb^{-1}$. And so $a = x^{-1}yb$ and $b = y^{-1}xa$, so that $a \,\mathcal{L}\, b$.

Conversely if $a \,\mathcal{L}\, b$, then either $a = b = 0$, or $a \neq 0$ and $a = x^{-1}yb$ for some $x^{-1}y \in Q$, $x, y \in S$, $x \,\mathcal{R}\, y$. Therefore $xa = yb \neq 0$ and so $a \,\lambda\, b$.

(5) It is clear that $\mathcal{R} \cap (S \times S) \subseteq \mathcal{R}^*$. To show that $\mathcal{R}^* \subseteq \mathcal{R} \cap (S \times S)$, let $a \,\mathcal{R}^*\, b$ in S. By (2), there exists y in S such that $ya \neq 0$. Hence $yb \neq 0$ by Lemma 2.2 of [8]. By Lemma 5.1.1 (ii), $aa^{-1} = y^{-1}y = bb^{-1}$ and therefore $a \,\mathcal{R}\, b$ in Q.

(6) The relation $\leqslant_{\mathcal{L}}$ can only occur within a single Brandt semigroup in Q. In a Brandt semigroup, $a \leqslant_{\mathcal{L}} b$ if and only if either $a = 0$ or $a \,\mathcal{L}\, b$. We then use (4) to get the result.

□

We now give Ghroda and Gould's characterisation of left primitive inverse semigroups of left I-quotients from [12].

Theorem 5.1.4 ([12, Theorem 3.1]). *A semigroup S is a left I-order in a primitive inverse semigroup Q if and only if S satisfies the following conditions:*

(A) S is categorical at 0;

(B) S is 0-cancellative;

(C) λ is transitive;

(D) $Sa \neq 0$ for all non-zero $a \in S$.

The original proof was constructive in nature. However, by applying Theorem 4.2.1, our proof bypasses this construction. Note that the construction in the original proof and the construction in the proof of Theorem 4.2.1 are identical for primitive inverse semigroups.

We start with proving the forward direction. Suppose that Q is exists, then S inherits Conditions (A) and (B) from Q. By Proposition 5.1.3 we have that Conditions (C) and (D) hold.

We will prove the other direction using Theorem 4.2.1. Let $\mathcal{R}' = \mathcal{R}^*$, and let $a \leqslant_l b$ if and only if $a \,\lambda\, b$ or $a = 0$. Note that $\mathcal{R}'$ is a left congruence, and $\mathcal{L}'$, defined as the equivalence relation associated with $\leqslant_l$, is equal to λ.

Lemma 5.1.5. *Let S satisfy Conditions (A) - (D). Then*

(i) the binary relation $\leqslant_l$ is a right compatible preorder;

(ii) $S/\mathcal{L}'$ is a meet semilattice under $\leqslant_l$ with

$$L'_a \wedge L'_b = \begin{cases} L'_a & \text{if } a \,\lambda\, b, \\ L'_0 & \text{if } a \not\lambda\, b. \end{cases}$$

Proof. (i) We start by proving reflexivity. We want that for all $a \in S$, the relation $a \,\lambda\, a$ holds. This is trivially true for $a = 0$ and for $a \neq 0$, we have $Sa \cap Sa = Sa \neq \{0\}$ by (D). So $a \,\lambda\, a$ for all $a \in S$ and we have that $a \leqslant_l a$ as required.

We now consider transitivity. Let $a \leqslant_l b$ and $b \leqslant_l c$. If $a = 0$, we have that $a \leqslant_l c$ immediately. If $a \neq 0$, then $a \leqslant_l b$ give us that $a \,\lambda\, b$, and so $b \neq 0$. Since b is non-zero, $b \leqslant_l c$ gives us that $b \,\lambda\, c$. Therefore using (C), $a \,\lambda\, b \,\lambda\, c$ implies that $a \,\lambda\, c$, and so $a \leqslant_l c$.

Lastly we consider right compatibility. Let $a \leqslant_l b$ and let $x \in S$. If $ax = 0$, we have that $ax \leqslant_l bx$ immediately. If $ax \neq 0$, then $a \neq 0$ and so $a \leqslant_l b$

implies that $a \,\lambda\, b$. Therefore $Sa \cap Sb \neq \{0\}$, and so there exists $u, v \in S$ such that $ua = vb \neq 0$. Since $ax \neq 0$, (A) gives us that $uax = vbx \neq 0$ and so $Sax \cap Sbx \neq \{0\}$. Therefore $ax \,\lambda\, bx$, which implies $ax \leqslant_l bx$.

(ii) (a) Let $a \,\lambda\, b$. We see that $L'_a = L'_b$, and so, by the definition of meet, we have $L'_a \wedge L'_b = L'_a$.

(b) Now let $a \not\lambda b$. We will prove that $L'_a \wedge L'_b = L'_0$. Firstly, by definition, we have $0 \leqslant_l a, b$. Secondly, if $h \leqslant_l a, b$ and $h \neq 0$, this would mean that $h \,\lambda\, a$ and $h \,\lambda\, b$. By (C), λ is transitive, and therefore $a \,\lambda\, b$, a contradiction. Therefore h is 0, and so $h \leqslant_l 0$. By definition, this mean that L'_0 is the meet of L'_a and L'_b.

□

We will use the following lemma repeatedly in our proof of (M1) - (M6).

Lemma 5.1.6. *Let S satisfy Conditions (A) - (D). Then $xa \neq 0$ implies that $x \,\mathcal{R}^*\, xa \,\lambda\, a$.*

Proof. Let $xa \neq 0$. We start by proving $x \,\mathcal{R}^*\, xa$. If $u, v \in S^1$ such that $ux = vx$, then obviously $uxa = vxa$. Conversely if $uxa = vxa \neq 0$, then $ux = vx$ by 0-cancellativity. On the other hand if $uxa = vxa = 0$, then by categoricity at 0, $ux = vx = 0$.

In addition, since $xa \neq 0$, we have $Sxa \cap Sa = Sxa \neq \{0\}$ by (D), so that $xa \,\lambda\, a$. □

We will now prove (M1) - (M6) with $\mathcal{R}'$ and $\leqslant_l$ defined as above.

(M1) Let $\alpha, \beta \in S$. We need $\gamma, \delta \in S$ such that $\gamma \,\mathcal{R}^* \delta \,\mathcal{R}^* \delta\beta = \gamma\alpha$ and $L'_\alpha \wedge L'_\beta = L'_{\gamma\alpha}$.

Case 1: Either $\alpha = \beta = 0$ or $\alpha \not\lambda \beta$. In either case, we have $L'_\alpha \wedge L'_\beta = L'_0$. Therefore, we can take $\gamma = \delta = 0$.

Case 2: α, β non-zero such that $\alpha \,\lambda\, \beta$. This means that $L'_\alpha = L'_\beta$, and so $L'_\alpha \wedge L'_\beta = L'_\alpha$. We know $S\alpha \cap S\beta \neq \{0\}$. Therefore there exists $\gamma, \delta \in S$ such that $\gamma\alpha = \delta\beta \neq 0$. By Lemma 5.1.6, $\gamma\alpha \neq 0$ implies that $\gamma \,\mathcal{R}^*\, \gamma\alpha \,\lambda\, \alpha$, and so $L'_{\gamma\alpha} = L'_\alpha$. Similarly, $\delta\beta \neq 0$ implies that $\delta \,\mathcal{R}^*\, \delta\beta$.

(M2) Let $L'_\alpha \wedge L'_\beta = L'_\gamma$ and let $\delta \in S$.

We need $L'_{\alpha\delta} \wedge L'_{\beta\delta} = L'_{\gamma\delta}$. From Lemma 5.1.5 (ii), we need only consider two cases:

Case 1: $\alpha \,\lambda\, \beta \,\lambda\, \gamma$. By Lemma 5.1.5 (i), λ is a right compatible relation, and therefore $\alpha\delta \,\lambda\, \beta\delta \,\lambda\, \gamma\delta$. By Lemma 5.1.5 (ii), this gives us that $L'_{\alpha\delta} \wedge L'_{\beta\delta} = L'_{\gamma\delta}$.

Case 2: $\alpha \not\lambda \beta$ and $\gamma = 0$. We want to prove that $L'_{\alpha\delta} \wedge L'_{\beta\delta} = L'_0$.

If $\alpha\delta \not\lambda \beta\delta$, then Lemma 5.1.5 (ii), we have $L'_{\alpha\delta} \wedge L'_{\beta\delta} = L'_0$ and we are done.

If $\alpha\delta \,\lambda\, \beta\delta$, then, by definition, either $\alpha\delta = \beta\delta = 0$ or $S\alpha\delta = S\beta\delta \neq \{0\}$.

If $\alpha\delta = \beta\delta = 0$, then $L'_{\alpha\delta} \wedge L'_{\beta\delta} = L'_0 \wedge L'_0 = L'_0$.

If $S\alpha\delta = S\beta\delta \neq \{0\}$, then there exists $u, v \in S$ such that $u\alpha\delta = v\beta\delta \neq 0$. By 0-cancellativity, $u\alpha = v\beta \neq 0$, and so $\alpha \,\lambda\, \beta$, giving a contradiction.

Therefore, in all cases, $L'_{\alpha\delta} \wedge L'_{\beta\delta} = L'_0 = L'_{\gamma\delta}$.

(M3) Let $\alpha, \beta \in S$. If $\alpha\beta = 0$, then by definition $\alpha\beta \leqslant_l \beta$. If $\alpha\beta \neq 0$, then by Lemma 5.1.6, we have $\alpha\beta \,\lambda\, \beta$, and so $\alpha\beta \leqslant_l \beta$ by the definition of $\leqslant_l$.

(M4) $\mathcal{R}^* \subseteq \mathcal{R}^*$.

(M5) Let $\gamma \,\mathcal{R}^*\, \gamma\alpha \,\lambda\, \alpha$, $\delta \,\mathcal{R}^*\, \delta\beta \,\lambda\, \beta$. We want to prove that $\gamma \,\lambda\, \delta$ if and only if $\alpha \,\mathcal{R}^*\, \beta$. Note that 0 is its own $\mathcal{R}^*$-class and its own λ-class, so $\gamma = \delta = 0$ implies that $\gamma\alpha = \delta\beta = 0$ and therefore $\alpha = \beta = 0$. Similarly $\alpha = \beta = 0$ implies that $\gamma = \delta = 0$. Therefore if any $\alpha, \beta, \gamma, \delta, \gamma\alpha, \delta\beta$ are zero we are done. Now assume these are all non-zero.

Firstly, let $\gamma \,\lambda\, \delta$. To show that $\alpha \,\mathcal{R}^*\, \beta$, assume that $x, y \in S^1$ such that $x\alpha = y\alpha$. We want to prove that $x\beta = y\beta$.

Case 1: $x = y = 1$. Then $x\beta = \beta = y\beta$.

Case 2: $x \in S$ and $y = 1$. Then $x\alpha = \alpha$. Since $\gamma \,\lambda\, \delta$ is non-zero, we know that there exists $u, v \in S$ such that $u\gamma = v\delta \neq 0$. We know that $\gamma\alpha$ is non-zero, and so we can use left multiply $x\alpha = \alpha$ by γ to obtain $\gamma x\alpha = \gamma\alpha \neq 0$. By 0-cancellativity, this gives us that $\gamma x = \gamma \neq 0$. We then left multiply by u to obtain $u\gamma x = u\gamma \neq 0$, and so $v\delta x = v\delta \neq 0$. By

0-cancellativity, this gives us that $\delta x = \delta \neq 0$, and so $\delta x\beta = \delta\beta \neq 0$. We then use 0-cancellativity once again to obtain $x\beta = \beta$. The case in which $y \in S$ and $x = 1$ is similar.

Case 3a: $x, y \in S$ and $x\alpha = y\alpha \neq 0$. By 0-cancellativity, $x = y$ and so $x\beta = y\beta$.

Case 3b: $x, y \in S$ and $x\alpha = y\alpha = 0$. We will prove $x\beta = 0$ by contradiction. If $x\beta \neq 0$, then by Lemma 5.1.6, $x\beta\,\lambda\,\beta$. Using the right compatibility of λ, $\gamma\,\lambda\,\delta$ implies that $\gamma\beta\,\lambda\,\delta\beta$, and therefore using (C), $x\beta\,\lambda\,\beta\,\lambda\,\delta\beta\,\lambda\,\gamma\beta$. Since $x\beta \neq 0$, this means that there exists $u, v \in S$ such that $ux\beta = v\gamma\beta \neq 0$. By 0-cancellativity this gives us that $ux = v\gamma \neq 0$ and therefore $ux\alpha = v\gamma\alpha$. This gives us our contradiction, because $v\gamma \neq 0$ and $\gamma\alpha \neq 0$ implies that $v\gamma\alpha \neq 0$ by (A), but since $x\alpha = 0$, we have $ux\alpha = 0$. Therefore $x\beta = 0$. Similarly $y\beta = 0$.

We now consider the converse. Let $\alpha\,\mathcal{R}^*\,\beta$. Then, since $\mathcal{R}^*$ is a left congruence $\gamma\alpha\,\mathcal{R}^*\,\gamma\beta$, and therefore, as $\gamma\alpha \neq 0$, $\gamma\beta \neq 0$. We then apply Lemma 5.1.6 to get $\gamma\beta\,\lambda\,\beta$, which in turn means that $\gamma\beta\,\lambda\,\delta\beta$. Since $\gamma\beta \neq 0$, this means that $S\gamma\beta \cap S\delta\beta \neq \{0\}$, and so there exists $u, v \in S$ such that $u\gamma\beta = v\delta\beta \neq 0$. We use (B) to give us that $u\gamma = v\delta \neq 0$, and therefore $\gamma\,\lambda\,\delta$.

(M6) Let $\alpha\,\lambda\,\gamma\alpha$, $\beta\,\lambda\,\gamma\beta$, $\gamma\alpha = \gamma\beta$. We want $\alpha = \beta$.

Case 1: $\gamma\alpha = \gamma\beta = 0$. Since λ is transitive and $\{0\}$ is a λ-class, this gives us that $\alpha = \beta = 0$.

Case 2: $\gamma\alpha = \gamma\beta \neq 0$. Using 0-cancellativity, we have $\alpha = \beta$.

Therefore, applying Theorem 4.2.1, we see that S has a semigroup of straight left I-quotients, Q, such that $\mathcal{R}^Q \cap (S \times S) = \mathcal{R}'$ and $\leqslant_{\mathcal{L}^Q} \cap\, (S \times S) = \leqslant_l$. We will now prove that Q is a primitive inverse semigroup.

Let e be a non-zero idempotent of Q and let f be an idempotent of Q such that $f \leqslant e$. By Lemma 2.1.3, this means that $f \leqslant_{\mathcal{L}^Q} e$.

Since S is a straight left I-order in Q, we know that S intersects every $\mathcal{L}$-class of Q. Therefore there exists $s, t \in S$ such that $s\,\mathcal{L}^Q\,f$ and $t\,\mathcal{L}^Q\,e$. Therefore $s \leqslant_l t$. By the definition of $\leqslant_l$, this means that either $s = 0$ or $s\,\lambda\,t$. If $s = 0$, then $f = 0$. If $s\,\lambda\,t$, then $s\,\mathcal{L}^Q\,t$, and so $e\,\mathcal{L}^Q\,f$. Since there is a unique idempotent in each $\mathcal{L}$-class of an inverse semigroup, this gives us $e = f$. Therefore e is a primitive idempotent.

Since every non-zero idempotent of Q is primitive, Q is a primitive inverse semigroup.

5.2 Left ample left I-orders in their inverse hulls

In this section, we prove Theorem 3.7 of [10] using part of Theorem 4.2.1 and Corollary 4.4.1. This result gives a necessary and sufficient condition for a left ample semigroup to be a left I-order in its inverse hull.

In Section 2.3, we showed that there is an embedding of a left ample semigroup S into the symmetric inverse semigroup $\mathcal{I}_S$. We take the inverse hull $\Sigma(S)$ of S to be the inverse subsemigroup of $\mathcal{I}_S$ generated by $\operatorname{im}\theta_S$, where θ_S is the embedding of S into the symmetric inverse semigroup $\mathcal{I}_S$ as defined in Section 2.3. Where convenient we identify S with its image under θ_S in $\Sigma(S)$. We begin with the following useful lemma.

Lemma 5.2.1. *Let S be a left ample semigroup and let $a, b \in S$. Then $\rho_a \,\mathcal{R}\, \rho_b$ in $\Sigma(S)$ if and only $a \,\mathcal{R}^*\, b$ in S.*

Proof. Recall that $a \,\mathcal{R}^*\, b$ if and only if $a^+ = b^+$. We have that $\rho_a \,\mathcal{R}^{\Sigma(S)}\, \rho_b$ if and only if $\rho_a\rho_a^{-1} = \rho_b\rho_b^{-1}$ if and only if $\operatorname{dom}\rho_a = \operatorname{dom}\rho_b$. This is true exactly when $Sa^+ = Sb^+$, which is true if and only if $a^+ = b^+$. □

Lemma 5.2.2. *Let S be a left ample semigroup and let $a, b \in S$. Then $\rho_a \leqslant_{\mathcal{L}} \rho_b$ in $\Sigma(S)$ if and only if $Sa \subseteq Sb$.*

Proof. Suppose that $a, b \in S$. Using Lemma 2.2.6, we know that $\rho_a \leqslant_{\mathcal{L}} \rho_b$ in $\Sigma(S)$ if and only if $\rho_a = \rho_a\rho_b^{-1}\rho_b$. We can rewrite this as $\rho_a = \rho_a \operatorname{Id}_{\operatorname{im}\rho_b}$, which is true if and only if $\operatorname{im}\rho_a \subseteq \operatorname{im}\rho_b$. This is equivalent to $Sa \subseteq Sb$. □

We can therefore immediately see the meet structure of the $\mathcal{L}$-classes of $\Sigma(S)$.

Corollary 5.2.3. *Let S be a left ample semigroup. Then, for any $a, b, c \in S$, $L_{\rho_a} \wedge L_{\rho_b} = L_{\rho_c}$ if and only if $Sa \cap Sb = Sc$.*

Proof. Intersection of sets is the meet of set inclusion. □

Lemma 5.2.4 ([10, Lemma 2.4]). *Let S be a left ample semigroup, embedded (as a (2, 1)-algebra) in an inverse semigroup Q. If S is a left I-order in Q, then S is straight.*

Proof. Let $q = a^{-1}b \in Q$ where $a, b \in S$. Then

$$q = (a^+a)^{-1}(b^+b) = a^{-1}a^+b^+b = a^{-1}b^+a^+b = (b^+a)^{-1}(a^+b).$$

We have

$$a^+b\,\mathcal{R}^*\,a^+b^+ = b^+a^+\,\mathcal{R}^*\,b^+a$$

and so $a^+b\,\mathcal{R}^Q\,b^+a$ and S is straight. □

We now give the characterisation of left ample semigroups which are left I-orders in their inverse hulls from [10]. By saying that a semigroup S satisfies the (LC) condition we mean for any $a, b \in S$ there exists $c \in S$ such that $Sa \cap Sb = Sc$.

Theorem 5.2.5 ([10, Theorem 3.7]). *Let S be a left ample semigroup. Then $S\theta_S$ is a left I-order in its inverse hull if and only if S has the (LC) condition.*

Proof. Let S be a left ample semigroup such that $S\theta_S$ is a left I-order in its inverse hull $\Sigma(S)$. By Lemma 5.2.4, we know that $S\theta_S$ is straight in $\Sigma(S)$. By Lemma 2.2.8, we know that the $\mathcal{L}$-classes of $\Sigma(S)$ form a meet semilattice under $\leqslant_{\mathcal{L}}$. Since $S\theta_S$ intersects every $\mathcal{L}$-class of $\Sigma(S)$ by Lemma 3.1.3, this means that for any $a, b \in S$, there exists $c \in S$ such that $L_{\rho_a} \wedge L_{\rho_b} = L_{\rho_c}$ Using Corollary 5.2.3, this is equivalent to $Sa \cap Sb = Sc$. This is the (LC) condition.

Now let S be a left ample semigroup with the (LC) condition. By Corollary 4.4.1, we know that $S\theta_S$ is a straight left I-order in $\Sigma(S)$ if for all $b, c \in S$ there exists $u, v \in S$ such that

$$\rho_u\,\mathcal{R}^{\Sigma(S)}\,\rho_v \text{ and } \rho_b\rho_c^{-1} = \rho_u^{-1}\rho_v.$$

By Lemma 3.3.6, this is true if and only if

$$\rho_u\,\mathcal{R}^{\Sigma(S)}\,\rho_v\,\mathcal{R}^{\Sigma(S)}\,\rho_v\rho_c = \rho_u\rho_b \text{ and } L_{\rho_b} \wedge L_{\rho_c} = L_{\rho_u\rho_b}. \tag{5.1}$$

Note that since θ_S is an embedding, $\rho_v\rho_c = \rho_{vc}$ and $\rho_u\rho_b = \rho_{ub}$.

Let $b, c \in S$. By the (LC) condition there exists $w \in S$ such that

$$Sb \cap Sc = Sw.$$

Therefore there exists $x, y \in S$ such that

$$xb = yc = w.$$

We take $u = xb^+$ and $v = yc^+$. We see that

$$ub = xb^+b = xb = yc = yc^+c = vc. \tag{5.2}$$

Using, the fact that $\mathcal{R}^*$ is a left congruence, we see that $b^+ \mathcal{R}^* b$ implies that $ub^+ \mathcal{R}^* ub$. Therefore

$$u = xb^+ = xb^+b^+ = ub^+ \mathcal{R}^* ub. \tag{5.3}$$

Similarly

$$v = vc^+ \mathcal{R}^* vc. \tag{5.4}$$

Lastly, note that

$$Sb \cap Sc = Sw = Sxb = Sxb^+b = Sub. \tag{5.5}$$

We compare (5.2), (5.3), (5.4) and (5.4) with (5.1). Using Lemma 5.2.1 and Lemma 5.2.3, we see that (5.1) is satisfied. Therefore, by Corollary 4.4.1, we have that S is a left I-order in its inverse hull. $\square$

Theorem 5.2.5 gives a necessary and sufficient condition for a left ample semigroup to be a left I-order in its inverse hull. The question of when a left ample semigroup is a left I-order in other types of inverse semigroup remains an open question, but we cover two additional special cases in this thesis. The first is two-sided ample left I-orders, covered in Section 6.2, in which we consider left I-orders that are both left ample and right ample. The second is left ample left I-orders that intersect every $\mathcal{R}$-class of their semigroup of left I-quotients, covered in Section 7.2.

5.2.1 Exel and Steinberg's result on the inverse hull of a 0-left cancellative semigroup in the context of Ghroda and Gould's result on left ample left I-orders

We will now apply Theorem 5.2.5 to Exel and Steinberg's work on inverse hulls of 0-left cancellative semigroups [5], as an example of when semigroups of I-quotients are used in the literature implicitly. Note that as Theorem 5.2.5 is joint work of Ghroda and Gould, my work is not necessary in order to make this connection.

Exel and Steinberg's work is motivated by the study of certain C*-algebras associated with the inverse hull of 0-left cancellative semigroups. There are various, now standard, methods of constructing C*-algebras from inverse semigroups, such as Exel's tight C*-algebra [6] or Paterson's universal C*-algebra [28]. In [5], Exel and Steinberg consider the tight groupoid of the inverse hull of the path semigroup of a particular kind of directed graph, and they use this as their motivating example to study the inverse hulls of 0-left cancellative semigroups. The path semigroup of a graph has many other useful properties, which they use to get their strongest results; namely 0-categoricity, right reductivity and the existence of right local units.

We will be using the dual statements of the results in [5], in order to continue working with left I-orders instead of the corresponding work on right I-orders.

We start by defining the terms used.

Recall from Section 5.1 that a semigroup S with zero is defined to be categorical at 0 if for all $a, b, c \in S$, $ab \neq 0$ and $bc \neq 0$ implies $abc \neq 0$.

A semigroup S with zero is defined to be *0-right cancellative* if for all $a, b, c \in S$, $ab = ac \neq 0$ implies that $b = c$.

A semigroup S is called *left reductive* if $xs = xt$ for all $x \in S$ implies that $s = t$.

A semigroup S is said to have *left local units* if for every $s \in S$, there exists an idempotent element $e \in E(S)$ such that $es = s$.

We note that if S is a 0-right cancellative semigroup with left local units, then a non-zero element, s, of S has a unique left local unit, since if $es = s$ and $fs = s$

for $e, f \in E(S)$, then $es = fs \neq 0$ and therefore $e = f$.

Definition 5.2.6. If S is a 0-right cancellative semigroup with left local units, then for $s \in S \backslash \{0\}$, we denote by $\bar{s}$ the unique idempotent such that $\bar{s}s = s$.

We remind readers that S satisfies the (LC) condition if for any $a, b \in S$ there exists $c \in S$ such that $Sa \cap Sb = Sc$.

We consider the dual of Theorem 7.22 in [5].

Theorem 5.2.7 ([5, Theorem 7.22]). *Let T be a 0-right cancellative, left reductive semigroup that is categorical at zero, has left local units and satisfies the (LC) condition. Then the non-zero elements of the inverse hull are precisely those elements of the form $\rho_s^{-1}\rho_t$ with $\bar{s} = \bar{t}$.*

Assuming that S embeds into its inverse hull, this describes S as a left I-order. We want to show that this is obtainable from Ghroda and Gould's result on left ample left I-orders, and my re-working, Theorem 5.2.5.

Firstly, we must prove that the semigroup T described in Theorem 5.2.7 (which from now on we shall refer to as T) is left ample. We can do this without using the (LC) condition.

Lemma 5.2.8. *Let S be a 0-right cancellative, left reductive semigroup that is categorical at zero and has left local units. Then S is left ample.*

Proof. We will use Definition 2.3.1 to define a left ample semigroup as a semigroup where every $\mathcal{R}^*$-c1ass contains an idempotent, $E(S)$ is semilattice, and S satisfies the left ample condition.

Every $\mathcal{R}^*$-class contains an idempotent: We need that for every $s \in S$, there exists $s^+ \in E(S)$ such that $s \, \mathcal{R}^* \, s^+$. We claim that we can take

$$s^+ = \begin{cases} \bar{s} & \text{if } s \neq 0, \\ 0 & \text{if } s = 0. \end{cases}$$

Obviously 0 is an idempotent that is $\mathcal{R}^*$-related to 0.

Now let s be non-zero. By definition, $\bar{s}$ is an idempotent. We will prove $s \, \mathcal{R}^* \, \bar{s}$. Let $x, y \in S^1$ such that

$$x\bar{s} = y\bar{s}.$$

We can right multiply s and use $\bar{s}s = s$ to obtain

$$xs = ys.$$

On the other hand, let $x, y \in S^1$ such that

$$xs = ys.$$

Using $\bar{s}s = s$, we can rewrite this as

$$x\bar{s}s = y\bar{s}s.$$

If this is non-zero, we can use 0-right cancellativity to obtain $x\bar{s} = y\bar{s}$. Otherwise, we have

$$x\bar{s}s = y\bar{s}s = 0.$$

Since s is non-zero, we have $\bar{s}s = s \neq 0$. Using categoricity at 0, we therefore obtain

$$x\bar{s} = 0 = y\bar{s}.$$

Therefore, for all $x, y \in S^1$, $xs = ys$ if and only if $x\bar{s} = y\bar{s}$. By definition, this means $s\,\mathcal{R}^*\,\bar{s}$.

Left ample condition: We need that for every $a \in S$ and $e \in E(S)$,

$$(ae)^+a = ae.$$

If $ae = 0$, this is obvious. If $ae \neq 0$, then $(ae)^+ = \overline{ae}$. Therefore

$$\overline{ae}ae = ae = aee.$$

Since this is non-zero, we can use 0-right cancellativity to give us

$$\overline{ae}a = ae.$$

E(S) is a semilattice: We will prove this in two parts. These can be found in Proposition 3.13 and Proposition 3.15 of [5].

(i) Let $s \in S$ and $e \in E(S)$. Then $se \neq 0$ implies that $se = s$.

Proof. If $se \neq 0$, then $se = see \neq 0$. We can then use 0-right cancellativity to obtain $s = se$. $\square$

(ii) Let $e, f \in E(S)$. Then $e \neq f$ implies that $ef = 0$.

Proof. We prove the contrapositive. Let $ef \neq 0$. By (i), we have $ef = e$. We right multiply by e to obtain

$$efe = e^2 = e.$$

Since ef is non-zero we know that e is non-zero. Therefore efe is non-zero. Therefore fe is non-zero. By (i), this gives us that $fe = f$.

Since $ef = e$, we have that $Se \subseteq Sf$. Similarly, $fe = f$ implies that $Sf \subseteq Se$. Together this gives us that $Se = Sf$.

We will now prove that $xe = xf$ for all $x \in S$.

Let $x \in S$. Either $x \in Se = Sf$ or $x \notin Se = Sf$. If $x \in Se = Sf$, then Lemma 2.1.1 gives us $xe = x = xf$. If $x \notin Se = Sf$, then Lemma 2.1.1 gives us $xe \neq x$ and $xf \neq x$. We apply (i) to obtain $xe = 0 = xf$.

Thus $xe = xf$ for all $x \in S$. We apply left reductivity to obtain $e = f$. $\square$

From (ii), we easily get commutativity of idempotents. Let $e, f \in E(S)$. If $e = f$, then $ef = e^2 = fe$. If $e \neq f$, then $ef = 0 = fe$, by (ii). $\square$

Since we see that T is a left ample semigroup with the (LC) condition, we should be able to apply Theorem 5.2.5 directly. However, due to the broad use of the term ‘inverse hull’, it is perhaps prudent to check that the definition of inverse hull of T is the same in each paper first.

Ghroda and Gould’s [10] definition of the inverse hull of a left ample semigroup is defined in Section 2.3, but copied here for convenience. We define

$$\rho_a : Ta^+ \to Ta,\ s \mapsto sa.$$

In [10], Ghroda and Gould define the inverse hull of a left ample semigroup as the inverse subsemigroup of $\mathcal{I}_T$ generated by $T\rho$, where

$$\rho : T \to \mathcal{I}_T,\ a \mapsto \rho_a.$$

Clearly ρ_a maps zero to zero. Since T is categorical at zero, ρ_a maps only zero to zero, as we now show. Let $sa^+ \in Ta^+$ be non-zero and suppose $(sa^+)\rho_a = 0$. That is,

$$sa^+a = 0.$$

Since sa^+ is non-zero, a^+ is non-zero. By our definition of $^+$, a is non-zero and $a^+a = \bar{a}a = a$ is non-zero. Since sa^+ is non-zero and a^+a is non-zero, by 0-categoricity, sa^+a is non-zero leading to a contradiction.

Since ρ_a maps only zero to zero, we can consider ρ_a as $\rho_a : Ta^+\backslash\{0\} \to Ta\backslash\{0\}$, without changing the structure of the inverse hull. Note that $Ta^+\backslash\{0\} = T\bar{a}\backslash\{0\}$.

We now consider the definition of the inverse hull in [5]. As before, we use the dual definitions. Let

$$F_a = \{x \in T \mid xa \neq 0\}$$

and

$$E_a = Ta\backslash\{0\}.$$

We define

$$\theta_a : F_a \to E_a,\ s \mapsto sa.$$

In [5], Exel and Steinberg define the inverse hull of a 0-right cancellative semigroup as the inverse subsemigroup of $\mathcal{I}_T$ generated by $T\theta$, where

$$\theta : T \to \mathcal{I}_T,\ a \mapsto \theta_a.$$

We will prove that these two definitions are equal by showing that $\theta_a = \rho_a$ for all $a \in S$. We start with their domains.

Lemma 5.2.9. *Let T be 0-right cancellative, categorical at 0, and have left local units. Then $x \in F_a$ if and only if $x \in T\bar{a}\backslash\{0\}$.*

Proof. Let $x \in F_a$. That is let $x \in T$ such that $xa \neq 0$. Therefore

$$x\bar{a}a = xa \neq 0.$$

We use 0-right cancellativity, to obtain

$$x\bar{a} = x \neq 0.$$

Therefore $x \in T\bar{a}\backslash\{0\}$.

Conversely, let $x \in T\bar{a}\backslash\{0\}$. Therefore,

$$x = s\bar{a} \neq 0,$$

for some $s \in T$. Therefore

$$xa = s\bar{a}a$$

We know that $s\bar{a} \neq 0$ and $\bar{a}a = a \neq 0$. Therefore, by 0-categoricity, $xa = s\bar{a}a \neq 0$, and so $x \in F_a$. $\square$

Given that the domains of θ_a and ρ_a are equal and the mapping is the same, this gives us $\theta_a = \rho_a$ for all $a \in S$.

We can now apply Theorem 5.2.5 to T to obtain Theorem 5.2.7. Note that the fact that s and t can be chosen such that $\bar{s} = \bar{t}$ is exactly due to the fact that left ample left I-orders are always straight (Lemma 5.2.4).

Chapter 6

Right ample straight left I-orders

In this chapter, we characterise right ample and two-sided ample left I-orders. Note that right ample left I-orders are in no way dual to left ample left I-orders, as the dual of left ample left I-orders is right ample right I-orders.

Section 6.1 is devoted to the proof of Theorem 6.1.3, which gives necessary and sufficient conditions for a right ample semigroup to be a straight left I-order embedded into its semigroup of straight left I-quotients as a (2,1)-algebra.

In Section 6.2, we use Theorem 6.1.3 to prove Corollary 6.2.1, which gives a necessary and sufficient condition for an ample semigroup to be a left I-order embedded into its semigroup of left I-quotients as a (2,1,1)-algebra.

6.1 Right ample straight left I-orders - the general case

This section is dedicated to the proof of Theorem 6.1.3, which gives necessary and sufficient conditions for a right ample semigroup to be a straight left I-order embedded into its semigroup of straight left I-quotients as a (2,1)-algebra. We begin with the following useful lemma, which is the dual of Lemma 2.3.3.

Lemma 6.1.1. *Let S be a right ample semigroup. Then for all $a, b, x \in S$:*

(i) $bb^* = b$;

(ii) $(ab)^* = (a^*b)^*$;

(iii) $x(bx)^* = b^*x$; *and*

(iv) $b^* = (ab)^*$ *if and only if* $b = a^*b$.

In order to apply Theorem 4.2.1, we must find $\leqslant_{\mathcal{L}^Q}$ and the associated meet.

Lemma 6.1.2. *Let S be a right ample semigroup embedded as a unary semigroup into an inverse semigroup Q. Then for all $a, b, c, x \in S$,*

(i) $a \leqslant_{\mathcal{L}^Q} b$ *if and only if* $a^* = a^*b^*$ *if and only if* $a \leqslant_{\mathcal{L}^*} b$;

(ii) $L_a \wedge L_b = L_c$ *if and only if* $c^* = a^*b^*$; *and*

(iii) $L_a \wedge L_b = L_{xa}$ *if and only if* $ab^* = x^*a$.

Proof. Since S is embedded in Q in such a way that * is preserved, we have that for all $a \in S$, $a^* = a^{-1}a$.

(i) Since $a \,\mathcal{L}^Q\, a^*$ for all $a \in S$, we have that $a \leqslant_{\mathcal{L}^Q} b$ if and only if $a^* \leqslant_{\mathcal{L}^Q} b^*$. By Lemma 2.1.5, $a^* \leqslant_{\mathcal{L}^Q} b^*$ if and only if $a^* \leqslant_{\mathcal{L}^*} b^*$, which is equivalent to $a \leqslant_{\mathcal{L}^*} b$, since $a \,\mathcal{L}^*\, a^*$ for all $a \in S$. By Lemma 2.1.3, $a^* \leqslant_{\mathcal{L}^Q} b^*$ if and only if $a^* \leqslant b^*$, which is equivalent to $a^* = a^*b^*$, since idempotents commute in a right ample semigroup.

(ii) Lemma 2.2.8 gives the result.

(iii) Let $L_a \wedge L_b = L_{xa}$. Therefore, by (ii), we have $a^*b^* = (xa)^*$. Then, using Lemma 6.1.1 (i) and Lemma 6.1.1 (iii),

$$ab^* = aa^*b^* = a(xa)^* = x^*a.$$

On the other hand let $ab^* = x^*a$. Then, using Lemma 6.1.1 (ii),

$$(xa)^* = (x^*a)^* = (ab^*)^* = (a^*b^*)^* = a^*b^*,$$

and so by (ii), we have $L_a \wedge L_b = L_{xa}$.

□

We now introduce the main theorem of this section.

Theorem 6.1.3 (Right Ample Straight Left I-Orders)**.** *Let S be a right ample semigroup and let $\mathcal{R}'$ be a binary relation on S. Then S has a semigroup of straight left I-quotients, Q, such that S is embedded in Q as a unary semigroup and $\mathcal{R}^Q \cap (S \times S) = \mathcal{R}'$ if and only if $\mathcal{R}'$ is a left compatible equivalence relation such that S satisfies Conditions (A1) - (A3).*

(A1) For all $\alpha, \beta \in S$, there exists $\gamma, \delta \in S$ such that $\gamma\, \mathcal{R}'\, \delta\, \mathcal{R}'\, \delta\beta = \gamma\alpha$ and $\alpha\beta^* = \gamma^*\alpha$.

(A2) For all $\alpha, \beta, \gamma \in S$, $\gamma\alpha\, \mathcal{R}'\, \gamma\beta$ implies that $\gamma^*\alpha\, \mathcal{R}'\, \gamma^*\beta$.

(A3) $\mathcal{R}' \subseteq \mathcal{R}^*$.

Proof. We consider the forward implication first. Let S be a right ample straight left I-order in Q, such that S is embedded in Q as a unary semigroup, and let $\mathcal{R}' = \mathcal{R}^Q \cap (S \times S)$. We know that $\mathcal{R}'$ is a left congruence and that therefore $\gamma\alpha\, \mathcal{R}'\, \gamma\beta$ implies that $\gamma^{-1}\gamma\alpha\, \mathcal{R}'\, \gamma^{-1}\gamma\beta$, so (A2) is satisfied. Using Theorem 4.2.1, we know that (M1) and (M4) are satisfied, which are exactly (A1) and (A3) respectively, using Lemma 6.1.2 (iii).

We will prove the converse by proving each property in Theorem 4.2.1 with $\leqslant_l = \leqslant_{\mathcal{L}^*}$. Using the fact that $a\, \mathcal{L}^*\, a^*$ for all $a \in S$, along with Lemma 2.1.3 and Lemma 2.1.5, this means that $a \leqslant_l b$ if and only if $a^*b^* = a^*$, for $a, b, \in S$. Note that $\mathcal{L}' = \mathcal{L}^*$. We already know that $\leqslant_{\mathcal{L}^*}$ is a right compatible preorder.

By Lemma 2.2.9, we know that $L^*_{a^*} \wedge L^*_{b^*} = L^*_{a^*b^*}$. Therefore, using the fact that there is a unique idempotent in each $\mathcal{L}^*$-class, we have that

$$L^*_a \wedge L^*_b = L^*_c \text{ if and only if } c^* = a^*b^*.$$

We will now prove (M1) - (M6) with $\leqslant_l = \leqslant_{\mathcal{L}^*}$ in order to satisfy the conditions of Theorem 4.2.1.

(M1) Let $\alpha, \beta \in S$. Applying Property (A1), there exists $\gamma, \delta \in S$ such that $\gamma\, \mathcal{R}'\, \delta\, \mathcal{R}'\, \delta\beta = \gamma\alpha$ and $\alpha\beta^* = \gamma^*\alpha$. We can use Lemma 6.1.1 (ii), along with $\alpha\beta^* = \gamma^*\alpha$ to obtain

$$(\gamma\alpha)^* = (\gamma^*\alpha)^* = (\alpha\beta^*)^* = (\alpha^*\beta^*)^* = \alpha^*\beta^*.$$

Therefore $L^*_\alpha \wedge L^*_\beta = L^*_{\gamma\alpha}$.

(M2) Let $L^*_\alpha \wedge L^*_\beta = L^*_\gamma$. Then $\gamma^* = \alpha^*\beta^*$. Also, let $\delta \in S$. We use Lemma 6.1.1 (iii) twice, to get

$$\delta(\alpha\delta)^*(\beta\delta)^* = \alpha^*\delta(\beta\delta)^* = \alpha^*\beta^*\delta.$$

Therefore, using Lemma 6.1.1 (ii),

$$(\alpha^*\beta^*\delta)^* = (\delta(\alpha\delta)^*(\beta\delta)^*)^* = (\delta^*(\alpha\delta)^*(\beta\delta)^*)^* = \delta^*(\alpha\delta)^*(\beta\delta)^*. \tag{6.1}$$

Also, since $\alpha\delta \leqslant_l \delta$ which we will prove shortly in (M3), we have that

$$\delta^*(\alpha\delta)^* = (\alpha\delta)^*. \tag{6.2}$$

Lastly, using Lemma 6.1.1 (ii),

$$(\gamma\delta)^* = (\gamma^*\delta)^* = (\alpha^*\beta^*\delta)^*. \tag{6.3}$$

Putting all this together,

$$(\gamma\delta)^* \overset{(6.3)}{=} (\alpha^*\beta^*\delta)^* \overset{(6.1)}{=} \delta^*(\alpha\delta)^*(\beta\delta)^* \overset{(6.2)}{=} (\alpha\delta)^*(\beta\delta)^*,$$

which gives us that $L^*_{\alpha\delta} \wedge L^*_{\beta\delta} = L^*_{\gamma\delta}$.

(M3) By definition, we know that $\alpha\beta \leqslant_{\mathcal{L}^*} \beta$.

(M4) This is Property (A3).

(M5) Let $\gamma\,\mathcal{R}'\,\gamma\alpha\,\mathcal{L}^*\,\alpha$ and let $\delta\,\mathcal{R}'\,\delta\beta\,\mathcal{L}^*\,\beta$. We have that $\gamma\gamma^* = \gamma\,\mathcal{R}'\,\gamma\alpha$. Therefore we can use Property (A2) to obtain $\gamma^* = \gamma^*\gamma^*\,\mathcal{R}'\,\gamma^*\alpha$. We also have that $\gamma\alpha\,\mathcal{L}^*\,\alpha$, and so $(\gamma\alpha)^* = \alpha^*$. By Lemma 6.1.1 (iv) this is equivalent to $\alpha = \gamma^*\alpha$. Similarly $\delta^*\,\mathcal{R}'\,\delta^*\beta$ and $\beta = \delta^*\beta$.

Let $\gamma\,\mathcal{L}^*\,\delta$, and so $\gamma^* = \delta^*$. Therefore $\alpha = \gamma^*\alpha\,\mathcal{R}'\,\gamma^* = \delta^*\,\mathcal{R}'\,\delta^*\beta = \beta$.

Conversely, let $\alpha\,\mathcal{R}'\,\beta$. We see that $\gamma^*\,\mathcal{R}'\,\gamma^*\alpha = \alpha\,\mathcal{R}'\,\beta = \delta^*\beta\,\mathcal{R}'\,\delta^*$, and therefore using (A3), $\gamma^*\,\mathcal{R}^*\,\delta^*$. We know from Lemma 2.1.4, that since $E(S)$ is a semilattice, there can only be one idempotent in each $\mathcal{R}^*$-class, and so $\gamma^* = \delta^*$.

(M6) Let $\alpha \,\mathcal{L}^*\, \beta \,\mathcal{L}^*\, \gamma\alpha \,\mathcal{L}^*\, \gamma\beta$ and let $\gamma\alpha = \gamma\beta$. We have that

$$\alpha^* = \beta^* = (\gamma\alpha)^* = (\gamma\beta)^*,$$

and so we can use Lemma 6.1.1 (iv) to give us that $\alpha = \gamma^*\alpha$ and $\beta = \gamma^*\beta$. We then use the fact that $\gamma \,\mathcal{L}^*\, \gamma^*$, to give us that $\gamma\alpha = \gamma\beta$ implies that $\gamma^*\alpha = \gamma^*\beta$, and therefore $\alpha = \beta$.

Therefore, S with $\leqslant_l = \leqslant_{\mathcal{L}^*}$ satisfies the conditions of Theorem 4.2.1 and we can apply Theorem 4.2.1 to give us that S has a semigroup of straight left I-quotients, Q, such that $\mathcal{R}^Q \cap (S \times S) = \mathcal{R}'$ and $\leqslant_{\mathcal{L}^Q} \cap (S \times S) = \leqslant_{\mathcal{L}^*}$. Therefore $\mathcal{L}^Q \cap (S \times S) = \mathcal{L}^*$.

Let $a, b \in S$. We have that $a^* = b^*$ if and only if $a \,\mathcal{L}^*\, b$, which we see is true exactly when $a^{-1}a = b^{-1}b$ in Q. Therefore * is preserved. That is, S is embedded in Q as a (2,1)-algebra. $\square$

6.2 Two-sided ample left I-orders

Now we consider the two-sided ample case, where S is both right ample and left ample. If S is embedded in Q such that $^+$ and * are preserved, then by Lemma 2.3.4 and its dual, we have that $\mathcal{R}^Q \cap (S \times S) = \mathcal{R}^*$ and $\mathcal{L}^Q \cap (S \times S) = \mathcal{L}^*$. That is, $a \,\mathcal{R}^Q\, b$ if and only if $a^+ = b^+$ and $a \,\mathcal{L}^Q\, b$ if and only if $a^* = b^*$, for $a, b \in S$.

Corollary 6.2.1 (Two-sided ample left I-orders)**.** *Let S be a two-sided ample semigroup. Then S has a semigroup of left I-quotients such that $^+$ and * are preserved if and only if for all $b, c \in S$, there exists $u, v \in S$ such that*

$$ub = vc,\ u^+ = v^+ = (vc)^+,\ bc^* = u^*b. \tag{$\star$}$$

Note that in this case we get, perhaps, the best possible result, in that the Ore condition is sufficient to give us our result.

Proof. We first consider the forward implication. Let S be a two-sided ample semigroup with a semigroup of left I-quotients, Q, such that $^+$ and * are preserved. We know that $a \,\mathcal{R}^Q\, b$ if and only if $a^+ = b^+$ and $a \,\mathcal{L}^Q\, b$ if and only if $a^* = b^*$. By Lemma 5.2.4, we know that S is straight in Q. Therefore, by Theorem 6.1.3, Property (A1) is satisfied. Therefore ($\star$) is satisfied.

For the backward implication, we aim to apply Theorem 6.1.3 with $\mathcal{R}' = \mathcal{R}^*$. That is, $a \, \mathcal{R}' \, b$ if and only if $a^+ = b^+$. Note that this is a left congruence. We now prove Properties (A1) - (A3).

(A1) Satisfied by ($\star$).

(A2) Let $xa \, \mathcal{R}^* \, xb$. This means

$$(xa)^+ = (xb)^+.$$

We apply Lemma 2.3.3 (ii) to get

$$(xa^+)^+ = (xb^+)^+.$$

Right multiplying this by x gives us

$$(xa^+)^+x = (xb^+)^+x.$$

We then apply the left ample property to give us

$$xa^+ = xb^+.$$

By the definition of $\mathcal{L}^*$, we know that $x \, \mathcal{L}^* \, x^*$. Therefore, the above equation implies that

$$x^*a^+ = x^*b^+.$$

Therefore, applying $^+$ to both sides, we have

$$(x^*a^+)^+ = (x^*b^+)^+.$$

Then, Lemma 2.3.3 (ii) gives us

$$(x^*a)^+ = (x^*b)^+,$$

and so $x^*a \, \mathcal{R}^* \, x^*b$.

(A3) $\mathcal{R}' = \mathcal{R}^*$.

Therefore, Theorem 6.1.3 gives us that S has a straight left I-order, Q, such that * is preserved and $\mathcal{R}^Q \cap (S \times S) = \mathcal{R}^*$. Therefore, by Lemma 2.3.4, $^+$ is also preserved. $\square$

Chapter 7

Straight left I-orders that intersect every $\mathcal{R}$-class

In this chapter, we characterise straight left I-orders, S, with semigroups of straight left I-quotients, Q, such that S intersects every $\mathcal{R}$-class of Q.

We know that every straight left I-order, S, intersects every $\mathcal{L}$-class of any semigroup of straight left I-quotients, Q. By insisting that S also intersects every $\mathcal{R}$-class of Q, we are asking that S and Q have an even stronger relationship. This can lead to new results. We have already witnessed some examples of this in the preceding chapter, as a right ample straight left I-order, S, embedded as a unary in its semigroup of straight left I-quotients, Q, intersects every $\mathcal{R}$-class of Q.

Section 7.1 is devoted to the proof of Theorem 7.1.3, which gives necessary and sufficient conditions for a semigroup, S, to have a semigroup of straight left I-quotients, Q, such that S intersects every $\mathcal{R}$-class of Q.

In Section 7.2, we use Theorem 7.1.3 to prove Proposition 7.2.1, which gives necessary and sufficient conditions for a left ample semigroup, S, to have a semigroup of left I-quotients, Q, such that S intersects every $\mathcal{R}$-class of Q.

In Section 7.3, we use Theorem 7.1.3 to prove Proposition 7.3.1, which states that if a semigroup S is both a straight left I-order and a straight right I-order, then its semigroups of I-quotients are isomorphic if and only if their $\mathcal{R}$ and $\mathcal{L}$ relations restricted to S are equal.

7.1 Straight left I-orders that intersect every $\mathcal{R}$-class - the general case

This section is dedicated to the proof of Theorem 7.1.3, which gives necessary and sufficient conditions for a semigroup, S, to have a semigroups of straight left I-quotients, Q, such that S intersects every $\mathcal{R}$-class of Q. We will start with an important property of straight left I-orders that intersect every $\mathcal{R}$-class of their semigroups of straight left I-quotients.

Lemma 7.1.1. *Let S have a semigroup of straight left I-quotients, Q. Then S intersects every $\mathcal{R}$-class of Q if and only if for all $x \in S$, there exists $a \in S$ such that $x \,\mathcal{R}^Q\, xa \,\mathcal{L}^Q\, a$.*

Proof. Let S be a subsemigroup of an inverse semigroup Q such that S intersects every $\mathcal{R}$-class of Q. Let $x \in S$. Since Q is an inverse semigroup, we have that $x^{-1} \in Q$. Since S intersects every $\mathcal{R}$-class of Q, there exists an $a \in S$ such that $a \,\mathcal{R}^Q\, x^{-1}$. By Lemma 3.3.3, we have $x \,\mathcal{R}^Q\, xa \,\mathcal{L}^Q\, a$.

Now let S have a semigroup of straight left I-quotients, Q, such that for all $x \in S$, there exists $a \in S$ such that $x \,\mathcal{R}^Q\, xa \,\mathcal{L}^Q\, a$. Let $q \in Q$. Since Q is a semigroup of straight left I-quotients, there exists $x, y \in S$ such that $q = x^{-1}y$, with $x \,\mathcal{R}^Q\, y$. We have that

$$qq^{-1} = x^{-1}yy^{-1}x = x^{-1}x.$$

By our assumption, for every $x \in S$, there exists $a \in S$ such that $x \,\mathcal{R}^Q\, xa \,\mathcal{L}^Q\, a$. By Lemma 3.3.3, this means that $a \,\mathcal{R}^Q\, x^{-1}$. Therefore, there exists an $a \in S$ such that $a \,\mathcal{R}^Q\, x^{-1} \,\mathcal{R}^Q\, q$. Therefore, S intersects every $\mathcal{R}$-class of Q. □

We will be using Theorem 4.3.5 in this section, so we need to understand the $\mathcal{U}$ relation from Section 4.3 in this context. Note that although we will be using the $\mathcal{U}$ relation as a tool in the proof of Theorem 7.1.3, it will not appear explicitly in the statement of Theorem 7.1.3. We remind the reader that $\mathcal{U}$ is a ternary relation on an inverse semigroup, Q, which can be defined as

$$(b, c, u) \in \mathcal{U}^Q \text{ if and only if } u^{-1}u = bc^{-1}cb^{-1}.$$

Given that S is a subsemigroup of inverse semigroup, Q, with S intersecting every $\mathcal{R}^Q$-class, we can give $\mathcal{U}^Q$ restricted to S in terms of $\mathcal{R}^Q$ and $\mathcal{L}^Q$.

Lemma 7.1.2. *Let S be a subsemigroup of inverse semigroup, Q, such that S intersects every $\mathcal{R}$-class of Q and let $b, c, u \in S$. Then $(b, c, u) \in \mathcal{U}^Q$ if and only if there exists $m \in S$ such that*

$$c \,\mathcal{R}^Q\, cm \,\mathcal{L}^Q\, m \;\; \textit{and} \;\; u \,\mathcal{R}^Q\, ubm \,\mathcal{L}^Q\, bm.$$

In this case we will say that m witnesses $(b, c, u) \in \mathcal{U}^Q$.

Proof. Let $(b, c, u) \in \mathcal{U}^Q$. Since $c \in S$, we have $c^{-1} \in Q$. Therefore, using the fact that S intersects every $\mathcal{R}$-class of Q, we know that there exists an $m \in S$ such that $m \,\mathcal{R}^Q\, c^{-1}$, so that $mm^{-1} = c^{-1}c$. By Lemma 3.3.3, this gives us that $c \,\mathcal{R}^Q\, cm \,\mathcal{L}^Q\, m$. By applying $(b, c, u) \in \mathcal{U}^Q$, we get

$$u^{-1}u = bc^{-1}cb^{-1} = bmm^{-1}b^{-1} = (bm)(bm)^{-1}.$$

Therefore $bm \,\mathcal{R}^Q\, u^{-1}$, and we can apply Lemma 3.3.3 again, to obtain $u \,\mathcal{R}^Q\, ubm \,\mathcal{L}^Q\, bm$.

Conversely let

$$c \,\mathcal{R}^Q\, cm \,\mathcal{L}^Q\, m \;\text{ and }\; u \,\mathcal{R}^Q\, ubm \,\mathcal{L}^Q\, bm.$$

We apply Lemma 3.3.3 to both expressions to obtain $m \,\mathcal{R}^Q\, c^{-1}$ and $bm \,\mathcal{R}^Q\, u^{-1}$. Therefore

$$u^{-1}u = (bm)(bm)^{-1} = bmm^{-1}b^{-1} = bc^{-1}cb^{-1},$$

and so $(b, c, u) \in \mathcal{U}^Q$. □

We now introduce the main theorem of this chapter.

Theorem 7.1.3 (Straight Left I-Orders that Intersect every $\mathcal{R}$-class)**.** *Let S be a semigroup and let $\mathcal{R}'$ and $\mathcal{L}'$ be binary relations on S. Then S is a straight left I-order in an inverse semigroup Q with $\mathcal{R}^Q \cap (S \times S) = \mathcal{R}'$ and $\mathcal{L}^Q \cap (S \times S) = \mathcal{L}'$ such that S intersects every $\mathcal{R}^Q$-class if and only if $\mathcal{R}'$ is a left congruence, $\mathcal{L}'$ is a right congruence, and S satisfies the Conditions (R1) - (R6).*

(R1) For all $\alpha, \beta \in S$, there exists $\gamma, \delta, m \in S$ such that

$$\gamma\,\mathcal{R}'\,\delta\,\mathcal{R}'\,\delta\beta = \gamma\alpha,\ \beta\,\mathcal{R}'\,\beta m\,\mathcal{L}'\,m \text{ and } \gamma\,\mathcal{R}'\,\gamma\alpha m\,\mathcal{L}'\,\alpha m.$$

(R2) For all $\alpha \in S$, there exists $\gamma \in S$ such that $\gamma\,\mathcal{R}'\,\gamma\alpha\,\mathcal{L}'\,\alpha$.

(R3)(r) For all $\alpha, \beta, \gamma \in S$, $\alpha\,\mathcal{R}'\,\alpha\beta\gamma$ implies that $\alpha\,\mathcal{R}'\,\alpha\beta$.

(R3)(l) For all $\alpha, \beta, \gamma \in S$, $\alpha\beta\gamma\,\mathcal{L}'\,\gamma$ implies that $\beta\gamma\,\mathcal{L}'\,\gamma$.

(R4) $\mathcal{R}' \subseteq \mathcal{R}^*$

(R5) Let $\alpha, \beta, \gamma, \delta \in S$ such that $\gamma\,\mathcal{R}'\,\gamma\alpha\,\mathcal{L}'\alpha$ and $\delta\,\mathcal{R}'\,\delta\beta\,\mathcal{L}'\beta$. Then $\gamma\,\mathcal{L}'\,\delta$ if and only if $\alpha\,\mathcal{R}'\,\beta$.

(R6) For all $\alpha, \beta, \gamma \in S$, $\alpha\,\mathcal{L}'\,\beta\,\mathcal{L}'\,\gamma\alpha = \gamma\beta$ implies that $\alpha = \beta$.

Proof. We begin with the forward direction. Let S be a straight left I-order in an inverse semigroup Q that intersects every $\mathcal{R}^Q$-class and let $\mathcal{R}' = \mathcal{R}^Q \cap (S \times S)$ and $\mathcal{L}' = \mathcal{L}^Q \cap (S \times S)$. Obviously $\mathcal{R}'$ is a left congruence and $\mathcal{L}'$ is a right congruence. Now we prove (R1) - (R6). Note that, by Theorem 4.3.5, we know that Properties (U1) - (U11) are satisfied with $\mathcal{U}' = \mathcal{U}^Q \cap (S \times S \times S)$.

(R1) Using Lemma 7.1.2, this is (U1).

(R2) Applying (U1) to α twice, there exists $\gamma \in S$ such that $(\alpha, \alpha, \gamma) \in \mathcal{U}'$. By (U2), this means that $\gamma\,\mathcal{R}'\,\gamma\alpha\,\mathcal{L}'\,\alpha$.

(R3)(r) By definition of $\leqslant_{\mathcal{R}^Q}$, we know that

$$\alpha\beta\gamma \leqslant_{\mathcal{R}^Q} \alpha\beta \leqslant_{\mathcal{R}^Q} \alpha.$$

Since $\mathcal{R}^Q$ is the equivalence relation associated with $\leqslant_{\mathcal{R}^Q}$, we can use the anti-symmetric property of $\leqslant_{\mathcal{R}}$ on $\mathcal{R}$-classes to give us that $\alpha\,\mathcal{R}^Q\,\alpha\beta\gamma$ implies that $\alpha\,\mathcal{R}^Q\,\alpha\beta$.

(R3)(l) Dual of (R3)(r)

(R4) (U9)

(R5) (U10)

(R6) (U11)

We now consider the converse. Let S be a semigroup and let $\mathcal{R}'$ be a left congruence on S and $\mathcal{L}'$ be a right congruence on S such that S satisfies Conditions (R1) - (R6). Note the special application of (R5), in which $\gamma = \delta$ or $\alpha = \beta$, which will be used.

We define $\mathcal{U}'$ to be the ternary relation on S given by $(b, c, u) \in \mathcal{U}'$ if and only if there exists $m \in S$ such that

$$c\,\mathcal{R}'\,cm\,\mathcal{L}'\,m \text{ and } u\,\mathcal{R}'\,ubm\,\mathcal{L}'\,bm.$$

Our aim is to apply Theorem 4.3.5. The following lemma will give us some useful shortcuts for this proof.

Lemma 7.1.4. *Let S be a semigroup with a left congruence $\mathcal{R}'$, and a right congruence $\mathcal{L}'$, such that S satisfies (R1) - (R6). Then, for all $a, b, x, y \in S$,*

(i) $x\,\mathcal{R}'\,xa\,\mathcal{L}'\,a$ *and* $a\,\mathcal{R}'\,b$ *implies that* $x\,\mathcal{R}'\,xb\,\mathcal{L}'\,b$*;*

(ii) $x\,\mathcal{R}'\,xa\,\mathcal{L}'\,a$ *and* $x\,\mathcal{L}'\,y$ *implies that* $y\,\mathcal{R}'\,ya\,\mathcal{L}'\,a$*;*

(iii) $a\,\mathcal{R}'\,b$ *and* $xa\,\mathcal{L}'\,ya$ *implies that* $xb\,\mathcal{L}'\,yb$*.*

Proof. (i) Let $a, b, x, y \in S$ such that $x\,\mathcal{R}'\,xa\,\mathcal{L}'\,a$ and $a\,\mathcal{R}'\,b$. Using the fact that $\mathcal{R}'$ is a left congruence, we know that $a\,\mathcal{R}'\,b$ implies that $xa\,\mathcal{R}'\,xb$. Therefore $x\,\mathcal{R}'\,xa\,\mathcal{R}'\,xb$.

By (R2), there exists $y \in S$ such that $y\,\mathcal{R}'\,yb\,\mathcal{L}'\,b$. By (R5), we know that $a\,\mathcal{R}'\,b$ implies that $x\,\mathcal{L}'\,y$. Using the fact that $\mathcal{L}'$ is a right congruence, we know that $x\,\mathcal{L}'\,y$ implies that $xb\,\mathcal{L}'\,yb$. Therefore $xb\,\mathcal{L}'\,yb\,\mathcal{L}'\,b$.

Putting these two together, we have $x\,\mathcal{R}'\,xb\,\mathcal{L}'\,b$.

(ii) Let $a, x, y \in S$ such that $x\,\mathcal{R}'\,xa\,\mathcal{L}'\,a$ and $x\,\mathcal{L}'\,y$. Using the fact that $\mathcal{L}'$ is a right congruence, we know that $x\,\mathcal{L}'\,y$ implies that $xa\,\mathcal{L}'\,ya$. Therefore $ya\,\mathcal{L}'\,xa\,\mathcal{L}'\,a$.

Implicitly, by Property (R1) with $\beta = y$, there exists $b \in S$ such that $y\,\mathcal{R}'\,yb\,\mathcal{L}'\,b$. By (R5), we know that $x\,\mathcal{L}'\,y$ implies that $a\,\mathcal{R}'\,b$. Using the fact that $\mathcal{R}'$ is a left congruence, we know that $a\,\mathcal{R}'\,b$ implies that $ya\,\mathcal{R}'\,yb$. Therefore $y\,\mathcal{R}'\,yb\,\mathcal{R}'\,ya$.

Putting these two together, we have $y\,\mathcal{R}'\,ya\,\mathcal{L}'\,a$.

(iii) Let $a, b, x, y \in S$ such that $a\,\mathcal{R}'\,b$ and $xa\,\mathcal{L}'\,ya$. Applying (R1) to xa and ya, there exists $w, z, m \in S$ such that

$$w\,\mathcal{R}'z\,\mathcal{R}'zya = wxa,\ ya\,\mathcal{R}'\,yam\,\mathcal{L}'\,m \text{ and } w\,\mathcal{R}'\,wxam\,\mathcal{L}'\,xam.$$

Using $ya\,\mathcal{L}'\,xa$, (ii) gives us that $xa\,\mathcal{R}'\,xam\,\mathcal{L}'\,m$. We then use $xa\,\mathcal{R}'\,xam$ and (i) to obtain

$$w\,\mathcal{R}'\,wxa\,\mathcal{L}'\,xa. \tag{7.1}$$

We can then use the fact that $ya\,\mathcal{L}'\,xa\,\mathcal{L}'\,wxa = zya\,\mathcal{R}'\,z$, to get

$$z\,\mathcal{R}'\,zya\,\mathcal{L}'\,ya. \tag{7.2}$$

Since $\mathcal{R}'$ is a left congruence, $a\,\mathcal{R}'\,b$ implies that both $xa\,\mathcal{R}'\,xb$ and $ya\,\mathcal{R}'\,yb$. We can therefore apply (i) to both (7.1) and (7.2) to get

$$w\,\mathcal{R}'\,wxb\,\mathcal{L}'\,xb \text{ and } z\,\mathcal{R}'\,zyb\,\mathcal{L}'\,yb.$$

Using $a\,\mathcal{R}'\,b$, Property (R4) gives us that $zya = wxa$ implies that $zyb = wxb$. Therefore $xb\,\mathcal{L}'\,wxb = zyb\,\mathcal{L}'\,yb$.

$\square$

We now have the tools to prove (U1) - (U11) with with $\mathcal{R}' = \mathcal{R}'$, $\mathcal{L}' = \mathcal{L}'$ and $\mathcal{U}' = \mathcal{U}'$.

(U1) True by (R1) and the definition of $\mathcal{U}'$.

(U2) Let $(\beta, \beta, \alpha) \in \mathcal{U}'$. Therefore, by definition, there exists $m \in S$ such that

$$\beta\,\mathcal{R}'\,\beta m\,\mathcal{L}'\,m \text{ and } \alpha\,\mathcal{R}'\,\alpha\beta m\,\mathcal{L}'\,\beta m.$$

By Lemma 7.1.4 (i), we can use the fact that $\beta\,\mathcal{R}'\,\beta m$ to give us that $\alpha\,\mathcal{R}'\,\alpha\beta\,\mathcal{L}'\,\beta$.

On the other hand, let $\alpha\,\mathcal{R}'\,\alpha\beta\,\mathcal{L}'\,\beta$. Implicitly, by (R1), there exists $m \in S$ such that $\beta\,\mathcal{R}'\,\beta m\,\mathcal{L}'\,m$. By Lemma 7.1.4 (i), we can use the fact that $\beta\,\mathcal{R}'\,\beta m$ to give us that $\alpha\,\mathcal{R}'\,\alpha\beta m\,\mathcal{L}'\,\beta m$. Therefore m witnesses $(\beta, \beta, \alpha) \in \mathcal{U}'$.

(U3) Let $(\alpha, \beta, \gamma) \in \mathcal{U}'$ and $(\delta, \epsilon, \beta) \in \mathcal{U}'$. Then there exists $m, n \in S$ such that

$$\beta\,\mathcal{R}'\,\beta m\,\mathcal{L}'\,m \;\text{ and }\; \gamma\,\mathcal{R}'\,\gamma\alpha m\,\mathcal{L}'\,\alpha m$$

and

$$\epsilon\,\mathcal{R}'\,\epsilon n\,\mathcal{L}'\,n \;\text{ and }\; \beta\,\mathcal{R}'\,\beta\delta n\,\mathcal{L}'\,\delta n.$$

Using (R5), we know that $\beta\,\mathcal{L}'\,\beta$ implies that $m\,\mathcal{R}'\,\delta n$. Therefore, using the fact that $\mathcal{R}'$ is a left congruence, $\alpha m\,\mathcal{R}'\,\alpha\delta n$. We can then apply Lemma 7.1.4 (i) to $\gamma\,\mathcal{R}'\,\gamma\alpha m\,\mathcal{L}'\,\alpha m$ to obtain $\gamma\,\mathcal{R}'\,\gamma\alpha\delta n\,\mathcal{L}'\,\alpha\delta n$. Therefore, since $\epsilon\,\mathcal{R}'\,\epsilon n\,\mathcal{L}'\,n$ and $\gamma\,\mathcal{R}'\,\gamma\alpha\delta n\,\mathcal{L}'\,\alpha\delta n$, we see that n witnesses $(\alpha\delta, \epsilon, \gamma) \in \mathcal{U}'$.

(U4) Let $(\alpha\beta, \gamma, \delta) \in \mathcal{U}'$ and $\alpha\beta\,\mathcal{L}'\,\beta$. Therefore there exists an $m \in S$ such that

$$\gamma\,\mathcal{R}'\,\gamma m\,\mathcal{L}'\,m \;\text{ and }\; \delta\,\mathcal{R}'\,\delta\alpha\beta m\,\mathcal{L}'\,\alpha\beta m.$$

Using the fact that $\mathcal{L}'$ is a right congruence, $\alpha\beta\,\mathcal{L}'\,\beta$ implies that $\alpha\beta m\,\mathcal{L}'\,\beta m$. We apply (R3)(r) to $\delta\,\mathcal{R}'\,\delta\alpha\beta m$ to obtain $\delta\alpha\,\mathcal{R}'\,\delta\alpha\beta m$. Putting these two facts together, we have

$$\gamma\,\mathcal{R}'\,\gamma m\,\mathcal{L}'\,m \;\text{ and }\; \delta\alpha\,\mathcal{R}'\,\delta\alpha\beta m\,\mathcal{L}'\,\beta m.$$

Therefore m witnesses $(\beta, \gamma, \delta\alpha) \in \mathcal{U}'$.

(U5) Let $(\alpha, \beta, \gamma) \in \mathcal{U}'$ and $\beta\,\mathcal{L}'\,\delta$. Then there exists $m \in S$ such that

$$\beta\,\mathcal{R}'\,\beta m\,\mathcal{L}'\,m \;\text{ and }\; \gamma\,\mathcal{R}'\,\gamma\alpha m\,\mathcal{L}'\,\alpha m.$$

We apply Lemma 7.1.4 (ii) to $\beta\,\mathcal{R}'\,\beta m\,\mathcal{L}'\,m$ to obtain $\delta\,\mathcal{R}'\,\delta m\,\mathcal{L}'\,m$. Therefore m witnesses $(\alpha, \delta, \gamma) \in \mathcal{U}'$.

(U6) Let $(\alpha, \beta, \gamma) \in \mathcal{U}'$. Then there exists $m, n \in S$ such that

$$\beta\,\mathcal{R}'\,\beta m\,\mathcal{L}'\,m \;\text{ and }\; \gamma\,\mathcal{R}'\,\gamma\alpha m\,\mathcal{L}'\,\alpha m.$$

Firstly, let $(\alpha,\beta,\delta)\in\mathcal{U}'$. Then there exists $n\in S$ such that

$$\beta\,\mathcal{R}'\,\beta n\,\mathcal{L}'\,n \text{ and } \delta\,\mathcal{R}'\,\delta\alpha n\,\mathcal{L}'\,\alpha n.$$

We apply (R5) to give us that $\beta\,\mathcal{L}'\,\beta$ implies that $m\,\mathcal{R}'\,n$. Therefore, using the fact that $\mathcal{R}'$ is a left congruence, we have $\alpha m\,\mathcal{R}'\,\alpha n$. We then apply (R5) again to obtain $\gamma\,\mathcal{L}'\,\delta$.

Conversely, suppose that $\gamma\,\mathcal{L}'\,\delta$. Applying Lemma 7.1.4 (ii) to $\gamma\,\mathcal{R}'\,\gamma\alpha m\,\mathcal{L}'\,\alpha m$, we know that $\gamma\,\mathcal{L}'\,\delta$ implies that $\delta\,\mathcal{R}'\,\delta\alpha m\,\mathcal{L}'\,\alpha m$. Therefore m witnesses $(\alpha,\beta,\delta)\in\mathcal{U}'$.

(U7) Let $\gamma\alpha=\delta\beta\,\mathcal{R}'\,\delta$ and let $(\alpha,\beta,\gamma)\in\mathcal{U}'$. Then there exists $m\in S$ such that

$$\beta\,\mathcal{R}'\,\beta m\,\mathcal{L}'\,m \text{ and } \gamma\,\mathcal{R}'\,\gamma\alpha m\,\mathcal{L}'\,\alpha m.$$

Applying (R1) to β and α, we know that there exists $p,q,n\in S$ such that

$$p\,\mathcal{R}'\,q\,\mathcal{R}'\,q\alpha=p\beta,\ \alpha\,\mathcal{R}'\,\alpha n\,\mathcal{L}'\,n \text{ and } p\,\mathcal{R}'\,p\beta n\,\mathcal{L}'\,\beta n.$$

Since $\mathcal{R}'$ is a left congruence, $\alpha\,\mathcal{R}'\,\alpha n$ implies that $\gamma\alpha\,\mathcal{R}'\,\gamma\alpha n$. Therefore, using $\gamma\alpha=\delta\beta$ and $\delta\beta\,\mathcal{R}'\,\delta$, we have that $\delta\,\mathcal{R}'\,\delta\beta\,\mathcal{R}'\,\delta\beta n$. Applying (R1) to q and γ, we know that there exists $i,j,x\in S$ such that

$$i\,\mathcal{R}'\,j\,\mathcal{R}'\,j\gamma=iq,\ \gamma\,\mathcal{R}'\,\gamma x\,\mathcal{L}'\,x \text{ and } i\,\mathcal{R}'\,iqx\,\mathcal{L}'\,qx.$$

Comparing this with $\gamma\,\mathcal{R}'\,\gamma\alpha m\,\mathcal{L}'\,\alpha m$, we obtain $x\,\mathcal{R}'\,\alpha m$ by (R5). Therefore, we have $qx\,\mathcal{R}'\,q\alpha m$ by left compatibility. This means we can apply Lemma 7.1.4 (i) to $i\,\mathcal{R}'\,iqx\,\mathcal{L}'\,qx$ to obtain $i\,\mathcal{R}'\,iq\alpha m\,\mathcal{L}'\,q\alpha m$.

Since $q\alpha=p\beta$, this means that $ip\beta m\,\mathcal{L}'\,p\beta m$. Using $\beta\,\mathcal{R}'\,\beta m$, we can apply Lemma 7.1.4 (iii) to give us $ip\beta\,\mathcal{L}'\,p\beta$. Therefore

$$j\delta\beta=j\gamma\alpha=iq\alpha=ip\beta\,\mathcal{L}'\,p\beta.$$

Using the fact that $\mathcal{L}'$ is a right congruence, this means that we can right multiply by n to obtain

$$j\delta\beta n\,\mathcal{L}'\,p\beta n\,\mathcal{L}'\,\beta n.$$

Therefore by (R3)(l), $\delta\beta n\,\mathcal{L}'\,\beta n$. This mean that there exists $n \in S$ such that

$$\alpha\,\mathcal{R}'\,\alpha n\,\mathcal{L}'\,n \text{ and } \delta\,\mathcal{R}'\,\delta\beta n\,\mathcal{L}'\,\beta n.$$

Therefore $(\beta, \alpha, \delta) \in \mathcal{U}'$.

(U8) Let $(\alpha\beta, \beta, \gamma) \in \mathcal{U}'$. Then there exists $m \in S$ such that

$$\beta\,\mathcal{R}'\,\beta m\,\mathcal{L}'\,m \text{ and } \gamma\,\mathcal{R}'\,\gamma\alpha\beta m\,\mathcal{L}'\,\alpha\beta m.$$

Using the fact that $\mathcal{R}'$ is a left congruence, $\beta\,\mathcal{R}'\,\beta m$ implies that $\alpha\beta\,\mathcal{R}'\,\alpha\beta m$. Therefore by Lemma 7.1.4 (i), we have that

$$\gamma\,\mathcal{R}'\,\gamma\alpha\beta\,\mathcal{L}'\,\alpha\beta.$$

From our proof of (U2), we have already proved that this implies that $(\alpha\beta, \alpha\beta, \gamma) \in \mathcal{U}'$.

(U9) (R4)

(U10) (R5)

(U11) (R6)

We can therefore apply Theorem 4.3.5 to give us that S has a semigroup of straight left I-quotients, Q, such that $\mathcal{R}^Q \cap (S \times S) = \mathcal{R}'$, $\mathcal{L}^Q \cap (S \times S) = \mathcal{L}'$, and $\mathcal{U}^Q \cap (S \times S \times S) = \mathcal{U}'$.

Implicitly, by (R1), we can see that for all $c \in S$, there exists an $m \in S$ such that $c\,\mathcal{R}^Q\,cm\,\mathcal{L}^Q\,m$. Therefore, by Lemma 7.1.1, we know that S intersects every $\mathcal{R}$-class of Q. □

7.2 Left ample straight left I-orders that intersect every $\mathcal{R}$-class

Up until this point we have only been able to tell whether a left ample semigorup is a left I-order in its inverse hull. We will now provide a new result on left ample left I-orders in the case that they intersect every $\mathcal{R}$-class of their semigroup of left I-quotients.

Proposition 7.2.1. *Let S be a left ample semigroup and let $\mathcal{L}'$ be a binary relation on S. Then S is embedded as a unary semigroup in a semigroup of left I-quotients, Q, such that S intersects every $\mathcal{R}$-class of Q with $\mathcal{L}' = \mathcal{L}^Q \cap (S \times S)$ if and only if $\mathcal{L}'$ is a right congruence such that S satisfies (L1) - (L4).*

(L1) For all $\alpha, \beta \in S$, there exists $\gamma, \delta, m \in S$ such that

$$\gamma^+ = \delta^+ = (\delta\beta)^+,\ \delta\beta = \gamma\alpha,\ \beta\,\mathcal{L}'\,m^+ \ \text{ and } \ \gamma\,\mathcal{L}'(\alpha m)^+.$$

(L2) For all $\alpha, \gamma, \delta \in S$, $\gamma\alpha\,\mathcal{L}'\,\delta\alpha$ implies that $\gamma\alpha^+\,\mathcal{L}'\,\delta\alpha^+$.

(L3) $\mathcal{L}' \subseteq \mathcal{L}^*$.

(L4) For all $\alpha, \beta, \gamma \in S$, $\alpha\,\mathcal{L}'\,\beta\,\mathcal{L}'\,\gamma\alpha = \gamma\beta$ implies that $\alpha = \beta$.

We start the proof with this useful lemma.

Lemma 7.2.2. *Let S be a left ample semigroup with a right congruence, $\mathcal{L}'$, such that (L2) and (L3) are satisfied. Then $x\,\mathcal{L}'\,a^+$ if and only if $x\,\mathcal{R}^*\,xa\,\mathcal{L}'\,a$.*

Proof. Let $x\,\mathcal{L}'\,a^+$. Since $\mathcal{L}'$ is a right congruence, we can right multiply this by a to obtain

$$xa\,\mathcal{L}'\,a^+a = a.$$

Similarly, we can right multiply by a^+ to obtain

$$xa^+\,\mathcal{L}'\,a^+.$$

Therefore

$$x\,\mathcal{L}'\,a^+\,\mathcal{L}'\,xa^+ = (xa)^+x,$$

using Lemma 2.3.3 (iii) in the last step. Using Lemma 2.3.3 (i), this means that $x^+x\,\mathcal{L}'\,(xa)^+x$. We can then apply (L2) to this to obtain

$$x^+\,\mathcal{L}'\,(xa)^+x^+.$$

Using (L3), this means that $x^+ \,\mathcal{L}^*\, (xa)^+x^+$. There is only one idempotent in each $\mathcal{L}^*$-class of a left ample semigroup, so $x^+ = (xa)^+x^+$. We right multiply by x and apply Lemma 2.3.3 (iii), to obtain

$$x = (xa)^+x = xa^+.$$

Therefore $x^+ = (xa)^+$, by Lemma 2.3.3 (iv).

Conversely, suppose that $x \,\mathcal{R}^*\, xa \,\mathcal{L}'\, a$. This means that $x^+ = (xa)^+$, and therefore $x = xa^+$ by Lemma 2.3.3 (iv). We also have $xa \,\mathcal{L}'\, a = a^+a$, using Lemma 2.3.3 (i). We then apply (L2) to give us that $xa^+ \,\mathcal{L}'\, a^+a^+ = a^+$. Therefore

$$x = xa^+ \,\mathcal{L}'\, a^+.$$

□

We start the proof of Proposition 7.2.1 with the forward direction. Let S be a left ample semigroup embedded as a unary semigroup in a semigroup of left I-quotients, Q, such that S intersects every $\mathcal{R}$-class of Q. By Lemma 5.2.4, we know that Q is a semigroup of straight left I-quotients. Therefore Properties (R1) - (R6) are satisfied with $\mathcal{R}' = \mathcal{R}^Q \cap (S \times S)$ and $\mathcal{L}' = \mathcal{L}^Q \cap (S \times S)$. Since S is embedded as a unary semigroup in Q, this means that $a \,\mathcal{R}'\, b$ if and only if $a \,\mathcal{R}^*\, b$, which is equivalent to $a^+ = b^+$. We will now prove Properties (L1) - (L4).

(L1) Using Lemma 7.2.2, this is Property (R1).

(L2) Let $\alpha, \gamma, \delta \in S$ such that $\gamma\alpha \,\mathcal{L}'\, \delta\alpha$, and so $\gamma\alpha \,\mathcal{L}^Q\, \delta\alpha$. Using the fact that $\mathcal{L}^Q$ is a right congruence, this implies that $\gamma\alpha\alpha^{-1} \,\mathcal{L}^Q\, \delta\alpha\alpha^{-1}$. Since S is embedded as a unary semigroup in Q, we know that $\alpha\alpha^{-1} = \alpha^+ \in S$. Therefore $\gamma\alpha^+ \,\mathcal{L}'\, \delta\alpha^+$.

(L3) Since $\mathcal{L}' = \mathcal{L}^Q \cap (S \times S)$ and Q is an oversemigroup of S, then, by definition, $\alpha \,\mathcal{L}'\, \beta$ implies that $\alpha \,\mathcal{L}^*\, \beta$.

(L4) (R6)

We now prove the backwards direction of Proposition 7.2.1. Let S be a left ample semigroup with right congruence, $\mathcal{L}'$, such that (L1) - (L4) are satisfied. We take $\mathcal{R}'$ to be $\mathcal{R}^*$, i.e. $a\,\mathcal{R}'\,b$ if and only if $a^+ = b^+$.

We aim to apply Theorem 7.1.3. By definition $\mathcal{L}'$ is a right congruence. Since $\mathcal{R}' = \mathcal{R}^*$, we know that $\mathcal{R}'$ is a left congruence. We must prove Properties (R1) - (R6). We can prove each of these directly, except for (R3)(l) which is a little harder. We therefore leave (R3)(l) to the end in order to use the other properties in the proof.

(R1) Using Lemma 7.2.2, this is (L1).

(R2) Let $\alpha \in S$. By Lemma 7.2.2, we need an $\gamma \in S$ such that $\gamma\,\mathcal{L}'\,\alpha^+$. We can simply take $\gamma = \alpha^+$.

(R3)(r) Let $\alpha, \beta, \gamma \in S$ such that $\alpha^+ = (\alpha\beta\gamma)^+$. Using Lemma 2.3.3 (v) gives us both that $(\alpha\beta)^+(\alpha\beta\gamma)^+ = (\alpha\beta\gamma)^+$ and that $(\alpha\beta)^+\alpha^+ = (\alpha\beta)^+$. Then, using $\alpha^+ = (\alpha\beta\gamma)^+$ twice, we have

$$\alpha^+ = (\alpha\beta\gamma)^+ = (\alpha\beta)^+(\alpha\beta\gamma)^+ = (\alpha\beta)^+\alpha^+ = (\alpha\beta)^+,$$

and so $\alpha\,\mathcal{R}'\,\alpha\beta$.

(R4) $\mathcal{R}' = \mathcal{R}^*$.

(R5) Let $\alpha, \beta, \gamma, \delta \in S$ such that $\gamma\,\mathcal{R}'\,\gamma\alpha\,\mathcal{L}'\alpha$ and $\delta\,\mathcal{R}'\,\delta\beta\,\mathcal{L}'\beta$. By Lemma 7.2.2, this means that $\gamma\,\mathcal{L}'\alpha^+$ and $\delta\,\mathcal{L}'\,\beta^+$.

If $\gamma\,\mathcal{L}'\,\delta$, then $\alpha^+\,\mathcal{L}'\,\beta^+$. Since $\mathcal{L}' \subseteq \mathcal{L}^*$, this implies that $\alpha^+\,\mathcal{L}^*\,\beta^+$. Since there is a unique idempotent in each $\mathcal{L}^*$-class of a left ample semigroup, this gives us $\alpha^+ = \beta^+$.

Conversely, if $\alpha^+ = \beta^+$, then

$$\gamma\,\mathcal{L}'\,\alpha^+ = \beta^+\,\mathcal{L}'\,\delta.$$

(R6) (L4)

Property (R3)(l) is a little harder and we will use the next lemma to help us.

Lemma 7.2.3. *Let S be a left ample semigroup with a right congruence, $\mathcal{L}'$, such that (L1) - (L4) are satisfied. Then:*

(i) for all $a, b, x \in S$, $x \mathcal{R}^* xa \mathcal{L}' a$ *and* $a \mathcal{R}^* b$ *implies that* $x \mathcal{R}^* xb \mathcal{L}' b$*;*

(ii) for all $c, n, p, q, v \in S$, $p \mathcal{L}' c^+$, $q \mathcal{L}' n^+$ *and* $vpc \mathcal{L}' qc$ *implies that* $vpn \mathcal{L}' pn$*.*

Proof.

(i) Let $a, b, x \in S$ such that $x \mathcal{R}^* xa \mathcal{L}' a$ and $a \mathcal{R}^* b$. By Lemma 7.2.2, $x \mathcal{R}^* xa \mathcal{L}' a$ implies that $x \mathcal{L}' a^+$. Since we know that $a^+ = b^+$, this means that $x \mathcal{L}' b^+$. We then apply Lemma 7.2.2 again to get the required result.

(ii) Let $c, n, p, q, v \in S$ such that $p \mathcal{L}' c^+$, $q \mathcal{L}' n^+$ and $vpc \mathcal{L}' qc$. Using the fact that $\mathcal{L}'$ is a right congruence, $p \mathcal{L}' c^+$ implies that $pn^+ \mathcal{L}' c^+n^+$ Similarly, $q \mathcal{L}' n^+$ implies that $qc^+ \mathcal{L}' n^+c^+$. Putting these together gives us

$$qc^+ \mathcal{L}' n^+c^+ = c^+n^+ \mathcal{L}' pn^+. \tag{7.3}$$

Applying Lemma 7.2.2 to $p \mathcal{L}' c^+$ gives us $p \mathcal{R}^* pc$, i.e. $p^+ = (pc)^+$. Therefore $p = pc^+$ by Lemma 2.3.3 (iv).

We use (L2) to give us that $vpc \mathcal{L}' qc$ implies that $vpc^+ \mathcal{L}' qc^+$. Using (7.3) along with $p = pc^+$, we see that

$$vp = vpc^+ \mathcal{L}' qc^+ \mathcal{L}' pn^+.$$

We can then use the fact that $\mathcal{L}'$ is a right congruence and right multiply by n to obtain $vpn \mathcal{L}' pn$.

□

We now prove Property (R3)(l).

(R3)(l) Let $\alpha\beta\gamma \mathcal{L}' \gamma$. By (R2), there exists an $x \in S$ such that

$$x \mathcal{R}' x\alpha\beta\gamma \mathcal{L}' \alpha\beta\gamma.$$

Note that by (R3)(r), we know that $x\alpha\beta \mathcal{R}' x\alpha\beta\gamma$. Therefore, using $\alpha\beta\gamma \mathcal{L}' \gamma$, we have

$$x\alpha\beta \mathcal{R}' x\alpha\beta\gamma \mathcal{L}' \gamma. \tag{7.4}$$

We apply (R1) to β and $x\alpha\beta$ to get that there exists $u, v, m \in S$ such that

$$u\,\mathcal{R}'\,v\,\mathcal{R}'\,vx\alpha\beta = u\beta,\ x\alpha\beta\,\mathcal{R}'\,x\alpha\beta m\,\mathcal{L}'\,m \text{ and } u\,\mathcal{R}'\,u\beta m\,\mathcal{L}'\,\beta m. \tag{7.5}$$

Using (R5), comparing $x\alpha\beta\,\mathcal{R}'\,x\alpha\beta m\,\mathcal{L}'\,m$ to (7.4), we see that $m\,\mathcal{R}'\,\gamma$. Using the fact that $\mathcal{R}'$ is a left congruence, this implies that $\beta m\,\mathcal{R}'\,\beta\gamma$. Therefore, by Lemma 7.2.3 (i), $u\,\mathcal{R}'\,u\beta m\,\mathcal{L}'\,\beta m$ implies that

$$u\,\mathcal{L}'\,u\beta\gamma\,\mathcal{L}'\,\beta\gamma. \tag{7.6}$$

Implicitly, by Property (L1), we know there exists n such that $\beta\,\mathcal{L}'\,n^+$. By Lemma 7.2.2, we know that this means that

$$\beta\,\mathcal{R}'\,\beta n\,\mathcal{L}'\,n.$$

We aim to prove that

$$v\,\mathcal{R}'\,vx\alpha\beta n\,\mathcal{L}'\,x\alpha\beta n. \tag{$\star$}$$

We can prove the $\mathcal{R}'$ relation of ($\star$) quite simply. Since $\mathcal{R}'$ is a left congruence, we know that $vx\alpha\beta\,\mathcal{R}'\,vx\alpha\beta n$. Since we already know that $v\,\mathcal{R}'\,vx\alpha\beta$ from (7.5), this gives us $v\,\mathcal{R}'\,vx\alpha\beta n$.

In order to obtain the $\mathcal{L}'$ relation of ($\star$), we start by applying Lemma 7.2.2 to (7.4) to obtain $x\alpha\beta\,\mathcal{L}'\,\gamma^+$. Using $vx\alpha\beta = u\beta$ from (7.5) along with $u\beta\gamma\,\mathcal{L}'\,\beta\gamma$ from (7.6), we have

$$vx\alpha\beta\gamma = u\beta\gamma\,\mathcal{L}'\,\beta\gamma.$$

We also note that $\beta\,\mathcal{L}'\,n^+$. We can therefore apply Lemma 7.2.3 (ii) with $c = \gamma$, $n = n$, $p = x\alpha\beta$, $q = \beta$ and $v = v$ to obtain $vx\alpha\beta n\,\mathcal{L}'\,x\alpha\beta n$. Therefore ($\star$) is satisfied.

Using the fact that $\mathcal{R}'$ is a left congruence, $\beta\,\mathcal{R}'\,\beta n$ implies that $x\alpha\beta\,\mathcal{R}'\,x\alpha\beta n$. Therefore we can apply Lemma 7.2.3 (i) to ($\star$) to obtain

$$v\,\mathcal{R}'\,vx\alpha\beta\,\mathcal{L}'\,x\alpha\beta.$$

Using the fact that $\mathcal{L}'$ is a right congruence, $vx\alpha\beta\,\mathcal{L}'\,x\alpha\beta$ implies that

$vx\alpha\beta\gamma\,\mathcal{L}'\,x\alpha\beta\gamma$. Therefore, using Equations (7.6), (7.5) and (7.4), we have

$$\beta\gamma\,\mathcal{L}'\,u\beta\gamma = vx\alpha\beta\gamma\,\mathcal{L}'\,x\alpha\beta\gamma\,\mathcal{L}'\,\gamma.$$

We have now proven that S with relations $\mathcal{L}'$ and $\mathcal{R}'$ satisfies the conditions of Theorem 7.1.3. Applying Theorem 7.1.3 we have that S is a straight left I-order in inverse semigroup Q such that S intersects every $\mathcal{R}^Q$-class with $\mathcal{R}^Q \cap (S \times S) = \mathcal{R}'$. Since $\mathcal{R}^Q \cap (S \times S) = \mathcal{R}^*$, Lemma 2.3.4 gives us that S is embedded in Q as a unary semigroup.

7.3 Straight left I-orders which are also straight right I-orders

We are now very familiar with straight left I-orders. A semigroup, S, is a straight left I-order if there exists an inverse semigroup, Q, such that every element of Q can be written as $a^{-1}b$, where $a, b \in S$ and a and b are $\mathcal{R}$-related in Q.

There is a dual concept of a straight right I-order. We say that a semigroup, S, is a *straight right I-order* if there exists an inverse semigroup, P, such that every element of P can be written as dc^{-1}, where $c, d \in S$ and c and d are $\mathcal{L}$-related in P. For every result we have on left I-orders, there is a dual result on right I-orders.

This section proves that when a semigroup is both a straight left I-order and a straight right I-order, then its respective semigroups of straight I-quotients are in fact the same inverse semigroup if their $\mathcal{R}$ and $\mathcal{L}$ relations are equal.

By saying that two semigroups Q and P are isomorphic with respect to a shared subsemigroup S, we mean that there exists an isomorphism from Q to P, which is the identity map on S.

Proposition 7.3.1. *Let a semigroup S have a semigroup of straight left I-quotients Q and a semigroup of straight right I-quotients P. Then $Q \cong P$ with respect to S if and only if $\mathcal{R}^P \cap (S \times S) = \mathcal{R}^Q \cap (S \times S)$ and $\mathcal{L}^P \cap (S \times S) = \mathcal{L}^Q \cap (S \times S)$.*

Proof. The forward implication is obvious. If Q and P are isomorphic with respect to S, then their Green's relations restricted to S will be equal.

We now consider the backwards direction. We define

$$\mathcal{R}' = \mathcal{R}^P \cap S \times S = \mathcal{R}^Q \cap S \times S$$

and

$$\mathcal{L}' = \mathcal{L}^P \cap S \times S = \mathcal{L}^Q \cap S \times S.$$

For $a \in S$, we use a^{-1} to denote the inverse of a in P, and $a^{\dagger}$ to denote the inverse of a in Q. By saying that an element dc^{-1} of P is in standard form, we mean that $c, d \in S$ such that $c\,\mathcal{L}'\,d$. By saying that an element $a^{\dagger}b$ of Q is in standard form, we mean that $a, b \in S$ such that $a\,\mathcal{R}'\,b$.

By the dual of Lemma 3.1.3, we know that S intersects every $\mathcal{R}$-class of P. Let $x \in S$. Since $x^{-1} \in P$, we know that there exists an $a \in S$ such that $a\,\mathcal{R}^P\,x^{-1}$ in P. By Lemma 3.3.3, this implies that $x\,\mathcal{R}'\,xa\,\mathcal{L}'\,a$. Since $\mathcal{R}'$ and $\mathcal{L}'$ are also both restrictions of relations on Q, this means that for all $x \in S$, there exists $a \in S$ such that $x\,\mathcal{R}^Q\,xa\,\mathcal{L}^Q\,a$. Therefore S intersects every $\mathcal{R}$-class of Q by Lemma 7.1.1. We can therefore apply Theorem 7.1.3 to give us that S satisfies Properties (R1) - (R6). Dually, S also intersects every $\mathcal{L}$-class of P, and therefore we can apply the dual of Theorem 7.1.3 to give us that S satisfies the duals of Properties (R1) - (R6), which we label (R1)$'$ - (R6)$'$.

We want to construct an isomorphism from P to Q. Let dc^{-1} be an element of P in standard form. Applying (R1) to d and c, we have that there exists $a, b, m \in S$ such that

$$a\,\mathcal{R}'\,b\,\mathcal{R}'\,bc = ad, \;\; c\,\mathcal{R}'\,cm\,\mathcal{L}'\,m, \;\; \text{and} \;\; a\,\mathcal{R}'\,adm\,\mathcal{L}'\,dm.$$

We apply Lemma 7.1.4 (ii) to $d\,\mathcal{L}'\,c$ and $c\,\mathcal{R}'\,cm\,\mathcal{L}'\,m$ to obtain $d\,\mathcal{R}'\,dm\,\mathcal{L}'\,m$. We can then apply Lemma 7.1.4 (i) to $dm\,\mathcal{R}'\,d$ and $a\,\mathcal{R}'\,adm\,\mathcal{L}'\,dm$ to obtain $a\,\mathcal{R}'\,ad\,\mathcal{L}'\,d$.

To summarise, we know that for all $d, c \in S$ such that $d\,\mathcal{L}'\,c$, there exists $a, b \in S$ such that $a\,\mathcal{R}'\,b$, $ad = bc$ and $a\,\mathcal{R}'\,ad\,\mathcal{L}'\,d$.

We will define $\theta : P \to Q$. Let dc^{-1} be an element of P in standard form. We define θ as

$$(dc^{-1})\theta = a^{\dagger}b,$$

where

$$a\,\mathcal{R}'\,b,\ ad = bc \text{ and } a\,\mathcal{R}'\,ad\,\mathcal{L}'\,d. \tag{7.7}$$

Note such an a and b exists in S by above. Also, since $a\,\mathcal{R}'\,b$, $a^\dagger b$ is an element of Q in standard form. We will now prove that θ is an isomorphism.

$\boldsymbol{\theta}$ is well-defined:

Let dc^{-1} be an element of Q in standard form, and let

$$(dc^{-1})\theta = a^\dagger b \text{ and } (dc^{-1})\theta = p^\dagger q.$$

By definition, this means that

$$a\,\mathcal{R}'\,b,\ ad = bc \text{ and } a\,\mathcal{R}'\,ad\,\mathcal{L}'\,d, \tag{7.8}$$

and that

$$p\,\mathcal{R}'\,q,\ pd = qc \text{ and } p\,\mathcal{R}'\,pd\,\mathcal{L}'\,d. \tag{7.9}$$

We want that $a^\dagger b = p^\dagger q$ in Q. By Lemma 3.3.4, this is true if and only if there exists $x, y \in S$ such that

$$xa = yp,\ xb = yq,\ x\,\mathcal{R}'\,xa\,\mathcal{L}'\,a, \text{ and } y\,\mathcal{R}'\,yp\,\mathcal{L}'\,p. \tag{7.10}$$

Applying (R1) to a and p, we have that there exists $x, y, m \in S$ such that

$$x\,\mathcal{R}'\,y\,\mathcal{R}'\,yp = xa,\ p\,\mathcal{R}'\,pm\,\mathcal{L}'\,m \text{ and } x\,\mathcal{R}'\,xam\,\mathcal{L}'\,am. \tag{7.11}$$

Using property (R5), we can compare $p\,\mathcal{R}'\,pm\,\mathcal{L}'\,m$ from (7.11) to $p\,\mathcal{R}'\,pd\,\mathcal{L}'\,d$ from (7.9), to obtain $m\,\mathcal{R}'\,d$. Using the fact that $\mathcal{R}'$ is a left congruence, we have that $m\,\mathcal{R}'\,d$ implies that $am\,\mathcal{R}'\,ad\,\mathcal{R}'\,a$, using (7.8) in the last relation. We can then apply Lemma 7.1.4 (i) to $am\,\mathcal{R}'\,a$ and $x\,\mathcal{R}'\,xam\,\mathcal{L}'\,am$ from (7.11), to obtain $x\,\mathcal{R}'\,xa\,\mathcal{L}'\,a$.

We can apply (R5) to $a\,\mathcal{R}'\,ad\,\mathcal{L}'\,d$ from (7.8) and $p\,\mathcal{R}'\,pd\,\mathcal{L}'\,d$ from (7.9) to obtain $a\,\mathcal{L}'\,p$. We use this, along with $x\,\mathcal{R}'\,y$ and $xa = yp$ from (7.11), to see that $x\,\mathcal{R}'\,xa\,\mathcal{L}'\,a$ implies that $y\,\mathcal{R}'\,yp\,\mathcal{L}'\,p$.

Lastly, since $xa = yp$, we can right multiply by d, to get $xad = ypd$. Using $ad = bc$ and $pd = qc$, this gives us $xbc = yqc$. We then right multiply in Q by

$c^\dagger$, to obtain

$$xbcc^\dagger = yqcc^\dagger. \tag{7.12}$$

From (7.8), we have that $b\,\mathcal{R}'\,a\,\mathcal{R}'\,ad = bc$. Therefore, by Lemma 2.2.3, we have that $b = bcc^\dagger$. Similarly, (7.9) implies that $q\,\mathcal{R}'\,qc$. Therefore, by Lemma 2.2.3, we have that $q = qcc^\dagger$. We apply these two facts to (7.12) to obtain that $xb = yq$ in both Q and S.

Therefore, we see that (7.10) is satisfied and therefore θ is well-defined.

θ is a homomorphism:

Let dc^{-1} and kj^{-1} be elements of P in standard form, and let

$$(dc^{-1})\theta = a^\dagger b \text{ and } (kj^{-1})\theta = h^\dagger i.$$

By definition, this means that

$$a\,\mathcal{R}'\,b,\; ad = bc \text{ and } a\,\mathcal{R}'\,ad\,\mathcal{L}'\,d. \tag{7.13}$$

and that

$$h\,\mathcal{R}'\,i,\; hk = ij \text{ and } h\,\mathcal{R}'\,hk\,\mathcal{L}'\,k. \tag{7.14}$$

We want to prove that

$$((dc^{-1})(kj^{-1}))\theta = (dc^{-1})\theta(kj^{-1})\theta = (a^\dagger b)(h^\dagger i). \tag{7.15}$$

Applying (R1) to b and h, we have that there exists $x, y, m \in S$ such that

$$x\,\mathcal{R}'\,y\,\mathcal{R}'\,yh = xb,\; h\,\mathcal{R}'\,hm\,\mathcal{L}'\,m \text{ and } x\,\mathcal{R}'\,xbm\,\mathcal{L}'\,bm. \tag{7.16}$$

Using Property (R5), we can compare $h\,\mathcal{R}'\,hk\,\mathcal{L}'\,k$ from (7.14) and $h\,\mathcal{R}'\,hm\,\mathcal{L}'\,m$ from (7.16), to obtain $m\,\mathcal{R}'\,k$. Using the fact that $\mathcal{R}'$ is a left congruence, $m\,\mathcal{R}'\,k$ implies that $bm\,\mathcal{R}'\,bk$. Applying Lemma 7.1.4 (i) to $bm\,\mathcal{R}'\,bk$ and $x\,\mathcal{R}'\,xbm\,\mathcal{L}'\,bm$ from (7.16), we obtain

$$x\,\mathcal{R}'\,xbk\,\mathcal{L}'\,bk. \tag{7.17}$$

Considering (7.16), we see that $(b, h, x) \in \mathcal{U}^Q$ by Lemma 7.1.2. Since we also have $x\,\mathcal{R}'\,y$ and $xb = yp$, we can apply Lemma 4.3.2 to obtain $bh^\dagger = x^\dagger y$ in Q.

Therefore, in Q,

$$(a^{\dagger}b)(h^{\dagger}i) = a^{\dagger}(bh^{\dagger})i = a^{\dagger}(x^{\dagger}y)i = (xa)^{\dagger}(yi). \tag{7.18}$$

Note that since $\mathcal{R}'$ is left compatible, $a\,\mathcal{R}'\,b$ implies that $xa\,\mathcal{R}'\,xb$ and $h\,\mathcal{R}'\,i$ implies that $yh\,\mathcal{R}'\,yi$. Therefore, using the fact that $xb = yh$ from (7.16), we have

$$xa\,\mathcal{R}'\,xb = yh\,\mathcal{R}'\,yi. \tag{7.19}$$

And so, $(xa)^{\dagger}(yi)$ is an element of Q in standard form.

By applying the dual of (R1) to c and k, we know that there exists $v, u, n \in S$ such that

$$v\,\mathcal{L}'\,u\,\mathcal{L}'\,ku = cv,\ \ n\,\mathcal{R}'\,nc\,\mathcal{L}'\,c\ \text{ and }\ nk\,\mathcal{R}'\,nku\,\mathcal{L}'\,u. \tag{7.20}$$

Using $b\,\mathcal{R}'\,a$, $ad = bc$, and $c\,\mathcal{L}'\,d$, $a\,\mathcal{R}'\,ad\,\mathcal{L}'\,d$ implies that $b\,\mathcal{R}'\,bc\,\mathcal{L}'\,c$. Therefore, by (R5), $b\,\mathcal{L}'\,n$. Since $\mathcal{L}'$ is a right congruence, $b\,\mathcal{L}'\,n$ implies that $bt\,\mathcal{L}'\,nt$. We can then apply Lemma 7.1.4 (ii) to $nt\,\mathcal{R}'\,ntu\,\mathcal{L}'\,u$ from (7.20) to obtain

$$bk\,\mathcal{R}'\,bku\,\mathcal{L}'\,u. \tag{7.21}$$

Considering (7.20), we see that (c, k, v) is in the dual of $\mathcal{U}^P$ by the dual of Lemma 7.1.2. Since we also have $u\,\mathcal{L}'\,v$ and $ku = cv$, we can apply the dual of Lemma 4.3.2 to obtain $c^{-1}k = vu^{-1}$ in P. Therefore, in P,

$$(dc^{-1})(kj^{-1}) = d(c^{-1}k)j^{-1} = d(vu^{-1})j^{-1} = (dv)(ju)^{-1}. \tag{7.22}$$

Using the fact that $\mathcal{L}'$ is right compatible, we see that $d\,\mathcal{L}'\,c$ implies that $dv\,\mathcal{L}'\,cv$ and $k\,\mathcal{L}'\,j$ implies that $ku\,\mathcal{L}'\,ju$. Therefore, using $cv = ku$ from (7.20), we have

$$dv\,\mathcal{L}'\,cv = ku\,\mathcal{L}'\,ju. \tag{7.23}$$

Therefore $(dv)(ju)^{-1}$ is an element of P in standard form.

Consulting (7.18) and (7.22), we see that (7.15) is satisfied if and only if

$$\big((dv)(ju)^{-1}\big)\,\theta = (xa)^{\dagger}(yi).$$

By definition, this is true if and only if

$$xa\,\mathcal{R}'\,yi,\ xadv = yiju \text{ and } xa\,\mathcal{R}'\,xadv\,\mathcal{L}'\,dv. \tag{$\star$}$$

We already know that $xa\,\mathcal{R}'\,yi$ from (7.19).

Using $ad = bc$ from (7.13), $xb = yh$ from (7.16), $cv = ku$ from (7.20), and $hk = ij$ from (7.14), we see that

$$xadv = xbcv = yhku = yiju.$$

Using $bk\,\mathcal{L}'\,xbk$ from (7.17), we can apply Lemma 7.1.4 (ii) to $bk\,\mathcal{R}'\,bku\,\mathcal{L}'\,u$ from (7.21) to get

$$xbk\,\mathcal{R}'\,xbku\,\mathcal{L}'\,u.$$

Using $xb = yh$ from (7.16) and $hk = ij$ from (7.14), we have $xbk = yhk = yij$ Therefore, we can rewrite the above equation as

$$yij\,\mathcal{R}'\,yiju\,\mathcal{L}'\,u. \tag{7.24}$$

From (R3)(l), we know that $yiju\,\mathcal{L}'\,u$ implies that $ju\,\mathcal{L}'\,u$ Using the fact that $\mathcal{R}'$ is a left congruence, $i\,\mathcal{R}'\,ij$ from (7.14) implies that $yi\,\mathcal{R}'\,yij$. Therefore, we can use the relations in (7.24) to obtain

$$yi\,\mathcal{R}'\,yiju\,\mathcal{L}'\,ju.$$

Sinc $xa\,\mathcal{R}'\,yi$ from (7.19), $dv\,\mathcal{L}'\,ju$ from (7.23), and $xadv = yiju$, the above equation implies that

$$xa\,\mathcal{R}'\,xadv\,\mathcal{L}'\,dv.$$

Altogether, this gives us ($\star$). Therefore (7.15) is satisfied and θ is a homomorphism.

$\boldsymbol{\theta}$ preserves elements of S:

Let $s \in S$. We can write s as an element of P in standard form as $(sa)a^{-1}$, where $a \in S$ such that $s\,\mathcal{R}'\,sa\,\mathcal{L}'\,a$. We know that such an a exists by the dual of Lemma 4.2.3 (i).

Similarly, we can write s as an element of Q in standard form as $x^{\dagger}(xs)$, where

$x \in S$ such that $x \,\mathcal{R}'\, xs \,\mathcal{L}'\, s$. We know that such an x exists by Lemma 4.2.3 (i).

In order to prove that θ preserves elements of S, we need that

$$((sa)a^{-1})\theta = (x)^{\dagger}(xs),$$

which is true if and only if

$$x \,\mathcal{R}'\, xs, \; xsa = xsa \text{ and } x \,\mathcal{R}'\, xsa \,\mathcal{L}'\, sa. \tag{7.25}$$

We already know that $x \,\mathcal{R}'\, xs$ and that $xsa = xsa$. Finally, since $s \,\mathcal{R}'\, sa$, we can apply Lemma 7.1.4 (i) to $x \,\mathcal{R}'\, xs \,\mathcal{L}'\, s$ to get $x \,\mathcal{R}'\, xsa \,\mathcal{L}'\, sa$. We have therefore satisfied (7.25), and so θ preserves elements of S.

$\boldsymbol{\theta}^{-1}$ is a well-defined homomorphism:

We define $\theta^{-1} : Q \to P$ from elements of Q in standard form to elements of P in standard form. Let $a^{\dagger}b$ be an element of Q in standard form. Then we define $(a^{\dagger}b)\,\theta^{-1} = dc^{-1}$, where

$$d \,\mathcal{L}'\, c, \; ad = bc \text{ and } a \,\mathcal{R}'\, ad \,\mathcal{L}'\, d.$$

It is obvious that θ^{-1} is the inverse of θ.

By the exact dual of everything we have done so far, we know that d and c exist and that θ^{-1} is a well-defined homomorphism.

Therefore θ is surjective and onto, and therefore an isomorphism. $\square$

Chapter 8

Left I-orders with totally ordered idempotents

In this chapter, we consider semigroups of left I-quotients, Q, with *totally ordered idempotents*, that is, for every $e, f \in E(Q)$, either $e \leqslant f$ or $f \leqslant e$, using the natural ordering of idempotents. In this case, we say that $E(Q)$ forms a *chain*.

In Section 8.1, we consider the most general case of semigroups of left I-quotients with totally ordered idempotents. We prove Theorem 8.1.3, which gives necessary and sufficient conditions for a semigroup S to be a left I-order in an inverse semigroup with totally ordered idempotents.

In Section 8.2, we investigate left I-orders in inverse ω-semigroups. We find necessary and sufficient conditions for a semigroup to be a left I-order in an inverse ω-semigroup, along with the three special cases of inverse ω-semigroups: no kernel, simple and proper kernel.

8.1 Left I-orders in inverse semigroups having totally ordered idempotents - the general case

The aim of this section is to find necessary and sufficient conditions for a semigroup S to be a left I-order in an inverse semigroup with totally ordered idempotents.

Lemma 8.1.1. *Let Q be an inverse semigroup with totally ordered idempotents. Then $\leqslant_{\mathcal{L}}$ and $\leqslant_{\mathcal{R}}$ are both total preorders.*

Proof. Let $x, y \in Q$. We know that $x^{-1}x \, \mathcal{L}^Q \, x$ and $y^{-1}y \, \mathcal{L}^Q \, y$. Since $x^{-1}x$ and $y^{-1}y$ are idempotents and Q has totally ordered idempotents, either $x^{-1}x \leqslant y^{-1}y$ or $y^{-1}y \leqslant x^{-1}x$. Without loss of generality, let $x^{-1}x \leqslant y^{-1}y$. By Lemma 2.1.3, this implies that $x^{-1}x \leqslant_{\mathcal{L}^Q} y^{-1}y$. Therefore

$$x \, \mathcal{L}^Q \, x^{-1}x \leqslant_{\mathcal{L}^Q} y^{-1}y \, \mathcal{L}^Q \, y,$$

and so $x \leqslant_{\mathcal{L}^Q} y$.

The fact that $\leqslant_{\mathcal{R}}$ is a total preorder can be proved dually. □

Lemma 8.1.2. *Let S be a left I-order in Q with totally ordered idempotents. Then S is straight in Q.*

Proof. By Lemma 3.1.3, S is straight in Q if and only if S intersects every $\mathcal{L}$-class of Q. Let $q \in Q$. Our aim is to prove that q is $\mathcal{L}$-related to some element of S. Define $e = q^{-1}q \in E(Q)$. We know that $q \, \mathcal{L}^Q \, e$. Since S is a left I-order in Q, we have that $e = x^{-1}y$, where $x, y \in S$. By Lemma 8.1.1, $\leqslant_{\mathcal{R}}$ is a total order, so either $x \leqslant_{\mathcal{R}} y$ or $y \leqslant_{\mathcal{R}} x$.

If $x \leqslant_{\mathcal{R}} y$, then by Lemma 2.2.5, we have that $yy^{-1}x = x$. Using the fact that e is an idempotent, $e^{-1} = e$, so we have

$$e = ee^{-1} = x^{-1}yy^{-1}x = x^{-1}x.$$

Therefore $e \, \mathcal{L}^Q \, x \in S$.

If $y \leqslant_{\mathcal{R}} x$, then by Lemma 2.2.5, we have that $xx^{-1}y = y$. Therefore

$$e = e^{-1}e = y^{-1}xx^{-1}y = y^{-1}y.$$

Therefore $e \, \mathcal{L}^Q \, y \in S$.

In either case, $S \cap L_q = S \cap L_e \neq \varnothing$, and therefore S is straight in Q. □

We now introduce the main theorem of this section. We denote the equivalence relation associated with $\leqslant_l$ by $\mathcal{L}'$.

Theorem 8.1.3. *Let S be a semigroup and let $\mathcal{R}'$ and $\leqslant_l$ be binary relations on S. Then S is a left I-order in an inverse semigroup with totally ordered idempotents, Q, such that $\mathcal{R}^Q \cap (S \times S) = \mathcal{R}'$ and $\leqslant_{\mathcal{L}^Q} \cap (S \times S) = \leqslant_l$ if and only if $\mathcal{R}'$ is a left compatible equivalence relation, $\leqslant_l$ is a right compatible total preorder, and S satisfies Conditions (T1) - (T5).*

(T1) For all $\alpha, \beta \in S$ such that $\alpha \leqslant_l \beta$, there exists $\gamma, \delta \in S$ such that

$$\gamma\, \mathcal{R}' \delta\, \mathcal{R}' \delta\beta = \gamma\alpha \ \text{ and } \ \gamma\alpha\, \mathcal{L}'\, \alpha.$$

(T2) For all $\alpha, \beta \in S$, $\alpha\beta \leqslant_l \beta$.

(T3) $\mathcal{R}' \subseteq \mathcal{R}^*$.

(T4) Let $\alpha, \beta, \gamma, \delta \in S$ such that $\gamma\, \mathcal{R}'\, \gamma\alpha\, \mathcal{L}'\, \alpha$ and $\delta\, \mathcal{R}'\, \delta\beta\, \mathcal{L}'\, \beta$. Then $\gamma\, \mathcal{L}'\, \delta$ if and only if $\alpha\, \mathcal{R}'\, \beta$.

(T5) For all $\alpha, \beta, \gamma \in S$, $\alpha\, \mathcal{L}'\, \beta\, \mathcal{L}'\, \gamma\alpha = \gamma\beta$ implies that $\alpha = \beta$.

Proof. We start the proof of Theorem 8.1.3 by proving the forward implication. We assume that S has a semigroup of left I-quotients, Q, with totally ordered idempotents, and we label $\mathcal{R}^Q \cap (S \times S) = \mathcal{R}'$ and $\leqslant_{\mathcal{L}^Q} \cap (S \times S) = \leqslant_l$. From knowledge of Green's relations we know that $\mathcal{R}'$ is a left congruence and $\leqslant_l$ is a right compatible preorder. From Lemma 8.1.1, we know that $\leqslant_l$ is a total preorder.

By Lemma 8.1.2, we know that S is straight in Q. Therefore, by Theorem 4.2.1, we know that Properties (M1) - (M6) hold.

We now prove that Properties (T1) - (T5) are satisfied.

(T1) Let $\alpha, \beta \in S$ such that $\alpha \leqslant_l \beta$. Since (M1) is satisfied, there exists $\gamma, \delta \in S$ such that

$$\gamma\, \mathcal{R}' \delta\, \mathcal{R}' \delta\beta = \gamma\alpha \ \text{ and } \ L'_\alpha \wedge L'_\beta = L'_{\gamma\alpha}.$$

Since $\alpha \leqslant_l \beta$, we have that

$$L'_{\gamma\alpha} = L'_\alpha \wedge L'_\beta = L'_\alpha,$$

and so $\gamma\alpha \, \mathcal{L}' \, \alpha$.

(T2) (M3)

(T3) (M4)

(T4) (M5)

(T5) (M6)

This proves the forward implication of Theorem 8.1.3.

We now consider the converse. Let S be a semigroup, $\mathcal{R}'$ be a left compatible equivalence relation on S, and $\leqslant_l$ be a right compatible total preorder on S, such that S satisfies Conditions (T1) - (T5). We will prove that S is a left I-order, by showing that the relations $\mathcal{R}'$ and $\leqslant_l$ satisfy the conditions of Theorem 4.2.1.

Firstly, since $\leqslant_l$ is a total preorder, we have that for all $a, b \in S$ either $a \leqslant_l b$ or $b \leqslant_l a$. This implies that either $L'_a \wedge L'_b = L'_a$ or $L'_a \wedge L'_b = L'_b$, respectively. Therefore the $\mathcal{L}'$-classes of S form a meet semilattice under the associated partial order.

Note that since $\mathcal{L}'$ is the equivalence relation associated with $\leqslant_l$, the right compatibility of $\leqslant_l$ implies that $\mathcal{L}'$ is right compatible. We now need to prove Properties (M1) - (M6).

(M1) Let $\alpha, \beta \in S$. Since $\leqslant_l$ is a total preorder, either $\alpha \leqslant_l \beta$ or $\beta \leqslant_l \alpha$.

If $\alpha \leqslant_l \beta$, then, by (T1), there exists $\gamma, \delta \in S$ such that

$$\gamma \, \mathcal{R}' \delta \, \mathcal{R}' \delta\beta = \gamma\alpha \text{ and } \gamma\alpha \, \mathcal{L}' \, \alpha.$$

Since $\alpha \leqslant_l \beta$, we have $L'_\alpha \wedge L'_\beta = L'_\alpha = L'_{\gamma\alpha}$, proving (M1).

If $\beta \leqslant_l \alpha$, then, by (T1), there exists $\delta, \gamma \in S$ such that

$$\delta \, \mathcal{R}' \gamma \, \mathcal{R}' \gamma\alpha = \delta\beta \text{ and } \delta\beta \, \mathcal{L}' \, \beta.$$

Since $\beta \leqslant_l \alpha$, we have $L'_\alpha \wedge L'_\beta = L'_\beta = L'_{\delta\beta} = L'_{\gamma\alpha}$, again proving (M1).

(M2) Let $L'_\alpha \wedge L'_\beta = L'_\gamma$. Since $\leqslant_l$ is a total preorder, either $\alpha \leqslant_l \beta$ or $\beta \leqslant_l \alpha$. Without loss of generality, let $\alpha \leqslant_l \beta$. Therefore,

$$L'_\gamma = L'_\alpha \wedge L'_\beta = L'_\alpha,$$

and so $\gamma \,\mathcal{L}'\, \alpha$. Since $\mathcal{L}'$ is a right congruence, this implies that $\gamma\delta \,\mathcal{L}'\, \alpha\delta$.

Since $\leqslant_l$ is right compatible, $\alpha \leqslant_l \beta$ implies that $\alpha\delta \leqslant_l \beta\delta$, and therefore

$$L'_{\alpha\delta} \wedge L'_{\beta\delta} = L'_{\alpha\delta} = L'_{\gamma\delta}.$$

(M3) (T2)

(M4) (T3)

(M5) (T4)

(M6) (T5)

Therefore, we can apply Theorem 4.2.1 to obtain that S has a semigroup of straight left I-quotients, Q, such that $\mathcal{R}^Q \cap (S \times S) = \mathcal{R}'$ and $\leqslant_{\mathcal{L}^Q} \cap (S \times S) = \leqslant_l$.

We now prove that Q has totally ordered idempotents. Let $e, f \in E(S)$. Since S is straight in Q, then, by Lemma 3.1.3, we know that S intersects every $\mathcal{L}$-class of Q. Therefore there exists $s, t \in S$ such that $e \,\mathcal{L}^Q\, s$ and $f \,\mathcal{L}^Q\, t$. Since $\leqslant_l$ is a total preorder on S, either $s \leqslant_l t$ or $t \leqslant_l s$. Without loss of generality, let $s \leqslant_l t$. Using the fact that $\leqslant_{\mathcal{L}^Q} \cap (S \times S) = \leqslant_l$, this gives us that

$$e \,\mathcal{L}^Q\, s \leqslant_{\mathcal{L}^Q} t \,\mathcal{L}^Q\, f,$$

and so $e \leqslant_{\mathcal{L}^Q} f$. By Lemma 2.1.3, this means that $e \leqslant f$. Therefore the idempotents of Q are totally ordered. □

8.2 Inverse ω-semigroups of left I-quotients

In this section, we consider inverse ω-semigroups of left I-quotients. Inverse ω-semigroups are inverse semigroups whose idempotents form an inverse ω-chain. Since an inverse ω-chain is a type of chain, the idempotents in an inverse ω-semigroup are totally ordered.

Ghroda [11] gives necessary and sufficient conditions for a semigroup to have a bisimple inverse ω-semigroup of left I-quotients, extending the result of Gould's [15] categorisation of bisimple inverse ω-semigroups of left Fountain-Gould quotients.

In this section, we investigate left I-orders in general inverse ω-semigroups. Inverse ω-semigroups fall into one of three different types, depending on whether they have a kernel, and if they have a kernel on whether their kernel is proper.

In Subsection 8.2.1 we give necessary and sufficient conditions for a semigroup to be a left I-order in an inverse ω-semigroup with no kernel. Inverse ω-semigroups with no kernel are a type of Clifford semigroup, so we can easily characterise left I-orders in inverse ω-semigroups with no kernel using Corollary 3.2.12.

In Subsection 8.2.2, we consider left I-orders in general inverse ω-semigroups using Munn's [24] structure theorem. We prove Theorem 8.2.12, which gives necessary and sufficient conditions for a semigroup to be a left I-order in a general inverse ω-semigroup. We then do the same for three special cases: inverse ω-semigroups with kernel, simple inverse ω-semigroups, and inverse ω-semigroups with proper kernel.

Definition 8.2.1. An inverse ω-semigroup, Q, is an inverse semigroup whose idempotents form an inverse ω-chain; that is,

$$E(Q) = \{e_i \,|\, i \in \mathbb{N}^0\} \text{ with } e_0 > e_1 > e_2 > \dots .$$

This is a special case of the previous chapter, since the idempotents are totally ordered. Therefore, if S is a left I-order in an inverse ω-semigroup, Q, then S is straight in Q, by Lemma 8.1.2.

Munn [24] provides a structure result for inverse ω-semigroups.

Proposition 8.2.2 ([24]). *If Q is an inverse ω-semigroup then it is one of the following types:*

(1) Q is an inverse ω-chain of groups (if Q has no kernel),

(2) Q is a Bruck-Reilly semigroup over a finite chain of groups (if Q is simple),

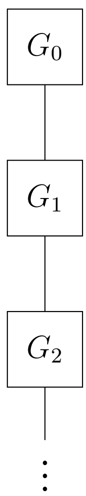
(3) Q is an ideal extension of a semigroup of Type (2) by a finite chain of groups (if Q has a proper kernel).

We consider inverse ω-semigroups of left I-quotients of Type (1) in Subsection 8.2.1. We consider inverse ω-semigroups of left I-quotients of all types in Subsection 8.2.2.

8.2.1 Inverse ω-semigroups of left I-quotients with no kernel

In this subsection, we characterise inverse ω-semigroups of left I-quotients of Type (1) from Proposition 8.2.2.

Figure 8.1: An ω-chain of groups



Corollary 8.2.3. *A semigroup S is a left I-order in an inverse ω-semigroup with no kernel if and only if S is an inverse ω-chain of right reversible, cancellative semigroups.*

Proof. Let S be a left I-order in an inverse ω-semigroup with no kernel, Q. By Proposition 8.2.2, Q is an inverse ω-chain Y of groups G_α, $\alpha \in Y$. By Corollary 3.2.12, S is an inverse ω-chain Y of right reversible, cancellative semigroups S_α, $\alpha \in Y$.

Conversely, let S be an inverse ω-chain Y of right reversible, cancellative semigroups, S_α, $\alpha \in Y$. By Corollary 3.2.12, S is a left I-order in Q, an inverse ω-chain Y of groups G_α, $\alpha \in Y$. By Proposition 8.2.2, Q is an inverse ω-semigroup with no kernel. $\square$

8.2.2 Inverse ω-semigroups of left I-quotients - the general case

In this subsection, we characterise inverse ω-semigroups of left I-quotients. We start by giving two structure theorems for inverse ω-semigroups with kernel.

Theorem 8.2.4 ([24, Theorem 4.11]). *A semigroup K is a simple inverse ω-semigroup if and only if $K \cong BR(T, \theta)$ for a finite chain of groups, T.*

Moreover, if T is a chain of d groups, then the number of $\mathcal{D}$-classes of K is d.

Theorem 8.2.5 ([24, Theorem 2.7]). *Let K be a simple inverse ω-semigroup with group of units G_0 and let C be a finite chain of groups disjoint from the non-unit elements of K, with group of units, G_0.*

Let $Q = C \cup K$, extending the multiplication of C and K as follows. Let $c \in C$ and $x \in K$. Then

(i) $cx = (c1_0)x$,

(ii) $xc = x(1_0c)$,

where 1_0 is the identity of G_0.

Then Q is an inverse ω-semigroup with kernel. Conversely, if Q is an inverse ω-semigroup with kernel, then Q is isomorphic to a semigroup constructed as above.

Note that by Theorem 8.2.4, if C is a group, then $G_0 \subseteq K$ and Q is simple.

This is a very complicated structure, so we devise convenient notation to refer to a semigroup of this type.

The semigroup $\omega(C, T, \theta)$.

Let T be a finite chain of d groups. We label the groups that comprise T as $F_0, F_1, \ldots, F_{d-1}$, with F_0 the largest and F_{d-1} the smallest. Therefore $T = \bigcup\limits_{\alpha=0}^{d-1} F_\alpha$ with associative multiplication satisfying $F_\alpha F_\beta \subseteq F_{\max\{\alpha,\beta\}}$.

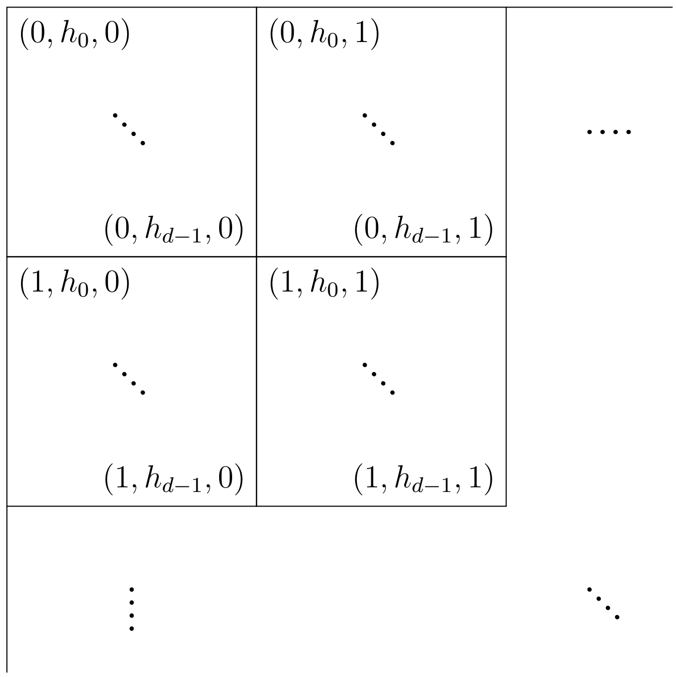
Let θ be an endomorphism on T such that $T\theta \subseteq F_0$. Let $K = BR(T, \theta)$ be the Bruck-Reilly semigroup over T with respect to θ. We remind readers from

Section 2.6 that this describes K as the semigroup $K = BR(T,\theta) = \mathbb{N}^0 \times T \times \mathbb{N}^0$ with multiplication

$$(m,a,n)(p,b,q) = \begin{cases} (m-n+p,(a\theta^{p-n})b,q) & \text{if } n<p, \\ (m,ab,q) & \text{if } n=p, \\ (m,a(b\theta^{n-p}),q-p+n) & \text{if } n>p. \end{cases}$$

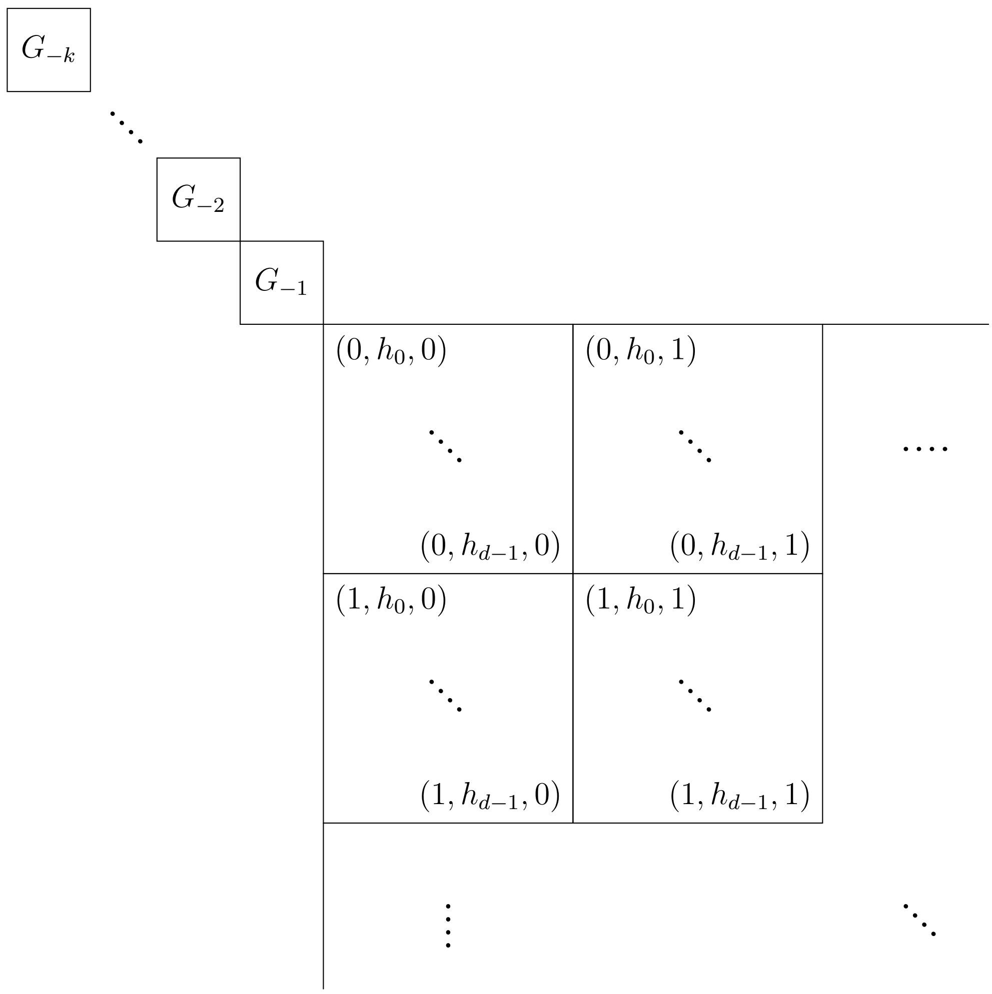
Note that units in K have the form $(0,h_0,0)$, where $h_0 \in F_0$.

Figure 8.2: A visual representation of $\omega(C,T,\theta)$ with $k=0$



Let C be a finite chain of $k+1$ groups. We label the groups that comprise C as $G_{-k}, G_{-k+1}, \ldots, G_{-1}, G_0$, with G_{-k} the largest and G_0 the smallest. We require that G_0 is the group of units of K, i.e. $G_0 = \{\, (0,h_0,0) \mid h_0 \in F_0 \,\}$. Note that $G_0 \cong F_0$. Therefore $C = \bigcup\limits_{i=-k}^{0} G_i$ with associative multiplication satisfying $G_iG_j \subseteq G_{\max\{i,j\}}$.

Figure 8.3: A visual representation of $\omega(C, T, \theta)$ with $k \neq 0$



Then $\omega(C, T, \theta)$ is then the union, $C \cup K$ extending the multiplication of C and K as follows:

(i) $g_i x = (g_i 1_0)x$,

(ii) $x g_i = x(1_0 g_i)$,

where $g_i \in G_i$, $x \in K$ and 1_0 is the identity of G_0.

By Theorem 8.2.5, a semigroup is an inverse ω-semigroup with kernel if and only if it is isomorphic to $\omega(C, T, \theta)$ for some appropriate C, T, θ.

By Theorem 8.2.4, if $k = 0$, then $\omega(C, T, \theta)$ is simple.

We give some properties of $\omega(C, T, \theta)$ in the following two lemmas. These are proved easily by referring to Lemma 2.2.1 and Lemma 2.2.6. For brevity, we label elements of F_α with the appropriate Greek letter subscript and elements of G_i with the appropriate Latin letter subscript.

Lemma 8.2.6. *Let $Q = \omega(C, T, \theta)$. Then*

(1) for all $g_i \in G_i$ and $g_j \in G_j$, $g_i \,\mathcal{R}^Q\, g_j$ if and only if $i = j$;

(2) for all $g_i \in G_i$ and $(m, h_\alpha, n) \in K$, $g_i \,\mathcal{R}^Q\, (m, h_\alpha, n)$ if and only if $i = 0$, $m = 0$, $n = 0$ and $\alpha = 0$;

(3) for all $(m, h_\alpha, n), (p, h_\beta, q) \in K$, $(m, h_\alpha, n) \,\mathcal{R}^Q\, (p, h_\beta, q)$ if and only if $m = p$ and $\alpha = \beta$.

Lemma 8.2.7. *Let $Q = \omega(C, T, \theta)$. Then*

(1) for all $x \in K$ and $c \in C$, $x \leqslant_{\mathcal{L}^Q} c$;

(2) for all $g_i \in G_i$ and $g_j \in G_j$, $g_i \leqslant_{\mathcal{L}^Q} g_j$ if and only if $i \geqslant j$;

(3) for all $(m, h_\alpha, n), (p, h_\beta, q) \in K$, $(m, h_\alpha, n) \leqslant_{\mathcal{L}^Q} (p, h_\beta, q)$ if and only if either $n > q$ or both $n = q$ and $\alpha \geqslant \beta$.

Lemmas 8.2.6 and 8.2.7 are stated as they are for convenience, but their duals also hold.

Recall the bicyclic monoid, $\mathcal{B}$, from Section 2.6. We introduce a generalisation of this semigroup.

Definition 8.2.8. We define the semigroup $\mathcal{B}_d$. Let d be any positive integer. Then $\mathcal{B}_d$ is defined by

$$\mathcal{B}_d = \{(m, n) \in \mathcal{B} : \ m = n \,(\text{mod}\, d)\},$$

where $\mathcal{B}$ is the bicyclic monoid.

We introduce a homomorphism from simple inverse ω-semigroups to $\mathcal{B}_d$. This homomorphism is mentioned in [24], but we will show that the semigroup operation is preserved here for completeness.

Lemma 8.2.9. *If Q is a simple inverse ω-semigroup, then there exists a homomorphism ϕ from Q to $\mathcal{B}_d$ for some d.*

Proof. By Theorem 8.2.4, we know that $Q = BR(T, \theta)$. In order to use the notation we have introduced, we will consider Q as $\omega(G_0, T, \theta)$, where G_0 is the group of units of K. We will label an element of F_α with a subscript α for clarity. For example, g_α is an element of F_α.

We define $\phi : Q \to \mathcal{B}_d$ as

$$(m, g_\alpha, n)\phi = (md + \alpha, nd + \alpha).$$

We need to prove that this is a homomorphism. Let $(m, g_\alpha, n), (p, h_\beta, q) \in Q$. Our aim is to prove that

$$((m, g_\alpha, n)(p, h_\beta, q))\, \phi = (m, g_\alpha, n)\phi(p, h_\beta, q)\phi. \tag{$\star$}$$

There are three cases: either $n = p$, $n > p$ or $n < p$.

Case 1: Let $n = p$. Then, by multiplying in the bicyclic monoid,

$$\begin{aligned}(m, g_\alpha, n)\phi(p, h_\beta, q)\phi &= (md + \alpha, nd + \alpha)(pd + \beta, qd + \beta) \\ &= (md + \max\{\alpha, \beta\}, qd + \max\{\alpha, \beta\}),\end{aligned}$$

since $n = p$ implies that $\max\{nd + \alpha, pd + \beta\} = nd + \max\{\alpha, \beta\}$. On the other hand, we have

$$((m, g_\alpha, n)(p, h_\beta, q))\, \phi = (m, g_\alpha h_\beta, q)\phi.$$

We know that $g_\alpha h_\beta \in F_{\max\{\alpha,\beta\}}$. Therefore

$$(m, g_\alpha h_\beta, q)\phi = (md + \max\{\alpha,\beta\}, qd + \max\{\alpha,\beta\}).$$

This gives us ($\star$).

Case 2: Let $n > p$. Then

$$\begin{aligned}(m, g_\alpha, n)\phi(p, h_\beta, q)\phi &= (md + \alpha, nd + \alpha)(pd + \beta, qd + \beta)\\ &= (md + \alpha, (q - p + n)d + \alpha),\end{aligned}$$

since $n > p$ implies that $nd + \alpha > pd + \beta$. On the other hand, we have

$$((m, g_\alpha, n)(p, h_\beta, q))\,\phi = (m, g_\alpha(h_\beta\theta^{n-p}), q - p + n)\phi.$$

Since $T\theta \subseteq F_0$, we know that $g_\alpha(h_\beta\theta^{n-p}) \in F_{\max\{\alpha,0\}} = F_\alpha$. Therefore

$$(m, g_\alpha(h_\beta\theta^{n-p}), q - p + n)\phi = (md + \alpha, (q - p + n)d + \alpha),$$

giving us ($\star$).

Case 3: Let $p > n$. This is dual to Case 2. □

We can conclude from Lemma 8.2.9 that $\mathcal{H}$ is a congruence on simple inverse ω-semigroups, and the $\mathcal{H}$-trivial semigroups of the form $\omega(G_0, T, \theta)$, where T has length d, are isomorphic to $\mathcal{B}_d$.

We will now do a same thing for all inverse ω-semigroups with kernel. We will see that $\mathcal{H}$ is a congruence in this more general case, and that the $\mathcal{H}$-trivial semigroups of the form $\omega(C, T, \theta)$, where C has length $k + 1$ and T has length d, will be isomorphic to $\mathcal{A}_{k,d}$, which we define in the following definition.

Definition 8.2.10. We define the semigroup $\mathcal{A}_{k,d}$ for d a positive integer and k a non-negative integer.

If $k = 0$, we define $\mathcal{A}_{k,d} = \mathcal{B}_d$. If k is positive, we adjoin k new elements to $\mathcal{B}_d$ to obtain $\mathcal{A}_{k,d} = \mathcal{B}_d \cup \{(j, j) \mid -k \leqslant j \leqslant -1\}$.

We then define the multiplication over all elements as

$$(a, b)(c, d) = (a - b + \max\{b, c\}, d - c + \max\{b, c\}).$$

This is consistent with the multiplication of $\mathcal{B}_d$. In this way, we can consider both $\mathcal{B}_d$ and $\mathcal{A}_{k,d}$ to be subsemigroups of $\mathbb{Z} \times \mathbb{Z}$ under the above multiplication.

Note that for $(a, b) \in \mathcal{B}_d$ and $j \leqslant 0$,

$$(a, b)(j, j) = (j, j)(a, b) = (a, b).$$

Lemma 8.2.11. *If Q is an inverse ω-semigroup with kernel, then there exists a homomorphism ϕ from Q to $\mathcal{A}_{k,d}$ for some positive integer d and non-negative integer k.*

Proof. By Theorem 8.2.5, we know that $Q = \omega(C, T, \theta)$, where $T = \bigcup_{\alpha=0}^{d-1} F_\alpha$ is a chain of d semigroups and $C = \bigcup_{i=-k}^{0} G_i$ is a chain of $k + 1$ semigroups.

We define $\phi : Q \to \mathcal{A}_{k,d}$. Since $Q = K \cup C$, we can define ϕ piecewise.

Let $(m, h_\alpha, n) \in K$. Then

$$(m, h_\alpha, n)\phi = (md + \alpha, nd + \alpha).$$

Let $g_i \in G_i$ with $-l \leqslant i \leqslant 0$. Then

$$g_i\phi = (i, i).$$

We must check that the image of elements of G_0 are well-defined, since these elements are in the intersection of C and K. Thinking of elements of G_0 as elements of C, ϕ maps them to $(0, 0)$ by the above definition. Thinking of elements of G_0 as elements of K, they have the form $(0, h_0, 0)$, for some $h_0 \in F_0$, and so ϕ maps them to $(0d + 0, 0d + 0) = (0, 0)$. This gives us well-definedness.

From the proof of Lemma 8.2.9, we know that ϕ restricted to K is a homomorphism.

We prove that ϕ restricted to C is a homomorphism. Let $g_i \in G_i$ and $g_j \in G_j$ and let $t = \max\{i, j\}$. Then

$$(g_i\phi)(g_j\phi) = (i, i)(j, j) = (t, t),$$

We know that $g_i g_j \in G_t$ by the definition of C. Therefore

$$(g_i g_j)\phi = (t,t).$$

Finally, let $g_i \in G_i$ and $(m, g_\alpha, n) \in K$. Then

$$g_i\phi\,(m, g_\alpha, n)\phi = (i,i)(md+\alpha, nd+\alpha) = (md+\alpha, nd+\alpha).$$

On the other hand,

$$g_i(m, g_\alpha, n) = (g_i 1_0)(m, g_\alpha, n).$$

We know that $g_i 1_0 \in G_0$ and therefore $g_i 1_0 = (0, h_0, 0)$ for some $h_0 \in F_0$. Therefore

$$g_i(m, g_\alpha, n) = (0, h_0, 0)(m, g_\alpha, n) = (m, (h_0\theta^m)g_\alpha, n)\,.$$

Since $h_0\theta^m \in F_0$ (whether or not $m = 0$), we have $(h_0\theta^m)g_\alpha \in F_\alpha$. Therefore

$$(g_i(m, g_\alpha, n))\,\phi = (m, (h_0\theta^m)g_\alpha, n)\,\phi = (md+\alpha, nd+\alpha).$$

The proof that $((m, g_\alpha, n)g_i)\,\phi = (m, g_\alpha, n)\phi g_i\phi$ is dual. $\square$

We introduce the main theorem of this subsection.

Theorem 8.2.12. *A semigroup S is a left I-order in an inverse ω-semigroup if and only if S satisfies the following conditions.*

(A) There is a homomorphism $\varphi : S \to \mathcal{A}_{k,d}$ for some $k \geqslant 0$, $d \geqslant 1$, such that, defining $s\varphi = (r(s), l(s))$, the image $l(S)$ is infinitely large.

(B) For $x, y, a \in S$,

  (i) $l(x), l(y) \geqslant r(a)$ and $xa = ya$ implies $x = y$,

  (ii) $r(x), r(y) \geqslant l(a)$ and $ax = ay$ implies $x = y$.

(C) For any $b, c \in S$ with $l(b) \geqslant l(c)$, there exists $u, v \in S$ such that

$$ub = vc,\ r(u) = r(v)\ \textit{and}\ l(u) = r(b).$$

Proof. We start with the forward implication. Let S be a left I-order in an inverse ω-semigroup, Q. Either Q has a kernel or Q has no kernel. We will deal with these two cases separately.

If Q has no kernel, then Q is an inverse ω-chain of groups, by Proposition 8.2.2. We label the groups that comprise Q as $G_0, G_1, G_2, \ldots$ with $G_iG_j \subseteq G_{\max\{i,j\}}$. For conciseness, we will label an element x with a subscript x_i to denote that $x_i \in G_i$. We will show that in this case S satisfies Properties (A), (B) and (C).

(A) We define $\phi : Q \to \mathcal{B}$ by

$$x_i\phi = (i, i),$$

for $x_i \in G_i$. Note that $r(a) = l(a)$ for all $a \in S$. Also $l(a) \geqslant l(b)$ if and only if $a \leqslant_{\mathcal{L}^Q} b$ by Lemma 2.4.4. We show that ϕ is a homomorphism. Let $x_i \in G_i$ and $y_j \in G_j$. Then

$$x_i\phi\, y_j\phi = (i, i)(j, j) = (\max\{i, j\}, \max\{i, j\}).$$

On the other hand, since $G_iG_j \subseteq G_{\max\{i,j\}}$, we know that $x_iy_j \in G_{\max\{i,j\}}$. Therefore

$$(x_iy_j)\phi = (\max\{i, j\}, \max\{i, j\}).$$

Therefore ϕ is a homomorphism. We restrict ϕ to S to get a homomorphism $\varphi : S \to \mathcal{B}$. Note that $\mathcal{B} = \mathcal{A}_{0,1}$, so φ is of the correct form. By the proof of Corollary 8.2.3, we know that S intersects every G_i. Therefore $l(S)$ is infinitely large.

(B) (i) Let $x, y, a \in S$ such that $l(x), l(y) \geqslant r(a)$ and $xa = ya$. We right multiply to obtain $xaa^{-1} = yaa^{-1}$. We know that $r(a) = l(a)$, for all $a \in S$. This gives us $l(x) \geqslant r(a) = l(a)$, and so $x \leqslant_{\mathcal{L}^Q} a$. By Lemma 2.2.6, this means that $xa^{-1}a = x$. Using the fact that $a^{-1}a = aa^{-1}$ in a Clifford semigroup, this gives us $xaa^{-1} = x$. Similarly, $l(y) \geqslant r(a)$ implies that $yaa^{-1} = y$. Therefore

$$x = xaa^{-1} = yaa^{-1} = y.$$

(ii) Dual of (B)(i)

(C) Let $b, c \in S$ such that $l(b) \geqslant l(c)$. Hence $b \leqslant_{\mathcal{L}^Q} c$. Since S is a left I-order in an inverse semigroup with totally ordered idempotents, we can use Property (T1) of Theorem 8.1.3 to give us that there exists $u, v \in S$

such that

$$u\,\mathcal{R}^Q\,v\,\mathcal{R}^Q\,vc = ub \text{ and } ub\,\mathcal{L}^Q\,b.$$

Since Q is a chain of groups, we see that $\mathcal{R}^Q = \mathcal{L}^Q$ with $x\,\mathcal{R}^Q\,y$ if and only if $r(x) = r(y)$. Therefore, we see that both $r(u) = r(v)$ and

$$l(u) = r(u) = r(ub) = l(ub) = l(b) = r(b).$$

Putting this all together, we have

$$ub = vc, \; r(u) = r(v) \text{ and } l(u) = r(b).$$

This proves the forward direction when Q has no kernel.

We now assume Q has a kernel. We know that $Q = \omega(C, T, \theta)$. We prove that Properties (A), (B) and (C) hold.

(A) From Lemma 8.2.11, we know there exists a homomorphism ϕ from Q to $\mathcal{A}_{k,d}$. Since S is a subsemigroup of Q, it follows that we can restrict ϕ to S to get a homomorphism $\varphi : S \to \mathcal{A}_{k,d}$.

Using Lemma 8.2.6 and Lemma 8.2.7, we see that defining $s\phi = (r(s), l(s))$, we have

$$a\,\mathcal{R}^Q\,b \text{ if and only if } r(a) = r(b)$$

and

$$a \leqslant_{\mathcal{L}^Q} b \text{ if and only if } l(a) \geqslant l(b).$$

Defining the function l in this way, l partitions the elements of Q into $\mathcal{L}^Q$-classes, and the image $l(Q)$ is infinitely large. Since S is a straight left I-order in Q, we know that S intersects every $\mathcal{L}$-class of Q. Therefore, the image $l(S)$ is also infinitely large.

(B) (i) Let $x, y, a \in S$ such that $l(x), l(y) \geqslant r(a)$ and $xa = ya$. We consider four cases:

Case 1: $l(x), l(y), r(a)$ all non-negative.

This implies that x, y and a are all elements of K. By right multiplication, we have that $xaa^{-1} = yaa^{-1}$. We start by calculating xaa^{-1}. Since a and x are both elements of K, we can write them as $a = (m, g_\alpha, n)$ and $x = (p, h_\beta, q)$. Then $l(x) = qd + \beta$ and $r(a) = md + \alpha$. From Proposition 2.6.1, we know that

$$aa^{-1} = (m, g_\alpha, n)(n, g_\alpha^{-1}, m) = (m, 1_\alpha, m),$$

where 1_α is the identity of F_α.

Since $l(x) \geqslant r(a)$, we have $qd + \beta \geqslant md + \alpha$. Since $\alpha, \beta \in [0, d-1]$, we have two cases: either $q > m$ or both $q = m$ and $\beta \geqslant \alpha$.

Case 1a: Let $q > m$. We calculate

$$xaa^{-1} = (p, h_\beta, q)(m, 1_\alpha, m) = (p, h_\beta(1_\alpha\theta^{q-m}), m - m + q).$$

Since θ is a homomorphism into F_0, we see that $1_\alpha\theta^{q-m} = 1_0$, the identity of T. Therefore

$$xaa^{-1} = (p, h_\beta, q) = x.$$

Case 1b: Let $q = m$ and $\beta \geqslant \alpha$. We calculate

$$xaa^{-1} = (p, h_\beta, q)(m, 1_\alpha, m) = (p, h_\beta 1_\alpha, q).$$

Since $\beta \geqslant \alpha$ as an integer, we know that $\beta \leqslant \alpha$ in the semilattice. Therefore $h_\beta 1_\alpha = h_\beta$ by Lemma 2.4.4. Therefore $xaa^{-1} = x$.

In either case, we have $xaa^{-1} = x$. We can obtain $yaa^{-1} = y$ similarly. Therefore,

$$x = xaa^{-1} = yaa^{-1} = y.$$

Case 2: $l(x), l(y)$ non-negative, $r(a)$ negative.

This implies that $x, y \in K$ and $a \notin K$. Therefore, using $xa = ya$, we have

$$x(1_0 a) = xa = ya = y(1_0 a).$$

Since $1_0 a \in G_0$, this gives us $r(1_0 a) = 0$. Therefore $l(x), l(y) \geqslant r(1_0 a)$ and we have reduced this to Case 1.

Case 3: One of $l(x), l(y)$ negative, the other non-negative.

Without loss of generality, let $l(x)$ be non-negative and let $l(y)$ be negative. Since $l(x), l(y) \geqslant r(a)$, we have that $r(a)$ is negative. Therefore, we have $x \in K$, $y \in G_j$ and $a \in G_k$, where j, k negative integers such that $j \geqslant k$. This implies that $xa \in K$, but $ya \in G_j$, and therefore $ya \notin K$. Since $xa = ya$, this leads to a contradiction.

Case 4: $l(x), l(y), r(a)$ all negative.

By the definition of φ, this means that x, y, a are all elements of the Clifford semigroup C, and $r(a) = l(a)$. This gives us $l(x) \geqslant r(a) = l(a)$, and so $x \leqslant_{\mathcal{L}^Q} a$. By Lemma 2.2.6, this means that $xa^{-1}a = x$. Using the fact that $a^{-1}a = aa^{-1}$ in a Clifford semigroup, this gives us $xaa^{-1} = x$. Similarly, $l(y) \geqslant r(a)$ implies that $yaa^{-1} = y$. Therefore

$$x = xaa^{-1} = yaa^{-1} = y.$$

(ii) Dual of (B)(i).

(C) Let $b, c \in S$ such that $l(b) \geqslant l(c)$. Using Lemma 8.2.7, we see that this implies that $b \leqslant_{\mathcal{L}^Q} c$. Since S is a left I-order in an inverse semigroup with totally ordered idempotents, we can use Property (T1) of Theorem 8.1.3 to give us that there exists $u, v \in S$ such that

$$u \,\mathcal{R}^Q\, v \,\mathcal{R}^Q\, vc = ub \;\text{ and }\; ub \,\mathcal{L}^Q\, b.$$

By Lemma 8.2.6 and Lemma 8.2.7, this gives us that

$$r(u) = r(v), \; r(v) = r(vc) \;\text{ and }\; l(ub) = l(b). \tag{8.1}$$

Since $ub = vc$, we have that

$$r(ub) = r(vc) = r(v) = r(u). \tag{8.2}$$

Since $\varphi : S \to \mathcal{A}_{k,d}$ is a homomorphism, we have

$$(r(ub), l(ub)) = (r(u), l(u))(r(b), l(b)) = (r(u) - l(u) + t, \, l(b) - r(b) + t),$$

where $t = \max\{l(u), r(b)\}$. Therefore, using (8.2), we have

$$r(u) = r(ub) = r(u) - l(u) + t,$$

and so $t = l(u)$. Since $l(ub) = l(b)$, by (8.1), we also have

$$l(b) = l(ub) = l(b) - r(b) + t,$$

and so $t = r(b)$. Therefore $l(u) = t = r(b)$. Putting this all together gives us

$$ub = vc, \ r(u) = r(v) \ \text{ and } \ l(u) = r(b).$$

This proves the forward implication. We now consider the converse. Let S satisfy Properties (A), (B) and (C). We start by proving that S satisfies the conditions of Theorem 8.1.3, with $\mathcal{R}'$ and $\leqslant_l$ defined by

$$a \,\mathcal{R}'\, b \ \text{ if and only if } \ r(a) = r(b) \tag{8.3}$$

and

$$a \leqslant_l b \ \text{ if and only if } \ l(a) \geqslant l(b). \tag{8.4}$$

Obviously $\mathcal{R}'$ is an equivalence relation and $\leqslant_l$ is a total order. We define $\mathcal{L}'$ as the equivalence relation associated with $\leqslant_l$. We prove that $\mathcal{R}'$ is left compatible and $\leqslant_l$ is right compatible.

Let $a, b, x \in S$ such that $a \,\mathcal{R}'\, b$, that is $r(a) = r(b)$. Using the fact that φ is a homomorphism, we have that

$$\begin{aligned}(r(xa), l(xa)) &= (r(x), l(x))(r(a), l(a)) \\ &= (r(x) - l(x) + \max\{l(x), r(a)\},\ l(a) - r(a) + \max\{l(x), r(a)\}).\end{aligned}$$

Therefore

$$r(xa) = r(x) - l(x) + \max\{l(x), r(a)\}.$$

Similarly

$$r(xb) = r(x) - l(x) + \max\{l(x), r(b)\}.$$

Since $r(a) = r(b)$, we know that

$$\max\{l(x), r(a)\} = \max\{l(x), r(b)\},$$

and therefore $r(xa) = r(xb)$ or, equivalently, $xa \,\mathcal{R}'\, xb$. Therefore $\mathcal{R}'$ is left compatible.

Now let $a, b, x \in S$ such that $a \leqslant_l b$, that is, $l(a) \geqslant l(b)$. Using the fact that φ is a homomorphism, we have that

$$\begin{aligned}(r(ax), l(ax)) &= (r(a), l(a))(r(x), l(x)) \\ &= (r(a) - l(a) + \max\{l(a), r(x)\},\ l(x) - r(x) + \max\{l(a), r(x)\}).\end{aligned}$$

Therefore

$$l(ax) = l(x) - r(x) + \max\{l(a), r(x)\}.$$

Similarly,

$$l(bx) = l(x) - r(x) + \max\{l(b), r(x)\}.$$

Since $l(a) \geqslant l(b)$, we know that

$$\max\{l(a), r(x)\} \geqslant \max\{l(b), r(x)\},$$

and therefore $l(ax) \geqslant l(bx)$ or, equivalently, $ax \leqslant_l bx$. Therefore $\leqslant_l$ is right compatible.

We now prove Properties (T1) - (T5) with $\mathcal{R}'$ and $\leqslant_l$ defined as in (8.3) and (8.4).

(T1) Let $\alpha, \beta \in S$ such that $\alpha \leqslant_l \beta$. By (C), there exists $\gamma, \delta \in S$ such that

$$\gamma\alpha = \delta\beta,\ r(\gamma) = r(\delta) \text{ and } l(\gamma) = r(\alpha).$$

Since $l(\gamma) = r(\alpha)$, we have that

$$(r(\gamma\alpha), l(\gamma\alpha)) = (r(\gamma), l(\gamma))\,(r(\alpha), l(\alpha)) = (r(\gamma), l(\alpha))\,.$$

Therefore $l(\gamma\alpha) = l(\alpha)$. Also, using $\gamma\alpha = \delta\beta$ and $r(\gamma) = r(\delta)$, we have that

$$r(\delta\beta) = r(\gamma\alpha) = r(\gamma) = r(\delta).$$

Putting this all together, we have

$$\gamma\,\mathcal{R}'\delta\,\mathcal{R}'\delta\beta = \gamma\alpha \text{ and } \gamma\alpha\,\mathcal{L}'\,\alpha.$$

(T2) Let $\alpha, \beta \in S$. By definition, and a now familiar argument,

$$l(\alpha\beta) = l(\beta) - r(\beta) + \max\{l(\alpha), r(\beta)\}.$$

We see that $\max\{l(\alpha), r(\beta)\} - r(\beta)$, is either zero or positive. Therefore, for all $\alpha, \beta \in S$, we have $l(\alpha\beta) \geqslant l(\beta)$, and hence $\alpha\beta \leqslant_l \beta$.

(T3) Let $a, b \in S$ such that $r(a) = r(b)$. We want to prove that $a \mathcal{R}^* b$. Let $x, y \in S^1$ such that $xa = ya$. It is sufficient to prove $xb = yb$. If $x = y = 1$ this is obviously true, so we only need to consider two cases without loss of generality: either $x, y \in S$, or $x \in S$ and $y = 1$.

Firstly let $x \in S$ and $y = 1$. Then $xa = a$. Therefore

$$l(xa) = l(a) - r(a) + \max\{l(x), r(a)\} = l(a),$$

so $\max\{l(x), r(a)\} = r(a)$, i.e. $r(a) \geqslant l(x)$. Therefore, using $xa = a$ again,

$$r(xa) = r(x) - l(x) + \max\{l(x), r(a)\} = r(x) - l(x) + r(a) = r(a),$$

so $r(x) = l(x)$.

Applying Property (C) with $b = c = a$, we have that there exists $X \in S$ such that $l(X) = r(a)$. We know that $l(X) = r(a) \geqslant l(x)$. Therefore we can apply Property (C) with $b = X$ and $c = x$, to obtain $u, v \in S$ such that

$$uX = vx, \; r(u) = r(v) \; \text{ and } \; l(u) = r(X).$$

Therefore

$$\begin{aligned}(r(uX), l(uX)) &= (r(u), l(u))\,(r(X), l(X)) \\ &= (r(u), l(X)) = (r(u), r(a))\,,\end{aligned} \tag{8.5}$$

using $l(X) = r(a)$ in the last equality. Since ϕ is a homomorphism, we can use $r(x) = l(x)$ to obtain

$$\begin{aligned}(r(vx), l(vx)) &= (r(v) - l(v) + \max\{l(v), r(x)\}, l(x) - r(x) + \max\{l(v), r(x)\}) \\ &= (r(v) - l(v) + \max\{l(v), r(x)\}, \max\{l(v), r(x)\})\,.\end{aligned} \tag{8.6}$$

Since $uX = vx$, we can compare the first variable of (8.5) and (8.6), and use $r(u) = r(v)$ to obtain

$$r(v) = r(u) = r(uX) = r(vx) = r(v) - l(v) + \max\{l(v), r(x)\}.$$

Therefore $\max\{l(v), r(x)\} = l(v)$. Comparing the second variable of (8.5) and (8.6), we can then obtain

$$r(a) = l(uX) = l(vx) = l(x) - r(x) + \max\{l(v), r(x)\} = l(v), \tag{8.7}$$

using $r(x) = l(x)$. Since $xa = a$, we can left multiply to obtain $vxa = xa$. By (8.7), $l(vx) = l(v) = r(a)$, and so we can apply Property (B)(i) to obtain $vx = v$. We right multiply by b to get $vxb = vb$. Using $r(x) - l(x)$, we have that

$$r(xb) = r(x) - l(x) + \max\{l(x), r(b)\} = \max\{l(x), r(b)\}.$$

Therefore $r(xb) \geqslant r(b)$. Also $r(b) = r(a) = l(v)$ by (8.7). Putting this together $r(xb), r(b) \geqslant l(v)$, and so we can apply Property (B)(ii) to $vxb = vb$ to obtain $xb = b$.

Now let $x, y \in S$ with $xa = xb$. We split this into three cases: Either $l(x)$ and $l(y)$ are both larger than $r(a)$, $l(x)$ and $l(y)$ are both smaller than $r(a)$, or one is larger and one is smaller.

Case 1: Let $l(x), l(y) \geqslant r(a)$.

We can use Property (B)(i) to give us that $xa = ya$ implies that $x = y$. Clearly then $xb = yb$.

Case 2: Let $l(x), l(y) \leqslant r(a)$.

Since $l(x) \leqslant r(a)$, we have that

$$r(xa) = r(x) - l(x) + \max\{l(x), r(a)\} = r(x) - l(x) + r(a). \tag{8.8}$$

Similarly, since $l(y) \leqslant r(a)$, we have

$$r(ya) = r(y) - l(y) + r(a). \tag{8.9}$$

We know that $xa = ya$, so that $r(xa) = r(ya)$. Therefore, comparing (8.8) with (8.9), we have that

$$r(x) - l(x) = r(y) - l(y). \tag{8.10}$$

Applying Property (C) with $b = c = a$, we have that there exists $X \in S$ such that $l(X) = r(a)$. We know that $l(X) = r(a) \geqslant l(x)$. Therefore we can apply Property (C) with $b = X$ and $c = x$, to obtain $u, v \in S$ such that

$$uX = vx, \ r(u) = r(v) \ \text{ and } \ l(u) = r(X).$$

Therefore, using $l(X) = r(a)$,

$$\begin{aligned} (r(uX), l(uX)) &= (r(u), l(u))\,(r(X), l(X)) \\ &= (r(u), l(X)) = (r(u), r(a))\,. \end{aligned} \tag{8.11}$$

Since ϕ is a homomorphism,

$$(r(vx), l(vx)) = (r(v) - l(v) + \max\{l(v), r(x)\}, l(x) - r(x) + \max\{l(v), r(x)\})\,. \tag{8.12}$$

Since $uX = vx$, comparing the first variable of (8.11) and (8.12) and using $r(u) = r(v)$ gives us

$$r(v) = r(u) = r(uX) = r(vx) = r(v) - l(v) + \max\{l(v), r(x)\}.$$

Therefore $\max\{l(v), r(x)\} = l(v)$. Comparing the second variable of (8.11) and (8.12), we can then obtain

$$\begin{aligned} r(a) = l(uX) = l(vx) &= l(x) - r(x) + \max\{l(v), r(x)\} \\ &= l(x) - r(x) + l(v). \end{aligned} \tag{8.13}$$

Therefore

$$l(v) = r(x) - l(x) + r(a) \geqslant r(x), \tag{8.14}$$

since $r(a) \geqslant l(x)$. We can combine this with (8.10) to obtain

$$l(v) = r(y) - l(y) + r(a) \geqslant r(y), \tag{8.15}$$

since $r(a) \geqslant l(y)$. We can then calculate

$$l(vy) = l(y) - r(y) + \max\{l(v), r(y)\} = l(y) - r(y) + l(v) = r(a), \quad (8.16)$$

using (8.15), (8.10) and (8.14). Since $xa = ya$, we can left multiply to obtain $vxa = vya$. Using (8.13) and (8.16), we have that $l(vy) = r(a) = l(vx)$. Therefore, we can apply Property (B)(i) to obtain $vx = vy$. We right multiply by b to get $vxb = vyb$.

Using $r(b) = r(a) \geqslant l(x)$, we have

$$r(xb) = r(x) - l(x) + \max\{l(x), r(b)\} = r(x) - l(x) + r(b).$$

Using $r(a) = r(b)$ and (8.14), this implies that

$$r(xb) = r(x) - l(x) + r(a) = l(v).$$

Similarly, using $r(b) = r(a) \geqslant l(y)$, we have

$$r(yb) = r(y) - l(y) + \max\{l(y), r(b)\} = r(y) - l(y) + r(b).$$

Using $r(b) = r(a)$ and (8.15), this implies that

$$r(yb) = r(y) - l(y) + r(a) = l(v).$$

Using $r(yb) = r(xb) = l(v)$, we can apply Property (B)(ii) to $vxb = vyb$ to obtain $xb = yb$.

Case 3: Let one of $l(x)$ and $l(y)$ be greater than or equal to $r(a)$ and the other be less than or equal to $r(a)$.

Without loss of generality, let $l(x) \geqslant r(a) \geqslant l(y)$. This gives us

$$l(xa) = l(a) - r(a) + \max\{l(x), r(a)\} = l(a) - r(a) + l(x).$$

and

$$l(ya) = l(a) - r(a) + \max\{l(y), r(a)\} = l(a).$$

Since $xa = ya$, we must have $l(xa) = l(ya)$. Therefore

$$l(a) - r(a) + l(x) = l(a).$$

This means that $l(x) = r(a)$ and we have reduced this to Case 2.

(T4) Let $\alpha, \beta, \gamma, \delta \in S$ such that $\gamma \,\mathcal{R}'\, \gamma\alpha \,\mathcal{L}'\, \alpha$ and $\delta \,\mathcal{R}'\, \delta\beta \,\mathcal{L}'\, \beta$. Then

$$(r(\gamma\alpha),\, l(\gamma\alpha)) = (r(\gamma),\, l(\alpha)) \text{ and } (r(\delta\beta),\, l(\delta\beta)) = (r(\delta),\, l(\beta))\,.$$

We know that

$$(r(\gamma\alpha), l(\gamma\alpha)) = (r(\gamma) - l(\gamma) + t,\, l(\alpha) - r(\alpha) + t)\,,$$

where $t = \max\{l(\gamma), r(\alpha)\}$. Therefore $(r(\gamma\alpha),\, l(\gamma\alpha)) = (r(\gamma),\, l(\alpha))$ implies that $l(\gamma) = t = r(\alpha)$. Similarly, $(r(\delta\beta),\, l(\delta\beta)) = (r(\delta),\, l(\beta))$ implies that $l(\delta) = r(\beta)$.

We can now see that $l(\gamma) = l(\delta)$ if and only if $r(\alpha) = r(\beta)$.

(T5) Let $\alpha, \beta, \gamma \in S$ such that $\alpha \,\mathcal{L}'\, \beta \,\mathcal{L}'\, \gamma\alpha = \gamma\beta$. Then

$$l(\gamma\alpha) = l(\alpha),\, l(\gamma\beta) = l(\beta) \text{ and } \gamma\alpha = \gamma\beta.$$

We know that

$$l(\gamma\alpha) = l(\alpha) - r(\alpha) + \max\{l(\gamma), r(\alpha)\}.$$

Therefore, since $l(\gamma\alpha) = l(\alpha)$, we have

$$\max\{l(\gamma), r(\alpha)\} = r(\alpha),$$

and so $r(\alpha) \geqslant l(\gamma)$. Similarly $l(\gamma\beta) = l(\beta)$ implies that $r(\beta) \geqslant l(\gamma)$.

We can then apply Property B(ii) to obtain $\alpha = \beta$.

We can now apply Theorem 8.1.3 to give us that S is a left I-order in an inverse semigroup with totally ordered idempotents, Q, such that for all $a, b \in S$, $a \,\mathcal{R}^Q\, b$ if and only if $r(a) = r(b)$ and $a \leqslant_{\mathcal{L}^Q} b$ if and only if $l(a) \geqslant l(b)$. Note that this implies that $a \mathcal{L}^Q b$ if and only if $l(a) = l(b)$. By Lemma 8.1.2 we also have that S is straight in Q.

We now need to prove that Q is an inverse ω-semigroup. We know that Q has totally ordered idempotents, so we only need that Q has a maximal idempotent and no minimal idempotent. We will do this by

Let $e \in E(Q)$. By Lemma 3.1.3, S intersects every $\mathcal{L}$-class of Q. Therefore, there exists $a_e \in S$ such that $a_e \mathcal{L}^Q e$. We define a function $L : E(Q) \to \mathbb{Z}$ by

$$L(e) = l(a_e), \text{ where } a_e \in S \text{ such that } a_e \mathcal{L}^Q e.$$

We know that such an a_e exists by above. The function L is well-defined, since $a_e, b_e \in S$ with $a_e \mathcal{L}^Q b_e \mathcal{L}^Q e$ implies that $l(a_e) = l(b_e)$.

We will now prove that L is injective. We see that $L(e) = L(f)$ implies that $a_e, a_f \in S$ such that $a_e \mathcal{L}^Q e$, $a_f \mathcal{L}^Q f$, and $l(a_e) = l(a_f)$. Therefore $f \mathcal{L}^Q a_e \mathcal{L}^Q a_f \mathcal{L}^Q f$. Since Q is inverse, there is a unique idempotent in each $\mathcal{L}$-class of Q. Therefore $e = f$.

Let $e, f \in E(Q)$, and let $a_e, a_f \in S$ such that $a_e \mathcal{L}^Q e$ and $a_f \mathcal{L}^Q f$. Using Lemma 2.1.3, we see that

$$e \leqslant f \iff e \leqslant_{\mathcal{L}^Q} f \iff a_e \leqslant_{\mathcal{L}^Q} a_f \iff l(a_e) \geqslant l(a_f) \iff L(e) \geqslant L(f).$$

Additionally, we can use the fact that L is an injective function, to obtain

$$e < f \iff L(e) > L(f). \tag{8.17}$$

Therefore natural ordering of the idempotents is a subset of the ordering of the integers. Now we just need that Q has a maximum idempotent and no minimum idempotent.

From the structure of $\mathcal{A}_{k,d}$, we know that the smallest possible value of $l(S)$, and therefore $L(E(Q))$, is $-k$. Let e be the element of $E(Q)$ such that $L(e)$ is the smallest value. By (8.17), e is the maximum idempotent of Q.

We prove that there is no minimum idempotent by contradiction. Assume that Q has a minimal idempotent, f. Let $a_f \in S$ such that $a_f \mathcal{L}^Q f$. By Property (A), $l(S)$ is an infinitely large subset of the integers. Therefore, the fact that

$l(S)$ has a minimum value implies that $l(S)$ has no maximum. Therefore there exists $t \in S$ such that $l(t) > l(a_f)$. Therefore $L(t^{-1}t) > L(a_f)$, and by (8.17), $t^{-1}t < f$. Contradiction.

Therefore Q is an inverse ω-semigroup. □

The previous theorem characterises all inverse ω-semigroups of left I-quotients. We now consider some special cases of inverse ω-semigroups. Since we have already dealt with inverse ω-semigroups of left I-quotients without kernel in Corollary 8.2.3, we start by characterising inverse ω-semigroups of left I-quotients with kernel.

Corollary 8.2.13. *A semigroup S is a left I-order in an inverse ω-semigroup with kernel if and only if S satisfies the following conditions.*

(A) There is a homomorphism $\varphi : S \to \mathcal{A}_{k,d}$ for some $k \geqslant 0$, $d \geqslant 1$, such that, defining $s\varphi = (r(s), l(s))$, the image $l(S)$ is infinitely large and there exists $x \in S$ such that $r(x) \neq l(x)$.

(B) For $x, y, a \in S$,

(i) $l(x), l(y) \geqslant r(a)$ and $xa = ya$ implies $x = y$,

(ii) $r(x), r(y) \geqslant l(a)$ and $ax = ay$ implies $x = y$.

(C) For any $b, c \in S$ with $l(b) \geqslant l(c)$, there exists $u, v \in S$ such that

$$ub = vc, \ r(u) = r(v) \ \text{and} \ l(u) = r(b).$$

Proof. First, let S be a left I-order in an inverse ω-semigroup with kernel Q. We know that $Q = \omega(C, T, \theta)$. From Lemma 8.2.11, we know there exists a homomorphism ϕ from Q to $\mathcal{A}_{k,d}$, defined by

$$(m, h_\alpha, n)\phi = (md + \alpha, nd + \alpha),$$

for $(m, h_\alpha, n) \in K$ and

$$g_i\phi = (i, i),$$

for $g_i \in G_i$. We can restrict ϕ to S to obtain a homomorphism $\varphi : S \to \mathcal{A}_{k,d}$.

Assume that $r(x) = l(x)$ for all $x \in S$. We know that there exists an element of the kernel of Q, (m, h_α, n), such that $m \neq n$. Since S is a left I-order in Q, there exists $s, t \in S$ such that

$$(m, h_\alpha, n) = s^{-1}t.$$

Since $r(x) = l(x)$ for all $x \in S$, we can write $r(s) = l(s) = i$ and $r(t) = l(t) = j$. We know that homomorphisms between inverse semigroups preserve inverses, so we can apply ϕ to both sides of this equation to obtain

$$(md + \alpha, nd + \alpha) = (i, i)^{-1}(j, j) = (i, i)(j, j) = (\max\{i, j\}, \max\{i, j\}),$$

which is a contradiction since $m \neq n$. Therefore, there exists an $x \in S$ such that $r(x) \neq l(x)$. By the proof of Theorem 8.2.12, we know that S satisfies the rest of the conditions.

Now let S satisfy Properties (A) - (C). By Theorem 8.2.12, we know that S has an inverse ω-semigroup of left I-quotients, Q. Also, by the use of Theorem 8.1.3 in the proof of Theorem 8.2.12, we know that for $a, b \in S$,

$$a\,\mathcal{R}^Q\,b \text{ if and only if } r(a) = r(b) \text{ and } a\,\mathcal{L}^Q\,b \text{ if and only if } l(a) = l(b).$$

We will now prove that Q has a kernel by contradiction.

Assume that Q does not have a kernel. By Proposition 8.2.2, we know that Q is a chain of groups. Therefore, if $a \in S$, then a is in a subgroup of Q. By Green's Theorem, this implies that $a\,\mathcal{H}^Q\,a^2$, or equivalently, $r(a^2) = r(a)$ and $l(a^2) = l(a)$. We know that

$$\left(r(a^2), l(a^2)\right) = (r(a), l(a))\,(r(a), l(a)) = (r(a) - l(a) + t, l(a) - r(a) + t)\,,$$

where $t = \max\{r(a), l(a)\}$. Therefore, $r(a^2) = r(a)$ implies that $t = l(a)$ and $l(a^2) = l(a)$ implies that $t = r(a)$. Together this means that $r(a) = l(a)$ for all $a \in S$. This contradicts Property (A). $\square$

We now characterise simple inverse ω-semigroups of left I-quotients.

Corollary 8.2.14. *A semigroup S is a left I-order in a simple inverse ω-semigroup if and only if S satisfies the following conditions.*

(A) There is a homomorphism $\varphi : S \to \mathcal{B}_d$ for some $d \geqslant 1$, such that, defining $s\varphi = (r(s), l(s))$, the image $l(S)$ is infinitely large and there exists $x \in S$ such that $(r(x), l(x)) \in (\{0\} \times \mathbb{N}) \cup (\mathbb{N} \times \{0\})$.

(B) For $x, y, a \in S$,

(i) $l(x), l(y) \geqslant r(a)$ and $xa = ya$ implies $x = y$,

(ii) $r(x), r(y) \geqslant l(a)$ and $ax = ay$ implies $x = y$.

(C) For any $b, c \in S$ with $l(b) \geqslant l(c)$, there exists $u, v \in S$ such that

$$ub = vc, \ r(u) = r(v) \ \text{and} \ l(u) = r(b).$$

Proof. Let S be a left I-order in a simple inverse ω-semigroup Q. By Theorem 8.2.4, we know that $Q \cong BR(T, \theta)$ for a finite chain of groups, T. By Lemma 8.2.9, we know that there exists a homomorphism ϕ from Q to $\mathcal{B}_d$, defined by

$$(m, h_\alpha, n)\phi = (md + \alpha, nd + \alpha),$$

for $(m, h_\alpha, n) \in Q$.

We could also write Q as $Q = \omega(C, T, \theta)$, where $C = G_0$. Considering Q this way, we can see that the homomorphism above is equal to the homomorphism from the proof of Corollary 8.2.13. Using Lemma 8.2.6 and Lemma 8.2.7, we see that defining $s\phi = (r(s), l(s))$, we have $a \,\mathcal{R}^Q\, b$ if and only if $r(a) = r(b)$ and $a \leqslant_{\mathcal{L}^Q} b$ if and only if $l(a) \geqslant l(b)$.

We will now prove that there exists $x \in S$ such that

$$(r(x), l(x)) \in (\{0\} \times \mathbb{N}) \cup (\mathbb{N} \times \{0\}).$$

Consider $(0, h_0, 1) \in Q$, where $h_0 \in F_0$. Since Q is a semigroup of straight left I-quotients of S, we can write $(0, h_0, 1)$ as

$$(0, h_0, 1) = s^{-1}t, \tag{8.18}$$

where $s, t \in S$ with $s \,\mathcal{R}^Q\, t$. That is, $r(s) = r(t)$. Applying ϕ to the left side of (8.18) gives us

$$(0, h_0, 1)\phi = (0, d).$$

Applying ϕ to the right side of (8.18) gives us

$$\begin{aligned}(s^{-1}t)\phi = (s\phi)^{-1}(t\phi) &= (r(s), l(s))^{-1}(r(t), l(t)) \\ &= (l(s), r(s))(r(t), l(t)) = (l(s), l(t)).\end{aligned}$$

Therefore $l(s) = 0$ and $l(t) = d$. If $(r(s), l(s)) \in \mathbb{N} \times \{0\}$, we are done by taking $x = s$. If not, then $r(s) = 0$, and therefore $r(t) = r(s) = 0$. As a result, $(r(t), l(t)) \in \{0\} \times \mathbb{N}$, and we are done by taking $x = t$.

Corollary 8.2.13 then proves the rest of Conditions (A) - (C).

Now let S satisfy Properties (A) - (C). Note that there exists $x \in S$ such that $r(x) \neq l(x)$. By Corollary 8.2.13, we know that S has an inverse ω-semigroup of left I-quotients with kernel, Q. Also, by the application of Theorem 8.1.3 in the proof of Corollary 8.2.13, we know that for $a, b \in S$,

$$a\,\mathcal{R}^Q\,b \text{ if and only if } r(a) = r(b) \text{ and } a \leqslant_{\mathcal{L}^Q} b \text{ if and only if } l(a) \geqslant l(b).$$

We will now prove that Q is simple by contradiction.

Assume that Q is not simple. By Theorem 8.2.5, this means that $Q = \omega(C, T, \theta)$, where C is a finite chain of at least two groups. Let G_{-k} be the largest group of C. Note that $G_{-k} \nsubseteq K$, where K is the kernel of Q.

Case 1: Let $(r(x), l(x)) \in \mathbb{N} \times \{0\}$. We see that $r(x) \neq l(x)$. Therefore, by the proof of Corollary 8.2.13, x is not in a subgroup of Q. Using the structure of Q, we see that this implies that x is an element of the kernel, K. Since K is an inverse semigroup, this means that $x^{-1}x$ is also in K.

Since $l(x) = 0$, x is the element of S with the smallest possible value of $l(S)$. By the proof of Theorem 8.2.12, this implies that $x^{-1}x$ is the maximal idempotent. Therefore $x^{-1}x \in G_{-k} \nsubseteq K$. Contradiction.

Case 2: Let $(r(x), l(x)) \in \{0\} \times \mathbb{N}$. We see that $r(x) \neq l(x)$. Therefore, by the proof of Corollary 8.2.13, x is not in a subgroup of Q. Using the structure of Q we see that this implies that x is an element of the kernel, K. Since K is an inverse semigroup, this means that xx^{-1} is also in K.

Since S is a straight left I-order in Q, we can write

$$xx^{-1} = u^{-1}v,$$

where $u, v \in S$ with $u \,\mathcal{R}^Q\, v$. By Lemma 3.3.6, this implies that $u \,\mathcal{R}^Q\, ux$ and $L_x \wedge L_x = L_{ux}$. We can rewrite this as $r(u) = r(ux)$ and $l(x) = l(ux)$. We see that

$$(r(ux), l(ux)) = (r(u) - l(u) + t, l(x) - r(x) + t),$$

where $t = \max\{l(u), r(x)\}$. Therefore, since $(r(ux), l(ux)) = (r(u), l(x))$, this implies that $l(u) = t = r(x) = 0$.

Since $l(u) = 0$, u is the element of S with the smallest possible value of $l(S)$. By the proof of Theorem 8.2.12, this implies that $u^{-1}u$ is the maximal idempotent. Therefore $u^{-1}u \in G_{-k} \nsubseteq K$ We see that

$$\begin{aligned} u^{-1}u &= u^{-1}uu^{-1}u = u^{-1}vv^{-1}u = (u^{-1}v)(u^{-1}v)^{-1} \\ &= (xx^{-1})(xx^{-1})^{-1} = xx^{-1}xx^{-1} = xx^{-1}. \end{aligned}$$

Therefore $xx^{-1} = u^{-1}u \in G_{-k} \nsubseteq K$. Contradiction. □

By Theorem 8.2.4, we can use Corollary 8.2.14 with $d = 1$ to obtain the bisimple case. This is similar, but not identical to Theorem 3.1 of [15].

Corollary 8.2.15. *A semigroup S is a left I-order in a bisimple inverse ω-semigroup if and only if S satisfies the following conditions.*

(A) There is a homomorphism $\varphi : S \to \mathcal{B}$ such that, defining $s\varphi = (r(s), l(s))$, the image $l(S)$ is infinitely large and there exists $x \in S$ such that $(r(x), l(x)) \in (\{0\} \times \mathbb{N}) \cup (\mathbb{N} \times \{0\})$.

(B) For $x, y, a \in S$,

- *(i) $l(x), l(y) \geqslant r(a)$ and $xa = ya$ implies $x = y$,*
- *(ii) $r(x), r(y) \geqslant l(a)$ and $ax = ay$ implies $x = y$.*

(C) For any $b, c \in S$ with $l(b) \geqslant l(c)$, there exists $u, v \in S$ such that

$$ub = vc, \; r(u) = r(v) \; \text{ and } \; l(u) = r(b).$$

We now characterise inverse ω-semigroups of left I-quotients of Type (3) from Proposition 8.2.2.

Corollary 8.2.16. *A semigroup S is a left I-order in an inverse ω-semigroup with proper kernel if and only if S satisfies the following conditions.*

(A) There is a homomorphism $\varphi : S \to \mathcal{A}_{k,d}$ for some $k \geqslant 1$, $d \geqslant 1$, such that, defining $s\varphi = (r(s), l(s))$, the image $l(S)$ is infinitely large and includes a negative number, and there exists $x \in S$ such that $r(x) \neq l(x)$.

(B) For $x, y, a \in S$,

(i) $l(x), l(y) \geqslant r(a)$ and $xa = ya$ implies $x = y$,

(ii) $r(x), r(y) \geqslant l(a)$ and $ax = ay$ implies $x = y$.

(C) For any $b, c \in S$ with $l(b) \geqslant l(c)$, there exists $u, v \in S$ such that

$$ub = vc, \ r(u) = r(v) \ \text{and} \ l(u) = r(b).$$

Proof. Let S be a left I-order in an inverse ω-semigroup with proper kernel, Q. By Theorem 8.2.5, we know that $Q = \omega(C, T, \theta)$, where C is a finite chain of at least two groups. Let G_{-k} be the largest group of C. Note that k is a positive integer and therefore $G_{-k} \nsubseteq K$, where $K = BR(T, \theta)$ is the kernel of Q.

By Lemma 8.2.11, we know that there exists a homomorphism ϕ from Q to $\mathcal{A}_{k,d}$, defined by

$$(m, h_\alpha, n)\phi = (md + \alpha, nd + \alpha),$$

for $(m, h_\alpha, n) \in K$, and

$$g_i\phi = (i, i),$$

for $g_i \in G_i$. This is the same homomorphism from the proof of Corollary 8.2.13.

Since S is a straight left I-order in Q, Lemma 3.1.3 tells us that S intersects every $\mathcal{L}$-class of Q. We have that G_{-k} is its own $\mathcal{L}^Q$-class. Therefore there exists $s \in S \cap G_{-k}$. We see that

$$l(s) = -k,$$

and so $l(S)$ includes a negative number.

Corollary 8.2.13 then proves the rest of Conditions (A) - (C).

Now let S satisfy Properties (A) - (C). By Corollary 8.2.13, we know that S has an inverse ω-semigroup of left I-quotients with kernel, Q. Also, by the use of

Theorem 8.1.3 in the proof of Corollary 8.2.13, we know that for $a, b \in S$,

$$a \,\mathcal{R}^Q\, b \text{ if and only if } r(a) = r(b) \text{ and } a \leqslant_{\mathcal{L}^Q} b \text{ if and only if } l(a) \geqslant l(b).$$

Since $\leqslant_{\mathcal{L}^Q}$ is a total order, the meet structure of the $\mathcal{L}^Q$-classes is completely determined by $\leqslant_{\mathcal{L}^Q}$. Therefore, $\varphi : S \to \mathcal{A}_{k,d}$ from Condition (A), satisfies the conditions of Theorem 3.3.7. We apply Theorem 3.3.7 to obtain that φ lifts to a homomorphism $\bar{\varphi} : Q \to \mathcal{A}_{k,d}$.

We will now prove that the kernel of Q is proper. This is equivalent to proving that Q is not simple, which we will do by contradiction.

Assume that Q is simple. We know that there exists an $x \in S$ such that $r(x) \neq l(x)$. By the definition of $\mathcal{A}_{k,d}$, we see that $x\varphi$ is in the $\mathcal{B}_d$ part of $\mathcal{A}_{k,d}$. We also know that there exists an $s \in S$ such that $l(s)$ is negative. By the definition of $\mathcal{A}_{k,d}$, we see that $s\varphi$ is not in $\mathcal{B}_d$. Since Q is simple, we know that x and s are $\mathcal{J}$-related. Therefore there exists $p, q \in Q$ such that

$$s = pxq.$$

We apply $\bar{\varphi}$ to this to obtain

$$s\varphi = (p\bar{\varphi})x\varphi(q\bar{\varphi}).$$

We see that $\mathcal{B}_d$ is an ideal of $\mathcal{A}_{k,d}$. Therefore, since $x\varphi \in \mathcal{B}_d$, we have

$$s\varphi = (p\bar{\varphi})x\varphi(q\bar{\varphi}) \in \mathcal{B}_d.$$

However, we also know that $s\varphi \notin \mathcal{B}_d$, giving us a contradiction. $\square$

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