Embedded cobordisms, motion groupoids and topological quantum field theories.

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Abstract

Topological phases of matter are a particular class of phases of matter which are potentially of interest in the construction of quantum computers. Examples are given by fractional quantum Hall states. Topological quantum field theories (TQFTs), and generalisations of TQFTs, are mathematical constructions that axiomatise the properties of topological phases. In this thesis we are motivated by the aim of understanding possible statistics of generalised quasiparticles (loops or strings in 3-dimensions, for example), in topological phases of arbitrary dimension.

In 2-dimensional topological phases, the worldlines of monotonic evolutions of point particles, which start and end in the same configuration, can be modelled by the braid groups. The braid group has several different topological realisations, each with possible generalisations. In particular it has realisations as a mapping class group and as a motion group. In Chapter 4 we construct for each manifold M its *motion groupoid* Mot_M, whose objects are the power set of M, and a *mapping class groupoid* MCG_M with the same object class. These generalise the classical definition of a motion group and mapping class group associated to a pair of a manifold and a subset. The classical definitions can be recovered by considering the automorphisms of the corresponding object. Our motivating aim is to frame questions that inform the modelling of the worldlines of particles in topological phases. These include questions about the skeletons of these categories, and about monoidal structures. But our constructions also frame technical questions that we *answer* here, such as the following. For a chosen manifold M we explicitly construct a functor $\mathsf{F}\colon \mathrm{Mot}_M \to \mathrm{MCG}_M$ and prove that this is an isomorphism if π_0 and π_1 of the appropriate space of self-homeomorphisms of M is trivial. In particular we have an isomorphism in the physically important case $M = [0,1]^n$ with fixed boundary, for any $n \in \mathbb{N}$.

In Chapter 5 we are motivated by the construction of embedded TQFTs. These are functors from some choice of embedded cobordism category, which models the worldlines of particles in topological phases, into **Vect**. We construct a category HomCob, and a family of functors $\mathsf{Z}_G\colon \mathrm{HomCob} \to \mathbf{Vect}$, one for each finite group G. The category HomCob has equivalence classes of cospans of topological spaces as morphisms. This is a very general construction, making it possible to later fix a choice of a categorical model of particle worldlines, and obtain a TQFT by precompsing Z_G with a functor into HomCob. Roughly, such a functor can be realised by taking the complement of the particle worldlines in the ambient space. Notice we do not require that the complement be modelled as a manifold. We also give an interpretation of the functor Z_G showing that it is explicitly calculable. The construction is a generalisation of an untwisted version of Dijkgraaf-Witten.

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Chapter 1

Introduction

The content of this work looks like pure maths, but we are motivated and informed by physics. We begin by attempting to give enough of the physical picture, and directions to more complete references, to give the reader an idea of our motivation, and thus an understanding of the choices we make in this thesis.

In Section 1.1 we cover the background, and then in Section 1.2 we explain the work covered in this thesis, and give a thesis overview.

1.1 Physical background

Our motivation can be concisely stated as 'modelling the statistics of generalised quasiparticle excitations in topological phases', so this is what we aim to make sense of here.

1.1.1 Quasiparticles

The formalism of quantum mechanics allows for the possibility of precisely two types of point particle in 3-dimensions, bosons and fermions (see e.g. [DV11, Sec.6.3]). It will be most useful for us to consider this from the point of view of the path integral formalism of quantum mechanics, developed by Feynman [HF65]. Suppose a quantum system evolves in time from an initial particle configuration, to a final particle configuration. The path integral formalism says that the change to the wave function describing the system, caused by such an evolution, is given by a sum over all possible paths of the particles from the initial configuration to the final configuration. Here path is not yet a mathematically well-defined notion, rather just an allowed particle trajectory in a given physical setting. We refer to particle trajectories in spacetime as worldlines.

Suppose now that physical space is well modelled by $\mathbb{R}^d$, where d is the spatial dimension. Suppose also that allowed particle trajectories are paths in the configuration space of the particles – for indistinguishable particles this is the set of all possible arrangements of particles in $\mathbb{R}^d$, where each particle occupies a different point, quotiented by permutations, and topologised as a subset of $(\mathbb{R}^d)^n$ where n is the number of particles. It can be shown that it is consistent with quantum mechanics to split the sum over paths further into summands, each consisting of a sum over all paths in a homotopy class and a coefficient known as a weight factor. Homotopy classes of paths in the configuration space are elements of the fundamental group of the configuration space, and the allowed weight factors are representations of the fundamental group (see [LD71; LM77]). These weight factors are what is meant by particle statistics.

In a 3-dimensional system consisting of two indistinguishable particles, there are only two homotopy classes of loops in the configuration space, one class that swaps the positions of the two particles, and one that contains the identity. Moreover following a trajectory which swaps the particles twice is homotopic to the identity. Hence there are two possible 1-dimensional representations, the trivial representation and the representation sending the swap to -1. These two choices of representations correspond to bosons and fermions. (Higher dimensional representations are shown to be excluded in [LD71].) In a system of N indistinguishable particles in 3 dimensions, particle statistics are representations of the symmetric group.

In a 2-dimensional system with two point particles, the path which takes one particle once around the other, which remains fixed, and returns it to its initial position is not homotopic to the identity. Hence the setup allows for point particles which have non-trivial braiding statistics, by which we mean a path in the relevant configuration space which swaps a pair of particles can change the wave function by a factor other than 1 or -1. In 2 dimensions, the statistics of a system of N indistinguishable particles are given by representations of the braid group, and they are no longer restricted to 1-dimensional representations. Such particles whose interchange can give *any* phase were called anyons by [Wil82a] (for further discussion of how anyons arise see also [Wil82b; LM77]). Note in particular that, after fixing a mathematical model of the worldlines of particles, the statistics of the particles are given by representations of the mathematical model. Trajectories which are topologically non-trivial is a requirement for non-trivial statistics, but it is not sufficient.

Electrons, protons and photons are all still bosons or fermions, even when confined to a plane. But if a system confined to a plane has *quasiparticles*, these may be anyons. Quasiparticle excitations in condensed matter systems are local excitations of the ground state. These emergent phenomena allow us to model the system as though it was made up of these emergent particles in a vacuum [Nay+08, Sec.II.A]. Anyons were considered in conformal field theory in [MS89], and in the context of discrete gauge theories in [Bai80; BDWP92]. Also *topological phases*, the fractional quantum Hall effect, for example, support anyons [ASW84; Hal84; MR91].

1.1.2 Topological phases

In condensed matter physics, the principle of emergence says that the properties of a system are determined by the arrangements of particles within the system [Wen17]. Phases of matter are equivalence classes of arrangements of particles which share certain physical properties [RW18]. Think, for example, of a glass containing ice and water. This physical system consists of regions in four distinct phases: the glass, the water, the ice and the air above the water. The densities, indices of refraction, the melting point of the ice and the boiling point of the water are all examples of properties which are uniform within each phase. A phase transition is an abrupt change in the physical properties of the system. A question then is how to classify phases. That is, what invariants can be used to determine if two systems are in the same or different phases.

A key result in answering this question was provided by Landau [Lan36; GL50]. Many phases can be classified by certain symmetry groups of the system, and a phase transition occurs when a symmetry is broken. An example is ferromagnets. Below the Curie temperature, all spins are aligned parallel to one another, and the material is permanently magnetic. Above the Curie temperature, the spins are randomly aligned, so the system gains rotational symmetry, and the material loses its magnetism [KW62].

The first experimental realisation of a system completely outside of the classification afforded by Landau theory came from the fractional quantum Hall effect [TSG82]. This effect is observed in 2-dimensional systems of electrons in a strong magnetic field, at low temperature. One of the unusual properties of the fractional quantum Hall effect can be observed by taking a resistance measurement. The measured resistance depends only on the order in which the voltage and current leads are connected around the edge of the sample, and smooth deformations of the positions do not change the measurement. This is in contrast with a resistance measurement on a sample of metal, for example, which depends on exactly where the leads are attached and on the size and shape of the sample.

A feature of fractional quantum Hall systems is the emergence of *topological* quasiparticles – as they are quantum systems, the evolution of these systems depends on the wordlines of these particles, but only up to their 'topology' [Nay+08] (we are being deliberately vague about the meaning of topology here). Fractional quantum Hall states, and the related chiral spin states were thus dubbed *topological phases* [Wen89].

We make common and practical assumption that physical systems are well-mathematically-modelled as living in ambient spaces that are manifolds (other assumptions are explored in [Sch79], for example, and references therein). Then we can say that precisely, a physical system is in a topological phase if its low-energy observable properties are invariant under diffeomorphisms of the spacetime manifold in which the system lives — see e.g. [Nay+08, Sec.3]. For further discussion of how such emergent topology can arise in physical, hence metric, systems we direct the reader to, for example [Fra13] and references therein.

The topological quasiparticles supported by a topological phase are an invariant of the phase, and the presence of topological quasiparticles is a sufficient condition for a system to be in a topological phase [Wen17]. Topological phases in 3 spatial dimensions support particles which are point like, as well as loop and string excitations which may be knotted and linked [Wen+18].

1.1.3 Topological quantum computation

Topological quantum computation refers to using a topological phase, which supports topological quasiparticles, to perform computation [Nay+08]. Computations are carried out by braiding quasiparticles around each other, and non-trivial operations are possible because the particles are anyons – thus have non-trivial statistics.

Decoherence describes the collapse of the wavefunction as a result of its interaction with the environment, and is a problem in other quantum computing models [WS06]. The operations in topological quantum computation are protected from these errors since they are sensitive only to the topology of the particle motion, hence remain unchanged by small perturbations [Kit03].

Topological phases have the property that there is an energy gap between the ground state and the first excited state, this is known as a gapped ground state. The presence of a gapped ground state means that, at low energy, essentially the only way a topological phase can move from one ground state to another is by braiding particles around each other. Moreover, to perform a computation, the particles only have to trace the correct braid. In other models of quantum computation, one has to take exceptional care to ensure that a given system evolution is actually the one performed. Hence topological quantum computers are theoretically completely protected from control errors [LP17]. For more on topological quantum computing, see for example [Kit03; Fre+03].

1.1.4 Topological quantum field theory

Topological quantum field theories (TQFTs) are mathematical constructions abstracting the properties of topological phases. Indeed, one way to define a topological phase is as a physical system whose low-energy effective field theory is a TQFT [Nay+08, Sec.III.A].

The first constituent part of a TQFT is a cobordism category. Locality in field theories implies that a global computation can be made on spacetime, by cutting spacetime into parts, each of which represents some finite time evolution of the system, calculating on the component parts, and then composing the results [Fre92]. Note that the diffeomorphism invariance of topological phases implies something stronger than this. We must have that the image of an evolution of the form $X \times [0,1]$, where X is any ambient space and $[0,1]$ represents time, must be the same as the image of $X \times J$ where J is any finite length interval. This implies the unit of time is unimportant in TQFT, and this has various implications, see [Nay+08, Sec.III.A.] for more. When making cuts we must retain, in each component part, sufficient information to capture the local interaction – the result of the computation should not depend on the choice of how to make cuts. Precisely what constitutes ‘sufficient’ here will depend on the field theory. We can ensure sufficient information is retained by giving conditions on the way we are allowed to make cuts in terms of some form of 'collaring'.

We again assume ambient space in physical systems is well-mathematically-modelled by a manifold. An $(n+1)$-dimensional *concrete cobordism* from an n-dimensional oriented smooth manifold X to an n-dimensional oriented smooth manifold Y, is an $(n+1)$-dimensional oriented manifold M equipped with an orientation preserving diffeomorphism $\phi\colon \bar{X} \sqcup Y \to \partial M$ (where the bar denotes the opposite orientation) [Lur09]. The collection of all cobordisms will, in general, be too large to be an interesting object of study. Thus we use the diffeomorphism invariance of topological phases to add an equivalence relation. A *cobordism* is an equivalence class of concrete cobordisms, where a pair of cobordisms are equivalent, roughly speaking, if there is a diffeomorphism between them which commutes with the maps into the boundary.

The question of how to impose a sufficiently strong equivalence, such that we obtain a manageable algebraic structure, will be a recurring theme of this thesis. By manageable, we will often mean a finitely generated category, such that we can give a presentation, and thus construct representations. Usually these categories are finitely generated only for a certain subcategory of the categories we construct, a specific choice of ambient space or particle type, for example.

Cobordisms can be composed via pushouts of representative concrete cobordisms, intuitively this can be thought of as gluing along the boundary – see [Koc04; Lur09], for example, for more. We note that there is not a unique way to globally define smooth structures on such compositions, although they are all represent the same cobordism. With this composition, cobordisms can be organised into a category [Mil65], we denote the category whose objects are n-dimensional manifolds by $\mathbf{Cob}_n$.

A TQFT is a functor from a cobordism category, mapping a manifold X to a $\mathbb{C}$ vector space, which we think of as the space of states, and a cobordism to a linear map. Although not originally written in terms of categories, this axiomatisation of TQFT is due to [Ati88]. To understand particles in TQFTs, we can add embeddings of submanifolds modelling the worldlines of the particles in cobordisms, as is the approach in [Wit89]. There are various ways to construct a category of embedded cobordisms, and in this thesis we will not try to explicitly fix a choice, rather construct a framework to investigate the implications of various choices. We will refer to all such categories as *embedded cobordism categories*. Here

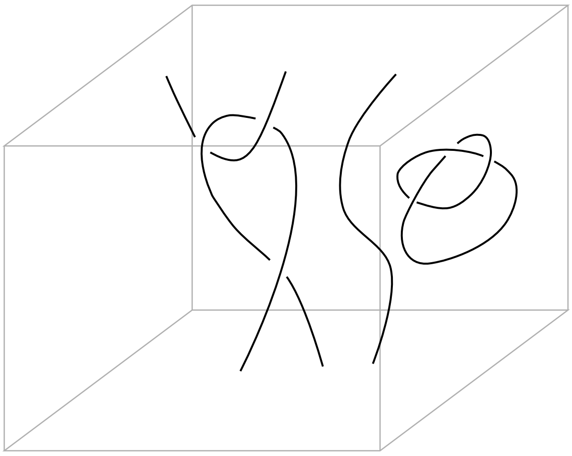

Figure 1.1: Schematic representing an example of a concrete morphism in the tangle category. The ambient space is $[0,1]^2$, and at $t = 0$, along the bottom boundary, there are three embedded point particles. As time progresses up the page the left most two particles braid with each other and then braid with two particles which are created, before annihilating. The right most particle does not braid with any other particles. There are also two creations giving four particles which are braided into a trefoil knot, and then pairwise annihilated. At $t = 1$, on the top boundary, there are three point particles.

we give some examples of concrete morphisms in various embedded cobordism categories.

Let M be a manifold representing some ambient space. It is common to restrict evolutions of spacetime to be of the form $M \times [0,1]$, so then worldlines of particles are submanifolds embedded in $M \times [0,1]$, such that the boundaries of the submanifolds are in $M \times \{0,1\}$. This is the approach taken in the tangle category, which corresponds to point particles in 2-dimensional space (see [Kas12]). This approach is also discussed in general in [Pic97], and in [BD95] where they are referred to as generalised tangle categories.

In Figure 1.1 we have a schematic representation of an example of a concrete morphism in the tangle category. We think of time as going up the page, so the bottom boundary corresponds to a configuration of three point particles in $[0,1]^2$, and similarly there are three particles at the top of the box. In the interior the particles braid and knot with each other as shown. We also have pairwise creations and annihilations of particles.

Figure 1.2 depicts an example of a concrete embedded cobordism, with $M = [0,1]^2$, and loop-like particles. At the bottom boundary of the box, there is a single loop particle. Progressing up the page, this loop splits into two, with one loop remaining inside the other. At the final time we have two nested loop particles. In Figure 1.3 we have a schematic representing loop particles in $[0,1]^3$. In 4-dimensions, it is possible for the loop particles to pass through each other as shown. It is also possible to have knotted or linked

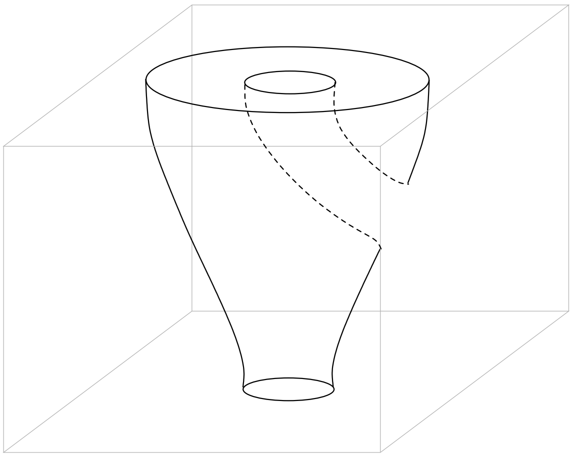

Figure 1.2: Schematic representing an embedded cobordism, with ambient space $[0,1]^2$. At the bottom boundary there is a copy of S^1 embedded in $[0,1]^2$. As time progresses up the page, the loop splits in two, and one loop remains inside the other, tracing an embedded submanifold in $[0,1]^3$. At the top the system is represented by an embedding of two copies of S^1 in $[0,1]^2$.

particles, whose worldlines form knotted surfaces.

It is also possible to construct embedded cobordism categories which allow for the worldlines of particles to be embedded in a non-trivial evolution of spacetime, this is related to the cobordism with defects framework in [CMS16]. We note however, that it is not straightforward to allow for particles, and worldlines of particles which are not manifolds in embedded cobordism categories. In Chapter 4, and the examples therein, we discuss a framework that allows for modelling particles in topological phases which are any subset of the ambient manifold.

1.2 Present work

Our motivating aim here is to construct a framework allowing for the study of the statistics of generalised quasiparticle excitations in topological phases, varying both the ambient manifold and the topology of the particles (loop or string excitations in a 3-ball, for example).

1.2.1 Motion Groupoids and mapping class groupoids

Modelling particle trajectories as paths in the appropriate configuration space leads to the result that the statistics of point particles in 2 dimensions are realised by representations of the braid groups, since the braid groups can be defined as homotopy classes of paths in

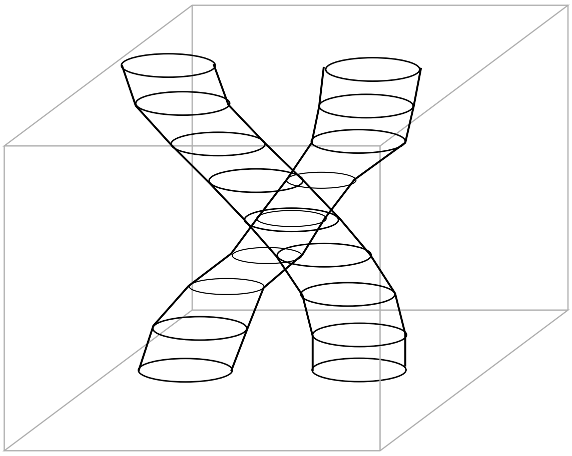

Figure 1.3: This schematic represents an embedded cobordism in 4-dimensions projected down a dimension. The bottom boundary represents two loop particles in 3-dimensions. We have then marked the image of each loop particle at various times progressing up the page. The left most particle shrinks and passes through the right most particle, and at the top boundary the particles have swapped positions.

the configuration space of point particles in 2 dimensions [BB05]. Similarly one also arrives at the result that the statistics of (unknotted, unlinked) loop particles in 3 dimensions are realised by representations of the loop braid groups [Dam17].

The tangle category, models point particles in 2 dimensions, is one of the main examples of an embedded cobordism category in the literature [Kas12]. One way to approach the study of embedded cobordism categories is to restrict initially to the simpler subcategories of isomorphisms. Isomorphisms in the tangle category are equivalence classes of monotonic embeddings of unit intervals in $\mathbb{R}^2 \times [0,1]$, and this subcategory can be shown to be isomorphic to the braid category (this follows from Theorem 12 of [Art50]). The braid category has the braid groups as automorphism groups. Physically this says that again, the braiding statistics of point particles in 2-dimensional space are again given by representations of the braid groups. One can also define ambient isotopy equivalence classes of monotonic embeddings of (unknotted and unlinked) loop-particles in $\mathbb{R}^3 \times [0,1]$, which is another way to define the loop braid groups [Dam17]. This is why the statistics of loop-particles in 3-dimensional topological phases give rise to representations of the loop braid group [BFMM19].

Part of our aim is to understand in general when the different ways of modelling the worldlines of particles lead to different algebraic structures, and thus different possible statistics.

The braid groups also have topological realisations as motion groups, and as mapping class groups. Our objective here is to generalise these topological constructions to, for each ambient manifold, a motion groupoid and a mapping class groupoid. The object set in each category is the power set of the underlying set of the ambient manifold. The objects model particle types, and thus this allows for particle types which are any subset of the ambient manifold. A novel aspect of our construction is that we work with groupoids – allowing for worldlines of particles which do not start and end in the same configuration. We also construct a functor between the motion groupoid and mapping class groupoid of a manifold M, and conditions for isomorphism.

We give a further mathematical introduction in 4.1.

1.2.2 Topological quantum field theories for cospan cobordisms

TQFTs give representations of embedded cobordism categories, thus statistics of particles in topological phases. Examples of TQFTs which are determined using homotopy invariants of the cobordisms include [Qui95; Yet92] and an untwisted version of Dijkgraaf-Witten [DW90]. Our motivating aim is to construct TQFTs using homotopy invariants for embedded cobordisms.

TQFTs will often factor through categories with stronger equivalence relations than those in cobordism categories. Often these categories will be easier to work with. Concrete cobordisms can be seen as cospans, i.e. diagrams of the form $i\colon X \to M \leftarrow Y : j$, considered as a kind of morphism from X to Y, with some conditions on the maps i and j. In Section 5.3 we define *homotopy cobordisms*; cospans of topological spaces, with a condition on the maps in terms of cofibrations. There is a canonical map from a concrete non-embedded cobordism to a homotopy cobordism, and we can map an embedded cobordism to a homotopy cobordism by taking the complement of the embedded space. Note that for this to work for all embedded cobordisms we might be interested in, we require the generality of working with topological spaces. This is because the complement of an embedded space, or even manifold, may not be itself a manifold. We quotient the *homotopy cobordisms* by a kind of homotopy equivalence of cospans and organise them into a category HomCob.

In Section 5.4 we construct a functor from HomCob into the category **Vect**, of vector spaces and linear maps. For this we follow the construction of [Yet92], working with fundamental groupoids, as opposed to triangulations. Our construction is more general than that of Yetter as we work with topological spaces, and in any dimension. A key aspect of our construction is that we give an interpretation working with the fundamental groupoid with respect to a finite set of basepoints, allowing for explicit calculation.

We note that although we construct a specific family of functors, HomCob is a useful source category to construct a larger class of TQFTs. We can consider functors which make use of higher homotopy groups, and algebraic structures which are models for higher n-types, such as crossed modules, for example.

1.2.3 Thesis overview

In Chapter 2 we collect some notation and conventions. In Chapter 3 we give preliminaries that we will need throughout the thesis, and fix notation. Here we introduce some well known constructions in unusual ways which we will find to be useful in what follows: the introduction of *magmoids* in Section 3.1 leading to categories, and obtaining the fundamental groupoid via the *path magmoid* in Section 3.3, for example.

In Chapter 4 we have the construction of the motion groupoid Mot_M and the mapping class groupoid MCG_M of a manifold M, the object class of each is the power set $\mathcal{P}(M)$ (Theorems 4.3.37 and 4.5.4). Picking a single object and looking at the automorphism group gives back respectively, the motion group and mapping class group of a manifold, subset pair, as given in [Gol81; Dam17]. We also have a version of the motion group fixing a distinguished subset (Theorem 4.3.47) and an equivalent theorem for the mapping class groupoid (Theorem 4.5.7).

In Theorem 4.6.1 we construct a functor from Mot_M to MCG_M. We also have Theorem 4.4.6 which says that there is an alternative congruence on motions which leads to the same groupoid, Mot_M. This will be necessary to prove Theorem 4.6.12, which gives conditions under which the aforementioned functor is an isomorphism. In Theorem 4.6.13 we have a version giving conditions for isomorphism of groupoids relative to some distinguished subset. We also give many examples demonstrating the richness of our construction.

In Chapter 5 we give the construction of a category CofCsp which has topological spaces as objects and *cofibrant cospans* as morphisms (Theorem 5.3.16). We then have Theorem 5.3.21 which proves there is a monoidal structure on CofCsp with monoidal product which, on objects, is given by disjoint union. Next we obtain the category HomCob (Theorem 5.3.32) as a subcategory of CofCsp with a finiteness condition on spaces, and show that the monoidal structure from CofCsp also makes HomCob a monoidal category (Theorem 5.3.34).

In Section 5.4 the main result is Theorem 5.4.24 which gives a functor $\mathsf{Z}_G\colon \mathrm{HomCob} \to \mathbf{Vect}_{\mathbb{C}}$. We then have Theorem 5.4.27, which gives an alternative interpretation of the map Z_G on objects, making explicit calculation possible. Finally we have Theorem 5.4.27 which gives another alternative interpretation of Z_G on objects, connecting our construction to others in the literature, e.g. [DW90].

Finally in Chapter 6 we have some conclusions and suggestions for future directions.

Chapter 2

General preliminaries

Here we collect the various general notation and conventions that we will need.

Definition 2.0.1. For X a set, $\mathcal{P}X$ denotes the power set of X.

Definition 2.0.2. Let X and Y be sets. A relation between X and Y is a an element $R \in \mathcal{P}(X \times Y)$. If $(x, y) \in R$ we write $x \sim y$. When $X = Y$ we say that R is a relation on X.

Given a set S and a relation ρ on S we write $\bar{\rho}$ for the reflexive, symmetric transitive closure of ρ. Given an equivalence relation $\sim$ on S we write $S/\sim$ for the corresponding set of equivalence classes. We will also write S/ρ for $S/\bar{\rho}$.

Definition 2.0.3. Let I be an indexing set. Given any family $(A_i)_{i \in I}$ of sets, the disjoint union $\bigsqcup_{i \in I} A_i$ is the set of all pairs (a, i) with $i \in I$ and $a \in A_i$.

A topological space (from now, a space) is a pair (X, τ) where X is a set and τ is a topology on X. We shall see topologies as collections of either open or closed sets, depending on what is most convenient. Often we will refer to both the set and the topology using just the symbol X.

Definition 2.0.4. We will fix notation for some topological spaces we will make use of. In each of the following cases we define the space as a subset and take the subspace topology:

- $\mathbb{I} = [0, 1] \subset \mathbb{R}$,
- $S^n = \{x \in \mathbb{R}^{n+1} \mid |x| = 1\}$, and
- $D^n = \{x \in \mathbb{R}^n \mid |x| \leq 1\}$.

Definition 2.0.5. Let X and X' be topological spaces. The underlying set of the product, denoted $X \times X'$, is the cartesian product $\{(x, y) \mid x \in X, y \in X'\}$. The topology on $X \times X'$ is the coarsest topology that makes the canonical projections $X \times X' \to X$ and $X \times X' \to X'$ continuous.

Chapter 3

Preliminaries

Here we introduce concepts that will be relevant throughout the thesis. We will focus on aspects that we will need and thus do not aim to give a complete picture, for example in Section 3.6 we discuss colimits but not limits. In each section we give references directing the reader to more complete approaches.

We spend some time on this section for number of reasons. One is to make this work accessible to a wide audience. Another is to take the opportunity to fix some non-standard notation that will be helpful. Finally we do this because we take some unusual routes to well known constructions that will be a useful warm up for later sections, using a *path magmoid* to construct the fundamental groupoid in Proposition 3.3.8 for example.

We start, in Sections 3.1 and 3.2, with magmoids, and magmoid congruences which may lead to categories. In Section 3.3 we have the fundamental groupoid, and in Section 3.4 the compact-open topology. In Section 3.5 we introduce adjunctions, which allows us to use the compact-open topology to obtain a partial lift of the classical product-hom adjunction in the category of sets (Lemma 3.5.16). In Section 3.6 we fix choices of colimits in the categories of sets, topological spaces and groupoids. Finally we have monoidal categories in Section 3.7.

3.1 Magmoids, categories and groupoids

In this work constructions of categories are a recurrent theme. Such constructions will often start from something concrete with a composition. Equivalence classes of these concrete things eventually become the morphisms of the constructed category. So it will be useful to have a general machinery for studying such constructions. We can think of the underlying idea of a category as sets of objects, morphisms and a non associative composition - a *magmoid.* We can then study congruences on these magmoids, some of which will lead to categories.

We are unaware of a reference to magmoids in the literature although the construction is a straightforward extension of the use of magmas for the underlying structure of a group. Everything else in this section can be found in e.g. [Mac71; Rie17; AHS90].

Definition 3.1.1. A magmoid M is a triple

$$\mathsf{M} = (Ob(\mathsf{M}), \mathsf{M}(-,-), \Delta_{\mathsf{M}})$$

consisting of

(I) a collection $Ob(\mathsf{M})$ of *objects*,

(II) for each pair $X, Y \in Ob(\mathsf{M})$ a collection $\mathsf{M}(X,Y)$ of *morphisms from* X *to* Y, and

(III) for each triple $X, Y, Z \in Ob(\mathsf{M})$ a *c*omposition

$$\Delta_{\mathsf{M}} \colon \mathsf{M}(X,Y) \times \mathsf{M}(Y,Z) \to \mathsf{M}(X,Z).$$

We use $f \colon X \to Y$ to indicate that f is a morphism from X to Y and $f \in \mathsf{M}$ to indicate there exists a pair of objects $X, Y \in Ob(M)$ such that $f \in \mathsf{M}(X,Y)$.
Where convenient we will replace instances of $-$ in the triple with generic symbols.

Example 3.1.2. *The following are magmoids. In each case we give the objects and morphisms, the composition is then the usual composition of maps of each structure.*

(i) Set*: Objects are sets and morphisms from* X *to* Y *are all functions* $f \colon X \to Y$*.*

(ii) $\mathsf{Vect}_{\Bbbk}$*: Objects are vector spaces over the field* $\Bbbk$ *and morphisms from* V *to* W *are* $\Bbbk$*-linear maps* $f \colon V \to W$*.*

Example 3.1.3. *There is a magmoid*

$$\mathsf{Top} = (Ob(\mathsf{Top}), \mathsf{Top}(-,-), \circ)$$

where $Ob(\mathsf{Top})$ *is the set of all topological spaces, for* $X, Y \in Ob(\mathsf{Top})$, $\mathsf{Top}(X,Y)$ *is the set of continuous maps from* X *to* Y *and composition of maps is given by the composition of the underlying functions in* Set.

Example 3.1.4. *We will treat the following more thoroughly in Section 3.3. Let* X *be a topological space. Then there is a magmoid* $\mathfrak{P}X = (X, \mathfrak{P}X(-,-), \Gamma_{\frac{1}{2}})$ *where for a pair* $x, x' \in X$, $\mathfrak{P}(x,x')$ *is the set of paths from* x *to* x' *in* X *and* $\Gamma_{\frac{1}{2}}$ *is path composition.*

Definition 3.1.5. A magmoid $\mathsf{M} = (Ob(\mathsf{M}), \mathsf{M}(-,-), \Delta_{\mathsf{M}})$ is called reversible if for all pairs $N, N' \in Ob(\mathsf{M})$, there is a bijection

$$\mathrm{rev}\colon \mathsf{M}(N,N') \to \mathsf{M}(N',N).$$

Definition 3.1.6. A magmoid M is called small if $Ob(M)$ is a set and for each pair $X, Y \in Ob(\mathsf{M})$, $\mathsf{M}(X,Y)$ is a set.

Definition 3.1.7. Let M and M' be magmoids. A magmoid morphism $F\colon \mathsf{M} \to \mathsf{M}'$ is a map sending each object $X \in Ob(\mathsf{M})$ to an object $F(X) \in Ob(\mathsf{M}')$ and each morphism $f\colon X \to Y$ in M to a morphism $F(f)\colon F(X) \to F(Y)$ in M' such that for any pair of morphisms $f, g \in \mathsf{M}$

$$F(\Delta_{\mathsf{M}}(f,g)) = \Delta_{\mathsf{M}'}(F(f), F(g))$$

wherever $\Delta_{\mathsf{M}}(f,g)$ is defined.

Proposition 3.1.8. *Let* $\mathsf{M}, \mathsf{M}', \mathsf{M}''$ *be magmoids. There exists a partial composition of magmoid morphisms which sends a pair of magmoid morphisms* $F\colon \mathsf{M} \to \mathsf{M}'$ *and* $F'\colon \mathsf{M}' \to \mathsf{M}''$ *to* $F' \circ F\colon \mathsf{M} \to \mathsf{M}''$ *with*

$$F' \circ F(f\colon X \to Y) = F'(F(f))\colon F'(F(X)) \to F'(F(Y)).$$

Proof. It is straightforward to check that $F' \circ F$ is well defined and is a magmoid morphism. □

Definition 3.1.9. A magmoid morphism $F\colon \mathsf{M} \to \mathsf{M}'$ is full if for each $X, Y \in Ob(\mathsf{M})$, the induced map $\mathsf{M}(X,Y) \to \mathsf{M}'(\mathcal{F}(X), \mathcal{F}(Y))$ is surjective and faithful if the same map is injective.

Let $\mathsf{M} = (Ob(\mathsf{M}), \mathsf{M}(-,-), *_{\mathsf{M}})$ be a magmoid. We will find it convenient to have an alternative notation for composition, for which we use function order. For morphisms $f{:}X \to Y$ and $g{:}Y \to Z$ in M we define

$$*_{\mathsf{M}}(f,g) = g *_{\mathsf{M}} f.$$

We now give the familiar definition of a category in terms of a magmoid.

Definition 3.1.10. A <u>category</u> is a quadruple

$$\mathcal{C} = (Ob(\mathcal{C}), \mathcal{C}(-,-), *_{\mathcal{C}}, 1_{-})$$

consisting of a magmoid $(Ob(\mathcal{C}), \mathcal{C}(-,-), *_{\mathcal{C}})$ and

(IV) for each $X \in Ob(\mathcal{C})$ a distinguished morphism $1_X \in \mathcal{C}(X,X)$ called the *identity*,

such that the following axioms are satisfied.

($\mathcal{C}$1) **Identity:** for any morphism $f{:}X \to Y$, we have $1_Y *_{\mathcal{C}} f = f = f *_{\mathcal{C}} 1_Y$.

($\mathcal{C}$2) **Associativity:** for any triple of morphisms $f{:}X \to Y$, $g{:}Y \to Z$ and $h{:}Z \to W$ we have $h *_{\mathcal{C}} (g *_{\mathcal{C}} f) = (h *_{\mathcal{C}} g) *_{\mathcal{C}} f$.

We refer to $(Ob(\mathcal{C}), \mathcal{C}(-,-), *_{\mathcal{C}})$ as the underlying magmoid of $\mathcal{C}$. By abuse of notation we refer also to the underlying magmoid as $\mathcal{C}$.

Proposition 3.1.11. *There exist categories with underlying magmoids* Set *and* $\mathsf{Vect}_{\Bbbk}$*, defined in Example 3.1.2. In each case the identities are the usual identities for each object. We denote these by* **Set** *and* $\mathbf{Vect}_{\Bbbk}$ *respectively.*

Proof. It is straightforward to check axioms $\mathcal{C}1$ and $\mathcal{C}2$. □

Proposition 3.1.12. *There exists a category with underlying magmoid* Top*, defined in Example 3.1.3. For a space X, the identity is the map which is the identity in* **Set** *on the underlying set of X, we denote this* $\mathrm{id}_X{:}X \to X$.

Proof. Axioms $\mathcal{C}1$ and $\mathcal{C}2$ follow directly from the corresponding axioms for the underlying maps in **Set**. □

Proposition 3.1.13. *There is a category* $\mathbf{Mag} = (Ob(\mathbf{Mag}), \mathbf{Mag}(-,-), \circ, 1_{-})$ *where objects are all small magmoids,* $\mathbf{Mag}(\mathsf{M}, \mathsf{M}')$ *is the set of all magmoid morphisms from* M *to* M'*, composition is as in Proposition 3.1.8 and the identity on each magmoid is the magmoid morphism which is the* **Set** *identity on objects and morphisms.*

Proof. Proposition 3.1.8 gives that $\circ$ is a composition. It is immediate that this composition is associative and the described magmoid morphism is an identity. □

Proposition 3.1.14. *Let* $\mathcal{C} = (Ob(\mathcal{C}), \mathcal{C}(-,-), *_{\mathcal{C}}, 1_{-})$ *be a category. There is a category* $\mathcal{C}^{op} = (Ob(\mathcal{C}), \mathcal{C}^{op}(-,-), *_{\mathcal{C}^{op}}, 1_{-})$ *where for* $X, Y \in Ob(\mathcal{C})$*,* $\mathcal{C}^{op}(X,Y) = \mathcal{C}(Y,X)$ *and for composable morphisms* $f, g \in \mathcal{C}^{op}$ *we have* $f *_{\mathcal{C}^{op}} g = g *_{\mathcal{C}} f$*. This is called the opposite category.*

Proof. The associativity and identity axioms follow directly from the corresponding axioms in $\mathcal{C}$. □

Definition 3.1.15. A category $\mathcal{C}$ is called *finitely generated* if there exists a finite set X of morphisms (including identities) in $\mathcal{C}$ such that every morphism in $\mathcal{C}$ can be obtained by composing morphisms in X.

Note that this implies there are finitely many objects.

Proposition 3.1.16. *Let* $\mathcal{C}$ *and* $\mathcal{D}$ *be categories. Then there is a category*

$$\mathcal{C} \times \mathcal{D} = (Ob(\mathcal{C}) \times Ob(\mathcal{D}), \mathcal{C} \times \mathcal{D}(-,-), *_{\mathcal{C}\times\mathcal{D}}, 1_{X,Y} = (1_X, 1_Y))$$

where $\mathcal{C} \times \mathcal{D}(W \times X, Y \times Z) = \mathcal{C}(W,Y) \times \mathcal{D}(X,Z)$ *and* $(f', g') *_{\mathcal{C}\times\mathcal{D}} (f,g) = (f' *_{\mathcal{C}} f, g' *_{\mathcal{D}} g)$*. This is called the product category.*

Proof. Straightforward. □

Definition 3.1.17. We will say a category $\mathcal{C}$ is finite if the collection of all morphisms in $\mathcal{C}$ is a finite set.

A category $\mathcal{C}$ is called small if the collection of all morphisms in $\mathcal{C}$ is a set. Note this implies that $Ob(\mathcal{C})$ is a set, since the objects of any category are in bijective correspondence with the identity morphisms.

Definition 3.1.18. A morphism $f{:}X \to Y$ in a category $\mathcal{C}$ is an isomorphism if there exists an inverse morphism $g{:}Y \to X$ such that $g *_{\mathcal{C}} f = 1_X$ and $f *_{\mathcal{C}} g = 1_Y$.
If there exists an isomorphism X to Y we say that X and Y are isomorphic.

Definition 3.1.19. Let $\mathcal{C}$ and $\mathcal{C}'$ be categories. A functor $F{:}\mathcal{C} \to \mathcal{C}'$ is a magmoid morphism $\mathcal{C} \to \mathcal{C}'$ such that for any $X \in Ob(\mathcal{C})$

$$1_{F(X)} = F(1_X).$$

Definition 3.1.20. A bifunctor is a functor $F{:}C \times C' \to D$ whose domain is a product of two categories.

Proposition 3.1.21. *Let $\mathcal{C}$ be a category. There is a bifunctor*

$$Hom_{\mathcal{C}}{:}\mathcal{C}^{op} \times \mathcal{C} \to \mathbf{Set}$$

which sends

- *an object $(X, X') \in \mathcal{C}^{op} \times \mathcal{C}$ to the set of morphisms $\mathcal{C}(X, X')$,*
- *a pair of morphisms $f{:}X \to Y$ in $\mathcal{C}^{op}$ (so $f{:}Y \to X$ is a morphism in $\mathcal{C}$) and $g{:}X' \to Y'$ in $\mathcal{C}$ to the function $Hom_{\mathcal{C}}(f,g){:}\mathcal{C}(X,X') \to \mathcal{C}(Y,Y')$,*

$$(h{:}X \to X') \mapsto (g *_{\mathcal{C}} h *_{\mathcal{C}} f{:}Y \to Y').$$

This is called the hom bifunctor.

Proof. The pair $(1_X, 1_X) \in \mathcal{C}^{op} \times \mathcal{C}$ is mapped to the function $h \mapsto h$ from $\mathcal{C}(X,X)$ to $\mathcal{C}(X,X)$.
Let $(f,g){:}(X,X') \to (Y,Y')$ and $(f',g'){:}(Y,Y') \to (Z,Z')$ be morphisms in $\mathcal{C}^{op} \times \mathcal{C}$. The composition in $\mathcal{C}^{op} \times \mathcal{C}$ is $(f *_{\mathcal{C}} f', g' *_{\mathcal{C}} g)$ so we have $Hom_{\mathcal{C}}(f *_{\mathcal{C}} f', g' *_{\mathcal{C}} g) = Hom_{\mathcal{C}}(f',g') *_{\mathbf{Set}} Hom_{\mathcal{C}}(f,g)$ is the function $(h{:}X \to X') \mapsto (g' *_{\mathcal{C}} g *_{\mathcal{C}} h *_{\mathcal{C}} f *_{\mathcal{C}} f'{:}Z \to Z')$. □

Example 3.1.22. *Let V be a vector space over a field $\Bbbk$. Then $Hom_{\mathbf{Vect}_{\Bbbk}}(V, \Bbbk)$ is the set of linear maps from V to $\Bbbk$, which is the underlying set of the dual vector space of V. Let $V' \in Ob(\mathbf{Vect}_{\Bbbk})$ be a vector space, and $f{:}V' \to V$ a linear map. Then $Hom_{\mathbf{Vect}_{\Bbbk}}(f, 1_{\Bbbk})$ is* *the set map sending a linear map* $h: V \to \Bbbk$ *to* $hf: V' \to \Bbbk$*, this is the transpose of* f*.*

Definition 3.1.23. In analogy with the magmoid case, a functor $F: \mathcal{C} \to \mathcal{D}$ is full if for each $X, Y \in Ob(\mathcal{C})$, the induced map $\mathcal{C}(X,Y) \to \mathcal{D}(\mathcal{F}(X), \mathcal{F}(Y))$ is surjective and faithful if the same map is injective.

Proposition 3.1.24. *Let* $\mathcal{C}, \mathcal{C}'$ *and* $\mathcal{C}''$ *be categories and* $F: \mathcal{C} \to \mathcal{C}'$ *and* $F': \mathcal{C}' \to \mathcal{C}''$ *be functors. The composition* $\circ$ *of the magmoid morphisms* F *and* F' *given in Proposition 3.1.8, extends to a composition of functors.*

Proof. For each $X \in Ob(\mathcal{C})$, $F' \circ F(1_X) = F'(1_{F(X)}) = 1_{F'(F(X))} = 1_{F' \circ F(X)}$. □

We note that where we feel a more concise notation is helpful we may sometimes use the null composition symbol (i.e. just juxtaposition) for composition of functors and magmoid morphisms.

Proposition 3.1.25. *There is a category* $\mathbf{Cat} = (Ob(\mathbf{Cat}), \mathbf{Cat}(-,-), \circ, 1_-)$ *where objects are all small categories,* $\mathbf{Cat}(\mathcal{C}, \mathcal{C}')$ *is the set of all functors from* $\mathcal{C}$ *to* $\mathcal{C}'$*, composition is as in Proposition 3.1.24, and the identity* $1_{\mathcal{C}}$ *is the functor* $\mathcal{C} \to \mathcal{C}$ *which acts identically on objects and morphisms.*

Proof. The triple $(Ob(\mathbf{Cat}), \mathbf{Cat}(-,-), \circ)$ is a magmoid as Proposition 3.1.24 gives that $\circ$ is a composition. It is clear that $1_{\mathcal{C}}$ is an identity for each category $\mathcal{C}$. □

Definition 3.1.26. Let $\mathcal{C} = (Ob(\mathcal{C}), \mathcal{C}(-,-), *_{\mathcal{C}}, 1_-)$ be a category. A subcategory $\mathcal{S}$ of $\mathcal{C}$ consists of

- a subset $Ob(\mathcal{S}) \subseteq Ob(\mathcal{C})$,
- for each $X, Y \in Ob(\mathcal{S})$, a subset $\mathcal{S}(X,Y) \subseteq \mathcal{C}(X,Y)$,

such that,

- for all $X \in Ob(\mathcal{S})$, $1_X \in \mathcal{S}(X,X)$, and
- for all pairs of composable morphisms $f, g \in \mathcal{S}$, $g *_{\mathcal{C}} f \in \mathcal{S}$.

Note this implies $\mathcal{S} = (Ob(\mathcal{S}), \mathcal{S}(-,-), *_{\mathcal{C}}, 1_-)$ is a category.

Definition 3.1.27. A subcategory $\mathcal{S}$ of $\mathcal{C}$ is called full if, given any two objects $X, Y \in Ob(\mathcal{S})$, $\mathcal{S}(X,Y) = \mathcal{C}(X,Y)$.

A subcategory $\mathcal{S}$ of $\mathcal{C}$ is called wide if $Ob(\mathcal{S}) = Ob(\mathcal{C})$.

Definition 3.1.28. A groupoid $\mathcal{G}$ is a pentuple

$$\mathcal{G} = (Ob(\mathcal{G}), \mathcal{G}(-,-), *_{\mathcal{G}}, 1_{-}, (-) \mapsto (-)^{-1})$$

consisting of a small category $(Ob(\mathcal{G}), \mathcal{G}(-,-), *_{\mathcal{G}}, 1_{-})$, and

(V) for each pair $(X,Y) \in Ob(\mathcal{G}) \times Ob(\mathcal{G})$ a function

$$\begin{aligned}(-)^{-1}\colon \mathcal{G}(X,Y) &\to \mathcal{G}(Y,X)\\ f &\mapsto f^{-1}\end{aligned}$$

called the *inverse assigning* function;

such that the following is satisfied.

($\mathcal{G}$1) **Inverse:** for any morphism $f\colon X \to Y$, we have $f^{-1} *_{\mathcal{G}} f = 1_X$ and $f *_{\mathcal{G}} f^{-1} = 1_Y$.

Remark 3.1.1. A groupoid is precisely a small category in which all morphisms are isomorphisms.

Remark 3.1.2. We will see below that every group action leads to a groupoid, although groupoids arising from distinct group actions are not necessarily unique up to groupoid isomorphism.

Remark 3.1.3. Notice that a groupoid is necessarily reversible, although a reversible magmoid M does not imply the existence of a groupoid with underlying magmoid M.

Proposition 3.1.29. *Let* $G = (X, \circ_G, e_G)$ *be a group. Then*

$$\mathcal{G}_G = (\{*\}, \mathcal{G}_G(*,*), \circ_G, 1_* = e_G, g \mapsto g^{-1})$$

where $\mathcal{G}_G(*,*) = X$, *is a groupoid.*

Proof. Straightforward. □

Lemma 3.1.30. *Let* $(G, \circ_G, e_G)$ *be a group,* X *a set and* $\rho: G \times X \to X$ *a group action. Define for* x *and* x' *in* X,

$$X/\!/_\rho G(x, x') = \{(g, x, x') \mid \rho(g, x) = x'\}.$$

The pentuple

$$X/\!/_\rho G \;=\; (X,\; X/\!/_\rho G(-,-),\; \circ_G,\; e_G,\; g \mapsto g^{-1})$$

is a groupoid. That is,

(I) objects are elements of X*;*

(II) morphisms x *to* x' *are elements of* $X/\!/_\rho G(x, x')$*, denoted by triples* $(g, x, \rho(g, x))$ *where* $g \in G$*,* $x \in X$ *and* $\rho(g, x) = x'$*;*

(III) triples $(g_1, x, \rho(g_1, x))$ *and* $(g_2, x', \rho(g_2, x'))$ *are composable if* $\rho(g_1, x) = x'$*, and then the composite is* $(g_2 \circ_G g_1, x, \rho(g_2 \circ_G g_1, x))$*;*

(IV) the identity for any object $x \in X$ *is the triple* $(e_G, x, \rho(e_G, x)) = (e_G, x, x)$

(V) the inverse of a morphism $(g, x, \rho(g, x))$ *is the morphism* $(g^{-1}, \rho(g, x), \rho(g^{-1}, \rho(g, x)))$ $= (g^{-1}, \rho(g, x), x)$*.*

We call $X/\!/_\rho G$ *the* <u>*action groupoid*</u> *of the action of* G *on* X*.*

Remark 3.1.4. Note that in the last three entries of the tuple in the previous lemma we gave only information about what happens to the group element in each morphism. We do this to keep notation readable. We will take the same liberty in future constructions without the subsequent clarification. It will be clear what should happen to the relevant objects from the composition, identity and inverse axioms.

Proof. $(\mathcal{C}1)$ Let $(g, x, \rho(g, x))$ be a morphism. Then $(g \circ e_G, x, \rho(g \circ e_G, x)) = (g, x, \rho(g, x))$ and $(e_G \circ g, x, \rho(e_G \circ g, x)) = (g, x, \rho(g, x))$.
$(\mathcal{C}2)$ The composition is associative by the group associativity of G.
$(\mathcal{G}1)$ For any morphism $(g, x, \rho(g, x))$ with $\rho(g, x) = x'$, we have $(g \circ g^{-1}, x', \rho(g \circ g^{-1}, x')) = (e_G, x', x')$ and $(g^{-1} \circ g, x, \rho(g^{-1} \circ g, x)) = (e_G, x, x)$. ☐

Remark 3.1.5. Let $\mathcal{G}$ be a groupoid. By abuse of notation we will refer also to the underlying magmoid as $\mathcal{G}$. Note that the identities and inverses of $\mathcal{G}$ are uniquely determined from the underlying magmoid of $\mathcal{G}$.

Definition 3.1.31. Let G be a groupoid, a subgroupoid $\mathcal{S}$ of $\mathcal{G}$ is defined analogously to the category case with the additional condition that

- for all $f \in \mathcal{S}$, $f^{-1} \in \mathcal{S}$.

Proposition 3.1.32. *Let $\mathcal{G}$ and $\mathcal{G}'$ be groupoids and $F{:}\,\mathcal{G} \to \mathcal{G}'$ a magmoid morphism. Then we have*

- *for any $X \in Ob(\mathcal{G})$, $1_F(X) = F(1_X)$, and*
- *for any morphism $f \in \mathcal{G}$, $F(f^{-1}) = (F(f))^{-1}$.*

The says that a magmoid morphism preserves the structure of a groupoid without any additional conditions.

In analogy with the category case, we refer to a magmoid morphism between groupoids as a functor.

Proof. We have that $F(1_X) *_{\mathcal{G}'} F(1_X) = F(1_X *_{\mathcal{G}} 1_X) = F(1_X)$. Since $\mathcal{G}'$ is a groupoid, we can compose with $F(1_X)^{-1}$ and hence $F(1_X) = 1_{F(X)}$.

Suppose $f{:}\,X \to Y$ is a morphism in $\mathcal{G}$. Then $1_{F(X)} = F(1_X) = F(f^{-1} *_{\mathcal{G}} f) = F(f^{-1}) *_{\mathcal{G}'} F(f)$ so $F(f^{-1})$ is a left inverse for $F(f)$. We can similarly show it is a right inverse and so, by uniqueness of inverses, $F(f^{-1}) = F(f)^{-1}$. □

Remark 3.1.6. There is not a similar result for categories: a magmoid morphism on the underlying magmoids of a pair of categories can fail to be a functor.

Proposition 3.1.33. *There is a category* $\mathbf{Grpd} = (Ob(\mathbf{Grpd}), \mathbf{Grpd}(-,-), \circ, 1_{-})$ *where objects are all small groupoids,* $\mathbf{Grpd}(\mathcal{G}, \mathcal{G}')$ *is the set of all functors from $\mathcal{G}$ to $\mathcal{G}'$, composition is as in Proposition 3.1.8 and the identity $1_{\mathcal{G}}$ is the functor $\mathcal{G} \to \mathcal{G}$ which acts identically on objects and morphisms.*

Proof. It is immediate from the definition that this is a subcategory of **Cat**, and hence a category. In fact it is a full subcategory of **Cat**. □

3.2 Magmoid congruence

Often the magmoids we construct are too large to be interesting objects of study themselves. Here we introduce congruences and quotient magmoids, our main tool for obtaining a category from a magmoid. Congruences are families of relations on the morphism sets of magmoids. Note in particular that the object set is always fixed. Allowing equivalence relations on objects in magmoids potentially leads to extra morphisms, and so is not really a quotient in the usual sense. In Chapter 4 we will be particularly interested in cases for which we obtain a finitely generated category.

As with the previous section we are unaware of a reference that explicitly discusses congruences of structures which are not yet categories. However, if we take a category and quotient the underlying magmoid by a congruence the quotient magmoid can also be given a categorical structure in a canonical way (Proposition 3.2.3), in this case we obtain the same quotient category as in Chapter 2 of [Mac71].

Definition 3.2.1. A congruence C on a magmoid $\mathsf{M} = (Ob(\mathsf{M}), \mathsf{M}(-,-), \Delta_{\mathsf{M}})$ consists of, for each pair $X, Y \in Ob(\mathsf{M})$ an equivalence relation $R_{X,Y}$ on $\mathsf{M}(X,Y)$, such that $f' \in [f]$ and $g' \in [g]$ implies $\Delta_{\mathsf{M}}(f', g') \in [\Delta_{\mathsf{M}}(f,g)]$ where defined.

Definition 3.2.2. Let $\mathsf{M} = (Ob(\mathsf{M}), \mathsf{M}(-,-), \Delta_{\mathsf{M}})$ be a magmoid and C a congruence on M. The quotient magmoid of M by C is $\mathsf{M}/C = (Ob(\mathsf{M}), \mathsf{M}(X,Y)/R_{X,Y}, \Delta_{\mathsf{M}/C})$ where for each triple $X, Y, Z \in Ob(\mathsf{M}/C)$

$$\begin{aligned}\Delta_{\mathsf{M}/C} : \mathsf{M}/C(X,Y) \times \mathsf{M}/C(Y,Z) &\to \mathsf{M}/C(X,Z) \\ ([f],[g]) &\mapsto [\Delta_{\mathsf{M}}(f,g)].\end{aligned}$$

(That the composition is well defined follows directly from the definition of a congruence.)

In practice we will use the notation for the composition in M to denote also the composition M/C.

Proposition 3.2.3. *Suppose $\mathcal{C} = (Ob(\mathcal{C}), \mathcal{C}(X,Y), *_{\mathcal{C}}, 1_{-})$ is a category. For any congruence C on $(Ob(\mathcal{C}), \mathcal{C}(X,Y), *_{\mathcal{C}})$, we have that $\mathcal{C}/C = (Ob(\mathcal{C}), \mathcal{C}(X,Y)/R_{X,Y}, *_{\mathcal{C}/C}, [1_{-}])$, where composition is defined analogously to the magmoid case, is a category.*
We call this $\mathcal{C}/C$ the quotient category of $\mathcal{C}$ by C.

Proof. ($\mathcal{C}1$) For all $[f]: X \to Y$ we have

$$[f] *_{\mathcal{C}/C} [1_X] = [f *_{\mathcal{C}} 1_X] = [f] = [1_Y *_{\mathcal{C}} f] = [1_Y] *_{\mathcal{C}/C} [f].$$

($\mathcal{C}2$) Let $[f], [g], [h]$ be composable morphisms in $\mathcal{C}/C$. Then

$$[h] *_{\mathcal{C}/C} ([g] *_{\mathcal{C}/C} [f]) = [h *_{\mathcal{C}} g *_{\mathcal{C}} f] = ([h] *_{\mathcal{C}/C} [g]) *_{\mathcal{C}/C} [f]. \qquad \square$$

Proposition 3.2.4. *Suppose* $\mathcal{G} = (Ob(\mathcal{G}), \mathcal{G}(X,Y), *_{\mathcal{G}}, 1_{-}, (-)^{-1})$ *is a groupoid. For any congruence* C *on* $(Ob(\mathcal{G}), \mathcal{G}(X,Y), *_{\mathcal{G}})$*, we have that*

$$\mathcal{G}/C = (Ob(\mathcal{G}), \mathcal{G}(X,Y)/R_{X,Y}, *_{\mathcal{G}/C}, [1_{-}], [(-)^{-1}]),$$

where composition is defined analogously to the magmoid case, is a groupoid. We call $\mathcal{G}/C$ *the quotient groupoid of* $\mathcal{G}$ *by* C*.*

Proof. We have from Proposition 3.2.3 that $\mathcal{G}/C$ is a category. It remains only to check ($\mathcal{G}1$). Any $[f] \in \mathcal{G}/C(X,Y)$ has inverse $[f^{-1}]$ since

$$[f^{-1}] *_{\mathcal{G}/C} [f] = [f^{-1} *_{\mathcal{G}} f] = [1_X], \text{ and } \qquad [f] *_{\mathcal{G}/C} [f^{-1}] = [f *_{\mathcal{G}} f^{-1}] = [1_Y]. \square$$

Lemma 3.2.5. *Let* M *be a magmoid and* C *a congruence on* M*. There is an induced quotient morphism* $Q: \mathsf{M} \to \mathsf{M}/C$ *which is the identity on objects and which sends morphisms to their equivalence class under* C*.*

Proof. Let $f \in \mathsf{M}(X,Y)$ be a morphism. It is immediate from the definition of M/C that $Q(f)$ is a morphism from X to Y.

For composable morphisms $f, g \in \mathsf{M}$ we have

$$Q(\Delta_{\mathsf{M}}(f,g)) = [\Delta_{\mathsf{M}}(f,g)] = \Delta_{\mathsf{M}/C}([f],[g])$$

by the definition of $\Delta_{\mathsf{M}/C}$, so Q is a magmoid morphism. $\square$

Proposition 3.2.6. *Let* M *and* M' *be magmoids and* $F: \mathsf{M} \to \mathsf{M}'$ *a magmoid morphism.*

For any pair $X, Y \in Ob(\mathsf{M})$ *there is a relation on* $\mathsf{M}(X,Y)$ *given by* $f \sim g$ *if* $F(f) = F(g)$*, which is easily seen to be an equivalence relation. Suppose in addition* F *restricts to the identity on the set of objects, then all such equivalence relations give a congruence on* M*. This is called the* fibre congruence *of* F*.*

Proof. Let f', g' be composable morphisms in M with $F(f') = F(f)$ and $F(g') = F(g)$, hence $f' \in [f]$ and $g' \in [g]$ under the fibre congruence of F. We have

$$F(\Delta_{\mathsf{M}}(f',g')) = \Delta_{\mathsf{M}'}(F(f'),F(g')) = \Delta_{\mathsf{M}'}(F(f),F(g)) = F(\Delta_{\mathsf{M}}(f,g))$$

which implies $\Delta_{\mathsf{M}}(f',g') \in [\Delta_{\mathsf{M}}(f,g)]$. □

It is immediate from the construction that, for a congruence C on a magmoid M, the fibre congruence of the quotient morphism $Q\colon \mathsf{M} \to \mathsf{M}/C$ is precisely C.

Definition 3.2.7. Let M be a magmoid, C a congruence, and $Q\colon \mathsf{M} \to \mathsf{M}/C$ the induced quotient morphism. If there exists a magmoid M' and full, non-identity magmoid morphisms $G\colon \mathsf{M} \to \mathsf{M}'$ and $H\colon \mathsf{M}' \to \mathsf{M}/C$ such that $Q = H \circ G$, we say that the fibre congruence of Q has a factor. If Q has no factor we say that the fibre congruence of Q is minimal.

Definition 3.2.8. Let M be a magmoid and $R = \{R_{X,Y}\}_{X,Y \in Ob(\mathsf{M})}$ a collection of relations on the sets $\mathsf{M}(X,Y)$. Then let $\bar{R}$ be the closure of R to a congruence, this means we take the reflexive, symmetric, transitive closure of each $R_{X,Y}$ and insist that for any composition $\Delta_{\mathsf{M}}(f,g) \sim \Delta_{\mathsf{M}}(f',g')$ if $f \sim f'$ and $g \sim g'$.

Definition 3.2.9. Let G be a directed graph and $F(G)$ the free category generated by G. To give a presentation of a category $\mathcal{C}$ is to give a directed graph G and family of relations R on the morphism sets $F(G)(-,-)$, such that the quotient groupoid $F(G)/\bar{R}$ is isomorphic to $\mathcal{C}$.

A presentation of a groupoid is similarly defined.

3.2.1 Normal subgroupoids

Often we will find it convenient to study congruences by passing through a factor. In particular we will work with factors which are groupoids. For any groupoid $\mathcal{G}$ we can construct a congruence on $\mathcal{G}$ from a subgroupoid which is *normal* and thus obtain a quotient groupoid, mirroring quotienting groups by normal subgroups. We make this explicit here.

Everything in this section can be found in Section 1.4.3 of [Bro+99].

Definition 3.2.10. Let $\mathcal{G}$ be a groupoid and $\mathcal{H}$ a wide subgroupoid. Then $\mathcal{H}$ is said to be normal if for any morphism $h{:}\, Y \to Y$ in $\mathcal{H}$ and any $g{:}\, X \to Y$ in $\mathcal{G}$ we have $g^{-1} *_{\mathcal{G}} h *_{\mathcal{G}} g{:}\, X \to X$ is in $\mathcal{H}$.
We say $\mathcal{H}$ is totally disconnected if for any $X, Y \in Ob(\mathcal{H})$ with $X \neq Y$ we have $\mathcal{H}(X,Y) = \emptyset$.

Lemma 3.2.11. *Let $\mathcal{G}$ be a groupoid and $\mathcal{H}$ a normal, totally disconnected subgroupoid. For each $X, Y \in Ob(\mathcal{G})$ and $g, g' \in \mathcal{G}(X,Y)$ the relation $g \sim g'$ if $g'^{-1} *_{\mathcal{G}} g \in \mathcal{H}$ is an equivalence relation on $\mathcal{G}(X,Y)$. Moreover all such relations are a congruence on $\mathcal{G}$.*

Proof. We first check that $\sim$ defines an equivalence relation on each $\mathcal{G}(X,Y)$. Let $g, g', g'' \in \mathcal{G}(X,Y)$ with $g \sim g'$ and $g' \sim g''$. Reflexivity holds since we have $g^{-1} *_{\mathcal{G}} g = 1_X \in \mathcal{H}$ since $\mathcal{H}$ contains all identities. Symmetry holds since $g'^{-1} *_{\mathcal{G}} g \in \mathcal{H}$ implies $(g'^{-1} *_{\mathcal{G}} g)^{-1} = g^{-1} *_{\mathcal{G}} g' \in \mathcal{H}$ since $\mathcal{H}$ contains all inverses. For transitivity we have $g'^{-1} *_{\mathcal{G}} g \in \mathcal{H}$ and $g''^{-1} *_{\mathcal{G}} g' \in \mathcal{H}$, hence $g''^{-1} *_{\mathcal{G}} g = (g''^{-1} *_{\mathcal{G}} g') *_{\mathcal{G}} (g'^{-1} *_{\mathcal{G}} g) \in \mathcal{H}$ by closure.

We now check that $\sim$ is a congruence. Suppose we have $f, f' \in \mathcal{G}(X,Y)$ with $f \sim f'$ and $g, g' \in G(Y,Z)$ with $g \sim g'$, so $f'^{-1} *_{\mathcal{G}} f \in \mathcal{H}$ and $g'^{-1} *_{\mathcal{G}} g \in \mathcal{H}$. We show $g *_{\mathcal{G}} f \sim g' *_{\mathcal{G}} f'$. We have

$$\begin{aligned}
(g' *_{\mathcal{G}} f')^{-1} *_{\mathcal{G}} (g *_{\mathcal{G}} f) &= f'^{-1} *_{\mathcal{G}} g'^{-1} *_{\mathcal{G}} g *_{\mathcal{G}} f \\
&= f'^{-1} *_{\mathcal{G}} (f *_{\mathcal{G}} f^{-1}) *_{\mathcal{G}} g'^{-1} *_{\mathcal{G}} g *_{\mathcal{G}} f \\
&= (f'^{-1} *_{\mathcal{G}} f) *_{\mathcal{G}} (f^{-1} *_{\mathcal{G}} g'^{-1} *_{\mathcal{G}} g *_{\mathcal{G}} f)
\end{aligned}$$

which is in $\mathcal{H}$ using closure and normality of $\mathcal{H}$. Hence $g *_{\mathcal{G}} f \sim g' *_{\mathcal{G}} f'$.

Remark 3.2.1. Note that this is the weakest congruence such that all morphisms of the form $h{:}\, X \to X$ in $\mathcal{H}$ are equivalent to the appropriate identity.

3.3 $\mathbb{I}$, paths $\mathbf{Top}(\mathbb{I}, X)$ and the fundamental groupoid

In this section we first construct a magmoid of paths and then add a congruence such that the quotient groupoid is the fundamental groupoid (Proposition 3.3.8). Some careful constructions of the fundamental groupoid can be found in the literature for example in [Die08] and [Bro06], although our magmoid approach is non-standard and we will use (more radical versions of) similar ideas repeatedly in Chapter 4 so we think this 'warm up' is worthwhile. Here we also discuss the relationship between fundamental groupoids obtained by varying a finite number of basepoints which will be necessary for our TQFT construction in Chapter 5 (Lemmas 3.3.13 and 3.3.14).

Throughout the rest of this thesis we will use path-equivalence alongside several other equivalence relations so we introduce some careful notation here.

Definition 3.3.1. Let X be a topological space. An element of $\mathbf{Top}(\mathbb{I}, X)$ is called a path in X i.e. the set of all paths in X is $\{\gamma{:}\, \mathbb{I} \to X \mid \gamma \text{ is continuous}\}$.
We will use γ_t for $\gamma(t)$, and we say γ is a path from x to x', denoted $\gamma{:}\, x \to x'$, when $\gamma_0 = x$ and $\gamma_1 = x'$. For $x, x' \in X$, let

$$\mathfrak{P}X(x, x') = \{\gamma{:}\, \mathbb{I} \to X \mid \gamma \in \mathbf{Top}(\mathbb{I}, X),\ \gamma_0 = x,\ \gamma_1 = x'\}.$$

Proposition 3.3.2. *Let X be a topological space. For any $x, x', x'' \in X$, there exists a composition*

$$\begin{aligned} \Gamma_{\frac{1}{2}}{:}\, \mathfrak{P}X(x, x') \times \mathfrak{P}X(x', x'') &\to \mathfrak{P}X(x, x'') \\ (\gamma, \gamma') &\mapsto \gamma'\gamma \end{aligned}$$

with

$$(\gamma'\gamma)_t = \begin{cases} \gamma_{2t} & 0 \le t \le 1/2, \\ \gamma'_{2(t-1/2)} & 1/2 \le t \le 1. \end{cases} \tag{3.1}$$

(Note the convention to choose distinguished point $t = 1/2$ and the null composition symbol here.)

Proof. We check that, for any $\gamma \in \mathfrak{P}X(x, x')$ and $\gamma' \in \mathfrak{P}X(x', x'')$, $\gamma'\gamma \in \mathfrak{P}X(x, x'')$. We have $\gamma_1 = \gamma'_0$ so Equation (3.1) defines an continuous map. Notice $(\gamma'\gamma)_0 = \gamma_0 = x$ and $(\gamma'\gamma)_1 = \gamma'_1 = x''$, so the composition is well defined. □

Remark 3.3.1. We find the above convention for ordering path composition to be more convenient as we will later want to map paths to functions.

Definition 3.3.3. Let X be a topological space. Define the magmoid

$$\mathfrak{P}X = (X, \mathfrak{P}X(-,-), \Gamma_{\frac{1}{2}}).$$

Proposition 3.3.4. *Let X be a topological space. For any $x, x' \in X$, there is a bijection*

$$\begin{aligned} \mathrm{rev}\colon \mathfrak{P}X(x,x') &\to \mathfrak{P}X(x',x) \\ \gamma &\mapsto \gamma^{\mathrm{rev}} \end{aligned}$$

where $\gamma^{\mathrm{rev}}_t = \gamma_{1-t}$. Hence $\mathfrak{P}X$ is reversible.

Proof. It is straight forward to see that the automorphism of $\mathbb{I}$ given by $t \mapsto 1-t$ is continuous. It follows that γ^{rev} is continuous. The map rev is self-inverse, thus a bijection. □

Definition 3.3.5. Let X be a topological space. Define a relation on $\mathfrak{P}X(x,x')$ as follows. Suppose we have paths $\gamma, \gamma' \in \mathfrak{P}X(x,x')$, then $\gamma \overset{p}{\sim} \gamma'$ if there exists a continuous map $H\colon \mathbb{I} \times \mathbb{I} \to X$ such that

- for all $t \in \mathbb{I}$, $H(t,0) = \gamma(t)$,
- for all $t \in \mathbb{I}$, $H(t,1) = \gamma'(t)$, and
- for all $s \in \mathbb{I}$, $H(0,s) = x$ and $H(1,s) = x'$.

Notation: We call such an H a path-homotopy from γ to γ'.

Proposition 3.3.6. *Let X be a topological space. For each pair $x, x' \in X$, $\overset{p}{\sim}$ is an equivalence relation on $\mathfrak{P}X(x,x')$.*

Notation: If $\gamma \overset{p}{\sim} \gamma'$ we say γ and γ' are path-equivalent . We use $[\gamma]_p$ for the path-

equivalence class of γ. Where we feel it simplifies the exposition, we may also use γ for the path-equivalence class of γ.

Proof. We show that $\overset{p}{\sim}$ is reflexive, symmetric and transitive. Let $\gamma \in \mathfrak{P}X(x,x')$, $\gamma' \in \mathfrak{P}X(x,x')$ and $\gamma'' \in \mathfrak{P}X(x,x')$ be paths with $\gamma \overset{p}{\sim} \gamma'$ and $\gamma' \overset{p}{\sim} \gamma''$.

The relation is reflexive since the function $H(t,s) = \gamma(t)$ is a path-homotopy from γ to γ. By assumption, there exists a path-homotopy, say $H_{\gamma,\gamma'}$, from γ to γ'. The function $H_{\gamma',\gamma}(t,s) = H_{\gamma,\gamma'}(t,1-s)$ is a path-homotopy from γ' to γ, hence the relation is symmetric. By assumption, there also exists a path-homotopy, say $H_{\gamma',\gamma''}$, from γ' to γ''. The function

$$H_{\gamma,\gamma''}(t,s) = \begin{cases} H_{\gamma,\gamma'}(t,2s) & 0 \le s \le \frac{1}{2} \\ H_{\gamma',\gamma''}(t,2(s-\frac{1}{2})) & \frac{1}{2} \le s \le 1. \end{cases}$$

is a path-homotopy from γ to γ'', so $\overset{p}{\sim}$ is transitive. □

Lemma 3.3.7. *Let X be a topological space. The equivalence relations $(\mathfrak{P}X(x,x'), \overset{p}{\sim})$ for each $x, x' \in X$ are a congruence on $\mathfrak{P}X$.*

Proof. Suppose $\gamma, \gamma' \in \mathfrak{P}X(x,x')$ are path-equivalent and so there exists a path homotopy, say $H_{\gamma,\gamma'}$ from γ to γ'. And suppose $\delta, \delta' \in \mathfrak{P}X(x',x'')$ are path-equivalent and so there exists a path homotopy, say $H_{\delta,\delta'}$ from δ to δ'. Notice $H_{\gamma,\gamma'}(1,s) = H_{\delta,\delta'}(0,s) = x'$ and so the function

$$H(t,s) = \begin{cases} H_{\gamma,\gamma'}(2t,s) & 0 \le t \le \frac{1}{2} \\ H_{\delta,\delta'}(2(t-\frac{1}{2}),s) & \frac{1}{2} \le t \le 1 \end{cases}$$

is a homotopy from $\delta\gamma$ to $\delta'\gamma'$. □

Spanier [Spa89] and Brown [Bro06] were among the first to consider fundamental groupoids.

Proposition 3.3.8. *Let X be a topological space. There exists a groupoid*

$$\pi(X) \ = \ \mathfrak{P}X/\overset{p}{\sim} = \ (X, \mathfrak{P}X(-,-)/\overset{p}{\sim}, \Gamma_{\frac{1}{2}}, [e_x]_p, [\gamma^{\text{rev}}]_p)$$

with underlying magmoid as in Definition 3.3.3. Here the identity morphism $[e_x]_p$ at each object x is the path-equivalence class of the constant path $\gamma_t = x$ for all $t \in \mathbb{I}$. The inverse of a morphism $[\gamma]_p$ from x to x' is the path-equivalence class of $\gamma^{\text{rev}}_t = \gamma_{1-t}$.
This is the fundamental groupoid of X.

Proof. ($\mathcal{G}1$) Suppose $\gamma \in \mathfrak{P}X(x, x')$, the following function is a path homotopy from $e_x\gamma$ to γ:

$$H_{id}(t,s) = \begin{cases} \gamma_{\frac{t}{\frac{s}{2}+\frac{1}{2}}} & 0 \leq t \leq \frac{s}{2} + \frac{1}{2} \\ x & \frac{s}{2} + \frac{1}{2} \leq t \leq 1. \end{cases}$$

The case for $\gamma e_{x'}$ is very similar.

($\mathcal{G}2$) The following function is a path homotopy $\gamma''(\gamma'\gamma)$ to $(\gamma''\gamma')\gamma$:

$$H_{ass}(t,s) = \begin{cases} \gamma_{\frac{t}{\frac{s}{4}+\frac{1}{4}}} & 0 \leq t \leq \frac{s}{4} + \frac{1}{4} \\ \gamma'_{4(t-\frac{s}{4}-\frac{1}{4})} & \frac{s}{4} + \frac{1}{4} \leq t \leq \frac{s}{4} + \frac{1}{2} \\ \gamma''_{\frac{t-\frac{s}{4}-\frac{1}{2}}{\frac{1}{2}-\frac{s}{4}}} & \frac{s}{4} + \frac{1}{2} \leq t \leq 1. \end{cases}$$

($\mathcal{G}3$) The following function is a homotopy $\gamma^{\text{rev}}\gamma$ to e_x:

$$H_{in}(t,s) = \begin{cases} \gamma_{2t} & 0 \leq t \leq \frac{1}{2} - \frac{s}{2} \\ \gamma_{1-s} & \frac{1}{2} - \frac{s}{2} \leq t \leq \frac{1}{2} + \frac{s}{2} \\ \gamma_{1-2(t-\frac{1}{2})} & \frac{1}{2} + \frac{s}{2} \leq t \leq 1. \end{cases}$$

A similar homotopy gives $\gamma\gamma^{\text{rev}} \overset{p}{\sim} e_x$. □

Lemma 3.3.9. *There is a functor* $\pi\colon \mathbf{Top} \to \mathbf{Grpd}$ *which sends a space* X *to the fundamental groupoid* $\pi(X)$ *and is defined on morphisms as follows. Let* $f\colon X \to Y$ *be a continuous map,* $\pi(f)\colon \pi(X) \to \pi(Y)$ *is given by* $\pi(f)(x) = f(x)$ *for a point* $x \in X = Ob(\pi(X))$ *and by* $\pi(f)([\gamma]_{\mathrm{p}}) = [f \circ \gamma]_{\mathrm{p}}$ *for a path* γ *in* X.

Proof. We first check the functor is well defined. Suppose $[\gamma]_{\mathrm{p}} \in \pi(X)$ is an equivalence class of paths with $\gamma, \gamma' \in [\gamma]_{\mathrm{p}}$. Then there is a homotopy, H say, from γ to γ'. So $f \circ \gamma \sim f \circ \gamma'$ via the homotopy $f \circ H$.

If f is the identity map on a space X, it is immediate from the definition that $\pi(f)$ is the identity functor on $\pi(X)$.

That π preserves composition follows from associativity of function composition in **Set**. □

Definition 3.3.10. Let X be a topological space and $A \subseteq X$ a subset. The fundamental groupoid of X with respect to A is the full subgroupoid of $\pi(X)$ with object set A, denoted $\pi(X, A)$.

We refer to A as the set of basepoints.

We have $\pi(X, X) = \pi(X)$. Let X be a path-connected topological space and $x \in X$ be a point, we have that $\pi(X)(x, x)$ is the fundamental group based at $x \in X$. For any $A' \subseteq A$, there is an inclusion $\iota: \pi(X, A') \to \pi(X, A)$.

Definition 3.3.11. Let X and A be topological spaces, A is called *representative* in X if A contains a point in every path-component of X. (The nomenclature (X, A) is a 0-connected pair is also used.)

Lemma 3.3.12. *Suppose $f: X \to Y$ is a surjection and A is a representative subset of X, then $f(A)$ is representative in Y.*

Proof. Let $y \in Y$ be any point. We must construct a path from y to an element of $f(A)$. Let $y' \in f^{-1}(y)$ be any preimage, then there exists a path γ from y' to a point in A and $f \circ \gamma$ is a path from y to an element of $f(A)$. □

We will need the following results about the fundamental groupoid with finite sets of basepoints in Chapter 5.

Lemma 3.3.13. *Let $\mathcal{G}$ be a groupoid, X a topological space, $X_0 \subseteq X$ a finite subset and $y \in X \smallsetminus X_0$ any point. Given a groupoid map $f: \pi(X, X_0) \to \mathcal{G}$, a path $\gamma: x \to y$ where $x \in X_0$ and a morphism $g: f(x) \to \mathsf{g}$ in $\mathcal{G}$ with $\mathsf{g} \in Ob(\mathcal{G})$, there exists a unique $F: \pi(X, X_0 \cup \{y\}) \to \mathcal{G}$ extending f such that*

- *the diagram*

$$\begin{tikzcd} & \pi(X, X_0 \cup \{y\}) \arrow[dr, "F"] & \\ \pi(X, X_0) \arrow[ur, "\iota"] \arrow[rr, "f"'] & & \mathcal{G} \end{tikzcd} \tag{3.2}$$

  commutes, where ι is the inclusion map, and

- *$F(\gamma) = g$.*

Proof. First we construct such an F. On objects we have,

$$F(a) = \begin{cases} \mathsf{g}, & \text{if } a = y \\ f(a), & \text{otherwise.} \end{cases}$$

For a path-equivalence class $\phi: a \to y$ with $a \in X_0$ we must have

$$F(\phi) = F(\gamma\gamma^{-1}\phi) = F(\gamma)F(\gamma^{-1}\phi) = gf(\gamma^{-1}\phi)$$

Arguing similarly for all cases we have that for a morphism $\phi: a \to b$,

$$F(\phi) = \begin{cases} gf(\gamma^{-1}\phi), & \text{if } a \in X_0,\, b = y \\ f(\phi\gamma)g^{-1}, & \text{if } a = y,\, b \in X_0 \\ gf(\gamma^{-1}\phi\gamma)g^{-1}, & \text{if } a = y,\, b = y. \end{cases}$$

Notice that in each case F is inferred from the conditions set out in the theorem and by functoriality. This gives uniqueness. Now it remains to check that functoriality is always preserved, i.e. for any two paths $\phi, \phi' \in \pi(X, X_0 \cup \{y\})$ we have $F(\phi')F(\phi) = F(\phi'\phi)$. We check this case by case.

(I) If we have $\phi : a \to y, a \in X_0$, $\phi' : y \to b$, $b \in X_0$, then

$$F(\phi')F(\phi) = f(\phi'\gamma)g^{-1}gf(\gamma^{-1}\phi) = f(\phi'\gamma\gamma^{-1}\phi) = f(\phi'\phi) = F(\phi'\phi).$$

(II) If we have $\phi: y \to y$, $\phi': y \to b$, $b \in X_0$, then

$$F(\phi')F(\phi) = f(\phi'\gamma)g^{-1}gf(\gamma^{-1}\phi\gamma)g^{-1} = f(\phi'\gamma\gamma^{-1}\phi\gamma)g^{-1} = F(\phi'\phi).$$

(III) If we have $\phi: a \to y$, $a \in X_0$, $\phi': y \to y$, then

$$F(\phi')F(\phi) = gf(\gamma^{-1}\phi'\gamma)g^{-1}gf(\gamma^{-1}\phi) = gf(\gamma^{-1}\phi'\gamma\gamma^{-1}\phi) = gf(\gamma^{-1}\phi'\phi) = F(\phi'\phi).$$

(IV) If we have $\phi: y \to y$, $\phi': y \to y$, then

$$\begin{aligned} F(\phi')F(\phi) = gf(\gamma^{-1}\phi'\gamma)g^{-1}gf(\gamma^{-1}\phi\gamma)g^{-1} &= gf(\gamma^{-1}\phi'\gamma\gamma^{-1}\phi\gamma)g^{-1} \\ &= gf(\gamma^{-1}\phi'\phi\gamma)g^{-1} = F(\phi'\phi). \end{aligned}$$

There are another four cases which can be checked similarly. □

Lemma 3.3.14. *Let X be a topological space, G a group, $X_0 \subseteq X$ a finite representative* *subset and $y \in X$ a point with with $y \notin X_0$. There is a non-canonical bijection of sets*

$$\Theta_\gamma : \mathbf{Grpd}(\pi(X,X_0), \mathcal{G}_G) \times \mathcal{G}_G \to \mathbf{Grpd}(\pi(X, X_0 \cup \{y\}), \mathcal{G}_G)$$
$$(f,g) \mapsto F$$

where γ is a choice of a path from some $x \in X_0$ to y and F is the extension along γ and g as described in Lemma 3.3.13.
(Recall $\mathcal{G}_G = (\{*\}, \mathcal{G}_G(*,*), \circ_G, e_G, g \mapsto g^{-1})$ from Proposition 3.1.29.)

Proof. First notice that any $g \in \mathcal{G}_G$ satisfies the conditions of Lemma 3.3.13 since $\mathcal{G}_G$ has only one object.

The map Θ_γ has inverse which sends a map $f' \in \mathbf{Grpd}(\pi(X, X_0 \cup \{y\}), G)$ to the pair $(f' \circ \iota, f'(\gamma))$ where $\iota : \pi(X,X_0) \to \pi(X, X_0 \cup \{y\})$ is the inclusion. □

3.4 The compact-open topology on sets Top(X,Y)

At this point we change pace somewhat and discuss the compact-open topology on morphism sets in **Top**. We will use this topology to construct our motion groupoids in Chapter 4. In addition to its intuitive naturality (see Propositions 3.4.4-3.4.6), the compact-open topology allows us to find a partial lift of the classical product-hom adjunction in **Set** to an adjunction in **Top** (Theorem 3.5.16). We discuss the compact-open topology here, in particular, so that once we discuss adjunctions in the next section, we have the machinery in place to construct this product-hom adjunction in Section 3.5.1.

Everything in this section can be found in [Hat02]. Here we give the definition and some results to aid intuition.

Definition 3.4.1. Given a set X, and a subset Y of $\mathcal{P}X$ with $\cup_{A \in Y} A = X$, we write $\overline{Y}$ for the topology closure of Y. Hence the open sets in the topological space $(X, \overline{Y})$ are unions of finite intersections of elements in Y. We say that Y is a subbasis of (X,τ) if $\overline{Y} = \tau$. (Note that $\tau = \overline{Y}$ does not in general determine Y.)

Definition 3.4.2. A neighbourhoods basis of (X,τ) at $x \in X$ is a subset $B \subseteq \tau$, whose members are called basic neighbourhoods of x, such that every neighbourhood[1] of x contains an element of B.

Definition 3.4.3. Let (X,τ_X) and (Y,τ_Y) be topological spaces, then the compact-open topology τ_{XY}^{co} on **Top**(X,Y) has subbasis

$$b_{XY} = \{B_{XY}(K,U) | K \subseteq X \text{ is compact}, U \in \tau_Y\}$$

where

$$B_{XY}(K,U) = \{f\colon X \to Y \,|\, f(K) \subseteq U\}\,.$$

That is $\tau_{XY}^{co} = \overline{b_{XY}}$. [2]

We will use capital **TOP**(X,Y) to indicate the morphism set **Top**(X,Y) considered as a space with the compact open topology, so

$$\mathbf{TOP}(X,Y) = (\mathbf{Top}(X,Y), \tau_{XY}^{co}).$$

Proposition 3.4.4. *If X is the space with a single point then the τ_{XY}^{co} is the same in the obvious sense as the topology on Y.*

Proof. The maps $X \to Y$ can be labelled by their image in Y. The only compact set $K \subseteq X$ is the single point set X. For any $U \in \tau_Y$, the elements of the set of maps $B_{XY}(K,U)$ can be labelled by elements of U which correspond to the image of the point. □

Proposition 3.4.5. *If X a is space of n points with the discrete topology, τ_{XY}^{co} is the same in the obvious sense as the topology on $Y^n = Y \times \ldots \times Y$, the product of Y with itself n times.*

Proof. Maps $X \to Y$ are tuples $(y_1,\ldots,y_n) \in Y^n$ where y_i is the image of $x_i \in X$ and $i \in \{1,\ldots,n\}$. All subsets of X are compact so we have

$$B_{XY}(K,U) = \{(y_1,\ldots,y_n) \,|\, y_i \in U \text{ if } x_i \in K\}$$

---

[1] Our convention is that a neighbourhood of x is a subset of X containing an open set containing x.

[2] There are two conventions for the compact-open topology: the one written here (which is the classical one) and the one where we additionally impose that each K in $B_{XY}(K,U)$ be Hausdorff. For example [May99, Chapter 5] takes the latter convention. This creates an a priori smaller set of open sets in the function space. However they coincide for Hausdorff topological spaces.

which is the subset of Y^n with i^{th} component U if $x_i \in K$ and Y otherwise. Hence elements of the subbasis of τ^{co}_{XY} are open sets in the product topology.

Basis elements in the topology on Y^n are obtained from the compact open topology as follows. Let V be a basis open set in the topology on Y^n, then V is of the form $V_1 \times \ldots \times V_n$. Now

$$B_{XY}(\{x_i\}, V_i) = \{(y_1, \ldots, y_n) | y_i \in V_i\},$$

and $\cap_i B_{XY}(\{x_i\}, V_i) = V^n$. □

Proposition 3.4.6. (A.13 in [Hat02]) *Let X be a compact space and Y a metric space with metric d. Then*
(i) the function

$$d'(f,g) := \sup_{x \in X} d(f(x), g(x))$$

is a metric on $\mathbf{Top}(X,Y)$; and
(ii) the compact open topology on $\mathbf{Top}(X,Y)$ is the same as the one defined by the metric d'.

Proof. See A.13 in [Hat02]. □

3.5 Forgetful functors, natural transformations and adjunctions

Here we recall some results about forgetful funtors, natural transformations and adjunctions; giving examples and fixing notation that will be useful later. A non exhaustive list of references for the topics covered here is [Per19; AHS90; Rie17].

Many examples of adjunctions will come from forgetful functors. A forgetful functor is a general term for a functor which forgets structure.

Proposition 3.5.1. *There is a forgetful functor from* **Cat** *to* **Mag***, which sends a category $\mathcal{C} = (Ob(\mathcal{C}), \mathcal{C}(-,-), *_{\mathcal{C}}, 1_{-})$ to the magmoid $(Ob(\mathcal{C}), \mathcal{C}(-,-), *_{\mathcal{C}})$ and which sends a functor to its underlying magmoid morphism.*

Proof. This is immediate from the definitions. □

Example 3.5.2. *There is a forgetful functor* $\mathrm{U_T}\colon\mathbf{Top}\to\mathbf{Set}$ *sends a space to its underlying set and a continuous map to its underlying function.*

Example 3.5.3. *There is a forgetful functor* $\mathrm{U}_{\mathcal{G}}\colon\mathbf{Grpd}\to\mathbf{Set}$ *which sends a groupoid* $\mathcal{G}$ *to the set* $Ob(\mathcal{G})$ *and a functor to the corresponding set map determined by its action on objects.*

Example 3.5.4. *There is a forgetful functor* $\mathrm{U}_{\mathrm{V}_{\Bbbk}}\colon\mathbf{Vect}_{\Bbbk}\to\mathbf{Set}$ *which sends a vector space* V *to the set of all vectors and a linear map to the corresponding set map determined by its action on vectors.*

Definition 3.5.5. A *concrete category* is a category $\mathcal{C}$ equipped with a faithful functor $F\colon\mathcal{C}\to\mathbf{Set}$.

Example 3.5.6. *The forgetful functors* $\mathrm{U_T}$*,* $\mathrm{U}_{\mathcal{G}}$ *and* $\mathrm{U}_{\mathrm{V}_{\Bbbk}}$ *from Examples 3.5.2, 3.5.3 and 3.5.4 are faithful and thus* **Top***,* **Grpd** *and* $\mathbf{Vect}_{\Bbbk}$ *are concrete categories when equipped with* $\mathrm{U_T}$*,* $\mathrm{U}_{\mathcal{G}}$ *and* $\mathrm{U}_{\mathrm{V}_{\Bbbk}}$ *respectively.*

Definition 3.5.7. Let $\mathcal{C}$ and $\mathcal{D}$ be categories and $F,G\colon\mathcal{C}\to\mathcal{D}$ functors. A natural transformation $\eta\colon F\to G$ consists of

- for each object $X\in\mathcal{C}$ a morphism $\eta_X\colon F(X)\to G(X)$ in $\mathcal{D}$

such that for any morphism $f\colon X\to Y$ in $\mathcal{C}$ the following square of morphisms in $\mathcal{D}$ commutes.

$$\begin{tikzcd}
F(X) \arrow[r, "F(f)"] \arrow[d, "\eta_X"'] & F(Y) \arrow[d, "\eta_Y"] \\
G(X) \arrow[r, "G(f)"'] & G(Y)
\end{tikzcd}$$

A natural isomorphism is a natural transformation $\eta\colon F\to G$ such that every η_X is an isomorphism in $\mathcal{D}$.

Given a functor $F\colon\mathcal{C}\to\mathcal{D}$ one can ask if this is an isomorphism in **Cat**, although in some cases it is useful to consider a weaker notion. Let X be a **Set** and $F(X)$ the free group generated by X. Then the underlying set of $F(X)$ is not X. However, if we consider the set of all group homomorphisms from $F(X)$ to another group G, then there is a canonical bijection with the set of all functions X to the underlying set of the group G. Adjunctions generalise this phenomenon.

We write the following definition in a form that is only applicable to *locally small categories* – categories for which all collections of morphisms $\mathcal{C}(X,Y)$ are sets. Adjunctions can be defined for categories which are not locally small, although we will not need that generality here.

Definition 3.5.8. An adjunction consists of a pair of functors $F{:}\,\mathcal{C} \to \mathcal{D}$ and $G{:}\,\mathcal{D} \to \mathcal{C}$ together with, for each $X \in Ob(\mathcal{C})$ and $Y \in Ob(\mathcal{D})$, a set-bijection

$$\phi_{X,Y}{:}\,\mathcal{D}(F(X),Y) \xrightarrow{\sim} \mathcal{C}(X,G(Y))$$

natural in both variables. That is for any morphisms $f{:}\,X' \to X$ in $\mathcal{C}$ and $g{:}\,Y \to Y'$ in $\mathcal{D}$ the following square commutes

$$\begin{tikzcd}
\mathcal{D}(F(X),Y) \arrow[r, "\phi_{X,Y}"] \arrow[d, "Hom_{\mathcal{C}}(f,G(g))"'] & \mathcal{C}(X,G(Y)) \arrow[d, "Hom_{\mathcal{D}}(F(f),g)"] \\
\mathcal{D}(F(X'),Y') \arrow[r, "\phi_{X',Y'}"'] & \mathcal{C}(X',G(Y'))
\end{tikzcd}$$

where Hom is as in Proposition 3.1.21.

We refer to F as the left adjoint and G as the right adjoint.

Forgetful functors often have adjoints. The following adjunctions can be found, for example, in [Per19, Ch. 4].

Lemma 3.5.9. *Consider the forgetful functor* $\mathrm{U_T}{:}\,\mathbf{Top} \to \mathbf{Set}$ *introduced in Example 3.5.2.*
(I) There is a functor $\mathrm{F_T}{:}\,\mathbf{Set} \to \mathbf{Top}$ *which sends a set* X *to the space with underlying set* X *and the discrete topology and sends a function to the map which has the same action on the underlying sets.*
(II) The functor $\mathrm{F_T}$ *is left adjoint to* $\mathrm{U_T}$*.*

Proof. (I) Let X be any set, and S a topological space. Any function $f{:}\,\mathrm{F_T}(X) \to S$ is continuous as $\mathrm{F_T}$ has the discrete topology, and so the image of any function $g{:}\,X \to Y$ is a continuous map $\mathrm{F_T}(g){:}\,\mathrm{F_T}(X) \to \mathrm{F_T}(Y)$, thus $\mathrm{F_T}$ is well defined. Clearly $\mathrm{F_T}$ sends identities to identities. Preservation of composition follows immediately from the definition.

(II) Consider a continuous map $f \in \mathbf{Top}(\mathrm{F_T}(X), S)$, then $\mathrm{U_T}(f)$ is the function on the underlying sets, and given a function $g \in \mathbf{Set}(X, \mathrm{U_T}(S))$, then the function g defines a continuous map $\mathrm{F_T}(X) \to S$, so we have a bijective correspondence. It is straightforward to check naturality. □

Lemma 3.5.10. *Consider again the forgetful functor* $\mathrm{U_T}\colon \mathbf{Top} \to \mathbf{Set}$ *introduced in Example 3.5.2.*

(I) There is a functor $\mathrm{G_T}\colon \mathbf{Set} \to \mathbf{Top}$ *which sends a set X to the space with underlying set X and the indiscrete topology and sends a function to the map which has the same action on the underlying sets.*

(II) The functor $\mathrm{G_T}$ *is right adjoint to U_T.*

Proof. (I) For sets X and Y, let $f\colon X \to Y$ be a function. Then $f^{-1}(Y) = X$ and $f^{-1}(\varnothing) = \varnothing$ so f defines a continuous function from $\mathrm{G_T}(X)$ to $\mathrm{G_T}(Y)$. Thus $\mathrm{G_T}$ is well defined. Clearly $\mathrm{G_T}$ sends identities to identities. Preservation of composition follows immediately from the definition.

(II) This is similar to the previous lemma, the appropriate isomorphism sends functions to continuous maps which act in the same way on the underlying set. □

Proposition 3.5.11. *Let X be a set, there is a groupoid which has object set X and exactly one morphism (x, y) from x to y for any pair $x, y \in X$, with the composition defined by $(x, y)(y, z) = (x, z)$. The identity morphisms are (x, x) and $(x, y)^{-1} = (y, x)$. This is called the indiscrete groupoid and denoted $\Delta(X)$.*

Proof. Straightforward. □

Lemma 3.5.12. *(I) There is a functor* $\Delta\colon \mathbf{Set} \to \mathbf{Grpd}$ *which sends a set X to $\Delta(X)$, and sends a function $f\colon X \to Y$ to the unique functor from $\Delta(f)\colon \Delta(X) \to \Delta(Y)$ which acts as f on objects.*

(II) The functor $\mathrm{U}_{\mathcal{G}}\colon \mathbf{Grpd} \to \mathbf{Set}$ *introduced in Example 3.5.3 has right adjoint* $\Delta\colon \mathbf{Set} \to \mathbf{Grpd}$.

Proof. Straightforward. □

Lemma 3.5.13. *Consider the functor* $\mathrm{U}_{\mathrm{V}_{\Bbbk}} : \mathbf{Vect}_{\Bbbk} \to \mathbf{Set}$ *as in Example 3.5.4.*

(I) There is a functor $\mathrm{F}_{\mathrm{V}_{\Bbbk}} : \mathbf{Set} \to \mathbf{Vect}_{\Bbbk}$ *which sends a set* X *to the free vector space over* X *and a function to the unique linear map between the corresponding free vector spaces.*

(II) The functor $\mathrm{F}_{\mathrm{V}_{\Bbbk}}$ *is left adjoint to* $\mathrm{U}_{\mathrm{V}_{\Bbbk}} : \mathbf{Vect}_{\Bbbk} \to \mathbf{Set}$*.*

Proof. Straightforward. □

3.5.1 The space $\mathbf{TOP}(X, Y)$ and the product-hom adjunction

Here we recall the construction of a partial product-hom adjunction in the category **Top**, this is Theorem 3.5.16. The adjunction holds subject to some conditions which are not too restrictive for us. In particular, the compact-open topology allows us to define a right-adjoint to the functor $- \times K : \mathbf{Top} \to \mathbf{Top}$ (see Lemma 3.5.14), when K is a locally compact Hausdorff space. (The case $K = [0,1]$ was one of the examples given in the original reference on adjoint functors [Kan58, pp 294].)

We will see in Chapter 4 that this adjunction will be crucial in understanding the connection between the generalised motions we construct and embedded cobordisms. We will also need the fact that the product $- \times \mathbb{I}$ preserves colimits in **Top** for many of the proofs in Chapter 5.

We also give a first example of the utility of the product-hom adjunction in the context of the fundamental groupoid; paths in the fundamental groupoid are equivalent if and only if there is a path between them in the space of paths (Lemma 3.5.19).

Lemma 3.5.14. *Fix a topological space* Y*. We can define a functor* $- \times Y : \mathbf{Top} \to \mathbf{Top}$ *as follows. A space* X *is sent to the product space* $X \times Y$*. A continuous map* $f : X \to X'$ *is sent to the map* $f \times id : X \times Y \to X' \times Y$, $(x, y) \mapsto (f(x), y)$*. We will refer to this as the* *product functor.*

Proof. We must show that, for a map $f : X \to X'$, $f \times id$ is a continuous map $X \times Y$ to $X' \times Y$. Let $U' \times V$ be a basis open set in $X' \times Y$. Then the preimage under $f \times id$ is $f^{-1}(U') \times V$ which is open since f is continuous. It is clear that the product functor preserves the identity and respects the composition. □

Lemma 3.5.15. *Fix a topological space* Y*. We can define a functor* $\mathbf{Top}(Y, -) : \mathbf{Top} \to$

Top *as follows. A space Z is sent to the space* $\mathbf{TOP}(Y,Z)$. *A continuous map* $f\colon Z \to Z'$ *is sent to* $f\circ -\colon \mathbf{TOP}(Y,Z) \to \mathbf{TOP}(Y,Z')$, $g \mapsto f\circ g$. *We will refer to this as the* *top-hom functor*.

Note the relation to the hom functor introduced in Proposition 3.1.21.

Proof. We must show that $f\circ -$ is a continuous map. Open sets in the subbasis of $\tau^{co}_{YZ'}$ are of the form $B_{YZ'}(K,U)$ for some K a compact set in Y and U an open set in Z'. The set $f^{-1}(U)$ is open in Z since f is a continuous map. Hence $B_{YZ}(K,f^{-1}(U))$ is an open set in τ^{co}_{YZ} .

We will show that the inverse image of $B_{YZ'}(K,U)$ under $f\circ -$ is $B_{YZ}(K,f^{-1}(U))$. For any $g \in B_{YZ}(K,f^{-1}(U))$ we have $f\circ g \in B_{YZ'}(K,U)$. Conversely suppose some $h \in B_{YZ'}(K,U)$ can be written in the form $f\circ g'$ for some $g' \in \mathbf{TOP}(Y,Z)$, then $g' \in B_{YZ}(K,f^{-1}(U))$. ☐

Lemma 3.5.16. *Let Y be a locally compact Hausdorff topological space. The product funtor* $-\times Y : \mathbf{Top} \to \mathbf{Top}$ *is left adjoint to the top-hom functor* $\mathbf{TOP}(Y,-)$. *In particular for objects $X,Y,Z \in$* **Top** *the usual hom-tensor adjunction from* **Set** *sending a set map* $f\colon X \to \mathbf{TOP}(Y,Z)$ *to* $\hat{f}\colon X\times Y \to Z$, $(x,y) \mapsto f(x)(y)$ *is well-defined in* **Top** *(i.e. $\hat{f}$ is continuous); and this gives a set map*

$$\Phi_{XZ}\colon \mathbf{Top}(X,\mathbf{TOP}(Y,Z)) \to \mathbf{Top}(X\times Y,Z)$$

that is a bijection, natural in the variables X and Z. [3]

Remark 3.5.1. By twisting $\Phi_{X,Y}$ with a homeomorphism $g\colon Y \to Y$, so $f\colon X \to \mathbf{TOP}(Y,Z)$ is sent to the map $(x,y) \mapsto f(x)(g(y))$, it can be shown that each $g \in \mathbf{Top}(Y,Y)$ leads to a distinct adjunction between the maps $-\times Y$ and $\mathbf{TOP}(Y,-)$. The proof proceeds exactly as for the untwisted case.

Proof. That we have a bijection of sets is proved in Proposition A.14 of [Hat02]. It remains to prove that this bijection is natural. Suppose we have continuous maps $\alpha\colon X' \to X$ and

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[3] There is in fact an adjustment of the compact open topology which, with an adjustment to the product, gives an adjunction without the need to restrict Y. See section 5.9 in [Bro06] for more.

$\beta\colon Z \to Z'$ then we must show we have a commuting diagram of the form

$$\begin{tikzcd}
\mathbf{Top}(X, \mathbf{TOP}(Y,Z)) \arrow[r, "\Phi_{XZ}"] \arrow[d, "Hom(\alpha,\beta\circ -)"'] & \mathbf{Top}(X \times Y, Z) \arrow[d, "Hom(\alpha\times\mathrm{id},\beta)"] \\
\mathbf{Top}(X', \mathbf{TOP}(Y,Z')) \arrow[r, "\Phi_{X'Z'}"'] & \mathbf{Top}(X' \times Y, Z')
\end{tikzcd}$$

where Hom is the hom-functor $\mathcal{C}^{op} \times \mathcal{C} \to \mathbf{Set}$ as in Proposition 3.1.21. Looking first at the left hand vertical arrow, a map $f\colon X \to \mathbf{TOP}(Y,Z)$ is sent to the map $X' \to \mathbf{TOP}(Y,Z')$, $x' \mapsto \beta \circ f(\alpha(x'))$, and then to $(x',y) \mapsto (\beta \circ f(\alpha(x')))(y)$ in $\mathbf{Top}(X' \times Y, Z')$. Going first along the top, a map f is sent to the map $X \times Y \to Z$, $(x,y) \mapsto f(x)(y)$ and then to the map $X' \times Y \to Z'$ defined by $(x',y) \mapsto \beta(f(\alpha(x'))(y)) = (\beta \circ f(\alpha(x')))(y)$. ☐

Corollary 3.5.17. *The function $K\colon \mathbf{TOP}(M,M) \times M \to M$ defined by $K(h,m) = h(m)$ is a continuous map.*

Proof. Let $X = \mathbf{TOP}(M,M)$, $Y = M$ and $Z = M$ and then consider the image of $\mathrm{id}\colon \mathbf{TOP}(M,M) \to \mathbf{TOP}(M,M)$ under $\Phi_{\mathbf{TOP}(M,M)M}$. ☐

Definition 3.5.18. Let X be a space. We call $\mathbf{TOP}(\mathbb{I},X)$ the path space of X.

Lemma 3.5.19. *Let X be a topological space. Let $\gamma, \gamma' \in \mathfrak{P}X(x,x')$ be paths. Then $\gamma \overset{p}{\sim} \gamma'$ if and only if there is a path $\tilde{H}\colon \mathbb{I} \to \mathbf{TOP}(\mathbb{I},X)$ such that $\tilde{H}(0) = \gamma$, $\tilde{H}(1) = \gamma'$ and for all $t \in \mathbb{I}$, $\tilde{H}(t) \in \mathfrak{P}X(x,x')$. In other words, paths in X are equivalent in the fundamental groupoid if and only if they are connected by a path in the path space of X.*

Proof. We have that $\mathbb{I}$ is a locally compact Hausdorff topological space so Theorem 3.5.16 gives that there is a bijection between continuous maps $\mathbb{I} \times \mathbb{I} \to X$ and continuous maps $\mathbb{I} \to \mathbf{TOP}(\mathbb{I},X)$. It is straightforward to check that the image of the set of path homotopies H from a path γ to a path γ' under this bijection is the set of paths γ to γ'. ☐

3.6 Colimits

Colimits will play an integral role in Chapter 5 so we use this section to review some key properties. We also fix representative colimits in the categories we will work with throughout the thesis.

The topics covered here can be found, for example, in [Per19, Ch.3].

3.6.1 Coproducts and pushouts

We start by defining two specific types of colimit that will be particularly useful.

Definition 3.6.1. Let $\mathcal{C}$ be a category. A coproduct of two objects $C_1, C_2 \in Ob(\mathcal{C})$ is a diagram

$$C_1 \xrightarrow{i_1} C \xleftarrow{i_2} C_2$$

of morphisms in $\mathcal{C}$, with the universal property that for any diagram

$$C_1 \xrightarrow{v_1} C' \xleftarrow{v_2} C_2$$

of morphisms in $\mathcal{C}$, there is a unique morphism $v: C \to C'$ such that $v *_{\mathcal{C}} i_1 = v_1$ and $v *_{\mathcal{C}} i_2 = v_2$. By abuse of language it is common to refer to C as a coproduct of C_1 and C_2. We may use $C_1 \sqcup C_2$ for a coproduct of C_1 and C_2.

As the following example shows, coproducts in a category do not always exist.

Example 3.6.2. *Let $\mathcal{C}$ be the category with $Ob(\mathcal{C}) = \{A, B\}$ and only identity morphisms, then a coproduct of A and B does not exist.*

If a coproduct of two objects in a category does exist there will generally be many choices of coproduct. It is straightforward to see that if $C_1 \xrightarrow{i_1} C \xleftarrow{i_2} C_2$ is a coproduct in a category $\mathcal{C}$, and there exists an isomorphism $u: C \to C'$ in $\mathcal{C}$, then $C_1 \xrightarrow{u *_{\mathcal{C}} i_1} C' \xleftarrow{u *_{\mathcal{C}} i_2} C_2$ is also a coproduct.

Even when there is a unique choice of object $C_1 \sqcup C_2$, there may be many choices of maps making $C_1 \to C_1 \sqcup C_2 \leftarrow C_2$ a coproduct. Although we do have the following lemma.

Lemma 3.6.3. *Let $\mathcal{C}$ be a category and $C_1, C_2 \in \mathcal{C}$. Suppose $C_1 \xrightarrow{i_1} C \xleftarrow{i_2} C_2$ and $C_1 \xrightarrow{v_1} C \xleftarrow{v_2} C_2$ are both coproducts. There is a unique morphism $v: C \to C'$ making the following* *diagram commute*

$$\begin{tikzcd}
 & C \arrow[dd, "v"] & \\
C_1 \arrow[ur, "i_1"] \arrow[dr, "v_1"'] & & C_2 \arrow[ul, "i_2"'] \arrow[dl, "v_2"] \\
 & C', &
\end{tikzcd}$$

and v is an isomorphism.

Proof. It follows from the universal property of the coproducts that there are unique morphisms $v: C \to C'$ and $v': C' \to C$ making the above diagram commute.

The universal property also gives that $v *_{\mathcal{C}} v'$ is the identity on C and $v' *_{\mathcal{C}} v$ is the identity on C', hence v, v' are isomorphisms. □

Where coproducts do exist, we are free to choose a representative element of each isomorphism class of coproducts to work with. Indeed we will *fix* representative elements for various categories so a coproduct is uniquely defined by giving two elements of the category. For example, in **Set** we will fix the representative coproduct of a pair $X, Y \in Ob(\mathbf{Set})$, to be the disjoint union

$$X \sqcup Y := (X \times \{1\}) \cup (Y \times \{2\}),$$

with the natural inclusions (i.e. those given by $\iota_i(x) := (x, i)$).

We check this really is a coproduct.

Lemma 3.6.4. *The diagram $X \xrightarrow{\iota_1} X \sqcup Y \xleftarrow{\iota_2} Y$ is a coproduct in* **Set**.

Proof. Suppose we have another diagram $X \xrightarrow{v_1} A \xleftarrow{v_2} Y$. Then a map $v: X \sqcup Y \to A$ is defined as follows. For all $x \in X$, $v(x,1) = v_1(x)$ and for all $y \in Y$, $v(y,2) = v_2(y)$. By construction the map v commutes with the v_i and ι_i, and is unique. □

Definition 3.6.5. Let $\mathcal{C}$ be a category. A <u>pushout</u> of two morphisms $f_1: C_0 \to C_1$ and $f_2: C_0 \to C_2$ in $\mathcal{C}$ is a diagram

$$\begin{tikzcd}
C_0 \arrow[r, "f_1"] \arrow[d, "f_2"'] & C_1 \arrow[d, "u_1"] \\
C_2 \arrow[r, "u_2"'] & C
\end{tikzcd}$$

which is commutative and has the universal property that for any other diagram

$$\begin{tikzcd}
C_0 \arrow[r, "i_1"] \arrow[d, "i_2"'] & C_1 \arrow[d, "v_1"] \\
C_2 \arrow[r, "v_2"'] & C',
\end{tikzcd}$$

there exists a unique $v\colon C \to C'$ such that $v *_{\mathcal{C}} u_1 = v_1$ and $v *_{\mathcal{C}} u_2 = v_2$.

Again, it is common to refer to C as the pushout.

As with coproducts, pushouts in a category do not always exist but, where they do exist, are unique up to canonical isomorphism (the proof is very similar to Lemma 3.6.3). We explain our convention for pushouts in **Set**.

Consider morphisms $f\colon Z \to X$ and $g\colon Z \to Y$ in **Set**. Then the diagram below:

$$\begin{tikzcd}
Z \arrow[r, "f"] \arrow[d, "g"'] & X \arrow[d, "p_X"] \\
Y \arrow[r, "p_Y"'] & X \sqcup_Z Y
\end{tikzcd}$$

is a pushout in **Set**. Here

$$X \sqcup_Z Y \ := (X \sqcup Y)/\sim,$$

where $\sim$ is the reflexive, symmetric, transitive closure of the relation

$$\big\{\big(\iota_1(f(z)), \iota_2(g(z))\big) \mid z \in Z\big\},$$

on $X \sqcup Y$, $p_X(x)$ is the equivalence class of $\iota_1(x)$ in $X \sqcup Y/\sim$ and $p_Y(y)$ is the equivalence class of $\iota_2(y)$ in $X \sqcup Y/\sim$.

Coproducts and pushouts also exist in the categories $\mathbf{Vect}_{\mathbb{k}}$, **Top** and **Grpd**. We fix representative coproducts and pushouts in the category **Top** in Section 3.6.3, and discuss colimits in **Grpd** in Section 3.6.4.

3.6.2 General colimits

We now recall the construction of a general colimit. We will only need to fix a general representative colimit in **Set**.

Definition 3.6.6. Let $\mathcal{C}$ be a category and $\mathbf{I}$ a small category. A functor $D{:}\,\mathbf{I} \to \mathcal{C}$ is called a diagram in $\mathcal{C}$ of shape $\mathbf{I}$.

Let $\mathbf{P} = \bullet \leftarrow \bullet \rightarrow \bullet$ be a category with three objects and two non identity morphisms as shown. Then a functor $\mathbf{P} \to \mathcal{C}$ for some category $\mathcal{C}$ is uniquely specified by drawing a diagram

$$C_1 \xleftarrow{f_1} C_0 \xrightarrow{f_2} C_2$$

in $\mathcal{C}$ of the same shape as $\mathbf{P}$, hence the nomenclature.

Definition 3.6.7. Let $\mathcal{C}$ be a category, $\mathbf{I}$ a small category and $D{:}\,\mathbf{I} \to \mathcal{C}$ a diagram in $\mathcal{C}$. A cocone is an object $C \in Ob(\mathcal{C})$ together with a family of morphisms

$$\psi = (\psi_i{:}\, D(i) \to C)_{i \in Ob(\mathbf{I})}$$

indexed by the objects in $\mathbf{I}$ such that for all morphisms $f{:}\, i \to j$ in $\mathbf{I}$ the following triangle commutes.

$$\begin{tikzcd}
D(i) \arrow[rr, "D(f)"] \arrow[dr, "\psi_i"'] & & D(j) \arrow[dl, "\psi_j"] \\
& C &
\end{tikzcd}$$

A colimit of D is a cocone (C, ϕ) with the universal property that for any other cocone (C', ψ) there exists a unique morphism $C \to C'$ making the following diagram commute for all morphisms $f{:}\, i \to j$ in $\mathbf{I}$.

$$\begin{tikzcd}
D(i) \arrow[rr, "D(f)"] \arrow[dr, "\phi_i"] \arrow[ddr, bend right, "\psi_i"'] & & D(j) \arrow[dl, "\phi_j"'] \arrow[ddl, bend left, "\psi_j"] \\
& C \arrow[d, dashed, "\exists!"] & \\
& C' &
\end{tikzcd}$$

We will refer to the object C as $colim(D)$.

Example 3.6.8. *A colimit of a diagram of shape* $\mathbf{P}$ *is a pushout.*

Example 3.6.9. *Let* $\mathbf{T}$ *be the category with two objects and no non identity morphisms, then a colimit of a diagram of shape* $\mathbf{T}$ *is a coproduct.*

Definition 3.6.10. Let $\mathbf{E} = \begin{tikzcd}\bullet \arrow[r, shift left] \arrow[r, shift right] & \bullet\end{tikzcd}$ be the category with two objects and two non identity morphisms as shown. A colimit of a diagram of shape E is called a coequaliser.

Proposition 3.6.11. *Let $f, g \colon X \to Y$ be functions in* $\mathbf{Set}$*. Then there exists a coequaliser*

$$\begin{tikzcd} X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \arrow[r, "p"] & Y/\sim, \end{tikzcd}$$

where $\sim$ is the reflexive, symmetric and transitive closure of the relation

$$\{(f(x), g(x)) \mid x \in N\}$$

on Y, and p is the canonical map sending $y \in Y$ to its equivalence class $[y] \in Y/\sim$.

Lemma 3.6.12. *Let $f_1 \colon C_0 \to C_1$ and $f_2 \colon C_0 \to C_2$ be morphisms in a category $\mathcal{C}$. Then*

$$\begin{tikzcd} C_0 \arrow[r, "f_1"] \arrow[d, "f_2"'] & C_1 \arrow[d, "p_1"] \\ C_2 \arrow[r, "p_2"'] & C \end{tikzcd}$$

is a pushout of f_1 and f_2 if and only if

$$\begin{tikzcd} C_0 \arrow[r, shift left, "i_1 *_{\mathcal{C}} f_1"] \arrow[r, shift right, "i_2 *_{\mathcal{C}} f_2"'] & C_1 \amalg C_2 \arrow[r, "p"] & C \end{tikzcd}$$

is a coequaliser, where p is the map obtained from applying the universal property of the coproduct to the maps $C_1 \xrightarrow{p_1} C \xleftarrow{p_2} C_2$.

Proof. Let $h \colon C_1 \amalg C_2 \to H$ be a morphism with $h *_{\mathcal{C}} i_1 *_{\mathcal{C}} f_1 = h *_{\mathcal{C}} i_2 *_{\mathcal{C}} f_2$. Then, using the universal property of the coproduct, h uniquely determines a pair of maps $h_1 \colon C_1 \to H$ and $h_2 \colon C_2 \to H$ with $h *_{\mathcal{C}} i_1 = h_1$ and $h *_{\mathcal{C}} i_2 = h_2$, and hence $h_1 *_{\mathcal{C}} f_1 = h_2 *_{\mathcal{C}} f_2$. Then there exists a unique map unique map $h' \colon C \to H$, with $p_1 *_{\mathcal{C}} h' = h_1$ and $p_2 *_{\mathcal{C}} h' = h_2$ if and only if h' is also the unique map satisfying $p *_{\mathcal{C}} h' = h$. □

We will fix the following representative colimit in $\mathbf{Set}$. Let $D \colon \mathbf{I} \to \mathbf{Set}$ be a diagram. Then $\amalg_{i \in Ob(\mathbf{I})} D(i)$ is the disjoint union as in Definition 2.0.3. For $i \in \mathbf{I}$, let $\tilde{\phi}_i \colon D(i) \to \amalg_{i \in \mathbf{I}} D(i)$

denote the map $x \mapsto (x,i)$. Consider the relation

$$R = \left\{ (\tilde{\phi}_i(x), \tilde{\phi}_j(D(f)(x))) \mid f\colon i \to j \in \mathbf{I} \right\}$$

on $\sqcup_{i\in\mathbf{I}} D(i)$. The colimit of D is given by

$$colim(D) = {}^{\sqcup_{i\in\mathbf{I}} D(i)}\!/_{\bar{R}}$$

with maps $\phi_i\colon D(i) \to {}^{\sqcup_{i\in\mathbf{I}} D(i)}\!/_{\sim}$ which send x to the equivalence class of $\tilde{\phi}_i(x)$.

The following theorem, which says that adjunctions interact nicely with colimits, will play a key role in Chapter 5.

Theorem 3.6.13. *([Rie17, Thm. 4.5.3]) Left adjoints preserve colimits. This means for any left adjoint $F\colon \mathcal{C} \to \mathcal{D}$, any diagram $D\colon \mathbf{I} \to \mathcal{C}$ then*

$$F(colim(D)) = colim(F(D)).$$

3.6.3 Colimits in Top

By Lemma 3.5.10 the forgetful functor $\mathrm{U_T}\colon \mathbf{Top} \to \mathbf{Set}$ is a left adjoint. Thus, by Theorem 3.6.13, $\mathrm{U_T}$ preserves colimits. This means coproducts and pushouts of diagrams in **Top** have the same underlying set as the coproducts and pushouts of their images in **Set**.

Let X and Y be spaces. Then

$$\tau_{X\sqcup Y} \coloneqq \left\{ U \subseteq X \sqcup Y \mid \iota_1^{-1}(U) \text{ is closed in } X \text{ and } \iota_2^{-1}(U) \text{ is closed in } Y \right\}$$

is a topology on $X \sqcup Y$. It is straightforward to prove that $(X \sqcup Y, \tau_{X\sqcup Y})$ is a coproduct in **Top** (see for example (3.1.2) of [Bro06]). We will use the notation indicated by the following diagram to refer to the map given by the universal property of the coproduct in **Top**.

$$\begin{tikzcd}
X \arrow[r, "\iota_1"] \arrow[dr, bend right, "i"'] & X \sqcup Y \arrow[d, dashed, "\exists!\langle i,j\rangle"] & Y \arrow[l, "\iota_2"'] \arrow[dl, bend left, "j"] \\
& M &
\end{tikzcd} \tag{3.3}$$

(If we have maps of spaces $i{:}\,X \to M$ and $j{:}\,Y \to N$, we will use $i \sqcup j$ for the obvious map $X \sqcup Y \to M \sqcup N$.)

Let X, Y, Z be topological spaces. Consider continuous maps $f{:}\,Z \to X$ and $g{:}\,Z \to Y$. The topology on $X \sqcup_Z Y$ which makes it into a pushout in **Top** is the following:

$$\tau_{X\sqcup_Z Y} := \{U \subseteq X \sqcup_Z Y \mid p_X^{-1}(U) \text{ is closed and } p_Y^{-1}(U) \text{ is closed}\}.$$

This topology can be equivalently defined as the finest topology on $X \sqcup_Z Y$ making p_X and p_Y continuous.

3.6.4 Colimits in Grpd

Our construction of a TQFT in Chapter 5 will rely on the fact that the pushout of two finitely generated groupoids is a finitely generated groupoid, which we prove in Theorem 3.6.21. We do this by explicitly constructing coequalisers in **Grpd**. Our main reference for this is [Hig71], although the parts on universal morphisms are also covered in [Bro06, Ch.8]. We note that everything done here can also be done in **Cat**.

The difficulty in constructing colimits of groupoids stems from the fact that the image of a functor of groupoids is often not a groupoid, as it is not closed. More precisely, suppose $F{:}\,\mathcal{G} \to \mathcal{H}$ is a functor of groupoids, and $g_1{:}\,w \to x$ and $g_2{:}\,y \to z$ are morphisms in $\mathcal{G}$. Then $g_2 * g_1$ is defined if and only if $x = y$ and then we must have $F(g_2 * g_1) = F(g_2) * F(g_1)$, i.e. $F(g_2) * F(g_1)$ must be the image of a morphism in $\mathcal{G}$. Suppose, however, that $x \neq y$ but $F(x) = F(y)$, then $F(g_2) * F(g_1)$ is defined in $\mathcal{H}$ but will not be the image of any single element in $\mathcal{G}$. A consequence is that it is possible for the coequaliser of finite groupoids to be infinite. This is illustrated by the following example.

Let $(\mathbb{Z}, +)$ denote the category with one object and morphisms labelled by elements of $\mathbb{Z}$, with composition given by addition in $\mathbb{Z}$.

Example 3.6.14. *Let $\{*\}$ be the groupoid with one object and only the identity morphism, and let $\mathbf{I}$ be the groupoid with two objects $\{a, b\}$ and one non-identity morphism from a to b. Let ι_a be the functor uniquely defined by $\iota_a(*) = a$, and ι_b the functor uniquely defined* *by* $\iota_b(*) = b$. *The following diagram is a coequaliser*

$$\begin{tikzcd}\{*\} \arrow[r, shift left, "\iota_a"] \arrow[r, shift right, "\iota_b"'] & \mathbf{I} \arrow[r, "p"] & (\mathbb{Z}, +),\end{tikzcd}$$

where p *is a functor which maps the only non-identity morphism to* $1 \in \mathbb{Z}$. *(Note* p *must send* $\{a\}$ *and* $\{b\}$ *to the only object in* $(\mathbb{Z}, +)$.*)*

Let $\mathcal{G}_1, \mathcal{G}_2 \in Ob(\mathbf{Grpd})$ be groupoids. Then we fix a representative coproduct, denoted $\mathcal{G}_1 \sqcup \mathcal{G}_2$ as follows. The object set of $Ob(\mathcal{G}_1 \sqcup \mathcal{G}_2) = Ob(\mathcal{G}_1) \sqcup Ob(\mathcal{G}_2)$, the coproduct in **Set**, and the morphism set in $\mathcal{G}_1 \sqcup \mathcal{G}_2$ is the coproduct in **Set** of the morphisms in $\mathcal{G}_1$ and the morphisms in $\mathcal{G}_2$, where for $f: w \to x$ in $\mathcal{G}_1$ and $g: y \to z$ in $\mathcal{G}_2$, (f, g) is a morphism from (w, y) to (x, z).

We proceed towards explicitly constructing coequalisers in **Grpd** by first introducing universal morphisms.

Let X be a set. Throughout this section we will also use X to denote the trivial groupoid with the set X as objects and only identity morphisms. The meaning will be clear from context.

Definition 3.6.15. Let $F: \mathcal{G} \to \mathcal{H}$ be a functor and denote by $Ob(F)$ the unique functor making the following square, where the vertical maps are inclusions, commute.

$$\begin{tikzcd}Ob(\mathcal{G}) \arrow[r, "Ob(F)"] \arrow[d, "\iota_{\mathcal{G}}"'] & Ob(\mathcal{H}) \arrow[d, "\iota_{\mathcal{H}}"] \\ \mathcal{G} \arrow[r, "F"] & \mathcal{H}\end{tikzcd}$$

Then F is called <u>universal</u> if this square is a pushout.

Lemma 3.6.16. *Let* X *be a set,* $\mathcal{G}$ *a groupoid and* $\sigma: Ob(\mathcal{G}) \to X$ *a function. There is a groupoid* $U_\sigma(\mathcal{G})$ *constructed as follows.*

(I) We have $Ob(U_\sigma(\mathcal{G})) = X$.

(II) For a pair $x, y \in X$ *a word of length* n *from* x *to* y *is a sequence*

$$a = a_n ... a_1$$

of morphisms $a_i: g_i \to g_i'$ *in* $\mathcal{G}$ *such that*

(i) for all $i = \{1, \dots, n-1\}$, $g_i' \neq g_{i+1}$,

(ii) for all $i = \{1, \dots, n-1\}$, $\sigma(g_i') = \sigma(g_{i+1})$,

(iii) $\sigma(g_1) = x$ *and* $\sigma(g_n') = y$,

(iv) for all $i \in \{1, \dots, n\}$, $a_i \neq 1_{g_i}$.

For a pair $x, y \in X$ *the set of morphisms* $U_\sigma(\mathcal{G})(x, y)$ *is the set all words from* x *to* y *when* $x \neq y$ *and in the case* $x = y$ *we also add the empty word which we will denote* 1_x. *(Notice that* $U_\sigma(\mathcal{G})(x, y)$ *may well be empty, and certainly will be in the case* $x \neq y$ *and* x *and* y *are not in the image of* σ.*)*

(III) Morphisms are composed by concatenating words and, where possible, evaluating compositions in $\mathcal{G}$ *cancelling identities.*

(IV) For $x \in X$, *the identity morphism is the empty word which we denote* 1_x.

(V) Suppose $a = a_n \dots a_1$ *is a word in* $U_\sigma(\mathcal{G})(x, y)$, *then it has inverse* $a^{-1} = a_1^{-1} ... a_n^{-1}$, *where* a_i^{-1} *is the inverse in* $\mathcal{G}$. *Notice that this is in* $U_\sigma(\mathcal{G})(y, x)$.

Proof. ($\mathcal{C}1$) It is immediate from the construction that the empty word acts as an identity under concatenation.

($\mathcal{C}2$) Evaluating compositions is associative because concatenation is associative and the composition in $\mathcal{G}$ is associative.

($\mathcal{G}1$) It is immediate from the construction that the described word is an inverse. □

Lemma 3.6.17. *Let* $\mathcal{G}$ *be a groupoid,* X *a set,* $\sigma: Ob(\mathcal{G}) \to X$ *a function and* $U_\sigma(\mathcal{G})$ *as constructed in Lemma 3.6.16. There is a functor* $\sigma': \mathcal{G} \to U_\sigma(\mathcal{G})$, *defined as follows. On objects* $\sigma' = \sigma$. *For a morphism* $a: g \to g'$ *in* $\mathcal{G}$ *we have* $\sigma'(a) = 1_{\sigma(g)}$ *if* $a = 1_g$ *and* $\sigma'(a) = a$, *considered as a length one word in* $U_\sigma(\mathcal{G})$, *otherwise. Note that* a *is a word from* g *to* g'.

Proof. First note that the all identities in $\mathcal{G}$ are mapped to identities in $U_\sigma(\mathcal{G})$ by construction.

Suppose $a: g \to g'$ and $a': g' \to g''$ are morphisms in $\mathcal{G}$. If $a = 1_g$ then $\sigma'(1_g *_{\mathcal{G}} a') = \sigma'(a') = a'$ which is the concatenation of a' with the empty word. If $a' = 1_g$, then similarly $\sigma(a *_{\mathcal{G}} a')$

is precisely the concatenation $\sigma'(a)\sigma'(a')$. If neither a, nor a' is an identity, then

$$\sigma'(a *_{\mathcal{G}} a') = a *_{\mathcal{G}} a'$$

which is precisely the concatenation $\sigma'(a)\sigma'(a')$ with all possible compositions in $\mathcal{G}$ evaluated. □

Lemma 3.6.18. *Let $\mathcal{G}$ be a groupoid, X a set and $\sigma\colon Ob(\mathcal{G}) \to X$ a function. The functor $\sigma'\colon \mathcal{G} \to U_\sigma(\mathcal{G})$ as constructed in Lemma 3.6.17 is universal.*

Proof. To prove σ' is universal we construct, for any groupoid $\mathcal{K}$ and functors τ and ϕ with $\tau *_{\mathcal{G}} \sigma = \phi *_{\mathcal{G}} \iota_{\mathcal{G}}$, a unique map ϕ^* making the following diagram commute.

$$\begin{tikzcd}
Ob(\mathcal{G}) \arrow[r, "\sigma"] \arrow[d, "\iota_{\mathcal{G}}"'] & X \arrow[d, "\iota_X"] \arrow[ddr, bend left, "\tau"] & \\
\mathcal{G} \arrow[r, "\sigma'"] \arrow[drr, bend right, "\phi"'] & U_\sigma(\mathcal{G}) \arrow[dr, dashed, "\phi^*"] & \\
 & & \mathcal{K}
\end{tikzcd}$$

We must have that on objects $x \in Ob(U_\sigma(\mathcal{G})) = X$, $\phi^*(x) = \tau(x)$ and that $\phi^*(1_x) = \tau(1_x)$. Now let a_1 be a word of length 1 in $U_\sigma(\mathcal{G})$, then a_1 is a morphism in $\mathcal{G}$ and, by commutativity, we must have $\phi^*(a_1) = \phi(a_1)$. For words of length n, $a = a_n \dots a_1$ in $U_\sigma(\mathcal{G})$, by functoriality we must have

$$\begin{aligned}
\phi^*(a) &= \phi^*(a_n) *_{\mathcal{K}} \dots *_{\mathcal{K}} \phi^*(a_1) \\
&= \phi(a_n) *_{\mathcal{K}} \dots *_{\mathcal{K}} \phi(a_1).
\end{aligned}$$

Notice for any $a_i\colon g_i \to g_i'$ and $a_{i+1}\colon g_{i+1} \to g_{i+1}'$ we have $\sigma(g_i') = \sigma(g_{i+1})$, hence $\tau(\sigma(g_i')) = \tau(\sigma(g_{i+1}))$ and so, by commutativity of the diagram, $\phi(g_i') = \phi(g_{i+1})$, so we have that ϕ^* is well defined. By construction composition is preserved on word concatenations. Composition is preserved also by evaluating compositions and removing identities because ϕ preserves composition. We have that ϕ^* is unique by construction. □

We now construct coequalisers in **Grpd**.

Since left adjoints preserve colimits (Theorem 3.6.13), and the forgetful functor $\mathrm{U}_{\mathcal{G}}\colon \mathbf{Grpd} \to \mathbf{Set}$ which sends a groupoid to its object set is a left adjoint (Lemma 3.5.12), we can find the object set of a coequaliser of a diagram D in **Grpd** by evaluating the set coequaliser of $\mathrm{U}_{\mathcal{G}} \circ D$ in **Set**.

Lemma 3.6.19. *Let* $f, g\colon \mathcal{G}_0 \to \mathcal{G}_1$ *be functors of groupoids and let* $\tilde{f}, \tilde{g}\colon Ob(\mathcal{G}_0) \to Ob(\mathcal{G}_1)$ *denote* $\mathrm{U}_{\mathcal{G}}(f)$ *and* $\mathrm{U}_{\mathcal{G}}(g)$ *respectively. Let* $\sigma\colon Ob(\mathcal{G}_1) \to Ob(\mathcal{G}_1)/\sim_{Ob}$ *be the coequaliser of* $\tilde{f}$ *and* $\tilde{g}$ *in* **Set**, *and let* $\sigma'\colon \mathcal{G}_1 \to U'_\sigma(\mathcal{G}_1)$ *denote the universal map constructed as in Lemma 3.6.17.*

For each pair $x, y \in Ob(\mathcal{G}_1)/\sim_{Ob}$ *let* $R_{x,y}$ *be the relation on* $U'_\sigma(\mathcal{G}_1)(x,y)$ *with*

$$(a_n \dots a_1, a'_n \dots a'_1) \in R_{x,y}$$

if there exists a morphism $b \in \mathcal{G}_0$ *such that for some* $i \in \{1, \dots, n\}$, $\sigma' f(b) = a_i$ *and* $\sigma' g(b) = a'_i$ *and for all other* $j \neq i$, $a_j = a'_j$. *(I) The collection of equivalence relations* $\bar{R} = (U'_\sigma(\mathcal{G}_1)(x,y), \bar{R}_{x,y})$ *is a congruence, hence there is a quotient groupoid* $U'_\sigma(\mathcal{G}_1)/\bar{R}$.

(II) The following diagram is a coequaliser

$$\begin{tikzcd} \mathcal{G}_0 \arrow[r, shift left, "\sigma' f"] \arrow[r, shift right, "\sigma' g"'] & U'_\sigma(\mathcal{G}_1) \arrow[r, "\gamma^*"] & U'_\sigma(\mathcal{G}_1)/\bar{R}, \end{tikzcd} \tag{3.4}$$

where γ^* *is the quotient functor induced by* $\bar{R}$.

Proof. (I) This is straightforward to check.
(II) Suppose we have a groupoid $\mathcal{H}$ and a map $\psi\colon U'_\sigma(\mathcal{G}_1) \to \mathcal{H}$ with $\psi *_{\mathcal{G}} \sigma' *_{\mathcal{G}} f = \psi *_{\mathcal{G}} \sigma' *_{\mathcal{G}} g$. Let a_1 and a'_1 be words of length 1 in $U'_\sigma(\mathcal{G}_1)$ and suppose there exists $b \in \mathcal{G}_0$ such that $\sigma' *_{\mathcal{G}} f(b) = a_1$ and $\sigma' *_{\mathcal{G}} g(b) = a'_1$. Then, by assumption, we must have $\psi(a_1) = \psi(a'_1)$.

Now suppose there exists words $a = a_n \dots a_1$ and $a' = a'_n \dots a'_1$ in $U'_\sigma(\mathcal{G}_1)$ and a morphism $b \in \mathcal{G}_0$ such that for some $i \in \{1, \dots, n\}$, $\sigma' f(b) = a_i$ and $\sigma' g(b) = a'_i$, and for all other $j \neq i$, $a_j = a'_j$. Then by functoriality we must have $\psi(a) = \psi(a')$.

Thus we arrive at precisely the relation described in the Lemma. So ψ must factor through $U'_\sigma(\mathcal{G}_1)/\bar{R}$, and the diagram is a coequaliser. □

Lemma 3.6.20. *Let $f, g \colon \mathcal{G}_0 \to \mathcal{G}_1$ be functors of groupoids. Then*

$$\begin{tikzcd} \mathcal{G}_0 \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & \mathcal{G}_1 \arrow[r, "\gamma^* \sigma'"] & U'_\sigma(\mathcal{G}_1)/\bar{R} \end{tikzcd} \tag{3.5}$$

is a coequaliser, where we use the notation of the previous Lemma.

Proof. We have from the Lemma 3.6.19 that (3.4) is a coequaliser. Suppose we have a functor $\psi \colon \mathcal{G}_1 \to \mathcal{H}$ with $\psi *_{\mathcal{G}} f = \psi * g$, then since $U'_\sigma(\mathcal{G}_1)$ is universal, there exists a unique map $\psi' \colon U'_\sigma(\mathcal{G}_1) \to \mathcal{H}$ with $\psi = \psi' \sigma'$ and hence using the universal property of the coequaliser a unique map $\Psi \colon U'_\sigma(\mathcal{G}_1)/\bar{R} \to \mathcal{H}$ making (3.4) commute. Then $\Psi \gamma^* \sigma' = \psi$ and so Ψ makes (3.5) commute.

Note that this is unique, since any $\Psi' \colon U'_\sigma(\mathcal{G}_1)/\bar{R} \to \mathcal{H}$ making (3.5) commute will also commute with ψ' and (3.4), and by the universal property of the coequaliser, this map is unique. □

We have now shown that we can construct a coequaliser in **Grpd** of any pair of maps. It is, in fact, possible to obtain all colimits in terms of from coproducts and coequalisers, although we won't need that level of generality here. Thus, having constructed the coequaliser, we now know that **Grpd** has all colimits.

Theorem 3.6.21. *Let $\mathcal{G}_0$ and $\mathcal{G}_1$ be finitely generated groupoids and $f \colon \mathcal{G}_0 \to \mathcal{G}_1$ and $g \colon \mathcal{G}_0 \to \mathcal{G}_1$ functors. The pushout of f and g is finitely generated.*

Proof. By Lemma 3.6.12 we can construct the pushout of f and g by finding the coequaliser of $\tilde{f} \colon \mathcal{G}_0 \to \mathcal{G}_1 \sqcup \mathcal{G}_2$ and $\tilde{g} \colon \mathcal{G}_0 \to \mathcal{G}_1 \sqcup \mathcal{G}_2$, where the tilde indicates composition with the maps into the coproduct. By Lemmas 3.6.17 and 3.6.16, equivalence classes of morphisms in the coequaliser are represented by words in $\mathcal{G}_1 \sqcup \mathcal{G}_2$. By construction $\mathcal{G}_1 \sqcup \mathcal{G}_2$ is finitely generated if $\mathcal{G}_1$ and $\mathcal{G}_2$ are, generated by the disjoint union of the generators of $\mathcal{G}_1$ and $\mathcal{G}_2$. Thus the coequaliser will be finitely generated. □

3.7 Monoidal categories

Here we recall the definition of monoidal and symmetric monoidal categories, and of functors preserving this extra structure. We also give examples that we will make use of later. A good reference for this section is [TV17].

Definition 3.7.1. A monoidal category is a pentuple

$$(\mathcal{C}, \otimes, \mathbb{1}, \alpha_{-,-,-}, \lambda_{-}, \rho_{-})$$

consisting of a category $\mathcal{C}$ and,

- a functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ called the *monoidal product*;
- an object $\mathbb{1} \in Ob(\mathcal{C})$ called the *monoidal unit*;
- for each triple of objects $X, Y, Z \in Ob(\mathcal{C})$, an isomorphism

  $$\alpha_{X,Y,Z} : (X \otimes Y) \otimes Z \to X \otimes (Y \otimes Z)$$

  called an *associator*;
- for each $X \in Ob(\mathcal{C})$ an isomorphism $\lambda_X : \mathbb{1} \otimes X \to X$ called a *left unitor*;
- for each $X \in Ob(\mathcal{C})$ an isomorphism $\rho_X : X \otimes \mathbb{1} \to X$ called a *right unitor*.

These are subject to the following constraints:

(M1) for all $W, X, Y, Z \in Ob(\mathcal{C})$ the following diagram, called the *pentagon identity* commutes:

$$\begin{tikzcd}
 & (W \otimes X) \otimes (Y \otimes Z) \arrow[dr, "\alpha_{W,X,Y\otimes Z}"] & \\
((W \otimes X) \otimes Y) \otimes Z \arrow[ur, "\alpha_{W\otimes X,Y,Z}"] \arrow[d, "\alpha_{W,X,Y}\otimes 1_Z"'] & & W \otimes (X \otimes (Y \otimes Z)) \\
(W \otimes (X \otimes Y)) \otimes Z \arrow[rr, "\alpha_{W,X\otimes Y,Z}"'] & & W \otimes ((X \otimes Y) \otimes Z), \arrow[u, "1_W \otimes \alpha_{X,Y,Z}"']
\end{tikzcd}$$

(M2) for all $X, Y \in Ob(\mathcal{C})$ the following diagram, called the *triangle identity*, commutes:

$$\begin{tikzcd}
(X \otimes \mathbb{1}) \otimes Y \arrow[rr, "\alpha_{X,\mathbb{1},Y}"] \arrow[dr, "\rho_X \otimes 1_Y"'] & & X \otimes (\mathbb{1} \otimes Y) \arrow[dl, "1_X \otimes \lambda_Y"] \\
 & X \otimes Y, &
\end{tikzcd}$$

(M3) all the associators and the left and right unitors are natural isomorphisms, that is for each morphism $f{:}\,X \to X'$ in $\mathcal{C}$, the following diagrams commute:

$$\begin{tikzcd}
\mathbb{1} \otimes X \arrow[r, "1_{\mathbb{1}} \otimes f"] \arrow[d, "\lambda_X"'] & \mathbb{1} \otimes X' \arrow[d, "\lambda_{X'}"] \\
X \arrow[r, "f"'] & X',
\end{tikzcd}$$

$$\begin{tikzcd}
X \otimes \mathbb{1} \arrow[r, "f \otimes 1_{\mathbb{1}}"] \arrow[d, "\rho_X"'] & X' \otimes \mathbb{1} \arrow[d, "\rho_{X'}"] \\
X \arrow[r, "f"'] & X',
\end{tikzcd}$$

and for all morphisms $f{:}\,X \to X'$, $g{:}\,Y \to Y'$ and $h{:}\,Z \to Z'$ in $\mathcal{C}$, the following diagram commutes:

$$\begin{tikzcd}
(X \otimes Y) \otimes Z \arrow[r, "(f \otimes g) \otimes h"] \arrow[d, "\alpha_{X,Y,Z}"'] & (X' \otimes Y') \otimes Z' \arrow[d, "\alpha_{X',Y',Z'}"] \\
X \otimes (Y \otimes Z) \arrow[r, "f \otimes (g \otimes h)"'] & X' \otimes (Y' \otimes Z').
\end{tikzcd}$$

Definition 3.7.2. An initial object in a category $\mathcal{C}$ is an object $I \in Ob(\mathcal{C})$ such that for any $X \in Ob(\mathcal{C})$, there exists a unique morphism $f{:}\,I \to X$.

Example 3.7.3. *In* **Set** *the empty set* $\varnothing$ *is an initial object. In* **Top** *the space with underlying set* $\varnothing$ *is an initial object.*

Proposition 3.7.4. *[Mac71, Sec. VII.1] If* $\mathcal{C}$ *is a category with all coproducts and an initial object then* $\mathcal{C}$ *becomes a monoidal category as follows. The monoidal product is the coproduct and monoidal unit the initial object.*

The associators are obtained by applying the universal property of the coproduct twice, it can be shown these are isomorphisms by constructing inverses in the same way. The unitors are obtained by applying the universal property of the coproduct to the pair $1_X{:}\,X \to X$ *and the unique map* $\mathbb{1} \to X$*. By construction, the map into the coproduct* $X \to X \otimes \mathbb{1}$ *(or* $X \to \mathbb{1} \otimes X$*) composed with the relevant unitor must commute with the identity, thus these are isomorphisms.*

This is called the cocartesian monoidal structure.

Proof. The triangle and pentagon identities can be proved by noting that objects in the same isomorphism class as a coproduct, are also coproducts of the same pair of objects, and the isomorphism connecting them is unique. It is straightforward to check the naturality diagrams.

The following three propositions are examples of cocartesian monoidal structures.

Proposition 3.7.5. *(I) There exists a bifunctor*

$$\sqcup : \mathbf{Set} \times \mathbf{Set} \to \mathbf{Set}$$

$$(f : W \to X, g : Y \to Z) \mapsto f \sqcup g : W \sqcup Y \to X \sqcup Z$$

where $f \sqcup g$ *is the map obtained using the universal property of the coproduct on the maps* $\iota_1 \circ f : W \to X \sqcup Z$ *and* $\iota_2 \circ g : Y \to X \sqcup Z$ *where* $X \xrightarrow{\iota_1} X \sqcup Z \xleftarrow{\iota_2} Z$ *is the coproduct given in Section 3.6.1.*
(II) There exists a monoidal category

$$(\mathbf{Top},\ \sqcup\ ,\ \varnothing\ ,\ \alpha^S_{X,Y,Z} : (X \sqcup Y) \sqcup Z \to X \sqcup (Y \sqcup Z)\ ,\ \lambda^S_X : \varnothing \sqcup X \to X\ ,\ \rho^S_X : X \sqcup \varnothing \to X)$$

where the associators and unitors are the obvious isomorphisms.

Proof. (I) It is immediate from the construction that $f \sqcup g$ is a map $W \sqcup Y \to X \sqcup Z$.
(II) This is precisely the monoidal structure described in Proposition 3.7.4. It is also straightforward to check each of the identities directly. □

Proposition 3.7.6. *(I) There exists a bifunctor*

$$\sqcup : \mathbf{Top} \times \mathbf{Top} \to \mathbf{Top}$$

$$(f : W \to X, g : Y \to Z) \mapsto f \sqcup g : W \sqcup Y \to X \sqcup Z$$

where $f \sqcup g$ *is the map obtained using the universal property of the coproduct on the maps* $\iota_1 \circ f : W \to X \sqcup Z$ *and* $\iota_2 \circ g : Y \to X \sqcup Z$*, where* $W \xrightarrow{\iota_1} W \sqcup Y \xleftarrow{\iota_2} Y$ *is the coproduct given in Section 3.6.3.*

(II) There exists a monoidal category

$$(\mathbf{Top},\ \sqcup\ ,\ \varnothing\ ,\ \alpha^T_{X,Y,Z}\colon (X \sqcup Y) \sqcup Z \to X \sqcup (Y \sqcup Z)\ ,\ \lambda^T_X\colon \varnothing \sqcup X \to X\ ,\ \rho^T_X\colon X \sqcup \varnothing \to X)$$

where the associators and unitors are the obvious isomorphisms.

Proof. As for Proposition 3.7.5. □

Proposition 3.7.7. *(I) There exists a bifunctor*

$$\otimes_{\Bbbk}\colon \mathbf{Vect}_{\Bbbk} \times \mathbf{Vect}_{\Bbbk} \to \mathbf{Vect}_{\Bbbk}$$

defined as follows. Let V and W be vector spaces, then $V \otimes_{\Bbbk} W = V \times X/\sim$ where $\sim$ is the closure to an equivalence of the relations: for all $k \in \Bbbk$, $v, v' \in V$ and $w, w' \in W$

- $(kv_1, v_2) \sim k(v_1, v_2) \sim (v_1, kv_2)$,
- $(v_1 + v_1', v_2) \sim (v_1, v_2) + (v_1', v_2)$,
- $(v_1, v_2 + v_2') \sim (v_1, v_2) + (v_1, v_2')$.

Given any $v \in V$ and $w \in W$, we use $v \otimes_{\Bbbk} w$ to denote the equivalence class $[(v, w)]$. For linear maps $S\colon V \to X$ and $T\colon W \to Y$ we define

$$\begin{aligned} S \otimes_{\Bbbk} T\colon V \otimes_{\Bbbk} W &\to X \otimes_{\Bbbk} Y \\ (v \otimes_{\Bbbk} w) &\mapsto S(v) \otimes_{\Bbbk} T(w). \end{aligned}$$

(II) There exists a monoidal category

$$(\mathbf{Vect}_{\Bbbk}, \otimes_{\Bbbk}, \Bbbk, \alpha^{\Bbbk}_{V,W,X}, \lambda^{\Bbbk}_V, \rho^{\Bbbk}_V).$$

where for all $v \in V$, $w \in W$, $x \in X$ and $k \in \Bbbk$, $\alpha^{\Bbbk}_{V,W,X}((v \otimes_{\Bbbk} w) \otimes_{\Bbbk} x) = (v \otimes_{\Bbbk} (w \otimes_{\Bbbk} x))$, $\lambda^{\Bbbk}_V(v \otimes k) = kv$ and $\rho^{\Bbbk}_V(k \otimes v) = kv$.

Proof. This is an example of a cocartesian monoidal category (see Proposition 3.7.4).

Remark 3.7.1. Using the relations in $V \otimes_{\Bbbk} W$, it is not hard to show that a basis for $V \otimes_{\Bbbk} W$ is given by elements of the form $v \otimes_{\Bbbk} w$ where $v \in V$ and $w \in W$ are basis elements.

Definition 3.7.8. A monoidal subcategory of a monoidal category $(\mathcal{C}, \otimes, \mathbb{1}, \alpha_{-,-,-}, \lambda_{-}, \rho_{-})$ is a pentuple $(\mathcal{D}, \otimes, \mathbb{1}, \alpha_{-,-,-}, \lambda_{-}, \rho_{-})$ such that

- $\mathcal{D}$ is a subcategory of $\mathcal{C}$,
- $\otimes$ restricts to a closed composition on $\mathcal{D}$,
- $\mathbb{1} \in \mathcal{D}$, and
- for all $X, Y, Z \in Ob(\mathcal{D})$ we have $\alpha_{X,Y,Z}, \lambda_X, \rho_X$ are in $\mathcal{D}$.

Definition 3.7.9. Let $(\mathcal{C}, \otimes, \mathbb{1}, \alpha_{-,-,-}, \lambda_{-}, \rho_{-})$ and $(\mathcal{D}, \otimes', \mathbb{1}', \alpha'_{-,-,-}, \lambda'_{-}, \rho'_{-})$ be monoidal categories. A monoidal functor is a functor $F\colon \mathcal{C} \to \mathcal{D}$ endowed with a morphism $F_0\colon \mathbb{1}' \to F(\mathbb{1})$ in $\mathcal{D}$ and with a natural transformation

$$F_2 = \left\{F_2(X,Y)\colon F(X) \otimes' F(Y) \to F(X \otimes Y)\right\}_{X,Y \in Ob(\mathcal{C})}$$

between the functors $F \otimes' F = \otimes' \circ (F \times F)\colon \mathcal{C} \times \mathcal{C} \to \mathcal{D}$ and $F \circ \otimes\colon \mathcal{C} \times \mathcal{C} \to \mathcal{D}$ such that for all $X, Y, Z \in Ob(\mathcal{C})$ the following three diagrams commute.

$$\begin{tikzcd}
(F(X) \otimes' F(Y)) \otimes' F(Z) \arrow[r, "\alpha'_{F(X),F(Y),F(Z)}"] \arrow[d, "F_2(X,Y) \otimes' 1_{F(Z)}"'] & F(X) \otimes' (F(Y) \otimes' F(Z)) \arrow[d, "1_{F(X)} \otimes' F_2(Y,Z)"] \\
F(X \otimes Y) \otimes' F(Z) \arrow[d, "F_2(X \otimes Y, Z)"'] & F(X) \otimes' F(Y \otimes Z) \arrow[d, "F_2(X, Y \otimes Z)"] \\
F((X \otimes Y) \otimes Z) \arrow[r, "F(\alpha_{X,Y,Z})"'] & F(X \otimes (Y \otimes Z))
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbb{1}' \otimes' F(X) \arrow[r, "\lambda'_{F(X)}"] \arrow[d, "F_0 \otimes 1_{F(X)}"'] & F(X) \\
F(\mathbb{1}) \otimes' F(X) \arrow[r, "F_2(\mathbb{1}, X)"'] & F(\mathbb{1} \otimes X) \arrow[u, "F(\lambda_X)"']
\end{tikzcd}$$

$$\begin{tikzcd}
F(X) \otimes' \mathbb{1} \arrow[r, "\rho'_{F(X)}"] \arrow[d, "1_{F(X)} \otimes F_0"'] & F(X) \\
F(X) \otimes F(\mathbb{1}) \arrow[r, "F_2(X, \mathbb{1})"'] & F(X \otimes \mathbb{1}) \arrow[u, "F(\rho'_X)"']
\end{tikzcd}$$

Definition 3.7.10. A strong monoidal functor is a monoidal functor F where F_0 and all maps in F_2 are isomorphisms.

Example 3.7.11. *Let* $(\mathcal{C},\otimes,\mathbb{1},\alpha_{-,-,-},\lambda_{-},\rho_{-})$ *be a monoidal category. Then the identity functor* $1_{\mathcal{C}}:\mathcal{C}\to\mathcal{C}$ *with* $(1_{\mathcal{C}})_0 = 1_{\mathbb{1}}:\mathbb{1}\to\mathbb{1}$ *and* $(1_{\mathcal{C}})_2(X,Y) = 1_{X\otimes Y}:X\otimes Y\to X\otimes Y$ *is a strong monoidal functor.*

Proposition 3.7.12. *There is an associative composition of monoidal functors which sends a pair* $F:\mathcal{C}\to\mathcal{D}$ *and* $G:\mathcal{D}\to\mathcal{E}$ *to the monoidal functor* $G\circ F:\mathcal{C}\to\mathcal{E}$ *with*

$$(G\circ F)_0 = G(F_0)G_0 \quad \textit{and} \quad (G\circ F)_2(X,Y) = G(F_2(X,Y))\circ G_2(F(X),F(Y))$$

for all $X,Y\in Ob(\mathcal{C})$.

Proof. Straightforward. □

Definition 3.7.13. A braided monoidal category is a six-tuple

$$(\mathcal{C},\otimes,\mathbb{1},\alpha_{-,-,-},\lambda_{-},\rho_{-},\beta_{-,-})$$

consisting of a monoidal category $(\mathcal{C},\otimes,\mathbb{1},\alpha_{-,-,-},\lambda_{-},\rho_{-})$ and a family of natural isomorphisms

$$\beta_{X,Y}:X\otimes Y\to Y\otimes X$$

for each pair $X,Y\in Ob(\mathcal{C})$ such that

$$\beta_{X,Y\otimes Z} = (1_Y\otimes\beta_{X,Z}) *_{\mathcal{C}} (\beta_{X,Y}\otimes 1_Z)$$
$$\beta_{X\otimes Y,Z} = (\beta_{X,Z}\otimes 1_Y) *_{\mathcal{C}} (1_X\otimes\beta_{Y,Z}).$$

Naturality of β means that for any morphisms $f:X\to X'$ and $g:Y\to Y'$ the following diagram commutes.

$$\begin{tikzcd}
X\otimes Y \arrow[r, "f\otimes g"] \arrow[d, "\beta_{X,Y}"'] & X'\otimes Y' \arrow[d, "\beta_{X',Y'}"] \\
Y\otimes X \arrow[r, "g\otimes f"'] & Y'\otimes X'
\end{tikzcd}$$

Such a family of natural isomorphisms is called a *braiding* on $(\mathcal{C},\otimes,\mathbb{1},\alpha_{-,-,-},\lambda_{-},\rho_{-})$.

When speaking about (braided) monoidal categories we may drop entries of the tuple corresponding to the natural isomorphisms, or even refer to a braided monoidal category as just $\mathcal{C}$ where $\mathcal{C}$ is the notation of the underlying category. We note however, that there will often be many (braided) monoidal categories with the same underlying category, monoidal product and monoidal unit.

Proposition 3.7.14. *There exists a braided monoidal category*

$$(\mathbf{Top}, \sqcup, \varnothing, \alpha^T_{X,Y,Z}, \lambda^T_X, \rho^T_X, \beta^T_{X,Y}: X \otimes Y \to Y \otimes X)$$

where $(\mathbf{Top}, \sqcup, \varnothing, \alpha^T_{X,Y,Z}, \lambda^T_X, \rho^T_X)$ *is as in Proposition 3.7.6 and the* $\beta_{X,Y}$ *are the obvious isomorphisms.*

Proof. Again it is straightforward to explicitly check each of the identities. □

Definition 3.7.15. A braided monoidal subcategory of a braided monoidal category $(\mathcal{C}, \otimes, \mathbb{1}, \alpha_{-,-,-}, \lambda_{-}, \rho_{-}, \beta_{-,-})$ is a six tuple $(\mathcal{D}, \otimes, \mathbb{1}, \alpha_{-,-,-}, \lambda_{-}, \rho_{-}, \beta_{-,-})$ such that $(\mathcal{D}, \otimes, \mathbb{1}, \alpha_{-,-,-}, \lambda_{-}, \rho_{-})$ is a monoidal subcategory of $(\mathcal{C}, \otimes, \mathbb{1}, \alpha_{-,-,-}, \lambda_{-}, \rho_{-})$ and for all $X, Y \in Ob(\mathcal{D})$, $\beta_{X,Y} \in \mathcal{D}$.

Definition 3.7.16. A braiding β on a monoidal category $(\mathcal{C}, \otimes, \mathbb{1}, \alpha_{-,-,-}, \lambda_{-}, \rho_{-})$ is called *symmetric* if for all pairs $X, Y \in Ob(\mathcal{C})$ we have

$$\beta_{Y,X} *_{\mathcal{C}} \beta_{X,Y} = 1_{X \otimes Y}: X \otimes Y \to X \otimes Y.$$

A symmetric monoidal category is a braided monoidal category $(\mathcal{C}, \otimes, \mathbb{1}, \alpha_{-,-,-}, \lambda_{-}, \rho_{-}, \beta_{-,-})$ such that β is symmetric.

Proposition 3.7.17. *The braided monoidal category*

$$(\mathbf{Top}, \sqcup, \varnothing, \alpha^T_{X,Y,Z}, \lambda^T_X, \rho^T_X, \beta^T_{X,Y}: X \otimes Y \to Y \otimes X)$$

is a symmetric monoidal category.

Proof. It is easy to see that $\beta^T_{Y,X} *_{\mathcal{C}} \beta^T_{X,Y} = 1_{X \otimes Y}$. □

Definition 3.7.18. A braided monoidal functor between braided categories $(\mathcal{C}, \beta_{-,-})$ and

$(\mathcal{C}', \beta'_{-,-})$ is a monoidal functor $F{:}\mathcal{C} \to \mathcal{C}'$ such that for all $X, Y \in Ob(\mathcal{C})$,

$$F_2(Y,X) \circ \beta'_{F(X),F(Y)} = F(\beta_{X,Y}) \circ F_2(X,Y).$$

Definition 3.7.19. A symmetric monoidal functor is a braided monoidal functor between symmetric monoidal categories.

Chapter 4

Motion groupoids and mapping class groupoids

4.1 Introduction

The braid group has several different realisations, each with very different flavours – see for example [BB05]. As discussed in Section 1.1, it has realisations as homotopy classes of paths in the configuration space of points in the 2-disk, D^2, and as monotonic embeddings of unit intervals in $D^2 \times [0,1]$. It has further topological realisations as a motion group of points in D^2, and as a mapping class group of marked points in D^2. Each construction consists of concrete elements with a composition, and then some equivalence. We note that none of the constructions are pairwise equivalent when considering the concrete elements. Each construction lends itself to a possible generalisation, and these generalisations may or may not lead to groups which are again isomorphic.

In Figure 4.1 we have some schematics illustrating some aspects of the bridges between these different realisations of the braid group. At the top we have a series of schematics representing boundary-fixing self-homeomorphisms of the disk, where the movement of the disk is illustrated by the marking of a polar grid. The schematics represent evenly spaced points along a path in an appropriate space of self-homeomorphisms of the disk. At the bottom we have a schematic of two point particles exchanging places. Notice that naively, this picture may represent an embedding of two unit intervals in the cylinder, although it could also represent the path in the space of self-homeomorphisms illustrated by the top schematic, which moves the point particles as shown. The latter is roughly a concrete morphism in the motion group. The schematic may also represent the path in the configuration space of points in the disk which moves the particles as shown. And the equivalence further complicates the picture. A concrete element of the mapping class group is a single self-homeomorphism, as opposed to a path. A map from the motion group to the mapping class group can be obtained by taking the endpoint of a path. For braids it is known that all these pictures are equivalent [BB05]. And for loop braid groups we have analogous equivalences between the different settings [Dam17].

Our objective here is to generalise the mathematical definitions of motion groups and mapping class groups away from braid groups: to different manifolds, to subsets which are not point like (loop or string excitations in a 3-ball, for example) and to evolutions that do not necessarily start and end in the same configuration. To allow for evolutions which do not start and end in the same configuration, we use the language of groupoids.

One long term aim is to understand if the realisation of braids as both isomorphisms in the tangle category (discussed in Section 1.1.4) and as motions, lifts to a more general connection between isomorphisms in embedded cobordism categories and the generalised motions discussed in this paper. Another is the connection between generalised motions and configuration spaces.

Motion groups of a manifold and submanifold pair were first rigorously studied by Dahm as a way to generalise braid groups [Dah36], and subsequently developed by Goldsmith [Gol72]. Mapping class groups of a manifold and submanifold pair similarly have origins in the study of braid groups [Bir16]. As already discussed, the braid group can be equivalently defined as the mapping class group or as the motion group of finite sets of points in the 2-disk [Bir16; Gol81], and the loop braid group can be obtained as the mapping class group or as the motion group of unlinked, unknotted loops in the 3-disk [BWC+07; Gol81; Dam17]. For further examples of the study of other motion groups in literature, see [Bul+19; DK19].

Here we construct, for a manifold M, its motion groupoid Mot_M, and its mapping class groupoid MCG_M. The object set of both is the power set of M. Looking at the automorphism group of a particular object in each case gives back the corresponding group. We then study the relationship between our two constructions. In particular we construct a functor

$$\mathsf{F}\colon \mathrm{Mot}_M \to \mathrm{MCG}_M$$

and give conditions for it to be an isomorphism of groupoids.

In this chapter we do all constructions in the topological category, following the motion group construction of e.g [Gol81]. We note that embedded cobordism categories are commonly constructed in the smooth setting, although configuration spaces are a topological construction, and also used to model worldlines of particles, as discussed above. Our aim is precisely to investigate the implications of different choices of assumptions. We expect a similar construction in the smooth category to be possible. For motion groups constructed in the smooth category see [BWC+07; QW21]. For unknotted, unlinked loop particles in 3-dimensions and for point particles in 2-dimensions the smooth and topological settings coincide [Wat72].

We will present some key examples to demonstrate the richness of our construction. For example, the groupoid framework allows us to think about skeletons. Note that the existence of a homeomorphism between subspaces, or indeed a homeomorphism of the ambient space sending one subspace to the other is not enough to ensure that the underlying sets are connected by a morphism in the motion groupoid. Alternatively we can find subsets which have isomorphic automorphism groups but which are not connected in the motion groupoid.

We also give examples to demonstrate the utility of the functor $\mathsf{F}\colon \mathrm{Mot}_M \to \mathrm{MCG}_M$. In particular we give examples for which the motion groupoid and mapping class groupoid are not isomorphic. We show that the boundary fixing motion groupoid and mapping class groupoid of D^n are isomorphic for all $n \in \mathbb{N}$, and that for S^1 the mapping class groupoid and motion groupoid are not isomorphic.

One of our objectives is to add a monoidal structure to the motion groupoid developed here. This will be addressed in a separate work. We intend to use this to prove a presentation for a full subcategory of the motion groupoid of points and unknotted, unlinked loops in the 3-disk, which is conjectured in Section 4.3.8. We also plan, in a future work, to address the relationship of the motion groupoid with isomorphisms in embedded cobordism categories.

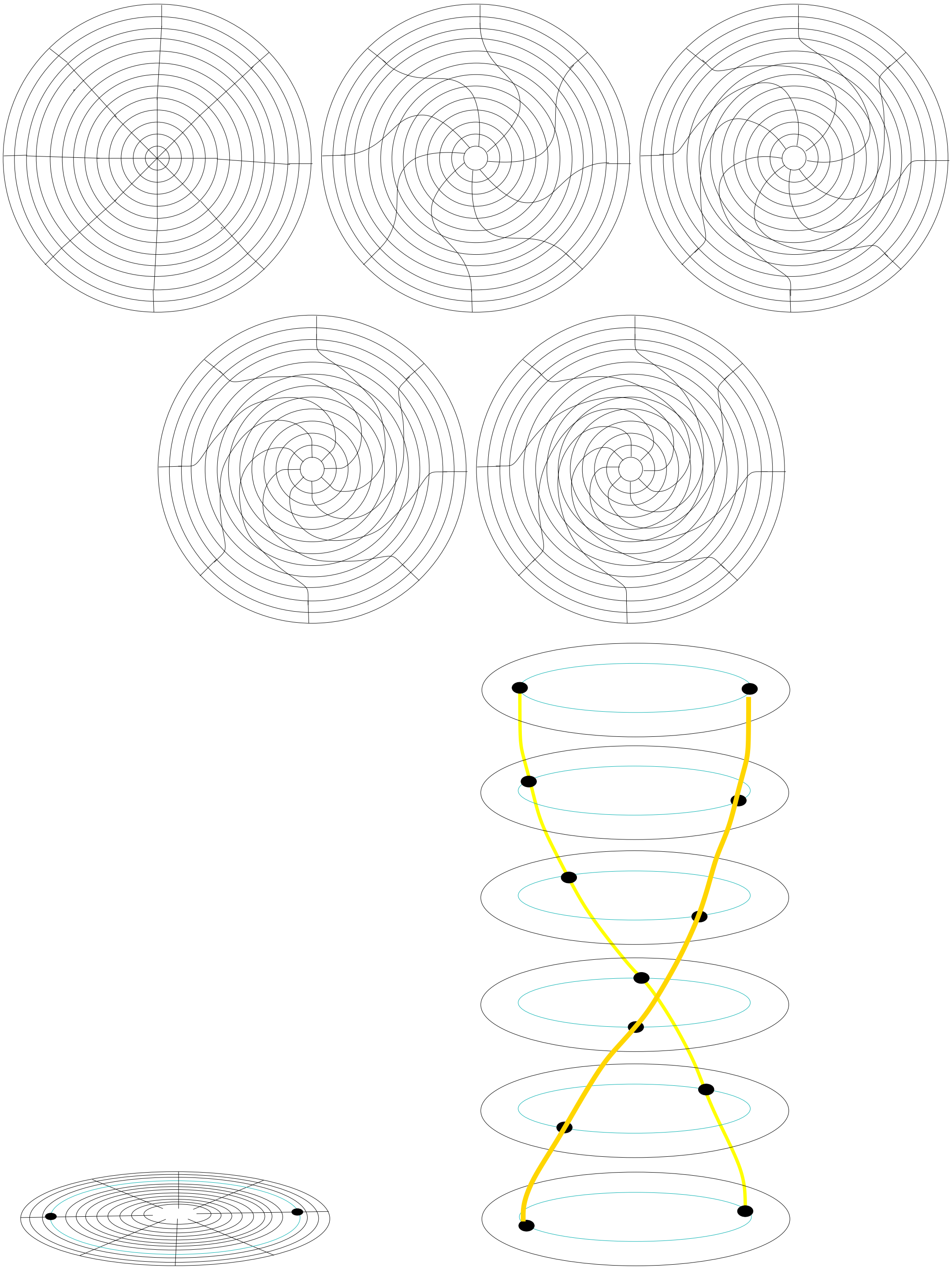


Figure 4.1: Top: Series of boundary-fixing self-homeomorphisms of the disk revealed by a marked polar grid. Bottom: We add a couple of marked points on the disk and watch them braid-exchange.

4.1.1 Chapter overview

In Section 4.2 we give the construction of a groupoid of self-homeomorphisms Homeo_M corresponding to a manifold M, with object class the power set $\mathcal{P}(M)$ (see Definition 4.2.4). This is a natural first step in our construction.

In Section 4.3 the main theorem is Theorem 4.3.37, the construction of the motion groupoid Mot_M of a manifold M, whose object class is again the power set $\mathcal{P}(M)$. We start by defining two magmoids of *motions* in M, which are paths in a space of self-homeomorphisms. We first quotient both magmoids by a congruence using path-homotopy, and show we obtain the same groupoid. We then quotient again by a normal subgroupoid of *stationary motions*, to obtain Mot_M. In Theorem 4.3.47, we consider a version of Theorem 4.3.37 where motions fix a distinguished subset of M, pointwise. This generalisation is important for us to explain the relationship of motion groupoids with braid groups and loop braid groups. Picking a single set in $\mathcal{P}(M)$ and looking at the group of automorphisms we get back the motion group constructed by Dahm [Dah36] and developed by Goldsmith [Gol72]. We also have Theorem 4.3.18 which says that motions are equivalent to homeomorphisms from $M \times [0,1]$ to $M \times [0,1]$ subject to some conditions. In Section 4.3.8 we have examples.

In Section 4.4 we discuss an alternative choice of congruence on the aforementioned groupoid of motions up to path-homotopy. The main result is Theorem 4.4.6 which says that this congruence leads again to the motion groupoid.

In Section 4.5 we construct the mapping class groupoid of a manifold M (Theorem 4.5.4). We obtain this as a quotient of the groupoid Homeo_M. Theorem 4.5.7 is a subset-fixing version. The automorphism group of an object in this category is the mapping class group of a pair, as described in [Dam17].

In Section 4.6 we construct a functor from the motion groupoid of a manifold to its mapping class groupoid (Theorem 4.6.12). We show that the restriction of this functor to automorphism groups is part of the long exact sequence of homotopy groups, following the ideas used in the group case by [Gol81]. This allows us to give conditions on the space of self-homeomorphisms of M under which we obtain an isomorphism between the motion groupoid of a manifold and its mapping class groupoid (Theorem 4.6.12). In Theorem 4.6.13 we have version relative to some distinguished subset. In Section 4.6.3 we give some examples demonstrating the use of the functor from Theorem 4.6.12.

Glossary

Top	The category of topological spaces and continuous maps.
Set	The category of sets and functions between sets.
τ^{co}_{XY}	The compact-open topology on the set $\mathbf{Top}(X,Y)$.
$\mathfrak{P}X(x,x')$	The subset of $\mathbf{Top}(\mathbb{I},X)$ of paths from x to x'.
$\overset{p}{\sim}$	Indicates paths related by path-equivalence, i.e. homotopy relative to the end-points, see Definition 3.3.5.
$[\gamma]_{\mathrm{p}}$	Equivalence class of a paths up to path-equivalence, see Definition 3.3.5.
$\pi(X)$	The fundamental groupoid of X.
$\pi(X,A)$	The fundamental groupoid of X with respect to a subset $A \subset X$ of basepoints.
$\mathbf{Top}^h(M,M)$	The submonoid of $\mathbf{Top}(M,M)$ containing homeomorphisms.
$\mathbf{TOP}^h(M,M)$	The set $\mathbf{Top}^h(M,M)$ equipped with subspace topology from τ^{co}_{MM}.
Homeo^A_M	Groupoid with $\mathcal{P}M$ as objects and triples (f,N,N') with f a self-homeomorphism fixing A pointwise, $N \subset M$ a subset and $f(N) = N'$ as morphisms, see Definition 4.2.4.
$\mathfrak{f}^A\colon N \curvearrowright N'$	Notation for morphisms in Homeo^A_M.
$Premot_M$	Set of all pre-motions in M, i.e. $f \in \mathbf{Top}(\mathbb{I},\mathbf{Top}^h(M,M))$ such that $f_0 = \mathrm{id}_M$, see Definition 4.3.1 .
Id_M	Pre-motion in M which is the path $f_t = \mathrm{id}_M$ for all t.
$f\colon N \hookrightarrow N'$	A motion from N to N' in the specified manifold, f is a pre-motion and $f_1(N) = N'$.
$\mathrm{Mt}_M(N,N')$	The set of all motions from N to N' in M.
$\mathbf{Mt}_M$	The set of all motions in M.
$\mathrm{Mt}_M/\overset{p}{\sim}$	Magmoid of motions up to path equivalence with $*$ or $\cdot$ composition, see Corollary 4.3.29.
$\overset{m}{\sim}$	Indicates motions related by motion-equivalence, see Proposition 4.3.36.
$[f\colon N \hookrightarrow N']_{\mathrm{m}}$	Equivalence class of a motion $f\colon N \hookrightarrow N'$ up to motion-equivalence, see Proposition 4.3.36.
$f^A\colon N \hookrightarrow N'$	A motion from N to N' fixing a distinguished subset A of the ambient manifold.
Mot^A_M	Groupoid with subsets of M as objects and motion-equivalence classes of A-fixing motions as morphisms, see Theorems 4.3.37 and 4.3.47.
$\overset{rp}{\sim}$	Indicates motions related by relative path-equivalence, see Definition 4.4.1.
$[f\colon N \hookrightarrow N']_{\mathrm{rp}}$	Equivalence class of a motion $f\colon N \hookrightarrow N'$ up to motion-equivalence, see Lemma 4.4.2.
$\overset{i}{\sim}$	Indicates self-homeomorphisms related by isotopy, see Definition 4.5.1.
$[\mathfrak{f}\colon N \curvearrowright N']_{\mathrm{i}}$	Equivalence class of a self-homeomorphism $\mathfrak{f}\colon N \curvearrowright N'$ up to isotopy, see Lemma 4.5.2.
MCG^A_M	Groupoid with subsets of M as objects and isotopy equivalence classes of A-fixing self-homeomorphisms as morphisms, see Theorems 4.5.4 and 4.5.7.
$\mathbb{I}$	The space $[0,1] \subset \mathbb{R}$ with the subset topology.
D^2	The 2-disk $\{x \in \mathbb{C} \mid \lvert x\rvert \leq 1\} \subset \mathbb{C}$ with the subset topology.
S^1	The circle $\{x \in \mathbb{C} \mid \lvert x\rvert = 1\} \subset \mathbb{C}$ with the subset topology.

4.2 Space of self-homeomorphisms of a space $\mathbf{TOP}^h(X,X)$

For any space X then $\mathbf{Top}(X,X)$ is a monoid, with identity the identity map. The subset of maps which are set bijections is a submonoid. Let $\mathbf{Top}^h$ be the subcategory of **Top** with the same objects as **Top** and morphisms which are homeomorphisms. (Note that the indicated subset is in fact closed.) Then $\mathbf{Top}^h(X,X)$ is the group of homeomorphisms $\mathfrak{f}: X \to X$. Denote by $\mathbf{TOP}^h(X,X)$ the subspace of $\mathbf{TOP}(X,X)$ with underlying set $\mathbf{Top}^h(X,X)$.

In Section 4.3 we will be interested in formalising how certain paths in $\mathbf{TOP}^h(M,M)$, where M is a manifold, induce 'motions' of subsets in M. As discussed in Section 4.1, one aim of this work is to establish a general relationship between motion groupoids and isomorphisms in embedded cobordism categories. For this we need to have a correspondence between paths in $\mathbf{TOP}^h(M,M)$ and homeomorphisms of $M \times \mathbb{I}$, which exists if M satisfies the conditions of the product-hom adjunction (Theorem 3.5.16), and $\mathbf{TOP}^h(M,M)$ is a topological group (this is proved in Theorem 4.3.18).

Here we begin by giving the precise conditions necessary such that, for a space X, $\mathbf{TOP}^h(X,X)$ a topological group. Then in Section 4.2.1 we restrict to the case $X = M$ is a manifold and organise the elements of $\mathbf{Top}^h(M,M)$ into a groupoid Homeo_M. In general this category is too large to be an interesting object of study itself but it is a natural first step in the construction that follows.

We have the following theorem giving conditions under which $\mathbf{TOP}^h(X,X)$ becomes a topological group. Notice that in this case we also have that that X satisfies the conditions of Theorem 3.5.16, the product-hom adjunction.

Theorem 4.2.1. *[Are46, Thm. 4] If X is a locally connected, locally compact Hausdorff space then $\mathbf{TOP}^h(X,X)$, the group of self-homeomorphisms of X with the subspace topology from τ^{co}_{XX}, is a topological group. (This means the composition $(\mathfrak{f},\mathfrak{g}) \mapsto \mathfrak{f} \circ \mathfrak{g}$ and the map $\mathfrak{f} \mapsto \mathfrak{f}^{-1}$ are both continuous.)*

Proof. See Section A.0.1. □

Lemma 4.2.2. *Let X be a space and $A \subset X$ a subset and let $\mathbf{Top}^h_A(X,X)$ denote the subset of $\mathbf{Top}^h(X,X)$ of homeomorphisms which fix A pointwise. Then $\mathbf{Top}^h_A(X,X)$ is a subgroup of $\mathbf{Top}^h(X,X)$.*

Proof. For all $a \in A$, we have $\mathrm{id}_X(a) = a$ so $\mathrm{id}_X \in \mathbf{Top}^h_A(X,X)$. Let $\mathfrak{f}, \mathfrak{g} \in \mathbf{Top}^h_A(X,X)$ then for all $a \in A$, $\mathfrak{f} \circ \mathfrak{g}(a) = \mathfrak{f}(a) = a$ and $\mathfrak{f}^{-1}(a) = a$. □

4.2.1 Action groupoid Homeo_M of the action of self-homeomorphisms on subsets

In this thesis, manifold means a Hausdorff topological manifold, which in particular is locally compact and locally connected.

Let M be a manifold possibly with boundary. Then we have that $\mathbf{TOP}^h(M,M)$ is a topological group and we can use the product-hom adjunction. Here we organise the elements of $\mathbf{Top}^h(M,M)$ into a groupoid Homeo_M, constructed as an action groupoid.

Lemma 4.2.3. *Let M be a manifold and $A \subseteq M$ a subset. There is a (left) group action*

$$\sigma^A \colon \mathbf{Top}^h_A(M,M) \times \mathcal{P}M \to \mathcal{P}M$$
$$(\mathfrak{f}, N) \mapsto \mathfrak{f}(N).$$

Proof. For any subset $N \subseteq M$, $\mathrm{id}_M(N) = N$ and for any $\mathfrak{f}, \mathfrak{g} \in \mathbf{Top}^h_A(M,M)$, $\mathfrak{f}(\mathfrak{g}(N)) = (\mathfrak{f} \circ \mathfrak{g})(N)$. □

Definition 4.2.4. Let M be a manifold and A a subset. By Lemma 3.1.30 there is an action groupoid, which we denote

$$\mathrm{Homeo}^A_M = \mathcal{P}M /\!/_{\sigma^A} \mathbf{Top}^h_A(M,M).$$

Explicitly the objects are $\mathcal{P}M$ and the morphisms are triples $(\mathfrak{f}, N, \mathfrak{f}(N))$ where

- $\mathfrak{f}$ is a homeomorphism $M \to M$,
- $\mathfrak{f}(N) = N'$,
- $\mathfrak{f}$ fixes A pointwise.

We will denote triples $(\mathfrak{f}, N, \mathfrak{f}(N)) \in \mathrm{Homeo}_M^A(N, N')$ as $\mathfrak{f}^A\colon N \rightsquigarrow N'$. In this notation the identity at each object N is $\mathrm{id}_M\colon N \rightsquigarrow N$, where id_M denotes the identity homeomorphism, and given a morphism $\mathfrak{f}^A\colon N \rightsquigarrow N'$ the inverse is the morphism $\mathfrak{f}^{-1A}\colon N' \rightsquigarrow N$.

We will use just Homeo_M to denote $\mathrm{Homeo}_M^{\varnothing}$, so morphism sets in Homeo_M are of the form $\mathrm{Homeo}_M(N, N')$ and we denote morphisms as $\mathfrak{f}\colon N \rightsquigarrow N'$.

Where convenient we will also use $\mathrm{Homeo}_M^A(N, N')$ to denote the set obtained by projecting to the first element of the triple. Then we have $\mathbf{Top}^h(M, M) = \mathrm{Homeo}_M(\varnothing, \varnothing) = \mathrm{Homeo}_M(M, M)$ and every $\mathrm{Homeo}_M^A(N, N') \subseteq \mathbf{Top}^h(M, M)$. Notice each $\mathfrak{f} \in \mathbf{Top}^h(M, M)$ will belong to many such $\mathrm{Homeo}_M^A(N, N')$.

Lemma 4.2.5. *Let M be a manifold and $A \subseteq M$ a fixed subset. For any subsets $N, N' \subseteq M$ we have*

$$\mathrm{Homeo}_M^A(N, N') \cong \mathrm{Homeo}_M^A(M \smallsetminus N, M \smallsetminus N').$$

Proof. Since any $\mathfrak{f}^A\colon N \rightsquigarrow N'$ is a bijection, $\mathfrak{f}(N) = N'$ iff $\mathfrak{f}(M \smallsetminus N) = M \smallsetminus N'$. □

Lemma 4.2.6. *Let M be a manifold and $A \subseteq M$ a subset. Each $\mathrm{Homeo}_M^A(N, N)$ becomes a topological subgroup of $\mathbf{TOP}^h(M, M)$.*

Proof. Suppose we have self-homeomorphisms $\mathfrak{f}^A\colon N \rightsquigarrow N$ and $\mathfrak{g}^A\colon N \rightsquigarrow N$, then $\mathfrak{f} \circ \mathfrak{g}(N) = \mathfrak{f}(N) = N$ and for all $a \in A$, $\mathfrak{f} \circ \mathfrak{g}(a) = \mathfrak{f}(a) = a$ so $\mathfrak{f} \circ \mathfrak{g}^A\colon N \rightsquigarrow N$ is in $\mathrm{Homeo}_M^A(N, N)$. Similarly $\mathfrak{f}^{-1A}\colon N \rightsquigarrow N$ is in $\mathrm{Homeo}_M^A(N, N)$. Note that a subgroup of a topological group is itself a topological group with the induced topology. □

Remark 4.2.1. There are various ways in which we could equip the subsets of M with extra structure. For example we could let N and N' be submanifolds of M equipped with an orientation and then consider homeomorphisms which preserve these orientations.

4.3 Motion groupoid Mot_M^A

In this section we construct the motion groupoid associated to a manifold.

The core topological ideas used in this section are present in [Gol81], and first appeared in [Dah36] (see also [Gol72]). Here Goldsmith constructs motion groups associated to a pair of a manifold and a subset.

We proceed by first defining *pre-motions* in a manifold M, and giving two choices of composition. At this point there are no 'objects', one choice of composition gives a magma, the other a group. We obtain motions by considering an action of pre-motions on $\mathcal{P}M$. The two compositions on motions give two magmoids, one of which can be given a groupoid structure. Under a congruence using path homotopy these magmoids become the same groupoid. This groupoid has, in general, uncountable morphism sets, and thus we add a further equivalence. By quotienting by the normal subgroupoid of *set-stationary motions*, we obtain the motion groupoid Mot_M (Theorem 4.3.37). The object set is the power set $\mathcal{P}M$ and the morphisms are equivalence classes of motions. Looking at the automorphism group at some $N \subseteq M$ gives back the motion group for the pair (M, N) as in [Gol81].

To make the notation more manageable we only give the full details of the proofs when working in Homeo_M. In Section 4.3.7 we also construct a version using Homeo_M^A, i.e fixing a distinguished choice of subset $A \subseteq M$. This leads to the motion groupoid Mot_M^A.

In Section 4.3.8 we have some examples which frame some of the questions that our construction allows us to ask. For example we can think about skeletons of our motion groupoids, or equivalently which subsets of a manifold M are connected in the motion groupoid. Alternatively we could look for subsets which are not connected by a morphism in the motion groupoid, but for which we do have isomorphic automorphism groups.

4.3.1 Pre-motions: paths in $\mathbf{Top}(\mathbb{I}, \mathbf{Top}^h(M, M))$

Here we define pre-motions and introduce two compositions.

Definition 4.3.1. Fix a manifold M. A pre-motion in M is a path in $\mathrm{Homeo}_M(\varnothing, \varnothing) = \mathbf{Top}^h(M, M)$ starting at id_M; i.e. a map $f \in \mathbf{Top}(\mathbb{I}, \mathbf{Top}^h(M, M))$ with $f_0 = \mathrm{id}_M$. We define notation for the set of all pre-motions in M,

$$Premot_M = \{f \in \mathbf{Top}(\mathbb{I}, \mathbf{Top}^h(M, M)) \mid f_0 = \mathrm{id}_M\}.$$

Example 4.3.2. *For any manifold M the path $f_t = \mathrm{id}_M$ for all t, is a pre-motion. We will denote this pre-motion Id_M.*

Example 4.3.3. *For $M = S^1$ (the unit circle) we may parameterise by $\theta \in \mathbb{R}/2\pi$ in the usual way. Consider the functions $\tau_\phi : S^1 \to S^1$ ($\phi \in \mathbb{R}$) given by $\theta \mapsto \theta + \phi$, and note that these are homeomorphisms. Then consider the path $f_t = \tau_{t\pi}$ ('half-twist'). This is a pre-motion.*

Lemma 4.3.4. *Let M be a manifold. For any pre-motion f in M, then $(f^{-1})_t = f_t^{-1}$ is a pre-motion.*

Proof. By Theorem 4.2.1 we have that $\mathbf{Top}^h(M,M)$ is a topological group, so the map $g \in \mathbf{Top}^h(M,M) \mapsto g^{-1} \in \mathbf{Top}^h(M,M)$ is continuous. It follows that the composition $t \mapsto f_t \mapsto f_t^{-1}$ is continuous. Notice also that $(f^{-1})_0 = \mathrm{id}_M^{-1} = \mathrm{id}_M$. □

Composition of pre-motions

The usual non-associative 'stack+shrink' composition of paths in $\mathbf{Top}(\mathbb{I},X)$ (see (3.1)) is a partial composition, precisely gf is a path if the end of the path f is the start of the path g. Now suppose $X = \mathbf{TOP}(Y,Y)$ for some space Y and $f,g \in \mathbf{Top}(\mathbb{I},\mathbf{TOP}(Y,Y))$. We can use the function composition in $\mathbf{TOP}(Y,Y)$ to construct paths $g_0 \circ f_t$ and $g_t \circ f_1$ which share an endpoint, and thus we can use the usual path composition on these modified paths.

Proposition 4.3.5. *Let Y be a space. There exists a composition*

$$\begin{aligned} *\colon \mathbf{Top}(\mathbb{I},\mathbf{TOP}(Y,Y)) \times \mathbf{Top}(\mathbb{I},\mathbf{TOP}(Y,Y)) &\to \mathbf{Top}(\mathbb{I},\mathbf{TOP}(Y,Y)) \\ (f,g) &\mapsto g * f \end{aligned}$$

where

$$(g * f)_t = \begin{cases} g_0 \circ f_{2t} & 0 \le t \le 1/2, \\ g_{2(t-1/2)} \circ f_1 & 1/2 \le t \le 1. \end{cases} \tag{4.1}$$

Proof. We check that for any $f,g \in \mathbf{Top}(\mathbb{I},\mathbf{TOP}(Y,Y))$, $g * f \in \mathbf{Top}(\mathbb{I},\mathbf{TOP}(Y,Y))$.

For any $\mathfrak{g} \in \mathbf{TOP}(Y,Y)$ the map $\mathfrak{g} \circ -\colon \mathbf{TOP}(Y,Y) \to \mathbf{TOP}(Y,Y)$, $\mathfrak{f} \mapsto \mathfrak{g} \circ \mathfrak{f}$ is continuous as for a subasis open set $B_{YY}(K,U)$ (see Definition 3.4.3) with $K \subseteq Y$ compact and $U \subseteq Y$ open we have $\mathfrak{g} \circ \mathfrak{f} \in B_{YY}(K,U) \iff \mathfrak{g}(\mathfrak{f}(K)) \subseteq U \iff \mathfrak{f} \in B_{YY}(K,\mathfrak{g}^{-1}(U))$ which is open.

Similarly for any $\mathfrak{g} \in \mathbf{TOP}(Y,Y)$ the map $- \circ \mathfrak{g} \colon \mathbf{TOP}(Y,Y) \to \mathbf{TOP}(Y,Y)$, $\mathfrak{f} \mapsto \mathfrak{f} \circ \mathfrak{g}$ is continuous as for a subbasis open set $B_{YY}(K,U)$ with $K \subseteq Y$ compact and $U \subseteq Y$ open we have $\mathfrak{f} \circ \mathfrak{g} \in B_{YY}(K,U) \iff \mathfrak{f}(\mathfrak{g}(K)) \subseteq U \iff \mathfrak{f} \in B_{YY}(\mathfrak{g}(K),U)$ which is open.

We also have that both functions agree at $t = 1/2$, hence Equation (4.1) defines a continuous map. □

Proposition 4.3.6. *Let M be a manifold. There exists a composition*

$$* \colon Premot_M \times Premot_M \to Premot_M$$
$$(f,g) \mapsto g * f$$

where

$$(g * f)_t = \begin{cases} f_{2t} & 0 \le t \le 1/2, \\ g_{2(t-1/2)} \circ f_1 & 1/2 \le t \le 1. \end{cases} \tag{4.2}$$

Proof. This is the restriction of the $*$ function of Proposition 4.3.5 to $Premot_M$ so we need only to check that $g * f \in Premot_M$. We have $(g * f)_0 = f_0 = \mathrm{id}_M$ and for all $t \in \mathbb{I}$, $(g * f)_t$ is a homeomorphism as it is the composition of two homeomorphisms. □

Remark 4.3.1. Notice that this means there is a magma $(Premot_M, *)$.

Given a manifold M, we can also define another composition of paths in $Premot_M$ which relies on the fact $\mathbf{TOP}(M,M)$ is a topological group.

Lemma 4.3.7. *Let M be a manifold. There is an associative composition*

$$\cdot \colon Premot_M \times Premot_M \to Premot_M$$
$$(f,g) \mapsto g \cdot f$$

where $(g \cdot f)_t = g_t \circ f_t$.

Proof. We first check that $g \cdot f$ is a path. This can be seen by rewriting as

$$\begin{aligned} I &\to \mathbf{TOP}^h(M,M) \times \mathbf{TOP}^h(M,M) \to \mathbf{TOP}^h(M,M) \\ t &\mapsto \qquad (f_t, g_t) \qquad \mapsto g_t \circ f_t. \end{aligned}$$

The map into the product is continuous because it is continuous on each projection and the second map is continuous because $\mathbf{TOP}^h(M,M)$ is a topological group by Theorem 4.2.1. Notice also that $(g \cdot f)_0 = g_0 \circ f_0 = \mathrm{id}_M$, so we have that $g \cdot f$ is a pre-motion. As in **Set**, composition of functions is associative and thus $\cdot$ is associative. □

Remark 4.3.2. There is a group $(Premot_M, \cdot)$ whose identity is Id_M and the inverse of $f \in Premot_M$ is f^{-1} as defined in Lemma 4.3.4.

Lemma 4.3.8. *Let M be a manifold and $f, g \in Premot_M$. Then $g * f \overset{p}{\sim} g \cdot f$.*

Before the proof, let us fix some conventions. Pre-motions are paths $\mathbb{I} \to \mathbf{TOP}^h(M,M)$ and then homotopies of paths are functions $H\colon \mathbb{I} \times \mathbb{I} \to \mathbf{TOP}^h(M,M)$. We will always think of the first copy of $\mathbb{I}$ in a homotopy as the one parameterising the pre-motion and will continue to use the parameter t. For the second copy of $\mathbb{I}$ which parameterises the homotopy we will use s.

Proof. The following function is a suitable path homotopy to prove the path-equivalence

$$H(t,s) = \begin{cases} g_{ts} \circ f_{2t(1-s)+ts} & 0 \le t \le \frac{1}{2}, \\ g_{2(t-1/2)(1-s)+ts} \circ f_{(1-s)+ts} & \frac{1}{2} \le t \le 1. \end{cases} \tag{4.3}$$

Notice $H(t,0) = (g * f)_t$, $H(t,1) = (g \cdot f)_t$ and for all $s \in \mathbb{I}$ we have $H(0,s) = g_0 \circ f_0 = \mathrm{id}_M$ and $H(1,s) = g_1 \circ f_1$. □

Remark 4.3.3. There are other choices of composition of pre-motion which assign paths g and f to a path which is path-homotopic to $g * f$ and $g \cdot f$. For example

$$(g *' f)_t = \begin{cases} g_{2t} & 0 \le t \le 1/2, \\ g_1 \circ f_{2(t-1/2)} & 1/2 \le t \le 1. \end{cases}$$

We can also generate from any pre-motion f, a pre-motion $\bar{f}$ which reverses the path.

Proposition 4.3.9. *Let M be a manifold. There exists a set map*

$$\begin{aligned} \bar{\ } : Premot_M &\to Premot_M \\ f &\mapsto \bar{f} \end{aligned}$$

with

$$\bar{f}_t \ = \ f_{(1-t)} \circ f_1^{-1}. \tag{4.4}$$

Proof. The path $f_{(1-t)}$ is continuous as it is the composition of the two continuous maps $t \mapsto 1 - t$ and $t \mapsto f_t$. By the same argument used in the proof Proposition 4.3.5, the composition with f_1^{-1} is continuous and so $\bar{f}$ is continuous. Also notice $\bar{f}_0 = f_1 \circ f_1^{-1} = \mathrm{id}_M$. □

Remark 4.3.4. The operation $f \mapsto \bar{f}$ is an involution, namely $\bar{\bar{f}} = f$.

Intuitively $\bar{f}$ is obtained from f by first changing the direction of travel along the path and then precomposing at each t with f_1^{-1} to force the reversed path to start at the identity. Notice also that for a pre-motion f, $\bar{f} * f = f^{rev} f$, with path composition as in (3.1) and the reverse path as in Proposition 3.3.8. Thus we have already shown in the proof of Proposition 3.3.8 that $\bar{f} * f \overset{p}{\sim} \mathrm{Id}_M$.

Proposition 4.3.10. *Let f and g be pre-motions in a manifold. Then $\overline{g * f} = \bar{f} * \bar{g}$.*

Proof. This is immediate from the definitions.

4.3.2 Motions: the action of pre-motions on subsets

We may think of a *magma action* as a group action without the identity condition, then $(Premot_M, *)$ acts on $\mathcal{P}M$ as $(f, N) \mapsto f_1(N)$. We can then obtain a *motion magmoid* where morphisms and composition are defined analogously to the groupoid case (Lemma 3.1.30). A *motion* is an element of this action magmoid.

Definition 4.3.11. Fix a manifold M. A motion in M is a triple $(f, N, f_1(N))$ consisting of a pre-motion $f \in Premot_M$ (Definition 4.3.1), a subset $N \subseteq M$ and the image of N at the endpoint of f, $f_1(N)$. (Note $f_1(N) = N'$ if and only if $f_1 \in \mathrm{Homeo}_M(N, N')$.)

We will denote such a triple by $f\colon N \rightarrowtail N'$ where $f_1(N) = N'$, and say it is a motion from N to N'. For subsets $N, N' \subseteq M$ we define

$$\mathrm{Mt}_M(N,N') = \{(f,N,f_1(N)) \text{ a motion in } M \,|\, f_1(N) = N'\}.$$

For each $N \subseteq M$ and $f \in \mathit{Premot}_M$ the triple $(f,N,f_1(N))$ is in exactly one $\mathrm{Mt}_M(N,N')$, so

$$\mathbf{Mt}_M = \bigcup_{N,N' \in \mathcal{P}M} \mathrm{Mt}_M(N,N') \quad \cong \; \mathit{Premot}_M \times \mathcal{P}M,$$

where the union is over all pairs $N, N' \subseteq M$.

As with Homeo_M, where convenient we will also use $\mathrm{Mt}_M(N,N')$ to denote the set obtained by projecting to the first element of the triple. Then each $f \in \mathit{Premot}_M$ will belong to many $\mathrm{Mt}_M(N,N')$.

The bar operation generates from any motion from N to N', a motion from N' to N.

Proposition 4.3.12. *Let M be a manifold. For any subsets $N, N' \subseteq M$ there is a set map*

$$\begin{aligned} \bar{\ } \colon \mathrm{Mt}_M(N,N') &\to \mathrm{Mt}_M(N',N) \\ f\colon N \rightarrowtail N' &\mapsto \bar{f}\colon N' \rightarrowtail N \end{aligned}$$

where $\bar{f}$ is as in Equation (4.4).

Proof. Proposition 4.3.9 gives that $\bar{f}$ is a pre-motion. Note that we have $\bar{f}_1(N') = f_0 \circ f_1^{-1}(N') = N$, hence $(\bar{f}\colon N' \rightarrowtail N) \in \mathrm{Mt}_M(N',N)$. □

Example 4.3.13. *For a manifold M, a subset $N \subseteq M$ and the pre-motion Id_M as in Example 4.3.2, $\mathrm{Id}_M\colon N \rightarrowtail N$ is a motion. We will call this the 'trivial motion' from N to N. Note that the pre-motion Id_M becomes a motion from N to N for any N, but not a motion from N to N' unless $N = N'$.*

Example 4.3.14. *The half-twist of S^1 (see Example 4.3.3) becomes a motion in S^1 from N to $\tau_\pi(N)$ for any $N \subseteq S^1$.*

4.3.3 Motions as maps from $M \times \mathbb{I}$, schematics and movie representations

In this section we give two further equivalent ways to define motions in a manifold M, in terms of certain maps from $M \times \mathbb{I}$. Equivalence Theorem 4.3.18 is significant because it indicates that we can connect to the cobordism picture of embeddings in $M \times \mathbb{I}$, as discussed in Section 1.1.4. (Note that the equivalences are still different so this does not immediately imply a functor between the two settings). The various definitions of motions lead us to some useful schematic representations, so we also discuss these here.

We begin by representing the space $\mathbf{TOP}^h(M,M)$, and elements of $\mathbf{Top}(\mathbb{I}, \mathbf{TOP}^h(M,M))$ schematically for arbitrary M. Figure 4.2 gives, schematically, two examples of motions in M. Here $\mathbf{TOP}^h(M,M)$ is represented as two disconnected regions of the plane, so the various $\mathrm{Homeo}_M(N,N')$s are possibly intersecting subregions. The blue path (a) represents a motion from N to N. Notice this is a path starting and ending in the same shaded region of $\mathrm{Homeo}_M(N,N)$. This is possible since $\mathrm{Homeo}_M(N,N)$ must contain the identity. (Although $\mathrm{Homeo}_M(N,N)$ may also have path connected components which do not contain the identity, as pictured.) The red path (b) is a motion from N to N' where $N \neq N'$.

Note a pre-motion corresponds to precisely one path in $\mathbf{TOP}^h(M,M)$, although many motions can have the same underlying pre-motion, thus to make such a diagram convey a motion it is necessary to explicitly state the subsets in addition to the schematic representation of the path.

We now give an interpretation of motions in a manifold M as a subset of $\mathbf{Top}(M \times \mathbb{I}, M)$.

Definition 4.3.15. Let M be a manifold and $N, N' \subset M$. Let $\mathrm{Mt}_M^{mov}(N,N') \subset \mathbf{Top}(M \times \mathbb{I}, M)$ denote the subset of elements $g \in \mathbf{Top}(M \times \mathbb{I}, M)$ such that:

(I) for all $t \in \mathbb{I}$, $g|_{M\times\{t\}}$ is a homeomorphism from $M \times \{t\}$ to M,

(II) for all $m \in M$, $g(m,0) = m$, and

(III) $g(N \times \{1\}) = N'$.

Lemma 4.3.16. *Let M be a manifold and $N, N' \subset M$. The restriction of the map*

$$\Phi \colon \mathbf{Top}(\mathbb{I}, \mathbf{TOP}(M,M)) \to \mathbf{Top}(M \times \mathbb{I}, M)$$

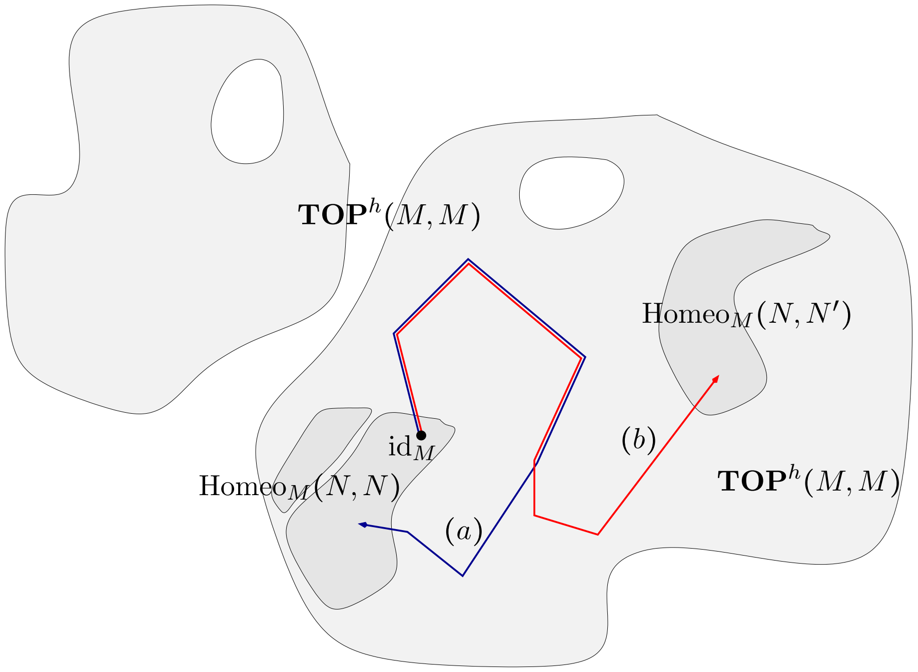


Figure 4.2: A schematic representation of $\mathbf{TOP}^h(M,M)$, for a fixed but arbitrary M, as a not-necessarily connected, not-necessarily simply-connected subset of $\mathbb{R}^2$. In practice we are only interested in the connected component of the point id_M. The blue line (a) is then a motion from N to N, and the red line (b) a motion from N to N'.

Φ *obtained by letting* $X = \mathbb{I}$ *and* $Y = Z = M$ *in Lemma 3.5.16 yields a bijection*

$$\mathrm{Mt}_M(N,N') \to \mathrm{Mt}_M^{mov}(N,N')$$

Proof. See Section A.0.2. □

Let M be a manifold and $f\colon N \rightsquigarrow N'$ a motion. Our next schematics are based on the 'movie presentations' of [CRS97]. A movie presentation of f consists of a number of pictures where each picture corresponds to a chosen value of t and shows the image of $N \subset M$ under f_t, ordered by $t \in \mathbb{I}$. We may also add 'grid line' subsets in M — these help to show the homeomorphism at t of M. See the top schematic of Figure 4.1. Here the relevant motion is of the from $f\colon \varnothing \rightsquigarrow \varnothing$ and the grid lines are a polar grid at $t = 0$. Movie presentations are used in [CRS97] for schematics representing the images of isotopies at various $t \in \mathbb{I}$, Lemma 4.3.16 gives that motions are precisely isotopies.

Next we give our second interpretation of motions in a manifold M as a subset of $\mathbf{Top}^h(M \times \mathbb{I}, M \times \mathbb{I})$.

Definition 4.3.17. Let M be a manifold and $N, N' \subset M$. Let $\mathrm{Mt}_M^{hom}(N, N') \subset \mathbf{Top}^h(M \times \mathbb{I}, M \times \mathbb{I})$ denote the subset of homeomorphisms $g \in \mathbf{Top}^h(M \times \mathbb{I}, M \times \mathbb{I})$ such that

(I) $g(m, 0) = (m, 0)$ for all $m \in M$,

(II) $g(M \times \{t\}) = M \times \{t\}$ for all $t \in \mathbb{I}$, and

(III) $g(N \times \{1\}) = N' \times \{1\}$.

Theorem 4.3.18. *Let M be a manifold and $N, N' \subset M$. There is a bijection*

$$\begin{aligned} \Theta\colon \mathrm{Mt}_M(N, N') &\to \mathrm{Mt}_M^{hom}(N, N'), \\ f &\mapsto ((m, t) \mapsto (f_t(m), t)). \end{aligned}$$

Proof. See Section A.0.2. □

Definition 4.3.19. [BZH13, Def.1.2] Two embeddings $f_0, f_1\colon X \to Y$ are ambient isotopic if there exists an isotopy

$$H\colon Y \times \mathbb{I} \to Y \times \mathbb{I}, \quad H(y, t) = (h_t(y), t),$$

with $f_1 = h_1 \circ f_0$ and $h_0 = \mathrm{id}_Y$.

Remark 4.3.5. From Theorem 4.3.18 it is straightforward to see that $\mathrm{Mt}_M(N, N')$ is non-empty if and only if N and N' are the images of ambient isotopic embeddings into M. Suppose we have an element $g \in \mathrm{Mt}_M^{hom}(N, N')$, there is a map $g'\colon M \to M$ defined by $g'(m) = p_0 \circ g(m, 1)$ where p_0 is the projection to the first coordinate. Then we have embeddings $\iota\colon N \to M$, the inclusion, and $g' \circ \iota\colon N \to M$. Definition 4.3.17 says precisely that there is an ambient isotopy between the inclusion ι and $g' \circ \iota$. An ambient isotopy in M between embeddings $f, g\colon N \to M$, is an element of $\mathrm{Mt}_M^{hom}(f(N), g(N))$.

We now introduce 'flare schematics'. These are to be understood as follows. A flare schematic represents the image of a monotonic homeomorphism $g\colon M \times \mathbb{I} \to M \times \mathbb{I}$ with $g(m, 0) = (m, 0)$ for all $m \in M$. By Theorem 4.3.18 this is a schematic for a pre-motion, and hence a motion for some appropriate choice of $N, N' \subset M$. In addition to the relevant subset $N \subset M$, we also mark chosen subsets of M whose images under g reveal the image

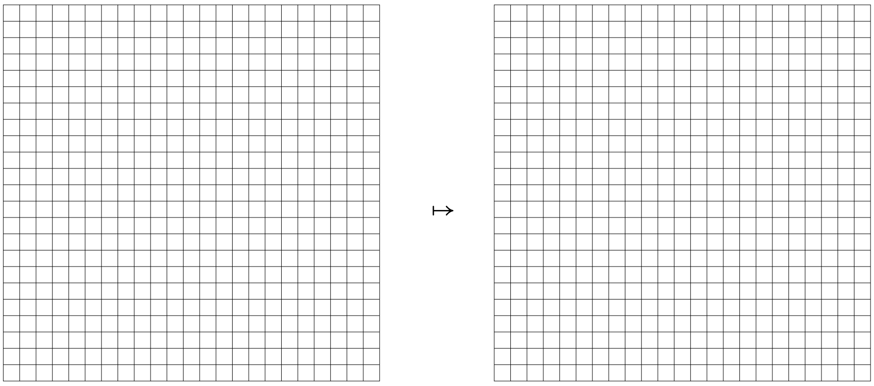


Figure 4.3: Flare schematic for the self-homeomorphism $\mathrm{id}_{\mathbb{I}\times\mathbb{I}}$. This is also the image under Θ of the constant path in $\mathrm{Id}_{\mathbb{I}} \in \mathbf{TOP}^h(\mathbb{I},\mathbb{I})$ starting at $\mathrm{id}_{\mathbb{I}}$.

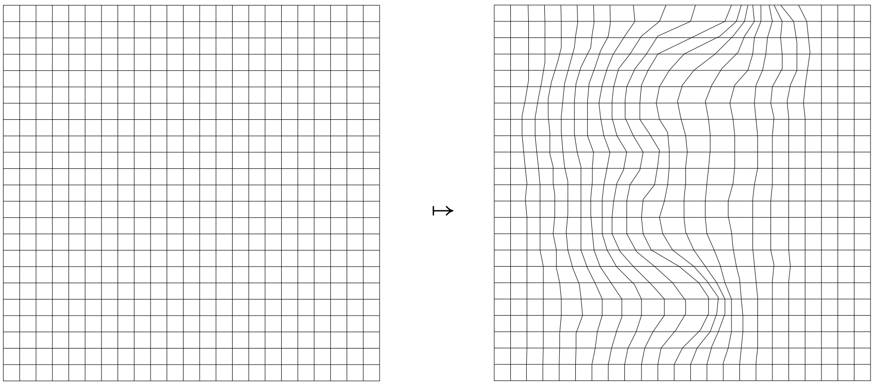


Figure 4.4: Flare line schematic for a non-identity 'y-monotonic' self-homeomorphism of $\mathbb{I}\times\mathbb{I}$. This homeomorphism restricts to the identity on the south, east and west but not the north part of the boundary. It is the image under Θ of a path in $\mathbf{TOP}^h(\mathbb{I},\mathbb{I})$ starting at $\mathrm{id}_{\mathbb{I}}$ (mapped to the southern edge) but not ending at $\mathrm{id}_{\mathbb{I}}$.

of M as $t \in \mathbb{I}$ increases.

Our first examples are Figures 4.3 and 4.4. In both figures the left square is just a reference image of $\mathbb{I}\times\mathbb{I}$. The ambient space $\mathbb{I}$ is oriented horizontally left to right. We choose marked points in M, in this case we have discrete points along the bottom boundary of the square. We mark the same subset of M at all $t \in \mathbb{I}$. The right hand figures represent the image of a homeomorphism $g\colon \mathbb{I}\times\mathbb{I} \to \mathbb{I}\times\mathbb{I}$, in Figure 4.3 this is the identity morphism $\mathrm{id}_{\mathbb{I}\times\mathbb{I}}$ and in Figure 4.4 we have a non-identity homeomorphism. We also see the image of these marked points under the homeomorphisms. The effect at each $t \in \mathbb{I}$ is seen ascending up the page. We call the resultant vertical indicator lines 'flares'. (The horizontal lines mark out snapshots of $\mathbb{I}$ as we progress along the path and so are merely a guide to the eye.)

In Figure 4.5 we have two more flare schematics corresponding to different motions in $\mathbb{I}$. Here we have omitted the reference image of $\mathbb{I}\times\mathbb{I}$, and only show the image of the

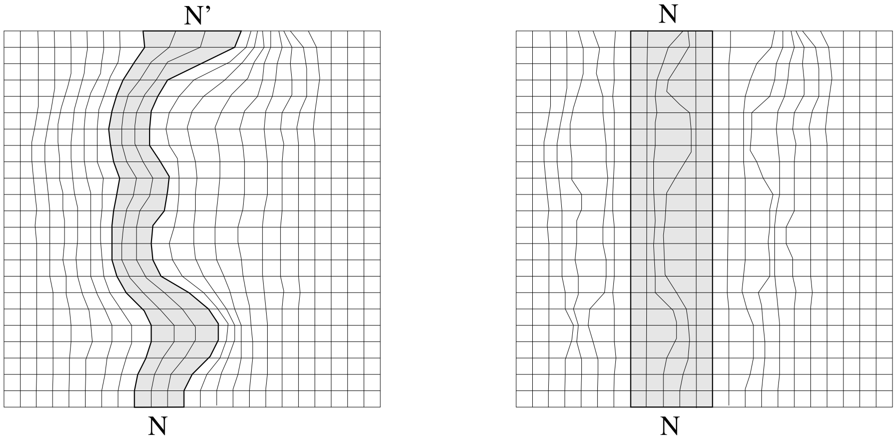


Figure 4.5: Schematic for motions (a) from N to N' and (b) from N to N in case $M = \mathbb{I}$, where N and N' are intervals in $\mathbb{I}$ and the grey shading represents the image of N at each $t \in \mathbb{I}$ progressing up the figure.

homeomorphism. These schematics represent motions from various intervals, so we mark these intervals in addition to the flares.

Figures 4.6, 4.7, 4.8 and 4.9 show paths of self-homeomorphisms of the circle $M = S^1$, realised as homeomorphisms $g\colon M \times \mathbb{I} \to M \times \mathbb{I}$. Again we include a reference figure on the left in each case. We put a marker set of eight points in S^1. The reference picture shows $S^1 \times \mathbb{I}$ with the product of each of the eight points with $\mathbb{I}$ marked. We have drawn $- \times \mathbb{I}$ radially, thus marked points become radial lines. The 'horizontal' lines we put in the $\mathbb{I}$ case merely to mark the passage of the t variable here become concentric circles.

We turn now to the paths themselves. The paths represented by Figures 4.7 and 4.6 both start at a different self-homeomorphism to the one in which they begin. The paths represented by Figures 4.8 and 4.9 instead both end at the same self-homeomorphism to the one in which they begin. The path in Fig.4.8 is contractible to the constant path. The path in Fig.4.9 is not.

We can also use our flare schematics to add some intuition to the construction $\bar{f}$. Notice that if we turn a flare schematic upside down (respectively inside-out in the S^1 case) it is not a flare schematic of a motion, because $f_{(1-t)}$ is not the identity at $t = 0$; but the initial f_1^{-1} in $\bar{f}$ 'fixes' this.

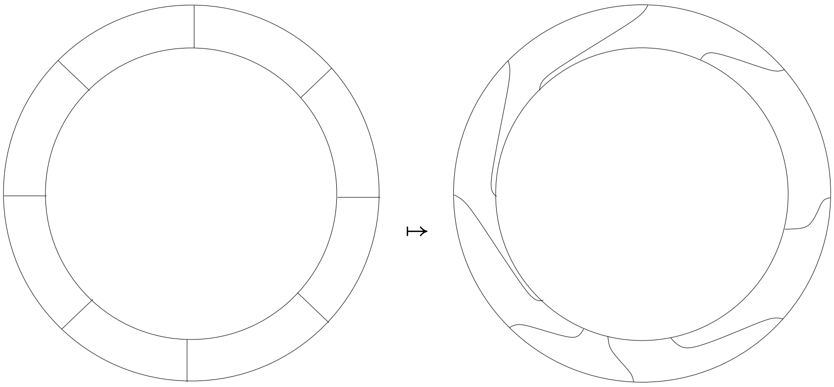


Figure 4.6: Illustration of a path of self-homeomorphisms of the circle $M = S^1$, realised as a homeomorphism $M \times \mathbb{I} \to M \times \mathbb{I}$. The circle is drawn together with eight points upon it, marked to reveal the space 'moving' under the path of self-homeomorphisms. In this case the $- \times \mathbb{I}$ is drawn radially, outside-to-inside, rather than bottom-to-top on the page (so the drawing scale changes with radial distance; while the angular coordinate does not). The path in $\mathbf{Top}^h(S^1, S^1)$ illustrated here does not end at the same homeomorphism in which it begins.

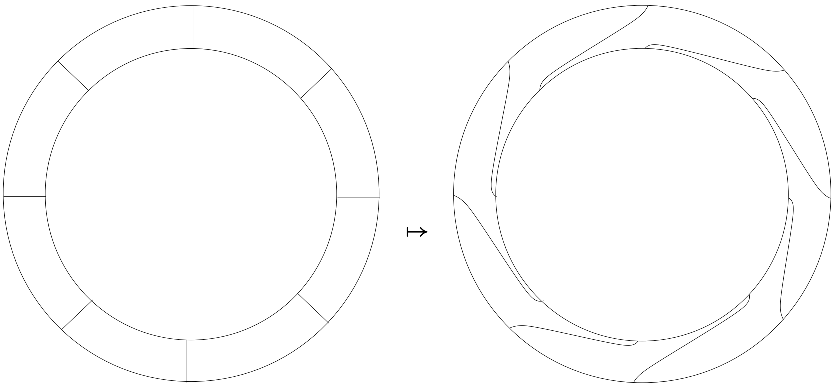


Figure 4.7: The path in $\mathbf{Top}^h(S^1, S^1)$ illustrated here does not end at the same homeomorphism at which it begins.

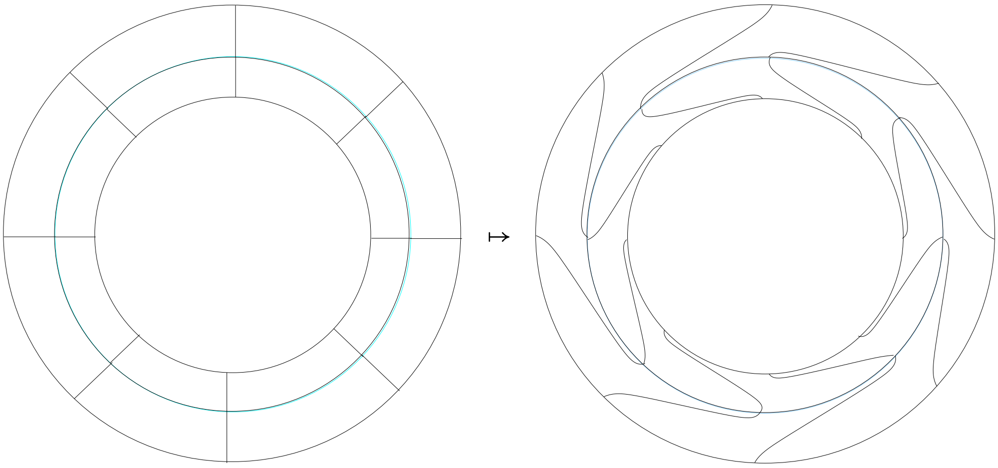


Figure 4.8: Illustration of a path of self-homeomorphisms of the circle $M = S^1$, realised as a homeomorphism $M \times \mathbb{I} \to M \times \mathbb{I}$. This path ends at the same self-homeomorphism at which it begins.

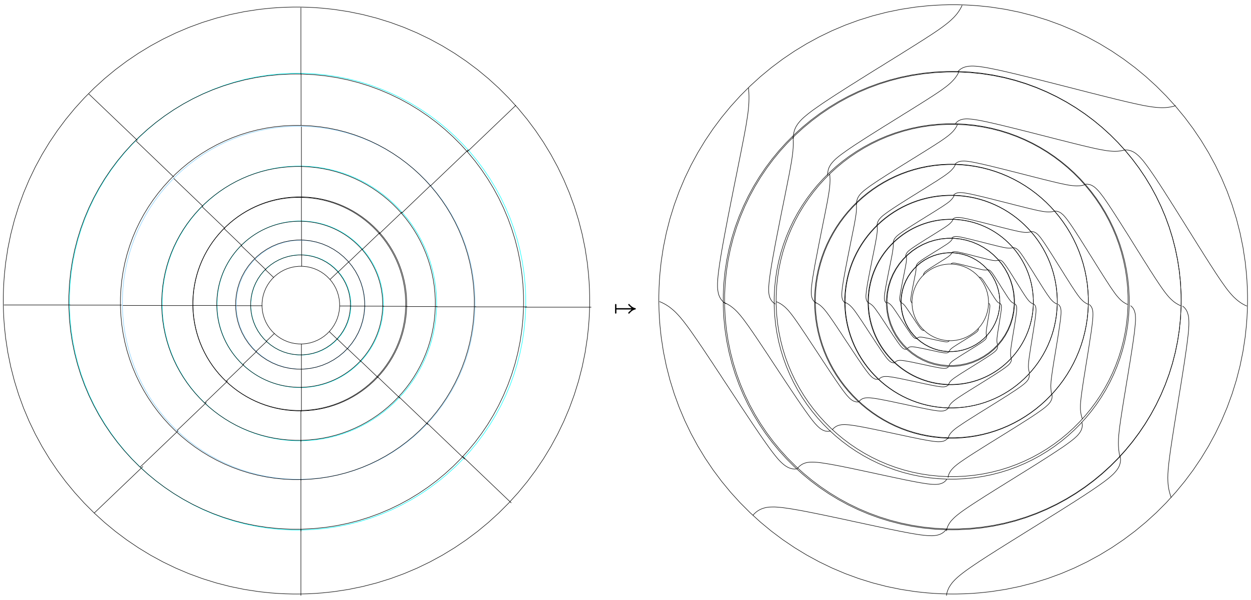


Figure 4.9: Illustration of a path of self-homeomorphisms of the circle $M = S^1$. Comparing with Fig.4.8, both paths can be taken to start at $\mathrm{Id}_{\mathbb{I}}$, and both finish at the same point.

4.3.4 Motion magmoids

The two compositions of pre-motions introduced in Section 4.3.2 become two distinct motion compositions mirroring how group compositions become compositions in the corresponding action groupoids. This leads to two magmoids with the same objects and morphisms.

Proposition 4.3.20. *Let M be a manifold. Then*

(I) for any subsets $N, N', N'' \subseteq M$ there exists a composition

$$\begin{aligned} *\colon \mathrm{Mt}_M(N,N') \times \mathrm{Mt}_M(N',N'') &\to \mathrm{Mt}_M(N,N'') \\ (f\colon N \rightsquigarrow N', g\colon N' \rightsquigarrow N'') &\mapsto (g\colon N' \rightsquigarrow N'') * (f\colon N \rightsquigarrow N') \end{aligned}$$

where $(g\colon N' \rightsquigarrow N'') * (f\colon N \rightsquigarrow N') = g * f\colon N \rightsquigarrow N''$ with $g * f$ as defined in Equation (4.2)*.*

(II) The triple

$$\mathrm{Mt}_M^* = (\mathcal{P}M, \mathrm{Mt}_M^*(-,-), *)$$

is a magmoid.

Proof. (I) From Proposition 4.3.6 we have that that $g * f$ is a pre-motion. We have also that $(g * f)_1(N) = g_1 \circ f_1(N) = g_1(N') = N''$, hence $g * f\colon N \rightsquigarrow N'' \in \mathrm{Mt}_M(N,N')$.

(II) This follows from (I). ☐

See Figure 4.10 for an example of composition in our flare schematic representation. Figure 4.10(a) simply shows the flare-schematics for two pre-motions in a formal stack — note that this is not itself a flare-schematic for a motion, since the indicative paths are not matched at the join. To turn this picture into a flare schematic, we must trace the images of the marked points along the bottom, throughout the whole schematic schematic. In Fig.4.10(b) we consider what happens when we move to motions. Choosing a subset along the bottom boundary and tracking it under the first pre-motion determines a choice of subset in the second motion such that paths of self-homeomorphisms become composable motions. Note that if we were to turn this into a flare schematic, the bold line representing the image of the chosen subset at each $t \in \mathbb{I}$ will remain the same.

Proposition 4.3.21. *The magmoid* Mt_M^* *is reversible.*

Proof. Proposition 4.3.12 gives a well defined map from $\mathrm{Mt}_M(N, N')$ to $\mathrm{Mt}_M(N', N)$ for any $N', N \in M$. This is a bijection by Remark 4.3.4. ☐

Proposition 4.3.22. *Let M be a manifold. There is a magmoid morphism*

$$\bar{} : \mathrm{Mt}_M^* \to \mathrm{Mt}_M^*$$
$$f : N \rightarrowtail N' \mapsto \bar{f} : N' \rightarrowtail N$$

where $\bar{f}$ is as in Equation (4.4)*.*

Proof. The map $\bar{}$ is well defined by Proposition 4.3.12. That $\bar{}$ preserves composition follows directly from Proposition 4.3.10. ☐

Lemma 4.3.23. *Let M be a manifold. Then (I) for any subsets $N, N', N'' \subseteq M$ there exists an associative composition*

$$\cdot : \mathrm{Mt}_M(N, N') \times \mathrm{Mt}_M(N', N'') \to \mathrm{Mt}_M(N, N'')$$
$$(f : N \rightarrowtail N', g : N' \rightarrowtail N'') \mapsto (g : N' \rightarrowtail N'') \cdot (f : N \rightarrowtail N')$$

where $(g\colon N' \rightsquigarrow N'') \cdot (f\colon N \rightsquigarrow N') = g \cdot f\colon N \rightsquigarrow N''$ *and* $(g \cdot f)_t = g_t \circ f_t$.

(II) The triple

$$Mt_M^{\cdot} = (\mathcal{P}M, Mt_M(-,-), \cdot)$$

is a magmoid.

Proof. (I) We have from 4.3.7 that $g \cdot f$ is a pre-motion, and that $\cdot$ is associative. Notice that $(g \cdot f)_1(N) = g_1 \circ f_1(N) = g_1(N') = N''$ so $(g \cdot f)_1 \in \mathrm{Homeo}_M(N, N'')$. So we have that $g \cdot f$ is a motion from N to N''.

(II) This follows from (I). □

Lemma 4.3.24. *Let M be a manifold. The pentuple*

$$(\mathcal{P}M, \mathrm{Mt}_M(-,-), \cdot, \mathrm{Id}_M, (f^{-1})_t = (f_t)^{-1})$$

is a groupoid.

Note that we give only the relevant pre-motions in the identity and inverse in order to shorten the notation. Explicitly, the magmoid $\mathrm{Mt}_M^{\cdot}$ becomes a groupoid whose identity at each object $N \in \mathrm{Mt}_M^{\cdot}$ is $\mathrm{Id}_M\colon N \to N$ and for any morphism $f\colon N \rightsquigarrow N'$, the inverse morphism is $f^{-1}\colon N' \rightsquigarrow N$ where $(f^{-1})_t = (f_t)^{-1}$.

Proof. Lemma 4.3.23 proves the action of $(Premot_M, \cdot)$ on $\mathcal{P}(M)$ defined by $(f, N) \mapsto f_1(N)$ preserves composition. We also have that for all $N \subseteq M$, $\mathrm{Id}_M(N) = N$. The described groupoid is precisely the action groupoid $P(M)/\!/_\sigma\,(Premot_M, \cdot)$. □

4.3.5 Path homotopy congruence on motion magmoids

Here we show that path-equivalence is a congruence on Mt_M^* and that the corresponding quotient magmoid is a groupoid. We then show the same equivalence is a congruence on $\mathrm{Mt}_M^{\cdot}$ and that the quotient magmoid is precisely the groupoid obtained from Mt_M^*.

Lemma 4.3.25. *Let M be a manifold.*

(I) For each pair $N, N' \subseteq M$ of subsets, $\overset{p}{\sim}$ is an equivalence relation on $\mathrm{Mt}_M(N, N')$ (see

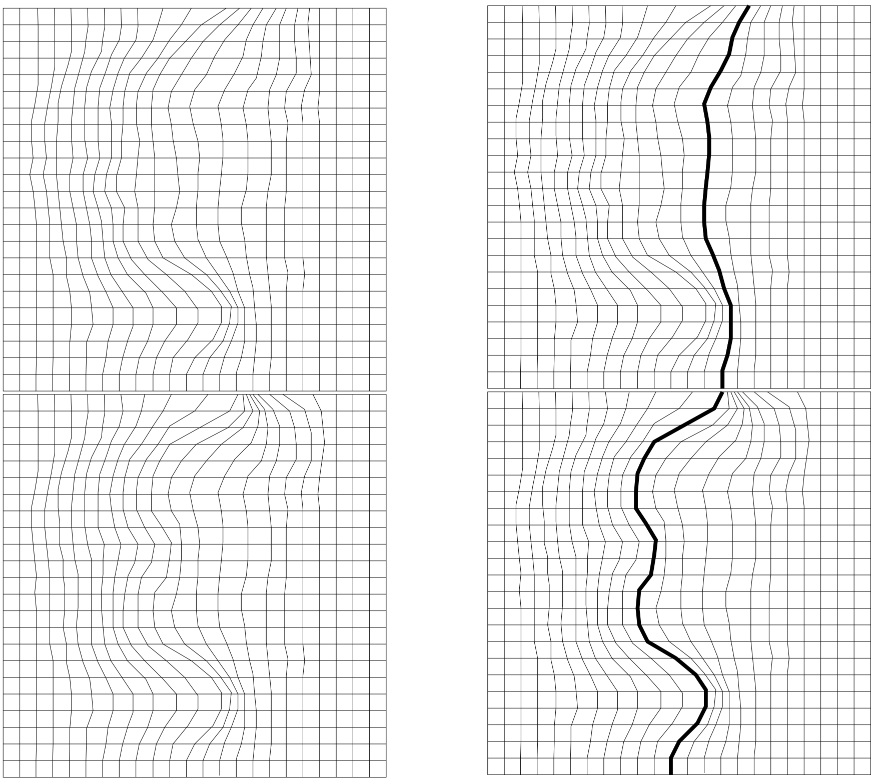


Figure 4.10: Schematic for composition of motions. (a) formal stack of pictures of paths; (b) formal stack of pictures of paths with a choice of subset in $\mathbb{I}$ that we track as $t \in \mathbb{I}$ progresses.

Definition 3.3.5 for the definition of $\overset{p}{\sim}$ *). In our notation this means*

$$(f\colon N \rightsquigarrow N') \sim (f'\colon N \rightsquigarrow N') \ \ \textit{if} \ \ f \overset{p}{\sim} f'.$$

(II) The equivalence relations $(\mathrm{Mt}_M(N,N'), \overset{p}{\sim})$ *for each pair* $N, N' \subseteq M$ *are a congruence on* Mt_M^*.

Notation: *We use* $[f\colon N \rightsquigarrow N']_p$ *for the path-equivalence class of* $f\colon N \rightsquigarrow N'$.

Proof. (I) We have that for any pair N, N', $\mathrm{Mt}_M(N,N') \subseteq \mathbf{Top}(\mathbb{I}, \mathbf{Top}(M,M))$, thus the proof that path-homotopy is an equivalence relation on $\mathbf{Top}(\mathbb{I}, \mathbf{Top}(M,M))$ (Proposition 3.3.6) is sufficient.

(II) Suppose we have pairs of equivalent motions $(f\colon N \rightsquigarrow N') \overset{p}{\sim} (f'\colon N \rightsquigarrow N')$ and $(g\colon N' \rightsquigarrow N'') \overset{p}{\sim} (g'\colon N' \rightsquigarrow N'')$. Then there exists a path homotopy, say H_f from f to f' and a path homotopy, say H_g from g to g'. Notice that, since path homotopies fix the endpoints, for all $s \in \mathbb{I}$ we have $H_f(1,s) = f_1$. Thus the map

$$H(t,s) = \begin{cases} H_f(2t,s) & 0 \le t \le 1/2 \\ H_g(2(t-1/2),s) \circ f_1 & 1/2 \le t \le 1 \end{cases}$$

is a path homotopy $g * f$ to $g' * f'$. □

Notice that, since path homotopies fix the endpoints, for any motion $f\colon N \rightsquigarrow N'$ and path-equivalence $f \overset{p}{\sim} f'$, $f'\colon N \rightsquigarrow N'$ is a motion.

Lemma 4.3.26. *Let M be a manifold. The pentuple*

$$\mathrm{Mt}^*_M/\overset{p}{\sim}\ = (\mathcal{P}M,\ \mathrm{Mt}_M(N,N')/\overset{p}{\sim},\ *,\ [\mathrm{Id}_M]_p,\ [f]_p \mapsto [\bar{f}]_p)$$

is a groupoid.

Proof. We have proved in Lemma 4.3.8 that $g * f \overset{p}{\sim} g \cdot f$, and by Lemma 4.3.24 $\cdot$ is associative and unital with unit Id_M. This is sufficient to prove $(\mathcal{C}1)$ and $(\mathcal{C}2)$. Since we are considering a different inverse to the inverse in the group $(Premot_M, \cdot)$, we prove this directly.

$(\mathcal{G}3)$ Note that for any morphism $[f\colon N \rightsquigarrow N']_p$, $\bar{f}\colon N' \rightsquigarrow N$ is well defined by Proposition 4.3.12. For any morphism $[f\colon N \rightsquigarrow N']_p$, the following function

$$H_{inv}(t,s) = \begin{cases} f_{2t(1-s)} & 0 \le t \le \frac{1}{2}, \\ f_{(1-2(t-1/2))(1-s)} & \frac{1}{2} \le t \le 1 \end{cases} \tag{4.5}$$

is a homotopy from $\bar{f} * f$ to Id_M. Observe that for each fixed s, the path $H_{\bar{f}*f}(t,s)$ starts at the identity, follows f until $f_{(1-s)}$, and then follows $f_{(1-t)}$ back to id_M. □

Remark 4.3.6. Note that $\mathrm{Mt}^*_M/\overset{p}{\sim}$ is the action groupoid $\mathcal{P}M/\!/_\sigma\left((Premot_M, *)/\overset{p}{\sim}\right)$ where $\sigma([f]_p, N) = f_1(N)$. The proof of Lemma 4.3.26 is essentially a proof that $(Premot_M, *)/\overset{p}{\sim}$ is a group.

Lemma 4.3.27. *Let M be a manifold. The relations* $(\mathrm{Mt}_M(N,N'), \overset{p}{\sim})$ *for each pair* $N, N' \subseteq M$ *are a magmoid congruence on* $\mathrm{Mt}^{\cdot}_M$.

Proof. By Lemma 4.3.8 $f \cdot g \overset{p}{\sim} f * g$ for all pre-motions, hence that $\overset{p}{\sim}$ is a congruence follows from Lemma 4.3.25. □

Lemma 4.3.28. *Let M be a manifold. The quotient magmoids $\mathrm{Mt}^{*}_{M}/\overset{p}{\sim}$ and $\mathrm{Mt}^{\cdot}_{M}/\overset{p}{\sim}$ are the same.*

Proof. By construction the two categories have the same objects and morphisms. By Lemma 4.3.8 the composition is the same up to path-equivalence. □

Lemma 4.3.29. *For a manifold M, then (I)*

$$\mathrm{Mt}^{\cdot}_{M}/\overset{p}{\sim} \;=\; (\mathcal{P}M, \mathrm{Mt}_M(N,N')/\overset{p}{\sim}, \cdot, [\mathrm{Id}_M]_p, [f]_p \mapsto [f^{-1}]_p)$$

and

$$\mathrm{Mt}^{*}_{M}/\overset{p}{\sim} \;=\; (\mathcal{P}M, \mathrm{Mt}_M(N,N')/\overset{p}{\sim}, *, [\mathrm{Id}_M]_p, [f]_p \mapsto [\bar{f}]_p).$$

are groupoids and (II) they are the same groupoid.

We will now denote this groupoid by just $\mathrm{Mt}_M/\overset{p}{\sim}$.

Proof. (I) Lemmas 4.3.26 gives that $\mathrm{Mt}^{*}_{M}/\overset{p}{\sim}$ is a groupoid. We have from Lemma 4.3.24 that $\mathrm{Mt}^{\cdot}_{M}$ is a groupoid, and by Proposition 3.2.4 the quotient is also a groupoid.
(II) Lemma 4.3.28 gives that the underlying magmoids are the same. By uniqueness of inverses and identities, they are the same groupoid. □

The previous lemma allows us to work with either of the compositions or inverses according to which simplifies each proof.

Let M be a manifold and N, N' be subsets of M. Given two motions $f, f' : N \rightsquigarrow N'$ such that $f_1 \neq f'_1$, then their path-homotopy classes (which we recall are relative to end-points) are different, so $[f : N \rightsquigarrow N']_p \neq [f' : N \rightsquigarrow N']_p$. From this we can see that the groupoid $\mathrm{Mt}_M/\overset{p}{\sim}$ typically has uncountable sets of morphisms.

In particular, let $M = D^2$ and $N \subset \mathrm{int}(D^2)$ be a finite set in the interior of D^2. Fix an $x \in \mathrm{int}(D^2) \smallsetminus N$. For any $y \in \mathrm{int}(D^2) \smallsetminus N$ there is a motion $f^y : N \rightsquigarrow N$ with $f^y_1(x) = y$ and $f^y_t(N) = N$ for all $t \in \mathbb{I}$ (using homogeneity of smooth manifolds). Note that $[f^y : N \rightsquigarrow$

$N]_\mathrm{p} \neq [f^{y'}: N \rightarrowtail N]_\mathrm{p}$ if $y \neq y'$, as $f^y_1(x) = y$ whereas $f^{y'}_1(x) = y'$. There are uncountably many choices of y, hence the set $\mathrm{Mt}_M / \overset{p}{\sim} (N,N)$ is uncountable.

Both the braid groups and the loop braid groups have presentations with a finite number of generators, thus are not uncountable. In the next section we impose a further quotient that will identify the motions f^y and $f^{y'}$.

4.3.6 The motion groupoid Mot_M: congruence induced by set-stationary motions

Given an ambient space M, we are interested in how movements of a subset are induced by movements of M. We aim to study these movements 'combinatorially', i.e. arranged into countable/finitely-generated classes. Accordingly, by imposing path equivalence, we have washed out some distinctions that do not affect the induced movement of subsets (else our sets are certainly larger than combinatorial). However, for general subsets, so far, we are still only allowing motions to be equivalent if their underlying paths share the same end point, and these sets can still be very large. Dahm's idea of 'motion groups' (partially) addresses this problem. Here we prove there is a lift of Dahm's idea to the groupoid setting.

We start by defining *N-stationary* motions, motions from N to N which leave N fixed setwise. We then show that N-stationary motions lead to a normal subgroupoid in $\mathrm{Mt}_M / \overset{p}{\sim}$ and hence induce a congruence. This leads to the motion groupoid Mot_M in Theorem 4.3.37.

Definition 4.3.30. Let M be a manifold, and $N \subset M$ a subset. A motion $f: N \rightarrowtail N$ in M is said to be $\underline{N\text{-stationary}}$ if $f_t \in \mathit{Homeo}_M(N,N)$ for all $t \in \mathbb{I}$. Define

$$\mathrm{SetStat}_M^N = \{ f: N \rightarrowtail N \in \mathrm{Mt}_M(N,N) \mid f_t \in \mathit{Homeo}_M(N,N) \text{ for all } t \in \mathbb{I} \}.$$

Example 4.3.31. *Let $M = D^2$, the 2-disk and let $N \subset M$ be a finite set of points. Then a motion $f: N \rightarrowtail N$ is N-stationary if and only if $f_t(x) = x$ for all $x \in N$ and $t \in \mathbb{I}$. More generally this holds if N is a totally disconnected subspace of M, e.g. $\mathbb{Q}$ in $\mathbb{R}$.*

Example 4.3.32. *Let $M = D^2$, the 2-disk. Consider the pre-motions which are homeo-*

morphisms of D^2 as shown in the top schematic of Figure 4.1. Now pick a subset $N \subset D^2$ which is any circle centred on the centre of the disk, i.e. the set of all points a fixed distance from the centre using the metric induced from the complex plane. Then these pre-motions become N-stationary motions from N to N.

Example 4.3.33. *The schematic Figure 4.5(b) represents an N-stationary motion from N to N in $\mathbb{I}$.*

Lemma 4.3.34. *Let M be a manifold. For $N, N' \subset M$ let $\mathrm{SetStat}_M(N,N)$ be the subset of $\mathrm{Mt}_M/\overset{p}{\sim}(N,N)$ of those classes that intersect $\mathrm{SetStat}_M^N$. Let $\mathrm{SetStat}_M(N,N') = \varnothing$ if $N \neq N'$. There is a totally disconnected, normal subgroupoid of $\mathrm{Mt}_M/\overset{p}{\sim}$,*

$$\mathrm{SetStat}_M = (\mathcal{P}M,\ \mathrm{SetStat}_M(N,N'),\ *,\ [\mathrm{Id}_M]_p,\ [f]_p \mapsto [\bar{f}]_p).$$

Note that

$$\mathrm{SetStat}_M(N,N') = \{[f\colon N \rightsquigarrow N']_p \mid \exists\ N\text{-stationary } f'\colon N \rightsquigarrow N' \in [f\colon N \rightsquigarrow N']_p\}.$$

Proof. First we will show that the tuple $\mathrm{SetStat}_M$ is a subgroupoid.

For each $N \subset M$ the identity $[\mathrm{Id}_M\colon N \rightsquigarrow N]_p$ is in $\mathrm{SetStat}_M(N,N)$ as for all $t \in \mathbb{I}$, $(\mathrm{Id}_M)_t(N) = \mathrm{id}_M(N) = N$.

For the existence of inverses observe that there is nothing to show if $N \neq N'$. For each $[x\colon N \rightsquigarrow N]_p \in \mathrm{SetStat}_M(N,N)$ with $x\colon N \rightsquigarrow N$ a N-stationary motion, the inverse $[\bar{x}\colon N \rightsquigarrow N]_p$ is in $\mathrm{SetStat}_M(N,N)$ since for all $t \in \mathbb{I}$, $\bar{x}_t(N) = x_{1-t} \circ x_1^{-1}(N) = x_{1-t}(N) = N$.

Let $[x\colon N \rightsquigarrow N]_p$ and $[x'\colon N \rightsquigarrow N]_p$ be in $\mathrm{SetStat}_M(N,N)$ with $x\colon N \rightsquigarrow N$ and $x'\colon N \rightsquigarrow N$ N-stationary. For all $t \in [0,1/2]$ we have that $(x' * x)_t(N) = x_t(N) = N$ and for $t \in [1/2,1]$ that $(x' * x)_t(N) = x'_t \circ x_1(N) = x'_t(N) = N$. Thus composition closes, and so $\mathrm{SetStat}_M$ is a groupoid.

Observe now that $\mathrm{SetStat}_M$ is totally disconnected and wide by construction.

Finally, we have that $\mathrm{SetStat}_M$ is normal as for any morphism $[f\colon N \rightsquigarrow N']_p \in \mathrm{Mt}_M/\overset{p}{\sim}$ and for $[x\colon N' \rightsquigarrow N']_p$ in $\mathrm{SetStat}_M(N',N')$, with $x\colon N' \rightsquigarrow N'$, N'-stationary, the following function

$$H(t,s) = f^{-1}_{t(1-s)+s} \circ x_t \circ f_{t(1-s)+s}$$

is a path homotopy from $f^{-1} \cdot x \cdot f$ to the path $f_1^{-1} \circ x_t \circ f_1$, which is an N-stationary motion. □

Proposition 4.3.35. *Let M be a manifold. Let $(f\colon N \rightarrowtail N') \overset{p}{\sim} (g\colon N \rightarrowtail N')$ be path-equivalent motions in M. Then $\bar{g} * f\colon N \to N$ is path-equivalent to an N-stationary motion.*

Proof. We have $\mathrm{Mt}_M/\overset{p}{\sim}$ is a groupoid by Lemma 4.3.26, thus with unique inverses. Hence $[f\colon N \rightarrowtail N']_{\mathrm{p}} = [g\colon N \rightarrowtail N']_{\mathrm{p}}$ implies $[f\colon N \rightarrowtail N']_{\mathrm{p}}^{-1} = [\bar{g}\colon N \rightarrowtail N']_{\mathrm{p}}$. This implies there is a path-homotopy H from $\bar{g} * f$ to Id_M, which is an N-stationary motion. □

Proposition 4.3.36. *For $N, N' \subset M$, denote by $\overset{m}{\sim}$ the relation*

$$f\colon N \rightarrowtail N' \overset{m}{\sim} g\colon N \rightarrowtail N' \quad \textit{if} \quad \bar{g} * f \in \mathrm{SetStat}_M(N,N)$$

on $\mathrm{Mt}_M(N,N')$. This is an equivalence relation.
We call this motion-equivalence and denote by $[f\colon N \rightarrowtail N']_m$ the motion-equivalence class of $f\colon N \rightarrowtail N'$.

Proof. Lemma 4.3.34 gives that $\mathrm{SetStat}_M$ is a normal subgroupoid of $\mathrm{Mt}_M/\overset{p}{\sim}$. Hence, by Lemma 3.2.11, there is a congruence $\overline{\mathrm{SetStat}_M}$ on $\mathrm{Mt}_M/\overset{p}{\sim}$ consisting of equivalence relations on $\mathrm{Mt}_M/\overset{p}{\sim}(N,N')$ given by

$$[f\colon N \rightarrowtail N']_{\mathrm{p}} \sim [g\colon N \rightarrowtail N']_{\mathrm{p}} \quad \text{if} \quad [\bar{g} * f]_{\mathrm{p}} \in \mathrm{SetStat}_M(N,N).$$

By Proposition 4.3.35 motions which are path-equivalent are motion-equivalent, thus $(\mathrm{Mt}_M(N,N')/\overset{p}{\sim})/\overline{\mathrm{SetStat}_M} = \mathrm{Mt}_M(N,N')/\overset{m}{\sim}$. □

Therefore we have:

Theorem 4.3.37. *Let M be a manifold. There is a groupoid*

$$\mathrm{Mot}_M = (\mathrm{Mt}_M/\overset{p}{\sim})/\overline{\mathrm{SetStat}_M} = (\mathcal{P}M,\ \mathrm{Mt}_M(N,N')/\overset{m}{\sim}, *, [\mathrm{Id}_M]_m,\ [f]_m \mapsto [\bar{f}]_m)$$

where

(I) objects are subsets of M*;*

(II) morphisms between subsets N, N' *are motion-equivalence classes* $[f\colon N \rightsquigarrow N']_m$ *of motions, explicitly*

$$f\colon N \rightsquigarrow N' \overset{m}{\sim} g\colon N \rightsquigarrow N' \text{ if } \bar{g} * f\colon N \rightsquigarrow N \in \mathrm{SetStat}_M(N,N);$$

(III) composition of morphisms is given by

$$[g\colon N' \rightsquigarrow N'']_m * [f\colon N \rightsquigarrow N']_m = [g * f\colon N \rightsquigarrow N'']_m$$

where

$$(g * f)_t = \begin{cases} f_{2t} & 0 \le t \le 1/2, \\ g_{2(t-1/2)} \circ f_1 & 1/2 \le t \le 1; \end{cases} \tag{4.6}$$

(IV) the identity at each object N *is the motion-equivalence class of* $\mathrm{Id}_M\colon N \rightsquigarrow N$*, where* $(\mathrm{Id}_M)_t(m) = m$ *for all* $m \in M$*;*

(V) the inverse for each morphism $[f\colon N \rightsquigarrow N']_m$ *is the motion-equivalence class of* $\bar{f}\colon N' \rightsquigarrow N$ *where* $\bar{f}_t = f_{(1-t)} \circ f_1^{-1}$. □

Remark 4.3.7. Using Lemma 4.3.29 we could also have written the composition in Mot_M to be $\cdot$ and the inverse of a motion f as $(f_t)^{-1} = (f^{-1})_t$ in Theorem 4.3.37.

Lemma 4.3.38. *Let* M *and* M' *be manifolds such that there exists a homeomorphism* $\psi\colon M \to M'$*. Then there is a isomorphism of categories*

$$\Psi\colon \mathrm{Mot}_M \to \mathrm{Mot}_{M'}$$

defined as follows. On objects $N \subset M$*,* $\Psi(N) = \psi(N)$*. For a motion* $f\colon N \rightsquigarrow N'$ *in* M*, let* $(\psi \circ f \circ \psi^{-1})_t = \psi \circ f_t \circ \psi^{-1}$*. Then* Ψ *sends the equivalence class* $[f\colon N \rightsquigarrow N']_m$ *to the equivalence class* $[\psi \circ f \circ \psi^{-1}\colon \psi(N) \to \psi(N')]_m$*.*

Proof. Notice $(\psi \circ f \circ \psi^{-1})_0 = \mathrm{id}_{M'}$ and $(\psi \circ f \circ \psi^{-1})_1(\psi(N)) = \psi \circ f_1 \circ \psi^{-1}(\psi(N)) =$

$\psi \circ f_1(N) = \psi(N')$.

We check Ψ is well defined. Suppose $f\colon N \rightsquigarrow N'$ and $f'\colon N \rightsquigarrow N'$ are equivalent motions in M. So there is a path homotopy $\bar{f}' * f$ to a path say x such that $x\colon N \rightsquigarrow N$ is an N-stationary motion, let us call this H. It is straightforward to check that the function $(\psi \circ H \circ \psi^{-1})(t,s) = \psi \circ H(t,s) \circ \psi^{-1}$ is a homotopy making $\overline{\Psi(f')} * \Psi(f)$ path-equivalent to $\psi \circ x \circ \psi^{-1}\colon \psi(N) \rightsquigarrow \psi(N)$ which is a $\psi(N)$-stationary motion.

We can define an inverse $\Psi^{-1}\colon \mathrm{Mot}_{M'} \to \mathrm{Mot}_M$ as follows. On objects $N \subset M'$, $\Psi^{-1}(N) = \psi^{-1}(N)$. Suppose we have a motion $f\colon N \rightsquigarrow N'$ in M', then Ψ^{-1} sends the equivalence class $[f\colon N \rightsquigarrow N']_{\mathrm{m}}$ to the equivalence class $[\psi^{-1} \circ f \circ \psi\colon \psi^{-1}(N) \to \psi^{-1}(N')]_{\mathrm{m}}$. □

Corollary 4.3.39. *Let M be a manifold and $N, N' \subset M$ subsets such that there exists a homeomorphism $\mathfrak{f}\colon M \to M$ with $\mathfrak{f}(N) = N'$. Then there is a group isomorphism*

$$\mathrm{Mot}_M(N,N) \xrightarrow{\cong} \mathrm{Mot}_M(N',N').$$

Proof. Letting $\psi = \mathfrak{f}$ in the previous theorem gives the isomorphism. □

Lemma 4.3.40. *Let M be a manifold. There is an involutive automorphism*

$$\Omega\colon \mathrm{Mot}_M \to \mathrm{Mot}_M$$

which sends an object $N \subset M$ to its complement $M \setminus N$ and which sends a morphism $[f\colon N \rightsquigarrow N']_m$ to $[f\colon M \setminus N \rightsquigarrow M \setminus N']_m$.

Proof. First notice that by Lemma 4.2.5, $f_1 \in \mathrm{Homeo}_M(M \setminus N, M \setminus N')$. We also need to check this functor is well defined. Suppose $f\colon N \rightsquigarrow N'$ and $f'\colon N \rightsquigarrow N'$ are motion-equivalent, so there is a path homotopy $f' * f$ to a stationary motion. So then $f\colon M \setminus N \rightsquigarrow M \setminus N' \overset{m}{\sim} f'\colon M \setminus N \rightsquigarrow M \setminus N'$ using the same homotopy. It is clear that Ω is self inverse. □

Example 4.3.41. *Let M be a manifold, then* $\mathrm{Mot}_M(M,M)$ *is trivial. This is because for any $f \in Premot_M$, $f\colon M \rightsquigarrow M$ is a motion, and it is M-stationary. By Lemma 4.3.40, also* $\mathrm{Mot}_M(\varnothing,\varnothing)$ *is trivial.*

4.3.7 Pointwise A-fixing motions

So far we have avoided working with A-fixing homeomorphisms to avoid overloading the notation and thus make the exposition clearer. Everything we have done so far could have been done by working instead with paths in $\mathrm{Homeo}_M^A(\varnothing,\varnothing)$. We have the following adjusted definitions.

Definition 4.3.42. Fix a manifold M and a subset $A \subseteq M$. An A-fixing pre-motion in M is a path in $\mathrm{Homeo}_M^A(\varnothing,\varnothing) = \mathbf{TOP}_A^h(M,M)$ starting at id_M (recall $f \in \mathbf{TOP}_A^h(M,M)$ is a self-homeomorphism with $f_t(a) = a$ for all $a \in A$); i.e. a path $f \in \mathbf{Top}(\mathbb{I}, \mathbf{TOP}_A^h(M,M))$ with $f_0 = \mathrm{id}_M$. We define notation for the set of all A-fixing pre-motions in M,

$$Premot_M^A \; = \; \{f \in \mathbf{Top}(\mathbb{I}, \mathbf{Top}_A^h(M,M)) \mid f_0 = id_M\}.$$

Definition 4.3.43. Let M be a manifold and $A \subseteq M$ a subset. An A-fixing motion in M is a triple $(f^A, N, f_1(N))$ consisting of an A-fixing pre-motion $f^A \in Premot_M^A$, a subset $N \subseteq M$ and the image of N at the endpoint of f^A, $f_1(N)$.

Notation: We will denote such a triple by $f^A \colon N \rightsquigarrow N'$ where $f_1(N) = N'$, and say it is an A-fixing motion from N to N'. For subsets $N, N' \subseteq M$ we define

$$\mathrm{Mt}_M^A(N,N') = \{(f^A, N, f_1(N)) \text{ a motion in } M \,|\, f_1(N) = N'\}.$$

In practice we will mostly be interested in the case $A = \partial M$.

Example 4.3.44. *All motions of $\mathbb{I}$ are $\partial\mathbb{I}$-fixing motions.*

Example 4.3.45. *The half-twist motions described in Example 4.3.14 are not A-fixing motions for any non-empty subset $A \subseteq S^1$.*

Example 4.3.46. *Let $M = D^2$ be the 2-disk. Consider the motions of the 2-disk represented schematically in Figure 4.1. These are ∂D^2 fixing motions.*

We have an analogous version of Theorem 4.3.37 working with A-fixing motions and considering equivalence as paths in $\mathbf{TOP}_A^h(M,M)$.

Theorem 4.3.47. *Let M be a manifold and $A \subseteq M$ a subset. We obtain a category*

$$\mathrm{Mot}_M^A = (\mathcal{P}M,\ \mathrm{Mt}_M^A(N,N')/\overset{m}{\sim}, *, [\mathrm{Id}_M]_m,\ [f^A]_m \mapsto [\bar{f}^A]_m).$$

Explicitly we have that A-fixing motions $f^A\colon N \curvearrowright N'$ and $g^A\colon N \curvearrowright N'$ are equivalent if $\bar{g} * f$ is path-equivalent to an A-fixing N-stationary motion as paths in $\mathrm{Homeo}_M^A(\varnothing,\varnothing)$.

Proof. Notice that if $f^A\colon N \curvearrowright N'$, $g^A\colon N' \curvearrowright N''$ are A-fixing motions then $\bar{f}$, f^{-1}, $g * f$ and $g \cdot f$ are all A-fixing motions. All motions constructed in homotopies required for the proof of Theorem 4.3.37 and associated lemmas are A-fixing if the input paths are A-fixing. Thus all proofs work in exactly the same way for A-fixing motions. ☐

Proposition 4.3.48. *Let M and M' be manifolds such that there exists a homeomorphism $\psi\colon M \to M'$. Then there is a isomorphism of categories*

$$\Psi\colon \mathrm{Mot}_M^A \to \mathrm{Mot}_{M'}^{\psi(A)}$$

defined as in Proposition 4.3.38.

Proof. We can use the same proof as in Proposition 4.3.38. ☐

Corollary 4.3.49. *Let M be a manifold and $A \subseteq M$ subset. Let $N, N' \subseteq M$ be subsets such that there exists a homeomorphism $\mathfrak{f}\colon M \to M$ with $\mathfrak{f}(N) = N'$ and $\mathfrak{f}(a) = a$ for all $a \in A$. Then there is a group isomorphism*

$$\mathrm{Mot}_M^A(N,N) \xrightarrow{\cong} \mathrm{Mot}_M^A(N',N').$$

Proof. As for Corollary 4.3.39. Note that for any motion $f^A\colon N \curvearrowright N$, the path $\mathfrak{f} \circ f \circ \mathfrak{f}$ fixes A pointwise. ☐

Lemma 4.3.50. *Let M be a manifold. There is an involutive automorphism*

$$\Omega\colon \mathrm{Mot}_M^A \to \mathrm{Mot}_M^A$$

which sends an object $N \subseteq M$ to it's complement $M \setminus N$ and which sends a morphism $[f^A\colon N \curvearrowright N']_m$ to $[f^A\colon M \setminus N \curvearrowright M \setminus N']_m$.

Proof. This is the same as for Lemma 4.3.40. ☐

Theorem 4.3.51. *Let* n *be a positive integer. Consider* $M = D^2$ *and* $A = \partial D^2$*. Given any finite subset* K*, with* n *elements, in the interior of* D^2*, then* $\mathrm{Mot}_{D^2}^{\partial D^2}(K,K)$ *is isomorphic to* B_n*, the braid group in* n *strands. In particular the image of the motion represented in Figure 4.1 is an elementary braid on two strands.*

Also if L *is an unlink in the interior of* D^3 *with* n *components then* $\mathrm{Mot}_{D^3}^{\partial D^3}(L,L)$ *is isomorphic to the extended loop braid group.*

Proof. This theorem is essentially in [Dah36] (Thm. II.1.2). A proof is contained in Remarks 4.6.2 and 4.6.3 in Section 4.6.3 below. Our argument makes use of the functor from the motion groupoid to the mapping class groupoid that we construct in Theorem 4.6.1. ☐

Remark 4.3.8. There are several different realisations of the braid group [BB05; Dam17]. In the proof of the previous theorem we use the realisation of B_n as a mapping class group. Let $k_1,\dots,k_n$ be distinct elements of $\mathbb{C}$. Let $K = \{k_1,\dots,k_n\}$. We can see B_n as an isomorphic group of the fundamental group B_n' of the configuration space:

$$C_n := \{(x_1,\dots,x_n)\in\mathbb{C}^n \mid i\neq j \implies x_i\neq x_i\}/S_n,$$

based at $[(k_1,\dots,k_n)]$.

An explicit isomorphism from $\mathrm{Mot}_{D^2}^{\partial D^2}(K,K)$ sends the class of a motion $f\colon K \rightsquigarrow K$ to the homotopy class of the closed path:

$$t\in[0,1]\mapsto[(f_t(k_1),\dots,f_t(k_n))]\in C_n.$$

In particular the equivalence class of any motion which moves two points as shown in the bottom schematic of Figure 4.1 will be sent to the generating element of the braid group on 2 strands.

It is straightforward to show this map is well defined, but harder to show injectivity and surjectivity, cf. [BB05, Theorem 1].

4.3.8 Examples

Here we will consider some examples which serve to illustrate some key aspects of the richness of the construction.

By Lemma 4.3.48 we have that if M and M' are homeomorphic manifolds, Mot_M and $\mathrm{Mot}_{M'}$ are isomorphic groupoids. Thus it is enough to consider one M for each homeomorphism class.

An interesting problem in each case is to give a characterisation of a skeleton. This is far from straightforward, even if we restrict to objects that are themselves manifolds. Note that subsets $N, N' \subset M$ being homeomorphic submanifolds is not a sufficient condition to ensure an isomorphism connecting them in the motion groupoid. For example, let $M = \mathbb{I}^2$. Let $N \subset \mathrm{int}(\mathbb{I}^2)$ be the circle of points a distance $1/4$ from the point $(1/2, 1/2)$. Let L be the point $(1/2, 1/2)$, and L' the point $(3/4, 3/4)$. Then $N \cup L$ and $N \cup L'$ are homeomorphic but $\mathrm{Mot}_{\mathbb{I}^3}(N \cup L, N \cup L') = \varnothing$. Examples 4.3.52-4.3.55 below discuss which objects are connected in the motion groupoids $\mathbb{I}$ and $\mathbb{R}$.

We can think of the aforementioned characterisation of the skeleton as looking for 'inner' isomorphisms, objects which are connected by an isomorphism in the motion groupoid. This allows us to compare these with 'outer' isomorphisms, by which we mean: for a manifold M, which N and N' have a constructible group isomorphism $\chi\colon \mathrm{Mot}_M(N, N) \to \mathrm{Mot}_M(N', N')$, but with $\mathrm{Mot}_M(N, N')$ empty? This is discussed Examples 4.3.57-4.3.62.

Observe that even in a skeleton most objects are undefinable so it is a good exercise to restrict to a full subgroupoid of particular interest. Given a subset Q of the object class $\mathcal{P}M$ of Mot_M^A we write $\mathrm{Mot}_M^A|_Q$ for the corresponding full subgroupoid.

In Section 4.3.8 we give a conjecture for a presentation of the the full subgroupoid of the motion groupoid of certain configurations of points and loops in $\mathbb{I}^3$. Here we see one benefit of working with a motion group$\underline{\text{oid}}$, as opposed to the motion group: we can often write a more simple presentation. Adding constraints on motions can make presentations more complicated. We can see this phenomenon by looking at the example of the braids. The pure braid group on n strands is a subset of the braid group on n strands but there is a simpler presentation of the braid group than of the pure braid group.

On $\mathrm{Mot}_{\mathbb{I}}$

Example 4.3.52. *Suppose* $N \subset \mathbb{I} \smallsetminus \{0,1\}$ *is a compact subset with a finite number of connected components. So* N *is a union of points and closed intervals. We can assign a word in* $\{a,b\}$ *to* N *as follows: each point in* N *is represented by an* a *and each interval by* b*, ordered in the obvious way using the natural ordering on* $\mathbb{I}$*. Let* $N' \subset \mathbb{I} \smallsetminus \{0,1\}$ *be another compact subset with a finite number of connected components. It is possible to construct even a piecewise linear motion from* N *to* N' *if the word assigned to* N *and* N' *is the same. And then* $|\mathrm{Mot}_{\mathbb{I}}(N,N')| = 1$*. Otherwise* $\mathrm{Mot}_{\mathbb{I}}(N,N') = \varnothing$*.*

Note homeomorphisms send boundary points to boundary points and interior points to interior points, so any continuous path of homeomorphisms $\mathbb{I} \to \mathbb{I}$ *fixes the boundary points. So if we instead consider, for example, finite subsets* $A, B \subseteq \mathbb{I}$ *we have* $|\mathrm{Mot}_{\mathbb{I}}(A,B)| = 1$ *if and only if* $A \cap \{0,1\} = B \cap \{0,1\}$ *and* A *and* B *have the same cardinality. Otherwise* $\mathrm{Mot}_{\mathbb{I}}(A,B) = \varnothing$*.*

Example 4.3.53. *If we consider non-compact subsets we must also pay attention to the embeddings. Suppose* $N = (1/4,1/2) \cup (1/2,3/4)$ *and* $N' = (1/4,3/8) \cup (5/8,3/4)$*, then* $\mathrm{Mot}_{\mathbb{I}}(N,N') = \varnothing$*.*

The automorphism group $\mathrm{Mot}_{\mathbb{I}}(N,N)$ for $N \subset \mathbb{I}$ with a finite number of connected components are always trivial. The following example shows this changes dramatically if more complicated subsets of $\mathbb{I}$ are considered.

Example 4.3.54. *Let* $M = \mathbb{I}$ *and* $N = \mathbb{I} \cap \mathbb{Q}$*, then* $\mathrm{Mot}_{\mathbb{I}}^{\partial\mathbb{I}}(N,N)$ *is uncountably infinite. This will be shown in Example 4.5.9 and Remark 4.6.4.*

On $\mathrm{Mot}_{\mathbb{R}}$

Example 4.3.55. *Let* $M = \mathbb{R}$*. There does not exist a motion* $f\colon \mathbb{Q} \rightarrowtail \mathbb{Z}$*. This can be seen by observing that there is no homeomorphism* $\theta\colon \mathbb{R} \to \mathbb{R}$ *sending* $\mathbb{Q}$ *to* $\mathbb{Z}$*, since homeomorphisms* $\mathbb{R} \to \mathbb{R}$ *must map dense subsets to dense subsets.*

Question: Let $N \neq N'$ be countable dense subsets of $\mathbb{R}$. Then does this imply the existence of a motion $f\colon N \rightarrowtail N'$ in $\mathbb{R}$?

Example 4.3.56. *Let $M = \mathbb{R}$. Then there is a group isomorphism $\phi\colon (\mathbb{Z},+) \xrightarrow{\cong} \mathrm{Mot}_{\mathbb{R}}(\mathbb{Z},\mathbb{Z})$ such that, for $n \in (\mathbb{Z},+)$, $\phi(n)$ is the motion-equivalence class of the motion $f\colon \mathbb{Z} \rightarrowtail \mathbb{Z}$ such that $f_t(x) = x + tn$.*

Relating automorphism groups

It may be useful to be able to obtain the automorphism group of an object in terms of the automorphism group of another object. If objects are connected in the motion groupoid then this is straightforward. Otherwise we may still be able to construct a canonical 'outer' isomorphism between automorphism groups, or we may be able to construct a group homomorphism. The following examples investigate this in various cases.

Example 4.3.57. *For any M, $\mathrm{Mot}_M(\varnothing,\varnothing) \cong \mathrm{Mot}_M(M,M)$ are trivial, each containing only the motion-equivalence class of the motion with underlying pre-motion Id_M. Also $\mathrm{Mot}_M(\varnothing, M) = \varnothing$ unless $M = \varnothing$.*

Example 4.3.57 is a special case of the following example.

Example 4.3.58. *Let M be any manifold, $N \subset M$ a subset and $N' = M \smallsetminus N$. Using Lemma 4.2.5 we have a group isomorphism $\mathrm{Mot}_M(N,N) \cong \mathrm{Mot}_M(N',N')$. In general there will not exist an inner isomorphism in $\mathrm{Mot}_M(N,N')$, although we can construct specific cases for which $\mathrm{Mot}_M(N,N') \neq \varnothing$. For example let $M = S^1$, and $\tau_{t\pi}\colon N \rightarrowtail \tau_\pi(N)$ as in Example 4.3.14. Then letting $N = [0,\pi) \subset S^1$, we have that this is a motion N to $N' = M \smallsetminus N$.*

Let M be a manifold. Let $h\colon M \to M$ be a homeomorphism and $S \subset M$ be a subset. Let $T = h(S)$, then $h\big(cl(S)\big) = cl(T)$. In particular if f is a pre-motion in M such that $f\colon S \rightarrowtail T$ is a motion, then $f\colon \mathrm{cl}(S) \rightarrowtail \mathrm{cl}(T)$ is a motion. Note that if $f\colon S \rightarrowtail S$ is N-stationary then $f\colon \mathrm{cl}(S) \rightarrowtail \mathrm{cl}(S)$ is also N-stationary. This again follows since if $h\colon M \to M$ is a homeomorphism sending S to S then $h\big(\mathrm{cl}(S)\big) = \mathrm{cl}(S)$. In particular it follows that for any subsets $S,T \subset M$, there is a mapping:

$$\Gamma^M_{S,T}\colon \mathrm{Mot}_M(S,T) \to \mathrm{Mot}_M\big(\mathrm{cl}(S), \mathrm{cl}(T)\big).$$

Note that this map is, in general, neither injective, nor surjective, as the following examples show.

Example 4.3.59. *Let $M = D^2 = \{x \in \mathbb{C} \,|\, |x| \leq 1\} \subset \mathbb{C}$. Let $N = [-a, a]$ be a closed interval in the real axis with $0 < a < 1$, and let $N' = (-a, a]$.*

There is a path in $\mathbf{Top}(\mathbb{I}, \mathbf{TOP}^h(D^2, D^2))$, which we label τ_π, such that $\tau_{\pi t}$ is a πt rotation of D^2. Now τ_π gives a motion from N to N, but not from N' to N'. Any motion from $f\colon N' \curvearrowright N'$ must satisfy $f_1(a) = a$.

A stationary motion $s\colon N \curvearrowright N$ must satisfy, for all $t \in \mathbb{I}$, $s_t(a) = a$ and $s_t(-a) = -a$, as there is no path of homeomorphisms from N to N starting at the identity and ending in a homeomorphism sending a to $-a$. Suppose $(f\colon N \curvearrowright N) \overset{m}{\sim} (\tau_\pi\colon N \curvearrowright N)$, then $\bar{f} * \tau_\pi \overset{p}{\sim} s$, where $s\colon N \curvearrowright N$ is some stationary motion. So we have $(\bar{f} * \tau_\pi)_1(a) = a$. We know $\tau_{\pi 1}(a) = -a$, so this implies $\bar{f}_1(-a) = a$, and hence $f_1(a) = -a$. So all $f \in [\tau_\pi]_m$ satisfy $f_1(a) = -a$. Hence $[\tau_\pi]_m$ has no preimage in $\mathrm{Mot}_M(N', N')$ under $\Gamma^{D^2}_{N,N'}$.

It is possible to show that if $x \in N$ is any point in the interior of N, and and $N'' = N \smallsetminus \{x\}$, then $\Gamma^{D^2}_{N,N''}\colon Mot_{D^2}(N'', N'') \xrightarrow{\sim} Mot_{D^2}(N, N)$ is a group isomorphism. Similarly $\Gamma^{D^2}_{N,(-a,a)}\colon Mot_{D^2}((-a, a), (-a, a)) \xrightarrow{\sim} Mot_{D^2}(N, N)$ is a group isomorphism.

Note also that none of the constructed subsets are isomorphic to each other in Mot_{D^2}.

Example 4.3.60. *Let $M = D^2$. Let N be a circle centred on the centre of the disk, and $N' = N \smallsetminus \{x\}$ where $x \in N$ is any point, so $N' = cl(N)$. Let $\tau_{2\pi}$ be the path in $\mathbf{TOP}^h(D^2, D^2)$, constructed analogously to τ_π in Example 4.3.59. Then we have that $\tau_{2\pi}\colon N \curvearrowright N$ and $\tau_{2\pi}\colon N' \curvearrowright N'$ are motions. Notice that $\tau_{2\pi t}(N) = N$ for all $t \in \mathbb{I}$, thus $\mathrm{Id}_M * \tau_{2\pi}\colon N \curvearrowright N$ is a stationary motion, and $(\tau_{2\pi}\colon N \curvearrowright N) \overset{m}{\sim} (\mathrm{Id}_M\colon N \curvearrowright N)$.*

For N', $\tau_{2\pi t}(N') \neq N'$ unless $t \in \{0, 1\}$, thus we do not obtain a motion-equivalence between $(\tau_{2\pi}\colon N' \curvearrowright N')$ and $(\mathrm{Id}_M\colon N' \curvearrowright N')$ in the same way. This would require a path-homotopy $\overline{\mathrm{Id}}_M * \tau_{2\pi}$ to a N'-stationary motion, and hence a path-homotopy $\tau_{2\pi}$ to an N'-stationary motion. Such a path-homotopy would imply the existence of a path-homotopy making the 2π rotation of S^1, considered as a motion from any point $y \in S^1$ to itself, motion equivalent to a $\{y\}$-stationary motion. We discuss this situation further in Section 4.6.3.

Notice N and N' are not connected in the motion groupoid as this would imply N home-

omorphic to N'*. Using the comment before previous the example, there is a map from* $\mathrm{Mt}_M(N',N')$ *to* $\mathrm{Mt}_M(N,N)$ *sending a motion to the motion with the same underlying pre-motion. There is a homomorphism* $\mathrm{Mot}_M(N',N') \to \mathrm{Mot}_M(N,N) \times \mathbb{Z}$ *constructed as follows. A representative motion* $f\colon N' \rightsquigarrow N'$ *is mapped to the product of the equivalence class of the motion* $f\colon N \rightsquigarrow N$*, and the number of* 2π *rotations of the point* x*, (where clockwise rotations correspond to positive numbers, and anti-clockwise to negative).*

In fact it can be shown that the group $\mathrm{Mot}_M(N,N)$ *is trivial.*

Example 4.3.61. *Let* $M = \mathbb{I}^3$ *and* $N \subset \mathbb{I}^3$ *a subset which is a Hopf link in the interior. Let* $N' = N \smallsetminus \{x\}$ *where* $x \in N$ *is any point. Then* $\mathrm{Mot}_M(N,N') = \varnothing$*. We can construct a homomorphism similar to the one constructed in the previous example.*

Let $K \subset \mathbb{I}^3$ *be the subset with* 2 *unknotted unlinked connected components homeomorphic to* S^1*. Then* $\mathrm{Mot}_M(N',K) = \varnothing$*.*

Let $a \subset N$ *be an arc in one component of the Hopf link, and* $N'' = N \smallsetminus a$*. Then* $\mathrm{Mot}_M(N',N'') = \varnothing$*, and* $\mathrm{Mot}_M(K,N'') = \varnothing$*.*

Example 4.3.62. *Let* M *be the torus* $T^2 = S^1 \times S^1$*, and let* $N = S^1 \times \{1\}$*. Let* N' *be the image of* N *under a Dehn twist about* $\{1\} \times S^1$*. Then the curves* N *and* N' *are not isotopic so there is no path* f *in* $\mathbf{TOP}^h(T^2,T^2)$*, starting in* id_{T^2} *and with* $f_1(N) = N'$*. However* $\mathrm{Mot}_{T^2}(N,N) \cong \mathrm{Mot}_{T^2}(N',N')$*. This is just a case of Corollary* 4.3.39*.*

Example 4.3.63. *Let* $M = S^3$*. If* K *and* K' *are non-isotopic knots in* S^3 *then we have* $\mathrm{Mot}_{S^3}(K,K') = \varnothing$*.*

Points and unknotted circles in $M = \mathbb{I}^3$

The setup

For $n \in \mathbb{N}$ denote by A_n the set of objects in $\mathrm{Mot}_{\mathbb{I}^3}^{\partial\mathbb{I}^3}$ which are subsets of $\mathbb{I}^3$ of the following form. Let $N \in A_n$, then N has n connected components which we label by a_i, $i \in \{1,\ldots,n\}$. Each a_i is either a circle or a point. If a_i is a point, then it is the point $((i-1/2)/n, 1/2, 1/2)$. If a_i is a circle, then it is a circle lying in the plane $y = 1/2$ with centre $((i-1/2)/n, 1/2, 1/2)$ and radius $1/4n$. See Figure 4.11 for an example. For a fixed $n \in \mathbb{N}$, we denote the full subgroupoid $\mathrm{Mot}_{\mathbb{I}^3}^{\partial\mathbb{I}^3}|_{A_n}$ by $\mathcal{A}_n$.

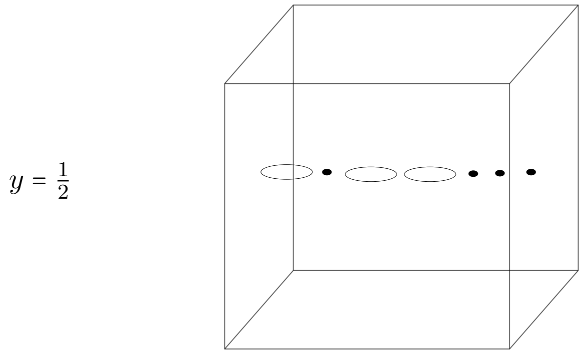


Figure 4.11: Example of positioning of three circles and four points in $\mathbb{I}^3$. This is an object in A_7.

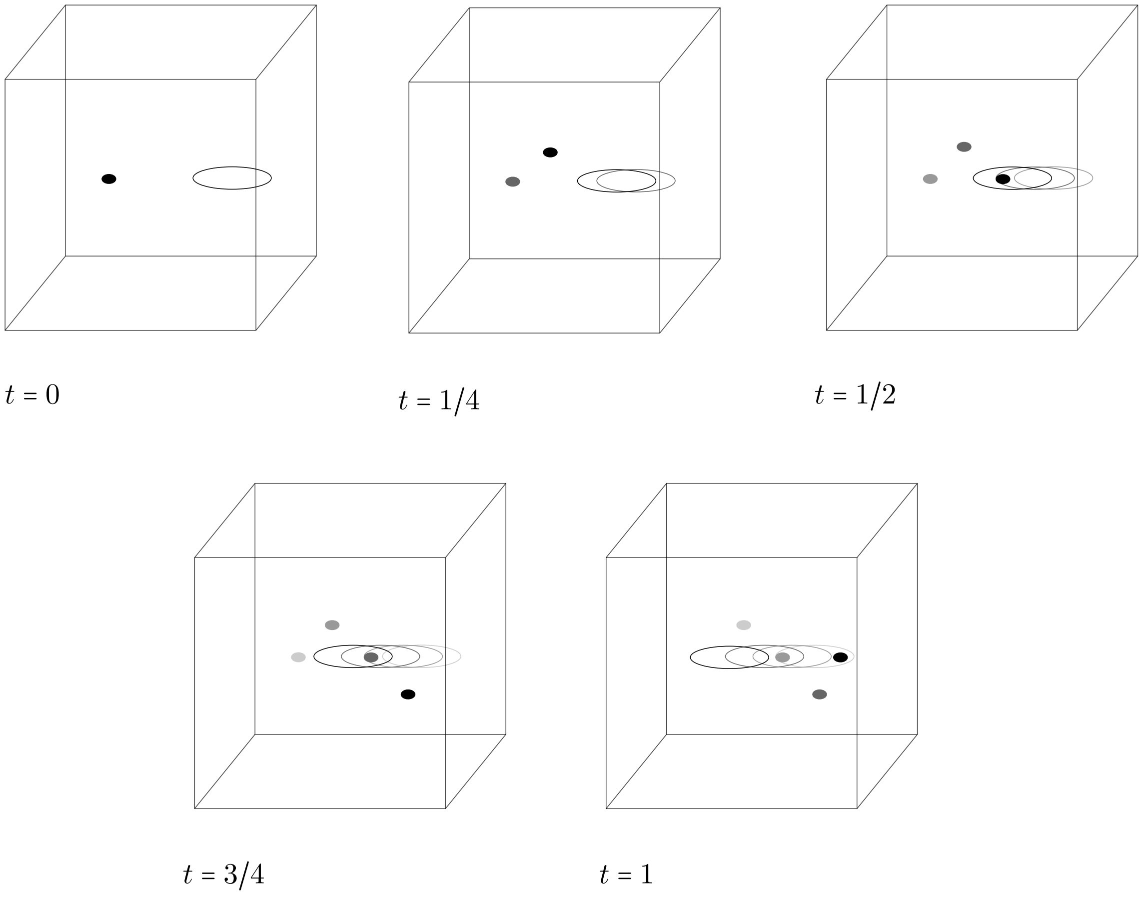
It will be clear that the set $\cup_n A_n$ can be given the structure of a free monoid, with generators $\cdot$ and $\circ$. Thus for example (fixing appropriate conventions) the element in Fig.4.11 is $\circ \cdot \circ \circ \cdot \cdot \cdot$.

Let us consider some motions in $\mathcal{A}_n$. First let $N \in A_2$ be the subset of $\mathbb{I}^3$ corresponding to $\circ\cdot$, as shown on the left hand side of Figure 4.12, labelled $t = 0$. Let $N' \in A_2$ be the subset $\cdot\circ$ of $\mathbb{I}^3$ with a_1 a loop and a_2 a point, as shown on the right hand side of Figure 4.12, labelled $t = 1$.

There is a continuous map $\varsigma : \mathbb{I}^3 \times \mathbb{I} \to \mathbb{I}^3$ that fixes $\partial \mathbb{I}^3$ and such that the circle prescribes a well-defined disk at in the $y = 1/2$ plane every t, and such that, looking at the image of N under ς, the point 'passes through' the plane, and indeed the disk, of the circle exactly once. That is, the image of the point lies in the disk of the image of the circle only at $t = 1/2$, approaching from above the plane, and departing below. We claim that such a motion may have a movie presentation as shown in Figure 4.12. Since this is not the crux of the present section, we leave it to intuition here to ensure that there exists such a smooth path of embeddings moving the subsets as described. Hence, using the isotopy extension theorem for smooth, compact submanifolds, (see for example [Hir12, Ch.8.1]) such a motion can be shown to exist. (Of course there are infinitely many such maps.) This yields a motion $\varsigma : N \rightsquigarrow N$.

There is another motion, with underlying map ϱ say, taking N to N' (again among infinitely many such) during which the circle again prescribes a well-defined disk at every t and the point does not pass through this disk.

We claim that the above characterisations are enough to ensure that the two motions are



$t = 0$ $t = 1/4$ $t = 1/2$

$t = 3/4$ $t = 1$

Figure 4.12: The images of subset $N \subset \mathbb{I}^3$ as shown at $t = 0$ under a motion in $\mathbb{I}^3$. The black markings indicate the position of N at the labelled time, while the gray markings indicate the position of N at earlier times.

not representatives of the same morphism.

There is also a motion τ from N to N which leaves the point fixed for all t but which rotates the circle by π around an axis in the $y = 1/2$ plane which passes through the centre of the circle. (If we added an orientation this would be a motion changing the orientation of the circle.) We claim this motion is not a representative of the same morphism as the identity.

A conjecture for a presentation for $\mathcal{A}_n$.

In this section we will construct an abstract category by giving objects, generating morphisms and relations and conjecture that this category is isomorphic to $\mathcal{A}_n$.

Definition 4.3.64. For any $n \in \mathbb{N}$, denote by $\mathcal{D}_n$ the category with objects, generating morphisms and relations as follows.

The objects of $\mathcal{D}_n$ are words of length n in $\{p, c\}$. Let $\{s_1, \dots, s_n\}$ denote the generators of the Coxeter presentation of the symmetric group, S_n. The generator s_i acts on $X \in Ob(\mathcal{D}_n)$ by permuting the ith and $i+1$th letters. We use X_{s_i} to denote the image of X under the action of s_i.

(Note that in what follows we find it convenient to label morphisms by a triple $f\colon X \to Y$, so there may exist a distinct morphism $f\colon W \to Z$ with $W \neq X$ and $Y \neq Z$, and f alone has no meaning.)

For each $X \in Ob(\mathcal{D}_n)$ we have generating morphisms $\rho_i\colon X \to X_{s_i}$ for each $i \in \{1, \dots, n-1\}$ subject to the following relations:

$$\begin{cases} \left(\rho_j\colon X_{s_i} \to X_{s_j s_i}\right) * \left(\rho_i\colon X \to X_{s_i}\right) = \left(\rho_i\colon X_{s_j} \to X_{s_i s_j}\right) * \left(\rho_j\colon X \to X_{s_j}\right) & |i-j| > 1 \\ \begin{array}{l} (\rho_i\colon X_{s_{i+1}s_i} \to X_{s_i s_{i+1} s_i}) * (\rho_{i+1}\colon X_{s_i} \to X_{s_{i+1}s_i}) * (\rho_i\colon X \to X_{s_i}) = \\ \quad (\rho_{i+1}\colon X_{s_i s_{i+1}} \to X_{s_{i+1}s_i s_{i+1}}) * (\rho_i\colon X_{s_{i+1}} \to X_{s_i s_{i+1}}) * (\rho_{i+1}\colon X \to X_{s_{i+1}}) \end{array} & i = 1, \dots, n-2 \\ (\rho_i\colon X_{s_i} \to X_{s_i s_i}) * (\rho_i\colon X \to X_{s_i}) = 1_X\colon X \to X & i = 1, \dots, n-1. \end{cases}$$

And also for each $X \in Ob(\mathcal{D}_n)$ such that the $i+1$th letter of X is c, generating morphisms

$\sigma_i : X \to X_{s_i}$ for each $i \in \{1, \dots, n-1\}$, subject to the following relations:

$$\left\{\begin{array}{ll}
\big(\sigma_j : X_{s_i} \to X_{s_j s_i}\big) * (\sigma_i : X \to X_{s_i}) = \big(\sigma_i : X_{s_j} \to X_{s_i s_j}\big) * \big(\sigma_j : X \to X_{s_j}\big) & |i-j| > 1 \\
\begin{array}{l}(\sigma_i : X_{s_{i+1} s_i} \to X_{s_i s_{i+1} s_i}) * (\sigma_{i+1} : X_{s_i} \to X_{s_{i+1} s_i}) * (\sigma_i : X \to X_{s_i}) = \\ \quad (\sigma_{i+1} : X_{s_i s_{i+1}} \to X_{s_{i+1} s_i s_{i+1}}) * (\sigma_i : X_{s_{i+1}} \to X_{s_i s_{i+1}}) * (\sigma_{i+1} : X \to X_{s_{i+1}})\end{array} & i = 1, \dots, n-2 \\
\big(\sigma_j : X_{s_i} \to X_{s_j s_i}\big) * (\rho_i : X \to X_{s_i}) = \big(\rho_i : X_{s_j} \to X_{s_i s_j}\big) * \big(\sigma_j : X \to X_{s_j}\big) & |i-j| > 1 \\
\begin{array}{l}(\rho_i : X_{s_{i+1} s_i} \to X_{s_i s_{i+1} s_i}) * (\rho_{i+1} : X_{s_i} \to X_{s_{i+1} s_i}) * (\sigma_i : X \to X_{s_i}) = \\ \quad (\sigma_{i+1} : X_{s_i s_{i+1}} \to X_{s_{i+1} s_i s_{i+1}}) * (\rho_i : X_{s_{i+1}} \to X_{s_i s_{i+1}}) * (\rho_{i+1} : X \to X_{s_{i+1}})\end{array} & i = 1, \dots, n-2 \\
\begin{array}{l}(\sigma_i : X_{s_{i+1} s_i} \to X_{s_i s_{i+1} s_i}) * (\sigma_{i+1} : X_{s_i} \to X_{s_{i+1} s_i}) * (\rho_i : X \to X_{s_i}) = \\ \quad (\rho_{i+1} : X_{s_i s_{i+1}} \to X_{s_{i+1} s_i s_{i+1}}) * (\sigma_i : X_{s_{i+1}} \to X_{s_i s_{i+1}}) * (\sigma_{i+1} : X \to X_{s_{i+1}})\end{array} & i = 1, \dots, n-2
\end{array}\right.$$

And also for each $X \in Ob(\mathcal{D}_n)$ such that the ith letter of X is c generating morphisms $\tau_i : X \to X$ for each $i \in \{1, \dots, n\}$ subject to the following relations:

$$\left\{\begin{array}{ll}
(\tau_j : X \to X) * (\tau_i : X \to X) = (\tau_i : X \to X) * (\tau_j : X \to X) & i \neq j \\
(\tau_i : X \to X) * (\tau_i : X \to X) = (1_X : X \to X) & i = 1, \dots, n \\
(\tau_j : X_{s_i} \to X_{s_i}) * (\sigma_i : X \to X_{s_i}) = (\sigma_i : X \to X_{s_i}) * (\tau_j : X \to X) & |i-j| > 1 \\
(\tau_j : X_{s_i} \to X_{s_i}) * (\rho_i : X \to X_{s_i}) = (\rho_i : X \to X_{s_i}) * (\tau_j : X \to X) & |i-j| > 1 \\
(\rho_i : X \to X_{s_i}) * (\tau_i : X \to X) = (\tau_{i+1} : X_{s_i} \to X_{s_i}) * (\rho_i : X \to X_{s_i}) & i = 1, \dots, n-1 \\
(\sigma_i : X \to X_{s_i}) * (\tau_i : X \to X) = (\tau_{i+1} : X_{s_i} \to X_{s_i}) * (\sigma_i : X \to X_{s_i}) & i = 1, \dots, n-1 \\
\begin{array}{r}(\sigma_i : X \to X_{s_i}) * (\tau_{i+1} : X \to X) = (\tau_i : X_{s_i} \to X_{s_i}) * (\rho_i : X \to X_{s_i}) * \\ (\sigma_i : X \to X_{s_i})^{-1} * (\rho_i : X \to X_{s_i})\end{array} & i = 1, \dots, n-1
\end{array}\right.$$

Note that there is a monoidal structure on $\cup_n Ob(\mathcal{D}_n)$ given by concatenation of words. The monoid is freely generated by p and c. Thus there is a monoid morphism $\phi_0 : \cup_n Ob(\mathcal{D}_n) \to \cup_n A_n$ given by $\phi_0(p) = \cdot$ and $\phi_0(c) = \circ$. Notice that this restricts to a function for each fixed n.

Conjecture 4.3.65. *For each $n \in \mathbb{N}$, there is a functor*

$$\phi : \mathcal{D}_n \to \mathcal{A}_n = \mathrm{Mot}_{\mathbb{I}^3}^{\partial \mathbb{I}^3}|_{A_n}$$

which is given by the following. A word $w \in Ob(\mathcal{D}_n)$ is mapped to the object in A_n, with a_i a point if the ith letter in w is p and a_i a circle if the ith letter is a c (i.e. the restriction of ϕ_0 as above).

We only give the images of certain generators in the $n = 2$ case, and leave it to the reader *to fully construct the morphism for each n. The morphism $\sigma_1 \colon pc \to cp$ is mapped to the motion equivalence class of the motion $\varsigma \colon N \rightsquigarrow N'$ described above. The morphism $\rho_1 \colon pc \to cp$ is mapped to the motion equivalence class of the motion $\varrho \colon N \rightsquigarrow N'$ described above. The morphism $\tau_2 \colon pc \to pc$ is mapped to the motion equivalence class of the motion $\tau \colon N \rightsquigarrow N$ described above.*

Conjecture 4.3.66. *For each $n \in \mathbb{N}$ the map*

$$\phi \colon \mathcal{D}_n \to \mathcal{A}_n$$

is an isomorphism. That is to say $\mathcal{D}_n$ is a presentation of $\mathcal{A}_n = \mathrm{Mot}_{\mathbb{I}^3}^{\partial \mathbb{I}^3}|_{A_n}$.

Note that if we further restrict to the subset $N \in A_n$ where all connected components are points, this is a presentation of the symmetric group and if we restrict to the subset $N' \in A_n$ where all connected components are circles this is a presentation of the nth extended loop braid group [Dam17, Prop. 3.14 and 3.16]. The presentation of the non-extended version of the loop braid group is obtained by considering the subgroupoids with all τ generators excluded. Topologically, this corresponds to allowing only orientation preserving homeomorphisms .

4.4 A useful alternative congruence leading to Mot_M

In this section we introduce an alternative equivalence relation on the sets $\mathrm{Mt}_M(N, N')$. We prove in Theorem 4.4.6 that this alternative equivalence relation is the same as the relation $\overset{m}{\sim}$ constructed in the previous section, and thus leads to the motion groupoid. This gives us another way to understand equivalence classes of motions.

The equivalence relation we use here is the relative path-equivalence used in the construction of the relative fundamental set of a pair of spaces. Thus it will allow us to use the relative homotopy long exact sequence to prove the relationship between motion groupoids and mapping class groupoids in Section 4.6.

Definition 4.4.1. Fix a manifold M. Define a relation on $\mathrm{Mt}_M(N, N')$ as follows. Let $f \colon N \rightsquigarrow N' \overset{rp}{\sim} g \colon N \rightsquigarrow N'$ if the motions $f \colon N \rightsquigarrow N'$ and $g \colon N \rightsquigarrow N'$ are relative path-homotopic. This means there exists a continuous map

$$H : \mathbb{I} \times \mathbb{I} \to \mathbf{TOP}^h(M, M)$$

such that

- for any fixed $s \in \mathbb{I}$, $t \mapsto H(t,s)$ is a motion from N to N',
- for all $t \in \mathbb{I}$, $H(t,0) = f_t$, and
- for all $t \in \mathbb{I}$, $H(t,1) = g_t$.

We call such a homotopy a relative path-homotopy.

Lemma 4.4.2. *Fix a manifold M. For each pair N, N', the relation $\overset{rp}{\sim}$ is an equivalence relation on $\mathrm{Mt}_M(N, N')$.*

Notation: *We call $\overset{rp}{\sim}$ classes relative path-equivalence classes and use $[f : N \rightsquigarrow N']_{rp}$ for the class of f.*

Proof. Let $f : N \rightsquigarrow N'$, $g : N \rightsquigarrow N'$ and $h : N \rightsquigarrow N'$ be motions. We can prove reflexivity by observing that the homotopy $H(t,s) = f_t$ for all $s \in \mathbb{I}$ is a relative path-homotopy from $f : N \rightsquigarrow N'$ to itself.

For symmetry let $H_{f,g}$ be a relative path-homotopy from $f : N \rightsquigarrow N'$ to $g : N \rightsquigarrow N'$. Then the function $H_{g,f}(t,s) = H_{f,g}(t, 1-s)$ is a relative path-homotopy from $g : N \rightsquigarrow N'$ to $f : N \rightsquigarrow N'$.

For transitivity let $H_{g,h}$ be a relative path-homotopy from $g : N \rightsquigarrow N'$ to $h : N \rightsquigarrow N'$. Then

$$H_{f,h}(t,s) = \begin{cases} H_{f,g}(t, 2s) & 0 \leq s \leq \frac{1}{2} \\ H_{g,h}(t, 2(s - \frac{1}{2})) & \frac{1}{2} \leq s \leq 1 \end{cases}$$

is a relative path-homotopy from $f : N \rightsquigarrow N'$ to $h : N \rightsquigarrow N'$. □

Proposition 4.4.3. *Let $(f : N \rightsquigarrow N') \overset{p}{\sim} (g : N \rightsquigarrow N')$ be path equivalent motions, then $(f : N \rightsquigarrow N') \overset{rp}{\sim} (g : N \rightsquigarrow N')$.*

Proof. A path-homotopy from f to g has fixed endpoint, thus is a relative path-homotopy $(f : N \rightsquigarrow N')$ to $(g : N \rightsquigarrow N')$. □

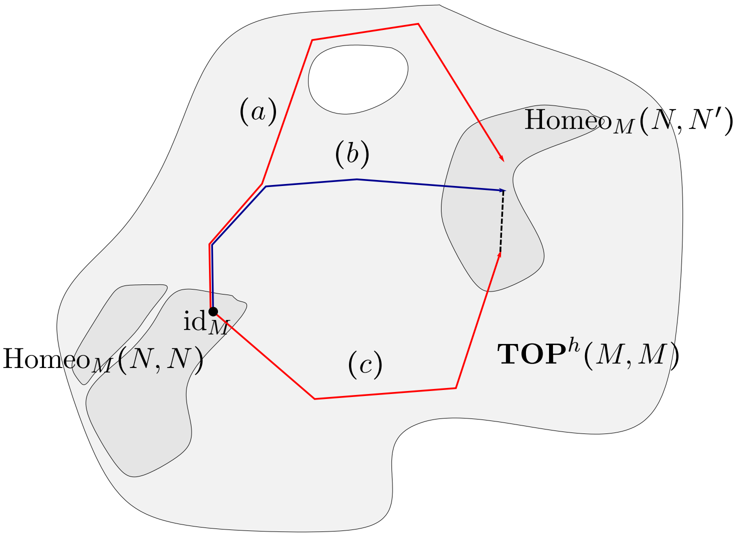


Figure 4.13: Let M be a manifold, and $N, N' \subset M$ subsets. Here we use the same schema used in Figure 4.2. The space $\mathbf{TOP}^h(M,M)$ is a connected region of the plane homeomorphic to $S^1 \times \mathbb{I}$. The paths labelled (a), (b) and (c) represent motions from N to N' in M. There is a relative path-homotopy from (b) to (c), but not from (a) to (b) or (c).

Figure 4.13 gives examples of relative path-homotopic, and non relative path-homotopic motions in our schema introduced in Figure 4.2.

Lemma 4.4.4. *Suppose we have relative path-equivalent motions* $(f\colon N \rightsquigarrow N') \overset{rp}{\sim} (f'\colon N \rightsquigarrow N')$*, then* $(f\colon N \rightsquigarrow N') \overset{m}{\sim} (f'\colon N \rightsquigarrow N')$*.*

Proof. Let H be a relative path-homotopy from $f\colon N \rightsquigarrow N'$ to $f'\colon N \rightsquigarrow N'$. We must show that $\bar{f}' * f\colon N \rightsquigarrow N$ is path-equivalent to a stationary motion from N to N.

Notice first that $H(1, 1-s)$ is a path f'_1 to f_1 which is in $\mathrm{Homeo}_M(N,N')$ for all t, we relabel this path as γ. We define $\tilde{\gamma}$ as $\tilde{\gamma} = \gamma \circ f_1'^{-1}$, so $\tilde{\gamma}\colon N' \rightsquigarrow N'$ is a stationary motion with $\tilde{\gamma}_1 = f_1 \circ f_1'^{-1}$.

We can use H to construct a path-homotopy from f to the path composition $\gamma f'$. Explicitly a suitable function is:

$$H_1(t,s) = \begin{cases} H(\frac{2t}{2-s}, s) & t \leq 1 - \frac{s}{2} \\ \gamma_{2t-1} & 1 - \frac{s}{2} \leq t. \end{cases}$$

For fixed $s \in \mathbb{I}$ the path $H_1(t,s)$ starts at the identity, traces the whole of the path $H(t,s)$ followed by the part of the path γ starting from $\gamma_{1-s} = H(1,s)$ and ending at γ_1. Note that the path composition, $\gamma f'$ is precisely the motion composition $\tilde{\gamma} * f'$, so $f \overset{p}{\sim} \tilde{\gamma} * f'$.

By gluing H_1 appropriately with the trivial homotopy, we have that $\bar{f}' * f$ is path-equivalent to $\bar{f}' * (\gamma * f')$.

Now using the normalcy of stationary motions proved in Lemma 4.3.34, we have that the motion $\bar{f}' * (\tilde{\gamma} * f') \colon N \rightsquigarrow N$ is path-equivalent to a stationary motion from N to N. ☐

Lemma 4.4.5. *Suppose we have motion-equivalent motions* $(f\colon N \rightsquigarrow N') \overset{m}{\sim} (f'\colon N \rightsquigarrow N')$, *then* $(f\colon N \rightsquigarrow N') \overset{rp}{\sim} (f'\colon N \rightsquigarrow N')$.

Proof. By uniqueness of inverses we have $(f\colon N \rightsquigarrow N') \overset{m}{\sim} (f'\colon N \rightsquigarrow N')$ implies $(\bar{f}\colon N' \rightsquigarrow N) \overset{m}{\sim} (\bar{f}'\colon N' \rightsquigarrow N)$. Thus we have a path-homotopy, say H, from $f' * \bar{f}$ to a stationary motion $\gamma\colon N' \rightsquigarrow N'$. Consider the following function:

$$H_1(t,s) = \begin{cases} f_{\frac{2t}{2-s}} & t \le 1 - \frac{s}{2} \\ \gamma_{2(t+\frac{s}{2}-1)} \circ f_1 & 1 - \frac{s}{2} \le t. \end{cases}$$

We have that $H_1(t,0)$ is the path f and $H_1(t,1)$ is the path $\gamma * f$. And for any fixed $s \in \mathbb{I}$ we have $H(0,s) = \mathrm{id}_M$ and $H(1,s)(N) = \gamma_s \circ f_1(N) = \gamma_s(N') = N'$. Also note H_1 is continuous as both functions agree when $t = 1 - \frac{s}{2}$. Hence we have that H_1 is a relative path-homotopy and $(f\colon N \rightsquigarrow N') \overset{rp}{\sim} (\gamma * f\colon N \rightsquigarrow N')$.

Using Lemma 4.3.8 we have that $\gamma * f \overset{p}{\sim} \gamma \cdot f$. Now $H_2(t,s) = H(t, 1-s) \circ f_t$ is a path-homotopy giving $\gamma \cdot f \overset{p}{\sim} (f' * \bar{f}) \cdot f$. Again using Lemma 4.3.8 $(f' * \bar{f}) \cdot f \overset{p}{\sim} f' \cdot \bar{f} \cdot f \overset{p}{\sim} f' \cdot (\bar{f} * f) \overset{p}{\sim} f' \cdot \mathrm{Id}_M$, so $\gamma * f \overset{p}{\sim} f'$.

Using Proposition 4.4.3 this implies $(\gamma * f\colon N \rightsquigarrow N') \overset{rp}{\sim} (f'\colon N \rightsquigarrow N')$ and hence we have $(f\colon N \rightsquigarrow N') \overset{rp}{\sim} (f'\colon N \rightsquigarrow N')$. ☐

Theorem 4.4.6. *For a manifold M and a motion $f\colon N \rightsquigarrow N'$ in M we have*

$$[f\colon N \rightsquigarrow N']_{rp} = [f\colon N \rightsquigarrow N']_m.$$

This means quotienting Mt_M *by relative path-equivalence leads to the same groupoid as quotienting by motion-equivalence.*

Proof. This follows from Lemmas 4.4.4 and 4.4.5. ☐

Remark 4.4.1. All proofs in this section work in exactly the same way restricting to A-fixing motions. Hence we also have that for a manifold M and an A-fixing motion $f^A\colon N \rightsquigarrow N'$

$$[f^A\colon N \rightsquigarrow N']_{\mathrm{rp}} = [f^A\colon N \rightsquigarrow N']_{\mathrm{m}}.$$

4.5 Mapping class groupoid MCG_M^A

4.5.1 The Mapping class groupoid MCG_M

In this section we construct the mapping class groupoid MCG_M associated to a manifold M. We do this by constructing a congruence on Homeo_M, so the morphisms in MCG_M are certain equivalence classes of self-homeomorphisms of M. Compare this with motions, which keep track of an entire path in $\mathbf{TOP}^h(M,M)$.

These are in general a simpler construction than motion groupoids and there are many known results already in the literature, e.g. [Bir16; FM11; Ham74; HT80].

Recall from Section 4.2 that for a manifold M and for subsets $N, N' \subseteq M$, morphisms in $\mathrm{Homeo}_M(N,N')$ are triples denoted $\mathfrak{f}\colon N \curvearrowright N'$ where $\mathfrak{f} \in \mathbf{Top}^h(M,M)$ and $\mathfrak{f}(N) = N'$. Where convenient we also think of the elements of $\mathrm{Homeo}_M(N,N')$ as the projection to the first coordinate of each triple i.e. $\mathfrak{f} \in \mathbf{Top}^h(M,M)$ such that $\mathfrak{f}(N) = N'$.

Definition 4.5.1. Let M be a manifold and $N, N' \subseteq M$. For any $\mathfrak{f}\colon N \curvearrowright N'$ and $\mathfrak{g}\colon N \curvearrowright N'$ in $\mathrm{Homeo}_M(N,N')$, $\mathfrak{f}\colon N \curvearrowright N'$ is said to be isotopic to $\mathfrak{g}\colon N \curvearrowright N'$, denoted by $\overset{i}{\sim}$, if there exists a continuous map

$$H\colon M \times \mathbb{I} \to M$$

such that

- for all fixed $s \in \mathbb{I}$, the map $m \mapsto H(m,s)$ is in $\mathrm{Homeo}_M(N,N')$,
- for all $m \in M$, $H(m,0) = \mathfrak{f}(m)$, and
- for all $m \in M$, $H(m,1) = \mathfrak{g}(m)$.

We call such a map an isotopy from $\mathfrak{f}\colon N \curvearrowright N'$ to $\mathfrak{g}\colon N \curvearrowright N'$.

Lemma 4.5.2. *Let M be a manifold. For all pairs $N, N' \subseteq M$, the relation $\overset{i}{\sim}$ is an equivalence relation on* $\mathrm{Homeo}_M(N,N')$.

Notation: *We call this equivalence relation* <u>*isotopy equivalence*</u>*. We denote the equivalence class of* $\mathfrak{f}\colon N \curvearrowright N'$ *up to isotopy equivalence as* $[\mathfrak{f}\colon N \curvearrowright N']_i$.

Proof. Let $\mathfrak{f}\colon N \curvearrowright N'$, $\mathfrak{g}\colon N \curvearrowright N'$ and $\mathfrak{h}\colon N \curvearrowright N'$ be in $\mathrm{Homeo}_M(N, N')$ with $(\mathfrak{f}\colon N \curvearrowright N') \overset{i}{\sim} (\mathfrak{g}\colon N \curvearrowright N')$ and $(\mathfrak{g}\colon N \curvearrowright N') \overset{i}{\sim} (\mathfrak{h}\colon N \curvearrowright N')$. Then there exists some isotopy, say $H_{\mathfrak{f},\mathfrak{g}}$, from $\mathfrak{f}\colon N \curvearrowright N'$ to $\mathfrak{g}\colon N \curvearrowright N'$ and an isotopy, say $H_{\mathfrak{g},\mathfrak{h}}$, from $\mathfrak{g}\colon N \curvearrowright N'$ to $\mathfrak{h}\colon N \curvearrowright N'$.

We first check reflexivity. Then the map $H(m, s) = \mathfrak{f}(m)$ for all $s \in \mathbb{I}$ is an isotopy from $\mathfrak{f}\colon N \curvearrowright N'$ to itself. For symmetry, $H(m, s) = H_{\mathfrak{f},\mathfrak{g}}(m, 1 - s)$ is an isotopy from $\mathfrak{g}\colon N \curvearrowright N'$ to $\mathfrak{f}\colon N \curvearrowright N'$. For transitivity,

$$H(m,s) = \begin{cases} H_{\mathfrak{f},\mathfrak{g}}(m, 2s) & 0 \le s \le \frac{1}{2} \\ H_{\mathfrak{g},\mathfrak{h}}(m, 2(s - \frac{1}{2}) & \frac{1}{2} \le s \le 1 \end{cases}$$

is an isotopy from $\mathfrak{f}\colon N \curvearrowright N'$ to $\mathfrak{h}\colon N \curvearrowright N'$. ☐

Lemma 4.5.3. *Let M be a manifold. The family of relations* $(\mathrm{Homeo}_M(N, N'), \overset{i}{\sim})$ *for all pairs $N, N' \subseteq M$ are a congruence on* Homeo_M.

Proof. We have that $\overset{i}{\sim}$ is an equivalence relation on each $\mathrm{Homeo}_M(N, N')$ from Lemma 4.5.2.

We check that the composition descends to a well defined composition on equivalence classes. Suppose there exists an isotopy, say $H_{\mathfrak{f},\mathfrak{f}'}$, from $\mathfrak{f}\colon N \curvearrowright N'$ to $\mathfrak{f}'\colon N \curvearrowright N'$ and another isotopy, say $H_{\mathfrak{g},\mathfrak{g}'}$ from $\mathfrak{g}\colon N \curvearrowright N'$ to $\mathfrak{g}'\colon N \curvearrowright N'$. Then

$$H(m, s) = H_{\mathfrak{g},\mathfrak{g}'}(m, s) \circ H_{\mathfrak{f},\mathfrak{f}'}(m, s)$$

is an isotopy from $\mathfrak{g} \circ \mathfrak{f}\colon N \curvearrowright N''$ to $\mathfrak{g}' \circ \mathfrak{f}'\colon N \curvearrowright N''$. ☐

Theorem 4.5.4. *Let M be a manifold. There is a groupoid*

$$\mathrm{MCG}_M = (\mathcal{P}M, \mathrm{Homeo}_M(N, N')/\overset{i}{\sim}, \circ, [\mathrm{id}_M]_i, [\mathfrak{f}]_i \mapsto [\mathfrak{f}^{-1}]_i).$$

We call this the <u>*mapping class groupoid*</u> *of M.*

Proof. This is the quotient $\mathrm{Homeo}_M/\overset{i}{\sim}$. Lemma 4.5.3 gives that $\overset{i}{\sim}$ is a congruence and Proposition 3.2.4 gives that the quotient of a groupoid by a congruence is still a groupoid with the given identity and inverse. □

Lemma 4.5.5. *Let M be a manifold. We have that as a set*

$$\mathrm{MCG}_M(N, N') = \pi_0(\mathrm{Homeo}_M(N, N')).$$

Here we are considering $\mathrm{Homeo}_M(N, N')$ *as a space which is possible by Lemma 4.2.6.*

Proof. Using Theorem 3.5.16 a continuous map $M \times \mathbb{I} \to M$ satisfying the conditions in Definition 4.5.1 corresponds to a path $\mathbb{I} \to \mathrm{Homeo}_M(N, N')$ from $\mathfrak{f}$ to $\mathfrak{g}$. □

Proposition 4.5.6. *We have*

$$\mathrm{MCG}_{S^1}(\varnothing, \varnothing) = \mathbb{Z}/2\mathbb{Z}.$$

Proof. Let $\mathfrak{f}, \mathfrak{g} \colon S^1 \to S^1$ be homeomorphisms that are connected by a path in $\mathbf{TOP}^h(S^1, S^1)$. Then $\mathfrak{f}$ and $\mathfrak{g}$ are homotopic, so they have the same degree, so either they are both orientation preserving or orientation reversing. It is proven in [Ham74, Theorem 1.1.2] that the space of orientation preserving homeomorphisms $S^1 \to S^1$ is homotopic to S^1, and in particular that it is path-connected. Since the identity map $S^1 \to S^1$ is orientation preserving, the path-component of the identity in $\mathbf{TOP}^h(S^1, S^1)$ is the set of orientation preserving homeomorphisms from S^1 to itself.

If $\mathfrak{f}$ and $\mathfrak{g}$ are orientation reversing, then $\mathfrak{f} \circ \mathfrak{g}^{-1}$ is orientation preserving, and hence can be connected by a path to id_{S^1}. It follows that $\mathfrak{f}$ and $\mathfrak{g}$ can be connected by a path in $\mathbf{TOP}^h(S^1, S^1)$.

In particular, $\mathbf{TOP}^h(S^1, S^1)$ has two path-components, containing respectively the orientation preserving and the orientation reversing homeomorphisms from S^1 to itself. Therefore the homomorphism $\pi_0(\mathrm{Homeo}_{S^1}(\varnothing, \varnothing)) \to \{\pm 1\} \cong \mathbb{Z}/2\mathbb{Z}$ induced by the degree homomorphism $\deg \colon \mathbf{Top}^h(S^1, S^1) = \mathrm{Homeo}_{S^1}(\varnothing, \varnothing) \to \{\pm 1\}$ is an isomorphism. □

4.5.2 Pointwise A-fixing mapping class groupoid MCG_M^A

Here we have a subset fixing version of the mapping class groupoid.

Theorem 4.5.7. *Let M be a manifold and $A \subseteq M$ a subset. There is a groupoid*

$$\mathrm{MCG}_M^A = (\mathcal{P}M, \mathrm{Homeo}_M^A(N,N')/\overset{i}{\sim}, \circ, [\mathrm{id}_M]_i, [\mathfrak{f}]_i \mapsto [\mathfrak{f}^{-1}]_i).$$

Note that in this case the first condition in Definition 4.5.1 becomes: for all $s \in \mathbb{I}$ the map $m \mapsto H(m,s)$ is in Homeo_M^A.

Proof. This is the quotient $\mathrm{Homeo}_M^A/\overset{i}{\sim}$. The proofs of Lemmas 4.5.2 and 4.5.3 proceed in the exactly the same way for A-fixing self-homeomorphisms. All constructed homotopies will be A-fixing for all $s \in \mathbb{I}$. Proposition 3.2.4 gives that the quotient of a groupoid by a congruence is still a groupoid. □

Proposition 4.5.8. *The morphism group* $\mathrm{MCG}_{D^2}^{\partial D^2}(\varnothing,\varnothing)$ *is trivial.*

Proof. (This follows from the Alexander trick [Ale23].) Suppose we have $\mathfrak{f}^{\partial D^2}\colon \varnothing \rightharpoondown \varnothing$ in D^2. Define

$$f_t(x) = \begin{cases} t\,\mathfrak{f}(x/t) & 0 \le |x| \le t, \\ x & t \le |x| \le 1. \end{cases}$$

Notice that $f_0 = \mathrm{id}_{D^2}$ and $f_1 = \mathfrak{f}$ and each f_t is continuous. Moreover:

$$H\colon D^2 \times \mathbb{I} \to D^2,$$
$$(x,t) \mapsto f_t(x)$$

is a continuous map. So we have constructed an isotopy from any boundary preserving self-homeomorphism of D^2 to id_{D^2}. □

Note that a lot more is true. The same argument gives that the space of maps $D^2 \to D^2$ fixing the boundary is contractible; see [Ham74].

Remark 4.5.1. Note that if K is a finite subset of $D^2 \setminus \partial D^2$ then the morphism group $\mathrm{MCG}_{D^2}^{\partial D^2}(K,K)$ is isomorphic to the braid group on $|K|$ strands. For discussion see [BB05; Bir16]. See also [Dam17] for a thorough exposition of how loop braid groups arise as morphisms groups of the form $\mathrm{MCG}_{D^3}^{\partial D^3}(L,L)$ where L consists of a set of unknotted loops contained in the interior of D^3.

Example 4.5.9. *Let* $M = \mathbb{I}$ *and* $N = \mathbb{I} \cap \mathbb{Q}$*. We show that* $\mathrm{MCG}_{\mathbb{I}}(N, N)$ *is uncountably infinite.*

We begin by choosing elements of $\mathbf{Top}^h(\mathbb{I}, \mathbb{I})$*. Choose points* $x, x' \in N \smallsetminus \{0, 1\}$*, then there is a unique piecewise linear, orientation preserving map with precisely two linear segments sending* x *to* x' *and moreover this map sends* N *to itself. Denote this by* $\phi_{xx'}$*. Let us fix the point* x*, then varying* x' *gives an infinite choice of maps* $\phi_{xx'}$*.*

We prove by contradiction that all such $\phi_{xx'}$ *represent distinct equivalence classes in* $\mathrm{MCG}_{\mathbb{I}}(N, N)$*. Let* $x, x', x'' \in N \smallsetminus \{0, 1\}$ *and suppose* $\phi_{xx'}: N \to N$ *is isotopic to* $\phi_{xx''}: N \to N$ *in* $\mathrm{MCG}_{\mathbb{I}}(N, N)$*. Then for all* $n \in N$ *we have a path* $\phi_{xx'}(n)$ *to* $\phi_{xx''}(n)$ *in* $\mathbb{Q}$*, and hence a path* $\phi_{xx'}(x) = x'$ *to* $\phi_{xx''}(x) = x''$*. But all paths* $\mathbb{I} \to \mathbb{Q}$ *are constant, which follows from the intermediate value theorem. Hence* $x' = x''$*. Therefore any pair of distinct maps of the described form are not isotopic.*

More generally a piecewise linear map can be defined as follows. Starting from $t = 0$*, each segment is be defined by choosing the upper bound* $t \in \{0, 1]$ *and the gradient (which is bounded by condition that the map is well defined). Repeating with the condition that the upper bound must be distinct from the upper bound of the previous section until* $t = 1$ *is chosen, defines a map. Choosing rational gradients, and rational bounds is sufficient to ensure such a map sends* N *to itself. By the same argument as above distinct such maps are non isotopic. Allowing for infinite segments, then this construction is countable product of countable sets, thus uncountable. (More precisely it has the cardinality of the continuum.)*

4.6 Functor from Mot_M^A to MCG_M^A

It is known that the braid groups and loop braid groups can be defined as mapping class groups, as well as as motion groups [Dam17; Dah36; Gol81; Bir16]. Here we generalise this by constructing, for any manifold M, a functor $\mathsf{F}: \mathrm{Mot}_M \to \mathrm{MCG}_M$ in Theorem 4.6.1. We prove this is an isomorphism if $\pi_1(\mathrm{Homeo}_M(\varnothing, \varnothing))$ and $\pi_0(\mathrm{Homeo}_M(\varnothing, \varnothing))$ are trivial. Precisely we prove F is full if and only if $\pi_0(\mathrm{Homeo}_M(\varnothing, \varnothing))$ is trivial, and that F is faithful if $\pi_1(\mathrm{Homeo}_M(\varnothing, \varnothing))$.

In Theorem 4.6.13 we state the version where we fix a distinguished subset $A \subseteq M$.

Theorem 4.6.1. *Let M be a manifold. There is a functor*

$$\mathsf{F}\colon \mathrm{Mot}_M \to \mathrm{MCG}_M$$

which is the identity on objects and on morphisms we have

$$\mathsf{F}\left([f\colon N \rightsquigarrow N']_m\right) = [f_1\colon N \curvearrowright N']_i.$$

Proof. We first check F is well defined. By Theorem 4.4.6 two motions $f\colon N \rightsquigarrow N'$ and $f'\colon N \rightsquigarrow N'$ are motion-equivalent if and only if they are relative path-equivalent, i.e. we have a relative path-homotopy:

$$H\colon \mathbb{I} \times \mathbb{I} \to \mathbf{TOP}^h(M, M).$$

Then $H(1,s)$ is a path f_1 to f_1' such that for all $s \in \mathbb{I}$, $H(1,s) \in \mathrm{Homeo}_M(N, N')$. Hence $f_1\colon N \curvearrowright N'$ and $f_1'\colon N \curvearrowright N'$ are isotopic.

We check F preserves composition. For $[f\colon N \rightsquigarrow N']_m$ and $[g\colon N' \rightsquigarrow N'']_m$ in Mot_M we have

$$\begin{aligned}
\mathsf{F}\left([g\colon N' \rightsquigarrow N'']_m * [f\colon N \rightsquigarrow N']_m\right) &= \mathsf{F}\left([g * f\colon N' \rightsquigarrow N'']_m\right) = [(g * f)_1\colon N \curvearrowright N'']_i \\
&= [g_1 \circ f_1\colon N \curvearrowright N'']_i = [g_1\colon N' \curvearrowright N'']_i \circ [f_1\colon N \curvearrowright N']_i \\
&= \mathsf{F}\left([g\colon N' \rightsquigarrow N'']_m\right) \circ \mathsf{F}\left([f\colon N \rightsquigarrow N']_m\right).
\end{aligned}$$

□

Lemma 4.6.2. *Let M be a manifold. The functor*

$$\mathsf{F}\colon \mathrm{Mot}_M \to \mathrm{MCG}_M$$

defined in Theorem 4.6.1 is full if and only if we have that $\pi_0(\mathrm{Homeo}_M(\varnothing, \varnothing), \mathrm{id}_M) = \pi_0(\mathbf{TOP}^h(M, M), \mathrm{id}_M)$ is trivial.

Proof. Suppose $\pi_0(\mathrm{Homeo}_M(\varnothing, \varnothing), \mathrm{id}_M)$ is trivial and let $[\mathfrak{f}\colon N \curvearrowright N']_i \in \mathrm{MCG}_M(N, N')$. Since $\mathrm{Homeo}_M(\varnothing, \varnothing)$ is path connected, there exists a path f with $f_0 = \mathrm{id}_M$ and $f_1 = \mathfrak{f}$. Since $\mathfrak{f}(N) = N'$, $f\colon N \rightsquigarrow N'$ is a motion and $\mathsf{F}([f\colon N \rightsquigarrow N']_m) = [\mathfrak{f}\colon N \curvearrowright N']_i$.

Now suppose $\pi_0(\mathrm{Homeo}_M(\varnothing,\varnothing),\mathrm{id}_M)$ is non-trivial. Let $\mathfrak{f}$ be a self-homeomorphism in a path component of $\mathrm{Homeo}_M(\varnothing,\varnothing)$ which does not contain the identity. Then $[\mathfrak{f}\colon\varnothing \curvearrowright \varnothing]_{\mathrm{i}} \in \mathrm{MCG}_M(\varnothing,\varnothing)$ and all representatives are in the same path connected component. Hence there is no motion with endpoint in $[\mathfrak{f}\colon\varnothing \curvearrowright \varnothing]_{\mathrm{i}}$. □

Example 4.6.3. *The functor* F *may be a surjection on some morphism sets and not on others.*

Let $M = S^3$*, then* $\mathrm{MCG}_{S^3}(\varnothing,\varnothing) = \mathbb{Z}/2\mathbb{Z}$ *corresponding to an orientation preserving and orientation reversing component (see [Hat78]) and so by the previous lemma* F *is not full.*

Consider $K \subset S^3$ *a knot which is not homeomorphic to its mirror image. Now* $\mathrm{MCG}_{S^3}(K,K)$ *contains only orientation preserving self-homeomorphisms, which are in the same connected component as the identity. Hence, by the first part of the proof of the previous lemma, the restriction* $\mathsf{F}\colon \mathrm{Mot}_{S^3}(K,K) \to \mathrm{MCG}_{S^3}(K,K)$ *is full.*

4.6.1 Long exact sequence of relative homotopy groups

To prove F is faithful if $\pi_1(\mathrm{Homeo}_M(\varnothing,\varnothing))$ we will use the homotopy long exact sequence. We briefly introduce this here, see [Hat02, Sec.4.1] or [May99, Ch.9] for a more thorough exposition.

Definition 4.6.4. Let $\mathbb{I}^{n-1}$ be the face of $\mathbb{I}^n$ with last coordinate 1 and let J^{n-1} be the closure of $\partial\mathbb{I}^n \smallsetminus \mathbb{I}^{n-1}$, i.e. the union of all remaining faces of $\mathbb{I}^n$.

Proposition 4.6.5. *Let* X *be a topological space and* $A \subseteq X$ *a subset and* $x_0 \in A$ *a point. For fixed* $n \geq 1$ *we define a relation on the set of continuous maps*

$$\gamma\colon (\mathbb{I}^n, \partial\mathbb{I}^n, J^{n-1}) \to (X, A, x_0)$$

as follows. We say $\gamma \sim \gamma'$ *if there exists* $H\colon \mathbb{I}^n \times \mathbb{I} \to X$ *such that*

- *for all* $s \in \mathbb{I}$, $H|_{\mathbb{I}^n \times \{s\}}$ *is a map* $(\mathbb{I}^n, \partial\mathbb{I}^n, J^{n-1}) \to (X, A, x_0)$,
- *for all* $x \in \mathbb{I}^n$, $H(x,0) = \gamma(x)$, *and*
- *for all* $x \in \mathbb{I}^n$, $H(x,1) = \gamma'(x)$.

This is an equivalence relation.

Notation: *We will call the set of such maps with the described equivalence the nth relative homotopy set and denote it* $\pi_n(X, A, x_0)$.

Proof. We omit this proof. It is similar to Lemma 4.4.2; see also [Hat02]. □

Lemma 4.6.6. *Let* M *be a manifoldand* $N \subseteq M$ *a subset. Then* $\mathrm{Mot}_M(N, N)$ *is precisely the relative fundamental set* $\pi_1(\mathrm{Homeo}_M(\varnothing, \varnothing), \mathrm{Homeo}_M(N, N), \mathrm{id}_M)$.

Proof. By projecting to the first element of the triple $\mathrm{Mot}_M(N, N)$ is the subset of paths $f \in (\mathbb{I}, \mathbf{Top}^h(M, M)) = (\mathbb{I}, \mathrm{Homeo}_M(\varnothing, \varnothing))$ such that $f_0 = \mathrm{id}_M$ and $f_1 \in \mathrm{Homeo}_M(N, N)$, up to relative path equivalence. This is precisely the definition of the relative fundamental set $\pi_1(\mathrm{Homeo}_M(\varnothing, \varnothing), \mathrm{Homeo}_M(N, N), \mathrm{id}_M)$. □

Notation: Due to the fact the two equivalences coincide on the sets we are interested in we will use $[\gamma]_{\mathrm{rp}}$ for the equivalence class of a continuous map γ in some relative homotopy set.

Lemma 4.6.7. *Let* X *be a topological space,* $A \subseteq$ *a subset and* $x_0 \in A$ *a point. For* $n \geq 2$, *given continuous maps* $\beta\colon (\mathbb{I}^n, \partial\mathbb{I}^n, J^{n-1}) \to (X, A, \{x_0\})$ *and* $\gamma\colon (\mathbb{I}^n, \partial\mathbb{I}^n, J^{n-1}) \to (X, A, \{x_0\})$, *Define*

$$(\gamma + \beta)(t_1, \ldots, t_n) = \begin{cases} \beta(2t_1, \ldots, t_n) & 0 \leq t_1 \leq \frac{1}{2}, \\ \gamma(2(t_1 - \frac{1}{2}), \ldots, t_n) & \frac{1}{2} \leq t_1 \leq 1. \end{cases}$$

Then there is a composition

$$\begin{aligned} +\colon \pi_n(X, A, \{x_0\}) \times \pi_n(X, A, \{x_0\}) &\to \pi_n(X, A, \{x_0\}) \\ (\gamma, \beta) &\mapsto \gamma + \beta. \end{aligned}$$

Proof. Note first that the two functions agree at $t_1 = 1/2$ as J^{n-1} is sent to x_0 under both α and β, hence $\alpha + \beta$ is continuous.

We now check the composition is well defined. Suppose $\beta, \beta'\colon (\mathbb{I}^n, \partial\mathbb{I}^n, J^{n-1}) \to (X, A, x_0)$ are equivalent in $\pi(X, A, \{x_0\})$ via some homotopy, say H_1. Similarly suppose γ and γ' are maps $(\mathbb{I}^n, \partial\mathbb{I}^n, J^{n-1}) \to (X, A, x_0)$ which are equivalent in $\pi(X, A, \{x_0\})$ via some homotopy, say H_2. Consider the function

$$H(t_1,\ldots,t_n,s) = \begin{cases} H_1(2t_1,\ldots,t_n,s) & 0 \le t_1 \le \frac{1}{2} \\ H_2(2(t_1-\frac{1}{2}),\ldots,t_n,s) & \frac{1}{2} \le t_1 \le 1. \end{cases}$$

Notice that since H_1 and H_2 are both relative path homotopies $H_1(J^{n-1}\times\{s\}) = H_2(J^{n-1}\times\{s\}) = \{x_0\}$ and the two component functions agree on $t = 1/2$. Hence H is a relative path-homotopy from $\gamma+\beta$ to $\gamma'+\beta'$. □

Lemma 4.6.8. *Let X be a topological space, $A \subseteq X$ a subset and $x_0 \in A$ a point. For $n \ge 2$ the set $\pi_n(X,A,\{x_0\})$ becomes a group with $+$. The identity is the equivalence class of the constant path $e_x(t) = \{x_0\}$. The inverse of $[\gamma]_{rp} \in \pi_n(X,A,\{x_0\})$ is the equivalence class of $(t_1,\ldots,t_n) \mapsto \gamma(1-t_1,\ldots,t_n)$.*

Proof. See [Hat02, Sec. 4.1]. □

Theorem 4.6.9. *(See for example [Hat02, Sec. 4.1].) Let $i\colon(A,\{x_0\}) \to (X,\{x_0\})$ and $j\colon(X,\{x_0\},\{x_0\}) \to (X,A,\{x_0\})$ be the inclusions. Then we define*

$$i_*^n\colon \pi_n(A,\{x_0\}) \to \pi_n(X,\{x_0\})$$
$$[\gamma]_p \mapsto [i\circ\gamma]_p$$

and

$$j_*^n\colon \pi_n(X,\{x_0\}) \to \pi_n(X,A,\{x_0\})$$
$$[\gamma]_p \mapsto [j\circ\gamma]_{rp}.$$

We also define a map which is the following restriction:

$$\partial^n\colon \pi_n(X,A,\{x_0\}) \to \pi_{n-1}(A,\{x_0\})$$
$$[\gamma]_{rp} \mapsto [\gamma|_{\mathbb{I}^{n-1}}]_p.$$

Note in particular that, for $n = 1$ we have

$$\partial^1 \colon \pi_1(X, A, \{x_0\}) \to \pi_0(A, \{x_0\}),$$
$$[\gamma]_{rp} \mapsto [\gamma(1)]_p.$$

Let X be a space, $A \subseteq X$ a subspace and $x_0 \in A$ a basepoint. There is a long exact sequence:

$$\ldots \to \pi_n(A, \{x_0\}) \xrightarrow{i_*^n} \pi_n(X, \{x_0\}) \xrightarrow{j_*^n} \pi_n(X, A, \{x_0\}) \xrightarrow{\partial^n} \pi_{n-1}(A, \{x_0\})$$
$$\xrightarrow{i_*^{n-1}} \ldots \xrightarrow{i_*^0} \pi_0(X, \{x_0\})$$

where exactness at the end of the sequence, where group structures are not defined, means the image of one map is equal to the set of maps sent to the homotopy class of the identity by the next. □

We note that the following long exact sequence generalises the sequence that appears in [Gol81].

Lemma 4.6.10. *Let M be a manifold and fix a subset $N \subseteq M$. Then we have a long exact sequence*

$$\ldots \to \pi_n(\mathrm{Homeo}_M(N, N), \mathrm{id}_M) \xrightarrow{i_*^n} \pi_n(\mathrm{Homeo}_M(\varnothing, \varnothing), \mathrm{id}_M) \xrightarrow{j_*^n}$$
$$\pi_n(\mathrm{Homeo}_M(\varnothing, \varnothing), \mathrm{Homeo}_M(N, N), \mathrm{id}_M) \xrightarrow{\partial^n} \pi_{n-1}(\mathrm{Homeo}_M(N, N), \mathrm{id}_M) \xrightarrow{i_*^{n-1}}$$
$$\ldots \xrightarrow{\partial^2} \pi_1(\mathrm{Homeo}_M(N, N), \mathrm{id}_M) \xrightarrow{i_*^1} \pi_1(\mathrm{Homeo}_M(\varnothing, \varnothing), \mathrm{id}_M)$$
$$\xrightarrow{j_*^1} \mathrm{Mot}_M(N, N) \xrightarrow{\mathsf{F}} \mathrm{MCG}_M(N, N) \xrightarrow{i_*^0} \pi_0(\mathrm{Homeo}_M(\varnothing, \varnothing), \mathrm{id}_M)$$

where all maps are group maps and F is the appropriate restriction of the functor defined in Theorem 4.6.1.

Proof. We have from Lemma 4.5.5 that $\mathrm{MCG}_M(N, N) = \pi_0(\mathrm{Homeo}_M(N, N), \mathrm{id}_M)$ and from Lemma 4.6.6 that $\mathrm{Mot}_M(N, N) = \pi_1(\mathrm{Homeo}_M(\varnothing, \varnothing), \mathrm{Homeo}_M(N, N), \mathrm{id}_M)$ as sets. Notice also that, as a set map, $\mathsf{F} \colon \mathrm{Mot}_M(N, N) \to \mathrm{MCG}_M(N, N)$ is precisely ∂^1. Hence by substituting $X = \mathrm{Homeo}_M(\varnothing, \varnothing)$, $A = \mathrm{Homeo}_M(N, N)$ and $x_0 = \mathrm{id}_M$ into Theorem 4.6.9 we get the exact sequence.

We have that F is a group map, as it is the restriction of a functor of groupoids. It remains to show that j_*^1 and i_*^0 become group maps. We check that j_*^1 preserves composition. Let g and f be paths from id_M to id_M in $\mathrm{Homeo}_M(\varnothing,\varnothing)$. Then the gf is a well defined pre-motion and it is precisely the pre-motion $g * f$ as $f_1 = id_M$. Hence we have

$$\begin{aligned} j_*^1([gf]_{\mathrm{p}}) = [gf\colon N \rightsquigarrow N]_{\mathrm{rp}} = [g * f\colon N \rightsquigarrow N]_{\mathrm{rp}} = [g\colon N \rightsquigarrow N]_{\mathrm{rp}} * [f\colon N \rightsquigarrow N]_{\mathrm{rp}} \\ = j_*^1([g]_{\mathrm{p}}) * j_*^1([f]_{\mathrm{p}}). \end{aligned}$$

The composition is in $\mathrm{MCG}_M(N,N)$ is composition of homeomorphisms, hence the composition is the same in the source and target of i_*^0, and i_*^0 is an inclusion. Thus composition is preserved. □

Lemma 4.6.11. *Suppose M is a manifold and fix a subset $N \subseteq M$. Suppose*

- $\pi_1(Homeo_M(\varnothing,\varnothing),\mathrm{id}_M)$ *is trivial, and*
- $\pi_0(Homeo_M(\varnothing,\varnothing),\mathrm{id}_M)$ *is trivial.*

Then there is a group isomorphism

$$\mathrm{F}\colon \mathrm{Mot}_M(N,N) \xrightarrow{\sim} \mathrm{MCG}_M(N,N).$$

Proof. Using the conditions of the lemma, the long exact sequence in Lemma 4.6.10 gives short exact sequence

$$1 \to Mot_M(N,N) \to MCG_M(N,N) \to 1.$$ □

4.6.2 Isomorphism from Mot_M^A to MCG_M^A

Here we give conditions under which the motion groupoid and the mapping class groupoid of a manifold are isomorphic categories.

Theorem 4.6.12. *Let M be a manifold. If*

- $\pi_1(Homeo_M(\varnothing,\varnothing),\mathrm{id}_M)$ *is trivial, and*
- $\pi_0(Homeo_M(\varnothing,\varnothing),\mathrm{id}_M)$ *is trivial,*

the functor

$$\mathsf{F}\colon Mot_M \to MCG_M,$$

defined in Theorem 4.6.1 is an isomorphism of categories.

Proof. Suppose $\pi_1(Homeo_M(\varnothing,\varnothing),\mathrm{id}_M)$ and $\pi_0(Homeo_M(\varnothing,\varnothing),\mathrm{id}_M)$ are trivial. We have from Lemma 4.6.2 that F is full. We check F is faithful. Let $[f\colon N \rightsquigarrow N']_{\mathrm{m}}$ and $[f'\colon N \rightsquigarrow N']_{\mathrm{m}}$ be in $\mathrm{Mot}_M(N,N')$. If

$$\mathsf{F}\left([f\colon N \rightsquigarrow N']_{\mathrm{m}}\right) = \mathsf{F}\left([f'\colon N \rightsquigarrow N']_{\mathrm{m}}\right),$$

then

$$\begin{aligned}[\mathrm{id}_M\colon N \rightharpoonup N]_{\mathrm{i}} = \mathsf{F}([f'\colon N \rightsquigarrow N']_{\mathrm{m}})^{-1} \circ \mathsf{F}([f\colon N \rightsquigarrow N']_{\mathrm{m}}) &= \mathsf{F}([f'\colon N \rightsquigarrow N']_{\mathrm{m}}^{-1} * [f\colon N \rightsquigarrow N']_{\mathrm{m}}) \\ &= \mathsf{F}([\bar{f}' * f\colon N \rightsquigarrow N]_{\mathrm{m}}).\end{aligned}$$

By Lemma 4.6.11 this is true if and only if

$$[\bar{f}' * f\colon N \rightsquigarrow N]_{\mathrm{m}} = [\mathrm{Id}_M\colon N \rightsquigarrow N]_{\mathrm{m}}$$

which is equivalent to saying $\mathrm{Id}_M * (\bar{f}' * f)$ is path-equivalent to a stationary motion, and hence that $\bar{f}' * f$ is path-equivalent to the stationary motion (since $\mathrm{Id}_M * (\bar{f}' * f) \overset{p}{\sim} \bar{f}' * f$). So we have $[f\colon N \rightsquigarrow N']_{\mathrm{m}} = [f'\colon N \rightsquigarrow N']_{\mathrm{m}}$. □

Remark 4.6.1. We note that it is possible for the functor F to restrict to a faithful functor on some subsets and not on others. See Example 3 in the next section.

We give a subset fixing version of the previous theorem.

Theorem 4.6.13. *Let M be a manifold. If*

- $\pi_1(Homeo_M^A(\varnothing,\varnothing),\mathrm{id}_M)$ *is trivial, and*
- $\pi_0(Homeo_M^A(\varnothing,\varnothing),\mathrm{id}_M)$ *is trivial*

the functor

$$\mathsf{F}\colon Mot_M^A \to MCG_M^A,$$

where F *defined analogously to Theorem 4.6.1, is an isomorphism of categories.*

Proof. The proof proceeds exactly as for the previous theorem. □

4.6.3 Examples using long exact sequence

Here we give examples of M for which F is an isomorphism, and examples for which it is not. Even when we do not have a category isomorphism, the long exact sequence of Lemma 4.6.10 will often be useful to obtain results about motion groupoids from results about mapping class groupoids.

Example 1: the disk D^m.

Proposition 4.6.14. *Let* D^2 *be the* 2*-disk as defined in Definition 5.3.6. Then we have an isomorphism*

$$\mathsf{F}\colon \mathrm{Mot}_{D^2}^{\partial D^2} \to \mathrm{MCG}_{D^2}^{\partial D^2}$$

with F *as in Theorem 4.6.1.*

Proof. We proved in Proposition 4.5.8 that $\mathrm{MCG}_{D^2}^{\partial D}(\varnothing,\varnothing) = \pi_0(\mathrm{Homeo}_{D^2}^{\partial D^2}(\varnothing,\varnothing),\mathrm{id}_M)$ is trivial. Also $\mathrm{Homeo}_{D^2}^{\partial D^2}(\varnothing,\varnothing)$ is contractible, see e.g. Theorem 1.1.3.2 of [Ham74]. Hence by Theorem 4.6.12 we have the result. □

Remark 4.6.2. As we recalled in Section 4.5.2, if K is a set with n-elements in the interior of D^2, then the morphism group $\mathrm{MCG}_{D^2}^{\partial D^2}(K,K)$ is isomorphic to the braid group in n strands. Hence the previous proposition implies that the group $\mathrm{Mot}_{D^2}^{\partial D^2}(K,K)$ is isomorphic to the braid group in n-strands. This isomorphism was (from what we know) first noticed in [Dah36; Gol81].

Remark 4.6.3. In fact, letting D^m be the m-dimensional disk, $Homeo_{D^m}^{\partial D^m}(\varnothing,\varnothing)$ contractible for all m. This follows from the Alexander Trick [Ale23]. Hence the same argument as for the $n = 2$ case proves that we have an isomorphism

$$\mathsf{F}\colon \mathrm{Mot}_{D^3}^{\partial D^3} \to \mathrm{MCG}_{D^3}^{\partial D^3}.$$

If L is an unlinked set of n loops in D^3, then this means that the loop braid group [Dam17; BWC+07] can either be defined as $\mathrm{MCG}_{D^3}^{\partial D^3}(L,L)$ or as $\mathrm{Mot}_{D^3}^{\partial D^3}(L,L)$. This latter isomorphism was also mentioned in [Dam17; Gol81].

We say a few words about what happens if we do not fix the boundary of the disk in the mapping class groupoid as we think it adds some nice intuition. Let $P_2 \subset D^2$ be a subset consisting of two points equidistant from the centre of the disk. Let $\tau_{2\pi}$ be the path in $\mathbf{TOP}^h(D^2, D^2)$ such that $\tau_{2\pi t}$ is a $2\pi t$ rotation of the disk.

The motion $\tau_\pi \colon P_2 \rightsquigarrow P_2$ represents a non-trivial equivalence class in Mot_{D^2}, and its end point also represents a non trivial element of MCG_{D^2}. Now consider the motion $\tau_\pi * \tau_\pi \colon P_2 \rightsquigarrow P_2$. It is intuitively clear this motion is non-trivial in Mot_{D^2} by considering the as its image as a homeomorphism $D^2 \times \mathbb{I} \to D^2 \times \mathbb{I}$, see Figure 4.14. A proof follows from the fact that the worldlines of the trajectory of the points in P_2 transcribe a non-trivial braid. However its endpoint is a 2π rotation, which clearly represents $[\mathrm{id}_{D^2} \colon P^2 \rightsquigarrow P^2]_{\mathrm{i}}$ in MCG_{D^2}.

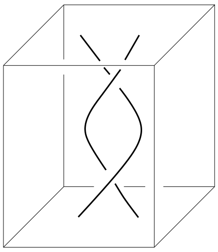


Figure 4.14: Movement of two points during motion $\tau_\pi * \tau_\pi \colon P_2 \rightsquigarrow P_2$ (see text), mapped into $\mathrm{Mot}_{\mathbb{I}^2}$, and represented as the image of a homeomorphism $\mathbb{I}^3 \to \mathbb{I}^3$.

In fact, the map $\mathsf{F} \colon \mathrm{Mot}_{D^2} \to \mathrm{MCG}_{D^2}$ is neither full nor faithful. The space Homeo_{D^2} is homotopy equivalent to $S^1 \sqcup S^1$, where the first connected component corresponds to orientation preserving homeomorphisms and the second orientation reversing (see Section 1.1 of [Ham74]). Hence we have that $\pi_1(Homeo_{D^2}(\varnothing, \varnothing), \mathrm{id}_{D^2}) = \mathbb{Z}$ where the single generating element corresponds to the 2π rotation. And $\pi_0(Homeo_{D^2}(\varnothing, \varnothing), \mathrm{id}_{D^2}) = \mathbb{Z}/2\mathbb{Z}$. So we have an exact sequence:

$$\ldots \to \pi_1(\mathrm{Homeo}_{D^2}(N,N), \mathrm{id}_{D^2}) \xrightarrow{i^1_*} \mathbb{Z} \to Mot_{D^2}(N,N) \to MCG_{D^2}(N,N) \to \mathbb{Z}/2\mathbb{Z}.$$

Remark 4.6.4. Using again that $Homeo_{D^m}^{\partial D^m}(\varnothing, \varnothing)$ contractible for all m we have an isomorphism

$$\mathsf{F} \colon \mathrm{Mot}_{\mathbb{I}}^{\{0,1\}} \to \mathrm{MCG}_{\mathbb{I}}^{\{0,1\}}$$

and since all motions in $\mathbb{I}$ are boundary fixing, an isomorphism

$$\mathsf{F}\colon \mathrm{Mot}_{\mathbb{I}} \to \mathrm{MCG}_{\mathbb{I}}^{\{0,1\}}.$$

All mapping classes considered in Example 4.5.9 are boundary fixing, thus the isomorphism implies $\mathrm{Mot}_{\mathbb{I}}(N,N)$ where $N = \mathbb{Q} \cap (0,1)$ is uncountably infinite.

Example 2: the 1-circle S^1. The unit circle S^1 is an example of very simple manifold with different motion and mapping class groupoids.

Let $P \subset S^1$ be a subset containing a single point in S^1. Similarly to the disk, there is a non-trivial morphism in $\mathrm{Mot}_{S^1}(P,P)$ represented by a 2π rotation of the circle, see Figure 4.15.

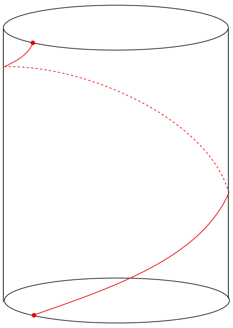


Figure 4.15: Example of motion of circle which is a 2π rotation carrying a point to itself.

We can prove this using the long exact sequence: Note that the connected component containing id_{S^1} of $\mathrm{Homeo}_{S^1}(P,P)$ is contractible, see section 1 of [Ham74]. In particular $\pi_1(\mathrm{Homeo}_{S^1}(P,P), \mathrm{id}_{S^1})$ is trivial. Also from [Ham74] we have that $S^1 \sqcup S^1$ is a strong deformation retract of $\mathrm{Homeo}_{S^1}(\varnothing,\varnothing)$, with the first copy of S^1 corresponding to oreintation preserving homeomorphisms and the second to orientation reversing. Hence the sequence becomes

$$\ldots \to \{1\} \to \mathbb{Z} \to Mot_{S^1}(P,P) \to MCG_{S^1}(P,P) \to \mathbb{Z}/2\mathbb{Z}.$$

The exact sequence gives an injective map $\mathbb{Z} \cong \pi_1(\mathrm{Homeo}_{S^1}(\varnothing,\varnothing), \mathrm{id}_{S^1}) \to Mot_{S^1}(P,P)$. Explicitly this sends $n \in \mathbb{Z}$ to the equivalence class of the pre-motion tracing a $2n\pi$ rotation of the circle S^1. The space $\mathrm{Homeo}_{S^1}(P,P)$ only has two connected components, consisting of orientations preserving and orientation reversing homeomorphisms of S^1 fixing P, each of which is connected. In particular it follows that the projection map $Mot_{S^1}(P,P) \to MCG_{S^1}(P,P) \cong \mathbb{Z}/2\mathbb{Z}$ is the trivial group map, since its image only contains isotopy equivalence classes of orientation preserving homeomorphisms. Hence the exact sequence becomes:

$$\ldots \to \{1\} \to \mathbb{Z} \xrightarrow{\cong} Mot_{S^1}(P,P) \xrightarrow{0} MCG_{S^1}(P,P) \xrightarrow{\cong} \mathbb{Z}/2\mathbb{Z}.$$

In particular the equivalence class of the 2π rotation of S^1 is non-trivial in $\mathrm{Mot}_{S^1}(P,P)$, even though its image in $MCG_{S^1}(P,P)$ is trivial.

Example 3: the 2-sphere. Let $M = S^2$ and P_2 be a subset containing 2 points in the sphere.

From Section 1.2 of [Ham74] we have the following,

$$\pi_1(\mathrm{Homeo}_{S^2}(P_2,P_2),\mathrm{id}_{S^2}) = \mathbb{Z}$$
$$\pi_1(\mathrm{Homeo}_{S^2}(\varnothing,\varnothing),\mathrm{id}_{S^2}) = \mathbb{Z}/2\mathbb{Z}$$
$$\pi_0(\mathrm{Homeo}_{S^2}(\varnothing,\varnothing),\mathrm{id}_{S^2}) = \mathbb{Z}/2\mathbb{Z}.$$

So the exact sequence becomes

$$\cdots \to \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to \mathrm{Mot}_{S^2}(P_2,P_2) \to \mathrm{MCG}_{S^2}(P_2,P_2) \to \mathbb{Z}/2\mathbb{Z}.$$

Also from [Ham74], the map $\pi_1(\mathrm{Homeo}_{S^2}(P_2,P_2),\mathrm{id}_{S^2}) \to \pi_1(\mathrm{Homeo}_{S^2}(\varnothing,\varnothing),\mathrm{id}_{S^2})$ is surjective, with the non trivial element in $\pi_1(\mathrm{Homeo}_{S^2}(\varnothing,\varnothing),\mathrm{id}_{S^2})$ represented by a path which maps $t \in \mathbb{I}$ to a $2\pi t$ rotation about some chosen axis. Hence the map $\mathbb{Z}/2\mathbb{Z} \to \mathrm{Mot}_{S^2}(P_2,P_2)$ is the zero map, and the same rotation is trivial in $\mathrm{Mot}_{S^2}(P_2,P_2)$.

This can be seen directly by choosing the points to be antipodal, say the north and south pole. Now consider a 2π rotation with axis through north and south pole. This is a path fixing both points, hence a stationary path which is equivalent to the identity.

Looking back at the exact sequence, we have that the map $\mathrm{Mot}_{S^2}(P_2,P_2) \to \mathrm{MCG}_{S^2}(P_2,P_2)$ is injective. From pg.50 of [FM11] we have that the subgroup of $\mathrm{MCG}_{S^2}(P_2,P_2)$ of orientation preserving mapping classes is isomorphic to $\mathbb{Z}/2\mathbb{Z}$. If $\mathfrak{f}$ and $\mathfrak{g}$ are orientation preserving, then $\mathfrak{g}^{-1} \circ \mathfrak{f}$ is orientation reversing, thus $\mathrm{Homeo}_{S^2}(P_2,P_2)$ has two isomorphic connected components, corresponding to orientation reversing and orientation preserving homeomorphisms. Thus we have that $\mathrm{MCG}_{S^2}(P_2, P_2) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. The non trivial element in the first copy of $\mathbb{Z}/2\mathbb{Z}$ is represented by a self-homeomorphism which swaps the points by an orientation preserving self-homeomorphism, and the non trivial element in the second component is represented by a self-homeomorphism swapping the two points with is orientation reversing. Hence a motion which swaps the two points represents a non trivial morphism in $\mathrm{Mot}_{S^2}(P_2, P_2)$.

Let $\mathrm{MCG}^+_{S^2}$ be the mapping class groupoid constructed using only orientation preserving homeomorphisms. Then we have a group isomorphism

$$\mathrm{Mot}_{S^2}(P_2, P_2) \simeq \mathrm{MCG}^+_{S^2}(P_2, P_2).$$

Note this does not extend to a category isomorphism. Considering instead the subset consisting of three points the groups are non isomorphic. Intuitively we can see this by arguing that we cannot place three points on the sphere such that any 2π rotation is a stationary motion. But as with the previous examples a 2π rotation of the sphere represents the identity morphism in the mapping class groupoid.

Chapter 5

Topological quantum field theories for homotopy cobordisms

5.1 Introduction

Our motivating aim here is the construction of representations of embedded cobordism categories, that is functors from some category of embedded cobordisms into the category $\mathbf{Vect}_{\mathbb{C}}$ of complex vector spaces and linear maps.

We discussed embedded cobordisms in Section 1.1.4, and directed the reader to various references giving detailed constructions. Thus here, rather than *specifying* a choice of embedded cobordism category, we will simply use EmbCob to denote some choice. We aim to give a construction sufficiently general that it does not depend on the particular choice.

We are interested in particular in functors from EmbCob to $\mathbf{Vect}_{\mathbb{C}}$ which are defined in terms of the homotopy of the complement of the embedding inside the ambient manifold. The non-embedded TQFTs of [Yet92; Kit03] and an untwisted version of [DW90] can all be shown to assign to a space Σ the vector space with basis homotopy classes of maps $\pi(\Sigma) \to G$. Each of these examples are generalised by [Qui95], which gives a class of TQFTs constructed using the 'homotopy content' of the manifolds involved, an invariant calculated using all homotopy groups. The approach of looking at the homotopy of the complement is taken in certain invariants of knots [CF63], Artin's representation of braids

[Bir16] and its lift to loop braids [DMM21].

Such a functor may factor through other categories. In some cases, these categories will be more convenient to work with. We will find it useful to study such functors by thinking about an intermediate category HomCob such that we have a composition

$$\mathsf{F}\colon \mathrm{EmbCob} \to \mathrm{HomCob} \to \mathbf{Vect}_{\mathbb{C}}$$

where F is a TQFT for EmbCob. Here the idea is that HomCob is some topologically defined category, but one which forgets some of the information contained in the category EmbCob, information that will not be seen by F, smooth structure, for example. We construct functors HomCob to $\mathbf{Vect}_{\mathbb{C}}$, and these can be turned into TQFTs by fixing a choice of embedded cobordism category. Here we construct the category HomCob and give family of functors HomCob to $\mathbf{Vect}_{\mathbb{C}}$, one associated to each finite group.

A concrete (embedded) cobordism can be seen as a cospan, that is, a diagram of shape $i\colon X \to M \leftarrow Y : j$, considered as a kind of morphism from X to Y. Here M represents some spacetime evolution, X the initial state of the system, and Y the final state, and i and j are maps from X and Y respectively into M. The category HomCob will be constructed in terms of concrete morphisms which are cospans of topological spaces. There is then a map from a concrete non-embedded cobordism by forgetting the smooth manifold structure, and from embedded cobordisms by taking the complement of the embedded manifold. Note that working with topological spaces means that we do not have to check the complement is itself a manifold, or can be turned into a manifold. Thus our construction is simplified and allows for greater generality in terms of the particle types and evolutions we can consider. Our first objective is to construct a category HomCob using cospans which will be useful for constructing homotopy invariant topological quantum field theories.

The ‘natural’ formalism for such constructions depends on one’s perspective, i.e. upon one’s aims. For example we have the categorical/‘join’ perspective following Benabou [Bén67]. One of Benabou’s archetypes is the bicategory $\mathrm{Sp}(V)$ of spans over a category V with pullbacks and a distinguished choice of pullback for each span. And a ‘dual’, $\mathrm{Cosp}(V)$ of cospans over a category with pushouts and choices. But this comes at a cost of inducing categories with properties that are undesirable in our setting, in particular the choice of identity cospan and failure of homotopy invariance. If one follows this line then a fix (<u>the</u> fix, essentially tautologically) is some form of 'collaring', which in this context means conditions on the maps in the cospans. This is the approach of [Mor09; Gra07]. These can be compared with [Fon15], for example, whose decorated cospans do not include a collaring.

Physically this collaring is the same as the conditions on the way we are allowed to make 'cuts' discussed in Section 1.1.4, and will be depend on the field theory. In the category of cobordisms, concrete morphisms are cospans $i{:}X \to M \leftarrow Y : j$ of smooth manifolds X, Y and M, with the condition that the map obtained using the universal property of the colimit, $\langle i, j\rangle{:} X \sqcup Y \to \partial M$, is a diffeomorphism. The axioms of TQFT give that, for a manifold X, the evolution $X \times \mathbb{I}$ and thus the cospan $\iota_0{:}X \to X \times \mathbb{I} \leftarrow X : \iota_1$ should be an identity. Thus to obtain a category of cobordisms, it is necessary to take cospans up to a notion of diffeomorphism of cospans. Notice that the equivalence required to obtain a category was essentially forced by the collaring. Equivalence up to diffeomorphism is also forced by issues with the smooth structure of a pushout of smooth manifolds, but this will not be so informative for our purposes.

Here we introduce cospans of topological spaces with the condition that the map obtained from the universal property of the coproduct is a closed cofibration. (Note these are cofibrations in the Strøm model structure on topological spaces (see [DS95]), and many of our results could alternatively be proved using tools from model categories.) Pushouts of cofibrations behave well with respect to homotopy, this is shown by a version of the van Kampen theorem due to Brown [Bro06], here it is Theorem 5.2.17. In this case the appropriate equivalence relation, to ensure the equivalence class of $\iota_0{:}X \to X \times \mathbb{I} \leftarrow X : \iota_1$ is an identity, is a notion of homotopy equivalence of cospans. We note that composing cospans of topological spaces via pushouts gives a magmoid; the composition itself does not necessitate an equivalence as in the case of smooth manifolds. We still have to choose an element of the diffeomorphism class of the pushout, but we can do this in a global way, unlike for smooth manifolds.

We also construct a functor into **Vect** from the category HomCob of *homotopy cobordisms*. The construction of our functor is largely based on the approach taken in [Yet92], although Yetter follows most of the construction with triangulations, whereas we work with the fundamental groupoid. The key novel part of our construction with respect to [Yet92] is our choice of source category.

5.1.1 Chapter Overview

In Section 5.2 we recall the definition of a cofibration, as well as some properties that we will make use of. We also have Corollary 5.2.18 which is a corollary of a version of the van Kampen theorem using cofibrations, due to [Bro06]. Our construction relies on this result.

In Section 5.3, we begin by constructing a magmoid whose objects are topological spaces and whose morphisms are *concrete cofibrant cospans*, which are cospans with some conditions. We then quotient by a congruence in terms of homotopy equivalences to obtain a category CofCsp which has *cofibrant cospans* as morphisms, this is Theorem 5.3.16. We then have Theorem 5.3.21 which proves there is a monoidal structure on CofCsp with monoidal product which, on objects, is given by disjoint union. Next we have the category HomCob (Theorem 5.3.32) which is a subcategory of CofCsp with a finiteness condition on spaces. In Theorem 5.3.34 we have a version with the monoidal structure from CofCsp.

We begin Section 5.4 by constructing, in Lemma 5.4.5, another magmoid which has as morphisms cospans of pairs of a topological space and a subset of basepoints, we call this $\mathsf{bHomCob}$. We then construct a magmoid morphism $\mathsf{Z}^!_G\colon \mathsf{bHomCob} \to \mathbf{Vect}_{\mathbb{C}}$ (Lemma 5.4.11), which depends on a finite group G. Under $\mathsf{Z}^!_G$, pairs (X, X_0) are mapped to the vector space with basis the set of maps $\pi(X, X_0)$ to G. We also give some examples. We then take a colimit over a diagram whose vertices are all allowed sets of basepoints. This leads to a map $\mathsf{Z}_G\colon Ob(\mathrm{HomCob}) \to Ob(\mathbf{Vect}_{\mathbb{C}})$, given in Definition 5.4.20. We then extend this to morphisms so we have, in Lemma 5.4.23, a magmoid morphism $\mathsf{Z}_G\colon \mathrm{HomCob} \to \mathbf{Vect}_{\mathbb{C}}$. In Theorem 5.4.24 we prove equivalence is preserved and thus we have a functor $\mathsf{Z}_G\colon \mathrm{HomCob} \to \mathbf{Vect}_{\mathbb{C}}$. In general the colimit construction is a global equivalence relation on an uncountably infinite set. Theorem 5.4.27 gives an alternative interpretation of $\mathsf{Z}_G(X)$, as the vector space with basis $\{f\colon \pi(X, X_0) \to G\}/\cong$ for some choice of basepoints X_0, where $\cong$ denotes functors up to natural transformation. This gives Z_G in terms of a local equivalence relation on a finite set. This result makes explicit calculation possible. It also follows that as a functor from Cob_n, the (non-embedded) cobordism category, our functor is an untwisted version of Dijkgraaf-Witten [DW90]. Finally we have Theorem 5.4.27 which says, for $X \in Ob(\text{HomCob})$, $\mathsf{Z}_G(X)$ is isomorphic to the vector space with basis $\{\pi(X) \to G\} / \cong$, hence our alternative construction leads to the same map on objects as in [Mor09].

5.2 Cofibrations in Top and a van Kampen theorem

In Section 5.3 we define a magmoid whose morphisms are cospans of cofibrations, and quotient by a congruence in terms of *cofibre homotopy equivalence*. Then, in Section 5.4, our TQFT construction will rely on a version of the van Kampen Theorem for cofibrations. Here we recall the results we will need. More detail on cofibrations can be found in [Die08, Ch. 5] or [Bro06, Ch. 7], the version of the van Kampen theorem reproduced here can be found in [Bro06, Thm. 9.1.2], and see [May99, Ch. 6] for cofibre homotopy equivalence.

5.2.1 Cofibrations

A cofibration can be thought of as a homotopically well behaved embedding. Specifically, an embedding $i: A \to X$ is a cofibration if we have both, that there is an open neighbourhood of the image which strongly deformation retracts onto $i(A)$, and that this retraction can be extended to a homotopy on the whole of X. In other words there is an open neighbourhood of the image which, up to a homotopy of X, is equivalent to the image. This characterisation of a cofibration is not immediately obvious from the below definition but we will see it is equivalent in Theorem 5.2.8.

One can think of the previous characterisation of a cofibration as a version of the Collar neighbourhood Theorem of a boundary of a manifold. To construct cobordism categories (see [Mil65]) the collar neighbourhood is required to prove the identity axiom. The cofibration condition we impose will play a similar role in our category construction, see Theorem 5.3.16.

The following definition is from [Die08, Sec. 5.1].

Definition 5.2.1. Let A and X be spaces. A map $i: A \to X$ has the homotopy extension property, with respect to the space Y, if for each homotopy $h: A \times \mathbb{I} \to Y$ and each map $f: X \to Y$ with $(f \circ i)(a) = h(a, 0)$ there exists a homotopy $H : X \times \mathbb{I} \to Y$, extending h, with $H(x,0) = f(x)$ and $H(i(a),t) = h(a,t)$. This is illustrated by the following diagram.

$$\begin{tikzcd}
 & X \arrow[dr, "\iota_0^X"'] \arrow[rrd, bend left, "f"] & & \\
A \arrow[ur, "i"] \arrow[dr, "\iota_0^A"'] & & X \times \mathbb{I} \arrow[r, dashed, "\exists H"] & Y \\
 & A \times \mathbb{I} \arrow[ur, "i \times \mathrm{id}_{\mathbb{I}}"] \arrow[rru, bend right, "h"'] & &
\end{tikzcd}$$

(Where for any space X, $\iota_0^X \colon X \to X \times \mathbb{I}$ is the map $x \mapsto (x,0)$.)

Definition 5.2.2. Let A and X be spaces. We say that $i \colon A \to X$ is a cofibration if i satisfies the homotopy extension property for all spaces Y. A closed cofibration is a cofibration with image a closed set. If $A \subseteq X$ is a subspace and the inclusion $\iota \colon A \to X$ is a cofibration, we say (X,A) is a cofibred pair.

Remark 5.2.1. Therefore a map $i \colon A \to X$ is a cofibration if and only if the following square

$$\begin{tikzcd}
 & X \arrow[dr, "\iota_0^X"] & \\
A \arrow[ur, "i"] \arrow[dr, "\iota_0^A"'] & & X \times \mathbb{I} \\
 & A \times \mathbb{I} \arrow[ur, "i \times \mathrm{id}_{\mathbb{I}}"'] &
\end{tikzcd}$$

is a *weak pushout*: a pushout without the uniqueness condition.

The following are well known results.

Lemma 5.2.3. *The composition of two cofibrations is a cofibration.*

Proof. Let A, X and Y be spaces and suppose $i \colon A \to X$ and $j \colon X \to Y$ are cofibrations. Let K be any space and $f \colon Y \to K$ and $h \colon A \times \mathbb{I} \to K$ be maps with $h(a,0) = f \circ i \circ j(a)$ for all $a \in A$. Consider the following diagram:

$$\begin{tikzcd}
 & A \times \mathbb{I} \arrow[dr, "i \times \mathrm{id}"] \arrow[rrrdd, bend left, "h"] & & & \\
A \arrow[ur, "i_0^A"] \arrow[dr, "i"] & & X \times \mathbb{I} \arrow[dr, "j \times \mathrm{id}"] \arrow[rrd, dashed, "h'"] & & \\
 & X \arrow[ur, "i_0^X"] \arrow[dr, "j"] & & Y \times \mathbb{I} \arrow[r, dashed, "H"] & K \\
 & & Y \arrow[ur, "i_0^Y"] \arrow[rru, bend right, "f"'] & &
\end{tikzcd}$$

Using that i is a cofibration we can extend the maps h and $f \circ j$ to a map $h' : X \times \mathbb{I} \to Z$. Now using that j is a cofibration we can extend maps f and h' to a map $H : Y \times \mathbb{I} \to Z$, hence $j \circ i : A \to Y$ is a cofibration. ☐

Lemma 5.2.4. *Every homeomorphism is a cofibration.*

Proof. Suppose X and X' are homeomorphic spaces via some homeomorphism $\phi : X \to X'$. Let K be any space, $f : X' \to K$ a map and $h : X \times \mathbb{I} \to K$ a homotopy such that for all $x \in X$ $h(x,0) = f \circ \phi(x)$. Then $h \circ (\phi^{-1} \times \mathrm{id})$ is a homotopy extending f. ☐

Proposition 5.2.5. *Let A and X be spaces and $i : A \to X$ a map which is a homeomorphism onto its image. Then i is a cofibration if and only if $(X, i(A))$ is a cofibred pair.*

Proof. Suppose $i : A \to X$ is a cofibration and K is any space. Consider a map $f : X \to K$ and a homotopy $h : i(A) \times \mathbb{I} \to K$ such that for all $x \in i(A)$, $h(x,0) = f(x)$. Applying the homotopy extension property to f and $h \circ (i \times \mathrm{id})$ gives a homotopy $H : X \times \mathbb{I} \to K$. This same H is also a homotopy extending h, and hence $(X, i(A))$ is a cofibred pair.

Suppose $i : A \to X$ a map which is a homeomorphism onto its image and $(X, i(A))$ is a cofibred pair. From Lemma 5.2.4 we have that the homeomorphism $A \to i(A)$, $a \mapsto i(a)$ is a cofibration and the inclusion $i(A) \to X$ is a cofibration by assumption,. Hence $i : A \to X$ is a composition of cofibrations, and so a cofibration by Lemma 5.2.3. ☐

Theorem 5.2.6. *(See for example [Str67, Th. 1]) Let A and X be spaces. If $i : A \to X$ is a cofibration then i is a homeomorphism onto $i(A)$ with the subspace topology (i.e. i is an embedding).* ☐

To prove specific pairs are cofibred we have the following two classical results.

Proposition 5.2.7. *(See for example [Die08, Prop. 5.1.2]) Let A be a closed subspace of X. The pair (X, A) is cofibred if and only if $X \times \{0\} \cup A \times \mathbb{I}$ is a retract of $X \times \mathbb{I}$.*

(Recall $N \subset M$ is a retract of M if there is a continuous map $r : M \to N$ such that $r(n) = (n)$ for all $n \in N$.) ☐

The following lemma characterises cofibrations as inclusions such that there is a neighbourhood of the image that deformation retracts onto, and hence is homotopy equivalent to the image. This is reminiscent of the Collar neighbourhood Theorem for smooth manifolds.

Theorem 5.2.8. *[Str67, Th. 2] Let A be a closed subspace of space X. Then (X, A) is a cofibred pair if and only if there exists*

i) a neighbourhood $U \subseteq X$ of A and a homotopy $H: U \times \mathbb{I} \to X$ such that for all $t \in \mathbb{I}$, $x \in U$ and $a \in A$, we have $H(x, 0) = x$, $H(a, t) = a$ and $H(x, 1) \in A$, and

ii) a map $\phi: X \to \mathbb{I}$ such that $A = \phi^{-1}(0)$ and $\phi(x) = 1$ for all $x \in X - U$. □

There is also a slightly more general version of the previous theorem (see [Str69, Lem. 4]) which characterises cofibred pairs in a similar way without restricting to closed subspaces. However we will only need the case of closed subspaces and this one will be easier to work with. Observe the following useful lemma about cofibrations and the coproduct in **Top**.

Lemma 5.2.9. *Let X and Y be topological spaces. The map $\iota_1: X \to X \sqcup Y$, $x \mapsto (x, 1)$ is a cofibration.*

Proof. Let K be any space. Suppose we have a homotopy $h: X \times \mathbb{I} \to K$ and map $f: X \sqcup Y \to K$ such that $h(x, 0) = f \circ \iota_1(x)$. We can define a map $H: (X \sqcup Y) \times \mathbb{I} \to K$ commuting with h and f as follows.

$$H((x,i),t) = \begin{cases} h(x,t) & \text{if } i = 1 \\ f(x) & \text{if } i = 2 \end{cases}$$

□

Examples of cofibrations

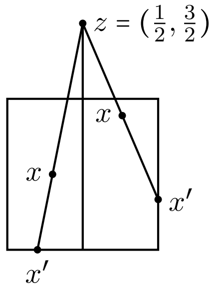
Here we give some key examples of cofibrations that will be useful later.

Example 5.2.10. *For any space X, the pairs (X, X) and $(X, \varnothing)$ are cofibred.*

Example 5.2.11. *The pair $(\mathbb{I}, \{0, 1\})$ is cofibred.*

Consider $\mathbb{I} \times \mathbb{I}$ as a subset of $\mathbb{R}^2$ and let $z = (\frac{1}{2}, \frac{3}{2}) \in \mathbb{R}^2$. For any $x \in \mathbb{I} \times \mathbb{I}$, let x' be the unique point of $\mathbb{I} \times 0 \cup \{0, 1\} \times \mathbb{I}$ such that z, x, x' are colinear. Then $\rho: x \mapsto x'$ is a retraction

$\mathbb{I} \times \mathbb{I}$ *to* $\mathbb{I} \times 0 \cup \{0,1\} \times \mathbb{I}$. *This is illustrated by the following figure.*



Example 5.2.12. *The pair* (D^n, S^{n-1}) *is cofibred.*

A retraction $r\colon D^n \to S^{n-1} \times \mathbb{I} \cup D^n \times 0$ *can be constructed in a similar way to the previous example, see [Die08, Ex. 2.3.5].*

The previous two examples are special cases of the following proposition for manifolds.

Proposition 5.2.13. *Let* M *be a smooth manifold with boundary. The inclusion* $i\colon \partial M \to M$ *is a cofibration.*

Proof. The Collar neighbourhood Theorem [Mil65, Cor. 3.5] says that there is a diffeomorphism $f\colon N \to \partial M \times [0,1)$, where N is a open neighbourhood of ∂M. Specifically we can choose N such that the we can identify the closure of N with $\partial M \times \mathbb{I}$, where $\partial M \cong \partial M \times \{0\}$. Then the function

$$\begin{aligned} H\colon (\partial M \times [0,1)) \times \mathbb{I} &\to M \\ ((n,s),t) &\mapsto (n, s(1-t)) \end{aligned}$$

is a homotopy satisfying condition (i) of Theorem 5.2.8. Define a map $\phi\colon M \to \mathbb{I}$ as follows.

$$\phi(m) = \begin{cases} s & \text{if } m = (n,s) \in \partial M \times [0,1] \\ 1 & \text{if } m \in M \setminus (\partial M \times [0,1)) \end{cases}$$

Notice that these definitions agree on the overlap so ϕ is continuous. Hence by Theorem 5.2.8 the inclusion $i\colon \partial M \to M$ is a cofibration. □

Recall that a smooth submanifold of M is a subset $N \subset M$ such that the identity inclusion $\iota\colon N \to M$ is a diffeomorphism onto its image and a topological embedding, as in [Lee03, Ch. 5]. A submanifold $N \subset M$ is neatly embedded if $\partial N \subset \partial M$.

We have the following stronger proposition.

Proposition 5.2.14. *Let N be a closed smooth submanifold of a smooth manifold M which is neatly embedded. Then the inclusion $\iota\colon N \to M$ is a cofibration.*

Proof. There exists a tubular neighbourhood $U \subset M$ of N [Hir12, Th. 6.3] which has the structure of a vector bundle. This allows us to see points in U as pairs (n, v) where $n \in N$ and v is a vector in the fibre over n. Further, this neighbourhood admits a Reimannian metric [Kos13, Thm. 3.2] and hence we can choose an $\epsilon > 0$ and take U to be the open neighbourhood obtained by taking (n, v) with $|v| < \epsilon$, and $\bar{U} \subset M$ the closed neighbourhood of N obtained by taking all $(n, v) \in U$ with $|v| \leq \epsilon$. Now, as in the previous proposition, we can define a homotopy

$$\begin{aligned} H\colon U \times \mathbb{I} &\to M \\ ((n,v),t) &\mapsto (n,(1-t)v) \end{aligned}$$

and a map $\phi\colon M \to \mathbb{I}$ with

$$\phi(m) = \begin{cases} |v|/\epsilon & \text{if } m = (n,v) \in \bar{U} \\ 1 & \text{if } m \in M \setminus \bar{U}. \end{cases}$$

□

In many cases it will be easier to find a CW complex structure than to prove we have a smooth manifold submanifold pair. In this case we have the following proposition.

Proposition 5.2.15. *Let X be a CW complex and let A be a subcomplex of X. Then the inclusion $i\colon A \to X$ is a closed cofibration.*

Proof. If (X, A) is a CW pair, then $X \times \{0\} \cup A \times \mathbb{I}$ is a deformation retract of $X \times \mathbb{I}$, this is [Hat02, Prop. 0.16]. □

5.2.2 A van Kampen Theorem for cofibrations

The following generalisation of the van Kampen Theorem, due to Brown [Bro06], says that pushouts are preserved by the fundamental groupoid functor if at least one of the maps we take the pushout over is a cofibration. Hence, in this case, we can obtain the fundamental groupoid of a pushout in **Top** as a pushout of groupoids.

Suppose we have spaces X_0, X_1 and X_2 and maps $f\colon X_0 \to X_1$ and $g\colon X_0 \to X_2$. Consider the pushout square:

$$\begin{tikzcd}
X_0 \arrow[r, "f"] \arrow[d, "g"'] & X_1 \arrow[d, "p_1"] \\
X_2 \arrow[r, "p_2"'] & X_1 \sqcup_{X_0} X_2
\end{tikzcd} \tag{5.1}$$

Now let A, B and C be representative subsets of X_0, X_1 and X_2 respectively, with $f(A) = B \cap f(X_0)$ and $g(A) \subseteq C$. Let $D = B \sqcup_A C$, the pushout of $f|_A\colon A \to B$ and $g|_A\colon A \to C$.

Lemma 5.2.16. *Under the above conditions conditions D is representative in $X_1 \sqcup_{X_0} X_2$ (See Definition 3.3.11 for representative).*

Proof. It is clear that $B \sqcup C$ is representative in $X_1 \sqcup X_2$. By Lemma 3.3.12 surjections send representative subsets to representative subsets, hence we have the result by considering the surjection $\langle p_1, p_2\rangle\colon X_1 \sqcup X_2 \to X_1 \sqcup_{X_0} X_2$. □

Theorem 5.2.17. *(See [Bro06, Thm. 9.1.2].) Now suppose in addition to the above conditions we take $X_0 \subseteq X_1$ and $f = \iota\colon X_0 \to X_1$ the inclusion map in (5.1). Then the following diagram is a pushout if (X_1, X_0) is a cofibred pair.*

$$\begin{tikzcd}
\pi(X_0, X_0 \cap B) \arrow[r, "\pi(\iota)"] \arrow[d, "\pi(g)"'] & \pi(X_1, B) \arrow[d, "\pi(p_1)"] \\
\pi(X_2, C) \arrow[r, "\pi(p_2)"'] & \pi(X_1 \sqcup_{X_0} X_2, D)
\end{tikzcd}$$

□

Corollary 5.2.18. *Now suppose in addition to the above conditions, we instead consider $f = i\colon X_0 \to X_1$ any cofibration in (5.1). Then the following square is a pushout.*

$$\begin{tikzcd}
\pi(X_0, A) \arrow[r, "\pi(i)"] \arrow[d, "\pi(g)"'] & \pi(X_1, B) \arrow[d, "\pi(p_1)"] \\
\pi(X_2, C) \arrow[r, "\pi(p_2)"'] & \pi(X_1 \sqcup_{X_0} X_2, D)
\end{tikzcd}$$

Proof. Using Proposition 5.2.5 and Theorem 5.2.6 we can separate the cofibration i into two maps, a homeomorphism $\tilde{i}\colon X_0 \to i(X_0)$ and a cofibred inclusion $\iota\colon i(X_0) \to X_1$. Consider the following commuting pushouts.

$$\begin{tikzcd}
i(X_0) \arrow[rr, "\iota"] \arrow[dd, "g\circ\tilde{i}^{-1}"'] & & X_1 \arrow[dd, "p_1"] \\
& X_0 \arrow[ul, "\tilde{i}"] \arrow[ur, "i"'] \arrow[dl, "g"] & \\
X_2 \arrow[rr, "p_2"'] & & X_1 \sqcup_{X_0} X_2
\end{tikzcd}$$

Notice that the pushout of g and i, and of $g\circ\tilde{i}^{-1}$ and ι, really is the same since $\tilde{i}$ is a homeomorphism. Choosing the subset $i(A)$ in $i(X_0)$ and keeping all other subsets as in the statement of the corollary, the outer pushout is preserved by the fundamental groupoid functor by Theorem 5.2.17. Hence, by functoriality, so is the inner pushout. □

5.2.3 Cofibre homotopy equivalence

We will require a notion of homotopy equivalence of spaces relative to maps from a shared space.

Definition 5.2.19. A space under A is a map $i\colon A\to X$. A map of spaces under A from $i\colon A\to X$ to $j\colon A\to Y$ is a map $f\colon X\to Y$ such that we have a commuting diagram

$$\begin{tikzcd}
& A \arrow[dl, "i"'] \arrow[dr, "j"] & \\
X \arrow[rr, "f"'] & & Y .
\end{tikzcd}$$

Suppose $f'\colon X\to Y$ is map under A from $i\colon A\to X$ to $j\colon A\to Y$. A homotopy under A from f to f' is a continuous map $H\colon X\times\mathbb{I}\to Y$ such that

- for all $a\in A$ and $t\in\mathbb{I}$, $H(i(a),t)=j(a)$,
- for all $x\in X$, $H(x,0)=f(x)$,
- for all $x\in X$, $H(x,1)=f'(x)$.

Proposition 5.2.20. *Define a relation on spaces under A as follows. We have $(i\colon A\to X)\sim(j\colon A\to Y)$ if there exists a map under A, $f\colon X\to Y$ from $i\colon A\to X$ to $j\colon A\to Y$, and a map of spaces under A, $f'\colon Y\to X$ from $j\colon A\to Y$ to $i\colon A\to X$ such that there exists a homotopy under A from $f\circ f'$ to the identity on Y and from $f'\circ f$ to the identity on X.*

This is an equivalence relation.

Proof. Note that $f \circ f'$ is a map under A from $j\colon A \to Y$ to $j\colon A \to Y$ since $f \circ f' \circ j(a) = f \circ i(a) = j(a)$. Similarly $f' \circ f$ is a map under A from $i\colon A \to X$ to $i\colon A \to X$. We omit the rest of the proof as it proceeds exactly as for the usual notion of homotopy equivalence. □

Definition 5.2.21. Given a space A, the equivalence relation described in Proposition 5.2.20 is called cofibre homotopy equivalence.

The following technical result, which justifies the choice of name, will be crucial for our construction. Proofs are present in [May99; Bro06; Str72].

Theorem 5.2.22. *(See for example [Bro06, Thm. 7.2.8].) Let A, X and Y be spaces. Let $i\colon A \to X$ and $j\colon A \to Y$ be cofibrations and let $f\colon X \to Y$ be a map such that $f \circ i = j$. Suppose that f is a homotopy equivalence from X to Y, then f is a cofibre homotopy equivalence $i\colon A \to X$ to $j\colon A \to Y$.* □

5.3 Homotopy cobordisms

In this section the main result is Theorem 5.3.34 which says that we have a symmetric monoidal category, HomCob, of *homotopy cobordisms.*

We proceed by first constructing a magmoid whose morphisms are *concrete cofibrant cospans*, which compose via pushouts. We then quotient by a congruence to obtain the category CofCsp (Theorem 5.3.16) and show that there exists a symmetric monoidal structure on CofCsp (Theorem 5.3.21). We add some finiteness conditions to the topological spaces in cospans to arrive at the category HomCob (Theorem 5.3.32) as a subcategory of CofCsp and show that it becomes a symmetric monoidal category with the same monoidal structure as CofCsp (Theorem 5.3.34).

We note that the congruence we use is chosen with the type of functor we will construct in the next section already in mind. We want our quotient category of CofCsp to be a manageable object to work with, but we don't want to make any morphisms equivalent that might have been mapped to different linear maps by the functor. We have already said that we are interested in functors which depend on the homotopy of the spaces, hence the congruence is a defined in terms of a suitable version of homotopy equivalence.

We note that our cospan categories deviate from those of e.g. [Fon15], in our choice of identity. For Fong, the category identity at an object X is the equivalence class of the cospan $\mathrm{id}_X: X \to X \leftarrow X : \mathrm{id}_X$. In a topological quantum field theory we require that any arbitrary time evolution of a state X is evaluated as the identity if the state X does not change. Hence we insist identities are the equivalence classes of cospans of the form $\iota_0^X : X \to X \times \mathbb{I} \leftarrow X : \iota_1^X$. As a result more work is required to prove that this is, in fact, an identity; see Theorem 5.3.16.

5.3.1 Magmoid of concrete cofibrant cospans CofCsp

Here we define *concrete cofibrant cospans*, construct a composition and organise them into a magmoid.

Definition 5.3.1. Let X, Y and M be spaces. A concrete cofibrant cospan from X to Y is a diagram $i: X \to M \leftarrow Y : j$ such that $\langle i, j\rangle : X \sqcup Y \to M$ is a closed cofibration. (The map $\langle i, j\rangle$ is obtained via the universal property of the coproduct, see Diagram (3.3).) For spaces $X, Y \in \mathbf{Top}$, we define the set of all concrete cofibrations

$$\mathsf{CofCsp}(X,Y) = \left\{ \begin{tikzcd} X \arrow[dr, "i"'] & & Y \arrow[dl, "j"] \\ & M & \end{tikzcd} \middle| \ \langle i, j\rangle \text{ is a closed cofibration} \right\}.$$

Remark 5.3.1. The previous definition forces the images of i and j to be disjoint since a cofibration is a homeomorphism onto its image (Theorem 5.2.6).

Example 5.3.2. *Let X be a space. The cospan $\mathrm{id}_X: X \to X \leftarrow X : \mathrm{id}_X$ is, in general, not a concrete cofibrant cospan. This is clear from the previous remark.*

Physically we expect the identity cospan to be (the equivalence class of) $\iota_0^X : X \to X \times \mathbb{I} \leftarrow X : \iota_1^X$.

Proposition 5.3.3. *For X a topological space, the cospan $\iota_0^X : X \to X \times \mathbb{I} \leftarrow X : \iota_1^X$ is a concrete cofibrant cospan.*

Proof. The complement of the image of $\langle \iota_0^X, \iota_1^X \rangle \colon X \sqcup X \to X \times \mathbb{I}$ is $X \times (0,1)$ which is open, so the image is a closed set.

We now show $\langle \iota_0^X, \iota_1^X \rangle \colon X \sqcup X \to X \times \mathbb{I}$ is a cofibration. Let K be any space and suppose we have a homotopy $h \colon (X \sqcup X) \times \mathbb{I} \to K$. By Theorem 3.5.16 the product with $\mathbb{I}$ preserves colimits, using this together with the universal property of the coproduct, the map $h \colon (X \sqcup X) \times \mathbb{I} \to K$ is uniquely defined by a pair of maps $h_0 \colon X \times \mathbb{I} \to K$ and $h_1 \colon X \times \mathbb{I} \to K$. Now suppose we have a map $f \colon X \times \mathbb{I} \to K$ such that for all $x \in X$ we have $h_0(x,0) = f(x,0)$ and $h_1(x,0) = f(x,1)$. (Notice this implies $h(\tilde{x},0) = f(\langle \iota_0^X, \iota_0^Y \rangle(\tilde{x}))$ for $\tilde{x} \in X \sqcup X$.)

We can construct a homotopy $H \colon (X \times \mathbb{I}) \times \mathbb{I} \to K$ which commutes with h and f as follows. Let $L = \{0,1\} \times \mathbb{I} \cup \mathbb{I} \times \{0\}$ be the subset of the unit square consisting of the two vertical edges and the bottom horizontal edge. Let $\Gamma \colon \mathbb{I} \times \mathbb{I} \to L$ be a retraction sending the unit square to the subset L, see Example 5.2.11. We denote elements of $X \times L \subset (X \times \mathbb{I}) \times \mathbb{I}$ as triples (x,s,t) and define $g \colon X \times L \to K$ as

$$g(x,s,t) = \begin{cases} f(x,s) & t = 0, \\ h_0(x,t) & s = 0, \\ h_1(x,t) & s = 1. \end{cases}$$

By assumption these agree on the overlap and so g is continuous. Now define $H \colon (X \times \mathbb{I}) \times \mathbb{I} \to K$ by $g(x, \Gamma(s,t))$. □

The following definition can be found in e.g. [Lur09], where it is referred to as a bordism.

Definition 5.3.4. An n-dimensional concrete cobordism from an $(n-1)$-dimensional smooth oriented manifold X to an $(n-1)$-dimensional smooth oriented manifold Y, is an n-dimensional smooth oriented manifold M equipped with an orientation preserving diffeomorphism $\phi \colon \bar{X} \sqcup Y \to \partial M$ (where the bar denotes the opposite orientation).

Proposition 5.3.5. *There is a canonical way to map a concrete cofibration to a concrete cofibrant cospan. Precisely, let X, Y and M be smooth oriented manifolds, and let M be a concrete cobordism from X to Y. Hence there exists a diffeomorphism $\phi \colon \bar{X} \sqcup Y \to \partial M$. Define maps $i(x) = \phi(x,0)$ and $j(y) = \phi(y,1)$. Then, using X, Y and M to denote the underlying topological spaces, $i \colon X \to M \leftarrow Y : j$ is a concrete cofibrant cospan.*

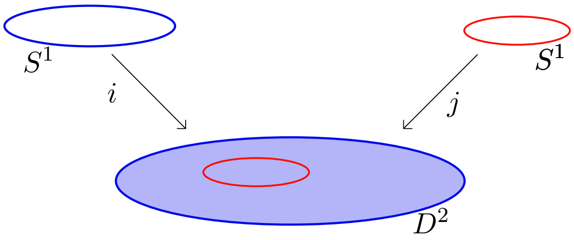


Figure 5.1: Here i is a diffeomorphism from S^1 to the boundary of D^2, and j is a smooth embedding of S^1 into the interior of the disk D^2. We have that $i: S^1 \to D^2 \leftarrow S^1 : j$ is a concrete cofibrant cospan (Proposition 5.3.6).

Proof. The pair $(M, \partial M)$ is cofibred by Proposition 5.2.13. The map $\langle i, j\rangle$ is a homeomorphism onto its image ∂M as ϕ is a diffeomorphism, hence using Proposition 5.2.5 $\langle i, j\rangle$ is a cofibration. The boundary ∂M is closed so $\langle i, j\rangle$ a closed cofibration. □

Proposition 5.3.6. *(See Figure 5.1.) There is a concrete cofibrant cospan* $i: S^1 \to D^2 \leftarrow S^1 : j$ *where* i *is a diffeomorphism sending* S^1 *to the boundary of* D^2*, and* j *is a smooth embedding of the* S^1 *into the interior of* D^2.

Proof. The map $\langle i, j\rangle: S^1 \sqcup S^1 \to D^2$ is the composition of a homeomorphism from $S^1 \sqcup S^1$ to $i(S^1) \sqcup j(S^1)$, and an inclusion $\iota: i(S^1) \sqcup j(S^1) \to D^2$. Proposition 5.2.14 gives that ι is a cofibration, by Proposition 5.2.5 the homeomorphism is a cofibration, and by Proposition 5.2.3 the composition is a cofibration. □

Example 5.3.7. *Consider the manifold* $\mathbb{I}^3$ *and let* M' *be an embedded submanifold as illustrated by the black part of Figure 5.2. Let* $M = \mathbb{I}^3 \setminus M'$, $X = (\mathbb{I}^2 \times \{0\}) \setminus (M \cap (\mathbb{I}^2 \times \{0\}))$ *and* $Y = (\mathbb{I}^2 \times \{1\}) \setminus (M \cap (\mathbb{I}^2 \times \{1\}))$*, i.e.* X *is the complement of* M' *in top boundary in the figure and* Y *the bottom boundary. There is a concrete cofibrant cospan* $i: X \to M \leftarrow Y : j$ *where* i *and* j *are subspace inclusions. We can see this by noticing that there are non-intersecting neighbourhoods of the top and bottom boundary of* M *are homeomorphic to* $X \times [0, \epsilon]$ *and* $Y \times [0, \epsilon']$ *with* $\epsilon, \epsilon' \in \mathbb{R}$*. Thus an* H *and* ϕ *satisfying the conditions of Theorem 5.2.8 can be constructed as in the proof of Proposition 5.2.13.*

Example 5.3.8. *There is a concrete cofibrant cospan as shown in Figure 5.3, and explained in the caption. Proposition 5.2.13 gives that* $\langle i, j\rangle$ *is a cofibration. Notice also that the boundary is a closed subset of* M.

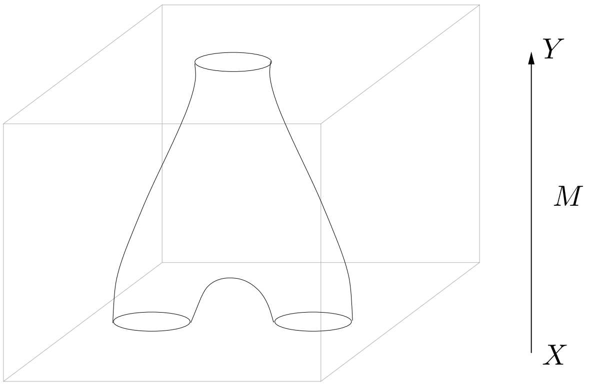


Figure 5.2: Here the grey lines represent the manifold $\mathbb{I}^3$, and the black lines represent an embedded submanifold $M' \subset \mathbb{I}^3$. Let X, be the complement of M' in the bottom boundary, $\mathbb{I}^2 \times \{0\}$, Y the complement in the top boundary, $\mathbb{I}^2 \times \{1\}$, and M the complement in $\mathbb{I}^3$. Then there is a concrete cofibrant cospan $i\colon X \to M \leftarrow Y : j$ where i and j are subspace inclusions.

Lemma 5.3.9. *For any pair $X, Y \in Ob(\mathbf{Top})$ there is a bijection*

$$\begin{tikzcd}[row sep=small, column sep=small]
\text{rev} : & & \mathsf{CofCsp}(X,Y) & & \rightarrow & & \mathsf{CofCsp}(Y,X) & \\
& X \arrow[dr, "i"'] & & Y \arrow[dl, "j"] & \mapsto & Y \arrow[dr, "j"'] & & X \arrow[dl, "i"] \\
& & M & & & & M. &
\end{tikzcd}$$

Proof. We first check rev is well defined. Suppose we have a map $h\colon (Y \sqcup X) \times \mathbb{I} \to K$ and a map $f\colon M \to K$ for any space K which satisfy the conditions of Definition 5.2.1. The map h canonically determines a map $h'\colon (X \sqcup Y) \times \mathbb{I} \to K$. The map $\langle i, j\rangle$ is a cofibration so we can apply the homotopy extension property to give a map $H\colon M \times \mathbb{I} \to K$ which extends f and h'. This H also commutes with f and h.

The image of $\langle j, i\rangle$ is the same as the image of $\langle i, j\rangle$ so it is a closed cofibration.

It is clear that rev is its own inverse, thus it is a bijection. □

Lemma 5.3.10. *If $i\colon X \to M \leftarrow Y : j$ is a concrete cofibrant cospan, then $i\colon X \to M$ and $j\colon Y \to M$ are closed cofibrations.*

Proof. The map $i\colon X \to M$ is equal to the composition $X \xrightarrow{\iota_1} X \sqcup Y \xrightarrow{\langle i,j\rangle} M$. The map ι_1 is a cofibration by Lemma 5.2.9 and the composition of cofibrations is a cofibration by Lemma 5.2.3, hence i is a cofibration.

We now prove that the image of X under the composition is closed in M. Here we use primes to denote images of $\langle i, j\rangle$. The map $\langle i, j\rangle$ is an embedding by Theorem 5.2.6, hence

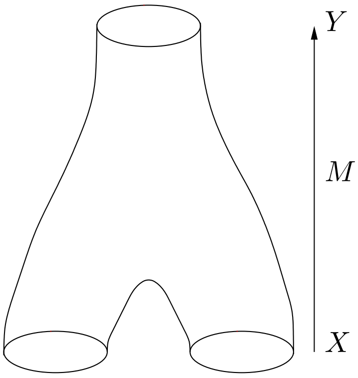


Figure 5.3: Let M be the represented manifold, let X be the bottom boundary and Y the top boundary. Then there is a concrete cofibrant cospan $i{:}X \to M \leftarrow Y :j$ where i and j are subspace inclusions.

a homeomorphism onto its image, and it is straightforward to see that $\iota_1(X)$ is closed in $X \sqcup Y$. Hence there exists an open $U \subseteq M$ with $U \cap (X \sqcup Y)' = (X \sqcup Y)' \smallsetminus \iota_1(X)'$. The image of $X \sqcup Y$ is closed since $\langle i, j\rangle$ is a closed cofibration, so $M \smallsetminus (X \sqcup Y)'$ is an open set. Thus there is an open set $M \smallsetminus (X \sqcup Y)' \cup U = M \smallsetminus \iota_1(X)'$, hence the image of X under $\langle i, j\rangle \circ \iota_1$ is closed.

The same argument gives that j is a closed cofibration. □

Lemma 5.3.11. *(I) For any spaces X,Y and Z in Ob(**Top**) there is a composition of concrete cofibrant cospans*

$$\bullet : \mathsf{CofCsp}(X,Y) \times \mathsf{CofCsp}(Y,Z) \to \mathsf{CofCsp}(X,Z)$$

$$\left(\begin{tikzcd}[column sep=small] X \arrow[dr, "i"'] & & Y \arrow[dl, "j"] \\ & M & \end{tikzcd} , \begin{tikzcd}[column sep=small] Y \arrow[dr, "k"'] & & Z \arrow[dl, "l"] \\ & N & \end{tikzcd} \right) \mapsto \begin{tikzcd}[column sep=small] X \arrow[dr, "\tilde{i}"'] & & Z \arrow[dl, "\tilde{l}"] \\ & M \sqcup_Y N & \end{tikzcd}$$

where $\tilde{i} = p_M \circ i$ and $\tilde{l} = p_N \circ l$ are obtained via the following diagram

$$\begin{tikzcd} X \arrow[dr, "i"'] & & Y \arrow[dl, "j"] \arrow[dr, "k"'] & & Z \arrow[dl, "l"] \\ & M \arrow[dr, "p_M"'] & & N \arrow[dl, "p_N"] & \\ & & M \sqcup_Y N, & & \end{tikzcd}$$

*the middle square of which is the pushout of $j{:}M \leftarrow Y \to N{:}k$ in **Top**.*

(*II*) *Hence there is a magmoid*

$$\mathsf{CofCsp} = (Ob(\mathbf{Top}), \mathsf{CofCsp}(-,-), \bullet).$$

Proof. We need to prove that $\langle \tilde{i}, \tilde{l} \rangle : X \sqcup Z \to M \sqcup_Y N$ is a closed cofibration. We first check the map is closed. The image of $\langle \tilde{i}, \tilde{l} \rangle$ is equal to $p_M(i(X)) \cup p_N(l(Y))$. Sets in $M \sqcup_Y N$ are closed if the preimage under p_M and p_N is closed in M and N respectively. By Proposition 5.2.5, $\langle i, j \rangle$ is a homeomorphism onto its image, hence we have $i(X) \cap j(Y) = \varnothing$. This implies $p_N^{-1}(p_M(i(X))) = \varnothing$, which is closed, and $p_M^{-1}(p_M(i(X))) = i(X)$ is closed by Lemma 5.3.10. Hence $p_M(i(X))$ is closed in $M \sqcup_Y N$. Similarly $p_N(l(Y))$ is closed.

We now check $\langle \tilde{i}, \tilde{l} \rangle$ is cofibration. Define J to be the map obtained by taking either route around the pushout square:

$$\begin{tikzcd}
 & Y \arrow[dl, "j"'] \arrow[dr, "k"] \arrow[dd, "J"] & \\
M \arrow[dr, "p_M"'] & & N \arrow[dl, "p_N"] \\
 & M \sqcup_Y N. &
\end{tikzcd}$$

We will prove that we have a cofibration $\big\langle \langle \tilde{i}, J \rangle, \tilde{l} \big\rangle : (X \sqcup Y) \sqcup Z \to M \sqcup_Y N$, then by Lemmas 5.2.3 and 5.2.9 we then have that the composition

$$X \sqcup Z \longrightarrow (X \sqcup Y) \sqcup Z \xrightarrow{\langle\langle \tilde{i}, J\rangle, \tilde{l}\rangle} M \sqcup_Y N \ ,$$

which is equal to $\langle \tilde{i}, \tilde{l} \rangle$, is a cofibration. Let K be a space and suppose we have maps $f : M \sqcup_Y N \to K$ and $h : ((X \sqcup Y) \sqcup Z) \times \mathbb{I} \to K$ satisfying the conditions of Definition 5.2.1. We construct a map $H : (M \sqcup_Y N) \times \mathbb{I} \to K$ extending f and h as follows. First note that by Theorem 3.5.16, the product with $\mathbb{I}$ preserves coproducts and thus we have canonical isomorphisms, $((X \sqcup Y) \sqcup Z) \times \mathbb{I} \cong ((X \times \mathbb{I}) \sqcup (Y \times \mathbb{I})) \sqcup (Z \times \mathbb{I})$ and $(M \sqcup_Y N) \times \mathbb{I} \cong M \times \mathbb{I} \sqcup_{Y \times \mathbb{I}} N \times \mathbb{I}$. By the universal property of the coproduct we have that the map h is in one to one correspondence with a triple of maps $h_X : X \times \mathbb{I} \to K$, $h_Y : Y \times \mathbb{I} \to K$ and $h_Z : Z \times \mathbb{I} \to K$. Now using the homotopy extension property of $\langle i, j \rangle$ on the maps $\langle h_X, h_Y \rangle$ and the restriction of f to M, we obtain a map $\mathcal{H}_L : M \times \mathbb{I} \to K$. Similarly we obtain a map $\mathcal{H}_R : N \times \mathbb{I} \to K$. These two homotopies agree on the images of $Y \times \mathbb{I}$ by construction so we can use the universal property of the pushout to obtain a map $\langle \mathcal{H}_L, \mathcal{H}_R \rangle : M \times \mathbb{I} \sqcup_{Y \times \mathbb{I}} N \times \mathbb{I} \to K$ which, precomposed with the canonical isomorphism $(M \sqcup_Y N) \times \mathbb{I} \cong M \times \mathbb{I} \sqcup_{Y \times \mathbb{I}} N \times \mathbb{I}$, is a homotopy extending h. □

Proposition 5.3.12. *The magmoid* CofCsp *is reversible.*

Proof. This follows from Proposition 5.3.9. □

5.3.2 Category of cofibrant cospans CofCsp

Notice that the composition in CofCsp is not strictly associative. Here we impose a congruence on concrete cofibrant cospans such that we obtain a category.

One option would be *cospan isomorphism*, by which we mean $i : X \to M \leftarrow Y : j$ is equivalent to $i' : X \to N \leftarrow Y : j'$ if there exists a homeomorphism $M \to N$ which commutes with the cospans. This is a direct analogue of the equivalence usually used for smooth manifold cobordisms in e.g. [Lur09]. This equivalence would be sufficient to give an associative composition. However it will not be sufficient to ensure the cospan $\iota_0^X : X \to X \times \mathbb{I} \leftarrow X : \iota_1^X$ behaves as an identity. (This is the image of a representative of the smooth manifold cobordism identity under the map described in Proposition 5.3.5.) One way to see this is by thinking about the cospan in Example 5.3.6: taking a pushout over S^1 to glue the cylinder $S^1 \times [0,1]$ to the interior of the disk will not give a space homeomorphic to the disk. Hence we use a stronger equivalence relation.

Definition 5.3.13. For each pair $X, Y \in Ob(\mathsf{CofCsp})$, we define a relation on $\mathsf{CofCsp}(X,Y)$ by

$$\left(\begin{tikzcd} X \arrow[dr, "i"'] & & Y \arrow[dl, "j"] \\ & M & \end{tikzcd}\right) \overset{ch}{\sim} \left(\begin{tikzcd} X \arrow[dr, "i'"'] & & Y \arrow[dl, "j'"] \\ & N & \end{tikzcd}\right)$$

if there exists a commuting diagram

$$\begin{tikzcd} & M \arrow[dd, "\psi"] & \\ X \arrow[ur, "i"] \arrow[dr, "i'"'] & & Y \arrow[ul, "j"'] \arrow[dl, "j'"] \\ & M' & \end{tikzcd}$$

where ψ is a homotopy equivalence.

Lemma 5.3.14. *The relation* $\overset{ch}{\sim}$ *an equivalence relation.*

We call the map ψ a cospan homotopy equivalence, and refer to an equivalence class of concrete cofibrant cospans as just a cofibrant cospan, denoted $[i\colon X \to M \leftarrow Y : j]_{\mathrm{ch}}$. Thus we have

$$\mathsf{CofCsp}/\overset{ch}{\sim}(X,Y) = \left\{ \left[\begin{tikzcd} X \arrow[dr, "i"'] & & Y \arrow[dl, "j"] \\ & M & \end{tikzcd} \right]_{\mathrm{ch}} \middle| \ \langle i,j\rangle \text{ is a closed cofibration} \right\}.$$

Proof. We can rewrite this relation in terms of a map $\psi\colon M \to M'$ under $X \sqcup Y$ from $\langle i,j\rangle\colon X \sqcup Y \to M$ to $\langle i',j'\rangle\colon X \sqcup Y \to M'$. Then since the maps $X \sqcup Y \to M$ are defined to be cofibrations, Theorem 5.2.22 gives that this relation is precisely cofibre homotopy equivalence of spaces under $X \sqcup Y$, thus is an equivalence relation by Proposition 5.2.20. □

Remark 5.3.2. The fact that, by Theorem 5.2.22, cospan homotopy equivalence is equivalent to cofibre homotopy equivalence of spaces under the disjoint union of the objects, will be vital to obtain a congruence from cospan homotopy equivalence. We could have instead defined cospan homotopy equivalence to be cofibre homotopy of spaces under the disjoint union of the objects. Then we would use Theorem 5.2.22 in the proof of the identity axiom instead.

Lemma 5.3.15. *For each pair* $X, Y \in \mathbf{Top}$ *the relations* $(\mathsf{CofCsp}(X,Y), \overset{ch}{\sim})$ *are a congruence on* CofCsp *and hence we have a magmoid*

$$\mathrm{CofCsp} = (Ob(\mathbf{Top}), \mathsf{CofCsp}/\overset{ch}{\sim}, \bullet).$$

Proof. The magmoid CofCsp is the quotient $\mathsf{CofCsp}/\overset{ch}{\sim}$, and we have from Lemma 5.3.14 that the $\overset{ch}{\sim}$ are equivalence relations for each pair $X, Y \in \mathbf{Top}$. Thus we only need to check that the relations respect composition.

Let $i\colon X \to M \leftarrow Y : j$ and $i'\colon X \to M' \leftarrow Y : j'$ be two representatives of the same cofibrant cospan from X to Y and similarly let $k\colon Y \to N \leftarrow Z : l$ and $k'\colon Y \to N' \leftarrow Z : l'$ be representatives of the same cofibrant cospan from Y to Z.

Using Theorem 5.2.22 we have the following commuting diagram where ϕ, ϕ', ψ and ψ' are cofibre homotopy equivalences between spaces under X, Y or Z as shown.

$$\begin{tikzcd}
 & & M \sqcup_Y N & & \\
 & M \arrow[ur, "p_M"] \arrow[d, bend right, "\phi"'] & & N \arrow[ul, "p_N"'] \arrow[d, bend right, "\psi"'] & \\
X \arrow[ur, "i"] \arrow[dr, "i'"'] & & Y \arrow[ul, "j"'] \arrow[dl, "j'"] \arrow[ur, "k"] \arrow[dr, "k'"'] & & Z \arrow[ul, "l"'] \arrow[dl, "l'"] \\
 & M' \arrow[u, bend right, "\phi'"'] \arrow[dr, "p_{M'}"'] & & N' \arrow[u, bend right, "\psi'"'] \arrow[dl, "p_{N'}"] & \\
 & & M' \sqcup_Y N' & &
\end{tikzcd}$$

This means there exists a homotopy under $X \sqcup Y$, say $H_\phi : M \times \mathbb{I} \to M$, from $\phi' \circ \phi$ to the identity and a homotopy under $Y \sqcup Z$, say $H_\psi : N \times \mathbb{I} \to N$, from $\psi' \circ \psi$ to the identity. And for all $y \in Y$ we have $H_\phi(j(y), t) = j(y)$ and $H_\psi(k(y), t) = k(y)$.

By the universal property of the pushout, the commuting pair $p_{M'} \circ \phi$ and $p_{N'} \circ \psi$ uniquely determine a map $F : M \sqcup_Y N \to M' \sqcup_Y N'$ making the diagram commute. We will show F is a homotopy equivalence.

We can similarly construct a map $F' : M' \sqcup_Y N' \to M \sqcup_Y N$ using the pair $p_M \circ \psi'$ and $p_N \circ \phi'$. Notice the maps $p_M \circ H_\phi \circ (j \times \mathrm{id}_{\mathbb{I}}) : Y \times \mathbb{I} \to M \sqcup_Y N$ and $p_N \circ H_\psi \circ (k \times \mathrm{id}_{\mathbb{I}}) : Y \times \mathbb{I} \to M \sqcup_Y N$ commute using that for all $y \in Y$ we have $H_\psi(k(y), t) = k(y)$ and $H_\phi(j(y), t) = j(y)$, and the commutativity of the diagram. Taking the product with $\mathbb{I}$ of the pushout of j and k is still a pushout, by Theorem 3.5.16. Using the universal property of this pushout on the maps $p_M \circ H_\phi$ and $p_N \circ H_\psi$ gives a map $(M \sqcup_Y N) \times \mathbb{I} \to M \sqcup_Y N$ which is a homotopy from $F' \circ F$ to the identity functor.

In the same way we can construct a homotopy $F \circ F'$ to the identity. □

Theorem 5.3.16. *The quadruple*

$$\mathrm{CofCsp} = \left(Ob(\mathbf{Top}) \,,\ \mathsf{CofCsp}(X,Y)/\overset{ch}{\sim} \,,\ \bullet \,,\ \left[\begin{tikzcd} X \arrow[dr, "\iota_0^X"'] & & X \arrow[dl, "\iota_1^X"] \\ & X \times \mathbb{I} & \end{tikzcd} \right]_{ch} \right)$$

is a category.

Proof. Note that $\left(Ob(\mathbf{Top}) \,,\ \mathsf{CofCsp}(X,Y)/\overset{ch}{\sim} \,,\ \bullet \right)$ is a magmoid by Lemma 5.3.15.

($\mathcal{C}1$) Note first that $\iota_0^X\colon X \to X \times \mathbb{I} \leftarrow X :\iota_1^X$ is a concrete cofibrant cospan by Proposition 5.3.3. Suppose we have a cofibrant cospan represented by $i\colon X \to M \leftarrow Y :j$. We will show there is a cospan homotopy equivalence from $(i\colon X \to M \leftarrow Y :j) \bullet (\iota_0^Y\colon Y \to Y \times \mathbb{I} \leftarrow Y :\iota_1^Y)$ to $i\colon X \to M \leftarrow Y :j$. Consider the following diagram.

$$\begin{tikzcd}
X \arrow[dr, "i"] & & Y \arrow[dl, "j"'] \arrow[dr, "\iota_0^Y"] & & Y \arrow[dl, "\iota_1^Y"'] \\
& M \arrow[dr, "p_M"'] \arrow[ddr, bend right, "id"'] & & Y \times \mathbb{I} \arrow[dl, "p_{Y\times\mathbb{I}}"] \arrow[ddl, bend left, "(y{,}t)\mapsto j(y)"] & \\
& & M \sqcup_Y (Y \times \mathbb{I}) \arrow[d, dotted, "\phi"] & & \\
& & M & &
\end{tikzcd}$$

The map ϕ is constructed using the universal property of the pushout. By construction ϕ commutes with the cospans $(i\colon X \to M \leftarrow Y :j) \bullet (\iota_0^Y\colon Y \to Y \times \mathbb{I} \leftarrow Y :\iota_1^Y)$ and $i\colon X \to M \leftarrow Y :j$. We claim ϕ is a homotopy equivalence with homotopy inverse p_M. It is immediate that $\phi \circ p_M = \mathrm{id}$.

We construct a homotopy $p_M \circ \phi \to \mathrm{id}$ as follows. Since $M \sqcup_Y (Y \times \mathbb{I})$ is a pushout, the map $p_M \circ \phi$ is uniquely determined by the pair of maps $M \to M \sqcup_Y (Y \times \mathbb{I})$, $m \mapsto p_M(m)$ and $Y \times \mathbb{I} \to M \sqcup_Y (Y \times \mathbb{I})$, $(y,t) \mapsto p_M(j(y))$, or equivalently $(y,t) \mapsto p_{Y\times\mathbb{I}}(\iota_0^Y(y))$. Similarly the identity is determined by the pair $M \to M \sqcup_Y (Y \times \mathbb{I})$, $m \mapsto p_M(m)$ and $Y \times \mathbb{I} \to M \sqcup_Y (Y \times \mathbb{I})$, $(y,t) \mapsto p_{Y\times\mathbb{I}}(y,t)$. The map $H_{Y\times\mathbb{I}}\colon (Y \times \mathbb{I}) \times \mathbb{I} \to M \sqcup_Y (Y \times \mathbb{I})$, $((y,t),s) \mapsto p_{Y\times\mathbb{I}}(y,ts)$ is a homotopy between the two maps from $Y \times \mathbb{I}$. And for M we can use the homotopy $H_M\colon M \times \mathbb{I} \to M \sqcup_Y (Y \times \mathbb{I})$, $(m,t) \mapsto p_M(m)$.

By Theorem 3.5.16 the product with $\mathbb{I}$ preserves pushouts. Notice that $H_M \circ (j \times \mathrm{id})\colon Y \times \mathbb{I} \to M \sqcup_Y (Y \times \mathbb{I})$ is $(y,t) \mapsto p_M(j(y))$ and $H_{Y\times\mathbb{I}} \circ (\iota_0^Y \times \mathrm{id})\colon Y \times \mathbb{I} \to M \sqcup_Y (Y \times \mathbb{I})$ is $(y,s) \mapsto p_{Y\times\mathbb{I}}(\iota_0^Y(y))$, so we can use the universal property of the pushout of $j \times \mathrm{id}$ and $\iota_0^Y \times \mathrm{id}$ to obtain a homotopy $\mathcal{H}\colon (M \sqcup_Y (Y \times \mathbb{I})) \times \mathbb{I} \to M \sqcup_Y (Y \times \mathbb{I})$ from $p_M \circ \phi$ to id.

We can similarly construct a cospan homotopy equivalence $(\iota_0^Y\colon Y \to Y \times \mathbb{I} \leftarrow Y :\iota_1^Y) \bullet (i\colon X \to M \leftarrow Y :j)$ to $i\colon X \to M \leftarrow Y :j$.

($\mathcal{C}2$) We now check that the composition is associative. Let $i\colon W \to M \leftarrow X :j$, $k\colon X \to N \leftarrow Y :l$ and $r\colon Y \to O \leftarrow Z :s$ be concrete cofibrant cospans. The two ways to compose these three cospans corresponds to taking a pushout first over X or first over Y as shown in the following diagram

$$\begin{tikzcd}
W \arrow[dr, "i"'] & & X \arrow[dl, "j"'] \arrow[dr, "k"'] & & Y \arrow[dl, "l"'] \arrow[dr, "r"'] & & Z \arrow[dl, "s"'] \\
& M \arrow[d] \arrow[ddrrr] & & N \arrow[dll] \arrow[drr] & & O \arrow[d] \arrow[ddlll] & \\
& M \sqcup_X N \arrow[dr] & & & & N \sqcup_Y O \arrow[dl] & \\
& & (M \sqcup_X N) \sqcup_Y O & & M \sqcup_X (N \sqcup_Y O)\,. & &
\end{tikzcd}$$

We can use the universal property of the pushout on the pair of maps $M \to M \sqcup_X (N \sqcup_Y O)$ and $N \to N \sqcup_Y O \to M \sqcup_X (N \sqcup_Y O)$ to obtain a map $M \sqcup_X N \to M \sqcup_X (N \sqcup_Y O)$. We can then apply the universal property again to this map $M \sqcup_X N \to M \sqcup_X (N \sqcup_Y O)$ and the map $O \to N \sqcup_Y O \to M \sqcup_X (N \sqcup_Y O)$ to obtain a map $(M \sqcup_X N) \sqcup_Y O \to M \sqcup_X (N \sqcup_Y O)$ which commutes with the diagram. In a similar way we can obtain an inverse $M \sqcup_X (N \sqcup_Y O) \to (M \sqcup_X N) \sqcup_Y O$. □

Let $i{:}\, X \to M \leftarrow Y :j$ and $k{:}\, Y \to N \leftarrow Z :l$ be concrete cofibrant cospans. In an attempt to avoid excessive notation, from here we may use i and l to refer also to the maps $\tilde{i} = p_M \circ i$ and $\tilde{l} = p_N \circ l$ obtained in the composition.

Proposition 5.3.17. *The map* $\mathrm{rev}{:}\, \mathsf{CofCsp}(X, Y) \to \mathsf{CofCsp}(Y, X)$ *from Proposition 5.3.9 extends to a functor*

$$\mathrm{rev} \;:\; \mathrm{CofCsp} \to \mathrm{CofCsp}$$

$$\left[\begin{tikzcd} X \arrow[dr, "i"'] & & Y \arrow[dl, "j"] \\ & M & \end{tikzcd}\right]_{ch} \mapsto \left[\begin{tikzcd} Y \arrow[dr, "j"'] & & X \arrow[dl, "i"] \\ & M. & \end{tikzcd}\right]_{ch}$$

Proof. Proposition 5.3.9 gives that rev is well defined. To show composition is preserved, let $i{:}\, X \to M \leftarrow Y :j$ and $k{:}\, Y \to N \leftarrow Z :l$ be concrete cofibrant cospans. Then the universal property of the pushout gives an isomorphism between $M \sqcup_Y N$ and $N \sqcup_Y M$, which gives a cospan homotopy equivalence from $\mathrm{rev}\big((k{:}\, Y \to N \leftarrow Z :l) \bullet (i{:}\, X \to M \leftarrow Y :j)\big)$ to $\mathrm{rev}(i{:}\, X \to M \leftarrow Y :j) \bullet \mathrm{rev}(k{:}\, Y \to N \leftarrow Z :l)$. □

Monoidal structure on CofCsp

We now construct a bifunctor on CofCsp and show there exists a symmetric monoidal category with underlying category CofCsp.

Lemma 5.3.18. *There is a bifunctor*

$$\begin{array}{rcl} \otimes : \mathrm{CofCsp}\times\mathrm{CofCsp} & \to & \mathrm{CofCsp} \\ \left(\left[\begin{tikzcd}[column sep=small] W \arrow[dr, "i"'] & & X \arrow[dl, "j"] \\ & M & \end{tikzcd}\right], \left[\begin{tikzcd}[column sep=small] Y \arrow[dr, "k"'] & & Z \arrow[dl, "l"] \\ & N & \end{tikzcd}\right]\right) & \mapsto & \left[\begin{tikzcd}[column sep=small] W\sqcup Y \arrow[dr, "i\sqcup k"'] & & X\sqcup Z \arrow[dl, "j\sqcup l"] \\ & M\sqcup N & \end{tikzcd}\right] \end{array}$$

where $i\sqcup j$ is the image of a pair of maps under the monoidal product on **Top** *as in Proposition 3.7.6.*

Proof. We first check that $i\sqcup k\colon W\sqcup Y\to M\sqcup N\leftarrow X\sqcup Z:j\sqcup l$ is a concrete cofibrant cospan. We will show that the map $\langle i\sqcup k, j\sqcup l\rangle\colon (W\sqcup Y)\sqcup(X\sqcup Z)\to M\sqcup N$ is a closed cofibration. Let K be some space and suppose we have maps $h\colon((W\sqcup Y)\sqcup(X\sqcup Z))\times\mathbb{I}\to K$ and $f\colon M\sqcup N\to K$ satisfying the axioms of Definition 5.2.1. By Theorem 3.5.16, the product with $\mathbb{I}$ preserves colimits so the map h uniquely determines a pair $h'\colon W\sqcup Y\to K$ and $h''\colon X\sqcup Z\to K$. Similarly the map f determines maps $f'\colon M\to K$ and $f''\colon N\to K$. We can use the homotopy extension property of $\langle i,j\rangle$ on the pair h' and f' to obtain a map $H'\colon M\times\mathbb{I}\to K$ and similarly of $\langle k,l\rangle$ on the pair h'' and f'' to obtain $H''\colon N\times\mathbb{I}\to K$. Now using Theorem 3.5.16 again, H' and H'' determine uniquely a map $H\colon(M\sqcup N)\times\mathbb{I}\to K$ extending f and h.

The image of $\langle i\sqcup k, j\sqcup l\rangle$ is the union of the images of $\langle i,j\rangle$ and $\langle k,l\rangle$, thus is closed.

We now check that the monoidal product is well defined. Suppose we have a concrete cofibrant cospan $i'\colon W\to M'\leftarrow X:j'$ which is cospan homotopy equivalent to $i\colon W\to M\leftarrow X:j$ via some cospan homotopy equivalence $\phi\colon M\to M'$ and similarly $k'\colon Y\to N'\leftarrow Z:l'$ equivalent to $k\colon Y\to N\leftarrow Z:l$ via $\psi\colon N\to N'$. Then there exist homotopy inverses ϕ' of ϕ

and ψ' of ψ. Then the following diagram commutes

$$\begin{tikzcd}
& M \sqcup N \arrow[dd, "\phi\sqcup\psi"] & \\
W \sqcup Y \arrow[ur, "i\sqcup k"] \arrow[dr, "i'\sqcup k'"'] & & X \sqcup Z \arrow[ul, "j\sqcup l"'] \arrow[dl, "j'\sqcup l'"] \\
& M' \sqcup N' &
\end{tikzcd}$$

and using the universal property of the coproduct on the appropriate homotopies it is straightforward to check that $\phi \sqcup \psi$ is a homotopy equivalence with homotopy inverse $\phi' \sqcup \psi'$.

Let X, Y be any spaces. The canonical isomorphism $(X \sqcup Y) \times \mathbb{I} \to (X \times \mathbb{I}) \times (Y \times \mathbb{I})$, which in particular is a homotopy equivalence, is sufficient to show that $\iota_0^X \sqcup \iota_0^Y : X \sqcup Y \to (X \times \mathbb{I}) \sqcup (Y \times \mathbb{I}) \leftarrow X \sqcup Y : \iota_0^X \sqcup \iota_1^Y$ is cospan homotopy equivalent to $\iota_0^{X\sqcup Y} : X \sqcup Y \to (X \sqcup Y) \times \mathbb{I} \leftarrow X \sqcup Y : \iota_1^{X \sqcup Y}$.

Finally we check that $\otimes$ preserves composition. Given two pairs of composable concrete cofibrant cospans, there are distinct cospans obtained from first appplying $\otimes$ and then composing and from composing and then applying $\otimes$. A commuting isomorphism is constructed between these cospans using the universal properties of the coproduct and the pushout. □

Lemma 5.3.19. *Let X and X' be spaces and $f : X \to X'$ a homeomorphism. Then the cospan*

$$\begin{tikzcd}
X \arrow[dr, "\iota_0^{X'} \circ f"'] & & X' \arrow[dl, "\iota_1^{X'}"] \\
& X' \times \mathbb{I} &
\end{tikzcd}$$

is a concrete cofibrant cospan and its cospan homotopy equivalence class is an isomorphism in CofCsp.

Proof. We first prove the cospan is a concrete cofibrant cospan. Note that the map $\langle \iota_0^{X'} \circ f, \iota_1^{X'} \rangle$ is equal to the composition

$$X \sqcup X' \xrightarrow{\langle f, \mathrm{id}_X \rangle} X' \sqcup X' \xrightarrow{\langle \iota_0^{X'}, \iota_1^{X'} \rangle} (X' \sqcup X') \times \mathbb{I}.$$

The first map is a homeomorphism; hence it is a cofibration by Lemma 5.2.4. The second map is the map from the coproduct corresponding to the concrete cofibrant cospan in Lemma 5.3.3. Hence the composition is a cofibration by Lemma 5.2.3. Since the first map is a homeomorphism, the image of the composition is equal to the image of the second map, so is closed by Lemma 5.3.3.

To see that the cospan homotopy equivalence class is an isomorphism notice that the composition

$$\left[\begin{tikzcd} X \arrow[dr, "\iota_0^{X'}\circ f"'] & & X' \arrow[dl, "\iota_1^{X'}"] \\ & X'\times\mathbb{I} & \end{tikzcd}\right] \cdot \left[\begin{tikzcd} X' \arrow[dr, "\iota_0^{X'}"'] & & X \arrow[dl, "\iota_1^{X'}\circ f"] \\ & X'\times\mathbb{I} & \end{tikzcd}\right]$$

is equivalent to $\iota_0^{X'}\circ f\colon X\to X'\times\mathbb{I}\leftarrow X:\iota_1^{X'}\circ f$ via the obvious isomorphism $X'\times\mathbb{I}\cong (X'\times\mathbb{I})\sqcup_{X'}(X'\times\mathbb{I})$, which is equivalent to $\iota_0^{X}\colon X\to X\times\mathbb{I}\leftarrow X:\iota_1^{X}$ via the homeomorphism $f\times\mathrm{id}\colon X\times\mathbb{I}\to X'\times\mathbb{I}$. □

Lemma 5.3.20. *Recall from Proposition 3.7.6 that* $(\mathbf{Top},\sqcup,\varnothing,\alpha^T_{X,Y,Z},\lambda^T_X,\rho^T_X)$ *is a monoidal category. There is a monoidal category*

$$(\mathrm{CofCsp}\ ,\ \otimes\ ,\ \varnothing\ ,\ \alpha_{X,Y,Z}\ ,\ \lambda_X\ ,\ \rho_X)$$

where $\otimes$ *is as in Lemma 5.3.18,*

- *for any spaces* $X,Y,Z\in Ob(\mathrm{CofCsp})$ *the associator* $\alpha_{X,Y,Z}\colon(X\sqcup Y)\sqcup Z\to X\sqcup(Y\sqcup Z)$ *is the cospan homotopy equivalence of the cospan*

$$\begin{tikzcd} (X\sqcup Y)\sqcup Z \arrow[dr, "\iota_0^{X\sqcup(Y\sqcup Z)}\circ\alpha^T_{X,Y,Z}"'] & & X\sqcup(Y\sqcup Z) \arrow[dl, "\iota_1^{X\sqcup(Y\sqcup Z)}"] \\ & (X\sqcup(Y\sqcup Z))\times\mathbb{I}, & \end{tikzcd}$$

- *for any space* $X\in Ob(\mathrm{CofCsp})$ *the left unitor* $\lambda_X\colon\varnothing\sqcup X\to X$ *is the cospan homotopy equivalence class of the cospan*

$$\begin{tikzcd} \varnothing\sqcup X \arrow[dr, "\iota_0^{X}\circ\lambda^T_X"'] & & X \arrow[dl, "\iota_1^{X}"] \\ & X\times\mathbb{I}, & \end{tikzcd}$$

- *for any space* $X\in Ob(\mathrm{CofCsp})$ *the right unitor* $\rho_X\colon X\sqcup\varnothing\to X$ *is the cospan homotopy* *equivalence class of the cospan*

$$\begin{tikzcd}
X \sqcup \varnothing \arrow[dr, "\iota_0^X \circ \rho_X^T"'] & & X \arrow[dl, "\iota_1^X"] \\
& X \times \mathbb{I}. &
\end{tikzcd}$$

Proof. First note that Lemma 5.3.19 gives that all associators and unitors are isomorphisms.

The proofs of the pentagon and triangle identities, and of naturality are similar, so we only give the proof of the triangle identity here.

We must construct a cospan homotopy equivalence from the cospan

$$\begin{tikzcd}
(X \sqcup \varnothing) \sqcup Y \arrow[dr, "\iota_0^{X \sqcup (\varnothing \sqcup Y)} \circ \alpha_{X,\varnothing,Y}^T"'] & & X \sqcup (\varnothing \sqcup Y) \arrow[dl, "\iota_1^{X \sqcup (\varnothing \sqcup Y)}"] \arrow[dr, "\iota_0^X \sqcup (\iota_0^Y \circ \lambda_Y^T)"'] & & X \sqcup Y \arrow[dl, "\iota_1^X \sqcup \iota_1^Y"] \\
& (X \sqcup (\varnothing \sqcup Y)) \times \mathbb{I} \arrow[dr] & & (X \times \mathbb{I}) \sqcup (Y \times \mathbb{I}) \arrow[dl] & \\
& & (X \sqcup (\varnothing \sqcup Y)) \times \mathbb{I} \sqcup_{X \sqcup (\varnothing \sqcup Y)} (X \times \mathbb{I}) \sqcup (Y \times \mathbb{I}), & &
\end{tikzcd}$$

to the cospan

$$\begin{tikzcd}
(X \sqcup \varnothing) \sqcup Y \arrow[dr, "(\iota_0^X \circ \rho_X^T) \sqcup \iota_0^Y"'] & & X \sqcup Y \arrow[dl, "\iota_1^X \sqcup \iota_1^Y"] \\
& (X \times \mathbb{I}) \sqcup (Y \times \mathbb{I}). &
\end{tikzcd}$$

By the universal property of the coproduct and Theorem 3.5.16, a map $f\colon (X \times \mathbb{I}) \sqcup ((\varnothing \times \mathbb{I}) \sqcup (Y \times \mathbb{I})) \to (X \times \mathbb{I}) \sqcup (Y \times \mathbb{I})$ is uniquely determined by

$$f_X\colon X \times \mathbb{I} \to (X \times \mathbb{I}) \sqcup (Y \times \mathbb{I})$$
$$(x, t) \mapsto ((x, t/2), 1)$$

and

$$f_Y\colon Y \times \mathbb{I} \to (X \times \mathbb{I}) \sqcup (Y \times \mathbb{I})$$
$$(y, t) \mapsto ((y, t/2), 2).$$

Similarly a map $g\colon (X \times \mathbb{I}) \sqcup (Y \times \mathbb{I}) \to (X \times \mathbb{I}) \sqcup (Y \times \mathbb{I})$ is determined by the pair

$$\begin{aligned} g_X\colon X \times \mathbb{I} &\to (X \times \mathbb{I}) \sqcup (Y \times \mathbb{I}) \\ (x,t) &\mapsto ((x, 1/2(t+1)), 1) \end{aligned}$$

and

$$\begin{aligned} g_Y\colon Y \times \mathbb{I} &\to (X \times \mathbb{I}) \sqcup (Y \times \mathbb{I}) \\ (y,t) &\mapsto ((y, 1/2(t+1)), 2). \end{aligned}$$

We have that $f \circ \iota_1^{X \sqcup (\varnothing \sqcup Y)} = g \circ \iota_0^X \sqcup (\iota_0^Y \circ \lambda_Y^T)$ commute, so by the universal property of the pushout, these maps determine a map

$$h\colon ((X \sqcup (\varnothing \sqcup Y)) \times \mathbb{I}) \sqcup_{X \sqcup (\varnothing \sqcup Y)} ((X \times \mathbb{I}) \sqcup (Y \times \mathbb{I})) \to (X \times \mathbb{I}) \sqcup (Y \times \mathbb{I})$$

which is a homeomorphism, and it is straightforward to check this commutes with the cospans, hence is a cospan homotopy equivalence. □

Theorem 5.3.21. *There is a symmetric monoidal category*

$$(\mathrm{CofCsp}\ ,\ \otimes\ ,\ \varnothing\ ,\ \alpha_{X,Y,Z}\ ,\ \lambda_X\ ,\ \rho_X\ ,\ \beta_X)$$

where $(\mathrm{CofCsp}, \otimes, \varnothing, \alpha_{X,Y,Z}, \lambda_X, \rho_X)$ *is as in Lemma 5.3.20, and for any spaces* $X, Y \in$ *Ob*(CofCsp) *the braiding* $\beta_{X,Y}\colon X \otimes Y \to Y \otimes X$ *is the cospan homotopy equivalence class of the cospan*

$$\begin{tikzcd}
Y \sqcup X \arrow[dr, "\iota_0^{Y \sqcup X} \circ \beta^T_{X,Y}"'] & & X \sqcup Y \arrow[dl, "\iota_1^{Y \sqcup X}"] \\
& (Y \sqcup X) \times \mathbb{I} &
\end{tikzcd}$$

where $\beta^T_{X,Y}$ *is the braiding in* **Top** *as in Proposition 3.7.14.*

By abuse of notation we will refer to this symmetric monoidal category as CofCsp.

Proof. As with the previous theorem, the proofs of all necessary identities are similar. Here we give the proof that β is symmetric.

We must construct a cospan homotopy equivalence from the cospan

$$\begin{tikzcd}
X \sqcup Y \arrow[dr, "\iota_0^{Y\sqcup X} \circ \beta^T_{X,Y}"'] & & Y \sqcup X \arrow[dl, "\iota_1^{Y\sqcup X}"] \arrow[dr, "\iota_0^{X\sqcup Y} \circ \beta^T_{Y,X}"'] & & X \sqcup Y \arrow[dl, "\iota_1^{X\sqcup Y}"] \\
& (Y \sqcup X) \times \mathbb{I} \arrow[dr] & & (X \sqcup Y) \times \mathbb{I} \arrow[dl] & \\
& & ((Y \sqcup X) \times \mathbb{I}) \sqcup_{Y \times X} ((X \sqcup Y) \times \mathbb{I}), & &
\end{tikzcd}$$

to the cospan

$$\begin{tikzcd}
X \sqcup Y \arrow[dr, "\iota_0^{X\sqcup Y}"'] & & X \sqcup Y \arrow[dl, "\iota_1^{X\sqcup Y}"] \\
& (X \sqcup Y) \times \mathbb{I}. &
\end{tikzcd}$$

Define maps

$$\begin{aligned} f_1 \colon (Y \sqcup X) \times \mathbb{I} &\to (X \sqcup Y) \times \mathbb{I} \\ (x,t) &\mapsto (\beta^T_{Y,X}(x), t/2) \end{aligned}$$

and

$$\begin{aligned} f_2 \colon (X \sqcup Y) \times \mathbb{I} &\to (X \sqcup Y) \times \mathbb{I} \\ (x,t) &\mapsto (x, 1/2(t+1)). \end{aligned}$$

Note that $f_1 \circ \iota_1^{Y \sqcup X} = f_2 \circ (\iota_0^{X \sqcup Y} \circ \beta^T_{Y,X})$, hence applying the universal property of the pushout determines a map

$$f \colon ((Y \sqcup X) \times \mathbb{I}) \sqcup_{Y \times X} ((X \sqcup Y) \times \mathbb{I}) \to (X \sqcup Y) \times \mathbb{I}.$$

Notice that f is a homeomorphism, and it is straightforward to check that it commutes with the cospans, and so is a cospan homotopy equivalence. □

Remark 5.3.3. Recall that $\mathbf{Top}^h$ is the wide subcategory of **Top** where all maps are homeomorphisms. There is a functor $\kappa \colon \mathbf{Top}^h \to \mathrm{CofCsp}$, which sends a homeomorphism $\mathfrak{f} \colon X \to Y$ to the cospan homotopy equivalence class of the cospan $\mathfrak{f} \colon X \to Y \times \mathbb{I} \leftarrow Y : \iota_1^Y$. It then follows that the triangle, pentagon and braiding identities commute in CofCsp as they are precisely the images of the corresponding identities in **Top**. This functor extends to a functor from the mapping class groupoid of a space X into CofCsp. This is why

TQFTs give representations of mapping class groups.

5.3.3 Category of homotopy cobordisms HomCob

Here we construct the category HomCob (Theorem 5.3.34), which we will use as the source category of the TQFT we construct in Section 5.4. We obtain HomCob as a full subcategory of CofCsp with a finiteness condition on spaces.

Definition 5.3.22. A space X is called *homotopically* 1*-finitely generated* if $\pi(X, A)$ is finitely generated for all finite sets of basepoints A.
Let χ denote the class of all homotopically 1-finitely generated spaces.

The following result says that, to check a space X is homotopically 1-finitely generated, it will be sufficient to find a single representative subset $A \subseteq X$ such that $\pi(X, A)$ is finitely generated.

Lemma 5.3.23. *If* $\pi(X, A)$ *is finitely generated for some finite representative set* A*, then* $\pi(X, A')$ *is finitely generated for all finite representative sets* A'*.*

Proof. Let $A = \{a_1, ..., a_N\}$ and $B = \{b_1, ..., b_M\}$ with $N, M \in \mathbb{N}$. The groupoid $\pi(X, A)$ is finitely generated so there exists a finite set of generating morphisms. Let $S = \{s_1, ..., s_K\}$ be a set of representative paths, such that taking path-equivalence classes of each path gives a set of generating morphisms for $\pi(X, A)$.

For each pair $\{n, m\}$ such that a_n and b_m are in the same path connected component, choose a path $\gamma_{n,m}\colon a_n \to b_m$. We denote the set of all such paths by Γ. Note that this is finite since A and B are finite. We will show that $\pi(X, B)$ is generated by the set by the set of path-equivalence classes of all paths of the form $\gamma_{n',m'} s \gamma_{n,m}^{-1}$, where $s \in S$ and $s_0 = a_n$ and $s_1 = a_{n'}$. Note this is again finite.

Let $t : b_n \to b_{n'}$ be any path. Choose m such that a_m is in the same path connected component as b_n, then $\tilde{t} = \gamma_{m,n'}^{-1} t \gamma_{m,n}\colon a_m \to a_m$ is a path and $\gamma_{m,n'} \tilde{t} \gamma_{m,n}^{-1} \overset{p}{\sim} t$. Now we have $\tilde{t} \sim p_L \dots p_2 p_1$, where $L \in \mathbb{N}$, with each $p_l = s_k$ for some $1 \leq k \leq K$, since the equivalence classes of the s_k generate $\pi(X, A)$. Hence $t \overset{p}{\sim} \gamma_{m,n'} p_L ... p_2 p_1 \gamma_{m,n}^{-1}$.

For each p_l, choose a path denoted $\gamma_{p_i} \in \Gamma$ such that $(\gamma_{p_i})_0 = (p_1)_1$, so $\gamma_{p_1}^{-1} \gamma_{p_1} \overset{p}{\sim} \mathrm{id}_{(p_1)_1}$. Now $t \overset{p}{\sim} \gamma_{m,n'} p_l \gamma_{p_{l-1}}^{-1} ... \gamma_{p_2} p_2 \gamma_{p_1}^{-1} \gamma_{p_1} p_1 \gamma_{m,n}^{-1}$, which is of the desired form. ☐

We will need the following result about homotopically 1-finitely generated spaces to ensure $\otimes$ restricts to a closed composition in HomCob.

Lemma 5.3.24. *If X and Y are homotopically 1-finitely generated spaces, then $X \sqcup Y$ is homotopically 1-finitely generated.*

Proof. Suppose X_0 and Y_0 are finite representative subsets of X and Y respectively. The images of X and Y in $X \sqcup Y$ are disjoint, hence there is an isomorphism $\pi(X \sqcup Y, X_0 \sqcup Y_0) \cong \pi(X, X_0) \amalg \pi(Y, Y_0)$ of groupoids given by sending a path equivalence class $[\gamma]$ to $([\gamma], 1)$ if γ is a path in X and to $([\gamma], 2)$ if γ is a path in Y. By Theorem 3.6.21 we have that $\pi(X, X_0) \amalg \pi(Y, Y_0)$ is finitely generated if and only if $\pi(X, X_0)$ and $\pi(Y, Y_0)$ are. By Lemma 5.3.23 this is sufficient. □

Lemma 5.3.25. *There exists a submagmoid*

$$\mathsf{HomCob} = (\chi, \mathsf{HomCob}(-,-), \bullet)$$

of CofCsp *where*

$$\mathsf{HomCob}(X,Y) = \left\{ \begin{tikzcd}[column sep=small] X \arrow[dr, "i"'] & & Y \arrow[dl, "j"] \\ & M & \end{tikzcd} \;\middle|\; \begin{array}{l} \langle i,j \rangle \textit{ is a closed cofibration, and} \\ X, Y \textit{ and } M \textit{ are homotopically } 1\textit{-finitely generated} \end{array} \right\}.$$

Morphisms in HomCob *are called* <u>*concrete homotopy cobordisms*</u>.

Proof. We check HomCob is closed under composition. Suppose $i{:}\, X \to M \leftarrow Y :j$ and $k{:}\, Y \to N \leftarrow Z :l$ are concrete homotopy cobordisms. Consider the pushout

$$\begin{tikzcd} & Y \arrow[dl, "j"'] \arrow[dr, "k"] & \\ M \arrow[dr] & & N \arrow[dl] \\ & M \sqcup_Y N. & \end{tikzcd}$$

We may choose finite representative subsets $Y_0 \subseteq Y$, $M_0 \subseteq M$ and $N_0 \subseteq N$ such that $j(Y_0) = M_0 \cap j(Y)$ and $k(Y_0) = N_0 \cap k(Y)$. Applying Corollary 5.2.18 the following square is also a pushout.

$$\begin{tikzcd}
 & \pi(Y, Y_0) \arrow[dl, "\pi(j)"'] \arrow[dr, "\pi(k)"] & \\
\pi(M, M_0) \arrow[dr] & & \pi(N, N_0) \arrow[dl] \\
 & \pi(M \sqcup_Y N, M_0 \sqcup_{Y_0} N_0) &
\end{tikzcd}$$

We have, from Theorem 3.6.21, that the pushout of finitely generated groupoids is finitely generated, so $\pi(M \sqcup_Y N, M_0 \sqcup_{Y_0} N_0)$ is finitely generated since $\pi(M, M_0)$ and $\pi(N, N_0)$ are. Hence the composition is a concrete homotopy cobordism. □

Example 5.3.26. *The concrete cofibrant cospan in Proposition 5.3.6 is a concrete homotopy cobordism, as the fundamental group of D^2 and S^1 are finitely generated.*

Example 5.3.27. *The concrete cofibrant cospan, $i: X \to M \leftarrow X : j$, in Example 5.3.8 is a concrete homotopy cobordism. We have $X \cong S^1 \sqcup S^1$, hence, letting X_0 be a subset with a single point in each copy path connected component, $\pi(X, X_0) \cong \mathbb{Z} \sqcup \mathbb{Z}$. Similarly $Y \cong S^1$, so $\pi(Y, \{y\}) \cong \mathbb{Z}$ for any $y \in Y$. The manifold M is a homotopy equivalent to the twice punctured disk, hence has fundamental group $\pi(M, \{m\}) \cong \mathbb{Z} * \mathbb{Z}$. Hence X, Y and M are homotopically 1-finitely generated.*

Example 5.3.28. *The concrete cofibrant cospan, $i: X \to M \leftarrow X : j$, in Example 5.3.7 is a concrete homotopy cobordism. The space X is homotopy equivalent to the disjoint union of two copies of the disk and a twice punctured disk. Thus, choosing $X_0 \subset X$ with a point in each connected component, we have $\pi(X, X_0)$ is finitely generated. The space Y is homotopy equivalent to the disjoint union of the circle and the disk thus, choosing Y_0 in the same way, we have $\pi(Y, Y_0)$ finitely generated. The space M is the disjoint union of a contractible space, and a space which is homotopy equivalent to a sphere with three lines from the boundary meeting at a point in the centre, and thus via a stereographic projection, homotopy equivalent to the twice punctured disk. Hence $\pi(M, M_0) \cong \mathbb{Z} * \mathbb{Z}$ for any choice of M_0 consisting of one basepoint in each connected component.*

Example 5.3.29. *Let Γ be a finite graph. Choose disjoint sets $V_1, V_2 \subseteq V(\Gamma)$ of vertices. Then $i: V_1 \to \Gamma \leftarrow V_2 : j$ is a concrete homotopy cobordism where i and j are inclusions.*

That the spaces are homotopically 1*-finitely generated can be seen by taking basepoints to be all vertices, and generating paths to be edges.*

Example 5.3.30. *Let M be a CW complex, and X and Y disjoint subcomplexes. Then $i\colon X \to M \leftarrow Y : j$, where i and j are inclusions, is a concrete homotopy cobordism. That the inclusions are cofibrations follows from Proposition 0.16 of [Hat02], and that finitely generated CW complexes have finite fundamental group is essentially Proposition 1.26 of [Hat02].*

Definition 5.3.31. A cofibrant cospan is called a homotopy cobordism if there exists a representative which is a concrete homotopy cobordism.
For homotopically 1-finitely generated spaces $X, Y \in \mathbf{Top}$ define

$$\mathrm{HomCob}(X,Y) = \left\{ \left[\begin{tikzcd} X \arrow[dr, "i"'] & & Y \arrow[dl, "j"] \\ & M & \end{tikzcd}\right]_{\mathrm{ch}} \;\middle|\; \begin{tikzcd} X \arrow[dr, "i"'] & & Y \arrow[dl, "j"] \\ & M & \end{tikzcd} \text{ is a concrete homotopy cobordism} \right\}.$$

Notice that if $i\colon X \to M \leftarrow Y : j$ is a concrete cofibrant cospan with all spaces homotopically 1-finitely generated, then it is clear from the definition of cospan homotopy equivalence that every cospan in the equivalence class also has all spaces homotopically 1-finitely generated.

Theorem 5.3.32. *There is a subcategory of* CofCsp *(see Theorem 5.3.16)*

$$\mathrm{HomCob} = \left(\chi, \mathrm{HomCob}(X,Y), \bullet, \left[\begin{tikzcd} X \arrow[dr, "\iota_0^X"'] & & X \arrow[dl, "\iota_1^X"] \\ & X \times \mathbb{I} & \end{tikzcd}\right]_{ch} \right)$$

with

- *all homotopically* 1*-finitely generated spaces as objects;*
- *for spaces $X, Y \in Ob(\mathrm{HomCob})$, morphisms in* $\mathrm{HomCob}(X,Y)$ *are homotopy cobordisms i.e. cospan homotopy equivalence classes (see Lemma 5.3.14) of cospans*

  $$\left[\begin{tikzcd} X \arrow[dr, "i"'] & & Y \arrow[dl, "j"] \\ & M & \end{tikzcd}\right]_{ch}$$

  with all spaces homotopically 1*-finitely generated and $\langle i, j \rangle$ a cofibration;*

- *composition is as follows*

$$\bullet : \mathrm{HomCob}(X,Y) \times \mathrm{HomCob}(Y,Z) \to \mathrm{HomCob}(X,Z)$$

$$\left(\left[\begin{tikzcd} X \arrow[dr, "i"'] & & Y \arrow[dl, "j"] \\ & M & \end{tikzcd}\right]_{ch}, \left[\begin{tikzcd} Y \arrow[dr, "k"'] & & Z \arrow[dl, "l"] \\ & N & \end{tikzcd}\right]_{ch} \right) \mapsto \left[\begin{tikzcd} X \arrow[dr, "\tilde{i}"'] & & Z \arrow[dl, "\tilde{l}"] \\ & M \sqcup_Y N & \end{tikzcd}\right]_{ch}$$

where $\tilde{i} = p_M i$ *and* $\tilde{l} = p_N l$ *are obtained via the pushout diagram*

$$\begin{tikzcd} X \arrow[dr, "i"'] & & Y \arrow[dl, "j"] \arrow[dr, "k"'] & & Z \arrow[dl, "l"] \\ & M \arrow[dr, "p_M"'] & & N \arrow[dl, "p_N"] & \\ & & M \sqcup_Y N; & & \end{tikzcd}$$

- *for a space* $X \in Ob(\mathrm{HomCob})$ *the identity morphism is the equivalence class of the cospan*

$$\begin{tikzcd} X \arrow[dr, "\iota_0^X"'] & & X \arrow[dl, "\iota_1^X"] \\ & X \times \mathbb{I}. & \end{tikzcd}$$

Proof. We have from Lemma 5.3.25 that HomCob = $(\chi, \mathrm{HomCob}(X,Y), \bullet)$ is a magmoid. Theorem 5.3.16 gives that $\bullet$ is a associative, and that the proposed identity is an identity of $\bullet$. It remains only to prove that the identity is in HomCob.

Let X be a homotopically 1-finitely generated space. Then $X \times \mathbb{I}$ is homotopy equivalent to X, and so $[\iota_0^X : X \to X \times \mathbb{I} \leftarrow X : \iota_1^X]$ is a homotopy cobordism. □

Proposition 5.3.33. *Let* $\mathbf{Cob}(n)$ *be the category where objects* $(n-1)$*-dimensional closed oriented smooth manifolds and morphisms are equivalence classes of concrete cobordisms (Definition 5.3.4) as in [Lur09, Ch. 1]. For all* $n \in \mathbb{N}$ *there is a functor*

$$\mathrm{Cob}_n : \mathbf{Cob}(n) \to \mathrm{HomCob}$$

which maps objects to their underlying space and maps a morphism to the equivalence class of the concrete cofibrant cospan which is the image of a representative cobordism under the mapping described in Proposition 5.3.5.

Proof. We first check that Cob_n is well defined. Chapter 6 of [Hir12] proves that compact smooth manifolds have the homotopy type of finite CW complexes (see in particular the start of Section 3 and Theorems 1.2 and 4.1). If we choose the set of basepoints to be the 0-cells of the corresponding CW complex, then the generators of the fundamental groupoid are the 1-cells and so the fundamental groupoids of smooth manifolds with a finite set of basepoints are finitely generated. If two concrete cobordisms are equivalent up to boundary preserving diffeomorphism then they are certainly equivalent up to cofibre homotopy equivalence using the same map. So we have that the functor is well defined.

Let X, Y, Z be a triple of objects in $\mathbf{Cob}(n)$ and $M\colon X \to Y$, $M'\colon Y \to Z$ a pair of cobordisms. Then we have maps $\phi\colon X \sqcup Y \to M$ and $\phi'\colon Y \sqcup Z \to M'$ between the underlying topological spaces. The image of the composition in $\mathbf{Cob}(n)$ is the cospan $i\colon X \to M \sqcup N/((y,0) \sim (y,1)) \leftarrow Z : j$ where $i(x) = \phi(x,0)$ and $j(y) = \phi'(z,1)$. This is precisely the composition of the images of $M\colon X \to Y$ and $M'\colon Y \to Z$ in HomCob.

The identity for a manifold X in $\mathbf{Cob}(n)$ is represented by the cylinder $X \times \mathbb{I}$ with $\langle \iota_0^X, \iota_1^X \rangle\colon \bar{X} \sqcup X \to X \times \mathbb{I}$, this clearly maps to a representative of the identity cospan of X. □

Monoidal structure on HomCob

The category HomCob becomes a symmetric monoidal category, just like CofCsp.

Theorem 5.3.34. *There is a symmetric monoidal subcategory*

$$(\text{HomCob}\ ,\ \otimes\ ,\ \varnothing\ ,\ \alpha_{X,Y,Z}\ ,\ \lambda_X\ ,\ \rho_X\ ,\ \tau_X)$$

of CofCsp. *Here* $\otimes$ *is as in Lemma 5.3.18, associators and unitors are as in Lemma 5.3.20 and braiding as in Lemma 5.3.21.*

Proof. The empty set is homotopically 1-finitely generated. For each pair of homotopically 1-finitely generated spaces, the disjoint union is homotopically 1-finitely generated by Lemma 5.3.24 so $\otimes$ sends a pair of homotopy cobordisms to a homotopy cobordism.

Using again Lemma 5.3.24 along with the fact that for any space X and finite $A \subseteq X$ we have $\pi(X, A) \cong \pi(X \times \mathbb{I}, A \times \{0\})$, the associators, unitors and braidings are all in HomCob. □

Proposition 5.3.35. *The functor* $Cob_n\colon \mathbf{Cob}_n \to \mathrm{HomCob}$ *as in Proposition 5.3.33 is symmetric strong monoidal with* $(Cob_n)_0 = [\varnothing\colon \varnothing \to \varnothing \leftarrow \varnothing : \varnothing]_{ch}$ *and* $(Cob_n)_2(X,Y) = \left[\iota_0^{X\sqcup Y}\colon X \sqcup Y \to (X\sqcup Y)\times \mathbb{I} \leftarrow X \sqcup Y : \iota_1^{X\sqcup Y}\right]_{ch}$.

Proof. Notice that the monoidal product $\otimes'$ in $\mathbf{Cob}(n)$ is given by disjoint union, thus we have $\otimes \circ (Cob_n \times Cob_n) = Cob_n \circ \otimes'$ and $(Cob_n)_2$ is the required natural transformation. It is straightforward to check all identities as $(Cob_n)_0$ and $(Cob_n)_2$ are identities and the functor Cob_n maps all associators, unitors and braidings to exactly the corresponding associators, unitors and braidings in HomCob. □

5.4 Topological quantum field theory construction

In this section we will explicitly construct a functor (Theorem 5.4.24)

$$\mathsf{Z}_G\colon \mathrm{HomCob} \to \mathbf{Vect}_{\mathbb{C}},$$

dependent on a choice G of finite group. Ultimately our functor will map a space X to the vector space with basis the set of maps $f\colon \pi(X) \to G$ up to natural transformation. The particular interest of our construction over and above this result twofold. Firstly we prove that this arises naturally as a colimit over all representative finite subsets $A \subseteq X$ of basepoints and maps $g\colon \pi(X,A) \to G$. Secondly we prove that this global equivalence over all subsets has an interpretation in terms of a local equivalence, taking maps $g\colon \pi(X,A) \to G$ up to natural transformation for some fixed choice of finite representative subset $A \subseteq X$. Thus our construction is explicitly calculable, see Example 5.4.36.

We begin by defining a magmoid morphism from a version of HomCob with basepoints, to $\mathsf{Vect}_{\mathbb{C}}$. We then use a colimit construction to remove the dependence on basepoints and arrive at the functor Z_G. We then show, in Section 5.4.4, that Z_G can be calculated on objects by choosing a fixed set of basepoints. In Section 5.4.5 we prove that Z_G is a symmetric monoidal functor.

5.4.1 Magmoid of based cospans

Let $\boldsymbol{\chi}$ denote the class of pairs of the form (X, X_0) where X is a homotopically 1-finitely generated space and X_0 is a representative finite subset of X. We will refer to the set X_0

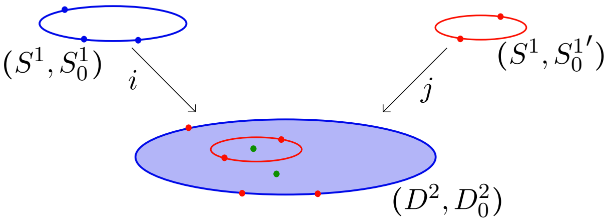


Figure 5.4: Here the dots represent basepoints, so the three blue points in the leftmost copy of S^1 are S^1_0 and and two red basepoints in the rightmost copy of S^1 are ${S^1}'_0$. The set of basepoints D^2_0 then contains $i(S^1_0) \cup j({S^1}'_0)$ as well as two extra basepoints marked in green which do not intersect $i(S^1) \cup j(S^1)$. Then $i\colon (S^1, S^1_0) \to (D^2, D^2_0) \leftarrow {S^1}'_0 : j$ is a concrete based homotopy cobordism.

as a set of basepoints.

Definition 5.4.1. Let X, Y be spaces and $A \subseteq X$ and $B \subseteq Y$ be subsets. A *map of pairs* $f\colon (X, A) \to (Y, B)$ is a map $f\colon X \to Y$ such that $f(A) \subseteq B$.

Definition 5.4.2. Let (X, X_0), (Y, Y_0) and (M, M_0) be pairs in $\boldsymbol{\chi}$. A *concrete based homotopy cobordism* from (X, X_0) to (Y, Y_0) is a diagram $i\colon (X, X_0) \to (M, M_0) \leftarrow (Y, Y_0) : j$ such that:

(i) $i\colon X \to M \to Y : j$ is a concrete homotopy cobordism.

(ii) i and j are maps of pairs.

(iii) $M_0 \cap i(X) = i(X_0)$ and $M_0 \cap j(Y) = j(Y_0)$.

For any pairs $(X, X_0), (Y, Y_0)$ with $X, Y \in \mathbf{Top}$ and $X_0 \subseteq X$, $Y_0 \subseteq Y$ finite representative subsets,

$$\mathsf{bHomCob}((X, X_0), (Y, Y_0)) = \left\{ \text{based homotopy cobordisms } \begin{tikzcd} (X, X_0) \arrow[dr, "i"'] & & (Y, Y_0) \arrow[dl, "j"] \\ & (M, M_0) & \end{tikzcd} \right\}.$$

Example 5.4.3. *Consider the concrete cofibrant cospan in Proposition 5.3.6, which is a homotopy cobordism (see Example 5.3.26). We can add basepoints to obtain a based homotopy cobordism as shown in Figure 5.4.*

Proposition 5.4.4. *Let $i\colon X \to M \leftarrow Y : j$ be a concrete homotopy cobordism, then there exists a based homotopy cobordism $i\colon (X, X_0) \to (M, M_0) \leftarrow (Y, Y_0) : j$ for some representative finite subsets X_0, Y_0 and M_0 of X, Y and M respectively.*

Proof. A suitable choice of X_0, Y_0 and M_0 is constructed as follows. Choose a point in each path-connected component of X, and let X_0 be the union of these points. Choose a

Y_0 similarly. Let M_0 be the union of $i(X_0)$ and $j(Y_0)$ along with a choice of point in each path-connected component not already containing a point in M_0.

Notice that X_0, Y_0 and M_0 are finite, as X, Y and M are homotopically 1-finitely generated, and thus contain a finite number of path-connected components. □

Of course for any manifold X consisting of a single path connected component, there is an uncountably infinite number of choices of $\{x\}$ such that $(X,\{x\}) \in \boldsymbol{\chi}$. Hence there will usually be many ways to obtain a based homotopy cobordism from a homotopy cobordism.

The following Lemma says that the composition $\bullet$ extends to a composition of based homotopy cobordisms.

Lemma 5.4.5. (*I*) *For any spaces X, Y and Z in $Ob(\mathbf{Top})$ there is a composition*

$$\bullet : \mathsf{bHomCob}((X,X_0),(Y,Y_0)) \times \mathsf{bHomCob}((Y,Y_0),(Z,Z_0)) \to \mathsf{bHomCob}((X,X_0),(Z,Z_0))$$

$$\left(\begin{tikzcd}[ampersand replacement=\&] (X,X_0) \arrow[dr, "i"'] \& \& (Y,Y_0) \arrow[dl, "j"] \\ \& (M,M_0) \& \end{tikzcd}, \begin{tikzcd}[ampersand replacement=\&] (Y,Y_0) \arrow[dr, "k"'] \& \& (Z,Z_0) \arrow[dl, "l"] \\ \& (N,N_0) \& \end{tikzcd}\right) \mapsto \begin{tikzcd}[ampersand replacement=\&] (X,X_0) \arrow[dr, "\tilde{i}"'] \& \& (Z,Z_0) \arrow[dl, "\tilde{l}"] \\ \& (M \sqcup_Y N, M_0 \sqcup_{Y_0} N_0) \& \end{tikzcd}$$

where $\tilde{i}: X \to M \sqcup_Y N \leftarrow Z : \tilde{l}$ is the composition $(i: X \to M \leftarrow Y : j) \bullet (k: Y \to N \leftarrow Z : l)$ with $\bullet$ as in Lemma 5.3.11, and $M_0 \sqcup_{Y_0} N_0$ the set pushout of $M_0 \xleftarrow{j} Y_0 \xrightarrow{k} Z_0$ (where we use j and k also for the obvious restrictions).
(*II*) *Hence there is a magmoid*

$$\mathsf{bHomCob} = (\boldsymbol{\chi}, \mathsf{bHomCob}(-,-), \bullet).$$

Proof. We check that the composition is well defined. We first show that $M_0 \sqcup_{Y_0} N_0$ is representative in $M \sqcup_Y N$.

Note that our fixed representatives of pushouts in **Top** have the same underlying set and set maps as the representative of the corresponding pushout of the underlying set maps in **Set**. Also j and k are homeomorphisms, by Lemma 5.2.4. Thus $M_0 \sqcup_{Y_0} N_0 \subseteq M \sqcup_Y N$ and $M_0 \sqcup_{Y_0} N_0 = p_M(M_0) \cup p_N(N_0)$ where p_M and p_N are as in Proposition 5.3.11.

Let $m \in M \sqcup_Y N$ be any point, then it has a preimage $p^{-1}(m)$ in M or N, and thus there is a path in M or N connecting $p^{-1}(m)$ to a point in M_0 or N_0. The image of this path under p_M or p_N connects m to a point in $M_0 \sqcup_{Y_0} N_0$.

We have from Proposition 5.3.32 that $i{:}X \to M \sqcup_Y N \leftarrow Z :l$ is a concrete homotopy cobordism.

Since $M_0 \sqcup_{Y_0} N_0 = p_M(M_0) \cup p_N(N_0)$, and $i(X_0) \subseteq M_0$ and $l(Z_0) \subseteq N_0$, we have $\tilde{i}(X_0) \subseteq M_0 \sqcup_{Y_0} N_0$ and $\tilde{l}(Z_0) \subseteq M_0 \sqcup_{Y_0} N_0$, thus $\tilde{i}$ and $\tilde{k}$ are maps of pairs.

The map $\langle i,j\rangle{:}X \sqcup Y \to M$ is a cofibration, hence by Theorem 5.2.6 it is a homeomorphism onto its image. This means $i(X) \cap j(Y) = \varnothing$ in M, and similarly $k(Y) \cap l(Z) = \varnothing$. Hence there is no equivalence on points in X or Z in the pushout. Thus $(M_0 \sqcup_{Y_0} N_0) \cap \tilde{i}(X) = \tilde{i}(X_0)$ follows directly from the fact that $(M_0) \cap i(X) = i(X_0)$ and similarly $(M_0 \sqcup_{Y_0} N_0) \cap \tilde{l}(Z) = \tilde{l}(Z_0)$. □

5.4.2 Magmoid morphism from $\mathsf{bHomCob}$ to $\mathsf{Vect}_{\mathbb{C}}$

Here we construct a magmoid morphism $\mathsf{Z}^!_G{:}\,\mathsf{bHomCob} \to \mathsf{Vect}_{\mathbb{C}}$.

Recall from Proposition 3.1.29 that there is a groupoid $\mathcal{G}_G$ obtained from any group G with morphisms the elements of G. Throughout this section, by abuse of notation we will use G for $\mathcal{G}_G$.

Definition 5.4.6. Let G be a group.

For a pair $(X, X_0) \in \boldsymbol{\chi}$, define

$$\mathsf{Z}^!_G(X, X_0) = \mathbb{C}\left(\mathbf{Grpd}\left(\pi(X, X_0), G\right)\right).$$

That is, $\mathsf{Z}^!_G(X, X_0)$ is the $\mathbb{C}$ vector space whose basis is the set of groupoid maps from the fundamental groupoid of X with respect to X_0 into G.

Example 5.4.7. *Let $X = S^1 \sqcup S^1$, and let $X_0 \subset X$ contain two points, one in each copy of S^1. We have $\pi(X, X_0) \cong \mathbb{Z} \sqcup \mathbb{Z}$. Hence maps from $\pi(X, X_0)$ to G are determined by pairs in $G \times G$, where the elements of G denote the image of the generating elements of each copy of $\mathbb{Z}$. So we have $\mathsf{Z}^!_G(X, X_0) \cong \mathbb{C}(G \times G)$.*

In the following example note that X_0 must be representative by the definition of $\boldsymbol{\chi}$, therefore we choose basepoints even in path components that are homotopically trivial.

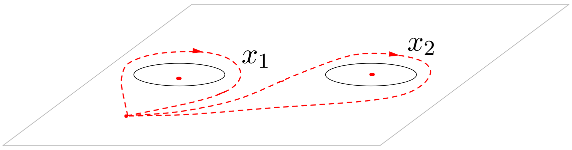


Figure 5.5: Let X be the complement in $\mathbb{I}^2$ of the embeddings of S^1 shown. Let X_0 be three basepoints as shown.

This will be necessary for the full calculation since, when considered as part of a cospan, these trivial components may have image in a homotopically non trivial component.

Example 5.4.8. *Let X and X_0 be as explained in the caption to Figure 5.5. Then $\pi(X,X_0) \cong \mathbb{Z} * \mathbb{Z}$ and maps from $\pi(X,X_0)$ to G are determined by pairs in $G \times G$, where the elements of G denote the image the equivalence classes of the loops marked x_1 and x_2 in the figure. So we have $\mathsf{Z}^{!}_{G}(X,X_0) \cong \mathbb{C}(G \times G)$.*

The vector spaces $\mathsf{Z}^{!}_{G}(X,X_0)$ has an intrinsic basis, the maps into G. We will define linear maps assigned to based homotopy cobordisms as matrices in terms of these bases.

Definition 5.4.9. Let $i\colon (X,X_0) \to (M,M_0) \leftarrow (Y,Y_0) : j$ be a concrete based homotopy cobordism. We define a matrix

$$\mathsf{Z}^{!}_{G}\left(\begin{tikzcd} (X,X_0) \arrow[dr, "i"'] & & (Y,Y_0) \arrow[dl, "j"] \\ & (M,M_0) & \end{tikzcd}\right) : \mathsf{Z}^{!}_{G}(X,X_0) \to \mathsf{Z}^{!}_{G}(Y,Y_0)$$

as follows. Let $f \in \mathsf{Z}^{!}_{G}(X,X_0)$ and $g \in \mathsf{Z}^{!}_{G}(Y,Y_0)$ be basis elements, then

$$\left\langle g \,\middle|\, \mathsf{Z}^{!}_{G}\left(\begin{tikzcd} (X,X_0) \arrow[dr, "i"'] & & (Y,Y_0) \arrow[dl, "j"] \\ & (M,M_0) & \end{tikzcd}\right) \middle|\, f \right\rangle = \left| \left\{ h : \pi(M,M_0) \to G \;\middle|\; \begin{tikzcd} \pi(X,X_0) \arrow[dr, "\pi(i)"] \arrow[ddr, bend right, "f"'] & & \pi(Y,Y_0) \arrow[dl, "\pi(j)"'] \arrow[ddl, bend left, "g"] \\ & \pi(M,M_0) \arrow[d, "h"] & \\ & G & \end{tikzcd} \right\} \right|. \tag{5.2}$$

In other words, the right hand side is the cardinality of the set of maps h making the diagram commute. Here we are using Dirac notation: (5.2) is the matrix element in the column corresponding to f and the row corresponding to g.

When we have already specified the relevant cospan, we will often use $\mathsf{Z}^{!}_{G}(M,M_0)$ for

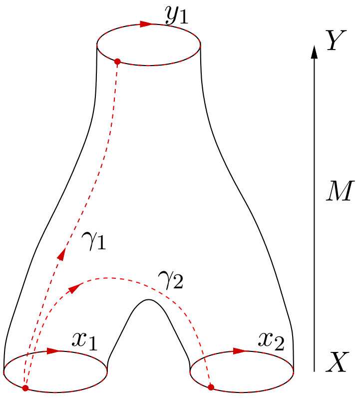


Figure 5.6: This figure represents the concrete cofibrant cospan from Example 5.3.8, so M is the represented manifold, and X and Y are the bottom and top boundary respectively, with the inclusion maps. The red points and lines show a possible choice of basepoints M_0 and generating paths. Let X_0 and Y_0 be the intersection of M_0 with X and Y respectively.

$\mathsf{Z}^!_G\,(i\colon(X,X_0)\to(M,M_0)\leftarrow(Y,Y_0)\colon j)$, and write the matrix elements as

$$\langle g\,|\,\mathsf{Z}^!_G(M,M_0)\,|\,f\rangle = \left|\left\{h\colon\pi(M,M_0)\to G\;|\;h|_{\pi(X,X_0)}=f\wedge h|_{\pi(Y,Y_0)}=g\right\}\right|,$$

where by $h|_{\pi(X,X_0)}$ we really mean the restriction of the map h to the image $\pi(i)(\pi(X,X_0))$.

Example 5.4.10. *Let $i\colon(X,X_0)\to(M,M_0)\leftarrow(Y,Y_0)\colon j$ be the based homotopy cobordism shown in Figure 5.6 with base points as marked. Note this is a homotopy cobordism as discussed in Examples 5.3.8 and 5.3.27. Note also that there is a finite number of marked points, and at least one in each connected component of X, Y and M.*

Now $\pi(Y,Y_0)\cong\mathbb{Z}$, where the isomorphism is realised by mapping the loop labelled y_1 in the figure to 1. Hence a map $g\colon\pi(Y,Y_0)\to G$ is uniquely determined by a choice of an element $g_1\in G$ with $f(\overset{p}{\sim}y_1)=g_1$. Thus we have $\mathsf{Z}^!_G(Y,Y_0)\cong\mathbb{C}(G)$.

Recall from Example 5.4.7 that $\mathsf{Z}^!_G(X,X_0)\cong\mathbb{C}(G\times G)$, where a pair (g_1,g_2) denotes the map $(g_1,g_2)(\overset{p}{\sim}x_1)=g_1$ and $(g_1,g_2)(\overset{p}{\sim}x_2)=g_2$.

*Let x be the basepoint which is in the loop labelled x_1. By Lemma 3.3.14, there is a bijection sending a map $h\in$ **Grpd**$(\pi(M,M_0),G)$ to a map $h'\in$ **Grpd**$(\pi(M,\{x\})\times G\times G$, which agrees with h on $\pi(M,\{x\})$ and where the first element of G corresponds to the image $h(\gamma_1)$ and the second to $h(\gamma_2)$. The space M is equivalent to the twice punctured disk, which has fundamental group isomorphic to the free product $\mathbb{Z}*\mathbb{Z}$. This isomorphism can be realised* *by sending the element represented by* x_1 *to the* 1 *in the first copy of* $\mathbb{Z}$ *and by* $\gamma_2^{-1}x_2\gamma_2$ *to the* 1 *in the second copy of* $\mathbb{Z}$*. Thus we can label elements in* $\mathbf{Grpd}(\pi(M,\{x\}),G)$ *by elements of* $G\times G$ *where* $a \in (a,b)$ *corresponds to the image of* $\overset{p}{\sim} x_1$*, and* b *the image of* $\overset{p}{\sim} \gamma_2^{-1}x_2\gamma_2$*. Hence a map in* $\mathbf{Grpd}(\pi(M,M_0),G)$ *is determined by a quadruple* $(a,b,c,d) \in G{\times}G{\times}G{\times}G$ *where* a *corresponds to the image of* x_1*,* b *to the image of* $\gamma_2^{-1}x_2\gamma_2$*, and* c *and* d *correspond to the images of* γ_1 *and* γ_2 *respectively.*

Choosing basis elements $(f_1,f_2) \in \mathsf{Z}^!_G(X,X_0)$ *and* $g_1 \in \mathsf{Z}^!_G(Y,Y_0)$ *the commutation condition in* (5.2) *gives conditions on allowed quadruples* $(a,b,c,d) \in G\times G\times G\times G$*. We have*

$$\begin{aligned}\langle (g_1) \mid \mathsf{Z}^!_G(M,M_0) \mid (f_1,f_2)\rangle &= \left|\left\{a,b,c,d \in G \mid (a, dbd^{-1}) = (f_1,f_2), c^{-1}bac = g_1\right\}\right| \\ &= \left|\left\{b,c,d \in G | dbd^{-1} = f_2, c^{-1}bf_1c = g_1\right\}\right| \\ &= \left|\left\{d \in G | c^{-1}d^{-1}f_2df_1c = g_1\right\}\right|.\end{aligned}$$

Lemma 5.4.11. *We have a magmoid morphism*

$$\mathsf{Z}^!_G\colon \mathsf{bHomCob} \to \mathsf{Vect}_{\mathbb{C}}.$$

Proof. It is immediate from the construction that the map is well defined. Thus we only need to check that composition is preserved. Let $i\colon (X,X_0) \to (M,M_0) \leftarrow (Y,Y_0) : j$ and $k\colon (Y,Y_0) \to (N,N_0) \leftarrow (Z,Z_0) : l$ be concrete based homotopy cobordisms. Let $f \in \mathsf{Z}^!_G(X,X_0)$ and $g \in \mathsf{Z}^!_G(Z,Z_0)$ be basis elements. The matrix element corresponding to these basis elements is given by counting maps h in the following diagram.

$$\begin{tikzcd}
\pi(X,X_0) \arrow[dr, "\pi(i)"'] \arrow[dddrr, bend right, "f"'] & & \pi(Y,Y_0) \arrow[dl, "\pi(j)"] \arrow[dr, "\pi(k)"'] & & \pi(Z,Z_0) \arrow[dl, "\pi(l)"] \arrow[dddll, bend left, "g"] \\
& \pi(M,M_0) \arrow[dr] & & \pi(N,N_0) \arrow[dl] & \\
& & \pi(M \sqcup_Y N, M_0 \sqcup_{Y_0} N_0) \arrow[d, "h"] & & \\
& & G & &
\end{tikzcd}$$

From Definition 5.4.2 and Lemma 5.3.10, the pushout of $(M,M_0) \xleftarrow{j} (Y,Y_0) \xrightarrow{k} (N,N_0)$ satisfies the conditions of Corollary 5.2.18. Hence the middle square of this diagram is a pushout. Hence each h is uniquely determined by a pair $h_1 \colon \pi(M,M_0) \to G$ and $h_2 \colon \pi(N,N_0) \to G$ such that the above diagram commutes. So we have

$$\begin{aligned}
&\langle g \,|\, \mathcal{F}^!_G(M \sqcup_Y N, M_0 \sqcup_{Y_0} N_0) \,|\, f \rangle \\
&\qquad = \left| \left\{ h_1, h_2 \,|\, h_1 \circ \pi(j) = h_2 \circ \pi(k) \wedge h_1|_{\pi(X,X_0)} = f \wedge h_2|_{\pi(Z,Z_0)} = g \right\} \right| \\
&\qquad = \sum_{\theta : \pi(Y,Y_0) \to G} \left| \left\{ h_1 \,|\, h_1|_{\pi(Y,Y_0)} = \theta \wedge h_1|_{\pi(X,X_0)} = f \right\} \right| \\
&\qquad\qquad\qquad\qquad \left| \left\{ h_2 \,|\, h_2|_{\pi(Y,Y_0)} = \theta \wedge h_2|_{\pi(Z,Z_0)} = g \right\} \right| \\
&\qquad = \sum_{\theta : \pi(Y,Y_0) \to G} \langle g | \mathcal{F}^!_G(N,N_0) | \theta \rangle \langle \theta | \mathcal{F}^!_G(M,M_0) | f \rangle
\end{aligned}$$

Now this is precisely the corresponding matrix element given by multiplying the matrices $\mathcal{F}^!_G(M,M_0)$ and $\mathcal{F}^!_G(N,N_0)$. □

The following lemma says that $\mathsf{Z}^!_G$ respects cospan homotopy equivalence.

Lemma 5.4.12. *Suppose we have concrete homotopy cobordisms $i \colon X \to M \leftarrow Y : j$ and $i' \colon X \to M' \leftarrow Y : j'$ which are equivalent up to cospan homotopy equivalence (as defined in Lemma 5.3.14). Then (by Theorem 5.2.22) we have homotopy equivalences $\psi \colon M \to M'$ and $\psi' \colon M' \to M$ which commute with the cospans. Choose sets of baspoints $X_0 \subseteq X$, $Y_0 \subseteq Y$, $M_0 \subseteq M$ such that $i \colon (X,X_0) \to (M,M_0) \leftarrow (Y,Y_0) : j$ is a based homotopy cobordism. Then*

$$\mathsf{Z}^!_G\left(\begin{tikzcd}[sep=small] (X,X_0) \arrow[dr, "i"'] & & (Y,Y_0) \arrow[dl, "j"] \\ & (M,M_0) & \end{tikzcd}\right) = \mathsf{Z}^!_G\left(\begin{tikzcd}[sep=small] (X,X_0) \arrow[dr, "i'"'] & & (Y,Y_0) \arrow[dl, "j'"] \\ & (M',M'_0) & \end{tikzcd}\right)$$

where $M'_0 = \psi(M_0)$.

Proof. Let $f \in \mathsf{Z}^!_G(X,X_0)$ and $g \in \mathsf{Z}^!_G(Y,Y_0)$ be basis elements and $h \colon \pi(M,M_0) \to G$ a map with $h|_{\pi(X,X_0)} = f$ and $h|_{\pi(Y,Y_0)} = g$. On the level of fundamental groupoids ψ and ψ' become inverse group isomorphisms making the following diagram commute.

$$\begin{tikzcd}
 & & \pi(M,M_0) \arrow[d, bend right, "\pi(\psi)"'] \arrow[dd, bend left=40, "h"] & & \\
\pi(X,X_0) \arrow[urr, "\pi(i)"] \arrow[drr, "\pi(i')"'] \arrow[ddrr, bend right, "f"'] & & & & \pi(Y,Y_0) \arrow[ull, "\pi(j)"'] \arrow[dll, "\pi(j')"] \arrow[ddll, bend left, "g"] \\
 & & \pi(M',M'_0) \arrow[u, bend right, "\pi(\psi')"'] \arrow[d, dashed, "h'"] & & \\
 & & G & &
\end{tikzcd}$$

For any such h we can obtain a map h' making the diagram commute by precomposing h with $\pi(\psi')$. Thus we have a set map

$$\Psi\colon \left\{h\colon \pi(M,M_0)\to G \,|\, h|_{\pi(X,X_0)} = f \wedge h|_{\pi(Y,Y_0)} = g\right\} \to \\ \left\{h'\colon \pi(M',M_0')\to G \,|\, h'|_{\pi(X,X_0)} = f \wedge h'|_{\pi(Y,Y_0)} = g\right\},$$

which has inverse given by precomposing with $\pi(\psi)$. Thus we have $\langle g\,|\,\mathsf{Z}^!_G(M,M_0)\,|\,f\rangle = \langle g\,|\,\mathsf{Z}^!_G(M',M_0')\,|\,f\rangle$ for all f,g. □

5.4.3 Functor from HomCob to $\mathbf{Vect}_{\mathbb{C}}$

The magmoid morphism $\mathsf{Z}^!_G$ depends on the choices of sets of basepoints. Also notice that there are many ways we could obtain a based cospan from the cospan representing the identity, $\iota_0^X\colon X \to X\times\mathbb{I} \leftarrow X :\iota_1^X$, and in general these based cospans will not give the identity matrix under $\mathsf{Z}^!_G$. In this section we take a colimit over $\mathsf{Z}^!_G$ for all choices of basepoints, and adjust the map on morphisms such that Z_G no longer depends on the sets of basepoints, and hence we can extend it to a functor from HomCob. We will find that removing the basepoint dependence will also solve the identity problem.

Varying the set of basepoints

Let $i\colon (X,X_0)\to(M,M_0)\leftarrow(Y,Y_0) :j$ be a concrete based homotopy cobordism. We first consider how changing the set of basepoints in the set M_0 changes $\mathsf{Z}^!_G(M,M_0)$.

If such a point exists, choose a point $m\in M\smallsetminus M_0$ such that $i\colon(X,X_0)\to(M,M_0\cup\{m\})\leftarrow(Y,Y_0) :j$ is also a concrete based homotopy cobordism. By Lemma 3.3.14, the set of maps $h'\colon\pi(M,M_0\cup\{m\})\to G$ is in bijective correspondence with the set of pairs of a map $h\colon\pi(M,M_0)\to G$ and an element of G, via the bijection Θ_γ. Let $f\in\mathsf{Z}^!_G(X,X_0)$ and $g\in\mathsf{Z}^!_G(Y,Y_0)$ be basis elements. Note that for all $h\colon\pi(M,M_0)\to G$ such that $h|_{\pi(X,X_0)} = f$ and $h|_{\pi(Y,Y_0)} = g$, the map h' obtained from a pair (h,g) using the map Θ_γ^{-1} as in the proof of Lemma 3.3.14 also satisfies $h'|_{\pi(X,X_0)} = f$ and $h'|_{\pi(Y,Y_0)} = g$. Hence by Lemma 3.3.14 for all pairs f,g we have that $\left\langle g\,\middle|\,\mathsf{Z}^!_G(M,M_0\cup\{m\})\,\middle|\,f\right\rangle = |G|\left\langle g\,\middle|\,\mathsf{Z}^!_G(M,M_0)\,\middle|\,f\right\rangle$, and hence that

$$\mathsf{Z}^!_G(M,M_0\cup\{m\}) = |G|\,\mathsf{Z}^!_G(M,M_0).$$

It follows that for all $M_0' \supseteq M_0$ we have $\mathsf{Z}^!_G(M, M_0') = |G|^{(|M_0'|-|M_0|)}\mathsf{Z}^!_G(M, M_0)$, and hence

$$|G|^{-|M_0'|}\mathsf{Z}^!_G(M, M_0') = |G|^{-|M_0|}\mathsf{Z}^!_G(M, M_0).$$

Now suppose instead there are no containment conditions between M_0' and M_0, then we can write

$$\mathsf{Z}^!_G(M, M_0' \cup M_0) = |G|^{(|M_0' \cup M_0|-|M_0|)}\mathcal{F}^!_G(M, M_0)$$

and

$$\mathsf{Z}^!_G(M, M_0' \cup M_0) = |G|^{(|M_0' \cup M_0|-|M_0'|)}\mathsf{Z}^!_G(M, M_0')$$

which together imply

$$|G|^{-|M_0|}\mathsf{Z}^!_G(M, M_0) = |G|^{-|M_0'|}\mathsf{Z}^!_G(M, M_0')$$

and that

$$|G|^{-(|M_0|-|X_0|)}\mathsf{Z}^!_G(M, M_0) = |G|^{-(|M_0'|-|X_0|)}\mathsf{Z}^!_G(M, M_0').$$

We have proven the following.

Lemma 5.4.13. *The linear map $\mathsf{Z}^{!!}_G$, assigning a linear map to a concrete based homotopy cobordism as follows*

$$\mathsf{Z}^{!!}_G\left(\begin{tikzcd}[row sep=small, column sep=small] (X,X_0) \arrow[dr, "i"'] & & (Y,Y_0) \arrow[dl, "j"] \\ & (M,M_0) & \end{tikzcd}\right) = |G|^{-(|M_0|-|X_0|)}\mathsf{Z}^!_G\left(\begin{tikzcd}[row sep=small, column sep=small] (X,X_0) \arrow[dr, "i"'] & & (Y,Y_0) \arrow[dl, "j"] \\ & (M,M_0) & \end{tikzcd}\right),$$

does not depend on the choice of subset $M_0 \subseteq M$. ☐

When the relevant cospan is clear, we will refer to the image as $\mathsf{Z}^{!!}_G(M, X_0, Y_0)$ to highlight the dependence on X_0 and Y_0.

In defining this new matrix we have included a term counting the cardinality of X_0. Some term counting basepoints in X or Y is necessary to ensure the new definition is still compatible with the composition; however we could have chosen $1/2(|X_0| + |Y_0|)$ for example, as is the convention in [Yet92], and avoided the asymmetry. The reason for our convention is that it allows us to work for longer in the basis set rather than moving to the $\mathbb{C}$ vector space, making calculation easier. We will highlight later where this becomes relevant (Remark 5.4.1).

Lemma 5.4.14. *Let* $i\colon (X,X_0) \rightarrow (M,M_0) \leftarrow (Y,Y_0) :j$ *and* $k\colon (Y,Y_0) \rightarrow (N,N_0) \leftarrow (Z,Z_0) :l$ *be concrete based homotopy cobordisms. Then*

$$\mathsf{Z}^{!!}_G\left(\begin{tikzcd}(Y,Y_0) \arrow[dr, "k"'] & & (Z,Z_0) \arrow[dl, "l"] \\ & (N,N_0) & \end{tikzcd}\right)\mathsf{Z}^{!!}_G\left(\begin{tikzcd}(X,X_0) \arrow[dr, "i"'] & & (Y,Y_0) \arrow[dl, "j"] \\ & (M,M_0) & \end{tikzcd}\right) = \mathsf{Z}^{!!}_G\left(\begin{tikzcd}(X,X_0) \arrow[dr, "i"'] & & (Y,Y_0) \arrow[dl, "j"] \\ & (M,M_0) & \end{tikzcd} \bullet \begin{tikzcd}(Y,Y_0) \arrow[dr, "k"'] & & (Z,Z_0) \arrow[dl, "l"] \\ & (N,N_0) & \end{tikzcd}\right)$$

where concatenation denotes composition of linear maps, or equivalently matrix multiplication.

Proof. We have

$$\begin{aligned}\mathsf{Z}^{!!}_G(M \sqcup_Y N, X_0, Z_0) &= |G|^{-(|M_0 \sqcup_{Y_0} N_0| - |X_0|)}\mathsf{Z}^{!}_G(M \sqcup_Y N, M_0 \sqcup_{Y_0} N_0)\\ &= |G|^{-(|M_0|+|N_0|-|Y_0|-|X_0|)}\mathsf{Z}^{!}_G(M,M_0)\mathsf{Z}^{!}_G(N,N_0)\\ &= |G|^{-(|M_0|-|X_0|)}\mathsf{Z}^{!}_G(M,M_0)|G|^{-(|N_0|-|Y_0|)}\mathsf{Z}^{!!}_G(N,N_0)\\ &= \mathsf{Z}^{!!}_G(M,X_0,Y_0)\mathsf{Z}^{!!}_G(N,Y_0,Z_0)\end{aligned}$$ □

using that, by 5.4.23, $\mathsf{Z}^{!}_G$ preserves composition.

Basepoint independent map from $Ob(\text{HomCob})$ to $Ob(\mathbf{Vect}_{\mathbb{C}})$

We focus here on the sets of basepoints in $Ob(\mathsf{bHomCob})$. Here we will move to using Greek subscripts to indicate varying choices of subsets, so for a space X, objects in $\mathsf{bHomCob}$ are pairs of the form (X, X_α). We will eventually show that we can choose just one subset to calculate our functor and will then switch back to the original notation.

We proceed by constructing, for a space $X \in Ob(\text{HomCob})$, a colimit in $\mathbf{Vect}_{\mathbb{C}}$ over a diagram with vertices the images under $\mathsf{Z}^{!}_G$ of all possible choices X_α such that $(X, X_\alpha) \in \chi = Ob(\mathsf{bHomCob})$.

Proposition 5.4.15. *Let* X *be a homotopically* 1*-finitely generated space. There is a subcategory of* $\mathbf{Set}$*,*

$$\mathbf{FinSet}^*(X) = (Ob(\mathbf{FinSet}^*(X)), \mathbf{FinSet}^*(X)(-,-), \circ, \mathrm{id})$$

where $Ob(\mathbf{FinSet}^*(X))$ *contains all* X_α *such that* $(X, X_\alpha) \in \chi$ *and* $\mathbf{FinSet}^*(X)(X_\alpha, X_\beta)$ *contains the inclusion* $\iota_{\alpha\beta}: X_\alpha \to X_\beta$ *if* $X_\alpha \subseteq X_\beta$*, otherwise* $\mathbf{FinSet}^*(X)(X_\alpha, X_\beta) = \varnothing$*.*

Proof. Note we have $\iota_{\alpha\alpha} = 1_{X_\alpha}: X_\alpha \to X\alpha$ in $\mathbf{FinSet}^*(X)(X_\alpha, X_\alpha)$. Suppose $X_\alpha, X_\beta, X_\gamma \in \mathbf{FinSet}^*(X)$, with $X_\alpha \subseteq X_\beta \subseteq X_\gamma$, then the composition of $\iota_{\alpha\beta}: X_\alpha \to X_\beta$ and $\iota_{\beta\gamma}: X_\beta \to X_\gamma$ is precisely the unique morphism in $\mathbf{FinSet}^*(X)(X_\alpha, X_\gamma)$. (This is the only case for which we have composable morphisms.) □

By abuse of notation, for an inclusion $\iota_{\alpha\beta}: X_\alpha \to X_\beta$ we will also write $\iota_{\alpha\beta}: \pi(X, X_\alpha) \to \pi(X, X_\beta)$ for the inclusion of groupoids.

Lemma 5.4.16. *There is a contravariant functor*

$$\mathcal{V}_X : \mathbf{FinSet}^*(X) \to \mathbf{Set}$$

constructed as follows. Let $X_\alpha, X_\beta \in Ob(\mathbf{FinSet}^*(X))$ *with* $X_\beta \subseteq X_\alpha$*. Let* $\mathcal{V}_X(X_\alpha) = \mathbf{Grpd}(\pi(X, X_\alpha), G)$*. For any* $v_\alpha \in \mathcal{V}_X(X_\alpha)$ *we have a commuting triangle*

$$\begin{tikzcd} \pi(X, X_\beta) \arrow[r, "\iota_{\beta\alpha}"] \arrow[dr, dashed, "v_\alpha \circ \iota_{\beta\alpha}"'] & \pi(X, X_\alpha) \arrow[d, "v_\alpha"] \\ & G. \end{tikzcd}$$

Now let $\mathcal{V}_X(\iota_{\beta\alpha}: X_\beta \to X_\alpha) = \phi_{\alpha\beta}$ *where* $\phi_{\alpha\beta} : \mathcal{V}_X(X_\alpha) \to \mathcal{V}_X(X_\beta)$*,* $v_\alpha \mapsto v_\alpha \circ \iota_{\alpha\beta}$*.*

Proof. We have $\mathcal{V}(1_{X_\alpha}: X_\alpha \to X_\alpha) = 1_{\mathcal{V}(X_\alpha)}: \mathcal{V}(X_\alpha) \to \mathcal{V}(X_\alpha)$. Suppose $X_\alpha, X_\beta, X_\gamma \in \mathbf{FinSet}^*(X)$, with $X_\gamma \subseteq X_\beta \subseteq X_\alpha$, then $\phi_{\alpha\beta} \circ \phi_{\beta\gamma} = \phi_{\alpha\gamma}$ since $(v_\alpha \circ \iota_{\beta\alpha}) \circ \iota_{\gamma\beta} = v_\alpha \circ (\iota_{\alpha\gamma})$. □

Lemma 5.4.17. *For any space* X *and* $X_\beta, X_\alpha \in Ob(\mathbf{FinSet}^*(X))$ *with* $X_\beta \subseteq X_\alpha$*,* $\phi_{\alpha\beta}$ *is a surjection.*

Proof. By Lemma 3.3.13 for any $v_\beta \in \mathcal{V}_X(X_\beta)$ we can extend to some $v_\alpha \in \mathcal{V}_X(X_\alpha)$ which is equal to v_β on the image $\iota_{\beta\alpha}(\pi(X, X_\beta))$ in $\pi(X, X_\alpha)$. □

The colimit over $\mathcal{V}_X$ consists of a family of commuting triangles diagrams of the form

$$\begin{tikzcd}
\mathcal{V}_X(X_\alpha) \arrow[rr, "\phi_{\alpha\beta}"] \arrow[dr, "\phi_\alpha"'] & & \mathcal{V}_X(X_\beta) \arrow[dl, "\phi_\beta"] \\
& colim(\mathcal{V}_X) &
\end{tikzcd}$$

for each pair $X_\beta \subseteq X_\alpha$. By abuse of notation we will use v_α for both $v_\alpha \in \mathcal{V}_X(X_\alpha)$ and its image in $\sqcup_{X_\alpha}\mathcal{V}_X(X_\alpha)$. Hence we have

$$colim(\mathcal{V}_X) = {}^{\sqcup_{X_\alpha}\mathcal{V}_X(X_\alpha)}/_{\sim}$$

where $\sim$ is the reflexive, symmetric and transitive closure of $v_\alpha \sim v_\beta$ if $\phi_{\alpha\beta}(v_\alpha) = v_\beta$. See Section 3.6 for more on colimits in **Set**. We use $[v_\alpha]$ to denote the equivalence class of v_α in $colim(\mathcal{V}_X)$. Hence we have $\phi_\alpha : \mathcal{V}_X(X_\alpha) \to colim(\mathcal{V}_X)$, $v_\alpha \mapsto [v_\alpha]$.

Notice that this relation is certainly not itself an equivalence. For example for any $X_\beta \subset X_\alpha$, with $v_\beta = v_\alpha \circ \iota_{\beta\alpha} : \pi(X, X_\beta) \to G$, then the relation says $v_\beta = \phi_{\alpha\beta}(v_\alpha) \sim v_\alpha$ but not $v_\alpha \sim v_\beta$ as there is no map $\phi_{\beta\alpha}$.

Lemma 5.4.18. *Let $\mathcal{V}_X : \mathbf{FinSet}^*(X) \to \mathbf{Set}$ be as in Lemma 5.4.16, then all maps $\phi_\alpha : \mathcal{V}_X(X_\alpha) \to colim(\mathcal{V}_X)$ are surjections.*

Proof. Fix some $\mathcal{V}_X(X_\alpha)$. We must show that every equivalence class $[v] \in colim(\mathcal{V}_X)$ has a representative in $\mathcal{V}_X(X_\alpha)$. Certainly $[v]$ has a representative v_β in some $\mathcal{V}_X(X_\beta)$. Let $X_\gamma = X_\alpha \cup X_\beta$ and choose $v_\gamma \in \mathcal{V}_X(X_\gamma)$ with $\phi_{\gamma\beta}(v_\gamma) = v_\beta$, which is always possible since $\phi_{\gamma\beta}$ is an epimorphism by Lemma 5.4.17. Now $v_\alpha = \phi_{\gamma\alpha}(v_\gamma)$ is a representative for $[v]$ since $v_\gamma \sim v_\beta$ and $v_\gamma \sim v_\alpha$. □

Lemma 5.4.19. *Let $\mathcal{V}_X : \mathbf{FinSet}^*(X) \to \mathbf{Set}$ be as in Lemma 5.4.16. The set $colim(\mathcal{V}_X)$ is finite.*

Proof. The groupoid $\pi(X, X_\alpha)$ is finitely generated since X is a homotopically 1-finitely generated space and G is finite, hence the $\mathcal{V}_X(X_\alpha)$ is finite for all X_α, so with Lemma 5.4.18 we have the result. □

The free functor $F_{V_\mathbb{C}}$ is a left adjoint to the forgetful functor $U_{V_\mathbb{C}}$ and so preserves colimits (see Lemma 3.5.13).

Definition 5.4.20. For $X \in \chi$ define

$$\mathsf{Z}_G(X) = colim(\mathcal{V}'_X) = \mathbb{C}(colim(\mathcal{V}_X))$$

where $\mathcal{V}'_X = F_{V_\mathbb{C}} \circ \mathcal{V}_X$ and $\mathcal{V}_X \colon \mathbf{FinSet}^*(X) \to \mathbf{Set}$ as in Lemma 5.4.16.

Magmoid morphism $\mathsf{Z}_G \colon \mathsf{HomCob} \to \mathsf{Vect}_\mathbb{C}$

The linear map $\mathsf{Z}^{!!}_G$ assigned to a based homotopy cobordism still depends on the basepoints in objects. Here we will adjust $\mathsf{Z}^{!!}_G$ such that it assigns to a based homotopy cobordism, $i \colon (X, X_0) \to (M, M_0) \leftarrow (Y, Y_0) : j$, a linear map $\mathsf{Z}_G(X) \to \mathsf{Z}_G(Y)$. We will then show that this linear map is also be independent of X_0 and Y_0.

Let $(X, X_\alpha), (X, X_\beta) \in \boldsymbol{\chi}$. In the previous section we constructed $\mathcal{V}_X \colon \mathbf{FinSet}^*(X) \to \mathbf{Set}$ (Lemma 5.4.16) which sends inclusions $\iota_{\beta\alpha} \colon X_\beta \to X_\alpha$ to maps $\phi_{\alpha\beta} \colon \mathcal{V}(X_\alpha) \to \mathcal{V}\big(X_\beta\big)$. Notice that $F_{V_\mathbb{C}} \circ \mathcal{V}(X_\alpha) = \mathcal{V}'(X_\alpha) = \mathsf{Z}^!_G(X, X_\alpha)$, so we have a map $F_{V_\mathbb{C}}(\phi_{\alpha\beta}) \colon \mathsf{Z}^!_G(X, X_\alpha) \to \mathsf{Z}^!_G(X, X_\beta)$. By abuse of notation we will also use $\phi_{\alpha\beta}$ to refer to the maps $F_{V_\mathbb{C}}(\phi_{\alpha\beta})$. In this section we will need to vary the input space in the construction of $\mathcal{V}_X$, thus we add a superscript denoting the space, so we have maps

$$\phi^X_{\alpha\beta} \colon \mathsf{Z}^!_G(X, X_\alpha) \to \mathsf{Z}^!_G\big(X, X_\beta\big).$$

Lemma 5.4.21. *Let $i \colon X \to M \leftarrow Y : j$ be a concrete homotopy cobordism. Then for any pair $X_\alpha, X_\beta \subseteq X$ with $X_\beta \subseteq X_\alpha$, and concrete based homotopy cobordisms $i \colon (X, X_\alpha) \to (M, M_{\alpha\alpha'}) \leftarrow (Y, Y_{\alpha'}) : j$ and $i \colon (X, X_\beta) \to (M, M_{\beta\alpha'}) \leftarrow (Y, Y_{\alpha'}) : j$, the following diagram commutes*

$$\begin{tikzcd}
\mathsf{Z}^!_G(X, X_\alpha) \arrow[rr, "\phi^X_{\alpha\beta}"] \arrow[dr, "\mathsf{Z}^{!!}_G(M, X_\alpha, Y_{\alpha'})"'] & & \mathsf{Z}^!_G\big(X, X_\beta\big) \arrow[dl, "\mathsf{Z}^{!!}_G(M, X_\beta, Y_{\alpha'})"] \\
& \mathsf{Z}^!_G(Y, Y_{\alpha'}) &
\end{tikzcd}.$$

That is, the maps $\mathsf{Z}^{!!}_G$ form a cocone over the vector spaces $\mathsf{Z}^!_G(X, X_\alpha)$ and the maps $\phi^X_{\alpha\beta}$.

Hence there is a unique map

$$d^M_{\alpha'} \colon \mathsf{Z}_G(X) \to \mathsf{Z}^!_G(Y, Y_{\alpha'}).$$

Proof. First suppose $X_\alpha = X_\beta \cup \{x\}$ for some $x \notin X_\beta$. Let $f \in \mathsf{Z}^!_G(X, X_\alpha)$ and $g \in \mathsf{Z}^!_G(Y, Y_{\alpha'})$ be basis elements. We have

$$\left\langle g \middle| \mathsf{Z}^{!!}_G(M, X_\alpha, Y_{\alpha'}) \middle| f \right\rangle = |G|^{-(|M_{\alpha\alpha'}| - |X_\alpha|)} \left\langle g \,\middle|\, \mathsf{Z}^!_G(M, M_{\alpha\alpha'}) \,\middle|\, f \right\rangle$$

and

$$\left\langle g \middle| \mathsf{Z}^{!!}_G(M, X_\beta, Y_{\alpha'}) \middle| \phi^X_{\alpha\beta}(f) \right\rangle = |G|^{-(|M_{\beta\alpha'}| - |X_\beta|)} \left\langle g \,\middle|\, \mathsf{Z}^!_G(M, M_{\beta\alpha'}) \,\middle|\, \phi^X_{\alpha\beta}(f) \right\rangle$$

for appropriate choices $M_{\alpha\alpha'}$ and $M_{\beta\alpha'}$. We may choose $M_{\alpha\alpha'} = M_{\beta\alpha'} \cup \{x\}$. There is a map from $\left\langle g \,\middle|\, \mathsf{Z}^!_G(M, M_{\alpha\alpha'}) \,\middle|\, f \right\rangle$ to $\left\langle g \,\middle|\, \mathsf{Z}^!_G(M, M_{\beta\alpha'}) \,\middle|\, \phi^X_{\alpha\beta}(f) \right\rangle$ given by taking the restriction of a map $h \colon \pi(M, M_{\alpha\alpha'}) \to G$ to $h' = h|_{\pi(M, M_{\beta\alpha'})}$. Note also that if $h|_{\pi(X, X_\alpha)} = f$ and $h|_{\pi(Y, Y_{\alpha'})} = g$, then also $h'|_{\pi(X, X_\beta)} = \phi^X_{\alpha\beta}(f)$ and $h'|_{\pi(Y, Y_{\alpha'})} = g$.

This map has inverse given by extending any map $\tilde{h} : \pi(M, M_{\beta\alpha'}) \to G$ to map $\tilde{h}' : \pi(M, M_{\alpha\alpha'}) \to G$, which sends a path $\gamma : x \to x'$ in $\pi(X, X_\alpha)$, with $x' \in X_\beta$, to $f(\gamma)$, as in Lemma 3.3.13. If the map $\tilde{h}$ satisfies $\tilde{h}|_{\pi(X, X_\beta)} = \phi^X_{\alpha\beta}(f)$ and $\tilde{h}|_{\pi(Y, Y_{\alpha'})} = g$, then $\tilde{h}'|_{\pi(X, X_\alpha)} = f$ and $\tilde{h}'|_{\pi(Y, Y_{\alpha'})} = g$. Hence

$$\langle g \,|\, \mathcal{F}^!_G(M, M_{\alpha\alpha'}) \,|\, f \rangle = \langle g \,|\, \mathcal{F}^!_G(M, M_{\beta\alpha'}) \,|\, \phi^X_{\alpha\beta}(f) \rangle.$$

Also $|M_{\beta\alpha'}| - |X_\beta| = |M_{\alpha\alpha'} + 1| - |X_\alpha + 1| = |M_{\alpha\alpha'}| - |X_\alpha|$. The set $X_\alpha \setminus X_\beta$ is finite, so we can repeat the same process for all $\{x_1, ..., x_n\} \in X_\alpha \setminus X_\beta$. □

Remark 5.4.1. Notice that, had we defined the normalisation to be $|G|^{-(M_0 - 1/2(X_0 + Y_0))}$, as is Yetter's convention, the triangle in the previous Lemma would not be commutative. The fix is to redefine the maps $\phi_{\alpha\beta}$ in such a way that they no longer send basis elements to basis elements. This complicates the picture slightly. It is straighforward to see that each choice leads to the same image on any cospan of the form $\varnothing \to M \leftarrow \varnothing$.

Lemma 5.4.22. *Let* $i \colon X \to M \leftarrow Y : j$ *be a concrete homotopy cobordism. Fix a choice of* $Y_{\alpha'} \subseteq Y$ *such that* $(Y, Y_{\alpha'}) \in \boldsymbol{\chi}$. *For each pair* $X_\alpha, X_\beta \subseteq X$ *such that* $(X, X_\alpha), (X, X_\beta) \in \boldsymbol{\chi}$

we have the following diagram

$$\begin{tikzcd}
\mathsf{Z}^!_G(X, X_\alpha) \arrow[rr, "\phi^X_{\alpha\beta}"] \arrow[dr, "\phi^X_\alpha"'] \arrow[ddr, bend right, "\mathsf{Z}^{!!}_G(M, X_\alpha, Y_{\alpha'})"'] & & \mathsf{Z}^!_G(X, X_\beta) \arrow[dl, "\phi^X_\beta"] \arrow[ddl, bend left, "\mathsf{Z}^{!!}_G(M, X_\beta, Y_{\alpha'})"] \\
& \mathsf{Z}_G(X) \arrow[d, "d^M_{\alpha'}"'] & \\
& \mathsf{Z}^!_G(Y, Y_{\alpha'}) \arrow[dr, bend right, "\phi^Y_{\alpha'}"'] & \\
& & \mathsf{Z}_G(Y).
\end{tikzcd} \tag{5.3}$$

The assignment

$$\mathsf{Z}_G\left(\begin{tikzcd}[ampersand replacement=\&] X \arrow[dr, "i"'] \& \& Y \arrow[dl, "j"] \\ \& M \& \end{tikzcd}\right) = \phi^Y_{\alpha'} d^M_{\alpha'}$$

does not depend on the choice of $Y_{\alpha'}$.

As above, where we have given a cospan we will use the notation $\mathsf{Z}_G(M)$ *for* $\mathsf{Z}_G(i: X \to M \leftarrow Y : j)$.

Proof. We show that the following diagram commutes for any pair $Y_{\alpha'}, Y_{\beta'}$

$$\begin{tikzcd}
& \mathsf{Z}^!_G(X, X_\alpha) \arrow[dl, "\mathsf{Z}^{!!}_G(M, X_\alpha, Y_{\alpha'})"'] \arrow[dr, "\mathsf{Z}^{!!}_G(M, X_\alpha, Y_{\beta'})"] & \\
\mathsf{Z}^!_G(Y, Y_{\alpha'}) \arrow[rr, "\phi^Y_{\alpha'\beta'}"'] & & \mathsf{Z}^!_G(Y, Y_{\beta'})
\end{tikzcd}$$

This implies that $\phi^Y_{\alpha'\beta'}$ is a map of cocones and, by the universal property of the colimit that $\phi^Y_{\alpha'\beta'} d^M_{\alpha'} = d^M_{\beta'}$ and hence that $\phi^Y_{\alpha'} d^M_{\alpha'} = \phi^Y_{\beta'} \phi^Y_{\alpha'\beta'} d^M_{\alpha'} = \phi^Y_{\beta'} d^M_{\beta'}$.

Suppose first that $Y_{\alpha'} = Y_{\beta'} \cup \{y\}$ for some $y \notin Y_{\beta'}$ and let $f \in \mathsf{Z}^!_G(X, X_\alpha)$ and $g \in \mathsf{Z}^!_G(Y, Y_{\beta'})$ be a basis elements. The map $\phi^Y_{\alpha', \beta'}: \mathcal{V}_Y(Y_{\alpha'}) \to \mathcal{V}_Y(Y_{\beta'})$ is an epimorphism (by Lemma 5.4.17), so sends a subset of $\mathcal{V}_Y(Y_{\alpha'})$ to $g \in \mathcal{V}_Y(Y_{\beta'})$. Thus the matrix element $\langle f | \phi^Y_{\alpha'\beta'} \mathsf{Z}^{!!}_G(M, X_\alpha, Y_{\alpha'}) | g \rangle$ is the sum of the matrix elements in $\mathsf{Z}^{!!}_G(M, X_\alpha, Y_{\alpha'})$ corresponding to f and to each g' in the preimage ${\phi^{Y}_{\alpha'\beta'}}^{-1}(g)$. Hence we have

$$\begin{aligned}\big\langle g \,\big|\, \phi^{Y}_{\alpha'\beta'} \mathsf{Z}^{!!}_{G}(M, X_\alpha, Y_{\alpha'}) \,\big|\, f \big\rangle &= \sum_{g' \in \phi^{Y-1}_{\alpha'\beta'}(g)} \big\langle g' \,\big|\, \mathsf{Z}^{!!}_{G}(M, X_\alpha, Y_{\alpha'}) \,\big|\, f \big\rangle \\ &= |G|^{-(|M_{\alpha\alpha'}| - |X_\alpha|)} \sum_{g' \in \phi^{Y-1}_{\alpha\beta}} \big\langle g' \,\big|\, \mathsf{Z}^{!}_{G}(M, M_{\alpha\alpha'}) \,\big|\, f \big\rangle\end{aligned}$$

for an appropriate choice of $M_{\alpha\alpha'}$. Following the same argument as used in the previous lemma we may choose a subset $M_{\alpha\beta'}$ with $M_{\alpha\alpha'} = M_{\alpha\beta'} \cup \{y\}$ and then

$$\big\langle g \,\big|\, \mathsf{Z}^{!}_{G}(M, M_{\alpha\beta'}) \,\big|\, f \big\rangle = \big\langle g' \,\big|\, \mathsf{Z}^{!}_{G}(M, M_{\alpha\alpha'}) \,\big|\, f \big\rangle .$$

For every map $g : \pi(Y, Y_{\beta'}) \to G$, there will be precisely G maps in the preimage under $\phi^{Y}_{\alpha'\beta'}$, one for each choice of an element of G. This can be seen by noting that $\phi^{Y}_{\alpha'\beta'}$ is the composition of the bijection $\Theta^{-1}_{\gamma}\colon \mathbf{Grpd}(\pi(X, Y_{\alpha'}), G) \to \mathbf{Grpd}(\pi(Y, Y_{\beta'}), G) \times G$ in Lemma 3.3.14 with the projection to the first coordinate, for some choice of $\gamma\colon y \to y' \in M_{\alpha\beta'}$. Hence we have

$$\begin{aligned}\big\langle g \,\big|\, \phi^{Y}_{\alpha'\beta'} \mathsf{Z}^{!!}_{G}(M, X_\alpha, Y_{\alpha'}) \,\big|\, f \big\rangle &= |G|^{-(|M_{\alpha\alpha'}| - |X_\alpha|)} |G| \big\langle g \,\big|\, \mathsf{Z}^{!}_{G}(M, M_{\alpha\alpha'}) \,\big|\, f \big\rangle \\ &= |G|^{-(|M_{\alpha\beta'}| - |X_\alpha|)} \big\langle g \,\big|\, \mathsf{Z}^{!}_{G}(M, M_{\alpha\beta'}) \,\big|\, f \big\rangle \\ &= \big\langle g \big| \mathsf{Z}^{!!}_{G}(M, X_\alpha, Y_{\beta'}) \big| f \big\rangle .\end{aligned}$$

Now suppose $Y_{\alpha'} = Y_{\beta'} \cup \{y_1, ..., y_n\}$, then we similarly acquire one factor of $|G|$ and one factor $|G|^{-1}$ for each new point, hence $\phi^{Y}_{\alpha'\beta'} \mathsf{Z}^{!!}_{G}(M, X_\alpha, Y_{\alpha'}) = \mathsf{Z}^{!!}_{G}(M, X_\alpha, Y_{\beta'})$. ☐

Lemma 5.4.23. *We have a magmoid morphism*

$$\mathsf{Z}_G\colon \mathsf{HomCob} \to \mathsf{Vect}_{\mathbb{C}}$$

where Z_G *is given in Definition 5.4.20 and Lemma 5.4.22.*

Proof. Lemmas 5.4.13 and 5.4.22 give that Z_G is well defined.

We prove Z_G preserves composition. Suppose we have concrete homotopy cobordisms $i\colon X \to M \leftarrow Y :j$ and $k\colon Y \to N \leftarrow Z :l$. Let $Y_0 \subseteq Y$ and $Z_0 \subseteq Z$ be fixed finite representative subsets. Notice that for any finite representative subset $X_0 \subseteq X$, by Lemma 5.4.14, we have $\mathsf{Z}^{!!}_G(M \sqcup N, X_0, Z_0) = \mathsf{Z}^{!!}_G(N, Y_0, Z_0)\mathsf{Z}^{!!}_G(M, X_0, Y_0) = d^N_0 \phi^Y_0 \mathsf{Z}^{!!}_G(M, X_0, Y_0)$. Thus $d^N_0 \phi^Y_0 d^M_0 : \mathsf{Z}_G(X) \to \mathsf{Z}^!_G(Z, Z_0)$ is a map commuting with the cocone given by the maps $\mathsf{Z}^{!!}_G(M \sqcup N, X_0, Z_0)$. Hence by the uniqueness of the map obtained from the universal property of the colimit gives $d^N_0 \phi^Y_0 d^M_0 = d^{M \sqcup_Y N}_0$. Hence we have $\phi^Z_0 d^N_0 \phi^Y_0 d^M_0 = \phi^Z_0 d^{M \sqcup_Y N}_0$ and $\mathsf{Z}_G(N)\mathsf{Z}_G(M) = \mathsf{Z}_G(M \sqcup_Y N)$. □

The functor Z_G: HomCob → $\mathbf{Vect}_{\mathbb{C}}$

The following theorem says that Z_G becomes a functor from the category HomCob.

Theorem 5.4.24. *There is a functor*

$$\mathsf{Z}_G : \mathrm{HomCob} \to \mathbf{Vect}_{\mathbb{C}}$$

defined as follows.

- *For a space $X \in Ob(\mathrm{HomCob})$,*

  $$\mathsf{Z}_G(X) = \mathbb{C}\left(colim(\mathcal{V}_X)\right)$$

  where $\mathcal{V}_X$ is the diagram in **Set** *with vertices $\mathcal{V}_X(X_\alpha) = \mathbf{Grpd}\left(\pi(X, X_\alpha), G\right)$ for each finite representative subset $X_\alpha \subseteq X$ and edges $\phi_{\alpha\beta} : \mathcal{V}_X(X_\alpha) \to \mathcal{V}_X(X_\beta)$ whenever $X_\beta \subseteq X_\alpha$ sending each $f \in \mathcal{V}_X(X_\alpha)$ to $f \circ \iota_{\beta\alpha}$ where $\iota_{\beta\alpha} : \pi(X, X_\beta) \to \pi(X, X_\alpha)$ is the inclusion.*

- *For a homotopy cobordism $[i : X \to M \leftarrow Y : j]_{ch}$,*

  $$\mathsf{Z}_G\left(\left[\begin{tikzcd} X \arrow[dr, "i"'] & & Y \arrow[dl, "j"] \\ & M & \end{tikzcd}\right]_{ch}\right) = \mathsf{Z}_G\left(\begin{tikzcd} X \arrow[dr, "i"'] & & Y \arrow[dl, "j"] \\ & M & \end{tikzcd}\right) = \phi^Y_{\alpha'} d^M_{\alpha'} : \mathsf{Z}_G(X) \to \mathsf{Z}_G(Y)$$

  where $Y_{\alpha'} \subseteq Y$ is some choice of finite representative subset and, $\phi^Y_{\alpha'}$ and $d^M_{\alpha'}$ are as in Lemma 5.4.22.

Proof. We have from Lemma 5.4.23 that Z_G is a magmoid morphism so it remains only to check that Z_G does not depend on a choice of representative cospan and that it preserves identities. We will need a different interpretation of the colimit to prove that Z_G preserves identities, we do this in Lemma 5.4.35.

In Lemma 5.4.12 we show that $\mathsf{Z}^{!}_{G}$ does not depend on the representative homotopy cobordism we choose. It thus follows that $\mathsf{Z}^{!!}_{G}$ and hence Z_{G} do not depend on a choice of representative cospan. ☐

The following Lemma gives an alternative description of the image of the linear map a cospan is sent to under Z_G, in terms of a choice of based cospan.

Lemma 5.4.25. *Let $i\colon X \to M \leftarrow Y : j$ be a concrete homotopy cobordism, $i\colon (X, X_0) \to (M, M_0) \leftarrow (Y, Y_0) : j$ a choice of concrete based homotopy cobordism, and $[f] \in \mathsf{Z}_G(X)$ and $[g] \in \mathsf{Z}_G(Y)$ be basis elements (so $[f]$, for example, is an equivalence class in colim$(\mathcal{V}_X)$), then*

$$\begin{aligned}\langle [g] | \mathsf{Z}_G(M) | [f] \rangle &= |G|^{-(|M_0|-|X_0|)} \sum_{g \in \phi_0^{Y-1}([g])} \left| \left\{ h\colon \pi(M, M_0) \to G \,|\, h|_{\pi(X,X_0)} = f \wedge h|_{\pi(Y,Y_0)} = g \right\} \right| \\ &= |G|^{-(|M_0|-|X_0|)} \sum_{g \in \phi_0^{Y-1}([g])} \left\langle g \,\middle|\, \mathsf{Z}^{!!}_G(M, M_0) \,\middle|\, f \right\rangle\end{aligned}$$

where $\phi_0^Y \colon \mathsf{Z}^{!}_G(Y, Y_0) \to \mathsf{Z}_G(Y)$ is the map into colim$(\mathcal{V}'_Y)$; see Definition 5.4.20.

Proof. We will use notation as in 5.3. Since each map ϕ_0^Y is surjective (Lemma 5.4.18), we can find $d_0^M([f])$ by looking at $\mathsf{Z}^{!!}_G(M, X_0, Y_0)(f)$. Hence we have

$$d_0^M([f]) = \sum_{g \in \mathcal{V}_Y(Y_0)} \left\langle g \,\middle|\, \mathsf{Z}^{!!}_G(M, X_0, Y_0) \,\middle|\, f \right\rangle |g\rangle$$

and, choosing a basis element $[g] \in \mathsf{Z}_G(Y)$,

$$\begin{aligned}\langle [g] | \mathsf{Z}_G(M) | [f] \rangle &= \sum_{g \in \phi_0^{Y-1}([g])} \left\langle g \,\middle|\, \mathsf{Z}^{!!}_G(M, X_0, Y_0) \,\middle|\, f \right\rangle \\ &= |G|^{-(|M_0|-|X_0|)} \sum_{g \in \phi_0^{Y-1}([g])} \left\langle g \,\middle|\, \mathsf{Z}^{!!}_G(M, M_0) \,\middle|\, f \right\rangle \\ &= |G|^{-(|M_0|-|X_0|)} \sum_{g \in \phi_0^{Y-1}([g])} \left| \left\{ h\colon \pi(M, M_0) \to G \,|\, h|_{\pi(X,X_0)} = f \ \wedge \ h|_{\pi(Y,Y_0)} = g \right\} \right|.\end{aligned}$$

☐

Remark 5.4.2. The set of maps $\phi_0^{-1}([g])$ contains all maps $g'\colon \pi(Y, Y_0) \to G$ such that $g' \sim g$ where $\sim$ is the equivalence relation defined by the colimit. And since we are only counting the cardinality of maps h we can rewrite the map on morphisms as

$$\langle [g] | \mathsf{Z}_G(M) | [f] \rangle = |G|^{-(|M_0|-|X_0|)} \left| \left\{ h : \pi(M,M_0) \to G \,|\, h|_{\pi(X,X_0)} = f \wedge h|_{\pi(Y,Y_0)} \sim g \right\} \right| \tag{5.4}$$

where we have removed the sum and only insist maps h are equivalent to g on Y. In many cases, especially with the local equivalence obtained in following section, this will be the most useful formulation to use for calculations.

Example 5.4.26. *Let $i\colon X \to M \leftarrow Y : j$ be the homotopy cobordism shown in Figure 5.3. Note this is a homotopy cobordism from Examples 5.3.8 and 5.3.27. Using 5.4, we may choose to calculate the image of $\mathsf{Z}_G([i\colon X \to M \leftarrow Y : j]_{ch})$ using the based homotopy cobordism considered in Example 5.3.8. Using the results and notation from Example 5.3.8, we have*

$$\begin{aligned} \langle [(g_1)] \mid \mathsf{Z}_G(M) \mid [(f_1, f_2)] \rangle &= |G|^{-1} \langle (g_1) \mid \mathsf{Z}^!_G(M, M_0) \mid (f_1, f_2) \rangle \\ &= |G|^{-1} \, \left| \left\{ c, d \in G | c^{-1} d^{-1} f_2 d f_1 c \sim g_1 \right\} \right|. \end{aligned}$$

5.4.4 Writing the colimit in terms of a local equivalence

For a general homotopically 1-finitely generated space X it is unlikely to be straightforward to calculate the colimit constructed in the previous section. Usually there will be an uncountably infinite number of choices of finite representative subsets $X_\alpha \subseteq X$, and thus an uncountably infinite number of vertices in $\mathcal{V}_X$. However we did show, in Lemma 5.4.19, that $\mathsf{Z}_G(X)$ is finite dimensional for all X.

In this section we show that this global equivalence given by taking the colimit over all subsets, is the same as choosing a single subset and taking a local equivalence given by taking maps up to natural transformation.

This will allow us to prove, in Lemma 5.4.35, that Z_G preserves the identity. We will also need this interpretation of Z_G to prove, in Section 5.4.5, that Z_G is a monoidal functor.

Here we only need to work with a single space X, so with $\mathcal{V}_X$ as constructed in Lemma 5.4.16, we drop the subscript on $\mathcal{V}_X$, and the superscript on the ϕ^X. Consider the commuting diagram

$$\begin{tikzcd}
 & & colim(\mathcal{V}) & \\
{}^{\mathcal{V}(X_\alpha)}/_{\cong} \arrow[rru, dashed, bend left, "\hat{\phi}_\alpha"] & \mathcal{V}(X_\alpha) \arrow[l, "p_\alpha"'] \arrow[ru, bend left, "\phi_\alpha"] \arrow[rr, "\phi_{\alpha\beta}"'] & & \mathcal{V}(X_\beta), \arrow[lu, bend right, "\phi_\beta"']
\end{tikzcd} \tag{5.5}$$

where $\cong$ denotes taking maps up to natural isomorphism (it is straightforward to check this is an equivalence relation). The set map p_α sends a groupoid map in $\mathbf{Grpd}(\pi(X, X_\alpha), G)$ to its equivalence class in ${}^{\mathcal{V}(X_\alpha)}/_{\cong}$. The map $\hat{\phi}_\alpha : {}^{\mathcal{V}(X_\alpha)}/_{\cong} \to colim(\mathcal{V})$ is the canonical map sending an equivalence class to ϕ_α of some representative (it remains to check this is well defined).

Theorem 5.4.27. *For a space X, the map $\hat{\phi}_\alpha$ is an isomorphism. Hence, for a homotopically 1-finitely generated space $X \in \chi$*

$$\mathsf{Z}_G(X) = \mathbb{C}((\mathbf{Grpd}(\pi(X, X_0), G)/\cong),$$

for any choice $X_0 \subset X$ of finite representative subset, where $\cong$ denotes taking maps up to natural transformation.

Proof. Surjectivity follows directly from Lemma 5.4.18. We prove $\hat{\phi}_\alpha$ is well defined and injective in Lemmas 5.4.28 and 5.4.30 respectively. □

For a path $s : \mathbb{I} \to X$ in X, we will also use s to denote its path equivalence class in $\pi(X)$ and $s \overset{p}{\sim} s'$ to mean that $s' \in [s]_{\mathrm{p}}$.

Lemma 5.4.28. *Let $v_\alpha, v'_\alpha \in \mathcal{V}(X_\alpha)$ be two groupoid maps such that $p_\alpha(v_\alpha) = p_\alpha(v'_\alpha)$, then $\phi_\alpha(v_\alpha) = \phi_\alpha(v'_\alpha)$.*

Proof. There exists a subset $X_{\tilde{\alpha}} \subseteq X_\alpha$ containing precisely one basepoint in each path-connected component, and maps $\tilde{v}_\alpha, \tilde{v}'_\alpha : \pi(X, X_{\tilde{\alpha}}) \to G$ such that $\phi_{\alpha\tilde{\alpha}}(v_\alpha) = \tilde{v}_\alpha$ and $\phi_{\alpha\tilde{\alpha}}(v'_\alpha) = \tilde{v}'_\alpha$. We will show that $\tilde{v}_\alpha$ and $\tilde{v}'_\alpha$ are equivalent in the colimit, implying $v_\alpha \sim v'_\alpha$.

The idea of this proof is illustrated by the following diagram.

$$\begin{tikzcd}
 & \pi(X,X_\gamma) \arrow[ddr, "v_\gamma"'] & & \pi(X,X_\gamma) \arrow[ddl, "v'_\gamma"] & \\
\pi(X,X_{\tilde{\alpha}}) \arrow[ur] \arrow[drr, "\tilde{v}_\alpha"'] & & \pi(X,X_\beta) \arrow[ul] \arrow[ur] \arrow[d] & & \pi(X,X_{\tilde{\alpha}}) \arrow[ul] \arrow[dll, "\tilde{v}'_\alpha"] \\
 & & G & &
\end{tikzcd}$$

We use the morphisms in the natural transformation connecting v_α and v'_α to extend the map $\tilde{v}_\alpha$ to a map from $v_\gamma \colon \pi(X,X_\gamma) \to G$, where X_γ is a larger set of basepoints. We also trivially extend the map $\tilde{v}'_\alpha$ to $v'_\gamma \colon \pi(X,X_\gamma) \to G$, and show that these extensions have the same image under some $\phi_{\gamma\beta}$, and therefore are equivalent in the colimit.

The set $X_{\tilde{\alpha}}$ is finite so we can write $X_{\tilde{\alpha}} = \{x_1, ..., x_N\}$. Since v_α and v'_α are related by a natural transformation, for all points $x_n \in X_{\tilde{\alpha}}$ and for all equivalence classes of loops $s \colon x_n \to x_n$, the below square commutes.

$$\begin{tikzcd}
v_\alpha(x_n) \arrow[r, "v_\alpha(s)"] \arrow[d, "\eta_{x_n}"'] & v_\alpha(x_n) \arrow[d, "\eta_{x_n}"] \\
v'_\alpha(x_n) \arrow[r, "v'_\alpha(s)"'] & v'_\alpha(x_n)
\end{tikzcd}$$

Recall that the image of v_α and $v_{\alpha'}$ is a groupoid with one object, so the image on points is always the same. Hence the two maps must be the same on any path-components that have no non-trivial paths.

Choose another set of points $X_\beta = \{y_1, ..., y_N\}$ as follows. If there are no non-trivial loops based at x_n then $y_n = x_n$, otherwise choose $y_n \neq x_n$ and choose a path $t_n \colon x_n \to y_n$, with t_n the constant path if $x_n = y_n$. This is always possible since a non-trivial loop based at x_n must contain some $y_n \neq x_n$.

Let $X_\gamma = X_{\tilde{\alpha}} \cup X_\beta$. We define a map $v_\gamma \colon \pi(X,X_\gamma) \to G$ as follows. Let $v_\gamma|_{\pi(X,X_{\tilde{\alpha}})} = \tilde{v}_\alpha$, and $v_\gamma(t_n) = \eta_{x_n}$ unless t_n is the constant path, in which case $v_\gamma(t_n) = 1_G$. By Lemma 3.3.13 this completely defines v_γ. Notice $\phi_{\gamma\tilde{\alpha}}(v_\gamma) = \tilde{v}_\alpha$, hence $\tilde{v}_\alpha \sim v_\gamma$.

Define another map $v'_\gamma \colon \pi(X,X_\gamma) \to G$ by $v'_\gamma|_{\pi(X,X_{\tilde{\alpha}})} = \tilde{v}'_\alpha$ and $v'_\gamma(t_n) = 1_G$. We have $\phi_{\gamma\tilde{\alpha}}(v'_\gamma) = \tilde{v}'_\alpha$ and so $\tilde{v}'_\alpha \sim v'_\gamma$.

Now we check that $\phi_{\gamma\beta}(v_\gamma) = \phi_{\gamma\beta}(v'_\gamma)$, hence $v_\gamma \sim v'_\gamma$. Since X_β has only one point in each path-connected component we only need to check that v_γ and $v_{\gamma'}$ agree on loops. For any trivial $s: x_n \to x_n$ with $y_n = x_n$, we have $v_\gamma(s) = 1_G = \tilde{v}_\alpha(s) = \tilde{v}'_\alpha(s)$.

Now suppose $s: y_n \to y_n$ is any class of loops with $y_n \neq x_n$,

$$v_\gamma(s) = v_\gamma(t_n t_n^{-1} s t_n t_n^{-1}) = \eta_{x_n} \tilde{v}_\alpha(t_n^{-1} s t_n) \eta_{x_n}^{-1} = \tilde{v}'_\alpha(t_n^{-1} s t_n)$$

and similarly,

$$v'_\gamma(s) = v'_\gamma(t_n t_n^{-1} s t_n t_n^{-1}) = v'_\gamma(t_n) v'_\gamma(t_n^{-1} s t_n) v'_\gamma(t_n^{-1}) = \tilde{v}'_\alpha(t_n^{-1} s t_n).$$

Hence $\phi_{\gamma\beta}(v_\gamma) = \phi_{\gamma\beta}(v'_\gamma)$ so $v_\gamma \sim v'_\gamma$ and $\tilde{v}_\alpha \sim \tilde{v}'_\alpha$. □

Lemma 5.4.29. *For any finite representative subset X_α of a space X, $\pi(X, X_\alpha)$ and $\pi(X)$ are equivalent as categories.*

Proof. We have an inclusion $\iota_\alpha: \pi(X, X_\alpha) \to \pi(X)$. We define explicitly a map $r_\alpha: \pi(X) \to \pi(X, X_\alpha)$ as follows. For each $x \in X \setminus X_\alpha$, choose a point $y_x \in X_\alpha$ in the same path-connected component as x, and a path $t_x: x \to y_x$. If $x \in X_\alpha$ choose $y_x = x$ and t_x the trivial path. Now define

$$r_\alpha(x) = y_x$$

and for a path $s: x \to x'$ in $\pi(X)$

$$r_\alpha(s) = t_{x'} s t_x^{-1}$$

The composition $r_\alpha \iota_\alpha$ is equal to the identity and a natural transformation $\eta: \mathrm{id} \to \iota_\alpha r_\alpha$ is given by

$$\eta_x = t_x.$$

□

Lemma 5.4.30. *Let $v_\alpha, v'_\alpha \in \mathcal{V}(X_\alpha)$ be two maps such that $\phi_\alpha(v_\alpha) = \phi_\alpha(v'_\alpha)$. Then $p_\alpha(v_\alpha) = p_\alpha(v'_\alpha)$.*

Proof. The maps v_α and v'_α being equivalent in the colimit means there is some finite sequence of relations $v_\alpha = v_0 \sim v_1 \sim \ldots \sim v_N = v'_\alpha$ where $v_n \neq v_{n+1}$, and of maps

$v_n \colon \pi(X, X_n) \to G$ such that for each pair v_n, v_{n+1} we have one of the following two diagrams:

$$\begin{tikzcd}
\pi(X,X_n) \arrow[r, "\iota_{n,n+1}"] \arrow[dr, "v_n = v_{n+1}\iota_{n,n+1}"'] & \pi(X,X_{n+1}) \arrow[d, "v_{n+1}"] & \pi(X,X_{n+1}) \arrow[r, "\iota_{n+1,n}"] \arrow[dr, "v_{n+1} = v_n\iota_{n+1,n}"'] & \pi(X,X_n) \arrow[d, "v_n"] \\
 & G & & G
\end{tikzcd}$$

Then we have a commuting diagram of the following form

$$\begin{tikzcd}
 & \pi(X, X_{n,n+1}) \arrow[dd] & \\
\pi(X,X_n) \arrow[ur, "\iota'_n"] \arrow[dr, "v_n"'] & & \pi(X,X_{n+1}) \arrow[ul, "\iota'_{n+1}"'] \arrow[dl, "v_{n+1}"] \\
 & G &
\end{tikzcd}$$

where we let $X_{n,n+1}$ be the larger of X_n and X_{n+1} and one ι'_n and ι'_{n+1} is a strict inclusion and the other is the identity. The middle arrow is either v_n or v_{n+1}. Consider the below diagram

$$\begin{tikzcd}[column sep=small]
 & & & & & & & \pi(X) \arrow[ddlllllll, bend right, "r_0"'] \arrow[dllllll, bend right, "r_{0,1}"'] \arrow[ddlllll, bend right, "r_1"'] \arrow[ddl, bend right, "r_n"'] \arrow[d, "r_{n,n+1}"] \arrow[ddr, bend left, "r_{n+1}"] \arrow[ddrrrrr, bend left, "r_{N-1}"] \arrow[drrrrrr, bend left, "r_{N,N-1}"] \arrow[ddrrrrrrr, bend left, "r_N"] & & & & & & & \\
 & \pi(X,X_{0,1}) \arrow[ddrrrrrr] & & & \cdots & & & \pi(X,X_{n,n+1}) \arrow[dd] & & & \cdots & & & \pi(X,X_{N-1,N}) \arrow[ddllllll] & \\
\pi(X,X_0) \arrow[ur, "\iota'_0"] \arrow[drrrrrrr, bend right, "v_0"'] & & \pi(X,X_1) \arrow[ul, "\iota'_1"'] \arrow[urr] \arrow[drrrrr, bend right, "v_1"'] & & & & \pi(X,X_n) \arrow[ull] \arrow[ur, "\iota'_n"] \arrow[dr, "v_n"'] & & \pi(X,X_{n+1}) \arrow[ul, "\iota'_{n+1}"'] \arrow[urr] \arrow[dl, "v_n+1"] & & & & \pi(X,X_{N-1}) \arrow[ull] \arrow[ur, "\iota'_{N-1}"] \arrow[dlllll, bend left, "v_{N-1}"] & & \pi(X,X_N) \arrow[ul, "\iota'_N"'] \arrow[dlllllll, bend left, "v_N"] \\
 & & & & & & & G & & & & & & &
\end{tikzcd}$$

where the maps r and ι are as constructed in the proof of Lemma 5.4.29.

We will show there is a natural transformation $v_0 r_0$ to $v_N r_N$. Since $X_0 = X_N = X_\alpha$, $\iota_0 = \iota_N$ and hence $v_0 r_0 \cong v_N r_N$ implies $v_0 r_0 \iota_0 \cong v_N r_N \iota_N$. This implies $v_0 \cong v_N$ since $r_\beta \iota_\beta = \text{id}$ for all finite representative $X_\beta \subseteq X$ by Lemma 5.4.29.

We show all triangles in the diagram commute up to natural transformation. The bottom triangles commute exactly by the construction explained in the first part of the proof. Notice that $\iota'_n r_n = r_{n,n+1} \iota_n r_n$, where $\iota_n \colon \pi(X, X_n) \to \pi(X)$ is the inclusion. By Lemma 5.4.29

we have that $r_{n,n+1}\iota_n r_n \simeq r_{n,n+1}$. □

Example 5.4.31. *Let $X = S^1 \sqcup S^1$. Then, letting $X_0 \subset X$ be a subset with precisely one point in each connected component,* $\mathbf{Grpd}(\pi(X,X_0),G) = G \times G$ *as discussed in Example 5.4.7. Taking maps up to natural transformation corresponds to allowing taking each element of G up to conjugation, so we have $\mathsf{Z}_G(X) = \mathbb{C}(G/G \times G/G)$.*

Example 5.4.32. *Consider again Example 5.4.26, which in turn refers to Examples 5.3.8 and 5.3.27. The equivalence class $[g_1]$ in the basis of $\mathsf{Z}_G(Y)$ consists of all maps sending S^1 to something in the conjugacy class of g_1. This allows us to refine the result of Example 5.4.26 as follows.*

$$\begin{aligned}\langle[(g_1)] \mid \mathsf{Z}_G(M) \mid [(f_1,f_2)]\rangle &= |G|^{-1}|\left\{c,d \in G | c^{-1}d^{-1}f_2 d f_1 c \sim g_1\right\}| \\ &= \ |\left\{d \in G | d^{-1}f_2 d f_1 \sim g_1\right\}|.\end{aligned}$$

Example 5.4.33. *Let X be the embedding of two circles as shown in Figure 5.5. Then, letting $X_0 \subset X$ be the subset shown,* $\mathbf{Grpd}(\pi(X,X_0),G) = G \times G$ *as discussed in Example 5.4.8. Since all objects are mapped to the unique object in G, taking maps up to natural transformation is equivalent to conjugation by elements of G at each point, hence corresponds to taking the pairs up to simultaneous conjugation, so we have $\mathsf{Z}_G(X) = \mathbb{C}((G \times G)/G)$.*

Theorem 5.4.34. *Let $X \in \chi$ be a homotopically 1-finitely generated space. Then*

$$\mathsf{Z}_G(X) \cong \mathbb{C}(\mathbf{Grpd}(\pi(X),G)/\cong),$$

the set of groupoid maps up to natural transformation.

Proof. We have from Theorem 5.4.27 that $\mathsf{Z}_G(X) \cong \mathbb{C}(\mathbf{Grpd}(\pi(X,X_0),G)/\cong)$, for some finite representative set X_0.

We have from Lemma 5.4.29 that $\pi(X,X_0)$ and $\pi(X)$ are equivalent as categories. Let $\iota_0 \colon \pi(X,X_0) \to X$ and $r_0 \colon \pi(X) \to \pi(X,X_0)$ be as in the proof of Lemma 5.4.29.

Let $[f] \in \mathbf{Grpd}(\pi(X), G)/\cong$, then there is a map

$$\phi\colon \mathbf{Grpd}(\pi(X), G)/\cong \to \mathbf{Grpd}(\pi(X, X_0), G)/\cong$$
$$\phi([f]) \mapsto [f \circ \iota_0].$$

We show this map is well defined. Suppose $f' \in [f]$, so there is natural transformation, say η, from f to f'. Then $f' \circ \iota_0 \sim f \circ \iota_0$ using the restriction of η to $x \in X_0$.

This ϕ has inverse ϕ', where $\phi'(g) = g \circ r_0$. Again this map is well defined, this time if η is a natural transformation in $\mathbf{Grpd}(\pi(X, X_0), G)/\cong$, then maps $\eta_{r_0(x)}$ give the required natural transformation. □

We now use Theorem 5.4.27 to prove that identities are preserved.

Lemma 5.4.35. *The identity homotopy cobordism* $\left[\iota_0^X\colon X \to X \times \mathbb{I} \leftarrow X : \iota_1^X\right]_{ch}$ *for a space* X *is mapped to the identity matrix by* Z_G*.*

Proof. We will show that the matrix element

$$\langle [f] | \mathsf{Z}_G(X \times \mathbb{I}) | [g] \rangle .$$

is 1 if $[f] = [g]$ and 0 otherwise.

Let e_i denote the constant path at any point i. First note, there is an isomorphism

$$\pi\left(X \times \mathbb{I}, (X_0 \times \{0\}) \cup (X_0 \times \{1\})\right) \xrightarrow{\sim} \pi(X, X_0) \times \pi\left([0,1], \{0,1\}\right).$$

given by sending an equivalence class of paths to the pair containing the equivalence classes of each projection (see 6.4.4 in [Bro06]). Hence we have that $\left\langle f \,\middle|\, \mathsf{Z}_G^!(X \times \mathbb{I}) \,\middle|\, g \right\rangle$ is given by the cardinality of the set of maps

$$h\colon \pi(X, X_0) \times \pi\left([0,1], \{0,1\}\right) \to G$$

such that

$$h(\gamma, e_i) = \begin{cases} f(\gamma), i = 0, \\ g(\gamma), i = 1. \end{cases}$$

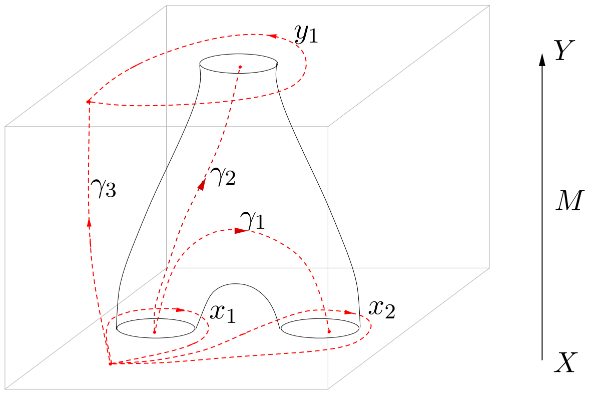


Figure 5.7: This figure represents the concrete cofibrant cospan from Example 5.3.7. The red points and lines show a possible choice of basepoints M_0 and paths. It can be seen that the equivalence classes of the marked paths generate $\pi(M, M_0)$. We can see X and Y are homotopically 1-finitely generated by considering the intersection of the marked points and paths with X and Y respectively. Thus it is a concrete homotopy cobordism.

Any pair in the product space can be written as a composition of pairs with only one non-identity component. The morphisms of $\pi\left([0,1],\{0,1\}\right)$ are generated by the equivalence class of the path $\mathrm{id}\colon[0,1]\to[0,1]$. Thus a map h is completely defined by specifying its action on pairs of the form (e_{x_j},id). Let $s\colon x_0\to x_1$ be a path in X with $x_0,x_1\in X_0$. Notice that

$$(e_{x_1},\mathrm{id}^{-1})(s,e_1)(e_{x_0},\mathrm{id})=(s,e_0)\implies h(e_{x_1},\mathrm{id})^{-1}g(s)h(e_{x_0},\mathrm{id})=f(s).$$

Hence such an h exists if and only if the $h(e_{x_i},\mathrm{id})$ are a natural transformation from f to g. By Theorem 5.4.27 this means the matrix element corresponding to f and g is zero unless $[f]=[g]$.

Now we consider the matrix element

$$\langle[f]|\mathsf{Z}_G(X\times\mathbb{I})|[f]\rangle\,.$$

A map h is a defined by a choice of $h(e_{x_0},\mathrm{id})\in G$ for each $x_0\in X$ and all choices define a natural transformation. Using the definition of Z_G from Lemma 5.4.25, we must sum over all $\left\langle f\,\middle|\,\mathsf{Z}^!_G(X\times\mathbb{I})\,\middle|\,f'\right\rangle$ with $f'\sim f$, which, by Theorem 5.4.27, means there is a natural transformation f to f'. Hence all choices of $h(e_{x_0},\mathrm{id})=g\in G$ will contribute to the sum. There are $|G|^{|X_0|}$ choices, then with the normalisation, the matrix element is 1. □

Example 5.4.36. *Consider the based homotopy cobordism shown in Figure 5.2. This represents a manifold M, which is the complement of the marked subset in $\mathbb{I}^3$ and X and*

Y are given by the bottom and top boundary respectively. This becomes a cospan with the inclusion maps. This is in fact a homotopy cobordism from Examples 5.3.7 and 5.3.28.

We calculate $\mathsf{Z}_G\big([i{:}X \to M \leftarrow j:Y]_{ch}\big)$. We choose to use the based homotopy cobordism shown in Figure 5.7 for calculation. The set M_0 consists of all marked points and X_0 and Y_0 consist of the intersection of M_0 with X and Y respectively.

We have from Example 5.4.32 that basis elements in $\mathsf{Z}_G(X)$ are given by equivalence classes $[(f_1, f_2)]$ where $f_1, f_2 \in G$ and $[]$ denotes simultaneous conjugation by the same element of G.

Basis elements in $\mathsf{Z}_G(Y)$ are given by elements of g taken up to conjugation, denoted $[g_1]$.

Let $x \in X$ be the basepoint which is in the connected component of X homotopy equivalent to the punctured disk, and $x' \in X$ some choice of basepoint in another connected component. By Lemma 3.3.14, there is a bijection sending a map $h \in \mathbf{Grpd}(\pi(M, M_0), G)$ to a map $h' \in \mathbf{Grpd}(\pi(M, \{x, x'\}) \times G \times G \times G$, which agrees with h on $\pi(M, \{x, x'\})$ and where the first element of G corresponds to the image $h(\gamma_1)$, the second to $h(\gamma_2)$, and the third to $h(\gamma_3)$. Now $\pi(M, \{x, x'\})$ is the disjoint union of the groupoids $\pi(M_1, \{x\})$ and $\pi(M_2, \{x'\})$ where M_1 is the path connected component of M containing x, and M_2 is the path connected component containing x'. The group $\pi(M_2, \{x'\})$ is trivial, so there is one unique map into G. The group $\pi(M_1, \{x\})$ is equivalent to the twice punctured disk (see Example 5.3.28), which has fundamental group isomorphic to the free product $\mathbb{Z} * \mathbb{Z}$. This isomorphism can be realised by sending the loop x_1 to the 1 in the first copy of $\mathbb{Z}$ and x_2 to the 1 in the second copy of $\mathbb{Z}$. Thus we can label elements in $\mathbf{Grpd}(\pi(M_1, \{x\}), G)$ by elements of $G \times G$ where $g_1 \in (g_1, g_2)$ corresponds to the image of x_1, and g_2 the image of x_2. Hence a map in $\mathbf{Grpd}(\pi(M, M_0), G)$ is determined by a five tuple $(a, b, c, d, e) \in G \times G \times G \times G \times G$ where a corresponds to the image of x_1, b to the image of x_2, and c, d and e correspond to the images of γ_1, γ_2 and γ_3 respectively. Hence we have

$$\begin{aligned}
\langle [g_1] | \mathsf{Z}_G(M) | [(f_1, f_2)] \rangle &= |G|^{-2} \left\{ a, b, c, d, e \in G \mid a = f_1, b = f_2, g_1 \sim ebae^{-1} \right\} \\
&= \left\{ e \in G \mid g_1 \sim ef_1 f_2 e^{-1} \right\} \\
&= \begin{cases} |G| & \text{if } g_1 \sim f_1 f_2 \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}$$

5.4.5 Monoidal functor $\mathsf{Z}_G\colon \text{HomCob} \to \mathbf{Vect}_{\mathbb{C}}$

We now show that the functor Z_G is symmetric monoidal. We will need the following lemma.

Lemma 5.4.37. *Let X and Y be homotopically 1-finitely generated spaces. There is a bijection*

$$\kappa\colon colim(\mathcal{V}_{X\sqcup Y}) \xrightarrow{\sim} colim(\mathcal{V}_X) \times colim(\mathcal{V}_Y)$$

where $\mathcal{V}_X$ is as in Lemma 5.4.16

Proof. For any subsets $X_\alpha \subseteq X$ and $Y_{\alpha'} \subseteq Y$, and points $x \in X_\alpha$ and $y \in Y_{\alpha'}$, we have that $\pi(X \sqcup Y, X_\alpha \sqcup Y_{\alpha'})(x,y)$ is empty. Thus there is an isomorphism of groupoids $\pi(X \sqcup Y, X_\alpha \sqcup Y_{\alpha'}) \xrightarrow{\sim} \pi(X, X_\alpha) \sqcup \pi(Y, Y_{\alpha'})$ and we have a bijection $\mathbf{Grpd}(\pi(X \sqcup Y, X_\alpha \sqcup Y_{\alpha'}), G) \xrightarrow{\sim} \mathbf{Grpd}(\pi(X, X_\alpha), G) \times \mathbf{Grpd}(\pi(Y, Y_{\alpha'}), G)$ sending a map to the appropriate pair of restrictions. Equivalently we have a bijection $\mathcal{V}_{X\sqcup Y}(X_\alpha \sqcup Y_{\alpha'}) \xrightarrow{\sim} \mathcal{V}_X(X_\alpha) \times \mathcal{V}_Y(Y_{\alpha'})$. Thus $colim(\mathcal{V}_{X\sqcup Y})$ is isomorphic to the colimit over the diagram with vertices of the form $\mathcal{V}_X(X_\alpha) \times \mathcal{V}_Y(Y_{\alpha'})$ and maps of the form $(\phi^X_{\alpha\beta}, \phi^Y_{\alpha'\beta'})$, which we denote $colim(\mathcal{V}_{X\sqcup Y})'$. We construct a bijection between $colim(\mathcal{V}_{X\sqcup Y})'$ and $colim(\mathcal{V}_X) \times colim(\mathcal{V}_Y)$.

Suppose $[(f,g)] = [(f',g')]$ in $colim(\mathcal{V}_{X\sqcup Y})'$ with $(f,g) \in \mathcal{V}_X(X_\alpha) \times \mathcal{V}_Y(Y_{\alpha'})$ and $(f',g') \in \mathcal{V}_X(X_\beta) \times \mathcal{V}_Y(Y_{\beta'})$. By the construction of the colimit, there is a sequence of sets $\mathcal{V}_X(X_0) \times \mathcal{V}_Y(Y_0), ..., \mathcal{V}_X(X_n) \times \mathcal{V}_Y(Y_n)$ with $\mathcal{V}_X(X_0) \times \mathcal{V}_Y(Y_0) = \mathcal{V}_X(X_\alpha) \times \mathcal{V}_Y(Y_{\alpha'})$ and $\mathcal{V}_X(X_n) \times \mathcal{V}_Y(Y_n) = \mathcal{V}_X(X_\beta) \times \mathcal{V}_Y(Y_{\beta'})$, and a sequence of maps $\phi_0, ..., \phi_{n-1}$ connecting (f,g) and (f',g') where each ϕ_i is either a map $\mathcal{V}_X(X_n) \times \mathcal{V}_Y(Y_n) \to \mathcal{V}_X(X_{n+1}) \times \mathcal{V}_Y(Y_{n+1})$ or a map $\mathcal{V}_X(X_{n+1}) \times \mathcal{V}_Y(Y_{n+1}) \to \mathcal{V}_X(X_n) \times \mathcal{V}_Y(Y_n)$. The projections of this sequence of maps give sequences of maps connecting f and f' in $colim(\mathcal{V}_X)$ and g and g' in $colim(\mathcal{V}_Y)$. Thus there is a well defined map

$$\begin{aligned}\kappa'\colon colim(\mathcal{V}_{X\sqcup Y})' &\to colim(\mathcal{V}_X) \times colim(\mathcal{V}_Y)\\ [(f,g)] &\mapsto ([f],[g]).\end{aligned}$$

It is easy to see this map is a surjection. To see that it is an injection, suppose now that $[f] = [f']$ in $colim(\mathcal{V}_X)$ and $[g] = [g']$ in $colim(\mathcal{V}_Y)$ then there are sequences $\phi^f_0, ..., \phi^f_n$ and $\phi^g_0, ..., \phi^g_n$ as in the proof of well definedness. Now the sequence given by $(\phi_0^f, \mathrm{id}), ..., (\phi_n^f, \mathrm{id}), (\mathrm{id}, \phi_0^g), ...(\mathrm{id}, \phi_1^g)$ is a sequence connecting (f,g) and (f',g') in $colim(\mathcal{V}_{X \sqcup Y})'$. □

Lemma 5.4.38. *The functor* $\mathsf{Z}_G\colon \mathrm{HomCob} \to \mathbf{Vect}_{\mathbb{C}}$ *(where* $\mathbf{Vect}_{\mathbb{C}}$ *has the monoidal structure from Lemma 3.7.7) endowed with* $(\mathsf{Z}_G)_0 = 1_{\mathbb{C}}\colon \mathbb{C} \to \mathbb{C}$ *and natural transformations*

$$(\mathsf{Z}_G)_2(X,Y)\colon \mathsf{Z}_G(X) \otimes_{\mathbb{C}} \mathsf{Z}_G(Y) \to \mathsf{Z}_G(X \sqcup Y)$$

which acts on basis elements as

$$[f] \otimes_{\mathbb{C}} [g] \mapsto \kappa^{-1}([f],[g])$$

with κ *as in Lemma 5.4.37, is strong monoidal.*

Proof. Notice $colim(\mathcal{V}_{\varnothing}) = \varnothing$ as $\mathcal{V}_{\varnothing}$ has just one vertex, the empty set, and no maps. Hence $\mathsf{Z}_G(\varnothing) = \mathbb{C}$ so $(\mathsf{Z}_G)_0$ is well defined.

The vector space $\mathsf{Z}_G(X) \otimes_{\mathbb{C}} \mathsf{Z}_G(Y)$ has a basis isomorphic to $colim(\mathcal{V}_X) \times colim(\mathcal{V}_Y)$. Thus the map $(\mathsf{Z}_G)_2(X,Y)$ is the linear extension of κ^{-1}, hence an isomorphism by Lemma 5.4.37.

The only complication in checking the associativity relation is understanding the image of the associator, $\mathsf{Z}_G(\alpha_{X,Y,Z})$. The proof is similar to the proof that the identity is preserved so we don't repeat it here but we have that on basis elements $\mathsf{Z}_G(\alpha_{X,Y,Z})((f \otimes_{\mathbb{C}} g) \otimes_{\mathbb{C}} h) = f \otimes_{\mathbb{C}} (g \otimes_{\mathbb{C}} h)$. Similarly we can check the unitality relations using that, on basis elements, we have $\mathsf{Z}_G(\lambda_X)(\varnothing \otimes_{\mathbb{C}} f) = f$ and $\mathsf{Z}_G(\rho_X)(f \otimes_{\mathbb{C}} \varnothing) = f$. □

Lemma 5.4.39. *The monoidal functor* $\mathsf{Z}_G\colon \mathrm{HomCob} \to \mathbf{Vect}_{\mathbb{C}}$ *is symmetric monoidal.*

Proof. As in the previous proof it is straightforward to check the relevant identity. □

Lemma 5.4.40. *The functor*

$$\tilde{\mathsf{Z}}_G = \mathsf{Z}_G \circ \mathrm{Cob}_n\colon \mathbf{Cob}(n) \to \mathbf{Vect}_{\mathbb{C}}$$

where Cob_n *is as in Proposition 5.3.33, is a TQFT for all* $n \in \mathbb{N}$ *, i.e. is a symmetric monoidal functor.*

Proof. We have from Propositions 5.3.33 and 5.3.35 that Cob_n is a symmetric monoidal functor into HomCob and from Theorem 5.4.24 and Lemma 5.4.39 that Z_G is a symmetric monoidal functor HomCob → $\mathbf{Vect}_{\mathbb{C}}$. □

Chapter 6

Conclusions

We conclude by summarising the work done in this thesis and outlining some possible future directions.

In Chapter 4 we constructed the motion groupoid and mapping class groupoid associated to a manifold, generalising motions groups [Dah36] and mapping class groups [Bir16]. We also gave the general relationship between these constructions, proving there is a groupoid isomorphism whenever the space of homeomorphisms of the manifold, with the compact-open topology, has only one path component, which has trivial π_1. One interesting future direction would be completing the relationship between the motion groupoid and generalisations of other topological definitions of the braid group. In particular the definition of braids as monotonic embeddings of intervals in $\mathbb{I}^3$. An important part of the bridge between the motion groupoid setting and the monotonic embedding setting is provided by the interpretation of motions as maps from $M \times \mathbb{I}$ to $M \times \mathbb{I}$ given in Section 4.3.3. This leads to a map from motions to embeddings. The work to be done consists of investigating what map this leads to on equivalence classes. We discuss in Section 1.1.4 that the n strand braid group can be realised as isomorphisms at the object consisting of n points in the tangle category. One motivating aim was to understand if the braid-tangle relation generalises to a more general relationship between motion groupoids and embedded cobordism categories. The final part required is to understand how isomorphisms in embedded cobordism categories relate to monotonic embeddings.

Another possible future direction is to give combinatorial presentations of certain motion subgroupoids corresponding to physically interesting configurations of subsets in manifolds, loops and simple links in a 3-ball, for example. One objective is to prove the presentation of the motion groupoid of loops and points conjectured in Section 4.3.8. We have already begun work on a construction of a monoidal structure for these groupoids, which will be a useful tool in proving presentations of these groupoids. Another physically interesting setting would be modifying the motion groupoid to allow for points on a graph. This is of interest because one system which may be useful for building a topological quantum computer consists of particles moving in systems of nanowires.

In Chapter 5 we give a family of functors from the category HomCob into $\mathbf{Vect}_{\mathbb{C}}$, which take as input a finite group G and which are constructed using the fundamental groupoid. Putting together this with the relationship between TQFTs and modular tensor categories (MTCs), an interesting question would be to characterise the modular tensor categories which correspond to TQFTs arising from our construction. Following a similar line, it would be interesting to investigate constructions using higher homotopical properties of the spaces involved, and again understand which MTCs arise in this way.

It would also be interesting to consider constructions which use information about the complement of a particle trajectory, but also of the embedding. This would be a generalisation of a kind of quandle invariant of knots, which is a stronger invariant than the knot group.

Another direction is to directly calculate the image of the functor on specific choices of spaces. In many cases it should be a relatively straightforward exercise to extend the functor Z_G to a functor from a motion groupoid and calculate its value on elements, thus obtaining representations of motion groupoids.

Appendix A

Appendices to Chapter 4

A.0.1 Proof of Theorem 4.2.1

This section follows the proof of Theorem 4 in [Are46].

Recall that a space X is said to be locally compact if each $x \in X$ has an open neighbourhood which is contained in a compact set. If X is Hausdorff, X is locally compact if and only if for each $x \in X$ and open set $U \subset X$ containing x, there exists an open set V containing x with $\bar{V}$ compact and $\bar{V} \subset U$ (where $\bar{V}$ is the closure of V) [Mun16, Thm 29.2].

Lemma A.0.1. *Let X be a locally compact Hausdorff space. Let $K \subset X$ be compact and $U \subset X$ be open with $K \subset U$. Then there exists an open set V with $K \subset V \subset \overline{V} \subset U$, where $\overline{V}$ is compact.*

Proof. Since X is locally compact Hausdorff, for every $x \in K$ there is an open set $V(x) \subset U$ with $\overline{V(x)} \subset U$ compact. The set of all $V(x)$ is a cover for K, and K is compact so there exists a finite subcover. Hence we have

$$K \subset \bigcup_{i \in \{1,\dots,n\}} V(x_i) \subset \bigcup_{i \in \{1,\dots,n\}} \overline{V(x_i)} \subset U$$

for some finite set $\{x_1, \dots x_n\} \subset K$. We can choose $V = \bigcup_{i \in \{1,\dots,n\}} V(x_i)$, noting that (since the union is finite) $\overline{V} = \bigcup_{i \in \{1,\dots,n\}} \overline{V(x_i)}$, and hence $\overline{V}$ is compact, since it is a finite union of compact subsets. □

Lemma A.0.2. *Let X be a locally compact Hausdorff space. Then the composition of homeomorphisms*

$$\begin{aligned}\circ\colon \mathbf{TOP}^h(X,X)\times\mathbf{TOP}^h(X,X) &\to \mathbf{TOP}^h(X,X)\\ (\mathfrak{f},\mathfrak{g}) &\mapsto \mathfrak{g}\circ\mathfrak{f}\end{aligned}$$

is continuous.

Proof. Let $B_{XX}(K,U)$ be an element of the subbasis of τ^{co}_{XX}. Now suppose $\mathfrak{h}\in B_{XX}(K,U)$ is in the image of $\circ$, so $\mathfrak{h}=\mathfrak{g}\circ\mathfrak{f}$ for some $\mathfrak{g},\mathfrak{f}\in\mathbf{Top}^h(X,X)$. We show that for all such $\mathfrak{h}$, we can construct an open set in $V\in\mathbf{Top}^h(X,X)\times\mathbf{Top}^h(X,X)$ with $(\mathfrak{f},\mathfrak{g})\in V$ and for all $(\mathfrak{f}',\mathfrak{g}')\in V$, $\mathfrak{g}'\circ\mathfrak{f}'\in B_{XX}(K,U)$.

We have $\mathfrak{g}(K)\subset\mathfrak{f}^{-1}(U)$, and so by Lemma A.0.1 there exists an open set W with $\mathfrak{g}(K)\subset W\subset\overline{W}\subset f^{-1}(U)$, and $\overline{W}$ compact. Now $B_{XX}(K,W)\times B_{XX}(\overline{W},U)$ is an open set containing $(\mathfrak{f},\mathfrak{g})$ and for any $(\mathfrak{f}',\mathfrak{g}')\in B_{XX}(K,W)\times B_{XX}(\overline{W},U)$, $\mathfrak{g}\circ\mathfrak{f}\in B_{XX}(K,U)$. □

There is a more general version of the previous Lemma where the maps are not necessarily homeomorphisms, see [Dug66, Thm.2].

Lemma A.0.3. *Let X be a locally connected, locally compact Hausdorff space. Then the sets $B_{XX}(L,U)$ where L is compact, connected and has non-empty interior, and U is open, form a subbasis for the compact open topology.*

Proof. Again we follow the argument in [Are46]. Let $\mathfrak{h}\in\mathbf{Top}^h(X,X)$. We show that for any $B_{XX}(K,U)$ containing $\mathfrak{h}$ where K is compact and U is open, there exists a subset of $B_{XX}(K,U)$ containing $\mathfrak{h}$ of the form $B_{XX}(L_1,U)\cap\cdots\cap B_{XX}(L_n,U)$ where each L_i is compact, connected and has non empty interior.

Since $\mathfrak{h}$ is continuous, for each $x\in K$ we can find an open set $V(x)$ containing x such that $\mathfrak{h}(V(x))\subset U$. Since X is locally compact and Hausdorff, we can then find another $V'(x)$, open in X, such that

$$x\in V'(x)\subset\overline{V'(x)}\subset V(x),$$

with $\overline{V'(x)}$ compact. Now since X is locally connected, there exists a connected open set $V''(x)$ such that $x\in V''(x)\subset V'(x)$. Also $\overline{V''(x)}$ is compact, since $\overline{V''(x)}\subset\overline{V'(x)}$ and closed subsets of compact spaces are compact. Furthermore $\overline{V''(x)} \subset V(x)$, so $\mathfrak{h}(\overline{V''(x)}) \subset U$.

The $V''(x)$ cover K and so there exists a finite subcover by $V(x_i)$ for some finite set of $x_i \in K$ with $i \in \{1,\dots,n\}$. Clearly:

$$\mathfrak{h} \in \bigcap_{i \in \{1,\dots,n\}} B_{XX}(\overline{V''(x_i)}, U) \subset B_{XX}(K,U).$$

□

Lemma A.0.4. *Let X be a locally connected, locally compact Hausdorff space. Then the inverse map*

$$\begin{aligned}(-)^{-1}\colon \mathbf{TOP}^h(X,X) &\to \mathbf{TOP}^h(X,X)\\ \mathfrak{f} &\mapsto \mathfrak{f}^{-1}\end{aligned}$$

is continuous.

Proof. Throughout the proof, only, we will put $(-)^{-1} = T$. So $T\colon \mathbf{TOP}^h(X,X) \to \mathbf{TOP}^h(X,X)$ is the function such that $T(\mathfrak{h}) = \mathfrak{h}^{-1}$.

By Lemma A.0.3, in order to prove that T is continuous, we only need to prove that the inverse images under T of sets of the form $B_{XX}(L,U)$, with L compact, connected and with non-empty interior, and U open, are open in $\mathbf{TOP}^h(X,X)$.

Let $L \subset X$ be compact, connected, and with a non-empty interior. Let U be open in X. We show that for any $\mathfrak{f}^{-1} \in B_{XX}(L,U)$, we can construct an open subset of $\mathbf{TOP}^h(X,X)$, containing $\mathfrak{f}$, which is a subset of $T^{-1}(B_{XX}(L,U))$.

Since $\mathfrak{f}^{-1}$ is a homeomorphism, it sends compact subsets to compact subsets. So $\mathfrak{f}^{-1}(L)$ is compact. Also $\mathfrak{f}^{-1}(L) \subset U$, since $\mathfrak{f}^{-1} \in B_{XX}(L,U)$.

Using Lemma A.0.1, we can choose an open set $V \subset X$ such that $f^{-1}(L) \subset V \subset \overline{V} \subset U$, with $\overline{V}$ compact, and then an open set $W \subset X$ with $\overline{V} \subset W \subset \overline{W} \subset U$, with $\overline{W}$ compact. In full:

$$\mathfrak{f}^{-1}(L) \subset V \subset \overline{V} \subset W \subset \overline{W} \subset U.$$

Therefore

$$\mathfrak{f}\big((X \smallsetminus V) \cap \overline{W}\big) = \big(X \smallsetminus \mathfrak{f}(V)\big) \cap \mathfrak{f}(\overline{W}) \subset (X \smallsetminus L) \cap f(U).$$

We can also choose an $x \in X$ such that $\mathfrak{f}(x) \in int(L)$ (where $int(L)$ is the interior of L). So there exists an open set (in $\mathbf{TOP}^h(X,X)$):

$$B_{XX}\left(\{x\}, int(L)\right) \cap B_{XX}\left((X \smallsetminus V) \cap \overline{W}, (X \smallsetminus L) \cap \mathfrak{f}(U)\right)$$

containing $\mathfrak{f}$, which we denote U_0. We claim that $U_0 \in T^{-1}(B_{XX}(L,U))$.

Let $\mathfrak{h} \in U_0$. We have: $\mathfrak{h}\left((X \smallsetminus V) \cap \overline{W}\right) \subset (X \smallsetminus L) \cap \mathfrak{f}(U)$. Taking complements and reversing the inclusion we have

$$L \cup (X \smallsetminus \mathfrak{f}(U)) \subset \mathfrak{h}(V \cup (X \smallsetminus \overline{W})) = \mathfrak{h}(V) \cup \mathfrak{h}(X \smallsetminus \overline{W}).$$

Now $\mathfrak{h}(V)$ and $\mathfrak{h}(X \smallsetminus \overline{W})$ are disjoint open sets, and L is connected[1], so either L is contained in $\mathfrak{h}(V)$ or L is contained in $\mathfrak{h}(X \smallsetminus \overline{W})$, but not both. We claim that $L \subset \mathfrak{h}(V)$.

Note that since $\mathfrak{h} \in B_{XX}\left(\{x\}, int(L)\right)$, we have $\mathfrak{h}(x) \in int(L)$. Since $\mathfrak{f}(x) \in int(L)$, by construction, we have $x \in \mathfrak{f}^{-1}(int(L)) \subset V$. So $\mathfrak{h}(x) \in \mathfrak{h}(V)$. So $L \cap \mathfrak{h}(V) \neq \varnothing$. So $L \subset \mathfrak{h}(V)$.

Since $L \subset \mathfrak{h}(V)$, we have $\mathfrak{h}^{-1}(L) \subset V \subset U$. Hence $\mathfrak{h}^{-1} \in B_{XX}(L,U)$. □

Proof. (Of Theorem 4.2.1) In Lemma A.0.2 we prove that the composition is continuous if X is locally compact Hausdorff. In Lemma A.0.4 we prove that the inverse map is continuous. □

A.0.2 Motions as maps from $M \times \mathbb{I}$

Definition A.0.5. Fix a manifold M. Let $Premot_M^{mov} \subset \mathbf{Top}(M \times \mathbb{I}, M)$ denote the subset of elements $g \in \mathbf{Top}(M \times \mathbb{I}, M)$ such that:

(I) for all $t \in \mathbb{I}$, $g|_{M \times \{t\}}$ is a homeomorphism $M \times \{t\} \to M$, and

(II) for all $m \in M$, $g(m,0) = m$.

---

[1] This is where the crucial fact that L can be chosen to be connected is used.

Lemma A.0.6. *The restriction of the map Φ obtained by letting $X = \mathbb{I}$ and $Y = Z = M$ in Lemma 3.5.16 yields a bijection*

$$Premot_M \xrightarrow{\sim} Premot_M^{mov}.$$

Proof. We have that Φ is a bijection so we just need to check that $\Phi(Premot_M) = Premot_M^{mov}$ and that $\Phi^{-1}(Premot_M^{mov}) = Premot_M$ where Φ^{-1} sends a map $g\colon M \times \mathbb{I} \to M$ to the map $t \mapsto (m \mapsto g(m,t))$.

Let $f \in Premot_M$ be a pre-motion. Then $\Phi(f)|_{M\times\{t\}} = f_t$ which is a homeomorphism and $\Phi(f)(m,0) = f_0(m) = m$.

Let $g \in Premot_M^{mov}$. Then $m \mapsto g(m,t)$ is a homeomorphism for all $t \in \mathbb{I}$ and $\Phi^{-1}(g)(0) = (m \mapsto g(m,0)) = id_M$. □

Proof. (Of Lemma 4.3.16) Notice first that each $\mathrm{Mt}_M^{mov}(N,N')$ is a subset of $Premot_M^{mov}$. We have from Lemma A.0.6 that Φ gives a bijection $Premot_M \cong Premot_M^{mov}$ so we only need to check that $\Psi(\mathrm{Mt}_M(N,N)) = \mathrm{Mt}_M^{mov}(N,N')$ and $\Psi^{-1}(\mathrm{Mt}_M^{mov}(N,N)) = \mathrm{Mt}_M(N,N')$.

Suppose $f\colon N \rightsquigarrow N'$ is a motion, then $\Phi(f)(N \times \{1\}) = f_1(N) = N'$. Suppose $f' \in \mathrm{Mt}_M^{mov}(N,N')$, then $\Phi^{-1}(f')_1(N) = f'(N \times \{1\}) = N'$. □

Definition A.0.7. Fix a manifold M. Consider $\mathbf{Top}^h(M \times \mathbb{I}, M \times \mathbb{I})$. Let

$$\begin{aligned} Premot_M^{hom} = \{ f \in \mathbf{Top}^h(M\times\mathbb{I}, M\times\mathbb{I}) \mid & f(m,0) = (m,0)\ \forall m \in M; \\ & f(M\times\{t\}) = M\times\{t\}\ \forall t \in \mathbb{I}\}. \end{aligned}$$

That is, $Premot_M^{hom} \subset \mathbf{Top}^h(M \times \mathbb{I}, M \times \mathbb{I})$ denotes the subset of homeomorphisms $g \in \mathbf{Top}^h(M \times \mathbb{I}, M \times \mathbb{I})$ such that

(I) $g(m,0) = (m,0)$ for all $m \in M$, and

(II) $g(M \times \{t\}) = M \times \{t\}$ for all $t \in \mathbb{I}$.

Notice that to prove the following we need both that $\mathrm{Homeo}_M(\varnothing,\varnothing)$ is a topological group and the product-hom adjunction of Lemma 3.5.16. [2]

---

[2] We use the fact that M is a manifold, so that $\mathrm{Homeo}_M(\varnothing,\varnothing)$ is a topological group. An alternative proof of this result that holds if M is compact (and not necessarily a manifold) follows from the fact that

Lemma A.0.8. *Let M be a manifold. There is a bijection*

$$\begin{aligned}\Theta\colon Premot_M &\to Premot_M^{hom},\\ f &\mapsto ((m,t)\mapsto (f_t(m),t)).\end{aligned}$$

Proof. We first check the Θ is well defined. Let $f \in Premot_M$. Then $\Theta(f)$ is continuous since the projection onto the first coordinate of the map $(m,t)\mapsto (f_t(m),t)$ is $\Phi(f)$ with Φ as defined in Lemma 3.5.16 and the projection on the second coordinate is clearly continuous. We have $\Theta(f)(m,0) = (f_0(m),0) = (m,0)$ and $\Theta(f)(M\times\{t\}) = f_t(M)\times\{t\} = M\times\{t\}$.

We now check $\Theta(f)$ is a homeomorphism. The map $(m,t)\mapsto (f_t(m),t)$ has inverse $(m,t)\mapsto (f_t^{-1}(m),t)$. Let us see that the inverse is continuous. We have that f is a pre-motion and so Lemma 4.3.4 gives that f^{-1} is a pre-motion, specifically it is a continuous map $\mathbb{I}\to \mathbf{TOP}(M,M)$. Hence $(m,t)\mapsto (f_t^{-1}(m),t)$, which is the image of f^{-1} under Θ, is continuous.

Consider the following map.

$$\begin{aligned}\Theta^{-1}\colon Premot_M^{hom} &\to Premot_M\\ g &\mapsto (t\mapsto (m\mapsto p_0\circ g(m,t))\end{aligned}$$

It is straightforward to check that for any $f\in Premot_M$ we have $\Theta^{-1}\circ\Theta(f) = f$ and that for any $g\in Premot_M^{hom}$ we have $\Theta\circ\Theta^{-1}(g) = g$. It remains to check that Θ^{-1} is well defined. Let $g\in Premot_M^{hom}$. The map $\Theta^{-1}(g)$ is continuous as it is equal to $(\Phi^{-1})(p_0\circ g)$, with Φ as in Lemma 3.5.16.

We have $(\Theta^{-1}(g))_0(m) = p_0\circ g(m,0) = m$ so $\Theta^{-1}(g)_0 = id_M$. For all $t\in\mathbb{I}$ the restriction $g|_{M\times\{t\}}$ is also a homeomorphism onto its image which, by (II), is $M\times\{t\}$. The projection $p_0\colon M\times\{t\}\to M$ is an isomorphism. Hence for all $t\in\mathbb{I}$, $\Theta^{-1}(g)_t = p_0\circ g|_{M\times\{t\}}$ is in $\mathrm{Homeo}_M(\varnothing,\varnothing)$. □

Proof. (Of Theorem 4.3.18) Notice each $\mathrm{Mt}_M^{hom}(N,N')$ is a subset of $Premot_M^{hom}$. Also Lemma A.0.8 gives that Θ yields a bijection $Premot_M \cong Premot_M^{hom}$, hence we only need

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any continuous bijection between compact Hausdorff spaces is a homeomorphism.

to check that $\Theta(\mathrm{Mt}_M(N,N')) \subset \mathrm{Mt}_M^{hom}(N,N')$ and $\Theta^{-1}(\mathrm{Mt}_M^{hom}(N,N')) \subset \mathrm{Mt}_M(N,N')$.

If $f\colon N \rightsquigarrow N'$ is a motion, then $\Theta(f)(N \times \{1\}) = f_1(N) \times \{1\} = N' \times \{1\}$. Now suppose $f'\colon M \times \mathbb{I} \to M \times \mathbb{I}$ is a homeomorphism with $f'(N \times \{1\}) = N' \times \{1\}$, then $\Theta^{-1}(f')_1(N) = p_0 \circ f'(N \times \{1\}) = p_0(N' \times \{1\}) = N'$, so $\Theta^{-1}(f)$ is in $\mathrm{Mt}_M(N,N')$. $\square$

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