Essays on Endogenous Growth and Innovation

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Abstract

This thesis consists of three essays on endogenous growth and innovation. Chapter 1 explores the interaction between productivity improvements and innovations by existing firms, and by more productive new firms. I develop a tractable endogenous growth model with capital accumulation in which growth is driven by innovation from incumbents and creative destruction by entrants. I demonstrate that capital accumulation, by affecting the level of incumbents' and entrants' R&D (which is an endogenous variable and the direct determinant of long-rung growth), plays an important role in stimulating economic growth. I also show the effect of policies on equilibrium productivity growth and provide a new perspective to the welfare analysis of models of innovation by both incumbents and entrants.

Chapter 2 studies how strengthening patent protection influences economic growth in a Schumpeterian endogenous growth model with capital accumulation. In contrast to the previous literature, which mostly considered patent policy in infinite-lifetime economies, this paper investigates the implications of patent policy in an overlapping generations framework. That allows me to study how heterogeneity in patent ownership across generations changes the implications of patent length and breadth for R&D-based growth.

The aim of Chapter 3 is to investigate interactive effects of intellectual property rights protection and monetary policy on economic growth. I develop an overlapping generations model with R&D-based growth in which IPR protection is introduced by considering patent breadth that determines firms' market power, while money demand is incorporated by imposing a cash-in-advance constraint on old age consumption expenditure. The demographic structure makes it possible to study inter-generational trade in patents and a life-cycle saving motive, thereby allowing the paper to contribute to the theory of optimal monetary and patent policy in a framework with R&D-based endogenous growth.

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Declaration

I declare that this thesis is a presentation of original work and I am the sole author. This work has not previously been presented for an award at this, or any other, University. All sources are acknowledged as References.

Introduction

This thesis contributes to the study of endogenous economic growth. Since Solow's pioneering study (1956), growth theory has experienced remarkable progress the last decades. The Solow model viewed technological change as exogenous factor that is determined outside the model, while the AK models were the first to attempt at endogenizing growth. These models were capable of producing long-run growth based on capital accumulation, but they disregarded the role of technological change caused by innovation. From the early 1990s, many authors proposed endogenous growth models in which innovation activity is viewed as the most important engine of economic growth. Well-known Romer (1990) and Grossman and Helpman (1991b) papers presented models in which innovation induces productivity growth by creating new variety of products. Aghion and Howitt (1992); Howitt and Aghion (1998) incorporated the Schumpeterian idea of creative destruction in the models in which growth is assumed to depend on, not the number of products, but on their quality. Besides endogenizing technological change, these models connect the technological progress with market structure, competition and intellectual property rights policy.

One of the criticisms levelled at innovation-based models is that they ignore capital accumulation as a source of growth. The common view in economic growth literature has been that the accumulation of capital contributes positively to growth in the short run, but in the long run only the rate of technological progress matters (Grossman and Helpman (1991a)). However, Howitt and Aghion (1998) demonstrate that this common view is mistaken. They present a Schumpeterian model that considers capital accumulation and innovation as equal elements in the growth process. Their model predicts productivity growth coming solely from creative destruction by new firms/entrants. At the same time, empirical evidence suggests that not only entrant's activity promotes productivity growth, but the contribution of incumbents to the growth is a substantial. On the other hand, the endogenous growth literature that study interaction incumbent and entrants in growth process (Acemoglu and Cao (2015), Klette and Kortum (2004)) ignores capital accumulation as a source of growth. Chapter 1 integrates innovation and capital into a single framework by constructing a model in which growth is driven by the incumbents' and entrants' innovations. Developed a tractable general equilibrium growth model is rich enough to investigate roles of capital accumulation and innovation in long-run growth, and the growth and welfare implication of innovation policy (R&D subsidies).

Innovation activity and, as a result, economic growth can be affected by various policies. In particular, patent policy has a significant impact on innovation by granting firms monopoly right to produce patented goods. Chapter 2 and 3 study the interaction of intellectual patent protection and heterogeneity in patent ownership by considering an overlapping generations framework. In Chapter 2, I examine the effect of patent policy on growth in an OLG endogenous growth model with vertical innovation. In general, patent length and patent breadth determine the degree of patent protection. I consider the two extreme cases of patent length — one-period patent protection and infinite patent length and show that heterogeneity in patent ownership affects the implications of patent policy, namely, under short duration of patent growth rate is higher that under long patent duration because of "crowding-out" effect. Turning to the breadth protection, I reveal that that incomplete patent breadth has two opposite effects on growth. On the one hand, loosening patent breadth by reducing patented intermediate goods price increases demand for the intermediate product, and, as a result, increases output. This stimulates aggregate investment, including investment in R&D, which promotes growth. On the other hand, an increase in demand for intermediate product leads to reallocation of investment towards physical capital by reducing R&D investment, and, consequently, reduces growth rate. If latter effect dominates the former, patent protection is not growth enhancing.

Chapter 3 continues the investigation of effect of intellectual property right on growth, but turns to the analysis of interactive effects of intellectual property right protection and monetary policy. I construct an OLG model with R&D-based growth along the line of Rivera-Batiz and Romer (1991). Within this framework, I embed a hybrid model, in which money is introduced by imposing a cash-in-advance constraint on old age consumption expenditure, whereas IPR protection is incorporated by considering patent breadth that determines firms' market power. I find that strengthening patent breadth raises the cost of holding money, thereby reducing investment in capital and research and thus growth. At the same time, tightening IPR moves investment from capital accumulation to research investment and, as a result, increases growth rate. All in all, the impact of these two contradicting effects on growth rate in favour of strengthening breadth protection. In turn, monetary expansion lowers the growth rate by reducing investment in R&D. Moreover, a stronger patent breadth weakens effects of monetary policy.

Chapter 1

Innovation by Incumbents and Entrants, Capital and Endogenous Growth

1.1 Introduction

A large literature has focused on understanding the determinants of long-run economic growth. One distinguishes two alternative strands to study endogenous growth: capital accumulation and "innovation-based" approaches. The former emphasizes the importance of investment in human and physical capital (Lucas (1988); Rebelo (1991)), while the latter is based on the idea that long-run growth relies on innovation activity (Aghion and Howitt (1992); Grossman and Helpman (1991a,b); Romer (1990); Aghion et al. (2015a); Jones (1995)). But, as has been indicated by Howitt and Aghion (1998), capital accumulation and innovation should be considered as complementary factors of growth process as each plays an important role.

A series of papers have considered innovation jointly with capital accumulation. Howitt and Aghion (1998) construct a hybrid model by introducing capital into the

Schumpeterian growth paradigm. They find that government policies, such as a subsidy to research and capital accumulation, will raise the rate of economic growth. A few other papers (Arnold (1998); Blackburn et al. (2000)) emphasize the importance of human capital accumulation and innovation by constructing model that incorporates both of these factors.

This paper integrates innovation and capital into a single framework by developing a tractable model in which growth is driven by the incumbents' as well as entrants' innovations; this is in contrast to the previous literature. I study an economy in which incumbents' and entrants' technological breakthroughs are the key drivers of growth, whereas the literature related to the capital-innovation integrating approach predicts productivity growth coming solely from creative destruction by new firms/entrants. Developing a model that generates productivity growth driven by continuing firms together with new entrants provides a richer framework for the analysis that is consistent with features of the empirical data that highlights that a large part of growth comes from productivity improvements by incumbents, although entrants also make contributions to productivity growth.

The purpose of this paper is to build a tractable framework to understand the link among capital accumulation, entrants' and incumbents' innovation, and endogenous growth. The model builds on innovation by both incumbents and entering firms framework in Klette and Kortum (2004) and Acemoglu and Cao (2015), and introduces capital accumulation as a determinant of long-run growth. The distinguishing feature of the model is the lab-equipment specification for R&D (where a composite final good is used as an input) rather than the employment of skilled or unskilled workers or scientists.

I prove the existence of a steady state equilibrium at which the economy grows at an endogenously determined constant rate. The key aspect about the steady state growth rate is that it is determined by both incumbents' and entrants' R&D activity.

In turn, the level of research is an endogenous variable and is affected by the incentive to accumulate capital.

The tractable model allows me to get a number of comparative static results. I find that the equilibrium response of an incumbent's research expenditure to an increase in the R&D subsidy rate to entrants is negative, while the response of entrants' research expenditure is positive, i.e. increasing the R&D subsidy rate to entrants will reduce incumbents' R&D, whereas entrants will spend more on research. The increase of R&D subsidy rate to incumbents encourages incumbents' research efforts, while the effect of this rate on entrants' R&D is the opposite. A capital subsidy reduces the cost of capital which raises the flow of profit to an innovator. This, in turn, increases the value of an innovation thereby spurs R&D investment. The model shows that a subsidy to an incumbent's research activity and to capital are beneficial for long term growth. The former result — a positive effect of R&D subsidies to an incumbent — is not surprising; the latter result — a positive effect of the capital subsidy rate — is more interesting. This result is consistent with the main result derived in Howitt and Aghion (1998), which leads to an important policy conclusion: it implies that, in general, a subsidy to capital accumulation can be as effective a way of promoting growth as a subsidy to R&D. A subsidy to an entrants has two opposite effects on growth. On the one hand, by encouraging more entry subsidy to an entrant's research activity lower profitability and value of incumbents that reduce incumbents R&D investment and, as result, lower economic growth. On the other hand, the cost reduction induced by the subsidy, increases the expected return to entrant. This encourages more entrants' research investment and tends to increase economic growth. If incumbents' contribution to growth is sufficiently low,[1] the latter effect dominates the former, i.e., a subsidy to entrants has positive effect on growth. In this case, my finding is in line with the Schumpeterian literature, in which encouraging entrants

[1]The probability of an incumbent being successful in innovation is sufficiently low. Note that if a flow rate of innovation by incumbent is sufficiently hight, the effect is ambiguous.

leads to an increase in economic growth. Note that in Acemoglu and Cao's model the growth rate is decreasing with subsidy to entrants (increasing in the tax on entrants) for a model with a linear R&D technology for incumbents (an effect which is opposite of the one in a standard Schumpeterian model), while for the general case they find the effect to be ambiguous.

My model brings to light several new insights regarding welfare analysis. Acemoglu and Cao (2015) indicate two differences between the decentralized equilibrium and the Pareto optimal allocation: the first corresponds to the "monopoly distortion effect", the second — to the "business stealing effect". I find additional distortions between the laissez-faire steady-state and optimal solution. First, following Hagedorn et al. (2007), I decompose the "intertemporal spillover effect" into a "passive business stealing", a "standing on shoulders", and a "consumption dilution" subeffects. The "passive business stealing effect" affects private firms: the monopolist's profit flow will be reduced by creative destruction that comes from innovation by entrants. This effect is negative consequence of creative destruction because this part promotes underinvestment. The "standing on shoulders effect", which can be interpreted as capturing the cumulative nature of knowledge, arises because the planner captures the rent from the next innovation into perpetuity, while a private firm receives the benefit only during one interval and after that rents will be obtained by the next innovator. Further, the social planner takes into account expenditures on incremental and radical innovations, whereas the private firm accounts for only its own R&D expenditure. This difference reflects the "consumption dilution effect": spending on radical innovations reduces resources available for consumption. In Howitt and Aghion (1998), the consumption dilution effect compensates two other effects, the "passive business stealing" and the "standing on shoulders" effects, but, in my model, this is not the case. The "passive business stealing" and "standing on shoulders" effects tend to generate too little incremental and radical research under laissez-faire, while the "consumption dilution" effect tends to make these innovations too large. Thus, the direction of the "intertemporal effect" is ambiguous, implying that the shift of surfaces is not uniquely determined. Second, social and private revenue differ, on the other hand, their costs also differ. These differences correspond to a "appropriability effect": the social planner appropriates the entire consumer surplus related to the good that is created. The private innovator of a new good captures only part of this surplus because markup of price over cost decreases sales from the optimal level were the good to be sold at its marginal cost. At the same time, the social cost exceeds the cost to a private firm because of market power of a private firm. This distortion corresponds to the "monopoly distortion effect". The "appropriability" and "monopoly distortion" effects work in opposite directions. The "appropriability" effect tends to generate insufficient research, whereas the "monopoly distortion" effect induces too much research under laissez-faire. Third, the active "business stealing effect" reflects the fact that the social planner internalizes the destruction of rents generated by radical innovations, i.e. the social planner takes into consideration that the social return from the previous innovation will be destroyed by a new innovation. This implies that in the social optimum there will be less radical innovation than under laissez-faire. Thus, the "business stealing" effect affects only research by entrants, making entrants more active under laisser-faire. Fourth, the "monopoly distortion effect" also can be seen in the capital equation and is absent in a model without a capital accumulation (e.g., the Acemoglu and Cao model). An monopolist gains from the lower capital cost at the cost of a household as supplier of capital. This effect tends to generate too little capital accumulation under laissez-faire. Overall, whether the growth rate under laisser-faire is greater or smaller than the optimal growth rate depends upon whether the steady-state level of research, by incumbents and entrants, for the decentralized economy is higher or lower than the socially optimal level for these variables. The "appropriability", "passive business stealing" and "standing on shoulders" effects tends to make laissez-faire incremental and radical innovation less than the optimal rate, whereas "monopoly distortion" effect tends to make it more than the optimal rate, and at the same time, by affecting only research by entrants, the "business stealing effect" tends to generate too much radical research in the decentralized economy. In case of linear return to research by the incumbents, the growth of the Pareto optimal allocation is greater than that under laissez-faire, while for a general specification of the technology the comparison is ambiguous.

Welfare analysis reveals that policy instruments — subsidies to incumbents' and entrants' research activity, and a subsidy to capital accumulation — may have different impact on welfare depending on the structural characteristics of the economy. The relationships between welfare and subsidies to research activity may be negative, or may be represented by an inverted U-shaped curve with maximum shifts either left or right. The latter suggests that increasing the subsidies may initially enhances welfare, but for further higher values of subsidy rates the welfare change becomes negative. A subsidy to capital initially improves welfare but further increases of the subsidy rate reduces welfare.

This paper contributes to the endogenous growth literature but mostly to a strand of it which consider the interaction between incumbents' and entrants' innovation in the growth process. It provides an extension to the workhorse Schumpeterian model to produce an model of endogenous growth that is very well suited answer important questions regarding economic policy.

The paper is organized as follows. Section 1.2 describes the model of the paper, defines the equilibrium notion and proves existence and uniqueness. Section 1.3 provides the analysis of the effects of policy on equilibrium growth. Section 1.4 characterizes the Pareto optimal allocation and compares it to the equilibrium allocation. Section 1.5 looks at welfare implications. Section 1.6 contains brief final remarks, while the Appendix contains several proofs and derivations.

1.2 Model

Time is continuous. The economy consists of households; a final good sector; and an intermediate goods sector that produces differentiated goods. The final good is storable in the form of capital. Final good is produced competitively and used as a consumption good and a capital good, and also as an input in R&D. Innovations will be quality improvements in intermediate goods.

Households are represented by a single agent that maximizes utility. The population is constant at L, and labor is supplied inelastically. The household owns firms, provides labor services in exchange for wages, accumulates physical capital, and rents it at a rental rate R_t to firms.

An important feature of the model is that all that is required for research is investment in equipment or in laboratories rather than the employment of skilled or unskilled workers or scientists (this is the so-called "lab equipment" model).

1.2.1 Final good producer

The final good is produced under perfect competition using the intermediate goods and labor, according to

$$Y_t = \frac{1}{1-\gamma}\int_0^1 q_t(j)F(x_t(j), L)dj, \tag{1.1}$$

where $x_t(j)$ is the quantity of intermediate good $j \in [0, 1]$, $q_t(j)$ is the quality of the latest version of intermediate product j, and $F : R_+^2 \to R_+$ is twice differentiable in x and L, $F_x(x, L) > 0$, $F_L(x, L) > 0$, $F_{xx}(x, L) < 0$, $F_{LL}(x, L) < 0$. Moreover, F is a constant-returns production function. Intermediate goods are necessary factors, $F(0, L) = 0$. Furthermore, assume that the production function is Cobb-Douglas $F(x, L) \equiv x^{1-\gamma}L^{\gamma}$, $\gamma \in (0, 1)$. The price of the final good is normalized to 1.

The final good producer solves the problem:

$$\max_{x_t(j),L_t} Y_t - \int_0^1 p_t(j)x_t(j)dj - w_tL_t \tag{1.2}$$

subject to (1.1), taking prices $p_t(j)$ and qualities $q_t(j)$ as given.

The first order condition (hereafter, FOC) for the maximization problem is:

$$p_t(j) = \frac{1}{1-\gamma}q_t(j)F_x(x_t(j), L) = x_t(j)^{-\gamma}q_t(j)L_t^{\gamma}, \tag{1.3}$$

$$w_t = \frac{1}{1-\gamma}\int_0^1 q_t(j)F_L(x_t(j), L)dj = \frac{\gamma}{1-\gamma}\left(\int_0^1 q_t(j)x_t(j)^{1-\gamma}dj\right)L_t^{\gamma-1}. \tag{1.4}$$

Equation (1.3) is the inverse demand function for intermediate goods, while (1.4) equalizes the marginal cost of employing labor to the value of the marginal product of labor.

1.2.2 Intermediate goods production and R&D

Incumbents

There is a continuum of intermediate firms that produce the differentiated quality-enhancing goods $x_t(j)$, $j \in [0, 1]$, which are produced with the input of capital $K_t(j)$.

Following Howitt and Aghion (1998), the production function is given by:

$$x_t(j, q) = K_t(j)/q_t(j), \tag{1.5}$$

where the division by $q_t(j)$ reflects that successive blueprints are produced by increasingly capital-intensive techniques.

The intermediate goods sector is monopolistic. The marginal cost of firms will depend on the rental rate of capital, R_t. Later it will be shown that the rental rate is the interest rate, r_t, plus the rate of depreciation, δ, and minus the subsidy rate to holding of capital, β_k:

$$R_t = r_t + \delta - \beta_k. \tag{1.6}$$

Taking into account that the inverse demand curve facing a monopolist is given by (1.3), the monopolist's maximization problem can be written as[2]

$$\pi_t(j) = \max_{x_t(j)} \Big(p_t(j)x_t(j) - R_t K_t(j)\Big) = \max_{x_t(j)} \Big(x_t(j)^{1-\gamma} L^{\gamma} - R_t x_t(j)\Big) q_t(j). \tag{1.7}$$

Solution $x_t(j)$ is independent of j: in equilibrium all intermediate producers supply the same quantity of goods:

$$x_t(j) \equiv x_t = \left(\frac{1-\gamma}{R_t}\right)^{\frac{1}{\gamma}} L. \tag{1.8}$$

Thus, the monopoly price is given by the standard formula with a constant markup over marginal cost:

$$p_t(j) = \frac{R_t q_t(j)}{1-\gamma}. \tag{1.9}$$

The aggregate demand for capital is given by:

$$K_t = \int_0^1 K_t(j)dj = \int_0^1 x_t(j) q_t(j) dj = x_t Q_t, \tag{1.10}$$

where $Q_t = \int_0^1 q_t(j)dj$ is an index of aggregate quality. Quantity x_t can, hence, be rewritten as $x_t = k_t L$, where $k_t \equiv \dfrac{K_t}{Q_t L}$ is the capital intensity.

[2]For a general CRS technology

$$\pi_t(j) = \max_{x_t(j)} \Big(p_t(j)x_t(j) - R_t K_t(j)\Big) = \max_{x_t(j)} \left(\frac{1}{1-\gamma} q_t(j) F_x(x_t(j), L) x_t(j) - R_t q_t(j) x_t(j)\right).$$

.

The first-order condition for profit maximization (1.7) can be expressed as:[3]

$$R(k_t) = k_t^{-\gamma}(1-\gamma). \tag{1.11}$$

Substituting for p_t from (1.3), $R(k_t)$ from (1.11) and $x_t = k_t L$ into (1.7) yields

$$\pi_t(j) = \pi(k_t) q_t(j) L, \tag{1.12}$$

where $\pi(k_t) = \gamma k_t^{1-\gamma}$.[4]

An innovation of product j at time t increases the quality of this product according to the following "quality ladder" for each good:

$$q_t(j) = \mu^{N_t(j)} q_s(j) \qquad \forall j, t, \tag{1.13}$$

where $\mu > 1$, an initial quality of product j is $q_0(j) \in R_+$, and N_t is a Poisson process — the number of incremental innovations between time $s \leq t$ and time t (s is the time at which this good was invented).

q_t is a stochastic process which obeys the stochastic differential equation of the form [5]:

$$dq_t = (\mu - 1) q_{t\_} dN_t.$$

[3]For a general case the rental rate of capital is given by $R_t = \frac{1}{1-\gamma}\Big(F_x(x_t, L) + x_t F_{xx}(x_t, L)\Big)$. Using $x_t = k_t L$, last equation can be rewritten as $R(k_t) = \frac{1}{1-\gamma}\Big(F_k(k_t, 1) + k_t F_{kk}(k_t, 1)\Big)$. It can been easily seen that for Cobb-Douglas case $F(k_t, 1) = k_t^{1-\gamma}$.

[4]For a general case $\pi(k_t) = \frac{1}{1-\gamma}\Big[F_k(k_t, 1)k_t - \big(F_k(k_t, 1) + k_t F_{kk}(k_t, 1)\big)k_t\Big] = -\frac{1}{1-\gamma} k_t^2 F_{kk}(k_t, 1)$.

[5]See Appendix.

Only an incumbent can generate incremental innovations.[6] Incumbent invests in research $S_t(j)q_t(j)$ units of the final good. The innovation on each good arrives at the Poisson rate $\zeta(S_t(j))$, where $S_t(j)$ is quality-adjusted level of research.

The Poisson arrival rate is strictly increasing in S, $\zeta'(\cdot) > 0$, that implies that higher R&D expenditure flows lead to a greater probability of a successful innovation, while $\zeta''(\cdot) < 0$ captures diminishing returns to R&D spending, and $\zeta(0) = 0$. Later on I will use the following functional form that satisfies the assumptions mentioned: $\zeta(S_t) = \zeta S_t^{1-\beta}$ with $\beta \in (0, 1)$.

Entrants

Entrants perform radical innovations.[7] Spending one unit of final good, radical innovations arrive at a rate $\frac{\psi(\tilde{S}_t(j))}{q_t(j)}$. Then with R&D expenditure of $\tilde{S}_t(j)q_t(j)$ units of the final good, the Poisson rate of entrants' innovation is equal to $\tilde{S}_t(j)\psi(\tilde{S}_t(j))$. I assume that $\psi(\tilde{S})$ is strictly decreasing, $\psi(\tilde{S}) \in C^1$, and $\tilde{S}\psi(\tilde{S})$ is strictly increasing in $\tilde{S}$, capturing the fact that the greater the entrant's R&D expenditure, the higher the probability of success in innovation. When entrants improve the product, the quality level jumps by $\tilde{\mu}q$: $q_t(j) = \tilde{\mu}q_{t_-}(j)$, where $\tilde{\mu} > \mu$. $\psi(\tilde{S}_t) = \psi\tilde{S}_t^{-\tilde{\beta}}$ with $\tilde{\beta} \in (0, 1)$ satisfies the assumptions and I will use this functional form below.

I assume that an entrant is not constrained by potential competition from previous innovators and charges the unconstrained monopoly price, i.e. innovations are drastic. Drastic innovations correspond to a sufficiently high value of $\tilde{\mu}$ such that entrant becomes monopolist after innovation. The value of $\tilde{\mu}$ should be such that final good producer is indifferent between buying intermediate goods from the entrant and from the previous innovator, i.e. $\Psi(p_t, Y) \leq \tilde{\Psi}(\tilde{p}_t, Y)$, where $\Psi(p_t, Y)$ is the cost of producing

[6] Following Acemoglu and Cao (2015) I assume that the cost of incremental research is infinite to the entrants. This is similar to the assumption in Barro and Sala-i Martin (2004), p.335, that the industry leader has a cost advantage in research.

[7] Although incumbents could have access to radical innovation technology, because of Arrow's replacement effect they would choose not to (because of free entry entrants make zero profit from this technology, whereas incumbents by replacing their own product would have negative profit).

Y units of the final good using the new/improved intermediate input at the monopoly price $p_t = \frac{R_t q_t}{1-\gamma}$, so that $\Psi(p_t, Y) = \frac{R_t q_t}{1-\gamma}\left(\frac{Y_t(1-\gamma)}{Q_t L^\gamma}\right)^{\frac{1}{1-\gamma}}$, while $\tilde{\Psi}(\tilde{p}_t, Y)$ is the cost of producing Y units of the final good using the intermediate good produced by the previous innovator at the marginal cost $\tilde{p}_t = \frac{R_t q_t}{\tilde{\mu}}$,[8] so that $\tilde{\Psi}(\tilde{p}_t, Y) = \frac{R_t q_t}{\tilde{\mu}}\left(\frac{Y(1-\gamma)\tilde{\mu}}{Q_t L^\gamma}\right)^{\frac{1}{1-\gamma}}$. It can be easily seen that the inequality $\Psi(p_t, Y) \leq \tilde{\Psi}(\tilde{p}_t, Y)$ is equivalent to

$$\tilde{\mu} \geq \left(\frac{1}{1-\gamma}\right)^{\frac{1}{\gamma}-1}, \tag{1.14}$$

so I assume the drastic innovation regime, i.e., entrants can charge the unconstrained monopoly price.

1.2.3 Household

The preferences of the representative household are defined by the utility function:

$$U(C_t) = \int_0^\infty e^{-\rho t}\frac{C_t^{1-\epsilon}-1}{1-\epsilon}dt, \tag{1.15}$$

where ρ is the discount rate, $\rho > 0$, C_t denotes consumption, and $\epsilon > 0$ is the coefficient of relative risk aversion.

The household holds its wealth in two forms: the first is capital which is rented by the intermediate good producers to produce the intermediate goods, the second is as the owner of firms that produce the intermediate goods where the value of these firms in aggregate is denoted by A_t, where $A_t = \int_0^1 V_t(j,q)dj$ and $V_t(j,q)$ is the net present value of monopolist with highest quality $q_t(j)$ in the j variety. The household faces an intertemporal consumption decision subject to a budget constraint and the

[8] Marginal cost of the firm with the next highest quality is $\tilde{p}_t = R_t q_{t-1} = \frac{R_t q_t}{\tilde{\mu}}$.

law of motion for capital:

$$\dot{A}_t = r_t A_t + w_t L_t + R_t K_t - C_t - I_t - T_t,$$
$$\dot{K}_t = I_t - (\delta - \beta_k) K_t,$$

where I_t is investment, and T_t a "lump sum" tax.

1.2.4 Equilibrium Characterization

Incumbent and entrant optimization The net present discounted value of an incumbent can be written as

$$V_t(j,q) = E_{t,q}\left[\int_t^T e^{-\int_t^s r(\tau)d\tau}\Big(\pi(k_s)q_s(j)L - S_s(j)(1-s_I)q_s(j)\Big)ds\right], \tag{1.16}$$

where $r(\tau)$ is the equilibrium market real interest rate, T is the time when a new firm enters, and s_I is the subsidy rate to R&D. The net present value of the monopolist with the highest quality q at time t for good j satisfies the Hamilton–Jacobi–Bellman equation (see the Appendix for the derivation):

$$r_t V_t(j,q) - \dot{V}_t(j,q) = \max_{S_t(j)\geq 0}\left\{\begin{array}{l} \pi(k_t)q_t(j)L - S_t(j)(1-s_I)q_t(j) \\ +\zeta(S_t(j))(V_t(j,\mu q) - V_t(j,q) \\ -\tilde{S}_t(j)\psi(\tilde{S}_t(j))V_t(j,q) \end{array}\right\}. \tag{1.17}$$

The first line in this equation is the gross profit net of the cost of R&D. The second line is the payoff from innovation, and final line gives the change in the firm's value due to creative destruction by entrants at the Poisson rate $\tilde{S}_t(j)\psi(\tilde{S}_t(j))$.

There is free entry into research.The value of each blueprint, the net present discounted value of an entrant's earning, for an entrant is $V_t(j,\tilde{\mu q})$, hence, by spending $q_t(j)$ units of final good on research, entrant generates the flow revenue

$\psi(\tilde{S}_t(j))V_t(j, \tilde{\mu}q)$. If there is free entry into research, as I assume, the flow of profit for an entrant must be zero, that is,

$$\psi(\tilde{S}_t(j))V_t(j, \tilde{\mu}q) - q_t(j)(1 - s_E) = 0. \tag{1.18}$$

This equation can be rewritten in complementary slackness form:

$$\begin{aligned} &\psi(\tilde{S}_t(j))V_t(j, \tilde{\mu}q) \leq q_t(j)(1 - s_E) && \text{and} \\ &\psi(\tilde{S}_t(j))V_t(j, \tilde{\mu}q) = q_t(j)(1 - s_E) && \text{if} \qquad \tilde{S}_t(j) > 0, \end{aligned} \tag{1.19}$$

i.e., as long as the cost of R&D is above revenue, the entrant will not invest in research. When the value of expected revenue exceeds the cost of research, the firm will increase R&D efforts up to the point at which the condition holds with equality.

Dynamic optimization by household As usual C_t obeys the Euler equation:[9]

$$\frac{\dot{C}_t}{C_t} = \frac{r_t - \rho}{\epsilon}. \tag{1.20}$$

The transversality conditions are

$$\lim_{t \to \infty} e^{-\int_0^t r(\tau) d\tau} A_t = 0, \tag{1.21}$$

$$\lim_{t \to \infty} e^{-\int_0^t r_\tau d\tau} K_t = 0. \tag{1.22}$$

Aggregate variables Aggregate output is divided between household consumption C_t, input to physical capital $\dot{K}_t + \delta K_t$, and total expenditure on R&D $\hat{S}_t$:

$$Y_t = C_t + \dot{K}_t + \delta K_t + \hat{S}_t. \tag{1.23}$$

[9]For derivation see Appendix.

The total expenditure on R&D is

$$\hat{S}_t = \int_0^1 (S_t(j) + \tilde{S}_t(j))q_t(j)dj. \tag{1.24}$$

Aggregate capital is given by (1.10), and Euler equation is given by (1.20).

Total output can be written as[10]

$$Y_t = \frac{1}{1-\gamma}\left(\int_0^1 q_t(j)x_t(j)^{1-\gamma}dj\right)L_t^\gamma = \frac{1}{1-\gamma}x_t^{1-\gamma}Q_tL^\gamma = \frac{1}{1-\gamma}k_t^{1-\gamma}Q_tL = \frac{1}{1-\gamma}K_t^{1-\gamma}Q_t^\gamma L^\gamma. \tag{1.25}$$

The government's budget constraint takes the form:

$$T_t = s_I\int_0^1 S_t(j)q_t(j)dj + s_E\int_0^1 \tilde{S}_t(j)q_t(j)dj + \beta_k\int_0^1 K_t(j)dj. \tag{1.26}$$

In the model describe above, a dynamic equilibrium can be defined as follows:

definition

Definition 1 *An equilibrium is a collection of time paths of functions:* [11]

$$\{C_t, K_t, \hat{S}_t, S_t, \tilde{S}_t, R_t, w_t, r_t, p_t, x_t, V_t(j,q)\}_{t=0}^\infty \tag{1.27}$$

that solve the final good, the intermediate good sectors and household problems, and all markets clear:

(i) C_t, K_t, $\hat{S}_t$ and T_t that satisfy (1.20), (1.10), (1.24) and (1.26) respectively;

(ii) prices $p_t(j)$ and quantities x_t of each intermediate good, cost of capital R_t and net present value of profits $V_t(j)$ are given by (1.9), (1.8), (1.6), (1.17) respectively;

[10] The intensive-form production function is given by $y_t = \frac{Y_t}{Q_tL} = \frac{1}{1-\gamma}F(k_t,1) \equiv \frac{1}{1-\gamma}f(k_t)$.

[11] While $q_t(j)$ is stochastic, average Q_t is deterministic (reasoning is the law of large numbers, hence, all aggregates are non-stochastic).

(iii) R&D expenditure for incumbents S_t and entrants $\tilde{S}_t$ that satisfy (1.17) and (1.19);

(iv) wage and interest rate w_t, r_t given by (1.4) and (1.20).

I am interested in the balanced growth paths (henceforth, BGP) which are those where all variables grow at a constant rate. First, I show that the growth rate of outputs, consumption, physical capital and technological progress (quality) are constant and equal:

$$g_Y = g_C = g_K = g_Q = g^*. \tag{1.28}$$

Along a BGP all variables grow at a constant rate, so that the interest rate is constant[12] and, by consequence, so is the cost of capital, $R_t = R$, so, by (1.8), $x_t = x$. Using this, K_t is proportional to Q_t, by (1.10), and so $g_K = g_Q$. When these facts are used in (1.25), I have $g_Y = g_Q = g_K$.

Using the "guess" that the value function V along BGP is linear in q, i.e. $V_t(j) = v_t q(j)$[13] for all j, the R&D by an incumbent j is defined as the solution to

$$\arg\max_S \{\zeta(S_t)(\mu - 1)v_t - S_t(1 - s_I)\}, \tag{1.29}$$

which is independent of j. Then, the first-order condition is given by

$$\zeta'(S_t)(\mu - 1)v_t = 1 - s_I \tag{1.30}$$

[12] This follows from the requirement that consumption grows at a constant rate.

[13] Here I use the fact that the quality of the intermediate good supplied by an incumbent is constant over time.

Notice that S_t depends on v_t, μ and s_I and does not depend on q. The entry condition (1.19) can be rewritten as

$$\tilde{\mu}\psi(\tilde{S})v_t = 1 - s_E, \tag{1.31}$$

The equation for aggregate research expenditure (1.24) gives

$$\hat{S}_t = (S_t + \tilde{S}_t)Q_t.$$

From the resource constraint (1.23), I have:

$$\frac{C_t + \hat{S}_t}{K_t} = -\delta - g_K + \frac{Y_t}{K_t}.$$

Because Y_t and K_t grow at same rate, $\frac{Y_t}{K_t}$ is constant, so that $\frac{C_t+\hat{S}_t}{K_t}$ is constant, hence, $C_t + \hat{S}_t$ grows at rate g_K and consequently, at rate g_Y, i.e. $g_{C+\hat{S}} = g_K = g_Y$. Then $g_{C+\hat{S}}$ can be expressed as

$$g_{C+\hat{S}} = \frac{\dot{C}_t}{C_t + \hat{S}_t} + \frac{\dot{\hat{S}}_t}{C_t + \hat{S}_t} = g_C \frac{C_t}{C_t + \hat{S}_t} + g_{\hat{S}} \frac{\hat{S}_t}{C_t + \hat{S}_t} = g_C + (g_{\hat{S}} - g_C)\frac{\hat{S}_t}{C_t + \hat{S}_t} = g_Y.$$

Since g_Y, g_C and $g_{\hat{S}}$ are constant along a BGP, $\frac{\hat{S}_t}{C_t+\hat{S}_t}$ is constant, and, hence, $g_{\hat{S}} = g_C$. Thus, it has been shown that

$$g_Y = g_K = g_Q = g_C = g_{\hat{S}.}$$

It follows from the equation for aggregate R&D expenditure (1.24) that, as $\hat{S}_t$ and Q_t grow at the same rate, $S_t + \tilde{S}_t$ is constant. From FOC (1.30) and the entry condition (1.31), R&D expenditures of incumbents S and entrants $\tilde{S}$ are increasing in v_t. Thus, if incumbents increase/decrease their research expenditure, entrants do the same.

These two factors implies that $S_t = S$ and $\tilde{S}_t = \tilde{S}$ are constant, and from (31) $v_t = v$ is also constant, i.e $V_t(j) = v_t q(j) = vq(j)$

Along BGP, the Hamilton-Jacobi-Bellman equation can be written as

$$r^* v = \max_{S \geq 0} \left(\pi(k^*)L - S(1 - s_I) + \zeta(S)(\mu - 1)v - \tilde{S}\psi(\tilde{S})v \right), \tag{1.32}$$

where k^* is defined by

$$R(k^*) = r^* + \delta - \beta_k, \tag{1.33}$$

and (1.32) holds with $S = S^*$, $\tilde{S} = \tilde{S}^*$, where S^* and $\tilde{S}^*$ denote the solutions to (1.30) and (1.31) respectively. It follows that v can be expressed as

$$v = \frac{\pi(k^*)L - S^*(1 - s_I)}{r^* + \tilde{S}^*\psi(\tilde{S}^*) - \zeta(S^*)(\mu - 1)}. \tag{1.34}$$

The growth rate of output is given by

$$g^* = \frac{\dot{Y}_t}{Y_t} = \frac{\dot{Q}_t}{Q_t}. \tag{1.35}$$

I now determine the growth of the average quality. To derive it I use the law of large numbers. Q_{t+dt} can be determined as

$$Q_{t+dt} = \int q_{t+dt} dj = \int_{\Upsilon_1} q_{t+dt} dj + \int_{\Upsilon_2} q_{t+dt} dj + \int_{\Upsilon_3} q_{t+dt} dj, \tag{1.36}$$

where Υ_1 is the set of product lines that experience an innovation by incumbents, Υ_2 is the set of product lines that experience an innovation by entrants and Υ_3 is the set of product lines that do not experience an innovation. The first term can be expressed as

$$\int_{\Upsilon_1} q_{t+dt} dj = E[\mu q_t(j) \mid \text{Innovate}] = \mu Q_t \zeta(S^*) dt.$$

A similar expression can be obtained for $\int_{\Upsilon_2} q_{t+dt} dj$ and $\int_{\Upsilon_3} q_{t+dt} dj$, so (1.36) can be rewritten as

$$Q_{t+dt} = \mu Q_t \zeta(S^*) dt + \tilde{\mu} Q_t \tilde{S}^* \psi(\tilde{S}^*) dt + (1 - \zeta(S^*) dt - \tilde{S}^* \psi(\tilde{S}^*) dt) Q_t + o(dt).$$

Subtracting Q_t from the right hand side and the left hand side, dividing by dt and taking $dt \to 0$ yields:

$$\frac{\dot{Q}_t}{Q_t} = \lim_{dt \to 0} \frac{Q_{t+dt} - Q_t}{dt} \frac{1}{Q_t} = \zeta(S^*)(\mu - 1) + \tilde{S}^* \psi(\tilde{S}^*)(\tilde{\mu} - 1). \tag{1.37}$$

Therefore, the growth rate is given by:

$$g^* = \zeta(S^*)(\mu - 1) + \tilde{S}^* \psi(\tilde{S}^*)(\tilde{\mu} - 1). \tag{1.38}$$

Thus, the BGP is characterized by the Euler equation, and equations (1.34), (1.33), (1.30),(1.31) and (1.38).

The following proposition establishes existence of a unique BGP equilibrium with the linear value function of incumbent $V_t(j) = vq$.

Proposition 1.1 *Assuming that innovation are drastic, (1.14),* $\epsilon \geq 1$ *and* $\rho + \delta - \beta_k > 0$ *holds, there exists unique BGP equilibrium with a linear value function of the incumbent.*

Proof See the Appendix.

Proposition 1.1 establishes the conditions for the existence of unique BGP equilibrium. Now I can perform comparative statics exercises.

1.3 Comparative statics

The BGP is characterized by the following system of equations:

$$R(k) = r^* + \delta - \beta_k, \tag{1.39}$$

$$v = \frac{\pi(k^*)L - S^*(1 - s_I)}{r^* + \tilde{S}^*\psi(\tilde{S}^*) - \zeta(S^*)(\mu - 1)}, \tag{1.40}$$

$$\zeta'(S^*)(\mu - 1)v = (1 - s_I), \tag{1.41}$$

$$\tilde{\mu}\psi(\tilde{S}^*)v \leq (1 - s_E) \qquad \text{with equality if } \tilde{S}^* > 0, \tag{1.42}$$

$$r^* = \rho + \epsilon g^*, \tag{1.43}$$

$$g^* = \zeta(S^*)(\mu - 1) + \tilde{S}^*\psi(\tilde{S}^*)(\tilde{\mu} - 1). \tag{1.44}$$

Substituting the expression for the quality-adjusted value v from the first-order condition (1.41) into (1.40), and using (1.43)-(1.44) gives a research "arbitrage" equation for an incumbent:

$$(1 - s_I) = \zeta'(S^*)(\mu - 1)\frac{\pi(k^*)L - S^*(1 - s_I)}{\rho + \epsilon g(S^*, \tilde{S}^*) + \tilde{S}^*\psi(\tilde{S}^*) - \zeta(S^*)(\mu - 1)}. \tag{1.45}$$

Similarly, substituting the free-entry condition (1.42) into (1.40), and using (1.43)-(1.44) yields a research "arbitrage" equation for an entrant:

$$(1 - s_E) = \tilde{\mu}\psi(\tilde{S}^*)\frac{\pi(k^*)L - S^*(1 - s_I)}{\rho + \epsilon g(S^*, \tilde{S}^*) + \tilde{S}^*\psi(\tilde{S}^*) - \zeta(S^*)(\mu - 1)}. \tag{1.46}$$

The numerator in (1.45)-(1.46) is the one-period flow of output from research, whereas the denominator is the rate at which an innovator discounts the stream of output. After these remarks, let me now consider the long-run impacts of a change in subsidies. The results of comparative statics in the balanced-growth equilibrium are summarized in Proposition 1.2.

Proposition 1.2 *(i) The BGP growth rate g depends positively on the incumbents' (s_I) and subsidy rate to capital β_K. The growth rate increases with subsidies to entrant research activity (s_E) when $\zeta(S) < \frac{\rho\beta(\tilde{\mu}-1)}{(\beta(\tilde{\mu}-2)+1))(\mu-1)}$; (ii) The increase of R&D subsidy rate to incumbents encourages incumbents' research efforts, i.e. $\frac{dS^*}{ds_I} > 0$, while the effect of this rate on the entrants' R&D is the opposite, $\frac{d\tilde{S}^*}{ds_I} < 0$; (iii) the equilibrium response of incumbents' research expenditure on an increase in the R&D subsidy rate to entrants is negative, $\frac{dS^*}{ds_E} < 0$, while the response of entrants' research expenditure is positive, $\frac{d\tilde{S}^*}{ds_E} > 0$; (iv) Incumbents and entrants research investment are increasing in the capital subsidy rate β_k.*

Proof See the Appendix.

By directly reducing the cost of R&D investment, a research subsidy to incumbents contributes by increasing the growth rate (equation (1.44)). However, there is an indirect effect of this policy instrument on growth that arises from the equilibrium change in the value of v (equation (1.40)). But the indirect effect is not compensated by the direct effect: Proposition 1.2 shows that growth is increasing in the subsidy rate s_I.

The result obtained with regard to the effect of a subsidy to capital on growth is consistent with the main result derived in Howitt and Aghion (1998), and this leads to an important policy conclusion. A capital subsidy reduces capital cost and raises the flow of profit $\pi(k)L$ to an innovator. This, in turn, increases the value of an innovation (equations (1.45) and (1.46)) thereby spurs economic growth. This result implies that a subsidy to capital accumulation can be as effective a way of promoting growth as a subsidy to R&D, and will have a constant effect on the economy's growth rate.

The effect of a subsidy to an entrants on growth depends on the probability of an incumbent being successful performing R&D, $\zeta(S)$: for a low success probability, the growth rate is increasing in the subsidy rate to entrants' research. The intuition for this result is the following. On the one hand, subsidy to an entrant's research activity by encouraging more entry, decreases profitability and value of incumbents, that in turn lowers incumbents R&D investment ($dS/ds_E < 0$) and reduces economic growth. On the other hand, the cost reduction induced by the subsidy, increases the expected return to entrant. This encourages more entrants research investment and tends to increase economic growth. If incumbents contribution to growth is sufficiently low, i.e., $\zeta(S) < \frac{\rho\beta(\tilde{\mu}-1)}{(\beta(\tilde{\mu}-2)+1))(\mu-1)}$ (see also (1.44)), then the latter effect dominates the former — a subsidy to entrants has positive effect on growth. If a flow rate of incumbents innovation, $\zeta(S)$, is sufficiently hight, the effect of subsidy to entrants R&D is ambiguous. Thus, for the case of low $\zeta(S)$, my finding is in line with the Schumpeterian literature, in which encouraging entrants leads to an increase in economic growth. Note that in Acemoglu and Cao's model the growth rate is decreasing with subsidy to entrants (increasing in the tax on entrants) for a model with a linear R&D technology for incumbents, while for the general case they find the effect to be ambiguous.

1.4 Pareto Optimal Allocation

How does this decentralized solution compare to the socially optimal allocation of resources? The answer can be obtained by first solving the following social planner problem for the economy.

A social planner maximizes the utility of a representative household by choices of consumption C_t^{sp} and R&D expenditures S_t^{sp} and $\tilde{S}_t^{sp}$:

$$max_{C_t,S_t,\tilde{S}_t} \int_0^\infty e^{-\rho t} \frac{C_t^{sp1-\epsilon} - 1}{1-\epsilon} dt \tag{1.47}$$

subject to: [14]

$$Y_t^{sp} = \frac{1}{1-\gamma}\int_0^1 q_t(j)F^{sp}(x_t(j), L)dj = C_t^{sp} + I_t^{sp} + \hat{S}_t^{sp}, \tag{1.48}$$

$$\dot{K}_t^{sp} = I_t^{sp} - \delta K_t^{sp}, \tag{1.49}$$

$$K_t^{sp} = \int_0^1 x_t^{sp}(j)q_t(j)dj, \tag{1.50}$$

$$\dot{Q}_t^{sp} = \Big(\zeta(S_t^{sp})(\mu - 1) + \tilde{S}_t^{sp}\psi(\tilde{S}_t^{sp})(\tilde{\mu} - 1)\Big)Q_t^{sp}. \tag{1.51}$$

The solution to social planner's problem is derived in the Appendix. Now, I can compare the social optimum with the steady state level of capital and research under laissez-faire. According to the results provided in the Appendix, the socially optimal level for k is given by

$$\frac{1}{1-\gamma}F_k(k^{sp}, 1) = (k^{sp})^{-\gamma} = \epsilon\Big(\zeta(S^{sp})(\mu - 1) + \tilde{S^{sp}}\psi(\tilde{S^{sp}})(\tilde{\mu} - 1)\Big) + \delta + \rho. \tag{1.52}$$

Let me consider the laissez-faire case with no subsidies ($s_I = s_E = \beta_k = 0$). Use (1.20), (1.33) and (1.38) to obtain $\epsilon\Big(\zeta(S^*)(\mu-1) + \tilde{S^*}\psi(\tilde{S^*})(\tilde{\mu}-1)\Big) + \delta + \rho = R(k^*)$. Recall the expression for the rental rate, $R(k^*) = \frac{1}{1-\gamma}\Big(F_k(k^*, 1) + k^*F_{kk}(k^*, 1)\Big)$, where for the Cobb-Douglas case $R(k^*) = (k^*)^{-\gamma}(1-\gamma)$. Hence, the equilibrium value of k^* for the decentralized economy satisfies

$$\begin{aligned}\frac{1}{1-\gamma}\Big(F_k(k^*, 1) + k^*F_{kk}(k^*, 1)\Big) &= (k^*)^{-\gamma}(1-\gamma) \\ &= \epsilon\Big(\zeta(S^*)(\mu - 1) + \tilde{S^*}\psi(\tilde{S^*})(\tilde{\mu} - 1)\Big) + \delta + \rho.\end{aligned} \tag{1.53}$$

[14] I assume here the same proportional level of research expenditure in each sector.

As shown in the Appendix, for the social planner's problem, the optimal allocation of resources to incremental research S^{sp} in steady state is given by:

$$1 = (\mu - 1)\zeta'(S^{sp})\frac{\frac{1}{1-\gamma}\Big(F(k^{sp},1) - F_k(k^{sp},1)k^{sp}\Big)L - (S^{sp} + \tilde{S}^{sp})}{\rho + g^{sp}\epsilon - \zeta(S^{sp})(\mu - 1) - \tilde{S}^{sp}\psi(\tilde{S}^{sp})(\tilde{\mu} - 1)} \tag{1.54}$$

with $\frac{1}{1-\gamma}\Big(F(k^{sp},1) - F_k(k^{sp},1)k^{sp}\Big) = \frac{1}{1-\gamma}(k^{sp})^{1-\gamma} - (k^{sp})^{1-\gamma} = \frac{\gamma}{1-\gamma}(k^{sp})^{1-\gamma}$.

Recall the expression for $\pi(k^*) = \frac{1}{1-\gamma}\Big(F_k(k^*,1)k^* - \big(F_k(k^*,1) + k^*F_{kk}(k^*,1)k^*\big)\Big)$, where for the Cobb-Douglas case $\pi(k^*) = (k^*)^{1-\gamma} - (1-\gamma)(k^*)^{1-\gamma} = \gamma(k^*)^{1-\gamma}$. Substituting the formulas for the quality-adjusted value of v from (1.34) and for r^* from Euler equation (1.20) into the first-order condition (1.30), and using the expression for $\pi(k^*)$, gives the equation determining the steady-state equilibrium level of an incumbent's research

$$1 = (\mu - 1)\zeta'(S^*)\frac{\frac{1}{1-\gamma}\Big(F_k(k^*,1)k^* - \big(F_k(k^*,1) + k^*F_{kk}(k^*,1)k^*\big)\Big)L - S^*}{\rho + g^*\epsilon - \zeta(S^*)(\mu - 1) + \tilde{S}^*\psi(\tilde{S}^*)}. \tag{1.55}$$

The Appendix shows that the socially optimal level for radical research $\tilde{S}^{sp}$ is given by:

$$1 = (\tilde{\mu} - 1)\Big(\psi(\tilde{S}^{sp}) + \tilde{S}^{sp}\psi'(\tilde{S}^{sp})\Big)\frac{\frac{1}{1-\gamma}\Big(F(k^{sp},1) - F_k(k^{sp},1)k^{sp}\Big)L - (S^{sp} + \tilde{S}^{sp})}{\rho + g^{sp}\epsilon - \zeta(S^{sp})(\mu - 1) - \tilde{S}^{sp}\psi(\tilde{S}^{sp})(\tilde{\mu} - 1)}. \tag{1.56}$$

The research "arbitrage" equation for a monopolist in the case of radical research is analogous to (1.55) and can be obtained by plugging the formula (1.34) for v into the free-entry condition (1.31):

$$1 = \tilde{\mu}\psi(\tilde{S}^*)\frac{\frac{1}{1-\gamma}\Big(F_k(k^*,1)k^* - \big(F_k(k^*,1) + k^*F_{kk}(k^*,1)k^*\big)\Big)L - S^*}{\rho + g^*\epsilon - \zeta(S^*)(\mu - 1) + \tilde{S}^*\psi(\tilde{S}^*)}. \tag{1.57}$$

First, let me explore the last four equations in detail. There are four differences between (1.54) and (1.55) (similarly, between (1.56) and (1.57)). The first is that the rates at which the social planner and a private firm discounts stream of output differ, namely, the social discount rate $\rho + g\epsilon - \zeta(S)(\mu - 1) - \tilde{S}\psi(\tilde{S})(\tilde{\mu} - 1)$ is less than the "private discount rate" $\rho + g\epsilon - \zeta(S)(\mu - 1) + \tilde{S}\psi(\tilde{S})$. This is the *"intertemporal spillover effect"*. This effect can be decomposed into subeffects in a manner to similar that done by Hagedorn et al. (2007) for the Aghion and Howitt (1992) and the Howitt and Aghion (1998) models. Hagedorn et. al distinguish three additional subeffects: the "passive business stealing effect", the "standing on shoulders effect", and the "consumption dilution effect". First, the *passive business stealing effect* affects private firms, namely, the last term $\tilde{S}\psi(\tilde{S})$ in the private discount rate reflects the probability that the monopolist's profit flow will be reduced by creative destruction that comes from innovation by entrants. This effect is the negative part associated with creative destruction because this part, as we see later, promotes underinvestment.

The *"standing on shoulders effect"*[15] arises because the planner captures the rent from the next innovation endlessly (in social discount rate this effect is expressed by the last negative term $-\tilde{S}\psi(\tilde{S})(\tilde{\mu} - 1)$), while a private firm receives benefit only during one interval and after that rents will be obtained by the next innovator.

The numerator in (1.54)-(1.57) is the one-period flow of new output that results from research. As can be seen, the social planner takes into account expenditures on incremental (S) and radical innovations ($\tilde{S}$), whereas the private firm consider only its own R&D expenditure, S. This difference reflects the *"consumption dilution effect"*: spending on radical innovations reduces resources available for consumption. Note that in Howitt and Aghion (1998), the "consumption dilution" effect compensates two other effects, passive business stealing and standing on shoulders effects. However, in my model this is not the case, namely, the net effect of passive business stealing

[15] This effect can be interpreted as capturing the cumulative nature of knowledge, i.e. new ideas builds on old ideas.

and standing on shoulders is $-\tilde{S}\psi(\tilde{S}) - \tilde{S}\psi(\tilde{S})(\tilde{\mu} - 1) = -\tilde{\mu}\tilde{S}\psi(\tilde{S})$, whereas the consumption dilution effect is associated with $-\tilde{S}$. Because the former term is in the denominator and the latter term is in the numerator, they work in opposite directions.

The "passive business stealing" and the "standing on shoulders" effects tend to generate too little incremental and radical research under laissez-faire because of which surface S^{sp} (defined by (1.54)) is shifted above the S surface (equation (1.55)), and the surface $\tilde{S}^{sp}$ (equation (1.56)) is shifted above $\tilde{S}$ (equation(1.57)), while the "consumption dilution" effect tends to make these innovations too large — the surface S^{sp} will be below the S surface and the $\tilde{S}^{sp}$ surface will be below the $\tilde{S}$ surface. Thus, the direction of the "intertemporal effect" is ambiguous, it implies that the shift of the surface S, determined by (1.54)-(1.55), and the surface $\tilde{S}$, determined by (1.56)-(1.57), are not uniquely determined.

The next difference is the first term in the numerator of a fraction. For the social planner this is $\frac{1}{1-\gamma}\Big(F(k,1) - F_k(k,1)k\Big)L = (\frac{1}{1-\gamma}(k)^{1-\gamma} - (k)^{1-\gamma})L$, while for a private monopolist, this is $\frac{1}{1-\gamma}\Big(F_k(k,1)k - \big(F_k(k,1) + kF_{kk}(k,1)k\big)\Big)L = ((k)^{1-\gamma} - (1-\gamma)(k)^{1-\gamma})L$. The given notations allow easily to see that social and private revenue differ, on the other hand, their costs also differ. The first of these differences corresponds to an *"appropriability effect"*: the social planner appropriates the entire consumer surplus related to the good that is created. The private innovator of a new good captures only part of this surplus because markup of price over cost decreases sales from the optimal level at which the good would be sold at its marginal cost. At the same time, the social cost exceeds the cost of a private firm because of the market power of a private firm. This distortion corresponds to the *"monopoly distortion effect"*.

The "appropriability" and the "monopoly distortion" effects work in opposite directions. The "appropriability" effect tends to generate insufficient research, S and $\tilde{S}$, under laissez-faire (the surface S^{sp} is above the S surface, and the surface $\tilde{S}^{sp}$ is above $\tilde{S}$), whereas the monopoly distortion effect induces too much research under laissez-faire (the surface S^{sp} is below the S surface, and the $\tilde{S}^{sp}$ surface below $\tilde{S}$ surface).

The other difference, *"active business stealing effect"*, can be seen in (1.56)-(1.57). The term in the second bracket on the right-hand side of (1.56), $\tilde{S}\psi'(\tilde{S})$, represents this effect. This term is negative, because the function $\psi(\tilde{S})$ is decreasing. It reflects the fact that the social planner internalizes the destruction of rents generated by radical innovations, i.e. the social planner takes into consideration that the social return from the previous innovation will be destroyed by a new innovation. This implies that in the social optimum there will be less radical innovation than under laissez-faire: the surface $\tilde{S}^{sp}$ will be below the surface $\tilde{S}$. Thus, the "active business stealing" effect affects only research by entrants, making entrants more active under laisser-faire.

Comparison of the equation for k in the social planner's problem (1.52) with that under laissez-faire (1.53) demonstrates the *monopoly distortion effect*, which is reflected by the term on the left hand side of these two equations. The monopolist gains from the lower capital cost at the cost of the household that supplies capital. This effect is absent in Acemoglu and Cao (2015). This effect tends to generate too little capital accumulation under laissez-faire, thus, the K surface, that is determined by (1.52)-(1.53), is below the K^{sp} surface (Figure 1.1).

To summarize, the "appropriability", the "passive business stealing" and the "standing on shoulders" effects tend to make laissez-faire incremental and radical innovation smaller than optimal, whereas the "monopoly distortion" effect tends to make them larger than optimal, and at the same time, by affecting only research by entrants, the "business stealing effect" tends to generate too much radical research in a decentralized economy. There are eight possible combination to compare the

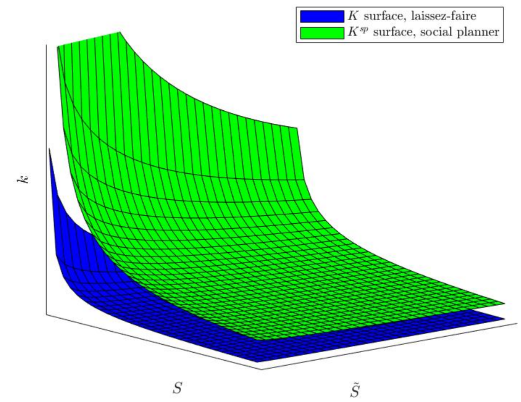

Figure 1.1: Social optimal vs laissez-faire (K surfaces)

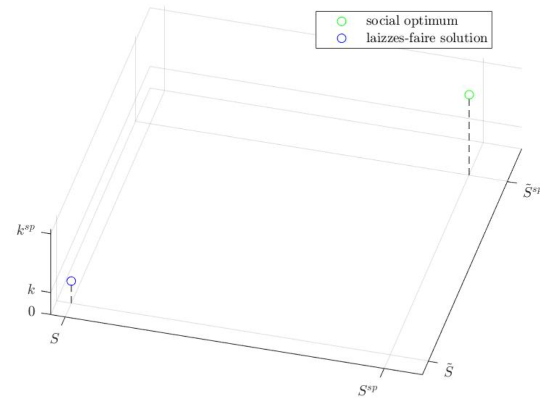

Figure 1.2: The laissez-faire solution vs social optimum (example, case 1)

social optimum with the steady-state decentralized equilibrium (case 1 is illustrated in Figure 1.2):

case 1: $S < S^{sp}$, $\tilde{S} < \tilde{S}^{sp}$, $k < k^{sp}$; case 2: $S < S^{sp}$, $\tilde{S} < \tilde{S}^{sp}$, $k > k^{sp}$;

case 3: $S < S^{sp}$, $\tilde{S} > \tilde{S}^{sp}$, $k < k^{sp}$; case 4: $S < S^{sp}$, $\tilde{S} > \tilde{S}^{sp}$, $k > k^{sp}$;

case 5: $S > S^{sp}$, $\tilde{S} < \tilde{S}^{sp}$, $k < k^{sp}$; case 6: $S > S^{sp}$, $\tilde{S} < \tilde{S}^{sp}$, $k > k^{sp}$;

case 7: $S > S^{sp}$, $\tilde{S} > \tilde{S}^{sp}$, $k < k^{sp}$; case 8: $S > S^{sp}$, $\tilde{S} > \tilde{S}^{sp}$, $k > k^{sp}$.

Whether the growth rate under laisser-faire will be greater or smaller than the optimal growth rate depends upon whether the steady-state level of research, S and $\tilde{S}$, for the decentralized economy are higher or lower than the socially optimal level, S^{sp} and $\tilde{S}^{sp}$. The "passive business stealing", the "standing on shoulders" and the "appropriability" effects tend to make growth rate under laissez-faire less than optimal, whereas the "monopoly distortion", the "active business stealing" and "consumption dilution" effects tend to make it greater than optimal. These effects act in opposite directions, the laissez-faire growth rate may be less or more than optimal.

The welfare effect of a subsidy to incumbent and entrant research will be positive when the economy is in case 1. At the same time in other cases the welfare effect will be unclear, because the economy either already enlarged with the capital stock,

$k > k^s$ (cases 2,4,6,8) or it is at a point where already too much research, incremental or radical (cases 3,5, and 7), is being carried out.

The welfare effect of a subsidy to capital accumulation will be positive if the economy is in case 1: a capital subsidy reduces capital cost and that raises the flow of profit $\pi(k)L$ to an innovator. This, in turn, increases the value of innovation thereby spurring technological progress. A subsidy to capital shifts the surface k towards the socially optimal level k^{sp}. But when the economy is in the other cases, the welfare effect of this subsidy is ambiguous: for cases 3, 5, and 7, such a policy measure will tend to increase the capital stock, however, R&D research, either incremental or radical, is in the sufficient level; for cases 2, 4, 6, 8 the capital stock is already expanded, however, the level of research (by incumbents or by entrants) can be increased.

I explore in detail the welfare effect of subsidies in the next section.

Special case: linear a flow rate of innovation by incumbents

As shown above, for a general specification of the technology comparison optimal growth rate and the growth rate under laissez-fare is ambiguous. However, when a flow rate of innovation by incumbents is linear, i.e. $\zeta(S) = \zeta S$, the growth rate under laissez-fare less than its socially optimal counterpart. This can be shown as follows.

The growth rate of the economy in this special case is given by[16]

$$g^* = \zeta S^{*a}(\mu - 1) + \tilde{S}^* \psi(\tilde{S}^*)(\tilde{\mu} - 1), \tag{1.58}$$

where $S^{*a} \equiv \frac{\int_0^1 S_t^*(j) q_t(j)}{Q_t}$ is the average BGP incumbents' research expenditure.

It can be easily seen that the socially optimal level k^{sp} (the analogous to (1.52)) is given by

$$(k^{sp})^{-\gamma} = \epsilon\Big(\zeta S^{sp}(\mu - 1) + \tilde{S^{sp}} \psi(\tilde{S^{sp}})(\tilde{\mu} - 1)\Big) + \delta + \rho, \tag{1.59}$$

[16]See the Appendix for a detailed derivation.

whereas the equilibrium value of k^* for the decentralized economy (the analogous to (1.53))

$$(k^*)^{-\gamma}(1-\gamma) = \epsilon\Big(\zeta S^{*a}(\mu - 1) + \tilde{S}^*\psi(\tilde{S}^*)(\tilde{\mu} - 1)\Big) + \delta + \rho. \tag{1.60}$$

Socially optimal level for incremental innovation S^{sp} (the analogous to (1.54)) satisfies

$$1 = (\mu - 1)\zeta \frac{\frac{\gamma}{1-\gamma}(k^{sp})^{1-\gamma}L - (S^{sp} + \tilde{S}^{sp})}{\rho + g^{sp}\epsilon - \zeta S^{sp}(\mu - 1) - \tilde{S}^{sp}\psi(\tilde{S}^{sp})(\tilde{\mu} - 1)}. \tag{1.61}$$

Steady-state equilibrium level of an R&D effort of incumbents, S^* (the analogous to (1.55)), is given by

$$1 = \frac{(\mu - 1)\zeta\gamma(k^*)^{1-\gamma}L}{\rho + g^*\epsilon + \tilde{S}^*\psi(\tilde{S}^*)}. \tag{1.62}$$

Using equations (1.59)-(1.62) I can compare the optimal growth rate, g^{sp}, and the equilibrium BGP growth rate, g^*.

Proposition 1.3 *When a flow rate of innovation by incumbents is linear then the growth rate under laissez-fare less than the optimal growth rate.*

Proof See the Appendix.

1.5 Welfare implications

Welfare in an economy can be expressed in terms of the stationary level of consumption, c:[17]

$$W = \int_0^\infty e^{-\rho t}\frac{(cQ_tL)^{1-\epsilon} - 1}{1-\epsilon}dt = \frac{1}{1-\epsilon}\left[\frac{(cQ_0L)^{1-\epsilon}}{\rho - g(1-\epsilon)} - \frac{1}{\rho}\right]. \tag{1.63}$$

[17] $c_t \equiv \frac{C_t}{Q_tL}$.

Let x denote any of the three subsidy rates: the subsidy rates to research, s_I, s_E, and a subsidy rate to the holding of capital, β_k. The change in steady-state welfare is a combination of the change in steady state consumption, c, and the steady-state growth rate, g:

$$\frac{\partial W}{\partial x} \propto (\rho - g(1-\epsilon))\frac{\partial c}{\partial x} + c(1-\epsilon)\frac{\partial g}{\partial x}. \tag{1.64}$$

A positive sign of the derivative means that the optimal policy is to increase the subsidy, whereas a negative sign implies that optimal policy is to reduce it. To determine the sign, I need to consider the sign of each term in (1.64). As has been shown in Proposition 1.2, $\frac{\partial g}{\partial x}$ is positive.[18] The derivative of consumption with respect to the subsidy rate is given by:

$$\frac{\partial c}{\partial x} = \frac{\partial k}{\partial x}(k^{-\gamma} - g) - \frac{\partial S}{\partial x} - \frac{\partial \tilde{S}}{\partial x} - \frac{\partial g}{\partial x}k. \tag{1.65}$$

The effect of subsidies on consumption is ambiguous.[19]

If steady-state consumption and growth change in the opposite directions, the value of the parameter ρ may determine the sign of the the welfare derivative: a low value of ρ reduces the weight of the consumption effect of a subsidy. Due to the ambiguity of the sign of the derivative of consumption the problem is not analytically tractable, thus, I appeal to numerical simulation with various empirically acceptable parameter values.

Table 1.1 reports baseline parameterization. The values of parameters ρ, δ, ϵ, and γ have been chosen within the standard ranges used in the literature (see Table 1.1); L is normalized to one. Parameters ζ and ψ is chosen to target growth $g = 2\%$ with 70% coming from incremental innovations and 30% from radical innovations as that

[18] $\frac{\partial g}{\partial s_E}$ is positive if $\zeta(S) < \frac{\rho\beta(\tilde{\mu}-1)}{(1-\beta)(\mu-1)}$.

[19] By using the implicit function theorem the same way as in Section 1.3, it can be shown that terms in (1.65) evolves in opposite direction, thus the sign of $\frac{\partial c}{\partial x}$ is ambiguous.

Table 1.1: Baseline parameterization

Symbol	Description	Value	Symbol	Description	Value
ρ	Discount rate	0.01	δ	Rate of depreciation	0.025
μ	Quality jump (incumbents)	1.2	ϵ	The coefficient of relative risk aversion	2
$\tilde{\mu}$	Quality jump (entrants)	1.75	γ	Labor's shares of output	2/3
ζ	Poisson coefficient for the flow rate of incremental innovation	0.16	β	Incumbent's R&D curvature	0.5
ψ	Poisson coefficient for the flow rate of radical innovation	0.012	$\tilde{\beta}$	Entrant's R&D curvature	0.5

corresponds with the empirical evidence: Akcigit et al. (2017) report the entrants' contribution to growth of around 30%, Bartelsman and Doms (2000) and Akcigit and Kerr (2018) report 25%, while Lentz and Mortensen (2008) report around 21%. The size of quality improvements is chosen $\mu = 1.2$ for incumbents and $\tilde{\mu} = 1.75$ for entrants as that captures the fact that innovation by new firms is radical and satisfies the assumption about innovation being drastic, which requires $\tilde{\mu} > 1.7$ (see (1.14)).

On the left panel of the Tables 1.2, 1.3 and 1.4, I report the simulation results for welfare using the baseline parametrization. The right panel of the tables provide the result with alternative model parameterizations.[20]

The results obtained reveal that policy instruments, subsidies to incumbents' and entrants' research activity, and subsidy to capital accumulation, may have different impact on welfare depending on the structural characteristics of the economy. Tables 1.2 and 1.3 show that the relationship between welfare and subsidies to research activity may be negative (the right panel of Table 1.2 for subsidy to incumbent, the left panel of Table 1.3 for subsidy to entrants) as well as may be represented as an inverted U-shaped curve with maximum shifts either left or right (the left panel of Table 1.2 for subsidy to incumbent, the right panel of Table 1.3 for subsidy to entrants). The latter suggests that increases of the subsidy may initially enhance welfare, but for higher values of the subsidy rate the welfare change becomes negative. The range of subsidy rate to capital accumulation is restricted by the condition in Proposition 1.1: $\rho + \delta - \beta_k > 0$. Table 1.4 suggests that the effect this policy instrument is

[20] I used various parametrizations for the simulations, the results obtained are similar to the ones reported in the Tables. All parameters have been chosen to be consistent with those in the literature.

always positive for different sets of parameters. Overall, a subsidy to capital initially improves welfare but further increases of the subsidy rate reduces welfare.

Table 1.2: Welfare effect of subsidy to incumbents research activity

Parameters: $\rho = 0.01$, $\mu = 1.2$, $\tilde{\mu} = 1.75$ $\zeta = 0.16$, $\psi = 0.012$, $\delta = 0.025$, $\epsilon = 2$ $\gamma = 2/3$, $\beta = 0.5$, $\tilde{\beta} = 0.5$		Parameters: $\rho = 0.08$, $\mu = 1.2$, $\tilde{\mu} = 2.5$ $\zeta = 0.08$, $\psi = 0.0008$, $\delta = 0.025$, $\epsilon = 2$ $\gamma = 0.2$, $\beta = 0.5$, $\tilde{\beta} = 0.5$	
subsidy rate, s_I	$\partial W/\partial s_I$	subsidy rate, s_I	$\partial W/\partial s_I$
0	0.951	0	-0.011
0.1	0.978	0.1	-0.012
0.2	1.025	0.2	-0.015
0.3	1.053	0.3	-0.020
0.4	1.030	0.4	-0.027
0.5	0.873	0.5	-0.039
0.6	0.315	0.6	-0.063
0.7	-0.116	0.7	-0.080

Table 1.3: Welfare effect of subsidy to entrants research activity

Parameters: $\rho = 0.01$, $\mu = 1.2$, $\tilde{\mu} = 1.75$ $\zeta = 0.16$, $\psi = 0.012$, $\delta = 0.025$, $\epsilon = 2$ $\gamma = 2/3$, $\beta = 0.5$, $\tilde{\beta} = 0.5$		Parameters: $\rho = 0.08$, $\mu = 1.2$, $\tilde{\mu} = 2$ $\zeta = 0.16$, $\psi = 0.012$, $\delta = 0.025$, $\epsilon = 2$ $\gamma = 0.9$, $\beta = 0.5$, $\tilde{\beta} = 0.5$	
subsidy rate, s_E	$\partial W/\partial s_E$	subsidy rate, s_E	$\partial W/\partial s_E$
0	-0.483	0	0.014
0.1	-0.579	0.1	0.014
0.2	-0.828	0.2	0.016
0.3	-1.242	0.3	0.016
0.4	-1.993	0.4	0.015
0.5	-3.553	0.5	0.008
0.6	-7.670	0.6	-0.019
0.7	-10.715	0.7	-0.039

Table 1.4: Welfare effect of subsidy to capital

Parameters: $\rho = 0.01$, $\mu = 1.2$, $\tilde{\mu} = 1.75$ $\zeta = 0.16$, $\psi = 0.012$, $\delta = 0.025$, $\epsilon = 2$ $\gamma = 2/3$, $\beta = 0.5$, $\tilde{\beta} = 0.5$		Parameters: $\rho = 0.08$, $\mu = 1.2$, $\tilde{\mu} = 2.5$ $\zeta = 0.08$, $\psi = 0.0008$, $\delta = 0.025$, $\epsilon = 2$ $\gamma = 0.2$, $\beta = 0.5$, $\tilde{\beta} = 0.5$	
subsidy rate, β_k	$\partial W/\partial \beta_k$	subsidy rate, β_k	$\partial W/\partial \beta_k$
0	43.129	0	0.356
0.005	43.212	0.015	0.329
0.01	43.347	0.03	0.280
0.015	43.408	0.045	0.239
0.02	43.373	0.06	0.203
0.025	43.209	0.075	0.171
0.03	42.875	0.09	0.144
0.035	42.660	0.105	0.132

1.6 Conclusion

This paper has developed a tractable general equilibrium growth model that is rich enough to investigate the growth and welfare implication of various subsidies. I prove the existence of the equilibrium and characterize its properties.

I find that a research subsidy to incumbents is beneficial for long term growth, while the effect of a subsidy to entrants research activity on growth depends on the probability of an incumbent being successful. The low probability of incumbents' success together with a subsidy encourage entrants to be more active and that eventually leads to an increase in economic growth. The effect of a subsidy to capital on growth is positive which is consistent with the finding in Howitt and Aghion (1998) and leads to the conclusion that capital accumulation can be as effective a way of stimulating growth as a subsidy to R&D.

I provide new perspectives to the welfare analysis of innovation by incumbents and entrants in comparison to Acemoglu and Cao (2015) by pointing out additional distortionary effects. Furthermore, as result of including capital accumulation, the model generates the new effect, the "monopoly distortion effect", that arises because a monopolist gains from the lower capital cost at the cost of household as supplier of capital. The "appropriability", the "passive business stealing" and the "standing on shoulders" effects tend to make laissez-faire incremental and radical innovation smaller than optimal, whereas the "monopoly distortion" effect tends to make it larger than optimal, and, at the same time, by affecting only research by entrants, the "business stealing effect" tends to generate too much radical research in a decentralized economy.

The welfare analysis demonstrates that policy instruments — subsidies to incumbents' and entrants' research activity, and a subsidy to capital accumulation — may have different impact on welfare depending on the structural characteristics of the economy.

A possible extension of the current work would be a study of the consequences of collateral constraints on the optimal growth path. My model could be used as a building block towards a quantitative model of innovation, financial frictions and endogenous growth. Such a model would be useful to assess the impact of innovation policy (R&D subsidies, patent policy), tax policy on innovation-led growth in the presence of financial constraints.

1.7 Appendix: Proofs and Derivations

Household optimization

The preferences of the representative household:

$$U(C_t) = \int_0^{\infty} e^{-\rho t} \frac{C_t^{1-\epsilon} - 1}{1-\epsilon} dt. \tag{1.66}$$

The household rents physical capital to firms which the household owns. The household maximizes utility from consumption subject to budget constraints and the law of motion for capital:

$$\dot{A}_t = r_t A_t + w_t L_t + R_t K_t - C_t - I_t - T_t, \tag{1.67}$$

$$\dot{K}_t = I_t - (\delta - \beta_k) K_t. \tag{1.68}$$

The household's maximization problem in the form of the current-value Hamiltonian is given by

$$H_t = u(C_t) + \lambda_{1t}(r_t A_t + w_t L_t + R_t K_t - C_t - I_t - T_t) + \lambda_{2t}(I_t - (\delta - \beta_k) K_t),$$

where λ_{1t}, λ_{2t} are co-state variables, the multipliers associated with the budget constraint and the law of motion for capital stock, and $u(C_t) = \frac{C_t^{1-\epsilon}-1}{1-\epsilon}$.

The optimality conditions are:

$$\frac{\partial H_t}{\partial C_t} = 0, \qquad \dot{\lambda}_{1t} = \rho \lambda_{1t} - \frac{\partial H_t}{\partial A_t}, \tag{1.69}$$

$$\frac{\partial H_t}{\partial I_t} = 0, \qquad \dot{\lambda}_{2t} = \rho \lambda_{2t} - \frac{\partial H_t}{\partial K_t}. \tag{1.70}$$

From (1.69)-(1.70) I get:

$$\frac{u''(C_t)\dot{C}_t}{u'(C_t)} = \frac{\dot{\lambda}_{1t}}{\lambda_{1t}}, \qquad \frac{\dot{\lambda}_{1t}}{\lambda_{1t}} = \rho - r_t, \tag{1.71}$$

$$\lambda_{1t} = \lambda_{2t}, \qquad \frac{\dot{\lambda}_{2t}}{\lambda_{2t}} = \rho - R_t + \delta - \beta_k. \tag{1.72}$$

Finally I obtain Euler equation:

$$\frac{\dot{C}_t}{C_t} = \frac{r_t - \rho}{\epsilon} \tag{1.73}$$

and the cost of capital

$$R_t = r_t + \delta - \beta_k. \tag{1.74}$$

Transversality condition becomes:

$$\lim_{t\to\infty} e^{-\rho t}\lambda_t A_t = \lim_{t\to\infty} e^{-\rho t}\lambda_0 e^{-\int_0^t (r_\tau - \rho)d\tau} A_t.$$

Consequently, $\lim_{t\to\infty} e^{-\int_0^t r_\tau d\tau} A_t = 0.$ [21] Similarly, $\lim_{t\to\infty} e^{-\int_0^t r_\tau d\tau} K_t = 0.$

Derivation of Hamilton-Jacobi-Bellman equation

I have a one-dimensional jump state process $\{q_t, t > 0\}$ which is the solution of the stochastic differential equation[22]

$$\frac{dq_t}{q_{t_-}} = (\mu - 1)dN_t^{(1)} + (\tilde{\mu} - 1)dN_t^{(2)} \tag{1.75}$$

[21] $A_t = \int_0^1 V_t(j, q)dj$.

[22] Consider a particular intermediate good, i.e. fix j and ignore the index j in notation further. For theoretical aspects, see Touzi (2013), Øksendal (2003), and Øksendal and Sulem (2007).

with initial $q_0 > 0$, where $\{N_t^{(1)}, t > 0\}$ and $\{N_t^{(2)}, t > 0\}$ are independent time-homogeneous Poisson processes with intensity $\zeta(S_t)$ and $\tilde{S}_t\psi(\tilde{S}_t)$ respectively, $N_t^{(1)}$ is the number of incremental innovations by incumbents, $N_t^{(2)}$ is the number of radical innovations by entrants. q_{t_-} is the notation for the left limit, $q_{t_-} \equiv \lim_{s\uparrow t} q(s)$. The solution of (1.75) is

$$q_t = q_0(\mu dN_t^{(1)} + \tilde{\mu} dN_t^{(2)}). \tag{1.76}$$

Given a subset U of R, denote the set of all measurable control processes valued in U by $\mathcal{U}$. Let[23]

$$f : [0,T) \times R \times U \to R \qquad \text{and} \qquad g : R \to R$$

be given function, and $T \in [0, \infty)$, where f is continuous. Define the gain function J on $[0,T] \times R \times U$ by:

$$J(t,q,u) := E\left[\int_t^T e^{-\int_t^s r(\tau)d\tau} f(s,q_s,u_s)ds + e^{-\int_t^T r(\tau)d\tau} g(q_T)1_{T<\infty}\right] \tag{1.77}$$

with control process u and $\{q_s, s \geq t\}$ given by (1.76).

Consider the optimization problem:

$$V(t,q) := \sup_{u\in\mathcal{U}} J(t,q,u) \qquad \text{for} \qquad (t,q) \in S, \tag{1.78}$$

where $S := [0,T) \times R_+$ is interior of the state space. Then according to the dynamic programming principle the value function $V(t,q)$ satisfies:

$$V(t,q) = \sup_{u\in\mathcal{U}} E_{t,q}\left[\int_t^T e^{-\int_t^s r(\tau)d\tau} f(s,q_s,u_s)ds + e^{-\int_t^s r(\tau)d\tau} V(T,q_T)\right] \tag{1.79}$$

[23] f is a running reward, g is a terminal reward.

with stopping time T.

Introduce the infinitesimal generator $\mathcal{L}^u$ associated to the process $\{q_t, t > 0\}$:

$$\mathcal{L}^u V(t,q) := -r(t)V(t,q) + \zeta(S_t)(V(t,\mu q) - V(t,q)) + \tilde{S}_t \psi(\tilde{S}_t)(V(t,\tilde{\mu} q) - V(t,q)). \tag{1.80}$$

The Hamiltonian-Jacobi-Bellman equation is the infinitesimal version of the dynamic programming principle:

$$\partial_t V(t,q) + \sup_{u \in \mathcal{U}} (\mathcal{L}^u V(t,q) + f(t,q,u)) = 0. \tag{1.81}$$

Applying formula (1.81) to the incumbent's problem, and using the fact that the incumbent's value drops to zero $V_t(t, \tilde{\mu} q_t) = 0$ when entrants innovate, I obtain [24]

$$r(t)V(t,q) - \dot{V}(t,q) = \max_{S(t) \geq 0} \left\{ \begin{array}{l} \pi(k)q(t)L - S(t)(1 - s_I)q(t) \\ +\zeta(S(t))(V(t, \mu q(t)) - V(t, q(t)) \\ -\tilde{S}(t)\psi(\tilde{S}(t))V(t, q(t)). \end{array} \right\} \tag{1.82}$$

[24] Using that $f(t,q,u) = \pi(t) - S(t)(1 - s_I)q(t)$ and that the terminal reward satisfies $g(q_T) = 0$. $\dot{V}(t,q) = 0$: $V(t,q)$ is constant over time along BGP, because $q(j)$ does not change over time — it is quality supplied by the incumbent, which remains constant while firm is incumbent.

Proof of Proposition 1.1

The BGP is characterized by the system of equations:

$$R(k) = r^* + \delta - \beta_k, \tag{1.83}$$

$$v = \frac{\pi(k^*)L - S^*(1-s_I)}{r^* + \tilde{S}^*\psi(\tilde{S}^*) - \zeta(S^*)(\mu - 1)}, \tag{1.84}$$

$$\zeta'(S^*)(\mu - 1)v = (1 - s_I), \tag{1.85}$$

$$\tilde{\mu}\psi(\tilde{S}^*)v \leq (1 - s_E) \qquad \text{with equality if } \tilde{S}^* > 0, \tag{1.86}$$

$$r^* = \rho + \epsilon g^*, \tag{1.87}$$

$$g^* = \zeta(S^*)(\mu - 1) + \tilde{S}^*\psi(\tilde{S}^*)(\tilde{\mu} - 1). \tag{1.88}$$

Use (1.11) and (1.83) in $\pi(k_t) = \gamma k_t^{1-\gamma}$ to define the function $\hat{\pi}(r^* + \delta - \beta_k) = \gamma(1-\gamma)^{\frac{1-\gamma}{\gamma}}(r^* + \delta - \beta_k)^{\frac{\gamma-1}{\gamma}}$. Inserting this into (1.84), and using the Euler equation (1.87) to eliminate r, gives the equation that determines the equilibrium value v:

$$\hat{\pi}\Big(\rho + \epsilon g^*(S^*, \tilde{S}^*) + \delta - \beta_k\Big)L = v \cdot \Big(\rho + \epsilon g^*(S^*, \tilde{S}^*) + \tilde{S}^*\psi(\tilde{S}^*) - \zeta(S^*)(\mu - 1) + S^*(1 - s_I)\Big), \tag{1.89}$$

where $\hat{\pi}\Big(\rho + \epsilon g^*(S^*, \tilde{S}^*) + \delta - \beta_k\Big)$ is obtained by plugging (1.87) into $\hat{\pi}(r^* + \delta - \beta_k)$. Now use the growth equation (1.88) and the definition of the function $\hat{\pi}$, to obtain

$$\begin{aligned}&\Big(\rho + \epsilon\Big(\zeta(S^*(v))(\mu - 1) + \tilde{S}^*(v)\psi(\tilde{S}^*(v))(\tilde{\mu} - 1)\Big) + \delta - \beta_k\Big)^{1-\frac{1}{\gamma}}\Theta L = \\ &\quad v\Big(\rho + (\epsilon - 1)\zeta(S^*(v))(\mu - 1) + (\epsilon(\tilde{\mu} - 1) + 1)\tilde{S}^*(v)\psi(\tilde{S}^*(v)) + S^*(v)(1 - s_I)\Big),\end{aligned} \tag{1.90}$$

where $\Theta \equiv \gamma(1-\gamma)^{\frac{1}{\gamma}-1}$, $S^*(v)$ is determined by the first-order condition (1.85), and $\tilde{S}^*(v)$ by the entry condition (1.86).

Using the functional forms for $\zeta(S) = \zeta S^{1-\beta}$ and $\psi(\tilde{S}) = \psi \tilde{S}^{-\beta}$, after substituting this into (1.85) and (1.86) respectively, the R&D expenditure of incumbents and entrants can be expressed as $S(v) = A^{\frac{1}{\beta}} v^{\frac{1}{\beta}}$, where $A = \frac{(\mu-1)(1-\beta)\zeta}{1-s_I}$ and $\tilde{S}(v) = B^{\frac{1}{\beta}} v^{\frac{1}{\beta}}$, with $B = \frac{\psi\tilde{\mu}}{1-s_E}$.

Substituting these values into the right-hand side of equation (1.90) yields:

$$F = v\rho + (\epsilon - 1)(\mu - 1)\zeta A^{\frac{1}{\beta}-1} v^{\frac{1}{\beta}} + (\epsilon(\tilde{\mu} - 1) + 1)\psi B^{\frac{1}{\beta}-1} v^{\frac{1}{\beta}} + A^{\frac{1}{\beta}} v^{\frac{1}{\beta}} (1 - s_I), \tag{1.91}$$

where the right hand side of (1.90) is denoted by F. It is clear that $\lim_{v\to 0} F = 0$ and $\lim_{v\to\infty} F = \infty$.

Let me show that F is strictly increasing in v when $\epsilon \geq \beta$ holds.

Taking the derivative of F with respect to v:

$$\frac{dF}{dv} = \rho + (\epsilon - 1)(\mu - 1)\frac{1}{\beta}\zeta A^{\frac{1}{\beta}-1} v^{\frac{1}{\beta}-1} + (\epsilon(\tilde{\mu} - 1) + 1)\frac{1}{\beta}\psi B^{\frac{1}{\beta}-1} v^{\frac{1}{\beta}-1} + \frac{1}{\beta} A^{\frac{1}{\beta}} v^{\frac{1}{\beta}-1}(1 - s_I). \tag{1.92}$$

$\frac{dF}{dv} > 0$ holds if $(\epsilon-1)(\mu-1)\frac{1}{\beta}\zeta A^{\frac{1}{\beta}-1} v^{\frac{1}{\beta}-1} + \frac{1}{\beta} A^{\frac{1}{\beta}} v^{\frac{1}{\beta}-1}(1-s_I) \geq 0$. It is straightforward to show that this holds for any $\epsilon \geq \beta$.[25]

The left hand side (LHS) of (1.90) is decreasing in v: $\zeta(S(v))$ and $\tilde{S}\psi(\tilde{S})$ are strictly increasing in v, $\left(\rho + \epsilon(\zeta(S(v))(\mu - 1) + (\tilde{\mu} - 1)\tilde{S}(v)\psi(\tilde{S}(v)) + \delta - \beta_k\right)^{1-\frac{1}{\gamma}}$ is decreasing function as composition of increasing and decreasing functions. Thus, the left-hand-side is decreasing in v.

[25] From $(\epsilon - 1)(\mu - 1)\frac{1}{\beta}\zeta A^{\frac{1}{\beta}-1} v^{\frac{1}{\beta}-1} + \frac{1}{\beta} A^{\frac{1}{\beta}} v^{\frac{1}{\beta}-1}(1 - s_I) = \frac{1}{\beta}\zeta A^{\frac{1}{\beta}-1} v^{\frac{1}{\beta}-1}\big((\epsilon - 1)(\mu - 1)\zeta A^{-1} + (1 - s_I)\big) \geq 0$, it follows that $\epsilon \geq -\frac{1-s_I}{(\mu-1)\zeta A^{-1}} + 1$, which after substituting into it values of A yields $\epsilon \geq \beta$.

Taking into account $\lim_{v\to 0}\zeta(S(v)) = 0$ and $\lim_{v\to 0}\tilde{S}\psi(\tilde{S}) = 0$, it is sufficient to have $\rho + \delta - \beta_k > 0$ holds to guarantee of the existence a positive equilibrium value of v.

Because the left hand side and the right hand side are continuous, then there exists a unique $v > 0$ which is a point of intersection.

Given v^*, the equilibrium values of S^* and $\tilde{S}^*$, the growth rate g^*, and the interest rate r^* are uniquely determined by the equations (1.85), (1.86), (1.88) and (1.87) respectively. The uniqueness of k^* follows directly from (1.83) and (1.87), $k^* = \left(\frac{\rho+g^*\epsilon+\delta-\beta_k}{1-\gamma}\right)^{-\frac{1}{\gamma}}$.

Finally, let me demonstrate that the transversality condition is satisfied when $\rho > (1-\epsilon)g^*$. The transversality condition along a BGP are

$$\lim_{t\to\infty} e^{-\int_0^t r(\tau)d\tau} K_t = 0,$$

$$\lim_{t\to\infty} e^{-\int_0^t r(\tau)d\tau} \int_0^1 V_t(j)dj = \lim_{t\to\infty} e^{-\int_0^t r(\tau)d\tau} vQ_t = 0,$$

that implies that $r > g$.[26] Because $r = \rho + \epsilon g$, the condition $r > g$ is satisfied if $\rho > g(1-\epsilon)$ which holds if $\epsilon \geq 1$.

Hence, the BGP with linear value function of incumbent exists and is uniquely determined.

[26] Using that along BGP $r_t = r$, $\frac{\dot{Q}_t}{Q_t} = g$.

Proof of Proposition 1.2

I derive the comparative statics results by using the implicit function theorem. The equilibrium values of S^*, $\tilde{S}^*$, g^*, and v^*, are determined by the system of equations:

$$H_1 = \frac{1-s_I}{\zeta'(S^*)(\mu-1)} - v^* = 0, \tag{1.93}$$

$$H_2 = \frac{1-s_E}{\tilde{\mu}\psi(\tilde{S}^*)} - v^* = 0, \tag{1.94}$$

$$H_3 = \zeta(S^*)(\mu-1) + \tilde{S}^*\psi(\tilde{S}^*)(\tilde{\mu}-1) - g^* = 0, \tag{1.95}$$

$$H_4 = \frac{(\rho + g^*\epsilon + \delta - \beta_k)^{1-\frac{1}{\gamma}}\Theta - S^*(1-s_I)}{\rho + g^*\epsilon + \tilde{S}^*\psi(\tilde{S}^*) - \zeta(S^*)(\mu-1)} - v^* = 0. \tag{1.96}$$

where $\Theta \equiv \gamma(1-\gamma)^{\frac{1}{\gamma}-1}$. The Jacobian matrix of the system is given by[27]

$$\mathcal{J} = \begin{bmatrix} \frac{\partial H_1}{\partial S} & \frac{\partial H_1}{\partial \tilde{S}} & \frac{\partial H_1}{\partial g} & \frac{\partial H_1}{\partial v} \\ \frac{\partial H_2}{\partial S} & \frac{\partial H_2}{\partial \tilde{S}} & \frac{\partial H_2}{\partial g} & \frac{\partial H_2}{\partial v} \\ \frac{\partial H_3}{\partial S} & \frac{\partial H_3}{\partial \tilde{S}} & \frac{\partial H_3}{\partial g} & \frac{\partial H_3}{\partial v} \\ \frac{\partial H_4}{\partial S} & \frac{\partial H_4}{\partial \tilde{S}} & \frac{\partial H_4}{\partial g} & \frac{\partial H_4}{\partial v} \end{bmatrix} = \begin{bmatrix} h_{11} & 0 & 0 & -1 \\ 0 & h_{22} & 0 & -1 \\ h_{31} & h_{32} & -1 & 0 \\ h_{41} & h_{42} & h_{43} & -1 \end{bmatrix}, \tag{1.97}$$

where $h_{11} = -\frac{(1-s_I)\zeta''(S)}{(\mu-1)(\zeta'(S))^2} > 0$; $h_{22} = -\frac{(1-s_E)\psi'(\tilde{S})}{\tilde{\mu}(\psi(S))^2} > 0$; $h_{31} = (\mu-1)\zeta'(S) > 0$; $h_{32} = (\tilde{S}\psi(\tilde{S}))'(\tilde{\mu}-1) > 0$. For ease of notation denote the first term in the numerator of equation (1.96) by $\Gamma_1 \equiv (\rho + g^*\epsilon + \delta - \beta_k)^{1-\frac{1}{\gamma}}\Theta - S^*(1-s_I)$ and the denominator by $\Gamma_2 \equiv \rho + g^*\epsilon + \tilde{S}^*\psi(\tilde{S}^*) - \zeta(S^*)(\mu-1)$, then $h_{41} = \frac{-(1-s_I)\Gamma_2 + \Gamma_1\zeta'(S)(\mu-1)}{\Gamma_2^2}$; $h_{42} = \frac{-(\tilde{S}\psi(\tilde{S}))'\Gamma_1}{\Gamma_2^2}$; $h_{43} = \frac{(\rho+g^*\epsilon+\delta-\beta_k)^{-\frac{1}{\gamma}}(1-\frac{1}{\gamma})\Theta\epsilon\Gamma_2 - \Gamma_1\epsilon}{\Gamma_2^2}$.

Let me determine the sign of these last three elements of the matrix. First, consider the sign of Γ_2. From the transversality condition $\rho + g\epsilon > g = \zeta(S)(\mu - 1) + \tilde{S}\psi(\tilde{S})(\tilde{\mu}-1) > \zeta(S)(\mu-1) - \tilde{S}\psi(\tilde{S})$, so $\Gamma_2 > 0$. Using (1.96), the fact that equilibrium value of v^* is positive and $\Gamma_2 > 0$, I can see that $\Gamma_1 > 0$. These facts yield $h_{42} < 0$ and $h_{43} < 0$. Using (1.93) and (1.96) $h_{41} = -\frac{1-s_I}{\Gamma_2} + \frac{\Gamma_1\zeta'(S)(\mu-1)}{\Gamma_2^2} =$

[27]Hereafter, to ease the notation I drop the "*".

$-\frac{v\zeta'(S)(\mu-1)}{\Gamma_2}+\frac{v\zeta'(S)(\mu-1)}{\Gamma_2}=0$. Using these results, a straightforward calculation shows that the determinant of J is positive, $\Delta>0$.

Let me now explore the comparative-statics relatitionships between the R&D subsidy rate to incumbent, s_I, and incumbents' and entrants' research activity, and growth.

The effect of subsidies to incumbents on incumbents' research expenditure can be derived by using the implicit function theorem:

$$\frac{dS}{ds_I}=-\frac{1}{\Delta}\begin{vmatrix}\frac{\partial H_1}{\partial s_I} & \frac{\partial H_1}{\partial \tilde{S}} & \frac{\partial H_1}{\partial g} & \frac{\partial H_1}{\partial v}\\ \frac{\partial H_2}{\partial s_I} & \frac{\partial H_2}{\partial \tilde{S}} & \frac{\partial H_2}{\partial g} & \frac{\partial H_2}{\partial v}\\ \frac{\partial H_3}{\partial s_I} & \frac{\partial H_3}{\partial \tilde{S}} & \frac{\partial H_3}{\partial g} & \frac{\partial H_3}{\partial v}\\ \frac{\partial H_4}{\partial s_I} & \frac{\partial H_4}{\partial \tilde{S}} & \frac{\partial H_4}{\partial g} & \frac{\partial H_4}{\partial v}\end{vmatrix}=-\frac{1}{\Delta}\begin{vmatrix}-\frac{1}{(\mu-1)\zeta'(S)} & 0 & 0 & -1\\ 0 & h_{22} & 0 & -1\\ 0 & h_{32} & -1 & 0\\ \frac{S}{\Gamma_2} & h_{42} & h_{43} & -1\end{vmatrix}. \tag{1.98}$$

It is straightforward to show that the determinant in (1.98) is negative, and thus, $\frac{dS}{ds_I}>0$.

The effect on an entrant's R&D of changes in subsidies to incumbents is given by the following expression:

$$\frac{d\tilde{S}}{ds_I}=-\frac{1}{\Delta}\begin{vmatrix}\frac{\partial H_1}{\partial S} & \frac{\partial H_1}{\partial s_I} & \frac{\partial H_1}{\partial g} & \frac{\partial H_1}{\partial v}\\ \frac{\partial H_2}{\partial S} & \frac{\partial H_2}{\partial s_I} & \frac{\partial H_2}{\partial g} & \frac{\partial H_2}{\partial v}\\ \frac{\partial H_3}{\partial S} & \frac{\partial H_3}{\partial s_I} & \frac{\partial H_3}{\partial g} & \frac{\partial H_3}{\partial v}\\ \frac{\partial H_4}{\partial S} & \frac{\partial H_4}{\partial s_I} & \frac{\partial H_4}{\partial g} & \frac{\partial H_4}{\partial v}\end{vmatrix}=-\frac{1}{\Delta}\begin{vmatrix}h_{11} & -\frac{1}{(\mu-1)\zeta'(S)} & 0 & -1\\ 0 & 0 & 0 & -1\\ h_{31} & 0 & -1 & 0\\ 0 & \frac{S}{\Gamma_2} & h_{43} & -1\end{vmatrix}. \tag{1.99}$$

The determinant in (1.99) is

$$\begin{vmatrix}h_{11} & -\frac{1}{(\mu-1)\zeta'(S)} & 0 & -1\\ 0 & 0 & 0 & -1\\ h_{31} & 0 & -1 & 0\\ 0 & \frac{S}{\Gamma_2} & h_{43} & -1\end{vmatrix}=-h_{11}\frac{S}{\Gamma_2}+h_{31}h_{43}\Big(-\frac{1}{(\mu-1)\zeta'(S)}\Big). \tag{1.100}$$

Substituting the value of h_{11}, h_{31} and h_{43} into (1.100), I get $-h_{11}\frac{S}{\Gamma_2} + h_{31}h_{43}\Big(- \frac{1}{(\mu-1)\zeta'(S)}\Big) = -\frac{(1-s_I)\zeta''(S)}{(\mu-1)(\zeta'(S))^2}\frac{S}{\Gamma_2} - \frac{(\rho+g^*\epsilon+\delta-\beta_k)^{-\frac{1}{\gamma}}(1-\frac{1}{\gamma})\Theta\epsilon\Gamma_2-\Gamma_1\epsilon}{\Gamma_2^2} = \frac{\Gamma_1}{\Gamma_2^2}\Big(- \frac{\zeta''(S)S}{\zeta'(S)} - \epsilon\Big) - \frac{(\rho+g^*\epsilon+\delta-\beta_k)^{-\frac{1}{\gamma}}(1-\frac{1}{\gamma})\Theta\Gamma_2}{\Gamma_2} > 0$,[28] since $\zeta''(S) < 0$ and $1 - \frac{1}{\gamma} < 0$. Thus, the determinant in (1.99) is positive, and so $\frac{d\tilde{S}}{ds_I} < 0$.

To show that effect on growth of s_I is positive I consider:

$$\frac{dg}{ds_I} = -\frac{1}{\Delta}\begin{vmatrix} \frac{\partial H_1}{\partial S} & \frac{\partial H_1}{\partial \tilde{S}} & \frac{\partial H_1}{\partial s_I} & \frac{\partial H_1}{\partial v} \\ \frac{\partial H_2}{\partial S} & \frac{\partial H_2}{\partial \tilde{S}} & \frac{\partial H_2}{\partial s_I} & \frac{\partial H_2}{\partial v} \\ \frac{\partial H_3}{\partial S} & \frac{\partial H_3}{\partial \tilde{S}} & \frac{\partial H_3}{\partial s_I} & \frac{\partial H_3}{\partial v} \\ \frac{\partial H_4}{\partial S} & \frac{\partial H_4}{\partial \tilde{S}} & \frac{\partial H_4}{\partial s_I} & \frac{\partial H_4}{\partial v} \end{vmatrix} = -\frac{1}{\Delta}\begin{vmatrix} h_{11} & 0 & -\frac{1}{(\mu-1)\zeta'(S)} & -1 \\ 0 & h_{22} & 0 & -1 \\ h_{31} & h_{32} & 0 & 0 \\ 0 & h_{42} & \frac{S}{\Gamma_2} & -1 \end{vmatrix}. \tag{1.101}$$

It is easy to see that the determinant in (1.101) is negative, so $\frac{dg}{ds_I} > 0$.

Next, consider the effect of subsidies to entrants. The relationship between s_E and S is defined by

$$\frac{dS}{ds_E} = -\frac{1}{\Delta}\begin{vmatrix} \frac{\partial H_1}{\partial s_E} & \frac{\partial H_1}{\partial \tilde{S}} & \frac{\partial H_1}{\partial g} & \frac{\partial H_1}{\partial v} \\ \frac{\partial H_2}{\partial s_E} & \frac{\partial H_2}{\partial \tilde{S}} & \frac{\partial H_2}{\partial g} & \frac{\partial H_2}{\partial v} \\ \frac{\partial H_3}{\partial s_E} & \frac{\partial H_3}{\partial \tilde{S}} & \frac{\partial H_3}{\partial g} & \frac{\partial H_3}{\partial v} \\ \frac{\partial H_4}{\partial s_E} & \frac{\partial H_4}{\partial \tilde{S}} & \frac{\partial H_4}{\partial g} & \frac{\partial H_4}{\partial v} \end{vmatrix} = -\frac{1}{\Delta}\begin{vmatrix} 0 & 0 & 0 & -1 \\ -\frac{1}{\tilde{\mu}\psi(\tilde{S})} & h_{22} & 0 & -1 \\ 0 & h_{32} & -1 & 0 \\ 0 & h_{42} & h_{43} & -1 \end{vmatrix}. \tag{1.102}$$

$\frac{dS}{ds_E} < 0$ follows from the fact that the determinant in (1.102) is positive.

The equilibrium response of R&D expenditure is given by

$$\frac{d\tilde{S}}{ds_E} = -\frac{1}{\Delta}\begin{vmatrix} \frac{\partial H_1}{\partial S} & \frac{\partial H_1}{\partial s_E} & \frac{\partial H_1}{\partial g} & \frac{\partial H_1}{\partial v} \\ \frac{\partial H_2}{\partial S} & \frac{\partial H_2}{\partial s_E} & \frac{\partial H_2}{\partial g} & \frac{\partial H_2}{\partial v} \\ \frac{\partial H_3}{\partial S} & \frac{\partial H_3}{\partial s_E} & \frac{\partial H_3}{\partial g} & \frac{\partial H_3}{\partial v} \\ \frac{\partial H_4}{\partial S} & \frac{\partial H_4}{\partial s_E} & \frac{\partial H_4}{\partial g} & \frac{\partial H_4}{\partial v} \end{vmatrix} = -\frac{1}{\Delta}\begin{vmatrix} h_{11} & 0 & 0 & -1 \\ 0 & -\frac{1}{\tilde{\mu}\psi(\tilde{S})} & 0 & -1 \\ h_{31} & 0 & -1 & 0 \\ 0 & 0 & h_{43} & -1 \end{vmatrix}. \tag{1.103}$$

[28]Here I use $\frac{1-s_I}{(\mu-1)\zeta'(S)} = \frac{\Gamma_1}{\Gamma_2}$ which follows from (1.93) and (1.96).

Straightforward calculation yields that determinant in (1.103) is negative, thus, $\frac{d\tilde{S}}{ds_E} > 0$.

Without imposing additional restrictions, the effect on growth of the subsidy rate s_E is ambiguous. The derivative of g with respect to s_E is given by:

$$\frac{dg}{ds_E} = -\frac{1}{\Delta}\begin{vmatrix} \frac{\partial H_1}{\partial S} & \frac{\partial H_1}{\partial \tilde{S}} & \frac{\partial H_1}{\partial s_E} & \frac{\partial H_1}{\partial v} \\ \frac{\partial H_2}{\partial S} & \frac{\partial H_2}{\partial \tilde{S}} & \frac{\partial H_2}{\partial s_E} & \frac{\partial H_2}{\partial v} \\ \frac{\partial H_3}{\partial S} & \frac{\partial H_3}{\partial \tilde{S}} & \frac{\partial H_3}{\partial s_E} & \frac{\partial H_3}{\partial v} \\ \frac{\partial H_4}{\partial S} & \frac{\partial H_4}{\partial \tilde{S}} & \frac{\partial H_4}{\partial s_E} & \frac{\partial H_4}{\partial v} \end{vmatrix} = -\frac{1}{\Delta}\begin{vmatrix} h_{11} & 0 & 0 & -1 \\ 0 & h_{22} & -\frac{1}{\tilde{\mu}\psi(\tilde{S})} & -1 \\ h_{31} & h_{32} & 0 & 0 \\ 0 & h_{42} & 0 & -1 \end{vmatrix}. \tag{1.104}$$

The determinant in (1.104) is

$$\begin{vmatrix} h_{11} & 0 & 0 & -1 \\ 0 & h_{22} & -\frac{1}{\tilde{\mu}\psi(\tilde{S})} & -1 \\ h_{31} & h_{32} & 0 & 0 \\ 0 & h_{42} & 0 & -1 \end{vmatrix} = \frac{1}{\tilde{\mu}\psi(\tilde{S})}\begin{vmatrix} h_{11} & 0 & -1 \\ h_{31} & h_{32} & 0 \\ 0 & h_{42} & -1 \end{vmatrix} = \frac{1}{\tilde{\mu}\psi(\tilde{S})}\Big(- h_{11}h_{32} - h_{31}h_{42}\Big). \tag{1.105}$$

The sign of this determinant and, as a result, the sign of $\frac{dg}{ds_E}$, is determined by the sign of $-h_{11}h_{32} - h_{31}h_{42}$. If $-h_{11}h_{32} - h_{31}h_{42} < 0$ then the BGP growth rate depends positively on s_E. Some algebra yields:[29]

$$- h_{11}h_{32} - h_{31}h_{42} = (\tilde{S}\psi(\tilde{S}))'\frac{\Gamma_1}{\Gamma_2^2\zeta'(S)}\Big[\zeta''(S)(\tilde{\mu}-1)\Gamma_2 + (\zeta'(S))^2(\mu-1)\Big]. \tag{1.106}$$

Thus, the sign of (1.106) depends on the sign of the term in brackets and has to be negative to get a positive relationship between the growth rate and the entrants' subsidy rate. Using the functional form $\zeta(S) = \zeta S^{1-\beta}$, it is straightforward to show that $\zeta''(S) = -\beta\zeta'(S)/S$ and $\zeta'(S) = (1-\beta)\zeta(S)/S$. Substituting this into $\zeta''(S)(\tilde{\mu}-1)\Gamma_2 + (\zeta'(S))^2(\mu-1) < 0$ allows me to reformulate the condition as $-\frac{\zeta'(S)}{S}\big(\beta(\tilde{\mu} -$

[29] Notice that using (1.93) and (1.96) yields $\frac{1-s_I}{(\mu-1)\zeta'(S)} = \frac{\Gamma_1}{\Gamma_2}$.

$1)\Gamma_2 - \zeta(S)(\mu-1)(1-\beta)\big) < 0$. Taking into account that $\zeta'(S) > 0$, this condition can be rewritten as

$$\beta(\tilde{\mu}-1)\Gamma_2 - \zeta(S)(\mu-1)(1-\beta) > 0. \tag{1.107}$$

Noticing that $\Gamma_2 > \rho - \zeta(S)(\mu-1)$, I get

$$\beta(\tilde{\mu}-1)\Gamma_2 - \zeta(S)(\mu-1)(1-\beta) > \beta(\tilde{\mu}-1)\Big(\rho - \zeta(S)(\mu-1)\Big) - \zeta(S)(\mu-1)(1-\beta). \tag{1.108}$$

The last inequality implies that (1.106) holds if $\zeta(S) < \frac{\rho\beta(\tilde{\mu}-1)}{(\beta(\tilde{\mu}-2)+1))(\mu-1)}$.

Finally, let me demonstrate that S, $\tilde{S}$, and g are increasing in the subsidy rate to capital β_k:

$$\frac{dS}{d\beta_k} = -\frac{1}{\Delta}\begin{vmatrix} \frac{\partial H_1}{\partial \beta_k} & \frac{\partial H_1}{\partial \tilde{S}} & \frac{\partial H_1}{\partial g} & \frac{\partial H_1}{\partial v} \\ \frac{\partial H_2}{\partial \beta_k} & \frac{\partial H_2}{\partial \tilde{S}} & \frac{\partial H_2}{\partial g} & \frac{\partial H_2}{\partial v} \\ \frac{\partial H_3}{\partial \beta_k} & \frac{\partial H_3}{\partial \tilde{S}} & \frac{\partial H_3}{\partial g} & \frac{\partial H_3}{\partial v} \\ \frac{\partial H_4}{\partial \beta_k} & \frac{\partial H_4}{\partial \tilde{S}} & \frac{\partial H_4}{\partial g} & \frac{\partial H_4}{\partial v} \end{vmatrix} = -\frac{1}{\Delta}\begin{vmatrix} 0 & 0 & 0 & -1 \\ 0 & h_{22} & 0 & -1 \\ 0 & h_{32} & -1 & 0 \\ -\frac{(\rho+g\epsilon+\delta-\beta_k)^{-\frac{1}{\gamma}}(1-\frac{1}{\gamma})\theta}{\Gamma 2} & h_{42} & h_{43} & -1 \end{vmatrix}, \tag{1.109}$$

$$\frac{d\tilde{S}}{d\beta_k} = -\frac{1}{\Delta}\begin{vmatrix} \frac{\partial H_1}{\partial S} & \frac{\partial H_1}{\partial \beta_k} & \frac{\partial H_1}{\partial g} & \frac{\partial H_1}{\partial v} \\ \frac{\partial H_2}{\partial S} & \frac{\partial H_2}{\partial \beta_k} & \frac{\partial H_2}{\partial g} & \frac{\partial H_2}{\partial v} \\ \frac{\partial H_3}{\partial S} & \frac{\partial H_3}{\partial \beta_k} & \frac{\partial H_3}{\partial g} & \frac{\partial H_3}{\partial v} \\ \frac{\partial H_4}{\partial S} & \frac{\partial H_4}{\partial \beta_k} & \frac{\partial H_4}{\partial g} & \frac{\partial H_4}{\partial v} \end{vmatrix} = -\frac{1}{\Delta}\begin{vmatrix} h_{11} & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 \\ h_{31} & 0 & -1 & 0 \\ 0 & -\frac{(\rho+g\epsilon+\delta-\beta_k)^{-\frac{1}{\gamma}}(1-\frac{1}{\gamma})\theta}{\Gamma 2} & h_{43} & -1 \end{vmatrix}, \tag{1.110}$$

$$\frac{dg}{d\beta_k} = -\frac{1}{\Delta}\begin{vmatrix} \frac{\partial H_1}{\partial S} & \frac{\partial H_1}{\partial \tilde{S}} & \frac{\partial H_1}{\partial \beta_k} & \frac{\partial H_1}{\partial v} \\ \frac{\partial H_2}{\partial S} & \frac{\partial H_2}{\partial \tilde{S}} & \frac{\partial H_2}{\partial \beta_k} & \frac{\partial H_2}{\partial v} \\ \frac{\partial H_3}{\partial S} & \frac{\partial H_3}{\partial \tilde{S}} & \frac{\partial H_3}{\partial \beta_k} & \frac{\partial H_3}{\partial v} \\ \frac{\partial H_4}{\partial S} & \frac{\partial H_4}{\partial \tilde{S}} & \frac{\partial H_4}{\partial \beta_k} & \frac{\partial H_4}{\partial v} \end{vmatrix} = -\frac{1}{\Delta}\begin{vmatrix} h_{11} & 0 & 0 & -1 \\ 0 & h_{22} & 0 & -1 \\ h_{31} & h_{32} & 0 & 0 \\ 0 & h_{42} & -\frac{(\rho+g\epsilon+\delta-\beta_k)^{-\frac{1}{\gamma}}(1-\frac{1}{\gamma})\theta}{\Gamma 2} & -1 \end{vmatrix}. \tag{1.111}$$

Straightforward calculations yield $\frac{dS}{d\beta_k} > 0$, $\frac{d\tilde{S}}{d\beta_k} > 0$ and $\frac{dg}{d\beta_k} > 0$.

Derivation of the social optimum

The social planner's problem reads as follows

$$max_{C_t,S_t,\tilde{S}_t} \int_0^{\infty} e^{-\rho t}\frac{C_t^{1-\epsilon}-1}{1-\epsilon}dt \tag{1.112}$$

subject to:[30]

$$\dot{K}_t = Y_t - \delta K_t - C_t - \hat{S}_t = \frac{1}{1-\gamma}K_t^{1-\gamma}Q_t^{\gamma}L^{\gamma} - \delta K_t - C_t - (S_t+\tilde{S}_t)Q_t, \tag{1.113}$$

$$\dot{Q}_t = \Big(\zeta(S_t)(\mu-1) + \tilde{S}_t\psi(\tilde{S}_t)(\tilde{\mu}-1)\Big)Q_t. \tag{1.114}$$

[30] At any t, the social planner ensures an efficient production. This requires, as in laissez-faire case, that $x_t(j) = x_t$, i.e all sectors produce the same quantuty of goods. That implies that the condition for equilibrium on the capital market $x_t = k_t L = \frac{K_t}{Q_t}$ and production function can be written as $Y_t = \frac{1}{1-\gamma}\int_0^1 q_t(j)F(x_t(j),L)dj = \frac{1}{1-\gamma}F(K_t,Q_tL)$, where for Cobb-Douglas case $Y_t = \frac{1}{1-\gamma}K_t^{1-\gamma}Q_t^{\gamma}L^{\gamma}$. By plugging the equation for capital accumulation (1.49) into the market clearing equation for the final good, (1.48), I get (1.113).

The current-value Hamiltonian for this problem is

$$\begin{aligned} H(C_t, S_t, \tilde{S}_t, K_t, Q_t, \eta_{1t}, \eta_{2t}) = \frac{C_t^{1-\epsilon} - 1}{1-\epsilon} \\ + \eta_{1t}\Big(\frac{1}{1-\gamma}K_t^{1-\gamma}Q_t^{\gamma}L^{\gamma} - \delta K_t - C_t - (S_t + \tilde{S}_t)Q_t\Big) \\ + \eta_{2t}\Big(\zeta(S_t)(\mu - 1) + \tilde{S}_t\psi(\tilde{S}_t)(\tilde{\mu} - 1)\Big)Q_t. \end{aligned} \tag{1.115}$$

The necessary first-order conditions are

$$\frac{\partial H_t}{\partial C_t} = C_t^{-\epsilon} - \eta_{1t} = 0, \tag{1.116}$$

$$\dot{\eta}_{1t} = -\frac{\partial H_t}{\partial K_t} + \rho\eta_{1t} = -\eta_{1t}\Big(K_t^{-\gamma}Q_t^{\gamma}L^{\gamma} - \delta\Big) + \rho\eta_{1t}, \tag{1.117}$$

$$\frac{\partial H_t}{\partial S_t} = -\eta_{1t}Q_t + \eta_{2t}(\mu - 1)\zeta'(S_t)Q_t = 0, \tag{1.118}$$

$$\frac{\partial H_t}{\partial \tilde{S}_t} = -\eta_{1t}Q_t + \eta_{2t}(\tilde{\mu} - 1)(\psi(\tilde{S}_t) + \tilde{S}_t\psi'(\tilde{S}_t))Q_t = 0, \tag{1.119}$$

$$\begin{aligned} \dot{\eta}_{2t} &= -\frac{\partial H_t}{\partial Q_t} + \rho\eta_{2t} \\ &= -\eta_{1t}\Big(\frac{\gamma}{1-\gamma}K_t^{1-\gamma}Q_t^{\gamma-1}L^{\gamma} - (S_t + \tilde{S}_t)\Big) - \eta_{2t}\Big(\zeta(S_t)(\mu - 1) + \tilde{S}_t\psi(\tilde{S}_t)\Big) + \rho\eta_{2t}. \end{aligned} \tag{1.120}$$

Notice that in the social planner's problem the growth rate is $g = \zeta(S)(\mu - 1) + \tilde{S}\psi(\tilde{S})(\tilde{\mu} - 1)$, which is the same expression as in the decentralized economy. It follows from this and (1.116) that

$$\frac{\dot{\eta}_{1t}}{\eta_{1t}} = -\epsilon\frac{\dot{C}_t}{C_t} = -\epsilon\Big(\zeta(S)(\mu - 1) + \tilde{S}\psi(\tilde{S})(\tilde{\mu} - 1)\Big). \tag{1.121}$$

I can rewrite (1.117) as

$$\frac{\dot{\eta}_{1t}}{\eta_{1t}} = -K_t^{-\gamma}Q_t^{\gamma}L^{\gamma} + \delta + \rho. \tag{1.122}$$

Using the first expression for $\frac{\dot{\eta}_{1t}}{\eta_{1t}}$ from (1.121) and $F(k_t, 1) = k_t^{1-\gamma}$, I have that

$$\frac{\dot{C}_t}{C_t} = \frac{K_t^{-\gamma} Q_t^{\gamma} L^{\gamma} - \delta - \rho}{\epsilon} = \frac{k_t^{-\gamma} - \delta - \rho}{\epsilon} = \frac{\frac{1}{1-\gamma} F_k(k_t, 1) - \delta - \rho}{\epsilon}. \tag{1.123}$$

The expression for $\frac{\dot{C}_t}{C_t}$ from (1.121) gives the equation for the socially optimal level of k:

$$\frac{1}{1-\gamma} F_k(k, 1) = \epsilon\Big(\zeta(S)(\mu - 1) + \tilde{S}\psi(\tilde{S})(\tilde{\mu} - 1)\Big) + \delta + \rho. \tag{1.124}$$

From (1.118) and (1.119) it follows that:[31]

$$\frac{\eta_{1t}}{\eta_{2t}} = (\mu - 1)\zeta'(S_t), \qquad \frac{\eta_{1t}}{\eta_{2t}} = (\tilde{\mu} - 1)(\psi(\tilde{S}_t) + \tilde{S}_t\psi'(\tilde{S}_t)), \qquad \frac{\dot{\eta_{1t}}}{\eta_{1t}} = \frac{\dot{\eta_{2t}}}{\eta_{2t}}. \tag{1.125}$$

Combining first two equation from (1.125), I get the condition that balances innovations by incumbents and by entrants for the social planner:

$$(\mu - 1)\zeta'(S_t) = (\tilde{\mu} - 1)(\psi(\tilde{S}_t) + \tilde{S}_t\psi'(\tilde{S}_t)). \tag{1.126}$$

Dividing (1.120) by η_{2t} and substituting $\frac{\dot{\eta_{1t}}}{\eta_{1t}} = \frac{\dot{\eta_{2t}}}{\eta_{2t}} = -\epsilon g$ yields

$$\rho - \frac{\eta_{1t}}{\eta_{2t}}\Big(\frac{\gamma}{1-\gamma} K_t^{1-\gamma} Q_t^{\gamma-1} L^{\gamma} - (S_t + \tilde{S}_t)\Big) - \Big(\zeta(S_t)(\mu - 1) + \tilde{S}_t\psi(\tilde{S}_t)(\tilde{\mu} - 1)\Big) = -\epsilon g. \tag{1.127}$$

Noticing that $\frac{\gamma}{1-\gamma} K_t^{1-\gamma} Q_t^{\gamma-1} L^{\gamma} = \frac{1}{1-\gamma}\big(F(k_t, 1) - F_k(k_t, 1)k_t\big)L$,[32] and substituting for $\frac{\eta_{1t}}{\eta_{2t}}$ from the first equation in (1.125), determines the socially optimal level for

[31] The last equation follows from $\ln \eta_{2t} + \ln(\mu - 1)\zeta'(S_t) = \ln \eta_{1t}$, or alternatively, $\ln \eta_{2t} + \ln(\tilde{\mu} - 1)(\psi(\tilde{S}_t) + \tilde{S}_t\psi'(\tilde{S}_t)) = \ln \eta_{1t}$.

[32] It follows from $\frac{\gamma}{1-\gamma} K_t^{1-\gamma} Q_t^{\gamma-1} L^{\gamma} = \frac{1}{1-\gamma} F_{QL}(K, OL) = \frac{1}{1-\gamma}\big(F(k_t, 1) - F_k(k_t, 1)k_t\big)L$.

incremental research S:

$$1 = (\mu - 1)\zeta'(S)\frac{\frac{1}{1-\gamma}\Big(F(k,1) - F_k(k,1)k_t\Big)L - (S + \tilde{S})}{\rho + g\epsilon - \zeta(S)(\mu - 1) - \tilde{S}\psi(\tilde{S})(\tilde{\mu} - 1)}. \tag{1.128}$$

Proceeding in the same way, but now using the second equation in (1.125) instead of the first, determines the socially optimal level for radical research $\tilde{S}$:

$$1 = (\tilde{\mu} - 1)\Big(\psi(\tilde{S}) + \tilde{S}\psi'(\tilde{S})\Big)\frac{\frac{1}{1-\gamma}\Big(F(k,1) - F_k(k,1)k_t\Big)L - (S + \tilde{S})}{\rho + g\epsilon - \zeta(S)(\mu - 1) - \tilde{S}\psi(\tilde{S})(\tilde{\mu} - 1)}. \tag{1.129}$$

Derivation of the laissez-faire solution and the social optimum for linear $\zeta(\cdot)$

In linear case, from HJB equation (1.17) the equilibrium condition that satisfies the optimality and market clearing is given by[33]

$$\zeta(V_t(j, \mu q) - V_t(j, q)) \le q_t(j),$$
$$\zeta(V_t(j, \mu q) - V_t(j, q)) = q_t(j) \quad \text{if} \quad S_t(j) > 0. \tag{1.130}$$

By appealing to symmetry, and dropping the index j,[34] the equlibrium value function satisfies

$$\zeta(V(\mu q) - V(q)) = q, \tag{1.131}$$

[33] Recall that in Section 1.4 I consider the case with no subsidies.

[34] Using that $q(j)$ does not change over time — it is quality supplied by the incumbent, which remains constant while firm is incumbent.

so than value function has to be linear in q, i.e. $V = vq$. Then (1.131) can be rewritten as

$$v = \frac{1}{\zeta(\mu - 1)}. \tag{1.132}$$

The entry condition for entrants is the same as for general case (see (1.19)).[35] Using linearity of value function and (1.132) gives[36]

$$\tilde{S}_t(j) = \tilde{S} = \psi^{-1}\Big(\frac{\zeta(\mu - 1)}{\tilde{\mu}}\Big). \tag{1.133}$$

Along the BGP HJB, (1.17), implies that

$$v = \frac{\pi(k^*)L}{r^* + \tilde{S}^*\psi(\tilde{S}^*)}. \tag{1.134}$$

The analogous to the law of motion of Q_t (1.37) for linear case is given by

$$g_Q = \frac{\dot{Q}_t}{Q_t} = \zeta S_t^a(\mu - 1) + \tilde{S}_t\psi(\tilde{S}_t)(\tilde{\mu} - 1), \tag{1.135}$$

where $S_t^a \equiv \frac{\int_0^1 S_t(j)q_t(j)}{Q_t}$ is the average incumbents' research expenditure at time t. Let me show that the growth rate of the economy is g_Q.

It can be shown that $g_Y = g_k = g_Q = g_C = g_{\hat{S}}$,[37] where the total expenditure on R&D is given by

$$\hat{S}_t = \int_0^1 S_t(j) + \tilde{S}_t(j))q_t(j)dj = \int_0^1 S_t(j)q_t(j)dj + \tilde{S}Q_t. \tag{1.136}$$

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[35] Using linearity of the value function, the entry condition (1.19) can be rewritten as $\psi(S_t(j)) = \frac{1}{\tilde{\mu}v}$.

[36] Note that for entrants, symmetry is a property of the equilibrium, not an assumption.

[37] The steps of proof is the same as for general case (see Section 1.2.4).

Rewriting and rearranging (1.136) yields

$$\frac{\int_0^1 S_t(j)q_t(j)dj}{Q_t} = \frac{\hat{S}_t}{Q_t} - \tilde{S}, \tag{1.137}$$

which shows that $\frac{\int_0^1 S_t(j)q_t(j)dj}{Q_t}$ is constant along BGP. Therefore, the aggregate incumbents research expenditures $\int_0^1 S_t(j)q_t(j)dj$ are proportional to Q_t. Thus, the BGP growth rate of the economy is given by

$$g^* = \zeta S^{*a}(\mu - 1) + \tilde{S}^* \psi(\tilde{S}^*)(\tilde{\mu} - 1), \tag{1.138}$$

where $S^{*a} \equiv \frac{\int_0^1 S_t^*(j)q_t(j)}{Q_t}$ is the average BGP incumbents' research expenditure.

Let me show that there exist a unique equilibrium with growth g^*, (1.138). By (1.133) $\tilde{S}^*$ is uniquely determined and is strictly positive. Combining (1.132) and (1.134) gives the BGP interest rate

$$r^* = \zeta(\mu - 1)\gamma k^{*1-\gamma} L - \tilde{S}^* \psi(\tilde{S}^*). \tag{1.139}$$

This equation, together with (1.33) yields:

$$k^{*-\gamma}(1-\gamma) = \zeta(\mu - 1)\gamma k^{*1-\gamma} L - \tilde{S}^* \psi(\tilde{S}^*) + \delta - \beta_k. \tag{1.140}$$

The left hand side (LHS) of (1.140) is decreasing in k^*, and $\lim_{k \to 0} LHS = \infty$, $\lim_{k \to \infty} LHS = 0$, while the right hand side in increasing in k^*, and $\lim_{k \to 0} RHS = -\tilde{S}^* \psi(\tilde{S}^*) + \delta - \beta_k$, $\lim_{k \to 0} RHS = \infty$. Thus, there exist a unique k that solves (1.140). From Euler equation (1.20) and equation (1.139) growth rate can be also expressed as

$$g^* = \frac{\zeta(\mu - 1)\gamma k^{*1-\gamma} L - \tilde{S}^* \psi(\tilde{S}^*) - \rho}{\epsilon}. \tag{1.141}$$

Using (1.138) and (1.141), I have that the average BGP incumbents' expenditure

$$S^{*a} = \frac{\zeta(\mu-1)\gamma k^{*1-\gamma}L - \tilde{S}^*\psi(\tilde{S}^*)(\epsilon(\mu-1)+1) - \rho}{\epsilon\zeta(\mu-1)}. \tag{1.142}$$

S^{*a} is strictly positive if[38]

$$k^{*1-\gamma} > \frac{\tilde{S}^*\psi(\tilde{S}^*)(\epsilon(\mu-1)+1)+\rho}{\zeta(\mu-1)\gamma L}, \tag{1.143}$$

Analogous to the case of a general specification of the technology, the transversality condition is satisfied if $\rho > g(1-\epsilon)$, which holds if $\epsilon \geq 1$. There to be positive aggregate growth follows from (1.138) and strictly positivity of S^{a*} and $\tilde{S}^*$.

Substituting $\zeta(S) = \zeta S$ into (1.52)-(1.55) and using (1.138) and (1.134) for the case of decentralized economy, (1.52)-(1.55) can be rewritten as follows. The socially optimal level k^{sp} (the analogous to (1.52)) is given by

$$(k^{sp})^{-\gamma} = \epsilon\Big(\zeta S^{sp}(\mu-1) + \tilde{S^{sp}}\psi(\tilde{S^{sp}})(\tilde{\mu}-1)\Big) + \delta + \rho, \tag{1.144}$$

whereas the equilibrium value of k^* for the decentralized economy (the analogous to (1.53))

$$(k^*)^{-\gamma}(1-\gamma) = \epsilon\Big(\zeta S^{*a}(\mu-1) + \tilde{S}^*\psi(\tilde{S}^*)(\tilde{\mu}-1)\Big) + \delta + \rho. \tag{1.145}$$

Socially optimal level for incremental innovation S^{sp} (the analogous to (1.54)) satisfies

$$1 = (\mu-1)\zeta\frac{\frac{\gamma}{1-\gamma}(k^{sp})^{1-\gamma}L - (S^{sp}+\tilde{S}^{sp})}{\rho + g^{sp}\epsilon - \zeta S^{sp}(\mu-1) - \tilde{S}^{sp}\psi(\tilde{S}^{sp})(\tilde{\mu}-1)}. \tag{1.146}$$

[38]Note that assumption is expressed in terms of k which is an endogenous variable. However, it is easy to find sets of parameters for which the assumption is satisfied. Note also that from (1.133) $\tilde{S}^*$ is function of parameters.

Combining (1.132) and (1.134), using Euler equation (1.20) and $\pi(k^*) = \gamma(k^*)^{1-\gamma}$ yields steady-state equilibrium level of an R&D effort of incumbents, S^* (the analogous to (1.55))

$$1 = \frac{(\mu - 1)\zeta\gamma(k^*)^{1-\gamma}L}{\rho + g^*\epsilon + \tilde{S}^*\psi(\tilde{S}^*)}. \tag{1.147}$$

Proof of Proposition 1.3

In liner case, the condition that balanced innovations by incumbents and entrants for the social planner becomes (the analogous (1.126))

$$(\mu - 1)\zeta = (\tilde{\mu} - 1)(\psi(\tilde{S}^{sp}) + \tilde{S}^{sp}\psi'(\tilde{S}^{sp})), \tag{1.148}$$

whereas this condition for case of decentralized economy is[39]

$$(\mu - 1)\zeta = \tilde{\mu}\psi(\tilde{S}). \tag{1.149}$$

Noticing that $\tilde{S}^{sp}\psi'(\tilde{S}^{sp}) = \tilde{\beta}\psi(\tilde{S}^{sp})$,[40] (1.148) can be rewritten as

$$(\mu - 1)\zeta = (\tilde{\mu} - 1)(1 - \tilde{\beta})\psi(\tilde{S}^{sp}). \tag{1.150}$$

Combining (1.149) and (1.150), I obtain

$$\psi(\tilde{S}^{sp}) > \frac{(\mu - 1)\zeta}{\tilde{\mu}} = \psi(\tilde{S}), \tag{1.151}$$

[39]This equation is obtained by combining (1.132) and the entry condition.

[40]Recall that the functional form for $\psi(\tilde{S}) = \psi\tilde{S}^{\tilde{\beta}}$ with $\tilde{\beta} \in (0, 1)$

which implies $\tilde{S}^{sp} < \tilde{S}$, since $\psi(\tilde{S})$ is decreasing in $\tilde{S}$. Rearranging (1.146) and plugging (1.150) into it yields

$$g^{sp} = \frac{(\mu-1)\zeta\frac{\gamma}{1-\gamma}(k^{sp})^{1-\gamma}L + \tilde{\beta}\tilde{S}^{sp}\psi(\tilde{S}^{sp})(\tilde{\mu}-1) - \rho}{\epsilon}. \tag{1.152}$$

By rearranging (1.147) I get

$$g^* = \frac{(\mu-1)\zeta\gamma(k^*)^{1-\gamma}L - \tilde{S}^*\psi(\tilde{S}^*) - \rho}{\epsilon}. \tag{1.153}$$

Substituting (1.144) and (1.145) into (1.152) and (1.153) respectively and rearranging it gives

$$f_1 \equiv g^{sp}\epsilon - (\mu-1)\zeta\frac{\gamma}{1-\gamma}(g^{sp}\epsilon + \delta + \rho)^{1-\frac{1}{\gamma}}L - \tilde{\beta}\tilde{S}^{sp}\psi(\tilde{S}^{sp})(\tilde{\mu}-1) + \rho = 0, \tag{1.154}$$

$$f_2 \equiv g^*\epsilon - (\mu-1)\zeta\frac{\gamma}{1-\gamma}(1-\gamma)^{\frac{1}{\gamma}}(g^*\epsilon + \delta + \rho)^{1-\frac{1}{\gamma}}L + \tilde{S}^*\psi(\tilde{S}^*) + \rho = 0. \tag{1.155}$$

Notice that f_1 is strictly increasing in g^{sp} and concave, goes to $-\infty$ as $g^{sp} = 0$, and goes to $+\infty$ as g^{sp} goes to $+\infty$. Function f_2 behaves the same way with regard g^*.[41] So that, function f_1 (and f_2) crosses the horizontal axis form below and this is only one crossing point. Therefore, for any $\bar{g} < g^{sp}$, $f_1(\bar{g}) < f_1(g^{sp})$, so that $g^* < g^{sp}$ as long as $f_1(g^*) < f_1(g^{sp}) = 0$.

[41] To see this, note that function $x^{1-\frac{1}{\gamma}}$ dominates the linear function x and any constant when $x \to 0$, while the linear function grows faster than $x^{1-\frac{1}{\gamma}}$ and any constant for $x \to \infty$. From (1.150) and (1.133) $\tilde{S}^{sp}$ and $\tilde{S}$ are functions of parameters, particularly, $\tilde{S}^{sp} = \left(\frac{(\tilde{\mu}-1)(1-\tilde{\beta})\psi}{(\mu-1)\zeta}\right)^{\frac{1}{\beta}}$ and $\tilde{S} = \left(\frac{\tilde{\mu}\psi}{(\mu-1)\zeta}\right)^{\frac{1}{\beta}}$.

Rearranging (1.155) and noticing that $(1-\gamma)^{\frac{1}{\gamma}} < 1$ gives

$$-\tilde{S}^*\psi(\tilde{S}^*) - \rho = g^*\epsilon - (\mu-1)\zeta\frac{\gamma}{1-\gamma}(1-\gamma)^{\frac{1}{\gamma}}(g^*\epsilon+\delta+\rho)^{1-\frac{1}{\gamma}}L > g^*\epsilon - (\mu-1)\zeta\frac{\gamma}{1-\gamma}(g^*\epsilon+\delta+\rho)^{1-\frac{1}{\gamma}}L. \quad (1.156)$$

Using obtained inequality I get

$$f_1(g^*) = g^*\epsilon - (\mu-1)\zeta\frac{\gamma}{1-\gamma}(g^*\epsilon+\delta+\rho)^{1-\frac{1}{\gamma}}L - \tilde{\beta}\tilde{S}^*\psi(\tilde{S}^*)(\tilde{\mu}-1) + \rho < -\tilde{S}^*\psi(\tilde{S}^*) - \rho - \tilde{\beta}\tilde{S}^*\psi(\tilde{S}^*)(\tilde{\mu}-1) + \rho = -(1+\tilde{\beta}(\tilde{\mu}-1))\tilde{S}^*\psi(\tilde{S}^*) < 0. \quad (1.157)$$

So that $f_1(g^*) < f_1(g^{sp})$ and, consequently, $g^* < g^{sp}$.

Chapter 2

Patents, Growth and Capital in an OLG framework

2.1 Introduction

It is widely believed that stronger patent protection should promote innovation, and as a result, economic growth. However, some empirical studies (e.g., Qian (2007), Lerner (2009))[1] indicate the possibilities of a nonmonotonic relationship between patent protection and growth (IPR and innovation). The present paper studies how strengthening patent protection influences economic growth in a Schumpeterian endogenous growth model with capital accumulation. In contrast to the previous literature, which mostly considers patent policy in infinite-lifetimes economy, this paper investigates the implications of patent policy in an overlapping generations (OLG) framework that allows to analyse how heterogeneity in patent ownership across generations can change the implication of patent policy. In this study I consider the effects of the duration of patents and patent breadth protection on growth and innovation. By concentrating on two extreme cases of patent length, the one-period and the infinite

[1] Using panel data, Aghion et al. (2005) find evidence of an inverted-U relationship between competition and innovation. Weakening patent protection in current model implies high competition.

patent life systems, I establish that a short patent duration enhances innovation. This can be explained as follows. Under one-period patent life, investment of the young is allocated to physical capital and research, whereas under infinite patent length protection, the young agent allocates their investment in physical capital, research and purchasing of patents from the older generation, thereby reducing the amount of investment in research. R&D expenditures spurs growth, thus, the growth rate with one-period protection is higher than that under infinite patent life.

Analysis of the implications of patent breadth protection reveals that loosening patent breadth affects growth in two opposing directions. Incomplete breadth reduces the price of patented intermediate goods, which leads to increased demand for the intermediate product, and, as a result, to increasing output. In its turn, this stimulates aggregate investment, including investment in R&D, which promotes growth. On the other hand, an increase in the quantity of the intermediate product leads to reallocation of investment towards physical capital by reducing research investment, and, as a result, reduces growth rate.

My paper relates to the literature on optimal patent protection. Starting with Nordhaus (1969, 1972) and Scherer (1972), works that study optimal patent length in a static partial equilibrium models, many researchers address their work to the different implications of patent policy (Goh and Olivier (2002), Iwaisako and Futagami (2013), Chu et al. (2016), etc.). However, only a few papers study the growth implications of patents in the OLG framework.

Chou and Shy (1993) construct an OLG endogenous growth model of variety expansion (with no physical capital) and show the existence of "crowding-out effect" for long period patent protection[2] that appears when a young agent purchases patents from the old agent rather than invest in new innovation. Specifically, they reveal that investment in R&D with one-period patent life is higher than under infinite

[2]Chou and Shy define the crowding-out effect as a situation where "part (or all) of the savings is allocated to purchasing existing monopoly firms rather than the construction of new firms".

patent life, that is, a short patent duration stimulates innovation. Sorek (2011) explores patent policy and endogenous growth for a quality-ladder OLG economy. His model uses the Grossman and Helpman (1991b) model of vertical innovation without physical capital with intermediate goods produced by labor. The main focus of the paper is on the qualitative effect of the value of the inter-temporal elasticity of substitution (IES). He shows that the effect of loosening patent breadth protection on growth and research depends on the length of the patent and the IES, specifically, weakening patent protection raises R&D investment and growth for IES less than one and prevents R&D investment and growth for IES exceeding one. In the logarithmic utility function case (IES equal to one) lagging breadth protection has no effect on investment in research and therefore a growth. By studying the effect of patent length he reveals that, depending on the IES, R&D investment and growth can be either higher or lower under one-period patent length than under infinite patent life. Diwakar et al. (2019) study the implications of patent policy for welfare and growth by constructing a variety expansion model with capital accumulation in an overlapping generations economy. Their paper establishes that growth is higher under incomplete patent protection. Additionally, their research demonstrates that shortening patent length is more effective than loosening patent breadth in spurring growth.

This paper contributes to the literature on patent policy and economic growth by investigating the implication of patent policy in a quality-ladder OLG economy with physical capital. To the best of my knowledge only Sorek (2011) provides such an analysis for an OLG economy with Schumpeterian structure with creative destruction; but unlike his study, I consider both quality improvements and capital accumulation that offers new insights in the analysis of the effects of imperfect patent protection on growth and innovation.

The paper is organized as follows. Section 2.2 describes the model, defines an equilibrium and proves existence and uniqueness for both the one-period and the infinite patent life cases, and investigates the effects of patent life regime on growth. Section 2.3 develops similar results with a focus on the protection granted by patent breadth and establishes mechanisms through which loosening breadth protection affects growth. Section 2.4 provides final remarks, while the Appendix contains the proofs.

2.2 The Model

I develop an overlapping generations model with R&D-based Schumpeterian growth as in Aghion and Howitt (2007). Time is discrete, starts at 0, and extends infinitely into the future. Each generation consists of L new individual agents who live for two periods. For simplicity I assume that there is no population growth, $L_t = L = 1$ for all t. There are two production sectors: the final good sector and the intermediate sector. Intermediate producers is patent-holding firms that produce and sell the differentiated good. I consider two instruments in influencing the degree of patent protection: patent duration and patent breadth. In Section 2.2 I focus on patent length that represents how long patentee can exclusively produce and sell the good. In Section 2.3 I introduce policy variable η that representing of breadth of patent and explore the effect of both patent length and patent breadth on growth, innovation and welfare.

2.2.1 Production sectors

Final goods producer

The final good is storable, in the form of capital, and the intermediate products are produced with capital. The final good is produced competitively according to the following production function:

$$Y_t = \frac{1}{1-\gamma}\left(\int_0^1 q_t(j)x_t(j)^{1-\gamma}dj\right)L_t^{\gamma}, \tag{2.1}$$

where $q_t(j)$ is the quality and $x_t(j)$ is the amount of intermediate good $j \in [0,1]$, $\gamma \in (0,1)$. The labor supply L of the entire economy is used in production of the final good, labor is supplied inelastically, and I set $L = 1$. The price of the final good is normalized to 1.

Profit maximization by the final good producer implies that demand for the intermediate goods of the highest available quality is

$$x_t(j) = p_t(j)^{-\frac{1}{\gamma}}q_t(j)^{\frac{1}{\gamma}}, \tag{2.2}$$

and the wage rate at time t is

$$w_t = \frac{\gamma}{1-\gamma}\left(\int_0^1 q_t(j)x_t(j)^{1-\gamma}dj\right). \tag{2.3}$$

Intermediate goods production and innovation

The optimization problem of an intermediate good producer is solved by backward induction: first, I derive the optimal price for a good, assuming that it has already been invented, then, I determine the optimal research expenditure.

Pricing, profit and production Each intermediate product is produced according the production function:

$$x_t(j,q) = K_t(j)/q_t(j), \tag{2.4}$$

where $K_t(j)$ is the amount of capital used as input. The division by $q_t(j)$ in (2.4) reflects that successive blueprints are produced by increasingly capital intensive techniques.

Each intermediate good producer is a monopolist in her sector. Her cost is

$$R_t K_t(j) = R_t q_t(j) x_t(j), \tag{2.5}$$

where R_t is the rental rate of capital.

The monopolist takes the demand for intermediate good $x_t(j)$ as given and maximizes her profit

$$\pi_t(j) = \max_{p_t(j)} (p_t(j) - R_t q_t(j)) x_t(j), \tag{2.6}$$

subject to demand function (2.2). Differentiate this equation to obtain $\frac{\partial \pi_t}{\partial p_t} = x_t(j) + \big(p_t(j) - R_t q_t(j)\big) \frac{\partial x_t}{\partial p_t} = 0$, so that $p_t(j) = \frac{-x_t}{\partial x_t / \partial p_t} + R_t q_t(j)$.

That yields the monopoly price:[3]

$$p_t(j) = \frac{R_t q_t(j)}{1 - \gamma}. \tag{2.7}$$

That implies the equilibrium quantity of each intermediate product is

$$x_t(j) \equiv x_t = \left(\frac{1 - \gamma}{R_t} \right)^{\frac{1}{\gamma}}. \tag{2.8}$$

[3] I assume that firm that create a new type of differentiated good can obtain a patent that allows to produce and sell good monopolistically.

The supply for capital is the predetermined capital stock K_t and the demand is the sum of demand for capital of each sector:

$$K_t = \int_0^1 K_t(j)dj = \int_0^1 x_t(j)q_t(j) = \left(\frac{1-\gamma}{R_t}\right)^{\frac{1}{\gamma}} Q_t, \tag{2.9}$$

where $Q_t = \int_0^1 q_t(j)dj$ is an index of aggregate quality. Let $k_t \equiv \dfrac{K_t}{Q_t}$ denote capital intensity. Then from (2.9), the equilibrium rental rate is a decreasing function of the capital-technological knowledge ratio k_t:

$$R_t = k_t^{-\gamma}(1-\gamma). \tag{2.10}$$

Now I can express quantities of intermediate goods in intensive form, that is $x_t = k_t$. Substituting this and (2.7), (2.10) into (2.6) gives

$$\pi_t(j) = \hat{\pi}(k_t)q_t(j), \tag{2.11}$$

where $\hat{\pi}(k_t) = \gamma k_t^{1-\gamma}$.

Determination of R&D effort Each period there is one innovator/entrepreneur in each sector. Innovation takes one period. Entrepreneur spends the final good in research and creates a new improved version of the intermediate product if she succeeds. Specifically, the quality of the intermediate good in each sector j is

$$q_t(j) = \begin{cases} \mu q_{t-1}(j) & \text{with probability } \phi(\tilde{R}_t(j)) \\ q_{t-1}(j) & \text{with probability } 1-\phi(\tilde{R}_t(j)), \end{cases} \tag{2.12}$$

where the probability of successful innovation $\phi(\tilde{R}_t(j))$ depends on the amount of $\tilde{R}_t(j)$ of final output spent on research, and $\mu > 1$.

If innovation is successful, the entrepreneur in that sector will become the monopolist. She will choose the R&D expenditure $\tilde{R}_t(j)$ that maximizes her net benefit:

$$\max_{\tilde{R}_t(j)} \phi(\tilde{R}_t(j))\pi_t(\mu q_{t-1}(j)) - \tilde{R}_t(j)q_{t-1}(j), \tag{2.13}$$

where $\pi_t(\mu q_{t-1}(j))$ is entrepreneur's profit if she succeeds. The innovation intensity depends positively on the amount of R&D expenditure $\tilde{R}_t(j)$ and takes the Cobb-Douglas functional form:

$$\phi(\tilde{R}_t(j)) = \zeta \tilde{R}_t(j)^{\frac{1}{2}}, \tag{2.14}$$

where the parameter ζ reflects the productivity of the research sector. The first-order condition for the above maximization problem is:

$$\phi'(\tilde{R}_t(j))\hat{\pi}(k_t)\mu = 1.$$

Combining this with (2.14) gives the equilibrium quality-adjusted level of research and research intensity:

$$\tilde{R}_t(j) \equiv \tilde{R}_t = \frac{(\zeta\gamma\mu)^2}{4} k_t^{2(1-\gamma)}, \tag{2.15}$$

$$\phi(\tilde{R}_t(j)) \equiv \phi(\tilde{R}_t) = \frac{\zeta^2\gamma\mu}{2} k_t^{1-\gamma}. \tag{2.16}$$

The parameters of the model ζ and γ should satisfy the condition $\frac{\zeta^2\gamma\mu}{2}k_t^{1-\gamma} < 1$ that guarantees that the innovation rate $\phi(\tilde{R}_t)$ is between 0 and 1.

Note that the probability of innovation $\phi(\tilde{R}_t)$ is the same in all sectors regardless of the starting level of quality $q_t(j)$. This property permits a simple characterization of the aggregate growth rate in the economy.

2.2.2 Consumption decisions

The preferences of agents are represented by the logarithmic utility function:

$$U(C_{1,t}, C_{2,t+1}) = u(C_{1,t}) + \rho u(C_{2,t+1}) = \ln C_{1,t} + \rho \ln C_{2,t+1}, \tag{2.17}$$

where $\rho \in (0,1)$ is the discount factor, $C_{1,t}$ denotes the consumption of an individual born at time t when young (at date t), and $C_{2,t+1}$ is this individual's consumption at date $t+1$ in his old age.

I consider two extreme patent lengths: infinite patent length, $T = \infty$, and one-period patent life, $T = 1$.

Infinite patent length During the first period of life, an agent receives wage income from inelastically supplying one unit of labor which can be used to consume the final good and to invest. Under infinite patent length an agent diversifies her investment in three forms: physical capital K_{t+1}, which is available for production in period $t+1$, investment in research $\tilde{R}_{t+1}^{(agg)}$, which will be used for R&D at time $t+1$, and buying "old" patents from the older generation $V_t^{o(agg)}$. A young agent buys the "old" (patent created by the old/previous generation) patent to hedge against the risk of possible R&D failure. When old, at period $t+1$, an agent's incomes come from different sources: the rental from her physical capital stock, the monopoly profit from patented intermediate goods and income from selling the patent. The budget constraints are

$$C_{1,t} + K_{t+1} + \tilde{R}_{t+1}^{(agg)} + V_t^{o(agg)} = w_t, \tag{2.18}$$

$$C_{2,t+1} = (R_{t+1} + (1-\delta))K_{t+1} + \phi(\tilde{R}_{t+1})(\pi_{t+1}^{n(agg)} + V_{t+1}^{n(agg)}) + (1 - \phi(\tilde{R}_{t+1}))(\pi_{t+1}^{o(agg)} + V_{t+1}^{o(agg)}). \tag{2.19}$$

where $\tilde{R}^{(agg)}_{t+1} = \int_0^1 \tilde{R}_{t+1}(j)q_t(j)dj = \tilde{R}_{t+1}Q_t$ is aggregate research expenditure; $\pi^{n(agg)}_{t+1} = \gamma k^{1-\gamma}_{t+1}\mu Q_t$ and $V^{n(agg)}_{t+1}$ are, respectively, aggregate profit obtained in time $t+1$ and the value of monopoly firms ("new" patent) that will be sold to the next generation if innovation is successful. Whereas, if invention fails the agent receives the aggregate profit $\pi^{o(agg)}_{t+1} = \gamma k^{1-\gamma}_{t+1}Q_t$ and sells the "old" patents $V^{o(agg)}_{t+1}$.

By treating the consumer's optimization problem as an asset pricing problem, I get the first-order condition

$$\frac{u'(C_{1,t})}{\rho u'(C_{2,t+1})} = R_{t+1} + (1-\delta)$$
$$= \frac{\phi(\tilde{R}_{t+1})\left(\pi^{n(agg)}_{t+1} + V^{n(agg)}_{t+1}\right)}{\tilde{R}^{(agg)}_{t+1}} = \frac{(1-\phi(\tilde{R}_{t+1}))\left(\pi^{o(agg)}_{t+1} + V^{o(agg)}_{t+1}\right)}{V^{o(agg)}_t}. \tag{2.20}$$

Using (2.20), I can rewrite (2.19), the budget constraint when old, as

$$C_{2,t+1} = \frac{u'(C_{1,t})}{\rho u'(C_{2,t+1})}\left[K_{t+1} + \tilde{R}_{t+1} + V^{o(agg)}_t\right]. \tag{2.21}$$

Using (2.18), the budget constraint when young, in (2.21) I have

$$C_{2,t+1} = \frac{u'(C_{1,t})}{\rho u'(C_{2,t+1})}\left[w_t - C_{1,t}\right]. \tag{2.22}$$

so that

$$C_{1,t} = \frac{w_t}{1+\rho}, \tag{2.23}$$

since $u(C_t) = \ln C_t$.

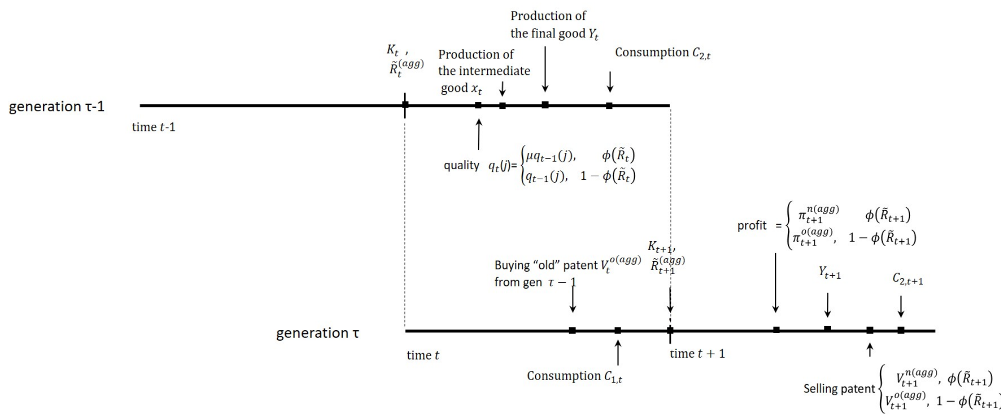

Figure 2.1: The timing of events, case $T = \infty$

Now it is immediate from (2.18) that

$$K_{t+1} + \tilde{R}_{t+1} + V_t^{o(agg)} = \frac{\rho}{1+\rho} w_t. \tag{2.24}$$

Note that $\pi_t^{o(agg)} = \frac{1}{\mu}\pi_t^{n(agg)}$, thus, it can be easily seen that $V_t^{o(agg)} = \frac{1}{\mu}V_t^{n(agg)}$. Substituting this into (2.20) yields

$$V_t^{o(agg)} = \frac{1}{\mu}\frac{1-\phi(\tilde{R}_{t+1})}{\phi(\tilde{R}_{t+1})}\tilde{R}_{t+1}^{(agg)}. \tag{2.25}$$

Now substituting $\tilde{R}_{t+1}^{(agg)} = \frac{(\zeta\gamma\mu)^2}{4}k_{t+1}^{2(1-\gamma)}Q_t$ into the last equation, and using (2.16), after some algebraic manipulations, I get

$$V_t^{o(agg)} = \big(\frac{\gamma}{2}k_{t+1}^{1-\gamma} - \frac{(\zeta\gamma)^2\mu}{4}k_{t+1}^{2(1-\gamma)}\big)Q_t. \tag{2.26}$$

Now I can summarize the timing of events in the case of infinite patent length. This is depicted in Figure 2.1.

Under one-period patent protection an agent invests in physical capital and R&D. Thus, agents will not buy “old” patents from the older generation as they are worthless: intermediate good producers collect a monopoly profit only for one period and become competitive thereafter. It can be easily seen that for the case of one-period patent protection equation (2.24) modifies to:

$$K_{t+1} + \tilde{R}_{t+1} = \frac{\rho}{1+\rho} w_t. \tag{2.27}$$

2.2.3 Equilibrium and growth

Definition A competitive equilibrium is a sequence of quantities $\{C_{1,t}, C_{2,t}, K_t, Y_t, x_t, Q_t, \tilde{R}_t^{(agg)}, V_t^{(agg)}, \pi_t^{(agg)}\}_{t=0}^{\infty}$ and prices $\{w_t, r_t, p_t\}_{t=0}^{\infty}$ such that i) consumers maximize utility subject to their intertemporal budget constraint taking prices as given; ii) the final good producers maximizes profits choosing labor and intermediate inputs; iii) intermediate good producers maximize net expected benefit by choosing a level of R&D expenditures and maximize profit choosing the price at which to sell invented goods, and iv) all markets clear.

I focus on steady-state equilibrium that is defined in the usual way as an equilibrium in which the capital-quality ratio $k_t = k \equiv \frac{K_t}{Q_t}$ is constant. The steady state for k_t corresponds to a balanced growth path for the original variables. A balanced growth path is defined as a path along which all variables grow at a constant rate and growth rates are equal across variables.

The equation for final output, (2.1), can be rewritten as $Y_t = \frac{1}{1-\gamma} k_t^{1-\gamma} Q_t$, that gives $g_Y = (1-\gamma)g_k + \gamma g_Q$, then $g_Y = g_Q$. The final good is devoted to consumption and to investment in physical capital and research, $Y_t = C_t + K_{t+1} + \tilde{R}_{t+1}^{(agg)}$, where $C_t = C_{1,t} + C_{2,t}$. Using that $w_t = \frac{\gamma}{1-\gamma} k_t^{1-\gamma} Q_t$, (2.23) can be rewritten as $C_{1,t} = \frac{1}{1+\rho}\frac{\gamma}{1-\gamma} k_t^{1-\gamma} Q_t$. Substitution (2.20), (2.23) and expression for w_t into (2.22) gets $C_{2,t+1} = \frac{\rho}{1+\rho}\frac{\gamma}{1-\gamma} k_t^{1-\gamma}(R_{t+1} + (1-\delta))Q_t$. Since along BGP $R_t = R$ (see (2.9)), clearly that $g_{c_1} = g_{c_2} = g_C = g_Q$.

Let Z_t denote $Z_t = K_{t+1} + \tilde{R}^{(agg)}_{t+1}$. From the resource constraint I have that

$$g_Y = g_{C+Z} = g_C\left(\frac{C_t}{C_t + Z_t}\right) + g_Z\left(\frac{Z_t}{C_t + Z_t}\right) = g_C + (g_Z - g_C)\left(\frac{Z_t}{C_t + Z_t}\right). \quad (2.28)$$

Since $g_Q = g_K = g_{\tilde{R}^{(agg)}}$,[4] $g_Y = g_Q = g_Z$. As g_Y, g_C, and g_Z are constant along BGP and (2.28) holds for all t, $\frac{Z_t}{C_t+Z_t}$ is also constant along BGP, therefore C_t and Z_t grow at the same rate. Finally, I have $g^* = g_Y = g_Q = g_C = g_K = g_{\tilde{R}^{(agg)}}$.

Thus, the growth rate of the economy is

$$g_t = g_Q = \frac{Q_t - Q_{t-1}}{Q_{t-1}}.$$

Average total quality is

$$Q_t = \int_0^1 \big(\phi(\tilde{R})\mu q_{t-1} + (1 - \phi(\tilde{R}))q_{t-1}\big)dj = Q_{t-1} + \phi(\tilde{R})(\mu - 1)Q_{t-1}$$

that implies:

$$g_t = \phi(\tilde{R}_t)(\mu - 1) = \frac{\zeta^2\gamma\mu(\mu - 1)}{2}k_t^{1-\gamma}. \quad (2.29)$$

The BGP growth rate is

$$g = \phi(\tilde{R})(\mu - 1) = \frac{\zeta^2\gamma\mu(\mu - 1)}{2}k^{1-\gamma}, \quad (2.30)$$

where k is steady-state capital intensity. Clearly, the result will be similar for the case of one-period patent length.

The law of motion for economy for the case of infinite patent length is derived as follows. Using (2.29), K_{t+1} can be expressed as $K_{t+1} = \frac{K_{t+1}}{Q_{t+1}}Q_{t+1} = k_{t+1}(g_{t+1}+1)Q_t = \big(\frac{\zeta^2\gamma\mu(\mu-1)}{2}k_{t+1}^{2-\gamma} + k_{t+1}\big)Q_t$.

[4]See (2.9) and recall $\tilde{R}^{(agg)}_{t+1} = \tilde{R}_{t+1}Q_t$.

Then (2.24) can be rewritten as

$$\underbrace{\frac{\zeta^2\gamma\mu(\mu-1)}{2}k_{t+1}^{2-\gamma}+k_{t+1}}_{K_{t+1}/Q_t}+\underbrace{\frac{(\zeta\gamma\mu)^2}{4}k_{t+1}^{2(1-\gamma)}}_{\tilde{R}_{t+1}^{agg}/Q_t}+\underbrace{\frac{\gamma}{2}k_{t+1}^{1-\gamma}-\frac{(\zeta\gamma)^2\mu}{4}k_{t+1}^{2(1-\gamma)}}_{V_t^{old(agg)}/Q_t}=\underbrace{\frac{\gamma}{1-\gamma}\frac{\rho}{1+\rho}k_t^{1-\gamma}}_{\frac{\rho}{1+\rho}w_t/Q_t}. \tag{2.31}$$

By simplifying the previous equation I have:

$$\frac{(\zeta\gamma\mu)^2}{4}(1-\frac{1}{\mu})k_{t+1}^{2(1-\gamma)}+\frac{\zeta^2\gamma\mu(\mu-1)}{2}k_{t+1}^{2-\gamma}+\frac{\gamma}{2}k_{t+1}^{1-\gamma}+k_{t+1}=\frac{\gamma}{1-\gamma}\frac{\rho}{1+\rho}k_t^{1-\gamma}, \tag{2.32}$$

which is the fundamental law of motion of my economy for the case of infinite patent length.

By modifying (2.27) I obtain the law of motion of the economy for the case of one-period patent protection:

$$\underbrace{\frac{\zeta^2\gamma\mu(\mu-1)}{2}k_{t+1}^{2-\gamma}+k_{t+1}}_{K_{t+1}/Q_t}+\underbrace{\frac{(\zeta\gamma\mu)^2}{4}k_{t+1}^{2(1-\gamma)}}_{\tilde{R}_{t+1}^{agg}/Q_t}=\underbrace{\frac{\gamma}{1-\gamma}\frac{\rho}{1+\rho}k_t^{1-\gamma}}_{\frac{\rho}{1+\rho}w_t/Q_t}. \tag{2.33}$$

Let me now verify that there exists a unique steady state. The following proposition states the result.

Let Ψ be the parameter space: $\psi = \{\gamma, \delta, \zeta, \mu, \rho\}$, and $\psi \in \Psi$.

Proposition 2.1 *(i) Under infinite patent length, there exists a unique steady-state equilibrium as long as* $\psi \in \{\Psi_1, \Psi_2\}$ *with (disjoint) subspaces of the parameter space*

$\Psi_1, \Psi_2 \subset \Psi$*:*

$$\Psi_1 = \{\psi \in \Psi \mid \frac{1-\rho}{1+\rho} < \gamma \leq \frac{1}{2}\},$$
$$\Psi_2 = \{\psi \in \Psi \mid \gamma > \frac{1}{2} \text{ and } 2\rho - (1-\gamma)(1+\rho)\big(1 + (2\gamma - 1)\zeta^2\mu(\mu - 1)\big) \geq 0\}. \tag{2.34}$$

(ii) Under one period patent length there exists a unique steady-state equilibrium whenever $\psi \in \{\Psi_3, \Psi_4\}$ *with (disjoint) subspaces of the parameter space* $\Psi_3, \Psi_4 \subset \Psi$*:*

$$\Psi_3 = \{\psi \in \Psi \mid \gamma \leq \frac{1}{2}\},$$
$$\Psi_4 = \{\psi \in \Psi \mid \gamma > \frac{1}{2} \text{ and}$$
$$2\rho - (1-\gamma)(1+\rho)(2\gamma - 1)\zeta^2\mu^2 \geq 0 \text{ and } \gamma - 2\gamma^2\mu + 2\mu - 2 \geq 0\}. \tag{2.35}$$

Proof The proof is in the Appendix.

Assumption 1 *Based on Proposition 2.1, I assume hereafter that*

(i) Under infinite patent length ($T = \infty$*)* $\psi \in \{\Psi_1, \Psi_2\}$ *with* Ψ_1 *and* Ψ_2 *(*$\Psi_1, \Psi_2 \subset \Psi$*)are defined in (2.34).*

(ii) Under one period patent length ($T = 1$*)* $\psi \in \{\Psi_3, \Psi_4\}$ *with* Ψ_3 *and* Ψ_4 *(*$\Psi_3, \Psi_4 \subset \Psi$*)are defined in (2.35).*

The law of motion of the economy under infinite patent length is determined by (2.31). The proof of Proposition 2.1 demonstrates that the system determined by (2.31) has a unique steady state with positive k if the parameters of the model satisfy Assumption 1. Given k, the growth rate g is uniquely determined by (2.30).

Proposition 2.2 *(i) R&D investment is higher under one-period patent protection than under infinite patent life ("patent crowding-out effect"). (ii) Under one-period patent length growth is higher than under infinite protection.*

Proof The proof is in the Appendix.

Part (i) of Proposition 2.2 establishes that a short patent duration enhances innovation. This can be explained by the following. Under one-period patent life the crowding-out effect does not occur, since investment of the young is allocated to physical capital and research, whereas with infinite patent, the young allocate their investment in physical capital, research and to purchasing of patents from the older generation, thereby reducing the amount of investment in research. In its turn, R&D expenditures spur growth, so the growth rate with one-period protection is higher than that under infinite patent life, as stated in part (ii) of Proposition 2.2.

2.3 Patent breadth

2.3.1 Patent protection and growth

In Section 2.2 the existence of uniquely determined BGP has been proved for the case when patentee can charge unconstrained monopolistic price. In this section I assume that government controls the degree of patent protection by using patent breadth along with patent length. Here I follow Goh and Olivier (2002)[5] to introduce policy variable η that representing of patent breadth. Patent breadth determines how high a price markup each monopolist can charge. The patentee maximizes profit by charging price $p_t^{\eta}(j) = \eta p_t(j) = \frac{\eta R_t q_t(j)}{1-\gamma}$ with $\eta \in (1-\gamma, 1)$. In the case of a narrow patent breadth, the maximum price that the patentee can charge coincides with marginal cost $R_t q_t(j)$ (so that $\eta = 1-\gamma$), while a broader breadth of patents raises the maximum price that the patentee can charge: the unconstrained monopolistic price in this case is $p_t(j) = \frac{R_t q_t(j)}{1-\gamma}$ (when $\eta = 1$).

[5]Similar modelling approach for patent breadth protection was used (among others) by Iwaisako and Futagami (2013), and Futagami and Iwaisako (2007), Chu et al. (2016).

Under patent breadth policy, the equilibrium quantity of the intermediate good (2.2) modifies to:

$$x_t^\eta(j) \equiv x_t^\eta = k_t = \left(\frac{1-\gamma}{\eta R_t}\right)^{\frac{1}{\gamma}}, \tag{2.36}$$

whereas the monopolist's profit (2.6) becomes:

$$\pi_t^\eta(j) = (\gamma+\eta-1)\frac{1}{\eta}k_t^{1-\gamma}q_t(j) = \theta k_t^{1-\gamma}q_t(j), \tag{2.37}$$

where $\theta \equiv (\gamma+\eta-1)\frac{1}{\eta}$. Accordingly, equilibrium quality-adjusted level of research and research intensity modify to:

$$\tilde{R}_t^\eta(j) \equiv \tilde{R}_t^\eta = \frac{(\zeta\theta\mu)^2}{4}k_t^{2(1-\gamma)}, \tag{2.38}$$

$$\phi(\tilde{R}^\eta(j)) \equiv \phi(\tilde{R}^\eta) = \frac{\zeta^2\theta\mu}{2}k_t^{1-\gamma}. \tag{2.39}$$

The growth rate is now expressed as[6]

$$g_t^\eta = \frac{\zeta^2\theta\mu(\mu-1)}{2}k_t^{1-\gamma}. \tag{2.40}$$

Finally, I obtain the law of motion of the economy under imperfect breadth protection with infinite patent length:

$$\frac{(\zeta\theta\mu)^2}{4}(1-\frac{1}{\mu})k_{t+1}^{2(1-\gamma)} + \frac{\zeta^2\theta\mu(\mu-1)}{2}k_{t+1}^{2-\gamma} + \frac{\theta}{2}k_{t+1}^{1-\gamma} + k_{t+1} = \frac{\gamma}{1-\gamma}\frac{\rho}{1+\rho}k_t^{1-\gamma}, \tag{2.41}$$

[6]The BGP growth rate is given by:

$$g^\eta = \frac{\zeta^2\theta\mu(\mu-1)}{2}k^{1-\gamma}.$$

which is analogous to (2.31) for the complete breadth protection, i.e., $\eta = 1$; and, under one period patent length I get the analogue of (2.33) which is:

$$\frac{\zeta^2\theta\mu(\mu-1)}{2}k_{t+1}^{2-\gamma} + k_{t+1} + \frac{(\zeta\theta\mu)^2}{4}k_{t+1}^{2(1-\gamma)} = \frac{\gamma}{1-\gamma}\frac{\rho}{1+\rho}k_t^{1-\gamma}. \tag{2.42}$$

As in Section 2.2.3, I now establish the existence of a unique steady state for two cases, $T = \infty$ and $T = 1$, in the following proposition.

Proposition 2.3 *(i) Under infinite patent length there exists a unique steady-state equilibrium as long as $\psi \in \{\tilde{\Psi}_1, \tilde{\Psi}_2\}$ with (disjoint) subspaces of the parameter space $\tilde{\Psi}_1, \tilde{\Psi}_2 \subset \Psi$:*

$$\tilde{\Psi}_1 = \{\psi \in \Psi \mid \gamma \leq \frac{1}{2} \textit{ and } 2\rho\gamma - (1-\gamma)(1+\rho)(1-\frac{1-\gamma}{\eta}) > 0\}, \tag{2.43}$$

$$\tilde{\Psi}_2 = \{\psi \in \Psi \mid \gamma > \frac{1}{2} \textit{ and }$$
$$2\rho\gamma^2 - (1-\gamma)(1+\rho)\big(1-\frac{1-\gamma}{\eta}\big)\big(1+(2\gamma-1)\zeta^2\mu(\mu-1)\big)\big(1-\frac{1-\gamma}{\eta}\big) \geq 0\}. \tag{2.44}$$

(ii) Under one period patent length there exists a unique steady-state equilibrium whenever $\psi \in \Psi_3, \tilde{\Psi}_4\}$ with (disjoint) subspaces of the parameter space $\Psi_3, \tilde{\Psi}_4 \subset \Psi$:

$$\Psi_3 = \{\psi \in \Psi \mid \gamma \leq \frac{1}{2}\}, \tag{2.45}$$

$$\tilde{\Psi}_4 = \{\psi \in \Psi \mid \gamma > \frac{1}{2} \textit{ and } 2\rho\gamma^2 - (1-\gamma)(1+\rho)(2\gamma-1)\zeta^2\mu^2\big(1-\frac{1-\gamma}{\eta}\big)^2 \geq 0$$
$$\textit{and } (2-\gamma)(\mu-1) - (2\gamma-1)\mu\big(1-\frac{1-\gamma}{\eta}\big) \geq 0\}. \tag{2.46}$$

Proof The proof is in the Appendix.

Proposition 2.3 is the direct analogue of Proposition 2.1.

Assumption 2 *Based on Proposition 2.3, I assume hereafter that*

(i) Under infinite patent length ($T = \infty$) $\psi \in \{\tilde{\Psi}_1, \tilde{\Psi}_2\}$ with $\tilde{\Psi}_1$ and $\tilde{\Psi}_2$ ($\tilde{\Psi}_1, \tilde{\Psi}_2 \subset \Psi$) are defined in (2.43) and (2.44).

(ii) Under one period patent length ($T = 1$) $\psi \in \Psi_3, \tilde{\Psi}_4\}$ with Ψ_3 and $\tilde{\Psi}_4$ ($\Psi_3, \tilde{\Psi}_4 \subset \Psi$) are defined in (2.45) and (2.46).

Having conditions for the existence of steady-state allows me to compare R&D efforts and the growth rates with incomplete breadth protection for two cases ($T = \infty$ and $T = 1$).

Proposition 2.4 *Under infinite and one-period patent length (i) For sufficiently low steady state k, $k \in (0, \epsilon)$ with $\epsilon > 0$ sufficiently small real number, weakening patent breadth stimulates investment in research and enhances growth. (ii) If $k \in (\epsilon, \infty)$, the opposite results apply. In particular, incomplete breadth protection lowers research investment and reduces the growth rate.*

Proof The proof is in the Appendix.

The economic intuition behind results in Proposition 2.4 is the following. Relaxing patent protection has two opposite effect on growth. On the one hand, if the stock of knowledge (Q) significantly higher than the stock of physical capital (K) ($k \in (0, \epsilon)$[7]), incomplete breadth reduces the price of patented intermediate goods, p^η, which leads to increased demand for the intermediate product used in the final goods sector, x^η, and, as a result, to increased output, Y. In its turn, this stimulates aggregate investment including investment in R&D. Increasing research investment promotes growth. On the other hand, if the stock of knowledge (Q) much lower than the stock of physical capital (K) ($k \in (\epsilon, \infty)$), an imperfect IPR lowers the price of patented intermediate goods, which increases the amount of of intermediate goods used in the final sector, that, in its turns, direct investment towards physical capital by reducing research investment.

[7]Recall that $k = K/Q$

The next proposition compares research investment and growth for two different patent durations.

Proposition 2.5 *(i) R&D investment is higher under one period patent protection than under infinite patent length. (ii) Under one period patent length growth rate is higher than under infinite patent length.*

Proof The proof is in the Appendix.

Proposition 2.5 is the direct analogue of Proposition 2.2. This proposition demonstrates the presence of the crowding-out effect for the case with incomplete breadth protection, and also shows that higher growth under incomplete patent breadth is achieved with short patent duration with the same reasoning as in Proposition 2.2.

2.4 Conclusion

In this paper I explore how patent protection affects economic growth in an OLG quality-ladder endogenous growth model with capital accumulation. I focus on the two extreme cases of patent length, one-period patent protection and infinite patent length, and reveal the existence of the "crowding-out effect" for an OLG Schumpeterian model which consists in reallocation of investment resources away from R&D towards purchase of patents from the older generation. This effect does not occur in the one-period patent life case that leads to higher growth rate under one-period patent length. I show that relationship between patent breadth and economic growth is nonmonotonic.

2.5 Appendix

Proof of Proposition 2.1

(i) **Case** $T = \infty$ The law of motion of the economy under infinite patent length is determined by (2.31):

$$\frac{\zeta^2\gamma\mu(\mu-1)}{2}k_{t+1}^{2-\gamma} + k_{t+1} + \frac{(\zeta\gamma\mu)^2}{4}k_{t+1}^{2(1-\gamma)} + \frac{\gamma}{2}k_{t+1}^{1-\gamma} - \frac{(\zeta\gamma)^2\mu}{4}k_{t+1}^{2(1-\gamma)} = \frac{\gamma}{1-\gamma}\frac{\rho}{1+\rho}k_t^{1-\gamma}.$$

Let $A \equiv \frac{(\zeta\gamma\mu)^2}{4}$, $B \equiv \frac{\zeta^2\gamma\mu(\mu-1)}{2}$, $C \equiv \frac{\gamma}{2}$, $D \equiv \frac{\gamma}{1-\gamma}\frac{\rho}{1+\rho}$. Then last equation can be rewritten as

$$A(1-\frac{1}{\mu})k_{t+1}^{2(1-\gamma)} + Bk_{t+1}^{2-\gamma} + Ck_{t+1}^{1-\gamma} + k_{t+1} = Dk_t^{1-\gamma}. \tag{2.47}$$

I show that the system characterized by (2.47) has a unique steady state with positive k. Plugging in $k_t = k_{t+1} = k$ into (2.47), I have:

$$h_{T=\infty}(k) \equiv A(1-\frac{1}{\mu})k^{2(1-\gamma)} + Bk^{2-\gamma} + (C-D)k^{1-\gamma} + k = 0. \tag{2.48}$$

If $C - D \geq 0$ then $h_{T=\infty}(k)$ is strictly increasing, and taking into account that $h_{T=\infty}(0) = 0$, the equation (2.48) does not have non-zero solution.

Consider the case $C - D < 0$. $h_{T=\infty}(0) = 0$, and $\lim_{k\to\infty} h_{T=\infty}(k) = \infty$, since $k^{2-\gamma}$ and $k^{2(1-\gamma)}$ grows faster than $k^{1-\gamma}$. Differentiating $h_{T=\infty}(k)$ with respect to k gives

$$\frac{\partial h_{T=\infty}}{\partial k} = (2-2\gamma)A(1-\frac{1}{\mu})k^{1-2\gamma} + (2-\gamma)Bk^{1-\gamma} + (1-\gamma)(C-D)k^{-\gamma} + 1. \tag{2.49}$$

For sufficiently low steady-state k ($k \to 0$), $k^{-\gamma}$ dominates the functions $k^{1-\gamma}$ and $k^{1-2\gamma}$,[8] therefore $\partial h_{T=\infty}/\partial k < 0$; while for $k \to \infty$, $k^{1-\gamma}$ and $k^{1-2\gamma}$ grows faster than $k^{-\gamma}$, so $\partial h_{T=\infty}/\partial k > 0$. The second derivative of $h_{T=\infty}(k)$ with respect to k is given by

$$\frac{\partial^2 h_{T=\infty}}{\partial k^2} = (2-2\gamma)(1-2\gamma)A(1-\frac{1}{\mu})k^{-2\gamma} + (1-\gamma)(2-\gamma)Bk^{-\gamma} - \gamma(1-\gamma)(C-D)k^{-\gamma-1}. \tag{2.50}$$

There are two cases, $\gamma \leq \frac{1}{2}$ and $\gamma > \frac{1}{2}$.

For $\gamma \leq \frac{1}{2}$, all terms in (2.50) are positive, so that $\partial^2 h_{T=\infty}/\partial k^2 > 0$, therefore, $\partial h_{T=\infty}/\partial k < 0$ is not feasible once $\partial h_{T=\infty}/\partial k$ has become positive, and, as a result, there is no second steady-state. Combining facts about signs of first and second derivative, the curve $h_{T=\infty}(k)$ must cross the horizontal axis from below, i.e., at this point $\partial h_{T=\infty}/\partial k > 0$ must hold, and this point is only one crossing point. Hence, if $\gamma \leq \frac{1}{2}$[9] and $C - D < 0$,[10] that is, $\frac{1-\rho}{1+\rho} < \gamma \leq \frac{1}{2}$, there exists a unique (non-zero) steady state for the system in (2.47).

In the case where $\gamma > \frac{1}{2}$, let me find conditions under which the second derivative (2.50) is positive. First, consider $k \in (0, 1]$. Since the second term in (2.50) is positive, to show that the second derivative is positive, $\partial^2 h_{T=\infty}/\partial k^2 > 0$, it suffices to have sum of the first and third terms in (2.50) is non-negative, i.e., $(2-2\gamma)(1-2\gamma)A(1-\frac{1}{\mu})k^{-2\gamma} - \gamma(1-\gamma)(C-D)k^{-\gamma-1} \geq 0$. For $k \in (0, 1]$, $k^{-\gamma-1} \geq k^{-2\gamma}$, thus, this sum will be positive if $-\gamma(1-\gamma)(C-D) \geq (2-2\gamma)(2\gamma-1)A(1-\frac{1}{\mu})$. Substituting the value of coefficients A, C and D and rearranging it I have that for $k \in (0, 1]$, $\partial^2 h_{T=\infty}/\partial k^2 > 0$ whenever $2\rho - (1-\gamma)(1+\rho)\big(1 + (2\gamma-1)\zeta^2\mu(\mu-1)\big) \geq 0$.

[8] For $k \to 0$ $(2-2\gamma)A(1-\frac{1}{\mu})k^{1-2\gamma} + (2-\gamma)Bk^{1-\gamma} + (1-\gamma)(C-D)k^{-\gamma} + 1 = (1-\gamma)(C-D)k^{-\gamma} + o(k^{-\gamma})$.

[9] This implies that $1-\gamma \geq \frac{1}{2}$. A capital share this large is reasonable if capital is interpreted in a broad sense (Mankiw et al. (1992), Barro and Sala-i Martin (2004)).

[10] Substituting the value of coefficients C and D into $C - D < 0$ yields $\frac{1-\rho}{1+\rho} < \gamma$.

Second, I have to determine the sign of the second derivative (2.50) when $k \in [1, \infty)$. Since the third term in (2.50) is positive, to have $\partial^2 h_{T=\infty}/\partial k^2 > 0$ it suffices to show that sum of first two terms is non-negative, i.e., $(2-2\gamma)(1-2\gamma)A(1-\frac{1}{\mu})k^{-2\gamma} + (1-\gamma)(2-\gamma)Bk^{-\gamma} \geq 0$. Note that for $k \in [1, \infty)$, $k^{-\gamma} \geq k^{-2\gamma}$, then from $(1-\gamma)(2-\gamma)B \geq (2-2\gamma)(2\gamma-1)A(1-\frac{1}{\mu})$[11] follows immediately that the sum is non-negative. Hence, when $k \in [1, \infty)$ the second derivative, $\partial^2 h_{T=\infty}/\partial k^2$, is positive.

Define two (disjoint) subspaces of the parameter space $\Psi_1, \Psi_2 \subset \Psi$:

$$\Psi_1 = \{\psi \in \Psi \mid \frac{1-\rho}{1+\rho} < \gamma \leq \frac{1}{2}\},$$
$$\Psi_2 = \{\psi \in \Psi \mid \gamma > \frac{1}{2} \text{ and } 2\rho - (1-\gamma)(1+\rho)\big(1 + (2\gamma-1)\zeta^2\mu(\mu-1)\big) \geq 0\}.$$

To summarize, under infinite patent length there exists a unique steady-state equilibrium as long as $\psi \in \{\Psi_1, \Psi_2\}$.

(ii) **Case** $T = 1$ The evolution of k_t in case of one-period patent length is given by (2.33):

$$\frac{\zeta^2\gamma\mu(\mu-1)}{2}k_{t+1}^{2-\gamma} + k_{t+1} + \frac{(\zeta\gamma\mu)^2}{4}k_{t+1}^{2(1-\gamma)} = \frac{\gamma}{1-\gamma}\frac{\rho}{1+\rho}k_t^{1-\gamma}.$$

Similar to the case of infinite patent length this equation can be rewritten as

$$Ak_{t+1}^{2(1-\gamma)} + Bk_{t+1}^{2-\gamma} + k_{t+1} = Dk_t^{1-\gamma}. \tag{2.51}$$

The steady state is obtained as the solution to

$$h_{T=1}(k) \equiv Ak^{2(1-\gamma)} + Bk^{2-\gamma} - Dk^{1-\gamma} + k = 0. \tag{2.52}$$

[11] Substituting the value of coefficients A and B in this inequality and rearranging it yields $\gamma^2 \leq 1$.

Equations (2.52) and (2.48) differ only by the term $Ck^{1-\gamma}$ in (2.48) and factor $1-\frac{1}{\mu}$ of the first term in (2.48), so the proof steps for the current case $T=1$ will be similar to the case $T=\infty$. Taking the derivative of $h_{T=1}(k)$ with respect to k, I get

$$\frac{\partial h_{T=1}}{\partial k} = (2-2\gamma)Ak^{1-2\gamma} + (2-\gamma)Bk^{1-\gamma} - (1-\gamma)Dk^{-\gamma} + 1. \tag{2.53}$$

Note that for $k \to 0$, $\partial h_{T=1}/\partial k < 0$, while for $k \to \infty$, $\partial h_{T=1}/\partial k > 0$. The second derivative of $h_{T=1}(k)$ with respect to k is

$$\frac{\partial^2 h_{T=1}}{\partial k^2} = (2-2\gamma)(1-2\gamma)Ak^{-2\gamma} + (1-\gamma)(2-\gamma)Bk^{-\gamma} + \gamma(1-\gamma)Dk^{-\gamma-1}. \tag{2.54}$$

As in case $T=\infty$, there are two possible cases, $\gamma \leq \frac{1}{2}$ and $\gamma > \frac{1}{2}$.

For $\gamma \leq \frac{1}{2}$, all terms in (2.54) are positive, so that $\partial^2 h_{T=1}/\partial k^2 > 0$. Thus, $h_{T=1}(k)$ must cross the horizontal axis from below, i.e., at this point $\partial h_{T=1}/\partial k > 0$ must hold, and this point is only one crossing point. Hence, for $\gamma \leq \frac{1}{2}$ there exists a unique (non-zero) steady state for the system in (2.51).

For the case $\gamma > \frac{1}{2}$, first consider $k \in (0,1]$. To show that the second derivative is non-negative, $\partial^2 h_{T=1}/\partial k^2 > 0$, it suffices to have sum of the first and third terms in (2.54) non-negative, i.e., $(2-2\gamma)(1-2\gamma)Ak^{-2\gamma} + \gamma(1-\gamma)Dk^{-\gamma-1} \geq 0$. For $k \in (0,1]$, $k^{-\gamma-1} \geq k^{-2\gamma}$, therefore, this sum will be positive if $\gamma(1-\gamma)D \geq (2-2\gamma)(2\gamma-1)A$. Substituting the value of coefficients A and D into this inequality and rearranging yields that $\partial^2 h_{T=1}/\partial k^2 > 0$ as long as $2\rho - (1-\gamma)(1+\rho)(2\gamma-1)\zeta^2\mu^2 \geq 0$.

Next, let me examine the case when $\gamma > \frac{1}{2}$ and $k \in [1,\infty)$. To have $\partial^2 h_{T=\infty}/\partial k^2 > 0$, it suffices to have the sum of the first two term be non-negative, i.e., $(2-2\gamma)(1-2\gamma)Ak^{-2\gamma} + (1-\gamma)(2-\gamma)Bk^{-\gamma} \geq 0$. For $k \in [1,\infty)$, $k^{-\gamma} \geq k^{-2\gamma}$, so that $\partial^2 h_{T=\infty}/\partial k^2 > 0$ follows from $(1-\gamma)(2-\gamma)B \geq (2-2\gamma)(2\gamma-1)A$, which, after substituting the value of coefficients A and B and rearranging, becomes $\gamma - 2\gamma^2\mu + 2\mu - 2 \geq 0$.

Define two (disjoint) subspaces of the parameter space $\Psi_3, \Psi_4 \subset \Psi$:

$$\Psi_3 = \{\psi \in \Psi \mid \gamma \leq \frac{1}{2}\},$$
$$\Psi_4 = \{\psi \in \Psi \mid \gamma > \frac{1}{2} \text{ and}$$
$$2\rho - (1-\gamma)(1+\rho)(2\gamma-1)\zeta^2\mu^2 \geq 0 \text{ and } \gamma - 2\gamma^2\mu + 2\mu - 2 \geq 0\}.$$

Thus, under one period protection there exists a unique steady-state equilibrium as long as $\psi \in \{\Psi_3, \Psi_4\}$.

Proof of Proposition 2.2

Denote the steady state in the case of one-period protection by k_1 and the steady-state in the case of infinite patent protection by k_∞, i.e., k_1 is the intersection point of $h_{T=1}(k)$ and horizontal axis, while k_∞ is intersection point of $h_{T=\infty}(k)$ and horizontal axis. It has been shown in Proposition 2.1 that the function $h_{T=1}(k)$ (and $h_{T=\infty}(k)$) crosses the horizontal axis from below and this is only one crossing point. Therefore, for any $\bar{k} < k_1$, $h_{T=1}(\bar{k}) < h_{T=1}(k_1) = 0$. So $k_\infty < k_1$ as long as $h_{T=1}(k_\infty) < h_{T=1}(k_1) = 0$. Since

$$h_{T=1}(k_\infty) = Ak_\infty^{2(1-\gamma)} + Bk_\infty^{2-\gamma} - Dk_\infty^{1-\gamma} + k_\infty, \tag{2.55}$$

I have to show that $h_{T=1}(k_\infty) = Ak_\infty^{2(1-\gamma)} + Bk_\infty^{2-\gamma} - Dk_\infty^{1-\gamma} + k_\infty < 0$. But, from (2.48), $h_{T=\infty}(k_\infty) = A(1-\frac{1}{\mu})k_\infty^{2(1-\gamma)} + Bk_\infty^{2-\gamma} + (C-D)k_\infty^{1-\gamma} + k_\infty = 0$. Therefore, $h_{T=1}(k_\infty) = h_{T=\infty}(k_\infty) + \frac{1}{\mu}Ak_\infty^{2(1-\gamma)} - Ck_\infty^{1-\gamma} = \frac{1}{\mu}Ak_\infty^{2(1-\gamma)} - Ck_\infty^{1-\gamma}$ has to be negative, i.e., $\frac{1}{\mu}Ak_\infty^{2(1-\gamma)} - Ck_\infty^{1-\gamma} < 0$.

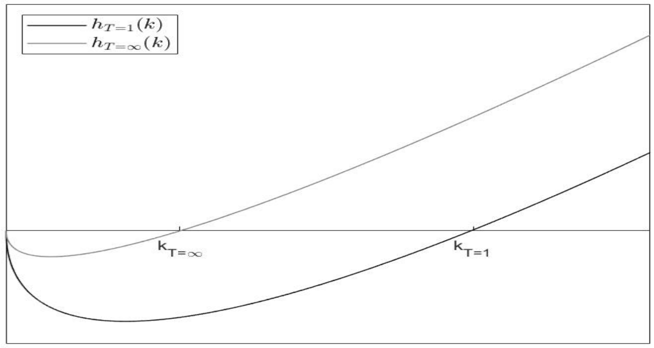

Figure 2.2: Comparison of steady-state values of k under infinite patent length and one-period patent protection ($k_{T=1} > k_{T=\infty}$)

Modifying last inequality and substituting into it the values of coefficients A and C, I have:

$$k_\infty < \left(\frac{C}{A}\mu\right)^{\frac{1}{1-\gamma}} = \left(\frac{2}{\zeta^2\gamma\mu}\right)^{\frac{1}{1-\gamma}}, \tag{2.56}$$

that is, $k_\infty < k_1$ as long as $k_\infty < \left(\frac{2}{\zeta^2\gamma\mu}\right)^{\frac{1}{1-\gamma}}$, but it is the assumption[12] that ensures that innovation intensity is between 0 and 1. Thus, it has been shown that $k_\infty < k_1$. Figure 2.2 illustrates the determination and comparison of steady-state values of k for cases $T = \infty$ and $T = 1$.

(i) Since R&D investment is strictly increasing in k (see (2.15)[13]), $\tilde{R}^{(agg)}_{T=1} > \tilde{R}^{(agg)}_{T=\infty}$

(ii) The growth rate is strictly increasing in k (see (2.30)) that implies that $g_{T=1} > g_{T=\infty}$.

[12] Assumption is made immediately after equation (2.16). Notice that assumption is expressed in terms of k which is an endogenous variable. However, it is easy to find sets of parameters for which the assumption is satisfied.

[13] The quality-adjusted level of research in BGP is given by $\tilde{R} = \frac{(\zeta\gamma\mu)^2}{4}k^{2(1-\gamma)}$

Proof of Proposition 2.3

(i) **Case** $T = \infty$ The law of motion of the economy under infinite patent length is determined by (2.41):

$$\frac{(\zeta\theta\mu)^2}{4}(1-\frac{1}{\mu})k_{t+1}^{2(1-\gamma)} + \frac{\zeta^2\theta\mu(\mu-1)}{2}k_{t+1}^{2-\gamma} + \frac{\theta}{2}k_{t+1}^{1-\gamma} + k_{t+1} = \frac{\gamma}{1-\gamma}\frac{\rho}{1+\rho}k_t^{1-\gamma}.$$

Let $A^\eta \equiv \frac{(\zeta\theta\mu)^2}{4}$, $B^\eta \equiv \frac{\zeta^2\theta\mu(\mu-1)}{2}$, $C^\eta \equiv \frac{\theta}{2}$, $D \equiv \frac{\gamma}{1-\gamma}\frac{\rho}{1+\rho}$. Now I can rewrite dynamic of k as:

$$A^\eta(1-\frac{1}{\mu})k_{t+1}^{2(1-\gamma)} + B^\eta k_{t+1}^{2-\gamma} + C^\eta k_{t+1}^{1-\gamma} + k_{t+1} = Dk_t^{1-\gamma}.$$

The steady state is determined as the solution to

$$h_{T=\infty}^\eta(k) \equiv A^\eta(1-\frac{1}{\mu})k^{2(1-\gamma)} + B^\eta k^{2-\gamma} + (C^\eta - D)k^{1-\gamma} + k = 0. \tag{2.57}$$

Note that equation (2.48) from the proof of Proposition 2.1, which is the case $\eta = 1$, and (2.57) differ only by the coefficients A^η, B^η and C^η, in particular, in the current case, $\eta \in (1-\gamma, 1)$, γ is replaced by θ, where $\theta \equiv (\gamma+\eta-1)\frac{1}{\eta}$.

The first derivative of $h_{T=\infty}^\eta(k)$ with respect to k is

$$\frac{\partial h_{T=\infty}^\eta}{\partial k} = (2-2\gamma)A^\eta(1-\frac{1}{\mu})k^{1-2\gamma} + (2-\gamma)B^\eta k^{1-\gamma} + (1-\gamma)(C^\eta - D)k^{-\gamma} + 1. \tag{2.58}$$

By following the same reasoning as for $h_{T=\infty}(k)$ in Proposition 2.1, for sufficiently low steady-state k ($k \to 0$), $\partial h_{T=\infty}/\partial k < 0$, while for $k \to \infty$, $\partial h_{T=\infty}/\partial k > 0$. The second derivative of $h^{\eta}_{T=\infty}(k)$ with respect to k is given by

$$\frac{\partial^2 h^{\eta}_{T=\infty}}{\partial k^2} = (2-2\gamma)(1-2\gamma)A^{\eta}(1-\frac{1}{\mu})k^{-2\gamma} + (1-\gamma)(2-\gamma)B^{\eta}k^{-\gamma} - \gamma(1-\gamma)(C^{\eta}-D)k^{-\gamma-1}. \tag{2.59}$$

As in the proof of Proposition 2.1 (i), I have to consider two cases, namely $\gamma \leq \frac{1}{2}$ and $\gamma > \frac{1}{2}$.

Case $\gamma \leq \frac{1}{2}$. It has been shown in the proof of Proposition 2.1 (i), a unique steady state equilibrium exists if $\gamma \leq \frac{1}{2}$ and $C - D < 0$. Taking into account the observation about coefficients in both cases, the second condition under which unique steady state equilibrium exists can be written as $C^{\eta} - D < 0$. Substituting the value of coefficients, I get that under infinite patent length there exists a unique steady state equilibrium as long as $\gamma \leq \frac{1}{2}$ and $2\rho\gamma - (1-\gamma)(1+\rho)(1-\frac{1-\gamma}{\eta}) > 0$.

Case $\gamma > \frac{1}{2}$. Analogously to the proof of Proposition 2.1 (i), consider two cases: $k \in (0,1]$ and $k \in [1,\infty)$.

The same reasoning as in Proposition 2.1 (i) can be applied to determine conditions under which the second derivative is non-negative when $k \in (0,1]$. The condition $-\gamma(1-\gamma)(C^{\eta}-D)k^{-\gamma-1} \geq (2-2\gamma)(2\gamma-1)A^{\eta}(1-\frac{1}{\mu})$ guarantees that $\partial^2 h^{\eta}_{T=\infty}/\partial k^2 > 0$. Substituting the value of coefficients A^{η}, C^{η} and D into this inequality and rearranging it yields that $\partial^2 h^{\eta}_{T=\infty}/\partial k^2 > 0$ as long as $2\rho\gamma^2 - (1-\gamma)(1+\rho)\big(1-\frac{1-\gamma}{\eta}\big)\big(1+(2\gamma-1)\zeta^2\mu(\mu-1)\big)\big(1-\frac{1-\gamma}{\eta}\big) \geq 0$.

By following the same reasoning as in Proposition 2.1 (i), to ensure that the sign of second derivative (2.59) is positive when $k \in [1,\infty)$, I have to show that $(1-\gamma)(2-\gamma)B^{\eta}k^{-\gamma} \geq (2-2\gamma)(2\gamma-1)A^{\eta}(1-\frac{1}{\mu})$. Substituting the value of coefficients A^{η} and B^{η} into this inequality and rearranging it gives $3\eta(\gamma-1)+2\gamma^2-3\gamma+1 < 0$. Noticing that the first term in this expression is negative, $3\eta(\gamma-1) < 0$ and recalling that I consider case $\gamma > \frac{1}{2}$ that implies $2\gamma^2 - 3\gamma + 1 < 0$, gives that $3\eta(\gamma-1) + 2\gamma^2 - 3\gamma + 1 < 0$ always holds. Thus, when $k \in [1,\infty)$, the second derivative, $\partial^2 h^{\eta}_{T=\infty}/\partial k^2$, is positive.

Define two (disjoint) subspaces of the parameter space $\tilde{\Psi}_1, \tilde{\Psi}_2 \subset \Psi$:

$$\tilde{\Psi}_1 = \{\psi \in \Psi \mid \gamma \leq \frac{1}{2} \text{ and } 2\rho\gamma - (1-\gamma)(1+\rho)(1 - \frac{1-\gamma}{\eta}) > 0\},$$
$$\tilde{\Psi}_2 = \{\psi \in \Psi \mid \gamma > \frac{1}{2} \text{ and}$$
$$2\rho\gamma^2 - (1-\gamma)(1+\rho)\big(1 - \frac{1-\gamma}{\eta}\big)\big(1 + (2\gamma - 1)\zeta^2\mu(\mu-1)\big)\big(1 - \frac{1-\gamma}{\eta}\big) \geq 0\}.$$

Thus, under infinite patent length there exists a unique steady-state equilibrium as long as $\psi \in \{\tilde{\Psi}_1, \tilde{\Psi}_2\}$.

(ii) **Case** $T = 1$ Similarly to the proof of (i), I can rewrite the law of motion of k from equation (2.42) as

$$A^\eta k_{t+1}^{2(1-\gamma)} + B^\eta k_{t+1}^{2-\gamma} + k_{t+1} = D k_t^{1-\gamma}.$$

The solution of following equation determines the steady state:

$$h_{T=1}^\eta(k) \equiv A^\eta k^{2(1-\gamma)} + B^\eta k^{2-\gamma} - D k^{1-\gamma} + k = 0. \tag{2.60}$$

It is clear that the proof for the current case $\eta \in (1-\gamma, 1)$ will be similar to the proof of Proposition 2.1 (ii) in which $\eta = 1$. Equation (2.60) and (2.52) differ only by the coefficients A^η and B^η. As in Proposition 2.1 (ii), I have to ensure that the second derivative of $h^{\eta}_{T=1}(k)$ with respect to k is positive.[14] There are two possible cases, $\gamma \leq \frac{1}{2}$ and $\gamma > \frac{1}{2}$.

For $\gamma \leq \frac{1}{2}$, there exists a unique (non-zero) steady state for the system (2.60).[15]

For the case $\gamma > \frac{1}{2}$ and $k \in (0, 1]$, the result will follow if I show $\gamma(1-\gamma)D \geq (2-2\gamma)(2\gamma-1)A^{\eta}$.[16] Substituting the value of coefficients A^{η}, and D into this inequality and rearranging it yields $\partial^2 h^{\eta}_{T=1} \partial k^2 > 0$ as long as $2\rho\gamma^2 - (1-\gamma)(1+\rho)(2\gamma-1)\zeta^2\mu^2\big(1-\frac{1-\gamma}{\eta}\big)^2 \geq 0$.

Consider the case $\gamma > \frac{1}{2}$ and $k \in [1, \infty)$. To have $\partial^2 h_{T=\infty}/\partial k^2 > 0$, it suffices to have $(1-\gamma)(2-\gamma)B^{\eta} \geq (2-2\gamma)(2\gamma-1)A^{\eta}$, which is after substituting the value of coefficients A^{η} and B^{η} and rearranging becomes $(2-\gamma)(\mu-1)-(2\gamma-1)\mu\big(1-\frac{1-\gamma}{\eta}\big) \geq 0$.

Define $\tilde{\Psi}_4 \subset \Psi$ as

$$\tilde{\Psi}_4 = \{\psi \in \Psi \mid \gamma > \frac{1}{2} \text{ and } 2\rho\gamma^2 - (1-\gamma)(1+\rho)(2\gamma-1)\zeta^2\mu^2\big(1-\frac{1-\gamma}{\eta}\big)^2 \geq 0$$
$$\text{and } (2-\gamma)(\mu-1) - (2\gamma-1)\mu\big(1-\frac{1-\gamma}{\eta}\big) \geq 0\}.$$

Thus, under one period protection there exists a unique steady-state equilibrium as long as $\psi \in \{\Psi_3, \tilde{\Psi}_4\}$.[17]

[14] Similarly to the proof of Proposition 2.1 (i), the first derivative of $h^{\eta}_{T=1}(k)$ with respect to k:

$$\frac{\partial h^{\eta}_{T=1}}{\partial k} = (2-2\gamma)A^{\eta}k^{1-2\gamma} + (2-\gamma)B^{\eta}k^{1-\gamma} - (1-\gamma)Dk^{-\gamma} + 1,$$

while the second derivative is given by

$$\frac{\partial^2 h^{\eta}_{T=1}}{\partial k^2} = (2-2\gamma)(1-2\gamma)A^{\eta}k^{-2\gamma} + (1-\gamma)(2-\gamma)B^{\eta}k^{-\gamma} + \gamma(1-\gamma)Dk^{-\gamma-1}.$$

[15] See the proof of Proposition 2.1 (ii).

[16] For reasoning, see the proof of Proposition 2.1 (ii).

[17] Ψ_3 has been defined in the proof of Proposition 2.1 (ii).

Proof of Proposition 2.4

Denote by $F_\infty(k)$ the function in (2.57) and by $F_1(k)$ the function in (2.60):

$$F_\infty(k) \equiv A^\eta(1-\frac{1}{\mu})k^{2(1-\gamma)} + B^\eta k^{2-\gamma} + (C^\eta - D)k^{1-\gamma} + k = 0, \tag{2.61}$$

$$F_1(k) \equiv A^\eta k^{2(1-\gamma)} + B^\eta k^{2-\gamma} - Dk^{1-\gamma} + k = 0, \tag{2.62}$$

where the coefficients A^η, B^η, C^η and D^η are defined in the proof of Proposition 2.3. Recall that $\theta \equiv (\gamma+\eta-1)\frac{1}{\eta}$, so that θ is increasing in η ($\frac{\partial\theta}{\partial\eta} = \frac{1-\gamma}{\eta^2} > 0$), therefore, A^η, B^η, C^η are strictly increasing functions in η.

Case $T = \infty$ First, consider the case with infinite patent length. By the implicit function theorem[18]

$$\frac{\partial k}{\partial \eta} = -\frac{\partial F_\infty/\partial\eta}{\partial F_\infty/\partial k}. \tag{2.63}$$

Taking the partial derivative of F_∞ with respect to η yields $\partial F_\infty/\partial\eta = \frac{\partial A^\eta}{\partial\eta}(1 - \frac{1}{\mu})k^{2(1-\gamma)} + \frac{\partial B^\eta}{\partial\eta}k^{2-\gamma} + \frac{\partial C^\eta}{\partial\eta}k^{1-\gamma} > 0$, since A^η, B^η, C^η is strictly increasing function in η. The partial derivative of F_∞ with respect to k is $\partial F_\infty/\partial k = A^\eta(1-\frac{1}{\mu})(2-2\gamma)k^{1-2\gamma} + B^\eta(2-\gamma)k^{1-\gamma} + (C^\eta - D)(1-\gamma)k^{-\gamma} + 1 > 0$, since F_∞ crosses the horizontal axis from below at steady state point k, i.e., $\partial F_\infty/\partial k$ (see Proposition 2.1 (i) for reasoning). Substituting these expressions into (2.63) gives $\frac{\partial k}{\partial\eta} < 0$, so that k^* is decreasing in η.

Taking the derivative of g (see (2.40)[19]) with respect to η:

$$\frac{\partial g}{\partial \eta} = \frac{\zeta^2\mu(\mu-1)k^{-\gamma}}{2}\Big(\frac{\partial\theta}{\partial\eta}k + \frac{\partial k}{\partial\eta}(1-\gamma)\theta\Big). \tag{2.64}$$

[18] Hereafter, to ease the notation I drop the superscript η.

[19] The BGP growth rate is given by $g^\eta = \frac{\zeta^2\theta\mu(\mu-1)}{2}k^{1-\gamma}$.

The sign of $\partial g/\partial\eta$ depends on the sign of expression inside the brackets in (2.64). Let me find conditions under which this expression is negative that can be written as:

$$\frac{\partial\theta}{\partial\eta} < -\frac{\partial k}{\partial\eta}\frac{(1-\gamma)\theta}{k}. \tag{2.65}$$

Substituting explicit expression for $\frac{\partial k}{\partial\eta}$ from (2.63) into (2.65) yields:

$$\frac{(\zeta\theta\mu)^2}{2}(1-\frac{1}{\mu})(1-\gamma)k^{2(1-\gamma)} + \frac{\zeta^2\theta\mu(\mu-1)}{2}(2-\gamma)k^{2-\gamma} + (1-\gamma)\Big(\frac{\theta}{2} - \frac{\gamma}{1-\gamma}\frac{\rho}{1+\rho}\Big)k^{1-\gamma} + k <$$
$$\frac{(\zeta\theta\mu)^2}{2}(1-\frac{1}{\mu})(1-\gamma)k^{2(1-\gamma)} + \frac{\zeta^2\theta\mu(\mu-1)}{2}(1-\frac{1}{\mu})k^{2-\gamma} + (1-\frac{1}{\mu})\frac{\theta}{2}k^{1-\gamma}, \tag{2.66}$$

which, after some algebraic manipulations, can be rewritten as:

$$\frac{\gamma\rho}{1+\rho}k^{-\gamma} - \frac{\zeta^2\theta\mu(\mu-1)}{2}k^{1-\gamma} - 1 > 0. \tag{2.67}$$

Denote this function by $f(k) \equiv \frac{\gamma\rho}{1+\rho}k^{-\gamma} - \frac{\zeta^2\theta\mu(\mu-1)}{2}k^{1-\gamma} - 1$. Note that for $k \to 0$, $k^{-\gamma}$ dominates two other functions in (2.67), namely, $k^{1-\gamma}$ and constant;[20] for $k \to \infty$, $k^{1-\gamma}$ and $const = 1$ grow faster than $k^{-\gamma}$; $\lim_{k\to 0} f(k) = \infty$, and $\lim_{k\to\infty} f(k) = -\infty$. Differentiating $f(k)$ with respect to k gives

$$\frac{\partial f(k)}{\partial k} = -\frac{\gamma^2\rho}{1+\rho}k^{-\gamma-1} - \frac{\zeta^2\theta\mu(\mu-1)(1-\gamma)}{2}k^{-\gamma} - 1,$$

i.e., $\partial f(k)/\partial k < 0$ everywhere. This facts demonstrate that there exists $\epsilon > 0$ such that for $k \in (0,\epsilon)$, $f(k) > 0$, i.e., (2.67) holds. Therefore, the expression inside the brackets in (2.64) is negative, which yields $\partial g/\partial\eta < 0$, i.e., $g^{\eta<1} > g^{\eta=1}$. In turn, for $k \in (\epsilon,\infty)$, $f(k) < 0$, which implies $\partial g/\partial\eta > 0$.

[20]For $k \to 0$, $\frac{\gamma\rho}{1+\rho}k^{-\gamma} - \frac{\zeta^2\theta\mu(\mu-1)}{2}k^{1-\gamma} - 1 = \frac{\gamma\rho}{1+\rho}k^{-\gamma} + o(k^{-\gamma})$.

Differentiating of $\tilde{R}$ (see (2.38)[21]) with respect to η:

$$\frac{\partial \tilde{R}}{\partial \eta} = \frac{\theta(\zeta\mu)^2 k^{1-2\gamma}}{2}\Big(\frac{\partial \theta}{\partial \eta}k + \frac{\partial k}{\partial \eta}(1-\gamma)\theta\Big). \tag{2.68}$$

Note that expression inside the brackets in (2.68) is the same as inside the brackets in (2.64), which gives that for $k \in (0, \epsilon)$, $\partial \tilde{R}/\partial \eta < 0$, $\tilde{R}^{\eta<1} > \tilde{R}^{\eta=1}$, i.e., weakening patent breadth spurs investment in research for infinite period patent length as well as for one-period patent duration. For $k \in (\epsilon, \infty)$, $\partial \tilde{R}/\partial \eta > 0$, i.e., innovation is increasing in patent breadth η.

Case $T = 1$ Equation (2.61) for the case $T = \infty$ and equation (2.62) for the case $T = 1$ differ only by the term $C^\eta k^{1-\gamma}$ and factor $(1 - \frac{1}{\mu})$ of the first term in (2.61). Note that over the derivation of (2.66), terms with coefficients C^η and $A^\eta(1 - \frac{1}{\mu})$ cancel out, so that the resulting equation (2.66) is the same as for the case $T = \infty$, that implies the same result as for case $T = \infty$.

Proof of Proposition 2.5

As has been noticed in the proof of Proposition 2.3, the case for loosening patent breadth protection, $\eta \in (1-\gamma, 1)$, differs from the case with complete patent protection, $\eta = 1$, only by the coefficients of dynamic equation of k.

It is clear that for the current case, $\eta \in (1-\gamma, 1)$, the proof steps will be the same as for Proposition 2.2,[22] thereby, $k_1^\eta > k_\infty^\eta$, which implies (for same reason as in Proposition 2.2 (i)) $\tilde{R}^\eta_{T=1} > \tilde{R}^\eta_{T=\infty}$ and, (ii) $g^\eta_{T=1} > g^\eta_{T=\infty}$.

[21] The quality-adjusted level of research in BGP is given by $\tilde{R}^\eta(j) = \frac{(\zeta\theta\mu)^2}{4} k^{2(1-\gamma)}$.

[22] The proof is the same as for Proposition 2.3, except that γ is replaced by θ in the coefficients A^η, B^η, C^η, and in equilibrium quality-adjusted level of research, $\tilde{R}^\eta$ (2.38, the BGP level), research intensity $\phi(\tilde{R}^\eta)$ (2.39, the BGP level), and growth rate g^η (2.40, the BGP growth rate). Note that γ is also replaced by θ in the assumption used in Proposition 2.2, namely, the assumption becomes $\frac{\zeta^2\theta\mu}{2} k^{1-\gamma} < 1$.

Chapter 3

Monetary Policy and Intellectual Property Rights Protection in an OLG Economy with Endogenous Growth

3.1 Introduction

The complexity of relationships between inflation and economics growth has been explored in various empirical and theoretical studies. A series of empirical papers (Barro (1991), Fischer (1993) and Bruno and Easterly (1998)) found a negative relationship between growth and inflation. A large theoretical literature has also investigated the impact of inflation on growth in the long run. In an early paper in this area, Sidrauski (1967) found that inflation rate has no effect on either the growth rate or the steady-state rate of output (money is superneutral). However, Tobin (1965)[1]

[1]Tobin's model is based on the one-sector neoclassical growth model of Solow and Swan, whereas Sidrauski's money-in-the-utility function model is based on Ramsey's paper on optimal savings behaviour and has been the first formulation of a monetary growth model in an explicitly optimizing framework.

presents a model in which inflation has a positive effect on growth, assuming that money is a substitute for capital. Other authors (Stockman (1981) and Cooley and Hansen (1989)) construct the models in which inflation rates affect steady state capital/output ratios but not growth rates.

Later a various of endogenous growth models have been proposed with varying results. Chari et al. (1995), using four types of endogenous growth models (one-sector model with a linear production function (AK); a generalization of the linear model that endogenizes the relative price of capital (two-sector); a model which emphasizes human capital accumulation (Lucas); model with spillover effects in the accumulation of physical capital (Romer)), and Dotsey and Sarte (2000), in an AK model with uncertainty, propose endogenous growth models with cash-in-advance constraints and find very small effect of inflation on growth. At the same time, Gomme (1993), using a human capital model with a cash-in-advance constraint, and Haslag (1998), using an AK model with money used for bank reserves, find a significant effect of inflation on growth.

A more recent and growing literature has investigated the growth effect of monetary policy in the framework with innovation-based endogenous growth. Marquis and Reffett (1994), Chu et al. (2012), Chu and Lai (2013), Chu and Cozzi (2014), Chu et al. (2015) study the effect of monetary policy on economic growth in the R&D-based growth models (Schumpeterian quality ladder and Romer's expending-variety growth models).

For example, Marquis and Reffett (1994) incorporates cash-in-advance constraint and a transaction-service sector into model with horizontal innovation (Romer-style model). They find that higher inflation reduces growth through reallocation of human capital form the production of final good and R&D to transaction services.

Following Sidrauski (1967), Chu and Lai (2013) incorporate money demand into a quality-ladder model formulated by Grossman and Helpman (1991b) using a money-in-utility specification and analyse how the elasticity of substitution between consumption and the real money balance affects the growth effect of inflation. They find that if consumption and the real money balance are complements (substitutes), reducing money growth increases (decreases) output growth in an R&D-based growth model.

In an OLG framework, money has been introduced along different lines. Weiss (1980), Drazen (1981), Abel (1987) introduce money in the utility function in order to study optimal monetary policy, monetary neutrality and Tobin effects.[2] As an alternative, one can introduce cash-in-advance constraints (see Schönfelder (1992), Hahn and Solow (1995), Crettez et al. (1999)).[3]

This study also relates to the literature on economic growth and patent policy. Starting with Nordhaus (1972), many authors have explored the different implication of patent policy in infinitely-lived homogeneous agents' framework (see Goh and Olivier (2002), Futagami and Iwaisako (2007), Furukawa (2007), Iwaisako and Futagami (2013), etc.). In the OLG framework, some studies also examine the growth implications of patents. For example, Chou and Shy (1993) focus on effect patent length on growth using model with expanding variety (without capital), while Sorek (2011) analyses the effect of length and breadth on growth in Schumpeterian style model, in which effects depend on elasticity of inter-temporal substitution. Using a two period OLG model of an expanding-variety growth with physical capital, Diwakar et al. (2019) show that weakening patent protection enhances economic growth.

Much less attention has been given to the interactive effects of monetary and patent policies in an theoretical framework for endogenous growth. An exceptional study with this respect, by Chu et al. (2012), examine an interaction between the effects of monetary policy and IPR protection policy on growth. They develop a

[2]As has been noticed by Crettez et al. (2002a), weakness of this approach is that the reasons money is introduced in utility function are not detailed.

[3]See Crettez et al. (2002b) for presentation of drawbacks and advantages of this approach.

monetary hybrid endogenous growth model of infinitely-lived agents in which R&D and capital accumulation are both engines of long-run economic growth. Chu et al. include in the model investment good production that determines the growth rate of physical capital and consider the knowledge-driven R&D specification, where R&D activities require labor inputs. They show that monetary expansion hurts economic growth by reducing R&D and capital accumulation, whereas the effect of intellectual property rights on economic growth is ambiguous due to a trade-off between R&D and capital accumulation. However, Chu et al. paper, unlike the present chapter, consider economies of infinitely-lived homogeneous agents.

To the best of my knowledge, there are no existing studies that investigate the effects of monetary and patent policies and their interaction in the OLG framework. The demographic structure makes it possible to study inter-generational trade in patents and a life-cycle saving motive, thereby allowing the paper to contribute to the theory of optimal monetary and patent policy in a framework with innovation-based endogenous growth.

This paper investigates interactive effects of intellectual property rights protection and monetary policy on economic growth. I develop an overlapping generations model with R&D-based growth as in Rivera-Batiz and Romer (1991). Intellectual property rights protection is introduced in the model by considering patent breadth that determines the firms' market power, while the money demand is incorporated by imposing a cash-in-advance constraint on old age consumption expenditure.

Using a monetary endogenous growth model, I examine how the strengthening patent protection influences growth. Patent protection determines market structure that in turn affects the effects of monetary policy on economic growth. The results show that strengthening patent protection increases the growth rate of output. The reason is as follows. Strengthening patent protection affects the growth in two opposite directions. First, a larger patent breadth raises the price of intermediate goods, which in turn increases the profit. Under the lab-equipment R&D specification the value of innovation is independent of patent breadth. As a result, an increase in monopoly profits increases the equilibrium rental rate. The higher rental rate motivates young to save less, i.e., in their young age agents have less incentives to invest in patents and capital. Moreover, a broader patent protection raises real balances held by agent thereby lowers investment in capital accumulation and research. This is a negative effect of patent protection growth. On the other hand, strengthening patent breadth increases monopolistic profits, providing more incentives for research investment and, as a result, increases growth rate. All in all, the impact of these two contradicting effects on growth rate in favour of strengthening breadth protection. Turning to the effect of monetary policy on growth, I find that monetary expansion raises the cost of holding money and, as a result, households reduce R&D investment and capital accumulation, which in turn decreases the growth. The analysis of the interactive effects of intellectual property rights and monetary policy shows that a larger breadth of patents mitigates the negative effect of money growth, that is, the increasing of protection breadth weakens the negative effect of money growth on growth rate of output.

The rest of the chapter is organized as follows. Section 3.2 presents the dynamic equilibrium model and shows that there is a unique balanced growth path. In Section 3.3, I examine how monetary and patent policies affect the growth rate. Section 3.4 concludes.

3.2 The Model

I develop a two-period OLG model with lab-equipment R&D-based growth. It extends the product product variety model drawing mostly from Rivera-Batiz and Romer (1991) and Diwakar et al. (2019). Agents are endowed with one unit of labor that they supply in the first period of life and are retired during old age. Each generation consists of L new individual agents who live for two periods. There is no population growth, $L_t = L$ for all t. There are the final goods sector and the intermediate-goods sector that produce differentiated goods. I assume that firms that create a new type of differentiated goods can obtain a patent that allows them to produce and sell goods monopolistically. But, patent protection may be incomplete and the degree of patent protection depends on the authority's policies. The patent protection degree generally is determined by two instruments: patent length and patent breadth. For simplicity I assume that patents have infinite life and the government controls the degree of patent protection by using patent breadth.[4]

3.2.1 Production sectors

Final goods producer

The final good can be used for consumption and for investment in physical capital and patents. The final good is produced by perfectly competitive firms using labor and intermediate goods :

$$Y_t = L^{1-\alpha} \int_0^{N_t} x_t(j)^\alpha dj, \tag{3.1}$$

where $x_t(j)$ is the amount of the j type of intermediate good, N_t is a number of intermediates, $\alpha \in (0, 1)$. The labor supply L of entire economy is used in production of final goods and I set $L = 1$.

[4] I describe the patent protection in more detail in Section 3.3.

Profit maximization by the final good producer implies that demand for the intermediate good is[5]

$$p_t(j) = \alpha x_t(j)^{\alpha-1} L^{1-\alpha}, \tag{3.2}$$

$$x_t(j) = L\left(\frac{\alpha}{p_t(j)}\right)^{\frac{1}{1-\alpha}} \tag{3.3}$$

and the wage rate at time t is[6]

$$w_t = (1-\alpha)L^{-\alpha}\int_0^{N_t} x_t(j)^{\alpha} dj. \tag{3.4}$$

Intermediate goods production and innovation

Each intermediate product is produced according to a linear production function:

$$x_t(j) = K_t(j), \tag{3.5}$$

where $K_t(j)$ is the amount of capital used as input. A firm owning a patent sets its production level so as to maximize the profit subject to demand function (3.2). The profit is given by:

$$\pi_t(j) = \max_{x_t(j)} p_t(j)x_t(j) - R_t x_t(j), \tag{3.6}$$

where R_t is the rental price of capital. The rental rate is determined in the market for capital $R_t = r_t + \delta$, where δ is the rate of depreciation. Later, to simplify analysis I assume that capital fully depreciates after use so $\delta = 1$ that implies $1 + r_{t+1} = R_{t+1}$.

[5] $p_t(j)$ is the price of intermediate good $x_t(j)$ relative to final good.
[6] In units of final good.

The optimal quantity and price set by monopolistic firm are

$$x_t(j) = x_t = \alpha^{\frac{2}{1-\alpha}} L R_t^{\frac{1}{\alpha-1}}, \tag{3.7}$$

$$p_t(j) = p_t = \frac{1}{\alpha} R_t. \tag{3.8}$$

Thus, the maximum profit for monopolistic firm is

$$\pi_t(j) = \pi_t = \frac{1-\alpha}{\alpha} \alpha^{\frac{2}{1-\alpha}} L R_t^{\frac{\alpha}{\alpha-1}}. \tag{3.9}$$

The supply for capital is predetermined capital stock K_t and demand is sum of demand for capital of each sectors:

$$K_t = \int_0^{N_t} x_t(j) = x_t N_t = \alpha^{\frac{2}{1-\alpha}} L R_t^{\frac{1}{\alpha-1}} N_t. \tag{3.10}$$

Let me denote $k_t = K_t/(N_t L)$, then $x_t = k_t L$. From (3.7) equilibrium rental rate is decreasing function of k_t:

$$R_t = \alpha^2 k_t^{\alpha-1}. \tag{3.11}$$

Plugging (3.11) into (3.9) the firm's profit can be expressed as

$$\pi_t = (1-\alpha)\alpha k_t^{\alpha} L. \tag{3.12}$$

Note, that in this Section I consider complete patent protection, that is the case when innovator can charge monopoly markup and gets the full monopoly profits. In a later Section 3.3 I examine intermediate case where firms are forced prices lower than the monopoly price.

Variety expansion depends on the amount of final output that is used as input in research $\tilde{R}_t$:

$$N_{t+1} - N_t = \frac{\tilde{R}_t}{\psi}, \tag{3.13}$$

where ψ is the cost to create a new variety of intermediates.[7] I assume free entry, i.e., the research sector is perfectly competitive. The free entry condition implies that in equilibrium with positive research expenditure the worth of each new good is equal to ψ.

3.2.2 Consumption decisions

The preferences of agents are represented by a life-cycle utility function

$$U(c_{1,t}, c_{2,t+1}), \tag{3.14}$$

where $c_{1,t}$ and $c_{2,t+1}$ are consumption of goods during youth and old age, respectively. $U : R_+^2 \to R_+$ is differentiable, increasing and strictly concave, $\lim_{c_1 \to 0} U'_{c_1}(c_1, c_2) = \infty$, for all $c_2 > 0$, and $\lim_{c_2 \to 0} U'_{c_2}(c_1, c_2) = \infty$, for all $c_1 > 0$. During youth each agent supplies inelastically one unit of labor and is retired during old age.

Agents can invest their savings in a nominal asset — money M_t, and real assets — physical capital, investment in new and old patents, which will be productive at date $t+1$. Investment in new patents are investment in R&D to create a new variety, that using (3.13) can be expressed as $\psi(N_{t+1} - N_t)$. Young buys existing firms/old patents on producing goods invented by the old generation. Value of old patents can be expressed as ψN_t. All patents are assumed to be granted forever. In their old age agents receive monopoly profit of newly constructed firms and earn from selling these

[7] This is similar to the assumption in Section 6.1 in Barro and Sala-i Martin (2004) that the cost of inventing a new type of good does not change over time.

firms to young, $(\pi_{t+1} + \psi)(N_{t+1} - N_t)$. Moreover, old agents collect monopoly profit and earn from selling patents from firms that they bought in the previous period, $(\pi_{t+1} + \psi)N_t$.

The budget constraints writes in real terms:[8]

$$c_{1,t} + k_{t+1}N_{t+1}/L + \psi(N_{t+1} - N_t)/L + \psi N_t/L + \frac{M_t}{P_t} = w_t, \tag{3.15}$$

$$c_{2,t+1} = R_{t+1}k_{t+1}N_{t+1}/L + (\pi_{t+1} + \psi)(N_{t+1} - N_t)/L + (\pi_{t+1} + \psi)N_t/L + \frac{M_t}{P_{t+1}}. \tag{3.16}$$

Following Hahn and Solow (1995), I assume that a fraction of at least $0 < \mu < 1$ of consumption during old age is financed by money balances held at the beginning of old age:

$$M_t \geq \mu P_{t+1} c_{2,t+1}. \tag{3.17}$$

I study the case in which the cash-in-advance constraint is binding:

$$M_t = \mu P_{t+1} c_{2,t+1}. \tag{3.18}$$

This implies that real return on investment in assets that agent holds is not less than the real return on money holdings:[9]

$$R_{t+1} \geq \frac{P_t}{P_{t+1}}. \tag{3.19}$$

[8] Recall that capital fully depreciates after use, i.e., $\delta = 1$.
[9] I verify it in Proposition 3.2

Agents maximize their utility (3.14) subject to two budget constraints (3.15), (3.16) and the CIA constraint (3.18). The first-order condition yields

$$R_{t+1} = \frac{\pi_{t+1} + \psi}{\psi}. \tag{3.20}$$

The optimal consumer's choice necessarily satisfies

$$\frac{U'_{c_{1,t}}}{U'_{c_{2,t+1}}} = \frac{R_{t+1}}{1 - \mu + \mu R_{t+1} \frac{P_{t+1}}{P_t}}. \tag{3.21}$$

3.2.3 The government

The government can choose the money growth rate λ_t:

$$\bar{M}_t - \bar{M}_{t-1} = \lambda_t \bar{M}_{t-1}, \tag{3.22}$$

where $\bar{M}_t = LM_t$ is the total money stock at time t. Public expenditures are given by the government budget constraint:

$$P_t G_t = \lambda_t \bar{M}_{t-1}. \tag{3.23}$$

3.2.4 Equilibrium characterization

A competitive equilibrium can be defined as follows.

Definition 1 A competitive equilibrium is a sequence of quantities $\{c_{1,t}, c_{2,t}, K_t, Y_t, N_t, \tilde{R}_t\}_{t=0}^{\infty}$ and prices $\{w_t, R_t, p_t\}_{t=0}^{\infty}$ and a time path of policy $\{\lambda_t\}_{t=0}^{\infty}$ such that i) consumers maximize utility subject to their budget and cash-in advance constraints taking prices as given; ii) firms in the final good sector maximize profits choosing labor and intermediate inputs; iii) intermediate firms' behaviour is optimal; iv) the government budget constraint holds; v) all markets clear.

I focus on the features of the competitive equilibrium of the economy under the hypothesis that the monetary authority pegs the money growth at a constant rate.

Definition 2 A BGP for the economy is an equilibrium sequence where the variables grow at the same constant factor. In addition, the money growth rate is constant along the BGP.

The steady-state k^* is determined by (3.20), that using (3.11) and (3.12) can be rewritten as:

$$\alpha^2 k^{*\alpha-1} = \frac{(1-\alpha)\alpha k^{*\alpha} L + \psi}{\psi}. \tag{3.24}$$

Let me show that there exists a unique steady state.

Proposition 3.1 *There exist a unique steady-state equilibrium with capital intensity given by (3.24).*

Proof The left hand side (LHS) in expression (3.24) is decreasing in k^*, and $\lim_{k\to 0} LHS = \infty$, $\lim_{k\to\infty} LHS = 0$, whereas the right hand side (RHS) is increasing, and $\lim_{k\to 0} RHS = 1$, $\lim_{k\to\infty} RHS = \infty$. Thus, there exists exactly one crossing point, that is a unique k^* that solves (3.24).

Remark Note that solution of (3.24) k^* can be written as $k^* = \tilde{k}(\alpha, \psi, L)$, where the $\tilde{k}$ is function of α, ψ, L. For the specific, but generically accepted case $\alpha = 1/3$, the steady-state k^* can be explicitly determined.[10]

Next, let me derive the BGP growth rate. Firs, rewrite (3.1) and (3.4) respectively as

$$Y_t = k_t^{\alpha} L N_t, \tag{3.25}$$

$$w_t = (1-\alpha) k_t^{\alpha} N_t. \tag{3.26}$$

Let me denote

$$s_t = k_{t+1} N_{t+1} + \psi(N_{t+1} - N_t) + \psi N_t = (k_{t+1} + \psi) N_{t+1}, \tag{3.27}$$

then, using first-order condition (3.20),[11] substitution from consumer's second budget constraint (3.16) into the budget constraint when young (3.15) yields:

$$c_{1,t} + \big(c_{2,t+1} - \frac{M_t}{P_{t+1}}\big)\frac{1}{R_{t+1}} + \frac{M_t}{P_t} = w_t. \tag{3.28}$$

Plugging CIA constraint (3.18) into this equation I get the intertemporal budget constraint:

$$c_{1,t} + \frac{c_{2,t+1}}{R_{t+1}}\Big(1 - \mu + \mu R_{t+1}\frac{P_{t+1}}{P_t}\Big) = w_t. \tag{3.29}$$

[10] Equation (3.24) can be rewritten as $(1-\alpha)\alpha L k^* + \psi k^{*1-\alpha} - \alpha^2\psi = 0$. Substituting $1-\alpha = \frac{l-m}{l}$ and $k = x^l$ into this equation I get $(1-\alpha)\alpha L x^3 + \psi x^2 - \alpha^2\psi = 0$, that can be reduced to depressed cubics $t^3 + pt + q$, where $x = t - \frac{\psi}{3(1-\alpha)\alpha L}$, $p = -\frac{\psi^2}{(3(1-\alpha)\alpha L)^2}$, $q = \frac{2\psi^3}{27((1-\alpha)\alpha L)^3} - \frac{\alpha\psi}{(1-\alpha)}$ and can be solved with Cardano's method.

[11] Using (3.20) and the fact that in equilibrium $z_{it} = 1$, $j = 1, 2, 3$, the budget constraint when old can be rewritten as $c_{2,t+1} = R_{t+1}s_t + \frac{M_t}{P_{t+1}}$. Note, that using notation (3.27), the budget constraint when young is $c_{1,t} + s_t + \frac{M_t}{P_t} = w_t$.

For tractability assume the logarithmic utility specification:

$$U(c_{1,t}, c_{2,t+1}) = \ln c_{1,t} + \rho \ln c_{2,t+1}, \tag{3.30}$$

where $\rho \in (0,1)$ is discount factor. Under this assumption equation (3.21) can be rewritten as:

$$c_{1,t} = \frac{c_{2,t+1}}{\rho R_{t+1}} \Big(1 - \mu + \mu R_{t+1} \frac{P_{t+1}}{P_t} \Big). \tag{3.31}$$

Combining (3.29) and (3.31) yields

$$c_{1,t} = \frac{1}{\rho + 1} w_t. \tag{3.32}$$

Using the budget constraint when old (3.16) and CIA constraint (3.18) one can rewrite budget constraint (3.16) as:

$$c_{2,t+1} = \frac{1}{1-\mu} R_{t+1} s_t. \tag{3.33}$$

Notice that the money held by old agents at the beginning of period t is equal to the aggregate money supply at date $t-1$, $\bar{M}_{t-1} = LM_{t-1}$. Using (3.33) the money stock held by old agents at the beginning of period t is given by

$$\bar{M}_{t-1} = LM_{t-1} = L\mu P_t c_{2,t} = L\mu P_t \bigg(\frac{1}{1-\mu} R_t s_{t-1} \bigg). \tag{3.34}$$

Substituting from (3.11) and (3.27) into (3.34) and rearranging it I have:

$$\frac{\bar{M}_{t-1}}{LP_t} = \Gamma k_t^{\alpha-1}(k_t + \psi) N_t, \tag{3.35}$$

with $\Gamma = \frac{\mu \alpha^2}{1-\mu}$.

In the money market equilibrium demand is equal supply:

$$LM_t = \bar{M}_t = (1 + \lambda_t)\bar{M}_{t-1}. \tag{3.36}$$

The real balances held by each young agent at date t is

$$m_t = \frac{M_t}{P_t} = (1 + \lambda_t)\Gamma k_t^{\alpha-1}(k_t + \psi)N_t. \tag{3.37}$$

Government consumption is given by:

$$G_t = \lambda_t \frac{\bar{M}_{t-1}}{P_t} = \lambda_t \Gamma k_t^{\alpha-1}(k_t + \psi)LN_t. \tag{3.38}$$

Using (3.15), (3.26), (3.32) and (3.37) non-monetary aggregate saving can be expressed as:

$$s_t = w_t - c_t - \frac{M_t}{P_t} = \frac{\rho}{\rho+1}(1-\alpha)k_t^{\alpha}N_t - (1+\lambda_t)\Gamma k_t^{\alpha-1}(k_t+\psi)N_t. \tag{3.39}$$

Substitution for s_t from (3.27) into (3.39) gives the growth rate of product variety:

$$g_N + 1 = \frac{N_{t+1}}{N_t} = \frac{\frac{\rho}{\rho+1}(1-\alpha)k_t^{\alpha} - (1+\lambda_t)\Gamma k_t^{\alpha-1}(k_t+\psi)}{k_{t+1}+\psi}. \tag{3.40}$$

In the next proposition I show that there exists a unique BGP.

Proposition 3.2 *Assuming that* $\lambda < \bar{\lambda}$, $\mu \geq \underline{\mu}$, *and the cost of creating a new variety* ψ *is sufficiently low, there exist a unique BGP in which all variables grow at the rate* $g_N^* = g^*$, *where* $\bar{\lambda} = \frac{\rho}{\rho+1}\frac{1-\alpha}{\alpha+(1-\alpha)L}\frac{1-\mu}{\mu\alpha} - 1$, $\underline{\mu} = 1 - \frac{\rho+1}{\rho}\frac{(\alpha+(1-\alpha)L)\alpha}{1-\alpha}$.

Proof From (3.25), the growth rate of output can be expressed as

$$g_Y + 1 = \frac{Y_{t+1}}{Y_t} = \frac{N_{t+1}}{N_t} = g_N + 1. \tag{3.41}$$

Resource constraint is given by $Y_t = c_t L + s_t L + G_t = (c_{1t} + c_{2t})L + (k_{t+1} + \psi)N_{t+1}L + G_t$, where $c_t = c_{1t} + c_{2t}$. From (3.39) and (3.38) s_t and G_t are proportional to N_t, so that $g_S^* = g_G^* = g_N^*$. Let me show that consumption grows at the rate g_N^*. Conjecture that c_t grows by the same rate as N_t, i.e., $c_{t+1} = c_t(1 + g_N)$ that implies $c_t = c_0(1 + g_N)^t$. Substituting the equilibrium values for Y_t, c_t and s_t in the resource constraint I get:

$$k^{*\alpha}L = \frac{c_0}{N_0}L + (k^* + \psi)(g_N + 1)L + \lambda\Gamma k^{*\alpha-1}(k^* + \psi)L. \qquad (3.42)$$

Therefore, for the initial value of consumption

$$c_0 = \Big(k^{*\alpha} - (k^* + \psi)(g_N + 1) - \lambda\Gamma k^{*\alpha-1}(k^* + \psi)\Big)N_0, \qquad (3.43)$$

the paths $\{c_t, N_t\}_{t=0}^{\infty}$ satisfy the resource constraints with N_t and c_t grow at the same rate g^*:[12]

$$g^* + 1 = g_N + 1 = \frac{\frac{\rho}{\rho+1}(1-\alpha)k^{*\alpha}}{k^* + \psi} - (1 + \lambda)\Gamma k^{*\alpha-1}. \qquad (3.44)$$

Next, I have to find conditions that guarantee positive growth.

As ψ approaches 0, steady-state capital intensity k also approaches zero as can be seen from (3.24). Let me denote $\Theta \equiv \frac{\rho}{\rho+1}(1-\alpha) - (1+\lambda)\Gamma$ and rewrite equation (3.44):

$$g^* + 1 = \frac{\Theta k^{*\alpha} - (1+\lambda)\Gamma\psi k^{*\alpha-1}}{k^* + \psi}. \qquad (3.45)$$

[12] An alternative way to show that $g_c^* = g_N^*$ is as follows. From (3.32), (3.26) and (3.33),(3.27) c_{1t} and c_{2t} are proportional to N_t, so that $g_{c_{1t}} = g_{c_{2t}} = g_N$, and, consequently, $g_c = g_N$.

By rearranging the expression (3.24) to obtain ψ and plugging it into previous equation (3.45), after some algebraic manipulations, I get:

$$g^* + 1 = k^{*\alpha-1}(\Upsilon_1 + \Upsilon_2), \tag{3.46}$$

where $\Upsilon_1 = \frac{\theta\alpha^2-(1+\lambda)\Gamma(1-\alpha)\alpha L}{\alpha^2-k^{*1-\alpha}+(1-\alpha)\alpha L}$ that, for $k^* \to 0$, approaches to some constant, and, $\Upsilon_2 = \frac{\theta k^{*1-\alpha}}{-(\alpha^2-k^{*1-\alpha}+(1-\alpha)\alpha L)} \to 0$ when k^* approaches 0. So that, the expression in brackets (3.46), which is the sum of Υ_1, Υ_2, approaches some constant, $\frac{\theta\alpha^2-(1+\lambda)\Gamma(1-\alpha)\alpha L}{\alpha^2+(1-\alpha)\alpha L}$. Hence, the sign of growth rate depend on sign of numerator of Υ_1, i.e., I need to have $\theta\alpha^2 - (1+\lambda)\Gamma(1-\alpha)\alpha L > 0 \Leftrightarrow$ (1-α)$\alpha\frac{\rho}{\rho+1} - (1+\lambda)\Gamma(\alpha + (1-\alpha)L) > 0$ $\Leftrightarrow \quad \lambda < \bar{\lambda} = \frac{\rho}{\rho+1}\frac{1-\alpha}{\alpha+(1-\alpha)L}\frac{1-\mu}{\mu\alpha} - 1$. That implies that for sufficiently low ψ and $\lambda < \bar{\lambda}$ the growth rate is positive.

Finally, I have to provide the condition under which the CIA constraint is binding along equilibrium. This condition corresponds to the property that the rate of return on money is dominated by the rate of return on assets $\frac{P_t}{P_{t+1}} \leq R_{t+1}$.

Using (3.35) and (3.11) the last inequality can be rewritten as

$$\frac{R_{t+1}P_{t+1}}{P_t} = \alpha^2(1+\lambda_t)\frac{k_t^{\alpha-1}(k_t+\psi)N_t}{(k_{t+1}+\psi)N_{t+1}} \geq 1. \tag{3.47}$$

Using (3.27) and (3.39) I have

$$(k_{t+1}+\psi)N_{t+1} = (\frac{\rho}{\rho+1}(1-\alpha)k_t^{\alpha} - (1+\lambda_t)\Gamma k_t^{\alpha-1}(k_t+\psi))N_t. \tag{3.48}$$

Substituting it into (3.47) and rearranging it produces

$$\frac{\rho(1-\alpha)}{\alpha^2(\rho+1)}\frac{k_t^{\alpha}}{k_t^{\alpha-1}(k_t+\psi)} \leq (1+\lambda_t)(1+\frac{\Gamma}{\alpha^2}). \tag{3.49}$$

Denote the left-hand side and the right-hand side of (3.49) by LHS and RHS respectively. Substituting expression for ψ, that has been obtained by rearrangement of (3.24), $\psi = \frac{(1-\alpha)\alpha k^{\alpha} L}{\alpha^2 k^{\alpha-1}-1}$, into the left-hand side of (3.49), gives

$$\frac{\rho(1-\alpha)}{\alpha^2(\rho+1)} \frac{k_t^{\alpha}}{k_t^{\alpha-1}(k_t+\psi)} < \frac{\rho}{\rho+1} \frac{1-\alpha}{(\alpha+(1-\alpha)L)\alpha}. \tag{3.50}$$

Let me denote $A = \frac{\rho}{\rho+1}\frac{1-\alpha}{(\alpha+(1-\alpha)L)\alpha}$. In (3.50) I have shown that $LHS < A$. I found above that one of conditions that guarantees positive growth rate is $\lambda < \bar{\lambda}$, so using introduced notation $\lambda \in (0, A\frac{\alpha^2}{\Gamma} - 1)$.[13] Then, (3.49) is satisfied for $\lambda \in (0, \bar{\lambda})$ if $A \leq 1 + \frac{\Gamma}{\alpha^2}$. Given the expression of Γ and A, the condition under which the rate of return on asset dominates the rate of return on money is

$$\mu \geq \underline{\mu} = 1 - \frac{\rho+1}{\rho}\frac{(\alpha+(1-\alpha)L)\alpha}{1-\alpha}. \tag{3.51}$$

Proposition 3.3 *The growth rate is decreasing in the money growth rate λ.*

Proof Taking the derivative of (3.44) with respect to λ gives the result.

3.3 Patent breadth and growth

In the previous section, I prove the existence of uniquely determined BGP for the case when breadth protection is complete, that is the case when a patentee can charge an unconstrained monopolistic price. In this section, I analyse how changes in patent policy impacts the growth rate of output.

Patent breadth determines how high a price markup each monopolist can charge. I model patent breadth protection with the parameter η, which limits the ability of patent holders to charge the unconstrained monopolistic price. Thereby, the patent breadth is parametrized by the maximum price that the innovator can charge with the

[13] Note that $\bar{\lambda}$ can be expressed as $\bar{\lambda} = A\frac{\alpha^2}{\Gamma} - 1$.

patent. Similar modelling approach for patent breadth protection was used (among others) by Goh and Olivier (2002), Iwaisako and Futagami (2013), and Futagami and Iwaisako (2007), Chu et al. (2012).

The patentee maximizes profit by charging price $p(\eta) = \eta p_t(j)$ with $\eta \in (\alpha, 1)$. A patent breadth $\eta = \alpha$ is related to the case where innovator charges a competitive price and thereby makes no profits. A patent breadth $\eta = 1$ is related to the case where innovator is able to charge the monopoly markup and obtain full monopoly profit. A patent breadth with $\alpha < \eta < 1$ will be the cases where the price that innovator is forced to charge lower the monopoly price, but still higher competitive price and innovator is able to make profits.

Price charged by innovator with patent breadth η is $p_t(\eta) = \eta \frac{R_t}{\alpha}$. Incorporation of patent breadth modifies the equations for the level of intermediate output, (3.3), and, as result, level of profit (3.12) which become:

$$k_t^{\eta} = \left(\frac{\alpha^2}{\eta R_t} \right)^{\frac{1}{1-\alpha}}, \tag{3.52}$$

$$\pi_t^{\eta} = (\frac{\eta}{\alpha} - 1)\frac{\alpha^2}{\eta}(k_t^{\eta})^{\alpha} L. \tag{3.53}$$

Modifying the rest of the analysis and taking into account (3.37),(3.39),(3.40) allow to get the expression for the growth rate:

$$g^{*\eta} + 1 = \frac{\frac{\rho}{\rho+1}(1-\alpha)(k^{*\eta})^{\alpha}}{k^{*\eta} + \psi} - (1+\lambda)\Gamma(k^{*\eta})^{\alpha-1}\frac{1}{\eta}, \tag{3.54}$$

where $k^{*\eta}$ is steady state capital intensity and is determined by (3.20) which can be rewritten as[14]

$$\frac{\alpha^2}{\eta}(k^{*\eta})^{\alpha-1} = \frac{(\frac{\eta}{\alpha} - 1)\frac{\alpha^2}{\eta}(k^{*\eta})^{\alpha} L + \psi}{\psi}. \tag{3.55}$$

[14]This equation is analogue of equation (3.24).

Next, I derive the effect of changes in patent and monetary policies on growth rate.

Proposition 3.4 *The growth rate g is increasing in patent breadth η and decreasing in the money growth rate λ. Increasing patent breadth weakens the negative effect of the money growth rate on the economic growth rate.*

Proof[15] First, let me consider $\frac{\partial k^*}{\partial \eta}$. By the implicit function theorem

$$\frac{\partial k^*}{\partial \eta} = -\frac{\partial F/\partial \eta}{\partial F/\partial k^*}, \tag{3.56}$$

where F is obtained by rearranging (3.55):

$$F = (\frac{\eta}{\alpha} - 1)\frac{\alpha^2}{\eta}k^{*\alpha}L - \psi\frac{\alpha^2}{\eta}k^{*\alpha-1} + \psi = 0. \tag{3.57}$$

Taking the partial derivative of F with respect to η yields $\partial F/\partial \eta = \frac{\alpha^2}{\eta^2}k^{*\alpha}L(1+\frac{\psi}{k}) > 0$, with respect to k^* gives $\partial F/\partial k^* = \frac{\alpha^2}{\eta}k^{*\alpha-1}\Big((\eta-\alpha)L + (1-\alpha)\frac{\psi}{k})\Big) > 0$. Substituting these expressions into (3.56) I find

$$\frac{\partial k^*}{\partial \eta} = -\frac{k^*(1+\frac{\psi}{k^*})L}{\eta(1-\alpha)\Big(\frac{\eta-\alpha}{1-\alpha}L + \frac{\psi}{k^*}\Big)}, \tag{3.58}$$

so that k^* is decreasing in η.

Next, differentiating expression for the growth rate (3.54) with respect to η, I obtain:

$$\frac{\partial(g^*+1)}{\partial \eta} = \frac{\rho(1-\alpha)}{(\rho+1)(k^*+\psi)^2}\frac{\partial k^*}{\partial \eta}k^{*\alpha}\Bigg(\alpha(\frac{\psi}{k^*}+1)-1\Bigg) - \frac{1+\lambda}{\eta}\Gamma k^{*\alpha-1}\Bigg(-(1-\alpha)\frac{\partial k^*}{\partial \eta}k^{*-1} - \frac{1}{\eta}\Bigg). \tag{3.59}$$

The sign of the first term depends on the sign of the term in brackets, $\alpha(\frac{\psi}{k^*}+1)-1$. From (3.55), $\psi = \frac{(\eta-\alpha)\alpha k^{*\alpha}L}{\alpha^2 k^{*\alpha-1}-\eta}$, then $\frac{\psi}{k^*} \xrightarrow[k^*\to 0]{} \frac{(\eta-\alpha)L}{\alpha}$. Using this fact, $\alpha(\frac{\psi}{k^*}+1)-1 \xrightarrow[k^*\to 0]{}$

[15] To ease the notation hereafter I drop superscript η in this section.

$\eta - 1$,[16] which is non-positive since $\eta \in (\alpha, 1)$. Thus, because $\frac{\partial k^*}{\partial \eta} < 0$, the first term in (3.59) is non-negative.

Next, consider the second term in expression (3.59). The sign of this term depends on the sign of the term in the second brackets in (3.59). Let me assume that the term in the second brackets in (3.59) is non-negative, $\frac{(1+\frac{\psi}{k^*})}{\frac{\eta-\alpha}{1-\alpha}+\frac{\psi}{k^*}} \geq 1$. Substituting expression for $\partial k^*/\partial \eta$ into this inequality and rearranging it yields $\frac{(1+\frac{\psi}{k^*})}{\frac{\eta-\alpha}{1-\alpha}+\frac{\psi}{k^*}} \geq 1$ which always holds since $\eta \in (\alpha, 1)$. Hence, the second term in (3.59) is always non-positive.

Thus, the first term is increasing in patent breadth η, whereas the second is decreasing. Strengthening patent protection affects growth in two opposite directions. Intuitively, stronger patent breadth increases monopolistic profits, providing more incentives for R&D – this is the positive effect of IPR protection on growth. On the other hand, a larger patent breadth raises the price of intermediate goods, which in turn increases the profit. Under the lab-equipment R&D specification the value of innovation, ψ, is independent of the patent breadth . As a result, from (3.20), an increase in monopoly profits increases the equilibrium rental rate R_{t+1}. The higher rental rate motivates young to save less, i.e., in their young age agents have less incentives to invest in patents and capital.[17] Moreover, increasing η raises real balances held by agent thereby lowers investment in capital accumulation and research. This is a negative effect of patent protection on growth.

To find sign of $\frac{\partial(g^*+1)}{\partial \eta}$, I rewrite (3.59) using (3.58) as

$$\frac{\partial(g^*+1)}{\partial \eta} = \frac{\partial k^*}{\partial \eta} k^{*\alpha} \left[\frac{\rho(1-\alpha)}{(\rho+1)(k^*+\psi)^2} \left(\alpha(\frac{\psi}{k^*}+1) - 1 \right) + \frac{1+\lambda}{\eta} \frac{(1-\alpha)\Gamma}{k^{*2}} - \frac{1+\lambda}{\eta^2} \frac{\Gamma}{k^{*2}} \left(\frac{\eta(1-\alpha)(\frac{\eta-\alpha}{1-\alpha}+\frac{\psi}{k^*})}{1+\frac{\psi}{k^*}} \right) \right]. \quad (3.60)$$

[16]Hereafter it is without loss of generality to assume that $L \equiv 1$.

[17]Alternatively, this effect can be explained (see Diwakar et al. (2019)) as shift from old to young saving when patent breadth is incomplete. As indicated by Diwakar et al this effect is similar to income transfers from old to the young in Jones and Manuelli (1992) paper.

The sign of $\frac{\partial(g^*+1)}{\partial\eta}$ depends on sign of term in square brackets in (3.60). Let me show that this term is negative:

$$\frac{\rho(1-\alpha)}{(\rho+1)(k^*+\psi)^2}\left(\alpha(\frac{\psi}{k^*}+1)-1\right)+\frac{1+\lambda}{\eta}\frac{(1-\alpha)\Gamma}{k^{*2}}-\frac{1+\lambda}{\eta^2}\frac{\Gamma}{k^{*2}}\left(\frac{\eta(1-\alpha)(\frac{\eta-\alpha}{1-\alpha}+\frac{\psi}{k^*})}{1+\frac{\psi}{k^*}}\right)<0. \quad (3.61)$$

Rearranging (3.61) yields

$$\frac{1+\lambda}{\eta}\Gamma\left(\frac{1-\frac{\eta-\alpha}{1-\alpha}+\frac{\psi}{k^*})}{1+\frac{\psi}{k^*}}\right)<\frac{\rho}{(\rho+1)}\frac{k^{*2}}{(k^*+\psi)^2}\left(1-\alpha(\frac{\psi}{k^*}+1)\right). \quad (3.62)$$

The right-hand side of (3.62) is $\frac{\rho}{(\rho+1)}\frac{k^{*2}}{(k^*+\psi)^2}\left(1-\alpha(\frac{\psi}{k^*}+1)\right)\xrightarrow[k^*\to 0]{}\frac{\rho}{(\rho+1)}\left(\frac{\alpha}{\eta}\right)^2(1-\eta)$, while the left-hand side is $\frac{1+\lambda}{\eta}\Gamma\left(\frac{1-\frac{\eta-\alpha}{1-\alpha}+\frac{\psi}{k^*})}{1+\frac{\psi}{k^*}}\right)\xrightarrow[k^*\to 0]{}\frac{1+\lambda}{\eta}\Gamma\frac{\alpha-2\alpha^2-\eta\alpha}{(1-\alpha)\eta}$. From Proposition 3.2, $\bar{\lambda}=\frac{\rho}{(\rho+1)}\frac{\alpha(1-\alpha)}{\Gamma}-1$,[18] thus $1<1+\lambda<\frac{\rho}{(\rho+1)}\frac{\alpha(1-\alpha)}{\Gamma}$. Using this I get

$$\frac{1+\lambda}{\eta}\Gamma\frac{\alpha-2\alpha^2-\eta\alpha}{(1-\alpha)\eta}<\frac{\rho}{(\rho+1)}\frac{\alpha(\alpha-2\alpha^2-\eta\alpha)}{\eta^2}.$$

So that if

$$\frac{\rho}{(\rho+1)}\frac{\alpha(\alpha-2\alpha^2-\eta\alpha)}{\eta^2}<\frac{\rho}{(\rho+1)}\left(\frac{\alpha}{\eta}\right)^2(1-\eta). \quad (3.63)$$

then LHS of (3.62) less than RHS of (3.62). It can be easily shown[19] that (3.63) always holds, therefore the expression in square brackets in (3.60) are negative, and, as a result, $\frac{\partial(g^*+1)}{\partial\eta}>0$.

The second statement of the proposition follows straightforwardly from taking the derivative of (3.59) with respect to λ. Intuitively, monetary expansion (higher λ)

[18]Note that it can be easily shown that the condition for $\bar{\lambda}$ in the current case ($\eta\in(\alpha,1)$) is the same as in the case when $\eta=1$ in Section 3.2.

[19]Rearranging 3.63 yields $2\alpha^2>0$.

raises the cost of holding money and, consequently, households reduce R&D investment and capital accumulation, which in turn decreases the growth rate.

To see why a broader patent breadth mitigates the negative effect of money growth, note that the term in the second brackets in (3.59) is non-negative. So that, it is immediately clear from (3.59) that increasing η weakens the negative effect of money growth.

3.4 Conclusion

In this paper I examine the mutual effect of patent and monetary policy on economic growth. Several studies that analyze this effect focus on economies of infinitely-lived homogeneous household, while the implications of inter-generational trade in patents and life-cycle saving motive are unexplored.

I show that within the framework of an overlapping generation model with a cash-in-advance constraints strengthening patent protection increases the growth rate of output. Monetary expansion, on the other hand, decreases the growth rate of output. The magnitude of the last effect depends on the degree of patent protection. In particular, a broader breadth of protection weakens this effect.

The study has a number of extensions that appear to be an interesting area for future research. As one example, the model could be used to explore the impact of patent length. As another extension, I could consider knowledge-driven R&D specification, as well as incorporate a capital-producing sector that allows to examine the effect of intellectual property rights protection on innovation and accumulation of physical capital separately.

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