The Lagrangian multiform approach to integrable systems

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The candidate connfirms that the work submitted is his own, except where work which has formed part of jointly-authored publications has been included. The contribution of the candidate and the other authors to this work has been explicitly indicated below. The candidate confirms that appropriate credit has been given within the thesis where reference has been made to the work of others.

Chapters 2, 3 and 4 are based on the publications [1], [2] and [3] respectively. For the publications [1] and [3], the candidate was responsible for the concepts, proofs and initial write-up, whilst the co-authors offered corrections. The concepts of [2] were developed in discussion with the co-authors. The candidate was responsible for all proofs and, with the exception of the introduction, the initial write-up.

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Abstract

A Lagrangian multiform enables the multi-dimensional consistency of a set of PDEs to be captured at the variational level. We offer a new perspective on the multiform Euler-Lagrange equations in terms of the variational derivative of the exterior derivative of a Lagrangian multiform and present for the first time in their full generality the multiform Euler-Lagrange equations for discrete Lagrangian multiforms. Then, by considering the closure property of a Lagrangian multiform as a conservation law, we use Noether's theorem to show that every variational symmetry of a Lagrangian leads to a Lagrangian multiform. In doing so, we provide a systematic method for constructing Lagrangian multiforms for which the closure property and the multiform Euler-Lagrange both hold. We present three examples, including what was at the time the first known example of a continuous Lagrangian 3-form: a Lagrangian multiform for the Kadomtsev-Petviashvili equation. We show that the Zakharov-Mikhailov Lagrangian structure for integrable nonlinear equations derived from a general class of Lax pairs possesses a Lagrangian multiform structure. We show that, as a consequence of this multiform structure, we can formulate a variational principle for the Lax pair itself, a problem that to our knowledge was never previously considered. As an example, we present an integrable $N \times N$ matrix system that contains the AKNS hierarchy. Finally, we present, for the first time, a Lagrangian multiform for the complete Kadomtsev-Petviashvili (KP) hierarchy: a single variational object that generates the whole hierarchy and encapsulates its integrability. By performing a reduction on this Lagrangian multiform, we are able to obtain Lagrangian multiforms for the Gelfand-Dickey hierarchy of hierarchies comprising, amongst others, the Korteweg-de Vries and Boussinesq hierarchies.

Contents

Chapter 1

Introduction to continuous Lagrangian multiforms

1.1 Introduction

Multidimensional consistency is a key feature of integrable systems. This is the idea that the defining equations of the integrable system are members of compatible hierarchies of equations in terms of an, in principle, arbitrary number of independent variables, which can be simultaneously imposed on the same set of dependent variables. Alternatively this can be interpreted as the existence of an infinite hierarchy of symmetries for those equations. From this point of view, the integrable system is the collection of all these compatible equations, i.e., we consider the integrable system to be the entire multidimensionally consistent system of equations. The traditional variational approach involves a Lagrangian that is a volume form, i.e.,

$$\mathscr{L}(x, u^{(n)})\mathsf{d}x_1 \wedge \ldots \wedge \mathsf{d}x_k, \tag{1.1}$$

on a k-dimensional base manifold. We use the notation $u^{(n)}$ to represent the dependent variable u and its derivatives with respect to the independent variables x_i, up to the n^{th} order. This can only give as many equations of motion as there are components of u. A system of multidimensionally consistent equations can be represented by a set of Lagrangians, but this captures nothing of the integrability of the system. Lagrangian multiforms, first conceived of in [4], allow a compatible set of Lagrangians to be combined into a single variational object that not only yields the relevant compatible equations, but also encapsulates the multidimensional consistency of those equations[1]. A **Lagrangian multiform**

$$\mathsf{L} = \sum_{1 \leq i_1 < \ldots < i_k \leq N} \mathscr{L}_{(i_1 \ldots i_k)}(x, u^{(n)})\ \mathsf{d}x_{i_1} \wedge \ldots \wedge \mathsf{d}x_{i_k}. \tag{1.2}$$

is a k-form in an N dimensional base manifold with $k < N$, subject to the following variational principle. We require that any u that is a critical point of the action

$$S[u;\sigma] = \int_\sigma \mathsf{L}(x, u^{(n)}) \tag{1.3}$$

must be a critical point for all possible surfaces of integration σ. This is equivalent to the requirement that u must satisfy the **multiform Euler-Lagrange equations** given by $\delta \mathsf{dL} = 0$ (see Section 1.3.1). Furthermore we require that any interior deformation of the surface σ must leave the critical action S unchanged. In other words, on the equations defined by $\delta \mathsf{dL} = 0$, we require that the differential form L is closed, i.e., that $\mathsf{dL} = 0$.

Section 1.2 gives a brief overview of the early development of Lagrangian multiforms, and is followed by Section 1.3 which deals with the multiform Euler-Lagrange equations in greater detail. Chapter 2 is largely based on [1] and explores the link between variational symmetries and Lagrangian multiforms. In Chapter 3, based on [2], we present a Lagrangian multiform resulting from the Zakharov-Mikhailov Lagrangian [5] that reduces to give a Lagrangian multiform for a general class of Lax pairs. In Chapter 4, based on [3], we present a Lagrangian multiform for the complete KP hierarchy and perform a reduction on this multiform to obtain Lagrangian multiforms for the Gelfand-Dickey hierarchy.

[1]Both continuous and discrete Lagrangian multiforms were introduced in [4]. Our main focus will be continuous Lagrangian multiforms.

1.2 Early development of continuous Lagrangian multiform theory

In this section, we review what was known about Lagrangian multiforms prior to the commencement of this project. As already mentioned, Lagrangian multiforms were first introduced to the world in [4]. This first paper was mainly focused on discrete Lagrangian multiforms (in particular Lagrangian multiforms relating to the ABS equations) but continuous multiforms are discussed and two examples are given: the continuous 2-form

$$\sum_{i<j} \mathscr{L}_{(ij)} \mathsf{d}t_i \wedge \mathsf{d}t_j \tag{1.4}$$

with

$$\mathscr{L}_{(ij)} = \frac{1}{n_i n_j}\left(\frac{1}{2}(t_i^2 - t_j^2)\omega_{t_i t_j}^2 + (n_j^2\omega_{t_i}^2 - n_i^2\omega_{t_j}^2) + \frac{t_i^2 + t_j^2}{t_i^2 - t_j^2}(n_j\omega_{t_i} - n_i\omega_{t_j})^2\right) \tag{1.5}$$

for a non-autonomous system of mutually compatible linear PDEs, and also the continuous 2-form for the KdV generating PDE [6], with

$$\mathscr{L}_{(ij)} = \frac{1}{2}(t_i - t_j)\frac{U_{t_i t_j}^2}{U_{t_i}U_{t_j}} + \frac{1}{2(t_i - t_j)}\left(n_j^2\frac{U_{t_i}}{U_{t_j}} + n_i^2\frac{U_{t_j}}{U_{t_i}}\right). \tag{1.6}$$

It was observed that in order for a Lagrangian multiform to represent a multidimensionally consistent system, it must obey a closure relation; in the case of a 2-form this will be of the form

$$\mathrm{D}_{t_i}\mathscr{L}_{(jk)} + \mathrm{D}_{t_j}\mathscr{L}_{(ki)} + \mathrm{D}_{t_k}\mathscr{L}_{(ij)} = 0 \tag{1.7}$$

which must hold on the equations of motion of the multiform. If this relation holds then Stokes theorem tells us that the action

$$S[\omega;\sigma] = \int_\sigma \sum_{i<j} \mathscr{L}_{(ij)} \mathsf{d}t_i \wedge \mathsf{d}t_j \tag{1.8}$$

is only dependent on the boundary of the surface of integration σ, so it is invariant to deformations of the interior of σ. The requirement for a Lagrangian multiform to be closed (on the equations of the multiform) is explained through the perspective of a variational principle in [7] and again in [8], where the action functional S of a Lagrangian 1-form

$$\mathsf{L} = \mathscr{L}_{(1)}\mathsf{d}t_1 + \mathscr{L}_{(2)}\mathsf{d}t_2 \tag{1.9}$$

is evaluated over a parameterised curve $\Gamma : s \to (t_1, t_2)$. The action

$$\begin{aligned} S[u(t_1,t_2);\Gamma] &= \int_\Gamma \mathscr{L}_{(1)}\mathsf{d}t_1 + \mathscr{L}_{(2)}\mathsf{d}t_2 \\ &= \int_{s_0}^{s_1} \left(\mathscr{L}_{(1)}(t_1(s),t_2(s))\frac{\mathsf{d}t_1}{\mathsf{d}s} + \mathscr{L}_{(2)}(t_1(s),t_2(s))\frac{\mathsf{d}t_2}{\mathsf{d}s} \right)\mathsf{d}s \end{aligned} \tag{1.10}$$

is now considered a functional of both the dependent variable u and the curve of integration Γ. Γ can be deformed by letting $t_1 \to t_1 + \delta t_1$ and $t_2 \to t_2 + \delta t_2$ with

$$\delta t_1(s_0) = \delta t_1(s_1) = 0 \text{ and } \delta t_2(s_0) = \delta t_2(s_1) = 0 \tag{1.11}$$

i.e., the variations leave the end points fixed. Applying the usual variational formalism leads to

$$\frac{\partial \mathscr{L}_{(1)}}{\partial t_2} = \frac{\partial \mathscr{L}_{(2)}}{\partial t_1} \tag{1.12}$$

i.e. the condition that the 1-form is closed and that the action S is invariant under such variations of Γ. The multiform Euler-Lagrange equations (referred to as the generalized Euler-Lagrange equations) are introduced for the first time in [7] (and corrected in [8]) where they are derived by varying the dependent variable $u(t_1,t_2)$ in the multiform, and expressing derivatives of δu in terms of a component parallel to Γ and a component perpendicular to Γ. The case where L has no 2^{nd} order of higher derivatives of u is considered: the component parallel to Γ gives

$$\frac{\mathsf{d}}{\mathsf{d}s}\left(\frac{1}{\|\mathsf{d}\mathbf{t}/\mathsf{d}s\|^2}\left[\left(\frac{\mathsf{d}t_1}{\mathsf{d}s}\right)^2\frac{\partial\mathscr{L}_{(1)}}{\partial u_{t_1}}+\frac{\mathsf{d}t_1}{\mathsf{d}s}\frac{\mathsf{d}t_2}{\mathsf{d}s}\left(\frac{\partial\mathscr{L}_{(1)}}{\partial u_{t_2}}+\frac{\partial\mathscr{L}_{(2)}}{\partial u_{t_1}}\right)+\left(\frac{\mathsf{d}t_2}{\mathsf{d}s}\right)^2\frac{\partial\mathscr{L}_{(2)}}{\partial u_{t_2}}\right]\right)$$
$$-\frac{\partial\mathscr{L}_{(1)}}{\partial u}\frac{\mathsf{d}t_1}{\mathsf{d}s}-\frac{\partial\mathscr{L}_{(2)}}{\partial u}\frac{\mathsf{d}t_2}{\mathsf{d}s}=0 \tag{1.13}$$

whilst the component perpendicular to Γ gives

$$\left(\frac{\mathsf{d}t_2}{\mathsf{d}s}\right)^2\frac{\partial\mathscr{L}_{(2)}}{\partial u_{t_1}}+\frac{\mathsf{d}t_1}{\mathsf{d}s}\frac{\mathsf{d}t_2}{\mathsf{d}s}\left(\frac{\partial\mathscr{L}_{(1)}}{\partial u_{t_1}}-\frac{\partial\mathscr{L}_{(2)}}{\partial u_{t_2}}\right)-\left(\frac{\mathsf{d}t_1}{\mathsf{d}s}\right)^2\frac{\partial\mathscr{L}_{(1)}}{\partial u_{t_2}}=0. \tag{1.14}$$

Since these relations hold for any curve Γ it follows that, in addition to the usual Euler-Lagrange equations for $\mathscr{L}_{(1)}$ and $\mathscr{L}_{(2)}$,

$$\frac{\partial\mathscr{L}_{(2)}}{\partial u_{t_1}}=\frac{\partial\mathscr{L}_{(1)}}{\partial u_{t_2}}=0 \tag{1.15}$$

and

$$\frac{\partial\mathscr{L}_{(1)}}{\partial u_{t_1}}=\frac{\partial\mathscr{L}_{(2)}}{\partial u_{t_2}}. \tag{1.16}$$

These relations are generalized in [9] to the case of n component 1-forms depending n time components $t_1,\ldots,t_n$. Two examples of Lagrangian 1-forms are given in [8] where the closure relation and multiform Euler-Lagrange equations are used to obtain the potential terms of Lagrangians for the full elliptic Calogero-Moser system and the full elliptic case of the Ruijsenaars-Schneider model.

In [10] a Lagrangian 2-form

$$\mathsf{L}=\mathscr{L}_{(12)}\mathsf{d}t_1\wedge\mathsf{d}t_2+\mathscr{L}_{(23)}\mathsf{d}t_2\wedge\mathsf{d}t_3+\mathscr{L}_{(31)}\mathsf{d}t_3\wedge\mathsf{d}t_1 \tag{1.17}$$

(again with no 2^{nd} order of higher derivatives of u) is considered. The surface of integration σ is parameterised such that $\sigma:\mathbf{t}=\mathbf{t}(r,s),\ (r,s)\in\Omega\in\mathbb{R}^2$, so the action

$$\mathsf{S}[u,\sigma] = \int_\sigma \mathsf{L} = \int\int_\Omega \sum_{1\leq i<j}^{3} \Big(\mathscr{L}_{(ij)} \frac{\partial(t_i,t_j)}{\partial(r,s)}\Big) \mathsf{d}r\mathsf{d}s. \tag{1.18}$$

The closure relation

$$\mathrm{D}_{t_1} \mathscr{L}_{(23)} + \mathrm{D}_{t_2} \mathscr{L}_{(31)} + \mathrm{D}_{t_3} \mathscr{L}_{(12)} = 0 \tag{1.19}$$

is obtained by considering variations of the independent variables t_1, t_2 and t_3. The multiform Euler-Lagrange equations

$$\begin{aligned}\sum_{i<j} \Big[\frac{\partial(t_i,t_j)}{\partial(r,s)}\frac{\partial \mathscr{L}_{(ij)}}{\partial u} - \frac{\partial}{\partial r}\Big(\frac{\partial(t_i,t_j)}{\partial(r,s)}\frac{\mathbf{t}_s\times\mathbf{n}}{\|\mathbf{t}_r\times\mathbf{t}_s\|}\cdot\frac{\partial \mathscr{L}_{(ij)}}{\partial\nabla u}\Big)\\ + \frac{\partial}{\partial s}\Big(\frac{\partial(t_i,t_j)}{\partial(r,s)}\frac{\mathbf{t}_r\times\mathbf{n}}{\|\mathbf{t}_r\times\mathbf{t}_s\|}\cdot\frac{\partial \mathscr{L}_{(ij)}}{\partial\nabla u}\Big)\Big] = 0\end{aligned} \tag{1.20}$$

and

$$\sum_{i<j} \frac{\partial(t_i,t_j)}{\partial(r,s)}\mathbf{n}\cdot\frac{\partial \mathscr{L}_{(ij)}}{\partial\nabla u} = 0 \tag{1.21}$$

are obtained by considering variations of the dependent variable u parallel to σ and perpendicular to σ respectively. Here $\mathbf{n}$ is the unit normal vector to the surface $\mathbf{t}$ given by

$$\mathbf{n} = \frac{\mathbf{t}_r\times\mathbf{t}_s}{\|\mathbf{t}_r\times\mathbf{t}_s\|}. \tag{1.22}$$

It follows from (1.20) and (1.21) that, in addition to the usual Euler-Lagrange equations for each $\mathscr{L}_{(ij)}$

$$\frac{\partial \mathscr{L}_{(ij)}}{\partial u_{t_k}} = 0 \tag{1.23}$$

when $k \neq i,j$ and

$$\frac{\partial \mathscr{L}_{(ij)}}{\partial u_{t_i}} + \frac{\partial \mathscr{L}_{(jk)}}{\partial u_{t_k}} = 0 \tag{1.24}$$

for all distinct i,j,k. These relations hold for Lagrangian 2-forms depending only on the first jet. Suris and Vermeeren [11] generalized these results to Lagrangian multiforms[1] depending on arbitrarily high jets. Their methodology required the approximation of the smooth surface of integration σ with a stepped surface. In order to present their results we must introduce the following multi-index notation. We let $I = (i_1, \ldots, i_N)$ and define

$$u_I = \mathrm{D}_I\, u = \left(\prod_{\alpha=1}^{p} (\mathrm{D}_{x_\alpha})^{i_\alpha}\right) u. \tag{1.25}$$

We let $Ii_k = (i_1, \ldots, i_{k+1}, \ldots, i_N)$ and define $|I| = i_1 + \ldots + i_N$. Then, for Lagrangian 1-forms

$$\mathsf{L}^{(1)} = \sum_{i=1}^{N} \mathscr{L}_{(i)} \mathsf{d}t_i \tag{1.26}$$

Suris and Vermeeren derived the relations

$$\begin{aligned} \frac{\delta_i \mathscr{L}_{(i)}}{\delta u_I} &= 0 \quad \forall I \not\ni i \\ \frac{\delta_i \mathscr{L}_{(i)}}{\delta u_{Ii}} &= \frac{\delta_j \mathscr{L}_{(j)}}{\delta u_{Ij}} \quad \forall I \end{aligned} \tag{1.27}$$

where the variational derivative

$$\frac{\delta_i \mathscr{L}_{(i)}}{\delta u_I} = \sum_{\alpha \geq 0} (-1)^\alpha D_i^\alpha \frac{\partial \mathscr{L}_{(i)}}{\partial u_{Ii^\alpha}}. \tag{1.28}$$

In the case of Lagrangian 2-forms

$$\mathsf{L}^{(2)} = \sum_{1 \leq i < j}^{N} \mathscr{L}_{(ij)} \mathsf{d}t_i \wedge \mathsf{d}t_j, \tag{1.29}$$

they derive that the multiform Euler-Lagrange equations are given by

[1]Which they refer to a pluri-Lagrangian systems.

$$
\begin{aligned}
&\frac{\delta_{ij}\mathscr{L}_{(ij)}}{\delta u_I} = 0 \quad \forall I \not\ni i,j \\
&\frac{\delta_{ij}\mathscr{L}_{(ij)}}{\delta u_{Ij}} = \frac{\delta_{ik}\mathscr{L}_{(ik)}}{\delta u_{Ik}} \quad \forall I \not\ni i \\
&\frac{\delta_{ij}\mathscr{L}_{(ij)}}{\delta u_{Iij}} + \frac{\delta_{jk}\mathscr{L}_{(jk)}}{\delta u_{Ijk}} + \frac{\delta_{ki}\mathscr{L}_{(ki)}}{\delta u_{Iki}} = 0 \quad \forall I
\end{aligned}
\tag{1.30}
$$

where the variational derivative

$$
\frac{\delta_{ij}\mathscr{L}_{(ij)}}{\delta u_I} = \sum_{\alpha,\beta\geq 0} (-1)^{\alpha+\beta} D_i^\alpha D_j^\beta \frac{\partial \mathscr{L}_{(ij)}}{\partial u_{Ii^\alpha j^\beta}}. \tag{1.31}
$$

In the same paper, they identified that $\delta \mathsf{dL} = 0$ on critical points u of a Lagrangian multiform L, but did not identify that the equations given by $\delta \mathsf{dL} = 0$ are equivalent to the multiform Euler-Lagrange equations (i.e., those given in (1.27) for a 1-form and those given in (1.30) for a 2-form). They also present a Lagrangian multiform for the entire PKdV hierarchy; the first example of a Lagrangian multiform for an entire integrable hierarchy.

1.3 The multiform Euler-Lagrange equations

In this section we derive the multiform Euler-Lagrange equations for both continuous and discrete Lagrangian k-forms. First, we follow the argument given in [11] to show that the multiform Euler-Lagrange equations for a Lagrangian multiform L are given by $\delta \mathsf{dL} = 0$. We then demonstrate how the equations given by $\delta \mathsf{dL} = 0$ are equivalent to a set of equations in terms of variational derivatives (that include the usual Euler-Lagrange equation for each Lagrangian coefficient in the multiform). In the continuous case, this was first shown in [12]; here we present our own proof that first appeared in [1], as well as an alternative proof that makes explicit the link between the multiform Euler-Lagrange equations in terms of variational derivatives and the coefficients of dL. In the discrete case we present for the first time how the equations given by $\delta \mathsf{dL} = 0$ can be expressed in terms of variational derivatives.

1.3.1 The multiform Euler-Lagrange equations given by $\delta \mathsf{dL} = 0$

We let

$$\mathsf{L} = \sum_{1 \leq l_1 < \ldots < l_k \leq N} \mathscr{L}_{(l_1 \ldots l_k)} \, \mathsf{d}x_{l_1} \wedge \ldots \wedge \mathsf{d}x_{l_k}. \tag{1.32}$$

be a k-form on a manifold of N independent coordinates $x_1, \ldots, x_N$ and dependent variable u. We will show that u is a critical point of L over every surface of integration if and only if $\delta \mathsf{dL} = 0$ by following the argument given in [11]. We assume that L contains terms up to n^{th} order derivatives of u, (i.e. L depends on u_I with $|I| \leq n$). Let B be an arbitrary $k+1$ dimensional ball with surface ∂B. We consider the action functional S over the closed surface ∂B such that

$$S[u] = \oint_{\partial B} \mathsf{L} \tag{1.33}$$

We then apply Stokes' theorem to write S in terms of an integral over B:

$$S[u] = \int_B \mathsf{dL} \tag{1.34}$$

and we look for solutions of

$$\delta S = \int_B \delta \mathsf{dL} = 0 \tag{1.35}$$

Since this must hold for arbitrary variations (i.e. with no boundary constraints) for every ball B, it follows that u is a critical point of L if and only if the integrand $\delta \mathsf{dL} = 0$. If

$$\mathsf{L} = \sum_{1 \leq i_1 < \ldots < i_k \leq N} \mathscr{L}_{(i_1 \ldots i_k)} \, \mathsf{d}x_{i_1} \wedge \ldots \wedge \mathsf{d}x_{i_k}. \tag{1.36}$$

is a k-form on a manifold of N independent coordinates $x_1, \ldots, x_N$ and dependent variable u, then

$$\mathsf{dL} = \sum_{1 \leq i_1 < \ldots < i_{k+1} \leq N} A^{i_1 \ldots i_{k+1}} \mathsf{d}x_{i_1} \wedge \ldots \wedge \mathsf{d}x_{i_{k+1}} \tag{1.37}$$

where the $A^{i_1 \ldots i_{k+1}}$ depend on the $\mathscr{L}_{(i_1 \ldots i_k)}$ in the usual way, i.e.

$$A^{i_1\dots i_{k+1}} = \sum_{\alpha=1}^{k+1} (-1)^{k(\alpha+1)} \, \mathrm{D}_{x_{i_\alpha}} \mathscr{L}_{(i_{\alpha+1}\dots i_{k+1} i_1 \dots i_{\alpha-1})}. \tag{1.38}$$

The operator δ acts on $A^{i_1\dots i_{k+1}}$ to give

$$\delta A^{i_1\dots i_{k+1}} = \sum_I \frac{\partial A^{i_1\dots i_{k+1}}}{\partial u_I} \delta u_I, \tag{1.39}$$

where I is a multi-index as defined in (1.25), so

$$\begin{aligned}\delta \mathsf{dL} &= \sum_{1\le i_1<\dots<i_{k+1}\le N} \delta A^{i_1\dots i_{k+1}} \mathsf{d}x_{i_1} \wedge \dots \wedge \mathsf{d}x_{i_{k+1}} \\ &= \sum_{1\le i_1<\dots<i_{k+1}\le N} \sum_I \frac{\partial A^{i_1\dots i_{k+1}}}{\partial u_I} \delta u_I \wedge \mathsf{d}x_{i_1} \wedge \dots \wedge \mathsf{d}x_{i_{k+1}}.\end{aligned} \tag{1.40}$$

Therefore, the equations given by $\delta \mathsf{dL} = 0$ are precidcely the same as those given by

$$\frac{\partial A^{i_1\dots i_{k+1}}}{\partial u_I} = 0 \tag{1.41}$$

for all $1 \le i_1 < \dots < i_{k+1} \le N$ and all I.

1.3.2 The multiform Euler-Lagrange equations for a k-form in terms of variational derivatives

For a fixed $i_1, \dots, i_{k+1}$, we shall write $\mathscr{L}_{(\bar{\alpha})}$ to denote $\mathscr{L}_{(i_{\alpha+1}\dots i_{k+1} i_1 \dots i_{\alpha-1})}$. We define the variational derivative with respect to u_I *acting on* $\mathscr{L}_{(\bar{\alpha})}$

$$\frac{\delta \mathscr{L}_{(\bar{\alpha})}}{\delta u_I} = \sum_{\substack{J \\ j_{i_\alpha}=0}} (-\mathrm{D})_J \frac{\partial \mathscr{L}_{(\bar{\alpha})}}{\partial u_{IJ}}, \tag{1.42}$$

where I is the same N component multi-index introduced in (1.25) representing derivatives with respect to $x_1, \dots, x_N$, and the multi-index J is such that components $j_i = 0$ whenever $i \neq i_1, \dots, i_{k+1}$, i.e. J represents derivatives with respect to $x_{i_1}, \dots, x_{i_{k+1}}$. As a result, the operator

$$(-\mathrm{D})_J = \prod_{i=1}^{N} (-\mathrm{D}_{x_i})^{j_i}. \tag{1.43}$$

We define that $\dfrac{\delta \mathscr{L}_{(\bar{\imath})}}{\delta u_I} = 0$ in the case where any component of the multi-index I is negative. Note that by this definition, the variational derivative of $\mathscr{L}_{(i_{\alpha+1}\dots i_{k+1} i_1 \dots i_{\alpha-1})}$ with respect to u_I only sees derivatives of u_I with respect to the variables $x_{i_{\alpha+1}}, \dots x_{i_{k+1}}, x_{i_1} \dots, x_{i_{\alpha-1}}$, even though derivatives with respect to other variables may appear in $\mathscr{L}_{(i_{\alpha+1}\dots i_{k+1} i_1 \dots i_{\alpha-1})}$.

Theorem 1. *The dependent variable u is a critical point of the k-form L as defined in (1.36) if and only if for all $i_1, \dots i_{k+1}$ such that $1 \leq i_1 < \dots < i_{k+1} \leq N$, and for all I,*

$$\sum_{\alpha=1}^{k+1} (-1)^{\alpha k} \frac{\delta \mathscr{L}_{(\bar{\alpha})}}{\delta u_{I \setminus i_\alpha}} = 0 \tag{1.44}$$

In order to prove that these are the multiform EL equations, we will require the following lemma:

Lemma 2. *Let $1 \leq i_1 < \dots < i_{k+1} \leq N$ be fixed. For all multi-indices I,*

$$\frac{\partial \mathscr{L}_{(\bar{\alpha})}}{\partial u_I} = \sum_{\substack{J \\ j_i \leq 1 \\ j_{i_\alpha} = 0}} \mathrm{D}_J \frac{\delta \mathscr{L}_{(\bar{\alpha})}}{\delta u_{IJ}} \tag{1.45}$$

where the summation is over all multi-indices J as defined for (1.42), such that the i_α^{th} component of J is zero and the non-zero j_i are equal to 1.

Proof. We first notice that the partial derivative on the left hand side of (1.45) appears only once in the sum on the right hand side. We now need to show that all other terms that appear on the right hand side of (1.45), which are all of the form $\mathrm{D}_A \dfrac{\partial \mathscr{L}_{(\bar{\alpha})}}{\partial u_{IA}}$ for some multi-index A, sum to zero. To show this, we consider the term $\mathrm{D}_A \dfrac{\partial \mathscr{L}_{(\bar{\alpha})}}{\partial u_{IA}}$, and let r be the number of non-zero entries in A. We notice that this term appears exactly once when $|J| = 0$ with a factor of $(-1)^{|A|}$, exactly $\binom{r}{1}$ times with a factor of $(-1)^{|A|+1}$ when $|J| = 1$, exactly $\binom{r}{2}$

times with a factor of $(-1)^{|A|+2}$ when $|J|= 2$ etc... In total, this term appears with a factor of $\pm\sum_{i=0}^{r}(-1)^i\binom{r}{i}$. It can easily be seen that this sum is zero by considering the binomial expansion of $(1-1)^r$.

$\square$

Proof. (of Theorem 1) We have already shown that u is a critical point of L over every surface of integration if and only if $\delta\mathsf{dL}=0$. Since

$$\delta\mathsf{dL}=\sum_{1\leq i_1<\ldots<i_{k+1}\leq N}\sum_{I}\frac{\partial A^{i_1\ldots i_{k+1}}}{\partial u_I}\delta u_I\wedge\mathsf{d}x_{i_1}\wedge\ldots\wedge\mathsf{d}x_{i_{k+1}},\tag{1.46}$$

the set of equations given by $\delta\mathsf{dL}=0$ are equivalent to those given by

$$\frac{\partial A^{i_1\ldots i_{k+1}}}{\partial u_I}=0\tag{1.47}$$

for all $1\leq i_1<\ldots<i_{k+1}\leq N$ and for all I.

In order to proceed, we must show that, for any choice of $1\leq i_1<\ldots<i_{k+1}\leq N$, (1.47) holds if and only if $\forall I$,

$$\sum_{\alpha=1}^{k+1}(-1)^{\alpha k}\frac{\delta\mathscr{L}_{(\bar{\alpha})}}{\delta u_{I\setminus i_\alpha}}=0.\tag{1.48}$$

To do this, we first show that (1.48) holds for $|I|>n$. We then use an inductive argument to show that if (1.48) holds for $|I|>m$ then it also holds for $|I|=m$. The converse (that (1.48) $\implies$ (A.5)) is then easily seen from the intermediary steps of the proof.

We begin by (arbitrarily) fixing $1\leq i_1<\ldots<i_{k+1}\leq N$ and noticing that for $|I|\geq n+2$, (1.48) holds. In fact all terms are zero since, by definition, there are no $n+1^{th}$ order derivatives in our multiform. We now consider the relation $\dfrac{\partial A^{i_1\ldots i_{k+1}}}{\partial u_I}=0$ in the case where $|I|=n+1$. In this case we find that

$$\frac{\partial A^{i_1\ldots i_{k+1}}}{\partial u_I}=\sum_{\alpha=1}^{k+1}(-1)^{\alpha k+1}\frac{\partial\mathscr{L}_{(\bar{\alpha})}}{\partial u_{I\setminus i_\alpha}}\tag{1.49}$$

since there are no $n+1^{th}$ order derivatives in the $\mathscr{L}_{(\bar{\alpha})}$. By setting this equal to zero, we see that (1.48) holds in the case where $|I|= n+1$.

Our inductive hypothesis is that (1.48) holds for $|I|> m$. We now consider the relation $\dfrac{\partial A^{i_1\dots i_{k+1}}}{\partial u_I} = 0$ in the case where $|I|= m$. We now notice that

$$\begin{aligned}
\frac{\partial A^{i_1\dots i_{k+1}}}{\partial u_I} &= \sum_{\alpha=1}^{k+1}(-1)^{\alpha k+1}\frac{\partial}{\partial u_I}\,\mathrm{D}_{x_{i_\alpha}}\mathscr{L}_{(\bar\alpha)} \\
&= \sum_{\alpha=1}^{k+1}(-1)^{\alpha k+1}\left\{\frac{\partial \mathscr{L}_{(\bar\alpha)}}{\partial u_{I\backslash i_\alpha}} + \mathrm{D}_{x_{i_\alpha}}\frac{\partial \mathscr{L}_{(\bar\alpha)}}{\partial u_I}\right\} \\
&= \sum_{\alpha=1}^{k+1}(-1)^{\alpha k+1}\left\{\frac{\partial \mathscr{L}_{(\bar\alpha)}}{\partial u_{I\backslash i_\alpha}} + \sum_{\substack{J\\ j_i\le 1\\ j_{i_\alpha}=0}}\mathrm{D}_{Ji_\alpha}\frac{\delta \mathscr{L}_{(\bar\alpha)}}{\delta u_{IJ}}\right\} \\
&= \sum_{\alpha=1}^{k+1}(-1)^{\alpha k+1}\left\{\frac{\partial \mathscr{L}_{(\bar\alpha)}}{\partial u_{I\backslash i_\alpha}} + \sum_{\substack{J\\ j_i\le 1\\ j_{i_\alpha}=1}}\mathrm{D}_{J}\frac{\delta \mathscr{L}_{(\bar\alpha)}}{\delta u_{IJ\backslash i_\alpha}}\right\} \\
&= \sum_{\alpha=1}^{k+1}(-1)^{\alpha k+1}\left\{\frac{\partial \mathscr{L}_{(\bar\alpha)}}{\partial u_{I\backslash i_\alpha}}\right\} + \sum_{\substack{J\\ j_i\le 1\\ |J|>0}}\sum_{\substack{\alpha\\ j_{i_\alpha}>0}}(-1)^{\alpha k+1}D_J\frac{\delta \mathscr{L}_{(\bar\alpha)}}{\delta u_{IJ\backslash i_\alpha}}
\end{aligned} \tag{1.50}$$

where we have made use of (1.45) in the third line, re-labeled J in the fourth line and changed the order of the summation in the last. We now apply the inductive hypothesis to get

$$\begin{aligned}
\frac{\partial A^{i_1\dots i_{k+1}}}{\partial u_I} &= \sum_{\alpha=1}^{k+1}(-1)^{\alpha k+1}\left\{\frac{\partial \mathscr{L}_{(\bar\alpha)}}{\partial u_{I\backslash i_\alpha}}\right\} + \sum_{\substack{J\\ j_i\le 1\\ |J|>0}}\sum_{\substack{\alpha\\ j_{i_\alpha}=0}}(-1)^{\alpha k}\,\mathrm{D}_J\frac{\delta \mathscr{L}_{(\bar\alpha)}}{\delta u_{IJ\backslash i_\alpha}} \\
&= \sum_{\alpha=1}^{k+1}(-1)^{\alpha k+1}\left\{\frac{\partial \mathscr{L}_{(\bar\alpha)}}{\partial u_{I\backslash i_\alpha}} - \sum_{\substack{J\\ j_i\le 1\\ j_{i_\alpha}=0\\ |J|>0}}\mathrm{D}_J\frac{\delta \mathscr{L}_{(\bar\alpha)}}{\delta u_{IJ\backslash i_\alpha}}\right\} = 0.
\end{aligned} \tag{1.51}$$

Finally, we use (1.45) to express this as

$$\frac{\partial A^{i_1\dots i_{k+1}}}{\partial u_I} = \sum_{\alpha=1}^{k+1}(-1)^{\alpha k+1}\frac{\delta \mathscr{L}_{(\bar{\alpha})}}{\delta u_{I\setminus i_\alpha}} = 0 \tag{1.52}$$

and we have shown that (1.48) holds for $|I|= m$. By induction, it follows that (1.48) holds for all I. The converse can easily be seen to hold by following the steps taken in (1.50), (1.51) and (1.52) in reverse order.

We have shown that the multiform EL equations (1.44) for a given $1 \le i_1 < \ldots < i_{k+1} \le N$ are equivalent to $\delta A^{i_1\cdots i_{k+1}} = 0$ for the same $1 \le i_1 < \ldots < i_{k+1} \le N$. It follows that the multiform EL equations holding for all $1 \le i_1 < \ldots < i_{k+1} \le N$ is equivalent to $\delta \mathsf{dL} = 0$. □

This proof is an extension to k-forms of the proof of the multiform Euler-Lagrange equations for a Lagrangian 2-form that originally appeared in [2]. We reproduce the original proof in Appendix A.

1.3.3 Multiform Euler-Lagrange equations in terms of variational derivatives of dL

In this section, we present an alternative proof of the multiform Euler-Lagrange equations for a Lagrangian multiform

$$\mathsf{L} = \sum_{1\le i_1<\ldots<i_k\le N} \mathscr{L}_{(i_1\dots i_k)}\, \mathsf{d}x_{i_1}\wedge\ldots\wedge \mathsf{d}x_{i_k}, \tag{1.53}$$

that also gives explicitly the link between the equations in terms of variational derivatives of the $\mathscr{L}_{(i_1\dots i_k)}$ and the $A^{i_1\cdots i_{k+1}}$ defined by

$$\mathsf{dL} = \sum_{1\le i_1<\ldots<i_{k+1}\le N} A^{i_1\dots i_{k+1}}\mathsf{d}x_{i_1}\wedge\ldots\wedge \mathsf{d}x_{i_{k+1}}. \tag{1.54}$$

In terms of the $\mathscr{L}_{(i_1\dots i_k)}$,

$$A^{i_1\dots i_{k+1}} = \sum_{\alpha=1}^{k+1}(-1)^{\alpha+1}\, \mathrm{D}_{x_{i_\alpha}}\, \mathscr{L}_{(i_1\dots i_{\alpha-1}i_{\alpha+1}\dots i_{k+1})}. \tag{1.55}$$

We recall that the multiform Euler-Lagrange equations are given by $\delta \mathsf{dL} = 0$. We again use I to represent the N component multi-index introduced in Section 1.2, allowing us to express the multiform Euler-Lagrange equations given by $\delta \mathsf{dL} = 0$ in the form

$$\frac{\partial}{\partial u_I} A^{i_1 \dots i_{k+1}} = 0 \tag{1.56}$$

for all $1 \leq i_1 < \dots < i_{k+1}$ and all multi-indices I. For a fixed choice of $i_1 \dots i_{k+1}$, we shall again write $\mathscr{L}_{(\bar{\alpha})}$ to denote $\mathscr{L}_{(i_1 \dots i_{\alpha-1} i_{\alpha+1} \dots i_{k+1})}$. Again, we define

$$\frac{\delta \mathscr{L}_{(\bar{\alpha})}}{\delta u_I} = \sum_{\substack{J \\ j_{i_\alpha}=0}} (-\mathrm{D})_J \frac{\partial \mathscr{L}_{(\bar{\alpha})}}{\partial u_{IJ}}, \tag{1.57}$$

where the multi-index J is such that components $j_\alpha = 0$ whenever $\alpha \neq i_1, \dots, i_{k+1}$, i.e. J represents derivatives with respect to $x_{i_1}, \dots, x_{i_{k+1}}$ only. We define that $\frac{\delta \mathscr{L}_{(\bar{\alpha})}}{\delta u_I} = 0$ in the case where any component of the multi-index I is negative. The identity

$$\frac{\partial}{\partial u_I} \mathrm{D}_{x_i} = \frac{\partial}{\partial u_{I \setminus i}} + \mathrm{D}_{x_i} \frac{\partial}{\partial u_I} \tag{1.58}$$

tells us that

$$\frac{\partial}{\partial u_I} A^{i_1 \dots i_{k+1}} = \sum_{\alpha=1}^{k+1} (-1)^{\alpha+1} \left(\frac{\partial \mathscr{L}_{(\bar{\alpha})}}{\partial u_{I \setminus i_\alpha}} + \mathrm{D}_{x_{i_\alpha}} \frac{\partial \mathscr{L}_{(\bar{\alpha})}}{\partial u_I} \right) \tag{1.59}$$

so

$$\begin{aligned} \frac{\delta}{\delta u_I} A^{i_1 \dots i_{k+1}} &= \sum_J (-\mathrm{D})_J \frac{\partial}{\partial u_{IJ}} A^{i_1 \dots i_{k+1}} \\ &= \sum_J (-\mathrm{D})_J \sum_{\alpha=1}^{k+1} (-1)^{\alpha+1} \left(\frac{\partial \mathscr{L}_{(\bar{\alpha})}}{\partial u_{IJ \setminus i_\alpha}} + \mathrm{D}_{x_{i_\alpha}} \frac{\partial \mathscr{L}_{(\bar{\alpha})}}{\partial u_{IJ}} \right). \end{aligned} \tag{1.60}$$

Whenever $j_{i_\alpha} \neq 0$ in this sum, so J is of the form $K i_\alpha$ for some multi-index K, then

$$\pm(-\mathrm{D})_J \frac{\partial \mathscr{L}_{(\bar{\alpha})}}{\partial u_{IJ\backslash i_\alpha}} = \mp \mathrm{D}_{x_{i_\alpha}} (-\mathrm{D})_K \frac{\partial \mathscr{L}_{(\bar{\alpha})}}{\partial u_{IK}} \tag{1.61}$$

will appear in this sum. When $J = K$, the term

$$\pm(-\mathrm{D})_K \, \mathrm{D}_{x_{i_\alpha}} \frac{\partial \mathscr{L}_{(\bar{\alpha})}}{\partial u_{IK}} \tag{1.62}$$

will appear. These two terms cancel, so (1.60) simplifies to

$$\begin{aligned} \frac{\delta}{\delta u_I} A^{i_1 \dots i_{k+1}} &= \sum_{\alpha=1}^{k+1} \sum_{\substack{J \\ j_{i_\alpha}=0}} (-1)^{\alpha+1} (-\mathrm{D})_J \frac{\partial \mathscr{L}_{(\bar{\alpha})}}{\partial u_{IJ\backslash i_\alpha}} \\ &= \sum_{\alpha=1}^{k+1} (-1)^{\alpha+1} \frac{\delta \mathscr{L}_{(\bar{\alpha})}}{\delta u_{I\backslash i_\alpha}}. \end{aligned} \tag{1.63}$$

It follows that if (1.56) holds, then

$$\frac{\delta}{\delta u_I} A^{i_1 \dots i_{k+1}} = \sum_{\alpha=1}^{k+1} (-1)^{\alpha+1} \frac{\delta \mathscr{L}_{(\bar{\alpha})}}{\delta u_{I\backslash i_\alpha}} = 0. \tag{1.64}$$

We have shown that

$$\delta \mathsf{dL} = 0 \implies \frac{\delta}{\delta u_I} A^{i_1 \dots i_{k+1}} = \sum_{\alpha=1}^{k+1} (-1)^{\alpha+1} \frac{\delta \mathscr{L}_{(\bar{\alpha})}}{\delta u_{I\backslash i_\alpha}} = 0 \tag{1.65}$$

for all $1 \leq i_1 \leq \ldots \leq i_{k+1} \leq N$ and I. By the identity is given (1.45)

$$\frac{\partial A^{i_1 \dots i_{k+1}}}{\partial u_I} = \sum_{\substack{J \\ j_i \leq 1}} \mathrm{D}_J \frac{\delta A^{i_1 \dots i_{k+1}}}{\delta u_{IJ}} \tag{1.66}$$

and it follows that the converse to (1.100) also holds. We summarise this result in the following theorem:

Theorem 3. *For a differential k-form L as given in (1.53), and $A^{i_1 \dots i_{k+1}}$ as defined in (1.55),*

$$\frac{\delta}{\delta u_I} A^{i_1 \dots i_{k+1}} = \sum_{\alpha=1}^{k+1} (-1)^{\alpha+1} \frac{\delta \mathscr{L}_{(\bar{\alpha})}}{\delta u_{I \setminus i_\alpha}}. \tag{1.67}$$

The set of equations defined by

$$\frac{\delta}{\delta u_I} A^{i_1 \dots i_{k+1}} = 0 \tag{1.68}$$

for all $1 \leq i_1 \leq \ldots \leq i_{k+1} \leq N$ *and* I *is equivalent to the set of equations defined by* $\delta \mathsf{dL} = 0$.

Corollary 4. *A corollary of Theorem 7 is that*

$$\frac{\delta}{\delta u_{x_{i_\alpha}}} A^{i_1 \dots i_{k+1}} = (-1)^{\alpha+1} \frac{\delta \mathscr{L}_{(i_1 \dots i_{\alpha-1} i_{\alpha+1} \dots i_{k+1})}}{\delta u}, \tag{1.69}$$

so the usual Euler-Lagrange equations of each Lagrangian coefficient in L *can be expressed in terms of variational derivatives of the coefficients of* dL.

1.3.4 Discrete Lagrangian k-form EL equations

The discrete multiform Euler-Lagrange equations take a very similar form to their continuous counterparts. Unsurprisingly, much of this section closely mirrors the previous one where we considered the continuous case.

On a discrete manifold of N independent coordinates $n_1, \ldots, n_N$ and dependent variable u, we define the shift operator T_i such that

$$\mathrm{T}_i \, u(n_1, \ldots, n_i, \ldots, n_N) = u(n_1, \ldots, n_i + 1, \ldots, n_N) \tag{1.70}$$

and the discrete derivative D_i such that

$$\mathrm{D}_i \, u = \mathrm{T}_i \, u - u \tag{1.71}$$

We let

$$\mathsf{L} = \sum_{1 \leq i_1 < \ldots < i_k \leq N} \mathscr{L}_{(i_1 \dots i_k)} \; \mathsf{d} n_{i_1} \wedge \ldots \wedge \mathsf{d} n_{i_k}. \tag{1.72}$$

be a k-form, such that each $\mathscr{L}_{(i_1 \dots i_k)}$ depends on u and shifts of u up to order M (without loss of generality, we shall assume that there are no backward shifts). Therefore

$$\mathsf{dL} = \sum_{1 \le i_1 < \dots < i_{k+1} \le N} A^{i_1 \dots i_{k+1}} \mathsf{d}n_{i_1} \wedge \dots \wedge \mathsf{d}n_{i_{k+1}} \tag{1.73}$$

where the $A^{i_1 \dots i_{k+1}}$ depend on the $\mathscr{L}_{(i_1 \dots i_k)}$ in the usual way, i.e.

$$A^{i_1 \dots i_{k+1}} = \sum_{\alpha=1}^{k+1} (-1)^{k(\alpha+1)} \, \mathrm{D}_{i_\alpha} \, \mathscr{L}_{(i_{\alpha+1} \dots i_{k+1} i_1 \dots i_{\alpha-1})}. \tag{1.74}$$

For a fixed $i_1, \dots, i_{k+1}$, we shall write $\mathscr{L}_{(\bar{\alpha})}$ to denote $\mathscr{L}_{(i_{\alpha+1} \dots i_{k+1} i_1 \dots i_{\alpha-1})}$. We define the variational derivative with respect to u_I *acting on* $\mathscr{L}_{(\bar{\alpha})}$

$$\frac{\delta \mathscr{L}_{(\bar{\alpha})}}{\delta u_I} = \sum_{\substack{J \\ j_{i_\alpha}=0}} (\mathrm{T}^{-1})_J \frac{\partial \mathscr{L}_{(\bar{\alpha})}}{\partial u_{IJ}}, \tag{1.75}$$

where I is an N component multi-index $(i_1, \dots, i_N)$ representing shifts with respect to $n_1, \dots, n_N$ such that

$$u_I = \mathrm{T}_I \, u = \mathrm{T}_1^{i_1} \dots \mathrm{T}_N^{i_N} \, u. \tag{1.76}$$

The multi-indices J are such that components $j_i = 0$ whenever $i \neq i_1, \dots, i_{k+1}$, i.e. J represents shifts with respect to $n_{i_1}, \dots, n_{i_{k+1}}$. Note that by this definition, the variational derivative of $\mathscr{L}_{(i_{\alpha+1} \dots i_{k+1} i_1 \dots i_{\alpha-1})}$ only sees shifts of u_I with respect to the variables $n_{i_{\alpha+1}}, \dots n_{i_{k+1}}, n_{i_1} \dots, n_{i_{\alpha-1}}$, even though shifts on with respect to other variables may appear in $\mathscr{L}_{(i_{\alpha+1} \dots i_{k+1} i_1 \dots i_{\alpha-1})}$.

Theorem 5. *The dependent variable u is a critical point of the k-form L as defined in (1.72) if and only if for all $i_1, \dots i_{k+1}$ such that $1 \le i_1 < \dots < i_{k+1} \le N$, and for all I,*

$$\sum_{\alpha=1}^{k+1} (-1)^{\alpha k} \, \mathrm{T}_{i_\alpha} \frac{\delta \mathscr{L}_{(\bar{\alpha})}}{\delta u_{I \setminus i_\alpha}} = 0 \tag{1.77}$$

In order to prove that these are the multiform EL equations, we will require the following lemma:

Lemma 6. *Let $1 \leq i_1 < \ldots < i_{k+1} \leq N$ be fixed. For all multi-indices I,*

$$\frac{\partial \mathscr{L}_{(\bar{\alpha})}}{\partial u_I} = \sum_{\substack{J \\ j_i \leq 1 \\ j_{i_\alpha}=0}} (-\mathrm{T}^{-1})_J \frac{\delta \mathscr{L}_{(\bar{\alpha})}}{\delta u_{IJ}} \tag{1.78}$$

where the summation is over all multi-indices J as defined for (1.75), such that the i_α^{th} component of J is zero and the non-zero j_i are equal to 1.

Proof. We first notice that the partial derivative on the left hand side of (1.78) appears only once in the sum on the right hand side. We now need to show that all other terms that appear on the right hand side of (1.78), sum to zero. We note that all terms on the right hand side of (1.78) are of the form $\mathrm{T}_A \dfrac{\partial \mathscr{L}_{(\bar{\alpha})}}{\partial u_{IA}}$ for some multi-index A which is of the same form as the multi-index J as defined in (1.75). To show that these terms sum to zero, we consider the term $\mathrm{T}_A \dfrac{\partial \mathscr{L}_{(\bar{\alpha})}}{\partial u_{IA}}$ for an arbitrary A, and let r be the number of non-zero entries in A. We notice that this term appears exactly once when $|J|= 0$, exactly $\binom{r}{1}$ times with a factor of -1 when $|J|= 1$, exactly $\binom{r}{2}$ times when $|J|= 2$ etc... In total, this term appears with a factor of $\sum_{i=0}^{r}(-1)^i\binom{r}{i}$. It can easily be seen that this sum is zero by considering the binomial expansion of $(1-1)^r$.

□

Proof. (of Theorem 5)
In order for u to be a critical point of the multiform L, we require that u is a critical point of the action

$$S = \sum_{\Omega} \mathsf{L} \tag{1.79}$$

for any choice of the surface Ω. Following a similar approach to the continuous case, we now apply the discrete analogue Stokes' theorem as given in [13] and [14]. We let H be any $k+1$ dimensional hypercube in dimensions $n_{i_1}, \ldots, n_{i_{k+1}}$, and let ∂H be the surface of H. We define

$$S_H = \sum_{\partial H} \mathsf{L}. \tag{1.80}$$

Then u is a critical point of the multiform L if and only if u is a critical point of every S_H, i.e. if $\delta S_H = 0$ for all H. In order to proceed, first note that

$$S_H = \pm \mathrm{T}_I \sum_{\alpha=1}^{k+1} ((-1)^{k\alpha} \mathrm{D}_{i_\alpha} \mathscr{L}_{(\bar{\alpha})}) = \pm \mathrm{T}_I \, A^{i_1 \ldots i_{k+1}} = \sum_H \mathsf{dL}. \tag{1.81}$$

We note that since any closed surface B can be composed of hypercubes, (1.81) can be generalised to obtain

$$S_B = \sum_{\partial B} \mathsf{L} = \sum_B \mathsf{dL}, \tag{1.82}$$

the discrete analogue of Stokes theorem (which, under continuum limit, gives the continuous Stokes theorem). The requirement that $\delta S_H = 0$ for all H is equivalent to the requirement that for all $1 \leq i_1 \leq \ldots \leq i_{k+1} \leq N$ and for all I,

$$\frac{\partial}{\partial u_I} A^{i_1 \ldots i_{k+1}} = 0 \tag{1.83}$$

We could stop here, and use (1.83) as our multiform EL equations. Indeed, this is often the most convenient formulation to use. However, as we did in the continuous case, we will express this in terms of variational derivatives; by doing so we see more clearly the interplay between the constituent $\mathscr{L}_{(l_1 \ldots l_k)}$ and, for example, see that a consequence of (1.83) is that $\mathrm{E}(\mathscr{L}_{(l_1 \ldots l_k)}) = 0$ for each $\mathscr{L}_{(l_1 \ldots l_k)}$.

For the second part of this proof, we show that, for any choice of $1 \leq i_1 < \ldots < i_{k+1} \leq N$, (1.83) holds if and only if $\forall I$,

$$\sum_{\alpha=1}^{k+1} (-1)^{\alpha k} \, \mathrm{T}_\alpha \frac{\delta \mathscr{L}_{(\bar{\alpha})}}{\delta u_{I \setminus i_\alpha}} = 0. \tag{1.84}$$

To do this, we first show that (1.84) holds for $|I| > M$. We then use an inductive argument to show that if (1.84) holds for $|I| > m$ then it also holds for $|I| = m$. The converse (that (1.84) $\implies$ (1.83)) is then easily seen from the intermediary steps of the proof.

We begin by (arbitrarily) fixing $1 \leq i_1 < \ldots < i_{k+1} \leq N$ (along with the corresponding H), and noticing that for $|I| \geq M + 2$, (1.84) holds. In fact all terms are zero since, by definition, there are no $M+1^{th}$ order shifts in our multiform.

Our inductive hypothesis is that (1.84) holds for $|I|>m$. We now consider the relation $\frac{\partial S_H}{\partial u_I}=0$ in the case where $|I|=m$.

We now notice that

$$
\begin{aligned}
\frac{\partial S_H}{\partial u_I} &= \sum_{\alpha=1}^{k+1}(-1)^{\alpha k}\frac{\partial}{\partial u_I}\,\mathrm{D}_{i_\alpha}\mathscr{L}_{(\bar\alpha)} \\
&= \sum_{\alpha=1}^{k+1}(-1)^{\alpha k}\left\{\mathrm{T}_{i_\alpha}\frac{\partial \mathscr{L}_{(\bar\alpha)}}{\partial u_{I\setminus i_\alpha}} - \frac{\partial \mathscr{L}_{(\bar\alpha)}}{\partial u_I}\right\} \\
&= \sum_{\alpha=1}^{k+1}(-1)^{\alpha k}\left\{\mathrm{T}_{i_\alpha}\frac{\partial \mathscr{L}_{(\bar\alpha)}}{\partial u_{I\setminus i_\alpha}} - \sum_{\substack{J\\ j_i\le 1\\ j_{i_\alpha}=0}}(-\mathrm{T}^{-1})_J\frac{\delta \mathscr{L}_{(\bar\alpha)}}{\delta u_{IJ}}\right\} \\
&= \sum_{\alpha=1}^{k+1}(-1)^{\alpha k}\left\{\mathrm{T}_{i_\alpha}\frac{\partial \mathscr{L}_{(\bar\alpha)}}{\partial u_{I\setminus i_\alpha}} - \sum_{\substack{J\\ j_i\le 1\\ j_{i_\alpha}=1}}(-T^{-1})_{J\setminus i_\alpha}\frac{\delta \mathscr{L}_{(\bar\alpha)}}{\delta u_{IJ\setminus i_\alpha}}\right\} \\
&= \sum_{\alpha=1}^{k+1}(-1)^{\alpha k}\left\{\mathrm{T}_{i_\alpha}\frac{\partial \mathscr{L}_{(\bar\alpha)}}{\partial u_{I\setminus i_\alpha}}\right\} - \sum_{\substack{J\\ j_i\le 1\\ |J|>0}}\sum_{\substack{\alpha\\ j_{i_\alpha}>0}}(-1)^{\alpha k}(-\mathrm{T}^{-1})_{J\setminus i_\alpha}\frac{\delta \mathscr{L}_{(\bar\alpha)}}{\delta u_{IJ\setminus i_\alpha}}
\end{aligned}
\tag{1.85}
$$

where we have made use of (1.78) in the third line, re-labeled J in the fourth line and changed the order of the summation in the last. We now apply the inductive hypothesis to get

$$
\begin{aligned}
\frac{\partial S_H}{\partial u_I} &= \sum_{\alpha=1}^{k+1}(-1)^{\alpha k}\left\{ \mathrm{T}_{i_\alpha}\frac{\partial \mathscr{L}_{(\bar{\alpha})}}{\partial u_{I\backslash i_\alpha}}\right\} + \sum_{\substack{J\\ j_i\leq 1\\ |J|>0}}\ \sum_{\substack{\alpha\\ j_{i_\alpha}=0}}(-1)^{\alpha k}(-\mathrm{T}^{-1})_{J\backslash i_\alpha}\frac{\delta \mathscr{L}_{(\bar{\alpha})}}{\delta u_{IJ\backslash i_\alpha}}\\
&= \sum_{\alpha=1}^{k+1}(-1)^{\alpha k}\left\{ \mathrm{T}_{i_\alpha}\frac{\partial \mathscr{L}_{(\bar{\alpha})}}{\partial u_{I\backslash i_\alpha}} - \sum_{\substack{J\\ j_i\leq 1\\ j_{i_\alpha}=0\\ |J|>0}}(-\mathrm{T}^{-1})_J\frac{\delta \mathscr{L}_{(\bar{\alpha})}}{\delta u_{IJ\backslash i_\alpha}}\right\} = 0.
\end{aligned}
\tag{1.86}
$$

Finally, we use (1.78) to express this as

$$
\frac{\partial S_H}{\partial u_I} = \sum_{\alpha=1}^{k+1}(-1)^{\alpha k}\,\mathrm{T}_{i_\alpha}\frac{\delta \mathscr{L}_{(\bar{\alpha})}}{\delta u_{I\backslash i_\alpha}} = 0 \tag{1.87}
$$

and we have shown that (1.84) holds for $|I|= m$. By induction, it follows that (1.84) holds for all I. The converse can easily be seen to hold by following the steps taken in (1.85), (1.86) and (A.15) in reverse order.

We have shown that the multiform EL equations (1.77) for a given $1 \leq i_1 < \ldots < i_{k+1} \leq N$ are equivalent to $\delta S_H = 0$ for the same $1 \leq i_1 < \ldots < i_{k+1} \leq N$. It follows that if the multiform EL equations hold for all $1 \leq i_1 < \ldots < i_{k+1} \leq N$ then u is a critical point of the multiform L. $\square$

1.3.5 Discrete multiform Euler-Lagrange equations in terms of variational derivatives of dL

As we did for the continuous case, we present an alternative proof of the multiform Euler-Lagrange equations for a discrete Lagrangian multiform

$$
\mathsf{L} = \sum_{1\leq i_1<\ldots<i_k\leq N} \mathscr{L}_{(i_1\ldots i_k)}\,\mathsf{d}n_{i_1}\wedge\ldots\wedge\mathsf{d}n_{i_k}, \tag{1.88}
$$

that also gives explicitly the link between the equations in terms of variational derivatives of the $\mathscr{L}_{(i_1\ldots i_k)}$ and the $A^{i_1\ldots i_{k+1}}$ defined by

$$\mathsf{dL} = \sum_{1\leq i_1<\ldots<i_{k+1}\leq N} A^{i_1\ldots i_{k+1}} \mathsf{d}n_{i_1} \wedge \ldots \wedge \mathsf{d}n_{i_{k+1}}. \tag{1.89}$$

In terms of the $\mathscr{L}_{(i_1\ldots i_k)}$,

$$A^{i_1\ldots i_{k+1}} = \sum_{\alpha=1}^{k+1} (-1)^{\alpha+1}\, \mathrm{D}_{i_\alpha}\, \mathscr{L}_{(i_1\ldots i_{\alpha-1} i_{\alpha+1}\ldots i_{k+1})}. \tag{1.90}$$

We recall that the multiform Euler-Lagrange equations are given by $\delta\mathsf{dL} = 0$. We again use I to represent the N component multi-index introduced in (1.76), allowing us to express the multiform Euler-Lagrange equations given by $\delta\mathsf{dL} = 0$ in the form

$$\frac{\partial}{\partial u_I} A^{i_1\ldots i_{k+1}} = 0 \tag{1.91}$$

for all $1 \leq i_1 < \ldots < i_{k+1}$ and all multi-indices I. For a fixed choice of $i_1 \ldots i_{k+1}$, we shall again write $\mathscr{L}_{(\bar{\alpha})}$ to denote $\mathscr{L}_{(i_1\ldots i_{\alpha-1} i_{\alpha+1}\ldots i_{k+1})}$. Again, we define the variational derivative

$$\frac{\delta \mathscr{L}_{(\bar{\alpha})}}{\delta u_I} = \sum_{\substack{J \\ j_{i_\alpha}=0}} (\mathrm{T}^{-1})_J \frac{\partial \mathscr{L}_{(\bar{\alpha})}}{\partial u_{IJ}}, \tag{1.92}$$

where the multi-index J is such that components $j_\alpha = 0$ whenever $\alpha \neq i_1, \ldots, i_{k+1}$, i.e. J represents shifts with respect to $n_{i_1}, \ldots, n_{i_{k+1}}$ only. We define that $\dfrac{\delta \mathscr{L}_{(\bar{\alpha})}}{\delta u_I} = 0$ in the case where any component of the multi-index I is negative. The identity

$$\frac{\partial}{\partial u_I}\, \mathrm{T}_i = \mathrm{T}_i\, \frac{\partial}{\partial u_{I\setminus i}} \tag{1.93}$$

tells us that

$$\frac{\partial}{\partial u_I} A^{i_1\ldots i_{k+1}} = \sum_{\alpha=1}^{k+1} (-1)^{\alpha+1} \left(\mathrm{T}_{i_\alpha} \frac{\partial \mathscr{L}_{(\bar{\alpha})}}{\partial u_{I\setminus i_\alpha}} - \frac{\partial \mathscr{L}_{(\bar{\alpha})}}{\partial u_I} \right) \tag{1.94}$$

so

$$\begin{aligned}\frac{\delta}{\delta u_I} A^{i_1 \dots i_{k+1}} &= \sum_J (\mathrm{T}^{-1})_J \frac{\partial}{\partial u_{IJ}} A^{i_1 \dots i_{k+1}} \\ &= \sum_J (\mathrm{T}^{-1})_J \sum_{\alpha=1}^{k+1} (-1)^{\alpha+1} \left(\mathrm{T}_{i_\alpha} \frac{\partial \mathscr{L}_{(\bar{\alpha})}}{\partial u_{IJ \setminus i_\alpha}} - \frac{\partial \mathscr{L}_{(\bar{\alpha})}}{\partial u_{IJ}} \right).\end{aligned} \tag{1.95}$$

Whenever $j_{i_\alpha} \neq 0$ in this sum, so J is of the form $K i_\alpha$ for some multi-index K, then

$$\pm (\mathrm{T}^{-1})_J \, \mathrm{T}_{i_\alpha} \frac{\partial \mathscr{L}_{(\bar{\alpha})}}{\partial u_{IJ \setminus i_\alpha}} = \pm (\mathrm{T}^{-1})_K \frac{\partial \mathscr{L}_{(\bar{\alpha})}}{\partial u_{IK}} \tag{1.96}$$

will appear in this sum. When $J = K$, the term

$$\mp (\mathrm{T}^{-1})_K \frac{\partial \mathscr{L}_{(\bar{\alpha})}}{\partial u_{IK}} \tag{1.97}$$

will appear. These two terms cancel, so (1.95) simplifies to

$$\begin{aligned}\frac{\delta}{\delta u_I} A^{i_1 \dots i_{k+1}} &= \sum_{\alpha=1}^{k+1} \sum_{\substack{J \\ j_{i_\alpha}=0}} (-1)^{\alpha+1} (\mathrm{T}^{-1})_J \, \mathrm{T}_{i_\alpha} \frac{\partial \mathscr{L}_{(\bar{\alpha})}}{\partial u_{IJ \setminus i_\alpha}} \\ &= \sum_{\alpha=1}^{k+1} (-1)^{\alpha+1} \, \mathrm{T}_{i_\alpha} \frac{\delta \mathscr{L}_{(\bar{\alpha})}}{\delta u_{I \setminus i_\alpha}}.\end{aligned} \tag{1.98}$$

It follows that if (1.91) holds, then

$$\frac{\delta}{\delta u_I} A^{i_1 \dots i_{k+1}} = \sum_{\alpha=1}^{k+1} (-1)^{\alpha+1} \, \mathrm{T}_{i_\alpha} \frac{\delta \mathscr{L}_{(\bar{\alpha})}}{\delta u_{I \setminus i_\alpha}} = 0. \tag{1.99}$$

We have shown that

$$\delta \mathsf{dL} = 0 \implies \frac{\delta}{\delta u_I} A^{i_1 \dots i_{k+1}} = \sum_{\alpha=1}^{k+1} (-1)^{\alpha+1} \, \mathrm{T}_{i_\alpha} \frac{\delta \mathscr{L}_{(\bar{\alpha})}}{\delta u_{I \setminus i_\alpha}} = 0 \tag{1.100}$$

for all $1 \leq i_1 \leq \ldots \leq i_{k+1} \leq N$ and I. Lemma 6 tells us that

$$\frac{\partial A^{i_1 \dots i_{k+1}}}{\partial u_I} = \sum_{\substack{J \\ j_i \leq 1}} (-\,\mathrm{T}^{-1})_J \frac{\delta A^{i_1 \dots i_{k+1}}}{\delta u_{IJ}} \tag{1.101}$$

and it follows that the converse to (1.100) also holds. We summarise this result in the following theorem:

Theorem 7. *For a discrete differential k-form L as given in (1.88), and $A^{i_1 \dots i_{k+1}}$ as defined in (1.90),*

$$\frac{\delta}{\delta u_I} A^{i_1 \dots i_{k+1}} = \sum_{\alpha=1}^{k+1} (-1)^{\alpha+1}\, \mathrm{T}_{i_\alpha} \frac{\delta \mathscr{L}_{(\bar{\alpha})}}{\delta u_{I \setminus i_\alpha}}. \tag{1.102}$$

The set of equations defined by

$$\frac{\delta}{\delta u_I} A^{i_1 \dots i_{k+1}} = 0 \tag{1.103}$$

for all $1 \leq i_1 \leq \ldots \leq i_{k+1} \leq N$ and I is equivalent to the set of equations defined by $\delta \mathsf{dL} = 0$.

1.3.6 Semi-discrete multiform Euler-Lagrange equations

For both the continuous and discrete cases, we used Stokes' theorem to show that the multiform Euler-Lagrange equations are given by $\delta \mathsf{dL} = 0$. The semi-discrete analogue of Stokes' theorem is obtained by taking a partial continuum limit (i.e., in some, but not all of the independent variables) of (1.82). It then follows that the multiform Euler-Lagrange equations are again given by $\delta \mathsf{dL} = 0$. Also, by combining the proofs given above, it also follows that the equations given by $\delta \mathsf{dL} = 0$ in the semi-discrete case are equivalent to the equations given by

$$\sum_{\alpha=1}^{k+1} (-1)^{\alpha+1}\, \mathrm{T}_{i_\alpha} \frac{\delta \mathscr{L}_{(\bar{\alpha})}}{\delta u_{I \setminus i_\alpha}} = 0, \tag{1.104}$$

where T_{i_α} is taken to be the shift operator when x_{i_α} is a discrete variable and the identity operator in the case where x_{i_α} is a continuous variable and I is a multi-index representing derivatives in the continuous independent variables and shifts in the discrete independent variables.

1.4 L vs. dL

Throughout this thesis we take our working definition to be that a Lagrangian multiform L is a differential form where $\mathsf{dL} = 0$ on the equations defined by $\delta\mathsf{dL} = 0$. We generally declare that a Lagrangian multiform L is trivial if the multiform Euler-Lagrange equations are satisfied by every u or only by the zero function. If we did not discount such trivial Lagrangian multiforms, then every differential form with polynomial coefficients in u and derivatives thereof would be considered a valid Lagrangian multiform. Whilst it may be obvious that such examples should not be considered true Lagrangian multiforms, there are examples that lie in a grey area in between; where the multiform Euler-Lagrange equations place additional constraints on the Euler-Lagrange equations of the Lagrangians in the multiform, but with a light enough touch to still allow non-trivial solutions. If it were a condition of a Lagrangian multiform that the equations given by $\mathrm{E}(\mathscr{L}_{(ij)}) = 0$ (where E is the Euler operator) must not be in any way constrained, then the majority of the currently known Lagrangian multiforms would fall foul. For example, in the AKNS Lagrangian multiform we shall give in Chapter 2, the equations given by $\mathrm{E}(\mathscr{L}_{(ij)}) = 0$ where $1 < i, j$ are further constrained by the Euler-Lagrange equations arising from the $\mathscr{L}_{(1i)}$ and $\mathscr{L}_{(1j)}$ Lagrangians. Sometimes, as is the case for the Lagrangian multiform for the potential KdV hierarchy given in [11], we find that the most fundamental equations of the multiform arise from the multiform Euler-Lagrange equations of the form

$$\frac{\delta \mathscr{L}_{(1i)}}{\delta u_{t_1}} + \frac{\delta \mathscr{L}_{(ij)}}{\delta u_{t_j}} = 0 \tag{1.105}$$

rather than $\mathrm{E}(\mathscr{L}_{(ij)}) = 0$ which only gives us differentiated versions of these equations. This all leads to a degree of ambiguity as to what exactly any given Lagrangian multiform is a Lagrangian multiform for. An interesting example of such a Lagrangian multiform is the Lagrangian 2-form for the KdV generating PDE given in [4] where

$$\mathscr{L}_{(ij)} = \frac{1}{2}(t_i - t_j)\frac{U_{t_i t_j}^2}{U_{t_i}U_{t_j}} + \frac{1}{2(t_i - t_j)}\left(n_j^2\frac{U_{t_i}}{U_{t_j}} + n_i^2\frac{U_{t_j}}{U_{t_i}}\right). \tag{1.106}$$

In this case, the equation given by $\mathrm{E}(\mathscr{L}_{(ij)}) = 0$ is equivalent to

$$
\begin{aligned}
U_{t_it_it_jt_j} =& U_{t_it_it_j}\left(\frac{1}{t_i-t_j}+\frac{U_{t_it_j}}{U_{t_i}}+\frac{U_{t_jt_j}}{U_{t_j}}\right)+U_{t_it_jt_j}\left(\frac{1}{t_j-t_i}+\frac{U_{t_it_j}}{U_{t_j}}+\frac{U_{t_it_i}}{U_{t_i}}\right)\\
&+U_{t_it_i}\left(\frac{n_i^2}{(t_i-t_j)^2}\frac{U_{t_j}^2}{U_{t_i}^2}-\frac{U_{t_it_j}^2}{U_{t_i}^2}-\frac{1}{t_i-t_j}\frac{U_{t_it_j}}{U_{t_i}}\right)-U_{t_it_j}\frac{U_{t_it_i}u_{t_jt_j}}{U_{t_i}U_{t_j}}\\
&+U_{t_jt_j}\left(\frac{n_j^2}{(t_i-t_j)^2}\frac{U_{t_i}^2}{U_{t_j}^2}-\frac{U_{t_it_j}^2}{U_{t_j}^2}-\frac{1}{t_j-t_i}\frac{U_{t_it_j}}{U_{t_j}}\right)\\
&+\frac{n_i^2}{2(t_i-t_j)^3}\frac{U_{t_j}}{U_{t_i}}(U_{t_i}+U_{t_j}+2(t_j-t_i)U_{t_it_j})\\
&+\frac{n_j^2}{2(t_j-t_i)^3}\frac{U_{t_i}}{U_{t_j}}(U_{t_j}+U_{t_i}+2(t_i-t_j)U_{t_it_j})\\
&+\frac{1}{2(t_i-t_j)}U_{t_it_j}^2\left(\frac{1}{U_{t_i}}-\frac{1}{U_{t_j}}\right),
\end{aligned}
\tag{1.107}
$$

the generating PDE of the KdV hierarchy [6]. If we define

$$
A^{ijk} := (t_i-t_j)\frac{U_{t_it_j}}{U_{t_i}U_{t_j}}+(t_j-t_k)\frac{U_{t_jt_k}}{U_{t_j}U_{t_k}}+(t_k-t_i)\frac{U_{t_kt_i}}{U_{t_k}U_{t_i}}
\tag{1.108}
$$

and

$$
\begin{aligned}
B^{ijk} :=& U_{t_it_jt_k}-\frac{U_{t_it_k}U_{t_it_j}}{2U_{t_i}}-\frac{U_{t_it_j}U_{t_jt_k}}{2U_{t_j}}-\frac{U_{t_it_k}U_{t_jt_k}}{2U_{t_k}}\\
&-\frac{1}{2}U_{t_i}U_{t_j}U_{t_k}\left(\frac{n_i^2}{U_{t_i}^2(t_k-t_i)(t_i-t_j)}+\frac{n_j^2}{U_{t_j}^2(t_i-t_j)(t_j-t_k)}\right.\\
&\left.+\frac{n_k^2}{U_{t_k}^2(t_j-t_k)(t_k-t_i)}\right)
\end{aligned}
\tag{1.109}
$$

then

$$
\mathsf{dL} = \sum_{i<j<k} A^{ijk}B^{ijk}\mathsf{d}t_i\wedge\mathsf{d}t_j\wedge\mathsf{d}t_k.
\tag{1.110}
$$

As a result, all multiform Euler-Lagrange equations of this multiform are consequences of $A^{ijk}=0$ and $B^{ijk}=0$ (a surprising consequence is that (1.107), a 1+1 dimensional PDE is given by $\frac{\delta A^{ijk}B^{ijk}}{\delta U_{t_k}}$, when both A^{ijk} and B^{ijk} are $2+1$ dimensional). Therefore, it would be more accurate to describe this as a Lagrangian multiform for $A^{ijk}=0$ and $B^{ijk}=0$ rather than for the KdV generating PDE given in (1.107).

One way to avoid such ambiguity is to shift our focus away from the Lagrangian multiform L and instead consider its exterior derivative dL to be the main object of interest. As we shall see in all of the new examples given in this thesis, by factorising the coefficients of dL (as we did in terms of A^{ijk} and B^{ijk} for the KdV generating PDE), the fundamental equations of the Lagrangian multiform become apparent. This perspective is used in [15] to define Lagrangian multiforms [1]. Since our working definition of a Lagrangian multiform ($\mathsf{dL}=0$ on the equations defined by $\delta\mathsf{dL}=0$) only places conditions on dL, there is a strong case to be made that dL should be considered to be the main object of study, rather than the Lagrangian multiform L.

[1] We note that Vermeeren and Petrera allow a constant term in their definition, since their definition does not require that dL=0 on the multiform Euler-Lagrange equations

Chapter 2

Variational symmetries and Lagrangian multiforms

2.1 Variational symmetries and Noether's theorem

In this section, we shall make use of a version of Noether's (first) theorem as presented in [16], where proofs of all statements in this section can be found. We consider systems with p independent variables $x = (x_1, \ldots, x_p)$ and q dependent variables $u = (u^1, \ldots, u^q)^T$. In the rest of this paper, we will often use u to denote the collection of fields $u^1, \ldots, u^q$ or the vector $(u^1, \ldots, u^q)^T$.

2.1.1 Generalized and evolutionary vector fields

We consider vector fields of the form

$$\mathbf{v} = \sum_{i=1}^{p} \xi_i \frac{\partial}{\partial x_i} + \sum_{\alpha=1}^{q} \phi_\alpha \frac{\partial}{\partial u^\alpha} \tag{2.1}$$

We say that $\mathbf{v}$ is a **geometric vector field** if the ξ_i and ϕ_α depend only on x and u. If the ξ_i and ϕ_α depend also on derivatives of u, we say that $\mathbf{v}$ is a **generalized vector field**. If all of the ξ_i are zero, i.e.

$$\mathbf{v}_Q = \sum_{\alpha=1}^{q} Q_\alpha \frac{\partial}{\partial u^\alpha} \equiv Q \cdot \frac{\partial}{\partial u}\,, \tag{2.2}$$

we call $\mathbf{v}_Q$ an **evolutionary vector field** with **characteristic** $Q(x, u^{(n)}) = (Q_1(x, u^{(n)}), \ldots, Q_q(x, u^{(n)}))^T$, where $Q(x, u^{(n)})$ is taken to mean that Q may depend on x, u and derivatives of u. The prolongation of an evolutionary vector field $\mathbf{v}_Q$ takes the form

$$\mathrm{pr}\,\mathbf{v}_Q = \sum_{\alpha,J} \mathrm{D}_J\, Q_\alpha \frac{\partial}{\partial u_J^\alpha} \tag{2.3}$$

where we have used the multi-index notation where J is the ordered set $(j_1, \ldots, j_p)$ and

$$\mathrm{D}_J := \prod_{i=1}^{p} (\mathrm{D}_{x_i})^{j_i}\,, \quad \mathrm{D}_{x_i} = \frac{\partial}{\partial x_i} + \sum_{\alpha,J} u^\alpha_{Ji} \frac{\partial}{\partial u^\alpha_J}\,. \tag{2.4}$$

We shall write Ji^r to denote $(j_1, \ldots, j_i + r, \ldots, j_p)$, $J\backslash k^r$ to denote $(j_1, \ldots, j_k - r, \ldots, j_p)$ and $|J|$ to denote the sum $j_1 + \ldots + j_p$.

Every vector field $\mathbf{v}$ in the form of (2.1) has an associated evolutionary representative $\mathbf{v}_Q$ where

$$Q_\alpha = \phi_\alpha - \sum_{i=1}^{p} \xi_i u^\alpha_{x_i} \tag{2.5}$$

2.1.2 Variational symmetries

The vector field $\mathbf{v}$ is a variational symmetry of a Lagrangian $\mathscr{L}(x, u^{(n)})\mathsf{d}x_1 \wedge \ldots \wedge \mathsf{d}x_p$ if and only if

$$\mathrm{pr}\,\mathbf{v}(\mathscr{L}) + \mathscr{L}\,\mathrm{Div}\,\xi = \mathrm{Div}\,B \tag{2.6}$$

for some $B(x, u^{(n)}) = (B_1(x, u^{(n)}), \ldots, B_p(x, u^{(n)}))^T$. For an evolutionary vector $\mathbf{v}_Q$, this simplifies to

$$\mathrm{pr}\,\mathbf{v}_Q(\mathscr{L}) = \mathrm{Div}\,\tilde{B} \tag{2.7}$$

for some $\tilde{B}(x, u^{(n)}) = (\tilde{B}_1(x, u^{(n)}), \ldots, \tilde{B}_p(x, u^{(n)}))^T$. A generalized vector field $\mathbf{v}$ is a variational symmetry of $\mathscr{L}$ if and only if its evolutionary representative $\mathbf{v}_Q$ is.

Finding the variational symmetries of a given Lagrangian is a non-trivial exercise. Methods for doing so are covered in [16], [17], [18] and [19]. In our approach, we assume that such a variational symmetry is given (by applying one of those methods for instance) and we use it as our starting point to construct a Lagrangian multiform.

2.1.3 Noether's theorem

In order to introduce Noether's theorem, we will require the Euler operater E. We define the Euler operator E to be the q-component vector operator whose α^{th} component is E_α given by

$$\mathrm{E}_\alpha = \sum_J (-1)^{|J|}\,\mathrm{D}_J \frac{\partial}{\partial u^\alpha_J} \tag{2.8}$$

The sum is over all multi-indices $J = (j_1, \ldots, j_p)$. For a Lagrangian $\mathscr{L}$, $\mathrm{E}(\mathscr{L}) = 0$ gives the standard Euler Lagrange equations for $\mathscr{L}$. For example, in the case where $p = 2$, $q = 1$ and $\mathscr{L}$ contains terms up to the 2^{nd} jet,

$$\mathrm{E}(\mathscr{L}) = \frac{\partial \mathscr{L}}{\partial u} - \mathrm{D}_{x_1}\frac{\partial \mathscr{L}}{\partial u_{x_1}} - \mathrm{D}_{x_2}\frac{\partial \mathscr{L}}{\partial u_{x_2}} + \mathrm{D}^2_{x_1}\frac{\partial \mathscr{L}}{\partial u_{x_1x_1}} + \mathrm{D}_{x_1}\mathrm{D}_{x_2}\frac{\partial \mathscr{L}}{\partial u_{x_1x_2}} + \mathrm{D}^2_{x_2}\frac{\partial \mathscr{L}}{\partial u_{x_2x_2}}. \tag{2.9}$$

We say that the equations of motion given by $\mathrm{E}(\mathscr{L}) = 0$ are of maximal rank if the $q \times (p + q\binom{p+n}{n})$ Jacobian matrix

$$\mathsf{J}_{\mathrm{E}(\mathscr{L})} = \left(\frac{\partial\,\mathrm{E}_i(\mathscr{L})}{\partial x_j}, \frac{\partial\,\mathrm{E}_i(\mathscr{L})}{\partial u^\alpha_J}\right) \tag{2.10}$$

is of rank q (i.e. of maximal rank) on the equations of motion given by $\mathrm{E}(\mathscr{L}) = 0$.

Theorem 8. **[Noether]** *Let $\boldsymbol{v}_Q$ be an evolutionary vector field with characteristic Q and $\mathscr{L}$ a Lagrangian density, such that $\mathrm{E}(\mathscr{L})$ is of maximal rank. Then,*

$$\mathrm{pr}\,\boldsymbol{v}_Q(\mathscr{L}) = \mathrm{Div}\,B \quad \textit{for some } B \quad \Longleftrightarrow \quad Q \cdot \mathrm{E}(\mathscr{L}) = \mathrm{Div}\,P \quad \textit{for some } P\,. \tag{2.11}$$

where $Q \cdot \mathrm{E} = \sum_{\alpha=1}^{q} Q_\alpha \, \mathrm{E}_\alpha$.

The right hand side of (2.11) is the characteristic form of a conservation law. Since setting $\mathrm{E}(\mathscr{L}) = 0$ defines the equations of motion, this tells us that $\mathrm{Div}\, P = 0$ on the equations of motion - the usual form of a conservation law.

2.1.4 Finding the components of a divergence

If we are given that an expression $A(x, u^{(n)})$ is a divergence (i.e., $A = \mathrm{Div}\, B$ for some $B = (B_1, B_2, \ldots, B_p)^T$, it is often easy to find the components B_k by trial and error. This can also be done algorithmically using the homotopy operator [16]. First for multi-indices I and J we define

$$\binom{I}{J} = \frac{I!}{J!\,(I \backslash J)!} \tag{2.12}$$

where $I! = i_1!\, i_2! \ldots i_p!$, $J! = j_1!\, j_2! \ldots j_p!$ and $(I \backslash J)! = (i_1 - j_1)! \ldots (i_p - j_p)!$. We now define the higher Euler operators E_α^J such that:

$$\mathrm{E}_\alpha^J(A) = \sum_{I \supset J} \binom{I}{J} (-\,\mathrm{D})_{I \backslash J} \frac{\partial A}{\partial u_I^\alpha}. \tag{2.13}$$

We note that when $|J| = 0$, the higher Euler operator coincides with the Euler operator defined in (2.8). Then the homotopy operator H acts on A as follows:

$$\mathsf{H}(A(x, u^{(n)})) = \hat{B} = (\hat{B}_1, \ldots \hat{B}_p)^T \tag{2.14}$$

with each

$$\hat{B}_k = \int_0^1 \sum_{\alpha=1}^{q} \sum_I \frac{i_k + 1}{|I| + 1} \mathrm{D}_I (u^\alpha \, \mathrm{E}_\alpha^{Ik}(A)[\lambda u]) \mathsf{d}\lambda + \int_0^1 x_k A(\lambda x, 0) \mathsf{d}\lambda, \tag{2.15}$$

where i_k is the k^{th} component of I and the $[\lambda u]$ denotes replacing every u^α in $\mathrm{E}_\alpha^{Ik}(A)$ with λu^α. So long as $A(x, u^{(n)})$ is a divergence, the $\hat{B}$ defined in this manner is such that $A = \mathrm{Div}\, \hat{B}$.

Example 9. *We let* $A = u_{x_1x_2}u_{x_3}$. *Since* $\mathrm{E}(A) = 0$, *it follows that* A *is a divergence. The only non-zero higher Euler operators are*

$$\begin{aligned}
\mathrm{E}^{x_1}(A) &= -u_{x_2x_3}\\
\mathrm{E}^{x_2}(A) &= -u_{x_1x_3}\\
\mathrm{E}^{x_3}(A) &= u_{x_1x_2}\\
\mathrm{E}^{x_1x_2}(A) &= u_{x_3}.
\end{aligned} \tag{2.16}$$

Then (2.15) *tells us that*

$$\begin{aligned}
\hat{B}_1 &= \int_0^1 u(-\lambda u_{x_2x_3}) + \frac{1}{2}\,\mathrm{D}_{x_2}(u\lambda u_{x_3})d\lambda\\
&= \frac{1}{4}u_{x_2}u_{x_3} - \frac{1}{4}uu_{x_2x_3},
\end{aligned} \tag{2.17a}$$

$$\begin{aligned}
\hat{B}_2 &= \int_0^1 u(-\lambda u_{x_1x_3}) + \frac{1}{2}\,\mathrm{D}_{x_1}(u\lambda u_{x_3})d\lambda\\
&= \frac{1}{4}u_{x_1}u_{x_3} - \frac{1}{4}uu_{x_1x_3},
\end{aligned} \tag{2.17b}$$

and

$$\begin{aligned}
\hat{B}_3 &= \int_0^1 u\lambda u_{x_1x_2}d\lambda\\
&= \frac{1}{2}uu_{x_1x_2}.
\end{aligned} \tag{2.17c}$$

For these $\hat{B}_1$, $\hat{B}_2$ *and* $\hat{B}_3$,

$$\mathrm{D}_{x_1}\hat{B}_1 + \mathrm{D}_{x_2}\hat{B}_2 + \mathrm{D}_{x_3}\hat{B}_3 = u_{x_1x_2}u_{x_3} = A \tag{2.18}$$

as required.

2.2 Variational symmetries as Lagrangian multiforms

In this section, we shall take the well known results of the previous section, and apply them in the context of Lagrangian multiforms. We consider the Lagrangian density $\mathscr{L}$ on a manifold with p independent, and q dependent variables from the previous section. In order to be able to apply Noether's theorem, we require that the corresponding EL equations $\mathrm{E}(\mathscr{L}) = 0$ are of maximal rank. If we introduce a new independent variable x_{p+1}, independent of $x_1, \ldots, x_p$, and the vector field $\mathbf{w} = u_{x_{p+1}} \cdot \dfrac{\partial}{\partial u}$ then

$$\mathrm{pr}\,\mathbf{w}(\mathscr{L}) = \mathrm{D}_{x_{p+1}} \mathscr{L}. \tag{2.19}$$

Also, by reversing the integration by parts that was used to get from $\mathscr{L}$ to $\mathrm{E}(\mathscr{L})$ it follows that

$$u_{x_{p+1}} \cdot \mathrm{E}(\mathscr{L}) = \mathrm{D}_{x_{p+1}} \mathscr{L} + \mathrm{Div}\, A \tag{2.20}$$

for some A, where the x_{p+1} component of A is zero. If Q is the characteristic of a variational symmetry of $\mathscr{L}$ then Noether's theorem tells us that

$$Q \cdot \mathrm{E}(\mathscr{L}) = \mathrm{Div}\, P \tag{2.21}$$

for some P. Adding (2.20) and (2.21) gives us that

$$(u_{x_{p+1}} + Q) \cdot \mathrm{E}(\mathscr{L}) = \mathrm{Div}\, \tilde{P} \tag{2.22}$$

where $\tilde{P} = A + P$ so the x_{p+1} component of $\tilde{P}$ is $\mathscr{L}$. We use this idea to construct Lagrangian multiforms as follows.

Theorem 10. *Let $Q(x, u^{(n)})$ be the characteristic of a variational symmetry of the Lagrangian density $\mathscr{L}(x, u^{(n)})$ such that $\mathscr{L}$ and Q have no dependence on x_{p+1} or derivatives of u with respect to x_{p+1}. If $\tilde{Q} = u_{x_{p+1}} + Q$ then*

$$\tilde{Q} \cdot \mathrm{E}(\mathscr{L}) = \mathrm{Div}\, P \tag{2.23}$$

for some $P = (P_1, \dots P_p, P_{p+1})^T$*, and the* p*-form* L *such that*

$$\mathsf{L} = \sum_{i=1}^{p+1} \mathscr{L}_{(\bar{i})} \mathsf{d}x_{i+1} \wedge \dots \wedge \mathsf{d}x_{p+1} \wedge \mathsf{d}x_1 \wedge \dots \wedge \mathsf{d}x_{i-1} \quad \textit{with} \quad \mathscr{L}_{(\bar{i})} = (-1)^{ip} P_i \tag{2.24}$$

is a Lagrangian multiform. The $p+1$ *component of* P *is equivalent (i.e. equal modulo total derivatives) to* $\mathscr{L}$*.*

Remark 11. *A Lagrangian multiform arising from Theorem 10 requires a Lagrangian and a single variational symmetry. Since, in general, a single symmetry is not a sufficient condition for integrability, it follows that Theorem 10 can give us Lagrangian multiforms for non-integrable systems.*

Proof. The existence of a P that satisfies (2.23) and has $\mathscr{L}$ as its $p+1$ component follows from the introduction to this section, equations (2.19) to (2.22). Since Q is a symmetry of $\mathrm{E}(\mathscr{L})$ we know that the equations $\tilde{Q} = 0$ and $\mathrm{E}(\mathscr{L}) = 0$ are compatible in the sense that there exists a *general* common solution. Then

$$\mathsf{dL} = (-1)^p \operatorname{Div} P \; \mathsf{d}x_1 \wedge \dots \wedge \mathsf{d}x_{p+1}, \tag{2.25}$$

and it follows that $\delta\mathsf{dL} = 0$ is equivalent to the requirement that

$$\frac{\partial}{\partial u_I} \operatorname{Div} P = 0 \quad \forall I. \tag{2.26}$$

Using (2.23), this gives us that

$$\frac{\partial}{\partial u_I} \operatorname{Div} P = \left(\frac{\partial}{\partial u_I} \tilde{Q} \right) \cdot \mathrm{E}(\mathscr{L}) + \tilde{Q} \cdot \left(\frac{\partial}{\partial u_I} \mathrm{E}(\mathscr{L}) \right), \tag{2.27}$$

and since $\mathrm{E}(\mathscr{L})$ is of maximal rank (a requirement for Noether's theorem), the necessary and sufficient condition for $\delta\mathsf{dL} = 0$ is that both $\tilde{Q} = 0$ and $\mathrm{E}(\mathscr{L}) = 0$ hold simultaneously. From the form of (2.23), it is clear that $\mathsf{dL} = 0$ on solutions of either $\tilde{Q} = 0$ or $\mathrm{E}(\mathscr{L}) = 0$. □

Remark 12. *If* $\boldsymbol{v}_Q$ *is a variational symmetry of a Lagrangian* $\mathscr{L}\mathsf{d}x_1 \wedge \dots \wedge \mathsf{d}x_p$*, it is tempting to say that* $\boldsymbol{v}_{\tilde{Q}}$ *is also a variational symmetry of* $\mathscr{L}$ *since* $\operatorname{pr} \boldsymbol{v}_{\tilde{Q}}(\mathscr{L}) = \operatorname{Div} B$ *for some* B*. This is not quite correct since* B *contains a* x_{p+1} *component.*

However, $\boldsymbol{v}_{\tilde{Q}}$ is a variational symmetry of $\mathscr{L}\,\mathsf{d}x_1 \wedge \ldots \wedge \mathsf{d}x_p \wedge \mathsf{d}x_{p+1}$ (i.e., the same $\mathscr{L}$ but now integrated over the coordinates $x_1, \ldots, x_{p+1}$ instead of $x_1, \ldots, x_p$, giving an alternative perspective for what is happening in Theorem 10.

Remark 13. *Theorem 10 allows us to construct a $p+1$ dimensional Lagrangian multiform from a Lagrangian in p dimensions and a single variational symmetry. It is natural to consider whether, in the case where we have a set of l commuting variational symmetries, we can iterate the process to find a $p+l$ dimensional Lagrangian multiform, as was achieved for a class of 1-forms in [20]. In Section 2.2.3 we use Theorem 10 to obtain a multiform that incorporates the first three flows of the AKNS hierarchy. We also show why, in the case of a Lagrangian 2-form, it is always possible to obtain a $2+l$ dimensional Lagrangian 2-form from an autonomous polynomial Lagrangian $\mathscr{L}_{(12)}$ and a set of l commuting variational symmetries with autonomous polynomial characteristics. A similar argument can be used for autonomous polynomial k-forms for arbitrary k. Whether or not non-autonomous, non-polynomial systems can be extended through repeated application of Theorem 10 remains an open problem.*

Remark 14. *A Lagrangian $p+1$-form given by the components of $\tilde{P}$ in Theorem 10 gives $\mathrm{E}(\mathscr{L}) = 0$ and $\tilde{Q} = 0$ and consequences thereof as its multiform Euler-Lagrange equations. We can also consider the components of P as giving us a Lagrangian p-form, the multiform Euler-Lagrange equations of which will give us $\mathrm{E}(\mathscr{L}) = 0$ and $Q = 0$ and consequences thereof as its multiform Euler-Lagrange equations. However, unlike the Lagrangian multiform given by Theorem 10, $\mathscr{L}$ will not be one of the Lagrangians of the multiform.*

We note that P is not unique. Indeed, any change to P that is equivalent to adding an exact form to L will also satisfy (2.23). In addition, we can perform "integration by parts" on the left hand side of (2.23) and the remaining terms will still be a divergence, e.g.

$$\tilde{Q}\cdot\mathrm{E}(\mathscr{L}) \to -\,\mathrm{D}_x\,\tilde{Q}\cdot\mathrm{D}_x^{-1}\,\mathrm{E}(\mathscr{L}) \text{ and } \operatorname{Div} P \to \operatorname{Div}\tilde{P} = \operatorname{Div} P - \mathrm{D}_x(Q\cdot\mathrm{D}_x^{-1}\,\mathrm{E}(\mathscr{L})). \tag{2.28}$$

Such a transformation amounts to adding a double zero to one of the components of P so the resultant Lagrangian multiform will be essentially the same in that

$\delta \mathsf{dL} = 0$ will give the same equations of motion, and $\mathsf{dL} = 0$ will still hold on these equations of motion. This idea can be generalized further by noticing that the "integration by parts" can be carried out on any constituent part of $\tilde{Q} \cdot \mathrm{E}(\mathscr{L})$, e.g.

$$\tilde{Q}_i \, \mathrm{E}_i(\mathscr{L}) \rightarrow - \mathrm{D}_x \, \tilde{Q}_i \, \mathrm{D}_x^{-1} \, \mathrm{E}_i(\mathscr{L}), \tag{2.29}$$

whilst leaving the resultant multiform essentially unchanged. The $\tilde{Q}$ in (2.23) is in evolutionary form with respect to x_{p+1} i.e. it is in the form $u_{x_{p+1}} + Q(x, u^{(n)}) = 0$ where $Q(x, u^{(n)})$ does not contain x_{p+1} or derivatives of u with respect to x_{p+1}. If, by using the above operations we are able to put $\mathrm{E}(\mathscr{L})$ into evolutionary form with respect to some x_j, and neither x_j nor derivatives of u with respect to x_j appear in $\tilde{Q}$ then we can reverse the roles of $\tilde{Q}$ and $\mathrm{E}(\mathscr{L})$ whilst essentially leaving the resultant multiform unchanged. This idea forms the basis of the following theorem.

Theorem 15. *Consider the Lagrangian and variational symmetry as given in Theorem 10 and let $j \in \{1, \ldots, p\}$ be fixed. If there exist constants a_k and multi-indices J_k for $k = 1, \ldots, q$ where the $p+1$ and j components of each J_k are zero, such that*

$$a_k \, \mathrm{D}_{J_k}^{-1} \, \mathrm{E}_k(\mathscr{L}) = 0 \tag{2.30}$$

is in evolutionary form with respect to x_j, then the q components of $\mathrm{E}(\mathscr{L}_{(\bar{j})})$, up to re-ordering, are precisely the q expressions

$$\frac{1}{a_k} \, \mathrm{D}_{J_k} \, \tilde{Q}_k. \tag{2.31}$$

Proof. If there exist multi-indices J_k and constants a_k as described that put $\mathrm{E}(\mathscr{L})$ into evolutionary form with respect to x_j, then applying $a_k \, \mathrm{D}_{J_k}^{-1}$ to $\mathrm{E}_k(\mathscr{L})$ and $\frac{1}{a_k} \, \mathrm{D}_{J_k}$ to $\tilde{Q}_k$ in (2.23) amounts to performing integration by parts on the products $\tilde{Q}_k \, \mathrm{E}_k(\mathscr{L})$, i.e.

$$\frac{1}{a_k} \, \mathrm{D}_{J_k} \, \tilde{Q}_k . a_k \, \mathrm{D}_{J_k}^{-1} \, \mathrm{E}_k(\mathscr{L}) = \tilde{Q}_k \, \mathrm{E}_k(\mathscr{L}) + \mathrm{Div} \, C_k \tag{2.32}$$

for some C_k. We note that the j and $p+1$ components of C_k are zero since the j and $p+1$ components of each J_k are zero. It follows that

$$\sum_{k=1}^{q} \frac{1}{a_k} \mathrm{D}_{J_k} \tilde{Q}_k . a_k \mathrm{D}_{J_k}^{-1} \mathrm{E}_k(\mathscr{L}) = \mathrm{Div}\, \hat{P} \tag{2.33}$$

where $\hat{P} = P + \sum_{k=1}^{q} C_k$. Now that each $a_k \mathrm{D}_{J_k}^{-1} \mathrm{E}_k(\mathscr{L})$ is in evolutionary form, it follows from Noether's theorem that the corresponding characteristics represent variational symmetries of $\frac{1}{a_k} \mathrm{D}_{J_k} \tilde{Q}_k$, and by Theorem 10, $\mathscr{L}_{(\bar{j})}$ is the Lagrangian for $\frac{1}{a_k} \mathrm{D}_{J_k} \tilde{Q}_k$, $k = 1, \ldots, q$. □

It follows that the multiforms described by P and $\hat{P}$ in theorems 10 and 15 both have $\mathscr{L}_{(\bar{j})}$ and $\mathscr{L}$ as their j and $p+1$ components respectively, since the j and $p+1$ components of each C_k are zero.

2.2.1 The "zero" symmetry

Every Lagrangian multiform we know of that has been considered up to this point has related to integrable systems. However, it is not the case that Lagrangian multiforms only exist for integrable systems, since Theorem 10 applies to any Lagrangian with a variational symmetry. In fact, it turns out that every variational equation has at least one Lagrangian multiform description.

Using our construction, the requirements for a Lagrangian multiform are a Lagrangian density $\mathscr{L}(x, u^{(n)})$ and a variational symmetry $\mathbf{v}$. It is trivially true that the zero vector (i.e. $\mathbf{v}_Q$ where $Q = 0$) is a symmetry of every Lagrangian since $\mathbf{v}_Q(\mathscr{L}) = 0$. Letting $\tilde{Q} = u_{x_{p+1}} + Q = u_{x_{p+1}}$, it follows that

$$\tilde{Q} \cdot \mathrm{E}(\mathscr{L}) = \mathrm{Div}\, P \tag{2.34}$$

for some P, and it follows from Theorem 10 that P describes a Lagrangian multiform. Therefore every Lagrangian, regardless of integrability, fits into at least one Lagrangian multiform description.

This particular multiform could reasonably be described as semi-trivial, in that one of the equations of motion is simply $u_{x_{p+1}} = 0$. However, it does have a practical application relating to the inverse problem of finding a Lagrangian (if it exists) for a given equation of motion. Theorem 10 tells us that $\mathscr{L}$ is given by the x_{p+1} component of P. By applying the homotopy operator to P we find that

$$\begin{aligned}\mathscr{L} &= \int_0^1 u \cdot (\mathsf{E}^{x_{p+1}}(P)[\lambda u])\mathsf{d}\lambda \\ &= \int_0^1 u \cdot (\mathsf{E}(\mathscr{L})[\lambda u])\mathsf{d}\lambda,\end{aligned} \tag{2.35}$$

which is precisely the formula given in [16]. Also, the relation

$$\mathrm{E}(P \cdot Q) = \mathrm{D}_P^*(Q) + \mathrm{D}_Q^*(P), \tag{2.36}$$

where $\mathrm{D}_P(Q)$ is the Fréchet derivative of P acting on Q and D_P^* is the adjoint of D_P, can be applied to (2.34) in the case where $\tilde{Q} = u_{x_{p+1}}$ to derive the condition (also given in [16]) that an equation has a Lagrangian description if and only if its Fréchet derivative is self adjoint.

Remark 16. *Since we can apply Theorem 10 with any variational symmetry, many Lagrangians can fit into more that one Lagrangian multiform description. For example, if a given Lagrangian possesses time/space shift symmetries and rotational symmetries then we can obtain a Lagrangian multiform for each. However, unless the symmetries themselves describe mutually commuting flows, we cannot expect it to be possible to connect these multiforms descriptions to each other in any coherent way (i.e., as we are able to do in the case of the AKNS multiform in section 2.2.3). The latter point emphasises the distinction between multiforms as just described, and multiforms carrying information about the integrability of the equations of motion, which was the original intent of the notion of Lagrangian multiforms.*

Next, we shall give three examples of constructing Lagrangian multiforms from variational symmetries. All three systems considered come from well known integrable hierarchies - this simplifies the task of finding variational symmetries, since the required symmetries are other equations taken from the respective hierarchies.

2.2.2 The sine-Gordon equation

The sine-Gordon equation, $u_{x_1x_2} = \sin u$ with Lagrangian density

$$\mathscr{L}_{(12)} = \frac{1}{2}u_{x_1}u_{x_2} - \cos u \tag{2.37}$$

and variational symmetry $Q = u_{3x_1} + \frac{1}{2}u_{x_1}^3$ is given as an example in [16]. We can confirm that Q is a variational symmetry of $\mathscr{L}$ by checking that $\mathrm{pr}\,\mathbf{v}_Q\mathscr{L} = \mathrm{Div}\,P$ for some P. Indeed, we find that

$$\begin{aligned}\mathrm{pr}\,\mathbf{v}_Q\mathscr{L} =&\frac{1}{2}(u_{4x_1} + \frac{3}{2}u_{x_1}^2u_{x_1x_1})u_{x_2} + \frac{1}{2}(u_{3x_1x_2} + \frac{3}{2}u_{x_1}^2u_{x_1x_2})u_{x_1}\\ &+ (u_{3x_1} + \frac{1}{2}u_{x_1}^3)\sin u\\ =&\,\mathrm{D}_{x_1}(\frac{1}{2}u_{x_1}u_{x_1x_1x_2} - \frac{1}{2}u_{x_1x_1}u_{x_1x_2} + \frac{1}{2}u_{x_1x_1x_1}u_{x_2} + \frac{1}{4}u_{x_1}^3u_{x_2}\\ &+ u_{x_1x_1}\sin u - \frac{1}{2}u_{x_1}^2\cos u) + \mathrm{D}_{x_2}(\frac{1}{8}u_{x_1}^4).\end{aligned} \tag{2.38}$$

We now let $\tilde{Q} = u_{x_3} - Q$. In this case, $\tilde{Q} = 0$ is precisely the modified KdV equation which is known to be compatible with the sine-Gordon equation. By Theorem 10, the product

$$\tilde{Q}\cdot\mathrm{E}(\mathscr{L}) = (u_{x_3} - u_{3x_1} - \frac{1}{2}u_{x_1}^3)(\sin u - u_{x_1x_2}) = \mathrm{Div}\,\tilde{P}, \tag{2.39}$$

i.e. it is a divergence. If we write this product in terms of the components of $\tilde{P}$ we find that

$$\tilde{P} = \begin{pmatrix} -\frac{1}{2}u_{x_2}u_{x_3} + u_{x_1x_1}u_{x_1x_2} - u_{x_1x_1}\sin u + \frac{1}{2}u_{x_1}^2\cos u \\ -\frac{1}{2}u_{x_1}u_{x_3} - \frac{1}{2}u_{x_1x_1}^2 + \frac{1}{8}u_{x_1}^4 \\ \frac{1}{2}u_{x_1}u_{x_2} - \cos u \end{pmatrix} = \begin{pmatrix}\mathscr{L}_{(23)}\\ \mathscr{L}_{(31)}\\ \mathscr{L}_{(12)}\end{pmatrix} \tag{2.40}$$

satisfies (2.39), and is precicely the Lagrangian multiform for the sine-Gordon equation that was given in [21]. In accordance with Remark 14, we also consider

$$Q\cdot\mathrm{E}(\mathscr{L}) = (u_{3x_1} + \frac{1}{2}u_{x_1}^3)(\sin u - u_{x_1x_2}) = \mathrm{Div}\,P, \tag{2.41}$$

where

$$P = \begin{pmatrix} u_{x_1x_1}\sin u - \frac{1}{2}u_{x_1}^2\cos u - u_{x_1x_1}u_{x_1x_2} \\ \frac{1}{2}u_{x_1x_1}^2 - \frac{1}{8}u_{x_1}^4 \end{pmatrix} = \begin{pmatrix} \mathscr{L}_{(2)} \\ -\mathscr{L}_{(1)} \end{pmatrix}. \tag{2.42}$$

The multiform Euler-Lagrange equations for the Lagrangian 1-form

$$\mathsf{L} = \mathscr{L}_{(1)}\mathsf{d}x_1 + \mathscr{L}_{(2)}\mathsf{d}x_2 \tag{2.43}$$

are given by

$$\frac{\delta\mathscr{L}_{(1)}}{\delta u_{I\setminus 2}} - \frac{\delta\mathscr{L}_{(2)}}{\delta u_{I\setminus 1}} = 0 \tag{2.44}$$

for all I. Therefore, the non-zero multiform Euler-Lagrange equations are as follows:

$$\frac{\delta\mathscr{L}_{(1)}}{\delta u} = 0 \implies -\mathrm{D}_{x_1}(\frac{1}{2}u_{x_1}^3 + u_{3x_1}) = 0 \tag{2.45a}$$

$$\begin{aligned}\frac{\delta\mathscr{L}_{(2)}}{\delta u} = 0 \implies & \mathrm{D}_{x_1}^2(\sin u - u_{x_1x_2}) + \frac{3}{2}u_{x_1}^2(\sin u - u_{x_1x_2}) \\ & - \mathrm{D}_{x_2}(u_{3x_1} + \frac{1}{2}u_{x_1}^3) = 0\end{aligned} \tag{2.45b}$$

$$\frac{\delta\mathscr{L}_{(2)}}{\delta u_{x_1}} = 0 \implies \mathrm{D}_{x_2}(u_{3x_1} + \frac{1}{2}u_{x_1}^3) = 0 \tag{2.45c}$$

$$\frac{\delta\mathscr{L}_{(2)}}{\delta u_{x_1x_1}} = 0 \implies \sin u - u_{x_1x_2} = 0 \tag{2.45d}$$

$$\frac{\delta\mathscr{L}_{(1)}}{\delta u_{x_1}} - \frac{\delta\mathscr{L}_{(2)}}{\delta u_{x_2}} = 0 \implies u_{3x_1} + \frac{1}{2}u_{x_1}^3 = 0 \tag{2.45e}$$

$$\frac{\delta\mathscr{L}_{(1)}}{\delta u_{x_1x_1}} - \frac{\delta\mathscr{L}_{(2)}}{\delta u_{x_1x_2}} = 0 \implies -u_{x_1x_1} + u_{x_1x_1} = 0. \tag{2.45f}$$

As expected, all are consequences of $u_{3x_1} + \frac{1}{2}u_{x_1}^3 = 0$ and $\sin u - u_{x_1x_2} = 0$. We can view this Lagrangian 1-form as a reduction of the sine-Gordon Lagrangian 2-form under the constraint that no motion is allowed in the x_3 direction. We shall make further use of this type of reduction in Chapters 3 and 4.

2.2.3 The AKNS multiform

The first two flows of the AKNS hierarchy [22] were shown to possess a Lagrangian multiform structure in [2]. The $\mathscr{L}_{(x_1x_2)}$ and $\mathscr{L}_{(x_3x_1)}$ AKNS Lagrangians, (see e.g. [23]) are as follows:

$$\mathscr{L}_{(12)} = \frac{1}{2}(rq_{x_2} - qr_{x_2}) + \frac{i}{2}q_{x_1}r_{x_1} + \frac{i}{2}q^2r^2\,, \tag{2.46}$$

and

$$\mathscr{L}_{(31)} = \frac{1}{2}(qr_{x_3} - rq_{x_3}) + \frac{1}{8}(r_{x_1}q_{x_1x_1} - q_{x_1}r_{x_1x_1}) + \frac{3}{8}qr(rq_{x_1} - qr_{x_1})\,, \tag{2.47}$$

giving equations of motion

$$r_{x_2} = -\frac{i}{2}r_{x_1x_1} + ir^2q\,, \tag{2.48}$$

$$q_{x_2} = \frac{i}{2}q_{x_1x_1} - iq^2r \tag{2.49}$$

corresponding to the two components of $\mathrm{E}(\mathscr{L}_{(12)}) = 0$, and

$$r_{x_3} = \frac{3}{2}rqr_{x_1} - \frac{1}{4}r_{x_1x_1x_1}\,, \tag{2.50}$$

$$q_{x_3} = \frac{3}{2}qrq_{x_1} - \frac{1}{4}q_{x_1x_1x_1}\,, \tag{2.51}$$

corresponding to the two components of $\mathrm{E}(\mathscr{L}_{(31)}) = 0$. It is straightforward (but time consuming) to check that

$$\mathbf{v}_Q = (\frac{3}{2}qrq_{x_1} - \frac{1}{4}q_{x_1x_1x_1})\frac{\partial}{\partial q} + (\frac{3}{2}rqr_{x_1} - \frac{1}{4}r_{x_1x_1x_1})\frac{\partial}{\partial r} \tag{2.52}$$

is a variational symmetry of $\mathscr{L}_{(12)}$. In order to apply Theorem 10 we define

$$\tilde{Q} = \begin{pmatrix} q_{x_3} \\ r_{x_3} \end{pmatrix} - Q \tag{2.53}$$

and it follows that

$$\tilde{Q} \cdot E(\mathscr{L}_{(12)}) = \begin{pmatrix} q_{x_3} - \frac{3}{2}qrq_{x_1} + \frac{1}{4}q_{x_1x_1x_1} \\ \\ r_{x_3} - \frac{3}{2}rqr_{x_1} + \frac{1}{4}r_{x_1x_1x_1} \end{pmatrix} \cdot \begin{pmatrix} -r_{x_2} - \frac{i}{2}r_{x_1x_1} + ir^2q \\ \\ q_{x_2} - \frac{i}{2}q_{x_1x_1} + iq^2r \end{pmatrix} = \mathrm{Div}\, P \tag{2.54}$$

for some P. We find that

$$P = \begin{pmatrix} \mathscr{L}_{(23)} \\ \mathscr{L}_{(31)} \\ \mathscr{L}_{(12)} \end{pmatrix} \tag{2.55}$$

with

$$\begin{aligned} \mathscr{L}_{(23)} = &\frac{1}{4}(q_{x_2}r_{x_1x_1} - r_{x_2}q_{x_1x_1}) - \frac{i}{2}(q_{x_3}r_{x_1} + r_{x_3}q_{x_1}) + \frac{1}{8}(q_{x_1}r_{x_1x_2} - r_{x_1}q_{x_1x_2}) \\ &+ \frac{3}{8}qr(qr_{x_2} - rq_{x_2}) - \frac{i}{8}q_{x_1x_1}r_{x_1x_1} + \frac{i}{4}qr(qr_{x_1x_1} + rq_{x_1x_1}) \\ &- \frac{i}{8}(q^2r_{x_1}^2 + r^2q_{x_1}^2) + \frac{i}{4}qrq_{x_1}r_{x_1} - \frac{i}{2}q^3r^3. \end{aligned} \tag{2.56}$$

and $\mathscr{L}_{(12)}$ and $\mathscr{L}_{(31)}$ as given in (2.46) and (2.47) will satisfy (2.54). This gives us the Lagrangian multiform

$$\mathsf{L} = \mathscr{L}_{(12)}\ \mathsf{d}x_1 \wedge\ \mathsf{d}x_2 + \mathscr{L}_{(23)}\ \mathsf{d}x_2 \wedge\ \mathsf{d}x_3 + \mathscr{L}_{(31)}\ \mathsf{d}x_3 \wedge\ \mathsf{d}x_1, \tag{2.57}$$

for which $\mathsf{dL} = 0$ and $\delta\mathsf{dL} = 0$ as expected. This 3-component multiform was first derived in [2].

Extending the multiform to include the x_4 flow

We now follow a similar procedure to find the $\mathscr{L}_{(14)}$, $\mathscr{L}_{(24)}$ and $\mathscr{L}_{(34)}$ Lagrangians of the AKNS multiform, illustrating how our construction can be used to go beyond the first few terms in a Lagrangian multiform to include the higher flows of an integrable hierarchy. For the AKNS case, this means that we want to include the flow corresponding to the independent variable x_4 to produce the Lagrangian multiform

$$\begin{aligned}\mathsf{L}_{1234} =& \mathscr{L}_{(12)}\,\mathsf{d}x_1\wedge\,\mathsf{d}x_2+\mathscr{L}_{(13)}\,\mathsf{d}x_1\wedge\,\mathsf{d}x_3+\mathscr{L}_{(14)}\,\mathsf{d}x_1\wedge\,\mathsf{d}x_4\\ &+\mathscr{L}_{(23)}\,\mathsf{d}x_2\wedge\,\mathsf{d}x_3+\mathscr{L}_{(24)}\,\mathsf{d}x_2\wedge\,\mathsf{d}x_4+\mathscr{L}_{(34)}\,\mathsf{d}x_3\wedge\,\mathsf{d}x_4\end{aligned} \tag{2.58}$$

In order to find the $\mathscr{L}_{(14)}$, $\mathscr{L}_{(24)}$ and $\mathscr{L}_{(34)}$ we require our $\tilde{Q}$ to represent the x_4 flow of the hierarchy, i.e.

$$\tilde{Q}_4=\begin{pmatrix} q_{x_4}+i(\frac{3}{4}q^3r^2-\frac{1}{4}q^2r_{x_1x_1}-\frac{1}{2}qq_{x_1}r_{x_1}-qrq_{x_1x_1}-\frac{3}{4}rq_{x_1}^2+\frac{1}{8}q_{4x_1})\\ r_{x_4}-i(\frac{3}{4}q^2r^3-\frac{1}{4}r^2q_{x_1x_1}-\frac{1}{2}rq_{x_1}r_{x_1}-qrr_{x_1x_1}-\frac{3}{4}qr_{x_1}^2+\frac{1}{8}r_{4x_1})\end{pmatrix}. \tag{2.59}$$

The components of $\tilde{Q}_4$ are obtained by using the recursive procedure given in [24]. Theorem 10 tells us that

$$\tilde{Q}_4\cdot \mathrm{E}(\mathscr{L}_{(12)})=\mathrm{Div}\,P^{124} \tag{2.60}$$

where the components of P^{124} (with respect to x_1, x_2 and x_4) are found to be

$$P_4^{124}=\frac{1}{2}(rq_{x_2}-qr_{x_2})+\frac{i}{2}q_{x_1}r_{x_1}+\frac{i}{2}q^2r^2, \tag{2.61a}$$

$$\begin{aligned}P_2^{124}=&\frac{1}{2}(qr_{x_4}-rq_{x_4})+\frac{3i}{16}(q^2r_{x_1}^2+r^2q_{x_1}^2)+\frac{i}{4}qrq_{x_1}r_{x_1}+\frac{5i}{16}qr(qr_{x_1x_1}+rq_{x_1x_1})\\ &-\frac{i}{8}q_{x_1x_1}r_{x_1x_1}-\frac{i}{4}q^3r^3\end{aligned} \tag{2.61b}$$

and

$$
\begin{aligned}
P_1^{124} =& \frac{3}{8}q^2r^2(rq_{x_1} - qr_{x_1}) - \frac{i}{16}(q^2r_{x_1}r_{x_2} + r^2q_{x_1}q_{x_2}) - \frac{5i}{16}qr(qr_{x_1x_2} + rq_{x_1x_2}) \\
& - \frac{1}{8}qr(rq_{3x_1} - qr_{3x_1}) - \frac{1}{8}(q^2r_{x_1}r_{x_1x_1} - r^2q_{x_1}q_{x_1x_1}) - \frac{1}{8}q_{x_1}r_{x_1}(rq_{x_1} - qr_{x_1}) \\
& \frac{1}{4}qr(r_{x_1}q_{x_1x_1} - q_{x_1}r_{x_1x_1}) + \frac{3i}{8}qr(q_{x_1}r_{x_2} + r_{x_1}q_{x_2}) - \frac{i}{8}(q_{3x_1}r_{x_2} + r_{3x_1}q_{x_2}) \\
& + \frac{1}{16}(q_{3x_1}r_{x_1x_1} - r_{3x_1}q_{x_1x_1}) + \frac{i}{8}(q_{x_1x_1}r_{x_1x_2} + r_{x_1x_1}q_{x_1x_2}) - \frac{i}{2}(q_{x_1}r_{x_4} + r_{x_1}q_{x_4}).
\end{aligned} \tag{2.61c}
$$

We can now recognize $P_4^{124} = \mathscr{L}_{(12)}$ and we set $P_2^{124} = \mathscr{L}_{(41)}$ and $P_1^{124} = \mathscr{L}_{(24)}$, consistently with Theorem 10. From the construction of the coefficients, it follows immediately that for the multiform

$$
\mathsf{L}_{124} = \mathscr{L}_{(12)}\ \mathsf{d}x_1 \wedge\ \mathsf{d}x_2 + \mathscr{L}_{(24)}\ \mathsf{d}x_2 \wedge\ \mathsf{d}x_4 + \mathscr{L}_{(41)}\ \mathsf{d}x_4 \wedge\ \mathsf{d}x_1, \tag{2.62}
$$

the multiform EL equations are satisfied when both $\mathrm{E}(\mathscr{L}_{(12)}) = 0$ and $\mathrm{E}(\mathscr{L}_{(41)}) = 0$, and that $\mathsf{dL}_{124} = 0$ on these equations of motion.

To produce the rest of the coefficients needed for L_{1234}, we now use the same $\tilde{Q}_4$ together with $\mathscr{L}_{(13)}$ to define P^{134} such that

$$
\tilde{Q}_4 \cdot \mathrm{E}(\mathscr{L}_{(13)}) = \mathrm{Div}\, P^{134}\,. \tag{2.63}
$$

Then we find that the components of P^{134} (with respect to x_1, x_3 and x_4) are such that $P_4^{134} = \mathscr{L}_{(13)} = -\mathscr{L}_{(31)}$ given in (2.47), as expected from Theorem 10,

$$\begin{aligned}
P_1^{134} = \mathscr{L}_{(34)} =& \frac{i}{8}(q_{x_1x_1}r_{x_1x_3} + r_{x_1x_1}q_{x_1x_3}) - \frac{i}{8}(q_{3x_1}r_{x_3} + r_{3x_1}q_{x_3}) - \frac{i}{32}q_{3x_1}r_{3x_1} \\
&+ \frac{i}{32}(q^2r^2_{x_1x_1} + r^2q^2_{x_1x_1}) + \frac{i}{32}q^2_{x_1}r^2_{x_1} + \frac{3}{8}qr(rq_{x_4} - qr_{x_4}) \\
&+ \frac{9i}{32}q^4r^4 - \frac{3i}{16}q^2r^2(qr_{x_1x_1} + rq_{x_1x_1}) - \frac{i}{16}(q^2r_{x_1}r_{x_3} + r^2q_{x_1}q_{x_3}) \\
&- \frac{5i}{16}qr(qr_{x_1x_3} + rq_{x_1x_3}) + \frac{1}{4}(q_{x_1x_1}r_{x_4} - r_{x_1x_1}q_{x_4}) \\
&+ \frac{3i}{16}qr(q_{x_1}r_{3x_1} + r_{x_1}q_{3x_1}) + \frac{i}{16}qrq_{x_1x_1}r_{x_1x_1} \\
&- \frac{i}{16}q_{x_1}r_{x_1}(qr_{x_1x_1} + rq_{x_1x_1}) - \frac{15i}{16}q^2r^2q_{x_1}r_{x_1} \\
&+ \frac{3i}{8}qr(q_{x_1}r_{x_3} + r_{x_1}q_{x_3}) - \frac{1}{8}(q_{x_1}r_{x_1x_4} - r_{x_1}q_{x_1x_4})\,,
\end{aligned} \tag{2.64}$$

and $P_3^{134} = \mathscr{L}_{(41)}$ - identical to the $\mathscr{L}_{(41)}$ previously identified as P_2^{124}, given in (2.61b). Again, from the construction of the coefficients, it follows immediately that for the multiform

$$\mathsf{L}_{134} = \mathscr{L}_{(13)}\, \mathsf{d}x_1 \wedge\, \mathsf{d}x_3 + \mathscr{L}_{(34)}\, \mathsf{d}x_3 \wedge\, \mathsf{d}x_4 + \mathscr{L}_{(41)}\, \mathsf{d}x_4 \wedge\, \mathsf{d}x_1, \tag{2.65}$$

the multiform EL equations are satisfied when both $\mathrm{E}(\mathscr{L}_{(13)}) = 0$ and $\mathrm{E}(\mathscr{L}_{(41)}) = 0$, and also that $\mathsf{dL}_{134} = 0$ on these equations of motion. We are now able to form the 6 component Lagrangian multiform L_{1234} given in (2.58) and, as we would hope, the multiform EL equations are all consequences of $\mathrm{E}(\mathscr{L}_{(1i)}) = 0$ for $i \in \{2,3,4\}$, and $\mathsf{dL}_{1234} = 0$ on these equations. Therefore, in this case, we were able to incorporate two commuting variational symmetries to extend our multiform, but will this always be possible? Inspired by the AKNS example we have just carried out, we now examine this problem in the case where the $\mathscr{L}_{(12)}$ Lagrangian and variational symmetry characteristics are autonomous polynomials in the field variables and their derivatives.

Given that each L_{1ij} is determined from dL_{1ij}, we have the freedom to add any exact 2-form to L_{1ij} without affecting the multiform structure. As a result, the

$\mathscr{L}_{(1i)}, \mathscr{L}_{(ij)}$ and $\mathscr{L}_{(j1)}$ we obtain are not uniquely defined; this fact holds added significance when extending our multiform to include more than one commuting symmetry. When forming L_{123}, any choice of $\mathscr{L}_{(12)}, \mathscr{L}_{(23)}$ and $\mathscr{L}_{(31)}$ such that $\mathsf{d}\mathsf{L}_{123} = \tilde{Q} \cdot E(\mathscr{L}_{(12)})\mathsf{d}x_1 \wedge \mathsf{d}x_2 \wedge \mathsf{d}x_3$ will give us a valid multiform. When we then form L_{124}, we now require that the $\mathscr{L}_{(12)}$ is exactly the same as the one in L_{123}. This is not a problem, since we will always be able to make it so by adding an appropriate exact 2-form to L_{124}. Similarly, when we come to form L_{134}, it will always be possible to get the same $\mathscr{L}_{(13)}$ that was obtained in L_{123} by adding an appropriate exact 2-form. However, it is not entirely obvious that the $\mathscr{L}_{(14)}$ obtained at this stage will be exactly the same as the one in L_{124}. If the two $\mathscr{L}_{(14)}$ components were to differ by a total x_4 derivative then it would not be possible to correct this by adding an exact 2-form without also changing $\mathscr{L}_{(13)}$, which we don't want to do because it is already in the form we require.

In the case of a 2-form where $\mathscr{L}_{(12)}$ contains only x_1 and x_2 derivatives of u, it follows from the form of $\mathsf{d}\mathsf{L}_{12i}$, as given by Theorem 10, that the resulting $\mathscr{L}_{(i1)}$ Lagrangian need only contain first order derivatives of u with respect to x_i and no products of x_i derivatives of u. This is because, when applying Theorem 10 to obtain $\mathsf{d}\mathsf{L}_{12i}$, the only x_i derivatives of u that appear come from

$$u_{x_i} \cdot \mathrm{E}(\mathscr{L}_{(12)}). \tag{2.66}$$

When reversing the integration by parts that was used to obtain $\mathrm{E}(\mathscr{L}_{(12)})$ from $\mathscr{L}_{(12)}$, this becomes

$$\mathrm{D}_{x_i} \mathscr{L}_{(12)} + \mathrm{D}_{x_1} A_1 + \mathrm{D}_{x_2} A_2 \tag{2.67}$$

for some A_1 and A_2, and since all integration by parts was with respect to x_1 and x_2, A_1 and A_2 do not contain 2^{nd} or higher order derivatives with respect to x_i, or products of x_i derivatives of u. This, in conjunction with the multiform EL equations, in particular those of the form

$$\frac{\delta \mathscr{L}_{(12)}}{\delta u_{x_2}} = \frac{\delta \mathscr{L}_{(1i)}}{\delta u_{x_i}} \tag{2.68}$$

for $i > 1$, where

$$\frac{\delta \mathscr{L}_{(ij)}}{\delta u_I} = \sum_{q,r=0}^{\infty} (-1)^{q+r} \, \mathrm{D}_{x_i}^q \, \mathrm{D}_{x_j}^r \, \frac{\partial \mathscr{L}_{(ij)}}{\partial u_{I i^q j^r}} \tag{2.69}$$

tells us that, modulo total x_1 derivatives, all $\mathscr{L}_{(1i)}$ for $i > 2$ are of the form

$$\frac{\delta \mathscr{L}_{(12)}}{\delta u_{x_2}} u_{x_i} + \mathscr{F}_i \tag{2.70}$$

where $\mathscr{F}_i$ is some function that has no direct dependence on x_i derivatives of u. This guarantees that, for example, the $\mathscr{L}_{(14)}$ coming from L_{134} can be made to coincide with the one coming from L_{124}.

There is also the question of whether the multiform EL equations and closure relation that relate to dL_{234} will be satisfied on the equations of motion relating to $\mathscr{L}_{(12)}, \mathscr{L}_{(13)}$ and $\mathscr{L}_{(14)}$. To show that this is the case, we follow a similar argument to the one given in [11]. Once all of the $\mathscr{L}_{(1i)}$'s are consistently defined, we can form L_{1234} and it follows from

$$\mathsf{d}^2(\mathsf{L}_{1234}) = 0 \tag{2.71}$$

and the form of $\mathsf{dL}_{123}, \mathsf{dL}_{124}$ and dL_{134} in terms of the $\mathscr{L}_{(ij)}$ that

$$\mathrm{D}_{x_1}(\mathrm{D}_{x_2} \mathscr{L}_{(34)} - \mathrm{D}_{x_3} \mathscr{L}_{(24)} + \mathrm{D}_{x_4} \mathscr{L}_{(23)}) \tag{2.72}$$

has a double zero on the equations of motion. Then, since each $\mathscr{L}_{(ij)}$ is an autonomous polynomial, it follows that dL_{234} also has a double zero on the equations of motion, so all of the required relations will be satisfied. This argument can then be used iteratively to further extend the multiform to include higher flows relating to additional commuting variational symmetries. It is also possible to extend this argument to the case of autonomous polynomial systems in higher dimensions, but it remains an open problem to extend this argument to non-autonomous, non-polynomial systems.

The entire AKNS Lagrangian multiform using the recursion operator

The equations of the n^{th} flow of the AKNS hierarchy are given by

$$\begin{pmatrix} q_{x_n} \\ r_{x_n} \end{pmatrix} = R^{n-1} \begin{pmatrix} q_{x_1} \\ r_{x_1} \end{pmatrix} \tag{2.73}$$

where

$$R = i \begin{pmatrix} -q\, \mathrm{D}_{x_1}^{-1}\, r + \frac{1}{2}\, \mathrm{D}_{x_1} & -q\, \mathrm{D}_{x_1}^{-1}\, q \\ r\, \mathrm{D}_{x_1}^{-1}\, r & r\, \mathrm{D}_{x_1}^{-1}\, q - \frac{1}{2}\, \mathrm{D}_{x_1} \end{pmatrix} \tag{2.74}$$

is the recursion operator originally found by A. Lenard. Using this formulation of the AKNS hierarchy in conjunction with Theorem 10, we obtain

$$\mathrm{Div}\, P^{1ij} = \left(\begin{pmatrix} q_{x_j} \\ r_{x_j} \end{pmatrix} - R^{j-1} \begin{pmatrix} q_{x_1} \\ r_{x_1} \end{pmatrix} \right)^T \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \left(\begin{pmatrix} q_{x_i} \\ r_{x_i} \end{pmatrix} - R^{i-1} \begin{pmatrix} q_{x_1} \\ r_{x_1} \end{pmatrix} \right) \tag{2.75}$$

where

$$P^{1ij} = \begin{pmatrix} \mathscr{L}_{(ij)} \\ \mathscr{L}_{(j1)} \\ \mathscr{L}_{(1i)} \end{pmatrix}. \tag{2.76}$$

Therefore $\mathsf{H}(P^{1ij})$ where H is the homotopy operator given in (2.14) gives us an explicit formula for every Lagrangian in the multiform for the entire AKNS hierarchy. This formulation pre-dates the one given in [25], so is the first formulation of a Lagrangian multiform for the entire AKNS hierarchy.

In Appendix B we present some further Lagrangian multiforms relating to the AKNS hierarchy. The results in this appendix have now largely been superseded by the results in [25], hence their relegation to the appendix.

2.2.4 The KP multiform

In this section, we shall construct a Lagrangian multiform for the Kadomtsev-Petviashvili (KP) equation [26]. This is the first example of a Lagrangian multiform for an integrable PDE in $2+1$ dimensions. It is therefore a 3-form. A Lagrangian multiform for the discretised KP equation is given in [27]. Attempts to perform a continuum limit (see [28] for examples of such a procedure) in order to obtain a continuous Lagrangian multiform for the KP equation have, so far, been unsuccessful. In order to proceed, we take as our starting point the Lagrangians

$$\mathscr{L}_{(123)} = \frac{1}{2}v_{x_1x_1}v_{x_1x_3} - \frac{1}{2}v_{3x_1}^2 - \frac{1}{2}v_{x_1x_2}^2 + v_{x_1x_1}^3 \tag{2.77a}$$

$$\mathscr{L}_{(412)} = \frac{1}{2}v_{x_1x_1}v_{x_1x_4} - 2v_{3x_1}v_{x_1x_1x_2} - \frac{2}{3}v_{x_1x_2}v_{x_2x_2} + 4v_{x_1x_1}^2 v_{x_1x_2} \tag{2.77b}$$

where $v_{3x_1} = v_{x_1x_1x_1}$. These are based on the KP Hamiltonians given in [29], which are based on the formulation of [30]. In order to avoid non-local terms, these Lagrangians are given in terms of v such that $v_{x_1x_1} = q$, where q is the usual KP field variable. These Lagrangians give equations of motion

$$v_{3x_1x_3} - v_{x_1x_1x_2x_2} + v_{6x_1} + 6v_{3x_1}^2 + 6v_{x_1x_1}v_{4x_1} = 0, \tag{2.78a}$$

the first KP equation, and

$$v_{3x_1x_4} + 4v_{5x_1x_2} - \frac{4}{3}v_{x_13x_2} + 8v_{4x_1}v_{x_1x_2} + 24v_{3x_1}v_{x_1x_1x_2} + 16v_{x_1x_1}v_{3x_1x_2} = 0 \tag{2.78b}$$

the second KP equation respectively. It is straightforward (although time consuming) to check that setting Q equal to

$$D_{x_1}^{-3}(-v_{x_1x_1x_2x_2} + v_{6x_1} + 6v_{3x_1}^2 + 6v_{2x_1}v_{4x_1}) = -D_{x_1}^{-1}(v_{x_2x_2} + 3v_{x_1x_1}^2) + v_{3x_1} \tag{2.79}$$

gives a variational symmetry $\mathbf{v}_Q$ of the second KP equation (2.78b). This implies that

$$(v_{x_1x_1x_1x_4} + 4v_{5x_1x_2} - \frac{4}{3}v_{x_13x_2} + 8v_{4x_1}v_{x_1x_2} + 24v_{3x_1}v_{x_1x_1x_2} + 16v_{x_1x_1}v_{3x_1x_2})(v_{x_3} - D_{x_1}^{-1}(v_{x_2x_2} + 3v_{x_1x_1}^2) + v_{3x_1}) = \operatorname{Div} P \tag{2.80}$$

We use integration by parts (i.e. integrate the first bracket and differentiate the second bracket, both with respect to x_1) to remove non-local terms and get

$$
\begin{aligned}
&(v_{x_1x_1x_4} + 4v_{4x_1x_2} - \frac{4}{3}v_{3x_2} + 8v_{3x_1}v_{x_1x_2} + 16v_{x_1x_1}v_{x_1x_1x_2})(v_{x_1x_3} - v_{x_2x_2} \\
&+ 3v_{x_1x_1}^2 + v_{4x_1}) = \text{Div}\,\tilde{P}
\end{aligned}
\tag{2.81}
$$

As expected, $\tilde{P}$ describes a Lagrangian 3-form

$$
\begin{aligned}
\mathsf{L} =& \mathscr{L}_{(123)}\mathsf{d}x_1 \wedge \mathsf{d}x_2 \wedge \mathsf{d}x_3 + \mathscr{L}_{(234)}\mathsf{d}x_2 \wedge \mathsf{d}x_3 \wedge \mathsf{d}x_4 + \mathscr{L}_{(341)}\mathsf{d}x_3 \wedge \mathsf{d}x_4 \wedge \mathsf{d}x_1 \\
&+ \mathscr{L}_{(412)}\mathsf{d}x_4 \wedge \mathsf{d}x_1 \wedge \mathsf{d}x_2
\end{aligned}
\tag{2.82}
$$

with the $1, 2, 3$ and 4 components of $\tilde{P}$ corresponding to $-\mathscr{L}_{(234)}, \mathscr{L}_{(341)}, -\mathscr{L}_{(412)}$ and $\mathscr{L}_{(123)}$. The $\mathscr{L}_{(123)}$ and $\mathscr{L}_{(412)}$ Lagrangians are precisely those given in (2.77a) and (2.77b). We find that the $\mathscr{L}_{(234)}$ Lagrangian is given by

$$
\begin{aligned}
\mathscr{L}_{(234)} =& -\frac{1}{2}v_{x_1x_3}v_{x_1x_4} - 4v_{x_1x_3}v_{3x_1x_2} + 2v_{x_1x_1x_3}v_{x_1x_1x_2} - \frac{2}{3}v_{x_2x_2}v_{x_2x_3} + v_{x_2x_2}v_{x_1x_4} \\
&+ 4v_{x_2x_2}v_{3x_1x_2} - \frac{8}{3}v_{x_1x_2x_2}v_{x_1x_1x_2} - v_{3x_1}v_{x_1x_1x_4} + \frac{4}{3}v_{3x_1}v_{3x_2} - 4v_{3x_1}^2v_{x_1x_2} \\
&+ 8v_{x_1x_1}v_{3x_1}v_{x_1x_1x_2} + 8v_{x_1x_1}v_{x_1x_2}v_{x_2x_2} + \frac{4}{3}v_{x_1x_2}^3 - 8v_{x_1x_1}v_{x_1x_2}v_{x_1x_3} \\
&- 8v_{x_1x_1}^3v_{x_1x_2}
\end{aligned}
\tag{2.83}
$$

and the $\mathscr{L}_{(341)}$ Lagrangian is given by

$$
\begin{aligned}
\mathscr{L}_{(341)} =& \frac{2}{3}v_{x_2x_2}^2 + 2v_{4x_1}^2 - 2v_{3x_1}v_{x_1x_1x_3} - \frac{4}{3}v_{x_2x_2}v_{x_1x_3} - \frac{2}{3}v_{x_1x_2}v_{x_2x_3} + v_{x_1x_2}v_{x_1x_4} \\
&- \frac{4}{3}v_{x_1x_1x_2}^2 + \frac{4}{3}v_{3x_1}v_{x_1x_2x_2} + 12v_{x_1x_1}^2v_{4x_1} + 4v_{3x_1}^2v_{x_1x_1} - 4v_{x_1x_1}^2v_{x_2x_2} \\
&+ 4v_{x_1x_1}v_{x_1x_2}^2 + 4v_{x_1x_1}^2v_{x_1x_3} + 10v_{x_1x_1}^4.
\end{aligned}
\tag{2.84}
$$

It is clear from (2.81) that $\mathsf{dL} = 0$ when either the first (2.78a) or second (2.78b) KP equation holds. When both the first and second KP equations hold, the left hand side of (2.81) gives a double zero, so we also have that $\delta\mathsf{dL} = 0$. As a consequence, all of the multiform EL equations hold. This is the first ever example of a continuous Lagrangian 3-form.

2.2.5 Constructing 3-forms with more than 2 flows

In the case of an appropriate 1+1 dimensional integrable hierarchy (e.g. the AKNS hierarchy), we can use Theorem 10 to construct arbitrarily many terms of the corresponding Lagrangian 2-form. This is made possible because every Lagrangian $\mathscr{L}_{(ij)}$ appears in at least one dL_{1ij}. In the case of a 2+1 dimensional integrable hierarchy, Theorem 10 can be used to give each dL_{12ij}, which will give us expressions for Lagrangians of the form $\mathscr{L}_{(12i)}$, $\mathscr{L}_{(1ij)}$ and $\mathscr{L}_{(2ij)}$, but will not help us to find the $\mathscr{L}_{(ijk)}$ Lagrangians for $i, j, k > 2$. In the following example, we present an algorithmic method for finding such Lagrangians and use it to construct a Lagrangian multiform incorporating three flows in 2+1 dimensions.

A simple Lagrangian 3-form with 3 flows

This example is based on a stripped down version of the linearised KP hierarchy. We take our three equations to be

$$\begin{aligned} q_{xt_3} &= q_{yy} \\ q_{xt_4} &= -4q_{xxxy} \\ q_{xt_5} &= -10q_{xxyy} \end{aligned} \tag{2.85}$$

with Lagrangians

$$\begin{aligned}
\mathscr{L}_{(123)} &= \frac{1}{2} q_x q_{t_3} - \frac{1}{2} q_y^2 \\
\mathscr{L}_{(124)} &= \frac{1}{2} q_x q_{t_4} - 2 q_{xx} q_{xy} \\
\mathscr{L}_{(125)} &= \frac{1}{2} q_x q_{t_5} - 5 q_{xy}^2
\end{aligned} \tag{2.86}$$

respectively. Our aim is to find the Lagrangian coefficients for the multiform

$$\begin{aligned}
\mathsf{L}_{12345} = \; & \mathscr{L}_{(123)} \mathsf{d}x \wedge \mathsf{d}y \wedge \mathsf{d}t_3 + \mathscr{L}_{(124)} \mathsf{d}x \wedge \mathsf{d}y \wedge \mathsf{d}t_4 + \mathscr{L}_{(125)} \mathsf{d}x \wedge \mathsf{d}y \wedge \mathsf{d}t_5 \\
& + \mathscr{L}_{(134)} \mathsf{d}x \wedge \mathsf{d}t_3 \wedge \mathsf{d}t_4 + \mathscr{L}_{(135)} \mathsf{d}x \wedge \mathsf{d}t_3 \wedge \mathsf{d}t_5 + \mathscr{L}_{(145)} \mathsf{d}x \wedge \mathsf{d}t_4 \wedge \mathsf{d}t_5 \\
& + \mathscr{L}_{(234)} \mathsf{d}y \wedge \mathsf{d}t_3 \wedge \mathsf{d}t_4 + \mathscr{L}_{(235)} \mathsf{d}y \wedge \mathsf{d}t_3 \wedge \mathsf{d}t_5 + \mathscr{L}_{(245)} \mathsf{d}y \wedge \mathsf{d}t_4 \wedge \mathsf{d}t_5 \\
& + \mathscr{L}_{(345)} \mathsf{d}t_3 \wedge \mathsf{d}t_4 \wedge \mathsf{d}t_5
\end{aligned} \tag{2.87}$$

In order to apply Theorem 10, we note that

$$Q_4 = 4 q_{xxy} \tag{2.88}$$

is the characteristic of a variational symmetry of $\mathscr{L}_{(123)}$ and $\mathscr{L}_{(125)}$, and that

$$Q_5 = 10 q_{xyy} \tag{2.89}$$

is the characteristic of a variational symmetry of $\mathscr{L}_{(123)}$ and $\mathscr{L}_{(124)}$. Letting

$$\tilde{Q}_4 = q_{t_4} + 4 q_{xxy} \tag{2.90}$$

we find that

$$Q_4 \, \mathrm{E}(\mathscr{L}_{(123)}) = (q_{t_4} + 4 q_{xxy})(-q_{xt_3} + q_{yy}) = \mathrm{Div}\, P_{1234} \tag{2.91}$$

where

$$P_{1234} = \begin{pmatrix} -\frac{1}{2}q_{t_3}q_{t_4} - 2q_{xt_3}q_{xy} + 4q_{xy}q_{yy} \\ -2q_{xt_3}q_{xx} + q_{t_4}q_y - 2q_{xy}^2 \\ -\frac{1}{2}q_xq_{t_4} + 2q_{xx}q_{xy} \\ \frac{1}{2}q_xq_{t_3} - \frac{1}{2}q_y^2 \end{pmatrix}. \tag{2.92}$$

Similarly, letting

$$\tilde{Q}_5 = q_{t_5} + 10q_{xyy} \tag{2.93}$$

we find that

$$Q_5\,\mathrm{E}(\mathscr{L}_{(123)}) = (q_{t_5} + 10q_{xyy})(-q_{xt_3} + q_{yy}) = \mathrm{Div}\,P_{1235} \tag{2.94}$$

where

$$P_{1235} = \begin{pmatrix} -\frac{1}{2}q_{t_3}q_{t_5} + 5q_{yy}^2 \\ -10q_{xt_3}q_{xy} + q_{t_5}q_y \\ -\frac{1}{2}q_xq_{t_5} + 5q_{xy}^2 \\ \frac{1}{2}q_xq_{t_3} - \frac{1}{2}q_y^2 \end{pmatrix} \tag{2.95}$$

and also that

$$Q_5\,\mathrm{E}(\mathscr{L}_{(124)}) = (q_{t_5} + 10q_{xyy})(-q_{xt_4} + q_{xxxy}) = \mathrm{Div}\,P_{1245} \tag{2.96}$$

where

$$P_{1245} = \begin{pmatrix} -\frac{1}{2}q_{t_4}q_{t_5} - 4q_{t_5}q_{xxy} + 2q_{xt_5}q_{xy} - 40q_{xxy}q_{xyy} \\ -10q_{xt_4}q_{xy} + 2q_{xt_5}q_{xx} + 20q_{xxy}^2 \\ -\frac{1}{2}q_xq_{t_5} + 5q_{xy}^2 \\ \frac{1}{2}q_xq_{t_4} - 2q_{xx}q_{xy} \end{pmatrix}. \tag{2.97}$$

In addition to the Lagrangians given in (2.86), we now also have

$$
\begin{aligned}
\mathscr{L}_{(134)} &= -2q_{xt_3}q_{xx} + q_{t_4}q_y - 2q_{xy}^2 \\
\mathscr{L}_{(135)} &= -10q_{xt_3}q_{xy} + q_{t_5}q_y \\
\mathscr{L}_{(145)} &= -10q_{xt_4}q_{xy} + 2q_{xt_5}q_{xx} + 20q_{xxy}^2 \\
\mathscr{L}_{(234)} &= \frac{1}{2}q_{t_3}q_{t_4} + 2q_{xt_3}q_{xy} - 4q_{xy}q_{yy} \\
\mathscr{L}_{(235)} &= \frac{1}{2}q_{t_3}q_{t_5} - 5q_{yy}^2 \\
\mathscr{L}_{(245)} &= \frac{1}{2}q_{t_4}q_{t_5} + 4q_{t_5}q_{xxy} - 2q_{xt_5}q_{xy} + 40q_{xxy}q_{xyy}.
\end{aligned}
\tag{2.98}
$$

As expected, our three applications of Theorem 10 have not given us $\mathscr{L}_{(345)}$. We might hope that we could obtain $\mathscr{L}_{(345)}$ by applying the $\tilde{Q}_5$ symmetry to the $\mathscr{L}_{(134)}$ or $\mathscr{L}_{(234)}$ Lagrangians, or that the $\tilde{Q}_4$ symmetry could be applied to the $\mathscr{L}_{(135)}$ or $\mathscr{L}_{(235)}$ Lagrangians. However, the presence of alien derivatives (e.g., y derivarives in $\mathscr{L}_{(134)}$, x derivatives in $\mathscr{L}_{(234)}$) prevents Theorem 10 from working. For this simple example, it is possible to find a $\mathscr{L}_{(345)}$ that works through a process of trial and error; we find that by setting

$$
\mathscr{L}_{(345)} = 10q_{xt_3}q_{yt_4} - 2q_{xt_5}q_{xt_3} - 10q_{t_4}q_{xyt_3} - 4q_{xy}q_{yt_5} - \frac{1}{5}q_{t_5}^2, \tag{2.99}
$$

the multiform EL equations of L_{12345} are (2.85) and corollaries thereof, and $\mathsf{dL}_{12345} = 0$ on the equations of motion.

When dealing with Lagrangians that are not as simple as these (e.g., those of the KP hierarchy), it is not realistic to find such Lagrangians through guesswork, so a more algorithmic method is required. We notice that the $\mathscr{L}_{(345)}$ we have obtained is not unique. If we add or subtract terms that have a double zero on the equations of the multiform (e.g. $(\tilde{Q}_4)_{xx}\,\mathrm{E}(\mathscr{L}_{(123)})$, $(\tilde{Q}_4)_x\tilde{Q}_5$, $\tilde{Q}_5^2$) to $\mathscr{L}_{(345)}$, then dL_{12345} will still have a double zero on the desired equations. By adding or subtracting such terms, we can obtain a unique $\widetilde{\mathscr{L}}_{(345)}$ that contains no products of t_3, t_4 and t_5 derivatives. This $\widetilde{\mathscr{L}}_{(345)}$ can also be obtained from the closure of L as follows. We know from the closure of L that

$$\mathrm{D}_x\,\widetilde{\mathscr{L}}_{(345)} - \mathrm{D}_{t_3}\,\mathscr{L}_{(145)} + \mathrm{D}_{t_4}\,\mathscr{L}_{(135)} - \mathrm{D}_{t_5}\,\mathscr{L}_{(134)} \tag{2.100}$$

has a double zero on the multiform Euler-Lagrange equations. Therefore

$$\mathrm{D}_x\,\widetilde{\mathscr{L}}_{(345)} = \mathrm{D}_{t_3}\,\mathscr{L}_{(145)} - \mathrm{D}_{t_4}\,\mathscr{L}_{(135)} + \mathrm{D}_{t_5}\,\mathscr{L}_{(134)} + A \tag{2.101}$$

where A has a double zero on the multiform Euler-Lagrange equations. Since $\mathrm{D}_x\,\widetilde{\mathscr{L}}_{(345)}$ contains no products of derivatives with respect to t_3, t_4 and t_5, it is easy to find the unique A that eliminates such products from $\mathrm{D}_{t_3}\,\mathscr{L}_{(145)} - \mathrm{D}_{t_4}\,\mathscr{L}_{(135)} + \mathrm{D}_{t_5}\,\mathscr{L}_{(134)}$ and hence we obtain $\mathrm{D}_x\,\widetilde{\mathscr{L}}_{(345)}$ which we can integrate to get $\widetilde{\mathscr{L}}_{(345)}$. In the case of the current example, we find that

$$\begin{aligned}\mathrm{D}_{t_3}\,\mathscr{L}_{(145)} - \mathrm{D}_{t_4}\,\mathscr{L}_{(135)} + \mathrm{D}_{t_5}\,\mathscr{L}_{(134)} =& -10q_{xt_4}q_{xyt_3} + 2q_{xt_5}q_{xxt_3} + 40q_{xxy}q_{xxyt_3}\\ & 10q_{xt_3}q_{xyt_4} - q_{t_5}q_{yt_4} - 2q_{xt_3}q_{xxt_5} + q_{t_4}q_{yt_5}\\ & - 4q_{xy}q_{xyt_5}.\end{aligned} \tag{2.102}$$

It is not a coincidence that there are no mixed time derivatives of q here (e.g., $q_{t_3t_4}$). This is a concequence of the multiform Euler-Lagrange equations of the form

$$\frac{\delta\mathscr{L}_{(ijk)}}{\delta q_{t_k}} - \frac{\delta\mathscr{L}_{(ijl)}}{\delta q_{t_l}} = 0. \tag{2.103}$$

The A that will remove products of t_3, t_4 and t_5 derivatives is

$$\begin{aligned}&-10(\mathrm{E}(\mathscr{L}_{(123)})_y(\tilde{Q}_4)_x + 2(\tilde{Q}_5)_x(\mathrm{E}(\mathscr{L}_{(123)})_x + 10\,\mathrm{E}(\mathscr{L}_{(123)})(\tilde{Q}_4)_{xy} + \tilde{Q}_5(\tilde{Q}_4)_y\\ &- 2\,\mathrm{E}(\mathscr{L}_{(123)})(\tilde{Q}_5)_{xx} - \tilde{Q}_4(\tilde{Q}_5)_y.\end{aligned} \tag{2.104}$$

As a result, we find that

$$\begin{aligned}\mathrm{D}_x \widetilde{\mathscr{L}}_{(345)} =& -10q_{t_4}q_{xyyy} + 4q_{t_5}q_{xxyy} - 4q_{xy}q_{xyt_5} - 20q_{xt_3}q_{xxxyy} - 10q_{yyy}q_{xt_4} \\ &+ 2q_{xt_5}q_{xyy} - 2q_{yy}q_{xxt_5} + 10q_{yy}q_{xyt_4} + 20q_{yy}q_{xxxyy} + 10q_{xyy}q_{yt_4} \\ &- 4q_{yt_5}q_{xxy} + 40q_{xxy}q_{xxyt_3} - 40q_{xxy}q_{xyyy} - 20q_{xxyy}q_{xxt_3} + 60q_{xxyy}q_{xyy} \\ &+ 40q_{xyt_3}q_{xxxy} - 40q_{yyy}q_{xxxy}\end{aligned} \tag{2.105}$$

which we integrate with respect to x to obtain

$$\begin{aligned}\widetilde{\mathscr{L}}_{(345)} =& -10q_{t_4}q_{yyy} + 4q_{t_5}q_{xyy} - 2q_{yy}q_{xt_5} - 4q_{yt_5}q_{xy} - 20q_{xt_3}q_{xxyy} + 10q_{yy}q_{yt_4} \\ &+ 40q_{xxy}q_{xyt_3} + 20q_{yy}q_{xxyy} - 40q_{xxy}q_{yyy} + 20q_{xyy}^2.\end{aligned} \tag{2.106}$$

This $\widetilde{\mathscr{L}}_{(345)}$ differs from $\mathscr{L}_{(345)}$ by

$$10\,\mathrm{E}(\mathscr{L}_{(123)})(\tilde{Q}_4)_y - 2\,\mathrm{E}(\mathscr{L}_{(123)})(\tilde{Q}_5)_x - 10(\mathrm{E}(\mathscr{L}_{(123)}))_y\tilde{Q}_4 + \frac{1}{5}(\tilde{Q}_5)^2, \tag{2.107}$$

a double zero on the equations of the multiform. It follows from

$$\mathsf{d}^2\mathsf{L}_{12345} = 0 \tag{2.108}$$

that $\widetilde{\mathscr{L}}_{(345)}$ is such that

$$\mathrm{D}_y \widetilde{\mathscr{L}}_{(345)} - \mathrm{D}_{t_3} \mathscr{L}_{(245)} + \mathrm{D}_{t_4} \mathscr{L}_{(235)} - \mathrm{D}_{t_5} \mathscr{L}_{(234)} \tag{2.109}$$

also has a double zero on the equations of the multiform.

In Appendix C we apply the method outlined above to extend the KP multiform to include the t_5 flow.

2.3 Discrete and semi-discrete Lagrangian multiforms from variational symmetries

In this section, we look at the discrete and semi-discrete analogue of the results from the previous sections. As was the case when we considered the multiform Euler-Lagrange equations, we shall work on a discrete manifold of N independent coordinates $n_1, \ldots, n_N$ and dependent variable u, with shift operator T_i such that

$$\mathrm{T}_i\, u(n_1, \ldots, n_i, \ldots, n_N) = u(n_1, \ldots, n_i + 1, \ldots, n_N). \tag{2.110}$$

Letting J be the ordered set $(j_1, \ldots, j_N)$, we define

$$\mathrm{T}_J = \prod_{i=1}^{N} (T_i)^{j_i} \tag{2.111}$$

and

$$u_J = \mathrm{T}_J\, u. \tag{2.112}$$

We also define the discrete derivative D_i such that

$$\mathrm{D}_i\, u = \mathrm{T}_i\, u - u. \tag{2.113}$$

In this case, our Lagrangian density $\mathscr{L}$ can be a function of discrete coordinates $n_1, \ldots, n_N$, our dependent variable $u = (u_1, \ldots, u_q)$ and shifts of u (either positive or negative) up to some order m. The Euler-Lagrange equations for $\mathscr{L}$ are given by

$$\frac{\delta \mathscr{L}}{\delta u} = 0 \tag{2.114}$$

where

$$\frac{\delta}{\delta u} = \sum_J (T^{-1})_J \frac{\partial}{\partial u_J}. \tag{2.115}$$

In sections 2.3.1 and 2.3.2 we present some key results from [31, 32] relating to variational symmetries of discrete Lagrangians.

2.3.1 Variational symmetries of discrete Lagrangians

Analogously to the continuous case, it is natural to consider as a generator of a variational symmetry, a vector

$$\mathbf{v} = \sum_{i=1}^{N} \xi_i \frac{\partial}{\partial n_i} + \sum_{\alpha=1}^{q} \phi_\alpha \frac{\partial}{\partial u^\alpha} \tag{2.116}$$

where the ξ_i and ϕ_α depend on our discrete coordinates $n_1, \dots, n_N$, our dependent variable u and shifts thereof. However, the $\xi_i \dfrac{\partial}{\partial n_i}$ components of $\mathbf{v}$ represent a continuous deformation of the lattice points on our discrete manifold. As a consequence, unlike in the continuous setting, it is not possible to find an associated evolutionary vector $\mathbf{v}_Q$ that generates essentially the same variational symmetry. Although it is possible to find variational symmetries generated by vectors in the form of $\mathbf{v}$ in the discrete setting, they do not yield Lagrangian multiforms so, from here on, we shall only consider evolutionary vectors of the form

$$\mathbf{v}_Q = \sum_{\alpha=1}^{q} \phi_\alpha \frac{\partial}{\partial u^\alpha}, \tag{2.117}$$

which behave in almost exactly the same way as their continuous counterparts. The prolongation of such an evolutionary vector $\mathbf{v}_Q$,

$$\operatorname{pr} \mathbf{v}_Q = \sum_{\alpha=1}^{q} \sum_{J} (\phi_\alpha)_J \frac{\partial}{\partial (u^\alpha)_J}. \tag{2.118}$$

An evolutionary vector $\mathbf{v}_Q$ is a variational symmetry of a Lagrangian $\mathscr{L}$ if and only if

$$\operatorname{pr} \mathbf{v}_Q(\mathscr{L}) = \operatorname{Div} B = \sum_{i=1}^{N} \mathrm{D}_i \, B_i \tag{2.119}$$

where $B = (B_1, \dots, B_N)^T$ and each B_i is a function of the discrete coordinates $n_1, \dots, n_N$, the dependent variable u and shifts thereof. I.e., an evolutionary vector $\mathbf{v}_Q$ is a variational symmetry of a Lagrangian $\mathscr{L}$ if and only if $\operatorname{pr} \mathbf{v}_Q(\mathscr{L})$ is a discrete divergence.

2.3.2 Noether-type identities and a discrete analogue of Noether's theorem

Since, for any Lagrangian $\mathscr{L}$,

$$\delta \mathscr{L} = \delta u \cdot \frac{\delta \mathscr{L}}{\delta u} + \operatorname{Div} A_1 \tag{2.120}$$

for some A_1, it follows that

$$\operatorname{pr} \mathbf{v}_Q(\mathscr{L}) = Q \cdot \frac{\delta \mathscr{L}}{\delta u} + \operatorname{Div} A_2 \tag{2.121}$$

for some A_2. For example, if $\mathscr{L}$ depends only on u, $T_1 u = \widetilde{u}$ and $T_2 u = \widehat{u}$, then

$$\begin{aligned}
\operatorname{pr} \mathbf{v}_Q \mathscr{L} &= Q \cdot \frac{\partial \mathscr{L}}{\partial u} + \widetilde{Q} \cdot \frac{\partial \mathscr{L}}{\partial \widetilde{u}} + \widehat{Q} \cdot \frac{\partial \mathscr{L}}{\partial \widehat{u}} \\
&= Q \cdot (\frac{\partial \mathscr{L}}{\partial u} + \frac{\partial \underset{\sim}{\mathscr{L}}}{\partial u} + \frac{\partial \underset{\wedge}{\mathscr{L}}}{\partial u}) + \mathrm{D}_1 \left(Q \cdot \frac{\partial \underset{\sim}{\mathscr{L}}}{\partial u} \right) + \mathrm{D}_2 \left(Q \cdot \frac{\partial \underset{\wedge}{\mathscr{L}}}{\partial u} \right) \\
&= Q \cdot \frac{\delta \mathscr{L}}{\delta u} + \mathrm{D}_1 \left(Q \cdot \frac{\partial \underset{\sim}{\mathscr{L}}}{\partial u} \right) + \mathrm{D}_2 \left(Q \cdot \frac{\partial \underset{\wedge}{\mathscr{L}}}{\partial u} \right).
\end{aligned} \tag{2.122}$$

Identities such as this one are referred to in [32] as Noether-type identities. If $\mathbf{v}_Q$ is a variational symmetry of $\mathscr{L}$, then combining (2.119) and (2.121) we obtain that

$$Q \cdot \frac{\delta \mathscr{L}}{\delta u} = \operatorname{Div} C \tag{2.123}$$

where $C = B - A_2$, a discrete analogue of Noether's theorem.

2.3.3 Semi-discrete Lagrangians and symmetries

Once we restrict ourselves to the case of evolutionary symmetries (i.e., in the form of (2.117)), we find that discrete Lagrangians and symmetries behave in exactly the same way as their continuous counterparts. As a result, a semi-discrete Lagrangian, i.e., one that is a function of discrete coordinates $n_1, \ldots, n_N$, continuous coordinates $x_1, \ldots, x_M$ dependent variable $u = (u_1, \ldots, u_q)$ and shifts and derivatives of u, can be treated in a similar manner. Similarly, a variational symmetry where the ϕ_α are semi-discrete can be used. In the following section, we give an example of a semi-discrete Lagrangian multiform arising from variational symmetries.

2.3.4 Examples of semi-discrete Lagrangian multiforms arising from variational symmetries

We will now use the principles outlined above to construct a semi-discrete Lagrangian 1-form and a semi-discrete Lagrangian 2-form. We take our starting Lagrangian to be

$$\mathscr{L} = \tilde{u}u - \frac{1}{2}u_{t_1}^2 \tag{2.124}$$

This gives Euler-Lagrange equations

$$\mathrm{E}(\mathscr{L}) = u_{t_1t_1} + \tilde{u} + \underset{\sim}{u}. \tag{2.125}$$

We define the vector fields $X_2 = \eta_2\partial_u$ and $X_3 = \eta_3\partial_u$ where $\eta_2 = \tilde{u} - \underset{\sim}{u}$ and $\eta_3 = \tilde{\tilde{u}} - \underset{\sim}{\underset{\sim}{u}}$. These vector fields are variational symmetries of any quadratic Lagrangian that is summed in the $\sim$ direction. In particular, they are both variational symmetries of $\mathscr{L}$. In the case of X_2,

$$\mathrm{pr}\, X_2(\mathscr{L}) = \mathrm{D}_\sim(u^2 + \tilde{u}\underset{\sim}{u} - u_{t_1}\underset{\sim}{u}_{t_1}). \tag{2.126}$$

We also have the Noether type identity

$$\mathrm{pr}\, X_2(\mathscr{L}) = \eta_2\, \mathrm{E}(\mathscr{L}) + \mathrm{D}_\sim(\eta_2 \frac{\partial \underset{\sim}{\mathscr{L}}}{\partial u}) + \mathrm{D}_{t_1}(\eta_2 \frac{\partial \mathscr{L}}{\partial u_{t_1}}). \tag{2.127}$$

Combining (2.126) and (2.127) we obtain

$$\eta_2\, \mathrm{E}(\mathscr{L}) = \mathrm{D}_\sim(u^2 + \underset{\sim}{u}^2 - u_{t_1}\underset{\sim}{u}_{t_1}) + \mathrm{D}_{t_1}(\tilde{u}u_{t_1} - \underset{\sim}{u}u_{t_1}), \tag{2.128}$$

a two component Lagrangian 1-form with $\mathscr{L}_{(\sim)} = \underset{\sim}{u}u_{t_1} - \tilde{u}u_{t_1}$ and $\mathscr{L}_{(t_1)} = u^2 + \underset{\sim}{u}^2 - u_{t_1}\underset{\sim}{u}_{t_1}$. The full set of multiform Euler-Lagrange equations is as follows:

$$
\begin{aligned}
\frac{\delta \mathscr{L}_{(\sim)}}{\delta \underset{\sim}{u}} &= 0 \implies u_{t_1} - \underset{\sim}{u}_{t_1} = 0 \\
\frac{\delta \mathscr{L}_{(\sim)}}{\delta \underset{\sim}{u}_{t_1}} &= 0 \implies \underset{\sim}{\underset{\sim}{u}} - u = 0 \\
\frac{\delta \mathscr{L}_{(t_1)}}{\delta \underset{\sim}{u}} &= 0 \implies 2\underset{\sim}{u} + u_{t_1t_1} = 0 \\
\frac{\delta \mathscr{L}_{(t_1)}}{\delta u} &= 0 \implies 2u + \underset{\sim}{u}_{t_1} = 0 \\
\frac{\delta \mathscr{L}_{(\sim)}}{\delta u} - \mathrm{T}_{\sim} \frac{\delta \mathscr{L}_{(t_1)}}{\delta \underset{\sim}{u}_{t_1}} &\implies -\underset{\sim}{u}_{t_1} + \tilde{u}_{t_1} = 0 \\
\frac{\delta \mathscr{L}_{(\sim)}}{\delta \tilde{u}} - \mathrm{T}_{\sim} \frac{\delta \mathscr{L}_{(t_1)}}{\delta u_{t_1}} &\implies -u_{t_1} + u_{t_1} = 0.
\end{aligned}
\tag{2.129}
$$

As expected, all of these are consistent with the equations $\mathrm{E}(\mathscr{L}) = 0$ and $\eta_2 = 0$.

We now consider the vector field $\bar{X}_2 = \bar{\eta}_2 \partial_u$ where $\bar{\eta}_2 = u_{t_2} + \tilde{u} - \underset{\sim}{u}$ is semi-discrete. In this case,

$$
\mathrm{pr}\, \bar{X}_2(\mathscr{L}) = \mathrm{D}_{t_2} \mathscr{L} + \mathrm{D}_{\sim}(u^2 + \tilde{u}\underset{\sim}{u} - u_{t_1}\underset{\sim}{u}_{t_1}), \tag{2.130}
$$

so (in the case where the action associated with $\mathscr{L}$ is an integration over t_1 and t_2 and a sum over the $\sim$ variable) $\bar{X}_2$ is a variational symmetry of $\mathscr{L}$. We can then use the same Noether-type identity to obtain

$$
\bar{\eta}_2\, \mathrm{E}(\mathscr{L}) = \mathrm{D}_{\sim}(u^2 + \underset{\sim}{u}^2 - u_{t_1}\underset{\sim}{u}_{t_1} - \underset{\sim}{u}u_{t_2}) + \mathrm{D}_{t_1}(u_{t_1}u_{t_2} + \tilde{u}u_{t_1} - \underset{\sim}{u}u_{t_1}) + \mathrm{D}_{t_2} \mathscr{L}, \tag{2.131}
$$

giving us a Lagrangian 2-form where $\mathscr{L}_{(\sim t_1)} = \mathscr{L}$, $\mathscr{L}_{(t_1 t_2)} = u^2 + \underset{\sim}{u}^2 - u_{t_1}\underset{\sim}{u}_{t_1} - \underset{\sim}{u}u_{t_2}$ and $\mathscr{L}_{(t_2 \sim)} = u_{t_1}u_{t_2} + \tilde{u}u_{t_1} - \underset{\sim}{u}u_{t_1}$. We now consider the vector field $\bar{X}_3 = \bar{\eta}_3 \partial_u$ where $\bar{\eta}_3 = u_{t_3} + \tilde{\tilde{u}} - \underset{\sim}{\underset{\sim}{u}}$. We find that

$$\operatorname{pr}\bar{X}_3(\mathscr{L}) = \mathrm{D}_{t_3}\,\mathscr{L} + \mathrm{D}_{\sim}(\widetilde{u}u + \underset{\sim}{u}u + \widetilde{\widetilde{u}}\underset{\sim}{u} + \widetilde{u}\underset{\sim}{\underset{\sim}{u}} - \widetilde{u}_{t_1}\underset{\sim}{u}_{t_1} - u_{t_1}\underset{\sim}{\underset{\sim}{u}}_{t_1}), \tag{2.132}$$

so $\bar{X}_3$ is a variational symmetry of $\mathscr{L}$. Again using the same Noether-type identity, we obtain

$$\begin{aligned}\bar{\eta}_3\,\mathrm{E}(\mathscr{L}) =& \mathrm{D}_{t_3}\,\mathscr{L} + \mathrm{D}_{\sim}(\widetilde{u}u + \underset{\sim}{u}u + \underset{\sim}{\underset{\sim}{u}}\underset{\sim}{u} + \widetilde{u}\underset{\sim}{\underset{\sim}{u}} - \widetilde{u}_{t_1}\underset{\sim}{u}_{t_1} - u_{t_1}\underset{\sim}{\underset{\sim}{u}}_{t_1} - \underset{\sim}{u}u_{t_3}) \\ &+ \mathrm{D}_{t_1}(u_{t_1}u_{t_3} + u_{t_1}\widetilde{\widetilde{u}} - u_{t_1}\underset{\sim}{\underset{\sim}{u}}),\end{aligned} \tag{2.133}$$

giving us a Lagrangian 2-form where $\mathscr{L}_{(\sim t_1)} = \mathscr{L}$, $\mathscr{L}_{(t_1 t_3)} = \widetilde{u}u + \underset{\sim}{u}u + \underset{\sim}{\underset{\sim}{u}}\underset{\sim}{u} + \widetilde{u}\underset{\sim}{\underset{\sim}{u}} - \widetilde{u}_{t_1}\underset{\sim}{u}_{t_1} - u_{t_1}\underset{\sim}{\underset{\sim}{u}}_{t_1} - \underset{\sim}{u}u_{t_3}$ and $\mathscr{L}_{(t_3\sim)} = u_{t_1}u_{t_3} + u_{t_1}\widetilde{\widetilde{u}} - u_{t_1}\underset{\sim}{\underset{\sim}{u}}$. Finally, we note that for the $\mathscr{L}_{(t_2\sim)}$ given above,

$$\begin{aligned}\operatorname{pr}\bar{X}_3(\mathscr{L}_{(t_2\sim)}) =& \mathrm{D}_{t_3}\,\mathscr{L}_{(\sim t_2)} + \mathrm{D}_{\sim}(\widetilde{u}_{t_1}\underset{\sim}{u}_{t_2} + \widetilde{u}_{t_2}\underset{\sim}{u}_{t_1} + u_{t_1}\underset{\sim}{\underset{\sim}{u}}_{t_2} + u_{t_2}\underset{\sim}{\underset{\sim}{u}}_{t_1} + \widetilde{\widetilde{u}}\underset{\sim}{u}_{t_1} \\ &+ \widetilde{u}\underset{\sim}{\underset{\sim}{u}}_{t_1} + u\widetilde{u}_{t_1} + \underset{\sim}{u}u_{t_1} - u\underset{\sim}{u}_{t_1} - \underset{\sim}{u}\underset{\sim}{\underset{\sim}{u}}_{t_1} - \underset{\sim}{\underset{\sim}{u}}\widetilde{u}_{t_1} - \underset{\sim}{\underset{\sim}{\underset{\sim}{u}}}u_{t_1}),\end{aligned} \tag{2.134}$$

so $\bar{X}_3$ is a variational symmetry of $\mathscr{L}_{(t_2\sim)}$. Since $\mathscr{L}_{(t_2\sim)}$ is a function of u_{t_1}, u_{t_2}, $\widetilde{u}$ and $\underset{\sim}{u}$, we use the Noether-type identity

$$\begin{aligned}\operatorname{pr}\bar{X}_3(\mathscr{L}_{(t_2\sim)}) =& \bar{\eta}_3\,\mathrm{E}(\mathscr{L}_{(t_2\sim)}) + \mathrm{D}_{\sim}(\underset{\sim}{\bar{\eta}_3}\frac{\partial\mathscr{L}_{(t_2\sim)}}{\partial u} - \bar{\eta}_3\frac{\partial\mathscr{L}_{(t_2\sim)}}{\partial\underset{\sim}{u}}) \\ &+ \mathrm{D}_{t_1}(\bar{\eta}_3\frac{\partial\mathscr{L}_{(t_2\sim)}}{\partial u_{t_1}}) + \mathrm{D}_{t_2}(\bar{\eta}_3\frac{\partial\mathscr{L}_{(t_2\sim)}}{\partial u_{t_2}}).\end{aligned} \tag{2.135}$$

We note that $\mathscr{L}_{(t_2\sim)}$ contains "alien" t_1 derivatives, so $\bar{\eta}_3\,\mathrm{E}(\mathscr{L}_{(t_2\sim)})$ is taken to mean

$$\begin{pmatrix} \frac{\delta \mathscr{L}}{\delta u} \\ \frac{\delta \mathscr{L}}{\delta u_{t_1}} \end{pmatrix} \cdot \begin{pmatrix} \bar{\eta}_3 \\ (\bar{\eta}_3)_{t_1} \end{pmatrix}. \tag{2.136}$$

Combining (2.134) and (2.135) we obtain that

$$\begin{aligned} \bar{\eta}_3 \,\mathrm{E}(\mathscr{L}_{(t_2\sim)}) =\, & \mathrm{D}_{t_3}(\mathscr{L}_{(t_2\sim)}) + \mathrm{D}_{t_2}(-u_{t_1}u_{t_3} - u_{t_1}\tilde{\tilde{u}} + u_{t_1}\underset{\sim}{\underset{\sim}{u}}) \\ & + \mathrm{D}_{\sim}(\tilde{u}_{t_1}\underset{\sim}{u}_{t_2} + \tilde{u}_{t_2}\underset{\sim}{u}_{t_1} + u_{t_1}\underset{\sim}{\underset{\sim}{u}}_{t_2} + u_{t_2}\underset{\sim}{\underset{\sim}{u}}_{t_1} + \tilde{u}\underset{\sim}{\underset{\sim}{u}}_{t_1} + u\tilde{u}_{t_1} \\ & - u_{t_1}\tilde{u} + \underset{\sim}{u}u_{t_1} - u\underset{\sim}{u}_{t_1} - \underset{\sim}{u}\,\underset{\sim}{\underset{\sim}{u}}_{t_1} + \underset{\sim}{\underset{\sim}{u}}\,\underset{\sim}{u}_{t_1} - \underset{\sim}{\underset{\sim}{u}}\tilde{u}_{t_1} - u_{t_3}\underset{\sim}{u}_{t_1} - u_{t_1}\underset{\sim}{u}_{t_3}). \end{aligned} \tag{2.137}$$

This gives us a $\mathscr{L}_{(\sim t_3)}$ Lagrangian the same as the one obtained eariler and also a $\mathscr{L}_{(t_3 t_2)}$ Lagrangian. Thus, just like we did for the first three flows of the AKNS hierarchy, we have obtained a semi-discrete Lagrangian 2-form

$$\begin{aligned} \mathsf{L} =\, & \mathscr{L}_{(t_1\sim)}\mathsf{d}t_1 \wedge \mathsf{d}\!\sim + \mathscr{L}_{(t_2\sim)}\mathsf{d}t_2 \wedge \mathsf{d}\!\sim + \mathscr{L}_{(t_3\sim)}\mathsf{d}t_3 \wedge \mathsf{d}\!\sim \\ & + \mathscr{L}_{(t_1 t_2)}\mathsf{d}t_1 \wedge \mathsf{d}t_2 + \mathscr{L}_{(t_2 t_3)}\mathsf{d}t_2 \wedge \mathsf{d}t_3 + \mathscr{L}_{(t_1 t_3)}\mathsf{d}t_1 \wedge \mathsf{d}t_3 \end{aligned} \tag{2.138}$$

where

$$\begin{aligned} \mathscr{L}_{(\sim t_1)} =\, & \tilde{u}u - \frac{1}{2}u_{t_1}^2 \\ \mathscr{L}_{(\sim t_2)} =\, & \underset{\sim}{u}u_{t_1} - \tilde{u}u_{t_1} - u_{t_1}u_{t_2} \\ \mathscr{L}_{(\sim t_3)} =\, & u_{t_1}\underset{\sim}{\underset{\sim}{u}} - u_{t_1}\tilde{\tilde{u}} - u_{t_1}u_{t_3} \\ \mathscr{L}_{(t_1 t_2)} =\, & u^2 + \underset{\sim}{u}^2 - u_{t_1}\underset{\sim}{u}_{t_1} - \underset{\sim}{u}u_{t_2} \\ \mathscr{L}_{(t_2 t_3)} =\, & -\tilde{u}_{t_1}\underset{\sim}{u}_{t_2} - \tilde{u}_{t_2}\underset{\sim}{u}_{t_1} - u_{t_1}\underset{\sim}{\underset{\sim}{u}}_{t_2} - u_{t_2}\underset{\sim}{\underset{\sim}{u}}_{t_1} - \tilde{u}\underset{\sim}{\underset{\sim}{u}}_{t_1} - u\tilde{u}_{t_1} + u_{t_1}\tilde{u} \\ & - \underset{\sim}{u}u_{t_1} + u\underset{\sim}{u}_{t_1} + \underset{\sim}{u}\,\underset{\sim}{\underset{\sim}{u}}_{t_1} - \underset{\sim}{\underset{\sim}{u}}\,\underset{\sim}{u}_{t_1} + \underset{\sim}{\underset{\sim}{u}}\tilde{u}_{t_1} + u_{t_3}\underset{\sim}{u}_{t_1} + u_{t_1}\underset{\sim}{u}_{t_3} \\ \mathscr{L}_{(t_1 t_3)} =\, & \tilde{u}u + \underset{\sim}{u}u + \underset{\sim}{\underset{\sim}{u}}\,\underset{\sim}{u} + \tilde{u}\underset{\sim}{\underset{\sim}{u}} - \tilde{u}_{t_1}\underset{\sim}{u}_{t_1} - u_{t_1}\underset{\sim}{\underset{\sim}{u}}_{t_1} - \underset{\sim}{u}u_{t_3} \end{aligned} \tag{2.139}$$

that contains the three flows.

2.4 Conclusion

Given any Lagrangian and an associated variational symmetry, the method outlined in this chapter allows us to construct a Lagrangian multiform. As a consequence, we have shown that the existence of a Lagrangian multiform structure is not a sufficient condition for integrability. However, by linking Lagrangian multiforms to variational symmetries, existing results relating symmetries to integrability can now be applied to Lagrangian multiforms of the type described in this chapter. Whilst we have shown that every variational symmetry leads to a Lagrangian multiform, the question of when the converse holds remains an open problem.

Chapter 3

The Zakharov-Mikhailov Lagrangian multiform

3.1 The Zakharov-Mikhailov Lagrangian

Following the method of Zakharov and Mikhailov [5] we start from a $N \times N$ matrix Lax pair U and V and auxiliary problem

$$\Psi_\xi = U(\xi, \eta, \lambda)\Psi, \quad \Psi_\eta = V(\xi, \eta, \lambda)\Psi. \tag{3.1}$$

Henceforth, we shall commit an abuse of terminology and refer to the $N \times N$ matrix Ψ as the eigenfunction of the Lax pair. This gives rise to the compatibility condition

$$U_\eta - V_\xi + [U, V] = 0. \tag{3.2}$$

We assume that U and V are rational functions of λ with a finite number of distinct simple poles (the case where U and V have higher order poles is dealt with in [33]), so

$$U = U^0(\xi, \eta) + \sum_{i=1}^{N_1} \frac{U^i(\xi, \eta)}{\lambda - a_i}, \quad V = V^0(\xi, \eta) + \sum_{j=1}^{N_2} \frac{V^j(\xi, \eta)}{\lambda - b_j}. \tag{3.3}$$

giving the compatibility conditions

$$U^0_\eta - V^0_\xi + [U^0, V^0] = 0 \tag{3.4}$$

and

$$U^i_\eta + \left[U^i, V^0 + \sum_{j=1}^{N_2} \frac{V^j}{a_i - b_j}\right] = 0, \quad V^j_\xi + \left[V^j, U^0 + \sum_{i=1}^{N_1} \frac{U^i}{b_j - a_i}\right] = 0. \tag{3.5}$$

Equation (3.4) implies that U^0 and V^0 can be written in terms of an invertible matrix $g(\xi, \eta)$ such that

$$U^0 = g_\xi g^{-1}, \quad V_0 = g_\eta g^{-1}. \tag{3.6}$$

The matrices U^i and V^j are expressed as

$$U^i = \varphi^i \bar{U}^i (\varphi^i)^{-1}, \quad V^j = \psi^j \bar{V}^j (\psi^j)^{-1} \tag{3.7}$$

where $\bar{U}^i$ and $\bar{V}^j$ are the Jordan normal forms of U^i and V^j which depend only on ξ and η respectively. In order to show that $\bar{U}^i$ depends only on ξ, we let Y^i be the solution of

$$Y^i_\eta = V|_{\lambda=a_i} Y^i \tag{3.8}$$

then

$$\begin{aligned}\partial_\eta((Y^i)^{-1}U^iY^i) &= -(Y^i)^{-1}V|_{\lambda=a_i}U^iY^i + (Y^i)^{-1}[V|_{\lambda=a_i}, U^i]Y^i + (Y^i)^{-1}U^iV|_{\lambda=a_i}Y^i \\ &= 0,\end{aligned} \tag{3.9}$$

so $(Y^i)^{-1}U^iY^i$ is constant with respect to η. Since similarity transformations preserve eigenvalues, this tells us that the eigenvalues of U^i do not depend on η. Therefore the Jordan normal matrix $\bar{U}^i$ which has the same eigenvalues as U^i does not depend on η. Similarly $\bar{V}^j$ does not depend on ξ.

The ZM Lagrangian

$$\mathscr{L}_{(\xi\eta)} = \operatorname{tr}\left\{ \sum_{i=1}^{N_1} (\varphi^i)^{-1}(\varphi^i_\eta - g_\eta g^{-1}\varphi^i)\bar{U}^i - \sum_{j=1}^{N_2} (\psi^j)^{-1}(\psi^j_\xi - g_\xi g^{-1}\psi^j)\bar{V}^j \right. \\ \left. - \sum_{i=1}^{N_1}\sum_{j=1}^{N_2} \frac{\psi^j\bar{V}^j(\psi^j)^{-1}\varphi^i\bar{U}^i(\varphi^i)^{-1}}{a_i - b_j} \right\} \tag{3.10}$$

has Euler-Lagrange equations equivalent to the compatibility conditions (3.5). We find that

$$\frac{\delta\mathscr{L}}{\delta\varphi^i} = -(\varphi^i)^{-1}\left(\varphi^i_\eta - (g_\eta g^{-1} + \sum_{j=1}^{N_2} \frac{\psi^j\bar{V}^j(\psi^j)^{-1}}{a_i - b_j})\varphi^i\right)\bar{U}^i(\varphi^i)^{-1} \\ + \bar{U}^i(\varphi^i)^{-1}\left(\varphi^i_\eta - (g_\eta g^{-1} + \sum_{j=1}^{N_2} \frac{\psi^j\bar{V}^j(\psi^j)^{-1}}{a_i - b_j})\varphi^i\right)(\varphi^i)^{-1} \tag{3.11a}$$

and

$$\frac{\delta\mathscr{L}}{\delta\psi^j} = (\psi^j)^{-1}\left(\psi^j_\xi - (g_\xi g^{-1} + \sum_{i=1}^{N_1} \frac{\varphi^i\bar{U}^i(\varphi^i)^{-1}}{b_j - a_i})\psi^j\right)\bar{V}^j(\psi^j)^{-1} \\ - \bar{V}^j(\psi^j)^{-1}\left(\psi^j_\xi - (g_\xi g^{-1} + \sum_{i=1}^{N_1} \frac{\varphi^i\bar{U}^i(\varphi^i)^{-1}}{b_j - a_i})\psi^j\right)(\psi^j)^{-1} \tag{3.11b}$$

which, when we use (3.7) and set equal to zero are equivalent to (3.5).

Remark 17. *From the definition of φ^i and ψ^j in (3.7), it is clear that they are not-unique. As a result, (3.11a) is equivalent to the statement that*

$$(\varphi^i)^{-1}(\varphi^i_\eta - (g_\eta g^{-1} + \sum_{j=1}^{N_2} \frac{\psi^j\bar{V}^j(\psi^j)^{-1}}{a_i - b_j})\varphi^i) \tag{3.12}$$

can be any matrix that commutes with $\bar{U}^i$. A similar statement relating to $\bar{V}^j$ follows from (3.11b). However, the non-uniqueness of φ^i and ψ^j does not lead to any additional freedom on solutions of the system because, by (3.7), this freedom does not affect U^i and V^j.

We also find that

$$\begin{aligned}\frac{\delta \mathscr{L}}{\delta g} =& \sum_{i=1}^{N_1} \left\{ D_\eta(g^{-1}\varphi^i \bar{U}^i (\varphi^i)^{-1}) + g^{-1}\varphi^i \bar{U}^i (\varphi^i)^{-1} g^{-1} g_\eta \right\} \\ &- \sum_{j=1}^{N_2} \left\{ D_\xi(g^{-1}\psi^j \bar{V}^j (\psi^j)^{-1}) + g^{-1}\psi^j \bar{V}^j (\psi^j)^{-1} g^{-1} g_\xi \right\}.\end{aligned} \tag{3.13}$$

When we use (3.7) and set equal to zero this is equivalent to

$$\sum_{i=1}^{N_1} \left\{ U^i_\eta + [U^i, V^0] \right\} = \sum_{j=1}^{N_2} \left\{ V^j_\xi + [V^j, U^0] \right\} \tag{3.14}$$

which is a consequence of (3.5). Compatibility condition (3.4) follows directly from the form of U^0 and V^0 in terms of g, i.e. it is not a variational equation of this Lagrangian. Zakharov and Mikhailov made no reference in [5] to varying the fields $\bar{U}^i$ and $\bar{V}^j$ (although, in [33, 34], Dickey does vary the analog of these fields). We note that, in the ZM formulation, this would amount to varying a Jordan normal matrix.

Remark 18. *By letting $\Psi \to \Phi = g^{-1}\Psi$, letting $U^i \to \tilde{U}^i = g^{-1}U^i g$ and letting $V^j \to \tilde{V}^j = g^{-1}V^j g$ we can express the auxiliary problem (3.1) without U^0 and V^0 terms. This allows us, without loss of generality, to omit g from all ZM related Lagrangians from here on.*

We shall now change our perspective from the ZM construction, and consider the ZM Lagrangian

$$\mathscr{L}_{(\xi\eta)} = \mathrm{tr} \left\{ \sum_{i=1}^{N_1} (\varphi^i)^{-1}\varphi^i_\eta \bar{U}^i - \sum_{j=1}^{N_2} (\psi^j)^{-1}\psi^j_\xi \bar{V}^j - \sum_{i=1}^{N_1}\sum_{j=1}^{N_2} \frac{\psi^j \bar{V}^j (\psi^j)^{-1} \varphi^i \bar{U}^i (\varphi^i)^{-1}}{a_i - b_j} \right\} \tag{3.15}$$

as our fundamental object. We impose that $\bar{U}^i$ and $\bar{V}^j$ depend only on ξ and η respectively. We no longer impose that $\bar{U}^i$ and $\bar{V}^j$ are in Jordan normal form, and now consider them to be fundamental matrix-valued fields. We now take variational derivatives with respect to all field variables, including $\bar{U}^i$ and $\bar{V}^j$. The variational derivative with respect to $\bar{U}^i$ reads

$$\frac{\delta \mathscr{L}_{(\xi\eta)}}{\delta \bar{U}^i} = (\varphi^i)^{-1}\varphi^i_\eta - \sum_{j=1}^{N_2} \frac{(\varphi^i)^{-1}\psi^j \bar{V}^j (\psi^j)^{-1}\varphi^i}{a_i - b_j}. \tag{3.16}$$

We set this equal to zero and define

$$V^j = \psi^j \bar{V}^j (\psi^j)^{-1} \tag{3.17}$$

and

$$V = \sum_{j=1}^{N_2} \frac{V^j(\xi, \eta)}{\lambda - b_j} \tag{3.18}$$

to get that

$$\varphi^i_\eta = V|_{\lambda = a_i} \varphi^i \tag{3.19}$$

Similarly, by varying with respect to $\bar{V}^j$ and setting

$$U^i = \varphi^i \bar{U}^i (\varphi^i)^{-1} \tag{3.20}$$

and

$$U = \sum_{i=1}^{N_1} \frac{U^i(\xi, \eta)}{\lambda - a_i}, \tag{3.21}$$

we get that

$$\psi^j_\xi = U|_{\lambda = b_j} \psi^j. \tag{3.22}$$

These relations imply that

$$\begin{aligned} U^i_\eta = D_\eta(\varphi^i \bar{U}^i (\varphi^i)^{-1}) &= V|_{\lambda=a_i} \varphi^i \bar{U}^i (\varphi^i)^{-1} - \varphi^i \bar{U}^i (\varphi^i)^{-1} V|_{\lambda=a_i} \\ &= [V|_{\lambda=a_i}, U^i] \end{aligned} \tag{3.23}$$

and similarly

$$V^j_\xi = [U|_{\lambda=b_j}, V^j]. \tag{3.24}$$

We get precisely the relations (3.5). We have already seen in (3.11a) and (3.11b) that the variational derivatives with respect to φ^i and ψ^j also give us (3.5) and the variational derivative with respect to g gives us (3.14) - a corollary of (3.5). Therefore all of these variations give compatible equations.

Remark 19. *The variational derivative with respect to φ^i gives a weaker relation than the variational derivative with respect to $\bar{U}^i$. This is due to the non-uniqueness in the choice of φ^i when putting U^i into Jordan normal form. When we re-write these relations in terms of U^i, using (3.17) this non-uniqueness is removed and we get the same relations in both cases. A similar statement can be made regarding ψ^j and $\bar{V}^j$.*

3.1.1 Multidimensional Consistency

One main goal is to incorporate the ZM Lagrangian into a Lagrangian multiform, each component of which corresponds to two Lax matrices of a Lax multiplet. We will do so for the first nontrivial case of a Lax triplet (U, V, W). In order for this to be possible at all, a necessary property of the triplet is that it produces a multidimensionally consistent system. Indeed, we will see that a consequence of our construction is that the multiform Euler-Lagrange equations form such a consistent system. Therefore, let us introduce a third Lax matrix W and associated independent variable ν (giving a third part to the auxiliary problem (3.1) of the form $\Psi_\nu = W\Psi$). We require that all of the matrices U, V and W are functions of three independent variables ξ, η and ν. In addition to the relation

$$U_\eta - V_\xi + [U, V] = 0. \tag{3.25}$$

that arises when we sum and combine equations (3.5), we assume that we have similar relations

$$V_\nu - W_\eta + [V, W] = 0 \quad \text{and} \quad W_\xi - U_\nu + [W, U] = 0 \tag{3.26}$$

relating V and W, and W and U. In order to proceed, we assume that two of the three relations (e.g. the relations (3.26)) hold simultaneously and show that the arising compatibility conditions are consistent with the third relation (i.e. the relation (3.25)). If we view the relations (3.26) as definitions for the η and ξ derivatives of W then we must check that $D_\eta W_\xi - D_\xi W_\eta = 0$ when (3.25) holds:

$$\begin{aligned} D_\eta W_\xi - D_\xi W_\eta &= D_\eta(U_\nu + [U,W]) - D_\xi(V_\nu + [V,W]) \\ &= U_{\eta\nu} + [U_\eta, W] + [U, W_\eta] - V_{\xi\nu} - [V_\xi, W] - [V, W_\xi] \\ &= U_{\eta\nu} - V_{\xi\nu} + [U_\eta - V_\xi, W] + [U, W_\eta] - [V, W_\xi]. \end{aligned} \tag{3.27}$$

We use (3.26) again to write this as

$$\begin{aligned} & U_{\eta\nu} - V_{\xi\nu} + [U_\eta - V_\xi, W] + [U, V_\nu + [V,W]] - [V, U_\nu - [W,U]] \\ =& D_\nu(U_\eta - V_\xi + [U,V]) + [U_\eta - V_\xi, W] + [U,[V,W]] + [V,[W,U]]. \end{aligned} \tag{3.28}$$

By the Jacobi identity, this is equivalent to

$$D_\nu(U_\eta - V_\xi + [U,V]) + [U_\eta - V_\xi + [U,V], W] \tag{3.29}$$

which is zero whenever (3.25) holds.

3.1.2 A Lagrangian Multiform Structure

We now introduce the Lagrangian multiform

$$\mathsf{L} = \mathscr{L}_{(\xi\eta)} \mathsf{d}\xi \wedge \mathsf{d}\eta + \mathscr{L}_{(\eta\nu)} \mathsf{d}\eta \wedge \mathsf{d}\nu + \mathscr{L}_{(\nu\xi)} \mathsf{d}\nu \wedge \mathsf{d}\xi \tag{3.30}$$

where

$$\mathscr{L}_{(\xi\eta)} = \operatorname{tr}\left\{ \sum_{i=1}^{N_1} (\varphi^i)^{-1} \varphi^i_\eta \bar{U}^i - \sum_{j=1}^{N_2} (\psi^j)^{-1} \psi^j_\xi \bar{V}^j - \sum_{i=1}^{N_1}\sum_{j=1}^{N_2} \frac{\psi^j \bar{V}^j (\psi^j)^{-1} \varphi^i \bar{U}^i (\varphi^i)^{-1}}{a_i - b_j} \right\}, \tag{3.31a}$$

$$\mathscr{L}_{(\eta\nu)} = \operatorname{tr}\left\{ \sum_{j=1}^{N_2} (\psi^j)^{-1} \psi^j_\nu \bar{V}^j - \sum_{k=1}^{N_3} (\chi^k)^{-1} \chi^k_\eta \bar{W}^k - \sum_{j=1}^{N_2}\sum_{k=1}^{N_3} \frac{\chi^k \bar{W}^k (\chi^k)^{-1} \psi^j \bar{V}^j (\psi^i)^{-1}}{b_j - c_k} \right\} \tag{3.31b}$$

and

$$\mathscr{L}_{(\nu\xi)} = \mathrm{tr}\left\{ \sum_{k=1}^{N_3} (\chi^k)^{-1}\chi^k_\xi \bar{W}^k - \sum_{i=1}^{N_1} (\varphi^i)^{-1}\varphi^i_\nu \bar{U}^i - \sum_{k=1}^{N_3}\sum_{i=1}^{N_1} \frac{\varphi^i \bar{U}^i (\varphi^i)^{-1} \chi^k \bar{W}^k (\chi^k)^{-1}}{c_k - a_i} \right\}. \tag{3.31c}$$

We impose that the $\bar{U}^i$ only depend on ξ, the $\bar{V}^j$ only depend on η and the $\bar{W}^k$ only depend on ν. The multiform Euler-Lagrange equations of L correspond to the criticality of the action

$$S = \int_\sigma \mathsf{L} \tag{3.32}$$

simultaneously for every surface σ in the ξ, η, ν plane. Since this Lagrangian 2-form depends only on 1^{st} order derivatives of the field variables, the multiform Euler-Lagrange equations reduce to the following:

- **The standard Euler-Lagrange equations**

$$\begin{aligned} &\frac{\delta \mathscr{L}_{(\xi\eta)}}{\delta \varphi} = 0, \quad \frac{\delta \mathscr{L}_{(\xi\eta)}}{\delta \psi} = 0, \quad \frac{\delta \mathscr{L}_{(\xi\eta)}}{\delta \chi} = 0, \\ &\frac{\delta \mathscr{L}_{(\xi\eta)}}{\delta \bar{U}} = 0, \quad \frac{\delta \mathscr{L}_{(\xi\eta)}}{\delta \bar{V}} = 0, \quad \frac{\delta \mathscr{L}_{(\xi\eta)}}{\delta \bar{W}} = 0 \end{aligned} \tag{3.33a}$$

and similarly for $\mathscr{L}_{(\eta\nu)}$ and $\mathscr{L}_{(\nu\xi)}$.

- **The first jet one component Euler-Lagrange equations**

$$\frac{\delta \mathscr{L}_{(\xi\eta)}}{\delta \varphi_\nu} = 0, \quad \frac{\delta \mathscr{L}_{(\xi\eta)}}{\delta \psi_\nu} = 0 \tag{3.33b}$$

and similar relations for cyclic permutations of ξ, η and ν.

- **The first jet two component Euler-Lagrange equations**

$$\frac{\delta \mathscr{L}_{(\xi\eta)}}{\delta \varphi_\xi} + \frac{\delta \mathscr{L}_{(\eta\nu)}}{\delta \varphi_\nu} = 0, \quad \frac{\delta \mathscr{L}_{(\eta\nu)}}{\delta \varphi_\eta} + \frac{\delta \mathscr{L}_{(\nu\xi)}}{\delta \varphi_\xi} = 0, \quad \frac{\delta \mathscr{L}_{(\nu\xi)}}{\delta \varphi_\nu} + \frac{\delta \mathscr{L}_{(\xi\eta)}}{\delta \varphi_\eta} = 0 \tag{3.33c}$$

and similar relations with respect to ψ and χ.

Remark 20. *Since, in this case, the Lagrangian multiform* L *has no* 2^{nd} *or higher jet terms, the variational derivatives with respect to any given first jet term are just partial derivatives with respect to that term.*

Theorem 21. *For the Lagrangian multiform*

$$\mathsf{L} = \mathscr{L}_{(\xi\eta)} \mathsf{d}\xi \wedge \mathsf{d}\eta + \mathscr{L}_{(\eta\nu)} \mathsf{d}\eta \wedge \mathsf{d}\nu + \mathscr{L}_{(\nu\xi)} \mathsf{d}\nu \wedge \mathsf{d}\xi, \tag{3.34}$$

the relevant Euler-Lagrange equations (3.33a), (3.33b) *and* (3.33c) *yield the multidimensional system of equations given by* (3.5) *and the corresponding relations for the matrix* W. *Furthermore,* $\mathsf{dL} = 0$ *on solutions of the multiform Euler-Lagrange equations.*

Proof. We begin by confirming that the Multiform Euler-Lagrange equations (3.33a), (3.33b) and (3.33c) hold. From varying $\bar{U}$ and $\bar{V}$ in $\mathscr{L}_{(\xi\eta)}$ we get

$$\varphi^i_\eta = V|_{\lambda=a_i}\varphi^i \quad \text{and} \quad \psi^j_\xi = U|_{\lambda=b_j}\psi^j. \tag{3.35a}$$

From varying $\bar{V}$ and $\bar{W}$ in $\mathscr{L}_{(\eta\nu)}$ we get

$$\psi^j_\nu = W|_{\lambda=b_j}\psi^j \quad \text{and} \quad \chi^k_\eta = V|_{\lambda=c_k}\chi^k. \tag{3.35b}$$

From varying $\bar{W}$ and $\bar{U}$ in $\mathscr{L}_{(\nu\xi)}$ we get

$$\chi^k_\xi = U|_{\lambda=c_k}\chi^k \quad \text{and} \quad \varphi^i_\nu = W|_{\lambda=a_i}\varphi^i. \tag{3.35c}$$

From varying φ^i and ψ^j in $\mathscr{L}_{(\xi\eta)}$ we get

$$U^i_\eta + \left[U^i, \sum_{j=1}^{N_2} \frac{V^j}{a_i - b_j}\right] = 0 \quad \text{and} \quad V^j_\xi + \left[V^j, \sum_{i=1}^{N_1} \frac{U^i}{b_j - a_i}\right] = 0 \tag{3.36a}$$

which are corollaries of (3.35a). From varying ψ^j and χ^k in $\mathscr{L}_{(\eta\nu)}$ we get

$$V^j_\nu + \left[V^j, \sum_{k=1}^{N_3} \frac{W^k}{b_j - c_k}\right] = 0 \quad \text{and} \quad W^k_\eta + \left[W^k, \sum_{j=1}^{N_2} \frac{V^j}{c_k - b_j}\right] = 0 \tag{3.36b}$$

which are corollaries of (3.35b). From varying χ^k and φ^i in $\mathscr{L}_{(\nu\xi)}$ we get

$$W^k_\xi + \left[W^k, \sum_{i=1}^{N_1} \frac{U^i}{c_k - a_i}\right] = 0 \quad \text{and} \quad U^i_\nu + \left[U^i, \sum_{k=1}^{N_3} \frac{W^k}{a_i - c_k}\right] = 0 \tag{3.36c}$$

which are corollaries of (3.35c). Equations of the type given in (3.33b) are trivially satisfied since there are no ν derivatives in $\mathscr{L}_{(\xi\eta)}$, no ξ derivatives in $\mathscr{L}_{(\eta\nu)}$ and no η derivatives in $\mathscr{L}_{(\eta\nu)}$. Equations of the type given in (3.33c) are also trivially satisfied, in that they do not require the field variables to be critical points of the action in order to hold.

The validity of the relation $\mathsf{dL} = 0$ for the Lagrangian (3.34) on the solutions of the Euler-Lagrange equations is verified by direct computation. The Lagrangian 2-form $\mathscr{L}_{(\xi\eta)}\mathsf{d}\xi\wedge\mathsf{d}\eta+\mathscr{L}_{(\eta\nu)}\mathsf{d}\eta\wedge\mathsf{d}\nu+\mathscr{L}_{(\nu\xi)}\mathsf{d}\nu\wedge\mathsf{d}\xi$ is closed if and only if $D_\nu\mathscr{L}_{(\xi\eta)} + D_\xi\mathscr{L}_{(\eta\nu)} + D_\eta\mathscr{L}_{(\nu\xi)} = 0$ on solutions of the system.

$$\begin{aligned} &D_\nu\mathscr{L}_{(\xi\eta)} + D_\xi\mathscr{L}_{(\eta\nu)} + D_\eta\mathscr{L}_{(\nu\xi)} \\ &= \mathrm{tr}\left\{ \sum_{i=1}^{N_1}[(\varphi^i)^{-1}\varphi^i_\eta, (\varphi^i)^{-1}\varphi^i_\nu]\bar{U}^i + \sum_{j=1}^{N_2}[(\psi^j)^{-1}\psi^j_\nu, (\psi^j)^{-1}\psi^j_\xi]\bar{V}^j \right. \\ &\quad \left. + \sum_{k=1}^{N_3}[(\chi^k)^{-1}\chi^k_\xi, (\chi^k)^{-1}\chi^k_\eta]\bar{W}^k \right\} \end{aligned} \tag{3.37a}$$

$$- \mathrm{tr}\left\{ \sum_{i=1}^{N_1}\sum_{j=1}^{N_2} \frac{V^j_\nu U^i + V^j U^i_\nu}{a_i - b_j} + \sum_{j=1}^{N_2}\sum_{k=1}^{N_3} \frac{W^k_\xi V^j + W^k V^j_\xi}{b_j - c_k} + \sum_{k=1}^{N_3}\sum_{i=1}^{N_1} \frac{U^i_\eta W^k + U^i W^k_\eta}{c_k - a_i} \right\}. \tag{3.37b}$$

The first set of terms (part (3.37a)) are equivalent to

$$\mathrm{tr}\left\{ \sum_{i=1}^{N_1} \varphi^i_\eta(\varphi^i)^{-1}U^i_\nu + \sum_{j=1}^{N_2} \psi^j_\nu(\psi^j)^{-1}V^j_\xi + \sum_{k=1}^{N_3} \chi^k_\xi(\chi^k)^{-1}W^k_\eta \right\}. \tag{3.38}$$

We can use (3.36) to re-write this as

$$\mathrm{tr}\left\{\sum_{i=1}^{N_1}\sum_{k=1}^{N_3}\frac{U^i_\eta W^k}{c_k-a_i}+\sum_{j=1}^{N_2}\sum_{i=1}^{N_1}\frac{V^j_\nu U^i}{a_i-b_j}+\sum_{k=1}^{N_3}\sum_{j=1}^{N_2}\frac{W^k_\xi V^j}{b_j-c_k}\right\} \tag{3.39}$$

and we see that all of these terms will cancel with terms in part (3.37b). This gives us that

$$\begin{aligned} &D_\nu\mathscr{L}_{(\xi\eta)}+D_\xi\mathscr{L}_{(\eta\nu)}+D_\eta\mathscr{L}_{(\nu\xi)} \\ &\qquad=\mathrm{tr}\left\{\sum_{i=1}^{N_1}\sum_{j=1}^{N_2}\frac{V^jU^i_\nu}{b_j-a_i}+\sum_{j=1}^{N_2}\sum_{k=1}^{N_3}\frac{W^kV^j_\xi}{c_k-b_j}+\sum_{k=1}^{N_3}\sum_{i=1}^{N_1}\frac{U^iW^k_\eta}{a_i-c_k}\right\}. \end{aligned} \tag{3.40}$$

We use (3.36) again to re-write this as

$$\begin{aligned}\mathrm{tr}\Bigg\{\sum_{i,j,k=1}^{N_1,N_2,N_3}V^j[U^i,W^k]\bigg(&\frac{1}{(b_j-a_i)(c_k-a_i)}+\frac{1}{(c_k-b_j)(a_i-b_j)}\\ &+\frac{1}{(a_i-c_k)(b_j-c_k)}\bigg)\Bigg\},\end{aligned}$$

which is zero, since the sum of the three fractions is zero. □

The proof given above is precisely the one that appeared in [2]. In the following theorem we show how dL can be expressed as a sum of terms, each of which is factorised into expressions that are zero on the multiform Euler-Lagrange equations. It is then a very obvious consequence that $\mathsf{dL} = 0$ on the multiform Euler-Lagrange equations. In addition, the expression we give for dL shall be useful later on when reducing the ZM Lagrangian multiform in order to obtain a Lagrangian multiform for the ZM Lax par.

Theorem 22. *The explicit form of dL is as follows:*

$$\begin{aligned}
dL = \operatorname{tr}\Bigg\{ &\frac{1}{2}\sum_{i=1}^{N_1}\left(\varphi^i_\eta - \sum_{j=1}^{N_2}\frac{V^j\varphi^i}{a_i-b_j}\right)(\varphi^i)^{-1}\left(U^i_\nu + \sum_{k=1}^{N_3}\frac{[U^i,W^k]}{a_i-c_k}\right)\\
&-\frac{1}{2}\sum_{i=1}^{N_1}\left(\varphi^i_\nu - \sum_{k=1}^{N_3}\frac{W^k\varphi^i}{a_i-c_k}\right)(\varphi^i)^{-1}\left(U^i_\eta + \sum_{j=1}^{N_2}\frac{[U^i,V^j]}{a_i-b_j}\right)\\
&+\frac{1}{2}\sum_{j=1}^{N_2}\left(\psi^j_\nu - \sum_{k=1}^{N_3}\frac{W^k\psi^j}{b_j-c_k}\right)(\psi^j)^{-1}\left(V^j_\xi + \sum_{i=1}^{N_1}\frac{[V^j,U^i]}{b_j-a_i}\right)\\
&-\frac{1}{2}\sum_{j=1}^{N_2}\left(\psi^j_\xi - \sum_{i=1}^{N_1}\frac{U^i\psi^j}{b_j-a_i}\right)(\psi^j)^{-1}\left(V^j_\nu + \sum_{k=1}^{N_3}\frac{[V^j,W^k]}{b_j-c_k}\right)\\
&+\frac{1}{2}\sum_{k=1}^{N_3}\left(\chi^k_\xi - \sum_{i=1}^{N_1}\frac{U^i\chi^k}{c_k-a_i}\right)(\chi^k)^{-1}\left(W^k_\eta + \sum_{j=1}^{N_2}\frac{[W^k,V^j]}{c_k-b_j}\right)\\
&-\frac{1}{2}\sum_{k=1}^{N_3}\left(\chi^k_\eta - \sum_{j=1}^{N_2}\frac{V^j\chi^k}{c_k-b_j}\right)(\chi^k)^{-1}\left(W^k_\xi + \sum_{i=1}^{N_1}\frac{[W^k,U^i]}{c_k-a_i}\right)\Bigg\} d\xi\wedge d\eta\wedge d\nu
\end{aligned} \tag{3.41}$$

or equivalently,

$$\begin{aligned}
dL = \operatorname{tr}\Bigg\{ &\sum_{i=1}^{N_1}\left(\varphi^i_\eta - \sum_{j=1}^{N_2}\frac{V^j\varphi^i}{a_i-b_j}\right)(\varphi^i)^{-1}\left(U^i_\nu + \sum_{k=1}^{N_3}\frac{[U^i,W^k]}{a_i-c_k}\right)\\
&+\sum_{j=1}^{N_2}\left(\psi^j_\nu - \sum_{k=1}^{N_3}\frac{W^k\psi^j}{b_j-c_k}\right)(\psi^j)^{-1}\left(V^j_\xi + \sum_{i=1}^{N_1}\frac{[V^j,U^i]}{b_j-a_i}\right)\\
&+\sum_{k=1}^{N_3}\left(\chi^k_\xi - \sum_{i=1}^{N_1}\frac{U^i\chi^k}{c_k-a_i}\right)(\chi^k)^{-1}\left(W^k_\eta + \sum_{j=1}^{N_2}\frac{[W^k,V^j]}{c_k-b_j}\right)\Bigg\} d\xi\wedge d\eta\wedge d\nu.
\end{aligned} \tag{3.42}$$

Proof. Lines one and two of (3.41) are equal, as are lines three and four, as are lines five and six; this is readily seen by expanding and comparing terms. The equivalence of (3.41) and (3.42) follows. In order to show that (3.42) holds, we begin by expanding the 1^{st} line to get

$$\sum_{i=1}^{N_1} \mathrm{tr}\left\{\varphi^i_\eta(\varphi^i)^{-1}U^i_\nu+\sum_{k=1}^{N_3}\varphi^i_\eta(\varphi^i)^{-1}\frac{[U^i,W^k]}{a_i-c_k}-\sum_{j=1}^{N_2}\frac{V^jU^i_\nu}{a_i-b_j}-\sum_{j=1}^{N_2}\sum_{k=1}^{N_3}\frac{V^j[U^i,W^k]}{(a_i-b_j)(a_i-c_k)}\right\} \tag{3.43}$$

The identities

$$\mathrm{tr}\{\varphi^i_\eta(\varphi^i)^{-1}U^i_\nu\} = \mathrm{tr}\{\mathrm{D}_\nu((\varphi^i)^{-1}\varphi^i_\eta\bar{U}^i) - \mathrm{D}_\eta((\varphi^i)^{-1}\varphi^i_\nu\bar{U}^i)\} \tag{3.44}$$

and

$$\mathrm{tr}\{\varphi^i_\eta(\varphi^i)^{-1}[U^i,W^k]\} = \mathrm{tr}\{U^i_\eta W^k\} \tag{3.45}$$

follow from the definition that $U^i = \varphi^i\bar{U}^i(\varphi^i)^{-1}$ and allow us to express (3.43) as

$$\begin{aligned}\sum_{i=1}^{N_1}\mathrm{tr}\Bigg\{&\mathrm{D}_\nu((\varphi^i)^{-1}\varphi^i_\eta\bar{U}^i) - \mathrm{D}_\eta((\varphi^i)^{-1}\varphi^i_\nu\bar{U}^i) + \sum_{k=1}^{N_3}\frac{U^i_\eta W^k}{a_i-c_k} - \sum_{j=1}^{N_2}\frac{V^jU^i_\nu}{a_i-b_j}\\ &-\sum_{j=1}^{N_2}\sum_{k=1}^{N_3}\frac{V^j[U^i,W^k]}{(a_i-b_j)(a_i-c_k)}\Bigg\}.\end{aligned} \tag{3.46}$$

Expanding the whole of (3.42) in a similar manner we get

$$\begin{aligned}\mathsf{dL} = \mathrm{tr}\Bigg\{&\sum_{i=1}^{N_1}\Bigg(\mathrm{D}_\nu((\varphi^i)^{-1}\varphi^i_\eta\bar{U}^i) - \mathrm{D}_\eta((\varphi^i)^{-1}\varphi^i_\nu\bar{U}^i) + \sum_{k=1}^{N_3}\frac{U^i_\eta W^k}{a_i-c_k} - \sum_{j=1}^{N_2}\frac{V^jU^i_\nu}{a_i-b_j}\\ &-\sum_{j=1}^{N_2}\sum_{k=1}^{N_3}\frac{V^j[U^i,W^k]}{(a_i-b_j)(a_i-c_k)}\Bigg)\\ &+\sum_{j=1}^{N_2}\Bigg(\mathrm{D}_\xi((\psi^j)^{-1}\psi^j_\nu\bar{V}^j) - \mathrm{D}_\nu((\psi^j)^{-1}\psi^j_\xi\bar{V}^j) + \sum_{i=1}^{N_1}\frac{V^j_\nu U^i}{b_j-a_i} - \sum_{k=1}^{N_3}\frac{W^kV^j_\xi}{b_j-c_k}\\ &-\sum_{i=1}^{N_1}\sum_{k=1}^{N_3}\frac{W^k[V^j,U^i]}{(b_j-c_k)(b_j-a_i)}\Bigg)\end{aligned}$$

$$
\begin{aligned}
&+\sum_{k=1}^{N_3}\Bigg(\mathrm{D}_\eta((\chi^k)^{-1}\chi^k_\xi\bar{W}^k) - \mathrm{D}_\xi((\chi^k)^{-1}\chi^k_\eta\bar{W}^k) + \sum_{j=1}^{N_2}\frac{W^k_\xi V^j}{c_k-b_j} - \sum_{i=1}^{N_1}\frac{U^i W^k_\eta}{c_k-a_i}\\
&-\sum_{i=1}^{N_1}\sum_{j=1}^{N_2}\frac{U^i[W^k,V^j]}{(c_k-a_i)(c_k-b_j)}\Bigg)\Bigg\}\mathsf{d}\xi\wedge\mathsf{d}\eta\wedge\mathsf{d}\nu.
\end{aligned}
\tag{3.47}
$$

The sum of each term with a triple sum is zero, giving us that

$$
\begin{aligned}
\mathsf{dL} = \mathrm{tr}\Bigg\{ &\mathrm{D}_\nu\Bigg(\sum_{i=1}^{N_1}(\varphi^i)^{-1}\varphi^i_\eta\bar{U}^i - \sum_{j=1}^{N_2}(\psi^j)^{-1}\psi^j_\xi\bar{V}^j - \sum_{i=1}^{N_1}\sum_{j=1}^{N_2}\frac{\psi^j\bar{V}^j(\psi^j)^{-1}\varphi^i\bar{U}^i(\varphi^i)^{-1}}{a_i-b_j}\Bigg)\\
&+\mathrm{D}_\xi\Bigg(\sum_{j=1}^{N_2}(\psi^j)^{-1}\psi^j_\nu\bar{V}^j - \sum_{k=1}^{N_3}(\chi^k)^{-1}\chi^k_\eta\bar{W}^k - \sum_{j=1}^{N_2}\sum_{k=1}^{N_3}\frac{\chi^k\bar{W}^k(\chi^k)^{-1}\psi^j\bar{V}^j(\psi^i)^{-1}}{b_j-c_k}\Bigg)\\
&+\mathrm{D}_\eta\Bigg(\sum_{k=1}^{N_3}(\chi^k)^{-1}\chi^k_\xi\bar{W}^k - \sum_{i=1}^{N_1}(\varphi^i)^{-1}\varphi^i_\nu\bar{U}^i - \sum_{k=1}^{N_3}\sum_{i=1}^{N_1}\frac{\varphi^i\bar{U}^i(\varphi^i)^{-1}\chi^k\bar{W}^k(\chi^k)^{-1}}{c_k-a_i}\Bigg)\\
&\Bigg\}\mathsf{d}\xi\wedge\mathsf{d}\eta\wedge\mathsf{d}\nu\\
=\ &(\mathrm{D}_\nu\mathscr{L}_{(\xi\eta)} + \mathrm{D}_\xi\mathscr{L}_{(\eta\nu)} + \mathrm{D}_\eta\mathscr{L}_{(\nu\xi)})\mathsf{d}\xi\wedge\mathsf{d}\eta\wedge\mathsf{d}\nu = \mathsf{dL}.
\end{aligned}
\tag{3.48}
$$

□

Remark 23. *We notice that the N_3 pairs of expressions derived from L by varying $\bar{W}^k$,*

$$\chi^k_\xi = U|_{\lambda=c_k}\chi^k \quad \textit{and} \quad \chi^k_\eta = V|_{\lambda=c_k}\chi^k \tag{3.49a}$$

are precisely the auxiliary problem (3.1) with $\lambda = c_k$. Similarly, we can view the N_1 expressions involving φ^i of the form

$$\varphi^i_\eta = V|_{\lambda=a_i}\varphi^i \quad \textit{and} \quad \varphi^i_\nu = W|_{\lambda=a_i}\varphi^i \tag{3.49b}$$

that come from varying $\bar{U}^i$ as an auxiliary problem based on V and W with $\lambda = a_i$ and the N_2 expressions involving ψ^j of the form

$$\psi^j_\nu = W|_{\lambda=b_j}\psi^j \quad \textit{and} \quad \psi^j_\xi = U|_{\lambda=b_j}\psi^j \tag{3.49c}$$

that come from varying $\bar{V}^j$ as an auxiliary problem based on W and U with $\lambda = b_j$.

3.1.3 Lagrangian for the ZM Lax Pair

Building on remark 23, in the case of the Lax pair (3.1) involving U and V with spectral parameter λ and associated coordinates ξ and η, we can view the spectral parameter λ as coming from a "ghost" direction ν as one of the poles of the associated Lax matrix W. In this case, the Lagrangian multiform (3.30) can be viewed as the Lagrangian for the Lax pair U and V, with the multiform Euler-Lagrange equations of the Lagrangian multiform including both the equations of motion of the Lax pair U and V and also the auxiliary problem (3.1). However, it would be just as valid to focus on V and W and consider ξ as the "ghost" direction, or to focus on W and U with η as the "ghost" direction, since the three Lax matrices U, V and W along with their respective associated coordinates ξ, η and ν all hold equal status within the multiform. Therefore, in the context of this Lagrangian multiform description, rather that considering a Lax pair as consisting of matrices U and V with spectral parameter λ, it is more satisfactory to consider the Lax triplet U, V and W.

In the **Lagrangian for the ZM Lax Pair** section of [2] we claim that an appropriate ZM Lagrangian multiform can be considered to be the Lagrangian for the ZM Lax Pair involving U and V if we impose that $W = 0$ on the multiform Euler-Lagrange equations. Here we shall perform a reduction on the ZM Lagrangian 2-form to obtain a 1-form thereby imposing that $W = 0$ at the Lagrangian multiform level. The resulting Lagrangian 1-form is a more elegant candidate for the Lagrangian for the ZM Lax Pair. We include the original version from [2] in Appendix D.

If we are only interested in the U, V auxiliary problem

$$\Psi_\xi = U(\xi, \eta, \lambda)\Psi, \quad \Psi_\eta = V(\xi, \eta, \lambda)\Psi, \tag{3.50}$$

and want to cast this in the multiform structure of Section 3.1.2 then it is necessary to temporarily introduce a "ghost" variable ν and require that all field variables now have a ν dependence. We must also introduce the additional Lax matrix W relating to the "ghost" direction ν. We then consider Lagrangian multiform $\mathsf{L}[\varphi, \psi, \chi, \bar{U}, \bar{V}, \bar{W}; \lambda]$ such that

$$
\begin{aligned}
\mathsf{L} =& \left(\sum_{i=1}^{N_1} (\varphi^i)^{-1} \varphi^i_\eta \bar{U}^i - \sum_{j=1}^{N_2} (\psi^j)^{-1} \psi^j_\xi \bar{V}^j - \sum_{i=1}^{N_1} \sum_{j=1}^{N_2} \frac{\psi^j \bar{V}^j (\psi^j)^{-1} \varphi^i \bar{U}^i (\varphi^i)^{-1}}{a_i - b_j} \right) d\xi \wedge d\eta \\
&+ \left(\sum_{j=1}^{N_2} (\psi^j)^{-1} \psi^j_\nu \bar{V}^j - \chi^{-1} \chi_\eta \bar{W} - \sum_{j=1}^{N_2} \frac{\chi \bar{W} \chi^{-1} \psi^j \bar{V}^j (\psi^j)^{-1}}{b_j - \lambda} \right) d\eta \wedge d\nu \\
&+ \left(\chi^{-1} \chi_\xi \bar{W} - \sum_{i=1}^{N_1} (\varphi^i)^{-1} \varphi^i_\nu \bar{U}^i - \sum_{i=1}^{N_1} \frac{\varphi^i \bar{U}^i (\varphi^i)^{-1} \chi \bar{W} \chi^{-1}}{\lambda - a_i} \right) d\nu \wedge d\xi.
\end{aligned}
\tag{3.51}
$$

This Lagrangian 2-form is special case of the multiform (3.30) where the matrix W has a single pole at λ. In accordance with Theorem 22 we obtain that $\mathsf{d}\mathsf{L} = \Omega \mathsf{d}\xi \wedge \mathsf{d}\eta \wedge \mathsf{d}\nu$ where

$$
\begin{aligned}
\Omega = \operatorname{tr} \Bigg\{ &- \sum_{i=1}^{N_1} \left(\varphi^i_\nu - \frac{W \varphi^i}{a_i - \lambda} \right) (\varphi^i)^{-1} \left(U^i_\eta + \sum_{j=1}^{N_2} \frac{[U^i, V^j]}{a_i - b_j} \right) \\
&+ \sum_{j=1}^{N_2} \left(\psi^j_\nu - \frac{W \psi^j}{b_j - \lambda} \right) (\psi^j)^{-1} \left(V^j_\xi + \sum_{i=1}^{N_1} \frac{[V^j, U^i]}{b_j - a_i} \right) \\
&+ \frac{1}{2} \left(\chi_\xi - \sum_{i=1}^{N_1} \frac{U^i \chi}{\lambda - a_i} \right) \chi^{-1} \left(W_\eta + \sum_{j=1}^{N_2} \frac{[W, V^j]}{\lambda - b_j} \right) \\
&- \frac{1}{2} \left(\chi_\eta - \sum_{j=1}^{N_2} \frac{V^j \chi}{\lambda - b_j} \right) \chi^{-1} \left(W_\xi + \sum_{i=1}^{N_1} \frac{[W, U^i]}{\lambda - a_i} \right) \Bigg\}.
\end{aligned}
\tag{3.52}
$$

Since we are only interested in the U, V Lax pair in the ξ, η coordinates, we wish to remove any dependence on ν from our multiform. To do so we set $\varphi^i_\nu = 0$ and $\psi^j_\nu = 0$ for all i and j to obtain

$$
\begin{aligned}
\tilde{\Omega} = \operatorname{tr}\Bigg\{ &\sum_{i=1}^{N_1}\left(\frac{W\varphi^i}{a_i-\lambda}\right)(\varphi^i)^{-1}\left(U^i_\eta + \sum_{j=1}^{N_2}\frac{[U^i,V^j]}{a_i-b_j}\right) \\
&-\sum_{j=1}^{N_2}\left(\frac{W\psi^j}{b_j-\lambda}\right)(\psi^j)^{-1}\left(V^j_\xi + \sum_{i=1}^{N_1}\frac{[V^j,U^i]}{b_j-a_i}\right) \\
&+\frac{1}{2}\left(\chi_\xi - \sum_{i=1}^{N_1}\frac{U^i\chi}{\lambda-a_i}\right)\chi^{-1}\left(W_\eta + \sum_{j=1}^{N_2}\frac{[W,V^j]}{\lambda-b_j}\right) \\
&-\frac{1}{2}\left(\chi_\eta - \sum_{j=1}^{N_2}\frac{V^j\chi}{\lambda-b_j}\right)\chi^{-1}\left(W_\xi + \sum_{i=1}^{N_1}\frac{[W,U^i]}{\lambda-a_i}\right)\Bigg\}.
\end{aligned}
\tag{3.53}
$$

We find that we can also express $\tilde{\Omega}$ as follows:

$$
\begin{aligned}
\tilde{\Omega} = \operatorname{tr}\Bigg\{ &\mathrm{D}_\eta\left(\chi^{-1}\chi_\xi\bar{W} - \sum_{i=1}^{N_1}\frac{\varphi^i\bar{U}^i(\varphi^i)^{-1}\chi\bar{W}\chi^{-1}}{\lambda-a_i}\right) \\
&-\mathrm{D}_\xi\left(\chi^{-1}\chi_\eta\bar{W} - \sum_{j=1}^{N_2}\frac{\chi\bar{W}\chi^{-1}\psi^j\bar{V}^j(\psi^i)^{-1}}{\lambda-b_j}\right)\Bigg\}.
\end{aligned}
\tag{3.54}
$$

It follows that if we define the Lagrangian 2-form

$$\tilde{\mathsf{L}} = \mathscr{L}_{(\xi)}\mathsf{d}\xi + \mathscr{L}_{(\eta)}\mathsf{d}\eta \tag{3.55}$$

where

$$\tilde{\mathscr{L}}_{(\xi)} = \operatorname{tr}\left\{\chi^{-1}\chi_\xi\bar{W} - \sum_{i=1}^{N_1}\frac{\chi\bar{W}\chi^{-1}\varphi^i\bar{U}^i(\varphi^i)^{-1}}{\lambda-a_i}\right\} \tag{3.56}$$

and

$$\tilde{\mathscr{L}}_{(\eta)} = \operatorname{tr}\left\{\chi^{-1}\chi_\eta\bar{W} - \sum_{j=1}^{N_2}\frac{\chi\bar{W}\chi^{-1}\psi^j\bar{V}^j(\psi^i)^{-1}}{\lambda-b_j}\right\} \tag{3.57}$$

then $\mathsf{d}\tilde{\mathsf{L}} = \tilde{\Omega}\,\mathsf{d}\eta\wedge\mathsf{d}\xi$ where $\tilde{\Omega}$ is as given in (3.53). As a result, the full set of multiform Euler-Lagrange equations give us

$$\chi_\xi = U\chi \quad \text{and} \quad \chi_\eta = V\chi \tag{3.58}$$

the auxiliary problem based on U and V,

$$U^i_\eta + [U^i, \sum_{j=1}^{N_2} \frac{V^j}{a_i - b_j}] = 0 \quad \text{and} \quad V^j_\xi + [V^j, \sum_{i=1}^{N_1} \frac{U^i}{b_j - a_i}] = 0 \tag{3.59}$$

the equations of motion for U^i and V^j and also that $\bar{W} = 0$. Therefore, the Lagrangian multiform (3.55) can be considered the Lagrangian (multiform) for the Lax pair U and V. We can summarise this result in the following theorem.

Theorem 24. *The Lagrangian 1-form $\mathsf{L}[\varphi, \psi, \chi, \bar{U}, \bar{V}, \bar{W}; \lambda]$ given by (3.55) is a Lagrangian for the Lax pair U and V. When we take the multiform Euler-Lagrange equations our equations of motion give us that $\bar{W} = 0$, the auxiliary problem*

$$\chi_\xi = U\chi \quad \textit{and} \quad \chi_\eta = V\chi \tag{3.60}$$

for U and V and the equations of motion

$$U^i_\eta + [U^i, \sum_{j=1}^{N_2} \frac{V^j}{a_i - b_j}] = 0 \quad \textit{and} \quad V^j_\xi + [V^j, \sum_{i=1}^{N_1} \frac{U^i}{b_j - a_i}] = 0 \tag{3.61}$$

corresponding to the compatibility conditions of this auxiliary problem.

3.2 Matrix AKNS Hierarchy

As a specific example of the general construction, we present here the case of the single-pole Lax pair which, with appropriate choice of variables, can be viewed as a generating model for the generalized, i.e. $N \times N$ matrix generalization, of the famous AKNS hierarchy of [22].

3.2.1 An Integrable $N \times N$ Hierarchy and its ZM Lagrangian

We begin by introducing co-ordinates x_i for $i = 0, \dots, \infty$ and we define the derivatives with respect to ξ and η such that

$$\partial_\xi = \sum_{i=0}^{\infty} \frac{1}{a^{i+1}} \frac{\partial}{\partial x_i} \quad \text{and} \quad \partial_\eta = \sum_{j=0}^{\infty} \frac{1}{b^{j+1}} \frac{\partial}{\partial x_j} \tag{3.62}$$

and apply this to form a Lax pair and auxiliary problem with a single simple pole

$$\Psi_\xi = \frac{U(\xi,\eta)}{\lambda - a}\Psi, \quad \Psi_\eta = \frac{V(\xi,\eta)}{\lambda - b}\Psi, \tag{3.63}$$

i.e. the ZM auxiliary problem with $N_1 = 1$ and $N_2 = 1$. This gives rise to the compatibility conditions

$$U_\eta = V_\xi = \frac{[V,U]}{a-b}. \tag{3.64}$$

By the ZM method outlined in Section 3.1, this has the Lagrangian

$$\mathscr{L}_{(\xi\eta)} = \mathrm{tr}\left\{\varphi^{-1}\varphi_\eta \bar{U} - \psi^{-1}\psi_\xi \bar{V} - \frac{\psi\bar{V}\psi^{-1}\varphi\bar{U}\varphi^{-1}}{a-b}\right\} \tag{3.65a}$$

We can now introduce the co-ordinate ν, the associated matrix $\bar{W}(\nu)$ and parameter c to form two further Lagrangians

$$\mathscr{L}_{(\eta\nu)} = \mathrm{tr}\left\{\psi^{-1}\psi_\nu \bar{V} - \chi^{-1}\chi_\eta \bar{W} - \frac{\chi\bar{W}\chi^{-1}\psi\bar{V}\psi^{-1}}{b-c}\right\} \tag{3.65b}$$

and

$$\mathscr{L}_{(\nu\xi)} = \mathrm{tr}\left\{\chi^{-1}\chi_\xi \bar{W} - \varphi^{-1}\varphi_\nu \bar{U} - \frac{\varphi\bar{U}\varphi^{-1}\chi\bar{W}\chi^{-1}}{c-a}\right\} \tag{3.65c}$$

to form the Lagrangian multiform

$$\mathscr{L}_{(\xi\eta)}\mathsf{d}\xi \wedge \mathsf{d}\eta + \mathscr{L}_{(\eta\nu)}\mathsf{d}\eta \wedge \mathsf{d}\nu + \mathscr{L}_{(\nu\xi)}\mathsf{d}\nu \wedge \mathsf{d}\xi. \tag{3.65d}$$

By Theorem 21, this Lagrangian multiform is closed on solutions of this system and has Multiform Euler-Lagrange equations that include (3.64) when we let $U = \varphi\bar{U}\varphi^{-1}$ and $V = \psi\bar{V}\psi^{-1}$, i.e. we have a Lagrangian multiform structure for this system.

Since, on the equations of motion, $U_\eta = V_\xi$, there exists a matrix H such that $U = H_\xi$ and $V = H_\eta$. Expressing (3.64) in terms of H, we get

$$H_{\xi\eta} = \frac{[H_\eta, H_\xi]}{a-b}. \tag{3.66}$$

A conventional Lagrangian that gives this expression directly was originally given in [35] and discussed further in [36]. When we expand the ξ and η derivatives in terms of the x_i co-ordinates this gives us

$$H_{x_i x_{j-1}} - H_{x_{i-1} x_j} = [H_{x_{j-1}}, H_{x_{i-1}}], \tag{3.67}$$

an integrable $N \times N$ matrix system [36, 37]. We will show that, in the 2×2 case, this contains the AKNS hierarchy; this particular case, and the underlying Kac-Moody algebra structure were treated in [24], where in particular the corresponding symplectic forms were given.

We define the matrix

$$Q_i := -\partial_{x_{i-1}} H \quad \text{for } i \geq 1 \tag{3.68}$$

so (3.67) becomes

$$\partial_{x_j} Q_i - \partial_{x_i} Q_j = [Q_j, Q_i] \tag{3.69}$$

and since partial derivatives of H with respect to the x_i co-ordinates commute, we also have that

$$\partial_{x_i} Q_j = \partial_{x_{j-1}} Q_{i+1}. \tag{3.70}$$

If we define Q_0 to be a constant matrix then (3.69) and (3.70) give us the additional relation

$$[Q_0, Q_{k+1}] + [Q_1, Q_k] = \partial_{x_1} Q_k. \tag{3.71}$$

The relations (3.69),(3.70) and (3.71)are used recursively to find Q_i for all i. In the case of the AKNS hierarchy, we take the Q_i to be 2×2 matrices and define

$$Q_0 = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}, \quad Q_1 = \begin{pmatrix} 0 & q \\ r & 0 \end{pmatrix} \tag{3.72}$$

where q and r are functions of the x_i co-ordinates. We are now able to follow the procedure outlined in [24] and use (3.69),(3.70) and (3.71) recursively to find the Q_i, e.g.

$$Q_2 = \frac{i}{2}\begin{pmatrix} -qr & q_{x_1} \\ -r_{x_1} & qr \end{pmatrix}, \quad Q_3 = -\frac{1}{4}\begin{pmatrix} qr_{x_1} - rq_{x_1} & q_{x_1x_1} - 2q^2r \\ r_{x_1x_1} - 2qr^2 & -qr_{x_1} + rq_{x_1} \end{pmatrix}, \quad \ldots \tag{3.73}$$

The equations of the AKNS hierarchy are given by

$$\partial_{x_N} Q_1 - \partial_{x_1} Q_N = [Q_N, Q_1] \tag{3.74}$$

i.e. equation (3.69) with $i = 1$.

3.3 Conclusion

Using the method outlined in this chapter, one is able to construct a Lagrangian multiform structure for systems with Lax pairs in the appropriate form, and in so doing, find a Lagrangian for the Lax pair itself. Lagrangians in the case of Lax pairs with higher-order poles were given by Dickey in [33], and it is to be expected that those can be extended to a Lagrangian multiform structure. The *generating PDEs* introduced in [6, 38] which are associated with non-isospectral Lax pairs, possess Lagrangians of the required form, cf. also [4]. Furthermore, we expect that the universal symplectic form of Krichever and Phong, [39, 40] associated with Lax operators could play a role in the construction of Lagrangians possessing a multiform structure.

Chapter 4

Lagrangian multiforms for Kadomtsev-Petviashvili (KP) and the Gelfand-Dickey hierarchy

4.1 Pseudodifferential operators

The main results in this chapter require the use of pseudodifferential operators. Here we give a brief summary based on [33, Chapter 1] and the references therein. We introduce the differential algebra $\mathcal{A}$ with generators $u_1, u_2, u_3, \dots$ and derivation D_x, the total derivative with respect to x, such that $\mathrm{D}_x\, u_\alpha^{(i)} = (u_\alpha^{(i)})_x = u_\alpha^{(i+1)}$, where $u_\alpha^{(0)} = u_\alpha$. Also, D_x obeys the Leibnitz rule $\mathrm{D}_x\, u_\alpha^{(i)} u_\beta^{(j)} = u_\alpha^{(i+1)} u_\beta^{(j)} + u_\alpha^{(i)} u_\beta^{(j+1)}$. Elements of $\mathcal{A}$ are polynomials with real or complex coefficients in the generators u_α and their derivatives of arbitrary order. The operator ∂ is defined such that for $f \in \mathcal{A}$,

$$\partial^k f = f\partial^k + \binom{k}{1} f'\partial^{k-1} + \binom{k}{2} f''\partial^{k-2} + \dots \tag{4.1}$$

where $f \in \mathcal{A}$, $f' = \mathrm{D}_x\, f$ and

$$\binom{k}{i} = \frac{k(k-1)\dots(k-i+1)}{i!}. \tag{4.2}$$

When $k > 0$ this sum naturally truncates, whereas when $k < 0$ the sum is infinite. Using these definitions for D_x and ∂, we note that for $f \in \mathcal{A}$, $\mathrm{D}_x f$ is also in $\mathcal{A}$, whereas ∂f is not, since $\partial f = \mathrm{D}_x f + f\partial$ which is an operator.

The ring of pseudodifferential operators $\mathcal{R}$ consists of elements

$$X = \sum_{i=-\infty}^{m} X_i \partial^i, \quad X_i \in \mathcal{A}. \tag{4.3}$$

Elements of $\mathcal{R}$ can be added (in the natural way) and multiplied term by term, moving all ∂s to the right hand side according to the commutation rule given in (4.1). Using the commutation rule (4.1), elements of $\mathcal{R}$ can also be written in the equivalent "left" form

$$X = \sum_{i=-\infty}^{m} \partial^i \tilde{X}_i, \quad \tilde{X}_i \in \mathcal{A}. \tag{4.4}$$

If the leading coefficient of X, X_m, is 1, then there exists a unique inverse X^{-1} also with leading coefficient 1, such that $XX^{-1} = X^{-1}X = 1$. There also exists a unique m^{th} root of X, $X^{1/m}$ starting with ∂. Then $X^{p/m} = (X^{1/m})^p$ and $(X^{1/m})^m = X$. We define $\mathcal{R}_+$ to be the set of all elements

$$X_+ = \sum_{i=0}^{m} X_i \partial^i \tag{4.5}$$

and $\mathcal{R}_-$ to be the set of all elements

$$X_- = \sum_{i=-\infty}^{-1} X_i \partial^i \tag{4.6}$$

The residue of a pseudodifferential operator, $\mathrm{res}\{X\} = X_{-1}$, is the coefficient of ∂^{-1} in X. We shall make use of two important properties relating to residues. Firstly,

$$\mathrm{res}\,\{X_+ Y\} = \mathrm{res}\,\{X_+ Y_-\} = \mathrm{res}\,\{X Y_-\}. \tag{4.7}$$

The second property we shall use is given on the following lemma.

Lemma 25. *The residue of a commutator of two pseudodifferential operators X and Y,*

$$\operatorname{res}\{[X,Y]\} = \mathrm{D}_x\, h \tag{4.8}$$

for some $h \in \mathcal{A}$, so is a total x derivative.

This lemma is given in [33, Chapter 1] but the proof contains errors that are corrected here.

Proof. We verify this for single term pseudodifferential operators $S = s\,\partial^m$ and $T = t\,\partial^n$. We shall use the notation $s^{(k)} = \mathrm{D}_x^k\, s$ and similarly for t. We first note that $\operatorname{res}\{[S,T]\}$ is only non-zero if one of m and n is greater than or equal to zero whilst the other is negative. Without loss of generality, we shall assume that $m \geq 0$ and $n < 0$. The product

$$ST = \sum_{k=0}^{\infty} \binom{m}{k} s t^{(k)} \partial^{m+n-k}. \tag{4.9}$$

so

$$\operatorname{res}\{ST\} = \binom{m}{m+n+1} s t^{(m+n+1)} \tag{4.10}$$

when $m+n+1 \geq 0$. Otherwise $\operatorname{res}\{ST\} = 0$ since $k \geq 0$ in (4.9). It follows that

$$\operatorname{res}\{[S,T]\} = \binom{m}{m+n+1} s t^{(m+n+1)} - \binom{n}{m+n+1} s t^{(m+n+1)}. \tag{4.11}$$

We notice that

$$\binom{m}{m+n+1} = \frac{m(m-1)\dots(-n)}{(m+n+1)!} \quad \text{and} \quad \binom{n}{m+n+1} = \frac{n(n-1)\dots(-m)}{(m+n+1)!} \tag{4.12}$$

so

$$\binom{n}{m+n+1} = (-1)^{m+n+1} \binom{m}{m+n+1}. \tag{4.13}$$

Then

$$
\begin{aligned}
&\operatorname{res}\{[S,T]\}\\
&=\binom{m}{m+n+1}\left(st^{(m+n+1)}+(-1)^{m+n}st^{(m+n+1)}\right)\\
&=\binom{m}{m+n+1}\left(st^{(m+n+1)}+s^{(1)}t^{(m+n)}-s^{(1)}t^{(m+n)}-s^{(2)}t^{(m+n-1)}+s^{(2)}t^{(m+n-1)}+\ldots\right.\\
&\left.\ldots-(-1)^{m+n}t^{(1)}s^{(m+n)}+(-1)^{m+n}t^{(1)}s^{(m+n)}+(-1)^{m+n}ts^{(m+n+1)}\right)
\end{aligned}
\tag{4.14}
$$

where, to get the expression on the second line we have added and subtracted $\sum_{\alpha=1}^{m+n} s^{(\alpha)}t^{(m+n+1-\alpha)}$. We recognise this as a total x derivative, so

$$
\operatorname{res}\{[S,T]\}=\binom{m}{m+n+1}\mathrm{D}_x\sum_{\alpha=0}^{m+n}(-1)^{\alpha}s^{(\alpha)}t^{(m+n-\alpha)}. \tag{4.15}
$$

It follows that, for general pseudodifferential operators X and Y, their residue, $\operatorname{res}\{[X,Y]\}$ can be expressed as the sum of total derivatives of the form given in (4.15) for pairs X_i and Y_j, so is a total x derivative. □

4.2 The KP hierarchy and its reduction to Gelfand-Dickey

4.2.1 The KP hierarchy

Here we give a brief summary of Sato's scheme [41] for the KP hierarchy [26]. We let

$$
L=\partial+u_1\partial^{-1}+u_2\partial^{-2}+\ldots=\partial+\sum_{\alpha=1}^{\infty}u_\alpha\partial^{-\alpha}. \tag{4.16}
$$

Using the notation L_+^i to represent $(L^i)_+$, for $i>0$

$$
L_{x_i}=[L_+^i,L] \tag{4.17}
$$

gives us the KP hierarchy. For each i, this produces an infinite set of PDEs containing derivatives with respect to x_i and x. From the case where $i = 1$, we see that $L_{x_1} = \mathrm{D}_x L$, allowing us to identify x_1 with x. A consequence of (4.17) is that

$$(L^n)_{x_i} = [L^i_+, L^n] \tag{4.18}$$

for all $n \geq 1$. This can be proved by induction on n. It follows that

$$\begin{aligned}(L^j_+)_{x_i} - (L^i_+)_{x_j} &= [L^i_+, L^j]_+ - [L^j_+, L^i]_+ \\ &= [L^i_+ - L^i, L^j]_+ + [L^i, L^j_+]_+ \\ &= [-L^i_-, L^j]_+ + [L^i, L^j_+]_+ \\ &= [-L^i_-, L^j_+]_+ + [L^i, L^j_+]_+ \\ &= [L^i_+, L^j_+].\end{aligned} \tag{4.19}$$

This gives us the "zero-curvature" equations for KP,

$$(L^j_+)_{x_i} - (L^i_+)_{x_j} = [L^i_+, L^j_+]. \tag{4.20}$$

For each $i, j > 0$, this produces a finite set of PDEs containing derivatives with respect to x_i, x_j and x. In the case where $i = 2$ and $j = 3$, (4.20) gives us

$$\begin{aligned}&3(u_1)_{x_2} = 3u_1^{(2)} + 6u_2^{(1)} \\ &3(u_1^{(1)})_{x_2} + 3(u_2)_{x_2} - 2(u_1)_{x_3} = u_1^{(3)} + 3u_2^{(2)} - 6u_1u_1^{(1)}.\end{aligned} \tag{4.21}$$

Letting $2u_1 = u$ and eliminating u_2, this gives us

$$3u_{x_2x_2} = (4u_{x_3} - u^{(3)} - 6uu^{(1)})_x, \tag{4.22}$$

the KP equation that gives its name to the hierarchy.

For a fixed choice of i and j, the PDEs given by (4.17) for i and j are not equivalent to the PDEs given by (4.20) for the same i and j, since (4.17) gives an infinite set of PDEs whilst (4.20) gives a finite one. However the set of PDEs given by (4.17) for all $i > 0$ is equivalent to the set of PDEs given by (4.20) for all $i, j > 0$. We have already shown that we can obtain (4.20) from (4.17). The following lemma relates to the converse.

Lemma 26. *The set of equations given by*

$$(L^j_+)_{x_i} - (L^i_+)_{x_j} = [L^i_+, L^j_+]. \tag{4.23}$$

for all $1 \leq i < j$ is equivalent to the set of equations given by

$$L_{x_i} = [L^i_+, L] \tag{4.24}$$

for all $i \geq 1$.

Proof. We have already shown that (4.24) for i and j implies (4.23) for the same i and j. To show that (4.23) for all $1 \leq i, j$ implies (4.24) for all $i \geq 1$, we consider (4.20) in the form

$$(L^j_+)_{x_i} - (L^i_+)_{x_j} = [L^i_+, L^j]_+ - [L^j_+, L^i]_+, \tag{4.25}$$

and without loss of generality assume that $j > i$. The first $j - i$ terms of this (i.e. the coefficients of ∂^k for k from $i - 1$ to $j - 2$) are identical to the first $j - i$ terms of

$$L^j_{x_i} = [L^i_+, L^j]. \tag{4.26}$$

We now let $j = n + 1$ in (4.26) and multiply from the left by L^{-n}, and from this we subtract (4.26) with $j = n$, multiplied on the left by L^{-n}, and on the right by L to obtain

$$L^{-n}(L^{n+1}_{x_i} - L^n_{x_i} L) = L^{-n}([L^i_+, L^{n+1}] - [L^i_+, L^n]L). \tag{4.27}$$

The left hand side of this is just L_{x_i}, whilst the right hand side simplifies to $[L^i_+, L]$. Therefore two copies of (4.20) with $j = n$ and $j = n+1$ gives us the first $n - i$ terms of

$$L_{x_i} = [L^i_+, L]. \tag{4.28}$$

Since n is arbitrary, we are able to obtain all terms of (4.17).

$\square$

In [1], a Lagrangian multiform incorporating a re-scaled version of (4.22) and the corresponding equation arising from (4.20) with $i = 2$ and $j = 4$ was presented with the following Lagrangian coefficients:

$$\mathscr{L}_{(123)} = \frac{1}{2}v_{x_1x_1}v_{x_1x_3} - \frac{1}{2}v_{3x_1}^2 - \frac{1}{2}v_{x_1x_2}^2 + v_{x_1x_1}^3 \tag{4.29a}$$

$$\mathscr{L}_{(412)} = \frac{1}{2}v_{x_1x_1}v_{x_1x_4} - 2v_{3x_1}v_{x_1x_1x_2} - \frac{2}{3}v_{x_1x_2}v_{x_2x_2} + 4v_{x_1x_1}^2v_{x_1x_2} \tag{4.29b}$$

$$\begin{aligned}\mathscr{L}_{(234)} = &-\frac{1}{2}v_{x_1x_3}v_{x_1x_4} - 4v_{x_1x_3}v_{3x_1x_2} + 2v_{x_1x_1x_3}v_{x_1x_1x_2} - \frac{2}{3}v_{x_2x_2}v_{x_2x_3} + v_{x_2x_2}v_{x_1x_4} \\ &+ 4v_{x_2x_2}v_{3x_1x_2} - \frac{8}{3}v_{x_1x_2x_2}v_{x_1x_1x_2} - v_{3x_1}v_{x_1x_1x_4} + \frac{4}{3}v_{3x_1}v_{3x_2} - 4v_{3x_1}^2v_{x_1x_2} \\ &+ 8v_{x_1x_1}v_{3x_1}v_{x_1x_1x_2} + 8v_{x_1x_1}v_{x_1x_2}v_{x_2x_2} + \frac{4}{3}v_{x_1x_2}^3 - 8v_{x_1x_1}v_{x_1x_2}v_{x_1x_3} \\ &- 8v_{x_1x_1}^3v_{x_1x_2}\end{aligned} \tag{4.29c}$$

$$\begin{aligned}\mathscr{L}_{(341)} =&\frac{2}{3}v_{x_2x_2}^2 + 2v_{4x_1}^2 - 2v_{3x_1}v_{x_1x_1x_3} - \frac{4}{3}v_{x_2x_2}v_{x_1x_3} - \frac{2}{3}v_{x_1x_2}v_{x_2x_3} + v_{x_1x_2}v_{x_1x_4} \\ &- \frac{4}{3}v_{x_1x_1x_2}^2 + \frac{4}{3}v_{3x_1}v_{x_1x_2x_2} + 12v_{x_1x_1}^2v_{4x_1} + 4v_{3x_1}^2v_{x_1x_1} - 4v_{x_1x_1}^2v_{x_2x_2} \\ &+ 4v_{x_1x_1}v_{x_1x_2}^2 + 4v_{x_1x_1}^2v_{x_1x_3} + 10v_{x_1x_1}^4.\end{aligned} \tag{4.29d}$$

where the dependent variable $v_{x_1x_1} = u$ has been used to eliminate non-local terms. These Lagrangians were found using the variational symmetries method outlined in the same paper. Although it is possible to extend this Lagrangian multiform to incorporate more flows of the hierarchy, the resultant Lagrangians become increasingly unwieldy. Also, as we progress up the hierarchy, an ever increasing number of non-local terms appear in the Lagrangians, and the Lagrangians grow very large very quickly. Expanding this multiform to include the x_5 flow results in Lagrangians that are many pages long (see Appendix C). Also, this approach does not yield an explicit formula for all of the constituent Lagrangians of the multiform for the complete hierarchy, so in order to obtain a multiform for the entire hierarchy, a different approach is needed.

4.2.2 The Gelfand-Dickey hierarchy as a reduction of KP

The n^{th} Gelfand-Dickey hierarchy [42] can be formulated as follows. We let

$$L_{GD} = \partial^n + v_{n-2}\partial^{n-2} + v_{n-3}\partial^{n-3} + \ldots + v_0 \tag{4.30}$$

and let

$$P_m = (L_{GD}^{m/n})_+. \tag{4.31}$$

We note that whilst L_{GD} is not a pseudodifferential operator, in general a fractional power of L_{GD} will be. The n^{th} Gelfand-Dickey hierarchy is then given by

$$(L_{GD})_{x_m} = [P_m, L_{GD}]. \tag{4.32}$$

In the case where $n = 2$, this gives the KdV hierarchy, whilst for $n = 3$ we get the Boussinesq hierarchy. We now consider the KP equation (4.18)

$$L^n_{x_m} = [L^m_+, L^n]. \tag{4.33}$$

In order to reduce the KP hierarchy to the n^{th} Gelfand-Dickey hierarchy we impose the constraint that $L^n_- = 0$. We note that

$$L^n_- = 0 \implies L^n = L^n_+, \tag{4.34}$$

an n^{th} order differential operator that we equate with L_{GD}. It follows that $L_{GD}^{1/n} = L$, so P_m is given by L^m_+, making (4.32) and (4.33) equivalent. We also note that $L^n_- = 0 \implies L^{kn}_- = 0$ for all $k \in \mathbb{Z}_+$, so (4.33) gives $L^n_{x_m} = 0$ whenever n divides

m. This is as expected since, by (4.32), $(L_{GD})_{x_m} = 0$ whenever P_m is an integer power of L_{GD}, which happens when n divides m.

4.3 A Lagrangian for the KP hierarchy

In this section, we present a Lagrangian for the KP hierarchy that was originally given in [43]. We define $\mathcal{A}_\varphi$ to be the differential algebra analogous to $\mathcal{A}$ with generators $\varphi_0, \varphi_1, \varphi_2, \dots$ (i.e. where elements of $\mathcal{A}_\varphi$ are differential polynomials in the generators φ_β), and we define $\mathcal{R}_\varphi$ to be the ring of pseudodifferential operators with coefficients in $\mathcal{A}_\varphi$. We define $\mathcal{R}_{\varphi+}$ and $\mathcal{R}_{\varphi-}$ analogously to $\mathcal{R}_+$ and $\mathcal{R}_-$. We make the dressing substitution

$$L = \phi\partial\phi^{-1} \tag{4.35}$$

where

$$\phi = 1 + \sum_{\beta=0}^{\infty} \varphi_\beta \partial^{-\beta-1}, \tag{4.36}$$

noting that because of the leading 1, a unique ϕ^{-1} exists. Expanding (4.35) we find that

$$L = \partial - \varphi_0'\partial^{-1} + (\varphi_0\varphi_0' - \varphi_1')\partial^{-2} + (\varphi_1\varphi_0' + \varphi_0\varphi_1' - (\varphi_0')^2 - \varphi_0^2\varphi_0' - \varphi_2')\partial^{-3} + \dots, \tag{4.37}$$

where φ_β' denotes the x derivative of φ_β. Equating coefficients with (4.16), we see that $u_1 = -\varphi_0'$, $u_2 = \varphi_0\varphi_0' - \varphi_1'$, $u_3 = \varphi_1\varphi_0' + \varphi_0\varphi_1' - (\varphi_0')^2 - \varphi_0^2\varphi_0' - \varphi_2'$ etc., giving an injective map from $\mathcal{A}$ to $\mathcal{A}_\varphi$.

The resulting KP equation in terms of ϕ is given by

$$\phi_{x_i} = -L_-^i\phi. \tag{4.38}$$

In order to show this, we invoke the idea of homogeneity in the sense of all terms of an expression carying equal weight. Let us consider this in the case of the KP equation

$$3u_{x_2x_2} = (4u_{x_3} - u^{(3)} - 6uu^{(1)})_x. \tag{4.39}$$

We begin by assigning a weight of 1 to the derivative with respect to x. On the left hand side of the equation, we see a $u_{x_2x_2}$ term, which we compare to the $u^{(4)}$ term on the right hand side. In order for these terms to have equal weight, an x_2 derivative must have weight 2. Similarly, by comparing the $u^{(1)}_{x_3}$ and $u^{(4)}$ terms, it follows that an x_3 derivative has weight 3. Finally by comparing $u^{(3)}$ and $uu^{(1)}$ we see that u carries weight 2. Whenever it is possible to assign weights in this manner such that all terms of an expression carry equal weight, we say that the expression is *homogeneous.*

Homogeneity can also be introduced directly on the level of the pseudodifferential operators. Applying this to the KP operator

$$L = \partial + u_1\partial^{-1} + u_2\partial^{-2} + \dots, \tag{4.40}$$

we again assign a weight of 1 to the derivative with respect to x, so the leading ∂ carries weight 1. In order for all terms to carry equal weight, it follows that u_1 has weight 2, u_2 has weight 3, and in general u_α has weight $\alpha + 1$. Similarly, the leading 1 of the operator

$$\phi = 1 + \varphi_0\partial^{-1} + \varphi_1\partial^{-2} + \dots \tag{4.41}$$

tells us that ϕ has weight 0, so φ_0 has weight 1, φ_1 has weight 2, and φ_β has weight $\beta + 1$ in order that each term has weight 0. In this chapter we only deal with homogeneous equations. With this in mind, we have the following lemma.

Lemma 27. *We let $L = \phi\partial\phi^{-1} \in \mathcal{R}_\varphi$. Then*

$$L_{x_i} = [L^i_+, L] \iff \phi_{x_i} = -L^i_-\phi. \tag{4.42}$$

Proof. Using that $L = \phi\partial\phi^{-1}$, the equation

$$L_{x_i} = [L^i_+, L] \tag{4.43}$$

becomes

$$[\phi_{x_i}\phi^{-1} - L_+^i, L] = 0, \tag{4.44}$$

This is equivalent to the statement that

$$\phi_{x_i}\phi^{-1} - L_+^i + f_i = 0 \tag{4.45}$$

for some f_i in $\mathcal{R}_\varphi$ such that $[L, f_i] = 0$. Letting $\tilde{f}_i = \phi^{-1} f_i \phi$, the requirement that $[L, f_i] = 0$ is equivalent to the requirement that $[\partial, \tilde{f}_i] = \mathrm{D}_x \tilde{f}_i = 0$. Therefore $\tilde{f}_i$ is a constant in $\mathcal{R}_\varphi$, so

$$\tilde{f}_i = \sum_{j=-\infty}^{m} \gamma_j \partial^j \tag{4.46}$$

for some m, where each γ_j is a constant in $\mathcal{A}_\varphi$ (i.e. a real or complex number), and consequently

$$f_i = \sum_{j=-\infty}^{m} \gamma_j L^j \tag{4.47}$$

for the same constants γ_j. In (4.45) we see that both $\phi_{x_i}\phi^{-1}$ and L_+^i are of weight i, so we require that f_i is also of weight i. Therefore, $\gamma_j = 0$ whenever $j \neq i$, so f_i is of the form $\gamma_i L^i$. When f_i takes this form, the coefficient of ∂^i in (4.45) is $\gamma_i - 1$, and setting this equal to zero gives us that $\gamma_i = 1$. Then (4.45) becomes

$$\phi_{x_i}\phi^{-1} + L_-^i = 0, \tag{4.48}$$

so the resulting equation for ϕ_{x_i} is

$$\phi_{x_i} = -L_-^i \phi. \tag{4.49}$$

I.e.,

$$L_{x_i} = [L_+^i, L] \implies \phi_{x_i} = -L_-^i \phi. \tag{4.50}$$

Conversely, we see that if (4.38) holds then

$$\begin{aligned} L_{x_i} &= (\phi\partial\phi^{-1})_{x_i} \\ &= \phi_{x_i}\partial\phi^{-1} - \phi\partial\phi^{-1}\phi_{x_i}\phi^{-1} \\ &= -L^i_-\phi\partial\phi^{-1} + \phi\partial\phi^{-1}L^i_- \\ &= [-L^i_-, L] \\ &= [L^i_+, L] \end{aligned} \tag{4.51}$$

so (4.38) implies (4.43). □

Corollary 28. *Lemmas 26 and 27 together tell us that the set of equations given by*

$$(L^j_+)_{x_i} - (L^i_+)_{x_j} = [L^i_+, L^j_+] \tag{4.52}$$

in $\mathcal{R}$ for all $1 \le i, j$ is equivalent to the set of equations given by

$$\phi_{x_i}\phi^{-1} + L^i_- = 0 \tag{4.53}$$

in $\mathcal{R}_\varphi$ for all $i \ge 1$.

We now consider a Lagrangian $\mathscr{L}_{(1ij)}\mathsf{d}x_1\wedge\mathsf{d}x_i\wedge\mathsf{d}x_j$ with $\mathscr{L}_{(1ij)} \in \mathcal{A}_\varphi$. For such a Lagrangian, we can take variational derivatives $\dfrac{\delta\mathscr{L}_{(1ij)}}{\delta\varphi_\beta}$ (i.e., the Euler operator with respect to φ_β acting on $\mathscr{L}_{(1ij)}$) to obtain expressions in $\mathcal{A}_\varphi$. However it is convenient to define the variational derivative with respect to the pseudodifferential operator ϕ,

$$\frac{\delta\mathscr{L}_{(1ij)}}{\delta\phi} = \sum_{\beta=0}^{\infty}\partial^\beta\frac{\delta\mathscr{L}_{(1ij)}}{\delta\varphi_\beta}. \tag{4.54}$$

According to this definition, $\dfrac{\delta\mathscr{L}_{(1ij)}}{\delta\phi}$ is a pseudodifferential operator in $\mathcal{R}_{\varphi+}$ that can be put in the usual form with all ∂s on the right using (4.1). The motivation for this definition is made clear by the following lemma.

Lemma 29. *If there exist h_1, h_2 and h_3 such that*

$$\delta\mathscr{L}_{(1ij)} = \operatorname{res}\{X\,\delta\phi\} + \mathrm{D}_x\, h_1 + \mathrm{D}_{x_i}\, h_2 + \mathrm{D}_{x_j}\, h_3 \tag{4.55}$$

for some $X \in \mathcal{R}_\varphi$*, then the variational derivative of* $\mathscr{L}_{(1ij)}$ *with respect to* ϕ*,*

$$\frac{\delta \mathscr{L}_{(1ij)}}{\delta \phi} = X_+ \tag{4.56}$$

Proof. Since $\delta\phi = \delta\varphi_0 \partial^{-1} + \delta\varphi_1 \partial^{-2} + \ldots$ has only negative powers of ∂, (4.55) is equivalent to

$$\delta \mathscr{L}_{(1ij)} = \operatorname{res}\{X_+\, \delta\phi\} + \mathrm{D}_x\, h_1 + \mathrm{D}_{x_i}\, h_2 + \mathrm{D}_{x_j}\, h_3. \tag{4.57}$$

We write X_+ in the "left" form described in equation (4.4), so

$$X_+ = \sum_{k=0}^{m} \partial^k \tilde{X}_k, \quad \tilde{X}_k \in \mathcal{A}_\varphi, \tag{4.58}$$

and consider the product of an arbitrary term in X_+ with an arbitrary term in $\delta\phi$. This will be of the form

$$\partial^n \tilde{X}_n\, \delta\varphi_m \partial^{-m-1} = \tilde{X}_n\, \delta\varphi_m \partial^{n-m-1} + \sum_{i=1}^{n} \binom{n}{i} \mathrm{D}_x^i (\tilde{X}_n\, \delta\varphi_m) \partial^{n-m-i-1} \tag{4.59}$$

and the only term on the right hand side that is not a total derivative is $\tilde{X}_n\, \delta\varphi_m \partial^{n-m-1}$. Therefore,

$$\delta \mathscr{L}_{(1ij)} = \operatorname{res}\{X_+\, \delta\phi\} + \mathrm{D}_x\, h_1 + \mathrm{D}_{x_i}\, h_2 + \mathrm{D}_{x_j}\, h_3 = \sum_{k=0}^{m} \tilde{X}_k\, \delta\varphi_k + \mathrm{D}_x\, \tilde{h}_1 + \mathrm{D}_{x_i}\, h_2 + \mathrm{D}_{x_j}\, h_3 \tag{4.60}$$

for some $\tilde{h}_1$, so the variational derivative

$$\frac{\delta \mathscr{L}_{(1ij)}}{\delta \varphi_k} = \tilde{X}_k \tag{4.61}$$

for $0 \leq k \leq m$ and is zero for $k > m$. It follows that

$$\frac{\delta \mathscr{L}_{(1ij)}}{\delta \phi} = \sum_{k=0}^{\infty} \partial^k \frac{\delta \mathscr{L}_{(1ij)}}{\delta \varphi_k} = \sum_{k=0}^{m} \partial^k \tilde{X}_k = X_+ \tag{4.62}$$

□

Following the formulation in [43], we introduce

$$\phi_p = 1 + p \sum_{\beta=0}^{\infty} \varphi_\beta \partial^{-\beta-1}. \tag{4.63}$$

where $p \in \mathbb{R}$.

Proposition 30. *The Lagrangian density*

$$\mathscr{L}_{(1ij)} = \operatorname{res}\left\{ -\int_0^1 p^{-1}[(\phi_p \partial^i \phi_p^{-1})_+, (\phi_p \partial^j \phi_p^{-1})_+]\phi_p^{-1} dp + \partial^j \phi^{-1}\phi_{x_i} - \partial^i \phi^{-1}\phi_{x_j} \right\} \tag{4.64}$$

gives Euler-Lagrange equations that are equivalent to the KP equation

$$(L_+^i)_{x_j} - (L_+^j)_{x_i} + [L_+^i, L_+^j] = 0. \tag{4.65}$$

It is important to note that where ∂ appears in this Lagrangian, it signifies an operator that acts on everything to its right, rather than the x derivative of whatever is immediately to its right. Also, even though ϕ consists of an infinite number of components, because this Lagrangian is a residue, only a finite number of these components actually feature. A proof that (4.64) gives the KP equation as its Euler-Lagrange equations is given in [43] and repeated here. We shall require the following lemma:

Lemma 31. *The following formula holds:*

$$\begin{aligned} &\delta \operatorname{res}\left\{ \int_0^p \tilde{p}^{-1}[(\phi_{\tilde{p}} \partial^i \phi_{\tilde{p}}^{-1})_+, (\phi_{\tilde{p}} \partial^j \phi_{\tilde{p}}^{-1})_+]\phi_{\tilde{p}}^{-1} d\tilde{p} \right\} \\ &= -\operatorname{res}\left\{[(\phi_p \partial^i \phi_p^{-1})_+, (\phi_p \partial^j \phi_p^{-1})_+]\delta\phi_p \ \phi_p^{-1}\right\} + \mathrm{D}_x h_1 \end{aligned} \tag{4.66}$$

with

$$\begin{aligned} h_1 = \int \int_0^p \tilde{p}^{-1} \operatorname{res}\{&[T[V,S],U] + [[T,U]_+S,V] + [U[V,S]_+,T] \\ &+ [UT,[V,S]_+] + [T[S,U],V] + [U,[T,V]_+S] + [V[S,U]_+,T] \\ &+ [VT,[S,U]_+] + [[U,V],TS] + [T,[U,V]S]\} d\tilde{p}\, dx. \end{aligned} \tag{4.67}$$

where $S = \phi_{\tilde{p}}^{-1}$, $T = \delta\phi_{\tilde{p}}\ \phi_{\tilde{p}}^{-1}$, $U = (\phi_{\tilde{p}}\partial^i\phi_{\tilde{p}}^{-1})_+$ *and* $V = (\phi_{\tilde{p}}\partial^j\phi_{\tilde{p}}^{-1})_+$. *This* h_1 *is local.*

The first part of this result is essentially the same as the one given by Dickey in [43]. However, Dickey does not give an explicit expression for h_1, since when considering a single Lagrangian, it is only necessary to show that it is a total x derivative. In the Lagrangian multiform case, we will require an expression for h_1, so it is included here.

Proof of Lemma 31. We proceed by taking the p derivative of

$$\begin{aligned}&\delta\,\mathrm{res}\left\{\int_0^p \tilde{p}^{-1}[(\phi_{\tilde{p}}\partial^i\phi_{\tilde{p}}^{-1})_+,(\phi_{\tilde{p}}\partial^j\phi_{\tilde{p}}^{-1})_+]\phi_{\tilde{p}}^{-1}\mathsf{d}\tilde{p}\right\}\\&+\mathrm{res}\,\{[(\phi_p\partial^i\phi_p^{-1})_+,(\phi_p\partial^j\phi_p^{-1})_+]\delta\phi_p\ \phi_p^{-1}\},\end{aligned} \tag{4.68}$$

multiplying by p, and using that $p\dfrac{\partial\phi_p}{\partial p} = \phi_p - 1$. This gives us

$$\begin{aligned}&\delta\,\mathrm{res}\,\{[(\phi_p\partial^i\phi_p^{-1})_+,(\phi_p\partial^j\phi_p^{-1})_+]\phi_p^{-1}\} + \mathrm{res}\,\{[(\phi_p\partial^i\phi_p^{-1})_+,(\phi_p\partial^j\phi_p^{-1})_+]\delta\phi_p\ \phi_p^{-2}\}\\&+\mathrm{res}\,\{(p\frac{\partial}{\partial p}[(\phi_p\partial^i\phi_p^{-1})_+,(\phi_p\partial^j\phi_p^{-1})_+])\delta\phi_p\ \phi_p^{-1}\}.\end{aligned} \tag{4.69}$$

Again using $p\dfrac{\partial\phi_p}{\partial p} = \phi_p - 1$ we find that

$$p\frac{\partial}{\partial p}(\phi_p\partial^i\phi_p^{-1})_+ = -[\phi_p^{-1},(\phi_p\partial^i\phi_p^{-1})_+]_+. \tag{4.70}$$

We shall also use that

$$\delta(\phi_p\partial^i\phi_p^{-1})_+ = [\delta\phi_p\ \phi_p^{-1},(\phi_p\partial^i\phi_p^{-1})_+]_+. \tag{4.71}$$

Letting $S = \phi_p^{-1}$, $T = \delta\phi_p\ \phi_p^{-1}$, $U = (\phi_p\partial^i\phi_p^{-1})_+$ and $V = (\phi_p\partial^j\phi_p^{-1})_+$, (4.69) is equivalent to

$$\begin{aligned}\mathrm{res}\,\{&[[T,U]_+,V]S + [U,[T,V]_+]S + [U,V]TS - [U,V]ST - [[S,U]_+,V]T\\&-[U,[S,V]_+]T\}\end{aligned} \tag{4.72}$$

In order to show that this is a total x derivative, we make use of (4.8), the property that the residue of a commutator is a total x derivative. We consider (4.72) two terms at a time. Firstly,

$$\begin{aligned}
&\operatorname{res}\{[[T,U]_+,V]S-[U,[S,V]_+]T\}\\
=&\operatorname{res}\{[T,U]_+[V,S]+[[T,U]_+S,V]+[T,U][V,S]_++[U[V,S]_+,T]+[UT,[V,S]_+]\}\\
=&\operatorname{res}\{[T,U][V,S]+[[T,U]_+S,V]+[U[V,S]_+,T]+[UT,[V,S]_+]\}\\
=&\operatorname{res}\{T[U,[V,S]]+[T[V,S],U]+[[T,U]_+S,V]+[U[V,S]_+,T]+[UT,[V,S]_+]\}.
\end{aligned}\tag{4.73}$$

Then

$$\begin{aligned}
&\operatorname{res}\{[U,[T,V]_+]S-[[S,U]_+,V]T\}\\
=&\operatorname{res}\{[T,V]_+[S,U]+[U,[T,V]_+S]+[T,V][S,U]_++[V[S,U]_+,T]+[VT,[S,U]_+]\}\\
=&\operatorname{res}\{[T,V][S,U]+[U,[T,V]_+S]+[V[S,U]_+,T]+[VT,[S,U]_+]\}\\
=&\operatorname{res}\{T[V,[S,U]]+[T[S,U],V]+[U,[T,V]_+S]+[V[S,U]_+,T]+[VT,[S,U]_+]\}.
\end{aligned}\tag{4.74}$$

Finally,

$$\begin{aligned}
&\operatorname{res}\{[U,V]TS-[U,V]ST\}\\
=&\operatorname{res}\{[U,V][T,S]\}\\
=&\operatorname{res}\{T[S,[U,V]]+[[U,V],TS]+[T,[U,V]S]\}.
\end{aligned}\tag{4.75}$$

Adding (4.73), (4.74) and (4.75) together, we notice that

$$\operatorname{res}\{T([U,[V,S]]+[V,[S,U]]+[S,[U,V]])\}=0 \tag{4.76}$$

by the Jacobi identity, so (4.72) is equal to

$$\begin{aligned}
&\operatorname{res}\{[T[V,S],U]+[[T,U]_+S,V]+[U[V,S]_+,T]+[UT,[V,S]_+]+[T[S,U],V]\\
&+[U,[T,V]_+S]+[V[S,U]_+,T]+[VT,[S,U]_+]+[[U,V],TS]+[T,[U,V]S]\}.
\end{aligned} \tag{4.77}$$

Since every term is the residue of a commutator, this is a total x derivative. We set h_1 equal to the local expression obtained by letting $p \to \tilde{p}$ in (4.77), integrating with respect to $\tilde{p}$ from 0 to p, integrating with respect to x and setting the constant of integration equal to zero (i.e., the expression given in (4.67)). It follows that, for this choice of h_1, (4.66) holds. $\square$

Proof of Proposition 30. We use Lemma 31 with $p = 1$ to obtain

$$\begin{aligned}
&\delta \operatorname{res}\left\{\int_0^1 p^{-1}[(\phi_p\partial^i\phi_p^{-1})_+,(\phi_p\partial^j\phi_p^{-1})_+]\phi_p^{-1}\mathsf{d}p\right\}\\
&=-\operatorname{res}\{[(\phi\partial^i\phi^{-1})_+,(\phi\partial^j\phi^{-1})_+]\delta\phi\,\phi^{-1}\}+\mathrm{D}_x(h_1|_{p=1}).
\end{aligned} \tag{4.78}$$

Variation of the rest of the Lagrangian (4.64) gives us

$$\begin{aligned}
&\delta\operatorname{res}\{\partial^j\phi^{-1}\phi_{x_i}-\partial^i\phi^{-1}\phi_{x_j}\}\\
=&\,\mathrm{D}_{x_i}\operatorname{res}\{\partial^j\phi^{-1}\delta\phi\}-\mathrm{D}_{x_j}\operatorname{res}\{\partial^i\phi^{-1}\delta\phi\}\\
&+\operatorname{res}\{\phi\partial^j\phi^{-1}\phi_{x_i}\phi^{-1}\delta\phi\,\phi^{-1}\}-\operatorname{res}\{\phi\partial^i\phi^{-1}\phi_{x_j}\phi^{-1}\delta\phi\,\phi^{-1}\}\\
&-\operatorname{res}\{\phi_{x_i}\partial^j\phi^{-1}\delta\phi\,\phi^{-1}\}+\operatorname{res}\{\phi_{x_j}\partial^i\phi^{-1}\delta\phi\,\phi^{-1}\}+\partial h_2\\
=&\,\mathrm{D}_{x_i}\operatorname{res}\{\partial^j\phi^{-1}\delta\phi\}-\mathrm{D}_{x_j}\operatorname{res}\{\partial^i\phi^{-1}\delta\phi\}\\
&+\operatorname{res}\{((L^i_+)_{x_j}-(L^j_+)_{x_i})\delta\phi\,\phi^{-1}\}+\mathrm{D}_x h_2,
\end{aligned} \tag{4.79}$$

where we have made use of (4.7) and the fact that $\delta\phi\,\phi^{-1}\in\mathcal{R}_-$ to obtain the the final expression. Combining (4.78) and (4.79) we get

$$\begin{aligned}
\delta\mathscr{L}_{(1ij)}&=\operatorname{res}\{((L^i_+)_{x_j}-(L^j_+)_{x_i}+[L^i_+,L^j_+])\delta\phi\,\phi^{-1}\}\\
&=\operatorname{res}\{\phi^{-1}((L^i_+)_{x_j}-(L^j_+)_{x_i}+[L^i_+,L^j_+])\delta\phi\}+\mathrm{D}_x h_3,
\end{aligned} \tag{4.80}$$

so

$$\frac{\delta \mathscr{L}_{(1ij)}}{\delta\phi} = \{\phi^{-1}((L_+^i)_{x_j} - (L_+^j)_{x_i} + [L_+^i, L_+^j])\}_+, \tag{4.81}$$

and when set equal to zero, this is equivalent to (4.20). □

Example 32. *The explicit form of $\mathscr{L}_{(123)}$ given by (4.64) is*

$$\begin{aligned}
\mathscr{L}_{(123)} =& - U_{xxx_3} + X_{x_2} - VU_{xx_2} - WU_{x_2} - VV_{x_2} - U^2U_{x_3} + VU_{x_3} + UU_{xx_3}\\
&+ U^2U_{xx_2} + UV_{x_3} + U^2V_{x_2} - UU_{xxx_2} - U^3U_{x_2} - UW_{x_2} - 2UV_{xx_2} - 3V_xU_{x_2}\\
&- 3U_{xx}U_{x_2} + 2U_xU_{x_3} - 3U_xV_{x_2} - 3U_xU_{xx_2} - W_{x_3} + U_{xxxx_2} - \frac{3}{2}UV_{xxx}\\
&- \frac{3}{2}U_{xxx}V - 3V_{xx}V - \frac{3}{2}{U_x}^2U^2 + 2U_{xxx}U^2 + 2V_{xx}U^2 + 2{U_x}^2V - \frac{1}{2}UU_{xxxx}\\
&- \frac{3}{2}U_xU_{xxx} - 3U_xV_{xx} - \frac{3}{2}U_{xx}U^3 + 2{U_x}^3 + 3W_{xx_2} - 2V_{xx_3} + 3V_{xxx_2}\\
&+ 5UU_xU_{x_2} + 2UVU_{x_2} + 3U_{xx}U_xU + 2U_{xx}VU,
\end{aligned} \tag{4.82}$$

where $U = \varphi_0$, $V = \varphi_1$, $W = \varphi_2$ and $X = \varphi_3$. This was calculated using Maple and PSEUDO [44]. Note that although X and Y appear in this Lagrangian, their presence is trivial in that they do not contribute to or feature in the resulting Euler-Lagrange equations. We can simplify $\mathscr{L}_{(123)}$ considerably by subtracting total derivatives to obtain the equivalent Lagrangian

$$\tilde{\mathscr{L}}_{(123)} = 3U_x^2U^2 - \frac{3}{2}U_{xx_2}U^2 + 3V_{xx}U^2 + \frac{5}{2}U_x^3 + U_xU_{x_3} + U_{xx}^2 - 3U_xV_{x_2} - 3U_xV_{xx} + 3V_x^2 \tag{4.83}$$

that gives identical Euler-Lagrange equations. The variational derivatives with respect to U and V are

$$\begin{aligned}
\frac{\delta \mathscr{L}_{(123)}}{\delta U} =& - 6U^2U_{xx} - 6UU_x^2 - 6UU_{xx_2} + 6UV_{xx} - 3U_xU_2 - 15U_xU_{xx}\\
&- 2U_{xx_3} + 2U_{xxxx} + 3V_{xx_2} + 3V_{xxx},\\
\frac{\delta \mathscr{L}_{(123)}}{\delta V} =& 6UU_{xx} + 6U_x^2 - 3U_{xxx} + 3U_{xx_2} - 6V_{xx},
\end{aligned} \tag{4.84}$$

giving us that

$$\begin{aligned}\frac{\delta \mathscr{L}_{(123)}}{\delta \phi} =& \partial \frac{\delta \mathscr{L}_{(123)}}{\delta V} + \frac{\delta \mathscr{L}_{(123)}}{\delta U} \\ =& \frac{\delta \mathscr{L}_{(123)}}{\delta V}\partial + \mathrm{D}_x \frac{\delta \mathscr{L}_{(123)}}{\delta V} + \frac{\delta \mathscr{L}_{(123)}}{\delta U} \\ =& (6UU_{xx} + 6U_x^2 - 3U_{xxx} + 3U_{xx_2} - 6V_{xx})\partial - U_{xxxx} + 6UU_{xxx} + 3U_{xxx_2} \\ & - 3V_{xxx} + -6U^2U_{xx} + 3U_xU_{xx} - 6UU_x^2 - 6UU_{xx_2} + 6UV_{xx} - 3U_{x_2}U_x \\ & - 2U_{xx_3} + 3V_{xx_2}\end{aligned} \tag{4.85}$$

Since the Euler Lagrange equations (4.81) have a pre-factor of ϕ^{-1}, we calculate

$$\begin{aligned}\left(\phi \frac{\delta \mathscr{L}_{(123)}}{\delta \phi}\right)_+ =& (6UU_{xx} + 6U_x^2 - 3U_{xxx} + 3U_{xx_2} - 6V_{xx})\partial - 3U_{x_2}U_x - 3UU_{xx_2} \\ & + 3U_{xxx_2} + 3V_{xx_2} - 2U_{xx_3} + 3UU_{xxx} + 3U_xU_{xx} - U_{xxxx} - 3V_{xxx}.\end{aligned} \tag{4.86}$$

Making the substitution $u_1 = -U_x$, $u_2 = UU_x - V_x$ (based on the expansion (4.37)), this becomes

$$(3u_1^{(2)} - 3(u_1)_{x_2} + 6u_2^{(1)})\partial + 2(u_1)_{x_3} - 3(u_1^{(1)})_{x_2} - 3(u_2)_{x_2} - 6u_1u_1^{(1)} + u_1^{(3)} + 3u_2^{(2)}. \tag{4.87}$$

Setting this equal to zero gives us equations that are equivalent to (4.21).

4.4 Lagrangian multiforms for the KP hierarchy

In this section we present two closely related Lagrangian multiform structures for the KP hierarchy. Let

$$\mathsf{M} = \sum_{1 \le i < j < k} \mathscr{L}_{(ijk)} \mathsf{d}x_i \wedge \mathsf{d}x_j \wedge \mathsf{d}x_k \tag{4.88}$$

be a differential 3-form. We shall define the coefficients $\mathscr{L}_{(ijk)}$ such that the PDEs defined by $\delta \mathsf{dM} = 0$ are the full set of equations of the KP hierarchy, and we shall show that on these equations $\mathsf{dM} = 0$. We define $P_{(ijkl)}$ such that

$$\mathsf{dM} = \sum_{1 \leq i < j < k < l} P_{(ijkl)} \mathsf{d}x_i \wedge \mathsf{d}x_j \wedge \mathsf{d}x_k \wedge \mathsf{d}x_l \tag{4.89}$$

and will show that each $P_{(1ijk)}$ has a double zero on the equations of the KP hierarchy, so the coefficients $P_{(1ijk)}$ will be of the form

$$\sum_{\gamma=1}^{n} A_\gamma B_\gamma \tag{4.90}$$

where each A_γ and B_γ is zero on the equations of the KP hierarchy. More specifically, the A_γ will be of the form

$$(L^i_+)_{x_j} - (L^j_+)_{x_i} + [L^i_+, L^j_+] \tag{4.91}$$

whilst the B_γ will be of the form

$$\phi_{x_i} \phi^{-1} + L^i_-, \tag{4.92}$$

giving us the required double zero. Then

$$\delta P_{(1ijk)} = \sum_{\gamma=1}^{n} \delta A_\gamma B_\gamma + A_\gamma \delta B_\gamma \tag{4.93}$$

so the equations given by $\delta P_{(1ijk)} = 0$ will be a subset of the equations of the KP hierarchy. In order for the equations given by $\delta P_{(1ijk)} = 0$ for all $1 < i, j, k$ to be the full set of equations of the KP hierarchy, we require that the factors A_γ and B_γ span the set of equations of the KP hierarchy, and also that the A_γ and B_γ are non-degenerate. Rather than show this directly, we will instead show the equivalent result that the full set of equations of the KP hierarchy arise from the Euler-Lagrange equations of the $\mathscr{L}_{(1ij)}$ Lagrangians. Then, for the $P_{(ijkl)}$ where $1 < i, j, k, l$ we will show that $\delta P_{(ijkl)} = 0$ on the equations of the KP hierarchy. Together, these results will show that the multiform Euler-Lagrange equations given by $\delta \mathsf{dM} = 0$ are a subset of the equations of the KP hierarchy, and include the entire KP hierarchy. It follows that the multiform Euler-Lagrange equations are precisely the equations of the KP hierarchy.

The factorised form of $P_{(1ijk)}$ in terms of the A_γ and B_γ would suggest that as well as giving us equations in the form

$$(L_+^i)_{x_j} - (L_+^j)_{x_i} + [L_+^i, L_+^j] = 0, \tag{4.94}$$

the multiform Euler-Lagrange equations should also include KP equations of the type

$$\phi_{x_i}\phi^{-1} + L_-^i = 0. \tag{4.95}$$

However, Corollary 28 tells us that the set of equations of the form of (4.94) for all $i, j > 0$ is equivalent to the set of equations of the form of (4.95) for all $i > 0$, so we are free to view either of these equivalent sets of equations as the complete set of multiform Euler-Lagrange equations for M.

4.4.1 A Lagrangian multiform for KP based on Dickey's Lagrangian

We define

$$\begin{aligned}\Gamma_{ijk} :=& \frac{1}{2}([\phi\partial^k\phi^{-1}\phi_{x_i}\phi^{-1}\phi_{x_j}, \phi^{-1}] + [\phi\partial^j\phi^{-1}\phi_{x_k}\phi^{-1}\phi_{x_i}, \phi^{-1}] + [\phi\partial^i\phi^{-1}\phi_{x_j}\phi^{-1}\phi_{x_k}, \phi^{-1}] \\ &- [\phi\partial^k\phi^{-1}\phi_{x_j}\phi^{-1}\phi_{x_i}, \phi^{-1}] - [\phi\partial^j\phi^{-1}\phi_{x_i}\phi^{-1}\phi_{x_k}, \phi^{-1}] - [\phi\partial^i\phi^{-1}\phi_{x_k}\phi^{-1}\phi_{x_j}, \phi^{-1}] \\ &+ [\phi_{x_j}, \partial^k\phi^{-1}\phi_{x_i}\phi^{-1}] + [\phi_{x_i}, \partial^j\phi^{-1}\phi_{x_k}\phi^{-1}] + [\phi_{x_k}, \partial^i\phi^{-1}\phi_{x_j}\phi^{-1}] \\ &- [\phi_{x_i}, \partial^k\phi^{-1}\phi_{x_j}\phi^{-1}] - [\phi_{x_k}, \partial^j\phi^{-1}\phi_{x_i}\phi^{-1}] - [\phi_{x_j}, \partial^i\phi^{-1}\phi_{x_k}\phi^{-1}]),\end{aligned} \tag{4.96}$$

$$\begin{aligned}\Delta_{ij,k} := -\int_0^1 p^{-1}(&[T[V,S],U] + [[T,U]_+S,V] + [U[V,S]_+,T] + [UT,[V,S]_+] \\ &+ [T[S,U],V] + [U,[T,V]_+S] + [V[S,U]_+,T] \\ &+ [VT,[S,U]_+] + [[U,V],TS] + [T,[U,V]S])\mathsf{d}p\end{aligned} \tag{4.97}$$

where $S = \phi_p^{-1}$, $T = (\phi_p)_{x_k}\phi_p^{-1}$, $U = (\phi_p\partial^i\phi_p^{-1})_+$ and $V = (\phi_p\partial^j\phi_p^{-1})_+$,

$$\Theta_{ij,k} := \frac{1}{2}([\phi_{x_k}\phi^{-1}, L_+^i L_-^j] + [L_-^j, L_+^i\phi_{x_k}\phi^{-1}] + [L_+^j\phi_{x_k}\phi^{-1}, L_-^i] + [L_+^j L_-^i, \phi_{x_k}\phi^{-1}]) \tag{4.98}$$

and

$$\Lambda_{ijk} := \frac{1}{2}([L_+^i L_-^j - L_+^j L_-^i, L^k] + [L_+^k L_-^i, L_+^j] + [L_+^i, L_+^k L_-^j] + [L_-^i, L^{j+k}] + [L^{i+k}, L_-^j]). \tag{4.99}$$

In these definitions, L is used as an abbreviation of $\phi\partial\phi^{-1}$, so all of the above are pseudodifferential operators whose coefficients are in terms of φ_β and their derivatives.

Theorem 33. *The 3-form*

$$\mathsf{M} = \sum_{1\leq i<j<k} \mathscr{L}_{(ijk)} \mathsf{d}x_i \wedge \mathsf{d}x_j \wedge \mathsf{d}x_k \tag{4.100}$$

with coefficients

$$\mathscr{L}_{(1jk)} = \operatorname{res}\left\{ -\int_0^1 p^{-1}[(\phi_p\partial^j\phi_p^{-1})_+, (\phi_p\partial^k\phi_p^{-1})_+]\phi_p^{-1}\mathsf{d}p + \partial^k\phi^{-1}\phi_{x_j} - \partial^j\phi^{-1}\phi_{x_k}\right\} \tag{4.101}$$

and

$$\mathscr{L}_{(ijk)} = \int \operatorname{res}\{\Gamma_{ijk} + \Delta_{ij,k} + \Delta_{jk,i} + \Delta_{ki,j} + \Theta_{ij,k} + \Theta_{jk,i} + \Theta_{ki,j} + \Lambda_{ijk}\}\mathsf{d}x \tag{4.102}$$

(with the constant of integration set to zero) when $i > 1$ is a Lagrangian multiform for the KP hierarchy. Each $\mathscr{L}_{(ijk)}$ is a local expression in the fields φ_β and their derivatives. The multiform Euler-Lagrange equations given by $\delta\mathsf{dM} = 0$ are the full set of equations of the KP hierarchy and consequences thereof. On the equations of the KP hierarchy, $\mathsf{dM} = 0$.

We have constructed $\mathscr{L}_{(ijk)}$ in this way so that

$$\mathsf{dM} = \sum_{1\le i<j<k<l} P_{(ijkl)} \mathsf{d}x_i \wedge \mathsf{d}x_j \wedge \mathsf{d}x_k \wedge \mathsf{d}x_l. \tag{4.103}$$

has a double zero on the equations of the KP hierarchy. In particular, this $\mathscr{L}_{(ijk)}$ is such that

$$\begin{aligned} P_{(1ijk)} =& - \mathrm{D}_{x_k} \mathscr{L}_{(1ij)} - \mathrm{D}_{x_i} \mathscr{L}_{(1jk)} + \mathrm{D}_{x_j} \mathscr{L}_{(1ik)} + \mathrm{D}_{x_1} \mathscr{L}_{(ijk)} \\ =& - \operatorname{res} \{ \frac{1}{2}((L^i_+)_{x_j} - (L^j_+)_{x_i} + [L^i_+, L^j_+])(\phi_{x_k}\phi^{-1} + L^k_-) \\ &+ \frac{1}{2}((L^j_+)_{x_k} - (L^k_+)_{x_j} + [L^j_+, L^k_+])(\phi_{x_i}\phi^{-1} + L^i_-) \\ &+ \frac{1}{2}((L^k_+)_{x_i} - (L^i_+)_{x_k} + [L^k_+, L^i_+])(\phi_{x_j}\phi^{-1} + L^j_-)\}. \end{aligned} \tag{4.104}$$

Before we can show this to be the case, we shall require a number of lemmas. Lemmas 34 and 35 are closely related to Dickey's computations to obtain the Euler-Lagrange equations of his KP Lagrangian that we reproduced in Section 4.3. Lemma 36 then re-arranges some of the resulting terms to get us closer to (4.104), whilst Lemma 37 gives us the terms in (4.104) that do not contain any x_i, x_j or x_k derivatives. Also, it is important to note that each of Γ_{ijk}, $\Delta_{ij,k}$, $\Theta_{ij,k}$ and Λ_{ijk} are expressed in terms of the residue of commutators. Therefore they are all total x derivatives so can be integrated with respect to x to obtain a local expression for $\mathscr{L}_{(ijk)}$.

Lemma 34. *The Γ_{ijk} defined in (4.96) is such that*

$$\begin{aligned} &\mathrm{D}_{x_i}(\partial^k\phi^{-1}\phi_{x_j} - \partial^j\phi^{-1}\phi_{x_k}) + \mathrm{D}_{x_j}(\partial^i\phi^{-1}\phi_{x_k} - \partial^k\phi^{-1}\phi_{x_i}) + \mathrm{D}_{x_k}(\partial^j\phi^{-1}\phi_{x_i} - \partial^i\phi^{-1}\phi_{x_j}) \\ &= \frac{1}{2}(-(L^k)_{x_j}\phi_{x_i} + (L^j)_{x_k}\phi_{x_i} - (L^i)_{x_k}\phi_{x_j} + (L^k)_{x_i}\phi_{x_j} - (L^j)_{x_i}\phi_{x_k} + (L^i)_{x_j}\phi_{x_k})\phi^{-1} \\ &\quad + \Gamma_{ijk}. \end{aligned} \tag{4.105}$$

Proof of Lemma 34.

$$
\begin{aligned}
&\mathrm{D}_{x_i}(\partial^k\phi^{-1}\phi_{x_j}-\partial^j\phi^{-1}\phi_{x_k})+\mathrm{D}_{x_j}(\partial^i\phi^{-1}\phi_{x_k}-\partial^k\phi^{-1}\phi_{x_i})+\mathrm{D}_{x_k}(\partial^j\phi^{-1}\phi_{x_i}-\partial^i\phi^{-1}\phi_{x_j})\\
&=\partial^k\phi^{-1}\phi_{x_j}\phi^{-1}\phi_{x_i}+\partial^i\phi^{-1}\phi_{x_k}\phi^{-1}\phi_{x_j}+\partial^j\phi^{-1}\phi_{x_i}\phi^{-1}\phi_{x_k}\\
&\quad-\partial^k\phi^{-1}\phi_{x_i}\phi^{-1}\phi_{x_j}-\partial^i\phi^{-1}\phi_{x_j}\phi^{-1}\phi_{x_k}-\partial^j\phi^{-1}\phi_{x_k}\phi^{-1}\phi_{x_i}.
\end{aligned}
\tag{4.106}
$$

We now use commutators to get this in the form $(L^i)_{x_j}\phi_{x_k}\phi^{-1}$:

$$
\begin{aligned}
=&\frac{1}{2}(-\phi\partial^k\phi^{-1}\phi_{x_i}\phi^{-1}\phi_{x_j}\phi^{-1}+\phi\partial^j\phi^{-1}\phi_{x_i}\phi^{-1}\phi_{x_k}\phi^{-1}-\phi\partial^i\phi^{-1}\phi_{x_j}\phi^{-1}\phi_{x_k}\phi^{-1}\\
&+\phi\partial^k\phi^{-1}\phi_{x_j}\phi^{-1}\phi_{x_i}\phi^{-1}-\phi\partial^j\phi^{-1}\phi_{x_k}\phi^{-1}\phi_{x_i}\phi^{-1}+\phi\partial^i\phi^{-1}\phi_{x_k}\phi^{-1}\phi_{x_j}\phi^{-1})\\
&+\frac{1}{2}(-\phi_{x_j}\partial^k\phi^{-1}\phi_{x_i}\phi^{-1}+\phi_{x_k}\partial^j\phi^{-1}\phi_{x_i}\phi^{-1}-\phi_{x_k}\partial^i\phi^{-1}\phi_{x_j}\phi^{-1}\\
&+\phi_{x_i}\partial^k\phi^{-1}\phi_{x_j}\phi^{-1}-\phi_{x_i}\partial^j\phi^{-1}\phi_{x_k}\phi^{-1}+\phi_{x_j}\partial^i\phi^{-1}\phi_{x_k}\phi^{-1})\\
&+\frac{1}{2}([\phi\partial^k\phi^{-1}\phi_{x_i}\phi^{-1}\phi_{x_j},\phi^{-1}]+[\phi\partial^j\phi^{-1}\phi_{x_k}\phi^{-1}\phi_{x_i},\phi^{-1}]+[\phi\partial^i\phi^{-1}\phi_{x_j}\phi^{-1}\phi_{x_k},\phi^{-1}]\\
&-[\phi\partial^k\phi^{-1}\phi_{x_j}\phi^{-1}\phi_{x_i},\phi^{-1}]-[\phi\partial^j\phi^{-1}\phi_{x_i}\phi^{-1}\phi_{x_k},\phi^{-1}]-[\phi\partial^i\phi^{-1}\phi_{x_k}\phi^{-1}\phi_{x_j},\phi^{-1}]\\
&+[\phi_{x_j},\partial^k\phi^{-1}\phi_{x_i}\phi^{-1}]+[\phi_{x_i},\partial^j\phi^{-1}\phi_{x_k}\phi^{-1}]+[\phi_{x_k},\partial^i\phi^{-1}\phi_{x_j}\phi^{-1}]\\
&-[\phi_{x_i},\partial^k\phi^{-1}\phi_{x_j}\phi^{-1}]-[\phi_{x_k},\partial^j\phi^{-1}\phi_{x_i}\phi^{-1}]-[\phi_{x_j},\partial^i\phi^{-1}\phi_{x_k}\phi^{-1}])\\
=&\frac{1}{2}(-(L^k)_{x_j}\phi_{x_i}+(L^j)_{x_k}\phi_{x_i}-(L^i)_{x_k}\phi_{x_j}+(L^k)_{x_i}\phi_{x_j}-(L^j)_{x_i}\phi_{x_k}+(L^i)_{x_j}\phi_{x_k})\phi^{-1}\\
&+\Gamma_{ijk}.
\end{aligned}
\tag{4.107}
$$

□

Lemma 35. *The $\Delta_{ij,k}$ defined in (4.97) is such that*

$$
\begin{aligned}
&\mathrm{D}_{x_k}\,\mathrm{res}\left\{-\int_0^1 p^{-1}[(\phi_p\partial^i\phi_p^{-1})_+,(\phi_p\partial^j\phi_p^{-1})_+]\phi_p^{-1}dp\right\}\\
&=\mathrm{res}\,\{[(\phi\partial^i\phi^{-1})_+,(\phi\partial^j\phi^{-1})_+]\phi_{x_k}\ \phi^{-1}\}+\mathrm{res}\{\Delta_{ij,k}\}
\end{aligned}
\tag{4.108}
$$

Proof of Lemma 35. Since each $\mathscr{L}_{(1ij)}$ is autonomous, we notice that $\mathrm{D}_{x_k}\mathscr{L}_{(1ij)}=\delta\mathscr{L}_{(1ij)}|_{\delta\phi=\phi_{x_k}}$. It follows from Lemma 31 that the left hand side of (4.108) is equal to

$$\operatorname{res}\{[(\phi\partial^i\phi^{-1})_+, (\phi\partial^j\phi^{-1})_+]\phi_{x_k}\ \phi^{-1}\} - \mathrm{D}_x\, h_1|_{\delta\phi_{\tilde{p}}=(\phi_{\tilde{p}})_{x_k}} \tag{4.109}$$

evaluated at $p = 1$. We note that $\operatorname{res}\{\Delta_{ij,k}\}$ as defined in (4.97) is precisely $-\mathrm{D}_x\, h_1|_{\delta\phi_{\tilde{p}}=(\phi_{\tilde{p}})_{x_k}}$ evaluated at $p = 1$. That is,

$$\begin{aligned}\Delta_{ij,k} := -\int_0^1 p^{-1}([T[V,S],U] + [[T,U]_+S,V] + [U[V,S]_+,T] + [UT,[V,S]_+]\\ + [T[S,U],V] + [U,[T,V]_+S] + [V[S,U]_+,T] + [VT,[S,U]_+]\\ + [[U,V],TS] + [T,[U,V]S])\mathsf{d}p\end{aligned} \tag{4.110}$$

with $S = \phi_p^{-1}$, $T = (\phi_p)_{x_k}\phi_p^{-1}$, $U = (\phi_p\partial^i\phi_p^{-1})_+$ and $V = (\phi_p\partial^j\phi_p^{-1})_+$. □

Lemma 36. *The $\Theta_{ij,k}$ defined in (4.98) is such that*

$$\operatorname{res}\{[L_+^i, L_+^j]\phi_{x_k}\phi^{-1}\} = \frac{1}{2}\operatorname{res}\{[L_+^i, L_+^j]\phi_{x_k}\ \phi^{-1} + (L_+^j)_{x_k}L_-^i - (L_+^i)_{x_k}L_-^j\} + \operatorname{res}\{\Theta_{ij,k}\}. \tag{4.111}$$

Proof of Lemma 36. Using the identity

$$0 = [L^i, L^j]_+ = [L_+^i, L_+^j] + [L_+^i, L_-^j]_+ + [L_-^i, L_+^j]_+, \tag{4.112}$$

we see that

$$
\begin{aligned}
&\operatorname{res}\{[L_+^i, L_+^j]\phi_{x_k}\phi^{-1}\}\\
=&\frac{1}{2}\operatorname{res}\{[L_+^i, L_+^j]\phi_{x_k}\phi^{-1}\} - \frac{1}{2}\operatorname{res}\{[L_+^i, L_-^j]\phi_{x_k}\phi^{-1} + [L_-^i, L_+^j]\phi_{x_k}\phi^{-1}\}\\
=&\frac{1}{2}\operatorname{res}\{[L_+^i, L_+^j]\phi_{x_k}\phi^{-1}\} + \frac{1}{2}\operatorname{res}\{L_+^i\phi_{x_k}\phi^{-1}L_-^j - \phi_{x_k}\phi^{-1}L_+^iL_-^j\\
&\quad + \phi_{x_k}\phi^{-1}L_+^jL_-^i - L_+^j\phi_{x_k}\phi^{-1}L_-^i + [\phi_{x_k}\phi^{-1}, L_+^iL_-^j] + [L_-^j, L_+^i\phi_{x_k}\phi^{-1}]\\
&\quad + [L_+^j\phi_{x_k}\phi^{-1}, L_-^i] + [L_+^jL_-^i, \phi_{x_k}\phi^{-1}]\}\\
=&\frac{1}{2}\operatorname{res}\{[L_+^i, L_+^j]\phi_{x_k}\ \phi^{-1} + (L_+^j)_{x_k}L_-^i - (L_+^i)_{x_k}L_-^j\}\\
&\quad + \frac{1}{2}\operatorname{res}\{[\phi_{x_k}\phi^{-1}, L_+^iL_-^j] + [L_-^j, L_+^i\phi_{x_k}\phi^{-1}] + [L_+^j\phi_{x_k}\phi^{-1}, L_-^i]\\
&\quad + [L_+^jL_-^i, \phi_{x_k}\phi^{-1}]\}\\
=&\frac{1}{2}\operatorname{res}\{[L_+^i, L_+^j]\phi_{x_k}\ \phi^{-1} + (L_+^j)_{x_k}L_-^i - (L_+^i)_{x_k}L_-^j\} + \operatorname{res}\{\Theta_{ij,k}\},
\end{aligned}
\tag{4.113}
$$

where

$$
\Theta_{ij,k} := \frac{1}{2}([\phi_{x_k}\phi^{-1}, L_+^iL_-^j] + [L_-^j, L_+^i\phi_{x_k}\phi^{-1}] + [L_+^j\phi_{x_k}\phi^{-1}, L_-^i] + [L_+^jL_-^i, \phi_{x_k}\phi^{-1}]).
\tag{4.114}
$$

□

Lemma 37. *The identity*

$$
\operatorname{res}\{[L_+^i, L_+^j]L_-^k + [L_+^j, L_+^k]L_-^i + [L_+^k, L_+^i]L_-^j\} = -2\operatorname{res}\{\Lambda_{ijk}\},
\tag{4.115}
$$

holds.

Proof of Lemma 37. We consider $\operatorname{res}\{[L^i, L^j]L^k\}$, (which is clearly zero) and express this in terms of the positive and negative parts of the powers of L:

$$
\begin{aligned}
0 = \operatorname{res}\{[L^i, L^j]L^k\} = \operatorname{res}\{&[L_+^i, L_+^j]L_-^k + [L_-^i, L_+^j]L_+^k + [L_+^i, L_-^j]L_+^k\\
&+[L_-^i, L_-^j]L_+^k + [L_+^i, L_-^j]L_-^k + [L_-^i, L_+^j]L_-^k\}
\end{aligned}
\tag{4.116}
$$

The first three terms on the right hand side of (4.116) can be written as

$$\begin{aligned}&\operatorname{res}\{[L^i_+, L^j_+]L^k_- + [L^j_+, L^k_+]L^i_- + [L^k_+, L^i_+]L^j_- \\ &\quad +[L^i_-, L^j_+L^k_+] + [L^k_+, L^j_+L^i_-] + [L^i_+L^j_-, L^k_+] + [L^i_+L^k_+, L^j_-]\}\end{aligned} \tag{4.117}$$

whilst the final three terms on the right hand side of (4.116) can be written as

$$\begin{aligned}&\operatorname{res}\{\frac{1}{2}([L^j_-, L^k_+] + [L^j_+, L^k_-])L^i_- + \frac{1}{2}([L^k_-, L^i_+] + [L^k_+, L^i_-])L^j_- + \frac{1}{2}([L^i_-, L^j_+] \\ &+[L^i_+, L^j_-])L^k_- + \frac{1}{2}([L^i_-, L^j_-L^k_+] + [L^k_+, L^j_-L^i_-] + [L^i_-L^j_-, L^k_+] + [L^i_-L^k_+, L^j_-] \\ &+[L^i_+L^j_-, L^k_-] + [L^i_+L^k_-, L^j_-] + [L^i_-, L^j_+L^k_-] + [L^k_-, L^j_+L^i_-]\}.\end{aligned} \tag{4.118}$$

By (4.112), this is equal to

$$\begin{aligned}&\frac{1}{2}\operatorname{res}\{-[L^j_+, L^k_+]L^i_- - [L^k_+, L^i_+]L^j_- - [L^i_+, L^j_+]L^k_- + [L^i_-, L^j_-L^k_+] \\ &\qquad + [L^k_+, L^j_-L^i_-] + [L^i_-L^j_-, L^k_+] + [L^i_-L^k_+, L^j_-] + [L^i_+L^j_-, L^k_-] \\ &\qquad + [L^i_+L^k_-, L^j_-] + [L^i_-, L^j_+L^k_-] + [L^k_-, L^j_+L^i_-]\}.\end{aligned} \tag{4.119}$$

Since (4.117) and (4.119) sum to zero, it follows that

$$\begin{aligned}&\operatorname{res}\{[L^i_+, L^j_+]L^k_- + [L^j_+, L^k_+]L^i_- + [L^k_+, L^i_+]L^j_-\} \\ &= -\operatorname{res}\{2[L^i_-, L^j_+L^k_+] + 2[L^k_+, L^j_+L^i_-] + 2[L^i_+L^j_-, L^k_+] + 2[L^i_+L^k_+, L^j_-] \\ &\quad + [L^i_-, L^j_-L^k_+] + [L^k_+, L^j_-L^i_-] + [L^i_-L^j_-, L^k_+] + [L^i_-L^k_+, L^j_-] \\ &\quad + [L^i_+L^j_-, L^k_-] + [L^i_+L^k_-, L^j_-] + [L^i_-, L^j_+L^k_-] + [L^k_-, L^j_+L^i_-]\}\end{aligned} \tag{4.120}$$

which simplifies to

$$\begin{aligned}&-\operatorname{res}\{[L^i_+L^j_- - L^j_+L^i_-, L^k] + [L^k_+L^i_-, L^j_+] + [L^i_+, L^k_+L^j_-] + [L^i_-, L^{j+k}] + [L^{i+k}, L^j_-]\} \\ &= -2\operatorname{res}\{\Lambda_{ijk}\}\end{aligned} \tag{4.121}$$

where

$$\Lambda_{ijk} := \frac{1}{2}([L^i_+ L^j_- - L^j_+ L^i_-, L^k] + [L^k_+ L^i_-, L^j_+] + [L^i_+, L^k_+ L^j_-] + [L^i_-, L^{j+k}] + [L^{i+k}, L^j_-]). \tag{4.122}$$

□

Proof of Theorem 33. Since Γ_{ijk}, $\Delta_{ij,k}$, $\Theta_{ij,k}$ and Λ_{ijk} are composed entirely of commutators, it follows from Lemma 25 that

$$\mathscr{L}_{(ijk)} = \int \operatorname{res}\{\Gamma_{ijk} + \Delta_{ij,k} + \Delta_{jk,i} + \Delta_{ki,j} + \Theta_{ij,k} + \Theta_{jk,i} + \Theta_{ki,j} + \Lambda_{ijk}\}\mathsf{d}x \tag{4.123}$$

is local. Since the multiform Euler-Lagrange equations arising from $\delta\mathsf{dM} = 0$ include the Euler-Lagrange equations of the $\mathscr{L}_{1ij}$, we know that the set of equations given by $\delta\mathsf{dM} = 0$ includes all KP equations of the form

$$(L^i_+)_{x_j} - (L^j_+)_{x_i} + [L^i_+, L^j_+] = 0. \tag{4.124}$$

By Corollary 28, $\delta\mathsf{dM} = 0$ also gives us KP equations of the form

$$\phi_{x_i} + L^i_-\phi = 0. \tag{4.125}$$

In order to proceed, we again use the notation $P_{(ijkl)}$ such that

$$\mathsf{dM} = \sum_{1\le i<j<k<l} P_{(ijkl)}\mathsf{d}x_i \wedge \mathsf{d}x_j \wedge \mathsf{d}x_k \wedge \mathsf{d}x_l. \tag{4.126}$$

Combining the results of Lemmas 34 to 37, we see that

$$\begin{aligned} P_{(1ijk)} =& - \mathrm{D}_{x_k}\mathscr{L}_{(1ij)} - \mathrm{D}_{x_i}\mathscr{L}_{(1jk)} + \mathrm{D}_{x_j}\mathscr{L}_{(1ik)} + \mathrm{D}_{x_1}\mathscr{L}_{(ijk)} \\ =& - \operatorname{res}\{\frac{1}{2}((L^i_+)_{x_j} - (L^j_+)_{x_i} + [L^i_+, L^j_+])(\phi_{x_k}\phi^{-1} + L^k_-) \\ &+ \frac{1}{2}((L^j_+)_{x_k} - (L^k_+)_{x_j} + [L^j_+, L^k_+])(\phi_{x_i}\phi^{-1} + L^i_-) \\ &+ \frac{1}{2}((L^k_+)_{x_i} - (L^i_+)_{x_k} + [L^k_+, L^i_+])(\phi_{x_j}\phi^{-1} + L^j_-)\}, \end{aligned} \tag{4.127}$$

and since equations of the form $(L^i_+)_{x_j} - (L^j_+)_{x_i} + [L^i_+, L^j_+] = 0$ and $\phi_{x_i}\phi^{-1} + L^i_- = 0$ are both equations of the KP hierarchy, P_{1ijk} has a double zero on the hierarchy.

In order to complete the proof, we must show that for

$$P_{(ijkl)} = \mathrm{D}_{x_i}\, \mathscr{L}_{(jkl)} - \mathrm{D}_{x_j}\, \mathscr{L}_{(ikl)} + \mathrm{D}_{x_k}\, \mathscr{L}_{(ijl)} - \mathrm{D}_{x_l}\, \mathscr{L}_{(ijk)}, \tag{4.128}$$

$\delta P_{(ijkl)} = 0$ and $P_{(ijkl)} = 0$ on the equations of the KP hierarchy. We require that $\delta P_{(ijkl)} = 0$ on the equations of the KP hierarchy in order to confirm that $\delta P_{(ijkl)} = 0$ does not define any equations that are not part of the KP hierarchy, and we require that $P_{(ijkl)} = 0$ in order that $\mathsf{dM} = 0$ on the equations of the hierarchy. To show this, we first note that from its definition in terms of the $\mathscr{L}_{(ijk)}$, $P_{(ijkl)}$ is a polynomial with no constant term, in $(\varphi^{(n)}_\beta)_I$ where n gives the order of derivative with respect to x and I is a multi-index representing derivatives with respect to x_i for $i > 1$. Also, since $\mathsf{d}^2\mathsf{M}$ is identically zero,

$$D_x P_{(ijkl)} = \mathrm{D}_{x_i}\, P_{(1jkl)} - \mathrm{D}_{x_j}\, P_{(1ikl)} + \mathrm{D}_{x_k}\, P_{(1ijl)} - \mathrm{D}_{x_l}\, P_{(1ijk)}. \tag{4.129}$$

This is an identity, so we do not require the φ_β to satisfy the equations of the KP hierarchy for this to hold. Since each of $P_{(1ijk)}$, $P_{(1ikl)}$, $P_{(1ijl)}$, and $P_{(1jkl)}$ has a double zero on the equations of the KP hierarchy, it follows that $\mathrm{D}_x\, P_{(ijkl)}$ also has a double zero on the equations of the KP hierarchy, and therefore that

$$\frac{\partial}{\partial(\varphi^{(n)}_\beta)_I}\, \mathrm{D}_x\, P_{(ijkl)} = 0 \tag{4.130}$$

for all I and n. Using the identity

$$\frac{\partial}{\partial(\varphi^{(n+1)}_\beta)_I}\, \mathrm{D}_x\, P_{(ijkl)} = \mathrm{D}_x\, \frac{\partial}{\partial(\varphi^{(n+1)}_\beta)_I} P_{(ijkl)} + \frac{\partial}{\partial(\varphi^{(n)}_\beta)_I} P_{(ijkl)} \tag{4.131}$$

we see that for a fixed choice of I, if n is the largest such that $(\varphi^{(n)}_\beta)_I$ appears in $P_{(ijkl)}$, then

$$\frac{\partial}{\partial(\varphi_\beta^{(n)})_I} P_{(ijkl)} = 0 \tag{4.132}$$

on the equations of the KP hierarchy. It also follows from (4.131) that, on the equations of the KP hierarchy, if

$$\frac{\partial}{\partial(\varphi_\beta^{(n)})_I} P_{(ijkl)} = 0 \quad \text{then} \quad \frac{\partial}{\partial(\varphi_\beta^{(n-1)})_I} P_{(ijkl)} = 0. \tag{4.133}$$

Therefore, on the equations of the KP hierarchy,

$$\frac{\partial}{\partial(\varphi_\beta^{(n)})_I} P_{(ijkl)} = 0 \tag{4.134}$$

for all I and n, so $\delta P_{(ijkl)} = 0$. Since $P_{(ijkl)}$ is autonomous, (4.134) tells us that

$$\mathrm{D}_{x_i} P_{(ijkl)} = 0 \quad \forall i > 0 \tag{4.135}$$

so $P_{(ijkl)}$ is constant, and since the KP hierarchy admits the zero solution, we conclude that this constant is zero, and $P_{(ijkl)} = 0$ on the equations of the KP hierarchy.

Thus, the set of equations defined by $\delta \mathsf{dM} = 0$ is precisely the full set of equations of the KP hierarchy, and on these equations, $\mathsf{dM} = 0$, so M is a Lagrangian multiform for the KP hierarchy. □

4.4.2 An alternative KP Lagrangian multiform

In the KP Lagrangian multiform of Theorem 33, we used Dickey's KP Lagrangian for the $\mathscr{L}_{(1ij)}$, and the Lagrangian defined in (4.102) for the $\mathscr{L}_{(ijk)}$ when $1 < i, j, k$. Here we present an alternative version of the KP Lagrangian multiform in which every Lagrangian is of the same type.

Theorem 38. *The differential 3-form*

$$\widetilde{\mathsf{M}} = \sum_{1 \le i < j < k} \widetilde{\mathscr{L}_{(ijk)}} \; \mathsf{d}x_i \wedge \mathsf{d}x_j \wedge \mathsf{d}x_k \tag{4.136}$$

where

$$\widetilde{\mathscr{L}}_{(ijk)} = \int \text{res}\left\{\Gamma_{ijk} + \Delta_{ij,k} + \Delta_{jk,i} + \Delta_{ki,j} + \Theta_{ij,k} + \Theta_{jk,i} + \Theta_{ki,j} + \Lambda_{ijk}\right\} dx \quad (4.137)$$

(i.e., the Lagrangian defined in (4.102)), is a Lagrangian multiform for the KP hierarchy.

Proof. We recall that in Section 4.2 we identified x_1 with x. For now we choose not to do so and treat them as separate co-ordinates. This allows us to consider a 3-form M_1 such that the coefficient of $\mathsf{d}x \wedge \mathsf{d}x_i \wedge \mathsf{d}x_j$ with $1 \leq i < j$ is Dickey's KP Lagrangian $\mathscr{L}_{(xij)}$, whilst the coefficient of $\mathsf{d}x_i \wedge \mathsf{d}x_j \wedge \mathsf{d}x_k$ with $1 \leq i < j < k$ is the Lagrangian $\mathscr{L}_{(ijk)}$ defined in (4.102). It then follows from the proof of Theorem 33 that this is also a Lagrangian multiform for the KP hierarchy. The multiform Euler-Lagrange equations for M_1 will be the multiform Euler-Lagrange equations of M plus an additional set of equations that tell us to equate derivatives with respect to x_1 with derivatives with respect to x, arising from equations of the form

$$(L_+)_{x_j} - (L_+^j)_{x_1} + [L_+, L_+^j] = 0, \quad (4.138)$$

and dM_1 will have a double zero on these equations. We now define M_2 to be the restriction of M_1 to a submanifold with co-ordinates $x_1, x_2, x_3, \ldots$, obtained by fixing $x = c$, a constant. It follows that dM_2 still has a double zero on this same set of equations. If we then equate x_1 with x in M_2, we get $\widetilde{\mathsf{M}}$ and it follows that $\mathsf{d}\widetilde{\mathsf{M}}$ has a double zero on the equations of the KP hierarchy. Therefore, the equations defined by $\delta \mathsf{d}\widetilde{\mathsf{M}} = 0$ are a subset of the equations of the KP hierarchy.

To complete the proof that $\widetilde{\mathsf{M}}$ is a Lagrangian multiform for the KP hierarchy, we must show that the equations defined by $\delta \mathsf{d}\widetilde{\mathsf{M}} = 0$ are precisely the full set of equations of the KP hierarchy. We shall do this by showing that the Euler-Lagrange equations of the $\mathscr{L}_{(1jk)}$ Lagrangians give us these equations.

We first consider the coefficient $P_{(xijk)}$ from dM_1.

$$
\begin{aligned}
P_{(xijk)} =& -\mathrm{D}_{x_k}\mathscr{L}_{(xij)} - \mathrm{D}_{x_i}\mathscr{L}_{(xjk)} + \mathrm{D}_{x_j}\mathscr{L}_{(xik)} + \mathrm{D}_x\mathscr{L}_{(ijk)} \\
=& -\operatorname{res}\{\frac{1}{2}((L^i_+)_{x_j} - (L^j_+)_{x_i} + [L^i_+, L^j_+])(\phi_{x_k}\phi^{-1} + L^k_-) \\
&+ \frac{1}{2}((L^j_+)_{x_k} - (L^k_+)_{x_j} + [L^j_+, L^k_+])(\phi_{x_i}\phi^{-1} + L^i_-) \\
&+ \frac{1}{2}((L^k_+)_{x_i} - (L^i_+)_{x_k} + [L^k_+, L^i_+])(\phi_{x_j}\phi^{-1} + L^j_-)\},
\end{aligned}
\tag{4.139}
$$

so in the case where $i = 1$ this becomes

$$
\begin{aligned}
P_{(x1jk)} =& -\mathrm{D}_{x_k}\mathscr{L}_{(x1j)} - \mathrm{D}_{x_1}\mathscr{L}_{(xjk)} + \mathrm{D}_{x_j}\mathscr{L}_{(x1k)} + \mathrm{D}_x\mathscr{L}_{(1jk)} \\
=& -\operatorname{res}\{\frac{1}{2}(-(L^j_+)_{x_1} + (L^j_+)_x)(\phi_{x_k}\phi^{-1} + L^k_-) \\
&+ \frac{1}{2}((L^j_+)_{x_k} - (L^k_+)_{x_j} + [L^j_+, L^k_+])(\phi_{x_1}\phi^{-1} + L_-) \\
&+ \frac{1}{2}((L^k_+)_{x_1} - (L^k_+)_x)(\phi_{x_j}\phi^{-1} + L^j_-)\}
\end{aligned}
\tag{4.140}
$$

since $L_+ = \partial$. If we equate x_1 and x in this expression then this becomes zero. This is obvious in the first and third line; for the second line, we note that $L_- = (\phi\partial\phi^{-1})_- = (\partial - \phi_x\phi^{-1})_- = -\phi_x\phi^{-1}$. We now define

$$
\bar{\mathscr{L}}_{(xij)} = \mathscr{L}_{(xij)}|_{x\to x_1}
\tag{4.141}
$$

and consider the 2-form

$$
\mathsf{L} = \bar{\mathscr{L}}_{(x1j)}\mathsf{d}x_1 \wedge \mathsf{d}x_j + \bar{\mathscr{L}}_{(x1k)}\mathsf{d}x_1 \wedge \mathsf{d}x_k + (\bar{\mathscr{L}}_{(xjk)} - \bar{\mathscr{L}}_{(1jk)})\mathsf{d}x_j \wedge \mathsf{d}x_k.
\tag{4.142}
$$

By construction, $\mathsf{dL} = -P_{(x1jk)}|_{x\to x_1} = 0$. Then, by Corollary 4, the variational derivative of each of the Lagrangian coefficients in L is zero. Therefore,

$$
\frac{\delta}{\delta\phi}(\bar{\mathscr{L}}_{(xjk)} - \bar{\mathscr{L}}_{(1jk)}) = 0
\tag{4.143}
$$

so

$$\frac{\delta\bar{\mathscr{L}}_{(1jk)}}{\delta\phi} = \frac{\delta\bar{\mathscr{L}}_{(xjk)}}{\delta\phi} = \{\phi^{-1}((L_+^i)_{x_j} - (L_+^j)_{x_i} + [L_+^i, L_+^j])\}_+. \tag{4.144}$$

Since $\bar{\mathscr{L}}_{(1jk)} = \widetilde{\mathscr{L}}_{(1jk)}$, all equations of the KP hierarchy are consequences of $\delta \mathsf{d}\widetilde{\mathsf{M}} = 0$, so $\widetilde{\mathsf{M}}$ is a Lagrangian multiform for the KP hierarchy.

□

4.5 Reduction to multiforms for the Gelfand-Dickey hierarchy

In order to reduce KP to the n^{th} Gelfand-Dickey hierarchy, we imposed the constraint that $L_-^n = 0$. Since, by (4.38), $\phi_{x_n} = -L_-^n\phi$, we can achieve this in the Lagrangian multiform by setting $\phi_{x_n} = 0$. A simple way to obtain a Lagrangian multiform for the n^{th} Gelfand-Dickey hierarchy is to leave the KP multiform obtained in Section 4.4 unchanged and impose this constraint on the Euler-Lagrange equations. A more satisfactory approach involves setting $\phi_{x_n} = 0$ in (4.127) to obtain

$$\begin{aligned} &\mathrm{D}_{x_n}\hat{\mathscr{L}}_{(1ij)} + \mathrm{D}_{x_i}\hat{\mathscr{L}}_{(1jn)} - \mathrm{D}_{x_j}\hat{\mathscr{L}}_{(1in)} - \mathrm{D}_{x_1}\hat{\mathscr{L}}_{(ijn)} \\ &\quad = \mathrm{res}\,\{\frac{1}{2}((L_+^i)_{x_j} - (L_+^j)_{x_i} + [L_+^i, L_+^j])L_-^k \\ &\qquad + \frac{1}{2}(-(L_+^n)_{x_j} + [L_+^j, L_+^n])(\phi_{x_i}\phi^{-1} + L_-^i) \\ &\qquad + \frac{1}{2}((L_+^n)_{x_i} + [L_+^n, L_+^i])(\phi_{x_j}\phi^{-1} + L_-^j)\}. \end{aligned} \tag{4.145}$$

If we can find Lagrangians $\hat{\mathscr{L}}_{(ijk)}$ such that (4.145) holds, then the constraint $L_-^n = 0$ will be naturally incorporated into the multiform Euler-Lagrange equations, giving us the n^{th} Gelfand-Dickey hierarchy. The $\hat{\mathscr{L}}$ are not uniquely defined by this expression, but a natural choice would be

$$\hat{\mathscr{L}}_{(1ij)} = 0, \tag{4.146a}$$

$$\hat{\mathscr{L}}_{(1in)} = \text{res}\left\{ -\int_0^1 p^{-1}[(\phi_p\partial^i\phi_p^{-1})_+, (\phi_p\partial^n\phi_p^{-1})_+]\phi_p^{-1}\mathsf{d}p + \partial^n\phi^{-1}\phi_{x_i}\right\}, \quad (4.146b)$$

$$\hat{\mathscr{L}}_{(1jn)} = \text{res}\left\{ -\int_0^1 p^{-1}[(\phi_p\partial^j\phi_p^{-1})_+, (\phi_p\partial^n\phi_p^{-1})_+]\phi_p^{-1}\mathsf{d}p + \partial^n\phi^{-1}\phi_{x_j}\right\}, \quad (4.146c)$$

and

$$\hat{\mathscr{L}}_{(ijn)} = \int \{\hat{\Gamma}_{ijn} + \Delta_{jn,i} + \Delta_{ni,j} + \Theta_{jn,i} + \Theta_{ni,j} + \Lambda_{ijn}\}\mathsf{d}x \qquad (4.146d)$$

with the constant of integration set to zero, where

$$\begin{aligned}\hat{\Gamma}_{ijn} = \frac{1}{2}\,\text{res}\,\{&[\phi\partial^n\phi^{-1}\phi_{x_i}\phi^{-1}\phi_{x_j}, \phi^{-1}] - [\phi\partial^n\phi^{-1}\phi_{x_j}\phi^{-1}\phi_{x_i}, \phi^{-1}] \\ &+[\phi_{x_j}, \partial^n\phi^{-1}\phi_{x_i}\phi^{-1}] - [\phi_{x_i}, \partial^n\phi^{-1}\phi_{x_j}\phi^{-1}]\}\end{aligned} \qquad (4.147)$$

is equal to Γ_{ijn} with $\phi_{x_n} = 0$. The KP multiform (4.88) reduces to

$$\mathsf{M}_{(n)} = \sum_{1\leq i<j} \hat{\mathscr{L}}_{(ijn)}\mathsf{d}x_i \wedge \mathsf{d}x_j \wedge \mathsf{d}x_n. \qquad (4.148)$$

This multiform does not contain any derivatives with respect to x_n, so does not allow any motion in the x_n direction, and is equivalent (i.e., produces identical multiform Euler-Lagrange equations) to

$$\hat{\mathsf{M}}_{(n)} = \sum_{1\leq i<j} \hat{\mathscr{L}}_{(ijn)}\mathsf{d}x_i \wedge \mathsf{d}x_j, \qquad (4.149)$$

a Lagrangian 2-form for the n^{th} Gelfand-Dickey hierarchy. As was the case for the KP Lagrangian multiform, a Lagrangian multiform with all coefficients in the form of (4.146d) is also a Lagrangian multiform for the n^{th} Gelfand-Dickey hierarchy.

4.6 Conclusion

The Lagrangian multiforms we have presented constitute, in our view, the first instance of establishing the integrability of the KP hierarchy at the Lagrangian level. In contrast to the Lagrangian multiform for KP hierarchy (up to the x_4 flow) that was presented in [1], we now have explicit formulae for the constituent Lagrangians of the Lagrangian multiform for the complete hierarchy, and the constituent Lagrangians are fully local. In addition, whilst for the Lagrangian multiform in [1] the x_1 and x_2 co-ordinates held a special status (i.e., were treated differently to the other co-ordinates), for the Lagrangian multiform presented here, only x_1 holds a special status. Aspirations for future work include obtaining a Lagrangian multiform for KP that treats every co-ordinate (including x) on an equal footing, and also to connect the continuous KP Lagrangian multiform from this chapter with the discrete KP Lagrangian multiform given in [27].

Chapter 5

Conclusion

5.1 Summary

The past four years have seen a considerable number of advances related to Lagrangian multiforms, both in terms of new examples of Lagrangian multiforms, and also a deepening of understanding of the mathematics that underpins them. The main new results in this thesis are as follows.

In Chapter 1 we give new proofs for the multiform Euler Lagrange equations for both continuous and discrete k-forms. In the continuous case, the multiform Euler-Lagrange equations for a Lagrangian k-form were first found in [12]. However this new proof establishes for the first time the equivalence between the multiform Euler-Lagrange equations in terms of the Lagrangian coefficients, and variational derivatives of the coefficients of dL. In the discrete and semi-discrete cases, the multiform Euler-Lagrange equations that we present are a new result.

In Chapter 2 we demonstrate the link between Lagrangian multiforms and variational symmetries that arises from Noether's theorem. The connection between Lagrangian multiforms from variational symmetries was first explored in the context of 1 and 2-forms in [20] and [21]. The approach presented in this chapter is more general in that it applies to Lagrangian forms of any order, and also applies in the discrete and semi-discrete context. By applying this approach to the KP hierarchy, we were able to obtain the first example of a continuous Lagrangian 3-form.

In Chapter 3 we show that the Lagrangian density proposed in [5] can be extended naturally to a Lagrangian 2-form structure. This makes the multidimensional consistency of the corresponding Zakharov-Mikhailov system manifest at the Lagrangian level. We also show that, our Lagrangian multiform leads to a variational formulation of the underlying Lax pair itself. In fact, the 2-form structure leads naturally to the Lagrangian description for a *Lax triplet* (or more generally a *Lax multiplet*), and thus we can recover the Lax pair from the Lagrangian multiforms associated with the Zakharov-Mikhailov Lagrangians.

In Chapter 4 we obtain a Lagrangian multiform for the complete KP hierarchy. This is the first ever example of a continuous Lagrangian 3-form for a complete integrable hierarchy. Then, based on the reduction of KP to the Gelfand-Dickey hierarchy, we perform a reduction on the KP Lagrangian multiform to obtain Lagrangian multiforms for each of the integrable hierarchies that comprise the Gelfand-Dickey hierarchy.

5.2 Outlook

The theory of Lagrangian multiforms is still in its infancy, with many aspects yet to be studied in any significant detail. New examples of Lagrangian multiforms for integrable systems are being found on a regular basis, and there is no reason to think that this will not continue. At the same time, the understanding of the theory behind Lagrangian multiforms continues to advance. In Chapter 2 we showed how Noether's theorem links variational symmetries and Lagrangian multiforms. There remains some scope to extend the main result of this chapter; for example, the Lagrangian multiform of Chapter 3 cannot be obtained using a variational symmetries approach, but can be obtained using a similar approach where the vector $\mathrm{pr}\,\mathbf{v}_{Q_i}\mathscr{L}_{(jk)}$ is not a divergence, but $\mathrm{pr}\,\mathbf{v}_{Q_i}\mathscr{L}_{(jk)} + \mathrm{pr}\,\mathbf{v}_{Q_j}\mathscr{L}_{(ki)} + \mathrm{pr}\,\mathbf{v}_{Q_k}\mathscr{L}_{(ij)}$ is. This type of extension to the ideas in Chapter 2 may lead naturally to the concept of a variational symmetry of a Lagrangian multiform, and thereby an extension of Noether's theorem to the Lagrangian multiform case.

The Lagrangian multiform obtained in Chapter 3 appears to be a rather self-contained result with little scope for further work. However the Zakharov–Mikhailov

Lagrangian was recently linked to Chern–Simons theory [45], opening the possibility of extending this result. Also, obtaining a discrete analogue of the Zakharov–Mikhailov Lagrangian multiform remains an open problem.

It is tempting to claim that, as a result of the Lagrangian multiform found in Chapter 4, the KP Lagrangian multiform is done. However, the Lagrangian multiform we obtained is rather cumbersome and also gives a special status to the x co-ordinate (in that derivatives with respect to x appear in all Lagrangian coefficients). As a result, there remains the scope to find an improved Lagrangian multiform for the KP hierarchy.

Looking more generally at Lagrangian multiforms, there are many avenues left to explore. For example, it has been proposed that the existence of a Lagrangian multiform with certain properties could be used as a definition of integrability. This might be done by linking existing results (e.g., those that relate the existence of sufficient symmetries to integrability) to Lagrangian multiforms, or perhaps it may be possible to show that any non-trivial Lagrangian multiform that is sufficiently large (i.e., that has enough non-zero Lagrangian coefficients) leads to an integrable system. Also, so far only tentative steps have been made to link Lagrangian multiforms to quantum setting via Feynman path integrals [46]. Much work remains to develop this idea more fully.

Appendix A

Proof of multiform Euler-Lagrange equations for a Lagrangian 2-form

The following is the proof of the multiform Euler-Lagrange equations for a Lagrangian 2-form as originally presented in [2].

We consider the Lagrangian 2-form

$$\mathsf{L} = \mathscr{L}_{(ij)}\mathsf{d}\xi_i \wedge \mathsf{d}\xi_j + \mathscr{L}_{(jk)}\mathsf{d}\xi_j \wedge \mathsf{d}\xi_k + \mathscr{L}_{(ki)}\mathsf{d}\xi_k \wedge \mathsf{d}\xi_i \tag{A.1}$$

which contains terms up to N^{th} order derivatives of $\boldsymbol{\varphi}$, (i.e. such that $|I|\leq N$). Let B be an arbitrary three dimensional ball with surface ∂B. We consider the action functional S over the closed surface ∂B such that

$$S[\boldsymbol{\varphi}] = \oint_{\partial B} \mathsf{L} \tag{A.2}$$

We then apply Stokes' theorem to write S in terms of an integral over B:

$$S[\boldsymbol{\varphi}] = \int_{B} \mathsf{dL} \tag{A.3}$$

and we look for solutions of

$$\delta S = \int_B \delta \mathsf{dL} = 0 \tag{A.4}$$

Since this must hold for arbitrary variations (i.e. with no boundary constraints) for every arbitrary ball B, it follows that on solutions $\boldsymbol{\varphi}$ of our system, $\delta\mathsf{dL} = 0$. Up to this point, we have used the same argument as the one given in the proof of Proposition 2.2 in [11]. The statement that $\delta\mathsf{dL} = 0$ is equivalent to the statement that

$$\frac{\partial \mathsf{dL}}{\partial \boldsymbol{\varphi}_I} = 0 \quad \forall\ I \tag{A.5}$$

The scheme of this proof from here is to first use (A.5) to show that

$$\frac{\delta \mathscr{L}_{(ij)}}{\delta \boldsymbol{\varphi}_{I\setminus k}} + \frac{\delta \mathscr{L}_{(jk)}}{\delta \boldsymbol{\varphi}_{I\setminus i}} + \frac{\delta \mathscr{L}_{(ki)}}{\delta \boldsymbol{\varphi}_{I\setminus j}} = 0 \tag{A.6}$$

holds for $|I| > N$. We then use an inductive argument to show that it holds in all cases by showing that if it holds for $|I| > M$ then it also holds for $|I| = M$.

We begin by noticing that for $|I| \geq N+2$, (A.6) holds. In fact all terms are zero since, by definition, there are no $N+1^{th}$ order derivatives in our multiform. We now consider the relation $\dfrac{\partial \mathsf{dL}}{\partial \boldsymbol{\varphi}_I} = 0$ in the case where $|I| = N+1$. In this case we find that

$$\frac{\partial \mathsf{dL}}{\partial \boldsymbol{\varphi}_I} = \frac{\partial \mathscr{L}_{(ij)}}{\partial \boldsymbol{\varphi}_{I\setminus k}} + \frac{\partial \mathscr{L}_{(jk)}}{\partial \boldsymbol{\varphi}_{I\setminus i}} + \frac{\partial \mathscr{L}_{(ki)}}{\partial \boldsymbol{\varphi}_{I\setminus j}} \tag{A.7}$$

since there are no $N+1^{th}$ order derivatives in $\mathscr{L}_{(ij)}$, $\mathscr{L}_{(jk)}$ and $\mathscr{L}_{(ki)}$. By setting this equal to zero, we see that (A.6) holds in the case where $|I| = N+1$.

Our inductive hypothesis is that (A.6) holds for $|I| > M$. We now consider the relation $\dfrac{\partial \mathsf{dL}}{\partial \boldsymbol{\varphi}_I} = 0$ in the case where $|I| = M$. We first make use of the easily verified relation

$$\frac{\partial (\mathrm{D}_i \mathscr{L}_{(jk)})}{\partial \boldsymbol{\varphi}_I} = \mathrm{D}_i \frac{\partial \mathscr{L}_{(jk)}}{\partial \boldsymbol{\varphi}_I} + \frac{\partial \mathscr{L}_{(jk)}}{\partial \boldsymbol{\varphi}_{I\setminus i}} \tag{A.8}$$

along with similar relations for $\mathscr{L}_{(ki)}$ and $\mathscr{L}_{(ij)}$ to get that

$$\frac{\partial \mathsf{dL}}{\partial \boldsymbol{\varphi}_I} = \frac{\partial \mathscr{L}_{(jk)}}{\partial \boldsymbol{\varphi}_{I\setminus i}} + \mathrm{D}_i \frac{\partial \mathscr{L}_{(jk)}}{\partial \boldsymbol{\varphi}_I} + \frac{\partial \mathscr{L}_{(ki)}}{\partial \boldsymbol{\varphi}_{I\setminus j}} + \mathrm{D}_j \frac{\partial \mathscr{L}_{(ki)}}{\partial \boldsymbol{\varphi}_I} + \frac{\partial \mathscr{L}_{(ij)}}{\partial \boldsymbol{\varphi}_{I\setminus k}} + \mathrm{D}_k \frac{\partial \mathscr{L}_{(ij)}}{\partial \boldsymbol{\varphi}_I} \tag{A.9}$$

when expressed in terms of $\mathscr{L}_{(ij)}$, $\mathscr{L}_{(jk)}$ and $\mathscr{L}_{(ki)}$. We now make use of the relation

$$\frac{\delta \mathscr{L}_{(ij)}}{\delta \boldsymbol{\varphi}_I} = \frac{\partial \mathscr{L}_{(ij)}}{\partial \boldsymbol{\varphi}_I} - \mathrm{D}_i \frac{\delta \mathscr{L}_{(ij)}}{\delta \boldsymbol{\varphi}_{Ii}} - \mathrm{D}_j \frac{\delta \mathscr{L}_{(ij)}}{\delta \boldsymbol{\varphi}_{Ij}} - \mathrm{D}_i \mathrm{D}_j \frac{\delta \mathscr{L}_{(ij)}}{\delta \boldsymbol{\varphi}_{Iij}} \tag{A.10}$$

along with similar relations for $\mathscr{L}_{(jk)}$ and $\mathscr{L}_{(ki)}$ to expand (A.9) to get that

$$\begin{aligned}\frac{\partial \mathsf{dL}}{\partial \boldsymbol{\varphi}_I} =& \frac{\partial \mathscr{L}_{(jk)}}{\partial \boldsymbol{\varphi}_{I\setminus i}} + \mathrm{D}_i \left(\frac{\delta \mathscr{L}_{(jk)}}{\delta \boldsymbol{\varphi}_I} + \mathrm{D}_j \frac{\delta \mathscr{L}_{(jk)}}{\delta \boldsymbol{\varphi}_{Ij}} + \mathrm{D}_k \frac{\delta \mathscr{L}_{(jk)}}{\delta \boldsymbol{\varphi}_{Ik}} + \mathrm{D}_j \mathrm{D}_k \frac{\delta \mathscr{L}_{(jk)}}{\delta \boldsymbol{\varphi}_{Ijk}} \right) \\ &+ \frac{\partial \mathscr{L}_{(ki)}}{\partial \boldsymbol{\varphi}_{I\setminus j}} + \mathrm{D}_j \left(\frac{\delta \mathscr{L}_{(ki)}}{\delta \boldsymbol{\varphi}_I} + \mathrm{D}_k \frac{\delta \mathscr{L}_{(ki)}}{\delta \boldsymbol{\varphi}_{Ik}} + \mathrm{D}_i \frac{\delta \mathscr{L}_{(ki)}}{\delta \boldsymbol{\varphi}_{Ii}} + \mathrm{D}_k \mathrm{D}_i \frac{\delta \mathscr{L}_{(ki)}}{\delta \boldsymbol{\varphi}_{Iki}} \right) \\ &+ \frac{\partial \mathscr{L}_{(ij)}}{\partial \boldsymbol{\varphi}_{I\setminus k}} + \mathrm{D}_k \left(\frac{\delta \mathscr{L}_{(ij)}}{\delta \boldsymbol{\varphi}_I} + \mathrm{D}_i \frac{\delta \mathscr{L}_{(ij)}}{\delta \boldsymbol{\varphi}_{Ii}} + \mathrm{D}_j \frac{\delta \mathscr{L}_{(ij)}}{\delta \boldsymbol{\varphi}_{Ij}} + \mathrm{D}_i \mathrm{D}_j \frac{\delta \mathscr{L}_{(ij)}}{\delta \boldsymbol{\varphi}_{Iij}} \right).\end{aligned} \tag{A.11}$$

Since, by our inductive hypothesis,

$$\mathrm{D}_i \mathrm{D}_j \mathrm{D}_k \left(\frac{\delta \mathscr{L}_{(jk)}}{\delta \boldsymbol{\varphi}_{Ijk}} + \frac{\delta \mathscr{L}_{(ki)}}{\delta \boldsymbol{\varphi}_{Iki}} + \frac{\delta \mathscr{L}_{(ij)}}{\delta \boldsymbol{\varphi}_{Iij}} \right) = 0 \tag{A.12}$$

the triple derivative terms in (A.11) can be removed. We then use our inductive hypothesis to rewrite (A.11) as

$$\begin{aligned}\frac{\partial \mathsf{dL}}{\partial \boldsymbol{\varphi}_I} =& \frac{\partial \mathscr{L}_{(jk)}}{\partial \boldsymbol{\varphi}_{I\setminus i}} - \mathrm{D}_i \left\{ \frac{\delta \mathscr{L}_{(ki)}}{\delta \boldsymbol{\varphi}_{Ii\setminus j}} + \frac{\delta \mathscr{L}_{(ij)}}{\delta \boldsymbol{\varphi}_{Ii\setminus k}} + \mathrm{D}_j \left(\frac{\delta \mathscr{L}_{(ki)}}{\delta \boldsymbol{\varphi}_{Ii}} + \frac{\delta \mathscr{L}_{(ij)}}{\delta \boldsymbol{\varphi}_{Iij\setminus k}} \right) \right. \\ &\left. + \mathrm{D}_k \left(\frac{\delta \mathscr{L}_{(ki)}}{\delta \boldsymbol{\varphi}_{Iki\setminus j}} + \frac{\delta \mathscr{L}_{(ij)}}{\delta \boldsymbol{\varphi}_{Ii}} \right) \right\} \\ &+ \frac{\partial \mathscr{L}_{(ki)}}{\partial \boldsymbol{\varphi}_{I\setminus j}} - \mathrm{D}_j \left\{ \frac{\delta \mathscr{L}_{(ij)}}{\delta \boldsymbol{\varphi}_{Ij\setminus k}} + \frac{\delta \mathscr{L}_{(jk)}}{\delta \boldsymbol{\varphi}_{Ij\setminus i}} + \mathrm{D}_k \left(\frac{\delta \mathscr{L}_{(ij)}}{\delta \boldsymbol{\varphi}_{Ij}} + \frac{\delta \mathscr{L}_{(jk)}}{\delta \boldsymbol{\varphi}_{Ijk\setminus i}} \right) \right. \\ &\left. + \mathrm{D}_i \left(\frac{\delta \mathscr{L}_{(ij)}}{\delta \boldsymbol{\varphi}_{Iij\setminus k}} + \frac{\delta \mathscr{L}_{(jk)}}{\delta \boldsymbol{\varphi}_{Ij}} \right) \right\}\end{aligned}$$

$$
\begin{aligned}
&+\frac{\partial \mathscr{L}_{(ij)}}{\partial \boldsymbol{\varphi}_{I\setminus k}} - \mathrm{D}_k \left\{ \frac{\delta \mathscr{L}_{(jk)}}{\delta \boldsymbol{\varphi}_{Ik\setminus i}} + \frac{\delta \mathscr{L}_{(ki)}}{\delta \boldsymbol{\varphi}_{Ik\setminus j}} + \mathrm{D}_i \left(\frac{\delta \mathscr{L}_{(jk)}}{\delta \boldsymbol{\varphi}_{Ik}} + \frac{\delta \mathscr{L}_{(ki)}}{\delta \boldsymbol{\varphi}_{Iki\setminus j}} \right) \right. \\
&\quad \left. + \mathrm{D}_j \left(\frac{\delta \mathscr{L}_{(jk)}}{\delta \boldsymbol{\varphi}_{Ijk\setminus i}} + \frac{\delta \mathscr{L}_{(ki)}}{\delta \boldsymbol{\varphi}_{Ik}} \right) \right\}.
\end{aligned} \tag{A.13}
$$

We use our inductive hypothesis again to simplify, and get that

$$
\begin{aligned}
\frac{\partial \mathsf{dL}}{\partial \boldsymbol{\varphi}_I} =& \frac{\partial \mathscr{L}_{(jk)}}{\partial \boldsymbol{\varphi}_{I\setminus i}} - \mathrm{D}_i \left\{ \frac{\delta \mathscr{L}_{(ki)}}{\delta \boldsymbol{\varphi}_{Ii\setminus j}} + \frac{\delta \mathscr{L}_{(ij)}}{\delta \boldsymbol{\varphi}_{Ii\setminus k}} + \mathrm{D}_k \left(\frac{\delta \mathscr{L}_{(ki)}}{\delta \boldsymbol{\varphi}_{Iki\setminus j}} \right) \right\} \\
&+ \frac{\partial \mathscr{L}_{(ki)}}{\partial \boldsymbol{\varphi}_{I\setminus j}} - \mathrm{D}_j \left\{ \frac{\delta \mathscr{L}_{(ij)}}{\delta \boldsymbol{\varphi}_{Ij\setminus k}} + \frac{\delta \mathscr{L}_{(jk)}}{\delta \boldsymbol{\varphi}_{Ij\setminus i}} + \mathrm{D}_i \left(\frac{\delta \mathscr{L}_{(ij)}}{\delta \boldsymbol{\varphi}_{Iij\setminus k}} \right) \right\} \\
&+ \frac{\partial \mathscr{L}_{(ij)}}{\partial \boldsymbol{\varphi}_{I\setminus k}} - \mathrm{D}_k \left\{ \frac{\delta \mathscr{L}_{(jk)}}{\delta \boldsymbol{\varphi}_{Ik\setminus i}} + \frac{\delta \mathscr{L}_{(ki)}}{\delta \boldsymbol{\varphi}_{Ik\setminus j}} + \mathrm{D}_j \left(\frac{\delta \mathscr{L}_{(jk)}}{\delta \boldsymbol{\varphi}_{Ijk\setminus i}} \right) \right\}.
\end{aligned} \tag{A.14}
$$

Finally, we use (A.10) to write this as

$$
\frac{\partial \mathsf{dL}}{\partial \boldsymbol{\varphi}_I} = \frac{\delta \mathscr{L}_{(jk)}}{\delta \boldsymbol{\varphi}_{I\setminus i}} + \frac{\delta \mathscr{L}_{(ki)}}{\delta \boldsymbol{\varphi}_{I\setminus j}} + \frac{\delta \mathscr{L}_{(ij)}}{\delta \boldsymbol{\varphi}_{I\setminus k}}. \tag{A.15}
$$

Since $\dfrac{\partial \mathsf{dL}}{\partial \boldsymbol{\varphi}_I} = 0$, we now have that (A.6) holds for $|I| = M$.

In this proof, we have shown that the multiform Euler-Lagrange equations are a consequence of $\delta \mathsf{dL} = \mathsf{0}$. It is clear from (A.15) that the converse is also true, i.e. if all of the multiform Euler-Lagrange equations are satisfied then $\delta \mathsf{dL} = \mathsf{0}$. Whilst this proof applies only to a Lagrangian 2-form, it is relatively straightforward to generalize this argument to get the multiform Euler-Lagrange equations for a Lagrangian k-form.

Appendix B

A fully native Lagrangian multiform for the AKNS hierarchy

B.1 Introduction

In [28] the concept of **alien derivatives** is discussed, and a Lagrangian component $\mathscr{L}_{ij}$ is described as i,j-native if it only contains derivatives with respect to x_i, x_j and also x_1 (where x_1 is given special status as the spatial co-ordinate). If it contains derivatives with respect to any other co-ordinates, then these are described as alien derivatives. We will use the term **fully native** to describe Lagrangian components $\mathscr{L}_{ij}$ that only contain derivatives with respect to x_i and x_j. For the Lagrangian multiform we have already obtained for the AKNS hierarchy, we notice that, whilst the $\mathscr{L}_{1i}$ components only contain derivatives with respect to x_1 and x_i (i.e. corresponding to the labelling of $\mathscr{L}_{1i}$), the $\mathscr{L}_{ij}$ components for $i, j > 1$ contain derivatives with respect to x_i, x_j and also x_1. Therefore, each component is already i,j-native, but only the $\mathscr{L}_{1i}$ components are fully native. In this section we show how the Flaschka–Newell–Ratiu (FNR) construction of the AKNS hierarchy [24] can be used to construct a fully native Lagrangian multiform for the AKNS hierarchy. The results in this section have now largely been superseded by those in [25].

B.2 The FNR construction of the AKNS hierarchy

We begin by defining

$$L = L_0 + \frac{1}{\lambda}L_1 + \frac{1}{\lambda^2}L_2 + \cdots \tag{B.1}$$

and

$$L^{(k)} = \sum_{i=0}^{k} \lambda^i L_{k-i}. \tag{B.2}$$

We let $L_0 = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$ and the subsequent L_i are obtained recursively using the relation

$$L_{x_1} = [L^{(1)}, L] \tag{B.3}$$

and the assumption that all integration constants are zero. The next few L_i are as follows:

$$\begin{gathered} L_1 = \begin{pmatrix} 0 & q \\ r & 0 \end{pmatrix}, \quad L_2 = \begin{pmatrix} -\frac{i}{2}qr & \frac{i}{2}q_{x_1} \\ -\frac{i}{2}r_{x_1} & \frac{i}{2}qr \end{pmatrix}, \\ L_3 = \begin{pmatrix} \frac{1}{4}(rq_{x_1} - qr_{x_1}) & -\frac{1}{4}q_{x_1x_1} + \frac{1}{2}q^2r \\ -\frac{1}{4}r_{x_1x_1} + \frac{1}{2}qr^2 & -\frac{1}{4}(rq_{x_1} - qr_{x_1}) \end{pmatrix}, \quad L_4 = \\ \begin{pmatrix} -\frac{3i}{8}q^2r^2 + \frac{i}{8}(r_{x_1x_1}q - q_{x_1}r_{x_1} + q_{x_1x_1}r) & \frac{i}{8}(6qrq_{x_1} - q_{x_1x_1x_1}) \\ -\frac{i}{8}(6qrr_{x_1} - r_{x_1x_1x_1}) & \frac{3i}{8}q^2r^2 - \frac{i}{8}(r_{x_1x_1}q - q_{x_1}r_{x_1} + q_{x_1x_1}r) \end{pmatrix}. \end{gathered} \tag{B.4}$$

Where q and r are the field variables of the system. The equations of motion for the k^{th} flow of the hierarchy are then obtained from the off diagonal entries in

$$L^{(1)}_{x_k} - L^{(k)}_{x_1} + [L^{(1)}, L^{(k)}] = 0. \tag{B.5}$$

It is apparent from this construction that every equation of motion will contain x_1 derivatives and this is why we end up with a native but not fully native Lagrangian multiform. The source of these x_1 derivatives is (B.3) which guarantees that each L_i will contain more x_1 derivatives than L_{i-1}.

B.3 The Kaup–Newell hierarchy

Instead of using (B.3) to generate the matrices L_i, we can use

$$L_{x_2} = [L^{(2)}, L]. \tag{B.6}$$

The first few L_i we obtain are:

$$\begin{aligned} L_1 &= \begin{pmatrix} 0 & q \\ r & 0 \end{pmatrix}, \quad L_2 = \begin{pmatrix} -\frac{i}{2}qr & s \\ t & \frac{i}{2}qr \end{pmatrix}, \quad L_3 = \begin{pmatrix} -\frac{i}{2}(sr+tq) & \frac{i}{2}q_{x_2} \\ -\frac{i}{2}r_{x_2} & \frac{i}{2}(sr+tq) \end{pmatrix} \\ L_4 &= \begin{pmatrix} \frac{i}{8}q^2r^2 + \frac{1}{4}(rq_{x_2} - qr_{x_2}) - \frac{i}{2}st & \frac{i}{2}s_{x_2} + \frac{1}{2}(q^2t + qrs) \\ -\frac{i}{2}x_{x_2} + \frac{1}{2}(r^2s + qrt) & -\frac{i}{8}q^2r^2 - \frac{1}{4}(rq_{x_2} - qr_{x_2}) + \frac{i}{2}st \end{pmatrix}. \end{aligned} \tag{B.7}$$

We now have four field variables, q, r, s and t. The off diagonal entries in

$$L^{(2)}_{x_k} - L^{(k)}_{x_2} + [L^{(2)}, L^{(k)}] = 0. \tag{B.8}$$

gives us the Kaup–Newell hierarchy [47]. For example, the x_1 flow is given by $Q^{(2)}_{\Delta 1} = 0, R^{(2)}_{\Delta 1} = 0, S^{(2)}_{\Delta 1} = 0$ and $T^{(2)}_{\Delta 1} = 0$ where

$$\begin{aligned} Q^{(2)}_{\Delta 1} &:= q_{x_1} + 2is \\ R^{(2)}_{\Delta 1} &:= r_{x_1} - 2it \\ S^{(2)}_{\Delta 1} &:= s_{x_1} - q_{x_2} - iq^2r \\ T^{(2)}_{\Delta 1} &:= x_{x_1} - r_{x_2} + iqr^2. \end{aligned} \tag{B.9}$$

By inspection, we find that these equations of motion are variational, coming from the Lagrangian

$$\mathscr{L}_{(12)} := \frac{i}{2}q^2r^2 - 2\,ist - \frac{1}{2}\,qr_{x_2} + \frac{1}{2}\,qt_{x_1} - \frac{1}{2}\,tq_{x_1} + \frac{1}{2}\,rq_{x_2} - \frac{1}{2}\,rs_{x_1} + \frac{1}{2}\,sr_{x_1}. \tag{B.10}$$

We can now use the variational symmetries approach given in Chapter 2 to find the Lagrangian coefficients of the multiform. We do this by using

$$\mathsf{dL}_{12i} = \mathrm{E}(\mathscr{L}_{12}) \cdot \begin{pmatrix} Q^{(2)}_{\Delta i} \\ R^{(2)}_{\Delta i} \\ S^{(2)}_{\Delta i} \\ T^{(2)}_{\Delta i} \end{pmatrix} \tag{B.11}$$

where

$$\mathsf{L}_{12i} = \mathscr{L}_{(12)} \mathsf{d}x_1 \wedge \mathsf{d}x_2 + \mathscr{L}_{(2i)} \mathsf{d}x_2 \wedge \mathsf{d}x_i + \mathscr{L}_{(1i)} \mathsf{d}x_1 \wedge \mathsf{d}x_i \tag{B.12}$$

to obtain all $\mathscr{L}_{(1i)}$ and $\mathscr{L}_{(2i)}$. The remaining $\mathscr{L}_{(ij)}$ can then be obtained from

$$\mathsf{dL}_{1ij} = \mathrm{E}(\mathscr{L}_{2i}) \cdot \begin{pmatrix} Q^{(2)}_{\Delta j} \\ R^{(2)}_{\Delta j} \\ S^{(2)}_{\Delta j} \\ T^{(2)}_{\Delta j} \end{pmatrix} \tag{B.13}$$

using the $\mathscr{L}_{(2i)}$ obtained from (B.11). The first few Lagrangian coefficients are as follows:

$$\mathscr{L}_{(12)} := \frac{i}{2} q^2 r^2 - 2\, ist - \frac{1}{2}\, q r_{x_2} + \frac{1}{2}\, q t_{x_1} - \frac{1}{2}\, t q_{x_1} + \frac{1}{2}\, r q_{x_2} - \frac{1}{2}\, r s_{x_1} + \frac{1}{2}\, s r_{x_1} \tag{B.14}$$

$$\begin{aligned} \mathscr{L}_{(13)} := & \, irtq^2 + ir^2 sq - \frac{1}{8}\, r_{x_1} q^2 r + \frac{1}{8}\, q_{x_1} q r^2 - \frac{i}{4} q r_{x_1 x_2} + \frac{i}{4} r_{x_2} q_{x_1} + \frac{i}{4} q_{x_2} r_{x_1} \\ & - \frac{i}{4} r q_{x_1 x_2} - \frac{1}{2}\, q r_{x_3} + q_{x_2} t + \frac{1}{2}\, q_{x_3} r - r_{x_2} s + \frac{1}{2}\, s t_{x_1} - \frac{1}{2}\, t s_{x_1} \end{aligned} \tag{B.15}$$

$$\begin{aligned}\mathscr{L}_{(14)} := &-\frac{i}{4}q^3r^3 - \frac{i}{4}qt_{x_1x_2} + \frac{i}{4}t_{x_2}q_{x_1} + \frac{1}{2}\,q^2r_{x_2}r - \frac{1}{8}\,rq^2t_{x_1} + \frac{3}{8}\,tr_{x_1}q^2 - \frac{1}{4}\,rq_{x_1}qt\\ &-\frac{1}{2}\,r^2q_{x_2}q + \frac{1}{8}\,qr^2s_{x_1} + \frac{1}{4}\,qr_{x_1}rs - \frac{3}{8}\,sq_{x_1}r^2 - \frac{i}{4}sr_{x_1x_2} - \frac{i}{4}tq_{x_1x_2}\\ &-\frac{i}{2}r^2s^2 + \frac{i}{4}q_{x_2}t_{x_1} - \frac{i}{2}r_{x_2}q_{x_2} - \frac{i}{2}q^2t^2 + \frac{i}{4}r_{x_2}s_{x_1} + \frac{i}{4}s_{x_2}r_{x_1} - \frac{i}{4}rs_{x_1x_2}\\ &-\frac{1}{2}\,qr_{x_4} + \frac{1}{2}\,q_{x_4}r - t_{x_2}s + s_{x_2}t\end{aligned} \tag{B.16}$$

$$\begin{aligned}\mathscr{L}_{(23)} := &\frac{i}{2}q^2t^2 + irqst + \frac{i}{2}r^2s^2 - \frac{1}{8}\,q^2r_{x_2}r + \frac{1}{8}\,r^2q_{x_2}q - \frac{i}{4}qr_{x_2x_2} - \frac{i}{4}rq_{x_2x_2}\\ &-\frac{1}{2}\,qt_{x_3} + \frac{1}{2}\,q_{x_3}t + \frac{1}{2}\,rs_{x_3} - \frac{1}{2}\,r_{x_3}s + \frac{1}{2}\,t_{x_2}s - \frac{1}{2}\,s_{x_2}t\end{aligned} \tag{B.17}$$

$$\begin{aligned}\mathscr{L}_{(24)} := &-\frac{i}{4}rs_{x_2x_2} - \frac{i}{4}tq^3r^2 - \frac{i}{4}qt_{x_2x_2} - \frac{i}{4}sq^2r^3 - \frac{1}{8}\,rt_{x_2}q^2 + \frac{3}{8}\,tq^2r_{x_2} - \frac{1}{4}\,rqq_{x_2}t\\ &+\frac{1}{8}\,qs_{x_2}r^2 + \frac{1}{4}\,qrr_{x_2}s - \frac{3}{8}\,sr^2q_{x_2} + iqst^2 - \frac{i}{4}tq_{x_2x_2} - \frac{i}{4}sr_{x_2x_2} + itrs^2\\ &-\frac{1}{2}\,qt_{x_4} + \frac{1}{2}\,q_{x_4}t + \frac{1}{2}\,rs_{x_4} - \frac{1}{2}\,r_{x_4}s\end{aligned} \tag{B.18}$$

$$\begin{aligned}\mathscr{L}_{(34)} := &-\frac{1}{2}\,t_{x_2}tq^2 + \frac{1}{2}\,s_{x_2}sr^2 - \frac{1}{8}\,r_{x_2}q^3r^2 + \frac{1}{8}\,q_{x_2}q^2r^3 + \frac{3}{8}\,tr_{x_3}q^2 - \frac{3}{8}\,sq_{x_3}r^2\\ &-\frac{1}{8}\,rq^2t_{x_3} - \frac{1}{8}\,q_{x_4}qr^2 + \frac{1}{8}\,qr^2s_{x_3} + \frac{1}{8}\,r_{x_4}q^2r - \frac{i}{8}q^2{r_{x_2}}^2 - \frac{i}{4}qt_{x_2x_3}\\ &-\frac{i}{4}r_{x_2}q_{x_4} - \frac{i}{4}q_{x_2}r_{x_4} - \frac{i}{4}tq_{x_2x_3} - \frac{i}{4}rs_{x_2x_3} - \frac{i}{4}sr_{x_2x_3} - \frac{i}{2}t_{x_2}s_{x_2} - \frac{1}{2}\,t_{x_2}rsq\\ &+\frac{i}{4}qr_{x_2x_4} + \frac{i}{4}r_{x_2}s_{x_3} + \frac{i}{2}s^2t^2 + \frac{i}{32}q^4r^4 + \frac{i}{4}rq_{x_2x_4} + \frac{i}{4}s_{x_2}r_{x_3} + \frac{i}{4}q_{x_2}t_{x_3}\\ &+\frac{i}{4}t_{x_2}q_{x_3} - \frac{1}{2}\,st_{x_4} + \frac{1}{2}\,ts_{x_4} - \frac{i}{8}{q_{x_2}}^2r^2 - \frac{1}{4}\,rq_{x_3}qt + \frac{1}{4}\,qr_{x_3}rs + \frac{1}{2}\,s_{x_2}qtr\\ &+\frac{1}{2}\,r_{x_2}qst - \frac{1}{2}\,q_{x_2}rst - \frac{i}{2}q^3rt^2 - \frac{i}{2}qr^3s^2 + \frac{i}{4}r_{x_2}qq_{x_2}r - \frac{5}{4}\,itq^2r^2s.\end{aligned} \tag{B.19}$$

We notice that in this multiform, all $\mathscr{L}_{(2i)}$ coefficients are now fully native. However, with the exception of $\mathscr{L}_{(12)}$, the $\mathscr{L}_{(1i)}$s are no longer fully native. In fact, it appears that we are no closer to a fully native multiform that we were for the original AKNS multiform.

B.4 Beyond Kaup–Newell

We will now consider the hierarchy that comes from a x_3 based construction, i.e. where our L_is are found using

$$L_{x_3} = [L^{(3)}, L]. \tag{B.20}$$

The first few L_i we obtain are:

$$\begin{aligned} L_1 &= \begin{pmatrix} 0 & q \\ r & 0 \end{pmatrix}, \quad L_2 = \begin{pmatrix} -\frac{i}{2}qr & s \\ t & \frac{i}{2}qr \end{pmatrix}, \quad L_3 = \begin{pmatrix} -\frac{i}{2}(sr+tq) & u \\ v & \frac{i}{2}(sr+tq) \end{pmatrix} \\ L_4 &= \begin{pmatrix} \frac{i}{8}(q^2r^2 - 4qv - 4ru - 4st) & \frac{i}{2}q_{x_3} \\ -\frac{i}{2}r_{x_3} & -\frac{i}{8}(q^2r^2 - 4qv - 4ru - 4st) \end{pmatrix}. \end{aligned} \tag{B.21}$$

We now have six field variables, q, r, s, t, u and v and the off diagonal entries in

$$L^{(3)}_{x_k} - L^{(k)}_{x_3} + [L^{(3)}, L^{(k)}] = 0. \tag{B.22}$$

give us our equations of motion. In this case the x_1 flow is given by

$$\begin{aligned} Q^{(3)}_{\Delta 1} &:= q_{x_1} + 2is \\ R^{(3)}_{\Delta 1} &:= r_{x_1} - 2it \\ S^{(3)}_{\Delta 1} &:= s_{x_1} - iq^2r + 2iu \\ T^{(3)}_{\Delta 1} &:= x_{x_1} + iqr^2 - 2iv \\ U^{(3)}_{\Delta 1} &:= u_{x_1} - q_{x_3} - iq^2t - isqr \\ V^{(3)}_{\Delta 1} &:= v_{x_1} - r_{x_3} + ir^2s + iqrt, \end{aligned} \tag{B.23}$$

but these equations of motion (in the form given above) are not variational, so we cannot proceed as we did for the x_1 and x_2 based constructions of the hierarchy. However, it was shown in [23] that all systems of this type (i.e. the x_i version of this construction for ant x_i) are Hamiltonian, with equations of motion expressible in the form

$$\begin{pmatrix} q_j \\ r_j \\ \vdots \end{pmatrix} = D_{x_j}\underline{Q}^{(i)} = J^{(i)}\frac{\delta H_j^{(i)}}{\delta \underline{Q}^{(i)}} \tag{B.24}$$

for some Hamiltonian $H_j^{(i)}$, where $J^{(i)}$ is the matrix of Poisson brackets of the field variables. In the case of the x_3 construction,

$$J^{(3)} = \begin{pmatrix} \{q,q\} & \{q,r\} & \{q,s\} & \{q,t\} & \{q,u\} & \{q,v\} \\ \{r,q\} & \{r,r\} & \{r,s\} & \{r,t\} & \{r,u\} & \{r,v\} \\ \{s,q\} & \{s,r\} & \{s,s\} & \{s,t\} & \{s,u\} & \{s,v\} \\ \{t,q\} & \{t,r\} & \{t,s\} & \{t,t\} & \{t,u\} & \{t,v\} \\ \{u,q\} & \{u,r\} & \{u,s\} & \{u,t\} & \{u,u\} & \{u,v\} \\ \{v,q\} & \{v,r\} & \{v,s\} & \{v,t\} & \{v,u\} & \{v,v\} \end{pmatrix}. \tag{B.25}$$

These Poisson brackets can be evaluated in terms of the entries in the L_i matrices using the R-matrix approach given in [48]. If we denote the $(1,1)$ entry of L_i as a_i then

$$J^{(3)} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & a_0 \\ 0 & 0 & 0 & 0 & -a_0 & 0 \\ 0 & 0 & 0 & a_0 & 0 & a_1 \\ 0 & 0 & -a_0 & 0 & -a_1 & 0 \\ 0 & a_0 & 0 & a_1 & 0 & a_2 \\ -a_0 & 0 & -a_1 & 0 & -a_2 & 0 \end{pmatrix}. \tag{B.26}$$

In general, each $J^{(i)}$ has a_0s with alternating signs along the antidiagonal, a_1s with alternating signs along the -2-antidiagonal and generally, a_is with alternating signs along the $-2i$-antidiagonal, such that all a_is in the far right column have a positive sign. All other entries in $J^{(i)}$ are zero. We shall require the inverse of $J^{(i)}$; in the case of $J^{(3)}$,

$$J^{(3)^{-1}} = \begin{pmatrix} 0 & j_2 & 0 & j_1 & 0 & j_0 \\ -j_2 & 0 & -j_1 & 0 & -j_0 & 0 \\ 0 & j_1 & 0 & j_0 & 0 & 0 \\ -j_1 & 0 & -j_0 & 0 & 0 & 0 \\ 0 & j_0 & 0 & 0 & 0 & 0 \\ -j_0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{B.27}$$

where $j_0 = -\dfrac{1}{a_0}$ and subsequent j_K are given by

$$j_{K+1} = -\frac{1}{a_0}\sum_{i=0}^{K} j_{K-i}a_{i+1} \tag{B.28}$$

In general, each $J^{(i)^{-1}}$ has j_0s with alternating signs along the antidiagonal, j_1s with alternating signs along the 2-antidiagonal and generally, j_is with alternating signs along the $2i$-antidiagonal, such that all j_is in the top row have a positive sign. All other entries in $J^{(i)^{-1}}$ are zero.

Remark 39. *If we multiply (B.24) from the left by $J^{(i)^{-1}}$ then it is clear that the right hand side is variational. For the resulting equations of motion to be variational, we also require the left hand side to be variational. The results in [25] show that this is the case.*

In the case where $i = 3$, these same equations of motion can be obtained by evaluating $J^{(3)^{-1}}\underline{Q}^{(3)}_{\Delta i} = 0$ where

$$\underline{Q}^{(3)}_{\Delta i} = \begin{pmatrix} Q^{(3)}_{\Delta i} \\ R^{(3)}_{\Delta i} \\ S^{(3)}_{\Delta i} \\ T^{(3)}_{\Delta i} \\ U^{(3)}_{\Delta i} \\ V^{(3)}_{\Delta i} \end{pmatrix}. \tag{B.29}$$

and $J^{(3)^{-1}}\underline{Q}^{(3)}_{\Delta i} = \mathrm{E}(\mathscr{L}_{(i3)})$ for some Lagrangian $\mathscr{L}_{(i3)}$. We are now able to find the Lagrangian coefficients of our multiform since

$$\mathsf{dL}_{3ij} = \mathrm{E}(\mathscr{L}_{3i}) \cdot \underline{Q}^{(3)}_{\Delta j} = J^{(3)^{-1}}\underline{Q}^{(3)}_{\Delta i} \cdot \underline{Q}^{(3)}_{\Delta j}. \tag{B.30}$$

Remark 40. *$J^{(i)^{-1}}$ is a Hermitian matrix so*

$$\mathsf{dL}_{3ij} = J^{(3)^{-1}} \underline{Q}^{(3)}_{\Delta i} \cdot \underline{Q}^{(3)}_{\Delta j} = -J^{(3)^{-1}} \underline{Q}^{(3)}_{\Delta j} \cdot \underline{Q}^{(3)}_{\Delta i} = -\mathsf{dL}_{3ji} \tag{B.31}$$

and, more generally

$$\mathsf{dL}_{ijk} = J^{(i)^{-1}} \underline{Q}^{(i)}_{\Delta j} \cdot \underline{Q}^{(i)}_{\Delta k} = -J^{(i)^{-1}} \underline{Q}^{(i)}_{\Delta k} \cdot \underline{Q}^{(i)}_{\Delta j} = \mathsf{dL}_{ikj}. \tag{B.32}$$

The first six coefficients of the multiform obtained in this way are as follows:

$$\mathscr{L}_{(12)} := \frac{i}{2} q^2 r^2 - 2\, ist - \frac{1}{2}\, q r_{x_2} + \frac{1}{2}\, q t_{x_1} - \frac{1}{2}\, q_{x_1} t + \frac{1}{2}\, r q_{x_2} - \frac{1}{2}\, r s_{x_1} + \frac{1}{2}\, r_{x_1} s \tag{B.33}$$

$$\begin{aligned}\mathscr{L}_{(13)} :=\, & i q^2 r t + i q r^2 s - \frac{1}{8}\, q^2 r r_{x_1} + \frac{1}{8}\, q r^2 q_{x_1} - 2\, isv - 2\, iut - \frac{1}{2}\, q r_{x_3} + \frac{1}{2}\, q v_{x_1} \\ & - \frac{1}{2}\, q_{x_1} v + \frac{1}{2}\, r q_{x_3} - \frac{1}{2}\, r u_{x_1} + \frac{1}{2}\, r_{x_1} u + \frac{1}{2}\, s t_{x_1} - \frac{1}{2}\, s_{x_1} t\end{aligned} \tag{B.34}$$

$$\begin{aligned}\mathscr{L}_{(14)} :=\, & 2\, iqrst + \frac{i}{4} q_{x_3} r_{x_1} + \frac{i}{4} r_{x_3} q_{x_1} + \frac{i}{2} q^2 t^2 + \frac{i}{2} r^2 s^2 + q_{x_3} t - r_{x_3} s - \frac{i}{4} q r_{1,3} \\ & - \frac{i}{4} q^3 r^3 - \frac{i}{4} r q_{1,3} + i r q^2 v + i u q r^2 + \frac{1}{4}\, t q r q_{x_1} - \frac{1}{4}\, s q r r_{x_1} + \frac{1}{8}\, s_{x_1} q r^2 \\ & - \frac{1}{8}\, r_{x_1} q^2 t + \frac{1}{8}\, q_{x_1} r^2 s - \frac{1}{8}\, t_{x_1} q^2 r - 2\, iuv + \frac{1}{2}\, t_{x_1} u - \frac{1}{2}\, t u_{x_1} + \frac{1}{2}\, s v_{x_1} \\ & - \frac{1}{2}\, s_{x_1} v + \frac{1}{2}\, r q_{x_4} - \frac{1}{2}\, q r_{x_4}\end{aligned} \tag{B.35}$$

$$\begin{aligned}\mathscr{L}_{(23)} :=\, & \frac{i}{2} r^2 s^2 - 2\, iuv + \frac{i}{2} q^2 t^2 - \frac{1}{8}\, q^2 r r_{x_2} + \frac{1}{8}\, q r^2 q_{x_2} + irqst - \frac{1}{2}\, q t_{x_3} \\ & + \frac{1}{2}\, q v_{x_2} - \frac{1}{2}\, q_{x_2} v + \frac{1}{2}\, q_{x_3} t + \frac{1}{2}\, r s_{x_3} - \frac{1}{2}\, r u_{x_2} + \frac{1}{2}\, r_{x_2} u - \frac{1}{2}\, r_{x_3} s \\ & + \frac{1}{2}\, s t_{x_2} - \frac{1}{2}\, s_{x_2} t\end{aligned} \tag{B.36}$$

$$\begin{aligned}\mathscr{L}_{(24)} :=\, & isr^2u + itq^2v + itrs^2 + \frac{i}{4}q_{x_3}r_{x_2} + \frac{i}{4}r_{x_3}q_{x_2} - \frac{i}{4}tq^3r^2 - \frac{i}{4}sq^2r^3 + iqst^2 \\ & + \frac{1}{4}\,tqrq_{x_2} - \frac{1}{4}\,sqrr_{x_2} - \frac{1}{8}\,q^2rt_{x_2} - \frac{i}{4}rq_{x_2x_3} + \frac{1}{8}\,r^2sq_{x_2} + isqrv + ituqr \\ & - \frac{1}{8}\,q^2tr_{x_2} + \frac{1}{8}\,qr^2s_{x_2} + \frac{1}{2}\,t_{x_2}u + \frac{1}{2}\,sv_{x_2} - \frac{1}{2}\,s_{x_2}v + q_{x_3}v - \frac{1}{2}\,tu_{x_2} - r_{x_3}u \\ & + \frac{1}{2}\,rs_{x_4} - \frac{1}{2}\,r_{x_4}s - \frac{1}{2}\,qt_{x_4} + \frac{1}{2}\,q_{x_4}t - \frac{i}{4}qr_{x_2x_3}\end{aligned} \tag{B.37}$$

$$\begin{aligned}\mathscr{L}_{(34)} :=\, & -\frac{i}{4}tsq^2r^2 - \frac{1}{2}\,tu_{x_3} + \frac{1}{2}\,sv_{x_3} - \frac{1}{2}\,vs_{x_3} + \frac{1}{2}\,ut_{x_3} + \frac{1}{2}\,s_{x_4}t - \frac{1}{2}\,st_{x_4} \\ & + \frac{1}{2}\,ru_{x_4} - \frac{1}{2}\,qv_{x_4} + \frac{1}{2}\,q_{x_4}v - \frac{1}{2}\,r_{x_4}u + \frac{i}{2}s^2t^2 + \frac{i}{32}q^4r^4 + \frac{i}{2}r^2u^2 \\ & + \frac{i}{2}q^2v^2 - \frac{i}{4}vq^3r^2 - \frac{i}{4}uq^2r^3 - \frac{1}{8}\,qr^2q_{x_4} + ivuqr + ivqst + iurst \\ & - \frac{1}{8}\,q^2r_{x_3}t - \frac{1}{8}\,q^2rt_{x_3} - \frac{i}{4}rq_{x_3x_3} - \frac{i}{4}qr_{x_3x_3} + \frac{1}{8}\,qr^2s_{x_3} + \frac{1}{8}\,q^2rr_{x_4} \\ & + \frac{1}{8}\,q_{x_3}r^2s + \frac{1}{4}\,tqrq_{x_3} - \frac{1}{4}\,sqrr_{x_3}.\end{aligned} \tag{B.38}$$

On this occasion, not only are all of the $\mathscr{L}_{(3i)}$s fully native, but so is $\mathscr{L}_{(12)}$.

We now consider the x_4 construction, with the L_i matrices obtained from

$$L_{x_4} = [L^{(4)}, L], \tag{B.39}$$

and the first four given by:

$$\begin{aligned} & L_1 = \begin{pmatrix} 0 & q \\ r & 0 \end{pmatrix}, \quad L_2 = \begin{pmatrix} -\frac{i}{2}qr & s \\ t & \frac{i}{2}qr \end{pmatrix}, \quad L_3 = \begin{pmatrix} -\frac{i}{2}(sr+tq) & u \\ v & \frac{i}{2}(sr+tq) \end{pmatrix} \\ & L_4 = \begin{pmatrix} \frac{i}{8}(q^2r^2 - 4qv - 4ru - 4st) & w \\ x & -\frac{i}{8}(q^2r^2 - 4qv - 4ru - 4st) \end{pmatrix}\end{aligned} \tag{B.40}$$

where w and x are two additional variables. We obtain our equations of motion from

$$L^{(4)}_{x_k} - L^{(k)}_{x_4} + [L^{(4)}, L^{(k)}] = 0, \tag{B.41}$$

and find the multiform coefficients by reconstructing L_{4ij} from

$$\mathsf{dL}_{4ij} = {J^{(4)}}^{-1} \underline{Q}^{(4)}_{\Delta i} \cdot \underline{Q}^{(4)}_{\Delta j}. \tag{B.42}$$

The first six multiform coefficients are as follows:

$$\mathscr{L}_{(12)} := \frac{i}{2} q^2 r^2 - 2\,ist - \frac{1}{2}\,q r_{x_2} + \frac{1}{2}\,q t_{x_1} - \frac{1}{2}\,q_{x_1} t + \frac{1}{2}\,q_{x_2} r - \frac{1}{2}\,r s_{x_1} + \frac{1}{2}\,r_{x_1} s \tag{B.43}$$

$$\begin{aligned}\mathscr{L}_{(13)} := &- irq^2 t - iqr^2 s + \frac{1}{8}\,q^2 r r_{x_1} - \frac{1}{8}\,q q_{x_1} r^2 + 2\,isv + 2\,itu + \frac{1}{2}\,q r_{x_3} - \frac{1}{2}\,q v_{x_1} \\ &+ \frac{1}{2}\,q_{x_1} v - \frac{1}{2}\,r q_{x_3} + \frac{1}{2}\,r u_{x_1} - \frac{1}{2}\,r_{x_1} u - \frac{1}{2}\,s t_{x_1} + \frac{1}{2}\,s_{x_1} t\end{aligned} \tag{B.44}$$

$$\begin{aligned}\mathscr{L}_{(14)} := &\frac{i}{2} q^2 t^2 + \frac{i}{2} r^2 s^2 + irq^2 v + iqr^2 u + \frac{1}{4}\,t q q_{x_1} r - \frac{1}{4}\,s q r r_{x_1} - 2\,isx - \frac{i}{4} q^3 r^3 \\ &- 2\,itw - \frac{1}{8}\,q^2 r t_{x_1} + 2\,itqrs + \frac{1}{2}\,r_{x_1} w - \frac{1}{2}\,s_{x_1} v + \frac{1}{2}\,t_{x_1} u - \frac{1}{2}\,t u_{x_1} \\ &+ \frac{1}{2}\,s v_{x_1} + \frac{1}{2}\,r q_{x_4} - \frac{1}{2}\,r w_{x_1} - \frac{1}{2}\,q r_{x_4} + \frac{1}{2}\,q x_{x_1} - \frac{1}{2}\,q_{x_1} x + \frac{1}{8}\,q r^2 s_{x_1} \\ &+ \frac{1}{8}\,q_{x_1} r^2 s - \frac{1}{8}\,q^2 r_{x_1} t - 2\,iuv\end{aligned} \tag{B.45}$$

$$\begin{aligned}\mathscr{L}_{(23)} := &- \frac{i}{2} r^2 s^2 - \frac{i}{2} q^2 t^2 + 2\,iuv + \frac{1}{8}\,q^2 r r_{x_2} - \frac{1}{8}\,q q_{x_2} r^2 - iqstr + \frac{1}{2}\,q t_{x_3} \\ &- \frac{1}{2}\,q v_{x_2} + \frac{1}{2}\,v q_{x_2} - \frac{1}{2}\,t q_{x_3} - \frac{1}{2}\,r s_{x_3} + \frac{1}{2}\,r u_{x_2} - \frac{1}{2}\,u r_{x_2} + \frac{1}{2}\,s r_{x_3} \\ &- \frac{1}{2}\,s t_{x_2} + \frac{1}{2}\,t s_{x_2}\end{aligned} \tag{B.46}$$

$$\begin{aligned}\mathscr{L}_{(24)} := &-\frac{i}{4}sq^2r^3 - 2\,ivw - 2\,iux - \frac{1}{8}\,q^2rt_{x_2} - \frac{1}{8}\,q^2r_{x_2}t + \frac{1}{8}\,qr^2s_{x_2} + itqru \\ &+ isqrv + \frac{1}{2}\,r_{x_2}w - \frac{1}{2}\,s_{x_2}v + \frac{1}{2}\,t_{x_2}u + \frac{1}{2}\,sv_{x_2} + \frac{1}{2}\,rs_{x_4} - \frac{1}{2}\,rw_{x_2} \\ &- \frac{1}{2}\,qt_{x_4} + \frac{1}{2}\,qx_{x_2} - \frac{1}{2}\,q_{x_2}x + \frac{1}{2}\,tq_{x_4} - \frac{1}{2}\,tu_{x_2} - \frac{1}{2}\,sr_{x_4} + \frac{1}{8}\,q_{x_2}r^2s \\ &- \frac{i}{4}tq^3r^2 + itrs^2 + itq^2v + iqst^2 + isr^2u + \frac{1}{4}\,tqq_{x_2}r - \frac{1}{4}\,sqrr_{x_2}\end{aligned} \tag{B.47}$$

$$\begin{aligned}\mathscr{L}_{(34)} := &-\frac{i}{32}q^4r^4 - \frac{1}{4}\,tq_{x_3}qr + \frac{1}{4}\,sqr_{x_3}r - ivqru - ivqst - iurst - \frac{i}{2}q^2v^2 \\ &- \frac{i}{2}r^2u^2 + \frac{1}{2}\,q_{x_3}x - \frac{1}{2}\,r_{x_3}w + \frac{1}{2}\,rw_{x_3} + \frac{1}{2}\,qv_{x_4} - \frac{1}{2}\,qx_{x_3} + \frac{1}{2}\,tu_{x_3} \\ &+ \frac{1}{2}\,st_{x_4} - \frac{1}{2}\,sv_{x_3} - \frac{1}{2}\,ru_{x_4} + \frac{1}{2}\,vs_{x_3} - \frac{1}{2}\,vq_{x_4} - \frac{1}{2}\,ut_{x_3} + \frac{1}{2}\,ur_{x_4} \\ &- \frac{1}{2}\,ts_{x_4} + \frac{i}{4}tsq^2r^2 - \frac{1}{8}\,q_{x_3}sr^2 - \frac{1}{8}\,s_{x_3}qr^2 + \frac{1}{8}\,r_{x_3}tq^2 - \frac{1}{8}\,q^2rr_{x_4} \\ &+ \frac{1}{8}\,qq_{x_4}r^2 + \frac{1}{8}\,t_{x_3}q^2r + 2\,iwx - \frac{i}{2}s^2t^2 + \frac{i}{4}vq^3r^2 + \frac{i}{4}uq^2r^3\end{aligned} \tag{B.48}$$

In this case, all Lagrangian coefficients are fully native. In addition we notice that for $i, j \leq 3$, the Lagrangian coefficients are identical to those from the x_3 construction. This leads to the following two theorems:

Theorem 41. *The Lagrangian multiform based on a x_k FNR construction has fully native Lagrangian coefficients $\mathscr{L}_{(ij)}$ whenever $i, j \leq k$.*

Theorem 42. *Given two Lagrangian multiforms based on x_k and x_l FNR constructions respectively, if $k < l$ then both multiforms have the same Lagrangian coefficients $\mathscr{L}_{(ij)}$ whenever $i, j \leq k$.*

In order to prove these theorems, we will introduce the following notations. We shall label the $2k$ field variables that arise in the x_k construction as $q^{(k)1}, \ldots, q^{(k)k}$, $r^{(k)1}, \ldots, r^{(k)k}$ (so in the x_2 (Kaup–Newell) construction, we now label the field variables $q^{(2)1}, q^{(2)2}, r^{(2)1}, r^{(2)2}$ instead of q, r, s, t). Similarly, we shall denote the equations of motion for the x_i flow, as they arise from the x_k FNR construction as $Q^{(k)j}_{\Delta i} = 0, R^{(k)j}_{\Delta i} = 0$ for $j = 1, \ldots, k$. We will require the following lemma:

Lemma 43. *For $j < i$, $Q^{(i)l}_{\Delta j}$ has no x_i derivatives if $l + j - i < 1$. If $l + j - i \geq 1$ then the only x_i derivative in $Q^{(i)l}_{\Delta j}$ appears as $-q^{(i)l+j-i}_{x_i}$ with no additional coefficient. Similarly, for $j < i$, $R^{(i)l}_{\Delta j}$ has no x_i derivatives if $l + j - i < 1$. If $l + j - i \geq 1$ then the only x_i derivative in $R^{(i)l}_{\Delta j}$ appears as $-r^{(i)l+j-i}_{x_i}$ with no additional coefficient.*

Proof. This follows from the FNR construction that is used to derive the $Q^{(k)j}_{\Delta i}$ and $R^{(k)j}_{\Delta i}$. □

Proof. (**of Theorem 41.**) We now consider

$$\mathsf{dL}_{ijk} = {J^{(i)}}^{-1} \underline{Q}^{(i)}_{\Delta j} \cdot \underline{Q}^{(i)}_{\Delta k} = \mathrm{D}_{x_i} \mathscr{L}_{(jk)} + \mathrm{D}_{x_j} \mathscr{L}_{(ki)} + \mathrm{D}_{t_k} \mathscr{L}_{(ij)} \tag{B.49}$$

in the case where $i > j, k$ and notice that since the components of $\underline{Q}^{(i)}_{\Delta j}$ and $\underline{Q}^{(i)}_{\Delta k}$ only contain first order derivatives of any of the field variables with respect to x_j and t_k, and contain no products of derivatives of the field variables with respect to x_j and x_k, it is not possible for $\mathscr{L}_{(ki)}$ to contain any x_j derivatives, or for $\mathscr{L}_{(ij)}$ to contain any x_k derivatives, so these two Lagrangian coefficients are both fully native. In order to show that $\mathscr{L}_{(jk)}$ is also fully native when $j, k < i$, we will show that dL_{ijk} does not contain any products of derivatives of any of the field variables with respect to x_i (e.g. there is no $q^{(i)j}_{x_i} q^{(i)k}_{x_i}$, $q^{(i)j}_{x_i} r^{(i)k}_{x_i}$, $r^{(i)j}_{x_i} r^{(i)k}_{x_i}$ term in dL_{ijk}). We can express dL_{ijk} in terms of components as follows:

$$\mathsf{dL}_{ijk} = {J^{(i)}}^{-1} \underline{Q}^{(i)}_{\Delta j} \cdot \underline{Q}^{(i)}_{\Delta k} = \sum_{m=0}^{i-1} j_m \sum_{n=0}^{i-m-1} \left(Q^{(i)n+1}_{\Delta j} R^{(i)i-m-n}_{\Delta k} - R^{(i)n+1}_{\Delta j} Q^{(i)i-m-n}_{\Delta k}\right) \tag{B.50}$$

We are interested in when a product of x_i derivatives may appear in (B.50) – by combining (B.50) with lemma 43, we see that the products of x_i derivatives are given by

$$\sum_{m=0}^{i-1} j_m \sum_{n=0}^{i-m-1} \left(q^{(i)n+j+1-i}_{x_i} r^{(i)k-m-n}_{x_i} - r^{(i)n+j+1-i}_{x_i} q^{(i)k-m-n}_{x_i}\right) \tag{B.51}$$

where $q_{x_i}^{(i)j}$ and $r_{x_i}^{(i)j}$ are taken to be zero when $j < 1$. We can adjust the summation limits to exclude these zeros so the products of x_i derivatives are given by

$$\sum_{m=0}^{i-1} j_m \sum_{n=i-j}^{k-m-1} (q_{x_i}^{(i)n+j+1-i} r_{x_i}^{(i)k-m-n} - r_{x_i}^{(i)n+j+1-i} q_{x_i}^{(i)k-m-n}). \tag{B.52}$$

Now for the right hand side of this sum, we re-label and reverse the order of summation by letting $n \to k + i - m - j - 1 - n$ and the sum becomes

$$\sum_{m=0}^{i-1} j_m \sum_{n=i-j}^{k-m-1} (q_{x_i}^{(i)n+j+1-i} r_{x_i}^{(i)k-m-n} - r_{x_i}^{(i)k-m-n} q_{x_i}^{(i)n+j+1-i}). \tag{B.53}$$

Therefore, there are no products of x_i derivatives in dL_{ijk} for $j, k < i$ so each $\mathscr{L}_{(jk)}$ for $j, k < i$ is fully native. $\square$

Appendix C

The three flow KP multiform using Theorem 10

In Section 2.2.4 we used variational symmetries to obtain a Lagrangian multiform for the first two flows (t_3 and t_4) of the KP hierarchy. Here we extend this multiform to include the t_5 flow. We begin by introducing the dependent variable r such that $r_{x_1x_1x_1} = q$ where q is the usual KP variable (so $r_{x_1} = v$, the variable we used in 2.2.4). In terms of r, the Lagrangians already obtained become

$$\mathscr{L}_{(123)} = \frac{1}{2}r_{3x_1}r_{x_1x_1x_3} - \frac{1}{2}r_{4x_1}^2 - \frac{1}{2}r_{x_1x_1x_2}^2 + r_{3x_1}^3, \tag{C.1}$$

$$\mathscr{L}_{(124)} = \frac{1}{2}r_{3x_1}r_{x_1x_1x_4} - 2r_{4x_1}r_{3x_1x_2} - \frac{2}{3}r_{x_1x_1x_2}r_{x_1x_2x_2} + 4r_{3x_1}^2r_{x_1x_1x_2}, \tag{C.2}$$

$$\begin{aligned}\mathscr{L}_{(134)} = &\frac{2}{3}r_{x_1x_2x_2}^2 + 2r_{5x_1}^2 - \frac{4}{3}r_{x_1x_2x_2}r_{x_1x_1x_3} - \frac{2}{3}r_{x_1x_2x_3}r_{x_1x_1x_2} + r_{x_1x_1x_2}r_{x_1x_1x_4}\\ &- \frac{4}{3}r_{3x_1x_2}^2 + \frac{4}{3}r_{4x_1}r_{x_1x_1x_2x_2} + 12r_{3x_1}^2r_{5x_1} + 4r_{4x_1}^2r_{3x_1} - 4r_{3x_1}^2r_{x_1x_2x_2}\\ &+ 4r_{3x_1}r_{x_1x_1x_2}^2 + 4r_{3x_1}^2r_{x_1x_1x_3} + 10r_{3x_1}^4 + 2r_{5x_1}r_{x_1x_1x_3},\end{aligned} \tag{C.3}$$

and

$$
\begin{aligned}
\mathscr{L}_{(234)} = {} & 2r_{3x_1x_3}r_{3x_1x_2} + 2r_{5x_1}r_{4x_1x_2} + 6r_{3x_1}^2 t_{x_1x_2} + 4r_{3x_1}^3 r_{x_1x_1x_2} \\
& - 4r_{4x_1}^2 r_{x_1x_1x_2} - 2r_{3x_1}^2 r_{x_2x_2x_2} - \frac{8}{3}r_{x_1x_1x_2x_2}r_{3x_1x_2} - \frac{2}{3}r_{x_2x_2x_2}r_{5x_1} \\
& + 4r_{3x_1}r_{x_1x_1x_2}r_{5x_1} + 2r_{5x_1}t_{x_1x_2} + 2r_{x_1x_1x_3}t_{x_1x_2} - 2r_{x_1x_2x_2}t_{x_1x_2} \\
& + 6r_{3x_1}^2 r_{4x_1x_2} + 2r_{4x_1}r_{x_1x_1x_2x_3} + 2r_{x_1x_2x_2}r_{4x_1x_2} - r_{4x_1}r_{3x_1x_4} \\
& + \frac{1}{2}r_{x_1x_1x_4}r_{5x_1} + \frac{3}{2}r_{3x_1}^2 r_{x_1x_1x_4} + 4r_{3x_1}r_{x_1x_1x_2}r_{x_1x_2x_2} - \frac{2}{3}r_{x_1x_2x_2}r_{x_1x_2x_3} \\
& - \frac{2}{3}r_{x_1x_1x_3}r_{x_2x_2x_2} + \frac{2}{3}r_{x_1x_2x_2}r_{x_2x_2x_2} + \frac{4}{3}r_{4x_1}r_{x_1x_2x_2x_2} + \frac{4}{3}r_{x_1x_1x_2}^3 \\
& + 8r_{3x_1}r_{4x_1}r_{3x_1x_2} - 4r_{3x_1}r_{x_1x_1x_2}r_{x_1x_1x_3} + \frac{1}{2}r_{x_1x_2x_2}r_{x_1x_1x_4}.
\end{aligned}
\tag{C.4}
$$

The $\mathscr{L}_{(125)}$, based on a Hamiltonian given in [29] is given by

$$
\begin{aligned}
\mathscr{L}_{(125)} = {} & \frac{1}{2}r_{3x_1}r_{x_1x_1x_5} - \frac{3}{2}r_{5x_1}^2 - \frac{5}{6}r_{x_1x_2x_2}^2 - 5r_{3x_1x_2}^2 \\
& + 15r_{4x_1}^2 r_{3x_1} - \frac{15}{2}r_{3x_1}^4 + 5r_{3x_1}^2 r_{x_1x_2x_2} + 5r_{3x_1}r_{x_1x_1x_2}^2.
\end{aligned}
\tag{C.5}
$$

With the Lagrangians in this form we obtain the Euler Lagrange equations

$$
\mathrm{E}(\mathscr{L}_{(123)}) = -6r_{3x_1}r_{6x_1} - 18r_{4x_1}r_{5x_1} - r_{5x_1x_3} + r_{4x_1x_2x_2} - r_{8x_1} \tag{C.6}
$$

$$
\begin{aligned}
\mathrm{E}(\mathscr{L}_{(124)}) = {} & -16r_{3x_1}r_{5x_1x_2} - 8r_{x_1x_1x_2}r_{6x_1} - 40r_{4x_1x_2}r_{4x_1} \\
& - 32r_{5x_1}r_{3x_1x_2} - r_{5x_1x_4} + \frac{4}{3}r_{3x_13x_2} - 4r_{7x_1x_2}
\end{aligned}
\tag{C.7}
$$

$$
\begin{aligned}
\mathrm{E}(\mathscr{L}_{(125)}) = {} & 540r_{3x_1}r_{4x_1}r_{5x_1} + 180r_{4x_1}^3 - 60r_{4x_1}r_{3x_1x_2x_2} - 80r_{4x_1x_2}r_{3x_1x_2} - 40r_{5x_1}r_{x_1x_1x_2x_2} \\
& + 210r_{6x_1}r_{5x_1} + 90r_{3x_1}^2 r_{6x_1} - 10r_{x_1x_2x_2}r_{6x_1} - 20r_{x_1x_1x_2}r_{5x_1x_2} - r_{5x_1x_5} \\
& - 30r_{3x_1}r_{4x_1x_2x_2} + 120r_{4x_1}r_{7x_1} + \frac{5}{3}r_{x_1x_14x_2} + 30r_{3x_1}r_{8x_1} - 10r_{6x_1x_2x_2} \\
& + 3r_{10x_1}.
\end{aligned}
\tag{C.8}
$$

In order to apply Theorem 10, we shall require the four times integrated with respect to x_1 versions of these. Defining e_i such that $(e_i)_{4x_1} = \mathrm{E}(\mathscr{L}_{(12i)})$ we find that

$$e_3 = -3r_{3x_1}^2 - r_{x_1x_1x_3} + r_{x_1x_2x_2} - r_{5x_1}, \tag{C.9}$$

$$e_4 = -r_{x_1x_1x_4} - 4r_{4x_1x_2} - 8r_{3x_1}r_{x_1x_1x_2} + \frac{4}{3}r_{3x_2} - 4t_{x_1x_2}, \tag{C.10}$$

and

$$\begin{aligned} e_5 = &-r_{x_1x_1x_5} + 3r_{7x_1} + \frac{5}{3}s_{4x_2} - 10r_{3x_1x_2x_2} + 15r_{4x_1}^2 + 30r_{3x_1}r_{5x_1} \\ &+ 30r_{3x_1}^3 - 10r_{3x_1}r_{x_1x_2x_2} - 5r_{x_1x_1x_2}^2 - 5t_{x_2x_2} - 10u_{x_2}. \end{aligned} \tag{C.11}$$

These integrated versions of the Euler-Lagrange equations contain a number of non-localities, labelled s, t and u, such that $s_{x_1} = r$, $t_{x_1x_1} = r_{3x_1}^2$ and $u_{x_1} = r_{3x_1}r_{x_1x_1x_2}$. These non-localities shall feature in the Lagrangians we obtain for the KP multiform. We usually insist that all Lagrangian coefficients in a Lagrangian multiform are local expressions, so it is debatable whether or not the multiform we present here is a true Lagrangian multiform. However, the only non-localities that appear in the Lagrangians are the ones that appear in the e_i.

Using the above expressions for e_3, e_4 and e_5, Theorem 10 tells us that

$$e_3(e_5)_{x_1} = \mathrm{D}_{x_5}\mathscr{L}_{(123)} - \mathrm{D}_{x_3}\mathscr{L}_{(125)} + \mathrm{D}_{x_2}\mathscr{L}_{(135)} - \mathrm{D}_{x_1}\mathscr{L}_{(235)}, \tag{C.12}$$

which allows us to find that

$$\begin{aligned} \mathscr{L}_{(135)} = &-\frac{5}{3}r_{x_1x_2x_2}r_{x_1x_2x_3} - \frac{5}{3}r_{x_1x_1x_3}r_{x_2x_2x_2} - 10r_{3x_1x_3}r_{3x_1x_2} + r_{x_1x_1x_2}r_{x_1x_1x_5} \\ &+ \frac{5}{3}r_{x_1x_2x_2}r_{x_2x_2x_2} + \frac{5}{3}r_{4x_1}r_{x_1x_2x_2x_2} + \frac{5}{3}r_{x_1x_1x_2x_2}r_{3x_1x_2} + 7r_{5x_1}r_{4x_1x_2} \\ &+ 15r_{3x_1}^2t_{x_1x_2} + 20r_{3x_1}^3r_{x_1x_1x_2} - 10r_{3x_1}r_{x_1x_1x_2}r_{5x_1} - 70r_{3x_1}r_{4x_1}r_{3x_1x_2} \\ &- 20r_{4x_1}^2r_{x_1x_1x_2} + 10r_{3x_1}r_{x_1x_1x_2}r_{x_1x_1x_3} + 5r_{x_1x_2x_3}r_{3x_1}^2 - 10r_{x_1x_3}r_{3x_1}r_{3x_1x_2} \\ &- 5r_{3x_1}^2r_{x_2x_2x_2} + 10r_{3x_1}r_{3x_1x_2}r_{x_2x_2} + \frac{10}{3}r_{x_1x_1x_2}^3 \end{aligned} \tag{C.13}$$

and

$$
\begin{aligned}
\mathscr{L}_{(235)} = {}& \frac{5}{3} r_{x_1x_2x_2x_2} r_{3x_1x_2} - \frac{5}{3} r_{x_2x_2x_2} r_{x_1x_2x_3} + \frac{5}{3} r_{x_2x_2x_2x_2} r_{4x_1} - \frac{1}{2} r_{x_1x_1x_3} r_{x_1x_1x_5} \\
& + 30 r_{3x_1} r_{x_1x_1x_3} r_{5x_1} - 30 r_{x_1x_2x_2} r_{3x_1} r_{5x_1} - 30 r_{4x_1} r_{3x_1} r_{3x_1x_3} - 18 r_{3x_1} r_{4x_1} r_{6x_1} \\
& - 10 r_{4x_1x_2} r_{3x_1} r_{x_1x_1x_2} - 30 r_{x_1x_1x_2} r_{3x_1x_2} r_{4x_1} - 10 r_{x_1x_1x_3} r_{3x_1} r_{x_1x_2x_2} \\
& + \frac{15}{2} t_{x_1x_2}^2 + r_{x_1x_2x_2} r_{x_1x_1x_5} + 3 r_{6x_1} r_{x_1x_1x_2x_2} - 3 r_{7x_1} r_{x_1x_2x_2} \\
& - 20 r_{4x_1}^2 r_{x_1x_2x_2} - 30 r_{3x_1x_2}^2 r_{3x_1} - 30 r_{3x_1}^2 r_{3x_1x_2x_2} + 18 r_{4x_1}^2 r_{5x_1} + 9 r_{3x_1}^2 r_{7x_1} \\
& - 5 r_{4x_1} t_{x_1x_2x_2} + 24 r_{3x_1} r_{5x_1}^2 - 5 r_{x_1x_3} t_{x_1x_2x_2} - 3 r_{6x_1} r_{3x_1x_3} + 3 r_{7x_1} r_{x_1x_1x_3} \\
& - r_{4x_1} r_{3x_1x_5} - 5 r_{x_1x_1x_2}^2 r_{x_1x_1x_3} + 10 r_{x_1x_1x_2}^2 r_{x_1x_2x_2} + 10 r_{x_1x_2x_2}^2 r_{3x_1} \\
& - 10 r_{x_1x_1x_3} r_{3x_1x_2x_2} + 3 r_{5x_1} r_{4x_1x_3} - 3 r_{3x_1x_2x_2} r_{5x_1} + 10 r_{x_1x_2x_2} r_{3x_1x_2x_2} \\
& - 40 r_{3x_1}^3 r_{x_1x_2x_2} + 30 r_{x_1x_1x_3} r_{3x_1}^3 + 90 r_{3x_1}^3 r_{5x_1} + 3 r_{5x_1} r_{7x_1} + 5 r_{x_2x_2} t_{x_1x_2x_2} \\
& + 15 r_{4x_1}^2 r_{x_1x_1x_3} + \frac{7}{2} r_{4x_1x_2}^2 - \frac{25}{6} r_{x_1x_1x_2x_2}^2 + \frac{5}{6} r_{x_2x_2x_2}^2 - \frac{3}{2} r_{6x_1}^2 + 54 r_{3x_1}^5 .
\end{aligned}
\tag{C.14}
$$

We use Theorem 10 again to get that

$$
e_4(e_5)_{x_1} = \mathrm{D}_{x_5} \mathscr{L}_{(124)} - \mathrm{D}_{x_4} \mathscr{L}_{(125)} + \mathrm{D}_{x_2} \mathscr{L}_{(145)} - \mathrm{D}_{x_1} \mathscr{L}_{(245)}, \tag{C.15}
$$

allowing us to find

$$
\begin{aligned}
\mathscr{L}_{(145)} = {}& \frac{40}{9} r_{x_2} r_{x_1x_2x_2} r_{4x_1x_2} - \frac{20}{9} r r_{x_1x_1x_2} r_{3x_13x_2} - \frac{410}{3} r r_{5x_1} r_{8x_1} - \frac{140}{27} r r_{3x_1x_2} r_{x_1x_13x_2} \\
& - \frac{10}{3} r_{x_1x_1x_2x_2}^2 + \frac{10}{9} r_{3x_2}^2 + \frac{20}{9} r_{x_1x_1} r_{x_1x_1x_2} r_{x_13x_2} + \frac{20}{27} r_{x_1x_1} r_{3x_2} r_{3x_1x_2} + \frac{38}{3} r_{x_1} r_{3x_1} r_{9x_1} \\
& - \frac{20}{3} r_{3x_2} t_{x_1x_2} - \frac{5}{3} r_{3x_2} r_{x_1x_1x_4} - \frac{40}{27} r r_{4x_1} r_{x_14x_2} + 40 t_{x_1x_2} r_{x_1x_1x_2} r_{3x_1} + 80 r_{4x_1x_2} r_{3x_1} r_{x_1x_1x_2} \\
& - 40 r_{x_1x_2x_2} r_{3x_1} r_{5x_1} + 16 r_{3x_1} r_{4x_1} r_{6x_1} + 40 r_{x_2x_2} r_{4x_1} r_{5x_1} - 34 r_{x_1x_1} r_{4x_1} r_{7x_1} \\
& - 56 r_{x_1x_1} r_{5x_1} r_{6x_1} + 10 r_{3x_1} r_{x_1x_1x_2} r_{x_1x_1x_4} - 10 r_{x_1x_4} r_{3x_1} r_{3x_1x_2} + 90 r_{x_1} r_{5x_1} r_{7x_1} \\
& + 120 r_{x_2x_2} r_{3x_1}^2 r_{4x_1} + 10 t_{x_1x_2}^2 + 84 r_{3x_1}^5 + 18 r_{4x_1x_2}^2 - 6 r_{6x_1}^2 - \frac{40}{3} r r_{x_1x_1x_2x_2} r_{3x_1x_2x_2} \\
& - \frac{5}{3} r_{x_1x_2x_2} r_{x_1x_2x_4} + \frac{20}{3} r_{x_13x_2} r_{3x_1x_2} + \frac{4}{3} r_{x_1x_2x_2} r_{x_1x_1x_5} + \frac{2}{3} r_{x_1x_2x_5} r_{x_1x_1x_2} \\
& - \frac{38}{3} r r_{3x_1} r_{10x_1} - \frac{40}{9} r r_{x_1x_2x_2} r_{4x_1x_2x_2} - \frac{178}{3} r r_{4x_1} r_{9x_1} - \frac{40}{27} r r_{3x_1} r_{x_1x_14x_2} \\
& - \frac{20}{9} r r_{x_13x_2} r_{4x_1x_2} + \frac{20}{27} r r_{3x_2} r_{5x_1x_2} - \frac{40}{9} r_{x_2x_2} r_{x_1x_1x_2} r_{3x_1x_2} - \frac{40}{9} r_{x_2x_2} r_{x_1x_2x_2} r_{4x_1}
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{100}{9}r_{3x_2}r_{x_1x_1x_2}r_{3x_1}-\frac{40}{9}r_{x_1}r_{3x_1x_2}r_{x_13x_2}-\frac{40}{9}r_{x_2x_2}r_{3x_1}r_{x_1x_1x_2x_2}+\frac{80}{27}r_{x_2}r_{x_13x_2}r_{4x_1}\\
&+\frac{80}{27}r_{x_2}r_{3x_1}r_{x_1x_13x_2}+\frac{140}{3}r_{x_1}r_{4x_1}r_{8x_1}+\frac{80}{9}r_{x_2}r_{3x_1x_2}r_{x_1x_1x_2x_2}-\frac{20}{9}r_{x_1}r_{x_1x_1x_2}r_{x_1x_13x_2}\\
&+\frac{40}{9}r_{x_2}r_{x_1x_1x_2}r_{3x_1x_2x_2}-\frac{38}{3}r_{x_1x_1}r_{3x_1}r_{8x_1}-\frac{40}{27}r_{x_1}r_{3x_1}r_{x_14x_2}-\frac{20}{27}r_{x_1}r_{3x_2}r_{4x_1x_2}\\
&+\frac{80}{27}r_{x_1x_2}r_{3x_1}r_{x_13x_2}-202rr_{6x_1}r_{7x_1}-4r_{x_1x_1x_5}r_{3x_1}^2-420r_{4x_1}^2r_{3x_1}^2\\
&+5r_{x_1x_2x_4}r_{3x_1}^2+60r_{x_1x_1x_2}^2r_{3x_1}^2+20r_{4x_1x_2}t_{x_1x_2}+16r_{3x_1}r_{5x_1}^2-16r_{4x_1}^2r_{5x_1}\\
&+10r_{3x_1}^2r_{7x_1}-4r_{3x_1x_2x_2}r_{5x_1}+56r_{x_1}r_{6x_1}^2-2r_{5x_1}r_{x_1x_1x_5}-10r_{3x_1x_2}r_{3x_1x_4}
\end{aligned}
\tag{C.16}
$$

and

$$
\begin{aligned}
\mathscr{L}_{(245)}={}&-\frac{38}{3}r_{x_2}r_{3x_1}r_{9x_1}-\frac{140}{3}r_{x_2}r_{4x_1}r_{8x_1}-\frac{38}{3}rr_{3x_1}r_{9x_1x_2}-\frac{5}{3}r_{3x_2}r_{x_1x_2x_4}\\
&+\frac{2}{3}r_{x_1x_2x_2}r_{x_1x_2x_5}-\frac{20}{3}r_{x_13x_2}r_{x_1x_1x_2x_2}+\frac{40}{3}r_{3x_2}r_{3x_1x_2x_2}+\frac{4}{3}r_{3x_2}r_{x_1x_1x_5}\\
&+\frac{80}{9}r_{3x_1}r_{x_1x_2x_2}r_{3x_2}+\frac{40}{9}r_{x_2}r_{3x_1x_2}r_{x_13x_2}+\frac{20}{27}rr_{4x_2}r_{4x_1x_2}-\frac{1}{2}r_{x_1x_1x_4}r_{x_1x_1x_5}\\
&-\frac{40}{27}r_{x_1}r_{3x_1x_2}r_{4x_2}-\frac{40}{27}r_{x_2x_2}r_{3x_1}r_{x_13x_2}-\frac{40}{27}rr_{3x_1}r_{x_15x_2}-4t_{x_1x_2}r_{x_1x_1x_5}\\
&-2r_{4x_1}r_{x_1x_1x_2x_5}-12r_{5x_1x_2}r_{6x_1}+12r_{4x_1x_2}r_{7x_1}-10r_{x_1x_1x_4}r_{3x_1x_2x_2}-2r_{3x_1x_2}r_{3x_1x_5}\\
&+3r_{x_1x_1x_4}r_{7x_1}-3r_{3x_1x_4}r_{6x_1}+3r_{4x_1x_4}r_{5x_1}+15r_{4x_1}^2r_{x_1x_1x_4}+12r_{7x_1}t_{x_1x_2}\\
&+48r_{3x_1}^2r_{6x_1x_2}-56r_{x_2}r_{6x_1}^2-40r_{x_1x_1x_2}r_{5x_1}^2-4r_{3x_1x_2x_2}r_{4x_1x_2}-4r_{3x_2}r_{7x_1}+4r_{x_13x_2}r_{6x_1}\\
&-4r_{x_1x_13x_2}r_{5x_1}+60t_{x_1x_2}r_{4x_1}^2-5r_{x_1x_4}t_{x_1x_2x_2}-20t_{x_1x_2}r_{3x_1x_2x_2}-5r_{x_1x_1x_4}r_{x_1x_1x_2}^2\\
&-2r_{4x_1x_2}r_{x_1x_1x_5}+30r_{3x_1}^3r_{x_1x_1x_4}+120r_{3x_1}^3t_{x_1x_2}+180r_{3x_1}^4r_{x_1x_1x_2}-20r_{x_1x_1x_2}^2t_{x_1x_2}\\
&+\frac{76}{3}r_{x_1x_2}r_{3x_1}r_{8x_1}+\frac{20}{27}r_{x_2}r_{3x_2}r_{4x_1x_2}+\frac{76}{3}r_{x_1}r_{3x_1x_2}r_{8x_1}-\frac{20}{9}rr_{x_1x_1x_2}r_{x_1x_14x_2}\\
&+\frac{20}{9}r_{x_1x_1}r_{3x_2}r_{x_1x_1x_2x_2}+\frac{76}{3}r_{x_1}r_{3x_1}r_{8x_1x_2}+\frac{40}{9}r_{x_1}r_{x_13x_2}r_{x_1x_1x_2x_2}-\frac{40}{9}rr_{3x_1x_2}r_{x_14x_2}\\
&-\frac{140}{3}rr_{4x_1x_2}r_{8x_1}+\frac{20}{27}rr_{3x_2}r_{4x_1x_2x_2}-\frac{38}{3}rr_{3x_1x_2}r_{9x_1}-\frac{40}{9}rr_{x_1x_2x_2}r_{3x_13x_2}\\
&+\frac{40}{9}r_{x_1}r_{x_1x_2x_2}r_{x_1x_13x_2}+68r_{x_1}r_{4x_1x_2}r_{7x_1}+120r_{x_2x_2}r_{3x_1}^2r_{3x_1x_2}-40t_{x_1x_2}r_{x_1x_2x_2}r_{3x_1}\\
&-480r_{3x_1}^2r_{3x_1x_2}r_{4x_1}+112r_{x_1}r_{6x_1}r_{5x_1x_2}+120t_{x_1x_2}r_{3x_1}r_{5x_1}+40r_{x_1x_1x_2x_2}r_{3x_1}r_{3x_1x_2}\\
&+40r_{x_2x_2}r_{4x_1}r_{4x_1x_2}-64r_{x_1x_1}r_{4x_1}r_{6x_1x_2}-14r_{3x_1}r_{x_1x_1x_2}r_{7x_1}+48r_{3x_1}r_{3x_1x_2}r_{6x_1}\\
&-16r_{3x_1}r_{4x_1}r_{5x_1x_2}-88r_{x_1x_1x_2}r_{4x_1}r_{6x_1}+112r_{x_1x_2}r_{5x_1}r_{6x_1}+\frac{20}{9}r_{x_1x_1}r_{x_1x_1x_2}r_{4x_2}\\
&-\frac{40}{9}r_{x_1x_1}r_{x_13x_2}r_{x_1x_2x_2}-\frac{40}{9}r_{x_1x_2}r_{x_1x_1x_2}r_{x_13x_2}+\frac{40}{27}r_{x_2}r_{3x_1}r_{x_14x_2}
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{200}{27} r r_{x_1 3x_2} r_{3x_1x_2x_2} - \frac{100}{9} r r_{x_1x_1x_2x_2} r_{x_1x_1 3x_2} - \frac{40}{27} r_{x_1} r_{3x_2} r_{3x_1x_2x_2} - \frac{140}{3} r r_{4x_1} r_{8x_1x_2} \\
&- \frac{40}{27} r_{x_1x_2} r_{3x_2} r_{3x_1x_2} + \frac{20}{9} r_{x_2} r_{x_1x_1x_2} r_{x_1x_1 3x_2} - 90 r r_{5x_1x_2} r_{7x_1} - 64 r_{x_1x_1} r_{4x_1x_2} r_{6x_1} \\
&- 40 r_{x_1x_2x_2} r_{3x_1}^2 r_{x_1x_1x_2} + 240 r_{3x_1}^2 r_{x_1x_1x_2} r_{5x_1} - 104 r_{x_1x_1} r_{5x_1} r_{5x_1x_2} - 10 r_{x_1x_1x_4} r_{3x_1} r_{x_1x_2x_2} \\
&+ 68 r_{x_1x_2} r_{4x_1} r_{7x_1} + 56 r_{4x_1} r_{3x_1x_2} r_{5x_1} - 38 r_{x_1x_1} r_{3x_1} r_{7x_1x_2} - 38 r_{x_1x_1} r_{3x_1x_2} r_{7x_1} \\
&- 30 r_{3x_1x_4} r_{3x_1} r_{4x_1} - 90 r_{x_2} r_{5x_1} r_{7x_1} + 112 r_{x_1} r_{5x_1} r_{6x_1x_2} + 30 r_{x_1x_1x_4} r_{3x_1} r_{5x_1} \\
&+ 68 r_{x_1} r_{4x_1} r_{7x_1x_2} - 40 r_{3x_2} r_{3x_1} r_{5x_1} - 40 r_{x_1x_2x_2} r_{3x_1x_2} r_{4x_1} + 152 r_{3x_1} r_{5x_1} r_{4x_1x_2} \\
&- 40 r_{x_1x_2x_2} r_{3x_1} r_{4x_1x_2} - 112 r r_{6x_1} r_{6x_1x_2} - 90 r r_{5x_1} r_{7x_1x_2} + \frac{20}{3} r_{4x_2} r_{3x_1x_2} + \frac{20}{3} r_{3x_2} r_{x_1x_1x_2}^2
\end{aligned}
\tag{C.17}
$$

In order to find $\mathscr{L}_{(345)}$ we follow the method outlined in Section 2.2.5. We calculate

$$
\mathrm{D}_{x_3} \mathscr{L}_{(145)} - \mathrm{D}_{x_4} \mathscr{L}_{(135)} + \mathrm{D}_{x_5} \mathscr{L}_{(134)} \tag{C.18}
$$

and remove all products of x_3, x_4 and x_5 derivatives by adding terms that are double zeros on the KP equations. We then integrate the resulting expression with respect to x_1 to obtain

$$
\begin{aligned}
\mathscr{L}_{(345)} = &\frac{76}{3} r_{8x_1} r_{3x_1x_3} r_{x_1} + \frac{10}{3} r_{4x_2} r_{6x_1} - \frac{200}{3} r_{3x_1} r_{x_1x_1x_2} r_{x_1x_1 3x_2} \\
&- \frac{40}{9} r_{x_1x_2x_2} r_{x_1x_2} r_{x_1x_1x_2x_3} - \frac{40}{3} s_{4x_2} u_{x_2} + 5 r_{3x_2x_4} t_{x_1} + 10 u_{x_2} r_{x_1x_2x_4} + 48 r_{3x_1}^2 r_{6x_1x_3} \\
&- 56 r_{x_3} r_{6x_1}^2 + 80 r_{5x_1}^2 r_{x_1x_1x_3} + 480 r_{3x_1}^3 r_{4x_1}^2 + 1020 r_{3x_1}^4 r_{5x_1} + 5 r_{4x_1}^2 r_{x_1x_2x_4} \\
&+ 30 r_{x_1 4x_2} r_{3x_1}^2 - 36 r_{5x_1} r_{3x_1x_2}^2 - 4 r_{x_1x_1x_2x_2x_3} r_{5x_1} + 4 r_{x_1x_2x_2x_3} r_{6x_1} + 40 u_{x_2} t_{x_2x_2} \\
&+ 7 r_{5x_1x_2} r_{3x_1x_4} + 3 r_{6x_1x_2} r_{x_1x_1x_4} + 4 r_{x_1x_1x_2}^2 r_{x_1x_1x_5} - 4 r_{4x_1}^2 r_{x_1x_1x_5} + 2 r_{4x_1x_5} r_{x_1x_2x_2} \\
&+ 2 r_{5x_1} r_{4x_1x_5} - 10 r_{6x_1} t_{x_1x_2x_2} - 7 r_{4x_1x_2} r_{4x_1x_4} + 16 r_{4x_1x_2x_2} r_{x_1x_1x_2x_2} + 40 u_{x_2} r_{3x_1x_2x_2} \\
&- 132 r_{5x_1}^2 r_{x_1x_2x_2} - 4 r_{6x_1x_2x_2} r_{x_2x_2} + 24 r_{5x_1x_2x_2} r_{x_1x_2x_2} + 18 r_{3x_1}^2 r_{9x_1} - 5 r_{x_1x_1x_2}^2 r_{x_1x_2x_4} \\
&- 6 r_{3x_1x_3} r_{8x_1} + 12 r_{4x_1x_3} r_{7x_1} + 12 r_{6x_1x_2x_2} t_{x_1} - 72 r_{3x_1}^2 r_{5x_1x_2x_2} + 4 r_{6x_1x_2x_2} r_{4x_1} \\
&+ 36 r_{4x_1x_2} r_{3x_1x_2x_3} + 8 r_{6x_1x_2} r_{4x_1x_2} + 10 r_{x_1 4x_2} r_{x_1x_1x_3} - 120 r_{3x_1}^4 r_{x_1x_2x_2} \\
&- 20 r_{x_1x_1 4x_2} t_{x_1} + \frac{56}{3} r_{3x_1x_2x_2}^2 - \frac{80}{27} r_{x_3} r_{x_1 3x_2} r_{3x_1x_2} + \frac{20}{27} r_{x_3} r_{3x_2} r_{4x_1x_2} \\
&+ \frac{20}{3} r_{x_1x_2x_3} r_{x_1x_1 3x_2} - \frac{40}{9} r_{x_1x_2x_2x_3} r_{3x_1} r_{x_2x_2} - \frac{40}{9} r_{3x_1} r_{x_1x_2x_2} r_{x_2x_2x_3} \\
&- 112 r r_{6x_1} r_{6x_1x_3} - \frac{40}{9} r_{x_1x_2x_2x_3} r_{x_1x_2} r_{x_1x_1x_2} + 80 r_{3x_1} r_{5x_1x_2} u - 90 r r_{5x_1} r_{7x_1x_3} \\
&+ 480 r_{3x_1}^2 r_{3x_1x_2} u - 80 r_{4x_1} r_{4x_1x_2} u - 90 r r_{5x_1x_3} r_{7x_1} + 80 r_{x_1x_1x_2} r_{x_1x_1x_2x_2} u \\
&+ 80 r_{3x_1x_2} r_{5x_1} u - 56 r_{5x_1} r_{x_1x_1x_2} r_{4x_1x_2} + 40 r_{3x_1} r_{5x_1x_2} t_{x_2} + 10 r_{3x_1} r_{x_1x_4} r_{5x_1x_2}
\end{aligned}
$$

$$
\begin{aligned}
&-96r_{3x_1}r_{4x_1}r_{4x_1x_2x_2}+68r_{4x_1}r_{x_1}r_{7x_1x_3}+120r_{3x_1}^2r_{3x_1x_3}r_{x_2x_2}-64r_{4x_1}r_{x_1x_1}r_{6x_1x_3}\\
&+10r_{x_1x_1x_2}r_{x_1x_1x_2x_2}r_{x_1x_4}+68r_{4x_1}r_{x_1x_3}r_{7x_1}-40r_{3x_1x_2}r_{4x_1x_2}r_{x_2x_2}\\
&-40r_{3x_1x_3}r_{x_1x_1x_2}r_{3x_1x_2}-38r_{3x_1}r_{x_1x_1}r_{7x_1x_3}-90r_{x_3}r_{5x_1}r_{7x_1}-80r_{x_1x_1x_2}^2r_{x_1x_2x_2}r_{3x_1}\\
&-120r_{3x_1}^2t_{x_1}r_{x_1x_1x_2x_2}-204r_{3x_1}r_{5x_1}r_{3x_1x_2x_2}+40r_{3x_1}^2r_{x_2x_2}r_{x_1x_1x_2x_2}-16r_{4x_1}r_{5x_1x_3}r_{3x_1}\\
&+92r_{4x_1}r_{x_1x_1x_3}r_{6x_1}+46r_{3x_1}r_{x_1x_1x_3}r_{7x_1}+48r_{4x_1}r_{3x_1x_2}r_{4x_1x_2}+112r_{x_1}r_{5x_1x_3}r_{6x_1}\\
&-40r_{x_1x_1x_2}r_{x_13x_2}t_{x_1}+8r_{3x_1}r_{x_1x_1x_5}r_{5x_1}+240r_{3x_1}^2r_{3x_1x_2}t_{x_2}+112r_{x_1x_3}r_{5x_1}r_{6x_1}\\
&+112r_{x_1}r_{5x_1}r_{6x_1x_3}-40r_{4x_1x_3}r_{x_1x_2x_2}r_{3x_1}+10r_{5x_1}r_{3x_1x_2}r_{x_1x_4}+176r_{3x_1}r_{3x_1x_2}r_{5x_1x_2}\\
&-60r_{x_1x_1x_3}r_{4x_1}r_{x_1x_1x_2x_2}-10r_{4x_1}r_{4x_1x_2}r_{x_1x_4}+40r_{3x_1}^2r_{x_1x_2x_3}r_{x_1x_1x_2}\\
&-20r_{x_1x_1x_3}r_{x_1x_2x_2}r_{5x_1}+40r_{4x_1}r_{3x_1x_2x_2}r_{x_1x_3}+156r_{3x_1}r_{5x_1}r_{7x_1}+52r_{4x_1}r_{5x_1}r_{x_1x_1x_2x_2}\\
&-10r_{x_1x_1x_2}r_{x_1x_2x_2}r_{x_1x_1x_4}+10r_{3x_1}r_{4x_1x_2}r_{x_1x_1x_4}-40r_{3x_1x_2}r_{4x_1}r_{x_1x_2x_3}\\
&-40r_{3x_1}^2r_{x_1x_1x_2x_2}r_{x_1x_3}-64r_{x_1x_1x_2}r_{3x_1x_2}r_{6x_1}+60r_{3x_1}^2r_{x_1x_4}r_{3x_1x_2}+80r_{x_1x_1x_2}^2r_{3x_1}r_{x_1x_1x_3}\\
&+40r_{4x_1x_3}r_{x_2x_2}r_{4x_1}+44r_{3x_1}r_{6x_1}r_{x_1x_1x_2x_2}+30r_{4x_1}r_{3x_1x_2}r_{x_1x_1x_4}-36r_{3x_1}r_{4x_1}r_{8x_1}\\
&-4r_{3x_1}r_{4x_1}r_{3x_1x_5}-80r_{3x_1}^2r_{3x_1x_2}s_{3x_2}+360r_{x_1x_1x_3}r_{3x_1}r_{4x_1}^2-480r_{3x_1}r_{4x_1}^2r_{x_1x_2x_2}\\
&-12r_{3x_1x_3}r_{3x_1}r_{6x_1}-60t_{x_1x_2x_2}r_{3x_1}r_{4x_1}+80r_{x_1x_1x_2}r_{3x_1}r_{3x_1x_2x_3}+240r_{3x_1}r_{4x_1}^2r_{5x_1}\\
&-40r_{x_1x_1x_2}r_{5x_1}t_{x_1x_2}+40r_{x_1x_1x_2}r_{x_1x_1x_2x_2}t_{x_2}-38r_{3x_1x_3}r_{x_1x_1}r_{7x_1}+20r_{3x_1}r_{4x_1}r_{4x_2}\\
&-10r_{3x_1}r_{5x_1}r_{x_1x_2x_4}-40r_{4x_1}r_{4x_1x_2}t_{x_2}+120r_{3x_1x_2x_2}r_{4x_1}t_{x_1}-100r_{3x_1}r_{x_1x_1x_3}r_{3x_1x_2x_2}\\
&-40r_{x_1x_2x_2}r_{x_1x_1x_2x_2}t_{x_1}-160r_{3x_1}^2r_{4x_1}r_{x_1x_1x_2x_2}+30r_{4x_1}r_{x_1x_1x_2}r_{3x_1x_4}\\
&+420r_{3x_1}^2r_{x_1x_1x_3}r_{5x_1}-400r_{3x_1}^2r_{x_1x_2x_2}r_{5x_1}-40r_{3x_1}^2r_{x_1x_1x_3}r_{x_1x_2x_2}-660r_{3x_1x_3}r_{3x_1}^2r_{4x_1}\\
&+24r_{3x_1}r_{x_1x_1x_2}r_{6x_1x_2}-176r_{4x_1}r_{6x_1}r_{x_1x_2x_2}-104r_{x_1x_1}r_{5x_1}r_{5x_1x_3}-40r_{4x_1}r_{3x_1x_2x_2}r_{x_2x_2}\\
&+152r_{4x_1x_3}r_{3x_1}r_{5x_1}-64r_{4x_1}r_{3x_1x_3}r_{5x_1}+40r_{3x_1x_2}r_{5x_1}t_{x_2}+40r_{3x_1}r_{4x_1x_2}t_{x_1x_2}\\
&+10r_{3x_1}r_{x_1x_1x_2}r_{4x_1x_4}+60r_{3x_1}r_{3x_1x_2}r_{3x_1x_4}-64r_{4x_1x_3}r_{x_1x_1}r_{6x_1}+120r_{3x_1x_2}r_{4x_1x_2}t_{x_1}\\
&+80r_{3x_1}^2r_{x_1x_1x_2}r_{4x_1x_2}-10r_{5x_1}r_{x_1x_1x_2}r_{x_1x_1x_4}+60r_{4x_1}r_{5x_1}r_{6x_1}-84r_{3x_1}r_{x_1x_2x_2}r_{7x_1}\\
&+56r_{4x_1}r_{x_1x_1x_2}r_{5x_1x_2}+40r_{x_1x_1x_2x_2}r_{4x_1}r_{x_1x_2x_2}+40r_{3x_1x_2}r_{4x_1x_2}r_{x_1x_3}\\
&-20r_{x_1x_2x_2}r_{4x_1}r_{3x_1x_3}+40r_{x_1x_1x_2x_3}r_{3x_1x_2}r_{3x_1}-30r_{3x_1}^2r_{x_1x_1x_2}r_{x_1x_1x_4}\\
&-40r_{3x_1}r_{x_1x_1x_2}^2r_{5x_1}+68r_{4x_1x_3}r_{x_1}r_{7x_1}-\frac{20}{3}s_{4x_2}r_{3x_1x_2x_2}+\frac{40}{9}r_{x_1x_1x_2x_2}r_{x_2}r_{x_1x_1x_2x_3}\\
&-\frac{5}{3}r_{x_1x_2x_2x_4}r_{3x_1x_2}-\frac{20}{3}r_{x_1x_14x_2}r_{x_1x_3}-\frac{20}{9}r_{x_3}r_{x_1x_1x_2}r_{x_1x_13x_2}-\frac{40}{27}r_{3x_2x_3}r_{3x_1x_2}r_{x_1}\\
&-\frac{40}{9}r_{x_1x_2x_2}^3+\frac{76}{3}r_{8x_1}r_{x_1x_3}r_{3x_1}-\frac{5}{3}r_{3x_2x_4}r_{x_2x_2}+\frac{40}{3}r_{x_1x_1x_2}r_{x_13x_2}r_{x_2x_2}\\
&-\frac{140}{3}r_{8x_1}rr_{4x_1x_3}-\frac{20}{3}t_{x_1x_2x_3}s_{3x_2}+\frac{25}{3}r_{x_1x_1x_2x_2}r_{x_1x_1x_2x_4}+\frac{80}{27}r_{x_13x_2x_3}r_{3x_1}r_{x_2}\\
&+\frac{40}{9}r_{x_1x_2x_2x_3}r_{x_2}r_{3x_1x_2}-\frac{40}{9}r_{x_2x_2}r_{x_1x_1x_2}r_{x_1x_1x_2x_3}-\frac{4}{3}r_{x_1x_2x_2x_5}r_{x_1x_3}\\
&-\frac{40}{9}r_{3x_1x_3}r_{x_1x_2x_2}r_{x_2x_2}-\frac{40}{9}rr_{x_1x_2x_2}r_{3x_1x_2x_2x_3}-\frac{80}{27}r_{x_13x_2x_3}rr_{3x_1x_2}\\
&-\frac{140}{3}r_{8x_1x_3}rr_{4x_1}+\frac{20}{27}rr_{3x_2}r_{4x_1x_2x_3}-\frac{80}{9}r_{x_1x_1x_2x_2}r_{x_1x_1x_2x_2x_3}r-\frac{8}{3}r_{3x_1x_2}r_{x_1x_1x_2x_5}\\
&+\frac{40}{9}r_{x_1x_1x_2x_2x_3}r_{x_2}r_{x_1x_1x_2}+\frac{80}{9}r_{x_1x_1x_2}r_{x_1x_2x_2}r_{x_1x_2x_3}-\frac{80}{27}rr_{x_13x_2}r_{3x_1x_2x_3}
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{40}{9} r_{x_1x_2x_2x_3} r r_{3x_1x_2x_2} + \frac{40}{9} r^2_{x_1x_2x_2} r_{x_1x_1x_3} - \frac{20}{9} r_{x_1x_1x_2} r_{x_1x_13x_2x_3} r + 210 r^4_{4x_1} \\
&+ 10 r^4_{x_1x_1x_2} + 10 t^2_{x_2x_2} + 6 r^2_{7x_1} + 120 r^3_{5x_1} + 16 r^2_{5x_1x_2} + 120 r^6_{3x_1} + 40 u^2_{x_2} \\
&- \frac{38}{3} r_{9x_1} r_{x_3} r_{3x_1} + \frac{380}{3} r_{3x_1x_2x_2} r_{3x_1} r_{x_1x_2x_2} + \frac{40}{9} r_{x_1x_13x_2} r_{3x_2} - \frac{20}{3} r_{x_1x_2x_2x_3} r_{x_1x_1x_2x_2} \\
&+ \frac{40}{3} r_{x_1x_2x_2} r_{x_1x_1x_2x_2} r_{x_2x_2} + 20 t_{x_2x_2} r_{3x_1x_2x_2} + 10 r_{x_14x_2} r_{5x_1} - 5 r_{x_2x_2x_4} t_{x_1x_2} \\
&- 2 r_{6x_1} r_{3x_1x_5} + 20 r^2_{x_1x_2x_2} r_{5x_1} - 6 r_{x_1x_2x_2} r_{9x_1} - 10 u_{x_2x_2} r_{x_1x_4} + 180 r_{x_1x_1x_3} r^4_{3x_1} \\
&- 24 r_{5x_1x_2x_2} r_{x_1x_1x_3} - 6 r_{8x_1} r_{6x_1} - 12 r_{5x_1x_3} r_{6x_1} + 4 r_{6x_1x_2x_2} r_{x_1x_3} - 16 r_{4x_1x_2x_2} r_{6x_1} \\
&- 60 r_{3x_1} r^2_{6x_1} + 72 r^2_{4x_1} r_{7x_1} + 40 r_{x_2x_2x_3} r^3_{3x_1} - 100 r^2_{3x_1x_2} r_{x_1x_1x_3} - 4 r_{6x_1x_2} r_{x_1x_2x_3} \\
&+ 4 r_{5x_1x_2} r_{x_1x_1x_2x_3} + 5 t_{x_2x_2} r_{x_1x_2x_4} + 40 r^2_{3x_1} r^2_{x_1x_2x_2} - 24 r_{5x_1x_2x_2} r_{5x_1} + 60 r^2_{4x_1} r^2_{x_1x_1x_2} \\
&- 20 r^3_{3x_1} r_{x_1x_2x_4} - 24 r^2_{4x_1} r_{3x_1x_2x_2} + 300 r^2_{3x_1} r^2_{3x_1x_2} + 252 r^3_{3x_1} r_{7x_1} - 4 r_{3x_1x_2x_2} r_{4x_1x_3} \\
&+ 16 r^3_{3x_1} r_{x_1x_1x_5} + 20 t_{x_1x_2x_3} t_{x_2} + 20 r_{3x_1x_2} t_{x_1x_2x_3} - 3 r_{6x_1} r_{x_1x_1x_2x_4} - 380 r^3_{3x_1} r_{3x_1x_2x_2} \\
&+ 900 r^2_{3x_1} r^2_{5x_1} + 20 r_{x_2x_2x_3} r^2_{4x_1} - 15 t_{x_1} t_{x_1x_2x_4} - 40 r_{3x_1x_2} u_{x_2x_2} + 20 r_{3x_1x_2x_3} t_{x_1x_2} \\
&- 16 r_{4x_1x_2x_2} r_{3x_1x_3} - 2 r_{4x_1x_3} r_{x_1x_1x_5} + 72 r_{3x_1} r^2_{4x_1x_2} + 6 r_{x_1x_1x_3} r_{9x_1} + 10 t_{x_1x_2x_2} r_{x_1x_1x_2x_2} \\
&- 40 u_{x_2x_2} t_{x_2} + 6 r^2_{3x_1} r_{4x_1x_5} - 4 r_{x_1x_2x_2x_5} t_{x_1} + 6 r_{9x_1} r_{5x_1} - 10 r_{3x_1x_3} t_{x_1x_2x_2} + 6 r_{x_1x_1x_2x_2} r_{8x_1} \\
&- 12 r_{3x_1x_2x_2} r_{7x_1} - 7 r_{5x_1} r_{3x_1x_2x_4} - 10 r_{x_14x_2} r_{x_1x_2x_2} - \frac{2}{3} r_{3x_1x_5} r_{x_1x_1x_2x_2} - \frac{80}{3} r_{x_1x_13x_2} r_{4x_1x_2} \\
&+ \frac{40}{9} r^2_{x_13x_2} - \frac{40}{3} r_{x_1x_3} r_{x_1x_1x_2} r_{x_13x_2} + \frac{40}{9} r_{x_1x_1x_2} r_{x_1x_1x_2x_2} r_{x_2x_3} + \frac{20}{9} r_{3x_2x_3} r_{x_1x_1x_2} r_{x_1x_1} \\
&+ \frac{20}{9} r_{x_1x_1x_2x_3} r_{3x_2} r_{x_1x_1} - \frac{80}{9} r_{x_1x_1x_2x_2} r_{x_1x_3} r_{x_1x_2x_2} + \frac{40}{3} r_{x_1x_1x_2} r_{3x_2} r_{5x_1} \\
&+ \frac{80}{27} r_{3x_1x_3} r_{x_2} r_{x_13x_2} + \frac{40}{9} r_{x_1x_2x_2} r_{3x_1x_2x_3} r_{x_2} - \frac{80}{3} r_{x_1x_1x_2} r_{x_13x_2} r_{4x_1} + \frac{2}{3} r_{x_1x_2x_3} r_{x_1x_2x_5} \\
&- \frac{40}{3} r_{3x_1} r_{3x_2} r_{4x_1x_2} - \frac{40}{3} r_{3x_1x_2} r_{5x_1} s_{3x_2} + \frac{20}{3} r_{x_2x_2} r_{x_1x_14x_2} - \frac{38}{3} r_{9x_1} r_{3x_1x_3} r \\
&- \frac{20}{3} r_{x_1x_1x_2x_3} r_{x_13x_2} + \frac{40}{9} r_{x_1x_2x_2} r_{x_2x_3} r_{3x_1x_2} + \frac{5}{3} r_{3x_2x_4} r_{x_1x_3} + \frac{260}{3} r^2_{3x_1x_2} r_{x_1x_2x_2} \\
&+ \frac{280}{3} r_{x_1x_1x_2} r_{3x_1x_2} r_{x_1x_1x_2x_2} + \frac{40}{3} r_{4x_1} r_{4x_1x_2} s_{3x_2} - \frac{40}{3} r_{x_1x_1x_2} r_{x_1x_1x_2x_2} s_{3x_2} \\
&- \frac{40}{27} r_{3x_1x_3} r r_{x_14x_2} + 80 r_{x_1x_1x_2} r_{3x_1x_2} r_{x_2x_2} r_{3x_1} - 80 r_{3x_1} r_{x_1x_1x_2} r_{3x_1x_2} r_{x_1x_3} \\
&- 240 r_{3x_1} r_{x_1x_1x_2} r_{3x_1x_2} t_{x_1} - 80 r_{3x_1} r_{4x_1} r_{x_1x_1x_2} r_{3x_1x_2} + \frac{40}{9} r_{x_1x_1x_2x_2x_3} r_{x_1x_2x_2} r_{x_1} \\
&- 80 u u_{x_2x_2} + 40 u t_{x_1x_2x_3} + \frac{40}{3} u_{x_2x_2} s_{3x_2} - \frac{40}{9} r_{x_3} r_{x_1x_2x_2} r_{3x_1x_2x_2} \\
&- \frac{25}{3} r_{x_1x_13x_2} r_{x_1x_1x_4} - \frac{5}{3} r_{x_13x_2} r_{3x_1x_4} - \frac{40}{27} r_{x_1x_3} r_{3x_2} r_{3x_1x_2} + \frac{40}{9} r_{x_1x_1x_2x_2} r_{x_1x_2x_2x_3} r_{x_1} \\
&- \frac{20}{9} r_{x_2x_2x_3} r^2_{x_1x_1x_2} - \frac{200}{3} r_{3x_1} r_{x_13x_2} r_{3x_1x_2} - \frac{140}{3} r_{8x_1} r_{x_3} r_{4x_1} - \frac{40}{27} r_{x_3} r_{3x_1} r_{x_14x_2} \\
&- \frac{40}{3} r_{3x_1} r_{5x_1x_2} s_{3x_2} - \frac{40}{9} r^2_{x_1x_1x_2x_2} r_{x_3} - \frac{20}{3} r_{x_1x_14x_2} r_{4x_1} + \frac{80}{3} r_{3x_1} r^2_{x_1x_1x_2x_2}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{5}{3}r_{3x_1x_2x_2}r_{x_1x_2x_4}+\frac{4}{3}r_{x_1x_2x_2x_5}r_{x_2x_2}-\frac{5}{3}s_{4x_2}r_{x_1x_2x_4}+\frac{76}{3}r_{8x_1x_3}r_{x_1}r_{3x_1}\\
&+\frac{20}{27}r_{3x_2x_3}rr_{4x_1x_2}-\frac{40}{27}r_{x_14x_2x_3}rr_{3x_1}+\frac{80}{27}r_{3x_1}r_{x_13x_2}r_{x_2x_3}-\frac{40}{27}r_{3x_2}r_{3x_1x_2x_3}r_{x_1}\\
&+\frac{10}{3}r_{4x_2}r_{3x_1x_3}-\frac{40}{9}r_{x_1x_2x_2x_3}r_{x_1x_2x_2}r_{x_1x_1}-\frac{40}{3}r_{x_13x_2}r_{5x_1x_2}-\frac{10}{3}r_{4x_2}r_{x_1x_1x_2x_2}\\
&-\frac{38}{3}rr_{3x_1}r_{9x_1x_3}-\frac{40}{3}r_{x_1x_13x_2}t_{x_1x_2}+\frac{20}{9}r_{x_1x_1x_3}r_{3x_2}r_{x_1x_1x_2}-\frac{20}{9}r_{x_1x_13x_2}r_{x_1x_1x_2x_3}r\\
&+\frac{8}{3}r_{x_1x_1x_5}r_{3x_1x_2x_2}+\frac{20}{3}r_{3x_2x_3}r_{3x_1x_2}-\frac{20}{3}t_{x_2x_2}s_{4x_2}+\frac{10}{9}s_{4x_2}^2+\frac{40}{3}r_{x_1x_1x_2}^2r_{3x_1x_2x_2},
\end{aligned}
\tag{C.19}
$$

giving us a Lagrangian multiform up to the x_5 flow of the KP hierarchy.

Appendix D

Lagrangian for the ZM Lax Pair - version from [2]

If we are only interested in the U, V auxiliary problem

$$\Psi_\xi = U(\xi, \eta, \lambda)\Psi, \quad \Psi_\eta = V(\xi, \eta, \lambda)\Psi, \tag{D.1}$$

and want to cast this in the multiform structure of Section 3.1.2 then it is necessary to introduce a "ghost" variable ν and require that all field variables now have a ν dependence. We must also introduce the additional Lax matrix W relating to the "ghost" direction ν. These are required in the Lagrangian in order to have a closed 2-form. The multiform Euler-Lagrange equations from such a Lagrangian 2-form will have a ν dependence. We will go on to show that any set of ν dependent solutions can be reduced to a set of ν independent solutions, thereby obtaining precisely the auxiliary problem (D.1) and the associated compatibility conditions depending only on ξ and η from our Lagrangian 2-form.

We take our Lagrangian $\mathsf{L}[\varphi, \psi, \chi, \bar{U}, \bar{V}, \bar{W}; \lambda]$ to be

$$
\begin{aligned}
\mathsf{L} =& \Big(\sum_{\imath=1}^{N_1} (\varphi^i)^{-1} \varphi^i_\eta \bar{U}^i - \sum_{j=1}^{N_2} (\psi^j)^{-1} \psi^j_\xi \bar{V}^j - \sum_{\imath=1}^{N_1} \sum_{j=1}^{N_2} \frac{\psi^j \bar{V}^j (\psi^j)^{-1} \varphi^i \bar{U}^i (\varphi^i)^{-1}}{a_i - b_j} \Big) \mathsf{d}\xi \wedge \mathsf{d}\eta \\
&+ \Big(\sum_{j=1}^{N_2} (\psi^j)^{-1} \psi^j_\nu \bar{V}^j - \chi^{-1} \chi_\eta \bar{W} - \sum_{j=1}^{N_2} \frac{\chi \bar{W} \chi^{-1} \psi^j \bar{V}^j (\psi^j)^{-1}}{b_j - \lambda} \Big) \mathsf{d}\eta \wedge \mathsf{d}\nu \\
&+ \Big(\chi^{-1} \chi_\xi \bar{W} - \sum_{\imath=1}^{N_1} (\varphi^i)^{-1} \varphi^i_\nu \bar{U}^i - \sum_{\imath=1}^{N_1} \frac{\varphi^i \bar{U}^i (\varphi^i)^{-1} \chi \bar{W} \chi^{-1}}{\lambda - a_i} \Big) \mathsf{d}\nu \wedge \mathsf{d}\xi.
\end{aligned}
\tag{D.2}
$$

This Lagrangian 2-form is special case of the multiform (3.30) where the matrix W has a single pole at λ. In accordance with Theorem 21 the multiform equations of motion given by this multiform are

$$\chi_\xi = U\chi \quad \text{and} \quad \chi_\eta = V\chi \tag{D.3a}$$

$$\varphi^i_\eta = V|_{\lambda=a_i}\varphi^i \quad \text{and} \quad \varphi^i_\nu = W|_{\lambda=a_i}\varphi^i \tag{D.3b}$$

$$\psi^j_\nu = W|_{\lambda=b_j}\psi^j \quad \text{and} \quad \psi^j_\xi = U|_{\lambda=b_j}\psi^j \tag{D.3c}$$

and corollaries thereof, including

$$U^i_\eta + \left[U^i, \sum_{j=1}^{N_2} \frac{V^j}{a_i - b_j}\right] = 0 \quad \text{and} \quad V^j_\xi + \left[V^j, \sum_{i=1}^{N_1} \frac{U^i}{b_j - a_i}\right] = 0 \tag{D.4a}$$

$$V^j_\nu + \left[V^j, \frac{W^1}{b_j - \lambda}\right] = 0 \quad \text{and} \quad W^1_\eta + \left[W^1, \sum_{j=1}^{N_2} \frac{V^j}{\lambda - b_j}\right] = 0 \tag{D.4b}$$

$$W^1_\xi + \left[W^1, \sum_{i=1}^{N_1} \frac{U^i}{\lambda - a_i}\right] = 0 \quad \text{and} \quad U^i_\nu + \left[U^i, \frac{W^1}{a_i - \lambda}\right] = 0. \tag{D.4c}$$

At this stage, our equations of motion contain ν which does not feature in the U, V Lax pair. However, if the matrices $\bar{U}$, $\bar{V}$, $\bar{W}$, φ^i, ψ^j and χ satisfy these equations, then there is also a solution with the same $\bar{U}$ and $\bar{V}$ but with $\bar{W} = 0$.

In this case the second equation of (D.3b) and the first equation of (D.3c) tell us that φ^i and ψ^j no longer depend on ν, i.e. we can think of these as the φ^i and ψ^j of the original solution, with $\nu = \nu_0$, a constant. The first equation of (D.3b) and the second equation of (D.3c) are simply the definitions of φ^i and ψ^j which hold for $\nu = \nu_0$. Then (D.3a) is precisely the auxiliary problem for U and V, which no longer depends upon ν. Thus, the only remaining relations that are non-zero are

$$\chi_\xi = U\chi \quad \text{and} \quad \chi_\eta = V\chi \tag{D.5}$$

the auxiliary problem based on U and V,

$$\varphi^i_\eta = V|_{\lambda=a_i}\varphi^i \quad \text{and} \quad \psi^j_\xi = U|_{\lambda=b_j}\psi^j \tag{D.6}$$

the defining relations for φ^i and ψ^j and

$$U^i_\eta + [U^i, \sum_{j=1}^{N_2} \frac{V^j}{a_i - b_j}] = 0 \quad \text{and} \quad V^j_\xi + [V^j, \sum_{i=1}^{N_1} \frac{U^i}{b_j - a_i}] = 0 \tag{D.7}$$

the equations of motion for U^i and V^j. All of these relations now only depend upon ξ and η. Therefore, the Lagrangian multiform (D.2) can be considered the Lagrangian for the Lax pair U and V. We can summarise this result in the following theorem.

Theorem 44. *The Lagrangian 2-form* $\mathsf{L}(\varphi, \psi, \chi, \bar{U}, \bar{V}, \bar{W}, g; \lambda)$ *given by* (D.2) *is a Lagrangian for the Lax pair U and V. When we take the multiform Euler-Lagrange equations and set $\bar{W} = 0$ our equations of motion are the auxiliary problem*

$$\chi_\xi = U\chi \quad \textit{and} \quad \chi_\eta = V\chi \tag{D.8}$$

for U and V and the equations of motion

$$U^i_\eta + [U^i, \sum_{j=1}^{N_2} \frac{V^j}{a_i - b_j}] = 0 \quad \textit{and} \quad V^j_\xi + [V^j, \sum_{i=1}^{N_1} \frac{U^i}{b_j - a_i}] = 0 \tag{D.9}$$

corresponding to the compatibility conditions of this auxiliary problem.

Appendix E

The equations for ϕ directly from the KP multiform

In Section 4.4 of Chapter 4 we concluded that, since the KP Lagrangian multiform M contains Dickey's KP Lagrangian $\mathscr{L}_{(1ij)}$ for all i and j, the multiform Euler Lagrange equations given by $\delta \mathsf{dM} = 0$ include the KP equations of the type

$$(L_+^i)_{x_j} - (L_+^j)_{x_i} + [L_+^i, L_+^j] = 0. \tag{E.1}$$

We then used Corollary 28 to say that these equations are equivalent to the equations of the form

$$\phi_{x_i}\phi^{-1} + L_-^i = 0 \tag{E.2}$$

that also appear in the factorised form of $P_{(1ijk)}$. Here we show directly that the multiform Euler Lagrange equations given by given by $\delta \mathsf{dM} = 0$ also give us these ϕ equations.

Firstly we note that $\phi_{x_i}\phi^{-1} + L_-^i$ appears in the factorised form of $P_{(1ijk)}$ as the residue of its product with $(L_+^j)_{x_k} - (L_+^k)_{x_j} + [L_+^j, L_+^k]$. Since the highest power of ∂ to appear in $(L_+^j)_{x_k} - (L_+^k)_{x_j} + [L_+^j, L_+^k]$ is $\max(j,k) - 2$, we can only hope to get equations in the form of (E.2) truncated after the $\partial^{1-\max(j,k)}$ term from $\delta P_{(1ijk)} = 0$. In the following paragraphs, we shall demonstrate that the equations arising from $\delta P_{(1ijk)} = 0$ will, as a minimum, give us equations in the form of (E.2) up to the ∂^{i-k} term. It follows that the full multiform Euler-Lagrange equations given by $\delta \mathsf{dM} = 0$ give us all equations in the form of (E.2).

To confirm that the multiform Euler-Lagrange equations $\delta \mathsf{dM} = 0$ do give us equations of the type (E.2), we consider $\delta P_{(1ijk)}$ where $i < j < k$. The factorised form of $P_{(1ijk)}$ given in (4.127) can be written as

$$\frac{1}{2}\operatorname{res}\{A^{(ij)}B^{(k)} + A^{(jk)}B^{(i)} + A^{(ki)}B^{(j)}\} \tag{E.3}$$

where, $A^{(ij)} = (L^i_+)_{x_j} - (L^j_+)_{x_i} + [L^i_+, L^j_+]$, $B^{(k)} = \phi_{x_k}\phi^{-1} + L^k_-$ etc. Then

$$\delta P_{(1ijk)} =$$
$$\frac{1}{2}\operatorname{res}\{\delta A^{(ij)}B^{(k)} + A^{(ij)}\delta B^{(k)} + \delta A^{(jk)}B^{(i)} + A_{(jk)}\delta B^{(i)} + \delta A^{(ki)}B^{(j)} + A^{(ki)}\delta B^{(j)}\}. \tag{E.4}$$

We already have $A^{(ij)} = 0$, $A^{(jk)} = 0$ and $A^{(ki)} = 0$ from the Euler-Lagrange equations of $\mathscr{L}_{(1ij)}$, $\mathscr{L}_{(1jk)}$ and $\mathscr{L}_{(1ki)}$ respectively, so working modulo these equations, (E.4) becomes

$$\delta P_{(1ijk)} = \frac{1}{2}\operatorname{res}\{\delta A^{(ij)}B^{(k)} + \delta A^{(jk)}B^{(i)} + \delta A^{(ki)}B^{(j)}\}. \tag{E.5}$$

In order to proceed, we shall use the notation $A^{(ij)}_n$ and $B^{(k)}_n$ to represent the coefficient of ∂^n in $A^{(ij)}$ and $B^{(k)}$ respectively. We note that for all i, j and k, $A^{(ij)} \in \mathcal{R}_{\varphi+}$ and $B^{(k)} \in \mathcal{R}_{\varphi-}$. Therefore,

$$\begin{aligned}\operatorname{res}\{\delta A^{(ij)}B^{(k)}\} =& \operatorname{res}\{(\delta A^{(ij)}_{j-2}\partial^{j-2} + \ldots + \delta A^{(ij)}_1\partial + \delta A^{(ij)}_0)(B^{(k)}_{-1} + B^{(k)}_{-2} + \ldots)\}\\ =&\delta A^{(ij)}_{j-2}(\partial^{j-2}B^{(k)}_{-1} + \ldots + \partial B^{(k)}_{-(j-2)} + B^{(k)}_{-(j-3)})\\ &+ \delta A^{(ij)}_{j-3}(\partial^{j-3}B^{(k)}_{-1} + \ldots + \partial B^{(k)}_{-(j-3)} + B^{(k)}_{-(j-4)}) + \ldots\\ &+ \delta A^{(ij)}_1(\partial B^{(k)}_{-1} + B^{(k)}_{-2}) + \delta A^{(ij)}_0 B^{(k)}_{-1}.\end{aligned} \tag{E.6}$$

This means that, if each of the $\delta A^{(ij)}_l$, $\delta A^{(jk)}_m$ and $\delta A^{(ki)}_n$ are linearly independent, the multiform Euler-Lagrange equations arising from $\delta P_{(1ijk)}$ will give us the truncated forms of $B^{(i)}$, $B^{(j)}$ and $B^{(k)}$. However, it turns out that not all of the $\delta A_l^{(ij)}$, $\delta A_m^{(jk)}$ and $\delta A_n^{(ki)}$ are linearly independent. For example $4\delta A_1^{(23)} + 3\delta A_2^{(42)} = 0$. We shall proceed by showing that sufficient of the $\delta A_n^{(ij)}$ are linearly independent that the multiform Euler-Lagrange equations for the entire multiform, given by $\delta \mathsf{dM} = 0$, give us the full (i.e., not truncated) equations $B^{(k)}$.

We consider the terms in $A^{(ij)}$ in which $(\varphi_1)_{x^\nu x_j}$ features. In $A_0^{(ij)}$, a linear term that is a multiple of $(\varphi_1)_{x^{i-1}x_j}$ appears. Also we see non linear terms featuring $(\varphi_1)_{x^\nu x_j}$ for $\nu = 1, \ldots, i-2$. Similarly, we find $(\varphi_1)_{x^{i-2}x_j}$ appearing linearly in $A_1^{(ij)}$ as well as all lower order derivatives of the form $(\varphi_1)_{x^\nu x_j}$ that appear in non-linear terms. This pattern continues down to to $(\varphi_1)_{xx_j}$ appearing in $A_{i-2}^{(ij)}$. The other instances of $(\varphi_1)_{x^\nu x_j}$ appearing linearly in $P_{(1ijk)}$ will be $(\varphi_1)_{x^{k-1}x_j}$ appearing in $A_0^{(jk)}$, $(\varphi_1)_{x^{k-2}x_j}$ appearing in $A_1^{(jk)}$ and so on, up to $(\varphi_1)_{xx_j}$ appearing in $A_{k-2}^{(jk)}$. The same pattern continues, that if $(\varphi_1)_{x^\eta x_j}$ appears linearly in $A_\xi^{(jk)}$, then $(\varphi_1)_{x^\nu x_j}$ for $\nu = 1, \ldots, \eta - 1$ will feature in non-linear terms in $A_\xi^{(jk)}$. Therefore, $(\varphi_1)_{x^{k-1}x_j}, (\varphi_1)_{x^{k-2}x_j}, \ldots, (\varphi_1)_{x^i x_j}$ appear only once as linear terms in $P_{(1ijk)}$, so $(\delta\varphi_1)_{x^{k-1}x_j}, (\delta\varphi_1)_{x^{k-2}x_j}, \ldots, (\delta\varphi_1)_{x^i x_j}$ appear only once as linear terms in $\delta P_{(1ijk)}$. Then, the multiform Euler-Lagrange equation that arises from setting the coefficient of $(\delta\varphi_1)_{x^{k-1}x_j}$ equal to zero will give $B_{-1}^{(i)} = 0$. The multiform Euler-Lagrange equation that arises from setting the coefficient of $(\delta\varphi_1)_{x^{k-2}x_j}$ equal to zero will give $B_{-2}^{(i)} + F_1 B_{-1}^{(i)} = 0$ where F_1 is some polynomial of the φ_α and their derivatives, arising from the non-linear term featuring $(\delta\varphi_1)_{x^{k-2}x_j}$ in $A_1^{(jk)}$. The multiform Euler-Lagrange equation that arises from setting the coefficient of $(\delta\varphi_1)_{x^{k-3}x_j}$ equal to zero will give $B_{-3}^{(i)} + F_2 B_{-2}^{(i)} + F_3 B_{-1}^{(i)} = 0$ where F_2 and F_3 are again polynomials in the φ_α and their derivatives, and so on. The set of equations given by setting all of the coefficients of $(\delta\varphi_1)_{x^{k-1}x_j}, (\delta\varphi_1)_{x^{k-2}x_j}, \ldots, (\delta\varphi_1)_{x^i x_j}$ equal to zero will therefore give us equations equivalent to $B_{-n}^{(i)} = 0$ for $n = 1, \ldots, k-i$. It follows that the equation $B_{-n}^{(i)} = 0$ is a consequence of the multiform Euler-Lagrange equations given by $\delta P_{(1ijk)} = 0$ whenever $k \geq n - i$, and therefore the full set of multiform Euler-Lagrange equations given by $\delta \mathsf{dM} = 0$ gives us precisely $A^{(ij)} = 0$, $B^{(k)} = 0$ and consequences thereof for all i, j and k.

Appendix F

Explicit form of the KP Lagrangian multiform from Chapter 4

Here we present the first four Lagrangians of the KP Lagrangian multiform M and $\tilde{\mathsf{M}}$ as defined in Chapter 4, expressed in terms of the φ_β that constitute ϕ. In order to avoid notational confusion over the use of subscripts, we let $U = \varphi_0$, $V = \varphi_1$, $W = \varphi_2$ and $X = \varphi_3$. The following Lagrangians were found using Maple and PSEUDO [44]. In order to obtain $\mathscr{L}_{(234)}$, a Maple procedure based on (4.15) was used.

$$\begin{aligned}\mathscr{L}_{(123)} = &-U_{xxx_3} + X_{x_2} - VU_{xx_2} - WU_{x_2} - VV_{x_2} - U^2U_{x_3} + VU_{x_3} + UU_{xx_3} + U^2U_{xx_2} \\ &+ UV_{x_3} + U^2V_{x_2} - UU_{xxx_2} - U^3U_{x_2} - UW_{x_2} - 2UV_{xx_2} - 3V_xU_{x_2} - 3U_{xx}U_{x_2} \\ &+ 2U_xU_{x_3} - 3U_xV_{x_2} - 3U_xU_{xx_2} - W_{x_3} + U_{xxxx_2} - \frac{3}{2}UV_{xxx} - \frac{3}{2}U_{xxx}V - 3V_{xx}V \\ &- \frac{3}{2}{U_x}^2U^2 + 2U_{xxx}U^2 + 2V_{xx}U^2 + 2{U_x}^2V - \frac{1}{2}UU_{xxxx} - \frac{3}{2}U_xU_{xxx} - 3U_xV_{xx} \\ &- \frac{3}{2}U_{xx}U^3 + 2{U_x}^3 + 3W_{xx_2} - 2V_{xx_3} + 3V_{xxx_2} + 5UU_xU_{x_2} + 2UVU_{x_2} + 3U_{xx}U_xU \\ &+ 2U_{xx}VU,\end{aligned} \tag{F.1}$$

$$
\begin{aligned}
\tilde{\mathscr{L}}_{(123)} = {} & 2U^2U_{xxx}+3UU_xU_{xx}+2U_x^3+\frac{1}{2}U_{x_2}V_x-\frac{1}{2}U_xV_{x_2}-2U^2U_{xx_2}+\frac{3}{2}VU_{xx_2}+\frac{3}{2}UU_{xxx_2} \\
& +2U_xU_{xx_2}-\frac{3}{2}VU_{xxx}-\frac{3}{2}UV_{xxx}-\frac{3}{2}U^3U_{xx}-3U_xV_{xx}-\frac{3}{2}U_xU_{xxx}-\frac{1}{2}U_{x_2}U_{xx} \\
& -\frac{1}{2}UU_{xxxx}-UU_xU_{x_2}-UU_{xx_3}+2VU_x^2+2U^2V_{xx}-3VV_{xx}-\frac{3}{2}U^2U_x^2+\frac{3}{2}UV_{xx_2} \\
& +2UU_{xx}V,
\end{aligned}
\tag{F.2}
$$

$$
\begin{aligned}
\mathscr{L}_{(134)} = {} & -6U_{xx}V_{xxx}-\frac{3}{2}U_xU_{xxxxx}-5U_xV_{xxxx}-6U_xW_{xxx}-4V_xU_{xxxx}+U_{xxxxx_2}+40V_xU_xU_{xx} \\
& -6WV_{xxx}-12V_xV_{xxx}-4U_xW_{x_2}+Y_{x_2}+UW_{x_3}-4V_{x_2}V_x-6V_{x_2}U_{xx}+8U_{x_2}{U_x}^2 \\
& -4U_{x_2}W_x-6U_{x_2}V_{xx}-4U_{x_2}U_{xxx}+\frac{14}{3}U^2V_{xxxx}+2U^2U_{xxxxx}-2U^5U_{xx}+\frac{96}{5}U^2{U_x}^3 \\
& +\frac{12}{5}U^4V_{xx}+\frac{24}{5}U^4U_{xxx}-4U^4{U_x}^2-\frac{21}{2}U^2{U_{xx}}^2-6U^3V_{xxx}-U_{xxxx_3}-3U^3W_{xx} \\
& -6W_{xx}W-6U_{xx}W_{xx}-2U_{xx}U_{xxxx}-\frac{3}{2}UV_{xxxxx}-\frac{9}{2}U^3U_{xxxx}+UU_{xxx_3}+2UV_{xx_3} \\
& +V_{x_3}V+U_{xx_3}V+WU_{x_3}-U_{xx_3}U^2-V_{x_3}U^2-3UW_{xx_2}+U_{x_3}U^3-UX_{x_2} \\
& -VU_{xxx_2}-U_{xx_2}W+U^2U_{xxx_2}-8U_xV_{xx_2}-4U_{xx_2}V_x-6U_{xx_2}U_{xx}-4U_xU_{xxx_2} \\
& +3U_{x_3}V_x+3U_{x_3}U_{xx}+3V_{x_3}U_x+3U_{xx_3}U_x-3UV_{xxx_2}-UU_{xxxx_2}+U^2W_{x_2} \\
& -VW_{x_2}-V_{x_2}U^3-V_{x_2}W+U_{x_2}U^4+U_{x_2}V^2-U_{x_2}X+2U^2V_{xx_2}-2VV_{xx_2} \\
& -U_{xx_2}U^3+\frac{24}{5}U^3U_xV_x+24U^3U_xU_{xx}-5U_{x_3}U_xU-2U_{x_3}UV-12V_xW_{xx}+20U_x{U_{xx}}^2 \\
& +16U_x{V_x}^2+\frac{34}{3}{U_x}^2U_{xxx}+8{U_x}^2V_{xx}-2UW_{xxxx}-3{U_x}^4+2U_{xx_2}UV+7U_{xx_2}U_xU \\
& +9U_{x_2}UU_{xx}+7U_{x_2}UV_x+2U_{x_2}WU+6U_{x_2}U_xV-9U_{x_2}U_xU^2-3U_{x_2}U^2V \\
& -X_{x_3}+16UU_xW_{xx}+\frac{46}{3}U_{xxx}U_xV+7V_{x_2}U_xU+2V_{x_2}UV+\frac{70}{3}UU_xV_{xxx}+8U_xVV_{xx} \\
& +\frac{41}{3}UU_xU_{xxxx}+4U_{xx}WV+12U_xU_{xx}W-12UVU_xV_x-42UVU_xU_{xx}-6VW_{xxx} \\
& -2U_{xxxx}W+12{U_{xx}}^2V+6UU_{xx}V_{xx}+12UU_{xxx}U_{xx}-60U{U_x}^2U_{xx}+8U_xWV_x \\
& +16U_{xx}VV_x+4UV_xV_{xx}+\frac{28}{3}UU_{xxx}V_x-33U{U_x}^2V_x+12UVV_{xxx}-\frac{1}{2}UU_{xxxxxx} \\
& +4UV_{xx}W+\frac{22}{3}UVU_{xxxx}+8UVW_{xx}+4UU_{xxx}W-6UV^2U_{xx}-6U{U_x}^2W \\
& -27U^2U_{xxx}U_x+\frac{36}{5}U^3VU_{xx}+\frac{48}{5}U^2V{U_x}^2+6W_{xxx_2}+4V_{xxxx_2}-3W_{xx_3}-3V_{xxx_3} \\
& +4X_{xx_2}-9U^2V_xU_{xx}-6U^2V_{xx}V-3U^2WU_{xx}-15U^2V_{xx}U_x-12U^2U_{xxx}V \\
& +4U^2W_{xxx}-3{U_x}^3V+4U_{xxx}V^2-5VV_{xxxx}-\frac{3}{2}VU_{xxxxx},
\end{aligned}
\tag{F.3}
$$

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\begin{aligned}
\tilde{\mathscr{L}}_{(134)} = & -3U_x^3V - 4U_x^2U^4 + 16U_{xx}V_xV - 5VV_{xxxx} + 2UU_{xxxx_3} + 8UVW_{xx} - 6VW_{xxx} \\
& -6U_xW_{xxx} - 6U_x^2UW - \frac{9}{2}U^3U_{xxxx} - 6U^3V_{xxx} + 2U_{xx_3}W + 24U_{xx}U_xU^3 - 2V_{x_3}U_{xx} \\
& + \frac{24}{5}U_{xxx}U^4 + \frac{28}{3}UV_xU_{xxx} - 6U_{xx}V_{xxx} + 8U_x^2V_{xx} + 16U_xUW_{xx} - 3U^3W_{xx} \\
& - 2U_{xx}U_{xxxx} - 3U^2WU_{xx} - \frac{3}{2}VU_{xxxxx} + 20U_xU_{xx}^2 - 6U_{xx}W_{xx} - 2UW_{xxxx} \\
& - U_xW_{x_3} + 2U^2U_{xxxxx} + \frac{24}{5}U_xV_xU^3 - 42U_{xx}U_xUV - 2U_xU_{xx_4} + 3U^3U_{xx_3} \\
& + 3UV_{xxx_3} + 3VU_{xxx_3} + 4U_{xxx_3}U_x + 4U^2W_{xxx} - 4U^2U_{xxx_3} + 2V_{xx_3}U_x + 2VV_{xx_3} \\
& - \frac{3}{2}U_xU_{xxxxx} - U_{xx}UU_{x_3} - 2WU_{xxxx} + 2U^2U_{xx_4} - 12U_xUVV_x + 6U_{xx_3}V_x \\
& - 2U^5U_{xx} + 16V_x^2U_x + 2UW_{xx_3} + 2U_{xx_3}U_{xx} + 4U_{xxx}V^2 + 12U_{xx}^2V + U_{x_3}W_x \\
& - 12V_xW_{xx} - 4U_{xxxx}V_x - \frac{1}{2}V_xU_{x_4} + 8U_xVV_{xx} - 27U^2U_xU_{xxx} + 12V_{xxx}UV \\
& + \frac{14}{3}U^2V_{xxxx} + \frac{96}{5}U^2U_x^3 + 4UWV_{xx} + \frac{46}{3}U_{xxx}U_xV - U_{x_3}U_{xxx} + 12U_{xx}U_xW \\
& - 5U_xV_{xxxx} - 33UV_xU_x^2 + \frac{22}{3}U_{xxxx}UV - 6WV_{xxx} - \frac{21}{2}U^2U_{xx}^2 - 60U_x^2UU_{xx} \\
& + UU_xU_{x_4} + 3U^2U_xU_{x_3} + 8U_xV_xW + \frac{34}{3}U_x^2U_{xxx} - \frac{3}{2}UV_{xx_4} - \frac{3}{2}UU_{xxx_4} + 4U_{xxx}UW \\
& - \frac{7}{3}UU_{x_3}V_x - 12V_{xxx}V_x + 4UV_xV_{xx} - \frac{16}{3}UU_{xx_3}V - \frac{3}{2}VU_{xx_4} + \frac{70}{3}U_xV_{xxx}U \\
& - \frac{4}{3}VU_xU_{x_3} - 12U_{xxx}VU^2 - 3U_x^4 - 6U^2VV_{xx} - 15U^2U_xV_{xx} + \frac{48}{5}U_x^2VU^2 - \frac{35}{3}UU_xU_{xx_3} \\
& - \frac{1}{3}UU_xV_{x_3} - 6UV^2U_{xx} - \frac{4}{3}U_x^2U_{x_3} - 9U_{xx}V_xU^2 - \frac{8}{3}U^2V_{xx_3} + \frac{36}{5}U^3VU_{xx} \\
& + 40U_xU_{xx}V_x - 6WW_{xx} + 4VWU_{xx} + 12UU_{xxx}U_{xx} + 6U_{xx}UV_{xx} + \frac{41}{3}UU_xU_{xxxx} \\
& + \frac{12}{5}U^4V_{xx} + \frac{1}{2}U_{xx}U_{x_4} - \frac{1}{2}UU_{xxxxxx} + \frac{1}{2}U_xV_{x_4} - \frac{3}{2}UV_{xxxxx},
\end{aligned}
\tag{F.4}
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$$
\begin{aligned}
\mathscr{L}_{(142)} = {}& 6U^3U_{xxx} + 4U^3V_{xx} - \frac{24}{5}U^3{U_x}^2 - \frac{16}{5}U^4U_{xx} + 2U_{xx}U_{xxx} - U_{xxxxx_2} + 4U_{xx}V_{xx} \\
& - 16VU_{xx}U_x - \frac{20}{3}UU_{xx}V_x - \frac{16}{3}VV_xU_x - 16UU_{xxx}U_x - \frac{44}{3}UV_{xx}U_x + 4U_xW_{x_2} \\
& + U_{xxx_3} + U^2U_{x_3} - VU_{x_3} - UU_{xx_3} - UV_{x_3} - 2U_xU_{x_3} + W_{x_3} - Y_{x_2} + 4V_{x_2}V_x \\
& + 6V_{x_2}U_{xx} - 8U_{x_2}{U_x}^2 + 4U_{x_2}W_x + 6U_{x_2}V_{xx} + 4U_{x_2}U_{xxx} + 2V_{xx_3} + 3VU_{xxxx} \\
& + 8VV_{xxx} + 4VW_{xx} - \frac{8}{3}W{U_x}^2 + 12U{U_x}^3 - 6U{U_{xx}}^2 + 4V_{xx}W - 4U^2U_{xxxx} \\
& - \frac{20}{3}U^2V_{xxx} - \frac{8}{3}U^2W_{xx} + 2U_{xxx}W - \frac{8}{3}V^2U_{xx} - \frac{8}{3}UU_{xx}W - \frac{28}{3}UU_{xxx}V - 8UV_{xx}V \\
& + 8UV{U_x}^2 + 4U^2V_xU_x + 3UW_{xx_2} + UX_{x_2} + VU_{xxx_2} + U_{xx_2}W - U^2U_{xxx_2} \\
& + 8U_xV_{xx_2} + 4U_{xx_2}V_x + 6U_{xx_2}U_{xx} + 4U_xU_{xxx_2} + 3UV_{xxx_2} + UU_{xxxx_2} - U^2W_{x_2} \\
& + VW_{x_2} + V_{x_2}U^3 + V_{x_2}W - U_{x_2}U^4 - U_{x_2}V^2 + U_{x_2}X - 2U^2V_{xx_2} + 2VV_{xx_2} \\
& + U_{xx_2}U^3 + 3U_xU_{xxxx} + 4U_xW_{xx} + 8U_xV_{xxx} + 4U_{xxx}V_x + 8V_{xx}V_x - \frac{32}{3}V_x{U_x}^2 \\
& - 16{U_x}^2U_{xx} - 2U_{xx_2}UV - 7U_{xx_2}U_xU - 9U_{x_2}UU_{xx} - 7U_{x_2}UV_x - 2U_{x_2}WU \\
& - 6U_{x_2}U_xV + 9U_{x_2}U_xU^2 + 3U_{x_2}U^2V - 7V_{x_2}U_xU - 2V_{x_2}UV + UU_{xxxxx} + 3UV_{xxxx} \\
& - 6W_{xxx_2} - 4V_{xxxx_2} - 4X_{xx_2} + 2UW_{xxx} + 22U^2U_{xx}U_x + 8U^2VU_{xx},
\end{aligned}
\tag{F.5}
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$$
\begin{aligned}
\tilde{\mathscr{L}}_{(142)} = {}& 6U^3U_{xxx} + \frac{1}{3}UU_xV_{x_2} + \frac{7}{3}UU_{x_2}V_x - \frac{16}{3}U_xV_xV - \frac{20}{3}U_{xx}UV_x - 2UU_{xxxx_2} \\
& - \frac{8}{3}V^2U_{xx} + 8U^2VU_{xx} - 2U_{xx_2}W + 2V_{x_2}U_{xx} - 16U_x^2U_{xx} + \frac{8}{3}U^2V_{xx_2} \\
& + 3V_{xxxx}U + 2W_{xxx}U + \frac{4}{3}VU_xU_{x_2} - \frac{8}{3}U_x^2W + 4U^3V_{xx} + 12UU_x^3 \\
& + U_xW_{x_2} + \frac{4}{3}U_x^2U_{x_2} + 4WV_{xx} + 3VU_{xxxx} + 8VV_{xxx} + 4VW_{xx} - 3U^3U_{xx_2} \\
& - 3UV_{xxx_2} - 3VU_{xxx_2} - 4U_{xxx_2}U_x + 4U^2U_{xxx_2} - 2V_{xx_2}U_x - 2VV_{xx_2} \\
& + U_{xx}UU_{x_2} - 6U_{xx_2}V_x - 2UW_{xx_2} - 2U_{xx_2}U_{xx} + 2U_{xx}U_{xxx} - U_{x_2}W_x \\
& + 8U_xV_{xxx} - 4U^2U_{xxxx} + 2WU_{xxx} + 4U_{xx}V_{xx} + 4U_{xxx}V_x + U_{x_2}U_{xxx} \\
& + 3U_xU_{xxxx} + 8V_xV_{xx} - 8UVV_{xx} - 16U_{xx}U_xV - 3U^2U_xU_{x_2} - \frac{32}{3}U_x^2V_x \\
& - \frac{8}{3}UWU_{xx} + \frac{35}{3}UU_xU_{xx_2} - \frac{28}{3}U_{xxx}UV - 16U_{xxx}UU_x + UU_{xx_4} + 4U_xW_{xx} \\
& + \frac{16}{3}UU_{xx_2}V - \frac{44}{3}UU_xV_{xx} - \frac{24}{5}U^3U_x^2 + 22U^2U_xU_{xx} + 4U_xV_xU^2 \\
& - \frac{16}{5}U^4U_{xx} + UU_{xxxxx} - \frac{8}{3}U^2W_{xx} - 6U_{xx}^2U - \frac{20}{3}U^2V_{xxx} + 8UU_x^2V,
\end{aligned}
\tag{F.6}
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\begin{aligned}
\mathscr{L}_{(234)} = & -12U_{x_2}UVU_{xx} - 9U^2U_{xx_2}V_x + 14U_xV_xV_{xx} + 4UU_xW_{xxx} + U_xVV_{xxx} + 2UU_{xx}^2V \\
& + 3UU_xV_{xxxx} + UU_xU_{xxxxx} + 12U_xV_xW_x + 4U_xU_{xx}U_{xxx} + 6U_xVW_{xx} + UU_{xx_2}U_{x_3} \\
& + U_{x_2}V_{x_3}U - U_{x_2}U_{xx_3}U - UV_{x_2}U_{x_3} - U_{xxxx_2}U_{xx} + \frac{14}{3}U^2V_{xxxx_2} - 3U_{xx_2}X_x \\
& + 8U_xW_{xx_2}U + \frac{18}{5}U_{x_2}U^3V_x + 6V_xV_{xx_3} - 5UV_{xx}U_{xxx} - \frac{8}{3}UU_{xx_3}W + \frac{2}{3}U_x^2V_{x_3} \\
& + 2U_{xxx_3}U_{xx} - 6U_x^2U_{xx_3} - 3UV_{x_2}U_x^2 + 8UU_x^2U_{x_3} + 6U^2U_{x_3}U_{xx} - 8U_{xx_3}UU_{xx} \\
& + 4U_{x_3}V_xU^2 - 2U_{xxx}U_{x_3}U + 6U_{x_2}V_xU_{xx} + \frac{23}{3}U_{xxx_2}U_x^2 - 2W_{x_2}UU_{xx} + 3W_{x_2}U^2U_x \\
& - \frac{1}{2}U_{xx}U_{xxxxx} - VU_{xx}U_{xxx} - \frac{1}{2}U_{xx_4}U_{xx} + \frac{6}{5}U^3V_{x_2}U_x - \frac{8}{3}U_xWU_{x_3} - \frac{1}{2}U_{x_2}U_{xx_4} \\
& - \frac{3}{2}VU_{xxx_4} + 11U_{xx_2}VV_x + U_{x_2}U_{xxx_3} - 2V_{x_3}U_{xx_2} + 2U_{xx_3}V_{x_2} + W_{x_2}U_{x_3} - U_{xxx_2}U_{x_3} \\
& - U_{x_2}W_{x_3} + 8UVW_{xx_2} - \frac{7}{2}U_xV_{xxxx_2} - 2UU_{xx}W_{xx} - 6WW_{xx_2} - \frac{4}{3}U_{x_3}W_xU \\
& + 6UU_{xx}U_xW + UU_{xxx}U_xV + 6UV_{xx}U_xV - 29UU_xV_xU_{xx} + 16U^2U_xU_{xx_3} \\
& - 12UU_xU_{xxx_3} - 8UV_{xx_3}V + 8U^2U_{xx_3}V - 6U_xV^2U_{xx} + \frac{25}{3}U_{xxx_2}UU_{xx} - 6UV_{x_2}U_xV \\
& - 33UU_{xx_2}U_xV - 6U_{x_2}VUV_x - 4U_{x_2}U^4U_x - \frac{3}{2}V_{xx_4}U_x + 6U_x^3W + 11U_{xx_2}U_xW \\
& + 2V_{x_2}VU_{xx} - \frac{3}{2}UV_{xxx_4} - 6V_{xx_2}U^2U_x + 5V_{xx_2}U_xV + V_{xx_2}UV_x + 13U_{xx_2}VU_{xx} \\
& + V_{xx_2}UU_{xx} + 8UU_{xx_2}W_x - \frac{8}{3}U_{xx_3}V^2 + 2U_{x_2}VW_x - \frac{9}{2}U^3U_{xxxx_2} - 3VV_{xx_4} \\
& + 4V_{xx_3}W + 4W_{xx_3}V + 8V_{xxx_3}V + 2U^2V_{xx_4} + 3U_{xxxx_3}V - U_{xxx_4}U_x + 2U^2U_{xxx_4} \\
& + U_{x_4}U_x^2 - \frac{22}{3}U_xU_{x_3}V_x - \frac{1}{2}V_{xx_2}U_{xxx} - \frac{5}{2}U_{xx}V_{xxx_2} - 4U_{xxx_2}W_x - 3U_{x_2}U^2W_x \\
& - \frac{8}{3}U^2W_{xx_3} - 2V_{xx_2}V_{xx} + 3V_{xx_2}U_x^2 + UU_{xx}U_{xxxx} - 3U_x^3U_{x_2} - 4U_xU_{xx}U_{x_3} \\
& - \frac{3}{2}V_xU_{xx_4} - 2U^5U_{xx_2} - 6U^3V_{xxx_2} + 3V_x^2U_{xx} - 7U_x^2V_{xxx} - 2U_x^2U_{xxxx} + 8U_x^3U_{xx} \\
& - V_{x_4}U_xU - 27UU_{xx_2}U_x^2 + 2U_{x_4}VU_x + U_{x_4}V_xU + 2U_{xx_4}U_xU + 2U_{xx_4}UV \\
& - \frac{1}{2}U_{xxxx_4}U + 10UU_{xx_2}V_{xx} + U_{x_4}U_{xx}U + 14U_{xxx_2}U_xV + 8U_{xxx_2}UV_x + 2U_{x_2}UW_{xx} \\
& + 6U_{x_2}VV_{xx} - 2W_{x_2}U_xV - 2W_{x_2}UV_x - 6U_{x_2}U_x^2V + U_{x_2}UU_{xxxx} + 12U_x^2U^2V_x \\
& - 6UU_xV_x^2 - 2W_{xxxx_2}U - 6U_x^2V_xV + 6VU_{xx}W_x - 18UU_xU_{xx}^2 + V_xVU_{xxx} \\
& - 2U_{xxx}U^2V_x + \frac{2}{3}U_{xx}^2U_{x_2} + \frac{20}{3}U_{xxx}UU_{xx_2} - 6U^2U_{x_2}V_{xx} + 4U_{xx}U^2V_{xx} - 6V_xVV_{xx} \\
& + 3UV_xU_{xxxx} - 4U_xW_{xxx_2} + 4U_{xx}^3 - 2V_x^3 + 4U^3V_{xx_3} + 2U_{xxx_3}W - 3V_{xx_2}W_x \\
& - \frac{28}{3}U_{xx_3}UV_x + \frac{2}{3}V_{x_3}UU_{xx} - \frac{24}{5}U^3U_{x_3}U_x - 2WU_{xxxx_2} + 2U^2U_{xxxxx_2} - 4UU_x^2U_{xxx} \\
& - 5V_{xxx}U^2U_x - \frac{16}{5}U^4U_{xx_3} + 5U_{x_2}U_xW_x - \frac{3}{2}U^3U_{xx_4} + 6U_{x_2}U_xV_{xx} + 8U_xU_{xx}V_{xx}
\end{aligned}
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\begin{aligned}
&+11U_xV_xU_{xxx} + 9U_xU_{xx}W_x + 4U_{x_2}V_xW - \frac{3}{2}UV_{xxxxx_2} - 2W_{xx_2}U_{xx} - \frac{9}{2}U_{xxx}U^2U_{x_2}\\
&+ 8U_{x_3}UU_xV + \frac{36}{5}U^3U_{xx_2}V + 3UV_{xxxx_3} + 2UW_{xxx_3} - 12U_{xxx_2}U^2V + 4U^2W_{xxx_2}\\
&+ \frac{24}{5}U_{xxx_2}U^4 - \frac{1}{2}U_{xxxxxx_2}U - \frac{4}{3}U_xVV_{x_3} + 2UV_{x_2}W_x + 14UU_xV_{xxx_2} - 6W_{xx}U^2U_x\\
&+5UV_xV_{xxx}+6UV_xW_{xx}+4UV_{xx_2}W+5U_{x_2}V_x^2-U_{xxx_2}U_{xxx}-\frac{3}{2}U_{x_4}U_xU^2+\frac{29}{3}UU_xU_{xxxx_2}\\
&+4U_xV_{x_2}W-U_x^2VU_{xx}+4U_{xx_2}VW+4U_{xx}U_{x_2}W+UU_{xxxxx_3}-4U^2U_{xxxx_3}+\frac{31}{3}U_{xx_2}U_xU_{xx}\\
&-2W_{x_2}U_x^2-7V_xV_{xxx_2}+4U_{xxx_2}UW-2VU_{xx}V_{xx}-6WV_{xxx_2}-5U_{xx_2}W_{xx}+6U^3U_{xxx_3}\\
&+5U_{xxx_3}V_x+2V_{x_2}U_xV_x+\frac{84}{5}U^3U_{xx_2}U_x+4W_xU_{xx_3}+2U_xU_{xxxx_3}+2U_xW_{xx_3}+4V_{xx}U_{xx_3}\\
&+U_{xx_3}U_{xxx}+2V_{xx_3}U_{xx}+5V_{xxx_3}U_x-6UU_{xx_2}V^2-\frac{5}{2}U_{xx_2}V_{xxx}-\frac{7}{2}U_{xxx_2}V_{xx}+\frac{1}{3}UV_{x_2}U_{xxx}\\
&-3U_{xxxx}U^2U_x+15U_{xx_2}U_xV_x-6UU_x^2V_{xx}+\frac{48}{5}U_{x_2}U^2U_xV-3U_xWU_{xxx}-6U_xWV_{xx}\\
&-15U_{xx}UU_xU_{x_2}-15U_{x_2}UU_xV_x-5VV_{xxxx_2}-6VW_{xxx_2}-8U_x^2W_{xx}-W_xV_{xxx}+2V_{xx}W_{xx}\\
&+2V_{xx}V_{xxx}+3W_{xx}U_{xxx}-\frac{5}{2}U_{xxxx_2}V_x-\frac{16}{3}U_{x_3}VU_{xx}-\frac{20}{3}U^2V_{xxx_3}-3U^3W_{xx_2}+4U_{xxx_2}V^2\\
&+\frac{1}{2}U_{x_2}V_{x_4} + 12V_{xxx_2}UV - 2UW_xU_{xxx} - 6UU_{xx}X_x + 2U_{xxx}U_xU^3 + 3U_{xxx}U^2U_{xx}\\
&+\frac{12}{5}U^4V_{xx_2} - \frac{39}{2}U_{xxx_2}U^2U_x - 6W_{xx_2}V_x - U_{xxxxx_2}U_x - \frac{38}{3}U_xU_{xx_3}V + \frac{66}{5}U^2U_x^2U_{x_2}\\
&+\frac{36}{5}U_{xx}U^3U_{x_2}-6U^2V_{xx_2}V-2U_{xx}W_{xxx}+10V_xU_{xx}^2+6X_xV_{xx}+3X_xU_{xxx}-4V_xW_{xxx}\\
&-3V_xV_{xxxx}-V_xU_{xxxxx}-6W_xW_{xx}-6U_x^2X_x+2V_{xxx}U_{xxx}-2UU_{xxx}^2-2UV_{xx}^2+18U_{xx}U_x^2U^2\\
&-4U_x^3U^3-2U_{xx}^2U^3-3U^2U_{xx_2}W-6UU_x^2W_x+\frac{4}{3}U_xW_{x_3}U+\frac{7}{3}U_{x_2}UV_{xxx}+\frac{1}{2}U_{xxx}U_{xxxx}\\
&-\frac{28}{3}U_{xxx_3}UV - 4U_{x_3}VV_x - \frac{3}{2}VU_{xxxxx_2} - \frac{1}{2}V_{x_2}U_{x_4} - 6U_{x_2}U_xUW - \frac{10}{3}U_{x_3}UV_{xx}\\
&+\frac{22}{3}U_{xxxx_2}UV-\frac{22}{3}UU_xV_{xx_3}-\frac{3}{2}U_{xx}V_{xxxx}-\frac{27}{2}U^2U_{xx_2}U_{xx}+\frac{5}{3}U_{x_2}U_xU_{xxx}-\frac{1}{2}U_{xx_2}U_{xxxx}\\
&+\frac{10}{3}U_{x_2}VU_{xxx} + \frac{1}{2}V_{xx}U_{xxxx} + \frac{1}{2}U_{xx_2}U_{x_4}.
\end{aligned}
\tag{F.7}
$$

The Lagrangian $\tilde{\mathscr{L}}_{(234)}$ is identical to $\mathscr{L}_{(234)}$. From the Lagrangians given here, it appears that $\tilde{\mathscr{L}}_{(1ij)}$ gives a shorter Lagrangian than $\mathscr{L}_{(1ij)}$. In general, the difference between $\tilde{\mathscr{L}}_{(1ij)}$ and $\mathscr{L}_{(1ij)}$ can be expressed as the sum of a total x_i derivative and a total x_j derivative.

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