



The Centralizer of the Cartan Subalgebra in the Universal Enveloping Algebra

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To my mother and my father.

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Abstract

In the Lie theory, to give a classification of simple weight modules for a simple finite dimensional Lie algebra is an old open problem that has been successfully solved only for the Lie algebra $\mathfrak{sl}_2(K)$ by P. Gabriel in the 50s. This problem is essentially equivalent to another old open problem of classifying all simple modules of the centralizer of a Cartan subalgebra in the universal enveloping algebra of the Lie algebra. Generators for the centralizer were known for many years, but the defining relations were known only for $\mathfrak{sl}_2(K)$. Without defining relations, it is not possible to classify simple modules. One of the goals of this thesis is to present explicit finite sets of generators and defining relations for the centralizer in the case of the Lie algebra $\mathfrak{sl}_3(K)$. The properties of the centralizer are studied. In particular, its center and basis are given, the Gelfand-Kirillov dimension is computed, and simple 1-dimensional modules of the centralizer are classified. A review of known results on simple weight modules and the centralizer is given.

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Chapter 1

Introduction

1.1 Motivation

Representation theory of semisimple Lie algebras plays an important role in mathematics and physics and has been attracting a lot of attention of many mathematicians and physicists recently. One of the important problems arising in the representation theory of semisimple Lie algebras is to classify all the **simple weight** modules for semisimple Lie algebras. It is a **classical open problem** in the representation theory of Lie algebras from the 1950s. It is a difficult problem that has only been solved for the smallest simple Lie algebra $\mathfrak{sl}_2(K)$ by P. Gabriel in 1959 [15] (see [7, Theorem 7.8.16] for details). There are many results about simple weight modules, see for example [2, 3, 5, 7, 16], and recently some progress has been made towards the classification of simple weight modules for semisimple Lie algebras, see for example [11, 12, 13]. However, the problem of classifying these objects is still open.

The key aim of my research is to study the structure of the centralizer C of the Cartan subalgebra of the Lie algebra $\mathfrak{sl}_3(K)$ in the universal enveloping algebra $U(\mathfrak{sl}_3)$ of the Lie algebra $\mathfrak{sl}_3(K)$. Our interest in the centralizer C derives primarily from the fact that simple weight \mathfrak{sl}_3 -modules are completely determined by the simple C -modules. The 8-dimensional Lie algebra $\mathfrak{sl}_3(K)$ of 3×3 traceless matrices is the second smallest simple Lie algebra besides $\mathfrak{sl}_2(K)$, and the third smallest semisimple Lie algebra besides $\mathfrak{sl}_2(K)$ and $\mathfrak{sl}_2(K) \oplus \mathfrak{sl}_2(K)$.

1.2 The main results of the thesis

This thesis is essentially dedicated to the study of the structure of the centralizer C of the Cartan subalgebra of the Lie algebra $\mathfrak{sl}_3(K)$ in the universal enveloping algebra $U(\mathfrak{sl}_3)$. In particular, we first give sets of generators and defining relations of the centralizer C . Then we find some properties of the centralizer C and describe the center of

the algebra C . Finally, we present results that classify all the 1-dimensional C -modules.

Apart from $\mathfrak{sl}_2(K)$, this is the only simple Lie algebra for which the defining relations are known, and the defining relations are one of the main results of this thesis. The results define explicitly the generators and defining relations for the centralizer C of the Cartan subalgebra of the universal enveloping algebra $U(\mathfrak{sl}_3)$ of the Lie algebra $\mathfrak{sl}_3(K)$. It opens a way of classifying simple weight \mathfrak{sl}_3 -modules which is an old open problem in Lie theory that has significant applications in physics. This is the first results on generators and defining relations for the centralizer C of the Cartan subalgebra of the universal enveloping algebra of a simple Lie algebra. It shows the complexity of the algebra C and why the problem of classification of simple weight modules is a difficult problem. The next step is to classify all the finite dimensional simple C -modules in a similar fashion as the 1-dimensional case is done. Then to produce interesting classes of infinite dimensional C -modules, and finally to classify simple weight \mathfrak{sl}_3 -modules.

Let K denote an algebraically closed field of characteristic zero, \mathcal{G} a semisimple Lie algebra over K , \mathcal{H} a fixed Cartan subalgebra, $U = U(\mathcal{G})$ the universal enveloping algebra of the Lie algebra \mathcal{G} and

$$C = \{u \in U(\mathcal{G}) \mid Hu = uH, \text{ for all } H \in \mathcal{H}\}$$

the **centralizer** of \mathcal{H} in $U = U(\mathcal{G})$. A \mathcal{G} -module V is called a **weight module** if

$$V = \bigoplus_{\lambda \in \mathcal{H}^*} V_\lambda$$

where $\mathcal{H}^* = \text{Hom}_K(\mathcal{H}, K)$ and $V_\lambda = \{v \in V \mid hv = \lambda(h)v \text{ for all } h \in \mathcal{H}\}$. If $V_\lambda \neq \{0\}$, we call V_λ a **weight space**.

Theorem 1.2.1 ([4]). *Every weight space V_λ of a simple weight module V for the Lie algebra \mathcal{G} is a simple module over the centralizer C of \mathcal{H} in $U = U(\mathcal{G})$.*

Therefore, our strategy towards the classification of all simple weight \mathcal{G} -modules is to investigate the structure of the centralizer C . The first step of our strategy is to find defining relations for the centralizer C (generators are known for many years, [4]). Until recently, this was done only for $\mathfrak{sl}_2(K)$ where the centralizer is a polynomial algebra $C = K[H, Z]$ where H is a Cartan element ($\mathcal{H} = KH$) and Z is the Casimir element (the center of $U(\mathfrak{sl}_2)$ is $Z(U(\mathfrak{sl}_2)) = K[Z]$).

For a simple Lie algebra \mathcal{G} , finding defining relations for the centralizer C is an old long standing open problem and one of the main obstacles in the way of classifying simple weight \mathcal{G} -modules which is also an old long standing open problem. Very recently, Professor Vladimir Bavula and I found the defining relations for the centralizer in the case of $\mathcal{G} = \mathfrak{sl}_3(K)$.

The prime characteristic case is a completely different case from the case zero as all simple weight modules over a simple Lie algebra are finite dimensional. The future

classification of simple C -modules will certainly depends on whether the field K is algebraically closed or not as the classification of simple 1-dimensional C -modules shows.

Generators and defining relations for the centralizer C of $U(\mathfrak{sl}_3)$. The Lie algebra $\mathfrak{sl}_3(K)$ has a basis

$$\{X_1 = E_{12}, X_2 = E_{23}, X_3 = E_{13}, Y_1 = E_{21}, Y_2 = E_{32}, Y_3 = E_{31}, \\ H_1 = E_{11} - E_{22}, H_2 = E_{22} - E_{33}\}$$

with E_{ij} being the usual matrix units.

The center of the algebra $U = U(\mathfrak{sl}_3)$ is a polynomial algebra $Z(U) = K[Z_1, Z_2]$ in two generators (see [7, Theorem 7.3.8(ii)]). By [10, page 424], these two generators can be taken to be

$$Z_1 = \frac{1}{18}(H_1^2 + H_2^2 + H_1H_2) + \frac{1}{6}(H_1 + H_2) + \frac{1}{6}(Y_1X_1 + Y_2X_2 + Y_3X_3), \\ Z_2 = -\frac{2}{9}H_1^3 + \frac{2}{9}H_2^3 - \frac{1}{3}H_1^2H_2 + \frac{1}{3}H_1H_2^2 - H_1^2 + H_2^2 - H_1 + H_2 - Y_1X_1H_1 - Y_3X_3H_1 \\ + 2Y_2X_2H_1 - 2Y_1X_1H_2 + Y_3X_3H_2 + Y_2X_2H_2 - 3Y_2Y_1X_3 - 3Y_3X_1X_2 + 3Y_2X_2 \\ - 3Y_1X_1 + 3Y_3X_3.$$

Let

$$c_1 = E_{21}E_{12}, c_2 = E_{32}E_{23}, c_3 = E_{31}E_{13}, c = E_{32}E_{21}E_{13} \text{ and } \bar{c} = E_{31}E_{12}E_{23}.$$

In the next theorem, the centralizer C of the Cartan subalgebra \mathcal{H} in $U = U(\mathfrak{sl}_3)$ is described by generators and defining relations. Moreover, some properties of the centralizer C are given. Recall that the universal enveloping algebra $U = U(\mathfrak{sl}_3)$ admits the standard filtration $U = \bigcup_{i \geq 0} U_i$ with respect to the canonical generators of $\mathfrak{sl}_3(K)$ (by the total degree of the canonical generators). For element $a, b \in U$, let $[a, b] = ab - ba$.

Theorem 1.2.2. *(Theorem 5.2.1) Let C be the centralizer of the Cartan subalgebra \mathcal{H} in $U = U(\mathfrak{sl}_3)$.*

1. *The algebra C is generated by the elements $H_1, H_2, c_1, c_2, c_3, c$ and \bar{c} subject to the*

defining relations:

$$\begin{aligned}
[c_1, c_2] &= [c_2, c_3] = [c_3, c_1] = c - \bar{c}, \\
[c_1, c] &= c_1(c_2 - c_3) + (H_1 - 1)c + \bar{c} - H_1c_3, \\
[c_1, \bar{c}] &= -(c_2 - c_3)c_1 - (H_1 - 1)\bar{c} - c + H_1c_3, \\
[c_2, c] &= c_2(c_3 - c_1) + H_2c, \\
[c_2, \bar{c}] &= -(c_3 - c_1)c_2 - H_2\bar{c}, \\
[c_3, c] &= c_3(c_1 - c_2 + H_1) - (H_1 + H_2 + 2)c + 2\bar{c}, \\
[c_3, \bar{c}] &= -(c_1 - c_2 + H_1)c_3 + (H_1 + H_2 + 2)\bar{c} - 2c, \\
c\bar{c} &= (c_3 + H_1 + H_2)c_1c_2 + (-c_1 + c_2 + c_3 + H_2)\bar{c}, \\
\bar{c}c &= (c_2 + H_2)(c_1 + H_1)c_3 + (-c_1 + c_2 + c_3 + H_2)c,
\end{aligned}$$

and the elements H_1 and H_2 belong to the center of the algebra C .

2. $C = \bigoplus_{\alpha_1, \dots, \gamma \in \mathbb{N}} KH_1^{\alpha_1} H_2^{\alpha_2} c_1^{\beta_1} c_2^{\beta_2} c_3^{\beta_3} c^\gamma \oplus \bigoplus_{\alpha_1, \dots, \gamma \in \mathbb{N}} KH_1^{\alpha_1} H_2^{\alpha_2} c_1^{\beta_1} c_2^{\beta_2} c_3^{\beta_3} \bar{c}^\gamma$.
3. The algebra C is an almost commutative, finitely generated, Noetherian domain of Gelfand-Kirillov dimension $\text{GK}(C) = 6$.
4. Let $\{C_i = C \cap U_i\}_{i \geq 0}$ be the induced filtration on the algebra C . Then the associated graded algebra $\text{gr}(C) = \bigoplus_{i \geq 0} C_i/C_{i-1}$ (where $C_{-1} = 0$) is isomorphic to the factor algebra

$$P/(c\bar{c} - c_1c_2c_3)$$

where $P = K[H_1, H_2, c_1, c_2, c_3, c, \bar{c}]$ is a polynomial algebra in seven variables, and

$$\text{gr}(C) = \bigoplus_{\alpha_1, \dots, \gamma \in \mathbb{N}} KH_1^{\alpha_1} H_2^{\alpha_2} c_1^{\beta_1} c_2^{\beta_2} c_3^{\beta_3} c^\gamma \oplus \bigoplus_{\alpha_1, \dots, \gamma \in \mathbb{N}} KH_1^{\alpha_1} H_2^{\alpha_2} c_1^{\beta_1} c_2^{\beta_2} c_3^{\beta_3} \bar{c}^\gamma.$$

Ideas/structure of the proof. (i) The fact that the elements $H_1, H_2, c_1, c_2, c_3, c$ and \bar{c} generate the algebra C were known for many years. These generators can be derived using the basic cycles in $U(\mathfrak{sl}_3)$ (see section 4.3 for details).

Now, straightforward lengthy calculations show that the relations in statement (1) hold (careful and detailed calculations are given in the Appendix).

(ii) Statement (2) holds: Statement (2) follows from the statement (i) and the fact that $\text{gr}(C) \subseteq \text{gr}(U)$.

(iii) To prove that the relations in statement (1) are defining relations of the centralizer C , let C' be an abstract algebra with the same generators as C but the defining relations are as in statement (1).

Then the algebra C is a factor algebra of C' , $C = C'/I$, where I is an ideal of C' . We have to show that $I = 0$, i.e. there are no extra defining relations for the centralizer C . The main ideas to prove this fact are

- to use the PBW Theorem,
- to use the Gelfand-Kirillov dimension,
- to use filtrations on U and C ,
- to use associated graded algebras $\text{gr}(U)$ and $\text{gr}(C)$.

A basis and the center of the centralizer C . In the above theorem, the sets of generators and defining relations of the centralizer C are explicit but very complicated and involved. Therefore, we use the knowledge of the center of the universal enveloping algebra of the Lie algebra $\mathfrak{sl}_3(K)$ to simplify the existing sets of generators and defining relations. Additionally, we give an explicit basis for the centralizer C . Let us consider the elements

$$p := 6Z_1 - \frac{1}{3}\beta \quad \text{where} \quad \beta := H_1^2 + H_2^2 + H_1H_2 + 3(H_1 + H_2), \quad (1.1)$$

$$\begin{aligned} q &:= -\frac{1}{3}Z_2 + \frac{1}{27}(H_2 - H_1)\left(2\beta + 3(H_1H_2 + H_1 + H_2 + 3p + 3)\right) + p \\ &= -\frac{1}{3}Z_2 + \frac{1}{27}(H_2 - H_1)\left(2(H_1^2 + H_2^2) + 5H_1H_2 + 9(H_1 + H_2 + p + 1)\right) + p. \end{aligned} \quad (1.2)$$

Clearly, p and q belong to the center of the algebra C .

Theorem 1.2.3. *(Theorem 5.2.3) Let C be the centralizer of the Cartan subalgebra \mathcal{H} in $U = U(\mathfrak{sl}_3)$. Then*

1. *The algebra C is generated by the elements H_1, H_2, p, q, c_1, c_2 , and c subject to the defining relations:*

$$\begin{aligned} [c_1, c_2] &= 2c + (H_2 + 2)c_1 - H_1c_2 - q, \\ [c_1, c] &= (H_1 - 2)c + c_1(c_1 + 2c_2 + H_1 - H_2 - p - 2) + 2H_1c_2 - H_1p + q, \\ [c_2, c] &= (H_2 + 4)c - c_2(c_2 + 2H_1 - p) - 2c_1(c_2 - H_2 - 2) - 2q, \\ c^2 &+ \left(c_1(H_2 + 4) - c_2(H_1 - 2) + 2H_1 - H_2 - q - p - 4\right)c + \left((H_2 + 2)c_1 + H_1H_2 \right. \\ &+ 2H_1 - 2H_2 - 2q - 4\left.)c_1 + \left((-c_1 - 2H_1 + 2)c_2 - (c_1 + 3H_1 - p - 2)c_1 - H_1H_2 \right. \right. \\ &\left. \left. + 2H_1p - 2H_1^2 - q - 2p\right)c_2 + H_1H_2p + 2H_1p - 2H_1q + pq + 2q = 0, \end{aligned}$$

and the elements H_1, H_2, p and q are central elements of the algebra C .

$$2. \quad C = \bigoplus_{\alpha_i, \beta_j \in \mathbb{N}, \varepsilon \in \{0,1\}} KH_1^{\alpha_1} H_2^{\alpha_2} p^{\alpha_3} q^{\alpha_4} c_1^{\beta_1} c_2^{\beta_2} c^\varepsilon.$$

The next theorem is a description of the center of the algebra C .

Theorem 1.2.4. (Theorem 5.3.5) *Let C be the centralizer of the Cartan subalgebra \mathcal{H} in $U = U(\mathfrak{sl}_3)$. Then the center of the algebra C is a polynomial algebra*

$$K[H_1, H_2, Z_1, Z_2] = K[H_1, H_2, p, q]$$

where Z_1 and Z_2 are the generators of the center $Z(U) = K[Z_1, Z_2]$.

Classification of simple 1-dimensional C -modules. Now, we start our investigation of the classification of all the 1-dimensional C -modules. We show that a complete characterization of this family of 1-dimensional C -modules is determined by finding a complete list of solutions to the following system of equations:

$$\begin{aligned} (c - c_3)H_1 &= (c_3 - c_2)c_1, \\ H_2c &= c_2(c_1 - c_3), \\ c^2 + (c_1 - c_2 - c_3 + H_2)c - c_1c_2c_3 - (H_1 + H_2 + 2)c_1c_2 + 2c_2c_3 &= 0. \end{aligned}$$

A complete list of solutions to the above system of equations is partitioned into four disjoint classes as shown by the next theorem. As a result, we have a classification of all the 1-dimensional C -modules.

Theorem 1.2.5. (Theorem 5.4.3) *The solutions in the field K to the above system of equations are given as follows:*

1. *Suppose that $c - c_3 = 0$ and $c = 0$, i.e. $c = c_3 = 0$.*
 - (a) *If $c_1 = 0$, then the elements H_1 , H_2 and c_2 are arbitrary.*
 - (b) *If $c_1 \neq 0$, then necessarily $c_2 = 0$ and the elements H_1 , H_2 and $c_1 \neq 0$ are arbitrary.*
2. *Suppose that $c - c_3 = 0$ and $c \neq 0$.*
 - (a) *If $c_1 = 0$, then $H_2 = -c_2$ and the elements H_1 , c_2 and $c = c_3 \neq 0$ are arbitrary.*
 - (b) *If $c_1 \neq 0$, then $H_1 = -c_1$, $H_2 = c_1 - c$, and the elements $c_1 \neq 0$ and $c = c_2 = c_3 \neq 0$ are arbitrary.*
3. *Suppose that $c - c_3 \neq 0$ and $c = 0$.*
 - (a) *If $c_2 = 0$, then $H_1 = -c_1$, $c = c_2 = 0$ and the elements H_2 , c_1 and $c_3 \neq 0$ are arbitrary.*
 - (b) *If $c_2 \neq 0$, then $H_1 = c_2 - c_1$, $H_2 = -c_2$, $c = 0$, $c_3 = c_1$ and the elements $c_1 \neq 0$ and $c_2 \neq 0$ are arbitrary.*

4. If $c - c_3 \neq 0$ and $c \neq 0$, then the above system is equivalent to the system

$$\begin{aligned} H_1 &= (c - c_3)^{-1}c_1(c_3 - c_2), \\ H_2 &= c^{-1}c_2(c_1 - c_3), \\ (c - c_2)(c + c_1 - c_3)(c^2 - c_3c - c_1c_2c_3) &= 0. \end{aligned}$$

- (a) If $c - c_2 = 0$, then $H_1 = -c_1$, $H_2 = c_1 - c_3$, $c_2 = c$, $c_3 \neq c$ and the elements c_1 , c_3 and $c \neq 0$ are arbitrary such that $c_3 \neq c$.
- (b) If $c - c_2 \neq 0$ and $c + c_1 - c_3 = 0$, then $H_1 = c_2 - c_3$, $H_2 = -c_2$, $c_1 = c_3 - c$ and the elements c_2 , c_3 and c are arbitrary such that $c \neq c_2$, $c \neq c_3$ and $c \neq 0$.
- (c) Suppose that $c - c_2 \neq 0$, $c + c_1 - c_3 \neq 0$ and $c^2 - c_3c - c_1c_2c_3 = 0$.
- i. If $c_3 = -4c_1c_2$, then $H_1 = -2c_1 - \frac{1}{2}$, $H_2 = -2c_2 - \frac{1}{2}$, $c_3 = -4c_1c_2$, $c = -2c_1c_2$, and the elements $c_1 \neq 0$, $-\frac{1}{2}$ and $c_2 \neq 0$, $-\frac{1}{2}$ are arbitrary.
- ii. If $c_3 \neq -4c_1c_2$, then
- $$c_{\pm} = \frac{1}{2}(c_3 \pm \sqrt{c_3(c_3 + 4c_1c_2)}), \quad H_1 = (c_{\pm} - c_3)^{-1}c_1(c_3 - c_2) \text{ and}$$
- $$H_2 = c_{\pm}^{-1}c_2(c_1 - c_3), \text{ and the elements } c_1 \neq 0, c_2 \neq 0 \text{ and } c_3 \neq 0 \text{ are}$$
- arbitrary such that $c_3 \neq -4c_1c_2$ and $c_1 \neq (c_2 + 1)c_3$.

1.3 Structure of the thesis

The contents of this thesis is summarized as follows:

In chapter 2, first we review some of the most important notions and results on Lie algebras and their representation theory with an emphasis on semisimple finite dimensional Lie algebras over an algebraically closed field of characteristic zero. Then we introduce some standard results on the Gelfand-Kirillov dimension of finitely generated algebras. Moreover, we present some basic properties of the Gelfand-Kirillov dimension of algebras and modules.

Chapter 3 is dedicated to studying the structure of simple weight modules over finite dimensional semisimple Lie algebras. In this chapter, we review some results of Fernando's approach [9] to classifying simple modules that decompose into direct sums of finite dimensional weight spaces. He reduces the classification of simple modules that decompose into direct sums of finite dimensional weight spaces to the classification of torsion-free modules with finite dimensional weight spaces. In addition, Fernando shows that if \mathcal{G} is a simple Lie algebra that admits a torsion-free \mathcal{G} -module with a finite dimensional weight space, then \mathcal{G} is either of type A or of type C .

In chapter 4, we study the structure of basic cycles. The importance of basic cycles is that they form a generating set for the centralizer of the Cartan subalgebra in the universal enveloping algebra of a simple Lie algebra. In this chapter, we briefly review some results obtained by Britten and Lemire [4].

Chapters 5 and 6 contain the main results of this thesis. In chapter 5, we investigate the structure of the centralizer C of the Cartan subalgebra of the Lie algebra $\mathfrak{sl}_3(K)$ in the universal enveloping algebra $U(\mathfrak{sl}_3)$. In particular, we first give sets of generators and defining relations of the centralizer C and find an explicit basis for the centralizer C . Then we describe the center of the algebra C . Finally, we classify all the 1-dimensional C -modules. In chapter 6, we provide detailed calculations of the results in chapter 5 and the code in the computer algebra system SINGULAR to verify the results of our computations.

1.4 Notation

Unless stated otherwise, the following notation remain fixed throughout this thesis. K denotes an algebraically closed field of characteristic zero. All algebras and vector spaces are over the ground field K . The symbols \mathbb{Z} and \mathbb{N} denote the set of integers and the set of nonnegative integers, respectively. The symbol \subset indicates a proper inclusion. The notation $\dim_K(-)$ refers to the vector space dimension over the field K . A ring A is unital, associative and not necessarily commutative unless stated otherwise. Module means a left module. A module M over a ring A assumed to be unital, i.e. $1m = m$ for each $m \in M$. The notation ${}_A M$ indicates a (left) module M over the ring A . A submodule L of an A -module M is called a **direct summand** of M if

$$M = L \oplus N$$

for some submodule N of M . An element $x \in A$ is **central** if

$$xa = ax \text{ for all } a \in A.$$

The set of all central elements is called the **center** of A , denoted $Z(A)$. An element $x \in A$ is **left regular** if $ax = 0$ implies $a = 0$ for $a \in A$. Similarly, a right regular element is defined. A **regular element** is an element that is both left and right regular. A **multiplicative set** in a ring A is a subset $S \subseteq A$ such that $1 \in S$, $0 \notin S$ and S is closed under multiplication. A left ideal I of a ring A is called **essential** if for all nonzero left ideals J of A , one has $I \cap J \neq 0$. If A is a ring and G is a group, then the **group ring**, denoted AG , is defined to be a free left A -module with the elements of G as its basis and with multiplication given by the rule

$$a_1 g_1 \cdot a_2 g_2 = a_1 a_2 g_1 g_2 \text{ where } a_1, a_2 \in A, g_1, g_2 \in G,$$

together with bilinearity.

The n^{th} **Weyl algebra** over K denoted by $A_n(K)$ is defined to be the K -algebra with $2n$ generators $x_1, \dots, x_n, \partial_1, \dots, \partial_n$ subject to the defining relations:

$$\begin{aligned} \partial_i x_j - x_j \partial_i &= \delta_{ij}, \text{ the Kronecker delta,} \\ x_i x_j - x_j x_i &= \partial_i \partial_j - \partial_j \partial_i = 0, \end{aligned}$$

for all $i, j = 1, \dots, n$.

Let S be a monoid. A K -algebra A is called **S -graded** if

$$A = \bigoplus_{s \in S} A_s$$

for K -modules A_s such that $A_s A_{s'} \subseteq A_{s+s'}$ for all $s, s' \in S$. A **graded homomorphism** $\varphi : A \rightarrow B$ of S -graded algebras is an algebra homomorphism satisfying $\varphi(A_s) \subseteq B_s$ for all $s \in S$. An ideal I of an S -graded algebra A is called a **graded ideal** if

$$I = \sum (I \cap A_s).$$

A module V over an S -graded algebra A is graded if

$$V = \bigoplus_{s \in S} V_s \quad \text{and} \quad A_s V_{s'} \subseteq V_{s+s'} \quad \text{for all } s, s' \in S.$$

A filtration of a K -algebra A is a family $\{A_n\}_{n \geq 0}$ of subgroups of the additive group of A such that

1. $A_i \subseteq A_j$ for $i < j$,
2. $A_i A_j \subseteq A_{i+j}$ for all i, j , and
3. $A = \bigcup A_i$.

In this case, A is called a **filtered algebra**. Similarly, we can define a filtration of a module V over a filtered algebra A as a family of additive subgroups $\{V_n\}_{n \geq 0}$ of V such that

1. $V_i \subseteq V_j$ for $i < j$,
2. $A_i V_j \subseteq V_{i+j}$ for all i, j , and
3. $V = \bigcup V_i$.

In this case, V is called a **filtered module**. If A is a K -algebra with a filtration $\{A_n\}$, then we can construct a graded ring

$$\text{gr}(A) := \bigoplus_{i \geq 0} A_i / A_{i-1} \quad \text{where } A_{-1} := 0.$$

The multiplication in $\text{gr}(A)$ is defined by

$$(a + A_{i-1})(b + A_{j-1}) = ab + A_{i+j-1} \quad \text{for all } a, b \in A \text{ and all } i, j.$$

The algebra $\text{gr}(A)$ is called the **associated graded algebra**. Similarly, if M is a module over the algebra A with a filtration $\{M_n\}$, then the K -vector space

$$\text{gr}(M) := \bigoplus_{i \geq 0} M_i/M_{i-1} \text{ where } M_{-1} := 0,$$

is a $\text{gr}(A)$ -module where

$$(a + M_{i-1})(m + M_{j-1}) = am + M_{i+j-1} \text{ for all } a \in M_i \text{ and } m \in M_j \text{ and all } i, j.$$

It is called the **associated graded module** over the algebra $\text{gr}(A)$.

Chapter 2

Background

2.1 Lie algebras and their representation theory

The purpose of this section is to provide some background material on Lie algebras and their representation theory. The basic references we use in this thesis for Lie algebras and their representations are the book of Dixmier [7] and the book of Humphrey [16]. We refer the reader for details and proofs of most results in this section to these two references.

Unless stated otherwise, K is an **algebraically closed field of characteristic zero** and \mathcal{G} is a **finite dimensional Lie algebra** over K .

2.1.1 Review of Lie algebras

Definition 2.1.1. A **Lie algebra** \mathcal{G} is a vector space over a field K together with a bilinear map $\mathcal{G} \times \mathcal{G} \rightarrow \mathcal{G}$, denoted $(x, y) \rightarrow [x, y]$ that satisfies:

1. $[x, x] = 0$ for all $x \in \mathcal{G}$, and
2. $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$ (**Jacobi identity**) for all $x, y, z \in \mathcal{G}$.

The product $[x, y]$ is called the **bracket** of x and y .

Definition 2.1.2. Let I and J be vector subspaces of a Lie algebra \mathcal{G} over K , and let $[I, J]$ denote the set of all linear combinations of elements of the form $[a, b]$, where $a \in I$ and $b \in J$. A vector subspace \mathcal{G}' of \mathcal{G} is called a **Lie subalgebra** if

$$[\mathcal{G}', \mathcal{G}'] \subseteq \mathcal{G}'.$$

A vector subspace I of \mathcal{G} is called an **ideal** of the Lie algebra \mathcal{G} if $[\mathcal{G}, I] \subseteq I$. If I is an ideal of \mathcal{G} , then the vector space \mathcal{G}/I with the bracket defined by

$$[x + I, y + I] = [x, y] + I \quad \text{for all } x, y \in \mathcal{G},$$

is a Lie algebra, called the **quotient Lie algebra** of \mathcal{G} by I .

We now provide some interesting examples of Lie algebras that are relevant to the content of the rest of this thesis.

- Examples 2.1.3.** 1. Any associative algebra A over a field K may be transformed into a Lie algebra by replacing its product with $[x, y] = xy - yx$ for all $x, y \in A$.
2. Let $\text{End}(V)$ be the algebra of endomorphisms of a vector space V . Let

$$[x, y] = xy - yx$$

for all $x, y \in \text{End}(V)$. With respect to this product, $\text{End}(V)$ becomes a Lie algebra, denoted $\mathfrak{gl}(V)$.

3. Let $M_n(K)$ be the set of all $n \times n$ matrices over a field K . Let

$$[A, B] = AB - BA$$

for all $A, B \in M_n(K)$. With respect to this product, $M_n(K)$ becomes a Lie algebra, denoted $\mathfrak{gl}_n(K)$ and called the **general linear Lie algebra**.

4. Let $\mathfrak{sl}_n(K) = \{A \in M_n(K) \mid \text{tr}(A) = 0\}$ be the subset of $M_n(K)$ of all matrices with zero trace (where $\text{tr}(A) = \sum_{i=1}^n a_{ii}$). Then $\mathfrak{sl}_n(K)$ is a Lie subalgebra of $\mathfrak{gl}_n(K)$, and it is called the **special linear Lie algebra**. In particular, the Lie algebra $\mathfrak{sl}_2(K)$ has a basis

$$e = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad f = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad h = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

An easy computation in $M_2(K)$ yields the following set of relations:

$$[h, e] = 2e, \quad [h, f] = -2f, \quad [e, f] = h.$$

Definition 2.1.4. Let \mathcal{G}' be a Lie subalgebra of a Lie algebra \mathcal{G} . The **normalizer** of \mathcal{G}' in \mathcal{G} is defined by $N_{\mathcal{G}}(\mathcal{G}') = \{a \in \mathcal{G} \mid [a, \mathcal{G}'] \subseteq \mathcal{G}'\}$.

Notice that if \mathcal{G}' is a Lie subalgebra of the Lie algebra \mathcal{G} , then $N_{\mathcal{G}}(\mathcal{G}')$ is a Lie subalgebra of \mathcal{G} .

Definition 2.1.5. Given Lie algebras \mathcal{G}_1 and \mathcal{G}_2 over K , a linear map $\varphi : \mathcal{G}_1 \rightarrow \mathcal{G}_2$ is called a **homomorphism** of Lie algebras if

$$\varphi([x, y]) = [\varphi(x), \varphi(y)] \text{ for all } x, y \in \mathcal{G}_1.$$

A homomorphism φ is called a **monomorphism** if $\ker(\varphi) = 0$, an **epimorphism** if $\text{im}(\varphi) = \mathcal{G}_2$ and an **isomorphism** if it is bijective, i.e. if $\ker(\varphi) = 0$ and $\text{im}(\varphi) = \mathcal{G}_2$.

Definition 2.1.6. Let \mathcal{G}_1 and \mathcal{G}_2 be Lie algebras. Then the vector space $\mathcal{G}_1 \oplus \mathcal{G}_2$ provided with the bracket

$$[(x, x'), (y, y')] = ([x, y], [x', y']) \quad \text{for } x, y \in \mathcal{G}_1, \quad x', y' \in \mathcal{G}_2$$

is a Lie algebra, called the **direct sum** of the Lie algebras \mathcal{G}_1 and \mathcal{G}_2 . In a similar manner, we can define the direct sum of a finite number of Lie algebras.

Definition 2.1.7. A **derivation** of the Lie algebra \mathcal{G} is a linear map $\delta : \mathcal{G} \rightarrow \mathcal{G}$ such that

$$\delta([x, y]) = [\delta(x), y] + [x, \delta(y)] \quad \text{for all } x \text{ and } y \text{ in } \mathcal{G}.$$

Remarks 2.1.8. 1. Let $\text{Der}(\mathcal{G})$ be the set of all derivations of the Lie algebra \mathcal{G} .

Then $\text{Der}(\mathcal{G})$ contains the zero map, and it is closed under addition and scalar multiplication. Hence $\text{Der}(\mathcal{G})$ is a Lie subalgebra of the Lie algebra $\text{gl}(\mathcal{G})$, for if δ and δ' are derivations, then so is $[\delta, \delta'] = \delta\delta' - \delta'\delta$.

2. If \mathcal{G} is a Lie algebra, the maps $\text{ad}_x : \mathcal{G} \rightarrow \mathcal{G}$ for $x \in \mathcal{G}$ defined by $\text{ad}_x(y) := [x, y]$ are derivations of \mathcal{G} ; these derivations are called **inner derivations** of \mathcal{G} , all other are called **outer**.

Definition 2.1.9. Let \mathcal{G} be a Lie algebra. A Lie homomorphism $\varphi : \mathcal{G} \rightarrow \text{gl}(V)$ is called a **representation** of \mathcal{G} , where V is a vector space over K .

Remark 2.1.10. The dimension of the vector space V in Definition 2.1.9 is called the dimension of the representation φ .

Definition 2.1.11. Let \mathcal{G} be a Lie algebra. A vector space V together with an operation $\mathcal{G} \times V \rightarrow V$, $((x, v) \mapsto xv)$ is called a **Lie module** for \mathcal{G} , or simply a **\mathcal{G} -module**, if the following conditions are satisfied:

1. $(ax + by)v = a(xv) + b(yv)$,
2. $x(av + bw) = a(xv) + b(xw)$, and
3. $[x, y]v = x(yv) - y(xv)$,

for all elements $x, y \in \mathcal{G}, v, w \in V$ and $a, b \in K$.

Remarks 2.1.12. 1. Every Lie algebra \mathcal{G} is actually a \mathcal{G} -module over itself under the action $(x, y) \mapsto \text{ad}_x(y) := [x, y]$. This action is called the **adjoint \mathcal{G} -module**.

2. If $\varphi : \mathcal{G} \rightarrow \text{gl}(V)$ is a representation of \mathcal{G} , then V can be viewed as a \mathcal{G} -module via the action $xv = \varphi(x)(v)$. Conversely, if V is a \mathcal{G} -module, then $\varphi(x)(v) = xv$ defines a representation $\varphi : \mathcal{G} \rightarrow \text{gl}(V)$ of \mathcal{G} . Therefore, we may use the notations of modules and representations interchangeably throughout the remainder of this thesis.

Given a \mathcal{G} -module M , a **submodule** N of M is a subset of M such that N is a \mathcal{G} -module. If N is a submodule of a \mathcal{G} -module M , then the quotient vector space M/N with the action of \mathcal{G} defined by

$$x(m + N) = xm + N$$

is a \mathcal{G} -module, called a **quotient module**. A \mathcal{G} -module M is said to be **simple** if 0 and M are the only submodules of M . A **homomorphism of \mathcal{G} -modules** M and W is a linear map $\psi : M \rightarrow W$ such that

$$\psi(xm) = x\psi(m) \text{ for all } x \in \mathcal{G}, m \in M.$$

If the homomorphism $\psi : M \rightarrow W$ is bijective, then ψ is called an **isomorphism of \mathcal{G} -modules**.

Definition 2.1.13 ([7, p.5, 1.2.6]). Let φ be a representation of \mathcal{G} in V . A **Jordan–Hölder series** for the representation V is a series of submodules (V_0, V_1, \dots, V_n) such that

$$V = V_0 \supset V_1 \supset \dots \supset V_n = 0$$

and V_i/V_{i+1} is simple for $i = 0, 1, \dots, n - 1$.

Proposition 2.1.14 ([7, p.5, 1.2.6]). *If φ is a finite dimensional representation of a Lie algebra \mathcal{G} , then a Jordan–Hölder series exists. However, it may not exist if φ is an infinite dimensional representation of \mathcal{G} .*

The next proposition will be used in the classification of simple 1-dimensional modules over the centralizer C of the Cartan subalgebra in $U(\mathfrak{sl}_3)$ (see section 5.4). Especially, in producing the defining relations of the abelianization of the algebra C (where all commutators are put to zero).

Proposition 2.1.15 ([7, p.7, 1.2.12]). *A 1-dimensional module over a Lie algebra \mathcal{G} over a field K can be identified with a linear map $\rho : \mathcal{G} \rightarrow K$ such that $\rho([\mathcal{G}, \mathcal{G}]) = 0$.*

The next lemma is a key result in proving that the sum of weight subspaces in a weight module is a direct sum. This fact reduces the problem of classification of simple weight modules to a problem of classification of all simple modules over the centralizer.

Lemma 2.1.16 ([7, p.7, 1.2.13]). *Let V be a \mathcal{G} -module, \mathcal{B} a Lie subalgebra of \mathcal{G} and $\lambda \in \mathcal{B}^* = \text{Hom}_K(\mathcal{B}, K)$. Then the set*

$$V_\lambda = \{v \in V \mid xv = \lambda(x)v \text{ for all } x \in \mathcal{B}\}$$

is a vector subspace of V and the sum of all V_λ is a direct sum.

Definition 2.1.17 ([7, p.7, 1.2.10]). Let V be a \mathcal{G} -module. An element $v \in V$ is an **invariant** of the \mathcal{G} -module V if $\mathcal{G}v = 0$. The set of all invariant elements of V is denoted by $V^\mathcal{G}$.

If V and W are \mathcal{G} -modules, and $V \otimes W$ is the tensor product over K of the underlying vector spaces, then $V \otimes W$ becomes a \mathcal{G} -module by setting

$$x(v \otimes w) = xv \otimes w + v \otimes xw \quad \text{for all } x \in \mathcal{G}, v \in V, \text{ and } w \in W.$$

Let V and W be \mathcal{G} -modules. The vector space $\text{Hom}_K(V, W)$ of linear maps is a \mathcal{G} -module via the action

$$(x\rho)(v) = x\rho(v) - \rho(xv) \quad \text{for all } x \in \mathcal{G}, \rho \in \text{Hom}_K(V, W), \text{ and } v \in V.$$

In particular, the \mathcal{G} -module homomorphisms from V to W are simply the elements of the \mathcal{G} -module $\text{Hom}_K(V, W)$ which are invariant (*i.e.* $\text{Hom}_{\mathcal{G}}(V, W) = \text{Hom}_K(V, W)^{\mathcal{G}}$) (see [7, p.8, 1.2.15]).

If V is a \mathcal{G} -module, then the dual vector space $V^* = \text{Hom}_K(V, K)$ becomes a \mathcal{G} -module via the action

$$(x\rho)(v) = -\rho(xv) \quad \text{for all } x \in \mathcal{G}, \rho \in V^*, v \in V.$$

The \mathcal{G} -module V^* is called the **dual \mathcal{G} -module** of V .

Let V and W be finite dimensional \mathcal{G} -modules. There is a vector space isomorphism

$$\text{Hom}_K(V, W) \rightarrow V^* \otimes W \quad \text{defined by } \rho \mapsto \sum_i v_i^* \otimes \rho(v_i)$$

where $\{v_i\}$ and $\{v_i^*\}$ are bases of V and V^* , respectively. This isomorphism is in fact a \mathcal{G} -module isomorphism.

Definition 2.1.18. Define a sequence of ideals of \mathcal{G} (the **derived series**) by

$$\mathcal{G}^{(0)} = \mathcal{G}, \mathcal{G}^{(1)} = [\mathcal{G}, \mathcal{G}], \mathcal{G}^{(2)} = [\mathcal{G}^{(1)}, \mathcal{G}^{(1)}], \dots, \mathcal{G}^{(i)} = [\mathcal{G}^{(i-1)}, \mathcal{G}^{(i-1)}].$$

A Lie algebra \mathcal{G} is called **solvable** Lie algebra if $\mathcal{G}^{(n)} = 0$ for some n .

Definition 2.1.19. Define a sequence of ideals of \mathcal{G} (the **lower central series**) by

$$\mathcal{G}^0 = \mathcal{G}, \mathcal{G}^1 = [\mathcal{G}, \mathcal{G}], \mathcal{G}^2 = [\mathcal{G}, \mathcal{G}^1], \dots, \mathcal{G}^i = [\mathcal{G}, \mathcal{G}^{i-1}].$$

A Lie algebra \mathcal{G} is called a **nilpotent** Lie algebra if $\mathcal{G}^n = 0$ for some n .

Proposition 2.1.20. *Every nilpotent Lie algebra is solvable.*

Definition 2.1.21. Let \mathcal{G} be a Lie algebra and $x \in \mathcal{G}$. Then the element x is called **ad-nilpotent** if ad_x is a nilpotent endomorphism.

Theorem 2.1.22. (Engel) *If all elements of a Lie algebra \mathcal{G} are ad-nilpotent, then \mathcal{G} is a nilpotent Lie algebra.*

Theorem 2.1.23. (Lie's Theorem) *Let \mathcal{G} be a solvable subalgebra of the Lie algebra $\text{gl}(V)$ where V is a finite dimensional vector space. If $V \neq 0$, then V contains a common eigenvector for all the endomorphisms in \mathcal{G} .*

Corollary 2.1.24. *Let \mathcal{G} be a solvable subalgebra of the Lie algebra $\mathfrak{gl}(V)$ where V is a finite dimensional vector space. Then \mathcal{G} stabilizes some flag in V (in other words, the matrices of \mathcal{G} relative to a suitable basis of V are upper triangular).*

Definition 2.1.25. Let φ be a finite dimensional representation of \mathcal{G} . Let us consider a map

$$b(\cdot, \cdot) : \mathcal{G} \times \mathcal{G} \rightarrow K, \quad b(x, y) = \text{tr}(\varphi(x) \circ \varphi(y)) \text{ for all } x, y \in \mathcal{G}$$

where tr denotes the trace of a linear transformation. Then b is a symmetric bilinear form on \mathcal{G} and is said to be associated with φ . For all elements $x, y, z \in \mathcal{G}$,

$$b(x, y) = b(y, x) \quad \text{and} \quad b([x, y], z) = b(x, [y, z]).$$

Definition 2.1.26. Let \mathcal{G} be a finite dimensional Lie algebra. Then the bilinear form associated with the adjoint representation of \mathcal{G} is called the **Killing form** of \mathcal{G} .

The radical ideal.

Proposition 2.1.27. *Let \mathcal{G} be a Lie algebra. Then there is a unique maximal solvable ideal of \mathcal{G} that contains all the solvable ideals of \mathcal{G} .*

Definition 2.1.28. Let \mathcal{G} be a Lie algebra. The unique maximal solvable ideal of \mathcal{G} is called the **radical** of \mathcal{G} and denoted $\text{Rad}(\mathcal{G})$.

Proposition 2.1.29. *Let \mathcal{G} be a Lie algebra. Then $\text{Rad}(\mathcal{G})$ is the smallest ideal I of \mathcal{G} such that the radical of \mathcal{G}/I is zero.*

Semisimple Lie algebras.

Theorem 2.1.30. *Let \mathcal{G} be a Lie algebra. The following statements are equivalent:*

1. *The radical of \mathcal{G} is the zero ideal.*
2. *Every commutative ideal of \mathcal{G} is the zero ideal.*
3. *The Killing form of \mathcal{G} is nondegenerate.*

Definition 2.1.31. Let \mathcal{G} be a Lie algebra. If one of the statements of Theorem 2.1.30 is satisfied, then \mathcal{G} is called **semisimple**.

Proposition 2.1.32. *Let $\mathcal{G}_1, \dots, \mathcal{G}_m$ be Lie algebras. Then the direct sum $\mathcal{G}_1 \oplus \dots \oplus \mathcal{G}_m$ is semisimple if and only if the Lie algebras $\mathcal{G}_1, \dots, \mathcal{G}_m$ are semisimple.*

Proposition 2.1.33. *Let \mathcal{B} be an ideal of a semisimple Lie algebra \mathcal{G} , and let κ be the Killing form of \mathcal{G} . Then*

1. *the Lie subalgebra \mathcal{B} and the Lie algebra \mathcal{G}/\mathcal{B} are semisimple, and*

2. if J is the orthogonal subspace of \mathcal{B} with respect to the Killing form κ , then J is an ideal of \mathcal{G} which is complementary to \mathcal{B} , so that $\mathcal{G} = \mathcal{B} \oplus J$.

Theorem 2.1.34. Let \mathcal{G} be a semisimple Lie algebra. Then

1. $[\mathcal{G}, \mathcal{G}] = \mathcal{G}$, and
2. $\text{Der}(\mathcal{G}) = \text{ad}(\mathcal{G}) := \{\text{ad}_x \mid x \in \mathcal{G}\}$ (i.e. every derivation of the Lie algebra \mathcal{G} is an inner derivation).

Definition 2.1.35. Let \mathcal{G} be a Lie algebra. Then \mathcal{G} is called **simple** if $[\mathcal{G}, \mathcal{G}] \neq 0$, and \mathcal{G} has no ideals except 0 and \mathcal{G} .

Theorem 2.1.36. Let \mathcal{G} be a Lie algebra. Then \mathcal{G} is semisimple if and only if \mathcal{G} is a direct sum of simple Lie algebras.

Proposition 2.1.37. Let $\mathcal{G}_1, \dots, \mathcal{G}_m$ be simple Lie algebras and $L = \mathcal{G}_1 \oplus \dots \oplus \mathcal{G}_m$. Then the ideals of L are the direct sum of some of the Lie algebras \mathcal{G}_i . In particular, the Lie algebras $\mathcal{G}_1, \dots, \mathcal{G}_m$ are the nonzero minimal ideals of L .

Theorem 2.1.38. Let \mathcal{G} be a semisimple Lie algebra. Then every finite dimensional module over \mathcal{G} is a semisimple module (i.e. a direct sum of simple modules).

Definition 2.1.39. Let \mathcal{G} be a semisimple Lie algebra. A nilpotent Lie subalgebra \mathcal{H} of \mathcal{G} which is equal to its normalizer is called a **Cartan subalgebra** of the Lie algebra \mathcal{G} .

Remark 2.1.40. Every semisimple Lie algebra over a field of characteristic zero contains a Cartan subalgebra.

Theorem 2.1.41. Let K be an algebraically closed field of characteristic zero and let \mathcal{G} be a semisimple Lie algebra over K . Then any two Cartan subalgebras of \mathcal{G} , \mathcal{H}_1 and \mathcal{H}_2 , are conjugate (i.e. there exists an automorphism $\sigma : \mathcal{G} \rightarrow \mathcal{G}$ such that $\sigma(\mathcal{H}_1) = \mathcal{H}_2$).

Remark 2.1.42. Let K be an algebraically closed field of characteristic zero and let \mathcal{G} be a semisimple Lie algebra over K . Then the common dimension l of all Cartan subalgebras of \mathcal{G} is called the **rank** of \mathcal{G} .

Theorem 2.1.43. Let \mathcal{H} be a fixed Cartan subalgebra of a semisimple Lie algebra \mathcal{G} . We consider the adjoint representation $\text{ad}_{\mathcal{G}}(\mathcal{H})$ of \mathcal{H} in \mathcal{G} . Then

1. $\mathcal{G} = \bigoplus_{\lambda \in \mathcal{H}^*} \mathcal{G}_{\lambda}$ where $\mathcal{G}_{\lambda} = \{x \in \mathcal{G} \mid hx = \lambda(h)x \text{ for all } h \in \mathcal{H}\}$,
2. $[\mathcal{G}_{\lambda}, \mathcal{G}_{\mu}] \subseteq \mathcal{G}_{\lambda+\mu}$; in particular $[\mathcal{G}_0, \mathcal{G}_{\mu}] \subseteq \mathcal{G}_{\mu}$, and
3. $\mathcal{G}_0 = \mathcal{H}$.

Corollary 2.1.44. A Cartan subalgebra \mathcal{H} of a semisimple Lie algebra \mathcal{G} is a maximal nilpotent Lie subalgebra of \mathcal{G} .

Proposition 2.1.45. *With the notation of Theorem 2.1.43, let κ be the Killing form of \mathcal{G} . Then*

1. $\kappa(\mathcal{G}_\lambda, \mathcal{G}_\mu) = 0$ if $\lambda \neq -\mu$,
2. $\kappa(\mathcal{H}, \mathcal{G}_\mu) = 0$ if $\mu \neq 0$, and
3. if $x, y \in \mathcal{H}$, then $\kappa(x, y) = \sum_{\lambda \in \mathcal{H}^*} \dim_K(\mathcal{G}_\lambda) \lambda(x) \lambda(y)$.

The system of roots of a split semisimple Lie algebra.

Throughout this subsection, K denotes a field of characteristic zero unless otherwise specified.

Definition 2.1.46. A Cartan subalgebra \mathcal{H} of a finite dimensional Lie algebra \mathcal{G} over K is called **splitting** if for all $x \in \mathcal{H}$, the eigenvalues of $\text{ad}(x)$ belong to K . A finite dimensional semisimple Lie algebra \mathcal{G} over K is called a **split semisimple Lie algebra** if \mathcal{G} contains a splitting Cartan subalgebra.

Definition 2.1.47. Let \mathcal{H} be a splitting Cartan subalgebra of \mathcal{G} . The set of all nonzero elements $\lambda \in \mathcal{H}^*$ such that $\mathcal{G}_\lambda \neq 0$ is denoted by Φ , and an element $\lambda \in \Phi$ is called a **root** of \mathcal{G} relative to \mathcal{H} .

Remark 2.1.48. The set Φ is finite since \mathcal{G} is a finite dimensional Lie algebra.

Theorem 2.1.49. *Let \mathcal{G} be a split semisimple Lie algebra with Cartan subalgebra \mathcal{H} and the set of roots Φ , and let κ be the Killing form of \mathcal{G} . Then*

1. $\mathcal{G} = \mathcal{H} \oplus \bigoplus_{\lambda \in \Phi} \mathcal{G}_\lambda$, and $\dim_K(\mathcal{G}_\lambda) = 1$ for all $\lambda \in \Phi$,
2. the Cartan subalgebra \mathcal{H} is commutative. If $x \in \mathcal{G}_\lambda$ and $h \in \mathcal{H}$, then $[h, x] = \lambda(h)x$. If $\lambda, \mu \in \Phi$, then

$$[\mathcal{G}_\lambda, \mathcal{G}_\mu] \subseteq \mathcal{G}_{\lambda+\mu}.$$

If $\lambda \in \Phi$, then $-\lambda \in \Phi$, and $\mathcal{H}_\lambda = [\mathcal{G}_\lambda, \mathcal{G}_{-\lambda}]$ is a one dimensional vector subspace of \mathcal{H} ; there exists a unique $H_\lambda \in \mathcal{H}_\lambda$ such that $\lambda(H_\lambda) = 2$,

3. if $\lambda, \mu \in \mathcal{H}^*$ and $\lambda + \mu \neq 0$, then \mathcal{G}_λ and \mathcal{G}_μ are orthogonal with respect to the Killing form κ . The restriction of κ to $\mathcal{G}_\lambda \oplus \mathcal{G}_{-\lambda}$ (in particular, to $\mathcal{H} \oplus \mathcal{H}$) is nondegenerate. If $x, y \in \mathcal{H}$, then

$$\kappa(x, y) = \sum_{\lambda \in \Phi} \lambda(x) \lambda(y),$$

4. \mathcal{H}^* is spanned by the elements of Φ , and

5. if $\lambda \in \Phi$, and X_λ is any nonzero element of \mathcal{G}_λ , then there exists a unique element $Y_\lambda \in \mathcal{G}_{-\lambda}$ such that $[X_\lambda, Y_\lambda] = H_\lambda$. The vector subspace

$$S_\lambda = \mathcal{H}_\lambda + \mathcal{G}_\lambda + \mathcal{G}_{-\lambda}$$

is a Lie subalgebra of \mathcal{G} isomorphic to the Lie algebra $\mathfrak{sl}_2(K)$ via $X_\lambda \mapsto e$, $Y_\lambda \mapsto f$, $H_\lambda \mapsto h$.

Remarks 2.1.50. 1. Since the restriction of the Killing form of \mathcal{G} to $\mathcal{H} \oplus \mathcal{H}$ is nondegenerate, there exists for all $\lambda \in \mathcal{H}^*$ a unique $h_\lambda \in \mathcal{H}$ such that

$$\lambda(h) = \kappa(h_\lambda, h) \text{ for all } h \in \mathcal{H}.$$

The map $\lambda \mapsto h_\lambda$ is a vector space isomorphism of \mathcal{H}^* onto \mathcal{H} .

2. If we denote the Killing form of \mathcal{G} by $\langle -, - \rangle$, and for $\alpha, \beta \in \mathcal{H}^*$, we set

$$\langle \alpha, \beta \rangle = \langle h_\alpha, h_\beta \rangle = \alpha(h_\beta) = \beta(h_\alpha),$$

then $\langle -, - \rangle$ is a nondegenerate symmetric bilinear form on \mathcal{H}^* .

3. Let H_λ be as in part 2 of Theorem 2.1.49. We have $\langle h_\lambda, h_\lambda \rangle = \lambda(\lambda) \neq 0$, and

$$H_\lambda = \frac{2h_\lambda}{\langle h_\lambda, h_\lambda \rangle}, \text{ and hence } h_\lambda = \frac{2H_\lambda}{\langle H_\lambda, H_\lambda \rangle}.$$

4. If $\lambda \in \Phi$, and X_λ is any nonzero element of \mathcal{G}_λ , then we identify $KH_\lambda + \mathcal{G}_\lambda + \mathcal{G}_{-\lambda}$ with $\mathfrak{sl}_2(K)$ under the isomorphism in part 5 of Theorem 2.1.49.

Proposition 2.1.51. *Let \mathcal{G} be a split semisimple Lie algebra with Cartan subalgebra \mathcal{H} and the set of roots Φ , and let $\lambda, \mu \in \Phi$. Then*

1. the scalar $\mu(H_\lambda)$ is an integer $a_{\mu\lambda}$,
2. the set of integers k such that $\mu + k\lambda \in \Phi \cup \{0\}$ is an interval $[-r, s]$, where $r, s \geq 0$. We have $a_{\mu\lambda} = r - s$,
3. $\mu - a_{\mu\lambda}\lambda \in \Phi$,
4. if $\mu - \lambda \notin \Phi \cup \{0\}$, then $a_{\mu\lambda} \leq 0$, $r = 0$, $s = -a_{\mu\lambda}$,
5. if $\mu + \lambda \in \Phi$, then $[\mathcal{G}_\lambda, \mathcal{G}_\mu] = \mathcal{G}_{\lambda+\mu}$, and
6. if $\lambda \in \Phi$, then $-\lambda \in \Phi$, but no other scalar multiple of λ is a root.

Remarks 2.1.52. 1. The integers $a_{\mu\lambda}$ in part 1 of Proposition 2.1.51 are called the **Cartan integers**. If $\lambda \in \Phi$, then $a_{\lambda\lambda} = 2$. On the other hand,

$$a_{\mu\lambda} = \langle h_\mu, H_\lambda \rangle = 2 \frac{\langle h_\mu, h_\lambda \rangle}{\langle h_\lambda, h_\lambda \rangle} = 2 \frac{\langle \mu, \lambda \rangle}{\langle \lambda, \lambda \rangle} = 2 \frac{\langle H_\mu, H_\lambda \rangle}{\langle H_\mu, H_\mu \rangle}.$$

2. Let $\lambda \in \Phi$, and let s_λ denote the endomorphism of the vector space \mathcal{H}^* defined by

$$s_\lambda(\alpha) = \alpha - \alpha(H_\lambda)\lambda = \alpha - 2\frac{\langle \alpha, \lambda \rangle}{\langle \lambda, \lambda \rangle}\lambda.$$

Then $s_\lambda(\lambda) = -\lambda$, and s_λ is called the **reflection** relative to λ . We have $s_\lambda^2 = 1$, and s_λ preserve the form $\langle _, _ \rangle$ on \mathcal{H}^* . If $\mu \in \Phi$, then

$$s_\lambda(\mu) = \mu - a_{\mu\lambda}\lambda \in \Phi.$$

Therefore, $s_\lambda(\Phi) = \Phi$.

Definition 2.1.53. The **Weyl group** of \mathcal{G} , denoted \mathcal{W} , is the group of automorphisms of \mathcal{H}^* generated by the reflections s_λ for all $\lambda \in \Phi$.

Since $w(\Phi) = \Phi$ for all $w \in \mathcal{W}$, the Weyl group \mathcal{W} is a subgroup of the symmetric group on Φ . Since Φ is finite, the Weyl group \mathcal{W} is finite.

Definition 2.1.54. A subset Φ of the Euclidean space E (i.e. a finite dimensional real linear space with a symmetric positive definite inner product $(-, -)$) is called a **root system** in E if the following axioms are satisfied:

1. Φ is a finite subset that spans E and does not contain 0,
2. if $\alpha \in \Phi$, then the only scalar multiples of α in E are $\pm\alpha$,
3. if $\alpha, \beta \in \Phi$, then the reflection $\sigma_\alpha(\beta) := \beta - \frac{2(\beta, \alpha)}{(\alpha, \alpha)}\alpha \in \Phi$, and
4. if $\alpha, \beta \in \Phi$, then $a_{\beta\alpha} := \frac{2(\beta, \alpha)}{(\alpha, \alpha)} \in \mathbb{Z}$.

The elements of Φ are called **roots**, and $l = \dim(E)$ is called the **rank** of the root system Φ .

Definition 2.1.55. Let Φ be a root system of rank l in a Euclidean space E . A subset Δ of Φ is called a **base** if

1. Δ is a basis of E , and
2. each root β can be written as $\beta = \sum_{\alpha \in \Delta} k_\alpha \alpha$ with integral coefficients k_α all non-negative or all nonpositive.

The roots in the set Δ are called **simple**. In view of 1, $\text{Card}(\Delta) = l$. The root β is called **positive** (respectively, **negative**) if all $k_\alpha \geq 0$ (respectively, $k_\alpha \leq 0$). The set of all positive (respectively, negative) roots is denoted by Φ_+ (respectively, Φ_-). Clearly,

$$\Phi = \Phi_+ \amalg \Phi_-.$$

Remark 2.1.56. Notice that if \mathcal{G} is a split semisimple Lie algebra with a fixed Cartan subalgebra \mathcal{H} , then the set of roots Φ of \mathcal{G} (relative to \mathcal{H}) is a root system.

Definition 2.1.57. Let \mathcal{G} be a split semisimple Lie algebra with Cartan subalgebra \mathcal{H} and root system Φ , and choose a base Δ of Φ . Let Φ_+ (respectively, Φ_-) be the set of all positive (respectively, negative) roots of \mathcal{G} . Let

$$\mathfrak{n}_+ = \bigoplus_{\alpha \in \Phi_+} \mathcal{G}_\alpha \quad \text{and} \quad \mathfrak{n}_- = \bigoplus_{\alpha \in \Phi_-} \mathcal{G}_\alpha.$$

Then as a vector space, the direct sum

$$\mathcal{G} = \mathfrak{n}_- \oplus \mathcal{H} \oplus \mathfrak{n}_+$$

is called the **triangular decomposition** of \mathcal{G} . The Lie subalgebras of \mathcal{G} , $\mathfrak{b}_+ = \mathcal{H} \oplus \mathfrak{n}_+$ and $\mathfrak{b}_- = \mathcal{H} \oplus \mathfrak{n}_-$, are solvable Lie subalgebras of \mathcal{G} , and \mathfrak{b}_+ is called the **Borel subalgebra** relative to \mathcal{H} .

Proposition 2.1.58. *Assuming the notation of Definition 2.1.57,*

1. *if λ and μ are distinct roots in Φ , then $[\mathcal{G}_\lambda, \mathcal{G}_{-\mu}] = 0$,*
2. *the Lie subalgebras \mathfrak{n}_+ and \mathfrak{n}_- of \mathcal{G} are generated by \mathcal{G}_λ for $\lambda \in \Phi$ and $-\lambda \in \Phi$, respectively, and*
3. *the Lie subalgebras, \mathfrak{b}_+ and \mathfrak{b}_- , are maximal solvable Lie subalgebras of \mathcal{G} , and each one of them coincides with its normalizer in \mathcal{G} .*

Proposition 2.1.59. *Let K be an algebraically closed field of characteristic zero and let \mathcal{G} be a semisimple Lie algebra over K . Then*

1. *if \mathcal{G}' is a solvable Lie subalgebra of \mathcal{G} , then \mathcal{G}' lies in a Borel subalgebra of \mathcal{G} , and*
2. *if \mathcal{G}_1 and \mathcal{G}_2 are Borel subalgebras of \mathcal{G} , then there is a Cartan subalgebra \mathcal{H} of \mathcal{G} such that $\mathcal{H} \subseteq \mathcal{G}_1 \cap \mathcal{G}_2$.*

Proposition 2.1.60. *Let \mathcal{G} be a split semisimple Lie algebra with Cartan subalgebra \mathcal{H} . If w is an element of the Weyl group \mathcal{W} operating in \mathcal{H} , then there exists an automorphism $\sigma : \mathcal{G} \rightarrow \mathcal{G}$ such that $\sigma(\mathcal{H}) = w(\mathcal{H})$.*

Proposition 2.1.61. *Let K be an algebraically closed field of characteristic zero and let \mathcal{G} be a semisimple Lie algebra over K . Then any two Borel subalgebras, \mathcal{G}_1 and \mathcal{G}_2 , of \mathcal{G} are conjugate (i.e. there exists an automorphism $\sigma : \mathcal{G} \rightarrow \mathcal{G}$ such that $\sigma(\mathcal{G}_1) = \mathcal{G}_2$).*

Theorem 2.1.62. (Serre) *Fix a root system Φ with base $\Delta = \{\alpha_1, \dots, \alpha_n\}$. Let \mathcal{G} be the Lie algebra generated by $3n$ elements $\{x_i, y_i, h_i \mid 1 \leq i \leq n\}$ and subject to the defining relations:*

1. $[h_i, h_j] = 0$ ($1 \leq i, j \leq n$),

2. $[x_i, y_i] = h_i$, $[x_i, y_j] = 0$ if $i \neq j$,
3. $[h_i, x_j] = \langle \alpha_i, \alpha_j \rangle x_j$, $[h_i, y_j] = -\langle \alpha_j, \alpha_i \rangle y_j$,
4. $(\text{ad } x_i)^{1-\langle \alpha_i, \alpha_j \rangle}(x_j) = 0$ ($i \neq j$),
5. $(\text{ad } y_i)^{1-\langle \alpha_i, \alpha_j \rangle}(y_j) = 0$ ($i \neq j$).

Then \mathcal{G} is a (finite dimensional) semisimple algebra with a Cartan subalgebra spanned by the h_i and with corresponding root system Φ .

2.1.2 Universal enveloping algebras

In this subsection, K denotes an algebraically closed field of characteristic zero and \mathcal{G} a semisimple finite dimensional Lie algebra over K .

Construction of universal enveloping algebras.

Let \mathcal{G} be a finite dimensional Lie algebra over K . Let

$$\begin{aligned} T^0 &= K \cdot 1, & T^1 &= \mathcal{G}, \\ T^m &= \mathcal{G} \otimes \mathcal{G} \otimes \cdots \otimes \mathcal{G} \text{ (} m \text{ times)}. \end{aligned}$$

Define the **tensor algebra** on the vector space \mathcal{G} to be

$$T = T^0 \oplus T^1 \oplus \cdots \oplus T^n \oplus \cdots,$$

where the multiplication in T is tensor multiplication. Let J be the two sided ideal in T generated by all elements

$$x \otimes y - y \otimes x - [x, y],$$

with $x, y \in \mathcal{G}$. The associative algebra $U(\mathcal{G}) := T/J$ is called the **universal enveloping algebra** of the Lie algebra \mathcal{G} . The composite map φ of the maps $\mathcal{G} \rightarrow T \rightarrow U(\mathcal{G})$ is called the **canonical map** of \mathcal{G} into $U(\mathcal{G})$. However, it is not obvious for the moment that the canonical map φ is injective; this will be implied by PBW Theorem later. Notice that

$$\varphi(x)\varphi(y) - \varphi(y)\varphi(x) = \varphi([x, y])$$

for all $x, y \in \mathcal{G}$. Moreover, $U(\mathcal{G})$ has the following universal property.

Lemma 2.1.63. *Let φ be the canonical map of \mathcal{G} into $U(\mathcal{G})$, A be an associative algebra with unity and τ be a linear map of \mathcal{G} into A such that*

$$\tau(x)\tau(y) - \tau(y)\tau(x) = \tau([x, y])$$

for all $x, y \in \mathcal{G}$. Then there exists a unique homomorphism τ' of $U(\mathcal{G})$ into A such that $\tau'(1) = 1$ and $\tau' \circ \varphi = \tau$.

Theorem 2.1.64. (PBW Theorem)

Let \mathcal{G} be a Lie algebra over a field K , and $\{x_1, x_2, \dots, x_n\}$ be an ordered K -basis of \mathcal{G} . Then the set $\{x^\alpha := x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n} \mid \alpha = (\alpha_1, \alpha_2, \dots, \alpha_n) \in \mathbb{N}^n\}$ is a K -basis for the universal enveloping algebra $U(\mathcal{G})$.

Corollary 2.1.65. *The canonical map φ of \mathcal{G} into $U(\mathcal{G})$ is injective.*

The next proposition shows that a Lie algebra \mathcal{G} and its universal enveloping algebra $U(\mathcal{G})$ have the same representation theory.

Proposition 2.1.66. *Let \mathcal{G} be a Lie algebra and $U(\mathcal{G})$ be its universal enveloping algebra. Then every \mathcal{G} -module is a $U(\mathcal{G})$ -module, and vice versa.*

Sketch of proof. Let V be a \mathcal{G} -module and (ρ, V) be the correspondence representation, then by the universal property of $U(\mathcal{G})$ (stated in Lemma 2.1.63), there exists a unique associative algebra homomorphism $\rho' : U(\mathcal{G}) \rightarrow \text{End}(V)$ such that $\rho' \circ \varphi = \rho$. The existence of ρ' turns V into a module for the associative algebra $U(\mathcal{G})$. Conversely, every $U(\mathcal{G})$ -module becomes a \mathcal{G} -module by the existence of the injection map $\varphi : \mathcal{G} \rightarrow U(\mathcal{G})$.

□

Remark 2.1.67. This proposition says that the theory of representation of associative algebras can be used to understand the theory of representations of Lie algebras. A \mathcal{G} -module V corresponds to a Lie algebra homomorphism $\rho : \mathcal{G} \rightarrow \text{gl}(V)$. In general, the image of this homomorphism is not closed with respect to the composition of linear operators. This restricts the analysing of the structure of V . However, any module over the enveloping algebra $U(\mathcal{G})$ corresponds to a homomorphism $\rho : U(\mathcal{G}) \rightarrow \text{End}(V)$, and the image of this homomorphism is always closed with respect to the composition of operators. Therefore, the study of modules over $U(\mathcal{G})$ reduces to the study of the internal structure of the enveloping algebra $U(\mathcal{G})$. One disadvantage of this approach is that the enveloping algebra $U(\mathcal{G})$ is infinite dimensional even if \mathcal{G} is of finite dimension. Considering all the benefits that the universal enveloping algebra provide, this is a reasonable price to pay.

Filtration on $U(\mathcal{G})$ and the associated graded algebra of $U(\mathcal{G})$.

For $i \in \mathbb{N}$ denote by $U(\mathcal{G})_i$ the vector subspace of $U(\mathcal{G})$ generated by the products $x_1 x_2 \cdots x_t$, where $x_1, \dots, x_t \in \mathcal{G}$ and $t \leq i$. Notice that $U(\mathcal{G})_0 = K \cdot 1$ and $U(\mathcal{G})_i U(\mathcal{G})_j \subseteq U(\mathcal{G})_{i+j}$ for all $i, j \geq 0$. This gives us the following filtration on $U(\mathcal{G})$:

$$U(\mathcal{G}) = \bigcup_{i \geq 0} U(\mathcal{G})_i$$

which is called the **standard filtration** of $U(\mathcal{G})$. If u is a non-zero element of $U(\mathcal{G})$, the smallest integer n such that $u \in U(\mathcal{G})_n$ is called the **filtration** of u ; the elements of $U(\mathcal{G})_i$ are those of filtration at most i .

Remark 2.1.68. For all $i \geq 0$, $U(\mathcal{G})_i$ is a finite dimensional \mathcal{G} -submodule of $U(\mathcal{G})$. Therefore, if \mathcal{G} is a semisimple Lie algebra, then $U(\mathcal{G})$ is the sum of finite dimensional simple \mathcal{G} -submodules.

Let

$$\text{gr}(U(\mathcal{G})) = \bigoplus_{i \geq 0} U(\mathcal{G})_i / U(\mathcal{G})_{i-1}$$

be the associated graded algebra of $U(\mathcal{G})$ with respect to the standard filtration of $U(\mathcal{G})$ where $U(\mathcal{G})_{-1} := 0$. We have $U(\mathcal{G})_0 / U(\mathcal{G})_{-1} := K \cdot 1 = K$, and $U(\mathcal{G})_1 / U(\mathcal{G})_0$ can be canonically identified with \mathcal{G} .

Proposition 2.1.69 ([7, Proposition 2.3.6]). *The associated graded algebra $\text{gr}(U(\mathcal{G}))$ is isomorphic to a polynomial algebra in $\dim_K(\mathcal{G})$ variables.*

Sketch of proof. Let $\{x_1, \dots, x_n\}$ be a basis for the Lie algebra \mathcal{G} . Then the

$$x_1^{r_1} \cdots x_n^{r_n} \in U(\mathcal{G}) \quad \text{such that} \quad r_1 + \cdots + r_n \leq i$$

form a basis for $U(\mathcal{G})_i$. If $i, j \in \mathbb{N}$, $u \in U(\mathcal{G})_i$ and $v \in U(\mathcal{G})_j$, then $[u, v] \in U(\mathcal{G})_{i+j-1}$. Hence the algebra $\text{gr}(U(\mathcal{G}))$ is commutative. The canonical image of the standard monomials (the monomials which form the PBW basis) in $\text{gr}(U(\mathcal{G}))$ form a basis for $\text{gr}(U(\mathcal{G}))$. Since any monomial is a product of monomials of degree one, $\text{gr}(U(\mathcal{G}))$ is generated by monomials of degree one and hence we have an epimorphism from the polynomial algebra $K[x_1, \dots, x_n]$ to $\text{gr}(U(\mathcal{G}))$. This epimorphism is injective since the monomials in $K[x_1, \dots, x_n]$ are mapped to linearly dependent images in $\text{gr}(U(\mathcal{G}))$ of standard monomials from $\text{gr}(U(\mathcal{G}))$. \square

Corollary 2.1.70 ([7, Corollary 2.3.8, Corollary 2.3.9(ii)]). *1. The universal enveloping algebra $U(\mathcal{G})$ is Noetherian.*

2. The universal enveloping algebra $U(\mathcal{G})$ is a domain.

Sketch of proof. 1. Proposition 2.1.69 shows that $\text{gr}(U(\mathcal{G}))$ is Noetherian, and so $U(\mathcal{G})$ is Noetherian (see [18, Theorem 1.6.9]).

2. Let x' and y' be the canonical images of the nonzero elements x and y in $U(\mathcal{G})_i$ and $U(\mathcal{G})_j$, respectively. Then $x' \neq 0$ and $y' \neq 0$, and hence $x'y' \neq 0$. Therefore, $xy \notin U(\mathcal{G})_{i+j-1}$, and so $xy \neq 0$, which complete the proof. \square

The center of the enveloping algebra $U(\mathcal{G})$.

The center $Z(U(\mathcal{G}))$ of the enveloping algebra $U(\mathcal{G})$ is defined by

$$Z(U(\mathcal{G})) = \{z \in U(\mathcal{G}) \mid zu = uz \text{ for all } u \in U(\mathcal{G})\}.$$

Theorem 2.1.71. *The center $Z(U(\mathcal{G}))$ of the enveloping algebra $U(\mathcal{G})$ is isomorphic to the polynomial algebra in l variables, where l is the rank of \mathcal{G} .*

Proposition 2.1.72. *The center $Z(U(\mathcal{G}))$ of the enveloping algebra $U(\mathcal{G})$ acts on each simple \mathcal{G} -module M by scalars.*

2.1.3 Weight spaces

Let \mathcal{G} be a semisimple Lie algebra with a fixed Cartan subalgebra \mathcal{H} . Recall that if \mathcal{H} is a vector space over K , the dual space of \mathcal{H} is

$$\mathcal{H}^* = \text{Hom}_K(\mathcal{H}, K).$$

Definition 2.1.73. Let V be a \mathcal{G} -module and $\lambda \in \mathcal{H}^*$. We define

$$V_\lambda = \{v \in V \mid hv = \lambda(h)v \text{ for all } h \in \mathcal{H}\}.$$

If $V_\lambda \neq \{0\}$, we call V_λ a **weight space** and we say that λ is a **weight** of V . The nonzero elements of V_λ are called the **weight vectors** of weight λ .

Lemma 2.1.74. Let V be a \mathcal{G} -module, $\lambda \in \mathcal{H}^*$ and $\alpha \in \Phi$. Then

1. $\mathcal{G}_\alpha V_\lambda \subseteq V_{\alpha+\lambda}$, and
2. the sum $\bigoplus_{\mu \in \mathcal{H}^*} V_\mu$ is direct, and it is a \mathcal{G} -submodule of V .

2.1.4 Verma modules

Verma modules are important concepts in the representation theory of Lie algebras. For example, each finite dimensional simple module of a semisimple Lie algebra can be constructed as a quotient of a Verma module. They were introduced by D. N. Verma [21] in the 1960s. In this section, we show how to construct Verma modules for the universal enveloping algebra $U(\mathcal{G})$ where \mathcal{G} is a semisimple finite dimensional Lie algebra, and next we study some of their properties.

Notation.

In this subsection, K denotes an algebraically closed field of characteristic zero, \mathcal{G} a semisimple finite dimensional Lie algebra over K , \mathcal{H} a fixed Cartan subalgebra, Φ the corresponding root system, $\Delta = \{\theta_1, \dots, \theta_n\}$ a base for Φ , $\Phi_+ = \{\mu_1, \dots, \mu_m\}$ and Φ_- the set of positive and negative roots relative to Δ respectively, \mathcal{P} the set of weights of Φ , \mathcal{W} the Weyl group of Φ and $U(\mathcal{G})$ the universal enveloping algebra of the Lie algebra \mathcal{G} . Set

$$\begin{aligned} \mathfrak{n}_+ &= \bigoplus_{\alpha \in \Phi_+} \mathcal{G}_\alpha, & \mathfrak{n}_- &= \bigoplus_{\alpha \in \Phi_-} \mathcal{G}_\alpha, \\ \mathfrak{b}_+ &= \mathcal{H} \oplus \mathfrak{n}_+, & \mathfrak{b}_- &= \mathcal{H} \oplus \mathfrak{n}_-, \\ \delta &= \frac{1}{2} \sum_{\alpha \in \Phi_+} \alpha. \end{aligned}$$

Remark 2.1.75. Notice that $U(\mathfrak{b}_+)$ is a subalgebra of $U(\mathcal{G})$.

Definition 2.1.76. Let \mathcal{G} be a semisimple Lie algebra and let V be a \mathcal{G} -module. A nonzero vector $v \in V$ is called a **highest weight vector** if

$$\mathfrak{n}_+ v = 0,$$

and there exists $\lambda \in \mathcal{H}^*$ such that $h \cdot v = \lambda(h)v$ for all $h \in \mathcal{H}^*$. The unique weight $\lambda \in \mathcal{H}^*$ is called the **highest weight** of the highest weight vector v . A \mathcal{G} -module M is called a **highest weight module** if there exists a highest weight vector $v \in M$ that generates it.

Definition 2.1.77. Let $\lambda \in \mathcal{H}^*$ and consider a one dimensional \mathfrak{b}_+ -module $K_\lambda = K$, such that

$$(h + n)v = (\lambda - \delta)(h)v \text{ for all } n \in \mathfrak{n}_+, h \in \mathcal{H} \text{ and } v \in K.$$

The module

$$M(\lambda) = U(\mathcal{G}) \otimes_{U(\mathfrak{b}_+)} K_\lambda$$

is called the **Verma module** associated with \mathcal{G} , \mathcal{H} , Δ and λ .

Definition 2.1.78. Let M be a module over a Lie algebra \mathcal{G} . If every element z in the center $Z(U(\mathcal{G}))$ of the enveloping algebra $U(\mathcal{G})$ acts by a scalar $\chi_M(z)$, then the homomorphism

$$\chi_M : Z(U(\mathcal{G})) \rightarrow K, z \mapsto \chi_M(z)$$

is called the **central character** of M .

Basic properties of Verma modules.

Proposition 2.1.79. 1. The Verma module $M(\lambda)$ is a highest weight \mathcal{G} -module with the highest weight $\lambda - \delta$ and highest weight vector $1 \otimes 1$.

2. The weights of the \mathcal{G} -module $M(\lambda)$ have the form $\lambda - \delta - \sum_{\alpha \in \Delta} n_\alpha \alpha$, where the n_α are nonnegative integers.

3. $M(\lambda)$ is isomorphic to $U(\mathfrak{n}_-)$ as an \mathfrak{n}_- -module if we provide $U(\mathfrak{n}_-)$ with the left regular representation.

4. All highest weight \mathcal{G} -modules with the highest weight $\lambda - \delta$ are quotients of $M(\lambda)$.

5. There exists a unique maximal submodule of $M(\lambda)$, and its quotient $L(\lambda)$ is the unique simple \mathcal{G} -module with highest weight $\lambda - \delta$.

6. Every endomorphism of the \mathcal{G} -module $M(\lambda)$ is scalar.

7. The \mathcal{G} -module $M(\lambda)$ has a central character χ_λ .

Finite dimensional modules.

A weight $\lambda \in \mathcal{H}^*$ is called **integral** if all $\lambda(h_\alpha)$ are integers, and we call λ **dominant integral** if all $\lambda(h_\alpha)$ are nonnegative integers. The **set of dominant integral weights** is denoted by \mathcal{P}_+ .

Proposition 2.1.80. *Let \mathcal{G} be a semisimple finite dimensional Lie algebra and V is a finite dimensional \mathcal{G} -module. Then*

1. $V = \bigoplus_{\mu \in \mathcal{H}^*} V_\mu$, and
2. if μ is a weight of V and $w \in \mathcal{W}$, then $w(\mu)$ is a weight of V with $\dim(V_\mu) = \dim(V_{w(\mu)})$.

Proposition 2.1.81. *Let \mathcal{G} be a semisimple finite dimensional Lie algebra and V be a finite dimensional simple \mathcal{G} -module. Then*

1. there exists a unique $\lambda \in \mathcal{H}^*$ such that V is isomorphic to $L(\lambda + \delta)$,
2. $\lambda \in \mathcal{P}_+$ and $\dim(V_\lambda) = 1$, and
3. if μ is a weight of the module V , then $\mu = \lambda - \sum_{\alpha \in \Delta} n_\alpha \alpha$, where the n_α are nonnegative integers.

Theorem 2.1.82. *The map $\lambda \mapsto [L(\lambda + \delta)]$ gives a bijection between \mathcal{P}_+ and the set of isomorphism classes of finite dimensional simple \mathcal{G} -modules.*

The Harish-Chandra homomorphism.

The universal enveloping algebra $U(\mathcal{G})$ of the Lie algebra \mathcal{G} has a PBW basis consisting of all monomials of the form

$$u(\bar{\alpha}, \bar{\beta}, \bar{\gamma}) = X_{-\mu_m}^{\alpha_m} \cdots X_{-\mu_1}^{\alpha_1} X_{\mu_1}^{\beta_1} \cdots X_{\mu_m}^{\beta_m} H_{\theta_1}^{\gamma_1} \cdots H_{\theta_n}^{\gamma_n}$$

where $\alpha_i, \beta_i, \gamma_i \in \mathbb{N}$. If $H \in \mathcal{H}$, then

$$[H, u(\bar{\alpha}, \bar{\beta}, \bar{\gamma})] = ((\beta_m - \alpha_m)\mu_m + \cdots + (\beta_1 - \alpha_1)\mu_1)u(\bar{\alpha}, \bar{\beta}, \bar{\gamma}).$$

Let $\text{ad}_{U(\mathcal{G})}$ be the adjoint representation of $U(\mathcal{G})$, then

$$U(\mathcal{G}) = \bigoplus_{\lambda \in \mathcal{H}^*} U(\mathcal{G})_\lambda.$$

Moreover, if λ and μ are in \mathcal{H}^* , then $U(\mathcal{G})_\lambda U(\mathcal{G})_\mu \subseteq U(\mathcal{G})_{\lambda+\mu}$. Thus, $U(\mathcal{G})$ is an \mathcal{H}^* -graded algebra, graded by the family $\{U(\mathcal{G})_\lambda\}_{\lambda \in \mathcal{H}^*}$. The subspace $U(\mathcal{G})_0$ form a subalgebra of $U(\mathcal{G})$, and it is **the centralizer** C of the Cartan subalgebra \mathcal{H} in the universal enveloping algebra $U(\mathcal{G})$.

Recall that every Lie algebra \mathcal{G} can be embedded injectively into its universal enveloping algebra $U(\mathcal{G})$.

Lemma 2.1.83. *Let $I = U(\mathcal{G})\mathfrak{n}_+ \cap C$. Then*

1. $I = \mathfrak{n}_- U(\mathcal{G}) \cap C$, and I is a two sided ideal of C , and

2. $C = U(\mathcal{H}) \oplus I$.

From Lemma 2.1.83, the projection of C onto $U(\mathcal{H})$ with kernel I is an associative algebra homomorphism. It is called the **Harish-Chandra homomorphism**.

Recall that $U(\mathcal{H})$ can be viewed as the symmetric algebra $S(\mathcal{H})$ of \mathcal{H} and also as the algebra of polynomial functions on \mathcal{H}^* .

Proposition 2.1.84. *Let M be a \mathcal{G} -module generated by an element v such that $xv = 0$ for $x \in \mathfrak{n}_+$, $\lambda \in \mathcal{H}^*$, χ_M the central character of M and let φ be the Harish-Chandra homomorphism of C onto $U(\mathcal{H})$. Then $\chi_M(z) = (\varphi(z))(\lambda)$ for all $z \in Z(U(\mathcal{G}))$.*

Theorem 2.1.85. *Consider the automorphism γ_δ of the algebra $S(\mathcal{H})$ which maps the polynomial function f on \mathcal{H}^* to the function $\lambda \mapsto f(\lambda - \delta)$. Let C be the centralizer of \mathcal{H} in $U(\mathcal{G})$, φ the Harish-Chandra homomorphism of C onto $U(\mathcal{H}) = S(\mathcal{H})$ and $S(\mathcal{H})^\mathcal{W}$ the algebra of \mathcal{W} -invariant polynomials. Then*

$$\gamma_\delta \circ \varphi|_{Z(U(\mathcal{G}))} : Z(U(\mathcal{G})) \rightarrow S(\mathcal{H})^\mathcal{W}$$

is an isomorphism, independent of the choice of the base Δ .

The map $\gamma_\delta \circ \varphi$ of $Z(U(\mathcal{G}))$ onto $S(\mathcal{H})^\mathcal{W}$ in Theorem 2.1.85 is called the **Harish-Chandra isomorphism**.

Proposition 2.1.86. *Consider the action of the Weyl group \mathcal{W} on \mathcal{H}^* :*

$$w\lambda := w(\lambda + \delta) - \delta \quad \text{for } w \in \mathcal{W} \text{ and } \lambda \in \mathcal{H}^*.$$

If λ and λ' are in \mathcal{H}^* , then $\chi_\lambda = \chi_{\lambda'}$ if and only if $\mathcal{W}\lambda' = \mathcal{W}\lambda$.

Proposition 2.1.87. *If χ is a homomorphism of $Z(U(\mathcal{G}))$ into K , then there exists $\lambda \in \mathcal{H}^*$ such that $\chi_\lambda = \chi$.*

2.2 Gelfand-Kirillov dimension

In this section, we provide some basic results on Gelfand-Kirillov dimension. We follow closely the book of McConnell and Robson [18], a standard reference for Gelfand-Kirillov dimension.

2.2.1 Definition and examples

For any function f from \mathbb{N} to $R' = \{r \in \mathbb{R} \mid r \geq 1\}$, the number $\gamma(f) \in \mathbb{R}$, defined as

$$\gamma(f) := \inf\{r \in \mathbb{R} \mid f(n) \leq n^r \text{ for almost all } n\}$$

is called the **growth (or degree)** of f . If there does not exist $r \in \mathbb{R}$ such that $f(n) \leq n^r$ for almost all n , then $\gamma(f) = \infty$.

Lemma 2.2.1 ([18, Lemma 8.1.6]). *Let $f : \mathbb{N} \rightarrow \mathbb{R}'$. Then*

$$\inf\{r \in \mathbb{R} \mid f(n) \leq n^r \text{ for almost all } n\} = \limsup_{n \rightarrow \infty} \log_n(f(n)).$$

Lemma 2.2.2 ([18, Lemma 8.1.7]). *Let $f, g : \mathbb{N} \rightarrow \mathbb{R}'$. Then*

1. $\gamma(f + g) = \max\{\gamma(f), \gamma(g)\}$,
2. $\gamma(fg) \leq \gamma(f) + \gamma(g)$,
3. if $g(n) \leq f(an + b)$ for almost all n , where $a, b \in \mathbb{N}$, then $\gamma(g) \leq \gamma(f)$, and
4. if $f(n) = p(n)$ for almost all n , where p is a polynomial in $\mathbb{R}[x]$, then $\gamma(f) = \deg(p)$.

Definition 2.2.3. Let A be an algebra over a field K and M be a module over the algebra A . A filtration $\{A_n\}$ of A is **standard** if $A_n = A_1^n$ for all n and is called a **finite dimensional** filtration if $A_0 = K$ and $\dim_K(A_n) \leq \infty$ for all n . A filtration $\{M_n\}$ of M is **standard** if $M_n = A_n M_0$ for each n and is called a **finite dimensional** filtration if $\dim_K(M_n) \leq \infty$ for all n .

Remarks 2.2.4. Let A be a finitely generated algebra over a field K with generators x_1, \dots, x_n . Then A has a standard finite dimensional filtration $\{A_n\}$ with

$$A_0 = K, \quad A_1 = K + \sum_{i=1}^n Kx_i \quad \text{and} \quad A_n = A_1^n \text{ for all } n \geq 2.$$

Let M be a finitely generated module over the algebra A , $M = AM_0$ for a finite dimensional generating subspace M_0 of M . Then M has a standard finite dimensional filtration $\{M_n\}$ with $M_n = A_n M_0$ for all n .

Definition 2.2.5 ([18, Definition 8.1.11]). Let A be a finitely generated K -algebra and M be a finitely generated module over the algebra A . Then for any choice of standard filtration $\{A_n\}$ of A and $\{M_n\}$ of M the values

$$\text{GK}(A) := \gamma(\dim_K(A_n)) \quad \text{and} \quad \text{GK}(M) := \gamma(\dim_K(M_n))$$

are called the **Gelfand-Kirillov dimension** of the algebra A and the module M , respectively.

Lemma 2.2.6. *Let A be a finitely generated K -algebra and M be a finitely generated module over the algebra A . Then*

1. $\text{GK}(A) = \text{GK}({}_A A) = \text{GK}(A_A)$, and
2. if B is a subalgebra of A and N is a finitely generated B -submodule of M , then $\text{GK}({}_B N) \leq \text{GK}({}_A M)$.

- Remarks 2.2.7.** 1. Let A be a finitely generated K -algebra with a finite dimensional generating subspace V of A such that $1 \in V$. Then $\{A_n = V^n\}$ and hence $\text{GK}(A) := \gamma(\dim_K(V^n))$.
2. $\text{GK}(A)$ and $\text{GK}(M)$ are independent of the choice of the standard filtration of the algebra A or the generating subspace of the module M .
3. In general if A is any K -algebra, we define

$$\text{GK}(A) = \sup\{\text{GK}(S) \mid S \text{ is a finitely generated subalgebra of } A\}.$$

- Examples 2.2.8.** 1. Let A be a finite dimensional K -algebra. If we take $V = A$, then V is a generating subspace of A and $V^n = V$ for all $n \geq 1$. Hence $\dim_K(V^n) = \dim_K(A)$ for all $n \geq 1$. Thus $\text{GK}(A) = 0$.
2. Let $R = K[x_1, \dots, x_d]$ be the polynomial algebra over K in d variables. Let V be the subspace of R generated by $1, x_1, \dots, x_d$. Let $V^0 = K$ and V^n be the subspace of R generated by the monomials in $1, x_1, \dots, x_d$ of degree at most n . Then $\dim_K(V^n) = \binom{n+d}{d}$, a polynomial in n of degree d . Thus, $\text{GK}(R) = d$.
3. Let $A = A_n(K)$ be the Weyl algebra. Then $\text{GK}(A) = 2n$.
4. If A is the free associative algebra $K\langle x_1, \dots, x_m \rangle$ with $m \geq 2$, then $\text{GK}(A) = \infty$.

Proposition 2.2.9 ([18, Proposition 8.1.14]). *Let A be a finitely generated K -algebra with a standard finite dimensional filtration $\{A_n\}$ and M be a finitely generated module over the algebra A with a standard finite dimensional filtration $\{M_n\}$. Then*

$$\text{GK}(\text{gr}(A)) = \text{GK}(A) \quad \text{and} \quad \text{GK}(\text{gr}(M)) = \text{GK}(M).$$

Example 2.2.10. Let \mathcal{G} be a finite dimensional Lie algebra over a field K and $U(\mathcal{G})$ be its universal enveloping algebra. Recall that the algebra $U(\mathcal{G}) = \bigcup_{i \geq 0} U(\mathcal{G})_i$ admits the standard filtration with respect to the canonical generators of \mathcal{G} and the associated graded algebra

$$\text{gr}(U(\mathcal{G})) = \bigoplus_{i \geq 0} U(\mathcal{G})_i / U(\mathcal{G})_{i-1}$$

is a polynomial algebra in $\dim_K(\mathcal{G})$ variables where $U(\mathcal{G})_{-1} := 0$. Thus, the Gelfand-Kirillov dimension of $U(\mathcal{G})$ is $\text{GK}(U(\mathcal{G})) = \dim_K(\mathcal{G})$.

The rest of this subsection is devoted to recording that the possible values of the Gelfand-Kirillov dimension is

$$\{0\} \cup \{1\} \cup [2, \infty].$$

Definition 2.2.11. A K -algebra A is called **locally finite dimensional** if every finitely generated subalgebra of A is finite dimensional.

Proposition 2.2.12 ([18, Proposition 8.1.17 (i)]). *Let A be a K -algebra. Then $\text{GK}(A) = 0$ if and only if the algebra A is locally finite dimensional.*

Corollary 2.2.13. *If A is a K -algebra with $\text{GK}(A) \neq 0$, then $\text{GK}(A) \geq 1$.*

Proposition 2.2.14 ([18, Proposition 8.1.18]). *For any real number $r \geq 2$, there is a finitely generated K -algebra A with $\text{GK}(A) = r$.*

The existence of an algebra A with $1 < \text{GK}(A) < 2$ was an open problem for some years until Bergman [1] proved that there is no algebra A whose Gelfand-Kirillov dimension is a real number in the interval $(1, 2)$.

2.2.2 Some basic properties of the Gelfand-Kirillov dimension

Proposition 2.2.15 ([18, Proposition 8.2.2]). *If A is a K -algebra and B is a subalgebra or a homomorphic image of A , then $\text{GK}(B) \leq \text{GK}(A)$.*

Proposition 2.2.16 ([18, Proposition 8.2.3]). *Let A and A' be K -algebras. Then*

$$\max\{\text{GK}(A), \text{GK}(A')\} \leq \text{GK}(A \otimes_K A') \leq \text{GK}(A) + \text{GK}(A').$$

The next lemma shows that the second inequality stated in Proposition 2.2.16 is often equality.

Lemma 2.2.17. *Let A and A' be K -algebras. If B is a subalgebra of A' with a standard finite dimensional filtration $\{B_n\}$ such that*

$$\lim_{n \rightarrow \infty} \log_n(\dim_K(B_n)) = \text{GK}(B) = \text{GK}(A'),$$

then $\text{GK}(A \otimes_K A') = \text{GK}(A) + \text{GK}(A')$.

Proposition 2.2.18 ([18, Proposition 8.2.7]). *Let A be a K -algebra. Then*

1. $\text{GK}(A[x]) = \text{GK}(A) + 1$, where $A[x]$ is the polynomial ring in the variable x over A ,
2. $\text{GK}(M_n(A)) = \text{GK}(A)$, where $M_n(A)$ is the ring of all $n \times n$ matrices over A ,
3. $\text{GK}(A \otimes_K U(\mathcal{G})) = \text{GK}(A) + \dim_K(\mathcal{G})$ for any finite dimensional Lie algebra \mathcal{G} ,
4. $\text{GK}(A_n(A)) = \text{GK}(A) + 2n$, where $A_n(A)$ is the n^{th} Weyl algebra over A , and
5. $\text{GK}(AG) = \text{GK}(A)$, where AG is the group algebra of any finite group G .

Proposition 2.2.19 ([18, Proposition 8.2.9]). *Let A be K -algebra. Then*

1. if M is a finitely generated A -module, then

$$\mathrm{GK}(\mathrm{End}(M)) \leq \mathrm{GK}(A/\mathrm{ann}(M)) \leq \mathrm{GK}(A),$$

where $\mathrm{End}(M)$ is the ring of endomorphisms of M , and $\mathrm{ann}(M) = \{a \in A \mid aM = 0\}$ is the annihilator of the module M over the algebra A , and

2. if $A \subseteq A'$ with A' a K -algebra finitely generated as a left or right A -module, then $\mathrm{GK}(A') = \mathrm{GK}(A)$.

The next proposition shows that the Gelfand-Kirillov dimension is well behaved with respect to central localization.

Proposition 2.2.20 ([18, Proposition 8.2.13]). *Let S be a multiplicatively closed set of central regular elements in a K -algebra A . Then $\mathrm{GK}(A_S) = \mathrm{GK}(A)$ where $A_S = S^{-1}A$ is the localization of the algebra A at S .*

Theorem 2.2.21 ([18, Theorem 8.2.14]). *1. Let A be a commutative finitely generated K -algebra. Then A contains a subalgebra S which is isomorphic to a polynomial algebra $K[x_1, \dots, x_d]$ such that A is a finitely generated module over S and $\mathrm{GK}(A) = \mathrm{Kdim}(A) = d$ where $\mathrm{Kdim}(A)$ is the Krull dimension of A .*

2. *Let A be a commutative finitely generated K -algebra which is an integral domain. Then $\mathrm{GK}(A)$ equals the transcendence degree of the quotient field Q of A .*

As an application of the Proposition 2.2.20, we obtain the following corollary.

Corollary 2.2.22 ([18, Corollary 8.2.15]). *Let A be any K -algebra. Then*

$$\mathrm{GK}(A[x, x^{-1}]) = \mathrm{GK}(A) + 1,$$

where $A[x, x^{-1}]$ is the Laurent polynomial ring in the variable x over A , i.e. $A[x, x^{-1}]$ is a localization of $A[x]$ with respect to the multiplicative set $\{x^i \mid i \geq 0\}$.

2.2.3 Gelfand-Kirillov dimension of modules

Proposition 2.2.23 ([18, Proposition 8.3.2]). *Let A be a K -algebra and M be a module over the algebra A .*

1. *If $M = \sum M_i$ for some submodules M_i , then $\mathrm{GK}(M) = \sup\{\mathrm{GK}(M_i)\}$.*
2. *If $0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0$ is a short exact sequence of A -modules, then $\mathrm{GK}(M) \geq \max\{\mathrm{GK}(L), \mathrm{GK}(N)\}$.*
3. $\mathrm{GK}(M) \leq \mathrm{GK}(A)$.

4. If M_1, \dots, M_r are submodules of M such that $\bigcap M_i = 0$, then $\text{GK}(M) = \max\{\text{GK}(M/M_i)\}$.

5. If $IM = 0$ for an ideal I of A , then $\text{GK}({}_A M) \leq \text{GK}({}_{A/I} M)$.

Corollary 2.2.24 ([18, Corollary 8.3.3]). *Let A and A' be K -algebras.*

1. $\text{GK}(A \oplus A') = \max\{\text{GK}(A), \text{GK}(A')\}$.

2. Let I_1, \dots, I_r be ideals of the algebra A such that $\bigcap I_i = 0$. Then $\text{GK}(A) = \max\{\text{GK}(A/I_i)\}$.

Proposition 2.2.25 ([18, Proposition 8.3.5]). *Let A be a K -algebra. If*

$$0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0$$

is a short exact sequence of finitely generated A -modules with $L \simeq M$, then

$$\text{GK}(N) \leq \text{GK}(M) - 1.$$

The next corollary follows from the Proposition 2.2.25.

Corollary 2.2.26. *Let I be an ideal in a K -algebra A and assume that the ideal I contains a regular element of A . Then $\text{GK}(A/I) \leq \text{GK}(A) - 1$.*

Definition 2.2.27 ([18, Definition 6.8.4]). A **dimension function** δ for an algebra A assigns a value $\delta(M)$ to each finitely generated module M over A and satisfies the following properties:

1. $\delta(0) = -\infty$,
2. if $0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0$ is a short exact sequence of left A -modules, then $\text{GK}(M) \geq \max\{\text{GK}(L), \text{GK}(N)\}$, and
3. if $MP = 0$ for some (two sided) prime ideal P and M is a torsion module over A/P , then $\delta(M) + 1 \leq \delta(A/P)$.

If in property 2, $\delta(M) = \max\{\text{GK}(L), \text{GK}(N)\}$ is always true, then δ is called an **exact dimension function**.

Corollary 2.2.28 ([18, Corollary 8.3.6]). *Let A be a left Noetherian K -algebra and $\text{GK}(A) \leq \infty$.*

1. *If A is prime and I is an essential left ideal, then $\text{GK}(A/I) \leq \text{GK}(A) - 1$.*
2. *If M is a finitely generated left A -module, $PM = 0$ for some $P \in \text{Spec}(A)$ and M is torsion A/P -module, then $\text{GK}(M) \leq \text{GK}(A/P) - 1$.*

3. Gelfand-Kirillov dimension is a dimension function for the algebra A .
4. Let $P_0 \subset P_1 \subset \cdots \subset P_r$ be a chain of distinct prime ideals of the algebra A . Then

$$\mathrm{GK}(A) \geq \mathrm{GK}(A/P_0) \geq \mathrm{GK}(A/P_r) + r.$$

Proposition 2.2.29 ([18, Proposition 8.3.11]). *Let A be a finitely generated algebra over a field K with a standard finite dimensional filtration $\{A_n\}$ and assume that $\mathrm{gr}(A)$ is left Noetherian. Then the Gelfand-Kirillov dimension is exact for all short exact sequences of finitely generated left A -modules (i.e. if $0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0$ is a short exact sequence of left A -modules, then $\mathrm{GK}(M) = \max\{\mathrm{GK}(L), \mathrm{GK}(N)\}$).*

Corollary 2.2.30 ([18, Corollary 8.3.12]). *Let A be a finitely generated algebra over a field K with a standard finite dimensional filtration $\{A_n\}$ such that $\mathrm{gr}(A)$ is left Noetherian. Let P_1, \dots, P_m be the minimal prime ideals of A , and let E be the prime radical. Then $\mathrm{GK}(A) = \mathrm{GK}(A/E) = \max\{\mathrm{GK}(A/P_i) \mid i = 1, \dots, m\}$.*

2.2.4 Almost commutative algebras

Definition 2.2.31 ([18, Definition 8.4.2]). A finitely generated K -algebra A is called an **almost commutative algebra** if it has some standard finite dimensional filtration such that the associated graded algebra $\mathrm{gr}(A)$ is commutative.

Remark 2.2.32. Notice that an almost commutative algebra A is Noetherian since $\mathrm{gr}(A)$ is Noetherian.

Example 2.2.33. The Weyl algebra $A_n(K) = K\langle x_1, \dots, x_n, \partial_1, \dots, \partial_n \rangle$ is an almost commutative algebra with respect to the standard filtration associated by the set of generators $x_1, \dots, x_n, \partial_1, \dots, \partial_n$. The associated graded algebra $\mathrm{gr}(A_n(K))$ is a polynomial algebra $K[x_1, \dots, x_n, \partial_1, \dots, \partial_n]$.

Proposition 2.2.34 ([18, Proposition 8.4.3]). *A K -algebra A is almost commutative if and only if A is a homomorphic image of the enveloping algebra $U(\mathcal{G})$ of a finite dimensional Lie algebra \mathcal{G} .*

Proposition 2.2.35 ([18, Proposition 8.4.4]). *For all sufficiently large n , the following properties of the function $f : \mathbb{N} \mapsto \mathbb{Q}$, are equivalent:*

1. $f(n) = h(n)$ for some polynomial $h \in \mathbb{Q}[x]$ such that $\deg(h) = r$,
2. $f(n) = a_r \binom{n}{r} + \cdots + a_1 \binom{n}{1} + a_0$ with $a_i \in \mathbb{Q}$ and $a_r \neq 0$, and
3. $f(n+1) - f(n) = a_r \binom{n}{r-1} + \cdots + a_2 \binom{n}{1} + a_1$ with $a_i \in \mathbb{Q}$ and $a_r \neq 0$.

Moreover, if one of the above conditions holds, then

1. h and all a_i are uniquely determined by the function f ,
2. if $f(n) \in \mathbb{Z}$ for all sufficiently large n , then each $a_i \in \mathbb{Z}$, and
3. if $f(n) \in \mathbb{N}$ for all sufficiently large n , then $a_r \in \mathbb{N}$.

The Hilbert polynomial and the multiplicity of a module.

Theorem 2.2.36 ([18, Theorem 8.4.5]). *Let $A = \cup_{i \geq 0} A_i$ be an almost commutative K -algebra and let M be a finitely generated A -module with a standard finite dimensional filtration $\{M_n\}$. Then there exists a polynomial $h \in \mathbb{Q}[x]$ with $\deg(h) \leq \dim_K(A_1/A_0)$ such that $\dim_K(M_n) = h(n)$ for all sufficiently large n .*

Definition 2.2.37 ([18, Definition 8.4.6]). The polynomial h in Theorem 2.2.36 is called the **Hilbert polynomial** for the module M and is denoted by h_M .

- Remarks 2.2.38.**
1. Notice that $\deg(h_M) = \text{GK}(M)$ and this shows that $\deg(h_M)$ is independent of the choices of generating subspaces for A and M .
 2. By Theorem 2.2.36,

$$h(n) = a_r \binom{n}{r} + \cdots + a_1 \binom{n}{1} + a_0 \quad \text{with } a_r \in \mathbb{N}.$$

The number a_r is called the **multiplicity** of M denoted by $e(M)$.

The next corollary follows from Theorem 2.2.36.

Corollary 2.2.39 ([18, Corollary 8.4.6]). *Let A be an almost commutative K -algebra and let M be an A -module. Then the Gelfand-Kirillov dimension of the module M is nonnegative integer.*

The multiplicity is an additive function on short exact sequences in the sense of part 3 of Theorem 2.2.40.

Theorem 2.2.40 ([18, Theorem 8.4.8]). *Let $A = \cup_{i \geq 0} A_i$ be an almost commutative K -algebra and let $0 \rightarrow N \rightarrow M \rightarrow L \rightarrow 0$ be a short exact sequence of A -modules. Then*

1. *generating subspaces for the modules N, M and L can be chosen such that $h_M = h_N + h_L$,*
2. $\text{GK}(M) = \max\{\text{GK}(N), \text{GK}(L)\}$, *and*
3. *if $\text{GK}(N) = \text{GK}(L) = \text{GK}(M)$, then $e(M) = e(N) + e(L)$.*

Corollary 2.2.41 ([18, Corollary 8.4.9]). *Let A be an almost commutative K -algebra. Then the Gelfand-Kirillov dimension for the algebra A is an exact dimension function.*

Proposition 2.2.42 ([18, Proposition 8.4.10]). *Let $A = \cup_{i \geq 0} A_i$ and $B = \cup_{i \geq 0} B_i$ be almost commutative K -algebras with $B \subseteq A$, and let ${}_A M$ and ${}_B N$ be finitely generated modules with ${}_B N \triangleleft_B M$. Then*

1. $B_1 \subseteq A_d$ for some d ,
2. $\text{GK}({}_B N) \leq \text{GK}({}_A M)$, and
3. if $\text{GK}({}_B N) = \text{GK}({}_A M) = r$, then $e({}_B N) \leq d^r e({}_A M)$.

2.2.5 Somewhat commutative algebras

The aim of this section is to extend the class of almost commutative algebras to a larger class of algebras, namely the class of somewhat commutative algebras. The class of somewhat commutative algebras includes enveloping algebras and their factor algebras and includes numerous other algebras. It will be seen that most of the properties of Gelfand-Kirillov dimension of almost commutative algebras will also hold over somewhat commutative algebras.

Definition 2.2.43 ([18, Definition 8.6.3]). Let A be a filtered K -algebra. Then a filtration of an A -module M is called a **good filtration** if $\text{gr}(M)$ is a finitely generated module over $\text{gr}(A)$.

Lemma 2.2.44. *Let A be a filtered K -algebra. Then an A -module M has a good filtration if and only if it is finitely generated.*

Proposition 2.2.45 ([18, Proposition 8.6.5]). *Let A be a K -algebra with a finite dimensional filtration such that $\text{gr}(A)$ is a finitely generated K -algebra and let M be an A -module with a good finite dimensional filtration. Then*

$$\text{GK}(\text{gr}(A)) = \text{GK}(A) \quad \text{and} \quad \text{GK}(\text{gr}(M)) = \text{GK}(M).$$

Definition 2.2.46 ([18, Definition 8.6.6]). Let A and S be two rings such that $A \subseteq S$. We say that S is an **almost centralizing extension** of A if S is generated as a ring by x_1, \dots, x_t over A satisfying

1. $ax_i - x_i a \in A$ and
2. $x_i x_j - x_j x_i \in \sum_{k=1}^t A x_k + A$

for all $i, j \leq t$ and each $a \in A$.

Proposition 2.2.47 ([18, Proposition 8.6.7]). *Let A and S be K -algebras with S being an almost centralizing extension of A with generators x_1, \dots, x_t and let A have a finite dimensional filtration $\{A_n\}$ such that $\text{gr}(A)$ is finitely generated K -algebra. Let x^i denote the monomial $x_1^{i(1)} \dots x_t^{i(t)}$. Then*

1. the inclusion $A \hookrightarrow S$ induces a graded ring homomorphism $\Psi : \text{gr}(A) \rightarrow \text{gr}(S)$,
2. $\text{gr}(S)$ is finitely generated, and it is generated over $\text{im}(\Psi)$ by the central elements $\bar{x}_1, \dots, \bar{x}_t$,
3. $\text{GK}(A) \leq \text{GK}(S) \leq \text{GK}(A) + t$,
4. if $A \cap \sum\{Ax^i \mid i \neq 0\} = 0$, then $\Psi : \text{gr}(A) \rightarrow \text{gr}(S)$ is an inclusion, and
5. $\text{gr}(S)$ is the polynomial ring over $\text{gr}(A)$ in the central variables $\bar{x}_1, \dots, \bar{x}_t$ if and only if the set $\{x^i\}$ is a free basis for the module S over the algebra A . In this case, we have $\text{GK}(S) = \text{GK}(A) + t$.

Definition 2.2.48 ([18, Definition 8.6.9]). A K -algebra A is called a **somewhat commutative algebra** if it has a finite dimensional filtration such that the associated graded algebra $\text{gr}(A)$ is a commutative finitely generated K -algebra.

Remark 2.2.49. Notice that if A is an almost commutative K -algebra via a standard finite dimensional filtration $\{A_n\}$ and $\text{gr}(A)$ is the associated graded algebra, then $\text{gr}(A)$ is a finitely generated algebra, generated by A_1/A_0 . It follows from the definition of almost commutative algebras and the definition of somewhat commutative algebras that any almost commutative algebra is somewhat commutative.

An example of a somewhat commutative algebra which is not almost commutative is the Ore extension algebra $K[y][x; -y^2 \frac{\partial}{\partial y}]$ (see [18, 8.6.10 and 14.3.9] for details).

Proposition 2.2.50 ([18, Proposition 8.6.9]). *If A is a K -algebra being an almost centralizing extension of a somewhat commutative K -algebra, then A is somewhat commutative.*

Definition 2.2.51 ([18, Definition 8.6.12]). Let A be a K -algebra. The two filtrations $\{M_n\}$ and $\{M'_n\}$ of an A -module M are called **equivalent** if there are $i, j \in \mathbb{N}$ such that $M_{n-i} \subseteq M'_n \subseteq M_{n+j}$ for all n .

Proposition 2.2.52 ([18, Proposition 8.6.13]). *Let A be a filtered ring with $\text{gr}(A)$ being a finitely generated ring extension of A_0 , and let M be a finitely generated A -module. Then any two good filtrations of the module M are equivalent.*

Theorem 2.2.53 ([18, Theorem 8.6.18]). 1. *Let A be a K -algebra with a filtration such that $\text{gr}(A)$ is a commutative finitely generated K -algebra and let M be an A -module having good filtrations $\{M_n\}$ and $\{M'_n\}$. Then $\text{GK}(\text{gr}(M)) = \text{GK}(\text{gr}(M'))$.*

2. *If we choose standard finite dimensional filtrations of $\text{gr}(A)$, $\text{gr}(M)$ and $\text{gr}(M')$, then*

(a) *the leading terms of the Hilbert polynomials of $\text{gr}(M)$ and $\text{gr}(M')$ are equal, and*

(b) $e(\text{gr}(M)) = e(\text{gr}(M'))$.

Theorem 2.2.54 ([18, Theorem 8.6.19]). *Let A be a somewhat commutative K -algebra and let M be a finitely generated A -module having a good filtration. If we choose standard finite dimensional filtrations of $\text{gr}(A)$ and $\text{gr}(M)$ with h being the Hilbert polynomial of $\text{gr}(M)$, then*

1. $\deg(h) = \text{GK}(M) = \text{GK}(\text{gr}(M)) \in \mathbb{N}$. *The integer $\deg(h)$ does not depend on the choices of filtrations of A , $\text{gr}(A)$, M and $\text{gr}(M)$, and*
2. *if we choose a fixed filtration of A and $\text{gr}(A)$, then the leading term of h does not depend on the choices of filtrations of M and $\text{gr}(M)$.*

Proposition 2.2.55 ([18, Proposition 8.6.21]). *Let S be K -algebra. Then the following conditions are equivalent:*

1. *S has a standard filtration such that $\text{gr}(S)$ is a commutative finitely generated K -algebra, and*
2. *S is an almost centralizing extension of a commutative finitely generated K -algebra.*

Chapter 3

Weight modules over Lie algebras

Unless otherwise stated, the following notation remain fixed throughout this chapter:

- K is an algebraically closed field of characteristic zero;
- \mathcal{G} is a finite dimensional semisimple Lie algebra of rank n over K ;
- \mathcal{H} is a (fixed) Cartan subalgebra of \mathcal{G} and $\mathcal{H}^* := \text{Hom}_K(\mathcal{H}, K)$ is the dual space of \mathcal{H} ;
- $U = U(\mathcal{G})$ is the universal enveloping algebra of \mathcal{G} ;
- Φ is the set of roots of \mathcal{G} and $\Delta = \{\alpha_1, \dots, \alpha_n\}$ is the set of simple roots;
- $\Phi = \Phi_+ \amalg \Phi_-$ where Φ_+ are positive roots and $\Phi_- = -\Phi_+$ are negative roots of \mathcal{G} ;
- $\mathcal{G} = \mathfrak{n}_- \oplus \mathcal{H} \oplus \mathfrak{n}_+$ is the triangular decomposition and $\{X_\alpha \mid \alpha \in \Phi\}$ is the Chevalley basis for $\mathfrak{n}_- \oplus \mathfrak{n}_+$ ($\mathfrak{n}_\pm = \bigoplus_{\alpha \in \Phi_\pm} KX_\alpha$);
- $U = \bigcup_{i \in \mathbb{N}} U_i$ is the standard filtration on U (by the total degree of the canonical generators);
- $\text{gr}(U) = \bigoplus_{i \in \mathbb{N}} U_i/U_{i-1}$ is the associated graded algebra of U ;
- If \mathfrak{a} is a Lie algebra over K , then $\mathcal{M}(\mathfrak{a})$ stands for the set of all isomorphism classes of (left) $U(\mathfrak{a})$ -modules.

3.1 Introduction

Let K be an algebraically closed field of characteristic zero, \mathcal{G} a finite dimensional semisimple Lie algebra over K , \mathcal{H} a fixed Cartan subalgebra of \mathcal{G} , $\mathcal{H}^* = \text{Hom}_K(\mathcal{H}, K)$

the dual space of \mathcal{H} , $U(\mathcal{G})$ the universal enveloping algebra of \mathcal{G} and $\mathcal{M}(\mathcal{G}, \mathcal{H})$ the set of all isomorphism classes of finitely generated $U(\mathcal{G})$ -modules that decompose into direct sums of finite dimensional \mathcal{H} -weight spaces. Fernando's approach [9] to classifying $U(\mathcal{G})$ -modules in the set $\mathcal{M}(\mathcal{G}, \mathcal{H})$ that are simple is reduced to the classification of torsion-free $U(\mathcal{G})$ -modules in $\mathcal{M}(\mathcal{G}, \mathcal{H})$. A $U(\mathcal{G})$ -module $M \in \mathcal{M}(\mathcal{G}, \mathcal{H})$ is called a **torsion-free module** if $\dim_K(K[x].m) = \infty$, for every $x \in \mathcal{G} \setminus \mathcal{H}$ and $m \in M \setminus \{0\}$. In Fernando's work, torsion-free modules play a role similar to the role of highest weight spaces in the classification of simple highest weight modules. Furthermore, Fernando considers the question of determining the class of simple Lie algebras which admit a torsion-free module. He shows that if \mathcal{G} is a simple Lie algebra that admits a torsion-free $U(\mathcal{G})$ -module, then \mathcal{G} is either of type A or of type C .

3.2 Generalities

In this section, some results which are valid for any finite dimensional Lie algebra will be recalled.

Let \mathfrak{a} be a finite dimensional Lie algebra over K , $U = U(\mathfrak{a})$ the universal enveloping algebra of the Lie algebra \mathfrak{a} and $U(\mathfrak{a}) = \bigcup_{i \geq 0} U_i(\mathfrak{a})$ the standard filtration on U (by the total degree of the canonical generators). Recall that the associated graded algebra

$$\text{gr}(U(\mathfrak{a})) = \bigoplus_{i \geq 0} U_i(\mathfrak{a})/U_{i-1}(\mathfrak{a})$$

is isomorphic to the symmetric algebra $S(\mathfrak{a})$. The symmetric algebra $S(\mathfrak{a})$ can be viewed as the coordinate ring of the affine space \mathfrak{a}^* . If E is a subset of \mathfrak{a}^* , then

$$\mathcal{F}(E) = \{p \in S(\mathfrak{a}) \mid p(x) = 0 \text{ for every } x \in E\}$$

is an ideal of $S(\mathfrak{a})$. Let $I(S)$ be the set of all ideals in the algebra $S(\mathfrak{a})$ of the form $\mathcal{F}(E)$ for some $E \subseteq \mathfrak{a}^*$. If $J \in I(S)$, then let

$$\mathcal{V}(J) = \{x \in \mathfrak{a}^* \mid p(x) = 0 \text{ for every } p \in J\}.$$

If J is an ideal of the algebra $S(\mathfrak{a})$, then let $\sqrt{J} = \{p \in S(\mathfrak{a}) \mid p^n \in J \text{ for some } n \in \mathbb{N}\}$. If $\sqrt{J} = J$, then J is said to be a **radical ideal**. By Hilbert's zeros theorem ([23, Chapter VII, §3]),

1. the set of radical ideals in the algebra $S(\mathfrak{a})$ is equal to $I(S)$, and
2. the map $\mathcal{V} \mapsto \mathcal{F}(\mathcal{V})$ is a bijection between the set of all subvarieties of \mathfrak{a}^* and the set $I(S)$. The inverse map is $J \mapsto \mathcal{V}(J)$. Under this map, the prime ideals of the algebra $S(\mathfrak{a})$ are associated with irreducible subvarieties of \mathfrak{a}^* .

If $u \in U_j(\mathfrak{a}) \setminus U_{j-1}(\mathfrak{a})$, then the associated homogeneous polynomial $\text{gr}(u) \in S_j(\mathfrak{a}) = U_j(\mathfrak{a})/U_{j-1}(\mathfrak{a})$ is said to be the **symbol** of u . By using the canonical embedding $\mathfrak{a} \hookrightarrow S(\mathfrak{a})$, u will be identified with its symbol when $u \in \mathfrak{a}$. Let M be a finitely generated module over the enveloping algebra $U(\mathfrak{a})$, and let $M = U(\mathfrak{a})M_0$ for a finite dimensional generating subspace M_0 of M . Then the filtration $\{M_n\}$ of M is called **standard** if $M_n = U_n(\mathfrak{a})M_0$ for each n . The filtration $\{M_n\}$ of M is called a **good filtration** if $\text{gr}(M) = \bigoplus_{i \geq 0} M_i/M_{i-1}$ is a finitely generated module over $\text{gr}(U(\mathfrak{a}))$.

Lemma 3.2.1 ([17, Lemma 6.7]). *If $\{M_n\}$ is a standard filtration of a $U(\mathfrak{a})$ -module M , then $\{M_n\}$ is a good filtration on M .*

Let M be a finitely generated module over the algebra $U(\mathfrak{a})$ with any good filtration. Then $J(M) := \sqrt{\text{ann}(\text{gr}(M))}$ is a graded ideal in the algebra $S(\mathfrak{a})$. Notice that $J(M)$ is independent of the choice of the good filtration $\{M_n\}$ [14, p. 448]. Set $\mathcal{V}(M) = \mathcal{V}(J(M))$. $\mathcal{V}(M)$ is said to be the **associated variety of M** .

The submodules $M^{[S]}$ and $M^{(S)}$ of an \mathfrak{a} -module M .

Let x be an element in the algebra $U(\mathfrak{a})$, then the associative subalgebra of $U(\mathfrak{a})$ generated by x is denoted by $\langle x \rangle$.

Definition 3.2.2 ([9, p.760]). Let \mathfrak{a} be a Lie algebra over K , S a subset of the enveloping algebra $U(\mathfrak{a})$ and M a module over $U(\mathfrak{a})$. Let the subsets of M , $M^{[S]}$ and $M^{(S)}$, defined by

$$M^{[S]} = \{m \in M \mid \dim_K(\langle s \rangle.m) < \infty \text{ for every } s \in S\}$$

and

$$M^{(S)} = \{m \in M \mid \text{if } s \in S, \text{ then there exists } r = r(s, m) \in \mathbb{N} \text{ such that } s^r.m = 0\}.$$

Then

1. if $M^{[S]} = M$, then S is said to be **locally finite** on M . If the center $Z(U(\mathfrak{a}))$ of the enveloping algebra $U(\mathfrak{a})$ is locally finite on M , then M is said to be $Z(U(\mathfrak{a}))$ -finite.
2. if $M^{(S)} = M$, then S is said to be **locally nilpotent** on M .

For convenience, we use $M^{[s]}$ to denote $M^{\{\{s\}\}}$ for a single element s .

Remark 3.2.3. Note that if a subset S of the enveloping algebra $U(\mathfrak{a})$ is locally finite on a $U(\mathfrak{a})$ -module M , then $\text{gr}(M)$ is a subset of $J(M)$. Note also that since $J(M)$ is a graded ideal, $\{0\}$ is a subset of $\mathcal{V}(M)$.

The next lemma describes the case when the associated variety is equal to $\{0\}$.

Lemma 3.2.4 ([9, Lemma 2.1]). *If \mathfrak{a} is a finite dimensional Lie algebra over K , and M is a finitely generated module over the enveloping algebra $U(\mathfrak{a})$, then the following conditions are equivalent:*

1. The $U(\mathfrak{a})$ -module M is finite dimensional.
2. The associated variety $\mathcal{V}(M)$ is equal to $\{0\}$.

Any associative algebra A over K becomes a Lie algebra, denoted LA , via the bracket operation $[x, y] = xy - yx$ for all $x, y \in A$. Let M be a module over the enveloping algebra $U(\mathfrak{a})$ of a Lie algebra \mathfrak{a} . Then let U_M be the quotient algebra $U(\mathfrak{a})/\text{ann}(M)$. Let $x \mapsto \bar{x}$ be the natural projection from the enveloping algebra $U(\mathfrak{a})$ to the algebra U_M .

Lemma 3.2.5 ([9, Lemma 2.2]). *Let \mathfrak{a} be a Lie algebra over K , S a subset of the enveloping algebra $U(\mathfrak{a})$ and M a finitely generated $U(\mathfrak{a})$ -module. Then*

1. the set $M^{[S]}$ is a $U(\mathfrak{a})$ -submodule of the module M if the adjoint action of the image, \bar{S} , of the subset S in LU_M is locally finite on LU_M , and
2. the set $M^{(S)}$ is a $U(\mathfrak{a})$ -submodule of the module M if \bar{S} is locally nilpotent on LU_M .

The theorem of Gabber and its application.

The algebra $S(\mathfrak{a})$ has a Poisson bracket structure defined as follows.

Recall that the enveloping algebra $U = U(\mathfrak{a})$ of a Lie algebra \mathfrak{a} admits the standard filtration $U = \bigcup_{i \in \mathbb{N}} U_i$ and the associated graded algebra $\text{gr}(U) = \bigoplus_{i \in \mathbb{N}} \text{gr}_i(U)$ where $\text{gr}_i(U) = U_i/U_{i-1}$ and $U_{-1} = \{0\}$. Let ψ_i denote the projection of U_i onto $\text{gr}_i(U)$. Let $f \in \text{gr}_i(U)$ and $g \in \text{gr}_j(U)$, and choose $x \in U_i$ and $y \in U_j$ such that $\psi_i(x) = f$ and $\psi_j(y) = g$. The commutativity of $\text{gr}(U)$ implies that $xy - yx \in U_{i+j-1}$. Then the Poisson bracket $\{f, g\}$ of f and g is defined by

$$\{f, g\} = \psi_i(xy - yx).$$

The definition of the Poisson bracket $\{f, g\}$ is independent of the choices of x and y . This definition can be extended to nonhomogeneous elements of the algebra $S(\mathfrak{a})$ by using bilinearity. Notice that the algebra $S(\mathfrak{a})$ has a Lie algebra structure defined by this Poisson bracket that is an extension of the bracket operation on a . Also, notice that

$$\{fg, h\} = f\{g, h\} + \{f, h\}g \quad \text{for all } f, g, h \in S(\mathfrak{a}).$$

Theorem 3.2.6 (Gabber's theorem, [14]). *Let \mathfrak{a} be a finite dimensional Lie algebra over K , M a finitely generated module over $U(\mathfrak{a})$ and $\{M_n\}$ a good filtration on the module M . Then the graded ideal $J(M) = \sqrt{\text{ann}(\text{gr}(M))}$ of the algebra $S(\mathfrak{a})$ is closed under the Poisson bracket in the algebra $S(\mathfrak{a})$.*

Remark 3.2.7 ([9, Remark 2.6]). Note that $J(M)$ is a linear subspace of the algebra $S(\mathfrak{a})$ since $S(\mathfrak{a})$ is commutative. Hence, $J(M)$ is a Lie subalgebra of $S(\mathfrak{a})$, by Theorem 3.2.6.

Definition 3.2.8. A subvariety \mathcal{V} of \mathfrak{a}^* is called **involutive** if the ideal $\mathcal{F}(\mathcal{V})$ is closed under the Poisson bracket.

By Theorem 3.2.6, the associated variety of a finitely generated module M over the enveloping algebra $U(\mathfrak{a})$ is involutive.

Corollary 3.2.9 ([9, Corollary 2.7]). *The set $\mathfrak{a}[M] = \{s \in \mathfrak{a} \mid M^{[s]} = M\}$ of all elements in the Lie algebra \mathfrak{a} that are locally finite on the $U(\mathfrak{a})$ -module M is the largest Lie subalgebra of \mathfrak{a} that is locally finite on M .*

Remark 3.2.10 ([9, Remark 2.9]). The $U(\mathfrak{a})$ -module M is finite dimensional if and only if $\mathfrak{a}[M] = \mathfrak{a}$.

Pure $U(\mathfrak{a})$ -modules.

Definition 3.2.11. Let \mathfrak{a} be a finite dimensional Lie algebra over K , and let M be a $U(\mathfrak{a})$ -module. Then M is called a **pure module** if for every element $s \in \mathfrak{a}$, $M^{[s]}$ equals either $\{0\}$ or M .

- Remarks 3.2.12.**
1. Clearly, any simple module is a pure module.
 2. If \mathfrak{a}' is a Lie subalgebra of the Lie algebra \mathfrak{a} and M is a pure $U(\mathfrak{a}')$ -module, then the induced $U(\mathfrak{a})$ -module $U(\mathfrak{a}) \otimes_{U(\mathfrak{a}')} M$ is a pure module.

Corollary 3.2.13 ([9, Corollary 2.10]). *Let M be a finitely generated module over the enveloping algebra $U(\mathfrak{a})$. Then there exists a finite chain*

$$M = M_n \supseteq M_{n-1} \supseteq \dots \supseteq M_0 = 0$$

of $U(\mathfrak{a})$ -submodules of M such that M_i/M_{i-1} is a pure module for $i = 1, \dots, n$.

Corollary 3.2.14 ([9, Corollary 2.11]). *Any irreducible component of the associated variety $\mathcal{V}(M)$ of the module M is involutive.*

The inequality of Gabber and Joseph.

Let \mathfrak{a} be a finite dimensional Lie algebra over K , $\{U_i(\mathfrak{a})\}_{i \in \mathbb{N}}$ the standard filtration on the enveloping algebra $U(\mathfrak{a})$ and M a finitely generated module over $U(\mathfrak{a})$. Recall that if $\{M_n\}$ is a standard filtration on M , then the Gelfand-Kirillov dimension of M , $\text{GK}(M)$, equals the degree of the Hilbert polynomial of M . Therefore, $\text{GK}(M)$ may be identified with the dimension of the associated variety $\mathcal{V}(M)$. The next theorem is a result of Gabber and Joseph.

Theorem 3.2.15 ([9, Theorem 2.12]). *Let \mathfrak{a} be a finite dimensional Lie algebra over K , and let M be a finitely generated module over $U(\mathfrak{a})$. Let U_M denote the quotient algebra $U(\mathfrak{a})/\text{ann}(M)$, considered as a left module over $U(\mathfrak{a})$. Then*

$$\text{GK}(U_M) \leq 2\text{GK}(M).$$

3.3 Semisimple Lie algebras and parabolic induction

Most of the results in this section are generalizations of the properties of highest weight modules. A natural method to construct modules over a finite dimensional Lie algebra is via induction from parabolic subalgebras. Induction and restriction play a crucial role in the classification of simple modules.

Let \mathcal{G} be a semisimple Lie algebra, \mathcal{H} a fixed Cartan subalgebra of \mathcal{G} , Φ the corresponding root system, $\mathbb{Z}\Phi$ the root lattice of Φ , Δ a base for Φ and Φ_+ (respectively Φ_-) the set of positive (respectively negative) roots relative to Δ . Let S be a subset of Δ , and let Φ_S denote the subsystem $\Phi \cap \sum_{\alpha \in S} \mathbb{Z}\alpha$ of Φ . Then every parabolic subalgebra \mathfrak{p} of \mathcal{G} which contains the Cartan subalgebra \mathcal{H} is of the form $\mathfrak{p}_{\Delta,S} = \mathcal{G}_P \oplus \mathcal{H}$ where $P = \Phi_+ \cup \Phi_S$ for some base Δ of Φ and some subset S of Δ , and $\mathcal{G}_P = \bigoplus_{\lambda \in P} \mathcal{G}_\lambda$. If $|\Delta \setminus S| = 1$, then $\mathfrak{p}_{\Delta,S}$ is said to be a maximal parabolic subalgebra of \mathcal{G} . If B is a subset of Φ , then set $B^s = B \cap (-B)$ and $B^a = B \setminus (-B)$ (the symmetric part and antisymmetric part of B , respectively). Let $\mathcal{G} = \mathcal{H} \oplus \bigoplus_{\alpha \in \Phi} \mathcal{G}_\alpha$ be the root space decomposition of \mathcal{G} , and choose a Chevalley basis

$$\{X_\alpha \mid \alpha \in \Phi\} \cup \{H_i \in \mathcal{H} \mid 1 \leq i \leq n\}$$

for \mathcal{G} such that $X_\alpha \in \mathcal{G}_\alpha$ for every $\alpha \in \Phi$. The K -linear map $\nu : \mathcal{G} \rightarrow \mathcal{G}$, $X_\alpha \mapsto X_{-\alpha}$ and $H \mapsto -H$ for all $H \in \mathcal{H}$, is a Lie algebra automorphism of \mathcal{G} . The following notation will be fixed for the remainder of this part. Let $\mathfrak{p}, \mathfrak{p}^-$ be a pair of parabolic subalgebras of \mathcal{G} that contain \mathcal{H} such that $\nu(\mathfrak{p}) = \mathfrak{p}^-$. Choose a subset $S \subseteq \Delta$ of Φ such that $(\mathfrak{p}, \mathfrak{p}^-) = (\mathfrak{p}_{\Delta,S}, \mathfrak{p}_{-\Delta,-S})$. Let $\mathfrak{u} = \mathfrak{u}_{\Delta,S}$ (respectively $\mathfrak{u}^- = \mathfrak{u}_{-\Delta,-S}^-$) be the nilradical of \mathfrak{p} (respectively \mathfrak{p}^-), and let $\mathfrak{l} = \mathfrak{p} \cap \mathfrak{p}^-$. Let $B = \Phi_+ \cup \Phi_S$, so that $\mathfrak{u} = \mathcal{G}_{B^a}$, and $\mathfrak{l} = \mathcal{H} \oplus \mathcal{G}_{B^s}$. Let V be a $U(\mathfrak{l})$ -module, and let \mathfrak{u} act trivially on V . Then V becomes a \mathfrak{p} -module. Now, set

$$M_{\Delta,S}(V) = \text{ind}_{\mathfrak{p}}^{\mathcal{G}}(V) = U(\mathcal{G}) \otimes_{U(\mathfrak{p})} V.$$

Then $M_{\Delta,S}(V)$ is a $U(\mathcal{G})$ -module. Conversely, if M is a $U(\mathcal{G})$ -module, then

$$M_{\mathfrak{u}} = M_{\mathfrak{u}_{\Delta,S}} = \{m \in M \mid \mathfrak{u} \cdot m = 0\}$$

is a $U(\mathfrak{l})$ -module (see [9, Section 3, p. 764] for details).

Proposition 3.3.1 ([9, Proposition 3.1]). *With the notation above, if V is $U(\mathfrak{l})$ -module and M is a $U(\mathcal{G})$ -module, then there exist vector space isomorphisms:*

$$\Psi = \Psi_{M,V} : \text{Hom}_{U(\mathcal{G})}(M_{\Delta,S}(V), M) \cong \text{Hom}_{U(\mathfrak{l})}(V, M_{\mathfrak{u}_{\Delta,S}}).$$

Proposition 3.3.2 ([9, Proposition 3.3]). *Let V be a simple $U(\mathfrak{l})$ -module. Then there exists a unique maximal proper submodule in the $U(\mathcal{G})$ -module $M_{\Delta,S}(V)$ and a corresponding unique simple quotient $L_{\Delta,S}(V)$.*

Proof. Let W be a proper submodule of $M_{\Delta,S}(V)$. Then for any nonzero element $w \in W \cap (1 \otimes_K V)$,

$$\begin{aligned} U(\mathcal{G})w &= U(\mathcal{G})(U(\mathfrak{l})w) \\ &= U(\mathcal{G})(1 \otimes_K V) \quad \text{since } w \in 1 \otimes_K V \text{ and } V \text{ is a simple } U(\mathfrak{l})\text{-module} \\ &= M_{\Delta,S}(V). \end{aligned}$$

Thus, $W \cap (1 \otimes_K V) = \{0\}$. Let $W(V)$ be the sum of all proper submodules of $M_{\Delta,S}(V)$. Clearly, $W(V) \cap (1 \otimes_K V) = \{0\}$. Since $1 \otimes_K V \neq \{0\}$, $W(V)$ is a proper submodule of $M_{\Delta,S}(V)$. This implies that $W(V)$ is a maximal submodule of $M_{\Delta,S}(V)$, and $L_{\Delta,S}(V) = M_{\Delta,S}(V)/W(V)$ is a corresponding unique simple quotient $U(\mathcal{G})$ -module. \square

Let \mathcal{G} be a finite dimensional semisimple Lie algebra over K , $Z(U(\mathcal{G}))$ the center of the enveloping algebra $U(\mathcal{G})$ and $\widehat{Z}(U(\mathcal{G}))$ the set of all central characters of the enveloping algebra $U(\mathcal{G})$. If $\chi \in \widehat{Z}(U(\mathcal{G}))$, then the χ -primary component

$$M^\chi = \{m \in M \mid (z - \chi(z))^n \cdot m = 0 \text{ for all } z \in Z(U(\mathcal{G})), \text{ and some } n \in \mathbb{N}\}$$

of a $U(\mathcal{G})$ -module M is a $U(\mathcal{G})$ -submodule of M . Let $\text{ch}(M)$ denote the set

$$\{\chi \in \widehat{Z}(U(\mathcal{G})) \mid M^\chi \neq \{0\}\}$$

of all generalized central characters of a $U(\mathcal{G})$ -module M . By Quillen's Lemma [19], if M is a module over $U(\mathcal{G})$ of finite length, then M is $Z(U(\mathcal{G}))$ -finite, and the set $\text{ch}(M)$ is finite. If a $U(\mathcal{G})$ -module M is $Z(U(\mathcal{G}))$ -finite, then we have a direct sum decomposition

$$M = \bigoplus_{\chi \in \text{ch}(M)} M^\chi \tag{3.1}$$

of M in $\mathcal{M}(\mathcal{G})$. In addition, the sum in (3.1) is finite if M is a finitely generated $U(\mathcal{G})$ -module.

Now, set $\rho_{\Delta,S} = \frac{1}{2} \sum_{\lambda \in B^a} \lambda$, and let $\Psi_{\Delta,S}$ be the algebra automorphism of $Z(U(\mathcal{H})) (= S(\mathcal{H}) = U(\mathcal{H}))$ that restricts to $h \mapsto h - \rho_{\Delta,S}(h)$ on \mathcal{H} . Write $\varphi : U(\mathcal{G}) \rightarrow U(\mathfrak{l})$ and $\varphi' : U(\mathcal{G}) \rightarrow \mathfrak{u}^- U(\mathcal{G}) + U(\mathcal{G})\mathfrak{u}$ for the projection maps of the decomposition

$$U(\mathcal{G}) = U(\mathfrak{l}) \oplus (\mathfrak{u}^- U(\mathcal{G}) + U(\mathcal{G})\mathfrak{u})$$

of $U(\mathcal{G})$. The map φ restricts to a map $\varphi_{\Delta,S} : Z(U(\mathcal{G})) \rightarrow Z(U(\mathfrak{l}))$, and the map φ' restricts to a map $\varphi'_{\Delta,S} : Z(U(\mathcal{G})) \rightarrow U(\mathcal{G})\mathfrak{u}$, [22, p. 118].

If $S = \emptyset$, then set $\mathfrak{p}_\Delta = \mathfrak{p}_{\Delta,S}$, $\Psi_\Delta = \Psi_{\Delta,S}$ and $\varphi_\Delta = \varphi_{\Delta,S}$. Let the Harish-Chandra isomorphism $\Psi_\Delta \circ \varphi_\Delta : Z(U(\mathcal{G})) \rightarrow Z(U(\mathcal{H}))$ be denoted by $\Gamma_{\mathcal{G},\mathcal{H}}$ and the map $\chi_{\mathcal{G},\mathcal{H}} : \widehat{Z}(U(\mathcal{H})) \rightarrow \widehat{Z}(U(\mathcal{G}))$ defined by $\chi_{\mathcal{G},\mathcal{H}}(\alpha) = \alpha \circ \Gamma_{\mathcal{G},\mathcal{H}}$.

Notice that the Harish-Chandra isomorphisms of the Lie algebras \mathcal{G} and \mathfrak{l} satisfy the relations $\Psi_{\Delta,S}^{-1} \circ \Gamma_{\mathcal{G},\mathcal{H}} = \Gamma_{\mathfrak{l},\mathcal{H}} \circ \varphi_{\Delta,S}$.

Passing to characters, we have the following commutative diagram:

$$\begin{array}{ccc}
\widehat{Z}(U(\mathcal{H})) & \xrightarrow{\eta_{\Delta,S}} & \widehat{Z}(U(\mathcal{H})) \\
\downarrow \chi_{\mathfrak{l},\mathcal{H}} & & \downarrow \chi_{\mathcal{G},\mathcal{H}} \\
\widehat{Z}(U(\mathfrak{l})) & \xrightarrow{\widehat{\varphi}_{\Delta,S}} & \widehat{Z}(U(\mathcal{G}))
\end{array}$$

where $\eta_{\Delta,S}$ is the translation map $\alpha \mapsto \alpha + \rho_{\Delta,S}$, and $\widehat{\varphi}_{\Delta,S}$ is the map $\chi \mapsto \chi \circ \varphi_{\Delta,S}$.

Theorem 3.3.3 ([9, Theorem 3.4]). *The map $\chi_{\mathcal{G},\mathcal{H}} : \widehat{Z}(U(\mathcal{H})) \rightarrow \widehat{Z}(U(\mathcal{G}))$ is a surjective map, and the fibres of $\chi_{\mathcal{G},\mathcal{H}}$ are finite and invariant under the action of the Weyl group \mathcal{W} of \mathcal{G} .*

Remark 3.3.4. Note that from the commutative diagram above and Theorem 3.3.3, the fibres of $\widehat{\varphi}_{\Delta,S}$ are finite.

Proposition 3.3.5 ([9, Proposition 3.5]). *1. If $V \in \mathcal{M}(\mathfrak{l})$ is $Z(U(\mathfrak{l}))$ -finite, then the $U(\mathcal{G})$ -module $M_{\Delta,S}(V)$ is $Z(U(\mathcal{G}))$ -finite, and $\text{ch}(M_{\Delta,S}(V)) = \widehat{\varphi}_{\Delta,S}(\text{ch}(V))$. In particular, if $\text{ch}(V)$ is a finite set, then $\text{ch}(M_{\Delta,S}(V))$ is a finite set.*

2. Let M be a $U(\mathcal{G})$ -module such that M is $Z(U(\mathcal{G}))$ -finite. Then the $U(\mathfrak{l})$ -module $M_{\mathfrak{u}_{\Delta,S}}$ is $Z(U(\mathfrak{l}))$ -finite and $\widehat{\varphi}_{\Delta,S}(\text{ch}(M_{\mathfrak{u}_{\Delta,S}})) \subseteq \text{ch}(M)$. In particular, if $\text{ch}(M)$ is a finite set, then $\text{ch}(M_{\mathfrak{u}_{\Delta,S}})$ is a finite set.

Now, we equip $\widehat{Z}(U(\mathcal{G}))$ with a partial order which is determined by Δ and S . Let $Z(\mathfrak{l})$ be the center of \mathfrak{l} in the Lie algebra \mathfrak{l} , and let $\pi : \widehat{Z}(U(\mathcal{G})) \rightarrow Z(\mathfrak{l})^*$ be the restriction map $\mu \mapsto \mu|_{Z(\mathfrak{l})}$ induced by the inclusion $Z(\mathfrak{l}) \hookrightarrow Z(U(\mathfrak{l}))$. We introduce a partial order \leq among the elements of $\widehat{Z}(U(\mathfrak{l}))$ by letting $\chi_1 \leq \chi_2$ whenever $\pi(\chi_1) - \pi(\chi_2)$ is a sum of roots in the set $B^a = \{\lambda \in \Delta \mid \mathcal{G}_\lambda \subseteq \mathfrak{u}\}$. Let $\chi_1 < \chi_2$ mean $\chi_1 \leq \chi_2$ and $\chi_1 \neq \chi_2$.

Proposition 3.3.6 ([9, Proposition 3.6]). *Let $M \in \mathcal{M}(\mathcal{G})$ be generated by a $U(\mathfrak{l})$ -submodule V of $M_{\mathfrak{u}_{\Delta,S}}$. Then*

1. *there exists a unique $U(\mathfrak{l})$ -module homomorphism $\psi : M_{\Delta,S}(V) \rightarrow M$ such that $\psi(1 \otimes_K v) = v$ for every $v \in V$, and ψ is a surjective map,*
2. *$M = U(\mathfrak{u}^-)V$, and*
3. *if V is χ -primary for some $\chi \in \widehat{Z}(U(\mathfrak{l}))$, and $M_{\mathfrak{l}}$ is a $U(\mathfrak{l})$ -submodule of $\text{res}_{\mathfrak{l}}^{\mathcal{G}}(M)$ (the restriction of M to \mathfrak{l}), then we have a direct sum decomposition*

$$M_{\mathfrak{l}} = \bigoplus_{\chi' \in \widehat{Z}(U(\mathfrak{l}))} M_{\mathfrak{l}}^{\chi'} \quad (3.2)$$

of $M_{\mathfrak{l}}$ in $\mathcal{M}(\mathfrak{l})$. In addition, $\chi' \leq \chi$ whenever $\chi' \in \text{ch}(M_{\mathfrak{l}})$ and $M_{\mathfrak{l}}^{\chi} = V \cap M_{\mathfrak{l}}$. In particular, $(M_{\mathfrak{u}})^{\chi} = V$.

Recall that if V is a simple $U(\mathfrak{l})$ -module, then by Proposition 3.3.2, the $U(\mathcal{G})$ -module $M_{\Delta,S}(V)$ has a unique simple quotient $L_{\Delta,S}(V)$.

Proposition 3.3.7 ([9, Proposition 3.8]). *The maps $\beta_{\Delta,S} : V \mapsto L_{\Delta,S}(V)$ and $\delta_{\Delta,S} : M \mapsto M_{\mathfrak{u}_{\Delta,S}}$ give a bijective correspondence between the set of simple modules in $\mathcal{M}(\mathfrak{l})$ and the set of simple modules in $\mathcal{M}(\mathcal{G})$ that have nontrivial $\mathfrak{u}_{\Delta,S}$ -submodules.*

Proposition 3.3.8 ([9, Proposition 3.9]). *If M is a $U(\mathcal{G})$ -module of finite length, then the module $M_{\mathfrak{u}} \in \mathcal{M}(\mathfrak{l})$ is also of finite length.*

3.4 The category of weight modules.

Let \mathcal{G} be a finite dimensional semisimple Lie algebra over K , \mathcal{H} a fixed Cartan subalgebra of \mathcal{G} . Let V be a $U(\mathcal{G})$ -module and $\lambda \in \mathcal{H}^*$. We define

$$V_\lambda = \{v \in V \mid hv = \lambda(h)v \text{ for all } h \in \mathcal{H}\}.$$

In general, if $R \subseteq \mathcal{H}^*$, then let $V_R = \sum_{\lambda \in R} V_\lambda$. If $V_\lambda \neq \{0\}$, we call V_λ a **weight space**, and we say that λ is a **weight** of V . Let $\overline{\mathcal{M}}(\mathcal{G}, \mathcal{H})$ stand for the set of all isomorphism classes of $U(\mathcal{G})$ -modules that decompose into direct sums of weight spaces. If $V \in \overline{\mathcal{M}}(\mathcal{G}, \mathcal{H})$, then let $\text{wt}(V) := \{\lambda \in \mathcal{H}^* \mid V_\lambda \neq \{0\}\}$. Let $\mathcal{M}(\mathcal{G}, \mathcal{H})$ stands for the set of all isomorphism classes of finitely generated $U(\mathcal{G})$ -modules that decompose into direct sums of finite dimensional \mathcal{H} -weight spaces. The modules in the set $\mathcal{M}(\mathcal{G}, \mathcal{H})$ are called **$U(\mathcal{G})$ -weight modules**. Let Φ be the root system of \mathcal{G} , and choose a base Δ . Let \leq be the partial order on \mathcal{H}^* defined by Δ . Then a module V in $\overline{\mathcal{M}}(\mathcal{G}, \mathcal{H})$ is called Δ -highest weight module if there exists a $\lambda \in \text{wt}(V)$ such that $\alpha \leq \lambda$ for every $\alpha \in \text{wt}(V)$.

Lemma 3.4.1 ([9, Lemma 4.1]). *The classes $\overline{\mathcal{M}}(\mathcal{G}, \mathcal{H})$ and $\mathcal{M}(\mathcal{G}, \mathcal{H})$ are closed under taking submodules, quotients and tensoring with any finite dimensional $U(\mathcal{G})$ -module. If $V \in \overline{\mathcal{M}}(\mathcal{G}, \mathcal{H})$ and $R \in \mathcal{H}^*/Z\Phi$, then $V_R \in \overline{\mathcal{M}}(\mathcal{G}, \mathcal{H})$, and we have a direct sum decomposition*

$$V = \bigoplus_{R \in \mathcal{H}^*/Z\Phi} V_R \tag{3.3}$$

of V in $\overline{\mathcal{M}}(\mathcal{G}, \mathcal{H})$. If V is a finitely generated $U(\mathcal{G})$ -module, then the number of nonzero summands involved in the decomposition given in (3.3) is finite.

Lemma 3.4.2 ([9, Lemma 4.2]). *Let V be a module in $\overline{\mathcal{M}}(\mathcal{G}, \mathcal{H})$, and let the weight spaces of V be all finite dimensional. Then V is $Z(U(\mathcal{G}))$ -finite. If $V \in \mathcal{M}(\mathcal{G}, \mathcal{H})$, then the set of all generalized central characters, $\text{ch}(V)$, is finite.*

Lemma 3.4.3 ([9, Lemma 4.3]). *Let V be a module in $\overline{\mathcal{M}}(\mathcal{G}, \mathcal{H})$, and let the weight spaces of V be all finite dimensional. Then we have a direct sum decomposition*

$$V = \bigoplus_{R, \chi} V_R^\chi \tag{3.4}$$

of V in $\overline{\mathcal{M}}(\mathcal{G}, \mathcal{H})$ where the pairs (R, χ) run over a subset of $(\mathcal{H}^*/Z\Phi) \times \widehat{Z}(U(\mathcal{G}))$, and for each (R, χ) , V_R^χ is an χ -primary module such that $\text{wt}(V_R^\chi) \subseteq R$. If $V \in \mathcal{M}(\mathcal{G}, \mathcal{H})$, then the direct sum decomposition in (3.4) is a sum in $\mathcal{M}(\mathcal{G}, \mathcal{H})$, and the number of nonzero summands involved in this decomposition is finite.

Lemma 3.4.4 ([9, Lemma 4.5]). *Let $V \in \mathcal{M}(\mathcal{G}, \mathcal{H})$ be a simple module, and let $\mathcal{G} = \mathcal{G}_1 \oplus \mathcal{G}_2$ where \mathcal{G}_1 and \mathcal{G}_2 are ideals of \mathcal{G} . Set $\mathcal{H}_1 = \mathcal{G}_1 \cap \mathcal{H}$ and $\mathcal{H}_2 = \mathcal{G}_2 \cap \mathcal{H}$. Then there exist simple modules $V_1 \in \mathcal{M}(\mathcal{G}_1, \mathcal{H}_1)$ and $V_2 \in \mathcal{M}(\mathcal{G}_2, \mathcal{H}_2)$ such that $V = V_1 \otimes_K V_2$ as $U(\mathcal{G})$ -modules.*

Before we state the next lemma, we need some notation. Let $V \in \overline{\mathcal{M}}(\mathcal{G}, \mathcal{H})$, $\lambda \in \Phi$ and $s = X_\lambda$ (an element of the chosen Chevalley basis for \mathcal{G}). Set the $U(\mathcal{G})$ -module $V^{[\lambda]} = V^{[s]}$. If $V^{[\lambda]} = V$, then V is called λ -finite, and if $V^{[\lambda]} = \{0\}$, then V is called λ -free. If B is a subset of Φ , then let the subspaces $\bigoplus_{\lambda \in B} \mathcal{G}_\lambda$ and $\sum_{\lambda \in B} [\mathcal{G}_\lambda, \mathcal{G}_{-\lambda}]$ of \mathcal{G} be denoted by \mathcal{G}_B and \mathcal{H}_B , respectively. Let V be a finitely generated module in $\overline{\mathcal{M}}(\mathcal{G}, \mathcal{H})$. Then the set $F(V) = \{\lambda \in \Phi \mid V \text{ is } \lambda\text{-finite}\}$ is a closed subset of Φ (i.e. if $\lambda, \mu \in F(V)$ and $\lambda + \mu \in \Phi$, then $\lambda + \mu \in F(V)$). Now, we show that $\mathcal{H} \oplus \mathcal{G}_{F(V)} = L[V]$ where $L[V]$ is the set of all elements in the Lie algebra \mathcal{G} which are locally finite on V . It is clear that $\mathcal{H} \oplus \mathcal{G}_{F(V)} \subseteq L[V]$. On the other hand, notice that $\mathcal{H} \oplus \mathcal{G}_B = L[V]$ for some subset B of Φ since $L[V]$ is a Lie subalgebra of \mathcal{G} containing \mathcal{H} . But $\lambda \in F(V)$ if and only if $\mathcal{G}_\lambda \subseteq L[V]$, and hence $B = F(V)$. Therefore, $L[V] = \mathcal{H} \oplus \mathcal{G}_{F(V)}$.

Lemma 3.4.5 ([9, Lemma 4.6]). *If V is a finitely generated module in $\overline{\mathcal{M}}(\mathcal{G}, \mathcal{H})$, then $L[V] = \mathcal{H} \oplus \mathcal{G}_{F(V)}$. The $U(\mathcal{G})$ -module V is finite dimensional if and only if $F(V) = \emptyset$.*

Remark 3.4.6 ([9, Remark 4.7]). *If V is a module in $\overline{\mathcal{M}}(\mathcal{G}, \mathcal{H})$, then $V^{[\lambda]} = V^{(E_\lambda)}$ for every $\lambda \in \Phi$. In particular, X_λ is locally finite on V if and only if X_λ is locally nilpotent on V . To show this, notice that if $v \in V_\mu$ is a weight vector in V , then $X_\lambda \cdot v \in V_{\lambda+\mu}$. This implies that zero is the only eigenvalue of X_λ , in its action on V .*

Lemma 3.4.7 ([9, Lemma 4.8]). *Let $V \in \overline{\mathcal{M}}(\mathcal{G}, \mathcal{H})$, and let $\lambda \in \Phi$. Then*

1. $\text{wt}(V) + \mathbb{N}\lambda \subseteq \text{wt}(V)$ if V is λ -free, and
2. if $(\mu + \mathbb{N}\lambda) \cap \text{wt}(V)$ is a finite set for every $\mu \in \text{wt}(V)$, then V is λ -finite.

Definition 3.4.8. Let V be a module in $\overline{\mathcal{M}}(\mathcal{G}, \mathcal{H})$, and let \mathcal{G}' be a Lie subalgebra of \mathcal{G} . Then \mathcal{G}' is called **torsion-free** on V if $V^{[x]} = \{0\}$ for every $x \in \mathcal{G}' \setminus \mathcal{H}$. If $V^{[x]} = \{0\}$ for every $x \in \mathcal{G}' \setminus \mathcal{H}$, then V is called a **torsion-free module**.

Recall that if B is a subset of Φ , then $B^s = B \cap (-B)$ and $B^a = B \setminus (-B)$; $B = B^a \sqcup B^s$. B is called a **parabolic subset** if $B \sqcup (-B) = \Phi$, and B is a closed subset of Φ . A pair (B_1, B_2) of parabolic subsets of Φ is called a **pair of opposite parabolic subsets** if $B_1^s = B_2^s$ and $B_2^a = B_1^a$. A Lie subalgebra \mathfrak{p} of \mathcal{G} which contains the Cartan subalgebra \mathcal{H} is a parabolic subalgebra if and only if it is of the form $\mathcal{G}_B \oplus \mathcal{H}$ for some parabolic subset B of Φ . Moreover, $(\mathcal{G}_{B_1} \oplus \mathcal{H}, \mathcal{G}_{B_2} \oplus \mathcal{H})$ is a pair of opposite parabolic subalgebras of \mathcal{G} if and only if (B_1, B_2) is a pair of opposite parabolic subsets of Φ .

Lemma 3.4.9 ([9, Lemma 4.16]). *Let Φ be a disjoint union of subsets B_1 and B_2 . Then the following statements are equivalent:*

1. B_1 and B_2 are closed subsets of Φ .
2. $(B_1 \cup B_2^s, B_2 \cup B_1^s)$ is a pair of opposite parabolic subsets of Φ .
3. There exists a base Δ of Φ , a subset S of Δ and a decomposition of S into mutually orthogonal subsets S_1 and S_2 such that $B_1 = (\Phi_+ \setminus \Phi_S) \cup \Phi_{S_1}$ and $B_2 = (\Phi_- \setminus \Phi_S) \cup \Phi_{S_2}$ are the decompositions of B_1 and B_2 into their antisymmetric and symmetric parts.

The next proposition states that the action of the Lie algebra \mathcal{G} on a pure weight module V determines a decomposition of \mathcal{G} into a sum of Lie subalgebras which are either locally finite on V or torsion-free on V .

Proposition 3.4.10 ([9, Proposition 4.17]). *Let V be a pure module in $\mathcal{M}(\mathcal{G}, \mathcal{H})$. Then the Lie algebra \mathcal{G} has a unique pair $(\mathfrak{p}_V, \mathfrak{p}_V^-)$ of opposite parabolic subalgebras such that*

1. the nilradical \mathfrak{u}_V of \mathfrak{p}_V is locally nilpotent on V ,
2. the nilradical \mathfrak{u}_V^- of \mathfrak{p}_V^- is torsion-free on V , and
3. the common Levi factor $\mathfrak{l} = \mathfrak{p}_V \cap \mathfrak{p}_V^-$ of \mathfrak{p}_V and \mathfrak{p}_V^- decomposes into a direct sum $\mathfrak{l}_V = Z(\mathfrak{l}_V) \oplus \mathfrak{s}_V \oplus \mathfrak{t}_V$ of ideals where $Z(\mathfrak{l}_V)$ is the center of \mathfrak{l}_V , \mathfrak{s}_V is the largest $\text{ad}(\mathcal{H})$ -stable semisimple subalgebra of \mathcal{G} which is locally finite on V , and \mathfrak{t}_V is the largest $\text{ad}(\mathcal{H})$ -stable semisimple subalgebra of \mathcal{G} which is torsion-free on V .

Moreover, there exists a parabolic subalgebra \mathfrak{q}_V of \mathcal{G} such that

- (a) the Cartan subalgebra \mathcal{H} of \mathcal{G} is contained in \mathfrak{q}_V ,
- (b) the nilradical \mathfrak{d}_V of \mathfrak{q}_V is locally nilpotent on V , and
- (c) the $\text{ad}(\mathcal{H})$ -stable Levi complement \mathfrak{r}_V of \mathfrak{d}_V in \mathfrak{q}_V is torsion-free on V .

Proof. See [9, page 775]. □

Now, to classify simple modules in $\mathcal{M}(\mathcal{G}, \mathcal{H})$, we use Proposition 3.4.10. Let V be a simple module in $\mathcal{M}(\mathcal{G}, \mathcal{H})$. Then V is a pure module, and hence there exists a unique pair $(\mathfrak{p}, \mathfrak{p}^-)$ of opposite parabolic subalgebras of \mathcal{G} such that $(\mathfrak{p}, \mathfrak{p}^-)$ and V together satisfy the conditions 1, 2 and 3 of Proposition 3.4.10. Since V is simple, by Proposition 3.3.7, $V_{\mathfrak{u}}$ is a simple module in $\mathcal{M}(\mathfrak{l}, \mathcal{H})$ (notice that if the module V is torsion-free, then $\mathfrak{u} = \{0\}$, and hence $\mathfrak{l} = \mathcal{G}$ and $V_{\mathfrak{u}} = V$ in this case). In addition, V can be recovered from $V_{\mathfrak{u}}$ since $V \cong L_{\Delta, S}(V_{\mathfrak{u}})$. This implies that a simple module V in $\mathcal{M}(\mathcal{G}, \mathcal{H})$ is completely determined by the module $V_{\mathfrak{u}}$ in $\mathcal{M}(\mathfrak{l}, \mathcal{H})$ (i.e. the classification of simple modules in

$\mathcal{M}(\mathcal{G}, \mathcal{H})$ can be reduced to the question of classifying simple modules in $\mathcal{M}(\mathfrak{l}, \mathcal{H})$. Now, choose a base Δ of Φ and a subset S of Δ such that $(\mathfrak{p}_{\Delta, S}, \mathfrak{p}_{-\Delta, -S}) = (\mathfrak{p}, \mathfrak{p}^-)$ where $(\mathfrak{p}, \mathfrak{p}^-)$ is the pair of opposite parabolic subalgebras introduced in Proposition 3.4.10. Let $\mathfrak{s}, \mathfrak{t}$ and $Z(\mathfrak{l})$ be as described in Proposition 3.4.10.(3), and let $\mathfrak{r} = Z(\mathfrak{l}) \oplus \mathfrak{s}$. Then by Lemma 3.4.4, the decomposition $\mathfrak{l} = \mathfrak{r} \oplus \mathfrak{t}$ induces a decomposition of $V_{\mathfrak{u}}$ into a tensor product $V_1 \otimes_K V_2$ where V_1 is a simple $U(\mathfrak{r})$ -module, and V_2 is a simple $U(\mathfrak{t})$ -module. If V is in $\mathcal{M}(\mathfrak{l}, \mathcal{H})$ and $F(V) = \{\lambda \in \Phi \mid V \text{ is } \lambda\text{-finite}\}$, then by Lemma 3.4.5, V is a finite dimensional module if and only if $F(V) = \Phi$. This implies that V_1 is finite dimensional. Since \mathfrak{t} is torsion on V , \mathfrak{t} is torsion-free on V_2 . This result is summarized in the next theorem.

Theorem 3.4.11 ([9, Theorem 4.18]). *Let V be a simple module in $\mathcal{M}(\mathcal{G}, \mathcal{H})$. Let $\mathfrak{p}, \mathfrak{p}^-, \mathfrak{u}, \mathfrak{l}, Z(\mathfrak{l}), \mathfrak{s}$ and \mathfrak{t} as described in Proposition 3.4.10, and let \mathfrak{r} be the $\text{ad}(\mathcal{H})$ -stable reductive subalgebra $Z(\mathfrak{l}) \oplus \mathfrak{s}$ of \mathcal{G} . Let Δ be a base of Φ , and let S be a subset of Δ such that $(\mathfrak{p}_{\Delta, S}, \mathfrak{p}_{-\Delta, -S}) = (\mathfrak{p}, \mathfrak{p}^-)$. Then $V_{\mathfrak{u}}$ is a simple $U(\mathfrak{l})$ -module which decomposes into the tensor product $V_{\text{fin}} \otimes_K V_{\text{fr}}$ where V_{fin} is a simple finite dimensional $U(\mathfrak{r})$ -module and V_{fr} is a simple torsion-free module in $\mathcal{M}(\mathfrak{t}, \mathfrak{t} \cap \mathcal{H})$. Conversely, if the pair $(V_{\text{fin}}, V_{\text{fr}})$ is given, then the module V can be recovered as $L_{\Delta, S}(V_{\text{fin}} \otimes_K V_{\text{fr}})$.*

The next two results show that every module V in $\mathcal{M}(\mathcal{G}, \mathcal{H})$ is of finite length.

Lemma 3.4.12 ([9, Lemma 4.19]). *Let V be a torsion-free module in $\overline{\mathcal{M}}(\mathcal{G}, \mathcal{H})$, and let the weight spaces of V be all finite dimensional. Let*

$$V = \bigoplus_{R \in \mathcal{H}^*/Z\Phi} V_R$$

be the decomposition of V introduced in 3.4.1. Then the following statements are equivalent:

1. V is of finite length.
2. V is a finitely generated $U(\mathcal{G})$ -module.
3. The number of nonzero summand involved in the decomposition above is finite.

Theorem 3.4.13 ([9, Theorem 4.21]). *If V is a simple module in $\mathcal{M}(\mathcal{G}, \mathcal{H})$, then V is of finite length.*

Proposition 3.4.14 ([9, Proposition 5.1]). *Let V be a torsion-free module in $\mathcal{M}(\mathcal{G}, \mathcal{H})$. Then the Gelfand-Kirillov dimension of V equals the rank of the Lie algebra $[\mathcal{G}, \mathcal{G}]$.*

Theorem 3.4.15 ([9, Theorem 5.2]). *Let \mathcal{G} be a simple Lie algebra with a fixed Cartan subalgebra \mathcal{H} , and let V be a torsion-free module in $\mathcal{M}(\mathcal{G}, \mathcal{H})$. Then the Lie algebra \mathcal{G} is either of type A or of type C .*

Chapter 4

The centralizer of the Cartan subalgebra in the universal enveloping algebra

4.1 Introduction

In this chapter, we study the structure of basic cycles. The importance of basic cycles is that they form a generating set for the centralizer C of the Cartan subalgebra in the universal enveloping algebra of a simple Lie algebra. In particular, the fact that the centralizer C is a finitely generated algebra will become clear. We first briefly review some results about the structure of basic cycles of the simple Lie algebras A_{n-1} obtained by Britten and Lemire [4]. Then we provide a set of generators for the algebra C .

Throughout this chapter, K denotes an algebraically closed field of characteristic zero, \mathcal{G} a simple finite dimensional Lie algebra over K , \mathcal{H} a fixed Cartan subalgebra, Φ the corresponding root system, $\Delta = \{\theta_1, \dots, \theta_n\}$ a base for Φ , $\Phi_+ = \{\mu_1, \dots, \mu_m\}$ the set of positive roots relative to Δ , \mathcal{W} the Weyl group of Φ , $U(\mathcal{G})$ the universal enveloping algebra of the Lie algebra \mathcal{G} and

$$C = \{u \in U(\mathcal{G}) \mid Hu = uH, \text{ for all } H \in \mathcal{H}\}$$

the **centralizer** of \mathcal{H} in $U(\mathcal{G})$. The universal enveloping algebra $U(\mathcal{G})$ has a PBW basis consisting of all monomials of the form

$$u(\bar{\alpha}, \bar{\beta}, \bar{\gamma}) = X_{-\mu_m}^{\alpha_m} \cdots X_{-\mu_1}^{\alpha_1} X_{\mu_1}^{\beta_1} \cdots X_{\mu_m}^{\beta_m} H_{\theta_1}^{\gamma_1} \cdots H_{\theta_n}^{\gamma_n}$$

where $\alpha_i, \beta_i, \gamma_i \in \mathbb{N}$.

Definition 4.1.1. The element $u(\bar{\alpha}, \bar{\beta}, \bar{\gamma})$ in $U(\mathcal{G})$ is called a **cycle** if

$$\sum_{i=1}^m (\beta_i - \alpha_i) \mu_i = 0.$$

Example 4.1.2. Let $\{\alpha, \beta\}$ be a base for the root system of the Lie algebra $\mathfrak{sl}_3(K)$. Then the elements

$$H_\alpha, H_\beta, Y_\alpha X_\alpha, Y_{\alpha+\beta} X_\alpha X_\beta, Y_\beta X_\beta H_\alpha \text{ and } Y_{\alpha+\beta} Y_\beta X_\alpha X_\beta^2$$

are all cycles in $U(\mathfrak{sl}_3)$. On the other hand, the elements

$$X_\alpha, Y_\alpha H_\beta, Y_\beta X_\alpha \text{ and } Y_{\alpha+\beta} Y_\beta X_\alpha$$

are not cycles in $U(\mathfrak{sl}_3)$.

Definition 4.1.3 ([4, Definition 1.2]). Let $c = u(\bar{\alpha}, \bar{\beta}, \bar{\gamma})$ be a cycle in $U(\mathcal{G})$ with $\bar{\gamma} = 0$, and let $r(c)$ denotes the multiset of all roots (counting multiplicity) appearing as subscripts in c . Then c is called a **basic cycle** if $r(c)$ is non-empty, and no proper subset of $r(c)$ sums to 0. The cycles H_λ , where $\lambda \in \Delta$, will be called **trivial basic cycles**.

Example 4.1.4. To illustrate this, consider Example 4.1.2 again. Then the cycles H_α and H_β are trivial basic cycles, and the cycles $Y_\alpha X_\alpha$ and $Y_{\alpha+\beta} X_\alpha X_\beta$ are basic cycles. However, the cycles $Y_\beta X_\beta H_\alpha$ and $Y_{\alpha+\beta} Y_\beta X_\alpha X_\beta^2$ are not basic cycles since they can be written as product of two other cycles.

Definition 4.1.5 ([4, Definition 1.3]). Let c be a basic cycle in the centralizer C of \mathcal{H} in the universal enveloping algebra $U(\mathcal{G})$ with $r(c) = \{\lambda_1, \dots, \lambda_t\}$, and let

$$r(w(c)) := \{w(\lambda_1), \dots, w(\lambda_t)\} \quad \text{for some } w \in \mathcal{W}.$$

The number of negative roots in $r(c)$ is denoted by $\text{Neg}(r(c))$. Also, we define $\mathcal{M}(c)$ to be the minimum number of $\text{Neg}(r(w(c)))$ for all $w \in \mathcal{W}$ and $\mathcal{M}(\mathcal{G})$ to be the maximum number of $\mathcal{M}(c)$ for all basic cycles $c \in C$.

In 1972, Dixmier [8] conjectured that if \mathcal{G} is a simple finite dimensional Lie algebra over K , $\mathcal{M}(\mathcal{G}) = 1$. Van den Hombergh [20] verified that Dixmier's conjecture only holds for $A_n, B_2, B_3, D_4, D_5, G_2$ and E_6 , and it does not hold for the other simple Lie algebras.

Throughout this chapter, we denote $\epsilon_1, \dots, \epsilon_n$ as the standard basis of the vector space \mathbb{R}^n , and ϵ_{-i} denotes $-\epsilon_i$.

4.2 Basic cycles of A_{n-1}

This section concerns the concept of a circle representation for a basic cycle of A_{n-1} . By using this concept, it will be shown that $\mathcal{M}(A_n) = 1$ (Theorem 4.2.3).

The root system of A_{n-1} can be taken to be

$$\Phi_{A_{n-1}} = \{\epsilon_i - \epsilon_j \mid 1 \leq i, j \leq n, i \neq j\}.$$

A base for the root system $\Phi_{A_{n-1}}$ is given by

$$\Delta_{A_{n-1}} = \{\epsilon_i - \epsilon_{i+1} \mid i = 1, \dots, n-1\}.$$

Notation: Let $\varphi(a_i, a_j) = \epsilon_{a_i} - \epsilon_{a_j}$.

Theorem 4.2.1 ([4, Theorem 2.1]). *1. Let (a_1, \dots, a_k) be a sequence of distinct elements of $\{1, \dots, n\}$ for some $k \geq 2$. Then the set of roots*

$$c(a_1, \dots, a_k) := \{\varphi(a_1, a_2), \varphi(a_2, a_3), \dots, \varphi(a_k, a_1)\}$$

equals the set $r(c)$ for a basic cycle c in the centralizer $C_{A_{n-1}}$ of \mathcal{H} in the universal enveloping algebra of the simple Lie algebra of type A_{n-1} .

2. If c is a basic cycle in $C_{A_{n-1}}$, there exists a sequence (a_1, \dots, a_k) of distinct elements of $\{1, \dots, n\}$ for some $k \geq 2$ such that $r(c) = c(a_1, \dots, a_k)$.

Definition 4.2.2. If $r(c) = c(a_1, \dots, a_k)$, the sequence $c(a_1, \dots, a_k)$ is called a **circle representation** for the basic cycle c ; in this case we write

$$CR(c) = (a_1, \dots, a_k).$$

Theorem 4.2.3 ([20, Proposition 2]). $\mathcal{M}(A_{n-1}) = 1$.

Sketch of proof. The Weyl group $\mathcal{W}_{A_{n-1}}$ for the Lie algebra of type A_{n-1} is the symmetric group consists of all permutations on $\{\epsilon_i \mid 1 \leq i \leq n\}$ (see [16, page 64]). Hence, if c is a basic cycle in $C_{A_{n-1}}$, there exists an element $w \in \mathcal{W}_{A_{n-1}}$ with

$$r(w(c)) = \{w(\varphi_1), \dots, w(\varphi_k)\} = c(1, \dots, k).$$

Since $\epsilon_k - \epsilon_1$ is the only negative root in $c(1, \dots, k)$, $\mathcal{M}(A_{n-1}) = 1$. \square

4.3 A generating set for the centralizer of the Cartan subalgebra in $U(\mathfrak{sl}_3)$

In this section, we first show that the centralizer of the Cartan subalgebra in the universal enveloping algebra of a simple finite dimensional Lie algebra is generated by

all the basic cycles. Then we determine a generating set for the centralizer of the Cartan subalgebra in the universal enveloping algebra $U(\mathfrak{sl}_3)$. This set of generators will be used to study the centralizer in the next chapter.

If we consider $U(\mathcal{G})$ as a \mathcal{G} -module under the adjoint action, then $U(\mathcal{G})$ has a weight space decomposition

$$U(\mathcal{G}) = \bigoplus_{\lambda \in \mathcal{H}^*} U_\lambda \quad \text{with} \quad U_\lambda U_\gamma \subseteq U_{\lambda+\gamma}.$$

The weight space U_0 is a subalgebra of $U(\mathcal{G})$ equal to the centralizer C of \mathcal{H} in the universal enveloping algebra $U(\mathcal{G})$. If $H \in \mathcal{H}$, then by direct computation we have

$$\begin{aligned} [H, u(\bar{\alpha}, \bar{\beta}, \bar{\gamma})] &= [H, X_{-\mu_m}^{\alpha_m} \cdots X_{-\mu_1}^{\alpha_1} X_{\mu_1}^{\beta_1} \cdots X_{\mu_m}^{\beta_m} H_{\theta_1}^{\gamma_1} \cdots H_{\theta_n}^{\gamma_n}] \\ &= \left(\sum_{i=1}^m (\beta_i - \alpha_i) \mu_i \right) (X_{-\mu_m}^{\alpha_m} \cdots X_{-\mu_1}^{\alpha_1} X_{\mu_1}^{\beta_1} \cdots X_{\mu_m}^{\beta_m} H_{\theta_1}^{\gamma_1} \cdots H_{\theta_n}^{\gamma_n}) \\ &= \left(\sum_{i=1}^m (\beta_i - \alpha_i) \mu_i \right) u(\bar{\alpha}, \bar{\beta}, \bar{\gamma}). \end{aligned}$$

Therefore,

$$C = U_0 = \text{span} \left\{ u(\bar{\alpha}, \bar{\beta}, \bar{\gamma}) \mid \sum_{i=1}^m (\beta_i - \alpha_i) \mu_i = 0 \right\},$$

i.e. the centralizer C of \mathcal{H} in the universal enveloping algebra $U(\mathcal{G})$ is spanned as a linear subspace of $U(\mathcal{G})$ by all cycles in $U(\mathcal{G})$.

The following theorem shows that the centralizer C is generated by all the basic cycles.

Theorem 4.3.1. *Let C be the centralizer of the Cartan subalgebra \mathcal{H} in $U(\mathcal{G})$. Then the basic cycles generate the algebra C .*

Sketch of proof. Let C' be the subalgebra of C which is generated by all the basic cycles. Let u be a cycle in $U(\mathcal{G})$. We prove by induction on the degree of u that u is an element of C' . Since the algebra C is spanned by all cycles in $U(\mathcal{G})$, this will give the result.

If $\deg(u) = 0$ or $\deg(u) = 1$, then u is a basic cycle and hence $u \in C'$ trivially. Assume that $u \in C'$ for all u such that $\deg(u) \leq k$ with $k \geq 1$. Let z be a cycle with $\deg(z) = k + 1$. If z is basic, clearly $z \in C'$. If z is not basic, then there exist s and w such that z is the product of s and w when z, s and w are considered as commutative monomials. Then in the algebra C , it can be shown that

$$sw = z + \text{a linear combination of cycles of lower degrees.}$$

Now, by the induction hypothesis, the result follows. \square

We next give a set of generators for the centralizer of the Cartan subalgebra in the universal enveloping algebra $U(\mathfrak{sl}_3)$.

The Lie algebra $\mathfrak{sl}_3(K)$ has a basis

$$\{X_1 = E_{12}, X_2 = E_{23}, X_3 = E_{13}, Y_1 = E_{21}, Y_2 = E_{32}, Y_3 = E_{31}, \\ H_1 = H_\alpha = E_{11} - E_{22}, H_2 = H_\beta = E_{22} - E_{33}\}$$

with E_{ij} being the usual matrix units. The universal enveloping algebra $U(\mathfrak{sl}_3)$ has a PBW basis consisting of all monomials of the form

$$Y_3^{m_3} Y_2^{m_2} Y_1^{m_1} X_1^{n_1} X_2^{n_2} X_3^{n_3} H_1^{t_1} H_2^{t_2}$$

where $m_i, n_i, t_i \in \mathbb{N}$.

Theorem 4.3.2. *Let C be the centralizer of the Cartan subalgebra $\mathcal{H} = KH_1 \oplus KH_2$ in $U(\mathfrak{sl}_3)$. Then the algebra C is generated by the elements*

$$H_1 = H_\alpha = E_{11} - E_{22}, \quad H_2 = H_\beta = E_{22} - E_{33}, \\ c_1 = E_{21}E_{12} = Y_1X_1, \quad c_2 = E_{32}E_{23} = Y_2X_2, \quad c_3 = E_{31}E_{13} = Y_3X_3, \\ c = E_{32}E_{21}E_{13} = Y_2Y_1X_3 \quad \text{and} \quad \bar{c} = E_{31}E_{12}E_{23} = Y_3X_1X_2.$$

Sketch of proof. Clearly, the elements $H_1, H_2, c_1, c_2, c_3, c$ and \bar{c} are basic cycles in the algebra C . Let C' be the subalgebra of C which is generated by all the elements $H_1, H_2, c_1, c_2, c_3, c$ and \bar{c} . Recall that

$$C = \text{span} \left\{ Y_3^{m_3} Y_2^{m_2} Y_1^{m_1} X_1^{n_1} X_2^{n_2} X_3^{n_3} H_1^{t_1} H_2^{t_2} \mid (n_1 - m_1)\alpha + (n_2 - m_2)\beta + (n_3 - m_3)(\alpha + \beta) = 0 \right\}.$$

Now,

$$(n_1 - m_1)\alpha + (n_2 - m_2)\beta + (n_3 - m_3)(\alpha + \beta) = 0$$

if and only if

$$n_1 + n_3 = m_1 + m_3 \quad \text{and} \quad n_2 + n_3 = m_2 + m_3.$$

By using induction on $d = n_1 + n_2 + n_3 + m_1 + m_2 + m_3$, we can show all cycles in $U(\mathfrak{sl}_3)$ are in C' . Hence, $C = C'$. \square

Chapter 5

The centralizer of the Cartan subalgebra in $U(\mathfrak{sl}_3)$

The aim of this chapter is to find generators and defining relations of the centralizer C of the Cartan subalgebra of the Lie algebra $\mathfrak{sl}_3(K)$ in the universal enveloping algebra $U(\mathfrak{sl}_3)$ and to classify simple 1-dimensional C -modules.

5.1 Notation

The following notation will remain fixed in this chapter: K is an algebraically closed field of characteristic zero, $\mathcal{H} = KH_1 \oplus KH_2$ is the Cartan subalgebra of the Lie algebra $\mathfrak{sl}_3(K)$, $U = U(\mathfrak{sl}_3)$ is the universal enveloping algebra of the Lie algebra $\mathfrak{sl}_3(K)$ and

$$C = \{u \in U \mid H_i u = u H_i, i = 1, 2\}$$

is the centralizer of the Cartan subalgebra \mathcal{H} in U . b

5.2 The centralizer of the Cartan subalgebra in $U(\mathfrak{sl}_3)$

The center of $U(\mathfrak{sl}_3)$. Recall that the Lie algebra $\mathfrak{sl}_3(K)$ has a basis

$$\{X_1 = E_{12}, X_2 = E_{23}, X_3 = E_{13}, Y_1 = E_{21}, Y_2 = E_{32}, Y_3 = E_{31}, \\ H_1 = E_{11} - E_{22}, H_2 = E_{22} - E_{33}\}$$

with E_{ij} being the usual matrix units. We use the following ordering of the canonical basis of the Lie algebra $\mathfrak{sl}_3(K)$:

$$Y_3 > Y_2 > Y_1 > X_1 > X_2 > X_3 > H_1 > H_2.$$

The center of the algebra $U = U(\mathfrak{sl}_3)$ is a polynomial algebra $Z(U) = K[Z_1, Z_2]$ in two generators (see [7, Theorem 7.3.8(ii)]). By [10, page 424], these two generators can be taken to be

$$Z_1 = \frac{1}{18}(H_1^2 + H_2^2 + H_1H_2) + \frac{1}{6}(H_1 + H_2) + \frac{1}{6}(Y_1X_1 + Y_2X_2 + Y_3X_3), \quad (5.1)$$

$$\begin{aligned} Z_2 = & -\frac{2}{9}H_1^3 + \frac{2}{9}H_2^3 - \frac{1}{3}H_1^2H_2 + \frac{1}{3}H_1H_2^2 - H_1^2 + H_2^2 - H_1 + H_2 - Y_1X_1H_1 - Y_3X_3H_1 \\ & + 2Y_2X_2H_1 - 2Y_1X_1H_2 + Y_3X_3H_2 + Y_2X_2H_2 - 3Y_2Y_1X_3 - 3Y_3X_1X_2 + 3Y_2X_2 \\ & - 3Y_1X_1 + 3Y_3X_3. \end{aligned} \quad (5.2)$$

Filtration on the algebra C . Recall that the algebra $U = \bigcup_{i \geq 0} U_i$ admits the standard filtration with respect to the canonical generators of $\mathfrak{sl}_3(\bar{K})$ and the associated graded algebra $\mathrm{gr}(U) = \bigoplus_{i \geq 0} U_i/U_{i-1}$ is a polynomial algebra in 8 variables where $U_{-1} := 0$. In particular, the Gelfand-Kirillov dimension of U is $\mathrm{GK}(U) = 8$.

The subalgebra C of U admits the induced filtration

$$C = \bigcup_{i \geq 0} C_i \quad \text{where } C_i = C \cap U_i. \quad (5.3)$$

The associated graded algebra of C , $\mathrm{gr}(C) = \bigoplus_{i \geq 0} C_i/C_{i-1}$ is a subalgebra of the polynomial algebra $\mathrm{gr}(U)$. In particular, the algebra $\mathrm{gr}(C)$ is a commutative domain. Hence, the algebra C is an almost commutative domain.

For every nonzero element $u \in U$, there is a unique natural number d such that $u \in U_d \setminus U_{d-1}$. The number d is called the **degree** of the element u and is denoted by $\mathrm{deg}(u)$.

The canonical anti-involution of the universal enveloping algebra $U(\mathcal{G})$ of a Lie algebra \mathcal{G} . Let \mathcal{G} be a Lie algebra and $U(\mathcal{G})$ be its universal enveloping algebra. Then the K -linear map $(-)^* : \mathcal{G} \rightarrow \mathcal{G}, x \mapsto x^* = -x$ satisfies the equalities:

$$\begin{aligned} [x, y]^* &= [y^*, x^*] \quad \text{for all } x, y \in \mathcal{G}, \\ x^{**} &= x \quad \text{for all } x \in \mathcal{G}. \end{aligned}$$

The map $(-)^*$ induces the, so-called, **canonical anti-involution on the associative algebra $U(\mathcal{G})$**

$$(-)^* : U(\mathcal{G}) \rightarrow U(\mathcal{G}), a \mapsto a^*, \text{ i.e. for all } a, b \in U(\mathcal{G}) \quad (ab)^* = b^*a^* \text{ and } a^{**} = a. \quad (5.4)$$

The transposition map $(-)^t$. The transposition map

$$(-)^t : \mathfrak{gl}_n(K) \rightarrow \mathfrak{gl}_n(K), E_{ij} \mapsto E_{ji}$$

is an anti-isomorphism of order two of the Lie algebra $\mathfrak{gl}_n(K)$ ($[a, b]^t = [b^t, a^t]$ for all $a, b \in \mathfrak{gl}_n(K)$). It induces the anti-automorphism $(-)^t$ of order two of the universal enveloping algebra $U(\mathfrak{gl}_n(K))$ ($(uv)^t = v^t u^t$ for all $u, v \in U(\mathfrak{gl}_n(K))$).

Since $\mathfrak{sl}_n(K)^t = \mathfrak{sl}_n(K)$ and $\mathcal{H}^t = \mathcal{H}$, the transposition map is also an anti-isomorphism of order two of the Lie algebra $\mathfrak{sl}_n(K)$ and an anti-isomorphism of order two of its universal enveloping algebra $U = U(\mathfrak{sl}_n(K))$ and of the centralizer $C = C_U(\mathcal{H})$.

The automorphism ν of the Lie algebras $\mathfrak{gl}_n(K)$ on $\mathfrak{sl}_n(K)$. Let $M_n(K)$ be the (associative) algebra of $n \times n$ matrices over the field K . It is a Lie algebra $(\mathfrak{gl}_n(K) = M_n(K), [\cdot, \cdot])$ when $[a, b] = ab - ba$.

The K -linear map

$$\nu : \mathfrak{gl}_n(K) \rightarrow \mathfrak{gl}_n(K), E_{ij} \mapsto -E_{ji}$$

is a Lie algebra automorphism. Since $\nu(\mathfrak{sl}_n(K)) = \mathfrak{sl}_n(K)$, it induces a Lie algebra automorphism of the Lie algebra $\mathfrak{gl}_n(K)$ which is also denoted by ν . Let

$$\mathcal{H} = \bigoplus_{i=1}^{n-1} K(E_{ii} - E_{i+1, i+1})$$

be the **Cartan subalgebra** of $\mathfrak{sl}_n(K)$. Then $\nu(\mathcal{H}) = \mathcal{H}$ since $\nu(h) = -h$ for all $h \in \mathcal{H}$. The automorphism ν induces an automorphism of the universal enveloping algebra $U = U(\mathfrak{sl}_n(K))$ which is also denoted by ν ,

$$\nu : U(\mathfrak{sl}_n(K)) \rightarrow U(\mathfrak{sl}_n(K)), E_{ij} \mapsto -E_{ji} \ (i \neq j), E_{ii} - E_{i+1, i+1} \mapsto -(E_{ii} - E_{i+1, i+1}). \quad (5.5)$$

Since $\nu(\mathcal{H}) = \mathcal{H}$, $\nu(C) = C$ where $C = C_U(\mathcal{H})$ is the centralizer of the Cartan subalgebra \mathcal{H} in U . So, ν is also an automorphism of the algebra C . Clearly,

$$\nu \circ (-)^t = (-)^t \circ \nu = (-)^* \quad (5.6)$$

So, the maps ν and $(-)^t$ commute.

Let

$$\begin{aligned} H_1 &= E_{11} - E_{22}, H_2 = E_{22} - E_{33}, H_3 = H_1 + H_2, c_1 = E_{21}E_{12}, c_2 = E_{32}E_{23}, \\ c_3 &= E_{31}E_{13}, c = E_{32}E_{21}E_{13} \text{ and } \bar{c} = E_{31}E_{12}E_{23}. \end{aligned}$$

Then

$$\nu(H_i) = -H_i \text{ for } i = 1, 2, 3, \quad (5.7)$$

$$\nu(c_i) = c_i + H_i \text{ for } i = 1, 2, 3, \quad (5.8)$$

$$\nu(c) = -\bar{c} - c_1 + c_2 + c_3 + H_2, \quad (5.9)$$

$$\nu(\bar{c}) = -c - c_1 + c_2 + c_3 + H_2. \quad (5.10)$$

In more detail,

$$\begin{aligned}
 \nu(c_1) &= \nu(E_{21}E_{12}) = \nu(E_{21})\nu(E_{12}) = (-E_{12})(-E_{21}) = E_{12}E_{21} \\
 &= E_{21}E_{12} + [E_{12}, E_{21}] = c_1 + [E_{12}, E_{21}] = c_1 + H_1, \\
 \nu(c_2) &= \nu(E_{32}E_{23}) = \nu(E_{32})\nu(E_{23}) = (-E_{23})(-E_{32}) = E_{23}E_{32} \\
 &= E_{32}E_{23} + [E_{23}, E_{32}] = c_2 + [E_{23}, E_{32}] = c_2 + H_2, \\
 \nu(c_3) &= \nu(E_{31}E_{13}) = \nu(E_{31})\nu(E_{13}) = (-E_{13})(-E_{31}) = E_{13}E_{31} \\
 &= E_{31}E_{13} + [E_{13}, E_{31}] = c_3 + [E_{13}, E_{31}] = c_3 + H_1 + H_2 = c_3 + H_3, \\
 \nu(c) &= \nu(E_{32}E_{21}E_{13}) = \nu(E_{32})\nu(E_{21})\nu(E_{13}) = -E_{23}E_{12}E_{31} \\
 &= -E_{23}E_{31}E_{12} - E_{23}[E_{12}, E_{31}] \\
 &= -E_{31}E_{23}E_{12} - [E_{23}, E_{31}]E_{12} - E_{23}[E_{12}, E_{31}] \\
 &= -\left(E_{31}E_{23}E_{12} + [E_{23}, E_{31}]E_{12} + E_{23}[E_{12}, E_{31}]\right) \\
 &= -\left(E_{31}E_{23}E_{12} + E_{21}E_{12} - E_{23}E_{32}\right) = -\left(E_{31}E_{23}E_{12} + c_1 - E_{23}E_{32}\right) \\
 &= -\left(E_{31}E_{12}E_{23} + E_{31}[E_{23}, E_{12}] + c_1 - E_{23}E_{32}\right) \\
 &= -\left(E_{31}E_{12}E_{23} + E_{31}E_{13} + c_1 - E_{23}E_{32}\right) \\
 &= -\left(\bar{c} - c_3 + c_1 - E_{23}E_{32}\right) = -\left(\bar{c} - c_3 + c_1 - E_{32}E_{23} - [E_{23}, E_{32}]\right) \\
 &= -(\bar{c} - c_3 + c_1 - c_2 - H_2) = -\bar{c} - c_1 + c_2 + c_3 + H_2, \\
 \nu(\bar{c}) &= \nu(E_{31}E_{12}E_{23}) = \nu((E_{32}E_{21}E_{13})^t) = \nu(c^t) = \nu(c)^t \\
 &= (-\bar{c} - c_1 + c_2 + c_3 + H_2)^t = -c - c_1 + c_2 + c_3 + H_2, \text{ by (5.6).}
 \end{aligned}$$

The transposition map $(-)^t : C \rightarrow C$, $c \rightarrow c^t$ is an anti-involution such that $H_i^t = H_i$ and $c_i^t = c_i$ for $i = 1, 2, 3$; $c^t = \bar{c}$ and $\bar{c}^t = c$.

The Weyl group $W = S_n$ of $\mathfrak{sl}_n(K)$. Let S_n be the symmetric group on the set $\{1, 2, \dots, n\}$, i.e. S_n is the group of bijections/permutations on $\{1, 2, \dots, n\}$. The symmetric group S_n is a subgroup of the group of automorphisms of the algebra $M_n(K)$ where $\sigma(E_{ij}) = E_{\sigma(i)\sigma(j)}$ for all $\sigma \in S_n$.

• **The transposition (of a matrix) map commutes with elements of the symmetric group S_n .**

Since $\sigma(\mathcal{H}) = \mathcal{H}$ and $\sigma(\mathfrak{sl}_n(K)) = \mathfrak{sl}_n(K)$, the symmetric group S_n is also a subgroup of the group of automorphisms of the Lie algebras \mathcal{H} , $\mathfrak{sl}_n(K)$ and of the associative algebras $U(\mathcal{H})$, $U(\mathfrak{sl}_n(K))$ and $C = C_{U(\mathfrak{sl}_n)}(\mathcal{H})$.

Let $U = U(\mathfrak{sl}_3)$ and $C = C_U(\mathcal{H})$ be the centralizer of the Cartan subalgebra $\mathcal{H} = KH_1 \oplus KH_2$ (where $H_1 = E_{11} - E_{22}$ and $H_2 = E_{22} - E_{33}$) in U . Let $\sigma = (1\ 2\ 3) \in S_3$

be a cyclic permutation of $\{1, 2, 3\}$ ($\sigma(1) = 2, \sigma(2) = 3$ and $\sigma(3) = 1$). Notice that the automorphism $\sigma = (1\ 2\ 3)$ commutes with the transposition $(-)^t$. Then

$$\sigma(H_1) = H_2, \sigma(H_2) = -H_1 - H_2, \sigma^{-1}(H_1) = -H_1 - H_2, \sigma^{-1}(H_2) = H_1, \quad (5.11)$$

$$\sigma(c_1) = c_2, \sigma(c_2) = c_3 + H_1 + H_2, \sigma(c_3) = c_1 + H_1, \quad (5.12)$$

$$\sigma^{-1}(c_1) = c_3 + H_1 + H_2, \sigma^{-1}(c_2) = c_1, \sigma^{-1}(c_3) = c_2 + H_2. \quad (5.13)$$

Indeed,

$$\begin{aligned} \sigma(H_2) &= \sigma(E_{22} - E_{33}) = E_{33} - E_{11} = -H_1 - H_2, \\ \sigma(c_2) &= \sigma(E_{32}E_{23}) = E_{13}E_{31} = c_3 + [E_{13}, E_{31}] = c_3 + E_{11} - E_{33} = c_3 + H_1 + H_2, \\ \sigma(c_3) &= \sigma(E_{31}E_{13}) = E_{12}E_{21} = c_1 + [E_{12}, E_{21}] = c_1 + H_1. \end{aligned}$$

$$\sigma(c) = c + c_1 - c_2 + H_1, \quad (5.14)$$

$$\sigma(\bar{c}) = \bar{c} + c_1 - c_2 + H_1, \quad (5.15)$$

$$\sigma^{-1}(c) = c + c_1 - c_3, \quad (5.16)$$

$$\sigma^{-1}(\bar{c}) = \bar{c} + c_1 - c_3. \quad (5.17)$$

In more detail,

$$\begin{aligned} \sigma(c) &= \sigma(E_{32}E_{21}E_{13}) = E_{13}E_{32}E_{21} = c + [E_{13}, E_{32}]E_{21} + E_{32}[E_{13}, E_{21}] \\ &= c + E_{12}E_{21} - E_{32}E_{23} = c + c_1 + H_1 - c_2. \\ \sigma(\bar{c}) &= \sigma(c^t) = \sigma(c)^t = \bar{c} + c_1 - c_2 + H_1. \end{aligned}$$

Notice that $\sigma^3 = e$, and so $\sigma^{-1} = \sigma^2$. Now

$$\begin{aligned} \sigma^{-1}(c) &= \sigma^2(c) = \sigma(c + c_1 - c_2 + H_1) = c + c_1 - c_2 + H_1 + c_2 - (c_3 + H_1 + H_2) + H_2 \\ &= c + c_1 - c_3. \end{aligned}$$

$$\sigma^{-1}(\bar{c}) = \sigma^{-1}(c^t) = \sigma^{-1}(c)^t = (c + c_1 - c_3)^t = \bar{c} + c_1 - c_3.$$

Let $\tau = (1\ 2) \in S_3$, a transposition. So, $\tau(E_{ij}) = E_{\tau(i)\tau(j)}$ and $\tau(E_{12}) = E_{21}$. Then

$$\begin{aligned} \tau(H_1) &= -H_1, \tau(H_2) = H_1 + H_2, \tau(c_1) = c_1 + H_1, \tau(c_2) = c_3, \\ \tau(c_3) &= c_2, \tau(c) = \bar{c} \text{ and } \tau(\bar{c}) = c. \end{aligned}$$

Indeed,

$$\begin{aligned}
 \tau(H_1) &= \tau(E_{11} - E_{22}) = E_{22} - E_{11} = -H_1, \\
 \tau(H_2) &= \tau(E_{22} - E_{33}) = E_{11} - E_{33} = H_1 + H_2, \\
 \tau(c_1) &= \tau(E_{21}E_{12}) = E_{12}E_{21} = E_{21}E_{12} + [E_{12}, E_{21}] = c_1 + H_1, \\
 \tau(c_2) &= \tau(E_{32}E_{23}) = E_{31}E_{13} = c_3, \\
 \tau(c_3) &= \tau(E_{31}E_{13}) = E_{32}E_{23} = c_2, \\
 \tau(c) &= \tau(E_{32}E_{21}E_{12}) = E_{31}E_{12}E_{23} = \bar{c}, \\
 \tau(\bar{c}) &= \tau\tau(c) = c.
 \end{aligned}$$

Generators and defining relations for the centralizer C . By Theorem 5.2.1, the algebra C is described by generators and defining relations.

Theorem 5.2.1. *Let C be the centralizer of the Cartan subalgebra \mathcal{H} in $U = U(\mathfrak{sl}_3)$.*

1. *The algebra C is generated by the elements $H_1, H_2, c_1, c_2, c_3, c$ and \bar{c} subject to the defining relations:*

$$[c_1, c_2] = [c_2, c_3] = [c_3, c_1] = c - \bar{c}, \quad (5.18)$$

$$[c_1, c] = c_1(c_2 - c_3) + (H_1 - 1)c + \bar{c} - H_1c_3, \quad (5.19)$$

$$[c_1, \bar{c}] = -(c_2 - c_3)c_1 - (H_1 - 1)\bar{c} - c + H_1c_3, \quad (5.20)$$

$$[c_2, c] = c_2(c_3 - c_1) + H_2c, \quad (5.21)$$

$$[c_2, \bar{c}] = -(c_3 - c_1)c_2 - H_2\bar{c}, \quad (5.22)$$

$$[c_3, c] = c_3(c_1 - c_2 + H_1) - (H_1 + H_2 + 2)c + 2\bar{c}, \quad (5.23)$$

$$[c_3, \bar{c}] = -(c_1 - c_2 + H_1)c_3 + (H_1 + H_2 + 2)\bar{c} - 2c, \quad (5.24)$$

$$c\bar{c} = (c_3 + H_1 + H_2)c_1c_2 + (-c_1 + c_2 + c_3 + H_2)\bar{c}, \quad (5.25)$$

$$\bar{c}c = (c_2 + H_2)(c_1 + H_1)c_3 + (-c_1 + c_2 + c_3 + H_2)c, \quad (5.26)$$

and the elements H_1 and H_2 belong to the center of the algebra C . Using the automorphisms $\tau = (12)$ and σ^{-1} of the algebra C , the defining relations (5.18)–

(5.26) can be written in a more compact form where $d = c_2(c_3 - c_1)$:

$$[c_1, c_2] = [c_2, c_3] = [c_3, c_1] = c - \bar{c}, \quad (5.27)$$

$$[c_2, c] = d + H_2c, \quad (5.28)$$

$$[c_2, \bar{c}] = -[c_2, c]^t = -d - H_2\bar{c}, \quad (5.29)$$

$$[c_3, \bar{c}] = \tau([c_2, c]) = \tau(d) + \tau(H_2)\bar{c}, \quad (5.30)$$

$$[c_3, c] = \tau([c_2, \bar{c}]) = -\tau(d) - \tau(H_2)c, \quad (5.31)$$

$$[c_1, c] = \sigma^{-1}([c_2, c]) + \bar{c} - c = \sigma^{-1}(d) + \sigma^{-1}(H_2)(c + c_1 - c_3) + \bar{c} - c, \quad (5.32)$$

$$[c_1, \bar{c}] = -[c_1, c]^t = -\sigma^{-1}(d) - \sigma^{-1}(H_2)(\bar{c} + c_1 - c_3) + c - \bar{c}, \quad (5.33)$$

$$c\bar{c} = (c_3 + H_1 + H_2)c_1c_2 + (-c_1 + c_2 + c_3 + H_2)\bar{c}, \quad (5.34)$$

$$\bar{c}c = \tau(c\bar{c}). \quad (5.35)$$

2. $C = \bigoplus_{\alpha_1, \dots, \gamma \in \mathbb{N}} KH_1^{\alpha_1} H_2^{\alpha_2} c_1^{\beta_1} c_2^{\beta_2} c_3^{\beta_3} c^\gamma \oplus \bigoplus_{\alpha_1, \dots, \gamma \in \mathbb{N}} KH_1^{\alpha_1} H_2^{\alpha_2} c_1^{\beta_1} c_2^{\beta_2} c_3^{\beta_3} \bar{c}^\gamma$.
3. The algebra C is an almost commutative, finitely generated, Noetherian domain of Gelfand-Kirillov dimension $\text{GK}(C) = 6$.
4. Let $\{C_i = C \cap U_i\}_{i \geq 0}$ be the induced filtration on the algebra C . Then the associated graded algebra $\text{gr}(C) = \bigoplus_{i \geq 0} C_i/C_{i-1}$ (where $C_{-1} = 0$) is isomorphic to the factor algebra $P/(c\bar{c} - c_1c_2c_3)$ where $P = K[H_1, H_2, c_1, c_2, c_3, c, \bar{c}]$ is a polynomial algebra in 7 variables, and $\text{gr}(C) = \bigoplus_{\alpha_1, \dots, \gamma \in \mathbb{N}} KH_1^{\alpha_1} H_2^{\alpha_2} c_1^{\beta_1} c_2^{\beta_2} c_3^{\beta_3} c^\gamma \oplus \bigoplus_{\alpha_1, \dots, \gamma \in \mathbb{N}} KH_1^{\alpha_1} H_2^{\alpha_2} c_1^{\beta_1} c_2^{\beta_2} c_3^{\beta_3} \bar{c}^\gamma$.

Remark 5.2.2. By (5.18), $[c_1, c_2 + c_3] = [c_2, c_3 + c_1] = [c_3, c_1 + c_2] = 0$.

Proof. (i) **The relations in statement (1) hold** (for the convenience of the reader, more careful and detailed calculations are given in the Appendix): Let us show that (5.18)–(5.26) hold.

First, we show that (5.18) holds.

$$\begin{aligned} [c_1, c_2] &= [E_{21}E_{12}, E_{32}E_{23}] \\ &= \left([E_{21}, E_{32}]E_{23} + E_{32}[E_{21}, E_{23}] \right) E_{12} + E_{21} \left([E_{12}, E_{32}]E_{23} + E_{32}[E_{12}, E_{23}] \right) \\ &= -E_{31}E_{23}E_{12} + E_{21}E_{32}E_{13} = -\bar{c} - E_{31}[E_{23}, E_{12}] + c + [E_{21}, E_{32}]E_{13} \\ &= -\bar{c} + E_{31}E_{13} + c - E_{31}E_{13} = -\bar{c} + c. \end{aligned}$$

Now, applying the automorphism $\sigma = (1, 2, 3) \in S_3$ to the equality $[c_1, c_2] = c - \bar{c}$ and using the equalities (5.14) and (5.15) we have that

$$[c_2, c_3] = [\sigma(c_1), \sigma(c_2)] = \sigma([c_1, c_2]) = \sigma(c - \bar{c}) = c - \bar{c}.$$

Now, applying again σ , we have

$$[c_3, c_1] = \sigma([c_2, c_3]) = \sigma(c - \bar{c}) = c - \bar{c}.$$

So, the equalities (5.18) hold. Let us prove that the equality (5.21) holds.

$$\begin{aligned} [c_2, c] &= [E_{32}E_{23}, E_{32}E_{21}E_{13}] \\ &= \left([E_{32}, E_{32}]E_{21}E_{13} + E_{32}[E_{32}, E_{21}]E_{13} + E_{32}E_{21}[E_{32}, E_{13}] \right) E_{23} \\ &\quad + E_{32} \left([E_{23}, E_{32}]E_{21}E_{13} + E_{32}[E_{23}, E_{21}]E_{13} + E_{32}E_{21}[E_{23}, E_{13}] \right) \\ &= E_{32}E_{31}E_{13}E_{23} - E_{32}E_{21}E_{12}E_{23} + E_{32}H_2E_{21}E_{13} \\ &= E_{32} \left(E_{23}E_{31}E_{13} + [E_{31}, E_{23}]E_{13} + E_{31}[E_{13}, E_{23}] \right) \\ &\quad - E_{32} \left(E_{23}E_{21}E_{12} + [E_{21}, E_{23}]E_{12} + E_{21}[E_{12}, E_{23}] \right) + (H_2 + 2)E_{32}E_{21}E_{13} \\ &= c_2c_3 - c - c_2c_1 - c + (H_2 + 2)c \\ &= c_2(c_3 - c_1) + H_2c. \end{aligned}$$

By applying the transposition anti-automorphism to the equality (5.21) we obtain the equality (5.22):

$$[c_2, \bar{c}] = [c_2^t, c^t] = -[c_2, c]^t = -(c_2(c_3 - c_1) + H_2c)^t = -((c_3 - c_1)c_2 + H_2\bar{c}).$$

Now, applying the automorphism σ^{-1} to the equality (5.21) we have that

$$\begin{aligned} \sigma^{-1}([c_2, c]) &= [c_1, c + c_1 - c_3] = [c_1, c] + c - \bar{c}, \text{ by (5.18), hence} \\ [c_1, c] &= -c + \bar{c} + \sigma^{-1}(c_2(c_3 - c_1) + H_2c), \text{ by (5.21)} \\ &= -c + \bar{c} + c_1(c_2 + H_2 - c_3 - H_1 - H_2) + H_1(c + c_1 - c_3) \\ &= c_1(c_2 - c_3) + (H_1 - 1)c + \bar{c} - H_1c_3. \end{aligned}$$

Applying the transposition to the equality (5.19) we obtain (5.20). Similarly, applying the automorphism σ to the equality (5.21) and using (5.12) and (5.14) we have that

$$\sigma([c_2, c]) = [c_3 + H_1 + H_2, c + c_1 - c_2 + H_1] = [c_3, c] + 2(c - \bar{c}), \text{ by (5.18).}$$

Now, by (5.21), (5.12) and (5.14),

$$\begin{aligned} [c_3, c] + 2(c - \bar{c}) &= \sigma([c_2, c]) = \sigma(c_2(c_3 - c_1) + H_2c) \\ &= (c_3 + H_1 + H_2)(c_1 + H_1 - c_2) - (H_1 + H_2)(c + c_1 - c_2 + H_1) \\ &= c_3(c_1 - c_2 + H_1) - (H_1 + H_2)c, \text{ and (5.23) follows.} \end{aligned}$$

Applying the transposition to the equality (5.23) we obtain the equality (5.24). Let us show that the equality (5.25) holds.

$$\begin{aligned}
c\bar{c} &= E_{32}E_{21}E_{13}E_{31}E_{12}E_{23} \\
&= E_{21}E_{13}E_{31}E_{12}E_{32}E_{23} + [E_{32}, E_{21}E_{13}E_{31}E_{12}]E_{23} \\
&= \left(E_{13}E_{31}E_{21}E_{12} + [E_{21}, E_{13}E_{31}]E_{12}\right)c_2 + \left(E_{31}E_{13}E_{31}E_{12} - E_{21}E_{12}E_{31}E_{12}\right)E_{23} \\
&= \left((c_3 + H_1 + H_2)c_1 + E_{23}E_{31}E_{12}\right)c_2 + (c_3 - c_1)\bar{c} \\
&= \left((c_3 + H_1 + H_2)c_1 + c_1 - c_3 + \bar{c}\right)c_2 + (c_3 - c_1)\bar{c} \\
&= \left((c_3 + H_1 + H_2 + 1)c_1 - c_3 + \bar{c}\right)c_2 + (c_3 - c_1)\bar{c} \\
&= (c_3 + H_1 + H_2 + 1)c_1c_2 - c_3c_2 + \bar{c}c_2 + (c_3 - c_1)\bar{c} \\
&\stackrel{(5.22)}{=} (c_3 + H_1 + H_2 + 1)c_1c_2 - c_3c_2 + c_2\bar{c} + (c_3 - c_1)c_2 + H_2\bar{c} + (c_3 - c_1)\bar{c} \\
&= (c_3 + H_1 + H_2)c_1c_2 + (-c_1 + c_2 + c_3 + H_2)\bar{c}. \\
\bar{c}c &= \tau(c\bar{c}) = (c_2 - H_1 + H_1 + H_2)(c_1 + H_1)c_3 + (-c_1 - H_1 + c_3 + c_2 + H_1 + H_2)c \\
&= (c_2 + H_2)(c_1 + H_1)c_3 + (-c_1 + c_2 + c_3 + H_2)c.
\end{aligned}$$

(ii) $C = \sum_{\alpha_1, \dots, \gamma \in \mathbb{N}} KH_1^{\alpha_1} H_2^{\alpha_2} c_1^{\beta_1} c_2^{\beta_2} c_3^{\beta_3} c^\gamma + \sum_{\alpha_1, \dots, \gamma \in \mathbb{N}} KH_1^{\alpha_1} H_2^{\alpha_2} c_1^{\beta_1} c_2^{\beta_2} c_3^{\beta_3} \bar{c}^\gamma$: Statement (ii) follows from the statement (i) and the fact that $\text{gr}(C) \subseteq \text{gr}(U)$.

Let C' be an algebra that is generated by the elements/symbols $H_1, H_2, c_1, c_2, c_3, c$ and \bar{c} that satisfy the defining relations in statement (1) of the theorem. Then there is a natural epimorphism

$$f : C' \rightarrow C, H_1 \mapsto H_1, \dots, \bar{c} \mapsto \bar{c}.$$

(iii) $\text{GK}(C) = \text{GK}(C') = 6$: Since the algebra C is an epimorphic image of the algebra C' , we must have $\text{GK}(C) \leq \text{GK}(C')$. By statement (ii) and the defining relations of the algebra C' , $\text{GK}(C') \leq 6$. In more detail, the statement (ii) holds for the algebra C' as in the proof of the statement (ii). We define a filtration on the algebra C' that is defined by degrees of the canonical generators which are the same as the degrees of the canonical generators of the algebra C in the universal enveloping algebra U , i.e

$$\deg(H_i) = 1, \deg(c_j) = 2 \text{ and } \deg(c) = \deg(\bar{c}) = 3$$

where $i = 1, 2$ and $j = 1, 2, 3$. Then the associated graded algebra $\text{gr}(C')$ is a commutative algebra: the relations (5.18)–(5.24) yield that the elements c_1, c_2 and c_3 commute and they also commute with the elements c and \bar{c} ; the relations (5.25) and (5.26) yield the relations $c\bar{c} = c_1c_2c_3$ and $\bar{c}c = c_1c_2c_3$ (in particular, $c\bar{c} = \bar{c}c$). So, the algebra is a somewhat commutative algebra, and hence $\text{GK}(C') = \text{GK}(\text{gr}(C')) = \text{Kdim}(\text{gr}(C'))$. The algebra $\text{gr}(C')$ is a factor algebra of the polynomial algebra in seven variables by the relation $c\bar{c} - c_1c_2c_3$ (and possibly some other relations). Hence, $\text{Kdim}(\text{gr}(C')) \leq 7 - 1 = 6$, and we are done.

To finish the proof of the statement (iii), it suffices to show that the sum

$$S = \sum_{\alpha_1, \dots, \gamma \in \mathbb{N}} KH_1^{\alpha_1} H_2^{\alpha_2} c_1^{\beta_1} c_2^{\beta_2} c_3^{\beta_3} c^\gamma$$

is a direct sum (since then $\mathrm{GK}(C) \geq 6$ and we are done). The associated graded algebra $\mathrm{gr}(U)$ of the universal enveloping algebra U is isomorphic to the polynomial algebra in 8 variables where the elements of the canonical basis of the Lie algebra $\mathfrak{sl}_3(K)$ are the variables of the polynomial algebra. The images of the canonical generators of the centralizer C in the algebra $\mathrm{gr}(U)$ are monomials. We denote them by the same symbols as the original elements. So, it suffices to show that the sum S is a direct sum in $\mathrm{gr}(U)$, i.e. the monomials

$$H_1, H_2, c_1 = E_{21}E_{12}, c_2 = E_{32}E_{23}, c_3 = E_{31}E_{13}, \text{ and } c = E_{31}E_{12}E_{23}$$

are algebraically independent in the polynomial algebra $\mathrm{gr}(U)$. The factor algebra

$$R := \mathrm{gr}(U)/(E_{21} - 1, E_{32} - 1) = K[H_1, H_2, E_{12}, E_{13}, E_{23}, E_{31}]$$

is a polynomial algebra in 6 variables. The images of the elements $(H_1, H_2, c_1, c_2, c_3, c)$ under the epimorphism $\mathrm{gr}(U) \rightarrow R$ are given below,

$$(H_1, H_2, c_1, c_2, c_3, c) \mapsto (H_1, H_2, E_{12}, E_{23}, E_{31}E_{13}, E_{31}E_{12}E_{23}).$$

Let R' be the subalgebra of the polynomial algebra R which is generated by the elements $H_1, H_2, E_{12}, E_{23}, E_{31}E_{13}$, and $E_{31}E_{12}E_{23}$. Then the localization of the algebra R' at the multiplicative set generated by the elements $E_{12}, E_{23}, E_{31}E_{13}$, and $E_{31}E_{12}E_{23}$ is the algebra

$$K[H_1, H_2, E_{12}^{\pm 1}, E_{23}^{\pm 1}, E_{31}^{\pm 1}, E_{13}^{\pm 1}]$$

which has Krull dimension 6, and the result follows.

Notice that the degrees of the canonical generators of the algebra C with respect to the standard filtration $\{U_i\}_{i \geq 0}$ of the algebra U are as follows:

$$\deg(H_i) = 1, \deg(c_j) = 2 \text{ and } \deg(c) = \deg(\bar{c}) = 3$$

where $i = 1, 2$ and $j = 1, 2, 3$. Now, let us consider a filtration on the algebra

$$C' = \bigcup_{i \geq 0} C'_i$$

where the degrees of its canonical generators are as above. Then the degree of the right-hand side in the defining relations where the commutators are involved is strictly smaller than the degree of the commutators on the left-hand side. In the factor space, C'_6/C'_5 the last two relations are $c\bar{c} = c_1c_2c_3$ and $\bar{c}c = c_1c_2c_3$, respectively. Therefore, the associated graded algebra

$$\mathrm{gr}(C') = \bigoplus_{i \geq 0} C'_i/C'_{i-1}$$

is a commutative algebra. Furthermore, it is the factor algebra of the polynomial algebra $P = K[H_1, H_2, c_1, c_2, c_3, c, \bar{c}]$ modulo its prime ideal $(c\bar{c} - c_1c_2c_3)$, i.e. of $D = P/(c\bar{c} - c_1c_2c_3)$. In particular, the algebra D is a domain of Gelfand-Kirillov dimension

$$\text{GK}(D) = \text{GK}(P) - 1 = 7 - 1 = 6.$$

Since $\text{GK}(\text{gr}(C')) = \text{GK}(C') = 6$, by the statement (iii) and the algebra $\text{gr}(C')$ is an epimorphic image of the commutative finitely generated domain D ; the natural epimorphism $D \rightarrow \text{gr}(C')$ must be an isomorphism (since otherwise $\text{gr}(C') \simeq D/I$ for some nonzero ideal I of D and we would have a contradiction:

$$6 = \text{GK}(D) > \text{GK}(D/I) = \text{GK}(\text{gr}(C')) = \text{GK}(C') = 6).$$

So, we have proven the following statement.

(iv) $\text{gr}(C') \simeq D$ is a commutative domain and $\text{GK}(\text{gr}(C')) = \text{GK}(C') = 6$.

Now, statement (4) follows.

(v) $C' \simeq C$: By the very definition, the natural algebra epimorphism $f : C' \rightarrow C$ respects the filtrations on the algebras C' and C , i.e. $f(C'_i) = C_i$ for all $i \geq 0$. It induces the epimorphism of graded algebras

$$\bar{f} : \text{gr}(C') \rightarrow \text{gr}(C).$$

Since the commutative algebras $\text{gr}(C')$ and $\text{gr}(C)$ have the same Gelfand-Kirillov dimension 6 and the algebra $\text{gr}(C')$ is a domain; the epimorphism \bar{f} must be an isomorphism, i.e. $C' \simeq C$, and statements (1) and (3) follow. Now, statement (2) follows from statement (4). \square

Let us show that

$$[c, \bar{c}] = -\left((c_1 + H_1)c_3 + c\right)(H_2 + 2) - (c_2c_3 - c)H_1 + (c_1c_2 + \bar{c})(H_1 + H_2 + 2). \quad (5.36)$$

$$\begin{aligned} [c, \bar{c}] &= [E_{32}E_{21}E_{13}, E_{31}E_{12}E_{23}] \\ &= E_{31}E_{12}(E_{33} - E_{22})E_{21}E_{13} + E_{31}E_{32}(-H_1)E_{13}E_{23} + E_{32}E_{21}(H_1 + H_2)E_{12}E_{23} \\ &= (E_{12}E_{21}E_{31}E_{13} + E_{32}E_{21}E_{13})(-H_2 - 2) + (E_{32}E_{23}E_{31}E_{13} - E_{32}E_{21}E_{13})(-H_1) \\ &\quad + (E_{21}E_{12}E_{32}E_{23} + E_{31}E_{12}E_{23})(H_1 + H_2 + 2) \\ &= -\left((c_1 + H_1)c_3 + c\right)(H_2 + 2) - (c_2c_3 - c)H_1 + (c_1c_2 + \bar{c})(H_1 + H_2 + 2). \end{aligned}$$

Generators and defining relations for the centralizer C via $Z(U)$. Let us con-

sider the elements

$$p := 6Z_1 - \frac{1}{3}\beta \quad \text{where} \quad \beta := H_1^2 + H_2^2 + H_1H_2 + 3(H_1 + H_2), \quad (5.37)$$

$$\begin{aligned} q &:= -\frac{1}{3}Z_2 + \frac{1}{27}(H_2 - H_1)\left(2\beta + 3(H_1H_2 + H_1 + H_2 + 3p + 3)\right) + p \\ &= -\frac{1}{3}Z_2 + \frac{1}{27}(H_2 - H_1)\left(2(H_1^2 + H_2^2) + 5H_1H_2 + 9(H_1 + H_2 + p + 1)\right) + p. \end{aligned} \quad (5.38)$$

Clearly, p and q belong to the center of the algebra C .

The next theorem will be used in the next section. It simplifies the existing sets of generators and defining relations of the algebra C and helps us to find the center of C in the next section.

Theorem 5.2.3. *Let C be the centralizer of the Cartan subalgebra \mathcal{H} in $U = U(\mathfrak{sl}_3)$. Then*

1. *The algebra C is generated by the elements H_1, H_2, p, q, c_1, c_2 , and c (where p and q are defined in (5.37) and (5.38), respectively) subject to the defining relations:*

$$[c_1, c_2] = 2c + (H_2 + 2)c_1 - H_1c_2 - q, \quad (5.39)$$

$$[c_1, c] = (H_1 - 2)c + c_1(c_1 + 2c_2 + H_1 - H_2 - p - 2) + 2H_1c_2 - H_1p + q, \quad (5.40)$$

$$[c_2, c] = (H_2 + 4)c - c_2(c_2 + 2H_1 - p) - 2c_1(c_2 - H_2 - 2) - 2q, \quad (5.41)$$

$$\begin{aligned} c^2 &+ \left(c_1(H_2 + 4) - c_2(H_1 - 2) + 2H_1 - H_2 - q - p - 4\right)c + \left((H_2 + 2)c_1 + H_1H_2 \right. \\ &+ 2H_1 - 2H_2 - 2q - 4\left.)c_1 + \left((-c_1 - 2H_1 + 2)c_2 - (c_1 + 3H_1 - p - 2)c_1 - H_1H_2 \right. \right. \\ &+ 2H_1p - 2H_1^2 - q - 2p\left.)c_2 + H_1H_2p + 2H_1p - 2H_1q + pq + 2q = 0 \end{aligned} \quad (5.42)$$

and the elements H_1, H_2, p , and q are central elements of the algebra C .

$$2. \quad C = \bigoplus_{\alpha_i, \beta_j \in \mathbb{N}, \varepsilon \in \{0,1\}} KH_1^{\alpha_1} H_2^{\alpha_2} p^{\alpha_3} q^{\alpha_4} c_1^{\beta_1} c_2^{\beta_2} c^\varepsilon.$$

Proof. 1. The relations (5.39)–(5.42) are obtained from the relations (5.18)–(5.26) by replacing the pair (c_3, \bar{c}) by the pair (Z_1, Z_2) using the equalities (5.1) and (5.2). Then the eleven relations in part 1 of Theorem 5.2.1 collapse to the four relations of statement (1).

2. Statement (2) follows at once from Theorem 5.2.1.(2) and the definitions of the elements p, q (see (5.37) and (5.38)) and Z_1, Z_2 (see (5.1) and (5.2)). \square

5.3 The center of the algebra C

The aim of this section is to show that the center of the algebra C is a polynomial algebra $\mathcal{Z} = K[H_1, H_2, p, q]$, Theorem 5.3.5. We see that the algebra C comes with a filtration whose associated graded algebra admits a Poisson algebra structure. We show that the Poisson center of this Poisson algebra is \mathcal{Z} .

The filtration $F = \{F_i\}_{i \in \mathbb{N}}$ of the algebra C and the associated graded algebra $\text{gr}_F(C)$.

By part 2 of Theorem 5.2.3, the algebra C is a free (left and right) \mathcal{Z} -module

$$C = \bigoplus_{\beta_i \in \mathbb{N}, \varepsilon \in \{0,1\}} \mathcal{Z} c_1^{\beta_1} c_2^{\beta_2} c^\varepsilon$$

where $\mathcal{Z} = K[H_1, H_2, p, q]$ is a polynomial algebra that belongs to the center of the algebra C .

Let us consider the filtration $F = \{F_i\}_{i \in \mathbb{N}}$ on the algebra C where

$$F_i = \bigoplus \left\{ \mathcal{Z} c_1^{\beta_1} c_2^{\beta_2} c^\varepsilon \mid \beta_1 + \beta_2 + 2\varepsilon = i, \beta_1, \beta_2 \in \mathbb{N}, \varepsilon \in \{0, 1\} \right\}. \quad (5.43)$$

So, the filtration F is the deg_F -filtration where

$$\text{deg}_F(c_1) = \text{deg}_F(c_2) = 1, \text{deg}_F(c) = 2 \text{ and } \text{deg}_F(z) = 0$$

for all elements $z \in \mathcal{Z} \setminus \{0\}$. In particular, $F_0 = \mathcal{Z}$ and $F_1 = \mathcal{Z} \oplus \mathcal{Z}c_1 \oplus \mathcal{Z}c_2$. By the very definition, the filtration F is an ascending filtration,

$$F_0 \subseteq F_1 \subseteq \dots \subseteq F_i \subseteq \dots$$

By (5.39) – (5.42), $F_i F_j \subseteq F_{i+j}$ for all $i, j \geq 0$. Let $\text{gr}_F(C) := \bigoplus_{i \geq 0} F_i / F_{i-1}$ be the associated graded algebra where $F_{-1} := 0$. Recall that the Lie algebra

$$\mathcal{H}_3 := K\langle x, y, z \mid [x, y] = z, [x, z] = 0, [y, z] = 0 \rangle \quad (5.44)$$

is the **3-dimensional Heisenberg** Lie algebra.

Theorem 5.3.1. 1. $\text{gr}_F(C) = \bigoplus_{\alpha_i, \beta_j \in \mathbb{N}, \varepsilon \in \{0,1\}} K H_1^{\alpha_1} H_2^{\alpha_2} p^{\alpha_3} q^{\alpha_4} c_1^{\beta_1} c_2^{\beta_2} c^\varepsilon$.

2. The algebra $\text{gr}_F(C)$ is isomorphic to the tensor product of algebras $\mathcal{Z} \otimes \overline{\mathbb{H}}_3$ where $\mathcal{Z} = K[H_1, H_2, p, q]$ and $\overline{\mathbb{H}}_3 := K\langle c_1, c_2, c \mid [c_1, c_2] = 2c, [c_1, c] = 0, [c_2, c] = 0, c^2 = 0 \rangle$.

3. $Z(\text{gr}_F(C)) = \mathcal{Z} \oplus \mathcal{Z} \overline{\mathbb{H}}_3 c = \mathcal{Z} \oplus \bigoplus_{\beta_i \in \mathbb{N}} \mathcal{Z} c_1^{\beta_1} c_2^{\beta_2} c$.

Proof. 1. Statement (1) follows from the definition of the filtration F and Theorem 5.2.3.(2).

2. We denote by the same symbol the images of the generators H_1, H_2, p, q, c_1, c_2 and c of the algebra C in $\mathrm{gr}_F(C)$. Then the relations (5.37)–(5.42) yield the relations

$$[c_1, c_2] = 2c, [c_1, c] = 0, [c_2, c] = 0 \text{ and } c^2 = 0,$$

respectively. Therefore, the algebra $\mathrm{gr}_F(C)$ is an epimorphic image of the algebra $T := \mathcal{Z} \otimes \overline{\mathbb{H}}_3$. The algebra $\overline{\mathbb{H}}_3$ is isomorphic to the factor algebra $U(\mathcal{H}_3)/(z^2)$ of the universal enveloping algebra $U(\mathcal{H}_3)$ of the Heisenberg algebra \mathcal{H}_3 modulo the ideal generated by the element z^2 (see (5.44)). Clearly,

$$T = \bigoplus_{\beta_i \in \mathbb{N}, \varepsilon \in \{0,1\}} \mathcal{Z} c_1^{\beta_1} c_2^{\beta_2} c^\varepsilon$$

and the algebra $\mathrm{gr}_F(C)$ is an epimorphic image of the algebra T via

$$H_1 \mapsto H_1, \dots, c \mapsto c.$$

Therefore, $\mathrm{gr}_F(C) \simeq T$ by the direct sum $\mathrm{gr}_F(C) = \bigoplus_{\alpha_i, \beta_j \in \mathbb{N}, \varepsilon \in \{0,1\}} K H_1^{\alpha_1} H_2^{\alpha_2} p^{\alpha_3} q^{\alpha_4} c_1^{\beta_1} c_2^{\beta_2} c^\varepsilon$ and the definition of the algebra \mathcal{Z} .

3. By statement (2),

$$Z(\mathrm{gr}_F(C)) = Z(\mathcal{Z} \otimes \overline{\mathbb{H}}_3) = Z(\mathcal{Z}) \otimes Z(\overline{\mathbb{H}}_3) = \mathcal{Z} \otimes (K \oplus \overline{\mathbb{H}}_3 c) = \mathcal{Z} \oplus \mathcal{Z} \overline{\mathbb{H}}_3 c = \mathcal{Z} \oplus \bigoplus_{\beta_i \in \mathbb{N}} \mathcal{Z} c_1^{\beta_1} c_2^{\beta_2} c$$

where we have used the fact that $Z(\overline{\mathbb{H}}_3) = K \oplus \overline{\mathbb{H}}_3 c$. Clearly,

$$\overline{\mathbb{H}}_3 = \bigoplus_{\alpha_i \in \mathbb{N}} K c_1^{\alpha_1} c_2^{\alpha_2} \oplus \bigoplus_{\alpha_i \in \mathbb{N}} K c_1^{\alpha_1} c_2^{\alpha_2} c.$$

So, every element $a \in \overline{\mathbb{H}}_3$ is a unique sum

$$a = a_1(c_1, c_2) + ca_2(c_1, c_2)$$

where $a_s(c_1, c_2) = \sum_{\alpha_1, \alpha_2 \in \mathbb{N}} \lambda_{\alpha_1 \alpha_2}^s c_1^{\alpha_1} c_2^{\alpha_2}$ for unique scalars $\lambda_{\alpha_1 \alpha_2}^s \in K$ where $s = 1, 2$. Now, $a \in Z(\overline{\mathbb{H}}_3)$ iff

$$0 = [c_1, a] = 2c \frac{\partial a_1}{\partial c_2} \text{ and } 0 = [c_2, a] = -2c \frac{\partial a_1}{\partial c_1}$$

iff $0 = \frac{\partial a_1}{\partial c_2}$ and $0 = \frac{\partial a_1}{\partial c_1}$ iff $a_1 \in K$ since the field K has characteristic zero (and the scalars are the only polynomials in the variables c_1 and c_2 that are annihilated by the partial derivatives $\frac{\partial}{\partial c_1}$ and $\frac{\partial}{\partial c_2}$) iff $a \in K \oplus \overline{\mathbb{H}}_3 c$. \square

The Poisson algebra $\mathcal{P}_3 = K[c_1, c_2, c]$. Let $\mathcal{P}_3 = K[c_1, c_2, c]$ be a polynomial algebra in three indeterminates c_1, c_2 and c . The algebra \mathcal{P}_3 is a Poisson algebra where the Poisson bracket is given by the rule:

$$\{c_1, c_2\} = 2c, \quad \{c_1, c\} = c_1^2 + 2c_1c_2 \quad \text{and} \quad \{c_2, c\} = -c_2^2 - 2c_1c_2.$$

For an element $p \in \mathcal{P}_3$, let $\text{PC}(p) := \{q \in \mathcal{P}_3 \mid \{p, q\} = 0\}$, the **Poisson centralizer** of the element p in \mathcal{P}_3 .

Definition 5.3.2. Let A be a Poisson algebra with Poisson bracket $\{\cdot, \cdot\}$. The **Poisson center** of A is defined by

$$\text{PZ}(A) := \{a \in A \mid \{a, x\} = 0 \text{ for all } x \in A\}.$$

Proposition 5.3.3. 1. $\text{PC}(c) = K[c, \theta]$ where $\theta = c_1^2c_2 + c_1c_2^2$.

2. $\text{PZ}(\mathcal{P}_3) = K[\eta]$ where $\eta = c^2 - \theta = c^2 - c_1^2c_2 - c_1c_2^2$.

Proof. 1. Notice that $\mathcal{P}_3 = K[c] \otimes P_2$ where $P_2 = K[c_1, c_2]$.

Let $\partial := -\text{pad}_c = -\{\cdot, c\}$, the Poisson inner derivation of \mathcal{P}_3 which is determined by the element $-c \in \mathcal{P}_3$. Then $\partial(P_2) \subseteq P_2$ and

$$\partial = (c_1^2 + 2c_1c_2)\partial_1 - (c_2^2 + 2c_1c_2)\partial_2 \quad \text{where} \quad \partial_i = \frac{\partial}{\partial c_i}.$$

(i) $\text{PC}(c) = K[c] \otimes P_2^\partial$ **where** $P_2^\partial = \ker_{P_2}(\partial) := \{a \in P_2 \mid \partial(a) = 0\}$ **is the algebra of ∂ -constants in P_2 :** The equality is obvious.

Now, statement (1) follows from statement (ii).

(ii) $P_2^\partial = K[\theta]$: By direct calculation $\partial(\theta) = 0$, and so $P_2^\partial \supseteq K[\theta]$. The algebra P_2^∂ is a homogeneous subalgebra of the polynomial algebra P_2 w.r.t. the usual \mathbb{N} -grading on P_2 (by the total degree of the variables c_1 and c_2). Given

$$p = \sum_{i=s}^t \lambda_i c_1^i c_2^{n-i} \in P_2^\partial \setminus \{0\}$$

where $0 \leq s \leq t \leq n$, $\lambda_s \neq 0$ and $\lambda_t \neq 0$. Then

$$\begin{aligned} 0 = \partial(p) &= \sum_{i=s}^t \lambda_i (i(c_1 + 2c_2) - (n-i)(c_2 + 2c_1)) c_1^i c_2^{n-i} \\ &= \sum_{i=s}^t ((3i-2n)\lambda_i c_1^{i+1} c_2^{n-i} + (3i-n)\lambda_i c_1^i c_2^{n-i+1}) \\ &= (3s-n)\lambda_s c_1^s c_2^{n-s+1} + (3t-2n)\lambda_t c_1^{t+1} c_2^{n-t} + \sum_{s < i < t} ((3i-2n)\lambda_i + (3(i+1)-n)\lambda_{i+1}) c_1^{i+1} c_2^{n-i}. \end{aligned}$$

Therefore, $s = \frac{n}{3}$, $t = 2\frac{n}{3} = 2s$ and $(i-t)\lambda_i + (i+1-s)\lambda_{i+1} = 0$ for $s < i < t$. So, up to a multiplicative scalar there is only single solution for each $n = 3m$ where $m \geq 1$. For $n = 3$, the solution is θ . Then for each $m \geq 2$, the solution is θ^m , and statement (ii) follows.

2. By direct computation $\eta \in \mathrm{PZ}(\mathcal{P}_3)$, and so $\mathrm{PZ}(\mathcal{P}_3) \supseteq K[\eta]$.

By statement (1), $\mathrm{PZ}(\mathcal{P}_3) \subseteq K[c, \theta] = K[c, \eta]$. Let $f = f(\eta, c) \in \mathrm{PZ}(\mathcal{P}_3)$, then

$$0 = \{c_1, f\} = \frac{\partial f}{\partial c} \{c_1, c\},$$

and so $\frac{\partial f}{\partial c} = 0$, i.e. $f \in K[\eta]$. Therefore, $\mathrm{PZ}(\mathcal{P}_3) = K[\eta]$. \square

The filtration $G = \{G_i\}_{i \in \mathbb{N}}$ of the algebra C and the Poisson algebra $\mathrm{gr}_G(C)$.
By part 2 of Theorem 5.2.3, the algebra C is a free (left and right) \mathcal{Z} -module

$$C = \bigoplus_{\beta_i \in \mathbb{N}, \varepsilon \in \{0,1\}} \mathcal{Z} c_1^{\beta_1} c_2^{\beta_2} c^\varepsilon$$

where $\mathcal{Z} = K[H_1, H_2, p, q]$ is a polynomial algebra that belongs to the center of the algebra C .

Let us consider the filtration $G = \{G_i\}_{i \in \mathbb{N}}$ on the algebra C where

$$G_i = \bigoplus \left\{ \mathcal{Z} c_1^{\beta_1} c_2^{\beta_2} c^\varepsilon \mid 2\beta_1 + 2\beta_2 + 3\varepsilon = i, \beta_1, \beta_2 \in \mathbb{N}, \varepsilon \in \{0, 1\} \right\}. \quad (5.45)$$

So, the filtration G is the deg_G -filtration where

$$\mathrm{deg}_G(c_1) = \mathrm{deg}_G(c_2) = 2, \mathrm{deg}_G(c) = 3 \text{ and } \mathrm{deg}_G(\mathcal{Z}) = 0$$

for all elements $z \in \mathcal{Z} \setminus \{0\}$. In particular, $G_0 = G_1 = \mathcal{Z}$, $G_2 = \mathcal{Z} \oplus \mathcal{Z}c_1 \oplus \mathcal{Z}c_2$ and $G_3 = \mathcal{Z} \oplus \mathcal{Z}c_1 \oplus \mathcal{Z}c_2 \oplus \mathcal{Z}c$. By the very definition, the filtration G is an ascending filtration,

$$G_0 \subseteq G_1 \subseteq \dots \subseteq G_i \subseteq \dots$$

By (5.39) – (5.42), $G_i G_j \subseteq G_{i+j}$ and $[G_i, G_j] \subseteq G_{i+j-1}$ for all $i, j \geq 0$. The associated graded algebra $\mathrm{gr}_G(C) := \bigoplus_{i \geq 0} G_i / G_{i-1}$ (where $G_{-1} := 0$) is a commutative algebra that admits a Poisson bracket given by the rule: For all $u \in G_i / G_{i-1}$ and $v \in G_j / G_{j-1}$,

$$\{u, v\} := [u, v] + G_{i+j-2} \in G_{i+j-1} / G_{i+j-2}. \quad (5.46)$$

Proposition 5.3.4. 1. *The Poisson algebra $\mathrm{gr}_G(C) = K[H_1, H_2, p, q, c_1, c_2, c] / (\eta) = \mathcal{Z} \otimes_K \mathcal{P}$ (where $\eta = c^2 - c_1^2 c_2 - c_1 c_2^2$) is a tensor product of the Poisson algebras $(\mathcal{Z}, \{\cdot, \cdot\} = 0)$ and $\mathcal{P} = K[c_1, c_2, c] / (\eta)$ where the Poisson bracket on \mathcal{P} is given by the rule $\{c_1, c_2\} = 2c$, $\{c_1, c\} = c_1^2 + 2c_1 c_2$, and $\{c_2, c\} = -c_2^2 - 2c_1 c_2$. The algebras \mathcal{P} and $\mathrm{gr}_G(C)$ are domains.*

2. $\mathrm{PZ}(\mathcal{P}) = K$.

3. $\text{PZ}(\text{gr}_G(C)) = \mathcal{Z}$.

Proof. 1. Statement (1) follows from the definition of the filtration G and Theorem 5.2.3 (the algebra \mathcal{P} is a domain as the polynomial $\eta \in \mathcal{P}_3$ is an irreducible polynomial; then $\text{gr}_G(C) = \mathcal{Z} \otimes_K \mathcal{P} = \mathcal{P}[H_1, H_2, p, q]$ is a domain).

2. Notice that $\mathcal{P} = P_2 \oplus P_2 c$ where $P_2 = K[c_1, c_2]$. We keep the notation of the proof of Proposition 5.3.3. Then

$$\mathcal{P}^\partial = P_2^\partial \oplus P_2^\partial c = K[\theta] \oplus K[\theta]c = K[c^2] \oplus K[c^2]c = K[c]$$

since $P_2^\partial = K[\theta]$ (the statement (ii) of the proof of part 1 of Proposition 5.3.3) and $\eta = c^2 - \theta = 0$. Now, if $p \in \text{PZ}(\mathcal{P})$, then $p = p(c) \in K[c] = \mathcal{P}^\partial$, and so

$$0 = \{c_1, p(c)\} = p'(c)\{c_1, c\} = p'(c)(c_1^2 + 2c_1 c_2)$$

where $p'(c) = \frac{dp}{dc}$. Hence, $p'(c) = 0$ since \mathcal{P} is a domain, i.e. $p \in K$, and statement (2) follows.

3. By statements (1) and (2),

$$\text{PZ}(\text{gr}_G(C)) = \text{PZ}(\mathcal{Z} \otimes_K \mathcal{P}) = \text{PZ}(\mathcal{Z}) \otimes_K \text{PZ}(\mathcal{P}) = \mathcal{Z} \otimes_K K = \mathcal{Z}. \quad \square$$

Theorem 5.3.5. *Let C be the centralizer of the Cartan subalgebra \mathcal{H} in $U = U(\mathfrak{sl}_3)$. Then the center of the algebra C is a polynomial algebra $K[H_1, H_2, Z_1, Z_2] = K[H_1, H_2, p, q]$ where Z_1 and Z_2 are the Casimir elements of U (see (5.1) and (5.2)).*

Proof. Clearly, $\mathcal{Z} \subseteq Z(C)$. Suppose, to obtain a contradiction, that $\mathcal{Z} \neq Z(C)$. Then there is an element $\theta \in Z(C) \setminus \mathcal{Z}$. By Proposition 5.3.4, $\text{gr}_G(C) = \mathcal{Z} \otimes_K \mathcal{P}$, and so

$$\text{gr}(\theta) \in \text{PZ}(\text{gr}_G(C)) \setminus \mathcal{Z} = \mathcal{Z} \setminus \mathcal{Z} = \emptyset \quad (\text{by part 3 of Proposition 5.3.4, } \text{PZ}(\text{gr}_G(C)) = \mathcal{Z}),$$

which is a contradiction. \square

5.4 Classification of simple 1-dimensional C -modules

Let C be the centralizer of the Cartan subalgebra \mathcal{H} in $U = U(\mathfrak{sl}_3)$. In this section, we present a classification of all the 1-dimensional C -modules.

In order to classify all the 1-dimensional C -modules, we rewrite Theorem 5.2.1(1) using the following order of the variables

$$H_1^{\alpha_1} H_2^{\alpha_2} c_1^{\beta_1} c_2^{\beta_2} c_3^{\beta_3} c^{\gamma_1} \bar{c}^{\gamma_2} \quad \text{where } \alpha_1, \dots, \gamma_2 \in \mathbb{N},$$

in the monomials of the defining relations of the algebra C . Using this order is more convenient for solving the system of equations that we will see later in Proposition 5.4.2.

Theorem 5.4.1. *Let C be the centralizer of the Cartan subalgebra \mathcal{H} in $U = U(\mathfrak{sl}_3)$. Then the algebra C is generated by the elements $H_1, H_2, c_1, c_2, c_3, c$ and \bar{c} subject to the defining relations:*

$$[c_1, c_2] = [c_2, c_3] = [c_3, c_1] = c - \bar{c}, \quad (5.47)$$

$$[c_1, c] = c_1 c_2 + H_1 c - (H_1 + c_1) c_3 - c + \bar{c}, \quad (5.48)$$

$$[c_1, \bar{c}] = -H_1 \bar{c} - c_1 c_2 + (c_1 + H_1) c_3 + c - \bar{c}, \quad (5.49)$$

$$[c_2, c] = H_2 c - c_1 c_2 + c_2 c_3 + c - \bar{c}, \quad (5.50)$$

$$[c_2, \bar{c}] = -H_2 \bar{c} - c_1 c_2 - c_2 c_3 + c - \bar{c}, \quad (5.51)$$

$$[c_3, c] = -H_1 c - H_2 c + (c_1 - c_2 + H_1) c_3, \quad (5.52)$$

$$[c_3, \bar{c}] = H_1 \bar{c} + H_2 \bar{c} + (c_2 - c_1 - H_1) c_3 - 2c + 2\bar{c}, \quad (5.53)$$

$$c\bar{c} = c_1 c_2 c_3 + H_1 c_1 c_2 + H_2 c_1 c_2 + c_2 c - c_1 c + c_3 \bar{c} - H_2 c + 2c_1 c_2 - 2c_2 c_3, \quad (5.54)$$

$$\begin{aligned} \bar{c}c = & c_1 c_2 c_3 + H_1 H_2 c_3 + H_1 c_2 c_3 + H_2 c_1 c_3 + 2c_1 c_3 - 2c_2 c_3 + 2H_1 c_3 - c_1 c + c_2 c \\ & + c_3 \bar{c} - H_1 c - H_2 \bar{c} - H_1 \bar{c} + 2c - 2\bar{c}, \end{aligned} \quad (5.55)$$

and the elements H_1 and H_2 belong to the center of the algebra C .

Classification of simple 1-dimensional C -modules. For an algebra A , we denote by $[A, A]$ the ideal of A which is generated by all the commutators $[a, b] = ab - ba$ where $a, b \in A$. The factor algebra

$$A^{ab} := A/[A, A]$$

is called the **abelianization** of the algebra A . The algebra A^{ab} is a commutative algebra that is the ‘largest’ commutative factor algebra of A in the sense that if a factor algebra A/I is commutative, then $[A, A] \subseteq I$ and there is a natural epimorphism from A^{ab} to A/I .

Let $\widehat{C}(1\text{-dim})$ be the set of isomorphism classes of all 1-dimensional C -modules. Clearly, every 1-dimensional C -module is simple. Let $\mathcal{I}(C, 1\text{-codim})$ be the set of all ideals \mathfrak{m} of the algebra C of co-dimension 1, that is $\dim_K(C/\mathfrak{m}) = 1$. The map

$$\widehat{C}(1\text{-dim}) \rightarrow \mathcal{I}(C, 1\text{-codim}), \quad [M] \mapsto \mathrm{ann}_C(M) \quad (5.56)$$

is a bijection with inverse $\mathfrak{m} \rightarrow [C/\mathfrak{m}]$. Every 1-dimensional C -module is a 1-dimensional C^{ab} -module, and vice versa. The map

$$\widehat{C}(1\text{-dim}) \rightarrow \widehat{C^{ab}}(1\text{-dim}), \quad [M] \rightarrow [M] \quad (5.57)$$

is a bijection. Let $\mathrm{Max}(C^{ab})$ be the set of all maximal ideals of the algebra C^{ab} . The map

$$\widehat{C}(1\text{-dim}) \rightarrow \mathrm{Max}(C^{ab}), \quad [M] \rightarrow \mathrm{ann}_{C^{ab}}(M) \quad (5.58)$$

is a bijection provided the field K is algebraically closed.

Proposition 5.4.2 represents the commutative algebra C^{ab} via generators and defining relations.

Proposition 5.4.2. *The commutative K -algebra C^{ab} is generated by the elements $H'_1, H'_2, c'_1, c'_2, c'_3$ and c' subject to the defining relations:*

$$(c' - c'_3)H'_1 = (c'_3 - c'_2)c'_1, \quad (5.59)$$

$$H'_2c' = c'_2(c'_1 - c'_3), \quad (5.60)$$

$$c'^2 + (c'_1 - c'_2 - c'_3 + H'_2)c' - c'_1c'_2c'_3 - (H'_1 + H'_2 + 2)c'_1c'_2 + 2c'_2c'_3 = 0. \quad (5.61)$$

Proof. The proposition follows from Theorem 5.4.1. Clearly, the commutative algebra C^{ab} is generated by the elements $H'_1, H'_2, c'_1, c'_2, c'_3$ and c' (notice that (5.47) implies $c' - \bar{c}' = 0$ in the algebra C^{ab}) subject to the defining relations of the algebra C and relations $[a, b] = 0$ for all $a, b \in \{H'_1, H'_2, c'_1, c'_2, c'_3, c'\}$ where $a \neq b$. Equations (5.48) and (5.49) yield (5.59). Equations (5.50) and (5.51) yield (5.60). Equation (5.54) yields (5.61). Equations (5.52), (5.53) and (5.55) are redundant (Equations (5.52) and (5.60) yield (5.59); equations (5.53) and (5.60) yield (5.59); equations (5.55), (5.59) and (5.60) yield (5.61)). \square

The solutions in the field K to the system of equations (5.59), (5.60) and (5.61) are partitioned into four disjoint classes:

(I) $c' - c'_3 = 0$ and $c' = 0$,

(II) $c' - c'_3 = 0$ and $c' \neq 0$,

(III) $c' - c'_3 \neq 0$ and $c' = 0$,

(IV) $c' - c'_3 \neq 0$ and $c' \neq 0$.

Theorem 5.4.3 gives solutions to the system of equations (5.59), (5.60) and (5.61). As a result we have a classification of all the 1-dimensional C -modules and C^{ab} -modules, and a classification of all ideals of the algebras C and C^{ab} of co-dimension 1.

Theorem 5.4.3. *The solutions in the field K to the system of equations (5.59), (5.60) and (5.61) are given below for the Cases (I)–(IV), respectively:*

1. Suppose that $c' - c'_3 = 0$ and $c' = 0$, i.e. $c' = c'_3 = 0$.

(a) If $c'_1 = 0$, then the elements H'_1, H'_2 and c'_2 are arbitrary.

(b) If $c'_1 \neq 0$, then necessarily $c'_2 = 0$ and the elements H'_1 , H'_2 and $c'_1 \neq 0$ are arbitrary.

2. Suppose that $c' - c'_3 = 0$ and $c' \neq 0$.

(a) If $c'_1 = 0$, then $H'_2 = -c'_2$ and the elements H'_1 , c'_2 and $c' = c'_3 \neq 0$ are arbitrary.

(b) If $c'_1 \neq 0$, then $H'_1 = -c'_1$, $H'_2 = c'_1 - c'$, and the elements $c'_1 \neq 0$ and $c' = c'_2 = c'_3 \neq 0$ are arbitrary.

3. Suppose that $c' - c'_3 \neq 0$ and $c' = 0$.

(a) If $c'_2 = 0$, then $H'_1 = -c'_1$, $c' = c'_2 = 0$ and the elements H'_2 , c'_1 and $c'_3 \neq 0$ are arbitrary.

(b) If $c'_2 \neq 0$, then $H'_1 = c'_2 - c'_1$, $H'_2 = -c'_2$, $c' = 0$, $c'_3 = c'_1$ and the elements $c'_1 \neq 0$ and $c'_2 \neq 0$ are arbitrary.

4. If $c' - c'_3 \neq 0$ and $c' \neq 0$, then the system (5.59)–(5.61) is equivalent to the system

$$H'_1 = (c' - c'_3)^{-1}c'_1(c'_3 - c'_2), \quad (5.62)$$

$$H'_2 = c'^{-1}c'_2(c'_1 - c'_3), \quad (5.63)$$

$$(c' - c'_2)(c' + c'_1 - c'_3)(c'^2 - c'_3c' - c'_1c'_2c'_3) = 0. \quad (5.64)$$

(a) If $c' - c'_2 = 0$, then $H'_1 = -c'_1$, $H'_2 = c'_1 - c'_3$, $c'_2 = c'$, $c'_3 \neq c'$ and the elements c'_1 , c'_3 and $c' \neq 0$ are arbitrary such that $c'_3 \neq c'$.

(b) If $c' - c'_2 \neq 0$ and $c' + c'_1 - c'_3 = 0$, then $H'_1 = c'_2 - c'_3$, $H'_2 = -c'_2$, $c'_1 = c'_3 - c'$ and the elements c'_2 , c'_3 and c' are arbitrary such that $c' \neq c'_2$, $c' \neq c'_3$ and $c' \neq 0$.

(c) Suppose that $c' - c'_2 \neq 0$, $c' + c'_1 - c'_3 \neq 0$ and $c'^2 - c'_3c' - c'_1c'_2c'_3 = 0$.

i. If $c'_3 = -4c'_1c'_2$, then $H'_1 = -2c'_1 - \frac{1}{2}$, $H'_2 = -2c'_2 - \frac{1}{2}$, $c'_3 = -4c'_1c'_2$, $c' = -2c'_1c'_2$, and the elements $c'_1 \neq 0$, $-\frac{1}{2}$ and $c'_2 \neq 0$, $-\frac{1}{2}$ are arbitrary.

ii. If $c'_3 \neq -4c'_1c'_2$, then

$c'_\pm = \frac{1}{2}(c'_3 \pm \sqrt{c'_3(c'_3 + 4c'_1c'_2)})$, $H'_1 = (c'_\pm - c'_3)^{-1}c'_1(c'_3 - c'_2)$ and $H'_2 = c'^{-1}_\pm c'_2(c'_1 - c'_3)$, and the elements $c'_1 \neq 0$, $c'_2 \neq 0$ and $c'_3 \neq 0$ are arbitrary such that $c'_3 \neq -4c'_1c'_2$ and $c'_1 \neq (c'_2 + 1)c'_3$.

Proof. 1. If $c' - c'_3 = 0$ and $c' = 0$, then the system (5.59)–(5.61) is equivalent to the system

$$c'_1c'_2 = 0, \quad (5.65)$$

$$c'_1c'_2 = 0, \quad (5.66)$$

$$(H'_1 + H'_2 + 2)c'_1c'_2 = 0, \quad (5.67)$$

i.e. to the system $c'_1 c'_2 = 0$. Now, statement (1) is obvious.

2. If $c' - c'_3 = 0$ and $c' \neq 0$, then the system (5.59)–(5.61) is equivalent to the system

$$(c'_2 - c')c'_1 = 0, \quad (5.68)$$

$$H'_2 = c'^{-1}c'_2(c'_1 - c'), \quad (5.69)$$

$$(c'_1 + c'_2 + H'_2 - c'_1 c'_2)c' = (H'_1 + H'_2 + 2)c'_1 c'_2. \quad (5.70)$$

(a) If $c'_1 = 0$, then $H'_2 = -c'_2$ and the elements H'_1 , c'_2 and $c' = c'_3 \neq 0$ are arbitrary.

(b) If $c'_1 \neq 0$, then $c' = c'_2 = c'_3 \neq 0$, by (5.68). Hence, $H'_2 = c'_1 - c'$, by (5.68) (since $c' = c'_2$). Then the equation (5.70) is written as

$$c'_1 c'(H'_1 + c'_1) = 0,$$

and so $H'_1 = -c'_1$ (since $c'_1 \neq 0$ and $c' \neq 0$). So, if $c'_1 \neq 0$, then

$$H'_1 = -c'_1, \quad H'_2 = c'_1 - c',$$

and the elements $c'_1 \neq 0$ and $c' = c'_2 = c'_3 \neq 0$ are arbitrary.

3. Since $c' - c'_3 \neq 0$ and $c' = 0$, then $c'_3 \neq 0$ and the system (5.59)–(5.61) is equivalent to the system

$$H'_1 = c'^{-1}c'_1(c'_2 - c'_3), \quad (5.71)$$

$$c'_2(c'_1 - c'_3) = 0, \quad (5.72)$$

$$c'_2(c'_1 c'_3 + (H'_1 + H'_2 + 2)c'_1 - 2c'_3) = 0. \quad (5.73)$$

(a) If $c'_2 = 0$, then $H'_1 = -c'_1$, by (5.71), and the equations (5.72) and (5.73) are tautologies $0 = 0$. So, if $c'_2 = 0$, then $H'_1 = -c'_1$, $c' = c'_2 = 0$ and the elements H'_2 , c'_1 and $c'_3 \neq 0$ are arbitrary.

(b) If $c'_2 \neq 0$, then $c'_1 = c'_3 \neq 0$, by (5.72), and $H'_1 = c'_2 - c'_1$, by (5.71). Now, the equation (5.73) is written as

$$c'_1(H'_2 + c'_2) = 0,$$

and so $H'_2 = -c'_2$. So, if $c'_2 \neq 0$, then $H'_1 = c'_2 - c'_1$, $H'_2 = -c'_2$, $c' = 0$, $c'_3 = c'_1$ and the elements $c'_1 \neq 0$ and $c'_2 \neq 0$ are arbitrary.

4. Since $c' - c'_3 \neq 0$ and $c' \neq 0$ the equations (5.59) and (5.60) are equivalent to the equations (5.62) and (5.63), respectively. Now, we substitute the expression for the elements H'_1 and H'_2 (that are given in (5.62) and (5.63)) into (5.61) we obtain (5.64).

(a) If $c' - c'_2 = 0$, then $c'_2 = c' \neq 0$ and the equation (5.64) holds automatically, and the statement (a) follows.

(b) Suppose that $c' - c'_2 \neq 0$ and $c' + c'_1 - c'_3 = 0$. Then the equations (5.59) and (5.60) are equivalent to the equations

$$H'_1 = c'_2 - c'_3 \quad \text{and} \quad H'_2 = -c'_2.$$

Since $c' + c'_1 - c'_3 = 0$, the equation (5.64) holds automatically. Now, the statement (b) follows.

(c) Suppose that $c' - c'_2 \neq 0$, $c' + c'_1 - c'_3 \neq 0$ and $c'^2 - c'_3c' - c'_1c'_2c'_3 = 0$. Then the equation (5.64) holds automatically. The roots of the quadratic equation

$$c'^2 - c'_3c' - c'_1c'_2c'_3 = 0$$

are

$$c'_\pm = \frac{1}{2}(c'_3 \pm \sqrt{c'_3(c'_3 + 4c'_1c'_2)}).$$

Since $c' \neq 0$, we must have $c'_3 \neq 0$. Then the quadratic equation has double root if and only if $c'_3 = -4c'_1c'_2$.

(i) Suppose that $c'_3 = -4c'_1c'_2$, i.e. the quadratic polynomial $c'^2 - c'_3c' - c'_1c'_2c'_3$ has the double root

$$c' = \frac{1}{2}c'_3 = -2c'_1c'_2.$$

Since $c' \neq 0$, we have that $c'_1 \neq 0$ and $c'_2 \neq 0$. Then we plug the values $c'_3 = -4c'_1c'_2$ and $c' = -2c'_1c'_2$ into the expressions for H'_1 and H'_2 (see (5.62) and (5.63)), we obtain that

$$H'_1 = -2c'_1 - \frac{1}{2} \quad \text{and} \quad H'_2 = -2c'_2 - \frac{1}{2}.$$

Now, the inequality $c' \neq c'_2$ is equivalent to $c'_1 \neq -\frac{1}{2}$ (since $c' = -2c'_1c'_2$ and $c'_2 \neq 0$). The inequality $c' + c'_1 - c'_3 \neq 0$ is equivalent to $c'_2 \neq -\frac{1}{2}$ (since $c' = -2c'_1c'_2$, $c'_3 = -4c'_1c'_2$ and $c'_1 \neq 0$).

So, if $c'_3 = -4c'_1c'_2$, then

$$H'_1 = -2c'_1 - \frac{1}{2}, \quad H'_2 = -2c'_2 - \frac{1}{2}, \quad c'_3 = -4c'_1c'_2, \quad c' = -2c'_1c'_2,$$

and the elements $c'_1 \neq 0, -\frac{1}{2}$ and $c'_2 \neq 0, -\frac{1}{2}$ are arbitrary.

(ii) Suppose that $c'_3 \neq -4c'_1c'_2$, i.e. the quadratic polynomial $c'^2 - c'_3c' - c'_1c'_2c'_3$ has two distinct roots c'_+ and c'_- , see above. Then

$$c'_\pm = \frac{1}{2}(c'_3 \pm \sqrt{c'_3(c'_3 + 4c'_1c'_2)}), \quad H'_1 = (c_\pm - c'_3)^{-1}c'_1(c'_3 - c'_2) \quad \text{and} \quad H'_2 = c'^{-1}_\pm c'_2(c'_1 - c'_3).$$

Then $c'_\pm \neq 0$ if and only if $c'_3 \neq 0$ and $\pm\sqrt{1 + \frac{4c'_1c'_2}{c'_3}} \neq -1$ if and only if $c'_1 \neq 0$, $c'_2 \neq 0$ and $c'_3 \neq 0$.

Then $c'_\pm \neq c'_3$ if and only if $c'_3 \neq 0$ and $\pm\sqrt{1 + \frac{4c'_1c'_2}{c'_3}} \neq 1$ if and only if $c'_1 \neq 0$, $c'_2 \neq 0$ and $c'_3 \neq 0$.

Now, $c_\pm \neq c'_3 - c'_1$ if and only if

$$\pm\sqrt{c'_3(c'_3 + 4c'_1c'_2)} \neq c'_3 - 2c'_1$$

if and only if $c'_1 c'_2 c'_3 \neq c'_1 (c'_1 - c'_3)$ if and only if $c'_1 \neq (c'_2 + 1)c'_3$ (since $c'_1 \neq 0$).

So, if $c'_3 \neq -4c'_1 c'_2$, then

$$c'_\pm = \frac{1}{2}(c'_3 \pm \sqrt{c'_3(c'_3 + 4c'_1 c'_2)}), \quad H'_1 = (c'_\pm - c'_3)^{-1} c'_1 (c'_3 - c'_2) \quad \text{and} \quad H'_2 = c'^{-1}_\pm c'_2 (c'_1 - c'_3),$$

and the elements $c'_1 \neq 0$, $c'_2 \neq 0$ and $c'_3 \neq 0$ are arbitrary such that $c'_3 \neq -4c'_1 c'_2$ and $c'_1 \neq (c'_2 + 1)c'_3$. \square

Table 5.1: The solutions to the system of equations (5.59)–(5.61).

	H'_1	H'_2	c'_1	c'_2	c'_3	c'	Conditions
1	H'_1	H'_2	0	c'_2	0	0	No conditions
2	H'_1	H'_2	c'_1	0	0	0	$c'_1 \neq 0$
3	H'_1	$-c'_2$	0	c'_2	c'_3	c'_3	$c'_3 \neq 0$
4	$-c'_1$	$c'_1 - c'$	c'_1	c'	c'	c'	$c'_1 \neq 0, c' \neq 0$
5	$-c'_1$	H'_2	c'_1	0	c'_3	0	$c'_3 \neq 0$
6	$c'_2 - c'_1$	$-c'_2$	c'_1	c'_2	c'_1	0	$c'_1 \neq 0, c'_2 \neq 0$
7	$-c'_1$	$c'_1 - c'_3$	c'_1	c'	c'_3	c'	$c'_3 \neq c', c' \neq 0$
8	$c'_2 - c'_3$	$-c'_2$	$c'_3 - c'$	c'_2	c'_3	c'	$c' \neq c'_2, c' \neq c'_3, c' \neq 0$
9	$-2c'_1 - \frac{1}{2}$	$-2c'_2 - \frac{1}{2}$	c'_1	c'_2	$-4c'_1c'_2$	$-2c'_1c'_2$	$c'_1 \neq 0, -\frac{1}{2},$ $c'_2 \neq 0, -\frac{1}{2}$
10	$(c'_4 - c'_3)^{-1}c'_1(c'_3 - c'_2)$	$c'^{-1}_+c'_2(c'_1 - c'_3)$	c'_1	c'_2	c'_3	$c'_4 = \frac{1}{2}(c'_3 + \sqrt{c'_3(c'_3 + 4c'_1c'_2)})$	$c'_1 \neq 0, c'_2 \neq 0,$ $c'_3 \neq 0, c'_3 \neq -4c'_1c'_2,$ $c'_1 \neq (c'_2 + 1)c'_3$
11	$(c'_- - c'_3)^{-1}c'_1(c'_3 - c'_2)$	$c'^{-1}_-c'_2(c'_1 - c'_3)$	c'_1	c'_2	c'_3	$c'_- = \frac{1}{2}(c'_3 - \sqrt{c'_3(c'_3 + 4c'_1c'_2)})$	$c'_1 \neq 0, c'_2 \neq 0,$ $c'_3 \neq 0, c'_3 \neq -4c'_1c'_2,$ $c'_1 \neq (c'_2 + 1)c'_3$

Chapter 6

Appendix

In this appendix, we give more careful and detailed calculations of the eleven relations in part 1 of Theorem 5.2.1 and the four relations in part 1 of Theorem 5.2.3. Furthermore, we use the program SINGULAR to verify our computations that are really on the boundary of human ability to do it correctly (because of their complexity). By using SINGULAR, we verified all our complicated calculations, and they all turned out to be correct.

6.1 Generators and defining relations for the centralizer C

In this section, we provide in more details the calculations of the eleven relations in part 1 of Theorem 5.2.1.

Recall that

$$c_1 = Y_1X_1, c_2 = Y_2X_2, c_3 = Y_3X_3, c = Y_2Y_1X_3 \text{ and } \bar{c} = Y_3X_1X_2.$$

Let us show that the equalities (5.18)–(5.26) hold. First, we show that the equality (5.18) holds.

$$\begin{aligned} [c_1, c_2] &= [Y_1X_1, Y_2X_2] = [Y_1X_1, Y_2]X_2 + Y_2[Y_1X_1, X_2] \\ &= [Y_1, Y_2]X_1X_2 + Y_2Y_1[X_1, X_2] \\ &= -Y_3X_1X_2 - Y_2Y_1X_3 \\ &= -\bar{c} + c, \end{aligned}$$

$$\begin{aligned}
[c_2, c_3] &= [Y_2X_2, Y_3X_3] = [Y_2X_2, Y_3]X_3 + Y_3[Y_2X_2, X_3] \\
&= Y_2[X_2, Y_3]X_3 + Y_3[Y_2, X_3]X_2 \\
&= Y_2Y_1X_3 - Y_3X_1X_2 \\
&= -\bar{c} + c,
\end{aligned}$$

$$\begin{aligned}
[c_3, c_1] &= [Y_3X_3, Y_1X_1] = [Y_3X_3, Y_1]X_1 + Y_1[Y_3X_3, X_1] \\
&= Y_3[X_3, Y_1]X_1 + Y_1[Y_3, X_1]X_3 \\
&= -Y_3X_2X_1 + Y_1Y_2X_3 \\
&= -Y_3X_1X_2 - Y_3[X_2, X_1] + Y_2Y_1X_3 + [Y_1, Y_2]X_3 \\
&= -Y_3X_1X_2 + Y_3X_3 + Y_2Y_1X_3 - Y_3X_3 \\
&= -\bar{c} + c.
\end{aligned}$$

So, the equality (5.18) holds. Let us prove that the equality (5.19) holds.

$$\begin{aligned}
[c_1, c] &= [Y_1X_1, Y_2Y_1X_3] = [Y_1X_1, Y_2]Y_1X_3 + Y_2[Y_1X_1, Y_1]X_3 + Y_2Y_1[Y_1X_1, X_3] \\
&= [Y_1, Y_2]X_1Y_1X_3 + Y_2Y_1[X_1, Y_1]X_3 + Y_2Y_1[Y_1, X_3]X_1 \\
&= -Y_3X_1Y_1X_3 + Y_2Y_1H_1X_3 + Y_2Y_1X_2X_1 \\
&= -X_1Y_3Y_1X_3 - [Y_3, X_1]Y_1X_3 + Y_2H_1Y_1X_3 + Y_2[Y_1, H_1]X_3 + Y_1Y_2X_2X_1 \\
&\quad + [Y_2, Y_1]X_2X_1 \\
&= -X_1Y_3Y_1X_3 - Y_2Y_1X_3 + H_1Y_2Y_1X_3 + [Y_2, H_1]Y_1X_3 + 2Y_2Y_1X_3 \\
&\quad + Y_1Y_2X_1X_2 + Y_1Y_2[X_2, X_1] + Y_3X_2X_1 \\
&= -X_1Y_1Y_3X_3 - c + H_1c - c + 2c + Y_1X_1Y_2X_2 - Y_1Y_2X_3 + Y_3X_1X_2 \\
&\quad + Y_3[X_2, X_1] \\
&= -Y_1X_1Y_3X_3 - [X_1, Y_1]Y_3X_3 - c + H_1c - c + 2c + c_1c_2 - Y_2Y_1X_3 \\
&\quad - [Y_1, Y_2]X_3 + \bar{c} - c_3 \\
&= -c_1c_3 - H_1c_3 + H_1c + c_1c_2 - c + \bar{c} \\
&= c_1(c_2 - c_3) + (H_1 - 1)c + \bar{c} - H_1c_3.
\end{aligned}$$

Now, we show that the equality (5.20) holds.

$$\begin{aligned}
[c_1, \bar{c}] &= [Y_1 X_1, Y_3 X_1 X_2] = [Y_1 X_1, Y_3] X_1 X_2 + Y_3 [Y_1 X_1, X_1] X_2 + Y_3 X_1 [Y_1 X_1, X_2] \\
&= Y_1 [X_1, Y_3] X_1 X_2 + Y_3 [Y_1, X_1] X_1 X_2 + Y_3 X_1 Y_1 [X_1, X_2] \\
&= -Y_1 Y_2 X_1 X_2 - Y_3 H_1 X_1 X_2 + Y_3 X_1 Y_1 X_3 \\
&= -Y_1 Y_2 X_1 X_2 - H_1 Y_3 X_1 X_2 - [Y_3, H_1] X_1 X_2 + Y_3 X_1 X_3 Y_1 + Y_3 X_1 [Y_1, X_3] \\
&= -Y_1 Y_2 X_2 X_1 - Y_1 Y_2 [X_1, X_2] - H_1 Y_3 X_1 X_2 - Y_3 X_1 X_2 + Y_3 X_3 X_1 Y_1 + Y_3 X_1 X_2 \\
&= -Y_2 Y_1 X_2 X_1 - [Y_1, Y_2] X_2 X_1 - Y_2 Y_1 X_3 - [Y_1, Y_2] X_3 - H_1 Y_3 X_1 X_2 - \bar{c} \\
&\quad + Y_3 X_3 Y_1 X_1 + Y_3 X_3 [X_1, Y_1] + \bar{c} \\
&= -Y_2 X_2 Y_1 X_1 + Y_3 X_1 X_2 + Y_3 [X_2, X_1] - c + c_3 - H_1 \bar{c} - \bar{c} + c_3 c_1 + H_1 c_3 + \bar{c} \\
&= -c_2 c_1 + \bar{c} - c_3 - c + c_3 - H_1 \bar{c} - \bar{c} + c_3 c_1 + H_1 c_3 + \bar{c} \\
&= -c_2 c_1 - c - H_1 \bar{c} + c_3 c_1 + H_1 c_3 + \bar{c} \\
&= -(c_2 - c_3) c_1 - (H_1 - 1) \bar{c} - c + H_1 c_3.
\end{aligned}$$

Let us prove that the equality (5.21) holds.

$$\begin{aligned}
[c_2, c] &= [Y_2 X_2, Y_2 Y_1 X_3] = [Y_2 X_2, Y_2] Y_1 X_3 + Y_2 [Y_2 X_2, Y_1] X_3 + Y_2 Y_1 [Y_2 X_2, X_3] \\
&= Y_2 [X_2, Y_2] Y_1 X_3 + Y_2 [Y_2, Y_1] X_2 X_3 + Y_2 Y_1 [Y_2, X_3] X_2 \\
&= Y_2 H_2 Y_1 X_3 + Y_2 Y_3 X_2 X_3 - Y_2 Y_1 X_1 X_2 \\
&= H_2 Y_2 Y_1 X_3 + [Y_2, H_2] Y_1 X_3 + Y_2 X_2 Y_3 X_3 + Y_2 [Y_3, X_2] X_3 - Y_2 Y_1 X_2 X_1 \\
&\quad - Y_2 Y_1 [X_1, X_2] \\
&= H_2 c + 2c + c_2 c_3 - c - Y_2 X_2 Y_1 X_1 - c \\
&= H_2 c + c_2 c_3 - c_2 c_1 \\
&= c_2 (c_3 - c_1) + H_2 c.
\end{aligned}$$

Now, we show that the equality (5.22) holds.

$$\begin{aligned}
[c_2, \bar{c}] &= [Y_2X_2, Y_3X_1X_2] = [Y_2X_2, Y_3]X_1X_2 + Y_3[Y_2X_2, X_1]X_2 + Y_3X_1[Y_2X_2, X_2] \\
&= Y_2[X_2, Y_3]X_1X_2 + Y_3Y_2[X_2, X_1]X_2 + Y_3X_1[Y_2, X_2]X_2 \\
&= Y_2Y_1X_1X_2 - Y_3Y_2X_3X_2 - Y_3X_1H_2X_2 \\
&= Y_1Y_2X_1X_2 + [Y_2, Y_1]X_1X_2 - Y_3X_3Y_2X_2 - Y_3[Y_2, X_3]X_2 - Y_3H_2X_1X_2 \\
&\quad - Y_3[X_1, H_2]X_2 \\
&= Y_1X_1Y_2X_2 + \bar{c} - c_3c_2 + \bar{c} - H_2Y_3X_1X_2 - [Y_3, H_2]X_1X_2 - \bar{c} \\
&= c_1c_2 - c_3c_2 - H_2\bar{c} \\
&= -(c_3 - c_1)c_2 - H_2\bar{c}.
\end{aligned}$$

Let us prove that the equality (5.23) holds.

$$\begin{aligned}
[c_3, c] &= [Y_3X_3, Y_2Y_1X_3] = [Y_3X_3, Y_2]Y_1X_3 + Y_2[Y_3X_3, Y_1]X_3 + Y_2Y_1[Y_3X_3, X_3] \\
&= Y_3[X_3, Y_2]Y_1X_3 + Y_2Y_3[X_3, Y_1]X_3 + Y_2Y_1[Y_3, X_3]X_3 \\
&= Y_3X_1Y_1X_3 - Y_2Y_3X_2X_3 - Y_2Y_1(H_1 + H_2)X_3 \\
&= Y_3X_1X_3Y_1 + Y_3X_1[Y_1, X_3] - Y_3Y_2X_2X_3 - Y_2(H_1 + H_2)Y_1X_3 \\
&\quad - Y_2[Y_1, H_1 + H_2]X_3 \\
&= Y_3X_3X_1Y_1 + \bar{c} - Y_3Y_2X_3X_2 - (H_1 + H_2)Y_2Y_1X_3 - [Y_2, H_1 + H_2]Y_1X_3 \\
&\quad - 2Y_2Y_1X_3 + Y_2Y_1X_3 \\
&= Y_3X_3Y_1X_1 + Y_3X_3[X_1, Y_1] + \bar{c} - Y_3X_3Y_2X_2 - Y_3[Y_2, X_3]X_2 - (H_1 + H_2)c \\
&\quad - Y_2Y_1X_3 - 2Y_2Y_1X_3 - 2c + c \\
&= c_3c_1 + H_1c_3 + \bar{c} - c_3c_2 - (H_1 + H_2)c - 2c + 2\bar{c} \\
&= c_3(c_1 - c_2 + H_1) - (H_1 + H_2 + 2)c + 2\bar{c}.
\end{aligned}$$

Now, we show that the equality (5.24) holds.

$$\begin{aligned}
[c_3, \bar{c}] &= [Y_3 X_3, Y_3 X_1 X_2] = [Y_3 X_3, Y_3] X_1 X_2 + Y_3 [Y_3 X_3, X_1] X_2 + Y_3 X_1 [Y_3 X_3, X_2] \\
&= Y_3 [X_3, Y_3] X_1 X_2 + Y_3 [Y_3, X_1] X_3 X_2 + Y_3 X_1 [Y_3, X_2] X_3 \\
&= Y_3 (H_1 + H_2) X_1 X_2 + Y_3 Y_2 X_3 X_2 - Y_3 X_1 Y_1 X_3 \\
&= (H_1 + H_2) Y_3 X_1 X_2 + [Y_3, H_1 + H_2] X_1 X_2 + Y_2 Y_3 X_2 X_3 - X_1 Y_3 Y_1 X_3 \\
&\quad - [Y_3, X_1] Y_1 X_3 \\
&= (H_1 + H_2) \bar{c} + 2\bar{c} + Y_2 X_2 Y_3 X_3 + Y_2 [Y_3, X_2] X_3 - Y_1 X_1 Y_3 X_3 \\
&\quad - [X_1, Y_1] Y_3 X_3 - c \\
&= (H_1 + H_2) \bar{c} + 2\bar{c} + c_2 c_3 - 2c - c_1 c_3 - H_1 c_3 \\
&= (H_1 + H_2) \bar{c} + 2\bar{c} + c_2 c_3 - 2c - c_1 c_3 - H_1 c_3 \\
&= -(c_1 - c_2 + H_1) c_3 + (H_1 + H_2 + 2) \bar{c} - 2c.
\end{aligned}$$

Let us show that the equality (5.25) holds.

$$\begin{aligned}
c\bar{c} &= Y_2 Y_1 X_3 Y_3 X_1 X_2 = Y_2 X_3 Y_1 Y_3 X_1 X_2 + Y_2 [Y_1, X_3] Y_3 X_1 X_2 \\
&= X_3 Y_2 Y_3 Y_1 X_1 X_2 + [Y_2, X_3] Y_3 Y_1 X_1 X_2 + c_2 \bar{c} \\
&= X_3 Y_3 Y_1 Y_2 X_1 X_2 + X_3 Y_3 [Y_2, Y_1] X_1 X_2 - Y_1 X_1 Y_3 X_1 X_2 - [X_1, Y_1] \bar{c} + c_2 \bar{c} \\
&= Y_3 X_3 Y_1 X_1 Y_2 X_2 + [X_3, Y_3] Y_1 X_1 Y_2 X_2 + Y_3 X_3 Y_3 X_1 X_2 + [X_3, Y_3] Y_3 X_1 X_2 \\
&\quad - c_1 \bar{c} - H_1 \bar{c} + c_2 \bar{c} \\
&= c_3 c_1 c_2 + (H_1 + H_2) c_1 c_2 + c_3 \bar{c} + (H_1 + H_2) \bar{c} - c_1 \bar{c} - H_1 \bar{c} + c_2 \bar{c} \\
&= c_3 c_1 c_2 + (H_1 + H_2) c_1 c_2 - c_1 \bar{c} + c_2 \bar{c} + c_3 \bar{c} + H_2 \bar{c} \\
&= (c_3 + H_1 + H_2) c_1 c_2 + (-c_1 + c_2 + c_3 + H_2) \bar{c}.
\end{aligned}$$

Now, we prove that the equality (5.26) holds.

$$\begin{aligned}
\bar{c}c &= Y_3X_1X_2Y_2Y_1X_3 = X_1Y_3X_2Y_2Y_1X_3 + [Y_3, X_1]X_2Y_2Y_1X_3 \\
&= X_1X_2Y_2Y_1Y_3X_3 + X_1[Y_3, X_2]Y_2Y_1X_3 + c_2c \\
&= X_2X_1Y_2Y_1Y_3X_3 + [X_1, X_2]Y_2Y_1Y_3X_3 - Y_1X_1Y_2Y_1X_3 - [X_1, Y_1]Y_2Y_1X_3 + c_2c \\
&= X_2Y_2X_1Y_1Y_3X_3 + Y_3X_3Y_2Y_1X_3 + [X_3, Y_3]Y_2Y_1X_3 - c_1c - H_1c + c_2c \\
&= X_2Y_2Y_1X_1Y_3X_3 + X_2Y_2[X_1, Y_1]Y_3X_3 + c_3c + (H_1 + H_2)c - c_1c - H_1c + c_2c \\
&= Y_2X_2Y_1X_1Y_3X_3 + [X_2, Y_2]Y_1X_1Y_3X_3 + Y_2X_2H_1Y_3X_3 + [X_2, Y_2]H_1Y_3X_3 + c_3c \\
&\quad + (H_1 + H_2)c - c_1c - H_1c + c_2c \\
&= c_2c_1c_3 + H_2c_1c_3 + c_2H_1c_3 + H_2H_1c_3 + c_3c + (H_1 + H_2)c - c_1c - H_1c + c_2c \\
&= (c_2 + H_2)(c_1 + H_1)c_3 + (-c_1 + c_2 + c_3 + H_2)c.
\end{aligned}$$

6.2 Generators and defining relations for the centralizer C via $Z(U)$

In this section, we provide the calculations of the four relations in part 1 of Theorem 5.2.3.

The relations (5.39)–(5.42) are obtained from the relations (5.18)–(5.26) by replacing the pair (c_3, \bar{c}) by the pair (Z_1, Z_2) using the equalities (5.1) and (5.2) as follows.

From the equality (5.1) we obtain

$$c_3 = p - c_1 - c_2$$

where

$$p = 6Z_1 - \frac{1}{3}\left(H_1^2 + H_2^2 + H_1H_2 + 3(H_1 + H_2)\right),$$

and from the equality (5.2) we obtain

$$\bar{c} = q - c - (H_2 + 2)c_1 + H_1c_2$$

where

$$q = -\frac{1}{3}Z_2 + \frac{1}{27}(H_2 - H_1)\left(2\beta + 3(H_1H_2 + H_1 + H_2 + 3p + 3)\right) + p.$$

Now if we substitute the values of c_3 and \bar{c} in the relations (5.18)–(5.26), then the three relations in the equality (5.18) collapse to the relation

$$[c_1, c_2] = 2c + (H_2 + 2)c_1 - H_1c_2 - q,$$

the relations (5.19), (5.20), (5.23) and (5.24) collapse to the relation

$$[c_1, c] = (H_1 - 2)c + c_1(c_1 + 2c_2 + H_1 - H_2 - p - 2) + 2H_1c_2 - H_1p + q,$$

the relations (5.21) and (5.22) collapse to the relation

$$[c_2, c] = (H_2 + 4)c - c_2(c_2 + 2H_1 - p) - 2c_1(c_2 - H_2 - 2) - 2q$$

and the relations (5.25) and (5.26) collapse to the relation

$$\begin{aligned} & c^2 + \left(c_1(H_2 + 4) - c_2(H_1 - 2) + 2H_1 - H_2 - q - p - 4 \right) c \\ & + \left((H_2 + 2)c_1 + H_1H_2 + 2H_1 - 2H_2 - 2q - 4 \right) c_1 \\ & + \left((-c_1 - 2H_1 + 2)c_2 - (c_1 + 3H_1 - p - 2)c_1 - H_1H_2 + 2H_1p - 2H_1^2 - q - 2p \right) c_2 \\ & + H_1H_2p + 2H_1p - 2H_1q + pq + 2q = 0. \end{aligned}$$

6.3 SINGULAR code

Now we will use the computer algebra system SINGULAR in order to check the results of our calculations in the previous two sections.

These calculations are checked by the following code in SINGULAR:

```
ring r=0, (y(3),y(2),y(1),x(1),x(2),x(3),h(1),h(2)),(dp,C);
matrix D[8][8];
D[1,2]=0;D[1,3]=0;D[1,4]=-y(2);D[1,5]=y(1);D[1,6]=h(1)+h(2);
D[1,7]=-y(3);D[1,8]=-y(3);D[2,3]=-y(3);D[2,4]=0;D[2,5]=h(2);
D[2,6]=x(1);D[2,7]=y(2);D[2,8]=-2*y(2);D[3,4]=h(1);D[3,5]=0;
D[3,6]=-x(2);D[3,7]=-2*y(1);D[3,8]=y(1); D[4,5]=-x(3);
D[4,6]=0;D[4,7]=2*x(1);D[4,8]=-x(1);D[5,6]=0;D[5,7]=-x(2);
D[5,8]=2*x(2);D[6,7]=x(3);D[6,8]=x(3);D[7,8]=0;
def Us13=nc_algebra(1,D); setring Us13; Us13;
// coefficients: QQ
// number of vars: 8
//      block 1 : ordering dp
//      : names y(3) y(2) y(1) x(1) x(2) x(3) h(1) h(2)
//      block 2 : ordering C
// noncommutative relations:
//      x(1)y(3)=y(3)*x(1)-y(2)
//      x(2)y(3)=y(3)*x(2)+y(1)
//      x(3)y(3)=y(3)*x(3)+h(1)+h(2)
//      h(1)y(3)=y(3)*h(1)-y(3)
//      h(2)y(3)=y(3)*h(2)-y(3)
```

```

//      y(1)y(2)=y(2)*y(1)-y(3)
//      x(2)y(2)=y(2)*x(2)+h(2)
//      x(3)y(2)=y(2)*x(3)+x(1)
//      h(1)y(2)=y(2)*h(1)+y(2)
//      h(2)y(2)=y(2)*h(2)-2*y(2)
//      x(1)y(1)=y(1)*x(1)+h(1)
//      x(3)y(1)=y(1)*x(3)-x(2)
//      h(1)y(1)=y(1)*h(1)-2*y(1)
//      h(2)y(1)=y(1)*h(2)+y(1)
//      x(2)x(1)=x(1)*x(2)-x(3)
//      h(1)x(1)=x(1)*h(1)+2*x(1)
//      h(2)x(1)=x(1)*h(2)-x(1)
//      h(1)x(2)=x(2)*h(1)-x(2)
//      h(2)x(2)=x(2)*h(2)+2*x(2)
//      h(1)x(3)=x(3)*h(1)+x(3)
//      h(2)x(3)=x(3)*h(2)+x(3)
LIB "central.lib";
// ** loaded /usr/./singular/LIB/central.lib (4.1.2.0, Feb_2019)
// ** loaded /usr/./singular/LIB/poly.lib (4.1.2.0, Feb_2019)
// ** loaded /usr/./singular/LIB/ring.lib (4.1.2.0, Feb_2019)
// ** loaded /usr/./singular/LIB/primdec.lib (4.1.2.0, Feb_2019)
// ** loaded /usr/./singular/LIB/absfact.lib (4.1.2.0, Feb_2019)
// ** loaded /usr/./singular/LIB/triang.lib (4.1.2.0, Feb_2019)
// ** loaded /usr/./singular/LIB/matrix.lib (4.1.2.0, Feb_2019)
// ** loaded /usr/./singular/LIB/nctools.lib (4.1.2.0, Feb_2019)
// ** loaded /usr/./singular/LIB/random.lib (4.1.2.0, Feb_2019)
// ** loaded /usr/./singular/LIB/elim.lib (4.1.2.0, Feb_2019)
// ** loaded /usr/./singular/LIB/inout.lib (4.1.2.0, Feb_2019)
// ** loaded /usr/./singular/LIB/general.lib (4.1.2.0, Feb_2019)
poly z(1)=(1/6)*(y(1)*x(1)+y(2)*x(2)+y(3)*x(3))+(1/18)*(h(1)^2
+h(1)*h(2)+h(2)^2)+(1/6)*(h(1)+h(2));
inCenter(z(1));
1
poly z(2)=-(2/9)*(h(1)^3)+(2/9)*h(2)^3-(1/3)*h(1)^2*h(2)
+(1/3)*h(1)*h(2)^2-h(1)^2+h(2)^2-h(1)+h(2)-y(1)*x(1)*h(1)
-y(3)*x(3)*h(1)+2*y(2)*x(2)*h(1)-2*y(1)*x(1)*h(2)+y(3)*x(3)*h(2)
+y(2)*x(2)*h(2)-3*y(2)*y(1)*x(3)-3*y(3)*x(1)*x(2)+3*y(2)*x(2)
-3*y(1)*x(1)+3*y(3)*x(3);
inCenter(z(2));
1
poly c(1)= y(1)*x(1);
poly c(2)= y(2)*x(2);
poly c(3)= y(3)*x(3);
poly c= y(2)*y(1)*x(3);

```

```

poly g= y(3)*x(1)*x(2);
\\ checking the relations (5.18) – (5.26):
bracket(c(1),c(2)) == bracket(c(2),c(3));
1
bracket(c(2),c(3)) == bracket(c(3),c(1));
1
bracket(c(1),c) == c(1)*(c(2)-c(3))+h(1)-1*c+g-h(1)*c(3);
1
bracket(c(1),g) == -(c(2)-c(3))*c(1)-(h(1)-1)*g-c+h(1)*c(3);
1
bracket(c(2),c) == c(2)*(c(3)-c(1))+h(2)*c;
1
bracket(c(2),g) == -(c(3)-c(1))*c(2)-h(2)*g;
1
bracket(c(3),c) == c(3)*(c(1)-c(2)+h(1))-(h(1)+h(2)+2)*c+2*g;
1
bracket(c(3),g) == -(c(1)-c(2)+h(1))*c(3)+(h(1)+h(2)+2)*g-2*c;
1
c*g == ((c(3)+h(1)+h(2)+1)*c(1)-c(3)+g)*c(2)+(c(3)-c(1))*g;
1
g*c==((c(2)+h(2)+1)*(c(1)+h(1))-c(2)+c)*c(3)+(c(2)-c(1)-h(1))*c;
1
poly B= h(1)^2+h(2)^2+h(1)*h(2)+3*(h(1)+h(2));
poly p= 6*z(1)-(1/3)*B;
poly q= -(1/3)*z(2)+(1/27)*(h(2)-h(1))*(2*B+3*(h(1)*h(2)+h(1)
+h(2)+3*p+3))+p;
\\ checking the relations (5.39) – (5.42):
bracket(c(1),c(2)) == 2*c+(h(2)+2)*c(1)-h(1)*c(2)-q;
1
bracket(c(1),c) == (h(1)-2)*c+c(1)*(c(1)+2*c(2)+h(1)-h(2)-p-2)
+2*h(1)*c(2)-h(1)*p+q;
1
bracket(c(2),c) == (h(2)+4)*c-c(2)*(c(2)+2*h(1)-p)
-2*c(1)*(c(2)-h(2)-2)-2*q;
1
c^2+(c(1)*(h(2)+4)-c(2)*(h(1)-2)+2*h(1)-h(2)-q-p-4)*c
+((h(2)+2)*c(1)+h(1)*h(2)+2*h(1)-2*h(2)-2*q-4)*c(1)
+((-c(1)-2*h(1)+2)*c(2)-(c(1)+3*h(1)-p-2)*c(1)-h(1)*h(2)
+2*h(1)*p-2*h(1)^2-q-2*p)*c(2)+h(1)*h(2)*p+2*h(1)*p
-2*h(1)*q+p*q+2*q == 0;
1
\\ checking the relations (5.45) – (5.55):
bracket(c(1),c(2)) == bracket(c(2),c(3));
1

```

```

bracket(c(2),c(3)) == bracket(c(3),c(1));
1
bracket(c(1),c(2)) == c-g;
1
bracket(c(1),c) == c(1)*c(2)+h(1)*c-(h(1)+c(1))*c(3)-c+g;
1
bracket(c(2),c) == h(2)*c-c(1)*c(2)+c(2)*c(3)+c-g;
1
bracket(c(3),c) == -h(1)*c-h(2)*c+(h(1)+c(1)-c(2))*c(3);
1
bracket(c(1),g) == -h(1)*g-c(1)*c(2)+(c(1)+h(1))*c(3)+c-g;
1
bracket(c(2),g) == -h(2)*g+c(1)*c(2)-c(2)*c(3)+c-g;
1
bracket(c(3),g) == h(1)*g+h(2)*g+(c(2)-h(1)-c(1))*c(3)-2*c+2*g;
1
g*c == c(1)*c(2)*c(3)+h(1)*h(2)*c(3)+h(1)*c(2)*c(3)
+h(2)*c(1)*c(3)+2*c(1)*c(3)-2*c(2)*c(3)+2*h(1)*c(3)-c(1)*c
+c(2)*c+c(3)*g-h(1)*c-h(2)*g-h(1)*g+2*c-2*g;
1
c*g == c(1)*c(2)*c(3)+h(1)*c(1)*c(2)+h(2)*c(1)*c(2)+c(2)*c
-c(1)*c+c(3)*g-h(2)*c+2*c(1)*c(2)-2*c(2)*c(3);
1

```

Remark 6.3.1. In this code we set $x(1) = X_1$, $x(2) = X_2$, $x(3) = X_3$, $y(1) = Y_1$, $y(2) = Y_2$, $y(3) = Y_3$, $h(1) = H_1$, $h(2) = H_2$, $c(1) = c_1$, $c(2) = c_2$, $c(3) = c_3$, $g = \bar{c}$.

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