Pro-p Subgroups of Spin Groups and Quaternion Algebras

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Abstract

The main objective of this thesis is to classify pro-p subgroups of the spin groups of Clifford algebras defined on modules over p-adic rings using the method developed by Richard Pink. To accomplish this, we make certain changes to the original work which deals with pro-p subgroups of SL_2, and adapt the method to the Clifford algebra setting. Our main results consists of three cases of Pink's method on Clifford algebras: the Quaternion algebra ($n = 2$), and the Spin group of Clifford algebras with $n = 3, 4$. Later, we will discuss the reasons why the method stops working when $n > 4$, and a possible application on a certain Galois deformation problem in a work done by Gebhard Böckle.

Contents

Chapter 1

Introduction

Clifford algebras are a class of algebras generated from a vector space over a field K equipped with a quadratic form. This definition can also be extended to cover modules over commutative rings [11]. One can think of this as a way to generalise the construction of complex and quaternion numbers to arbitrary field K. Clifford algebras over the real and complex numbers have been studied extensively with the major result being the complete classification of their structures based on the properties of the quadratic forms attached to their generating vector spaces. They also possess a faithful representation into certain classes of square matrices over the same field which gives rise to a rich geometric interpretation. Thus it is no surprise that the theory of Clifford algebras has many far reaching applications ranging from geometry, particle physics to digital image processing.

In this thesis, we are interested in Clifford algebras over a field with p-adic structure such as the p-adic numbers. Specifically, we study a part of Clifford algebra where the coefficients of each element are from the ring of integers of such fields. To put it in another way, we want study the Clifford algebras defined on quadratic modules over said p-adic rings. In order to do this, we try to understand their spin groups, specifically, the pro-p subgroups of their spin groups. By modifying the method used by Richard Pink [23], we obtain the classification of pro-p subgroups for the spin groups of low dimensional Clifford algebras. Our main results are given by one-to-one correspondences from pro-p subgroups of the Clifford algebras with $n = 2$ and Spin groups with $n = 3, 4$ to additive subgroups of certain Lie algebras. This allows for a simple calculation of their lower central series which is a possible application that we will discuss in chapter 3.

In this introduction, we give a short review on the definition of Clifford algebra and necessary terms needed to understand our main result. We will end the chapter by stating a short version of our main theorem.

1.1 Quadratic Space

Through out this chapter, we fix V to be a finite dimensional vector space over a field K of characteristic not equal to 2 with a bilinear form $B : V \times V \to K$. We can define an *associated linear map* from V to its dual space V^* by

$$B^\flat : V \to V^*,\ v \mapsto B(v, \cdot).$$

Recall that a bilinear form is called *symmetric* if $B(v_1, v_2) = B(v_2, v_1)$ for all $v_1, v_2 \in V$. One can uniquely determine the symmetric bilinear form B by its associated quadratic form $Q_B(V) : V \to K$, $v \mapsto B(v, v)$ using the *polarization identity*

$$B(v, w) = \frac{1}{2}(Q_B(v + w) - Q_B(v) - Q_B(w)). \tag{1.1.0.1}$$

The *kernel* (or *radical*) of B, denoted $ker(B)$, is the subspace

$$ker(B) = \{v \in V | B(v, w) = 0 \text{ for all } w \in V\},$$

which is also the kernel of the linear map $B^\flat$. The bilinear form is called *non-degenerate* if $ker(B) = \bar{0}$, that is, if and only if $B^\flat$ is an isomorphism.

Definition 1.1.1. We refer to a vector space V together with a non-degenerate symmetric bilinear form B as a *quadratic vector space*.

When dealing with quadratic vector space, we often decompose it by orthogonality of its subspaces. Although we will not use them later on, we want to introduce some well-known results regarding the decomposition.

Definition 1.1.2. The *orthogonal group* $O(V)$ is the group

$$O(V) = \{A \in \mathrm{GL}(V) | B(Av, Aw) = B(v, w) \text{ for all } v, w \in V\}.$$

The subgroup of $O(V)$ with determinant 1 is called the *special orthogonal group*, denoted by $SO(V)$.

For any subspace $F \subset V$, the *orthogonal* or *perpendicular* subspace is defined as

$$F^\perp = \{v \in V | B(v, v_1) = 0 \text{ for all } v_1 \in F\}.$$

It is easy to see that $B^\flat(F^\perp) \subseteq V^*$ is the annihilator of F. This implies that

$$\dim F + \dim F^\perp = \dim V. \tag{1.1.2.1}$$

In particular, the following formulas hold for any $F, F_1, F_2 \subseteq V$

$$(F^\perp)^\perp = F,\ (F_1 \cap F_2)^\perp = F_1^\perp + F_2^\perp,\ (F_1 + F_2)^\perp = F_1^\perp \cap F_2^\perp.$$

For any subspace $F \subseteq V$, the restriction of B to F has the kernel $\ker(B|_{F\times F}) = F \cap F^{\perp}$.

Definition 1.1.3. A subspace $F \subseteq V$ is called a *quadratic subspace* if the restriction of B to F is non-degenerate, that is $ker(B|_{F\times F}) = F \cap F^{\perp} = 0$

Note that when B is non-degenerated, the equality $(F^{\perp})^{\perp} = F$ implies that the following are equivalent

1. F is quadratic.
2. $F^{\perp}$ is quadratic.
3. $F \oplus F^{\perp} = V$.

Next we will define what it means for a vector or a subspace to be isotropic.

Definition 1.1.4. A vector $v \in V$ is called *isotropic* if $B(v, v) = 0$. Otherwise, it is *non-isotropic.* A subspace $F \subseteq V$ is called *isotropic* if $B|_{F\times F} = 0$, that is $F \subseteq F^{\perp}$.

The polarization identity (1.1.0.1) implies that F is isotropic if and only if all of its vectors are isotropic. If F is isotropic, then from (1.1.2.1) we have that

$$\dim V = \dim F + \dim F^{\perp} \geq 2 \dim F.$$

Thus

$$\dim F \leq \frac{1}{2} \dim V.$$

An isotropic subspace is called *maximal* if it is not properly contained inside another isotropic subspace. That is, an isotropic subspace F is maximal if and only if it contains all isotropic elements of $F^{\perp}$. We say that F is *anisotropic* if it does not contain any non-zero isotropic vectors.

The next two propositions show that the dimension of any maximal isotropic subspace of V are fixed.

Proposition 1.1.5. *Let $F, F' \subseteq V$ be isotropic subspaces. The following three conditions are equivalent*

1. *$F + F'$ is quadratic,*
2. *$V = F \oplus (F')^{\perp}$,*
3. *$V = F' \oplus (F)^{\perp}$.*

Moreover, the conditions imply that $dimF = dimF'$, and for any isotropic subspace $F \subset V$, one can always find an isotropic subspace F' (possibly itself) satisfying these conditions.

Proposition 1.1.6. *Let $F, F' \subseteq V$ be maximal isotropic subspaces. Then following hold:*

1. *$ker(B|_{F+F'}) = F \cap F'$,*
2. *The images of F and F' in the quadratic vector space $(F + F')/(F \cap F')$ are maximal isotropic,*
3. *$\dim F = \dim F'$.*

Since the dimension of any maximal isotropic subspace of V is fixed, we can define the following index.

Definition 1.1.7. Let V be a quadratic vector space with a non-degenerate symmetric bilinear form B. *The Witt index* of B is the dimension of a maximal isotropic subspace of V.

Since $\dim F \leq \frac{1}{2} \dim V$ whenever F is isotropic, the maximum possible Witt index of B is $\frac{1}{2} \dim V$ when $\dim V$ is even, and $\frac{1}{2}(\dim V - 1)$ when $\dim V$ is odd.

Although we may not use them directly, we will end this subsection with the following two theorems which are fundamental to the theory of quadratic spaces.

Theorem 1.1.8 (Witt's Theorem)**.** *Suppose F, F' are subspaces of a quadratic space (V, B), such that there exists an isometric isomorphism $\phi : F \to F'$ where*

$$B(\phi(v), \phi(w)) = B(v, w),$$

for all $v, w \in F$. Then ϕ extends to an orthogonal transformation $A \in O(V)$.

Theorem 1.1.9 (Witt decomposition)**.** *Let (V, B) be a quadratic space. Then it admits a decomposition*

$$V = F \oplus F' \oplus W,$$

where F, F' are maximal isotropic, W is anisotropic, and $W^\perp = F \oplus F'$. Moreover, if $V = F_1 \oplus {F_1}' \oplus W_1$ is another such decomposition, then there exists a transformation $A \in O(V)$ such that

$$A(F) = F_1,\ A(F') = {F_1}',\ A(W) = W_1.$$

For the proofs, see chapter 1 in [19]. Further reference on the general theories of quadratic space can be found in [5], [15].

1.2 Clifford Algebra

We can now introduce our main object of interest. Let $T(V) = \bigoplus_{k\in\mathbb{Z}} T^kV$ denote the tensor algebra, where $T^kV = V \otimes ... \otimes V$ is the k-fold tensor product.

Definition 1.2.1. *A Clifford algebra* $\mathrm{Cl}(V, B)$ is the quotient

$$\mathrm{Cl}(V, B) = T(V)/F(V, B),$$

where $F(V, B) \subset T(V)$ is the two-sided ideal generated by elements of the form

$$u \otimes v + v \otimes u - 2B(v, u) \cdot 1,\ \ u, v \in V.$$

In the Clifford algebra, we write uv for the class of $u \otimes v$ from the quotient. Let $dim(V) = n$. Since $char(K) \neq 2$, there always exists an orthogonal basis $\{e_1, e_2, ..., e_n\}$ of V with respect to B, where $B(e_i, e_j) = 0$ for all $i \neq j$, (Prop 1.1 in [19]). This implies that $e_ie_j = -e_je_i$ for all $i \neq j$ and $B(e_i, e_i) = e_i^2$.

As we will see throughout the rest of this section, the anti-commutativity between the basis elements is one of the most distinct traits of Clifford algebras, and will play a crucial role in determining their structures.

Note that the inclusion $K \to T(V)$ on the tensor algebra naturally descends to an inclusion $K \to \mathrm{Cl}(V, B)$, and the inclusion $V \to T(V)$ descends to an inclusion $V \to \mathrm{Cl}(V, B)$.

Let $\alpha : V \to \mathrm{Cl}(V, B)$ be an inclusion described above. The Clifford algebra has the following universal property: given any unital associative algebra W over K and a linear map $f : V \to W$ satisfying

$$f(v_1)f(v_2) + f(v_2)f(v_1) = 2B(v_1, v_2) \cdot 1_W,\ \ \forall v_1, v_2 \in V,$$

then there exists a unique morphism of algebras $\phi : \mathrm{Cl}(V, B) \to W$ such that the following diagram commutes:

$$\begin{tikzcd} V \arrow[r, "\alpha"] \arrow[dr, "f"'] & \mathrm{Cl}(V,B) \arrow[d, dashed, "\phi"] \\ & W \end{tikzcd}$$

Fix an orthogonal basis $\{e_1, e_2, ..., e_n\}$ of V with respect to B. We can describe any element of $\mathrm{Cl}(V, B)$ in terms of these basis elements. First, we set up a few conventions. We write $e_{a_1}e_{a_2}...e_{a_i} = e_{a_1a_2...a_i}$ where $a_1, a_2, ..., a_i \in \{1, 2, ..., n\}$. For $A = a_1a_2...a_i$, if i is odd, we say e_A is an odd basis element, similarly we say e_A is even if i is even, and we count the empty product as even.

We will refer to the product of basis where the indexes of e_j are increasing, i.e., $a_1 < a_2 < ... < a_i$, as the standard form of expression. One can always arrange the basis into such form by swapping the basis elements in the product one pair at a time while multiplying the term with -1. For example,

$$e_{532} = (-1)^2 e_{325} = (-1)^3 e_{235},$$

so we write $-e_{235}$ instead of e_{532}. We can do this to every component of a Clifford element until all of its basis products follow the increasing index rule.

With all these conventions in place, an element x of the Clifford algebra can be described as

$$x = ce + \sum_{a=1}^{n} c_a e_a + \sum_{1 \le a_1 < a_2 \le n} c_{a_1 a_2} e_{a_1 a_2} + ... + c_{1...n} e_{1...n},$$

where the coefficients c, c_A's are in K, and e is the multiplicative identity, 1 in this case. Here the general terms are of the form:

$$c_{a_1 a_2 ... a_k}(e_{a_1 a_2 ... a_k}),$$

with $1 \le a_1 < a_2 < ... < a_k \le n$ and $k \le n$.

Therefore, when $\dim V = n$, we have $\dim \mathrm{Cl}(V, B) = 2^n$.

Example 1.2.2. These are some well-known examples of Clifford algebras.

1. When $K = \mathbb{R}$ and $n = 1$ with $(e_1)^2 = -1$, $\mathrm{Cl}(V, B) = \mathbb{C}$, the complex number. Here $i = e_1$.

2. When $K = \mathbb{R}$ and $n = 2$ with $(e_1)^2 = -1$ and $(e_2)^2 = -1$, $\mathrm{Cl}(V, B)$ is the Quaternion. Here $i = e_1$, $j = e_2$, $k = e_1 e_2$.

While the standard Clifford algebra is based on a vector space over a field, the definition can be extended to cover a module over any unital, associative, commutative ring, and the two versions of definitions are compatible (see [11]). In fact, our main result will concern the Clifford algebra defined on a module over commutative rings. For this we simply change the base structure from a quadratic space to a quadratic R-module.

Definition 1.2.3. Let R be a commutative ring. A *quadratic R-module* (M, B) is an R-module M together with a non-degenerate symmetric bilinear form $B : M \times M \to R$. When there is no confusion about the ring R, we simply refer to (M, B) as a *quadratic module*.

All the definitions regarding the R-module version of bilinear forms, quadratic forms and Clifford algebras can be found in [11] and [13].

Remark 1.2.4. By replacing a quadratic space (V, B) with a quadratic module (M, B), we can redo all the definitions of the Clifford algebra in terms of module. For the sake of conformity, we will carry on this chapter with the quadratic space definition, and start using the module version when we get to the part concerning our main result.

The Clifford algebra has a graded ring structure. First, we have a decomposition

$$\mathrm{Cl}(V, B) = \mathrm{Cl}^+(V, B) \oplus \mathrm{Cl}^-(V, B),$$

where

$$\mathrm{Cl}^+(V, B) = \langle e_A \mid \ e_A \text{ is even } \rangle$$

is the subspace generated by even basis elements, and

$$\mathrm{Cl}^-(V, B) = \langle e_A \mid \ e_A \text{ is odd } \rangle$$

is the subspace generated by odd basis elements.

Further, for $A = a_1a_2...a_n$ with $n \equiv k \mod 4$ we say that e_A is a *basis element of type* $\bar{\boldsymbol{k}}$ and we count empty product as type $\bar{\mathbf{0}}$. Let

$$\mathrm{Cl}^{\bar{\boldsymbol{k}}}(V, B) = \langle e_A \mid \ e_A \text{ of type } \bar{\boldsymbol{k}} \ \rangle$$

denote the subspace of $\mathrm{Cl}(V, B)$ generated by the basis elements of type $\bar{\boldsymbol{k}}$. Then we have a decomposition

$$\mathrm{Cl}(V, B) = \mathrm{Cl}^{\bar{\mathbf{0}}}(V, B) \bigoplus \mathrm{Cl}^{\bar{\mathbf{1}}}(V, B) \bigoplus \mathrm{Cl}^{\bar{\mathbf{2}}}(V, B) \bigoplus \mathrm{Cl}^{\bar{\mathbf{3}}}(V, B).$$

When a Clifford element is inside one of the subspace completely:

$$x \in \mathrm{Cl}^{\bar{\boldsymbol{k}}}(V, B),$$

we say that x is an *element of type* $\bar{\boldsymbol{k}}$. The reason we make this construction is so that their interactions with each involution that we will define next become clearer.

1.2.1 Conjugates

Let $x \in \mathrm{Cl}(V, B)$. We define the three main conjugations on Clifford algebra. Note that the scalar in R are always fixed under these maps.

Definition 1.2.5. First is the main involution, denoted by $\hat{x}$, which sends

$$e_i \mapsto -e_i,$$

wherever it appears in the components of x.

For example, if $x = e_1e_2e_3 + e_3e_4$, then

$$\begin{aligned}\hat{x} &= (-e_1)(-e_2)(-e_3) + (-e_3)(-e_4)\\ &= -e_1e_2e_3 + e_3e_4.\end{aligned}$$

Definition 1.2.6. Secondly, we have the transposition, denoted x^t, which reverses the order of the basis elements in a product, i.e.,

$$e_ie_{i+1}...e_{i+j} \mapsto e_{i+j}e_{i+j-1}...e_i,$$

wherever it appears in the components of x.

For example, if $x = e_1e_2e_3+e_3e_4$, then $x^t = e_3e_2e_1+e_4e_3$. As a convention, we always shuffle the basis product back to the standard form using the anti-commutativity law. Thus in this example, we have

$$x^t = (-1)^3e_1e_2e_3 + -e_3e_4 = -x.$$

Definition 1.2.7. Lastly, we can combine the first two maps to obtain the Clifford conjugation, as the main one we use in this thesis, denoted $\bar{x} = (\hat{x})^t$, which sends

$$e_ie_{i+1}...e_{i+j} \mapsto (-e_{i+j})(-e_{i+j-1})...(-e_i),$$

wherever it appears in the components of x.

For example, if $x = e_1e_2e_3 + e_3e_4$, then

$$\begin{aligned}\bar{x} &= (-e_3)(-e_2)(-e_1) + (-e_4)(-e_3)\\ &= -e_3e_2e_1 + e_4e_3\\ &= (-1)^3(-e_1e_2e_3) - e_3e_4\\ &= e_1e_2e_3 - e_3e_4.\end{aligned}$$

The following table shows the interaction between different conjugations and the subspace generated by each element type.

Type	$\bar{\mathbf{0}}$	$\bar{\mathbf{1}}$	$\bar{\mathbf{2}}$	$\bar{\mathbf{3}}$
$\hat{x}$	+	-	+	-
x^t	+	+	-	-
$\bar{x}$	+	-	-	+

The " + " means that a component does not change sign under the conjugation, and " - " means that the sign changes by multiplying the original term with -1.

In this thesis, we will use the conjugates to define a formula for norm or trace map on Clifford algebras and Spin groups which will aid us in our later calculations.

1.2.2 Lipschitz group

From this point on, we assume that the bilinear form B is non-degenerate and we will write $\mathrm{Cl}(V)$ instead of $\mathrm{Cl}(V, B)$. Let $\mathrm{Cl}(V)^\times$ denote the group of all invertible elements in $\mathrm{Cl}(V)$.

Definition 1.2.8. *The Lipschitz group* $\Omega(V, B) = \Omega(V)$ (or *Clifford group*, or *Clifford-Lipschitz groups*) to be the set of invertible elements $x \in \mathrm{Cl}(V)^\times$ that stabilizes the set of vectors in V under the twisted conjugation $v \mapsto \bar{x}vx^{-1}$. That is

$$\Omega(V) = \{x \in \mathrm{Cl}(V)^\times \mid \forall v \in V,\ \bar{x}vx^{-1} \in V\}.$$

The elements of $\Omega(V)$ are homogeneous in $\mathrm{Cl}(V) = \mathrm{Cl}^+(V) \oplus \mathrm{Cl}^-(V)$ grading, i.e., they are linear combinations of only odd, or only even elements. In fact, one can show that the elements of $\Omega(V)$ are always the products of the form

$$x = v_1 v_2 ... v_k,$$

where $v_1, v_2, ..., v_k \in V$ are non-isotropic vectors. (see Theorem 3.1 in [19]). We can define a following norm on $\Omega(V, B)$:

Definition 1.2.9. A *norm homomorphism* $N : \Omega(V) \to K^\times$ is given by

$$N(x) = \bar{x}x.$$

Note that some authors may define this as $x^t x$ but they differ only by some unit in $K^\times$. Observe that since every element $x \in \Omega(V)$ is of the form $x = v_1 v_2 ... v_k$, this norm always gives scalar in $K^\times$. Also, since elements of K are invariant under the conjugation, we have $N(\lambda x) = \lambda^2 N(x)$ for all $\lambda \in K$. Now, we can define a special subgroup inside $\Omega(V)$ which consists of elements of norm 1.

Definition 1.2.10. The *Pin group*, $\mathrm{Pin}(V)$, is the kernel of the norm homomorphism $N : \Omega(V) \to K^\times$.

$$\mathrm{Pin}(V) := \{x \in \Omega(V) \mid N(x) = 1\}.$$

Its intersection with $\mathrm{Cl}^+(V)$, the even part of the Clifford algebra, is called the *Spin group*, denoted $\mathrm{Spin}(V)$.

$$\mathrm{Spin}(V) := \{x \in \Omega(V) \cap \mathrm{Cl}^+(V) \mid N(x) = 1\}.$$

The Spin group is a part of the Pin group that only consists of even elements. In terms of the product mentioned above, this means that for all $x \in \mathrm{Spin}(V)$, $x = v_1 v_2 ... v_k$, where $v_1, v_2, ... v_k \in V$ are non-isotropic vectors and k is even. Our main results will involve the Spin group of Clifford algebra with $n = 3, 4$.

1.2.3 $Cl_{p,q}(\mathbb{R})$ and $Cl_n(\mathbb{C})$

The structure of a Clifford algebra can be categorized by the dimension of the underlying quadratic space (V, B) and the shape of its associated quadratic form Q_B. Thus the classification of Clifford algebras is simply the classification of quadratic forms. There are well-known results for the cases when $K = \mathbb{R}$ and $K = \mathbb{C}$ where they show remarkable periodical properties. In this subsection, we list a few low-dimensional cases and state the theorems regarding their periodical natures.

$K = \mathbb{R}$

Let V be a quadratic space over $\mathbb{R}$ with an associated quadratic form Q. Every non-degenerate quadratic form on a finite-dimensional real vector space can be written in the standard diagonal form:

$$Q(v) = v_1^2 + ... + v_p^2 - v_{p+1}^2 - ... - v_{p+q}^2,$$

where $p + q = n$ is the dimension of V. The pair of integers (p, q) is call the *signature* of the quadratic form. We call a real vector space V with a quadratic form of signature (p, q) as $\mathbb{R}^{p,q}$ and we will denote the Clifford algebra of $\mathbb{R}^{p,q}$ by $Cl_{p,q}(\mathbb{R})$. When there is no confusion, we will only write $Cl_{p,q}$ for $Cl_{p,q}(\mathbb{R})$.

The following propositions and theorems are well-known results on the periodicity, which can be found in all of the reference texts regarding Clifford algebra.

Proposition 1.2.11. *When $p + q \leq 2$, we have the following isomorphisms:*

$$\begin{aligned}
Cl_{0,1} &\cong \mathbb{C}, \\
Cl_{1,0} &\cong \mathbb{R} \oplus \mathbb{R}, \\
Cl_{0,2} &\cong \mathbb{H}, \\
Cl_{1,1} &\cong M_2(\mathbb{R}), \\
Cl_{2,0} &\cong M_2(\mathbb{R}).
\end{aligned}$$

Here $\mathbb{C}$ and $\mathbb{H}$ are viewed as algebra over $\mathbb{R}$, and $M_2(\mathbb{R})$ is the algebra of 2×2-matrices.

From this point on, we will denote $M_n(K)$ for the algebra of $n \times n$-matrices over a field K. The next proposition is the where the periodic structure comes in.

Proposition 1.2.12. *For all $m, p, q \geq 0$, the following isomorphisms hold:*

$$\begin{aligned}
Cl_{0,m+2} &\cong \mathbb{C}_{m,0} \otimes \mathbb{C}_{0,2}, \\
Cl_{m+2,0} &\cong \mathbb{C}_{0,m} \otimes \mathbb{C}_{2,0}, \\
Cl_{p+1,q+1} &\cong \mathbb{C}_{p,q} \otimes \mathbb{C}_{1,1}.
\end{aligned}$$

For proof, see [10]. These two propositions give rise to the periodicity theorem.

Theorem 1.2.13 (Cartan/Bott)**.** *For all $m \geq 0$, we have that*

$$Cl_{0,m+8} \cong \mathbb{C}_{0,m} \otimes \mathbb{C}_{0,8},$$
$$Cl_{m+8,0} \cong \mathbb{C}_{m,0} \otimes \mathbb{C}_{8,0},$$

Furthermore,

$$Cl_{0,8} \cong \mathbb{C}_{8,0} \cong M_{16}(\mathbb{R}).$$

For the proof, see [10].

$K = \mathbb{C}$

When $K = \mathbb{C}$, the structure of the Clifford algebra becomes much simpler due to the fact that any non-degenerate quadratic form on a finite-dimensional complex vector space is equivalent to the standard diagonal form

$$Q(v) = v_1^2 + v_2^2 + ... + v_n^2.$$

Thus the classification only depends on the dimension n up to isomorphism. We write $Cl_n(\mathbb{C})$ for the Clifford algebra over complex vector space V with dimension $\dim V = n$, or where no confusion may arise, simply $\mathbb{C}l_n$. We have the following periodicity theorem:

Theorem 1.2.14. *For all $m \geq 0$, we have that:*

$$\mathbb{C}l_{m+2} \cong \mathbb{C}l_m \otimes_{\mathbb{C}} \mathbb{C}l_2,$$

and $\mathbb{C}l_2 = M_2(\mathbb{C})$.

In fact, we can use this theorem to work out the structure for any dimension.

Corollary 1.2.15. *The following isomorphisms hold for $m \geq 0$.*

$$\mathbb{C}l_{2m} \cong M_{2^m}(\mathbb{C}),$$
$$\mathbb{C}l_{2m+1} \cong M_{2^m}(\mathbb{C}) \oplus M_{2^m}(\mathbb{C}).$$

For the proofs, see chapter 2 in [19].

Remark 1.2.16. Similar to the cases of $K = \mathbb{R}$ and $\mathbb{C}$, the classification of Clifford algebras over p-adic fields is determined by the dimensions their quadratic spaces and associated the quadratic forms. There is an extensive work regarding all cases over p-adic fields by Bertin Diarra [9].

1.2.4 Tower of Isomorphisms

Before we move on, let us introduce some interesting tricks that one can use to increase the scope of our main result.

We will define two types of isomorphisms between Clifford algebras of different dimensions. We start with some changes to notation. Only in this subsection, we will denote the Clifford algebra generated from a vector space V of dimension n over a field K by $Cl(n, K)$. We omit the notation of quadratic space (V, B) since it will change with each isomorphism in a process that will be explained later. The two types of isomorphism are categorized by the dimension of their starting Clifford algebras.

Type 1 $(n = 2k)$

It is well-known that there exist an isomorphism $Cl^+(n, K) \cong Cl(n-1, K)$ given by the map

$$Cl^+(n, K) \to Cl(n-1, K)\,,\ e_i e_n \mapsto e'_i,$$

for $i = 1, 2, ...n-1$.

This isomorphism maps the even part of Clifford algebra to a Clifford algebra of lower dimension. For example, consider a map from the even part of a quaternion algebra $Cl^+(2, K)$ down to complex algebra $Cl(1, K)$. The element $x \in Cl^+(2, K)$ is of the form

$$x = c_0 + c_1(e_1e_2).$$

We can reassign the basis $e_1e_2 = e'_1$ and view x as

$$x = c_0 + c_1e'_1.$$

This is a structure of Clifford algebra with $n = 1$, or the complex algebra.

Type 2 $(n = 2k + 1)$

There is an isomorphism $Cl(n, K) \cong Cl(n-1, K[W])$, where $K[W]$ is a field extension of K with $W = e_1e_2...e_n \in Cl(n, K)$.

There are many ways to define this isomorphism. We will illustrate with an example. Consider an isomorphism between a biquaternion algebra $Cl(3, K)$ and a quaternion algebra $Cl(2, K[w])$.

Suppose $Cl(3, K)$ is a biquaternion algebra over the field K with an associated quadratic space (V, B), where $\{e_1, e_2, e_3\}$ is the orthogonal basis of V and $e_1^2 = a$, $e_2^2 = b$, $e_3^2 = c$ determine the shape of B. Here $a, b, c \neq 0$. An element $x \in Cl(3, K)$ is of the form

$$x = c_0 + c_1e_1 + c_2e_2 + c_3e_3 + c_4(e_1e_2) + c_5(e_1e_3) + c_6(e_2e_3) + c_7(e_1e_2e_3).$$

Using the type 2 map, we will reassign the largest basis element $e_1e_2e_3 = w$ to a field extension $K[w]$ and treat it as a scalar. We then relabel the basis of the biquaternion as follow:

$$e_1e_2 = e_1' \ , \ e_1e_3 = e_2' \ , \ (-a)(e_2e_3) = e_3'.$$

With this, the rest of the basis become

$$e_1 = \frac{1}{abc}w(e_3') \ , \ e_2 = \frac{1}{ac}w(e_2') \ , \ e_3 = \frac{1}{ab}w(e_1').$$

So the element x can be viewed as

$$x = (c_0 + c_7w) + (c_4 - \frac{c_3}{ab}w)e_1' + (c_5 - \frac{c_2}{ac}w)e_2' + (-\frac{c_6}{a} + \frac{c_1}{abc}w)e_3'.$$

Observe that

$$e_1'e_2' = a(e_2e_3) = -e_3',$$

and the three new basis elements have the quaternion structure over a field extension $K[w]$, where $e_1' = i$, $e_2' = j$, and $e_3' = k$. In this way, we have an isomorphism $Cl(3, K) \to Cl(2, K[w])$.

Note that each time we move between Clifford algebras with these isomorphisms, their associated quadratic spaces change together with the dimensions, and their quadratic forms also change to facilitate the new set of orthogonal basis.

In this example, the new quadratic space (V', B') has $\{e_1', e_2', e_3'\}$ as an orthogonal basis with respect to B', attaining the values

$$(e_1')^2 = -ab \ , \ (e_2')^2 = -ac \ , \ (e_3')^2 = -a^2bc.$$

Observe that the total dimension "over K" goes down by half when passing through the type 1 map, and stays the same when going through type 2 map. Furthermore, we can combine these two types of isomorphisms by alternatively applying them and create an isomorphic from a subgroup of high dimensional Clifford algebra over a field K to a low dimensional Clifford algebra over an extension of K.

Although the calculation can be tedious and the result is limited, this process gives us a way to explore some parts of higher dimensional Clifford algebras by mapping them down to low dimensional Clifford algebras which our the results apply to.

We will finish off this introduction by discussing the setting we use for our main result which involves a Clifford algebra over p-adic rings, where p is odd prime, and state the main theorem.

1.3 Our Goal: The Main Theorem

In this thesis, instead of focusing on the types of Clifford algebras, we will be exploring the classification of their subgroups, specifically, the pro-p subgroups of their spin groups. We will mainly consider the Clifford algebra defined on a module over a ring of integers of p-adic field, but other rings with p-adic structure can work as well. To accomplish this goal, we follow the work of Richard Pink in his paper [23], where he classifies the pro-p subgroups of SL_2 over p-adic rings by mapping them to subgroups of Lie algebra sl_2.

Remark 1.3.1. Note that this matching is not done via the usual exponential map between a Lie group and its associated Lie algebra, but a certain linear map with an inverse. Thus this correspondence is simply a way to keep track of the pro-p subgroups through some additive algebras which, in this case, happens to be the Lie algebra associated to SL_2. In our main result, the additive algebra is also a Lie algebra, but not the one directly associated to the Clifford algebra we consider. We want to stress the point that Pink's method is by no means only limited to linking up a Lie group and its associated Lie algebra.

We found that his method can be adapt to work with some cases of Clifford algebra, namely, the Clifford algebra with $n = 2$, the Quaternion algebra, and the Spin group of Clifford algebras with $n = 3, 4$. The two cases of Spin group expand the original result which is restricted to only 2-by-2 matrices. The following is a short description of our main theorem for the case of $n = 4$. Note that the theorems for $n = 2$ and 3 are similar; we only need to change the setting to a different Spin group or Clifford algebra.

Let $S = Cl(M)$ be a Clifford algebra with $n = 4$ over a finitely generated quadratic R-module M, where M is free, and U its spin group. Let $\mathrm{Cl}^{\bar{\mathbf{2}}}(M)$ be the subspace of type $\bar{\mathbf{2}}$ elements of S, which is a Lie algebra with the commutator $[A, B] = AB - BA$ as the Lie bracket. To see this, consider the following: let $A, B \in \mathrm{Cl}^{\bar{\mathbf{2}}}(M)$. Then

$$\begin{aligned}
[A, B] &= AB - BA \\
&= AB - (-B)(-A) \\
&= AB - (\overline{B})\,(\overline{A}) \\
&= AB - \overline{(AB)}.
\end{aligned}$$

Write

$$AB = F_0 + F_2, \quad \text{where } F_0 \in \mathrm{Cl}^{\bar{\mathbf{0}}}(M) \text{ and } F_2 \in \mathrm{Cl}^{\bar{\mathbf{2}}}(M).$$

Then

$$[A, B] = F_0 + F_2 - (F_0 - F_2) = 2F_2 \in \mathrm{Cl}^{\bar{\mathbf{2}}}(M).$$

Therefore $\mathrm{Cl}^{\bar{\mathbf{2}}}(M)$ is closed under the Lie bracket.

The main result of our thesis is the following theorem.

Theorem. *There is a canonical one-to-one correspondence between all the pro-p subgroups* $\Gamma \subset U$ *and the pairs* (L, Δ) *with following properties:*
(i) L *is a closed additive subgroup of* $\mathrm{Cl}^{\bar{\mathbf{2}}}(M)$ *satisfying*

$$L^2 \equiv 0 \mod I, \text{ which implies } \bigcap_{n=1}^{\infty} L^n = \{0\}. \tag{1}$$

$$[L, L] \subset L. \tag{2}$$

$$\mathrm{tr}(L \cdot L) \cdot L \subset L. \tag{3}$$

(ii) Δ *is a closed subgroup of* $(L/[L,L], *)$ *such that the additive group* $L/[L,L]$ *is topologically generated by the subset* Δ*.*

Here $\mathrm{tr}(x) = \frac{1}{2}(x + \bar{x})$ for $x \in U$ and $\mathrm{tr}(L \cdot L)$ denote the closed additive subgroups of $\mathrm{Cl}^{\bar{\mathbf{2}}}(M)$ generated by the set $\{\mathrm{tr}(x),\, x \in L \cdot L\}$. All the definitions will be accounted for in the next chapter.

As a by-product from the process leading up to the theorems, our main results also provides the isomorphisms between subgroups of Clifford algebra and subgroups of Lie algebra, together with their lower central series, which enable calculations of Clifford elements to be carried out in the Lie algebra side instead. This process allows for an interesting possible application which we will discuss in later chapter. Now let us start on the main topic of our thesis: Pink's method.

Chapter 2

Main Result

2.1 The Original Pink's Method: The Case of SL_2

In [23], Pink establishes the classification of pro-p subgroups of SL_2 over a p-adic ring when p is an odd prime. The method revolves around a linear map $\theta : SL_2 \to sl_2$, defined by $x \mapsto x - \frac{\mathrm{tr}(x)}{2} \cdot Id$, where $\mathrm{tr}(x)$ is the usual trace of square matrices. This map has an inverse given by

$$\psi : sl_2 \to SL_2,\ \psi(y) = y + \sqrt{1 + \mathrm{tr}(y^2)/2}.$$

The core idea of this method can be summarized in steps as follows:

1. First, suppose there exist a Lie subalgebra $L \subset sl_2$ with a certain set of conditions then construct its low central series $L_{n+1} = [L, L_n]$.

2. Define matrix subgroups $H_n \subset SL_2$ as pre-image of L_n by

   $$H_n = \{x \in SL_2 \,|\, \theta(x) \in L_n,\ \mathrm{tr}(x) - 2 \in C\}$$

   where C is the additive subgroup generated by $\mathrm{tr}(L \cdot L)$, and show that the map $H_n \to L_n$, $x \mapsto \theta(x)$ is a homeomorphism.

3. From the opposite direction, pick a pro-p subgroups $\Gamma \subset SL_2$ and send them to sl_2 via θ. Then show that the additive subgroup generated by the resulting images, let us call it $\widetilde{L}$, always satisfies the conditions from the first step.

4. Put everything together by showing that if we start with $\widetilde{L}$ and go through the second step, the resulting $\widetilde{H}$ contains Γ and their lower central series coincide: for $n \geq 2$, $\Gamma_n = \widetilde{H}_n$.

This means that giving a pro-p subgroup of SL_2 is the same as giving an additive subgroup L of sl_2 with certain properties and vice versa.

Interestingly, Pink mentions that his main result cannot be directly translated to higher dimensional cases of SL_n, at least without some changes depending on n. He also gives a conjecture on what they may look like in the paper.

In this thesis, we adapt this method from the ground up to accommodate the Clifford algebra setting while maintaining the core structure of the proof. We obtain similar results for Clifford algebra with $n = 2$ and Spin group with $n = 3, 4$. In a way, this is Pink's method in higher dimensional matrices, but only on subgroups with specific shapes. However, we do not expect our results to coincide with the ones from Pink's conjecture, which deals with the whole matrices algebra, even if such cases were solved in the same dimension. This is due to the fact that the trace and norm we define here are only possible on Clifford algebra and Spin groups we choose, and not the whole matrix space. Thus the rest of the formulas must be drastically different to accommodate the norm and trace of normal matrix when $n > 2$.

We will now start on a part of the main results for Spin Group with $n = 4$. Since they are very similar in nature, the case of Spin group with $n = 3$ and the Quaternion algebra with $n = 2$ will be addressed in reference to this one.

2.2 The Main Result

2.2.1 The Case of Spin Group (n = 4)

Let R be a commutative semi-local ring with identity and I the intersection of its maximal ideals. We give R the I-adic topology, that is, the ideals I^n form a fundamental system of open neighbourhoods of zero. Then we assume that R is complete and compact with respect to this topology and that the quotient R/I is annihilated by an odd prime p. For the rest of this thesis, unless stated otherwise, we let p be an odd prime. The ring R is isomorphic to the inverse limit $\varprojlim R/I^n$. The examples we are interested in are flat algebras of finite degree over $\mathbb{Z}_p$, or over $\mathbb{Z}_p[[X]]$. In particular, we are mainly interested in an integer ring of a p-adic field.

Let (M, B) be a finitely generated quadratic R-module. We assume throughout that M is free, of dimension $n = 4$, and has an orthogonal basis $\{e_1, e_2, e_3, e_4\}$ with respect to B. Then let $S = \mathrm{Cl}(M)$ be a Clifford algebra defined on M, and U its Spin group. For compactness, we will write e_{1234} for $e_1e_2e_3e_4$.

With this setting we can introduce the following formula for a square root of Clifford element: for $\alpha \in I + Ie_{1234}$, where I is the intersection of maximal ideals of R, one can define a continuous map

$$\sqrt{1+\alpha} := \sum_{m=0}^{\infty} \binom{1/2}{m} \alpha^m,$$

which gives a unique solution to $\beta^2 = 1 + \alpha$ with $\beta \equiv 1 \mod I + Ie_{1234}$. To see this, we can use a commutative ring version of Hensel's lemma:

Theorem (Hensel's Lemma)**.** *Let $f(x) \in A[x]$, where A is a nonzero commutative ring complete with respect to a submultiplicative absolute value $|\cdot|$. If $a \in A$ satisfies $|f(a)| < 1$ and $f'(a) \in A^\times$, then there exists a unique $\beta \in A$ such that $f(\beta) = 0$ and $|\beta - a| < 1$.*

We can set $A = R + Re_{1234}$ which is complete with respect to the I-adic valuation. Let $f(x) = x^2 - (1+\alpha)$, where $\alpha \in I + Ie_{1234}$, and use $a = 1 \in A$. Note that $2 \in A^\times$ when p is an odd prime. The theorem gives uniqueness to the square root of $1+\alpha$ which is congruent to $1 \mod I + Ie_{1234}$ as defined above. For the proof of this theorem and further reference, see Theorem 10.5 in [7].

In this $n = 4$ case, we treat the basis element e_{1234} of the Clifford algebra as if it is a scalar to keep in line with the image of a trace map we will define next. It is important to note that we can do this precisely because e_{1234} commutes with everything in the Spin group.

An element $x \in U$ is of the form

$$x = c + c_1(e_1e_2) + c_2(e_1e_3) + c_3(e_1e_4) + c_4(e_2e_3) + c_5(e_2e_4) + c_6(e_3e_4) + c_7(e_{1234}),$$

where $e_1^2 = a_1$, $e_2^2 = a_2$, $e_3^2 = a_3$ and $e_4^2 = a_4$, for some non-zero scalars $a_1, ..., a_4 \in R$ depending on the quadratic form, and $c_1, ..., c_7 \in R$. We denote the conjugate of x by

$$\bar{x} = c - c_1(e_1e_2) - c_2(e_1e_3) - c_3(e_1e_4) - c_4(e_2e_3) - c_5(e_2e_4) - c_6(e_3e_4) + c_7(e_{1234}).$$

Since we are in the Spin group, we can define the following:

Definition 2.2.1. A *Spin norm* is a homomorphism $N : U \to R$ given by

$$N(x) = x\bar{x} \in R,$$

and a *trace* map $\mathrm{tr} : U \to R + Re_{1234}$ is given by

$$\mathrm{tr}(x) = \frac{1}{2}(x + \bar{x}) = c + c_7(e_{1234}).$$

By definition, for all $x \in U$, $x\bar{x} = 1$, and $\mathrm{tr}(x)$ commutes with everything in U. We now introduce the most crucial part of this method.

Definition 2.2.2. Define a map $\theta : U \to \mathrm{Cl}^{\bar{\mathbf{2}}}(M)$ by

$$\theta(x) = x - \mathrm{tr}(x) = c_1e_1e_2 + c_2e_1e_3 + c_3e_1e_4 + c_4e_2e_3 + c_5e_2e_4 + c_6e_3e_4.$$

This map sends element of U into a certain Lie algebra in the set of unit Clifford elements, which we will denote $\mathrm{Cl}^{\bar{\mathbf{2}}}(M)$, with the commutator $[A, B] = AB - BA$ as the Lie bracket. The $\mathrm{Cl}^{\bar{\mathbf{2}}}(M)$ is a subspace of elements of type $\bar{\mathbf{2}}$ in S. One can verify the following lemma by direct calculation. For clarity, we will use "$\cdot$" to denote a multiplication between a trace, which behaves like a scalar, and a Clifford element. In these instances, the trace commutes freely with any Clifford element.

Lemma 2.2.3. *Let x, $y \in U$. The following identities hold*

$$[x, y] = [\theta(x), y] = [x, \theta(y)] = [\theta(x), \theta(y)]. \tag{2.2.3.1}$$

$$[x, y] = \theta(xy) - \theta(yx). \tag{2.2.3.2}$$

$$2\,\theta(xy) = [\theta(x), \theta(y)] + 2\,\mathrm{tr}(x) \cdot \theta(y) + 2\,\mathrm{tr}(y) \cdot \theta(x). \tag{2.2.3.3}$$

$$\mathrm{tr}(xy) = \mathrm{tr}(\theta(x)\theta(y)) + \mathrm{tr}(x)\,\mathrm{tr}(y). \tag{2.2.3.4}$$

$$\mathrm{tr}(xy) = \mathrm{tr}(yx). \tag{2.2.3.5}$$

$$\mathrm{tr}(x)^2 = 1 + \mathrm{tr}(\theta(x)^2). \tag{2.2.3.6}$$

$$\mathrm{tr}(x^{-1}) = \mathrm{tr}(x)\,,\ \theta(x^{-1}) = -\theta(x). \tag{2.2.3.7}$$

$$2\,\mathrm{tr}(x) \cdot \theta(y) = \theta(xy) + \theta(x^{-1}y). \tag{2.2.3.8}$$

$$x, y \in \mathrm{Cl}^{\bar{\mathbf{2}}}(M) \implies 2\,\mathrm{tr}(xy) = xy + yx. \tag{2.2.3.9}$$

Further for $x, y, u, v \in \mathrm{Cl}^{\bar{\mathbf{2}}}(M)$, we have

$$\begin{aligned} 4\,\mathrm{tr}(xy) \cdot [u, v] = &[y, [x, [u, v]]] + [x, [y, [u, v]]] \\ &+ [[x, v], [u, y]] + [[y, v], [u, x]]. \end{aligned} \tag{2.2.3.10}$$

$$\begin{aligned} 4\,\mathrm{tr}([u, v]\,x) \cdot y = &[y, [x, [u, v]]] - [x, [y, [u, v]]] \\ &+ [[x, v], [u, y]] + [[y, v], [u, x]]. \end{aligned} \tag{2.2.3.11}$$

Proof. For the proof of this proposition, we will denote

$$\begin{aligned} x &= x_0 + x_2, \quad \text{where } x_0 \in \mathrm{Cl}^{\bar{\mathbf{0}}}(M) \text{ and } x_2 \in \mathrm{Cl}^{\bar{\mathbf{2}}}(M), \\ y &= y_0 + y_2, \quad \text{where } y_0 \in \mathrm{Cl}^{\bar{\mathbf{0}}}(M) \text{ and } y_2 \in \mathrm{Cl}^{\bar{\mathbf{2}}}(M). \end{aligned}$$

Consider the following calculations:

(2.2.3.1): By the commutativity of trace,

$$\begin{aligned} [\theta(x), \theta(y)] &= (x - \mathrm{tr}(x))(y - \mathrm{tr}(y)) - (y - \mathrm{tr}(y))(x - \mathrm{tr}(x)) \\ &= xy - \mathrm{tr}(x) \cdot y - x \cdot \mathrm{tr}(y) + \mathrm{tr}(x)\,\mathrm{tr}(y) \\ &\quad - yx + y \cdot \mathrm{tr}(x) + \mathrm{tr}(y) \cdot x - \mathrm{tr}(x)\,\mathrm{tr}(y) \\ &= xy - yx \\ &= [x, y]. \end{aligned}$$

Without loss of generality,

$$\begin{aligned}
[\theta(x), y] &= (x - \operatorname{tr}(x))(y) - (y)(x - \operatorname{tr}(x)) \\
&= xy - \operatorname{tr}(x) \cdot y - yx + y \cdot \operatorname{tr}(x) \\
&= xy - yx \\
&= [x, y].
\end{aligned}$$

(2.2.3.5): First we claim that $\operatorname{tr}(xy) = \operatorname{tr}(\bar{x}\bar{y})$.

To prove the claim,

$$\begin{aligned}
\operatorname{tr}(xy) &= \operatorname{tr}(x_0y_0 + x_0y_2 + x_2y_0 + x_2y_2) \\
&= x_0y_0 + \operatorname{tr}(x_2y_2)
\end{aligned}$$

Since $\bar{x} = x_0 - x_2$,

$$\begin{aligned}
\operatorname{tr}(\bar{x}\bar{y}) &= \operatorname{tr}(x_0y_0 - x_0y_2 - x_2y_0 + x_2y_2) \\
&= x_0y_0 + \operatorname{tr}(x_2y_2) \\
&= \operatorname{tr}(xy)
\end{aligned}$$

Thus the claim holds. Now consider

$$\begin{aligned}
\operatorname{tr}(xy) &= \operatorname{tr}(\overline{(xy)}) \\
&= \operatorname{tr}(\bar{y}\bar{x}) \\
&= \operatorname{tr}(yx)
\end{aligned}$$

(2.2.3.2): Since $\operatorname{tr}(xy) = \operatorname{tr}(yx)$ from (2.2.3.5),

$$\begin{aligned}
[\theta(x), \theta(y)] &= xy - yx \\
&= xy - \operatorname{tr}(xy) - yx + \operatorname{tr}(xy) \\
&= xy - \operatorname{tr}(xy) - (yx - \operatorname{tr}(yx)) \\
&= \theta(xy) - \theta(yx).
\end{aligned}$$

(2.2.3.8): Since $x^{-1} = \bar{x}$ and $\operatorname{tr}(\operatorname{tr}(x) \cdot y) = \operatorname{tr}(x)\operatorname{tr}(y)$,

$$\begin{aligned}
\theta(xy) + \theta(x^{-1}y) &= \theta(xy + \bar{x}y) \\
&= 2\,\theta(\frac{1}{2}(x + \bar{x})y) \\
&= 2\,\theta(\operatorname{tr}(x) \cdot y) \\
&= 2(\operatorname{tr}(x) \cdot y - \operatorname{tr}(x)\operatorname{tr}(y)) \\
&= 2\operatorname{tr}(x) \cdot (y - \operatorname{tr}(y)) \\
&= 2\operatorname{tr}(x) \cdot \theta(y).
\end{aligned}$$

(2.2.3.3): From (2.2.3.8),

$$\begin{aligned}2\,\mathrm{tr}(x)\cdot\theta(y)+2\,\mathrm{tr}(y)\cdot\theta(x) &= \theta(xy)+\theta(x^{-1}y)+\theta(yx)+\theta(y^{-1}x)\\ &= xy+yx-2\,\mathrm{tr}(xy),\end{aligned}$$

since $\theta(x^{-1}y)=\theta((y^{-1}x)^{-1})=-\theta(y^{-1}x)$ by (2.2.3.7).

But

$$xy+yx-2\,\mathrm{tr}(xy)=2\,\theta(xy)-[\theta(x),\theta(y)].$$

Therefore

$$2\,\mathrm{tr}(x)\cdot\theta(y)+2\,\mathrm{tr}(y)\cdot\theta(x)=2\,\theta(xy)-[\theta(x),\theta(y)].$$

(2.2.3.4):

$$\begin{aligned}\mathrm{tr}(xy) &= \mathrm{tr}(x_0y_0+x_0y_2+x_2y_0+x_2y_2)\\ &= x_0y_0+\mathrm{tr}(x_2y_2)\\ &= \mathrm{tr}(\theta(x)\theta(y))+\mathrm{tr}(x)\,\mathrm{tr}(y).\end{aligned}$$

(2.2.3.6): Since $\theta(x)\in\mathrm{Cl}^{\bar{\mathbf{2}}}(M)$, $\overline{\theta(x)}=-\theta(x)$. Therefore,

$$\begin{aligned}\mathrm{tr}(\theta(x)^2) &= \frac{1}{2}(\theta(x)^2+\overline{(\theta(x)^2)}\,)\\ &= \frac{1}{2}(\theta(x)^2+(\overline{\theta(x)})^2\,)\\ &= \frac{1}{2}(\theta(x)^2+(-\theta(x))^2)\\ &= \theta(x)^2.\end{aligned}$$

We have that

$$\begin{aligned}\mathrm{tr}(x)^2-\mathrm{tr}(\theta(x)^2) &= \mathrm{tr}(x)^2-\theta(x)^2\\ &= x_0^2-x_2^2\\ &= (x_0+x_2)(x_0-x_2)\\ &= (x)(\bar{x})\\ &= 1.\end{aligned}$$

(2.2.3.7): First part,

$$\begin{aligned}\mathrm{tr}(x^{-1}) &= \mathrm{tr}(\bar{x})\\ &= \mathrm{tr}(x_0-x_2)\\ &= x_0\\ &= \mathrm{tr}(x).\end{aligned}$$

Second part,

$$\begin{aligned}\theta(x^{-1}) &= \theta(\bar{x}) \\ &= \theta(x_0 - x_2) \\ &= -x_2 \\ &= -\theta(x).\end{aligned}$$

(2.2.3.9): Since $\bar{x} = -x$ for all $x \in \mathrm{Cl}^{\bar{\mathbf{2}}}(M)$,

$$\begin{aligned}2 \cdot \mathrm{tr}(xy) &= xy + \overline{(xy)} \\ &= xy + (\bar{y})(\bar{x}) \\ &= xy + yx.\end{aligned}$$

(2.2.3.10) and (2.2.3.11) can be verified with direct calculation.

□

Let us declare the following notations that we will use throughout this chapter. For closed additive subgroups $L, L' \subset \mathrm{Cl}^{\bar{\mathbf{2}}}(M)$ and a subset $C \subset R + Re_{1234}$, we will write $L \cdot L'$, $[L, L']$, $\mathrm{tr}(L)$, C^n, $C \cdot L$ to denote the closed additive subgroups of $Cl(M)$ generated by the the elements from the respective sets.

From Lie Algebra to Clifford Algebra

We fix an additive subgroup $L \subset \mathrm{Cl}^{\bar{\mathbf{2}}}(M)$ and let $C := \mathrm{tr}(L \cdot L) \subset R + Re_{1234}$. We will assume the following axioms about L, which we will later prove to be true for our particular cases. The axioms are:

$$L^2 \equiv 0 \mod I, \text{ which implies } \bigcap_{n=1}^{\infty} L^n = \{0\}. \tag{1}$$

$$[L, L] \subset L. \tag{2}$$

$$C \cdot L \subset L. \tag{3}$$

These axioms implies the following:

Proposition 2.2.4.

$$\bigcap_{n=1}^{\infty} C^n = \{0\}, \textit{ and } C \subset I + Ie_{1234}. \tag{2.2.4.1}$$

$$C \cdot C \subset C. \tag{2.2.4.2}$$

$$L \textit{ is a Lie subalgebra of } \mathrm{Cl}^{\bar{\mathbf{2}}}(M). \tag{2.2.4.3}$$

Proof. (2.2.4.1): From (2.2.3.9), for $x, y \in L$, $2\,\mathrm{tr}(xy) = xy + yx \in L^2$. This means $C \subset L^2$, and from axiom (1),

$$\bigcap_{n=1}^{\infty} C^n \subseteq \bigcap_{n=1}^{\infty} L^n = \{0\}.$$

By definition 2.2.1 and the fact that L^2 is contained in I, $C = \mathrm{tr}(L \cdot L) \subset I + Ie_{1234}$.

(2.2.4.2): The second assertion follows from axiom (3) since

$$C \cdot C = C \cdot \mathrm{tr}(L \cdot L) = \mathrm{tr}(C \cdot L \cdot L) \subset \mathrm{tr}(L \cdot L) = C.$$

(2.2.4.3): The last assertion follows directly from axiom (2). □

We define a descending sequence of closed additive subgroups by

$$L_1 = L \ , \ L_{n+1} = [L, L_n] \text{ for all } n \geq 1.$$

In the next proposition, we show that these L_n's are Lie subalgebras with certain properties.

Proposition 2.2.5.

$$\bigcap_{n=1}^{\infty} L_n = \{0\}. \tag{2.2.5.1}$$

$$L_{n+1} \subset L_n \ \textit{for all } n \geq 1. \tag{2.2.5.2}$$

$$[L_n, L_m] \subset L_{n+m} \ \textit{for all } n\,,\, m \geq 1. \tag{2.2.5.3}$$

$$C \cdot L_n \subset L_{n+2} \ \textit{for all } n \geq 2. \tag{2.2.5.4}$$

Proof. (2.2.5.1): The first assertion follows from axiom (1) since $L_n \subset L^n$ for all n.

(2.2.5.2): The second assertion follows from an induction on n:

$$L_{n+2} = [L, L_{n+1}] \subset [L, L_n] = L_{n+1},$$

and the case $n = 1$ follows from (2.2.4.3).

(2.2.5.3): For the third assertion, we use induction on n: the base case $n = 1$ is true for all m by the definition of L_m. Let $x \in L$, $y \in L_n$, and $z \in L_m$. For inductive step, the Jacobi identity and induction hypothesis imply that for all m and any $[[x, y], z] \in [L_{n+1}, L_m]$,

$$[[x, y], z] = [x, [y, z]] - [y, [x, z]] \in [L, [L_n, L_m]] + [L_n, [L, L_m]] \subset L_{n+m+1},$$

Note that when $n = m$, $[L_n, L_n] \subset L_{2n} \subset L_n$. Thus L_n is Lie subalgebra of L.

(2.2.5.4): The last assertion is, again, proved by induction on n: let $x, y, u \in L$ and $v \in L_{n-1}$, the base case $n = 2$ follows from the formula of (1.1.10):

$$\begin{aligned} 4\operatorname{tr}(xy) \cdot [u, v] &= [y, [x, [u, v]]] + [x, [y, [u, v]]] + [[x, v], [u, y]] + [[y, v], [u, x]] \\ &\in L_{n+2} \end{aligned}$$

Using the inductive hypothesis and definition of L_n, we obtain

$$C \cdot L_{n+1} = C \cdot [L, L_n] = [L, C \cdot L_n] \subset [L, L_{n+2}] = L_{n+3}.$$

$\square$

Next we define a class of subgroups of U in relation to L_n. For all $n \geq 1$,

$$H_n := \{x \in U \mid \theta(x) \in L_n,\ \operatorname{tr}(x) - 1 \in C\}.$$

These subgroups have the following properties.

Proposition 2.2.6.

$$\bigcap_{n=1}^{\infty} H_n = \{1\}. \tag{2.2.6.1}$$

For all $n \geq 1$,

$$H_{n+1} \subset H_n. \tag{2.2.6.2}$$

$$\textit{The map } H_n \to L_n,\ x \mapsto \theta(x) \textit{ is a homeomorphism.} \tag{2.2.6.3}$$

$$H_n \textit{ is a pro-}p \textit{ subgroup of } U. \tag{2.2.6.4}$$

$$H_n \textit{ is normalized by } H_1. \tag{2.2.6.5}$$

Proof. (2.2.6.3):

We start with the third assertion. Consider a continuous map

$$\psi : L_n \to H_n,\ \psi(y) := y + \sqrt{1 + \operatorname{tr}(y^2)}.$$

By the definition of C and (2.2.4.1), $\operatorname{tr}(y^2) \in C \subset I + Ie_{1234}$, thus the definition of a square root explained at the beginning of this section applies. Let $y \in L_n$. Since $\sqrt{1 + \operatorname{tr}(y^2)} \in R + Re_{1234}$, the map has image in U because

$$\begin{aligned} \psi(y)\overline{\psi(y)} &= (y + \sqrt{1 + \operatorname{tr}(y^2)})(-y + \sqrt{1 + \operatorname{tr}(y^2)}) \\ &= 1 + \operatorname{tr}(y^2) - y^2 \\ &= 1. \end{aligned}$$

The last equality comes from the fact that $\operatorname{tr}(y^2) = y^2$ for all $y \in \mathrm{Cl}^{\bar{\mathbf{2}}}(M)$.

Notice that $\sqrt{1+\operatorname{tr}(y^2)} \in \mathrm{Cl}^{\bar{\mathbf{0}}}(M)$ and $y \in \mathrm{Cl}^{\bar{\mathbf{2}}}(M)$. By the definition of the square root, $\operatorname{tr}(\psi(y)) = \sqrt{1+\operatorname{tr}(y^2)} \equiv 1 \mod I + Ie_{1234}$. Therefore $\operatorname{tr}(\psi(y)) - 1 \in C$. It is easy to see that

$$\theta(\psi(y)) = \psi(y) - \sqrt{1+\operatorname{tr}(y^2)} = y \in L_n,$$

and hence $\psi(y) \in H_n$.

Lastly, to check that ψ is the inverse of θ, consider for $x \in H_n$,

$$\begin{aligned}\psi(\theta(x)) &= \theta(x) + \sqrt{1+\operatorname{tr}(\theta(x)^2)} \\ &= \theta(x) + \operatorname{tr}(x) \\ &= x - \operatorname{tr}(x) + \operatorname{tr}(x) \\ &= x.\end{aligned}$$

The second equation is true because of (2.2.3.6): $\operatorname{tr}(x)^2 = 1 + \operatorname{tr}(\theta(x)^2)$, and the uniqueness of the square root explained at the beginning of 2.2.1. Both $\operatorname{tr}(x)$ and $\sqrt{1+\operatorname{tr}(\theta(x)^2)}$ are square roots of $1 + \operatorname{tr}(\theta(x)^2)$ which are congruent to 1 modulo $I + Ie_{1234}$, by the definition of H_n, and the way we define square root, respectively. Thus they must be equal since such a square root is unique.

(2.2.6.1): The first assertion follows from (2.2.5.1) and the third assertion shown above:

$$\bigcap_{n=1}^{\infty} L_n = \{0\} \implies \bigcap_{n=1}^{\infty} H_n = \{1\}.$$

(2.2.6.2): The second assertion is a direct consequence of (2.2.5.2):

$$L_{n+1} \subset L_n \implies H_{n+1} \subset H_n.$$

(2.2.6.4): For the fourth assertion, we first show that H_n is a subgroup.

Let $x, y \in H_n$. We know from (2.2.3.7) that $\theta(x^{-1}) = -\theta(x)$, therefore $x^{-1} \in H_n$. For the product, consider by (2.2.3.3) and Proposition 2.2.5,

$$\begin{aligned}2\,\theta(xy) &= [\theta(x), \theta(y)] + 2\ \operatorname{tr}(x) \cdot \theta(y) + 2\ \operatorname{tr}(y) \cdot \theta(x) \\ &\in [L_n, L_n] + L_n + C \cdot L_n \\ &\subset L_n.\end{aligned}$$

All that remains is to show that H_n is pro-p. Notice that H_n can be thought of as a module over a pro-finite ring R. By taking modulo I on all the coefficients of components in $x \in H_n$, the resulting element has the form

$$x = 1 + c_1'e_1e_2 + c_2'e_1e_3 + c_3'e_1e_4 + c_4'e_2e_3 + c_5'e_2e_4 + c_6'e_3e_4 + c_7'e_{1234},$$

where $c_i' \in R/I \cong \mathbb{F}_p$ for all $i = 1, 2, ..., 7$. Thus this quotient of H_n has the same cardinality as a subgroup of a finite dimensional vector space over $\mathbb{F}_p$, which can only be a power of p. Since I is the maximal ideal of R, we must have that H_n is pro-p.

(2.2.6.5): For the last assertion, we need to prove that for $x \in H_1$ and $y \in H_n$, we always have $\theta(xyx^{-1}) \in L_n$. Consider the following calculation:

$$\begin{aligned}
2 \cdot \theta(xyx^{-1}) &= 2 \cdot \theta(y + (xy - yx) \cdot x^{-1}) \\
&= 2 \cdot \theta(y) + \theta([x, y] \cdot x^{-1}) \\
&= 2 \cdot \theta(y) + [\theta([x, y]), \theta(x^{-1})] + \operatorname{tr}(x^{-1}) \cdot \theta([x, y]) \quad \text{(by 2.2.3.3)} \\
&= 2 \cdot \theta(y) + [[\theta(x), \theta(y)], \theta(x^{-1})] + \operatorname{tr}(x^{-1}) \cdot [\theta(x), \theta(y)] \quad \text{(by 2.2.3.1)} \\
&\in L_n + [[L, L_n], L] + (1 + C) \cdot [L, L_n] \\
&\in L_n \quad \text{(by Prop 2.2.5).}
\end{aligned}$$

This conclude the prove of Proposition 2.2.6. $\square$

In the next step, we want to observe the group structure of H_n/H_{n+1}. From this point on, we will make another convention: for $x \in H_n$ (or L_n), we denote its residue class in H_n/H_{n+1} (or L_n/L_{n+1}) respectively, by $\left|x\right|$.

Proposition 2.2.7. *(i) For all $n \geq 2$ the map*

$$H_n/H_{n+1} \to L_n/L_{n+1}, \quad \left|x\right| \mapsto \left|\theta(x)\right|$$

is a well-defined bicontinuous group isomorphism. In particular, H_n/H_{n+1} is abelian.

(ii) For all $x \in H_1$ and $y \in H_{n-1}$, we have that

$$\left|\theta(xyx^{-1}y^{-1})\right| = \left|[\theta(x), \theta(y)]\right| \tag{2.2.7.1}$$

in L_n/L_{n+1}, for all $n \geq 2$.

Proof. (i): For the first assertion, let x, $y \in H_n$. The formula from (2.2.3.3) gives us

$$2\,\theta(xy) = [\theta(x), \theta(y)] + 2 \operatorname{tr}(x) \cdot \theta(y) + 2 \operatorname{tr}(y) \cdot \theta(x).$$

Therefore

$$\begin{aligned}
2\,(\theta(xy) - \theta(x) - \theta(y)) &= [\theta(x), \theta(y)] + 2\,(\operatorname{tr}(x) - 1) \cdot \theta(y) + 2\,(\operatorname{tr}(y) - 1) \cdot \theta(x) \\
&\in [L_n, L_n] + C \cdot L_n \\
&\subset L_{n+1}. \quad \text{(by Prop 2.2.5)}
\end{aligned}$$

This implies that the map $H_n \to L_n/L_{n+1}$ defined by $x \mapsto \left|\theta(x)\right|$ is a group homomorphism. It is easy to see that H_{n+1} is its kernel because

$$x \in H_{n+1} \iff \theta(x) \in L_{n+1}.$$

By (2.2.6.3), this map is also surjective. Hence the map

$$H_n/H_{n+1} \to L_n/L_{n+1}, \ \ \big|x\big| \mapsto \big|\theta(x)\big|$$

is a well-defined group isomorphism. The abelian condition is obvious. The continuity condition follows from (2.2.6.3), and the fact that we have the standard quotient topology on H_n/H_{n+1}. This proves the first assertion.

(ii): For the second assertion, let $x \in H_1$ and $y \in H_{n-1}$, $n \geq 2$. Consider the following calculation

$$\begin{aligned} 2\,\theta(xyx^{-1}y^{-1}) &= 2\,\theta(1 + (xy - yx)(x^{-1}y^{-1})) \\ &= 2\,\theta([x,y](x^{-1}y^{-1})) \\ &= [\theta([x,y]), \theta(x^{-1}y^{-1})] + 2\,\mathrm{tr}(x^{-1}y^{-1}) \cdot \theta([x,y]) \ \ \text{(by 2.2.3.3)} \\ &= [[\theta(x), \theta(y)], \theta(x^{-1}y^{-1})] + 2\,\mathrm{tr}(x^{-1}y^{-1}) \cdot [\theta(x), \theta(y)]. \ \text{(by 2.2.3.1)} \end{aligned}$$

Thus we have the following difference:

$$\begin{aligned} &2\,(\theta(xyx^{-1}y^{-1}) - [\theta(x), \theta(y)]) \\ &= [[\theta(x), \theta(y)], \theta(x^{-1}y^{-1})] + 2(\mathrm{tr}(x^{-1}y^{-1}) - 1) \cdot [\theta(x), \theta(y)] \\ &\in [[L, L_{n-1}], L] + C \cdot [L, L_{n-1}] \\ &\subset L_{n+1}. \ \ \text{(by Prop 1.3)} \end{aligned}$$

This implies that $\Big|\theta(xyx^{-1}y^{-1})\Big| = \Big|[\theta(x), \theta(y)]\Big|$ in L_n/L_{n+1}, and we are done. □

Next we will to define a new binary operation on L_1/L_2. Keep in mind that here $x \in L_1/L_2$ still means $x + L_2$ where $x \in L_1$. First we start on $L_1 = L$.

Definition 2.2.8. We define a composition law $* : L \times L \to L$ by

$$x * y := (\sqrt{1 + \mathrm{tr}(x^2)}) \cdot y + (\sqrt{1 + \mathrm{tr}(y^2)}) \cdot x$$

The peculiar formula of this map will be explained in the proof of the following proposition.

Proposition 2.2.9. *(i) The induced map on the quotient*

$$L_1/L_2 \times L_1/L_2 \to L_1/L_2, (\big|x\big|, \big|y\big|) \mapsto \big|x * y\big|$$

is a well-defined composition law on the set L_1/L_2, which we will also denote by "$*$". Furthermore, the set $(L_1/L_2, *)$ is an abelian pro-p group with identity $\big|0\big|$.

(ii) The map

$$H_1/H_2 \to (L_1/L_2, *), \ \ \big|x\big| \mapsto \big|\theta(x)\big|$$

from the previous proposition is a well-defined bicontinuous group isomorphism onto $(L_1/L_2, *)$. In particular, H_1/H_2 is abelian.

Proof. (i): First let us address the formula for " $*$ " with the claim:

$$\left|\theta(xy)\right| = \left|\theta(x) * \theta(y)\right|$$

in L_1/L_2, for all x, $y \in H_1$. To prove this claim, one simply need to consider the identity (2.2.3.3),

$$2 \cdot \theta(xy) = [\theta(x), \theta(y)] + 2\,\mathrm{tr}(x) \cdot \theta(y) + 2\,\mathrm{tr}(y) \cdot \theta(x).$$

The first term on the right hand side is in L_2 by definition, and the formula of the inverse map ψ gives

$$\mathrm{tr}(x) = \sqrt{1 + \mathrm{tr}(\theta(x)^2)}.$$

So, the definition of " $*$ " gives

$$\theta(x) * \theta(y) = \mathrm{tr}(x) \cdot \theta(y) + \mathrm{tr}(y) \cdot \theta(x).$$

Putting them together we have the difference

$$2\,\theta(xy) - 2\,(\theta(x) * \theta(y)) = [\theta(x), \theta(y)] \in L_2,$$

and the claim is proved.

Next we will prove that " $*$ " is a well-defined composition law on L_1/L_2. Without loss of generality, by the symmetry of the formula, we only need to show that for all x, $y \in L_1$ and $u \in L_2$,

$$((x + u) * y) - (x * y) \in L_2.$$

By definition the difference is equal to

$$\left(\sqrt{1 + \mathrm{tr}((x+u)^2)} - \sqrt{1 + \mathrm{tr}(x^2)}\right) \cdot y - \left(\sqrt{1 + \mathrm{tr}(y^2)}\right) \cdot u.$$

The definition of C and the square root tell us that

$$\left(\sqrt{1 + \mathrm{tr}(y^2)}\right) \cdot u \in (C + 1) \cdot L_2,$$

and hence it is in L_2 by Proposition 2.2.5.

We need to show that the term in front of y:

$$\left(\sqrt{1 + \mathrm{tr}((x+u)^2)} - \sqrt{1 + \mathrm{tr}(x^2)}\right)$$

maps L_1 to L_2 when multiplying with y. To do this we can multiply it with the invertible element

$$\left(\sqrt{1 + \mathrm{tr}((x+u)^2)} + \sqrt{1 + \mathrm{tr}(x^2)}\right).$$

By the commutativity of trace and the definition of square root, it behaves like a scalar.

The term becomes

$$\begin{aligned}\mathrm{tr}((x+u)^2) - \mathrm{tr}(x^2) &= \mathrm{tr}(u^2 + 2xu) \\ &= \mathrm{tr}((u+2x)u) \\ &= \mathrm{tr}(u(u+2x)) \ \text{ by (2.2.3.5)}\end{aligned}$$

To show that this maps L_1 to L_2, we use the formula (2.2.3.11) and Proposition 2.2.5:

$$\begin{aligned}\mathrm{tr}(u(u+2x)) \cdot u &\in \mathrm{tr}(L_2 \cdot L_1) \cdot L_1 \\ &\subset L_4 \\ &\subset L_2.\end{aligned}$$

This is enough to show that " $*$ " is well-defined.

Next we want to show that it makes $(L_1/L_2, *)$ into an abelian group with identity $|0|$. Recall that the map

$$H_1 \to L_1/L_2,\ x \mapsto |\theta(x)|$$

is surjective by (2.2.6.3). The fact that H_1 is a group together with the claim we established above:

$$|\theta(xy)| = |\theta(x) * \theta(y)|,$$

implies that $(L_1/L_2, *)$ satisfies all the group axioms. It is easy to see from the formula of " $*$ " that it is abelian and $|0|$ is the identity. All that is left in the first assertion is to show that L_1/L_2 is pro-p. This will follow from the second assertion.

(ii): For the second assertion, we already established that the map

$$H_1 \to L_1/L_2,\ x \mapsto |\theta(x)|$$

is a continuous surjective homomorphism, and that its kernel is H_2. So we have a group isomorphism,

$$H_1/H_2 \to (L_1/L_2), |x| \mapsto |\theta(x)|.$$

However, L_1/L_2 here is still an additive group. By using our claim once again, we can consider the same group isomorphism

$$H_1/H_2 \to (L_1/L_2, *), |x| \mapsto |\theta(x)|,$$

but now with $(L_1/L_2, *)$ as a group in " $*$ " instead of addition. By the quotient topology on H_1/H_2, the map and its inverse are continuous. Finally, we know that H_1/H_2 is pro-p from (2.2.6.4), thus so is $(L_1/L_2, *)$ and we are done.

$\square$

Next consider a closed subgroup $\Gamma \subset H_1$ with the following property:

$$\text{The } \textit{additive} \text{ group } L_1/L_2 \text{ is topologically generated by the image } \theta(\Gamma) \tag{4}$$

The main result of this section is in determining the descending central series of Γ in relation to H_n. Define $\Gamma_1 = \Gamma$ and $\Gamma_{n+1} = (\overline{[\Gamma, \Gamma_n]})^{\boldsymbol{c}}$, the closure of the commutator subgroup $[\Gamma, \Gamma_n]$, for all $n \geq 1$. Then we have the following theorem.

Lemma 2.2.10. *For all $n \geq 2$, we have $\Gamma_n = H_n$.*

Proof. The key to this proof is the commutator relation (2.2.7.1):

$$\left|\theta(xyx^{-1}y^{-1})\right| = \left|[\theta(x), \theta(y)]\right|,$$

for all $x \in H_1$ and $y \in H_{n-1}$. First we will prove that $\Gamma_n \subset H_n$ using induction on n. For $n = 1$, it is true by our assumption of Γ. Suppose this holds for n, the relation (2.2.7.1) implies that

$$[\Gamma, \Gamma_n] \subset [H, H_n] \subset H_{n+1}.$$

But since H_{n+1} is close,

$$\Gamma_{n+1} = (\overline{[\Gamma, \Gamma_n]})^{c} \subset H_{n+1}$$

and the inductive step is done.

For the reverse inclusion $H_n \subset \Gamma_n$, first recall (2.2.6.1):

$$\bigcap_{n=1}^{\infty} H_n = \{1\}.$$

We fix $n \geq 2$. By the closeness of Γ_n,

$$\Gamma_n = \bigcap_{m \geq n}^{\infty} (\Gamma_n \cdot H_m).$$

Hence in order to obtain $H_n \subset \Gamma_n$, it is suffice to show that for all $m \geq n$,

$$\Gamma_n \cdot H_m = \Gamma_n \cdot H_{m+1}.$$

From Proposition 2.2.7, we have an isomorphism

$$H_n/H_{n+1} \to L_n/L_{n+1}, \ \ \left|x\right| \mapsto \left|\theta(x)\right|.$$

By the definition of H_n and L_n, the statement above is equivalent to

$$L_m = \theta(\Gamma_n \cap H_m) + L_{m+1},$$

for all $m \geq n$. Since Γ_n contains Γ_{n+1}, we only need to show the extremal case when $n = m$. Since we already shown $\Gamma_n \subset H_n$, the statement becomes

$$L_m = \theta(\Gamma_m) + L_{m+1},$$

for all $m \geq 2$. But we know from Proposition 2.2.7 that

$$\theta(\Gamma_m) + L_{m+1} \subset L_m.$$

Thus we are done after we prove the following claim:

For all $m \geq 1$, the additive subgroup L_m/L_{m+1} is topologically generated by the image of $\theta(\Gamma_m)$.

First, we introduce a new notation. Let Δ_m denote the closed additive subgroup of L_m/L_{m+1} generated by $\theta(\Gamma_m)$. We will prove that $\Delta_m = L_m/L_{m+1}$ by induction of m. For $m = 1$, this is our assumption of Γ. Suppose that the claim holds for m. The relation (2.2.7.1) implies that

$$\theta([\Gamma_1, \Gamma_m]) \equiv [\theta(\Gamma_1), \theta(\Gamma_m)] \mod L_{m+2}.$$

By the continuity of θ, the closure of the left hand side is just $\theta(\Gamma_{m+1})$. Since Δ_{m+1} is generated by $\theta(\Gamma_{m+1})$, we can also describe it using the right hand side of the congruence.

Observe that by (2.2.5.3), we can use the commutator and the congruence above to induce a well-defined continuous bilinear pairing

$$(L_1/L_2) \times (L_m/L_{m+1}) \to L_{m+1}/L_{m+2},\ (\big|u\big|, \big|v\big|) \mapsto \big|[u, v]\big|.$$

Thus we can view Δ_{m+1} as a subgroup of L_{m+1}/L_{m+2} generate by the image of $\Delta_1 \times \Delta_m$ under this pairing. By the inductive hypothesis, $\Delta_1 = L_1/L_2$ and $\Delta_m = L_m/L_{m+1}$.

On the other hand, the definition of L_{m+1}, together with the congruence, implies that L_{m+1}/L_{m+2} is also topologically generated by the image of this pairing. Hence we must have that the two structures are the same:

$$\Delta_{m+1} = L_{m+1}/L_{m+2},$$

and we have proved the claim. We have shown for each $n \geq 2$,

$$H_n \subset \Gamma_n.$$

Hence we have completed the proof that for all $n \geq 2$,

$$\Gamma_n = H_n.$$

This concludes the proof of lemma. □

From Clifford Algebra to Lie Algebra

In the previous part, we started our preparation with a subgroup of Lie algebra with certain properties then derived the results about a pro-p subgroup of the Clifford algebra which acts like its pre-image under the map θ.

Now we will begin from the opposite direction: we start with a pro-p subgroup of the Spin group and use its image under θ to derive a subgroup of Lie algebra which has all the properties we assume as axioms the first time around.

We will show that these two processes are mutually inverses which means we have a system for identifying pro-p subgroups of the Spin group, and their descending series, by subgroups of Lie algebra.

We will use similar notations to the previous subsection to keep track of their relationships. Let $\Gamma \subset U$ be a pro-p subgroup. By compactness of R, it is closed. Let $L \subset \mathrm{Cl}^{\mathbf{2}}(M)$ be the closed additive subgroup generated by $\theta(\Gamma)$. Again we denote $C := \mathrm{tr}(L \cdot L) \subset R + Re_{1234}$.

Proposition 2.2.11. *The setting above satisfies the axioms (1),(2) and (3).*

Proof. (1): For the first axiom, consider the fact that any pro-p subgroup of the Spin group must be of the form $1 + I \cdot (G)$, where (G) is a set of some elements. Note that this is a property carried over from the matrix algebra. So we have

$$\theta(\Gamma) = I \cdot (\theta(G)),$$

This means $L^2 \equiv 0 \mod I$ and, by using induction, we can derive

$$L^{2n} \equiv 0 \mod I^n.$$

Hence

$$\bigcap_{n=1}^{\infty} L^n = \{0\}$$

as desired.

(2): The second axiom, $[L, L] \subset L$, follows from the formulas (2.2.3.1) and (2.2.3.2):

$$[\theta(x), \theta(y)] = \theta(xy) - \theta(yx).$$

(3): For the last axiom, by (2.2.3.4) we have

$$C = \mathrm{tr}(L \cdot L) \subset \mathrm{tr}(\Gamma) + \mathrm{tr}(\Gamma)^2.$$

But (2.2.3.8) implies that $\mathrm{tr}(\Gamma) \cdot L \subset L$. Thus

$$C \cdot L \subset \mathrm{tr}(\Gamma) \cdot L + \mathrm{tr}(\Gamma)^2 \cdot L \subset L,$$

and we are done. $\square$

As in the first half, we define $L_1 := L$ and $L_2 := [L, L]$, and let

$$H_n := \{x \in U \mid \theta(x) \in L_n,\ \mathrm{tr}(x) - 1 \in C\}.$$

for $n = 1, 2$.

Proposition 2.2.12. *H_1 is a subgroup of U, H_2 is a normal subgroup of H_1, and H_1/H_2 is abelian.*

Proof. Because we have proved Proposition 2.2.11, we gain access to all the results we have established in the previous part. This proposition is true simply from the combination of Proposition 2.2.6 and Proposition 2.2.9. □

Classification of Pro-p Subgroup

The followings is the main theorem for the case of Spin group $n = 4$.

Lemma 2.2.13. *Γ is contained in H_1, and the commutator subgroup of Γ is H_2.*

Proof. For the first statement, we need to show that for every $\gamma \in \Gamma$,

$$\mathrm{tr}(\gamma) - 1 \in C.$$

By (2.2.3.6) we have that

$$\mathrm{tr}(\gamma)^2 = 1 + \mathrm{tr}(\theta(\gamma)^2) \equiv 1 \mod C.$$

So we either have $\mathrm{tr}(\gamma) \equiv 1 \mod C$, or $\mathrm{tr}(\gamma) \equiv -1 \mod C$.

Since Γ is pro-p, its shape as described in the proof of Proposition 2.2.11 dictates that

$$\mathrm{tr}(\gamma) \equiv 1 \mod C.$$

Since Γ is compact, it is a closed subgroup of H_1.

For the second statement, we need Γ to satisfy the condition (4) in order to use Lemma 2.2.10. This means we want the additive subgroup $L/[L, L]$ to be topologically generated by $\theta(\Gamma)$. But this true from the definition of L at the beginning of this part. □

Since H_1 and H_2 depend only on L, Lemma 2.2.13 implies that Γ is determined by L and the subgroup $\Gamma/H_2 \subset H_1/H_2$. Thus we go back full circle since we previously define L to depend only on Γ. This allows for the following theorem. Reminder that for $x \in L_n$, the term $|x|$ means the residue class of x in L_n/L_{n+1}

Theorem 2.2.14. *There is a canonical one-to-one correspondence between all the pro-p subgroups $\Gamma \subset U$ and the pairs (L, Δ) with following properties:*
(i) L is a closed additive subgroup of $\mathrm{Cl}^{\bar{\mathbf{2}}}(M)$ satisfying

$$L^2 \equiv 0 \mod I, \text{ which implies } \bigcap_{n=1}^{\infty} L^n = \{0\}. \tag{1}$$

$$[L, L] \subset L. \tag{2}$$

$$\mathrm{tr}(L \cdot L) \cdot L \subset L. \tag{3}$$

These three conditions imply that the formula

$$\big|x\big| * \big|y\big| := \Big|(\sqrt{1+\mathrm{tr}(x^2)}) \cdot y + (\sqrt{1+\mathrm{tr}(y^2)}) \cdot x\Big|$$

is a well-defined composition law on $L/[L, L]$, making it into an abelian pro-p group with identity $\big|0\big|$.
(ii) Δ is a closed subgroup of $(L/[L, L], *)$ such that the additive group $L/[L, L]$ is topologically generated by the subset Δ.

Proof. From the preparation up to this point, it is clear how the correspondence is defined. When Γ is given, we define L as in the beginning of Clifford-to-Lie part, which we will called the first process. The three properties are satisfied by Proposition 2.2.11. Then Δ is the closed subgroup of $(L/[L, L], *)$ corresponds to $\Gamma/H_2 \subset H_1/H_2$ under the isomorphism from Proposition 2.2.9. The generating conditions are satisfied by definition of L.

On the other hand, given a pair (L, Δ) with all the required properties. Then it satisfies the three axioms and we can restart the process we went through in the Lie-to-Clifford part, which we will call the second process, to obtain Γ. Here we have Δ as the subgroup of H_1 containing H_2 such that Γ/H_2 corresponds to Δ under the isomorphism from Proposition 2.2.9. By (2.2.6.2), H_1 is pro-p, and thus Γ is also pro-p.

Lastly, we will show that these two processes are mutually inverse. First lets start with $\Gamma \subset U$. Let (L, Δ) be the pair obtained from Γ through the first process. Suppose we define another pro-p subgroup Γ' associated to the pair (L, Δ) through the second process. Then H_2 must be contained in both Γ (by Lemma 2.2.13) and Γ' (by definition). Since both Γ/H_2 and Γ'/H_2 correspond to Δ under the same isomorphism in Proposition 2.2.9, we must have $\Gamma = \Gamma'$.

Conversely, if we start with a pair (L, Δ) satisfying all the conditions and obtain Γ from the second process. Suppose we define L' as the closed additive subgroup that is topologically generated by $\theta(\Gamma)$ as in the first process, then by definition L' contains $\theta(H_2)$ which is equal to $[L, L]$ by (2.2.6.3). Thus, by definitions,

$$L'/[L, L] = \Delta = \theta(\Gamma)/[L, L],$$

and hence $L' = L$. This concludes the proof. □

Remark 2.2.15. We want to point out an observation that Pink's method is very self-contained. Indeed, after defining a suitable norm, trace and θ map, the rest of Pink's method relies heavily on Lemma 2.2.3, and almost nothing else. It is not an exaggeration to say that the identities listed in the lemma dictate the direction of, and enable, all lines of proofs starting from the first proposition all the way to the main theorem.

Because of this reason, we suggest any readers who wish to explore Pink's method in other settings outside of the Clifford algebra or matrix algebra to look for the compatibility of these starting components as their top priority when considering any new structure.

2.2.2 The Case of Spin Group $(n = 3)$

For the this discussion, we denote the Spin group from the case of $n = 4$ by U_4, and for the case of $n = 3$ by U_3.

Since we are dealing with Spin group, the Spin norm $N(x)$ and trace map $\mathrm{tr}(x)$ can be defined in exactly the same way in both cases. Furthermore, we can consider an element

$$x = c + c_1(e_1e_2) + c_2(e_1e_3) + c_3(e_2e_3) \in U_3,$$

as if it is inside U_4 of some Clifford algebra with a quadratic form that gives rise to the same e_1^2, e_2^2, e_3^2 as the ones from U_3. Because of this, the formulas in Lemma 2.2.3 stay the same for $n = 3$ as well.

With this set up, the rest of the results follow, and we obtain a similar theorem for the case of Spin group with $n = 3$.

2.2.3 The Case of Quaternion Algebra $(n = 2)$

When $n = 2$, the Clifford algebra is also known as the quaternion algebra. The elements of $S = Cl(M)$ is of the form

$$x = c_0 + c_1(e_1) + c_2(e_2) + c_3(e_1e_2),$$

which is fundamentally different from the setup with Spin group. However, we will show that it gives rise to the same formulas.

We define the Clifford norm and a trace map by

$$N(x) = x\bar{x}, \quad \mathrm{tr}(x) = \frac{1}{2}(x + \bar{x}),$$

which coincide with the Spin norm and trace in the two previous cases. Instead of the Spin group, we will consider the subgroup of Clifford elements with norm 1,

$$U := \{x \in S \mid N(x) = 1\}.$$

In this case, we still give the same definition to the θ map

$$\theta(x) := x - \operatorname{tr}(x) = c_1(e_1) + c_2(e_2) + c_3(e_1e_2),$$

together with its inverse

$$\psi(y) := y + \sqrt{1 + \operatorname{tr}(y^2)}.$$

Note that now the image of θ is inside $\mathrm{Cl}^{\bar{\mathbf{1}}}(M) \bigoplus \mathrm{Cl}^{\bar{\mathbf{2}}}(M)$ instead of only $\mathrm{Cl}^{\bar{\mathbf{2}}}(M)$, but as long as $\operatorname{tr}(\theta(x)^2)$ lies inside a module of the form $I(...)$, which is implied from the starting axioms, we can define square root like before.

One can easily verify that the subspace $\mathrm{Cl}^{\bar{\mathbf{1}}}(M) \bigoplus \mathrm{Cl}^{\bar{\mathbf{2}}}(M)$, which can be viewed as a subgroup of the 2-by-2 matrix algebra, is closed under the commutator bracket $[A, B] = AB - BA$. Therefore it is also a Lie algebra, which we will again denote by $\mathrm{Cl}^{\bar{\mathbf{1}}}(M) \bigoplus \mathrm{Cl}^{\bar{\mathbf{2}}}(M)$.

Interestingly, despite these difference, the formulas in Lemma 2.2.3 stay the same for quaternion algebra, and thus the rest of the theorems and proofs do not change much outside of some notations and indexes. We will only provide a proof for the quaternion version of Lemma 2.2.3, and verify that ψ is the inverse of θ.

Lemma 2.2.16 (The Quaternion Case)**.** *Let x, $y \in U$. The following identities hold.*

$$[x, y] = [\theta(x), y] = [x, \theta(y)] = [\theta(x), \theta(y)]. \tag{2.2.16.1}$$

$$[x, y] = \theta(xy) - \theta(yx). \tag{2.2.16.2}$$

$$2\,\theta(xy) = [\theta(x), \theta(y)] + 2\operatorname{tr}(x) \cdot \theta(y) + 2\operatorname{tr}(y) \cdot \theta(x). \tag{2.2.16.3}$$

$$\operatorname{tr}(xy) = \operatorname{tr}(\theta(x)\theta(y)) + \operatorname{tr}(x)\operatorname{tr}(y). \tag{2.2.16.4}$$

$$\operatorname{tr}(xy) = \operatorname{tr}(yx). \tag{2.2.16.5}$$

$$\operatorname{tr}(x)^2 = 1 + \operatorname{tr}(\theta(x)^2). \tag{2.2.16.6}$$

$$\operatorname{tr}(x^{-1}) = \operatorname{tr}(x)\,,\ \theta(x^{-1}) = -\theta(x). \tag{2.2.16.7}$$

$$2\operatorname{tr}(x) \cdot \theta(y) = \theta(xy) + \theta(x^{-1}y). \tag{2.2.16.8}$$

$$x, y \in \mathrm{Cl}^{\bar{\mathbf{1}}}(M) \bigoplus \mathrm{Cl}^{\bar{\mathbf{2}}}(M) \implies 2\operatorname{tr}(xy) = xy + yx. \tag{2.2.16.9}$$

Further for $x, y, u, v \in \mathrm{Cl}^{\bar{\mathbf{1}}}(M) \bigoplus \mathrm{Cl}^{\bar{\mathbf{2}}}(M)$, we have

$$\begin{aligned} 4\operatorname{tr}(xy) \cdot [u, v] = {} & [y, [x, [u, v]]] + [x, [y, [u, v]]] \\ & + [[x, v], [u, y]] + [[y, v], [u, x]]. \end{aligned} \tag{2.2.16.10}$$

$$\begin{aligned} 4\operatorname{tr}([u, v]\,x) \cdot y = {} & [y, [x, [u, v]]] - [x, [y, [u, v]]] \\ & + [[x, v], [u, y]] + [[y, v], [u, x]]. \end{aligned} \tag{2.2.16.11}$$

Proof. For the proof, we denote

$$x = x_0 + x_2, \text{ where } x_0 \in \mathrm{Cl}^{\bar{\mathbf{0}}}(M) \text{ and } x_2 \in \mathrm{Cl}^{\bar{\mathbf{1}}}(M) \bigoplus \mathrm{Cl}^{\bar{\mathbf{2}}}(M),$$
$$y = y_0 + y_2, \text{ where } y_0 \in \mathrm{Cl}^{\bar{\mathbf{0}}}(M) \text{ and } y_2 \in \mathrm{Cl}^{\bar{\mathbf{1}}}(M) \bigoplus \mathrm{Cl}^{\bar{\mathbf{2}}}(M).$$

Notice that even though x_2 and y_2 are not entirely inside $\mathrm{Cl}^{\bar{\mathbf{2}}}(M)$ like in the proof of Lemma 2.2.3, they still commute with x_0 and y_0 in the same way. Furthermore, we also have $\bar{x}_2 = -x_2$ and $\bar{y}_2 = -y_2$. Thus the lines of proof are exactly the same as in Lemma 2.2.3.

□

Lastly, we will show that the formula

$$\psi(y) = y + \sqrt{1 + \mathrm{tr}(y^2)},$$

works as the inverse map to θ. It is easy to see that $\theta(\psi(y)) = y$.

Next, let $x \in U$. We then have

$$\begin{aligned} \psi(\theta(x)) &= \theta(x) + \sqrt{1 + (\mathrm{tr}(\theta(x)^2)} \\ &= \theta(x) + \mathrm{tr}(x) \\ &= x - \mathrm{tr}(x) + \mathrm{tr}(x) \\ &= x. \end{aligned}$$

The second equation is true because of (2.2.16.6): $\mathrm{tr}(x)^2 = 1 + \mathrm{tr}(\theta(x)^2)$, the fact that $\mathrm{tr}(x) \equiv 1 \mod I$, and the uniqueness of the square root explained in 2.2.1.

With all these preparations, the rest of the quaternion case proceeds in the same way as the Spin cases despite the differences in their elements.

In the next part, we will address the reasons why our main results stop at $n = 4$.

2.3 The Limitation of $n > 4$

Naturally, one question that may arise after obtaining such results is "What about the cases where $n > 4$?"

We believe that Pink's method as used in this thesis cannot be extend further than the Spin group of $n = 4$. The major problem lies in the definition of the trace map and its properties which are needed in order for the method to work.

To put simply, we want the trace map to have the following two properties:

1. Commute with any Clifford elements and is invariant under the conjugation. This condition is required so that the identities in Lemma 2.2.3 hold.
2. Allow for a separation of element types in the inverse map's formula

$$\psi(y) = y + \sqrt{1 + \mathrm{tr}(y^2)},$$

so that the norm of its image evaluates to 1.

The problem arises because these two properties always contradict each other in higher dimensional cases where $n > 4$.

In order to discuss these two properties, we first restate the settings of the main result. Let $\{e_1, e_2, ..., e_n\}$ be an orthogonal basis of a quadratic R-module (M, B) where R is a p-adic ring and $n \geq 5$. We will denote the Clifford algebra over M by $Cl(M)$, and its Spin group by $\mathrm{Spin}(M)$. We will only consider the set up for $\mathrm{Spin}(M)$ but similar argument applies for the set of unit in $Cl(M)$ as well.

2.3.1 Commutativity

Let us consider the commutativity of the trace map. We know that

$$\mathrm{Spin}(M) \subset \mathrm{Cl}^{\bar{\mathbf{0}}}(M) \bigoplus \mathrm{Cl}^{\bar{\mathbf{2}}}(M).$$

In order for the trace to remain invariant under the conjugation, $\overline{\mathrm{tr}(x)} = \mathrm{tr}(x)$, we must have $\mathrm{tr}(x) \in \mathrm{Cl}^{\bar{\mathbf{0}}}(M)$. We claim that when $n \geq 5$, the trace $\mathrm{tr}(x)$ commutes with all the elements in $\mathrm{Spin}(M)$ if and only if it is a scalar in R.

To illustrate, let us first assume $n = 5$. An element $x \in \mathrm{Spin}(M)$ is of the form

$$x = c_0 + (c_1)e_{12} + ... + (c_{10})e_{45} + (c_{11})e_{1234} + ... + (c_{15})e_{2345},$$

where the coefficients c_i's are scalars in R. Since we already established that $\mathrm{tr}(x) \in Cl^{\bar{\mathbf{0}}}$, the trace must be some linear combination of elements in a set

$$A = \{c_0, (c_{11})e_{1234}, ..., (c_{15})e_{2345}\} \subset \mathrm{Cl}^{\bar{\mathbf{0}}}(M).$$

Clearly, if $\mathrm{tr}(x) = c_0$, then the trace commutes with everything in $\mathrm{Spin}(M)$. Now, suppose $\mathrm{tr}(x)$ has any other element from A, without the loss of generality, says $(c_{11})e_{1234}$. Then there exists a basis e_{15} from some $x \in \mathrm{Spin}(M)$ such that

$$\begin{aligned}
(e_{15})(e_{1234}) &= (e_1e_5)(e_1e_2e_3e_4) \\
&= (-1)^4(e_1)(e_1e_2e_3e_4)(e_5) \\
&= (-1)^7(e_1e_2e_3e_4)(e_1e_5) \\
&= -(e_{1234})(e_{15}).
\end{aligned}$$

Thus $\mathrm{tr}(x)$ does not commute with e_{15}, and more importantly not with x. No matter what non-scalar element we choose from A, there exists an element in $\mathrm{Cl}^{\bar{\mathbf{2}}}(M)$ that does not commute with it. This is because we can always choose a pair of e_ie_j which has only one e_i in common with the element from A. For example, e_{1234} with e_{25}, e_{1245} with e_{34}, and so on. Therefore, if $\mathrm{tr}(x)$ commutes with everything in $\mathrm{Spin}(M)$, it cannot has any non-scalar elements from A. It is easy to see that this argument also holds for $n > 5$ and we have proved the claim.

2.3.2 Separation of Element Types

Next, let us explain the second property mentioned above. Once we have established a trace map on $\mathrm{Spin}(M)$, we can define $\theta(x) = x - \mathrm{tr}(x)$ which acts as a connection between the Spin group and a Lie algebra. The trick to this method is in its inverse:

$$\psi(y) = y + \sqrt{1 + \mathrm{tr}(y^2)}.$$

We need the two terms in this formula to be in two separated subspaces of $Cl(M)$ so that the calculation of Spin norm is possible and evaluates to 1.

Since y is an element of a Lie algebra which lies entirely in $\mathrm{Cl}^{\bar{\mathbf{2}}}(M)$, we already have

$$y \in \mathrm{Cl}^{\bar{\mathbf{2}}}(M).$$

The trace map comes in when we consider the square root. Here we want

$$\sqrt{1 + \mathrm{tr}(y^2)} \in \mathrm{Cl}^{\bar{\mathbf{0}}}(M).$$

By the definition of square root,

$$\sqrt{1+\alpha} = \sum_{m=0}^{\infty} \binom{1/2}{m} \alpha^m,$$

this condition is only possible when the images of the trace map $\mathrm{tr}(y^2)$ lies entirely inside $\mathrm{Cl}^{\bar{\mathbf{0}}}(M)$ as well.

When this condition is satisfies, the image $\psi(y)$ will have its Spin norm equals to

$$\begin{aligned}
\psi(y)\overline{\psi(y)} &= (\sqrt{1 + \mathrm{tr}(y^2)} + y)(\sqrt{1 + \mathrm{tr}(y^2)} - y) \\
&= (\sqrt{1 + \mathrm{tr}(y^2)})^2 - y^2) \\
&= 1 + \mathrm{tr}(y^2) - y^2 \\
&= 1
\end{aligned}$$

and thus it is inside $\mathrm{Spin}(M)$. The second equation holds because the commutativity of the trace. The last equation holds because $\mathrm{tr}(y^2) = y^2$ when we define $\mathrm{tr}(x) = \frac{1}{2}(x + \bar{x})$

and $y \in \mathrm{Cl}^{\bar{\mathbf{2}}}(M)$. In summary, we want one of the terms to be invariant under conjugation, while the other gets multiplied by -1 under conjugation.

The problem arises when $n \geq 5$ because the trace loses its commutativity unless we define $\mathrm{tr}(x) = c_0$, but doing so will ruin the calculation of Spin norm shown above since y will no longer be entirely inside $\mathrm{Cl}^{\bar{\mathbf{2}}}(M)$ and contain some components from $\mathrm{Cl}^{\bar{\mathbf{0}}}(M)$. This means y is neither invariant or goes negative under conjugation, and the calculation is no longer guaranteed to evaluate to 1. Thus we cannot make sure that the inverse map returns to $\mathrm{Spin}(M)$ and Pink's method, as we know it, breaks down.

2.3.3 Outside The Spin Group

We now explain why, a similar argument can be said about trying to do Pink's method outside of the Spin group. Notice that for $n = 3, 4$, we use only the Spin group for our main results instead of the whole set of Clifford elements with norm 1. Let us briefly address the two cases here.

When $n = 4$, going outside the Spin group will bring up a problem with trace map because the presence of type $\bar{\mathbf{1}}$ basis which does not commute with e_{1234}.

For example:

$$(e_1)(e_{1234}) = (-1)^3(e_{1234})(e_1).$$

This forces the trace to be pure scalar to preserve its commutativity, and thus disrupting the calculation of inverse map as shown previously.

When $n = 3$, the situation is a bit different. Even if we go outside the Spin group, there is no problem with the trace because the only option available is $\mathrm{tr}(x) = c_0$. However, the norm itself becomes a problem. Specifically, the relation (2.2.3.6):

$$\mathrm{tr}(x)^2 = 1 + \mathrm{tr}(\theta(x)^2)$$

no longer holds in this situation because the norm map defined by $N(x) = x\bar{x}$ is not guaranteed to be a scalar, and more importantly not equal to 1, when x is outside the Spin group. Without this, we will need to define a new norm and revise everything from the ground up to accommodate it. We are not certain if this is possible, so far all of our attempts have been unsuccessful. We believe that it will require a whole new method of its own that is different from Pink's. This concludes the section about our main results.

Chapter 3

Possible Application

The possible application we have in mind for the main result is as a calculating tool for a particular Galois deformation problem done by Gebhard Böckle [2], which, in fact, was the motivating problem of our thesis. While studying Böckle's Galois deformation problem, we notice the usage of Pink's result in the calculation of the Demuškin relation in SL_2. This compelled us to explore the possibilities of other structures that may be compatible with Pink's method in hope to expand Böckle's result. In the end, this search ended up being the main topic of our thesis.

We would like to use this chapter to provide a brief explanation of this deformation problem and point out the potential of applying our main results to Böckle's work in the Clifford algebra setting. Note that all the notations that appeared in this chapter have no relations to the ones used in our main results. Any repeated notations between the two chapters are simply due to the lack of character choices, and our attempt to preserve some of the original notations in Böckle's work for easy tracking. We apologise for any confusion that may occur.

3.1 Quick Review on Galois Deformation

Let $\mathcal{C}$ be the category of complete Noetherian local ring with residue field k, together with local ring homomorphisms as its arrows. Here every ring $R \in \mathcal{C}$ has a unique maximal ideal m_R and the quotient R/m_R is k.

Definition 3.1.1. Let $R_1, R_2 \in \mathcal{C}$ and $\rho : G \to \mathrm{GL}_2(R_1)$ be a representation and suppose there is a map (arrow) $h : R_2 \to R_1$.

1. A *lifting* (or a *lift*) $\tilde{\rho}$ of ρ to R_2 is a homomorphism $\tilde{\rho} : G \to \mathrm{GL}_2(R_2)$ such that $\overline{h} \circ \tilde{\rho} = \rho$, where $\overline{h}$ is a map obtained by applying h to each entry of the matrix in R_2.

2. Two lifts $\tilde{\rho}_1, \tilde{\rho}_2 : G \to \mathrm{GL}_2(R)$ of ρ are *strictly equivalent* if there is a matrix $M \in \mathrm{GL}_2(R)$ such that $\tilde{\rho}_1 = M\tilde{\rho}_2 M^{-1}$ and M reduces to identity matrix on $\mathrm{GL}_2(k)$.

3. An equivalence class of lifts of ρ to R is called a *deformation* of ρ to R.

Definition 3.1.2. Given a representation $\rho : G \to \mathrm{GL}_2(k)$, the pair $(\alpha, \mathcal{R})$ of *universal deformation* and *universal ring* is a deformation $[\alpha]$ of ρ to $\mathcal{R}$ such that for any ring $A \in \mathcal{C}$ with a deformation $[\rho_A] : G \to \mathrm{GL}_2(A)$ of ρ there exist a unique map $\phi : \mathcal{R} \to A$ as an arrow in $\mathcal{C}$ with $\overline{\phi} \circ \alpha = \beta$, for some β in the class $[\rho_A]$. That is to say the following diagram commutes:

$$\begin{tikzcd} G \arrow[r, "\alpha"] \arrow[dr, "\beta"'] & \mathrm{GL}_2(\mathcal{R}) \arrow[d, dashed, "\overline{\phi}"] \\ & \mathrm{GL}_2(A) \end{tikzcd}$$

where $\overline{\phi}$ is an induced map, applying ϕ to each entry of the matrix in $\mathcal{R}$.

This means that once we found the universal pair for ρ, we can construct a lifting to any ring $R \in \mathcal{C}$ with a homomorphism $\mathcal{R} \to R$. In this way, we can study deformations of the absolute Galois group conveniently and effectively.

Definition 3.1.3. We define the *functor* $\mathrm{Def}_\rho : \mathcal{C} \to \mathrm{Sets}$ from $\mathcal{C}$ to category of sets by

$$\mathrm{Def}_\rho(R) := \{\text{Deformations of } \rho \text{ to } R\}$$

sending R to the set of equivalence classes of lifts to R.

We can think of functor as a map between two categories. The problem of finding the universal pair $(\alpha, \mathcal{R})$ is the same as finding an object $\mathcal{R} \in \mathcal{C}$ and $\alpha \in \mathrm{Def}_\rho(\mathcal{R})$ with the universal property.

3.2 Böckle's Work

In his paper, "Demuškin Groups with Group Actions and Applications to Deformations of Galois Representations" [2], Gebhard Böckle solves a Galois deformation problem by determining the universal deformation ring of a residual representation

$$\rho : G_K \to \mathrm{GL}_2(k),$$

where k is a finite field of characteristic $p \neq 2$ and K is a local field of residue characteristic p. The result is done in the standard Mazur setup (see [18]), which is a very well-known method in Galois representation.

In essence, Böckle's main contribution to Mazur's method is in finding a shortcut that reduces the amount of works needed in order to obtain a homomorphism which acts as the universal lifting of ρ. Although the theoretical parts of his work do not require Pink's method for their proofs, the actual process of finding the universal map can be incredibly daunting without proper tools to aid the calculation. This is where

Pink's work, among many other options, comes into play. We believe there is a potential for our main results to work here in the similar fashion if one were to attempt Böckle's method in the Clifford algebra setting.

Do note that, in such event, one will need to make a complete overhaul of the method and change the all definitions from GL_2 to a Clifford algebra of choice. Though this is quite a feasible task since the Clifford algebra itself can be consider as a sub-structure inside the matrix algebra.

We will now explain a summary of Böckle's work. All the details of a process described below can be found in his paper [2].

Let E be a number field, $\overline{E}$ its separable algebraic closure, and k a finite field of characteristic $p \neq 2$. Then let

$$\rho : \mathrm{Gal}(\overline{E}/E) \to \mathrm{GL}_2(k)$$

be a Galois representation, we will denote by H its image inside $\mathrm{GL}_2(k)$, and denote by L the Galois extension of E corresponding to H. Here we will assume that H is solvable and contains elements of order p. From this assumption, we can also assume the H lies inside the set of upper triangular matrices.

Let U denote the set of unipotent elements of H. By lemma of Schur-Zassenhaus, we can pick a subgroup G of H of order prime to p such that $H = U \rtimes G$.

We let F be the fixed field of U in L, and $\mathrm{G}_F(p)$ the maximal pro-p quotient of $\mathrm{Gal}(\overline{E}/F)$ with $L(p)$ as its corresponding extension over F. This results in the following field extensions structure:

$$\begin{tikzcd}
\overline{E} \arrow[d, dash, dashed] \\
L(p) \arrow[d, dash] \arrow[dd, bend right, dash, "\mathrm{G}_F(p)"'] \\
L \arrow[d, dash, "U"] \arrow[dd, bend left, dash, "H"] \\
F \arrow[d, dash, "G"] \\
E
\end{tikzcd}$$

We will fix a lift of G to $\mathrm{GL}_2(W(k))$, where $W(k)$ is the ring of Witt vectors, which can be done through the Teichmuller lift. This exists because the pro-finite version of Schur-Zassenhaus applies to $\mathrm{GL}_2(W(k))$. Through this, G can be viewed inside any matrix group $\mathrm{GL}_2(R), R \in \mathcal{C}$. Next, we consider $\mathrm{Gal}(F/E)$, the pre-image of G on the Galois side, as a part of $\mathrm{Gal}(L(p)/E)$ that compliment $\mathrm{G}_F(p)$. This can be done, again, due to the pro-finite version of Schur-Zassenhaus. In this way, we have "G action", via conjugation, for the pro-finite extensions on both the matrix and Galois side. Later, when we say a homomorphism $\alpha : \mathrm{G}_F(p) \to \mathrm{GL}_2(R)$ "obeys" G action, we mean that taking G action before, or after, α gives rise to the same result. This process can be viewed as partially constructing a lifting for the original representation ρ by deciding the lift for the part from G beforehand. Thus the problem is now reduced to deciding how to lift the part from U.

We let $\{x_i : i \in \mathcal{I}\}$ denote a finite subset of $\mathrm{G}_F(p)$ such that elements of the form

$$\rho(x_i) = \begin{pmatrix} 1 & u_i \\ 0 & 1 \end{pmatrix},$$

generate U as an H-module where H acts by conjugation. By conjugating ρ if necessary, we will assume that $u_1 = 1$.

We prepare all these setup in order to construct a simpler functor to replace Def_ρ.

Define a functor $S : \mathcal{C} \to \mathrm{Sets}$ by

$$S(R) := \{\alpha \in \mathrm{Hom}_G(\mathrm{G}_F(p), \tilde{\Gamma}_2(R)) : \alpha(x_i) \equiv \begin{pmatrix} 1 & u_i \\ 0 & 1 \end{pmatrix} (mod\, m_R)\},$$

where $\tilde{\Gamma}_2(R)$ is the set of matrices in $\mathrm{GL}_2(R)$ that reduce to $\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$ in $\mathrm{GL}_2(k)$.

Böckle shows that the natural transformation $S \to Def_\rho$, sending, for each R, a map α to a class of lifting that agrees with α on U, is smooth and the induced map on tangent spaces $t_S \to t_{Def_\rho}$ is always an isomorphism (Prop 2.3 in [2]).

Furthermore, S is always representable. This allow us to study a simpler functor, S, instead of Def_ρ as we can obtain universal pair of one from another. So the problem is reduced to finding G-equivariant homomorphisms from $\mathrm{G}_F(p)$ to $\tilde{\Gamma}_2(R)$.

When $\mathrm{G}_F(p)$ is free, the problem can be solved simply by constructing a map that obeys both the G action and the definition of S.

In the case that $\mathrm{G}_F(p)$ is not free, it is known to be a *Demuškin group*, a group with certain properties which imply the following structure:

A Demuškin group D is a pro-p group with n generators, $x_1, x_2, ..., x_n$ and one relation of the form

$$x_1{}^q[x_1, x_2]...[x_{n-1}, x_n] = 0,$$

where q is a power of p and $[x_i, x_{i+1}]$ is the usual commutator of the two. When $p > 2$, it is well-known that n and q characterize D completely.

Remark 3.2.1. Note that D can be thought of as a quotient $\mathcal{F}/(r)$ where $\mathcal{F}$ is a free group with the same number of generators as D and (r) is the ideal generated by the relation. Because of the way Böckle set up this case, the x_i's here will be the same as those in definition of S.

In order to obtain the universal pair, we have to find a suitable homomorphism $\alpha \in \mathrm{Hom}_G(\mathrm{G}_F(p), \tilde{\Gamma}_2(R))$ such that images of x_i's in $\mathrm{GL}_2(R)$ respect both the G-action and the Demuškin relation. However, Böckle argues that such a map does not always exist (see Prop 3.6 in [2]). Thus he offers a solution to this problem by establishing the following proposition (Prop 3.8 in [2]).

Let $\mathcal{F}^{(i,q)}$ denote the i-th step of *the lower q-central series* defined by

$$\mathcal{F}^{(0,q)} = \mathcal{F} \quad \text{and} \quad \mathcal{F}^{(i,q)} = [\mathcal{F}^{(i-1,q)}, \mathcal{F}](\mathcal{F}^{(i-1,q)})^q.$$

Proposition (The Shortcut)**.** *Let P be a pro-p group with an action of G. If we have a homomorphism $\alpha : \mathcal{F} \to P$ such that $\alpha(r) \in \alpha(\mathcal{F}^{(3,q)})$ then there exists a homomorphism $\tilde{\alpha} : D \to P$ that agree with α modulo $\alpha(\mathcal{F}^{(2,q)})$.*

The above proposition gives us a weaker condition to look for in a homomorphism, i.e., we only need the image of Demuškin relation under this map to be inside $\alpha(\mathcal{F}^{(3,q)})$ instead of being the identity. It turns out, as Böckle has shown, that the map we obtain from this proposition is good enough a candidate for constructing the universal pair.

To summarise, with Böckle's method we can reduce the original problem down to finding a G-action-compatible map such that the images of the generators x_i's of $\mathrm{G}_F(p)$ satisfy the condition in the shortcut proposition above, then check for its universal property to be the universal pair. The rest of the theoretical work can be found in the referenced paper [2].

The only real challenge left is the actual process of checking the conditions for this shortcut. Here is where Pink's method comes in (see section 5 of [2]). Interestingly, Pink's result happens to have everything Böckle needs; isomorphisms between pro-p subgroups of SL_2 and additive subgroups of sl_2 allow for the Demuškin relation to be calculated in Lie algebra, hence replacing matrix multiplication by addition, and these connections remain intact even when we move into the quotient of lower central series. This method culminates in lemma 5.3 of Böckle's paper [2], where he gives explicit forms of images for the lift of ρ which he later proves to have universal property. In the end, he succeeds and obtain the universal deformation.

3.2.1 Possibility of Clifford Algebra Cases

We would like to use this opportunity to elaborate more on how this thesis came to be. When we first began our research into Böckle's method, we intended to solve the deformation problem in the setting of symplectic matrix, Sp_2. However, we quickly learned that even though all of the theoretical parts work out for Sp_2 case, we cannot obtain the universal map, or any lifting for that matter, without a way to calculate the Demuškin relation inside a lower central series. This was a problem because we could not extend Pink's method to our case of choice–the symplectic matrix.

Naturally, our question then shifted to "What other structures can we do Pink's method on?" We focused our search on the algebras with an established norm and trace map to imitate the determinant and trace of the matrix. After a long stretch of trial and error, we ended up with Clifford algebra. In a way, we have not wandered too far from its matrix root.

The set up in Böckle's work shown above can be modified into Clifford algebra setting quite conveniently due to the fact that Clifford algebra is closely related to the matrix algebra, where the matrix's determinant and trace can be replaced with Clifford norm and trace. Since the theoretical parts of Böckle's method mostly relies on the properties of Demuškin groups, and Galois groups in general, the structure we map the Galois representation into doesn't matter as long as we can find a way to calculate the Demuškin relation in lower central series of said structure. This is the reason why we believe that our main result can help solve the problem in Clifford algebra setting.

This means redefining Böckle's work from the ground up by replacing the original GL_2 with the Clifford algebras or spin groups, ones that we cover in the thesis, as the image space of our Galois representation.

In the end, we hope to acquire the following result: given a Galois representation that satisfies all the pre-requisites of Böckle's paper, one way to check if its image is solvable with elements of order p is by giving a faithful representation from the image in Clifford algebra to a matrix group, and see if it is contained in upper-triangular matrices, then our version of Pink's method and the shortcut lemma should allow us to find explicit images of at least one legitimate lifting, which we can then check for the universal property.

One important note to keep in mind is that the shortcut proposition does not guarantee the existence of a homomorphism that satisfies its condition. It is possible that, even with the powerful calculating tool like Pink's method, one may never find the suitable map at all. That being the case, we are hopeful in the possibility that others may obtain new results out of Böckle's problem in the Clifford algebra setting with the help of our work.

Conclusion

Pink's method is a unique and powerful way to obtain information about pro-p subgroups of an algebra over a p-adic ring. Its uniqueness comes from the fact that it only requires elementary definitions and calculations to arrive at a strong and encompassing result. Furthermore, the correspondence that occurs during this process can be used as a convenient calculating tool for other works such as Böckle's Demuškin paper.

However, as we have shown in the previous chapters, it does not always work with any given structure. In fact, this method is so tightly pieced together to the point where any deviations in its formulas and definitions can cause the whole process to breakdown. It is for this reason that we advice any readers who wish to adapt Pink's method onto a new structure to check the compatibility of the starting ingredients: trace, norm, the inverse map, and the identities in Lemma 2.2.3, above all else as it can be a good indicator on whether the structure in question has a potential for Pink's method.

Otherwise, one can also try to develop an entirely new method for their structure of choice with the original work as a base. Though we doubt that any such process can be as simple and elegant as the version we currently have.

This concludes our thesis.

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Declaration

I, the author, confirm that the Thesis is my own work. I am aware of the University's Guidance on the Use of Unfair Means. This work has not been previously been presented for an award at this, or any other, university.

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