Finiteness of compositions of localizations via fracture diagrams

Nicola Bellumat

Supervisor: Prof. Neil P. Strickland

Advisor: Prof. Sarah A. Whitehouse

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University of Sheffield

School of Mathematics and Statistics

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Abstract

In the last decades the study of the stable homotopy category made considerable progress due to the chromatic approach. The core idea of this programme is to decompose said category in simpler pieces which we can effectively understand and then recompose them together to reconstruct the global picture. This operation is accomplished via the Bousfield localizations with respect to the spectra $E(n)$ and $K(n)$, called respectively Johnson-Wilson spectrum and Morava K-theory. These two objects have crucial properties which encapsulate the information of complex orientation of spectra at height lesser or equal to n.

Having established the importance of these localization functors, it is not difficult to understand that we want to consider also their compositions and that some kind of regularity in these situations is desirable. There are classical results going in this direction: for example Ravenel showed that $L_{K(n)}L_{K(m)} = 0$ whenever $n > m$. Also, he proved that we have an equality between Bousfield classes $\langle E(n) \rangle = \bigvee_{i=0}^{n} \langle K(i) \rangle$. This should be read as some version of our intention of gluing back information after we decomposed it in smaller pieces.

This work aims to answer the following question: if we fix n as upper bound of the chromatic height, and consider localizations coming from wedges of $K(i)$'s, for $i \leq n$, are their compositions finitely many up to isomorphism? Not only we will provide a positive answer, but we will formulate it in an axiomatic framework which will allow us to propose the proof for any collection of localizations satisfying properties similar to the ones illustrated above. One of the key points of the proof is that we can reduce the composition of two iterated localizations to the combinatorics of a finite poset which models how they arise as homotopy limits of diagrams involving simple localizations.

Declaration

I, the author, confirm that the Thesis is my own work. I am aware of the University's Guidance on the Use of Unfair Means (www.sheffield.ac.uk/ssid/unfair-means). This work has not been previously been presented for an award at this, or any other, university. I declare that this Thesis has been written in conformity with the norms of the Code of Practice (www.sheffield.ac.uk/rs/code/plagiarism).

This work is largely based on the preprint [2], which is a joint work of me and my supervisor, Prof. Neil P. Strickland. I owe to him my introduction to chromatic homotopy theory, and more generally much of my academical formation during my Ph.D. at the University of Sheffield. I acknowledge that without his aid and direction it would not have been possible to achieve such result. The preprint is currently submitted to the journal "Algebraic & Geometric Topology" for publication.

I also need to mention Moritz Rahn (formerly Moritz Groth) for his contribution to the theory of derivators. While this branch of Pure Mathematics had been invented previously by multiple authors independently, he contributed enormously by providing a modern codification much easier to read and consult. My understanding of this language is due to his work and his personal guidance during my years at the University of Bonn. I do not claim any authorship on the basic framework of derivator theory which I adopted in this work.

Contents

Chapter 1

Basics of chromatic homotopy theory

1.1 Bousfield localization

Since Brown's representability results, the focus of algebraic topology shifted towards the study of spectra: these objects let us encode cohomology and homology theories via a sequence of topological spaces connected by structure homomorphisms which represent the suspension isomorphisms. These spectra can be collected, up to homotopy, in the stable homotopy category $\mathcal{SHC}$, whose properties are much more interesting than the ones of the homotopy theory of topological spaces. For example, it has a triangulated structure compatible with the tensor product induced by the smash product of spectra.

Much of the information about the homotopy type of an object $X \in \mathcal{SHC}$ is contained in its stable homotopy groups $\pi_n(X) = [\mathbb{S}^n, X] = \mathrm{Hom}_{\mathcal{SHC}}(\mathbb{S}^n, X)$, where $\mathbb{S}^n$ is the n-th suspension of $\mathbb{S}$, the sphere spectrum. In fact, a morphism $f \colon X \to Y$ is an isomorphism if and only if the induced $\pi_n(f)$ is invertible for any $n \in \mathbb{Z}$.

As conceptually pleasing these properties are, they do not allow us to perform any concrete computation: determining the stable homotopy groups of a general spectrum is not something which can be easily done. To provide a specific example we recall the following classical results about the stable homotopy groups of the sphere spectrum, which have not been completely determined up to this date.

Theorem 1.1.1.
- *The sphere spectrum $\mathbb{S}$ is connective, i.e. $\pi_k\mathbb{S} = 0$ for $k < 0$. Moreover, $\pi_0\mathbb{S} \cong \mathbb{Z}$.*
- *([33]) For any $k > 0$ the group $\pi_k\mathbb{S}$ is finite.*
- *([30]) Any $x \in \pi_k\mathbb{S}$ with $k > 0$ is nilpotent, that is $x^n = 0$ for some $n \in \mathbb{N}$.*

Also, we observe that by the tensor structure, since $\mathbb{S}$ coincides with the tensor unit, every spectrum X is a $\mathbb{S}$-module and $\pi_*(X)$ is a graded module over the ring $\pi_*\mathbb{S}$ which the previous results indicate to be very complicated (e.g. it is not Noetherian).

This indicates that the direct approach of understanding $\mathcal{SHC}$ by computing the homotopy groups of all the objects we want to study is destined to fail. This brings us to the question if we can simplify the picture is some way: can we reduce the information encoded into spectra, so we can effectively study them even if to provide partial data?

Going back to the setting of topological spaces, even in that situation the homotopy groups are objects which cannot be trivially determined. But we saw that homology theories provide other invariants which is more likely to be computed and Hurewicz theorem establishes a bridge between these two notions. For any spectrum E the functor $E_*(-) = \pi_*(E \wedge -)$ is an homology theory on $\mathcal{SHC}$, the previous discussion induces us to set up a theory in a way that spectra are determined not by their homotopy groups but by these E-homology groups and a morphism f is an isomorphism if and only if the induced map $E_*(f)$ is a bijection. This was accomplished by Bousfield who laid the foundations for the theory of localizations of $\mathcal{SHC}$ in his seminal paper [4].

Before explaining in details his results we need a few definitions.

Definition 1.1.2. We fix $E \in \mathcal{SHC}$. We say that a spectrum X is *E-acyclic* if $X \wedge E = 0$. A morphism of spectra $f \colon X \to Y$ is called an *E-equivalence* if the induced map $E_*(f)$ is an isomorphism. A spectrum Z is denoted *E-local* if $[f, Z]_*$ is an isomorphism for any E-equivalence f.

Remark 1.1.3. We recall that a spectrum Y is contractible if and only if its homotopy groups $\pi_n(Y)$ are zero for every $n \in \mathbb{Z}$. Then we have that X being E-acyclic is equivalent to $\pi_*(E \wedge X) = E_*(X) = 0$, i.e. its E-homology is zero. Consequently a morphism f is an E-equivalence if and only if its fiber is E-acyclic, therefore a spectrum Z is E-local if and only if $[X, Z] = 0$ for any E-acyclic spectrum X.

Theorem 1.1.4 ([4])**.** *For any spectrum E we can form a functor $L_E \colon \mathcal{SHC} \to \mathcal{SHC}$, together with a natural transformation $\eta \colon Id \Rightarrow L_E$ such that for every spectrum X we have a distinguished triangle*

$$C_E X \to X \xrightarrow{\eta_X} L_E X \to \Sigma C_E X$$

where η_X is an E-equivalence (equivalently $C_E X$ is E-acyclic) and $L_E X$ is E-local.

Moreover, the subcategory of E-local objects in $\mathcal{SHC}$ forms a colocalizing subcategory which can be identified with $L_E\mathcal{SHC}$, the essential image of the functor L_E. So we have a retraction of categories

$$\begin{tikzcd} L_E\mathcal{SHC} \arrow[r, hook] \arrow[rr, bend right, "Id"'] & \mathcal{SHC} \arrow[r, "L_E"] & L_E\mathcal{SHC}. \end{tikzcd}$$

This means that for a generic spectrum X we can present a new spectrum $L_E X$ which is initial among the E-local spectra equipped with a map from X. The theorem establishes that such a localization with respect of a generic homology theory E, characterized by the universal properties spelled out above, not always exists but can be formed in a functorial way. Using this functor we can project a spectrum to a subcategory of $\mathcal{SHC}$ which is hopefully easier to analyse. The following lemma confirms that after this operation the total information is not encapsulated any more by the stable homotopy groups, but by the E-homology.

Lemma 1.1.5 (E-local Whitehead theorem)**.** *Let $f \colon X \to Y$ be an E-equivalence between E-local spectra, then it is an equivalence.*

Proof. By definition, for any E-local spectrum Z the map $[f, Z]$ is an isomorphism. Thus Yoneda lemma applied to the category $L_E\mathcal{SHC}$ shows that f is an isomorphism in this category. Now the second claim of Theorem 1.1.4 implies f is an isomorphism in $\mathcal{SHC}$, therefore f is an equivalence of spectra. □

Another concept worth mentioning associated to these localizations is the notion of Bousfield class.

Definition 1.1.6. Let E, F be spectra. We denote by $\langle E \rangle$ the equivalence class of spectra X such that being X-acyclic is equivalent to being E-acyclic, that is for any spectrum Z we have $Z \wedge X = 0$ if and only if $Z \wedge E = 0$. This is called the *Bousfield class* of E.

We write $\langle E \rangle \leq \langle F \rangle$ if for any spectrum X being F-acyclic implies being E-acyclic, i.e. if $X \wedge F = 0$ then $X \wedge E = 0$.

We can use these classes to compare Bousfield localizations with respect to different spectra.

Lemma 1.1.7. *Let E, F be two spectra. If we have an inequality $\langle E \rangle \leq \langle F \rangle$ then being E-local implies being F-local, also for any spectrum X there exists a canonical E-equivalence $L_F X \to L_E X$. Thus $\langle E \rangle = \langle F \rangle$ if and only if there is a natural isomorphism $L_F \cong L_E$.*

Moreover, the wedge and smash product of spectra descend to operations on the Bousfield classes. In other words we have well defined classes

$$\langle E \rangle \vee \langle F \rangle = \langle E \vee F \rangle \qquad \langle E \rangle \wedge \langle F \rangle = \langle E \wedge F \rangle.$$

Proof. Just unravel the definitions and compare the classes of acyclic spectra. □

At this point we have only have to find the right homology theories which will let us deduce most of the information about the stable homotopy category. An important point is that after applying the localizations to the spectrum we are studying, we want to be able to recollect the partial data obtained from such localizations to a global result on the starting spectrum. Our ideal programme is to first decompose the spectra in smaller pieces, easier to compute, then glue all these pieces back together.

The identification of a family of homology theories which will allow us to realize this project and the study of their properties is what chromatic homotopy theory is about.

1.2 The Morava theories

After Quillen unravelled the connection between the geometry of formal groups and the information that spectra carry encoded in their complex orientation, one of the most successful approaches to study stable homotopy theory is to realize on it a filtration analogous to the one on the moduli stack of formal groups given by the height.

We recall that a ring spectrum E is complex orientable if the map $E^2(\mathbb{C}P^\infty) \to E^2(S^2) \cong E^0 = \pi_0(E)$, induced by the 2-cell inclusion $S^2 \hookrightarrow \mathbb{C}P^\infty$, is surjective. Then a choice of an orientation consists in a choice of an element $x \in E^2(\mathbb{C}P^\infty)$ such that its image is 1.

Fixed the orientation we can provide an isomorphism $E^*(\mathbb{C}P^\infty) \cong \pi_*(E)[\![x]\!]$ and similarly we have

$$E^*(\mathbb{C}P^\infty \times \mathbb{C}P^\infty) \cong \pi_*(E)[\![x \otimes 1, 1 \otimes x]\!].$$

Under these identifications the map $\mathbb{C}P^\infty \times \mathbb{C}P^\infty \to \mathbb{C}P^\infty$, representing the tensor product of complex line bundles, induces

$$\begin{aligned} E^*(\mathbb{C}P^\infty) &\to E^*(\mathbb{C}P^\infty \times \mathbb{C}P^\infty) \\ x &\mapsto F(x \otimes 1, 1 \otimes x) \end{aligned}$$

with $F(s,t)$ a power series in two variables with coefficients in $\pi_*(E)$. Since the operation of taking tensor products admits a unit (the trivial bundle), is commutative and associative we deduce F has the following corresponding properties

- $F(t, 0) = F(0, t) = t$;
- $F(s, t) = F(t, s)$;
- $F(F(s, t), u) = F(s, F(t, u))$;

which let us interpret it as some sort of abelian group operation. In fact we can write $s +_F t = F(s,t)$. A power series satisfying these equations is called *formal group law*.

This procedure let us associate to an oriented spectrum a formal group law, so it comes natural to ask if we can provide some kind of invariant to classify these.

It can be proved that if R is a $\mathbb{Q}$-algebra then any formal group law over R is isomorphic to the additive one $F_a(s,t) = s+t$. Instead, if the ring R is p-torsion the situation is much more varied: we have that the p-series of F, defined as $[p]_F(t) = t +_F \cdots +_F t$ with p summands, either is 0 or it factors as $[p]_F(t) = g(t^{p^n})$ for some $g \in R[\![t]\!]$ with $g'(0) \neq 0$ and a unique positive integer n. This quantity is denoted as the *height* of the formal group law, if $[p]_F(t) = 0$ then it is convention to say that F has *infinite* height.

Since the height is invariant under isomorphism we can use it to provide a stratification of formal group laws. In favourable conditions the height completely determines the isomorphism class: if R is an algebraically closed field of characteristic p then two formal group laws over it are isomorphic if and only if they have the same height.

Quillen proved that the stable homotopy group of complex cobordism spectrum MU realizes the Lazard ring L which classifies the formal group laws. Fixed a prime p, we have a splitting $MU_{(p)} \cong \bigvee_{n\in\mathbb{N}} \Sigma^{2d_n} BP$, where $BP_* \cong \mathbb{Z}_{(p)}[v_1, v_2, \dots]$ and $\{d_n\}_{n\in\mathbb{N}}$ is an appropriate increasing sequence of natural numbers, coming from the fact that over p-local rings any curve decomposes as sum of p-typical ones. Therefore, if we restrict to the p-local stable homotopy category $\mathcal{SHC}_{(p)}$, it is enough to work with this reduced version given by the Brown-Peterson spectrum.

From BP we can construct for every $n \in \mathbb{N}$, the spectra $E(n)$ and $K(n)$, the Johnson-Wilson theory and Morava K-theory respectively, characterized by

$$E(n)_* = \mathbb{Z}_{(p)}[v_1, \dots, v_n^{\pm 1}] \qquad K(n)_* = \mathbb{F}_p[v_n^{\pm 1}].$$

Informally, we should consider the parameters v_n to represent the information at height n. The localizations $L_{E(n)}$ and $L_{K(n)}$ on the stable homotopy category correspond to restricting the stack of formal groups to the open substack of formal groups of height $\leq n$ and taking the completion with respect to the locally closed substack of height exactly n respectively.

These spectra have been extensively studied and have exceptional properties which make them particularly apt to realize the project delineated above. We recollect the most important ones here. In the literature usually $L_{E(n)}$ is abbreviated as L_n, we also adopt this convention for sake of briefness. Since they are p-local, the localizations associated to them provide p-local spectra: therefore from now on we will restrict to consider $\mathcal{SHC}_{(p)}$, the p-local stable homotopy category, as we mentioned above.

Theorem 1.2.1 (Smash product theorem)**.** [35, Thm. 7.5.6] *The spectrum $E(n)$ is smashing, that is for any $X \in \mathcal{SHC}$*

$$L_n X \cong L_n \mathbb{S} \wedge X.$$

Theorem 1.2.2 ([34, Thm. 2.1])**.** *We have a decomposition of Bousfield classes*

$$\langle E(n) \rangle = \bigvee_{i=0}^{n} \langle K(i) \rangle.$$

This implies that for every n the localization map $X \to L_n X$ factors through the $E(n+1)$-localization

$$\begin{tikzcd} & X \arrow[dl] \arrow[dr] & \\ L_{n+1}X \arrow[rr] & & L_n X \end{tikzcd}$$

thus we obtain a tower of spectra of the following form

$$\begin{tikzcd}
 & & & X \arrow[dll] \arrow[dl] \arrow[d] \arrow[dr] \arrow[drr] & & \\
\arrow[r] & \cdots \arrow[r] & L_n X \arrow[r] & \cdots \arrow[r] & L_1 X \arrow[r] & L_0 X
\end{tikzcd}$$

which induces a map $X \to \underleftarrow{\operatorname{holim}} L_n X$. It comes natural to ask if, under the appropriate assumptions, this is an equivalence therefore we can recover X from the data of its $E(n)$-localizations.

Theorem 1.2.3 (Chromatic convergence theorem)**.** [35, Thm. 7.5.7] *If X is a finite spectrum the chromatic tower above converges, in the sense that the map*

$$X \to \underleftarrow{\operatorname{holim}} L_n X$$

is an equivalence.

This implies that to understand the sphere spectrum $\mathbb{S}$ we only have to first disassemble it in its chromatic pieces $L_n\mathbb{S}$, then we can recover it by taking the homotopy limit of the tower they form.

Another reassembly result which relates the localizations with respect to $E(n)$ and $K(n)$ is the chromatic fracture square.

Proposition 1.2.4. *Let X any spectrum, then for any natural number n we have a homotopy pull-back square*

$$\begin{tikzcd}
L_n X \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & L_{K(n)} X \arrow[d] \\
L_{n-1} X \arrow[r] & L_{n-1} L_{K(n)} X.
\end{tikzcd}$$

This should be read as the fact that for any spectrum X its $E(n)$-localization (representing the information at height $\leq n$) can be obtained from its other two localizations $L_{n-1}X$ (height $\leq n-1$) and $L_{K(n)}X$ (containing some information regarding the height n) after they are tuned by the gluing data $L_{n-1}L_{K(n)}X$. The most fitting algebraic analogue would be the p-local Hasse square

$$\begin{tikzcd}
\mathbb{Z}_{(p)} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathbb{Z}_p^{\wedge} \arrow[d] \\
\mathbb{Q} \arrow[r] & \mathbb{Q} \otimes \mathbb{Z}_p^{\wedge}
\end{tikzcd}$$

establishing that we can recover $\mathbb{Z}_{(p)}$ from its p-completion and its rationalization.

While the spectrum $K(n)$ is not smashing (for $n > 0$) it has other extremely good traits. Its main characteristic is that since its homotopy group $K(n)_*$ is a graded field the spectrum itself has a similar property.

Proposition 1.2.5. *For every $n \in \mathbb{N}$ the spectrum $K(n)$ is a skew field in the sense of* [20, Def. 3.7.1 (d)]*: that is, for any spectrum X which is a module over $K(n)$ we have a decomposition*

$$X \cong \bigvee_{i \in I} \Sigma^{k_i} K(n)$$

for some $k_i \in \mathbb{Z}$ with indexing set I.

Proof. We said $K(n)_*$ is a graded field in the sense that any of its homogeneous non-zero elements is invertible. As in the non-graded case this implies that every graded module over it splits as a direct sum of its copies (possibly shifted by degree): given M a graded $K(n)_*$-module then any basis of $\bigoplus_{i=0}^{|v_n|-1} M_i$ as $\mathbb{F}_p$-vector space is actually a $K(n)_*$-basis for M.

Applying this to $\pi_*(X)$ we obtain a decomposition

$$\pi_*(X) \cong \bigoplus_{i \in I} K(n)_{*-k_i}.$$

So we can fix a $K(n)_*$-basis $\{e_i\}_{i\in I}$ for $\pi_*(X)$ with $e_i \in \pi_{k_i}(X)$, these elements together with the module action let us produce maps

$$\Sigma^{k_i} K(n) \cong K(n) \wedge \Sigma^{k_i}\mathbb{S} \xrightarrow{1\wedge e_i} K(n) \wedge X \xrightarrow{\mu} X$$

which can be assembled into

$$\bigvee_{i\in I} \Sigma^{k_i} K(n) \to X.$$

By construction this is an isomorphism on the homotopy groups, so it is an equivalence of spectra. $\square$

A similar reasoning let us deduce the Künneth isomorphism, which is one of the main tools we have to compute $K(n)$-homology.

Proposition 1.2.6. *Let X, Y be two spectra, then we have an isomorphism*

$$K(n)_*(X \wedge Y) \cong K(n)_*(X) \otimes_{K(n)_*} K(n)_*(Y).$$

Proof. Applying Proposition 1.2.5 appropriately we get the following chain of isomorphisms

$$\begin{aligned} K(n)_*(X \wedge Y) = \pi_*(K(n) \wedge X \wedge Y) &\cong \pi_*\left(\bigvee_{i\in I} \Sigma^{k_i} K(n) \wedge Y\right) \cong \\ &\cong \bigoplus_{i\in I} K(n)_{*-k_i}(Y) \cong K(n)_*(X) \otimes_{K(n)_*} K(n)_*(Y). \end{aligned}$$

$\square$

Proposition 1.2.5 is also the key to proving the following crucial fact

Theorem 1.2.7 ([21, Thm. 7.5])**.** *The $K(n)$-local stable homotopy category $L_{K(n)}\mathcal{SHC}$ has no proper non-trivial localizing subcategories. That is if $\mathcal{C} \subseteq L_{K(n)}\mathcal{SHC}$ is a localizing subcategory, then $\mathcal{C}$ is either 0 or the whole category.*

This means that we cannot produce a non-trivial Bousfield localization on the category $L_{K(n)}\mathcal{SHC}$, thus for any spectrum X after forming its $K(n)$-localization $X \to L_{K(n)}X$ we cannot localize further to reduce $L_{K(n)}X$ to a simpler object.

We conclude this section by stating the classification of thick subcategories of finite p-local spectra.

Lemma 1.2.8 ([34, Thm. 2.11])**.** *Let F be a finite p-local spectrum, suppose we have $K(n)_*F = 0$ for some positive integer n. Then we also have $K(n-1)_*F = 0$.*

Definition 1.2.9. Let $n \in \mathbb{N}$. Let F be a finite p-local spectrum, we call it *of type n* if $K(n)_*F \neq 0$ and $K(n-1)_*F = 0$.

We denote by $\mathcal{C}_{\geq n}$ the thick full subcategory of $\mathcal{SHC}_{(p)}$ generated by finite spectra of type greater or equal to n, that is $F \in \mathcal{C}_{\geq n}$ if and only if $K(m)_*F = 0$ for all $m < n$. We also write $\mathcal{C}_{\geq\infty}$ for the trivial subcategory 0.

Remark 1.2.10. We observe that for any spectrum X the graded abelian group $[X, K(n)]_*$ admits a natural $K(n)_*$-action and we have an isomorphism of $K(n)_*$-modules

$$[X, K(n)]^* \cong \mathrm{Hom}_{K(n)_*}(K(n)_*X, K(n)_*).$$

Now if we consider DF the Spanier-Whitehead dual of a finite spectrum F we have $K(n)_*DF \cong [F, K(n)]_*$, therefore DF is of type n if and only if F is. We conclude all the categories $\mathcal{C}_{\geq n}$ are closed under Spanier-Whitehead dual.

Theorem 1.2.11 (Thick subcategory theorem)**.** [35, Thm. 3.4.3] *Let $\mathcal{T}$ be a thick subcategory of $\mathcal{SHC}^c_{(p)}$, the category of p-local compact spectra. Then $\mathcal{T} = \mathcal{C}_{\geq n}$ for some $n \in \mathbb{N} \cup \{\infty\}$.*

Remark 1.2.12. It is a consequence of the Periodicity theorem ([35, Thm. 1.5.4]) that the thick subcategories $\mathcal{C}_{\geq n}$ are all distinct.

Remark 1.2.13. Since $K(n)_*$ is a graded field, for any two $K(n)_*$-modules M, N if $M \neq 0$ and $M \otimes_{K(n)_*} N = 0$ then necessarily $N = 0$. This trivial observation can be used quite effectively in our study of spectra via the Künneth isomorphism and the existence of a spectrum of type n. That is, if X is a generic spectrum then we can fix $F(n)$ a spectrum of type n and see

$$K(n)_*(F(n) \wedge X) \cong K(n)_*F(n) \otimes_{K(n)_*} K(n)_*X$$

therefore X is $K(n)$-acyclic if and only if $F(n) \wedge X$ is. This argument lets us use $F(n)$ as test case for the acyclicity of a spectrum.

1.3 Compositions of localizations

In the previous section we explained why we are interested in the Morava spectra and why their Bousfield localizations are essential tools for modern algebraic topology. The next natural question is how much control can we have about successive compositions of such localization functors: do we get degenerate cases in which we have infinitely many different strings of compositions, or must there be stabilization phenomena which make impossible to obtain such convoluted situations?

We first propose some folklore results which seems to indicate some sort of regularity.

Proposition 1.3.1. *For any natural numbers m, n with $m < n$ we have $L_{K(n)}L_{K(m)} = 0$.*

Proof. We let X be a generic spectrum and we fix F a finite spectrum of type n, this means $K(m)_*(F) = 0$ while $K(n)_*F \neq 0$. Hence $F \wedge L_{K(m)}X \cong L_{K(m)}(F \wedge X) = 0$.

By Proposition 1.2.6 we have

$$K(n)_*(F) \otimes_{K(n)_*} K(n)_*(L_{K(m)}X) \cong K(n)_*(F \wedge L_{K(m)}X) \cong K(n)_*(L_{K(m)}(F \wedge X)) = 0.$$

By Remark 1.2.13 we deduce $L_{K(m)}X$ is $K(n)$-acyclic and the claim follows. □

Proposition 1.3.1 seems to indicate that raising the chromatic height we collapse the composition to zero. It also implies the following result, which indicates a similar simplification when we are considering a localization with respect to a wedge of Morava theories.

Proposition 1.3.2. *Consider natural numbers $m < t < n$, then the natural transformation*

$$L_{K(t)}L_{K(m)\vee K(n)} \Rightarrow L_{K(t)}L_{K(n)}$$

is an isomorphism.

Proof. The indicated map just comes from applying $L_{K(t)}$ to the natural transformation arising from the inequality of Bousfield classes $\langle K(n)\rangle \leq \langle K(m) \vee K(n)\rangle$.

Proposition 4.3.12 together with Proposition 1.3.1 imply we have for any spectrum X a chromatic square of the form

$$\begin{tikzcd} L_{K(m)\vee K(n)}X \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & L_{K(n)}X \arrow[d] \\ L_{K(m)}X \arrow[r] & L_{K(m)}L_{K(n)}X. \end{tikzcd}$$

Applying $L_{K(t)}$ we still get a homotopy pull-back square, but by Proposition 1.3.1 the lower row is constantly 0 hence the upper horizontal map must be an isomorphism. □

Remark 1.3.3. We postpone the proof of Proposition 4.3.12 which allows us to obtain the above chromatic fracture square, it is nothing specific of the Morava K-theories and we will prove a far more general result.

The results exposed up to now suggest that the compositions of localizations involving Morava theories should behave reasonably well if we set a limit for the maximum chromatic height. This intuition inspired the following conjecture, which is the first stepping stone of this work.

Definition 1.3.4. We fix a positive integer n. Let $A \subseteq \{0, 1, \ldots n-1\}$, we define

$$K(A) = \bigvee_{a \in A} K(a)$$

and set $\lambda_A = L_{K(A)}$. If $\mathbb{A} = (A_1, \ldots, A_l)$ is an l-uple of such subsets we denote by $\lambda_{\mathbb{A}}$ the composition $L_{K(A_1)}L_{K(A_2)} \ldots L_{K(A_l)}$.

Remark 1.3.5. It is immediate that $\lambda_{\{a\}} = L_{K(a)}$ and Theorem 1.2.2 implies $\lambda_{\{0,\ldots,k\}} = L_{E(k)}$ for any $k < n$.

Conjecture 1.3.6. *Fix the integer n as above, then all the possible iterated localizations $\lambda_{\mathbb{A}}$ are, up to natural isomorphism, only finitely many.*

The results proved up to this point already let us prove this conjecture for low values of n.

Example 1.3.7. If $n = 1$ the only possible composition we examine is the rationalization $L_{K(0)}$.

If $n = 2$, we are considering $L_{K(0)}$, $L_{K(1)}$ and $L_{E(1)}$: idempotency of localizations together with Theorem 1.2.2 and Proposition 1.3.1 let us compute all the possible compositions. These are $L_{K(1)}L_{K(0)} = 0$, $L_{K(i)}L_{E(1)} = L_{E(1)}L_{K(i)} = L_{K(i)}$ for $i = 0, 1$.

The case $n = 3$ can be solved with Proposition 1.3.2.

Even if Conjecture 1.3.6 is easy to formulate and understand it leaves a lot to be desired: for example we ask only the existence of a natural isomorphism establishing two compositions to be the same, but in fact what happens is that these iterated localizations are connected by a network or transformations arising from the homotopical universal properties characterizing the localizations. These are the maps we want to consider and determine if they are invertible.

We are going to improve Conjecture 1.3.6 in the following way:

1) for a generic spectrum X we will construct a diagram whose vertices represent these iterated compositions of localizations of X and the edges are the appropriate combinations of the natural transformations $\eta\colon Id \Rightarrow L_{K(A)}$.

2) We will form this diagram in a homotopy coherent manner: that is, if $\mathbb{Q}$ denotes its shape we will not provide a functor of $(\mathcal{SHC}_{(p)})^{\mathbb{Q}}$ but instead an element of the homotopy category of diagrams $\mathrm{Ho}(\mathrm{Sp}^{\mathbb{Q}})$, where Sp is a geometric model for the p-local stable homotopy category.

3) To verify the finiteness of the number of different localizations, we will show we can take as indexing diagram $\mathbb{Q}$ a finite poset. Moreover, we will endow $\mathbb{Q}$ with a binary operation which keeps track of the compositions of localizations.

To deal with the homotopy theoretical part of our proof and realize the claim 2) we will employ the theory of derivators, for which we will have to develop a few ad-hoc results not present in the literature. These will be presented in §3.7, 3.8 after we give a brief overview on the basics of derivators in §3.1-3.6.

We will show that such diagrams for different spectra X form the underlying category of a strong stable derivator $\mathcal{F}$ and the functoriality in X will be encoded via a theory of anafunctors, which we develop in §4.1, 4.2.

The first step towards constructing the diagram of 1) is that we can express $L_{K(A)}X$ as homotopy limit of a cube whose vertices are provided by compositions of localizations $L_{K(a)}$ for the various $a \in A$. This is a generalization of the usual chromatic fracture square and will be presented in §4.3.

In §4.5 we will finally construct the derivator $\mathcal{F}$ with the associated diagrams. The finiteness outcome will be achieved by proving a formula for the composition of two iterated localizations (Theorem 4.5.11) which will depend only on the combinatorics of the underlying poset $\mathbb{Q}$. This not only will complete the proof of Conjecture 1.3.6 but also provide us with a useful way to compute the composition of localizations completely independent of their geometric meaning, since this composition can be modelled by the operation on $\mathbb{Q}$.

Moreover, we will formulate our proof in an axiomatic framework so it will hold for any family of localizations satisfying a slight generalization of Proposition 1.3.1. Therefore the result will be true not exclusively for the chromatic case, even if it was our starting point and it is the most important example we can think of.

Chapter 2

Results and their formulations

2.1 The general setting

As we mentioned above, we are going to work with a well-behaved family of localizations on a category more general than the stable homotopy category. We make now explicit what this means and establish the setting of our discussion.

Definition 2.1.1. We fix a stable model category or a quasi-category $\mathcal{B}_0$, such that its homotopy category $\mathcal{B} = \mathrm{Ho}(\mathcal{B}_0)$ is a compactly generated triangulated category.

We also set an integer $n \geq 1$ and put $N = \{0, \ldots, n-1\}$ as before. Then we fix a family of homology theories $K(i)_* \colon \mathcal{B} \to \mathrm{Ab}_*$ for $i \in N$ and define $K(A)_* = \bigoplus_{a \in A} K(a)_*$ for any $A \subseteq N$. We let λ_A denote the localisation with respect to the localizing subcategory of $K(A)_*$-acyclics.

Remark 2.1.2. Even if we are interested only on the homotopy category $\mathcal{B}$ and its localizations, rather than the specific choice of its geometric model, we invoke the existence of such $\mathcal{B}_0$ so that $\mathcal{B}$ coincides with the underlying category of a strong, stable derivator $\mathcal{C}$. This condition is not strictly required for our argument to work, but it is the most common way to form a derivator. Also, the homology theories are usually established on the homotopy category or its model. Therefore, this comes more natural than asking for the existence of a derivator whose underlying category admits certain homology theories.

Remark 2.1.3. While the matter of the actual existence of the localizations λ_A is not trivial, it has been dealt in the literature exhaustively. One of the best classical references for the existence of Bousfield localizations in a general, well behaved triangulated category is [29, Ch. 9]. Margolis in [28, Ch. 7] presents a proof for the stable homotopy category, but it can be generalized to other (sufficiently nice) triangulated categories. This is the approach we adopted in [2, App. A].

It is worth mentioning that analogous existence results have been developed internal to the geometric models, for both settings of model categories and quasi-categories. Reference for these two approaches are [17] and [24, §5.2] respectively.

Definition 2.1.4. In the setting of Definition 2.1.1 we say the family of homology theories $K(i)_*$ satisfies the *fracture axiom* if for every A non-empty subset of N, and $b \in N$ with $b > \max(A)$, then $K(b)_* \lambda_A(X) = 0$ for all $X \in \mathcal{B}$. I.e. being $K(A)_*$-local implies being $K(b)_*$-acyclic.

Definition 2.1.5. We let $\mathbb{P} = \mathcal{P}(N)$ be the set of subsets of N, ordered by inclusion. For $A, B \in \mathbb{P}$, we write $A \angle B$ if $a \leq b$ for all $a \in A$ and $b \in B$.

Remark 2.1.6. Note that $A \angle B$ is vacuously satisfied if $A = \emptyset$ or $B = \emptyset$, and because of this, the relation is not transitive.

Lemma 2.1.7. *The fracture axiom implies the following (apparently more general) statement: if $A, B \in \mathbb{P}$ with $A \angle B$ and $K(B)_*(X) = 0$, then $K(B)_*(\lambda_A(X)) = 0$.*

Proof. If $A = \emptyset$ then $K(A)_* = 0$ and so $\lambda_A = 0$ and everything is trivial. We can thus assume that $A \neq \emptyset$, so $\max(A)$ is defined. The assumption $A \angle B$ then means that $b \geq \max(A)$ for all $b \in B$. We are given that $K(B)_*(X) = 0$, or in other words that $K(b)_*(X) = 0$ for all $b \in B$. We want to prove that $K(b)_*(\lambda_A(X)) = 0$. If $b > \max(A)$ then this is immediate from the fracture axiom. This just leaves the case where $b = \max(A)$, so $b \in A$. The map $X \to \lambda_A(X)$ is a $K(A)$-equivalence, so it is a $K(b)$-equivalence (because $b \in A$), and $K(b)_*(X) = 0$ by assumption, so $K(b)_*(\lambda_A(X)) = 0$ as required. $\square$

We next provide a criterion to verify the fracture axiom.

Proposition 2.1.8. *Let $\mathcal{B}$ be a stable homotopy category in the sense of* [20, Def. 1.1.4] *(so it is a closed tensor triangulated category with a set of strongly dualizable generators). Assume we have N as before and objects $K(i) \in \mathcal{B}$ representing the homology theories $K(i)_*(X) = \pi_*(K(i) \wedge X)$. Suppose we also have objects $F(i) \in \mathcal{B}$, and that the following conditions are satisfied:*

(a) $F(i)$ is strongly dualizable for all i.

(b) For $j < i$ we have $K(j) \wedge F(i) = 0$.

(c) For any object $X \in \mathcal{B}$ and any i we have $K(i) \wedge X = 0$ if and only if $K(i) \wedge F(i) \wedge X = 0$.

Then the fracture axiom is satisfied.

Proof. We take a subset $A \in \mathbb{P}$, to avoid trivial cases we assume it is non-empty. We suppose that $b > \max(A)$. We need to show that $K(b) \wedge \lambda_A(X) = 0$, by condition (c) this is the same as showing that $K(b) \wedge F(b) \wedge \lambda_A(X) = 0$. For this it will suffice to show that $F(b) \wedge \lambda_A(X) = 0$, or that the identity map of $F(b) \wedge \lambda_A(X)$ is zero which is equivalent to proving the adjoint map

$$DF(b) \wedge F(b) \wedge \lambda_A(X) \to \lambda_A(X)$$

is zero. Here $K(A) \wedge F(b) = 0$ by condition (b), so the source of the above map is $K(A)$-acyclic, whereas the target is $K(A)$-local; this implies that the map is zero as required. $\square$

The simplest example of a setting as in Definition 2.1.1 we can present is the following provided by algebra.

Example 2.1.9. Let $\mathcal{B} = D(\mathbb{Z}_{(p)})$ be the derived category of modules over $\mathbb{Z}_{(p)}$, and put

$$K(0) = \mathbb{Q} \qquad\qquad K(1) = \mathbb{Z}/p$$
$$F(0) = \mathbb{Z}_{(p)} \qquad\qquad F(1) = \mathbb{Z}/p.$$

It is then straightforward to check the hypotheses of Proposition 2.1.8, so the fracture axiom is satisfied.

$F(0)$ coincides with the unit of $\mathcal{B}$, while $F(1)$ is quasi-isomorphic to

$$\cdots \to 0 \to \mathbb{Z}_{(p)} \xrightarrow{p} \mathbb{Z}_{(p)} \to 0 \to \cdots$$

hence they are both perfect complexes. The equality $K(0) \wedge F(1) = \mathbb{Q} \otimes^{\mathbb{L}} \mathbb{Z}/p = 0$ comes from the fact that after tensoring with $\mathbb{Q}$ the above chain complex it becomes acyclic. Condition (c) is verified because $F(0)$ is the tensor unit and $F(1) \otimes^{\mathbb{L}} K(1) \cong \mathbb{Z}/p \oplus \Sigma\mathbb{Z}/p$ (to see this just apply $\mathbb{Z}/p \otimes -$ to the above complex).

Example 2.1.10. Now the motivating example. Let $\mathcal{B}_0$ denote the category of symmetric spectra of simplicial sets, equipped with the p-localisation of the usual model structure. Put $\mathcal{B} = \mathrm{Ho}(\mathcal{B}_0)$ (so this is the usual $\mathcal{SHC}_{(p)}$). For any $i \in N = \{0, \ldots, n-1\}$, let $K(i)$ denote the Morava K-theory spectrum of height i at the prime p, and let $F(i)$ be any finite p-local spectrum of type i. It is again straightforward to check the hypotheses of Proposition 2.1.8.

Assumptions (a) and (b) are satisfied by construction of our $F(i)$'s. (c) follows immediately from Remark 1.2.13.

Therefore, the fracture axiom is verified.

Example 2.1.11. Let $\mathcal{B} = \mathcal{SHC}_{(p)}$ again. As before $F(i)$ will denote any finite spectrum of type i. By the periodicity theorem [35, Thm. 1.5.4] it admits a v_i-self map, that is a map $f_i \colon \Sigma^{d_i} F(i) \to F(i)$ such that it is an isomorphism on $K(i)$-homology and for $j \neq i$ $K(j)_*(f_i)$ is nilpotent. We will be considering as objects representing our homology theories $T(i) = f_i^{-1} F(i)$ the i-th telescope.

Again (a) holds by construction of the objects $F(i)$. (b) follows from the fact that for $j < i$ we have $T(j) \wedge F(i) = (f_j \wedge 1)^{-1}(F(j) \wedge F(i))$. Observe $K(n)_*(f_j \wedge 1) \cong K(n)_*(f_j) \otimes 1$ must be nilpotent for every $n \in \mathbb{N}$. Thus, by an application of the nilpotence theorem [18, Thm. 3 (ii)], the morphism $f_j \wedge 1$ itself is nilpotent and the associated telescope must be trivial.

An alternative argument is that $f_j \wedge 1$ and 0 are both v_j-self maps for $F(j) \wedge F(i)$, so for the asymptotically uniqueness of these maps ([35, Thm. 1.5.4 (ii)]) there exists some $N \in \mathbb{N}$ such that $f_j^N \wedge 1 = 0$.

Finally, (c) follows from the following fact: we have $X \wedge T(n) = 0$ if and only if $X \wedge T(n)' = 0$ where $T(n)'$ is another telescope of a v_n-self map on a type n finite spectrum. Assuming this, we have only to notice that $T(n) \wedge F(n) = (f_n \wedge 1)^{-1}(F(n) \wedge F(n))$ is indeed a new n-th telescope. The claim is equivalent to the fact that the Bousfield class of $T(n)$ is independent of the choice of $F(n)$ and f_n. This is a well known fact in the literature, for a reference see [25, Lemma 2.1].

This example is relevant since the Bousfield localization associated to $\bigvee_{i=0}^{n} T(i)$ coincides with the finite localization associated to $E(n)$, usually denoted L_n^f. There exists a canonical comparison map $L_n^f \Rightarrow L_n$, determining whether this is an equivalence or not is the content of Telescope Conjecture.

Example 2.1.12. Let $G = C_{p^{n-1}}$ be the cyclic group of order p^{n-1}. We take $\mathcal{B}_0$ to be the category of orthogonal G-equivariant spectra with the stable model structure provided in [7, Thm. 1.2.22], so that $\mathcal{B}$ is the G-equivariant stable homotopy category.

We set H_i for $0 \leq i \leq n-1$ to be the subgroup of G of order p^{n-1-i}, so we have a sequence of inclusions

$$e = H_{n-1} \leq H_{n-2} \leq \cdots \leq H_1 \leq H_0 = G.$$

As it is usual in the literature, fixed $\mathcal{F}$ a family of subgroups of G closed under subgroups and conjugacy we denote by $E\mathcal{F}$ the unbased G-orthogonal space characterized, up to G-equivariant homotopy equivalence, by

$$E\mathcal{F}^K \simeq \begin{cases} \emptyset & \text{if } K \notin \mathcal{F} \\ * & \text{if } K \in \mathcal{F} \end{cases}$$

for all $K \leq G$ and its unreduced suspension by $\tilde{E}\mathcal{F}$. This is a pointed G-space with the following the property

$$\tilde{E}\mathcal{F}^K \simeq \begin{cases} * & \text{if } K \in \mathcal{F} \\ S^0 & \text{if } K \notin \mathcal{F} \end{cases}$$

for any $K \leq G$.

We define the families of subgroups $\mathcal{F}_i = \{K \leq G : K \leq H_i\} = \{H_j : j \geq i\}$ for $0 \leq i \leq n-1$, using these we can set

$$K(i) = \Sigma^\infty_+ G/H_i \wedge \Sigma^\infty \tilde{E}\mathcal{F}_{i+1} \qquad F(i) = \Sigma^\infty_+ G/H_i$$

where we mean $K(n-1) = \Sigma^\infty_+ G$. We notice that the $F(i)$'s are finite G-spectra ([7, Prop. 1.3.10]), so condition (a) is satisfied.

It is immediate to compute the geometric fixed points of these spectra

$$\Phi^{H_m}(F(i)) = \begin{cases} 0 & \text{if } m < i \\ F(i) & \text{if } m \geq i \end{cases} \qquad \Phi^{H_m}(K(i)) = \begin{cases} 0 & \text{if } m \neq i \\ \Sigma^\infty_+ G/H_i & \text{if } m = i. \end{cases}$$

Using these results and the fact that geometric fixed points commute with the smash product we see (b) holds.

To verify (c) we compute explicitly the smash product

$$F(i) \wedge T(i) = \Sigma^\infty_+ G/H_i \wedge \Sigma^\infty_+ G/H_i \wedge \Sigma^\infty \tilde{E}\mathcal{F}_{i+1}.$$

Since $\Sigma^\infty_+ G/H$ are cofibrant objects, for these factors the derived and non-derived smash products coincide thus

$$\Sigma^\infty_+ G/H_i \wedge \Sigma^\infty_+ G/H_i \wedge \Sigma^\infty \tilde{E}\mathcal{F}_{i+1} \cong \Sigma^\infty_+ (G/H_i \times G/H_i) \wedge \Sigma^\infty \tilde{E}\mathcal{F}_{i+1}.$$

We recall the double coset decomposition

$$G/H \times G/K \cong \coprod_{H \backslash G/K} G/(H^g \cap K)$$

which applied here for $H = K = H_i$ gives us the isomorphism

$$G/H_i \times G/H_i \cong \coprod_{G/H_i} G/H_i.$$

We deduce

$$F(i) \wedge K(i) \cong \bigvee_{G/H_i} \Sigma^\infty_+ G/H_i \wedge \Sigma^\infty \tilde{E}\mathcal{F}_{i+1} = \bigvee_{p^i} K(i)$$

and the last condition is ensured.

Before concluding we remark the following fact: unlike the previous examples, here the condition $[\mathbb{S}, K(i) \wedge X]_* = 0$ does not guarantee a priori that $K(i) \wedge X = 0$. That is the tensor unit, namely $\mathbb{S} = \Sigma^\infty_+ G/G$, is not a compact generator for $\mathcal{B}$. Instead we have to consider the whole set $\{\Sigma^\infty_+ G/H \;:\; H \leq G\}$ (see [7, Cor. 1.3.11]).

Nevertheless we define $K(i)_*(X) = [\mathbb{S}, K(i) \wedge X]_*$ and simplify this expression: the Wirthmüller isomorphism ([7, Thm. 2.1.10]) ensures that $\Sigma^\infty_+ G/H$ is self-dual thus

$$\begin{aligned} K(i)_*(X) = [\mathbb{S}, K(i) \wedge X]_* &= [\Sigma^\infty_+ G/H_i, \Sigma^\infty \tilde{E}\mathcal{F}_{i+1} \wedge X]_* = \\ &= \pi^{H_i}_*(\Sigma^\infty \tilde{E}\mathcal{F}_{i+1} \wedge X) = \pi_*(\Phi^{H_i} X) \end{aligned}$$

where the last isomorphism comes from [27, Ch. V, Prop. 4.17] after observing $\mathcal{F}_{i+1}$ coincides with the family of subgroups not containing H_i.

Therefore for any subgroup $H \leq G$ we have

$$\begin{aligned} [\Sigma^\infty_+ G/H, K(i) \wedge X]_* &= [\mathbb{S}, K(i) \wedge X \wedge \Sigma^\infty_+ G/H]_* = \\ &= \pi_*(\Phi^{H_i}(X \wedge \Sigma^\infty_+ G/H)) = \pi_*(\Phi^{H_i} X \wedge \Phi^{H_i}(\Sigma^\infty_+ G/H)) \end{aligned}$$

and $\Phi^{H_i}(\Sigma^\infty_+ G/H)$ is non-equivariantly either 0 or a finite coproduct of copies of $\mathbb{S}$.

We deduce that actually $K(i)_*(X) = 0$ if and only if $\phi^{H_i} X = 0$ (as non-equivariant spectrum) if and only if $K(i) \wedge X = 0$.

2.2 The main theorem

We proceed to give the statement of our main theorem. First we need to fix some notation regarding the iterated localizations.

Definition 2.2.1. Let $\mathcal{B}_0$, n and $K(i)_*$ like in Definition 2.1.1 and assume they satisfy the fracture axiom. Let $A = \{a_1 < a_2 < \cdots < a_k\}$ be an element of $\mathbb{P}$, then we define for every $X \in \mathcal{B}$

$$\phi_A(X) = \lambda_{\{a_1\}}\lambda_{\{a_2\}} \dots \lambda_{\{a_k\}}(X).$$

We notice that by the fracture axiom this is the only ordering of the localizations $\lambda_{\{a_i\}}$ which makes the composition not necessarily trivial.

If $\mathbb{A} = (A_1, \dots, A_l)$ is an l-uple of subsets of N we set

$$\lambda_{\mathbb{A}}(X) = \lambda_{A_1}\lambda_{A_2} \dots \lambda_{A_l}(X).$$

Definition 2.2.2. Let $\mathbb{Q}$ be the set of all subsets of $\mathbb{P}$ that are upwards closed, i.e. if $U \in \mathbb{Q}$ and $A, B \in \mathbb{P}$ with $A \subset B$ then $A \in U$ implies $B \in U$. We endow it with inverse inclusion ordering: that is for any two $U, V \in \mathbb{Q}$ we set $U \leq V$ if $V \subseteq U$. We define $u\colon \mathbb{P} \to \mathbb{Q}$ by $uA = \{B : A \subseteq B\}$, so u is a morphism of posets. We also define $v\colon \mathbb{P} \to \mathbb{Q}$ by $vA = \{B : B \cap A \neq \emptyset\}$, so v is order-reversing.

Remark 2.2.3. In $\mathbb{P}$, the smallest element is $\emptyset$ and the largest element is N. In $\mathbb{Q}$, the smallest element is $u\emptyset = \mathbb{P}$ and the largest element is $\emptyset$. Consider $\mathcal{P}'(N)$, the set of non-empty subsets of N, it is immediate that this coincides with $u\emptyset \setminus \{\emptyset\} \in \mathbb{Q}$. This is the second-smallest element in $\mathbb{Q}$: that is, if $U \in \mathbb{Q}$ and $U \neq u\emptyset$ then $u\emptyset \setminus \{\emptyset\} \leq U$. The element uN is second-largest in $\mathbb{Q}$, in the sense that every element $U \in \mathbb{Q}$ with $U \neq \emptyset$ satisfies $U \leq uN$.

Lemma 2.2.4. *There is a map* $*\colon \mathbb{Q} \times \mathbb{Q} \to \mathbb{Q}$ *of posets given by*

$$U * V = \{A \cup B \mid A \in U,\ B \in V,\ A \angle B\}.$$

This operation is associative, with

$$U * V * W = \{A \cup B \cup C \mid A \in U,\ B \in V,\ C \in W,\ A\angle B,\ A\angle C,\ B\angle C\},$$

and $u\emptyset$ *is a two-sided identity element. Moreover, it is distributive on both sides with respect to the union.*

Proof. Suppose that $A \in U$, $B \in V$, $A\angle B$ and $A \cup B \subseteq C$. We can then choose t such that $a \leq t$ for all $a \in A$, and $t \leq b$ for all $b \in B$. We put $A' = \{c \in C \mid c \leq t\}$ and $B' = \{c \in C \mid t \leq c\}$. Then $A \subseteq A'$ so $A' \in U$, and $B \subseteq B'$ so $B' \in V$. We also have $C = A' \cup B'$ with $A'\angle B'$, so $C \in U * V$. This proves that $U * V$ is closed upwards, so we have indeed defined a map $*\colon \mathbb{Q} \times \mathbb{Q} \to \mathbb{Q}$. It is clear that if $U \subseteq U'$ and $V \subseteq V'$ then $U * V \subseteq U' * V'$, so $*$ is indeed a morphism of posets. All remaining claims can also be trivially verified. $\square$

We are finally ready to state our results.

Theorem 2.2.5. *In the situation of Definition 2.1.1 and Definition 2.1.4 we denote by* $\mathcal{C}$ *the homotopy derivator associated to* $\mathcal{B}_0$*. Then there exists a strong stable derivator of fracture diagrams* $\mathcal{F}$*, which is a subderivator of the shifted derivator* $\mathcal{C}^{\mathbb{Q}}$ *and such that the restriction to* $u\emptyset$ *provides an equivalence* $u\emptyset^*\colon \mathcal{F} \to \mathcal{C}$*.*

Using this, we can define for every $U \in \mathbb{Q}$ *an anafunctor*

$$\theta_U\colon \mathcal{C} \xleftarrow{u\emptyset^*} \mathcal{F} \xrightarrow{U^*} \mathcal{C}.$$

These anafunctors satisfy the following composition law $\theta_U \circ \theta_V \cong \theta_{U*V}$*.*

We now unravel part of the information contained in this Theorem to make more clear how this result implies Conjecture 1.3.6.

Corollary 2.2.6. *For every object $X \in \mathcal{B}$ we can provide in a functorial manner a coherent diagram $Z \in Ho(\mathcal{B}_0^{\mathbb{Q}})$. If we denote its value at $U \in \mathbb{Q}$ by Z_U, then this diagram has the properties that $Z_{u\emptyset} \cong X$ and for any $A, B \in \mathbb{P}$ we have $Z_{uA} \cong \phi_A(X)$, $Z_{vB} \cong \lambda_B(X)$. Moreover, the isomorphism $Z_{vA*vB} \cong \lambda_A\lambda_B(X)$ implies that for any l-uple $(A_1, \dots, A_l)$ of elements of $\mathbb{P}$ the iterated localization $\lambda_{\mathbb{A}}(X)$ coincides with the value $Z_{vA_1*\cdots*vA_l}$. Since $\mathbb{Q}$ is a finite poset we deduce that these $\lambda_{\mathbb{A}}$ are finitely many up to isomorphism.*

Proof. We have just to consider the underlying level of the derivators and their anafunctors in Theorem 2.2.5.

The underlying category of $\mathcal{F}$ consists of diagrams $Z \in \mathrm{Ho}(\mathcal{B}_0^{\mathbb{Q}})$ with the property that if X denotes the initial vertex $Z_{u\emptyset}$ then value at the vertex uA is given by the iterated localization $\phi_A(X)$ (see Remark 4.5.6) and at vA we have the localization $\lambda_A(X)$ (by Proposition 4.5.10). Accordingly with this, the edge corresponding to the inequality $u\emptyset \le vA$ in $\mathbb{Q}$ can be identified with the $K(A)$-equivalence $X \to \lambda_A(X)$ and $u\emptyset \le uA$ provides the map $X \to \phi_A(X)$ coming from combining appropriately the natural transformations associated to the single localizations $\lambda_{\{a\}}$ for all the elements $a \in A$.

The anafunctors θ_U induce functors $(\theta_U)_e$ which can be considered generalizations of our iterated localizations in the sense that $(\theta_{uA})_e \cong \phi_A$ and $(\theta_{vA})_e \cong \lambda_A$.

By construction we have for any $Z \in \mathcal{F}(e)$ an isomorphism $(\theta_U)_e(Z_{u\emptyset}) \cong Z_U$, which let us conclude. $\square$

It is clear that after invoking this much derivator technology we have to spend a few words on it. Indeed, the next chapter will be dedicated to establishing the basic terminology and technical results we developed for our needs.

Chapter 3

Derivator theory

3.1 Motivations

Derivator theory was independently started by Gothendieck [14] and Heller [16], moreover it received analogous formulations later by Franke [8], Keller [22] and Maltsiniotis [26]. The core problem it tries to solve is the following: if we consider a triangulated category as defined by Verdier we have that the axioms guarantee only the existence of the cone of a morphism, but not its functionality. This has repercussions on the whole theory of homotopy (co)limits of the category in the sense that even when we can have results of existence of ad hoc versions of such objects they usually are not functorial and we can only hope to guarantee some sort of weak universal property. That is, we do not have the existence of canonical (i.e. unique) morphism arising from such properties.

The intuition behind the solution, which constitutes the core idea of derivators, is that the triangulated categories we want to study usually arise as homotopy categories of some geometric model, and on such model the construction of homotopy fiber can be done functorially. Let us consider the a simple example given by the derived category of an abelian category $\mathcal{A}$.

This is defined starting from $\mathrm{Ch}(\mathcal{A})$, the category of chain complexes in $\mathcal{A}$, after various procedures finalized to formally invert the quasi isomorphisms. As we stated, if we start with a morphism $f\colon X \to Y$ in $D(\mathcal{A})$ then its cone $C(f)$ is an object characterized by an exact triangle

$$X \xrightarrow{f} Y \to C(f) \to \Sigma X$$

such that if we have another morphism g determining a second cone and the two maps fit in a commutative diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & Y \arrow[d] \\ X' \arrow[r, "g"] & Y' \end{tikzcd}$$

then it can be extend to

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & Y \arrow[r] \arrow[d] & C(f) \arrow[r] \arrow[d, dashed, "\exists"] & \Sigma X \arrow[d] \\ X' \arrow[r, "g"] & Y' \arrow[r] & C(g) \arrow[r] & \Sigma X' \end{tikzcd}$$

in a non-unique way. Thus we cannot form a functor $C\colon D(\mathcal{A})^{[1]} \to D(\mathcal{A})$.

But if we started with a morphism $\tilde{f}$ in $\mathrm{Ch}(\mathcal{A})$ representing f then we have available the following procedure: the first step is to enlarge $\tilde{f}$ considered as an element of $\mathrm{Ch}(\mathcal{A})^{[1]}$ to a span

$$\begin{tikzcd} X \arrow[r, "\tilde{f}"] \arrow[d, "i"] & Y \\ CX & \end{tikzcd}$$

where CX is the usual cone chain complex given by $(CX)_n = X_n \oplus X_{n-1}$ and $i\colon X \to CX$ is given degree-wise by the inclusion in the first copy of the sum. The second step is forming the push-out of such diagram, so we get

$$\begin{tikzcd} X \arrow[r, "\tilde{f}"] \arrow[d] & Y \arrow[d, "\mathrm{cof}(\tilde{f})"] \\ CX \arrow[r] & C(\tilde{f}). \end{tikzcd}$$

It is a standard fact that the homotopy class of $C(\tilde{f})$ coincides with the above cone $C(f)$. The advantage of this construction is that it is completely functorial, so it allows us to define $C\colon D(\mathcal{A}^{[1]}) \to D(\mathcal{A})$.

This gives us the idea that the category $D(\mathcal{A}^{[1]})$ contains much more information than $D(\mathcal{A})^{[1]}$, in particular the extra data allow us to form the functorial cone construction we desire. Actually, this procedure can be generalized to provide functors $D(\mathcal{A}^I) \to D(\mathcal{A})$ providing homotopy limits and colimits of diagrams of shape I. To further confirm our picture, observe that the objects in $D(\mathcal{A}^I)$ are honest commuting diagrams of $\mathrm{Ch}(\mathcal{A})^I$, while the objects in $D(\mathcal{A})^I$ are diagrams commuting only up to homotopy.

This excursus tells us that instead of working just with the triangulated category $D(\mathcal{A})$ we should actually consider a whole collection of categories $D(\mathcal{A}^I)$ which should be related by functors $D(\mathcal{A}^I) \to D(\mathcal{A}^J)$ enabling us to construct all the homotopy limits and colimits we need.

The most simple way to do this is to encode all this information in a 2-functor

$$D\colon \mathrm{Dia}^{op} \to \mathrm{TCAT} \qquad I \mapsto D(\mathcal{A}^I)$$

subject to certain axioms. Here Dia is an appropriate full 2-subcategory of Cat, the 2-category of small categories, while TCAT denotes the subcategory determined by triangulated categories (not necessarily small) with 1-cells the exact functors. With Dia^{op} we indicate the 2-category obtained by formally reversing only the 1-cells, not the 2-cells.

This approach to homotopy theory was developed before the widespread use of the theory of model categories (first defined by Quillen) and ∞-categories (popularized by Lurie and Joyal) which are the most common ways used nowadays to perform computations on homotopy categories. Derivator theory is a much more simplistic approach since we do not work explicitly with a model where we can develop a well behaved theory of homotopy fibers and cofibers but rather we require the homotopy category to admit an extension to a diagram of triangulated categories which contains functor producing all the homotopy limits and colimits we want.

Since we could reduce our argument to work for any collection of homology theories satisfying the fracture axiom, regardless of other their properties and of the geometric model for the ambient category, the axiomatic approach offered by derivator theory is the most fitting to the generality of our proof.

3.2 Recollection on classical category theory

Before actually starting with derivator theory we will need some background on category theory. The aim of this section is to fix the notation and recollect basic facts about adjunctions and mate transformations.

Definition 3.2.1. An adjunction $(L, R)\colon \mathcal{A} \rightleftarrows \mathcal{B}$ can be defined in two equivalent ways. The most common one consists in a datum of bijections

$$\mathrm{Hom}_{\mathcal{B}}(LX, Y) \xrightarrow{\phi_{X,Y}} \mathrm{Hom}_{\mathcal{A}}(X, RY)$$

natural in $X \in \mathcal{A}$ and $Y \in \mathcal{B}$. This means that for every morphism $u\colon X \to X'$ and $v\colon Y \to Y'$ we have a commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{B}}(LX', Y) \arrow[r, "\phi_{X',Y}"] \arrow[d, "\mathrm{Hom}(Lu,v)"'] & \mathrm{Hom}_{\mathcal{A}}(X', RY) \arrow[d, "\mathrm{Hom}(u,Rv)"] \\
\mathrm{Hom}_{\mathcal{B}}(LX, Y') \arrow[r, "\phi_{X,Y'}"] & \mathrm{Hom}_{\mathcal{A}}(X, RY').
\end{tikzcd}$$

This is equivalent to providing two natural transformations $\eta\colon Id \Rightarrow RL$ and $\varepsilon\colon LR \Rightarrow Id$, called the *unit* and *counit* of the adjunction, which make the following two diagrams commute

$$\begin{tikzcd}
L \arrow[r, "L\eta"] \arrow[dr, "Id"'] & LRL \arrow[d, "\varepsilon L"] \\
& L
\end{tikzcd}
\qquad
\begin{tikzcd}
R \arrow[r, "\eta R"] \arrow[dr, "Id"'] & RLR \arrow[d, "R\varepsilon"] \\
& R
\end{tikzcd}$$

these are called the *triangular identities.*

Lemma 3.2.2. *The two notions proposed in Definition 3.2.1 are equivalent.*

Proof. If we start with the class of bijections ϕ then we set $\eta_X = \phi_{X,LX}(Id_{LX})$ and $\varepsilon_Y = \phi^{-1}_{RY,Y}(Id_{RY})$, from the naturality of ϕ we easily deduce such η and ε are natural as well. For any morphism $f\colon LX \to Y$ we form the commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{B}}(LX, LX) \arrow[r, "\phi_{X,LX}"] \arrow[d, "\mathrm{Hom}(LId_X,f)"'] & \mathrm{Hom}_{\mathcal{A}}(X, RLX) \arrow[d, "\mathrm{Hom}(Id_X,Rf)"] \\
\mathrm{Hom}_{\mathcal{B}}(LX, Y) \arrow[r, "\phi_{X,Y}"] & \mathrm{Hom}_{\mathcal{A}}(X, RY),
\end{tikzcd}$$

if we consider the element Id_{LX} in the upper left set then commutativity implies $\phi_{X,Y}(f) = Rf \circ \eta_X$ and similarly we can prove for any $g\colon X \to RY$ the formula $\phi^{-1}_{X,Y}(g) = \varepsilon_Y \circ Lg$. Therefore we deduce

$$Id_{LX} = \phi^{-1}_{X,LX}\phi_{X,LX}(Id_{LX}) = \phi^{-1}_{X,LX}(\eta_X) = \varepsilon_{LX} \circ L\eta_X$$

and the first triangular equality is proved, for the second we have to unravel $\phi_{RY,Y}\phi^{-1}_{RY,Y}(Id_{RY})$.

If instead we start with with a unit and counit, we can define the bijection $\phi_{X,Y}$ using the formula we established above

$$\begin{aligned}
\mathrm{Hom}_{\mathcal{B}}(LX, Y) &\xrightarrow{\phi_{X,Y}} \mathrm{Hom}_{\mathcal{A}}(X, RY) \\
f &\mapsto Rf \circ \eta_X.
\end{aligned}$$

The triangular equalities imply this has inverse given by

$$\begin{aligned}
\mathrm{Hom}_{\mathcal{A}}(X, RY) &\to \mathrm{Hom}_{\mathcal{B}}(LX, Y) \\
g &\mapsto \varepsilon_Y \circ Lg.
\end{aligned}$$

and naturality of the transformations implies the same property for this $\phi_{X,Y}$. □

Using the description with unit and counit let us formulate the following useful criterion.

Lemma 3.2.3. *Let $(L, R)\colon \mathcal{A} \rightleftarrows \mathcal{B}$ be an adjunction of functors.*

1. *The left adjoint L is fully faithful if and only if the associated unit $\eta\colon Id \Rightarrow RL$ is an isomorphism. If this is the case, an object $X \in \mathcal{B}$ lies in its essential image if and only if the counit $\varepsilon\colon LR \to Id$ evaluated at X is an isomorphism.*

2. *Dually R is fully faithful if and only if the counit ε is invertible. If this is the case then $Y \in \mathcal{A}$ is in its essential image if and only if η_Y is an isomorphism.*

Proof. We deal with the first case, the second follows by duality. By Lemma 3.2.2 we see we can describe $L\colon \mathrm{Hom}_{\mathcal{A}}(X, X') \to \mathrm{Hom}_{\mathcal{B}}(LX, LX')$ as the composition

$$\mathrm{Hom}_{\mathcal{A}}(X, X') \xrightarrow{\mathrm{Hom}(Id_X, \eta_{X'})} \mathrm{Hom}_{\mathcal{A}}(X, RLX') \xrightarrow{\phi^{-1}_{X,LX'}} \mathrm{Hom}_{\mathcal{B}}(LX, LX).$$

Since the second map is a bijection we deduce L is fully faithful if and only if $\mathrm{Hom}(Id_X, \eta_{X'})$ is a bijection for every $X, X' \in \mathcal{A}$, Yoneda lemma implies this is equivalent to $\eta_{X'}$ being a natural isomorphism.

For the second part of the claim we observe if ε_Y is an isomorphism then trivially $Y \cong LRY$ so Y lies in the essential image of L. Conversely, if we have an isomorphism $g\colon Y \to RY'$ then we can form the commutative square

$$\begin{tikzcd} Y \arrow[r, "g", "\sim"'] & LY' \\ LRY \arrow[u, "\varepsilon_Y"] \arrow[r, "\sim", "LRg"'] & LRLY'. \arrow[u, "\varepsilon_{LY'}"'] \end{tikzcd}$$

The assumption of η being invertible and the first triangle equality imply also $\varepsilon_{LY'}$ is an isomorphism, thus ε_Y must have this property as well. □

Lemma 3.2.4. *Let $(L, R), (L', R')\colon \mathcal{A} \rightleftarrows \mathcal{B}$ be two adjunctions. Then there exists a bijection between the natural transformations $\alpha\colon L' \Rightarrow L$ and $\beta\colon R \Rightarrow R'$ making the following square diagram commute*

$$\begin{tikzcd} \mathrm{Hom}_{\mathcal{B}}(LX, Y) \arrow[r, "\phi_{X,Y}"] \arrow[d, "\mathrm{Hom}(\alpha_X, Id_Y)"'] & \mathrm{Hom}_{\mathcal{A}}(X, RY) \arrow[d, "\mathrm{Hom}(Id_X, \beta_Y)"] \\ \mathrm{Hom}_{\mathcal{B}}(L'X, Y) \arrow[r, "\phi'_{X,Y}"] & \mathrm{Hom}_{\mathcal{A}}(X, R'Y). \end{tikzcd}$$

Proof. The claim is clear from Yoneda lemma and the fact that $\phi_{X,Y}$ and $\phi'_{X,Y}$ are bijections. Nevertheless, we write down explicitly the formulas which relate α and β.

If we fix $Y = LX$ and start from Id_{LX} in the upper left corner, then the commutativity of the diagram implies

$$\alpha_X = \varepsilon'_{LX} \circ L'\beta_{LX} \circ L'\eta_X$$

and dually setting $X = RY$ and comparing the two images of Id_{RY} under the maps of the diagram we get

$$\beta_Y = R'\varepsilon_Y \circ R'\alpha_{RY} \circ \eta'_{RY}.$$

□

Corollary 3.2.5. *In the situation of Lemma 3.2.4, the natural transformation α is invertible if and only if the corresponding β is.*

Definition 3.2.6. In the situation of Lemma 3.2.4 we call the two corresponding transformations *conjugate* or *total mates.*

Corollary 3.2.7. *If a functor has a left or right adjoint this is unique up to natural isomorphism.*

We now move our attention to the calculus of canonical mate transformations associated to squares inhabited by natural transformations. In the literature they are also denoted by Beck-Chevalley transformations.

Consider the diagram of functors

$$\begin{tikzcd}
\mathcal{A} \arrow[rd, phantom, "\Swarrow \alpha"] & \mathcal{B} \arrow[l, "r^*"'] \\
\mathcal{C} \arrow[u, "u^*"] & \mathcal{D} \arrow[l, "q^*"] \arrow[u, "v^*"']
\end{tikzcd}$$

and suppose u^* and v^* admit left adjoints, which we denote respectively by $u_!$ and $v_!$. Then we can extend horizontally the diagram as follows

$$\begin{tikzcd}
\mathcal{C} \arrow[rd, phantom, "\Swarrow \varepsilon"] & \mathcal{A} \arrow[l, "u_!"'] \arrow[rd, phantom, "\Swarrow \alpha"] & \mathcal{B} \arrow[l, "r^*"'] \arrow[rd, phantom, "\Swarrow \eta"] & \\
 & \mathcal{C} \arrow[u, "u^*"] \arrow[ul, bend left, "Id"] & \mathcal{D} \arrow[l, "q^*"] \arrow[u, "v^*"'] & \mathcal{B} \arrow[l, "v_!"] \arrow[ul, bend right, "Id"']
\end{tikzcd}$$

we denote such transformation by

$$\alpha_! = \varepsilon q^* v_! \circ u_! \alpha v_! \circ u_! r^* \eta.$$

Suppose instead that r^* and q^* have right adjoints, respectively r_* and q_*, then we can augment the diagram vertically in the following way

$$\begin{tikzcd}
\mathcal{B} \arrow[rd, phantom, "\Swarrow \eta"] & \\
\mathcal{A} \arrow[u, "r_*"] \arrow[rd, phantom, "\Swarrow \alpha"] & \mathcal{B} \arrow[l, "r^*"'] \arrow[ul, bend right, "Id"'] \\
\mathcal{C} \arrow[u, "u^*"] \arrow[rd, phantom, "\Swarrow \varepsilon"] & \mathcal{D} \arrow[l, "q^*"] \arrow[u, "v^*"'] \\
 & \mathcal{C} \arrow[u, "q_*"'] \arrow[ul, bend left, "Id"]
\end{tikzcd}$$

and call the composite transformation

$$\alpha_* = r_* u^* \varepsilon \circ r_* \alpha q_* \circ \eta v^* q_*.$$

We call these transformations the *canonical mates* associated to α.

Lemma 3.2.8. *Suppose that u^* and v^* admit left adjoints and r^*, q^* have right adjoints so that canonical mates $\alpha_!$ and α_* exist, then they are conjugate in the sense of Definition 3.2.6. Thus $\alpha_!$ is an isomorphism if and only if α_* is.*

Proof. We have just to verify that $\alpha_!$ and α_* satisfy the explicit relations written in the proof or Lemma 3.2.4. We first observe that if we have a pair of adjunctions

$$\mathcal{E} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{F} \underset{I}{\overset{H}{\rightleftarrows}} \mathcal{G}$$

then (HF, GI) forms a new adjunction with unit and counit given respectively by

$$Id \xRightarrow{\eta} GF \xRightarrow{G\eta F} GIHF \qquad HFGI \xRightarrow{H\varepsilon I} HI \xRightarrow{\varepsilon} Id.$$

We apply this to the adjunctions

$$\mathcal{B} \underset{v^*}{\overset{v_!}{\rightleftarrows}} \mathcal{D} \underset{q_*}{\overset{q^*}{\rightleftarrows}} \mathcal{C} \qquad \mathcal{B} \underset{r_*}{\overset{r^*}{\rightleftarrows}} \mathcal{A} \underset{u^*}{\overset{u_!}{\rightleftarrows}} \mathcal{C}.$$

Using the notation of Lemma 3.2.4 the two adjunctions we consider will be $L = q^*v_!$, $R = v^*q_*$ and $L' = u_!r^*$, $R' = r_*u^*$. We will be decorating the units and counits with letters to recall which adjunction they belong to.

In the end we have to show the formula

$$\alpha_! = \varepsilon q^*v_! \circ u_!r^*\alpha_*q^*v_! \circ u_!r^*\eta$$

where the above ε refers to the adjunction $L' \dashv R'$ and η is the unit of $L \dashv R$. To prove this equality we form the following diagram

$$\begin{tikzcd}
u_!r^* \arrow[r, "u_!r^*\eta^v"] \arrow[d, "u_!r^*\eta^v"'] & u_!r^*v^*v_! \arrow[r, "u_!r^*v^*\eta^q v_!"] \arrow[d, "u_!r^*v^*\eta^q v_!"'] & u_!r^*v^*q_*q^*v_! \arrow[d, "u_!r^*\eta^r v^*q_*q^*v_!"] \arrow[dl, equal] \\
u_!r^*v^*v_! \arrow[r, "u_!r^*v^*\eta^q v_!"] \arrow[d, "u_!\alpha v_!"'] & u_!r^*v^*q_*q^*v_! \arrow[d, "u_!\alpha q_*q^*v_!"] & u_!r^*r_*r^*v^*q_*q^*v_! \arrow[l, "u_!\varepsilon^r r^*v^*q_*q^*v_!"] \arrow[d, "u_!r^*r_*\alpha q_*q^*v_!"] \\
u_!u^*q^*v_! \arrow[d, "\varepsilon^u q^*v_!"'] & u_!u^*q^*q_*q^*v_! \arrow[l, "u_!u^*\varepsilon^q q^*v_!"'] \arrow[d, "u_!u^*\varepsilon^q q^*v_!"] & u_!r^*r_*u^*q^*q_*q^*v_! \arrow[l, "u_!\varepsilon^r u^*q^*q_*v_!"] \arrow[d, "u_!r^*r_*u^*\varepsilon^q q^*v_!"] \\
q^*v_! & u_!u^*q^*v_! \arrow[l, "\varepsilon^u q^*v_!"] & u_!r^*r_*u^*q^*v_!. \arrow[l, "u_!\varepsilon^r u^*q^*v_!"]
\end{tikzcd}$$

We notice that the left column is $\alpha_!$ and the composite of the maps on other three edges coincides with the right hand side of the previous equality (after unravelling the units and counits of the composite adjunctions).

All the subdiagrams commute trivially, except for two. The first is the lower triangle in the upper right square: this commutes by the triangular identity involving the left adjoint of $r^* \dashv r_*$. The second is the left square in the middle row: we rewrite it as

$$\begin{tikzcd}
u_!r^*v^*v_! \arrow[r, "u_!r^*v^*\eta^q v_!"] \arrow[d, "u_!\alpha v_!"'] & u_!r^*v^*q_*q^*v_! \arrow[d, "u_!\alpha q_*q^*v_!"] \\
u_!u^*q^*v_! \arrow[r, bend left, "u_!u^*q^*\eta^q v_!"] & u_!u^*q^*q_*q^*v_! \arrow[l, bend left, "u_!u^*\varepsilon^q q^*v_!"]
\end{tikzcd}$$

now it is immediate to see the upper part commutes, then we notice that by the triangle identity involving the left adjoint of $q^* \dashv q_*$ the two morphisms in the bottom row are inverses, so they can be interchanged. $\square$

A crucial property of canonical mates is that they are compatible with pasting of squares. Suppose we have a diagram of this shape

$$\begin{tikzcd}
\mathcal{A} & \mathcal{B} \arrow[l, "r^*"'] & \mathcal{E} \arrow[l, "s^*"'] \\
\mathcal{C} \arrow[u, "u^*"] \arrow[ur, phantom, "\Swarrow\alpha"] & \mathcal{D} \arrow[l, "q^*"] \arrow[u, "v^*"'] \arrow[ur, phantom, "\Swarrow\beta"] & \mathcal{F} \arrow[l, "t^*"] \arrow[u, "w^*"']
\end{tikzcd}$$

then we can compose appropriately the transformations α and β to get

$$\alpha \odot \beta = \alpha t^* \circ r^* \beta \colon r^* s^* w^* \Rightarrow u^* q^* t^*.$$

Lemma 3.2.9. *In the above situation, suppose that the appropriate adjoints exist to define $\alpha_!$ and $\beta_!$, then the mate $(\alpha \odot \beta)_!$ exists and we have*

$$(\alpha \odot \beta)_! = \beta_! \odot \alpha_!.$$

Dually, provided α_* and β_* exist, we have

$$(\alpha \odot \beta)_* = \beta_* \odot \alpha_*.$$

Proof. We show only the first statement, the second is proved similarly.

Consider the diagram

$$\begin{tikzcd}
\mathcal{C} & \mathcal{A} \arrow[l, "u_!"'] \arrow[dl, phantom, "\overset{\varepsilon}{\swarrow}"] & \mathcal{B} \arrow[l, "r^*"'] \arrow[dl, phantom, "\overset{\alpha}{\swarrow}"] & & & \\
& \mathcal{C} \arrow[ul, bend left, "Id"] \arrow[u, "u^*"'] & \mathcal{D} \arrow[l, "q^*"] \arrow[u, "v^*"'] \arrow[r, phantom, "\overset{\eta}{\swarrow}"] & \mathcal{B} \arrow[l, "v_!"] \arrow[ul, bend right, "Id"'] \arrow[dl, phantom, "\overset{\varepsilon}{\swarrow}"] & \mathcal{E} \arrow[l, "s^*"'] \arrow[dl, phantom, "\overset{\beta}{\swarrow}"] & \\
& & & \mathcal{D} \arrow[ul, bend left, "Id"] \arrow[u, "v^*"'] & \mathcal{F} \arrow[l, "t^*"] \arrow[u, "w^*"'] \arrow[r, phantom, "\overset{\eta}{\swarrow}"] & \mathcal{E} \arrow[l, "w_!"] \arrow[ul, bend right, "Id"']
\end{tikzcd}$$

and observe that when we compose all the natural transformations in the picture the two in the middle cancel out by a triangle equality. This means $\beta_! \odot \alpha_! = (\alpha \odot \beta)_!$. ☐

3.3 Basic definitions

We finally give the definition of a derivator.

Definition 3.3.1. We let PoSet be the 2-category of finite posets. Its objects are finite posets, considered as categories as usual, 1-cells consist of the classical morphisms of posets (i.e. maps of sets preserving the ordering) and we have a 2-cell $f \Rightarrow g$ if and only if $f \leq g$.

A *prederivator* consists of a strict 2-functor

$$\mathcal{D} \colon \mathrm{PoSet}^{op} \to \mathrm{CAT}$$

where CAT is the 2-category of all categories (not necessarily small) and PoSet^{op} is the 2-category obtained by formally reversing the 1-cells (but not the 2-cells) of PoSet.

Given a 1-cell $u \colon R \to T$, its image under the prederivator $\mathcal{D}(u) \colon \mathcal{D}(T) \to \mathcal{D}(R)$ will be denoted by u^* and we will call it the *restriction functor* associated to u.

Similarly for a 2-cell $\alpha \colon u \Rightarrow v$ we denote by $\alpha^* \colon u^* \Rightarrow v^*$ the image $\mathcal{D}(\alpha)$.

Remark 3.3.2. In full generality a (pre)derivator is defined on Cat, the 2-category of small categories, but we can restrict the source to a sub 2-category closed under all the constructions we need to develop our theory (e.g. finite product of categories, forming slice and coslice categories). For a reference see [5, §1.2].

For this work we choose to restrict to PoSet for two reasons: the first is that since we want to show finiteness of the composition of localizations we only need to index our diagrams over finite posets. The second is that we will need a technical result (Theorem 3.8.3) whose proof will make use of this finiteness condition on the indexing categories and does not hold for derivators taking values on the whole Cat. We will elaborate on this later, when it comes up.

Definition 3.3.3. We denote by e the poset with just one object. For $\mathcal{D}$ a prederivator the value $\mathcal{D}(e)$ is called its *underlying category*: this terminology comes from the fact that informally we think of $\mathcal{D}(R)$, for any $R \in \mathrm{PoSet}$, as a category of enriched diagrams of shape R over $\mathcal{D}(e)$.

More precisely, we can form a functor

$$\mathrm{dia}_R \colon \mathcal{D}(R) \to \mathcal{D}(e)^R$$

called the *diagram functor* as follows. For any element $r \in R$ we can define a functor $r \colon e \to R$ sending the unique object of e to r, so for any $X \in \mathcal{D}(R)$ we define the value $\mathrm{dia}_R(X)_r$ by r^*X. If we have a morphism $f \colon r \to t$ in R then it induces a natural transformation between the two evaluation functors $f \colon r \Rightarrow t$, thus we get $f^* \colon r^*X \to t^*X$ which we set to be $\mathrm{dia}_R(X)(f)$. It is immediate that this $\mathrm{dia}_R(X)$ is a functor $R \to \mathcal{D}(e)$ as we wanted.

When handling an object X of $\mathcal{D}(R)$ is much easier to think of it as a diagram, so we implicitly identify it with $\mathrm{dia}_R(X)$ but this is rather improper. Usually the functor dia_R is not an equivalence and with this passage we lose critical information.

Definition 3.3.4. A *derivator* $\mathcal{D}$ consists on a prederivator satisfying the following axioms.

(D1) $\mathcal{D}$ takes coproducts to products, i.e. given a finite family $\{R_i\}_{i \in I}$ of posets the canonical functor

$$\mathcal{D}\Big(\coprod_i R_i\Big) \to \prod_i \mathcal{D}(R_i)$$

must be an equivalence.

(D2) For any $R \in \mathrm{PoSet}$ the diagram functor dia_R is conservative. That is, a morphism f in $\mathcal{D}(R)$ is an isomorphism if and only if for every $r \in R$ its evaluation f_r is an isomorphism in $\mathcal{D}(e)$.

(D3) Given a functor $u \colon R \to T$ the restriction $u^* \colon \mathcal{D}(T) \to \mathcal{D}(R)$ must admit both a left and a right adjoint, which we denote respectively $u_!$ and u_*. These are usually called the *left* and *right Kan extensions* along u.

(D4) We require the Kan extensions to satisfy the following *Kan pointwise formulas*, which are totally analogous to the ones regulating Kan extensions in the classical setting of category theory.

Let u be as above, for any $t \in T$ we can form the slice category (u/t) and the coslice (t/u) with associated projection functors $p \colon (u/t) \to R$ and $q \colon (t/u) \to R$. These fit in the square diagrams

$$\begin{tikzcd} (u/t) \arrow[r, "p"] \arrow[d, "\pi_{(u/t)}"'] & R \arrow[d, "u"] \\ e \arrow[r, "t"'] \arrow[ur, phantom, "\swArrow"] & T \end{tikzcd} \qquad \begin{tikzcd} (t/u) \arrow[r, "q"] \arrow[d, "\pi_{(t/u)}"'] & R \arrow[d, "u"] \\ e \arrow[r, "t"'] \arrow[ur, phantom, "\neArrow"] & T \end{tikzcd}$$

which induce, via the calculus of mates, natural transformations

$$\pi_{(u/t)_!}p^* \Rightarrow t^*u_! \qquad t^*u_* \Rightarrow \pi_{(u/t)_*}q^*$$

which we require to be isomorphisms. We will explain in detail their definition below.

Remark 3.3.5. A bit of explanation is necessary to make sense of these axioms. (D1) and (D2) are not too difficult to grasp: these are just regularity conditions which come natural when we recall the slogan "objects of the derivators are enhanced diagrams". We want a diagram defined on $\coprod R_i$ to be determined by its restrictions to the connected components R_i, which is precisely what (D1) establishes.

An isomorphism f between two diagrams of shape R should be a pointwise isomorphism, i.e. f_r should be invertible for every $r \in R$. Which is exactly the content of (D2).

The most critical parts of the definition are axioms (D3) and (D4). The former postulates the existence of homotopical versions of the classical Kan extensions. For example, recall that if $\mathcal{A}$ is a category complete and cocomplete, fixed I a small category we can form the limit and colimit functors

$$\lim_I : \mathcal{A}^I \to \mathcal{A} \qquad \operatorname{colim}_I : \mathcal{A}^I \to \mathcal{A}$$

respectively as the right and left Kan extensions along the projection $\pi : I \to e$.

The idea is exactly the same for derivators: (D3) is the requirement for $\mathcal{D}(e)$ to be homotopically complete and cocomplete, by the existence of such Kan extensions. The weak universal properties which homotopy limits and colimits are required to have are encoded in the adjuctions $u^* \dashv u_*$ and $u_! \dashv u^*$.

(D4) is the condition that such extensions not only exist, but are well behaved and follow a formula we can use to perform computations. If we evaluate the two natural transformations at some $X \in \mathcal{D}(R)$, rewriting $\pi_{(u/t)_!}$ as $\underrightarrow{\operatorname{hocolim}}$ and $\pi_{(u/t)_*}$ as $\underleftarrow{\operatorname{holim}}$, we obtain the maps

$$\underrightarrow{\operatorname{hocolim}}_{(u/t)} p^* X \cong (u_! X)_t \qquad\qquad (u_* X)_t \cong \underleftarrow{\operatorname{holim}}_{(t/u)} q^* X$$

and now we can easily see the resemblance with the classical Kan formulas.

Example 3.3.6 (Represented derivators)**.** Let $\mathcal{A}$ be a category, then the most trivial example of prederivator we can provide is the *represented prederivator*

$$y(\mathcal{A}) : \mathrm{PoSet}^{op} \to \mathrm{CAT} \qquad R \mapsto \mathcal{A}^R = \mathrm{Fun}(R, \mathcal{A})$$

where the restriction u^* consists in the precomposition with u. It is immediate to see that the adjoints u_* and $u_!$ coincide with the classical right and left Kan extensions. Therefore $y(\mathcal{A})$ is a derivator if and only if $\mathcal{A}$ admits all limits and colimits of shape given by finite posets.

Example 3.3.7 (Homotopy derivators)**.** Let $\mathcal{C}$ be a model category or a quasi-category, then there exists a derivator

$$\mathcal{H}_{\mathcal{C}} : \mathrm{PoSet}^{op} \to \mathrm{CAT} \qquad R \mapsto \mathrm{Ho}(\mathcal{C}^R)$$

called the *homotopy derivator* associated to $\mathcal{C}$.

A complete proof of this is contained in [5, Thm. 6.11] for the case of $\mathcal{C}$ a model category: its core idea is that we can endow $\mathcal{C}^R$ with the Reedy model structure and work with this. In this model the class of weak equivalences W^R consists of the pointwise weak equivalences, thus we define $\mathrm{Ho}(\mathcal{C}^R)$ as the localization $\mathcal{C}^R[(W^R)^{-1}]$.

An easier reference is [10, Prop. 1.36] where we assume the model category $\mathcal{C}$ to be combinatorial. This hypothesis guarantees the existence of the injective and projective model structures on $\mathcal{C}^R$ which we can use for our arguments. Since for both models the class of weak equivalences again is W^R they provide equivalent homotopy categories, indeed it can be shown that the adjunction

$$(Id, Id) : \mathcal{C}^R_{\mathrm{proj}} \rightleftarrows \mathcal{C}^R_{\mathrm{inj}}$$

is a Quillen equivalence.

We observe this provides a nice example of the fact that the diagram functor is not an equivalence.

$$\mathrm{dia}_R \colon \mathrm{Ho}(\mathcal{C}^R) \to \mathrm{Ho}(\mathcal{C})^R$$

as we mentioned in § 3.1 the objects of the category $\mathrm{Ho}(\mathcal{C}^R)$ are elements of $\mathcal{C}^R$ i.e. diagrams commuting in $\mathcal{C}$, while objects in $\mathrm{Ho}(\mathcal{C})^R$ are diagrams commuting only up to homotopy. The functor dia_R can be considered as a forgetful functor which discards the data of coherence in $\mathcal{C}$.

Example 3.3.8 (Shifted derivators)**.** Suppose we start with a derivator $\mathcal{D}$ and we fix a finite poset P, then we can define a new assignment

$$\mathcal{D}^P \colon \mathrm{PoSet}^{op} \to \mathrm{CAT} \qquad R \mapsto \mathcal{D}(P \times R).$$

It is not difficult to show this provides a new derivator such that its underlying category coincides with $\mathcal{D}(P)$, it is called the *shift* of the derivator $\mathcal{D}$ by P.

We end the section with the application of the theory of canonical mates to derivators.

Definition 3.3.9. We consider a square diagram in PoSet commuting up to a natural transformation

$$\begin{tikzcd}
\mathcal{A} \arrow[r, "r"] \arrow[d, "u"'] \arrow[dr, phantom, "\swarrow\alpha"] & \mathcal{B} \arrow[d, "v"] \\
\mathcal{C} \arrow[r, "q"'] & \mathcal{E}
\end{tikzcd}$$

and let $\mathcal{D}$ be a derivator, we can apply it to the above square to get a new one

$$\begin{tikzcd}
\mathcal{D}(\mathcal{A}) \arrow[dr, phantom, "\swarrow\alpha^*"] & \mathcal{D}(\mathcal{B}) \arrow[l, "r^*"'] \\
\mathcal{D}(\mathcal{C}) \arrow[u, "u^*"] & \mathcal{D}(\mathcal{E}). \arrow[l, "q^*"] \arrow[u, "v^*"']
\end{tikzcd}$$

We observe that all the involved functors have a right and left adjoint, thus we can form the canonical mates $(\alpha^*)_!$ and $(\alpha^*)_*$. To keep the notation simple we will drop the upper asterisk of α, so we will denote them simply by $\alpha_!$ and α_*.

We say the starting square is *homotopy exact* if for any derivator $\mathcal{D}$ the associated canonical mates are isomorphisms.

Example 3.3.10 (The slice and coslice squares)**.** Let $u \colon R \to T$ be a morphisms of finite posets, then for any $t \in T$ we can form two square diagrams

$$\begin{tikzcd}
(u/t) \arrow[r, "p"] \arrow[d, "\pi"'] \arrow[dr, phantom, "\swarrow\alpha_t"] & R \arrow[d, "u"] \\
e \arrow[r, "t"'] & T
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
(t/u) \arrow[r, "q"] \arrow[d, "\pi"'] \arrow[dr, phantom, "\nearrow\beta_t"] & R \arrow[d, "u"] \\
e \arrow[r, "t"'] & T
\end{tikzcd}$$

which we call respectively the slice and coslice squares. Using this we can form the mates

$$(\alpha_t)_! \colon \underset{\substack{\longrightarrow \\ (u/t)}}{\mathrm{hocolim}}\, p^* \Rightarrow t^* u_! \qquad (\beta_t)_* \colon t^* u_* \Rightarrow \underset{\substack{\longleftarrow \\ (t/u)}}{\mathrm{holim}}\, q^*.$$

These are the canonical natural maps we mentioned in (D4): this axiom is correctly phrased as asking the two above squares to be homotopy exact for every $t \in T$.

Example 3.3.11. Let $\mathcal{D}$ be a derivator and suppose that we have a diagram as follows

$$\begin{tikzcd} \mathcal{D}(\mathcal{A}) & \mathcal{D}(\mathcal{B}) \arrow[l, "F"'] \arrow[dl, phantom, "\overset{\gamma}{\swarrow}\cong"] \\ \mathcal{D}(\mathcal{C}) \arrow[u, "u^*"] & \mathcal{D}(\mathcal{E}) \arrow[l, "G"] \arrow[u, "v^*"'] \end{tikzcd}$$

where F, G are equivalences and γ is a natural isomorphism, we claim that in this situation the associated canonical mates are invertible. It is enough to observe that γ_* is formed as the composition of the natural isomorphism γ with unit and counit of adjunctions involving the equivalences F and G respectively, hence they are isomorphisms as well.

We observe that if we have a square as

$$\begin{tikzcd} \mathcal{A} \arrow[r, "s"] \arrow[d, "u"'] & \mathcal{B} \arrow[d, "v"] \arrow[dl, phantom, "\overset{\alpha}{\swarrow}\cong"] \\ \mathcal{C} \arrow[r, "t"'] & \mathcal{E} \end{tikzcd}$$

where s, t are equivalences and α is invertible then by the 2-functoriality of $\mathcal{D}$ the induced restrictions s^*, t^* are also equivalences and α^* is again a natural isomorphism. Thus after applying $\mathcal{D}$ we are in the situation just discussed above. Therefore squares of this shape are homotopy exact.

3.4 Stable derivators

The structure of a derivator is not enough to provide a diagram of triangulated categories. We have to add additional assumptions, to make sense of them and explain how they provide a triangulated structure we first have to explain how we define the cone and fiber functors using a derivator. The basic idea is to mimic the procedure illustrated in § 3.1 to provide a functorial construction of the cone of a morphisms of chain complexes.

We first legitimate the name "extension" for the adjoints of the restrictions guaranteed by (D3): that is, we show that the Kan extensions along fully faithful functors are fully faithful functors.

Proposition 3.4.1. *Let $u\colon R \to T$ be a fully faithful functor and $\mathcal{D}$ a derivator, then the induced functors*

$$u_!, u_*\colon \mathcal{D}(R) \to \mathcal{D}(T)$$

are also fully faithful.

Moreover, the essential image of $u_!$ is given by the objects $Y \in \mathcal{D}(T)$ such that for all $t \in T \setminus R$ the morphism $(\varepsilon_Y)_t\colon (u_! u^* Y)_t \to Y_t$ is an isomorphism.

Dually X lies in the essential image of u_* if and only if $(\eta_Y)_t$ is an isomorphism for all such t's.

Proof. We deal the case of $u_!$, the other involving the right Kan extension is dual.

A concise proof can be formulated using the calculus of mates: if we consider the diagram

$$\begin{tikzcd} R \arrow[r, "Id"] \arrow[d, "Id"'] & R \arrow[d, "u"] \arrow[dl, phantom, "\overset{Id}{\swarrow}"] \\ R \arrow[r, "u"'] & T \end{tikzcd}$$

then we can form its canonical mate $Id_!$. Using the diagram

$$
\begin{tikzcd}
\mathcal{D}(R) & \mathcal{D}(R) \arrow[l, "Id"'] \arrow[dl, phantom, "\swarrow Id"] & \mathcal{D}(R) \arrow[l, "Id"'] \arrow[dl, phantom, "\swarrow Id"] & \\
& \mathcal{D}(R) \arrow[u, "Id"] \arrow[ul, bend left, "Id"] & \mathcal{D}(T) \arrow[l, "u^*"] \arrow[u, "u^*"] \arrow[ur, phantom, "\swarrow \eta"] & \mathcal{D}(R) \arrow[l, "u_!"] \arrow[ul, bend right, "Id"']
\end{tikzcd}
$$

we clearly see $Id_!$ reduces to just the unit $\eta \colon Id \Rightarrow u^* u_!$. By Lemma 3.2.3 we have to show this $Id_!$ is invertible, i.e. the starting square is homotopy exact.

By (D2) we can reduce to prove η_r is an isomorphism, for all $r \in R$. We consider the following pasting

$$
\begin{tikzcd}
(R/r) \arrow[r] \arrow[d] \arrow[dr, phantom, "\swarrow"] & R \arrow[r, "Id"] \arrow[d, "Id"'] \arrow[dr, phantom, "\swarrow Id"] & R \arrow[d, "u"] \\
e \arrow[r, "r"] & R \arrow[r, "u"] & T,
\end{tikzcd}
$$

Lemma 3.2.9 states that the induced mate coincides with η_r composed with the mate associated to a slice square, which is an isomorphism by (D4). Therefore we have to show the total square is homotopy exact.

This can be rewritten as

$$
\begin{tikzcd}
(R/r) \arrow[r, "s"] \arrow[d] \arrow[dr, phantom, "\swarrow Id"] & (u/u(r)) \arrow[r, "p"] \arrow[d, "\pi"'] \arrow[dr, phantom, "\swarrow"] & R \arrow[d, "u"] \\
e \arrow[r, "Id"] & e \arrow[r, "u(r)"] & T.
\end{tikzcd}
$$

The square on the right is a slice square: it is homotopy exact by (D4). Regarding the square on the left hand side: we observe that since u is fully faithful the induced functor $s \colon (R/r) \to (u/u(r))$ is an isomorphism of categories, thus also this square is homotopy exact by Example 3.3.11. We conclude the total square is homotopy exact as well.

The characterization of the essential image comes from the second part of the statements of Lemma 3.2.3. Apply it to $u_! \dashv u^*$: Y is in the essential image of $u_!$ if and only if ε_Y is an isomorphism. By (D2) this is equivalent to $(\varepsilon_Y)_t$ being invertible $\forall t \in T$.

Recall that the following triangle identity holds

$$Id = u^*\varepsilon \circ \eta u^* \colon u^* \Rightarrow u^* u_! u^* \Rightarrow u^*.$$

We proved η is invertible, hence also $u^*\varepsilon$ has this property. Thus ε_Y is an isomorphism if and only if $\forall t \in T \setminus R$ $(\varepsilon_Y)_t$ is an isomorphism. □

This means that for every object $X \in \mathcal{D}(R)$ the unit $X \to u^* u_! X$ is an isomorphism, hence on $R \subset T$ the Kan extension $u_! X$ coincides with the starting diagram X.

We now focus on particular kind of extensions which will be very useful.

Definition 3.4.2. Let $u \colon R \to T$ be a fully faithful functor, it is called

- a *cosieve* if for any $t \in T \setminus u(R)$ the slice category (u/t) is empty. That is, for all $r \in R$ if there is a morphism $u(r) \to t$ in T then t must lie in the image of u;

- a *sieve* if for any $t \in T \setminus u(R)$ the coslice (t/u) is empty. That is, for all $r \in R$ if we have a morphism $t \to u(r)$ in T then t must belong to the image of u.

Remark 3.4.3. From the axioms of derivators we can deduce that the underlying category $\mathcal{D}(e)$ admits both an initial object $\emptyset$ and a terminal object $*$. Reasoning with the shifts of the derivator we deduce actually all the categories $\mathcal{D}(R)$ have this property, for all R finite poset.

It is immediate from axiom (D4) and Proposition 3.4.1 that if we have a cosieve u then $u_!$ just extends the diagrams by adding $\emptyset$ as values in $T \setminus u(R)$. Dually, if u is a sieve u_* is just an extension by $*$.

Example 3.4.4. We gave the general definition of cosieve and sieve. In our case we will be considering morphisms between finite posets hence the slice and coslice admit explicit descriptions as

$$(u/t) \cong \{r \in R \mid u(r) \leq t\} \qquad (t/u) = \{r \in R \mid t \leq u(r)\}.$$

This implies a cosieve is just an inclusion of a terminal subposet in T and dually a sieve corresponds to a initial subposet.

We also observe that if $R \subset T$ is a cosieve then $T \setminus R \subset T$ is a sieve and vice versa.

Definition 3.4.5. Let $\mathcal{D}$ be a derivator, we say it is *pointed* if the underlying category is pointed. That is, the canonical map $\emptyset \to *$ in $\mathcal{D}(e)$ is an isomorphism. Notice by (D2) this implies that all the categories $\mathcal{D}(R)$ are pointed.

Corollary 3.4.6. *Let $u\colon R \to T$ be a cosieve. Let $\mathcal{D}$ be a pointed derivator. Then $u_!\colon \mathcal{D}(R) \to \mathcal{D}(T)$ is a left extension by zero and its essential image is given by the diagrams $Y \in \mathcal{D}(T)$ such that $Y_t = 0$ for all $t \in T \setminus R$.*

Dually if u is a sieve then u_* is a right extension by zero and its essential image admits the same characterization.

Proof. We suppose u is a cosieve, the other case is dual.

The claim about the extension by zero are immediate from the definition of pointed derivator and Remark 3.4.3.

The characterization of the essential image comes from Proposition 3.4.1. In the situation where u is a cosieve $(u/t) = \emptyset$ for any $t \in T \setminus R$, hence (D4) implies $Y_t \cong 0$. $\square$

We are finally ready to give the definition of cone $C\colon \mathcal{D}([1]) \to \mathcal{D}(e)$ in a pointed derivator $\mathcal{D}$.

As we explained an element $f \in \mathcal{D}([1])$ can be thought as some kind of coherent morphism $f\colon X \to Y$ between two objects $X, Y \in \mathcal{D}(e)$.

Consider the following chain of functors

$$[1] \xrightarrow{j} \ulcorner \xrightarrow{i_\ulcorner} \square$$

where $\square = [1] \times [1]$ and $\ulcorner$ consists of the span which can be identified with the upper left corner

$$\begin{tikzcd} (0,0) \arrow[r] \arrow[d] & (1,0) \\ (0,1) & \end{tikzcd}$$

of $\square$. The map j just identifies $[1]$ with the upper horizontal edge $(0,0) \to (1,0)$.

Observe j is a sieve, thus j_*X has as underlying diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \\ 0 & \end{tikzcd}$$

Now, the functor $(i_\ulcorner)_!$ consists of completing a span to a homotopy cocartesian square, so the image of j_*X under it has shape

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[d, "\mathrm{cof}(f)"] \\ 0 \arrow[r] & C(f). \end{tikzcd}$$

Therefore we define $C\colon \mathcal{D}([1]) \to \mathcal{D}(e)$ as the composition

$$\mathcal{D}([1]) \xrightarrow{j_*} \mathcal{D}(\ulcorner) \xrightarrow{(i_\ulcorner)_!} \mathcal{D}(\square) \xrightarrow{(1,1)^*} \mathcal{D}(e).$$

Observe that $(0,0)\colon e \to [1]$ is also a sieve, hence if we precompose C with $(0,0)_*$ we can define the suspension $\Sigma\colon \mathcal{D}(e) \to \mathcal{D}(e)$.

Dually we can define a fiber functor $F\colon \mathcal{D}([1]) \to \mathcal{D}(e)$ and a desuspension Ω which is right adjoint to Σ.

This gives an idea of how derivator theory works in practice: we started with a set of axioms which we used to manipulate formally diagrams by extending and restricting them in a way to perform the operations we desired.

We want to provide a triangulated structure by defining as exact triangles the ones arising from cofiber sequences like the one produced above to define $C(f)$. There are two problems though:

- first of all for $\mathcal{D}(e)$ to be a triangulated category we have to provide a triangle starting with f for every morphism in $\mathcal{D}(e)$. This is an element in $\mathcal{D}(e)^{[1]}$, while our construction started from an element of $\mathcal{D}([1])$ and we saw that these two categories are very different.

- A fact that is seen explicitly in the case of stable model categories is that cofiber sequences should coincide with fiber sequence. This does not necessarily hold for a general derivator, thus we should impose a condition ensuring it.

This makes clear the motivation behind the following two definitions.

Definition 3.4.7. Let $\mathcal{D}$ be a derivator, we say it is *strong* if for any $R \in \mathrm{PoSet}$ the partial diagram functor

$$\mathrm{dia}_{[1]}\colon \mathcal{D}(R \times [1]) \to \mathcal{D}(R)^{[1]}$$

is full and essentially surjective.

Also, if $\mathcal{D}$ is pointed we denote it *stable* if the adjunction $\Sigma \dashv \Omega$ is an equivalence.

Remark 3.4.8. There are actually several equivalent conditions defining when a derivator is stable: we refer to [12, Theorem 7.1]. This theorem establishes that the above condition is equivalent to the fact that all coherent squares are homotopy cocartesian if and only if they are homotopy cartesian. That is, if $i_\ulcorner\colon \ulcorner \to \square$ and $i_\lrcorner\colon \lrcorner \to \square$ are the inclusions of the upper left corner and right lower corner respectively, then the essential images of $(i_\ulcorner)_!$ and $(i_\lrcorner)_*$ coincide.

This gives all we need to construct the distinguished triangle starting with a morphism f in $\mathcal{D}(e)$. Using the strength of the derivator we lift f to an object $\tilde{f}$ of $\mathcal{D}([1])$. Now we consider the subposet $A \subset [2] \times [1]$ missing the elements $(1,1)$ and $(2,1)$: we have two inclusions

$$[1] \xrightarrow{a} A \xrightarrow{b} [2] \times [1]$$

where a identifies $0 < 1$ with $(0,0) \to (1,0)$. This is a sieve, hence the right extension $a_*\colon \mathcal{D}([1]) \to \mathcal{D}(A)$ is an extension by 0 and $a_*\tilde{f}$ has the form

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & Y \arrow[r] & 0 \\ 0 & & \end{tikzcd}$$

we then complete it by applying $b_!$ so we get

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & Y \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0 \arrow[r] & C(\tilde{f}) \arrow[r] & \Sigma X. \end{tikzcd}$$

The Kan formula (D4) implies that the new values at $(1,1)$ and $(2,1)$ are obtained by forming two homotopy push-out squares. Thus the they can be identified with $C(\tilde{f})$ and ΣX respectively.

If we now restrict to the non-zero part of the diagram and take its value under $\mathrm{dia}_{[3]}$ we obtain a triangle starting with f as we wanted. We actually declare the exact triangles to be the the ones isomorphic to triangles arising in this form.

Theorem 3.4.9. *Let $\mathcal{D}$ be a strong stable derivator. Then for any $R \in \mathrm{PoSet}$ the category $\mathcal{D}(R)$ admits a triangulated structure such that for the underlying category $\mathcal{D}(e)$ the collection of exact triangles is the one described above.*

Moreover, for any functor $u\colon R \to Q$ the associated restriction u^* admits a canonical natural isomorphism $u^*\Sigma \Rightarrow \Sigma u^*$ endowing it with the structure of exact functor. The same holds for the Kan adjunctions $u_!$ and u_*.

The conclusion is that $\mathcal{D}$ lifts to a 2-functor

$$\mathcal{D}\colon \mathrm{PoSet}^{op} \to \mathrm{TCAT}.$$

Proof. We first observe that the definitions of pointed, strong and stable derivators are closed under shift, so it is enough to prove $\mathcal{D}(e)$ is triangulated. All the others $\mathcal{D}(R)$ are covered by the shifts $\mathcal{D}^R$.

Verifying that the axioms of triangulated category are satisfied and showing $u^*, u_!, u_*$ with the associated canonical transformations preserve exact triangles necessitates careful work. We refer to [11] for a complete proof. □

3.5 Total fibers

We will need later to work with homotopy limits of diagrams of various shape. Thus we recall here basic facts about total fibers and their characterizations. Obviously what we will write has a dual application to homotopy colimits.

Definition 3.5.1. Let $R \in \mathrm{PoSet}$, we denote by $R^{\triangleleft}$ the poset obtained by adding a new minimal element which we denote by 0.

We let $i_R\colon R \to R^{\triangleleft}$ be the associated inclusion. Fixed $\mathcal{D}$ a pointed derivator, an object $X \in \mathcal{D}(R^{\triangleleft})$ will be called *cone* diagram. If it lies in the essential image of $(i_R)_*\colon \mathcal{D}(R) \to \mathcal{D}(R^{\triangleleft})$ it is a *limiting cone.*

Example 3.5.2. Let $R = \mathcal{P}'(S)$ be poset of non-empty subsets of some finite set S. Then $R^{\triangleleft}$ is isomorphic to $\mathcal{P}(S)$ where 0 is identified with $\emptyset$.

For these posets the limiting cones are also called *cartesian cubes*, if $|S| = 2$ they are *cartesian squares.*

Example 3.5.3. Let $v\colon \Box \to [1]$ be the projection $(i,j) \mapsto i$, we define a coherent diagram of $\mathcal{D}(\Box)$ to be *vertically constant* if and only if it is in the essential image of $v^*\colon \mathcal{D}([1]) \to \mathcal{D}(\Box)$.

We should think of such squares as the ones in the form

$$\begin{tikzcd} X \arrow[r] \arrow[d, "\cong"'] & Y \arrow[d, "\cong"] \\ X \arrow[r] & Y. \end{tikzcd}$$

Dually we can also define the horizontally constant squares.

Lemma 3.5.4. *Let $X \in \mathcal{D}(\Box)$, then it is vertically constant if and only if the morphisms $X_{(0,0)} \to X_{(0,1)}$ and $X_{(1,0)} \to X_{(1,1)}$ are isomorphism.*

Proof. Observe that the functor $b\colon [1] \mapsto \Box$ given by $i \mapsto (i,1)$ is a right adjoint of v. By the 2-functoriality of $\mathcal{D}$ we have b^* is a left adjoint to v^*, hence by uniqueness of adjoints we deduce v^* can be identified with b_*.

Since b is fully faithful now the claim follows from Proposition 3.4.1: X lies in the essential image of b_* only if the unit $\eta_X\colon X \to b_*b^*X$ is invertible at $(0,0)$ and $(1,0)$. But $\eta_{(0,0)}$ can be identified with

$$X_{(0,0)} \to (0,0)^*b_*b^*X \cong (0,0)^*v^*b^*X = X_{(0,1)}$$

and similarly $\eta_{(1,0)}$ is the map $X_{(1,0)} \to X_{(1,1)}$. $\Box$

Corollary 3.5.5. *Vertically constant squares are homotopy cartesian.*

Proof. We saw in Lemma 3.5.4 that a coherent square is vertically constant if and only if it is in the essential image of b_*. The claim immediately follows from the factorization of b

$$[1] \xrightarrow{k} \lrcorner \xrightarrow{i_\lrcorner} \Box$$

where k is the inclusion of the botton row in the lower right corner of the square $\Box$. $\Box$

Clearly we want the limiting cones to be the diagrams X which present X_0 as homotopy limit of the part of the diagram lying on R, this is what the next proposition confirms.

Proposition 3.5.6. *In the situation of Definition 3.5.1, given an object $X \in \mathcal{D}(R^\triangleleft)$ we can provide a canonical natural map $X_0 \to \underset{\longleftarrow\, A}{\operatorname{holim}}\, i_A^* X$ such that it is an isomorphism if and only if X is a limiting cone.*

Proof. By Proposition 3.4.1 X lies in the essential image of $(i_R)_*$ if and only if $(\eta_X)_0$ is an isomorphism. We observe η is the mate Id_* associated to the commutative square

$$\begin{tikzcd} R \arrow[r, "i_R"] \arrow[d, "i_R"'] & R^\triangleleft \arrow[d, "Id"] \\ R^\triangleleft \arrow[r, "Id"'] \arrow[ur, phantom, "\nearrow Id"] & R^\triangleleft. \end{tikzcd}$$

Arguing as in Proposition 3.4.1 we can consider instead the mate of the pasting

$$\begin{tikzcd} (0/i_R) \arrow[r] \arrow[d, "\pi"'] & R \arrow[r, "i_R"] \arrow[d, "i_R"] & R^\triangleleft \arrow[d, "Id"] \\ e \arrow[r, "0"'] \arrow[ur, phantom, "\nearrow"] & R^\triangleleft \arrow[r, "Id"'] \arrow[ur, phantom, "\nearrow Id"] & R^\triangleleft \end{tikzcd}$$

which differs from η_0 only by an isomorphism introduced as the mate of the newly attached coslice square. Under the isomorphism $(0/i_R) \cong R$ the outer square becomes

$$\begin{tikzcd} R \arrow[r, "i_R"] \arrow[d, "\pi"'] & R^{\lhd} \arrow[d, "Id"] \\ e \arrow[r, "0"] \arrow[ur, phantom, "\overset{\alpha}{\nearrow}"] & R^{\lhd} \end{tikzcd}$$

recalling that $\pi_* = \underset{\longleftarrow}{\mathrm{holim}}_R$ we get $(\alpha_*)_X \colon X_0 \to \underset{\longleftarrow}{\mathrm{holim}}_R (i_R)^* X$ which is the canonical transformation we wanted. $\square$

While the criterion of Proposition 3.5.6 confirms our intuition, usually checking explicitly if such canonical map is an isomorphism is not immediate. The idea behind the definition of total fiber is to present an obstruction for the diagram to be a limiting cone.

Definition 3.5.7. Let R be a finite poset, $\mathcal{D}$ a strong stable derivator. Consider the poset $(R^{\lhd})^{\lhd}$, we denote by $-\infty$ the minimal element of this double construction while 0 refers to the minimal element of $R^{\lhd} \subset (R^{\lhd})^{\lhd}$. Therefore, for any $r \in R$ we have inequalities $-\infty < 0 < r$.

We consider the functor $l_R = (i_R)^{\lhd} \colon R^{\lhd} \to (R^{\lhd})^{\lhd}$ mapping R to itself via the identity and sending 0 to $-\infty$, thus only 0 does not belong to the image of l_R. Define $c \colon [1] \to (R^{\lhd})^{\lhd}$ by $0 \mapsto -\infty$ and $1 \mapsto 0$.

Let X be cone diagram, its *total fiber* is defined as

$$\mathrm{tfib}(X) = F(c^*(l_R)_* X).$$

Example 3.5.8. First of all, let us present a concrete example to better visualize $(R^{\lhd})^{\lhd}$. We take $R = \lrcorner$, the lower right corner of the square $\square = [1] \times [1]$

$$\begin{tikzcd} & (1,0) \arrow[d] \\ (0,1) \arrow[r] & (1,1). \end{tikzcd}$$

Clearly $R^{\lhd} = \square$: this comes from identifying 0 with the top left corner $(0,0)$. Thus $(R^{\lhd})^{\lhd}$ has the following shape

$$\begin{tikzcd} -\infty \arrow[dr] & & \\ & (0,0) \arrow[r] \arrow[d] & (1,0) \arrow[d] \\ & (0,1) \arrow[r] & (1,1). \end{tikzcd}$$

If X is a square diagram

$$\begin{tikzcd} X_{(0,0)} \arrow[r] \arrow[d] & X_{(1,0)} \arrow[d] \\ X_{(0,1)} \arrow[r] & X_{(1,1)} \end{tikzcd}$$

then $(l_{\lrcorner})_* X$ is formed by adding at position 0 the homotopy pullback or the span lying over $\lrcorner$ and placing $X_{(0,0)}$ at $-\infty$.

$$\begin{tikzcd}
X_{(0,0)} \arrow[dr] & & \\
& P \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X_{(1,0)} \arrow[d] \\
& X_{(0,1)} \arrow[r] & X_{(1,1)}.
\end{tikzcd}$$

This way we produced a comparison map $f\colon X_{(0,0)} \to P$ as a coherent morphism, thus we can form its fiber and we define it to be $\mathrm{tfib}(X) = F(f)$.

Since $\mathcal{D}$ is stable it can be shown that a f is an isomorphism (that is, the underlying diagram $\mathrm{dia}_{[1]}(f)$ is invertible in $\mathcal{D}(e)$) if and only if its fiber is zero.

Therefore the idea is that X is a limiting cone if and only if $\mathrm{tfib}(X) = 0$.

Proposition 3.5.9. *In the setting of Definition 3.5.7, the diagram X is a limiting cone if and only if* $\mathrm{tfib}(X) = 0$.

Proof. In the proof of Proposition 3.5.6 we saw that the cone is limiting if and only if evaluating at X the canonical mate associated to

$$\begin{tikzcd}
R \arrow[r, "i_R"] \arrow[d, "\pi"'] \arrow[dr, phantom, "\overset{\alpha}{\nearrow}"] & R^{\triangleleft} \arrow[d, "Id"] \\
e \arrow[r, "0"'] & R^{\triangleleft}
\end{tikzcd}$$

we get an isomorphism. We can use it to form the pasting

$$\begin{tikzcd}
R \arrow[r, "i_R"] \arrow[d, "\pi"'] \arrow[dr, phantom, "\overset{\alpha}{\nearrow}"] & R^{\triangleleft} \arrow[r, "Id"] \arrow[d, "Id"] \arrow[dr, phantom, "\nearrow"] & R^{\triangleleft} \arrow[d, "l_R"] \\
e \arrow[r, "0"'] & R^{\triangleleft} \arrow[r, "l_R"'] & (R^{\triangleleft})^{\triangleleft}
\end{tikzcd}$$

the right square is homotopy exact since l_R is fully faithful (Proposition 3.4.1). Thus $\alpha_*(X)$ is an isomorphisms if and only if evaluating the mate of this pasting at X we get an isomorphism.

Observe this can be rewritten as the vertical pasting

$$\begin{tikzcd}
R \arrow[r, "i_R"] \arrow[d, "\pi"] \arrow[dr, phantom, "\nearrow"] & R^{\triangleleft} \arrow[d, "l_R"] \\
e \arrow[r, "0"] \arrow[d] \arrow[dr, phantom, "\nearrow"] & (R^{\triangleleft})^{\triangleleft} \arrow[d, "Id"] \\
e \arrow[r, "-\infty"] & (R^{\triangleleft})^{\triangleleft}
\end{tikzcd}$$

where the above natural transformation is induced by the inequalities $0 < r$ for all $r \in R$ and the lower one by $-\infty < 0$. We notice the upper square is isomorphic to the slice square associated to $(0/l_R) \cong R$. If we unravel the mate associated to this decomposition we get

$$\begin{tikzcd}
\mathcal{D}(e) \arrow[dr, phantom, "\nearrow"] & \mathcal{D}(R) \arrow[l, "\pi_*"'] \arrow[r, "(i_R)^*"] \arrow[dr, phantom, "\nearrow"] & \mathcal{D}(R^{\triangleleft}) \arrow[dr, phantom, "\nearrow"] & \\
& \mathcal{D}(e) \arrow[u, "\pi^*"] \arrow[r, "0^*"] \arrow[ul, bend left, "Id"] \arrow[dr, phantom, "\nearrow"] & \mathcal{D}((R^{\triangleleft})^{\triangleleft}) \arrow[u, "(l_R)^*"'] & \mathcal{D}(R^{\triangleleft}) \arrow[l, "(l_R)_*"] \arrow[ul, bend right, "Id"'] \\
& \mathcal{D}(e) \arrow[u, "Id"] \arrow[r, "(-\infty)^*"] & \mathcal{D}((R^{\triangleleft})^{\triangleleft}) \arrow[u, "Id"'] &
\end{tikzcd}$$

and evaluating it at X we obtain the natural transformation

$$((l_R)_*X)_{-\infty} \to ((l_R)_*X)_0 \xrightarrow{\cong} \underset{\overleftarrow{R}}{\operatorname{holim}}(i_R)^*X.$$

Thus X being limiting is equivalent to $((l_R)_*X)_{-\infty} \to ((l_R)_*X)_0$ being an isomorphism.

Now the claim follows from [10, Prop. 4.5]. $\square$

The advantage of total fibers is that they are relatively easy to compute. We restrict to the case of Example 3.5.2 where we can provide a useful computational result.

In this situation we rename 0 by $\emptyset$, given the immediate identification $R^{\triangleleft} \cong \mathcal{P}(S)$. Suppose $s \in S$, then we can divide a cone $X \in \mathcal{D}(R^{\triangleleft})$ in two smaller cones by restricting along the inclusions

$$s_0\colon \mathcal{P}(S \setminus \{s\}) \to \mathcal{P}(S) \quad A \mapsto A, \qquad s_1\colon \mathcal{P}(S \setminus \{s\}) \to \mathcal{P}(S) \quad A \mapsto A \cup \{s\}.$$

We can relate the total fibers of s_0^*X and s_1^*X to the one of the whole diagram in the following way.

Proposition 3.5.10. *Let $\mathcal{D}$ be a stable derivator and S a finite set. Suppose that $s \in S$, and define the functors s_0, s_1 as above.*

If we take $X \in \mathcal{D}(\mathcal{P}(S))$ then its total fiber coincides with the fiber of the induced map bewteen the total fibers of the restrictions of the diagram along s_0 and s_1, in formulas

$$\operatorname{tfib}(X) \cong \operatorname{fib}(\operatorname{tfib}({s_0}^*X) \to \operatorname{tfib}({s_1}^*X)).$$

Proof. We first consider the composition of functors

$$j\colon \mathcal{P}(S) \xrightarrow{j_1} J_1 \xrightarrow{j_2} J_2$$

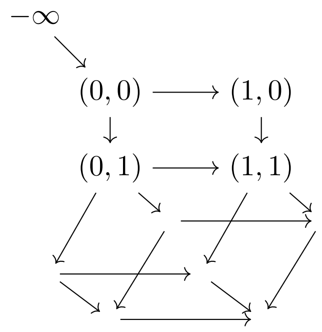
where $J_1 = \mathcal{P}(S)^{\triangleleft}$ consists of the poset $\mathcal{P}(S)$ to which we add an initial object here denoted i and $j_1 = l_{\mathcal{P}'(S)}$. J_2 is the push-out of the following diagram

$$\begin{tikzcd} \lrcorner \arrow[r] \arrow[d, "F"'] & \square^{\triangleleft} \arrow[d, "q"] \\ \mathcal{P}(S)^{\triangleleft} \arrow[r] & J_2 \end{tikzcd}$$

where the upper horizontal functor is just the inclusion. To describe F we denote as usual the elements of the square as

$$\begin{tikzcd} (0,0) \arrow[r] \arrow[d] & (1,0) \arrow[d] \\ (0,1) \arrow[r] & (1,1) \end{tikzcd}$$

and we name the initial point we add to this diagram by $-\infty$. Then F sends $(0,0)$ to i, $(0,1)$ to $\emptyset$ and $(1,1)$ to $\{s\}$, it is clear that the resulting poset inherits $-\infty$ as initial point. We give a complete picture of J_2 in the case $|S| = 3$ to facilitate its understanding.



Where the lower cube is $\mathcal{P}(S)$: the leftmost face is the restriction along s_0 while the rightmost one is the restriction along s_1. But the functor j_2 is not the inclusion induced by the push-out, instead we make j_2 map i to $-\infty$, $\emptyset$ to $(0,0)$ and $\{s\}$ to $(1,0)$ while the remaining values stay at their place.

We now can form the right Kan extension along these functor

$$j_*\colon \mathcal{D}(\mathcal{P}(S)) \xrightarrow{(j_1)_*} \mathcal{D}(J_1) \xrightarrow{(j_2)_*} \mathcal{D}(J_2)$$

and we compute j_*X. The description of $(j_1)_*X$ is immediate from Proposition 3.5.6: we are adding at the position $\emptyset$ the homotopy limit of the punctured cube while we move $X_\emptyset$ to i and the restriction along $i < \emptyset$ gives us the induced morphism from $X_\emptyset$ to the limit.

We now claim that for any $W \in \mathcal{D}(J_1)$ the diagram $(j_2)_*W$ adds the limits of the punctured faces s_0^*W and s_1^*W at the positions $(0,1)$ and $(1,1)$ respectively while shifting the values on the chain $i < \emptyset < \{s\}$ to $-\infty < (0,0) < (1,0)$. The latter claim follows immediately from the fully faithfulness of Kan extensions along fully faithful functors, for the former we use the detection result [10, Prop. 3.11]. Or more precisely, an immediate generalization of such criterion to higher dimensional cubes.

We start with the face parametrized by s_0: the only non-trivial assumption of the criterion we have to show is that the induced functor

$$\mathcal{P}(S \setminus \{s\}) \setminus \{\emptyset\} \to ((0,1)/j_2) \qquad A \mapsto A$$

is a left adjoint. Since the slice $((0,1)/j_2)$ can be identified with the subposet $\mathcal{P}(S) \setminus \{\emptyset, \{s\}\}$ we can easily present a right adjoint by

$$((0,1)/j_2) \to \mathcal{P}(S \setminus \{s\}) \setminus \{\emptyset\} \qquad A \mapsto A \setminus \{s\}.$$

Therefore $s_0^*(j_2)_*W$ is an limiting cone.

Now we deal with $s_1^*(j_2)_*W$ in the same way: again we only have to prove that the induced functor

$$\mathcal{P}(S \setminus \{s\}) \setminus \{\emptyset\} \to ((1,1)/j_2) \qquad A \mapsto A \cup \{s\}$$

is a left adjoint, but this time the slice $((1,1)/j_2)$ is given by the subsets A of S containing s such that $A \setminus \{s\} \neq \emptyset$, therefore the above functor is an isomorphism with inverse

$$((1,1)/j_2) \to \mathcal{P}(S \setminus \{s\}) \setminus \{\emptyset\} \qquad A \mapsto A \setminus \{s\}.$$

Thus $s_1^*(j_2)_*W$ is a limiting cone. From what we proved up to this point it is clear that j_*X is a diagram of the form

$$\begin{tikzcd}
X_\emptyset \arrow[dr] & & & & \\
& \lim X \arrow[rr] \arrow[d] & & X_{\{s\}} \arrow[d] & \\
& \lim s_0^* X \arrow[rr] \arrow[dr] \arrow[ddl] & & \lim s_1^* X \arrow[dr] \arrow[ddl] & \\
& & {} \arrow[rr] \arrow[ddl] & & {} \arrow[ddl] \\
{} \arrow[rr] \arrow[dr] & & {} \arrow[dr] & & \\
& {} \arrow[rr] & & {} &
\end{tikzcd}$$

where the the ones appearing in the upper part of the diagram are intended to be the limits of the appropriate punctured diagrams (we did not restrict X further just to keep the notation at minimum) while in the lower part X is left unchanged.

We denote by $f\colon \square \to J_2$ the restriction of q at $\square \subset \square^{\triangleleft}$. We claim that $f^* j_* X$ is a cartesian square diagram. To prove it, we consider another factorization of j as follows

$$\mathcal{P}(R) \xrightarrow{k_1} K \xrightarrow{k_2} K^{\triangleleft} = J_2.$$

Here K is defined as the push-out of

$$\begin{tikzcd}
{[1]} \arrow[r, "G"] \arrow[d, "H"'] & \square \\
\mathcal{P}(R) &
\end{tikzcd}$$

where G maps the poset $\{0 < 1\}$ to $(0,1) < (1,1)$ while H sends it to $\emptyset \subset \{s\}$. The functor k_1 takes $\emptyset$ to $(0,0)$, $\{s\}$ to $(1,0)$ and leaves the rest unchanged. k_2 just maps $(0,0)$ to $-\infty$ and is the identity on the other values of K.

Similarly to before $(k_1)_* X$ adds the limits of the punctured faces $s_0^* X$ and $s_1^* X$ at the values $(0,1)$ and $(1,1)$, while shifting $X_\emptyset$ and $X_{\{s\}}$ at the positions $(0,0)$ and $(1,0)$. Using again Proposition 3.5.6 we see $(k_2)_*(k_1)_* X$ just adds the limit of the punctured diagram on K at the position $(0,0)$, since $(k_2)_*(k_1)_* X \cong j_* X$ it is trivial to observe that this new limit is just $\lim X$. We now apply again the detection criterion to k_2 to prove $f^* j_* X$ is cartesian. This time we have to show that the functor induced by f

$$\lrcorner \to ((0,0)/k_2)$$

has a right adjoint. In this case $((0,0)/k_2)$ is the subposet of K obtained by removing the initial point $(0,0)$, thus we present the desired adjoint as

$$\begin{aligned}
((0,0)/k_2) &\to \lrcorner \\
(1,0), (0,1), (1,1) &\mapsto (1,0), (0,1), (1,1) \\
A &\mapsto \begin{cases} (0,1) & \text{if } s \notin A \\ (1,1) & \text{if } s \in A. \end{cases}
\end{aligned}$$

We now restrict our attention to $q^* j_* X = Y$. Since we are going to embed this diagram in the 3-dimensional cube $[1]^3$ we identify $\square^{\triangleleft}$ with an appropriate subposet via the functor $\square^{\triangleleft} \to [1]^3$ determined by

$$\begin{aligned}
-\infty &\mapsto (0,0,0) \quad (0,0) \mapsto (0,1,0) \quad (1,0) \mapsto (1,1,0) \\
(0,1) &\mapsto (0,1,1) \quad (1,1) \mapsto (1,1,1).
\end{aligned}$$

First we take the right Kan extension along $l\colon \square^{\lhd} \to C$, where C is obtained from $\square^{\lhd}$ by adding the point $(1,0,0)$. Using (D4) we have that $(l_*Y)_{(1,0,0)}$ is given by the limit of Y on the restriction along the slice $((1,0,0)/l)$, since this has $(1,0)$ as initial point $(l_*Y)_{(1,0,0)} \cong Y_{(1,0)} = X_{\{s\}}$. Now we take the left Kan extension along the inclusion $C \hookrightarrow [1]^3$ and again reasoning as in the previous point we get that the new diagram is obtained as constant extension by adding $X_\emptyset$ at $(0,0,1)$ and $X_{\{s\}}$ at $(1,0,1)$.

At the end we obtain the following coherent diagram $Z \in \mathcal{D}([1]^3)$

$$\begin{tikzcd}
& X_\emptyset \arrow[r] \arrow[dl] \arrow[dd, equal] & X_{\{s\}} \arrow[dl, equal] \arrow[dd, equal] \\
\lim X \arrow[r, crossing over] \arrow[dd] & X_{\{s\}} & \\
& X_\emptyset \arrow[r] \arrow[dl] & X_{\{s\}} \arrow[dl] \\
\lim s_0^* X \arrow[r] & \lim s_1^* X \arrow[from=2-2, crossing over] &
\end{tikzcd}$$

This can be seen as a morphism from the rear to the front face after exchanging the second and third coordinate and using $[1]^3 = [1]^2 \times [1]$.

We define $v\colon [1]^2 \times [1] \to [1]^2 \times \lrcorner$ as the shift of $[1] \to \lrcorner$ sending $0 < 1$ to $(1,0) < (1,1)$. v is a cosieve thus $v_!$ gives a left extension by zero, that is we are adding another square which is constantly zero and a morphism from this to the front face. Now take $i_\lrcorner\colon [1]^2 \times \lrcorner \to [1]^2 \times \square$ the shift of $\lrcorner \to \square$, the right Kan extension along this functor provides the homotopy pullback of a span of squares. We apply this construction to the diagram we obtained from X: that is we compute the fiber of Z seen as morphism in the last coordinate. Since the limit can be computed pointwise the new square we obtain is

$$\begin{tikzcd}
\mathrm{tfib} X \arrow[r] \arrow[d] & 0 \arrow[d] \\
\mathrm{tfib} s_0^* X \arrow[r] & \mathrm{tfib} s_1^* X
\end{tikzcd}$$

if we prove this is cartesian we can conclude. This follows from the fact that the pullback of cartesian squares is a cartesian square: to see it just consider the two factorizations of $\lrcorner \times \lrcorner \to \square \times \square$ as $\lrcorner \times \lrcorner \to \square \times \lrcorner \to \square \times \square$ and $\lrcorner \times \lrcorner \to \lrcorner \times \square \to \square \times \square$. In our case we proved explicitly that the front face is cartesian, the rear face and the square added by $v_!$ are cartesian since they are vertically constant (Corollary 3.5.5). $\square$

Remark 3.5.11. We observe that, if $\mathcal{D}$ is also strong, the above cartesian square

$$\begin{tikzcd}
\mathrm{tfib} X \arrow[r] \arrow[d] & \mathrm{tfib} s_0^* X \arrow[d] \\
0 \arrow[r] & \mathrm{tfib} s_1^* X
\end{tikzcd}$$

implies the existence of an exact triangle in $\mathcal{D}(e)$

$$\mathrm{tfib}\, X \to \mathrm{tfib}\, s_0^* X \to \mathrm{tfib}\, s_1^* X \to \Sigma\, \mathrm{tfib}\, X.$$

Corollary 3.5.12. *Let S be a finite set, let $\mathcal{D}$ be a strong stable derivator. Consider the diagram*

$$\begin{tikzcd}
& e \arrow[dl, "p"'] \arrow[dr, "s"] & \\
\mathcal{P}'(S) \arrow[rr, hook] & & \mathcal{P}(S)
\end{tikzcd}$$

of inclusions of the element S in $\mathcal{P}'(S)$ and $\mathcal{P}(S)$. Then for any $X \in \mathcal{D}(e)$ we have isomorphisms

$$\mathrm{tfib}(s_!X) \cong \Omega^{|S|}X \qquad \operatorname*{holim}_{\overleftarrow{\mathcal{P}'(S)}} p_!X \cong \Omega^{|S|-1}X.$$

Proof. Observe that, being s and p cosieves, $s_!X$ and $p_!X$ are coherent diagrams obtained from X by extending by zero.

We prove the first isomorphism by induction on $|S|$. If $|S| = 1$ then

$$\mathrm{tfib}(s_!X) = \mathrm{fib}(0 \to X) \cong \Omega X.$$

For $|S| \geq 2$ we can apply Proposition 3.5.10 to get

$$\mathrm{tfib}(s_!X) \cong \mathrm{fib}(\mathrm{tfib}(s_0^* s_! X) \to \mathrm{tfib}(s_1^* s_! X)) \cong \mathrm{tfib}(0 \to \Omega^{|S|-1}X) = \Omega^{|S|}X.$$

For the second claim observe that by definition of total fiber we have an exact triangle

$$\mathrm{tfib}(s_!X) \to 0 \to \operatorname*{holim}_{\overleftarrow{\mathcal{P}'(S)}} p_!X$$

thus $\operatorname*{holim}_{\overleftarrow{\mathcal{P}'(S)}} p_!X \cong \Sigma \mathrm{tfib}(s_!X) \cong \Omega^{|S|-1}X$. □

3.6 Two notions of homotopy equivalence on finite posets

We collect here some interactions of the homotopy theory of posets with derivators which will be useful later. The crucial point is that we want to give conditions to determine whether restricting along a map of posets preserves homotopy limits and colimits, this requires the two posets to have a "similar" shape in the sense that their homotopy type should be compatible in some way which we need to make precise.

Definition 3.6.1. Let R, T be two finite posets, we denote $\mathrm{PoSet}(R, T)$ the set of monotone maps from R to T. We endow this with the partial order given by $f \leq g$ if and only if $f(r) \leq g(r)$ in T for any $r \in R$. It is easy to see that this makes the category PoSet into a cartesian closed category.

Definition 3.6.2. Let R be a finite poset. We define a functor

$$\mathrm{PoSet} \xrightarrow{N} \mathrm{sSet} \xrightarrow{|-|} \mathrm{Top}$$

by composing the nerve functor with usual geometric realization of simplicial sets.

We denote this functor, with a slight abuse of notation, simply $|-|$ and call $|R|$ the *geometric realisation* of R. It can be proved that it preserves finite coproducts and finite limits. In particular, we can apply geometric realisation to the evaluation map $\mathrm{PoSet}(R, T) \times R \to T$, and then take adjoints, to obtain a continuous map

$$|\mathrm{PoSet}(R, T)| \to \mathrm{Top}(|R|, |T|).$$

Definition 3.6.3. For any finite poset R, we define $\pi_0(R)$ to be the quotient of R by the smallest equivalence relation such that $p \sim q$ whenever $p \leq q$. This is easily seen to be the same as the set of path components of $|R|$. It gives a functor from finite posets to finite sets, which preserves finite products and coproducts. It follows by a formal argument that we can construct a category $\mathrm{Ho}(\mathrm{PoSet})$ as quotient of PoSet with morphism sets $\pi_0(\mathrm{PoSet}(R, T))$. We call this the *strong homotopy category* of finite posets. It also follows that if f and g lie in the same equivalence class of $\pi_0(\mathrm{PoSet}(P,Q))$ then the resulting maps $|f|$ and $|g|$ are homotopic (by a straight-line homotopy, in the basic case where $f \leq g$ or $g \leq f$). Thus, geometric realisation descends to a functor

$$|-|\colon \mathrm{Ho}(\mathrm{PoSet}) \to \mathrm{Ho}(\mathrm{Top}).$$

Remark 3.6.4. Suppose we have morphisms $f\colon R \to T$ and $g\colon T \to R$ that are adjoint, in the sense that $f(r) \leq t$ if and only if $r \leq g(t)$. We then have (co)unit inequalities $1 \leq gf$ and $fg \leq 1$, showing that fg and gf give identities in the strong homotopy category, and thus that f and g are strong homotopy equivalences.

Definition 3.6.5. We say that R is *strongly contractible* if the projection map $\pi_R\colon R \to e$ is a strong homotopy equivalence.

Remark 3.6.6. We note that this is the case if R has a smallest element or a largest element. If m is the minimum of R then it provides a functor $m\colon e \to R$ left adjoint to π_R, dually a maximum n gives a right adjoint to π_R.

We also note that if R is a strongly contractible poset, then $|R|$ is a contractible space, since the geometric realization transfers homotopic maps to homotopic maps.

Definition 3.6.7. Consider a morphism $f\colon R \to T$ in PoSet, and note that $\pi_T f = \pi_R\colon R \to e$. For any derivator $\mathcal{C}$ and any $X, Y \in \mathcal{C}(e)$ we therefore get a map

$$f^*\colon \mathcal{C}(T)(\pi_T^* X, \pi_T^* Y) \to \mathcal{C}(R)(\pi_R^* X, \pi_R^* Y).$$

We say that f is a *$\mathcal{D}$-equivalence* if this map is always bijective. We also say that R is *$\mathcal{D}$-contractible* if the map $\pi_R\colon R \to e$ is a $\mathcal{D}$-equivalence, or equivalently the functor

$$\pi_R^*\colon \mathcal{C}(e) \to \mathcal{C}(R)$$

is full and faithful.

Remark 3.6.8. The definition of $\mathcal{D}$-equivalence is usually presented differently in the literature. In [5, §1.8] the following square diagram is considered

$$\begin{tikzcd} R \arrow[r, "\pi_R"] \arrow[d, "f"'] \arrow[dr, phantom, "\overset{Id}{\nearrow}"] & e \arrow[d, equal] \\ T \arrow[r, "\pi_T"] & e \end{tikzcd}$$

which gives the mate $Id_*\colon \pi_T^* \Rightarrow f_* \pi_R^*$. Applying the functor $(\pi_T)_*$ we get a natural transformation $\zeta\colon (\pi_T)_* \pi_T^* \Rightarrow (\pi_R)_* \pi_R^*$ and we declare f to be a $\mathcal{D}$-equivalence if this is an isomorphism.

To see these two conditions are the same observe that by the adjunctions $\pi^* \dashv \pi_*$ we can form a composition of maps

$$\begin{tikzcd} \mathcal{C}(T)(\pi_T^* X, \pi_T^* Y) \arrow[r, "f^*"] & \mathcal{C}(R)(\pi_R^* X, \pi_R^* Y) \arrow[d, "\cong"] \\ \mathcal{C}(e)(X, (\pi_T)_* \pi_T^* Y) \arrow[u, "\cong"] & \mathcal{C}(e)(X, (\pi_R)_* \pi_R^* Y) \end{tikzcd}$$

which we can show to coincide with $\mathcal{C}(e)(X, \zeta_Y)$. Now Yoneda lemma makes the equivalence clear.

We could also consider the diagram

$$\begin{tikzcd} R \arrow[r, "\pi_R"] \arrow[d, "f"'] \arrow[dr, phantom, "\swarrow Id"] & e \arrow[d, equal] \\ T \arrow[r, "\pi_T"] & e \end{tikzcd}$$

to get the mate $Id_!\colon f_!\pi_R^* \Rightarrow \pi_T^*$ and applying $(\pi_T)_!$ the natural transformation $(\pi_R)_!\pi_R^* \Rightarrow (\pi_T)_!\pi_T^*$. We can see that this being an isomorphism is equivalent to f^* inducing a bijection on the hom-sets similarly to before, using the adjunctions $\pi_! \dashv \pi^*$.

We chose to present the above condition as definition because it is easier to formulate and it is symmetric.

Remark 3.6.9. Since π_R^* is part of two adjunctions $(\pi_R)_! \dashv \pi_R^*$ and $\pi_R^* \dashv (\pi_R)_*$, by Lemma 3.2.3 R being $\mathcal{D}$-contractible is equivalent to the two natural transformations

$$\varepsilon\colon (\pi_R)_!\pi_R^* \Rightarrow Id \qquad \eta\colon Id \Rightarrow (\pi_R)_*\pi_R^*$$

being isomorphism.

This means that for any $X \in \mathcal{C}(e)$ the homotopy colimit (respectively limit) of $\pi_R^* X$, the constant diagram of shape R, must coincide with X.

Some authors (like [13]) use the term *homotopy contractible* rather than $\mathcal{D}$-contractible.

Example 3.6.10. Note that for $\mathcal{C}$ a represented derivator the condition of ε and η being isomorphisms is satisfied whenever R is connected, but in general the property if being $\mathcal{D}$-contractible is much more demanding.

For example, consider the poset C

$$\begin{tikzcd} a \arrow[r] \arrow[d] & b \\ c & d \arrow[l] \arrow[u] \end{tikzcd}$$

whose geometric realization coincides with the circle S^1. This poset is connected but not $\mathcal{D}$-contractible: take $\mathcal{C}$ to be the homotopy derivator associated to the Quillen model category of topological spaces. Then C being $\mathcal{D}$-contractible implies that for every space X the canonical map $\underrightarrow{\mathrm{hocolim}}\, \pi_C^* X \to X$ is an homotopy equivalence. For X a cofibrant space this homotopy colimit can be represented by the space $X \times S^1$ and it is clear this cannot be homotopic to X in full generality (take for example $X = S^1$).

Proposition 3.6.11. *If* $[f] = [g]$ *in* $\pi_0(\mathrm{PoSet}(R,T))$ *then*

$$f^* = g^*\colon \mathcal{C}(T)(\pi_T^* X, \pi_T^* Y) \to \mathcal{C}(R)(\pi_R^* X, \pi_R^* Y).$$

Thus, f is a $\mathcal{D}$-equivalence if and only if g is a $\mathcal{D}$-equivalence.

Proof. We can reduce easily to the case where $f \le g$. As $\mathcal{C}\colon \mathrm{PoSet}^{\mathrm{op}} \to \mathrm{CAT}$ is a strict 2-functor, the following diagram of categories and functors must commute on the nose:

$$\begin{tikzcd} \mathrm{PoSet}(R,T) \times \mathrm{PoSet}(T,e) \arrow[r, "\circ"] \arrow[d, "\mathcal{C}"] & \mathrm{PoSet}(R,e) \arrow[d, "\mathcal{C}"] \\ {[\mathcal{C}(T), \mathcal{C}(R)] \times [\mathcal{C}(e), \mathcal{C}(T)]} \arrow[r, "\circ"] & {[\mathcal{C}(e), \mathcal{C}(R)].} \end{tikzcd}$$

The inequality $f \le g$ gives a morphism $(f, \pi_T) \to (g, \pi_T)$ in the category $\mathrm{PoSet}(R,T) \times \mathrm{PoSet}(T,e)$, and this becomes the identity morphism of π_R in $\mathrm{PoSet}(R,e)$. This implies that for any $\alpha \in \mathcal{C}(T)(\pi_T^* X, \pi_T^* Y)$ we have a commutative square

$$
\begin{tikzcd}
f^*\pi_T^*X \arrow[r, "f^*\alpha"] \arrow[d, equal] & f^*\pi_T^*Y \arrow[d, equal] \\
g^*\pi_T^*X \arrow[r, "g^*\alpha"] & g^*\pi_T^*Y
\end{tikzcd}
$$

thus the claim is proved. □

Corollary 3.6.12. *If $f\colon R \to T$ is a strong homotopy equivalence, then it is a $\mathcal{D}$-equivalence. In particular:*

(a) If f has a left or right adjoint, then it is a $\mathcal{D}$-equivalence.

(b) If R is strongly contractible, then it is $\mathcal{D}$-contractible.

The importance of $\mathcal{D}$-contractible posets lies in the following criterion to determine whether a map is homotopy cofinal or final. We first recall the definition of these notions.

Definition 3.6.13.

(a) We say that a map $f\colon R \to T$ is *homotopy final* if the commuting square

$$
\begin{tikzcd}
R \arrow[r, "f"] \arrow[d, "\pi_R"'] \arrow[dr, phantom, "\swarrow Id"] & T \arrow[d, "\pi_T"] \\
e \arrow[r, "Id"] & e
\end{tikzcd}
$$

is homotopy exact. That is, the following canonical mate

$$\underrightarrow{\mathrm{hocolim}}_{R}\, f^*(X) = (\pi_R)_! f^*(X) = (\pi_T)_! f_! f^*(X) \to (\pi_T)_!(X) = \underrightarrow{\mathrm{hocolim}}_{T}\, X$$

is an isomorphism for all derivators $\mathcal{C}$ and all objects $X \in \mathcal{C}(T)$.

(b) Dually, we say that a map $f\colon R \to T$ is *homotopy cofinal* if the commuting square

$$
\begin{tikzcd}
R \arrow[r, "f"] \arrow[d, "\pi_R"'] \arrow[dr, phantom, "\nearrow Id"] & T \arrow[d, "\pi_T"] \\
e \arrow[r, "Id"] & e
\end{tikzcd}
$$

is homotopy exact, that is the associated mate

$$\underleftarrow{\mathrm{holim}}_{T}\, X = (\pi_T)_*(X) \to (\pi_T)_* f_* f^*(X) = (\pi_R)_* f^*(X) = \underleftarrow{\mathrm{holim}}_{R}\, f^*(X)$$

is an isomorphism for all derivators $\mathcal{C}$ and all objects $X \in \mathcal{C}(T)$.

The next proposition should clarify the relation between the classes of morphisms defined above.

Proposition 3.6.14. *If the map $f\colon R \to T$ is homotopy final or cofinal then it is a $\mathcal{D}$-equivalence.*

Proof. It follows immediately from Remark 3.6.8. We spell out the cofinal case, as the other is dual.

By definition f being homotopy cofinal is asking the canonical mate $Id_*\colon (\pi_T)_* \Rightarrow (\pi_R)_* f^*$ associated to the square in Definition 3.6.13 (b) to be an isomorphism. This mate has as conjugate $Id_!\colon f_!\pi_R^* \Rightarrow \pi_T^*$, so this also is an isomorphism. But applying $(\pi_T)_!$ to it we get the natural transformation $(\pi_R)!\pi_R^* \Rightarrow (\pi_T)_!\pi_T^*$ is invertible, establishing f is a $\mathcal{D}$-equivalence. $\square$

Proposition 3.6.15. *The map f is homotopy final if (t/f) is $\mathcal{D}$-contractible for all t, and this holds if f has a left adjoint. Dually, f is homotopy cofinal if (f/t) is $\mathcal{D}$-contractible for all t, and this holds if f has a right adjoint.*

Proof. We deal explicitly with the first claim: the second follows by duality.

By definition f is homotopy final if and only if the mate $Id_!$ associated to the square

$$\begin{tikzcd} R \arrow[r, "\pi_R"] \arrow[d, "f"'] \arrow[dr, phantom, "\nearrow Id"] & e \arrow[d, equal] \\ T \arrow[r, "\pi_T"] & e \end{tikzcd}$$

is an isomorphism. Equivalently we have show the conjugate mate, Id_*, is invertible. By (D2) this can be verified pointwise, so it is enough to show the following pasting of squares is homotopy exact for any $t \in T$.

$$\begin{tikzcd} (t/f) \arrow[r] \arrow[d, "\pi"] \arrow[dr, phantom, "\nearrow"] & R \arrow[r, "\pi_R"] \arrow[d, "f"'] \arrow[dr, phantom, "\nearrow Id"] & e \arrow[d, equal] \\ e \arrow[r, "t"] & T \arrow[r, "\pi_T"] & e \end{tikzcd}$$

The left square is a coslice square, and the whole composition coincides with

$$\begin{tikzcd} (t/f) \arrow[r, "\pi"] \arrow[d, "\pi"] \arrow[dr, phantom, "\nearrow Id"] & e \arrow[d, equal] \\ e \arrow[r, equal] & e \end{tikzcd}$$

and this square being homotopy exact is equivalent to (t/f) being $\mathcal{D}$-contractible.

If f is part of an adjunction $f \vdash g$ then the coslice (t/f) is naturally isomorphic to the coslice $(g(t)/Id_R)$. But this has an initial object, namely $(g(t), Id_{g(t)})$, so we conclude by Remark 3.6.6. $\square$

Remark 3.6.16. The content of [13, Rmk. 3.14] indicates that the converse implication of the above proposition holds: the functor f is homotopy final (respectively cofinal) if and only if every coslice (t/f) (respectively every slice (f/t)) is $\mathcal{D}$-contractible.

The upshot of the results of Heller and Cisinski is that a map f is a $\mathcal{D}$-equivalence if and only if the associated nerve $N(f)$ is a weak homotopy equivalence of simplicial sets. Therefore a poset R being $\mathcal{D}$-contractible is equivalent to being homotopy contractible in the classical sense.

These outcomes are highly non-trivial and it will not be needed here, so we do not elaborate further on the matter.

Proposition 3.6.17. *Consider a square*

$$\begin{tikzcd} P \arrow[r, "t"] \arrow[d, "u"'] \arrow[dr, phantom, "\nearrow \alpha"] & Q \arrow[d, "v"] \\ R \arrow[r, "w"] & S, \end{tikzcd}$$

and the resulting mate transformation $\alpha_*\colon w^*v_* \Rightarrow u_*t^*$. For any $r \in R$ we have coslice posets (r/u) and $(w(r)/v)$, and using t and α we can produce a morphism $t_r\colon (r/u) \to (w(r)/v)$. If this is homotopy cofinal for all r, then α_* is an isomorphism.

Proof. Again by (D2) we have to check $(\alpha_*)_r$ is an isomorphism for every $r \in R$ and this is equivalent to the square being homotopy exact after pasting on the left a coslice square

$$\begin{tikzcd}
(r/u) \arrow[r] \arrow[d, "\pi"] \arrow[dr, phantom, "\Nearrow"] & P \arrow[r, "t"] \arrow[d, "u"'] \arrow[dr, phantom, "\Nearrow\alpha"] & Q \arrow[d, "v"] \\
e \arrow[r, "r"] & R \arrow[r, "w"] & S.
\end{tikzcd}$$

Now we can rewrite the total square as

$$\begin{tikzcd}
(r/u) \arrow[r, "t_r"] \arrow[d, "\pi"] \arrow[dr, phantom, "\Nearrow Id"] & (w(r)/v) \arrow[r] \arrow[d, "\pi"'] \arrow[dr, phantom, "\Nearrow"] & Q \arrow[d, "v"] \\
e \arrow[r, equal] & e \arrow[r, "w(r)"] & S
\end{tikzcd}$$

where the right square is a coslice square and the left square is homotopy exact by the assumption of t_r being homotopy cofinal. $\square$

3.7 Barycentric subdivision

Definition 3.7.1. Recall a *chain* of R is a subset $\sigma \subset R$ such that the induced order on σ is total. If σ is nonempty, we define its dimension as $\dim(\sigma) = |\sigma| - 1$. We put

$$\begin{aligned}
s(R) &= \{\text{ nonempty chains } \sigma \subseteq R\} \\
s_d(R) &= \{\sigma \in s(R) \mid \dim(\sigma) = d\} \\
s_{\leq d}(R) &= \{\sigma \in s(R) \mid \dim(\sigma) \leq d\}.
\end{aligned}$$

Note that every nonempty chain σ has a largest element, which we denote by $\max(\sigma)$. We order $s(R)$ by inclusion, this makes $\max\colon s(R) \to R$ into a morphism of posets.

This provides a functor $s\colon \mathrm{PoSet} \to \mathrm{PoSet}$ and a natural transformation $\max\colon s(R) \to R$ (since for $\sigma \subseteq \sigma'$ we have $\max(\sigma) \leq \max(\sigma')$). However, if $f \leq g$ then it is not true in general that $s(f) \leq s(g)$: consider chains of dimension 0, we have $s(f)(\{r\}) \leq s(g)(\{r\})$ if and only if $f(r) = g(r)$ hence $s(f) \leq s(g)$ is equivalent to $f = g$.

Example 3.7.2. We should think of $s(R)$ as some sort of barycentric subdivision of the poset R: that is for any sequence of elements connected by inequalities $r_0 < r_1 < \cdots < r_d$ we have a corresponding chain $\{r_0, \ldots, r_d\}$ which encodes all the possible sequence of inequalities between the elements r_i's. This provides a subposet in $s(R)$ whose geometric realization is just Δ^d and the boundary $\partial\Delta^d$ is attached to $|R|$ along the sequence $r_0 < r_1 < \cdots < r_d$ in R, so we are not modifying to homotopy type of $|R|$.

As concrete example let us consider $[2] = \{0 < 1 < 2\}$, then we can visualize $s(R)$ as

$$\begin{tikzcd}
& 0 \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] & \\
01 \arrow[r, dash] \arrow[d, dash] & 012 \arrow[r, dash] \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] & 02 \arrow[d, dash] \\
1 \arrow[r, dash] & 12 \arrow[r, dash] & 2.
\end{tikzcd}$$

Proposition 3.7.3. *If $[f] = [g]$ in $\pi_0(\mathrm{PoSet}(R,T))$, then $[s(f)] = [s(g)]$. Thus, s descends to an endofunctor of the strong homotopy category.*

Proof. We can easily reduce to the case where $f \leq g$. We then choose a minimal element r_1 in R, then a minimal element r_2 in $R \setminus \{r_1\}$ and so on, giving an enumeration $R = \{r_1, \dots, r_{m-1}\}$ such that $r_a \not\leq r_b$ whenever $a > b$. We define $\phi\colon R \to [m] = \{0, \dots, m\}$ by $\phi(r_i) = i$, thus ϕ is injective and monotone, and 0 and m are not in the image. Then for $0 \leq k \leq m$ we define $u_k, v_k\colon s(R) \to s(T)$ by

$$u_k(\sigma) = \{f(r) \mid r \in \sigma,\ \phi(r) < k\} \cup \{g(r) \mid r \in \sigma,\ \phi(r) \geq k\}$$
$$v_k(\sigma) = \{f(r) \mid r \in \sigma,\ \phi(r) \leq k\} \cup \{g(r) \mid r \in \sigma,\ \phi(r) \geq k\}.$$

We find that $u_k(\sigma)$ and $v_k(\sigma)$ are nonempty chains in T, so we actually have maps $u_k, v_k\colon s(R) \to s(T)$ as claimed. It is also clear that $\sigma \subseteq \tau$ implies $u_k(\sigma) \subseteq u_k(\tau)$ and $v_k(\sigma) \subseteq v_k(\tau)$, so u_k and v_k are morphisms of posets. From the definitions it is immediate that $u_k, u_{k+1} \leq v_k$, which implies that all the maps u_k and v_k are homotopic. The equalities have $u_0 = s(g)$ and $u_m = s(f)$ let us conclude so $[s(f)] = [s(g)]$. □

Lemma 3.7.4. *The map* $\max\colon s(R) \to R$ *is homotopy cofinal (and so is a $\mathcal{D}$-equivalence).*

Proof. Fix $r \in R$; by Proposition 3.6.15 it will suffice to show that the poset

$$U = (\max / r) = \{\sigma \in s(R) \mid \max(\sigma) \leq r\}$$

is strongly contractible. Put

$$V = \{\sigma \in s(R) \mid \max(\sigma) = r\} = \{\sigma \in U \mid r \in \sigma\} \subseteq U.$$

As $\{r\}$ is smallest element in V, we see that V is strongly contractible. We can define a poset map $t\colon U \to V$ by $t(\tau) = \tau \cup \{r\}$, and we find that this is left adjoint to the inclusion $V \to U$, so the inclusion is a strong homotopy equivalence by Remark 3.6.4. It follows that U is also strongly contractible. □

The following proposition is the rephrasing in derivator language of a classical result about homotopy limits in simplicial or topological categories.

Proposition 3.7.5. *Let n be the maximum length of any chain in R. Then for all strong stable derivators $\mathcal{C}$ and objects $X \in \mathcal{C}(R)$ there is a natural tower*

$$\underset{\overleftarrow{R}}{\mathrm{holim}}(X) = T^n(X) \to T^{n-1}(X) \to \cdots \to T^0(X) \to T^{-1}(X) = 0$$

and natural distinguished triangles

$$\bigoplus_{\sigma \in s_d(R)} \Omega^d X_{\max(\sigma)} \to T^d(X) \to T^{d-1}(X).$$

Proof. Put $Y = \max^*(X) \in \mathcal{C}(s(R))$. Lemma 3.7.4 identifies $\underset{\longleftarrow\ R}{\mathrm{holim}}(X)$ with $\underset{\longleftarrow\ s(R)}{\mathrm{holim}}(Y)$, so we will work with Y from now on.

Let $j_d\colon s_{\leq d}(R) \to s(R)$ be the inclusion, and put $T^d(X) = \underset{\longleftarrow}{\mathrm{holim}}\, j_d^*(Y)$. Note that $T^n(Y) = \underset{\longleftarrow\ s(R)}{\mathrm{holim}}\, Y = \underset{\longleftarrow\ R}{\mathrm{holim}}\, X$.

Now we fix d and consider the object $Z = j_{\leq d}^*(Y)$ and the two inclusions $j\colon s_{\leq (d-1)}(R) \to s_{\leq d}(R)$ and $i\colon s_d(R) \to s_{\leq d}(R)$. Observe j is a sieve and i is its complementary cosieve, thus

[10, Example 4.25] gives a distinguished triangle $i_!i^*(Z) \to Z \to j_*j^*(Z)$. If we let $\pi\colon s_{\leq d}(R) \to e$ and apply π_*, we get a distinguished triangle $\pi_*i_!i^*(Z) \to T^d(X) \to T^{d-1}(X)$. Now note that $s_d(R)$ is a discrete poset: for $\sigma, \tau \in s_d(R)$ we can only have $\sigma \leq \tau$ if $\sigma = \tau$. Because of this and (D1), we see that $\mathcal{C}(s_d(R)) \simeq \prod_{\sigma \in s_d(R)} \mathcal{C}(e)$. Hence we can write $i^*(Z)$ as a coproduct of objects $W(\sigma) \in \mathcal{C}(s_d(R))$, where $W(\sigma)_\tau = 0$ for $\tau \neq \sigma$, and $W(\sigma)_\sigma = Z_\sigma = Y_\sigma = X_{\max(\sigma)}$. It will now suffice to identify $\pi_*i_!W(\sigma)$ with $\Omega^d Y_\sigma$. Since i is a cosieve, it follows that for $\tau \in s_{\leq d}(R)$ we have

$$(i_!W(\sigma))_\tau = \begin{cases} Y_\sigma & \text{if } \tau = \sigma \\ 0 & \text{otherwise.} \end{cases}$$

The poset $\{\tau \in s(R) \mid \tau \subseteq \sigma\}$ is naturally identified with $\mathcal{P}'(\sigma)$. Let $k\colon \mathcal{P}'(\sigma) \to s_{\leq d}(R)$ be the inclusion, which is a sieve. As the support of $i_!W(\sigma)$ is contained in the image of k, we see from Corollary 3.4.6 that the unit map $i_!W(\sigma) \to k_*k^*i_!W(\sigma)$ is an isomorphism. It follows that $\pi_*i_!W(\sigma)$ is the homotopy limit of $k^*i_!W(\sigma)$. By Corollary 3.5.12 this can be identified with $\Omega^d Y_\sigma$. Since $\mathcal{C}$ is stable Ω is an equivalence, hence it commutes with coproducts and we conclude. □

Corollary 3.7.6. *Let n be the maximum length of any chain in R. Then for all strong stable derivators $\mathcal{C}$ and objects $X \in \mathcal{C}(R)$ there is a natural diagram*

$$0 = T_{-1}(X) \to T_0(X) \to \cdots \to T_n(X) = \underset{R}{\underrightarrow{\operatorname{hocolim}}}(X)$$

and natural distinguished triangles

$$T_{d-1}(X) \to T_d(X) \to \bigoplus_{\sigma \in s_d(R)} \Sigma^d X_{\max(\sigma)}.$$

Proof. Apply the proposition to the dual derivator. □

Proposition 3.7.7. *Let $\mathcal{C}$ be a strong stable derivator. For any map $f\colon R \to T$ of finite posets, the functors f^*, $f_!$ and f_* all preserve arbitrary products and coproducts.*

Proof. The functors f^* and $f_!$ have right adjoints, so they preserve coproducts. The functors f^* and f_* have left adjoints, thus they preserve products. We need only to show that f_* preserves coproducts, by duality it will follow $f_!$ preserves products. Consider a family of objects $X_\alpha \in \mathcal{C}(R)$ and the resulting canonical map

$$h\colon \bigoplus_\alpha f_*(X_\alpha) \to f_*\left(\bigoplus_\alpha X_\alpha\right).$$

We want to prove that h is an isomorphism: by (D2) it will suffice to show that $t^*(h)$ is an isomorphism for all $t \in T$. We have already observed that t^* preserves coproducts, and we have the Kan formula (D4) expressing t^*f_* as a homotopy limit over the coslice (t/f). This reduces the matter to showing that all homotopy limit functors preserve coproducts. Note that the functor $\Omega\colon \mathcal{C}(e) \to \mathcal{C}(e)$ is an equivalence of categories, so it certainly preserves coproducts. It follows that all functors of the form $X \mapsto \Omega^d X_r$ also preserve coproducts. It then follows by induction that the functors T^d in Proposition 3.7.5 all preserve coproducts. By taking d sufficiently large, we see that homotopy limits preserve coproducts as required and this completes the proof. □

3.8 Thick subderivators

The aim of this section is to establish a correspondence between the thick subcategories of $\mathcal{C}(e)$ (for $\mathcal{C}$ a strong stable derivator) and inclusions of derivators $\mathcal{E} \subseteq \mathcal{C}$ which let us identify $\mathcal{E}(R)$ with a thick subcategory of $\mathcal{C}(R)$ for any $R \in \mathrm{PoSet}$.

We show that any thick subcategory $\mathcal{T} \subset \mathcal{C}(e)$ can be lifted to a derivator $\mathcal{E}$ with a morphism $\mathcal{E} \hookrightarrow \mathcal{C}$ given level-wise by inclusions.

Definition 3.8.1. Let $\mathcal{C}$ be a strong stable derivator. By a *thick subderivator* $\mathcal{E} \subseteq \mathcal{C}$ we mean a system of full subcategories $\mathcal{E}(R) \subseteq \mathcal{C}(R)$ for all R, such that:

(a) each category $\mathcal{E}(R)$ is closed under finite coproducts (including the empty coproduct, so $0 \in \mathcal{E}(R)$).

(b) Every category $\mathcal{E}(R)$ is closed under retracts: that is if we have two objects $X, Y \in \mathcal{C}(R)$ and morphisms $X \xrightarrow{a} Y \xrightarrow{b} X$ such that $ba = Id_X$ and $Y \in \mathcal{E}(R)$ then also $X \in \mathcal{E}(R)$. In particular, $\mathcal{E}(R)$ is closed under isomorphisms.

(c) For any morphism $u\colon R \to T$ of finite posets, we have $u^*\mathcal{E}(T) \subseteq \mathcal{E}(R)$, $u_!\mathcal{E}(R) \subseteq \mathcal{E}(T)$ and $u_*\mathcal{E}(R) \subseteq \mathcal{E}(T)$. More briefly, we say that the functors u^*, u_* and $u_!$ preserve $\mathcal{E}$.

Definition 3.8.2. Let $\mathcal{C}$ be a strong stable derivator, and let $\mathcal{T}$ be a thick subcategory of $\mathcal{C}(e)$. For any finite poset R and any $r \in R$ recall we have a corresponding map $r\colon e \to R$.

(a) We put
$$(\gamma_0\mathcal{T})(R) = \{X \in \mathcal{C}(R) \mid r^*X \in \mathcal{T} \text{ for all } r \in R\}.$$

(b) We let $(\gamma_1\mathcal{T})(R)$ denote the smallest thick subcategory of $\mathcal{C}(R)$ containing $r_!(\mathcal{T})$ for all $r \in R$.

(c) We let $(\gamma_2\mathcal{T})(R)$ denote the smallest thick subcategory of $\mathcal{C}(R)$ containing $r_*(\mathcal{T})$ for all $r \in R$.

Theorem 3.8.3. *The subcategories $(\gamma_i\mathcal{T})(R)$ are the same for $i = 0, 1, 2$ (so we will just write $(\gamma\mathcal{T})(R)$ in future). The map γ gives a bijection from thick subcategories of $\mathcal{C}(e)$ to thick subderivators of $\mathcal{C}$. Moreover, if $\mathcal{E}$ is a thick subderivator of $\mathcal{C}$, then $\mathcal{E}(R)$ is a thick subcategory of $\mathcal{C}(R)$ for all R.*

Remark 3.8.4. This result is the motivation for our choice to consider derivators defined only on finite posets, not on any small category. It is easy to see that the claim cannot hold for such a definition of derivator.

For a derivator $\mathcal{D}\colon \mathrm{Dia}^{op} \to \mathrm{CAT}$ defined on a general 2-category of diagrams Dia (see [5, §1.2]) usually (D1) is replaced by its stronger variant

(D1') for any indexing set I and any collection of small categories $\{A_i\}_{i \in I}$ such that $\coprod_i A_i \in \mathrm{Dia}$ the canonical functor
$$\mathcal{D}\Big(\coprod_i A_i\Big) \to \prod_i \mathcal{D}(A_i)$$
is an equivalence.

This with the other axioms implies the category $\mathcal{D}(R)$ admits products and coproducts over any indexing set I if this, considered as a discrete category, lies in Dia (see [11, Prop. 7.1]).

If Theorem 3.8.3 held even for a derivator $\mathcal{D}$ defined on Cat, then we would have that any $\mathcal{T}$ thick subcategory of $\mathcal{D}(e)$ would be closed under arbitrary products and coproducts, which is obviously false.

Imposing the categories of Dia to have a finite set of objects is not enough either. Fixed G a non-trivial group, we can consider $R = BG$ the category with just one object $*$ and its endomorphisms given by G. Then Theorem 3.8.3 would imply that if we have an object $X \in \mathcal{C}(R)$ whose associated element X_* lies in $\mathcal{T}$, then $\underset{\longrightarrow BG}{\operatorname{hocolim}} X \in \mathcal{T}$ which is again false in full generality.

The proof will be given after some preliminary lemmata.

Lemma 3.8.5. *If $\mathcal{E} \subseteq \mathcal{C}$ is a thick subderivator, then $\mathcal{E}(R)$ is a thick subcategory of $\mathcal{C}(R)$ for all R, and*

$$(\gamma_1\mathcal{E}(e))(R) \cup (\gamma_2\mathcal{E}(e))(R) \subseteq \mathcal{E}(R) \subseteq (\gamma_0\mathcal{E}(e))(R).$$

Proof. We saw in §3.4 that the triangulation of $\mathcal{C}(R)$ is defined in terms of operations of the form u^*, $u_!$, u_*, this is elaborated in detail in [10, §4.2]. From this it follows that $\mathcal{E}(R)$ is closed under the suspension functor and its inverse, and under cofibrations, so it is a thick subcategory. From the definitions we also know that the functors $r_!$ and r_* preserve $\mathcal{E}$, so $\mathcal{E}(R)$ contains the generators of $(\gamma_1\mathcal{E}(e))(R)$ and $(\gamma_2\mathcal{E}(e))(R)$, hence the first inclusion easily follows. On the other hand, the functors r^* also preserve $\mathcal{E}$ and this means that $\mathcal{E}(R) \subseteq (\gamma_0\mathcal{E}(e))(R)$. □

Lemma 3.8.6. *If $\mathcal{T}$ is a thick subcategory of $\mathcal{C}(e)$, then $\gamma_0\mathcal{T}$ is a thick subderivator of $\mathcal{C}$, with $(\gamma_0\mathcal{T})(e) = \mathcal{T}$.*

Proof. Suppose we have a morphism $u\colon R \to T$ of finite posets. It is clear from the definitions that $u^*((\gamma_0\mathcal{T})(T)) \subseteq (\gamma_0\mathcal{T})(R)$. We claim that also the functors u_* and $u_!$ preserve $\gamma_0\mathcal{T}$. In the case $T = e$ this follows from Proposition 3.7.5 and Corollary 3.7.6 using induction of the maximum length of the chains in R.

The general case can be reduced to the above: by (D4) we have isomorphisms

$$t^*u_!X \cong (\pi_{(u/t)})_!p^*X \qquad t^*u_*X \cong (\pi_{(t/u)})_*q^*X$$

for $p\colon (u/t) \to R$ and $q\colon (t/u) \to R$ the projections associated to the slice and coslice respectively. If we start from $X \in (\gamma_0\mathcal{T})(R)$ trivially $p^*X \in (\gamma_0\mathcal{T})((u/t))$ and $q^*X \in (\gamma_0\mathcal{T})((t/u))$.

It is immediate to check that $\gamma_0\mathcal{T}$ is closed under retracts, so $\gamma_0\mathcal{T}$ is a thick subderivator of $\mathcal{C}$. The relation $(\gamma_0\mathcal{T})(e) = \mathcal{T}$ is clear. □

Corollary 3.8.7. *If $\mathcal{T}$ is a thick subcategory of $\mathcal{C}(e)$, then $(\gamma_1\mathcal{T})(R) \cup (\gamma_2\mathcal{T})(R) \subseteq (\gamma_0\mathcal{T})(R)$ for all R.*

Proof. Lemma 3.8.6 allows us to apply Lemma 3.8.5 to the case $\mathcal{E} = \gamma_0\mathcal{T}$. □

Lemma 3.8.8. *If $\mathcal{T}$ is a thick subcategory of $\mathcal{C}(e)$, then all functors $u_!$ preserve $\gamma_1\mathcal{T}$, and all functors u_* preserve $\gamma_2\mathcal{T}$.*

Proof. Fix a map $u\colon R \to T$, and put $\mathcal{U} = \{X \in \mathcal{C}(R) \mid u_!(X) \in (\gamma_1\mathcal{T})(T)\}$. It is easy to see that $\mathcal{U}$ is a thick triangulated subcategory. As $u \circ r = u(r)\colon e \to T$ we see that $u_! \circ r_! \cong u(r)_!$, and it follows that $r_!(\mathcal{T}) \subseteq \mathcal{U}$ for all $r \in R$. It follows that $(\gamma_1\mathcal{T})(R) \subseteq \mathcal{U}$, which implies $u_!$ preserves $\gamma_1\mathcal{T}$ as required. Dually, we see that u_* preserves $\gamma_2\mathcal{T}$. □

Lemma 3.8.9. *If $\mathcal{T}$ is a thick subcategory of $\mathcal{C}(e)$, then $\gamma_0\mathcal{T} = \gamma_1\mathcal{T} = \gamma_2\mathcal{T}$.*

Proof. We will write $\Gamma_i = \gamma_i\mathcal{T}$ for brevity. It will be enough to prove that $\Gamma_0 = \Gamma_1$, as duality then gives $\Gamma_0 = \Gamma_2$. We already know from Corollary 3.8.7 that $\Gamma_1(R) \subseteq \Gamma_0(R)$ for all R, so it will suffice to prove that $\Gamma_0(R) \subseteq \Gamma_1(R)$. This is clear if R is empty, as (D1) implies $\mathcal{C}(\emptyset)$ is the terminal category e. If R is nonempty, we can choose a minimal element $a \in R$ and put

$T = R \setminus \{a\}$. Let $j\colon e \to R$ correspond to a, and let $i\colon T \to R$ be the inclusion. Consider an object $Y \in \Gamma_0(R)$: we must show it belongs to $\Gamma_1(R)$. Put $X = j_!j^*(Y)$, the constant diagram of value Y_a over R, and let Z be the cofibre of the counit map $X \to Y$. From Proposition 3.4.1 we have $j^*j_! = 1$ and so $j^*Z = 0$, so the support of Z is contained in $i(T)$. As i is a cosieve, we see that $Z \simeq i_!i^*(Z)$. Now $i^*(Z) \in \Gamma_0(T)$ and we can assume by induction that $\Gamma_0(T) \subseteq \Gamma_1(T)$, so $Z \in i_!\Gamma_1(T) \subseteq \Gamma_1(R)$. From the definitions we also have $j^*(X) \in \mathcal{T}$ thus $X \in \Gamma_1(R)$. As $\Gamma_1(R)$ is thick and contains X and Z, it also contains Y as required. □

Proof of Theorem 3.8.3. First we suppose that we start with a thick subcategory $\mathcal{T} \subseteq \mathcal{C}(e)$. Lemma 3.8.9 tells us that the $\gamma_i\mathcal{T}$ are all the same, so we can just write $\gamma\mathcal{T}$. Lemma 3.8.6 tells us that this is a thick subderivator, with $(\gamma\mathcal{T})(e) = \mathcal{T}$.

Suppose instead that we start with a thick subderivator $\mathcal{E} \subseteq \mathcal{C}$, and we put $\mathcal{T} = \mathcal{E}(e)$. Lemma 3.8.5 tells us that $\mathcal{E}(R)$ is a thick subcategory of $\mathcal{C}(R)$ for all R, and in particular that $\mathcal{T}$ is a thick subcategory of $\mathcal{C}(e)$. We can therefore apply Lemma 3.8.9 to $\mathcal{T}$ and combine this with Lemma 3.8.5 to see that $\mathcal{E} = \gamma\mathcal{T}$. □

Corollary 3.8.10. *Let $\mathcal{E}$ be a thick subderivator of $\mathcal{C}$, and let X be an object of $\mathcal{C}(R)$. Suppose that $R = \bigcup_i R_i$ for some family of subposets R_i. Then X lies in $\mathcal{E}(R)$ if and only if $X|_{R_i} \in \mathcal{E}(R_i)$ for all i.*

Proof. The identity $\mathcal{E} = \gamma\mathcal{E}(e)$ means that $X \in \mathcal{E}(R)$ if and only if $r^*(X) \in \mathcal{E}(e)$ for all $r \in R$. Similarly, $X|_{R_i} \in \mathcal{E}(R_i)$ if and only if $r^*(X) \in \mathcal{E}(e)$ for all $r \in R_i$. The claim is immediate from this. □

Lemma 3.8.11. *Let $\mathcal{C}$ be a strong stable derivator, and let $\mathcal{E}(R)$ be a thick subcategory of $\mathcal{C}(R)$ for all R. Suppose that for every $u\colon R \to T$, the functors $u_!$ and u^* preserve $\mathcal{E}$. Then $\mathcal{E} = \gamma\mathcal{E}(e)$, so in particular $\mathcal{E}$ is a thick subderivator.*

Proof. Put $\mathcal{E}' = \gamma\mathcal{E}(e) = \gamma_0\mathcal{E}(e) = \gamma_1\mathcal{E}(e)$. We now claim that $\mathcal{E}(R) \subseteq \mathcal{E}'(R)$ for all R. Using the description $\mathcal{E}' = \gamma_0\mathcal{E}(e)$, this reduces to the claim that $r^*\mathcal{E}(R) \subseteq \mathcal{E}(e)$ for all r, which is true because r^* preserves $\mathcal{E}$ by assumption. In the opposite direction, we know that the functors $r_!$ preserve $\mathcal{E}$, which means that $\mathcal{E}(R)$ contains all the generators of $\mathcal{E}'(R) = (\gamma_1\mathcal{E}(e))(R)$. As $\mathcal{E}(R)$ is assumed to be thick, it follows that $\mathcal{E}'(R) \subseteq \mathcal{E}(R)$. □

Definition 3.8.12. Let $\mathcal{T}$ be a triangulated category with coproducts. We say that an object $X \in \mathcal{T}$ is *compact* if the natural map $\bigoplus_\alpha[X, Y_\alpha] \to [X, \bigoplus_\alpha Y_\alpha]$ is an isomorphism for every family of objects Y_α. We write $\mathcal{T}_c$ for the full subcategory of compact objects (which is easily seen to be thick). If $\mathcal{T} = \mathcal{C}(R)$ for some derivator $\mathcal{C}$, then we will write $\mathcal{C}_c(R)$ rather than $\mathcal{C}(R)_c$. We say that $\mathcal{T}$ is *compactly generated* if

(a) The category $\mathcal{T}_c$ is essentially small (so there is a skeleton that has a set of objects, rather than a proper class); and

(b) $\mathcal{T}$ is the only thick subcategory of $\mathcal{T}$ that is closed under arbitrary coproducts and contains $\mathcal{T}_c$.

Lemma 3.8.13. *Let $\mathcal{T}$ be a triangulated category, let $\mathcal{G}$ be a set of objects of $\mathcal{T}$, and let $\mathcal{U}$ be the smallest thick subcategory containing $\mathcal{G}$. Then $\mathcal{U}$ is essentially small.*

Proof. Define full subcategories $\mathcal{U}_n$ as follows. Start with $\mathcal{U}_0 = \mathcal{G} \cup \{0\}$. Let $\mathcal{U}_{n+1}$ consist of $\bigcup_{k\in\mathbb{Z}} \Sigma^k\mathcal{U}_n$, together with a choice of cofibre for every morphism in $\mathcal{U}_n$, and a choice of splitting for every idempotent morphism in $\mathcal{U}_n$. Put $\mathcal{U}_\infty = \bigcup_n \mathcal{U}_n$. We then find that $\mathcal{U}_\infty$ has only a set of objects, and contains a representative of every isomorphism class in $\mathcal{U}$. □

Proposition 3.8.14. *$\mathcal{C}_c(R)$ is a thick subderivator of $\mathcal{C}$ (and so is the same as $\gamma\mathcal{C}_c(e)$).*

Proof. Put $\mathcal{E} = \gamma\mathcal{C}_c(e)$, which is a thick subderivator; if $\mathcal{C}_c(R)$ actually defines a thick subderivator of $\mathcal{C}$ then Theorem 3.8.3 immediately implies $\mathcal{C}_c = \mathcal{E}$.

If F is left adjoint to G and G preserves coproducts then for compact X we have

$$[FX, \bigoplus_\alpha Y_\alpha] = [X, G\bigoplus_\alpha Y_\alpha] = [X, \bigoplus_\alpha GY_\alpha] = \bigoplus_\alpha [X, GY_\alpha] = \bigoplus_\alpha [FX, Y_\alpha],$$

so FX is compact. Using this and Proposition 3.7.7, we see that $u_!$ and u^* preserve $\mathcal{C}_c$. The claim now follows from Lemma 3.8.11. $\square$

Corollary 3.8.15. *If $\mathcal{C}(e)$ is compactly generated, then $\mathcal{C}(R)$ is compactly generated for all R.*

Proof. First, we can choose a small skeleton $\mathcal{G}$ for $\mathcal{C}_c(e)$, and then put

$$\mathcal{G}(R) = \{r_!(X) \mid r \in R,\ X \in \mathcal{G}\} \subseteq \mathcal{C}_c(R).$$

From the description $\mathcal{C}_c(R) = (\gamma_1\mathcal{C}_c(e))(R)$ we see that $\mathcal{C}_c(R)$ is generated by $\mathcal{G}(R)$, and so is essentially small by Lemma 3.8.13.

Now let $\mathcal{T}$ be a localising subcategory of $\mathcal{C}(R)$ that contains $\mathcal{C}_c(R)$. Put

$$\mathcal{U} = \{X \in \mathcal{C}(e) \mid r_!(X) \in \mathcal{T} \text{ for all } r \in R\}.$$

This is easily seen to be a localising subcategory of $\mathcal{C}(e)$ containing $\mathcal{C}_c(e)$, so $\mathcal{U} = \mathcal{C}(e)$. From Theorem 3.8.3 it follows that $\gamma\mathcal{U} = \mathcal{C}$, so in particular $\mathcal{C}(R) = (\gamma_1\mathcal{U})(R)$. However, from the definition of $\mathcal{U}$ it is clear that $(\gamma_1\mathcal{U})(R) \subseteq \mathcal{T}$, so $\mathcal{T} = \mathcal{C}(R)$ as required. $\square$

Definition 3.8.16. We say that $\mathcal{C}$ is *compactly generated* if it satisfies the condition of Corollary 3.8.15.

Remark 3.8.17. Unfortunately for a derivator $\mathcal{C}$ defined only on finite posets the existence of arbitrary coproducts on $\mathcal{C}(R)$ is not guaranteed, so we cannot apply the above results.

The only examples of derivators of type PoSet admitting small coproducts we can think of are the ones obtained by restricting a derivator defined over Cat^{op}.

Anyway, the above results tell us that in the case $\mathcal{C}$ actually has coproducts if $\mathcal{C}(e)$ admits as set of compact generators $\mathcal{G}$, then $\mathcal{C}(R)$ has a set of compact generators given by

$$\{r_!X \mid r \in R, X \in \mathcal{G}\}$$

and moreover $\mathcal{C}_c$ defines a thick subderivator.

Definition 3.8.18. Let $\mathcal{U}$ be a thick subcategory of a triangulated category $\mathcal{T}$. We then write

$$\mathcal{U}^\perp = \{Y \in \mathcal{T} \mid [U, Y] = 0 \text{ for all } U \in \mathcal{U}\}$$
$${}^\perp\mathcal{U} = \{X \in \mathcal{T} \mid [X, U] = 0 \text{ for all } U \in \mathcal{U}\}.$$

Similarly, if $\mathcal{E}$ is a thick subderivator of a stable derivator $\mathcal{C}$, we put $\mathcal{E}^\perp(R) = \mathcal{E}(R)^\perp$ and $({}^\perp\mathcal{E})(R) = {}^\perp(\mathcal{E}(R))$.

Proposition 3.8.19. *$\mathcal{E}^\perp$ and ${}^\perp\mathcal{E}$ are thick subderivators, so $\mathcal{E}^\perp = \gamma(\mathcal{E}(e)^\perp)$ and ${}^\perp\mathcal{E} = \gamma({}^\perp\mathcal{E}(e))$.*

Proof. Suppose that $X \in ({}^\perp\mathcal{E})(R)$ and $u \colon R \to T$. For $V \in \mathcal{E}(T)$ we have $u^*(V) \in \mathcal{E}(R)$ and so $[u_!(X), V] = [X, u^*(V)] = 0$. From this we see that $u_!$ preserves ${}^\perp\mathcal{E}$. As u^* is left adjoint to u_* and u_* preserves $\mathcal{E}$, we see in the same way that u^* preserves ${}^\perp\mathcal{E}$. Therefore, it follows from Lemma 3.8.11 that ${}^\perp\mathcal{E}$ is a thick subderivator. A dual argument shows that $\mathcal{E}^\perp$ is also a thick subderivator. $\square$

3.9 Morphisms of derivators

We now provide a brief review of the appropriate notions of morphism of derivators and of natural transformation between between two such morphisms. Basically we want to be able to define an appropriate 2-category PDER of prederivators, and consequently the full 2-subcategory DER spanned by derivators. Since a prederivator consists in a strict 2-functor $\mathcal{C}\colon \mathrm{PoSet}^{op} \to \mathrm{CAT}$ the most obvious choice would be to set a morphism $F\colon \mathcal{C} \to \mathcal{D}$ to be a *pseudo-natural transformation*, and a 2-cell $\sigma\colon F \Rightarrow G$ should be a *modification* between the two morphisms. These two notions are taken from [3, Def. 7.5.2 and 7.5.3], but we proceed to spell them out here explicitly.

Definition 3.9.1. Let $\mathcal{C}, \mathcal{D}$ be two prederivators, a *morphism of derivators* between them $F\colon \mathcal{C} \to \mathcal{D}$ consists in a collection of functors $F_R\colon \mathcal{C}(R) \to \mathcal{D}(R)$ for any R finite poset, together with a collection of natural isomorphisms $\{\gamma^F_u\}$ (where $u\colon R \to T$ ranges over all the morphisms of posets) fitting in square diagrams as follows

$$\begin{tikzcd} \mathcal{C}(T) \arrow[r, "F_T"] \arrow[d, "u^*"'] \arrow[dr, phantom, "\overset{\gamma^F_u}{\swarrow}"] & \mathcal{D}(T) \arrow[d, "u^*"] \\ \mathcal{C}(R) \arrow[r, "F_R"'] & \mathcal{D}(R). \end{tikzcd}$$

Moreover, we require the collection $\{\gamma^F_u\}$ to satisfy the following compatibility conditions:

- $\gamma^F_{Id_R} = Id_{F_R}$;
- for any two composable non-decreasing maps of finite posets $R \xrightarrow{u} T \xrightarrow{v} S$ we have $\gamma^F_{vu} = \gamma^F_u \odot \gamma^F_v = \gamma^F_u v^* \circ u^* \gamma^F_v$, that is the pasting in the following diagram

$$\begin{tikzcd} \mathcal{D}(S) \arrow[r, "v^*"] & \mathcal{D}(T) \arrow[r, "u^*"] & \mathcal{D}(R) \\ \mathcal{C}(S) \arrow[r, "v^*"'] \arrow[u, "F_S"] \arrow[ur, phantom, "\overset{\gamma^F_v}{\searrow}"] & \mathcal{C}(T) \arrow[r, "u^*"'] \arrow[u, "F_T"] \arrow[ur, phantom, "\overset{\gamma^F_u}{\searrow}"] & \mathcal{C}(R) \arrow[u, "F_R"'] \end{tikzcd}$$

  coincides with γ^F_{vu};
- for any natural transformation $\alpha\colon u \Rightarrow v$ between two maps $u, v\colon R \to T$ the equality $F_R\alpha^* \circ \gamma^F_u = \gamma^F_v \circ \alpha^* F_T$ holds, that is the following two pastings coincide

$$\begin{tikzcd} \mathcal{C}(T) \arrow[r, "F_T"] \arrow[d, bend right=50, "v^*"'] \arrow[d, "u^*"] \arrow[d, phantom, shift right=3, "\overset{\alpha^*}{\Leftarrow}"] \arrow[dr, phantom, "\underset{\gamma^F_u}{\swarrow}"] & \mathcal{D}(T) \arrow[d, "u^*"] \\ \mathcal{C}(R) \arrow[r, "F_R"'] & \mathcal{D}(R) \end{tikzcd} \qquad \begin{tikzcd} \mathcal{C}(T) \arrow[r, "F_T"] \arrow[d, "v^*"'] \arrow[dr, phantom, "\overset{\gamma^F_v}{\swarrow}"] & \mathcal{D}(T) \arrow[d, "v^*"'] \arrow[d, bend left=50, "u^*"] \arrow[d, phantom, shift left=3, "\overset{\alpha^*}{\Leftarrow}"] \\ \mathcal{C}(R) \arrow[r, "F_R"'] & \mathcal{D}(R). \end{tikzcd}$$

We say F is a *strict* morphisms if $\gamma^F_u = Id$ for any u, i.e. F commutes with the restrictions on the nose.

Definition 3.9.2. Let $F, G\colon \mathcal{C} \to \mathcal{D}$ be two morphisms of prederivators. A *natural transformation* between them $\sigma\colon F \Rightarrow G$ consists in a collection of natural transformations $\sigma_R\colon F_R \Rightarrow G_R$ for every R finite poset which must be compatible with the isomorphisms $\{\gamma^F_u\}$, $\{\gamma^G_u\}$ in the following sense: for any map of posets $u\colon R \to T$ the following pastings must coincide

$$
\begin{tikzcd}
\mathcal{C}(T) \arrow[r, bend left, "F_T"] \arrow[r, bend right, "G_T"'] \arrow[r, phantom, "\sigma_T\Downarrow"] \arrow[d, "u^*"'] \arrow[dr, phantom, "\swarrow \gamma_u^G"] & \mathcal{D}(T) \arrow[d, "u^*"] \\
\mathcal{C}(R) \arrow[r, "G_R"'] & \mathcal{D}(R)
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{C}(T) \arrow[r, "F_T"] \arrow[d, "u^*"'] \arrow[dr, phantom, "\gamma_u^F \swarrow"] & \mathcal{D}(T) \arrow[d, "u^*"] \\
\mathcal{C}(R) \arrow[r, bend left, "F_R"] \arrow[r, bend right, "G_R"'] \arrow[r, phantom, "\Downarrow\sigma_R"] & \mathcal{D}(R).
\end{tikzcd}
$$

In formulas $\gamma_u^G \circ u^*\sigma_T = \sigma_R u^* \circ \gamma_u^F$.

Definition 3.9.3. We denote by PDER the 2-category whose objects are prederivators, the 1-cells morphisms of prederivators and 2-cells natural transformations between such morphisms.

We define DER to be the full 2-subcategory of PDER generated by the objects which are derivators.

We can use this arrangement to provide a notion of adjunction internal to the setting of (pre)derivators: as in Definition 3.2.1 we define an adjunction $(L, R)\colon \mathcal{C} \rightleftarrows \mathcal{D}$ to be a pair of morphisms going in opposite directions together with two 2-cells $\eta\colon Id \Rightarrow RL$ and $\varepsilon\colon LR \Rightarrow Id$ satisfying the triangular identities.

Specifying these data to every finite poset T it is immediate to see we obtain an adjunction $(L_T, R_T)\colon \mathcal{C}(T) \rightleftarrows \mathcal{D}(T)$ in the classical sense.

It comes natural to wonder if also the converse statement holds: given a collection of adjunctions (L_T, R_T) can we extend it to an adjunction between prederivators? The key step to answer this question is given by the next lemma.

Lemma 3.9.4. *Let $\mathcal{C}, \mathcal{D}$ be two prederivators and suppose that for any T finite poset we have an adjunction $(L_T, R_T)\colon \mathcal{C}(T) \rightleftarrows \mathcal{D}(T)$. Assume also that we fixed a family $\{\gamma_u^L\}$ of natural isomorphisms which let us assemble the functors $\{L_T\}$ into a morphism $L\colon \mathcal{C} \to \mathcal{D}$. Then for any $u\colon T \to S$ morphism of finite posets, there exists a unique natural transformation $\gamma_u^R\colon u^*R_S \Rightarrow R_T u^*$ which fits in the following diagram*

$$
\begin{tikzcd}
\mathrm{Hom}_{\mathcal{D}(S)}(L_S X, Y) \arrow[r, "\cong"] \arrow[d, "u^*"'] & \mathrm{Hom}_{\mathcal{C}(S)}(X, R_S Y) \arrow[d, "u^*"] \\
\mathrm{Hom}_{\mathcal{D}(T)}(u^* L_S X, u^* Y) \arrow[d, "-\circ(\gamma_u^L)^{-1}"'] & \mathrm{Hom}_{\mathcal{C}(T)}(u^* X, u^* R_S Y) \arrow[d, "\gamma_u^R\circ -"] \\
\mathrm{Hom}_{\mathcal{D}(T)}(L_T u^* X, u^* Y) \arrow[r, "\cong"] & \mathrm{Hom}_{\mathcal{C}(T)}(u^* X, R_T u^* Y).
\end{tikzcd}
$$

Furthermore, these γ_u^R's make $\{R_T\}$ into a lax morphism of derivators: that is, while not being necessarily invertible they satisfy the coherence conditions listed in Definition 3.9.1.

Proof. Even this proof boils down to basic diagram chasing: to determine γ_u^R we fix $X = R_S Y$ and compute the image of $Id_{u^*R_SY} \in \mathrm{Hom}_{\mathcal{C}(T)}(u^*R_SY, u^*R_SY)$ by going through the diagram in counter-clockwise motion.

Using the formulas of Lemma 3.2.2 it is easy to see that γ_u^R is determined as the composition

$$
u^*R_SY \xrightarrow{\eta_T u^* R_S} R_T L_T u^* R_S Y \xrightarrow{R_T(\gamma_u^L)^{-1}R_S} R_T u^* L_S R_S Y \xrightarrow{R_T u^* \varepsilon_S} R_T u^* Y.
$$

Naturality of the involved transformations guarantees that this γ_u^R is natural as well and it is the only possible candidate making the diagram commute.

The uniqueness ensures that the compatibility conditions of $\{\gamma_u^L\}$ are transferred to $\{\gamma_u^R\}$.

If $u = Id_S$ then the diagram reduces to

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{D}(S)}(L_S X, Y) \arrow[r, "\phi_{X,Y}"] \arrow[d, "Id"'] & \mathrm{Hom}_{\mathcal{C}(S)}(X, R_S Y) \arrow[d, "\gamma^R_{Id_S} \circ -"] \\
\mathrm{Hom}_{\mathcal{D}(S)}(L_S X, Y) \arrow[r, "\phi_{X,Y}"] & \mathrm{Hom}_{\mathcal{C}(S)}(X, R_S Y)
\end{tikzcd}$$

and $\gamma^R_{Id_S}$ can only be Id_{R_S}.

If we have two composable morphism of posets $T \xrightarrow{u} S \xrightarrow{v} P$ then we can form the commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{D}(P)}(L_P X, Y) \arrow[r, equal] \arrow[d, "v^*"'] & \mathrm{Hom}_{\mathcal{D}(P)}(L_P X, Y) \arrow[r, "\cong"] \arrow[d, "v^*"'] & \mathrm{Hom}_{\mathcal{C}(P)}(X, R_P Y) \arrow[d, "v^*"] \\
\mathrm{Hom}_{\mathcal{D}(S)}(v^* L_P X, v^* Y) \arrow[r, equal] \arrow[d, "u^*"'] & \mathrm{Hom}_{\mathcal{D}(S)}(v^* L_P X, v^* Y) \arrow[d, "- \circ (\gamma^L_v)^{-1}"'] & \mathrm{Hom}_{\mathcal{C}(S)}(v^* X, v^* R_P Y) \arrow[d, "\gamma^R_v \circ -"] \\
\mathrm{Hom}_{\mathcal{D}(T)}(u^* v^* L_P X, u^* v^* Y) \arrow[d, "- \circ (\gamma^L_v)^{-1}"'] & \mathrm{Hom}_{\mathcal{D}(S)}(L_S v^* X, v^* Y) \arrow[r, "\cong"] \arrow[d, "u^*"'] & \mathrm{Hom}_{\mathcal{C}(S)}(v^* X, R_S v^* Y) \arrow[d, "u^*"] \\
\mathrm{Hom}_{\mathcal{D}(T)}(u^* L_S v^* X, u^* v^* Y) \arrow[r, equal] \arrow[d, "- \circ (\gamma^L_u)^{-1}"'] & \mathrm{Hom}_{\mathcal{D}(T)}(u^* L_S v^* X, u^* v^* Y) \arrow[d, "- \circ (\gamma^L_u)^{-1}"'] & \mathrm{Hom}_{\mathcal{C}(T)}(u^* v^* X, u^* R_S v^* Y) \arrow[d, "\gamma^R_u \circ -"] \\
\mathrm{Hom}_{\mathcal{D}(T)}(L_T u^* v^* X, u^* v^* Y) \arrow[r, equal] & \mathrm{Hom}_{\mathcal{D}(T)}(L_T u^* v^* X, u^* v^* Y) \arrow[r, "\cong"] & \mathrm{Hom}_{\mathcal{C}(T)}(u^* v^* X, R_T u^* v^* Y).
\end{tikzcd}$$

The equality $\gamma^L_{vu} = \gamma^L_u \odot \gamma^L_v$ tells us that the left column is the one in the diagram of the claim associated to vu, we have to notice that similarly the column on the right can be rewritten as

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{C}(P)}(X, R_P Y) \arrow[r, equal] \arrow[d, "v^*"] & \mathrm{Hom}_{\mathcal{C}(P)}(X, R_P Y) \arrow[d, "v^*"] \\
\mathrm{Hom}_{\mathcal{C}(S)}(v^* X, v^* R_P Y) \arrow[r, equal] \arrow[d, "\gamma^R_v \circ -"] & \mathrm{Hom}_{\mathcal{C}(S)}(v^* X, v^* R_P Y) \arrow[d, "u^*"] \\
\mathrm{Hom}_{\mathcal{C}(S)}(v^* X, R_S v^* Y) \arrow[d, "u^*"] & \mathrm{Hom}_{\mathcal{C}(T)}(u^* v^* X, u^* v^* R_P Y) \arrow[d, "\gamma^R_v \circ -"] \\
\mathrm{Hom}_{\mathcal{C}(T)}(u^* v^* X, u^* R_S v^* Y) \arrow[r, equal] \arrow[d, "\gamma^R_u \circ -"] & \mathrm{Hom}_{\mathcal{C}(T)}(u^* v^* X, u^* R_S v^* Y) \arrow[d, "\gamma^R_u \circ -"] \\
\mathrm{Hom}_{\mathcal{C}(T)}(u^* v^* X, R_T u^* v^* Y) \arrow[r, equal] & \mathrm{Hom}_{\mathcal{C}(T)}(u^* v^* X, R_T u^* v^* Y)
\end{tikzcd}$$

to conclude $\gamma^R_{vu} = \gamma^R_u \odot \gamma^R_v$.

Now suppose we have two morphisms of posets $u, v\colon T \to S$ with a natural transformation $\alpha\colon u \Rightarrow v$. We want to show that $R_T \alpha^* \circ \gamma^R_u = \gamma^R_v \circ \alpha^* R_S$, we verify this equation directly by unravelling the formula for γ^R_u and γ^R_v provided at the start. After doing this we obtain the following diagram

$$\begin{tikzcd}
u^* R_S Y \arrow[r, "\eta_T u^* R_S"] \arrow[d, "\alpha^* R_S"'] & R_T L_T u^* R_S Y \arrow[r, "R_T (\gamma^L_u)^{-1} R_S"] \arrow[d, "R_T L_T \alpha^* R_S"'] & R_T u^* L_S R_S Y \arrow[r, "R_T u^* \varepsilon_S"] \arrow[d, "R_T \alpha^* L_S R_S"] & R_T u^* Y \arrow[d, "R_T \alpha^*"] \\
v^* R_S Y \arrow[r, "\eta_T v^* R_S"] & R_T L_T v^* R_S Y \arrow[r, "R_T (\gamma^L_v)^{-1} R_S"] & R_T v^* L_S R_S Y \arrow[r, "R_T v^* \varepsilon_S"] & R_T v^* Y.
\end{tikzcd}$$

We observe that the left square commutes by naturality of η_T, the right square by naturality of α^* and the one in the middle by the compatibility condition of the family of transformations $\{\gamma_u^L\}$. This lets us conclude. $\square$

This implies that for two morphisms $(L, R)\colon \mathcal{C} \rightleftarrows \mathcal{D}$ to form an adjunction of derivators there must be a strict relation between the associated natural isomorphisms γ^F and γ^G: fixed a family for one of the two morphisms the other is uniquely determined.

Moreover, it is not guaranteed that from a collection of natural isomorphism $\{\gamma_u^L\}$ the resulting γ_u^R's are invertible, we have to impose additional conditions to guarantee this.

If both $\mathcal{C}$ and $\mathcal{D}$ are derivators, the equality for γ_u^R stated at the start of the proof of Lemma 3.9.4 and the formulas of Lemma 3.2.4 imply that γ_u^R is conjugate to the natural transformation $((\gamma_u^L)^{-1})_!\colon u_! L_T \Rightarrow L_S u_!$, which is the pasting of the diagram

$$\begin{tikzcd}
\mathcal{C}(T) \arrow[r, "u_!"] \arrow[dr, bend right, "Id"'] & \mathcal{C}(S) \arrow[r, "L_S"] \arrow[d, "u^*"] \arrow[dl, phantom, "\Nearrow\eta"] & \mathcal{D}(S) \arrow[d, "u^*"] \arrow[dr, bend left, "Id"] \arrow[dl, phantom, "\Nearrow (\gamma_u^L)^{-1}"] & \\
& \mathcal{C}(T) \arrow[r, "L_T"'] & \mathcal{D}(T) \arrow[r, "u_!"'] \arrow[ur, phantom, "\Nearrow\varepsilon"] & \mathcal{D}(S).
\end{tikzcd}$$

This being an isomorphism means that L preserves left Kan extensions along u, if this is the case for any u then L is called *cocontinuous*.

Dually we say that a morphism of derivators $R\colon \mathcal{D} \to \mathcal{C}$ is *continuous* if it preserves right Kan extensions: more precisely the canonical mate $(\gamma_u^R)_*\colon R_S u_* \Rightarrow u_* R_T$ is invertible for any map of posets $u\colon T \to S$. We can similarly show it is conjugate to a transformation $L_T u^* \Rightarrow u^* L_S$ which is our candidate for $(\gamma_u^L)^{-1}$, provided it is invertible.

Proposition 3.9.5. *In the setting of Lemma 3.9.4 assume further $\mathcal{C}, \mathcal{D}$ to be derivators. Then the induced γ_u^R's make R into a morphism of derivators if and only if L is cocontinuous. Dually, if $\{R_T\}$ form a morphism of derivators R the associated transformation $L_T u^* \Rightarrow u^* L_S$ is always invertible if and only if R is continuous.*

Corollary 3.9.6. *Let $(L, R)\colon \mathcal{C} \rightleftarrows \mathcal{D}$ be an adjunction of derivators, then L is cocontinuous and R is continuous.*

At this point we can define an equivalence of (pre)derivators to be just an equivalence internal to the 2-category of (pre)derivators: i.e. a morphism of (pre)derivators that is part of an adjunction such that the associated unit and counit are invertible. Also in this case we provide a levelwise characterization.

Proposition 3.9.7. *Let $\mathcal{C}, \mathcal{D}$ be two prederivators, then a morphism $F\colon \mathcal{C} \to \mathcal{D}$ is an equivalence if and only if for any R finite poset F_R is an equivalence.*

Proof. The "only if" part is clear, we have to prove the inverse implication.

By assumption $F_R\colon \mathcal{C}(R) \to \mathcal{D}(R)$ is an equivalence, so we can fix an adjoint equivalence $G_R\colon \mathcal{D}(R) \to \mathcal{C}(R)$. Without loss of generality we can assume $F_R \dashv G_R$. By Lemma 3.9.4 we can provide natural transformations γ_u^G which make $\{G_R\}$ into a lax morphism.

We wrote down γ_u^G explicitly as appropriate composition of transformations coming from $(\gamma_u^F)^{-1}$, η_T and ε_S. All of these are invertible thus γ_u^G is a natural isomorphism as we wanted. $\square$

We end this section with a basic lemma which will be crucial in the following discussion.

Lemma 3.9.8. *Let $\mathcal{P}$ be a prederivator, $\mathcal{D}$ a derivator and $F\colon \mathcal{P} \to \mathcal{D}$ an equivalence of prederivators. Then $\mathcal{P}$ is also a derivator.*

Proof. The idea is to use the equivalence F to transfer the statements of the axioms from $\mathcal{P}$ to $\mathcal{D}$ where they hold. We only give a brief sketch of the proof, not carrying on all the computations.

To see $\mathcal{P}$ verifies (D1) we observe the following diagram commutes up to isomorphism

$$\begin{tikzcd}
\mathcal{P}(\coprod_i R_i) \arrow[r] \arrow[d, "F_{\coprod R_i}"'] \arrow[dr, phantom, "\overset{\cong}{\Nearrow}"] & \prod_i \mathcal{P}(R_i) \arrow[d, "\prod_i F_{R_i}"] \\
\mathcal{D}(\coprod_i R_i) \arrow[r] & \prod_i \mathcal{D}(R_i).
\end{tikzcd}$$

For a generic poset R we know F_R is an equivalence (hence conservative) and suitably compatible with the evaluations r^* for all $r \in R$: from this (D2) follows immediately.

Regarding (D3): we observe that for any $u\colon R \to T$ we have a diagram

$$\begin{tikzcd}
\mathcal{P}(R) \arrow[d, "F_R"'] & \mathcal{P}(T) \arrow[l, "u^*"'] \arrow[d, "F_T"] \arrow[dl, phantom, "\Nwarrow \gamma_u^F"] \\
\mathcal{D}(R) & \mathcal{D}(T) \arrow[l, "u^*"]
\end{tikzcd}$$

where the bottom functor has a left and a right adjoint ($u_!$ and u_* respectively), while the horizontal functors are equivalences. It is easy to cook up two functors going in the right direction composing only left adjoints or right adjoints, that these candidates are indeed the Kan extensions we need is a consequence of Corollary 3.2.5 (modulo tedious computations).

Finally for (D4) we show the Kan formula for left extensions holds for $\mathcal{P}$. The diagram

$$\begin{tikzcd}
 & \mathcal{D}((u/t)) & \mathcal{D}(R) \arrow[l, "p^*"'] & \\
\mathcal{D}(e) \arrow[ur, bend left, "\pi^*"] \arrow[r, phantom, "\Uparrow \gamma_\pi^F"] & \mathcal{P}((u/t)) \arrow[u, "F_{(u/t)}"] & \mathcal{P}(R) \arrow[l, "p^*"'] \arrow[u, "F_R"'] \arrow[ul, phantom, "\Swarrow \gamma_p^F"] \arrow[r, phantom, "\Swarrow \gamma_u^F"] & \mathcal{D}(T) \arrow[ul, bend right, "u^*"'] \\
 & \mathcal{P}(e) \arrow[ul, bend left, "F_e"] \arrow[u, "\pi^*"] & \mathcal{P}(T) \arrow[l, "t^*"] \arrow[u, "u^*"'] \arrow[ul, phantom, "\Swarrow \alpha^*"] \arrow[ur, bend right, "F_T"'] &
\end{tikzcd}$$

implies that the composite $\alpha^* \odot \gamma_p^F$ coincides, up to pasting with additional various γ^F, with the next diagram

$$\begin{tikzcd}
\mathcal{D}((u/t)) & \mathcal{D}(R) \arrow[l, "p^*"'] \\
\mathcal{D}(e) \arrow[u, "\pi^*"] & \mathcal{D}(T) \arrow[l, "t^*"'] \arrow[u, "u^*"'] \arrow[ul, phantom, "\Swarrow \alpha^*"] \\
\mathcal{P}(e) \arrow[u, "F_e"] & \mathcal{P}(T). \arrow[l, "t^*"] \arrow[u, "F_T"'] \arrow[ul, phantom, "\Swarrow \gamma_t^F"]
\end{tikzcd}$$

We observe that the mates associated to the various γ^F must be isomorphisms by Example 3.3.11. The total mate of the last pasting is an isomorphism by (D4) for $\mathcal{D}$ and the fact just explained. It follows that $\mathcal{P}(\alpha)_!$ is invertible after composing with $(\gamma_p^F)_!$: thus it had this property to begin with. $\square$

Chapter 4

Localizations as anafunctors

4.1 Anafunctors

We now introduce the notion of anafunctor: this is the way we adopted to avoid implicit choices when inverting equivalences in the constructions which we will form to prove the finiteness of the compositions of localizations. The idea is to introduce a calculus of fractions in bicategories and then conjugate it to invert the class of 1-cells given by equivalences. Our main reference for such calculus is [31], from which we take the following definition.

Definition 4.1.1. Let $\mathcal{C}$ be a bicategory and let W be a class of its morphisms. We say W *admits a right calculus of fractions* in $\mathcal{C}$ if the following conditions are satisfied.

BF1 All equivalences of $\mathcal{C}$ are in W.

BF2 W is closed under compositions: that is if $f\colon X \to Y$ and $g\colon Y \to Z$ are elements of W so is $g \circ f$.

BF3 For any $w\colon A \to X$ in W and any morphism $f\colon B \to X$ there exists a square commuting up to an invertible 2-cell

$$\begin{tikzcd} Y \arrow[r, "v"] \arrow[d, "g"'] \arrow[dr, phantom, "\overset{\cong}{\swarrow}"] & B \arrow[d, "f"] \\ A \arrow[r, "w"'] & X \end{tikzcd}$$

with $v \in W$. Moreover, we assume that in every class of such squares we have a preferred choice.

BF4 Let $w \in W$ and suppose we have a 2-cell $\alpha\colon wf \Rightarrow wg$, then there exists $v \in W$ and a 2-cell $\beta\colon fv \Rightarrow gv$ such that $\alpha v = w\beta$. Moreover, if α is invertible we require β to be invertible as well.

If v' and β' are another pair with such property, then we require the existence of two 1-cells u and u' such that $vu, v'u' \in W$ and we have a 2-cell $\varepsilon\colon vu \Rightarrow v'u'$ making the following diagram commute

$$\begin{tikzcd} fvu \arrow[r, "\beta u"] \arrow[d, "f\varepsilon"'] & gvu \arrow[d, "g\varepsilon"] \\ fv'u' \arrow[r, "\beta' u'"'] & gv'u'. \end{tikzcd}$$

BF5 W is closed under isomorphisms: if $f \in W$ and we have an invertible 2-cell $f \cong g$ then also g is in W.

In the situation of the above Definition we can define a new bicategory $\mathcal{C}[W^{-1}]$ by formally inverting the 1-cells in W. Its objects are the same as $\mathcal{C}$, its 1-cells are the spans $fw^{-1} = A \xleftarrow{w} X \xrightarrow{f} B$ where $w \in W$ and f is an arbitrary 1-cell.

Composition is given as follows: taken two spans $A \xleftarrow{w} X \xrightarrow{f} B$ and $B \xleftarrow{v} Y \xrightarrow{g}$ by BF3 there exists a preferred square associated to the span $X \xrightarrow{f} B \xleftarrow{v} Y$ which allows us to form a diagram

$$\begin{tikzcd}
 & & Z \arrow[dl, "u"'] \arrow[dr, "h"] & & \\
 & X \arrow[dl, "w"'] \arrow[dr, "f"'] & & Y \arrow[ll, phantom, "\overset{\cong}{\Longleftarrow}"] \arrow[dl, "v"] \arrow[dr, "g"] & \\
A & & B & & C
\end{tikzcd}$$

so we set the composite to be $gh(wu)^{-1}$.

A 2-cell between two morphisms fw^{-1} and gv^{-1} consists in an equivalence class of diagrams of the following shape

$$\begin{tikzcd}
 & X \arrow[dl, "w"'] \arrow[dr, "f"] & \\
A \arrow[r, phantom, "\overset{\alpha}{\Downarrow}"] & Z \arrow[u, "u_1"] \arrow[d, "u_2"'] \arrow[r, phantom, "\Downarrow_{\beta}"] & B \\
 & Y \arrow[ul, "v"] \arrow[ur, "g"'] &
\end{tikzcd}$$

where we require wu_1, vu_2 to be in W, α, β are 2-cells and the former must be an isomorphism.

Remark 4.1.2. Obviously Definition 4.1.1 is a generalization of the calculus of fractions introduced by Gabriel-Zisman to the context of bicategories. The core ideas are the same, we only have to modify the classical definition to take into account the 2-cells and the fact that in this new context composition is defined only up to natural isomorphism.

In fact, observe that 1-cells are not defined by equivalence classes of spans but directly as spans. This works because in a bicategory we consider two 1-cells to be morally the same if they are connected by an invertible 2-cell. Thus when we consider compute the composition of fractions, two different choices of squares invoked by BF3 give two results related by invertible 2-cells which are well behaved by the other axioms.

Thus, we are moving the problem of well definiteness of compositions to the level of 2-cells, which are identified according a precise equivalence relation. We do not write down such relation since it is very convoluted and we will not actually need the full potential of this theory of fractions in bicategory. We just mention that BF4 is needed to ensure this relation is transitive.

For the complete definitions and all the necessary verifications that this calculus of fractions is well defined we refer again to [31].

We present a criterion to determine the existence of a calculus of right fractions.

Definition 4.1.3. Let $\mathcal{C}$ be a bicategory, a square diagram

$$\begin{tikzcd} P \arrow[r, "f'"] \arrow[d, "g'"'] \arrow[dr, phantom, "\sigma \nearrow"] & C \arrow[d, "g"] \\ A \arrow[r, "f"] & B \end{tikzcd}$$

with σ invertible is defined as *bipullback* if for every other diagram commuting up to isomorphism

$$\begin{tikzcd} X \arrow[r, "k"] \arrow[d, "h"'] \arrow[dr, phantom, "\gamma \nearrow"] & C \arrow[d, "g"] \\ A \arrow[r, "f"] & B \end{tikzcd}$$

there exist a 1-cell $l\colon X \to P$ and invertible 2-cells $\alpha\colon g'l \Rightarrow h$ and $\beta\colon f'l \Rightarrow k$ such that the following diagram commutes

$$\begin{tikzcd} fg'l \arrow[r, "f\alpha"] \arrow[d, "\sigma l"'] & fh \arrow[d, "\gamma"] \\ gf'l \arrow[r, "g\beta"] & gk. \end{tikzcd}$$

That is, we can fill the second square as

$$\begin{tikzcd} X \arrow[dr, "l"] \arrow[drr, bend left, "k"] \arrow[ddr, bend right, "h"'] \arrow[dr, phantom, shift right=4, "\alpha\Downarrow"] \arrow[drr, phantom, shift left=2, "\Rightarrow\beta"] & & \\ & P \arrow[r, "f'"] \arrow[d, "g'"'] \arrow[dr, phantom, "\sigma \nearrow"] & C \arrow[d, "g"] \\ & A \arrow[r, "f"] & B. \end{tikzcd}$$

Moreover such filling is unique in the following sense: suppose we have another one given by $l'\colon X \to P$ and $\alpha'\colon g'l' \Rightarrow h$, $\beta'\colon f'l' \Rightarrow k$, then there exists a unique invertible 2-cell $\lambda\colon l' \Rightarrow l$ such that $\alpha \circ g'\lambda = \alpha'$ and $\beta \circ f'\lambda = \beta'$.

We say $\mathcal{C}$ *admits bipullbacks* if and only if every span

$$\begin{tikzcd} & C \arrow[d, "g"] \\ A \arrow[r, "f"] & B \end{tikzcd}$$

can be filled to a bipullback. In this case, for any 1-cell f we can consider the square diagrams

$$\begin{tikzcd} D_f \arrow[r] \arrow[d] \arrow[dr, phantom, "\sigma \nearrow"] & X \arrow[d, "f"] \\ X \arrow[r, "f"] & Y \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "Id"] \arrow[d, "Id"'] & X \arrow[d, "f"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

with the first a bipullback, so we have induced a 1-cell which we denote $\delta_f\colon X \to D_f$.

Proposition 4.1.4 ([36, Proposition 2.8])**.** *Let $\mathcal{C}$ be a bicategory such that all 2-cells are invertible. Suppose it admits bipullbacks and let W be a class of morphisms. Assume W satisfies the following conditions:*

BP1 *W contains all the equivalences;*

BP2 *W is closed under compositions;*

BP3 *W is stable under bipullbacks;*

BP4 *for any $w \in W$, $\delta_w \in W$;*

BP5 *W is closed under invertible 2-cells.*

Then W admits a right calculus of fractions in $\mathcal{C}$.

Remark 4.1.5. Let $\mathcal{C}$ be a bicategory admitting bipullbacks. Then it is easy to see that taking W to be the class of equivalences the conditions of Proposition 4.1.4 are satisfied. This guarantees that all the conditions of Definition 4.1.1 hold, except for BF4 when α is not invertible. Nevertheless, since equivalences admit inverses up to invertible 2-cells we can verify directly that even in this case BF4 is satisfied.

The point of Proposition 4.1.4 is that, if they are available and they are compatible with the chosen class W, bipullbacks offer an excellent choice of preferred squares as required by BF3.

Definition 4.1.6. In the situation of Remark 4.1.5 we call a 1-cell of the resulting bicategory of fractions $\mathcal{C}[W^{-1}]$ an *anafunctor.*

Example 4.1.7. Consider $\mathcal{C}$ = CAT the 2-category of locally small categories. Take C a category admitting finite products, then we can express the operation of forming the product of two elements as the anafunctor $C \times C \xleftarrow{a} P \xrightarrow{b} C$ where P is the category of pullback squares

$$\begin{tikzcd} T \arrow[r] \arrow[d] & A \arrow[d] \\ B \arrow[r] & * \end{tikzcd}$$

where $*$ is the terminal object and T is the product of A, B. Then $a\colon P \to C \times C$ sends this diagram to the pair (A, B), while $s\colon P \to C$ maps it to T.

Remark 4.1.8. Using the language of anafunctors we could rephrase all definitions of the usual categorical colimits and limits, by encoding their universal properties in appropriate categories of diagrams (like we did in Example 3.5.2). Usually this is not the adopted approach: classical colimits and limits are characterized by said properties and then treated as functors in practice. This is because in all the examples we want to consider or we have explicit constructions or we invert the reverse morphism of the respective anafunctor in virtue of the axiom of choice. This axiom in fact establishes that the datum of a fully faithful and essentially surjective functor is equivalent to an adjunction with invertible unit and counit.

4.2 Anafunctors for derivators

We want to conjugate Definition 4.1.6 in the case of the 2-category of derivators DER. So we have to prove it admits bipullbacks, at least along equivalences.

Lemma 4.2.1. *The 2-category DER admits bipullbacks along equivalences.*

Proof. Given a span $\mathcal{D} \xrightarrow{F} \mathcal{F} \xleftarrow{G} \mathcal{E}$ with G an equivalence of derivators we provide an explicit model for the bipullback. For a finite poset A we define $\mathcal{P}(A)$ to be the category with objects triples (X, Y, α) where $X \in \mathcal{D}(A)$, $Y \in \mathcal{E}(A)$ and $\alpha\colon F_A(X) \to G_A(Y)$ is an isomorphism in $\mathcal{F}(A)$. A morphism $(f, g)\colon (X, Y, \alpha) \to (X', Y', \beta)$ consists in a pair of morphisms $f\colon X \to X'$ and $g\colon Y \to Y'$ respectively in $\mathcal{D}(A)$ and $\mathcal{E}(A)$ such that the diagram

$$\begin{tikzcd}
F_A X \arrow[r, "F_A f"] \arrow[d, "\alpha"'] & F_A X' \arrow[d, "\beta"] \\
G_A Y \arrow[r, "G_A g"] & G_A Y'
\end{tikzcd}$$

commutes. Now we define the restriction along a functor $u\colon A \to B$ as follows: on objects we set $u^*(X, Y, \alpha) = (u^*X, u^*Y, \alpha_u)$ where the isomorphism α_u is the unique one making the following square commute

$$\begin{tikzcd}
F_A u^* X \arrow[r, "\alpha_u"] & G_A u^* Y \\
u^* F_B X \arrow[r, "u^*\alpha"] \arrow[u, "\gamma_u^F"] & u^* G_B Y \arrow[u, "\gamma_u^G"']
\end{tikzcd}$$

and on morphisms $u^*(f, g) = (u^*f, u^*g)$, it is immediate to show this pair defines a morphism $(u^*X, u^*Y, \alpha_u) \to (u^*X', u^*Y', \beta_u)$. This gives us a prederivator $\mathcal{P}$ and we can easily augment the starting span to a square commuting up to isomorphism

$$\begin{tikzcd}
\mathcal{P} \arrow[r, "P_2"] \arrow[d, "P_1"'] \arrow[dr, phantom, "\overset{\varphi}{\nearrow\!\!\nearrow}"] & \mathcal{E} \arrow[d, "G"] \\
\mathcal{D} \arrow[r, "F"'] & \mathcal{F}
\end{tikzcd}$$

where the morphism P_1, P_2 are given component-wise by the projections and $\varphi_{(X,Y,\alpha)} = \alpha$ defines an invertible modification from FP_1 to GP_2.

Using the fact that G is an equivalence we easily show P_1 to be an equivalence as well. Thus by Lemma 3.9.8 $\mathcal{P}$ is a derivator.

Given another square

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "K"] \arrow[d, "H"'] \arrow[dr, phantom, "\overset{\gamma}{\nearrow\!\!\nearrow}"] & \mathcal{E} \arrow[d, "G"] \\
\mathcal{D} \arrow[r, "F"'] & \mathcal{F}
\end{tikzcd}$$

we can easily form a filling $(L\colon \mathcal{C} \Rightarrow \mathcal{P}, \alpha\colon P_1 L \Rightarrow H, P_2 L \Rightarrow K)$ by

$$L_R\colon \mathcal{C}(R) \to \mathcal{P}(R) \quad Z \mapsto (H_R(Z), K_R(Z), \gamma_R(Z)\colon F_R H_R(Z) \to G_R K_R(Z))$$

and α and β can be taken to be identities. We should verify that this L actually defines a morphism of derivator, and that the filling provided in unique in the sense of Definition 4.1.3. These are just bothersome and trivial computations which we do not report here.

Therefore, we conclude $\mathcal{P}$ is a model for the bipullback of the starting span as claimed. □

Remark 4.2.2. It is not true that DER admits general bipullbacks, but the above proof shows clearly that the 2-category PDER of prederivators has this property.

The preferred squares we will use for the composition of fractions are the explicit bipullbacks constructed above.

Remark 4.2.3. We state here two easy consequences of the definition of the bicategory DER$[W^{-1}]$ which we will use automatically without mention.

First, suppose we have a commutative diagram of derivators as follows

$$\begin{tikzcd}
\mathcal{C} & \mathcal{X} \arrow[l, "F"', "\simeq"] \arrow[d, "G"] \arrow[dl, "J"] \\
\mathcal{Y} \arrow[u, "H", "\simeq"'] \arrow[r, "K"'] & \mathcal{D}
\end{tikzcd}$$

where F and H are equivalences, hence J is also an equivalence. The diagram then presents an isomorphism between the anafunctors GF^{-1} and KH^{-1}.

Second, suppose we fix an equivalence $F\colon \mathcal{X} \to \mathcal{C}$, and consider two functors $G_0, G_1\colon \mathcal{X} \to \mathcal{D}$. Then any natural transformation $\alpha\colon G_0 \Rightarrow G_1$ gives rise to a 2-cell $G_0F^{-1} \Rightarrow G_1F^{-1}$ between anafunctors

$$\begin{tikzcd}
 & \mathcal{X} \arrow[dl, "F"'] \arrow[dr, "G_0"] \arrow[dd, equal] & \\
\mathcal{C} & \arrow[r, phantom, "\alpha\Downarrow"] & \mathcal{D} \\
 & \mathcal{X} \arrow[ul, "F"] \arrow[ur, "G_1"'] &
\end{tikzcd}$$

and this is functorial in α.

4.3 The derivator of fully localizing cubes

We now establish our framework for Bousfield localisation in the context of derivators.

Definition 4.3.1. Let $\mathcal{C}$ be a derivator equivalent to the homotopy derivator associated to a stable model category (or stable quasi-category) with compactly generated homotopy category. This implies $\mathcal{C}$ is a compactly generated strong stable derivator.

Let $K\colon \mathcal{C}(e) \to \mathrm{Ab}$ be a homology theory. As usual, we extend this to a graded theory $K_*\colon \mathcal{C}(e) \to \mathrm{Ab}_*$ by $K_n(X) = K(\Sigma^{-n}X)$. For $X \in \mathcal{C}(R)$ define $K^R(X) = \bigoplus_{r\in R} K(X_r)$, and note that this is again a homology theory. Using Theorem 3.8.3 we see that the subcategories $\ker(K^R_*) \subseteq \mathcal{C}(R)$ form a localising subderivator of $\mathcal{C}$, which we will just call $\ker(K_*)$.

Now suppose we have an object $X \in \mathcal{C}([1]\times R)$. This gives a morphism $u\colon X_0 \to X_1$ in $\mathcal{C}(R)$ by taking the partial diagram along $[1]$. We say that X is a *localisation object* if $\mathrm{fib}(u) \in \ker(K^R_*)$ and $X_1 \in \ker(K^R_*)^\perp$. We write $\mathcal{L}(R)$ for the subcategory of localisation objects in $\mathcal{C}([1]\times R)$. This is clearly a subprederivator of $\mathcal{C}^{[1]}$, and we have a morphism $0^*\colon \mathcal{L} \to \mathcal{C}$ of prederivators.

Proposition 4.3.2. *$\mathcal{L}$ is a thick subderivator of $\mathcal{C}^{[1]}$, and $0^*\colon \mathcal{L} \to \mathcal{C}$ is an equivalence of derivators.*

Proof. We know from Theorem 3.8.3 and Proposition 3.8.19 that $\ker(K_*)$ and $\ker(K_*)^\perp$ are thick subderivators. Together with the result we recalled in Theorem 3.4.9, this implies that $\mathcal{L}$ is a thick subderivator of $\mathcal{C}^{[1]}$.

To prove our claim regarding $0^*\colon \mathcal{L} \to \mathcal{C}$ we show that the general level 0^*_R is an equivalence of categories.

Essential surjectivity: since $\mathcal{C}(R)$ is a compactly generated triangulated category with coproducts we can form the localization with respect to $\ker(K^R_*)$ and we get for any $X \in \mathcal{C}(R)$ a morphism $\eta\colon X \to LX$ which is an K^R_*-localization. Since $\mathcal{C}$ is strong there exists a coherent diagram $f \in \mathcal{C}([1]\times R)$ such that $\mathrm{dia}_{[1]} f \cong \eta$ in $\mathcal{C}(R)^{[1]}$, this clearly means that $0^*(f) \cong X$.

Fully faithfulness: we have to show that for any two coherent localizations $X\colon X_0 \to X_1$ and $Y\colon Y_0 \to Y_1$ the map

$$\mathrm{Hom}_{\mathcal{L}(R)}(X,Y) = \mathrm{Hom}_{\mathcal{C}([1]\times R)}(X,Y) \xrightarrow{0^*} \mathrm{Hom}_{\mathcal{C}(R)}(X_0,Y_0)$$

is a bijection.

We name the functors

$$0\colon R \to [1]\times R \quad r\mapsto (0,r) \qquad 1\colon R \to [1]\times R \quad r\mapsto (1,r).$$

Observe that we can form in $\mathcal{C}([1]\times R)$ the triangle

$$0_!0^*X \xrightarrow{\varepsilon_X} X \to Z \to \Sigma 0_!0^*X$$

where ε_X is the counit of the adjunction $0_! \dashv 0^*$ evaluated at X. The associated incoherent diagram in $\mathcal{C}(R)$ is

$$\begin{tikzcd} X_0 \arrow[r, "\cong"] \arrow[d, "\cong"'] & X_0 \arrow[r] \arrow[d, "X"] & 0 \arrow[d] \\ X_0 \arrow[r, "X"] & X_1 \arrow[r] & C(X) \end{tikzcd}$$

where $C(X)$ denotes the cone of the coherent morphism X. Therefore by Corollary 3.4.6 we can identify Z with the diagram $1_!C(X)$. Now we apply $\mathrm{Hom}_{\mathcal{C}([1]\times R)}(-,Y)$ to this triangle to get the long exact sequence

$$\dots \leftarrow \mathrm{Hom}_{\mathcal{C}([1]\times R)}(0_!0^*X,Y) \leftarrow \mathrm{Hom}_{\mathcal{C}([1]\times R)}(X,Y) \leftarrow \mathrm{Hom}_{\mathcal{C}([1]\times R)}(1_!C(X),Y) \leftarrow \dots$$

We have $\mathrm{Hom}_{\mathcal{C}([1]\times R)}(1_!C(X),Y) \cong \mathrm{Hom}_{\mathcal{C}(R)}(C(X),1^*Y) = 0$ since by construction $C(X)$ is K_*^R-acyclic and $1^*Y = Y_1$ is K_*^R-local. This implies that the map

$$(\varepsilon_X)^*\colon \mathrm{Hom}_{\mathcal{C}([1]\times R)}(X,Y) \to \mathrm{Hom}_{\mathcal{C}([1]\times R)}(0_!0^*X,Y)$$

is an isomorphism. Composing this with the bijection $\mathrm{Hom}_{\mathcal{C}([1]\times R)}(0_!0^*X,Y) \cong \mathrm{Hom}_{\mathcal{C}(R)}(0^*X,0^*Y)$ we obtain

$$0^*\colon \mathrm{Hom}_{\mathcal{C}([1]\times R)}(X,Y) \to \mathrm{Hom}_{\mathcal{C}(R)}(X_0,Y_0)$$

by Lemma 3.2.2, therefore 0_R^* induces isomorphisms on the hom-groups as we wanted. $\square$

We will sometimes need a slightly more general statement.

Lemma 4.3.3. *Suppose that $X,Y \in \mathcal{C}^{[1]}(R)$, giving morphisms $f\colon X_0 \to X_1$ and $g\colon Y_0 \to Y_1$ in $\mathcal{C}(R)$. Suppose that $\mathrm{fib}(f) \in \ker(K_*^R)$ and $Y_1 \in \ker(K_*^R)^\perp$. Then the map*

$$0^*\colon \mathcal{C}([1]\times R)(X,Y) \to \mathcal{C}(R)(X_0,Y_0)$$

is bijective.

Proof. These weakened hypotheses are all that was used in the proof of Proposition 4.3.2 $\square$

Definition 4.3.4. We define L_K to be the anafunctor $\mathcal{C} \xleftarrow{0^*} \mathcal{L} \xrightarrow{1^*} \mathcal{C}$, and call this *Bousfield localisation* with respect to K.

We now return to a framework similar to that of Definition 2.1.4: we assume that we have a compactly generated stable derivator $\mathcal{C}$ together with homology theories $K(i)$ on $\mathcal{C}(e)$ for $i \in N$, satisfying the fracture axiom. As in Section 2.2, we define $\mathbb{P}$ to be the poset of subsets of N (ordered by inclusion) and $\mathbb{Q}$ to be the poset of upward closed subsets of $\mathbb{P}$ (ordered by reverse inclusion). They are related by the morphism $u\colon \mathbb{P} \to \mathbb{Q}$ given by $uT = \{U \mid T \subseteq U\}$.

Definition 4.3.5. Consider a finite poset R and an object $X \in \mathcal{C}(\mathbb{P} \times R)$. Suppose that $t \in N$ and $U \subseteq N$ with $t < u$ for all $u \in U$. We then write tU for $\{t\} \cup U$, so we have $U < tU$ in $\mathbb{P}$, giving maps $f_{t,U,r} \colon X_{U,r} \to X_{tU,r}$ in $\mathcal{C}(e)$. We say that X is *(t, U)-localising* if $f_{t,U}$ is a $K(t)_*$-localisation. Equivalently, $X_{tU,r}$ should be $K(t)_*$-local, and the fibre of $f_{t,U,r}$ should be $K(t)_*$-acyclic for every $r \in R$. We also say that X is *fully localising* if it is (t, U)-localising for all t and U. We write $\mathcal{P}(R)$ for the full subcategory of fully localising objects in $\mathcal{C}(\mathbb{P} \times R)$. There is an evident inclusion $\emptyset \colon R \to \mathbb{P} \times R$, which gives a functor $\emptyset^* \colon \mathcal{P}(R) \to \mathcal{C}(R)$.

Example 4.3.6. Consider the case $n = 1$, thus $N = \{0\}$, and suppose that our derivator $\mathcal{C}$ arises from a stable model category $\mathcal{B}_0$. Under the isomorphism of posets $\mathbb{P} \cong [1]$ an object of $\mathcal{P}(R)$ is a diagram $X \colon [1] \times R \to \mathcal{C}_0$ such that the morphisms $X_{0r} \to X_{1r}$ are all localisations with respect to $K(0)$. Informally, we can therefore say that an object of $\mathcal{P}$ is a diagram of type $(X \to L_{K(0)}X)$. In the case $n = 2$ an object of $\mathcal{P}$ is essentially a diagram of the following type:

$$\begin{tikzcd} X \arrow[r] \arrow[d] & L_{K(0)}X \arrow[d] \\ L_{K(1)}X \arrow[r] & L_{K(0)}L_{K(1)}X \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r] \arrow[d] & \phi_0 X \arrow[d] \\ \phi_1 X \arrow[r] & \phi_{01} X \end{tikzcd}$$

The right hand diagram is just alternate notation for the left hand one. For $n = 3$, the diagram is as follows:

$$\begin{tikzcd} X \arrow[rr] \arrow[dr] \arrow[dd] & & \phi_0 X \arrow[dr] \arrow[dd] & \\ & \phi_1 X \arrow[rr] \arrow[dd] & & \phi_{01} X \arrow[dd] \\ \phi_2 X \arrow[rr] \arrow[dr] & & \phi_{02} X \arrow[dr] & \\ & \phi_{12} X \arrow[rr] & & \phi_{012} X \end{tikzcd}$$

where to keep the notation simple we write ϕ_A by just presenting the set A as a string of increasing integers.

Proposition 4.3.7. *$\mathcal{P}$ is a thick subderivator of $\mathcal{C}^{\mathbb{P}}$, and $\emptyset^* \colon \mathcal{P} \to \mathcal{C}$ is an equivalence of derivators.*

Proof. The claim is clear if $N = \emptyset$. If $N \neq \emptyset$, we let $n_0 \in N$ be the smallest element, so N can be decomposed as $\{n_0\} \amalg N_1$ say. This gives an obvious decomposition $\mathbb{P} = [1] \times \mathbb{P}_1$. We can define $\mathcal{P}_1 \subseteq \mathcal{C}^{\mathbb{P}_1}$ using N_1, and by induction we can assume that this is a thick subderivator with $\emptyset^* \colon \mathcal{P}_1 \to \mathcal{C}$ being an equivalence.

Now define $\mathcal{L}$ as in Definition 4.3.1, with respect to the homology theory $K(n_0)$ for the derivator $\mathcal{P}_1$. We find that

$$\mathcal{P}(R) = \{X \in \mathcal{L}(\mathbb{P}_1 \times R) \mid X_0 \in \mathcal{P}_1(R)\},$$

and the claim now follows from Proposition 4.3.2 together with the induction hypothesis. $\square$

Again, from the proof we can deduce a slightly more general statement.

Lemma 4.3.8. *Suppose that $X, Y \in \mathcal{C}^{\mathbb{P}}(R)$. Suppose that for t, U and r as before, the map $f_{t,U,r} \colon X_U \to X_{tU}$ is a $K(t)$-equivalence, and the object Y_{tU} is $K(t)$-local. Then the map*

$$\emptyset^* \colon \mathcal{C}(\mathbb{P} \times R)(X, Y) \to \mathcal{C}(R)(X_\emptyset, Y_\emptyset)$$

is bijective.

Proof. As above taking n_0 the minimum of N we get a decomposition $\mathbb{P} = [1] \times \mathbb{P}_1$ and arguing by induction we can assume the map $\mathcal{C}(\mathbb{P}_1 \times R)(X_0, Y_0) \to \mathcal{C}(R)(X_\emptyset, Y_\emptyset)$ is a bijection. Now we have just to compose this with $\mathcal{C}(\mathbb{P} \times R)(X, Y) \to \mathcal{C}(\mathbb{P}_1 \times R)(X_0, Y_0)$ which is an isomorphism by Lemma 4.3.3. $\square$

Definition 4.3.9. We finally adapt the iterated localizations of Definition 2.1.1 and Definition 2.2.1 to this setting. For $A \subseteq N$ define λ_A to be the anafunctor L_K of Definition 4.3.4 for the homology theory $K = \bigoplus_{a \in A} K(a)_*$.

We also set ϕ_A to be

$$\mathcal{C} \xleftarrow{\emptyset^*} \mathcal{P} \xrightarrow{A^*} \mathcal{C}.$$

Remark 4.3.10. Suppose that $A = \{a_1, \dots, a_r\}$ with $a_1 < \cdots < a_r$. It is then not hard to see from the definitions that in some sense we have

$$\phi_A \simeq L_{K(a_1)} \cdots L_{K(a_r)},$$

so that Definition 4.3.9 is a more precise version of Definition 2.2.1. A rigorous formulation with anafunctors will be given in Corollary 4.3.18.

Proposition 4.3.11. *The anafunctor $\phi_{\{a\}}$ is isomorphic to $L_{K(a)}$.*

Proof. Define $j\colon [1] \to \mathbb{P}$ by $j(0) = \emptyset$ and $j(1) = \{a\}$. This gives a morphism $j^*\colon \mathcal{C}^{\mathbb{P}} \to \mathcal{C}^{[1]}$. If we define $\mathcal{L}$ as in Definition 4.3.1 with respect to $K(a)$, we find that j^* restricts to give a morphism $\mathcal{P} \to \mathcal{L}$. This fits into a commutative diagram as follows:

$$\begin{tikzcd}
\mathcal{P} \arrow[r, "\{a\}^*"] \arrow[d, "\emptyset^*"', "\simeq"] \arrow[dr, "j^*"] & \mathcal{C} \\
\mathcal{C} & \mathcal{L} \arrow[l, "\simeq", "0^*"'] \arrow[u, "1^*"']
\end{tikzcd}$$

From this it is clear that j^* is an equivalence and the anafunctor $\phi_{\{a\}} = \{a\}^*(\emptyset^*)^{-1}$ is equivalent to $L_{K(a)} = 1^*(0^*)^{-1}$. $\square$

Now let j be the inclusion of $\mathbb{P}' = \mathbb{P} \setminus \{\emptyset\}$ in $\mathbb{P}$, and consider the fibration

$$\mathrm{tfib}(X) \to \emptyset^*(X) \to \underset{\mathbb{P}'}{\underleftarrow{\mathrm{holim}}}\, j^*(X) \to \Sigma\, \mathrm{tfib}(X).$$

We can now give a derivator formulation of the chromatic fracture argument.

Proposition 4.3.12. *For any $X \in \mathcal{P}(R)$, the above morphism $\emptyset^*(X) \to \underleftarrow{\mathrm{holim}}_{\mathbb{P}'}\, j^*(X)$ is a localisation with respect to $K(N)_* = \bigoplus_{i \in N} K(i)_*$.*

Proof. From the definition of $\mathcal{P}(R)$ we see that for all $T \in \mathbb{P}'$, the object $j^*(X)_T$ is local with respect to $K(\min(T))_*$ and thus with respect to $K(N)_*$. Proposition 3.8.19 tells us that the $K(N)_*$-local objects form a thick subderivator, so the object $LX = \underleftarrow{\mathrm{holim}}_{\mathbb{P}'}\, j^*(X)$ is $K(N)_*$-local. It will therefore suffice to show that the fibre $\mathrm{tfib}(X) = \mathrm{fib}(X_\emptyset \to LX)$ is $K(N)_*$-acyclic, or equivalently, that it is $K(i)_*$-acyclic for all i. Let Y_r be the total fibre of the subdiagram indexed by subsets of $\{r+1, \dots, n-1\}$, so $Y_{n-1} = 0$ and $Y_{-1} = \mathrm{tfib}(X)$. Let Z_r be the total fibre of the subdiagram indexed by subsets of $\{r, \dots, n-1\}$ containing r, so that $Y_r \to Z_r$ is a $K(r)_*$-localisation, and the fibre is Y_{r-1} by Proposition 3.5.10. This shows that Y_{i-1} is $K(i)_*$-acyclic. The fracture axiom then tells us that Z_{i-1} is also $K(i)_*$-acyclic, and since Y_{i-2} is the fibre of the map $Y_{i-1} \to Z_{i-1}$ it is again $K(i)_*$-acyclic. By iterating this, we find that Y_{-1} is $K(i)_*$-acyclic as required. $\square$

Definition 4.3.13. We say that an object $X \in \mathcal{C}(\mathbb{P} \times \mathbb{P} \times R)$ is *doubly localising* if

(a) X is fully localising relative to $\mathbb{P} \times R$, so it lies in $\mathcal{P}(\mathbb{P} \times R)$.

(b) The restriction to $\{\emptyset\} \times \mathbb{P} \times R \simeq \mathbb{P} \times R$ lies in $\mathcal{P}(R)$.

We write $\mathcal{P}_2(R)$ for the full subcategory of doubly localising objects in $\mathcal{C}(\mathbb{P} \times \mathbb{P} \times R)$.

Proposition 4.3.14. *$\mathcal{P}_2$ is a thick subderivator of $\mathcal{C}^{\mathbb{P}\times\mathbb{P}}$, and $(\emptyset, \emptyset)^* \colon \mathcal{P}_2 \to \mathcal{C}$ is an equivalence.*

Proof. The first claim is immediate from Theorem 3.8.3.

We can apply Proposition 4.3.7 to the shifted derivator $\mathcal{C}^{\mathbb{P}}$ and denote by $\mathcal{P}_s$ the resulting derivator of cubes of localizations. Then the two properties of the Definition 4.3.13 easily imply that we can restrict the associated equivalence to

$$\begin{array}{ccc} \mathcal{P}_s & \xrightarrow[\simeq]{\emptyset^*} & \mathcal{C}^{\mathbb{P}} \\ \uparrow & & \uparrow \\ \mathcal{P}_2 & \xrightarrow[\simeq]{\emptyset^*} & \mathcal{P} \end{array}$$

we have just compose the lower equivalence with $\emptyset^* \colon \mathcal{P} \to \mathcal{C}$ and the second claim is verified. $\square$

It is easy to see that $X \in \mathcal{C}(\mathbb{P} \times \mathbb{P} \times R) = \mathcal{C}^R(\mathbb{P} \times \mathbb{P})$ is doubly localising if and only if the following conditions are satisfied:

(a) For all a, A, B with $\{a\}\angle A$, the map $X_{A,B} \to X_{\{a\}\cup A,B}$ is a $K(a)_*$-localisation.

(b) For all b, B with $\{b\}\angle B$, the map $X_{\emptyset,B} \to X_{\emptyset,\{b\}\cup B}$ is a $K(b)_*$-localisation.

This essentially means that if $A = \{a_1 < \cdots < a_p\}$ and $B = \{b_1 < \cdots < b_q\}$ we must have

$$X_{A,B} = L_{K(a_1)} \cdots L_{K(a_p)} L_{K(b_1)} \cdots L_{K(b_q)} X_{(\emptyset,\emptyset)}.$$

In particular, we see that $X_{(A,B)} = 0$ unless $A\angle B$. This motivates the following construction.

Definition 4.3.15. We put $\mathbb{M} = \{(A, B) \in \mathbb{P} \times \mathbb{P} \mid A\angle B\}$, and define $\sigma \colon \mathbb{M} \to \mathbb{P}$ by $\sigma(A, B) = A \cup B$. We say that an object $X \in \mathcal{C}(\mathbb{M} \times R) = \mathcal{C}^R(\mathbb{M})$ is *doubly localising* if

(a) for all a, A, B with $\{a\}\angle A$ and $\{a\} \cup A\angle B$, the map $X_{A,B} \to X_{\{a\}\cup A,B}$ is a $K(a)_*$-localisation.

(b) For all b, B with $\{b\}\angle B$, the map $X_{\emptyset,B} \to X_{\emptyset,\{b\}\cup B}$ is a $K(b)_*$-localisation.

We write $\mathcal{P}'_2(R)$ for the subcategory of $\mathcal{C}(\mathbb{M} \times R)$ of doubly localising objects.

Proposition 4.3.16. *$\mathcal{P}'_2$ is a thick subderivator of $\mathcal{C}^{\mathbb{M}}$, and the inclusion* $\mathrm{inc} \colon \mathbb{M} \to \mathbb{P} \times \mathbb{P}$ *induces mutually inverse equivalences*

$$\mathcal{P}'_2 \xrightarrow{\mathrm{inc}_*} \mathcal{P}_2 \xrightarrow{\mathrm{inc}^*} \mathcal{P}'_2,$$

and the resulting morphism $(\emptyset, \emptyset)^* \colon \mathcal{P}'_2 \to \mathcal{C}$ is also an equivalence. Moreover, the map $\sigma \colon \mathbb{M} \to \mathbb{P}$ gives an equivalence $\sigma^* \colon \mathcal{P} \to \mathcal{P}'_2$ and thus an equivalence $\mathrm{inc}_* \circ \sigma^* \colon \mathcal{P} \to \mathcal{P}_2$.

Proof. We denote by $\mathcal{C}_{\mathbb{M}}$ the subderivator of $\mathcal{C}^{\mathbb{P}\times\mathbb{P}}$ spanned by the objects X such that $X_p = 0$ whenever $p \in (\mathbb{P}\times\mathbb{P}) \setminus \mathbb{M}$.

The subposet $\mathbb{M} \subseteq \mathbb{P}\times\mathbb{P}$ is a sieve, so Corollary 3.4.6 gives mutually inverse equivalences

$$\mathcal{C}(\mathbb{M}\times R) \xrightarrow{\mathrm{inc}_*} \mathcal{C}_{\mathbb{M}}(R) \xrightarrow{\mathrm{inc}^*} \mathcal{C}(\mathbb{M}\times R).$$

We have observed that if $X \in \mathcal{C}(\mathbb{P}\times R)$ is doubly localising then $X_{A,B} = 0$ for $(A,B) \notin \mathbb{M}$, so $X \in \mathcal{C}_{\mathbb{M}}(R)$. It follows that inc^* restricts to give an equivalence from $\mathcal{P}_2(R)$ to some subcategory of $\mathcal{C}(\mathbb{M}\times R)$, with inverse given by inc_*. It is easy to check that the relevant subcategory is $\mathcal{P}_2'(R)$. We have now seen that in the diagram

$$\mathcal{P}_2 \xrightarrow{\mathrm{inc}^*} \mathcal{P}_2' \xrightarrow{(\emptyset,\emptyset)^*} \mathcal{C},$$

the first map and the composite are both equivalences of (pre)derivators, so the second map is also an equivalence. From this and Lemma 3.9.8 it follows that $\mathcal{P}_2'$ is also a derivator. Finally, direct inspection of the definitions shows that $\sigma^* \colon \mathcal{C}^{\mathbb{P}} \to \mathcal{C}^{\mathbb{M}}$ restricts to $\sigma^* \colon \mathcal{P} \to \mathcal{P}_2'$, and $(\emptyset,\emptyset)^*\sigma^* = \emptyset^*$, which implies that $\sigma^* \colon \mathcal{P} \to \mathcal{P}_2'$ is an equivalence as well. $\square$

Proposition 4.3.17. *If $A \angle B$ then there is an equivalence of anafunctors $\phi_A\phi_B \simeq \phi_{A\cup B}$.*

Proof. Define $j_B \colon \mathbb{P} \to \mathbb{P}\times\mathbb{P}$ by $j_B(T) = (T,B)$. Consider the following diagram:

$$\begin{tikzcd}
\mathcal{C} & & \mathcal{P} \arrow[ll, "\emptyset^*"', "\simeq"] \arrow[dr, "A^*"] & \\
& \mathcal{P}_2 \arrow[ur, "j_B^*"] \arrow[rr, "(A,B)^*"] \arrow[dl, "\emptyset^*", "\simeq"'] \arrow[dd, "(\emptyset,\emptyset)^*"', "\simeq"] \arrow[dr, "\mathrm{inc}^*", "\simeq"'] & & \mathcal{C} \\
\mathcal{P} \arrow[uu, "B^*"] \arrow[dr, "\emptyset^*"', "\simeq"] & & \mathcal{P}_2' \arrow[ur, "(A,B)^*"] \arrow[dl, "(\emptyset,\emptyset)^*", "\simeq"'] & \\
& \mathcal{C} & & \mathcal{P}. \arrow[ll, "\emptyset^*", "\simeq"'] \arrow[ul, "\sigma^*"', "\simeq"] \arrow[uu, "(A\cup B)^*"']
\end{tikzcd}$$

Given that $\mathcal{C}$ is a strict 2-functor, we see that everything commutes on the nose. Several morphisms have been marked as equivalences; these are justified by Proposition 4.3.16. It follows that all routes from the middle bottom $\mathcal{C}$ to the middle right $\mathcal{C}$ give the same anafunctor up to equivalence. If we go clockwise around the edge of the diagram we get $\phi_A\phi_B$, and if we go anticlockwise we get $\phi_{A\cup B}$.

To be precise the composition of anafunctors $\phi_A\phi_B$ is given by the following composition of spans via the bipullback

$$\begin{tikzcd}
& & \mathcal{P}\times_{\mathcal{C}}\mathcal{P} \arrow[dl, "P_1"'] \arrow[dr, "P_2"] & & \\
& \mathcal{P} \arrow[dl, "\emptyset^*"'] \arrow[dr, "B^*"] & & \mathcal{P} \arrow[dl, "\emptyset^*"'] \arrow[dr, "A^*"] & \\
\mathcal{C} & & \mathcal{C} & & \mathcal{C}.
\end{tikzcd}$$

The upper left part of the previous diagram means that we can easily produce an isomorphism of anafunctors

$$\begin{tikzcd} & \mathcal{P} \times_{\mathcal{C}} \mathcal{P} \arrow[dl, "\emptyset^* P_1"'] \arrow[dr, "A^* P_2"] & \\ \mathcal{C} & & \mathcal{C} \\ & \mathcal{P}_2 \arrow[ul, "(\emptyset,\emptyset)^*"] \arrow[ur, "(A,B)^*"'] \arrow[uu, "L"] & \end{tikzcd}$$

where L is obtained using $\emptyset^*$ and j_B^*. ☐

Corollary 4.3.18. *Suppose that $A = \{a_1, \ldots, a_r\}$ with $a_1 < \cdots < a_r$. There is then an equivalence of anafunctors*

$$\phi_A \simeq L_{K(a_1)} \cdots L_{K(a_r)}.$$

Proof. This follows by induction using Propositions 4.3.11 and 4.3.17. The base case $A = \emptyset$ says that $\phi_\emptyset$ is equivalent to the identity, which is clear because $\phi_\emptyset = \emptyset^*(\emptyset^*)^{-1}$ by definition. ☐

4.4 The iterated chromatic localizations ϕ_A are distinct

Before carrying on our proof, we present an interesting digression about Example 2.1.10. That is, we show that in this case the various iterated localizations ϕ_A must be distinct.

Definition 4.4.1. We let BP_*-Mod be the category of graded BP_*-modules. We recall BP is the p-local Brown-Peterson spectrum, thus $BP_* \cong \mathbb{Z}_{(p)}[v_1, v_2, \ldots]$ where $|v_i| = 2(p^i - 1)$. We denote, for any positive integer m, by I_m the ideal of BP_* generated by $v_0, v_1, \ldots, v_{m-1}$, where $v_0 = p$. We define the functor

$$\phi_m^* \colon BP_*\text{-Mod} \to BP_*\text{-Mod} \qquad M \mapsto (v_m^{-1} M)^\wedge_{I_m}.$$

There is an evident natural map $\eta_m \colon M \to \phi_m^*(M)$, also we have a unique natural transformation μ making the following diagram commute

$$\begin{tikzcd} & M \otimes N \arrow[dl, "\eta_m \otimes \eta_m"'] \arrow[dr, "\eta_m"] & \\ \phi_m^*(M) \otimes \phi_m^*(N) \arrow[rr, "\mu"] \arrow[d, "\pi_k \otimes \pi_k"'] & & \phi_m^*(M \otimes N) \arrow[d, "\pi_k"] \\ (v_m^{-1} M)/I_m^k \otimes (v_m^{-1} N)/I_m^k \arrow[rr, "\mu"] & & (v_m^{-1}(M \otimes N))/I_m^k. \end{tikzcd}$$

These make ϕ_m^* into a lax monoidal functor. In particular, we deduce $\phi_m^*(BP_*)$ has a natural ring structure and $\phi_m^*(M)$ is a module over $\phi_m^*(BP_*)$.

For any finite subset $A \subset \mathbb{N}$ such that $A = \{a_1 < a_2 < \cdots < a_l\}$, we define the functor $\phi_A^* \colon BP_*\text{-Mod} \to BP_*\text{-Mod}$ by $\phi_A^* = \phi_{a_1}^* \phi_{a_2}^* \ldots \phi_{a_l}^*$. We can define natural transformations $\theta_{A,A'} \colon \phi_A^* \Rightarrow \phi_{A'}^*$ for any $A \subseteq A'$ by composing iteratively the maps η_m as follows. If $A' = \emptyset$ we must have $A = \emptyset$ hence $\phi_A^* = \phi_{A'}^* = Id$ and we set $\theta_{\emptyset,\emptyset} = Id$. Otherwise, we put $a = \min A'$ and $A_0' = A' \setminus \{a\}$ so that $\phi_{A'}^* = \phi_a^* \phi_{A_0'}^*$. If $a \notin A$ then $A \subseteq A_0'$, we can suppose $\theta_{A,A_0'}$ has already been defined and we put

$$(\theta_{A,A'})_M = (\phi_A^* M \xrightarrow{\theta_{A,A_0'}} \phi_{A_0'}^* M \xrightarrow{\eta_a} \phi_{A'}^* M).$$

If instead $a \in A$, we set $A_0 = A \setminus \{a\}$ and we must have $A_0 \subseteq A'_0$, hence $\phi^*_A = \phi^*_a \phi^*_{A_0}$ and we can assume we already established $\theta_{A_0, A'_0} \colon \phi^*_{A_0} \Rightarrow \phi^*_{A'_0}$ so we define $\theta_{A,A'} = \phi^*_a \theta_{A_0,A'_0} \colon \phi^*_A \Rightarrow \phi^*_{A'}$.

Remark 4.4.2. Using the naturality of the maps η_m it is easy to verify that for any triple of subsets $A \subseteq B \subseteq C$ we have $\theta_{B,C} \circ \theta_{A,B} = \theta_{A,C}$ and it is clear that $\theta_{A,A} = Id$. Also, the monoidal structures on the functors ϕ^*_a give rise to monoidal structures on the composite functors ϕ^*_A and the natural transformations $\theta_{A,A'}$ are compatible with these: the maps $\theta_{A,A'} \colon \phi^*_A(BP_*) \to \phi^*_{A'}(BP_*)$ are ring homomorphisms and in general $\phi^*_A(M) \to \phi^*_{A'}(M)$ is a morphism of $\phi^*_A(BP_*)$-modules.

Definition 4.4.3. Given $Q \subseteq \mathbb{N}$ we define

$$BP_Q = BP_*/(v_i : i \notin Q) = \begin{cases} \mathbb{F}_p[v_i : i \in Q] & \text{if } 0 \notin Q \\ \mathbb{Z}_{(p)}[v_i : i \in Q \setminus \{0\}] & \text{if } 0 \in Q. \end{cases}$$

In particular $BP_\emptyset = \mathbb{F}_p$. We observe that if Q is finite then BP_Q is a Noetherian graded ring.

Our first objective is to show that the functors ϕ^*_A are all distinct. This will be demonstrated by the following Proposition.

Proposition 4.4.4. *Fix $Q \subseteq \mathbb{N}$ and let $A \subseteq Q$ be a finite subset. Then we have $\{i \in \mathbb{N} : v_i \text{ is invertible in } \phi^*_A(BP_Q)\} = A$. Hence for different finite subsets $A, B \subseteq Q$ the two BP_*-modules $\phi^*_A(BP_Q)$ and $\phi^*_B(BP_Q)$ cannot be isomorphic.*

Before presenting the proof we will need some algebraic preliminaries about regular sequences in a (graded) ring.

Definition 4.4.5. Let R be a commutative ring with identity element, possibly graded. Let $x_1, \ldots, x_k$ be a sequence of elements of R, where in the graded case we assume these to be homogeneous. We say such sequence is *regular* if for any $i \leq k$ multiplication by x_i provides an injective morphism

$$x_i \colon R/(x_1, \ldots, x_{i-1}) \to R/(x_1, \ldots, x_{i-1}).$$

Remark 4.4.6. There are two important observations on this definition. The first one is that we do not require the quotients $R/(x_1, \ldots, x_{i-1})$ to be non-zero, in this case we have that the multiplication by x_{i-1} on $R/(x_1, \ldots, x_{i-2})$ induces an isomorphism. Hence x_{i-1} must be a unit on this quotient.

The second fact to be noted is that the regularity of the proposed sequence could depend on the ordering of the elements. As example consider $R = \mathbb{Z}[x, y]$ and $x_1 = 2, x_2 = 3x, x_3 = 3y$, this sequence is obviously regular but if we reorder it as $y_1 = 3x, y_2 = 3y, y_3 = 2$ then regularity does not hold any more.

Lemma 4.4.7. *Let R be a (graded) commutative Noetherian ring and $x_1, \ldots, x_k$ a regular sequence of (homogeneous) elements of R.*

(a) For every $S \subset R$ multiplicatively closed subset (of homogeneous elements) the same sequence is regular in the localization $S^{-1}R$.

(b) If $I \subset R$ is a (homogeneous) ideal then the sequence in question is regular in the completion $R^\wedge_I$.

(c) Let T be an R-algebra obtained by applying iteratively localizations and completions, in any order, then the sequence is again regular in T.

Proof. We observe that regularity can be expressed recursively as the existence of short exact sequences

$$0 \to R/(x_1, \ldots, x_{i-1}) \xrightarrow{x_i} R/(x_1, \ldots, x_{i-1}) \to R/(x_1, \ldots, x_i) \to 0$$

For point (a) we have just to recall that localization is an exact functor, hence after applying $- \otimes_R S^{-1}R$ we obtain the same sequences stating $x_1, \ldots, x_k$ is regular in the ring $S^{-1}R$.

For point (b) we need a bit more of machinery: the Artin-Rees lemma ensures that for a Noetherian ring R the completion with respect to I of finitely generated R-modules is given by tensoring with $R_I^\wedge$ ([1, Prop. 10.13]) and this ensures the morphism $R \to R_I^\wedge$ is flat ([1, Prop. 10.14]). Hence, as in (a) applying $- \otimes_R R_I^\wedge$ to the mentioned short exact sequences provides the same sequences for the ring $R_I^\wedge$.

Finally (c) follows by applying repeatedly (a) and (b), keeping in mind that the localization of a Noetherian ring is still Noetherian ([1, Prop. 7.3]) and the completion of a Noetherian ring remains Noetherian (another deep consequence of Artin-Rees, see [1, Thm. 10.26]). □

Lemma 4.4.8. *We fix a finite subset $A \subset \mathbb{N}$. Let $B \subseteq A$, then the elements $\{v_i : i \in A_{<\min B}\}$ (taken with the natural ordering) provide a regular sequence in the ring $\phi_B^*(BP_A)$, moreover we have $(\phi_B^*(BP_A))/I_{\min B} \neq 0$.*

Proof. We prove the claim by induction on $|B|$. In the starting case $B = \emptyset$ we set the convention $\min B = +\infty$ so $I_{\min B} = (v_i : i \in \mathbb{N})$. Since BP_A is a polynomial algebra, it is immediate to see the sequence provided by the v_i's for $i \in A$ is regular and clearly $BP_A/I_{\min B} = \mathbb{F}_p \neq 0$.

Suppose now $|B| \geq 1$, so we can write $B = \{b\} \cup B'$ for $b = \min B$ and $B' = B \setminus \{b\}$. We observe that BP_A is a Noetherian ring and $\phi_{B'}^*(BP_A)$ is obtained from it by applying recursively localizations and completions, thus arguing as in the proof of Lemma 4.4.7 (c) we conclude also $\phi_{B'}^*(BP_A)$ is Noetherian. By [1, Prop. 10.15 iii)] we have an isomorphism

$$(\phi_B^*(BP_A))/I_b = (v_b^{-1}\phi_{B'}^*(BP_A))_{I_b}^\wedge)/I_b \cong (v_b^{-1}\phi_{B'}^*(BP_A))/I_b$$

and since localization is an exact functor we have

$$(v_b^{-1}\phi_{B'}^*(BP_A))/I_b = (v_b^{-1}\phi_{B'}^*(BP_A))/(v_i : i \in A_{<b}) \cong v_b^{-1}(\phi_{B'}^*(BP_A)/(v_i : i \in A_{<b})).$$

From the surjection

$$\phi_{B'}^*(BP_A)/(v_i : i \in A_{<b}) \twoheadrightarrow \phi_{B'}^*(BP_A)/(v_i : i \in A_{<\min B'}) = \phi_{B'}^*(BP_A)/I_{<\min B'} \neq 0$$

we conclude $\phi_{B'}^*(BP_A)/(v_i : i \in A_{<\min B}) \neq 0$.

By inductive assumption $\{v_i : i \in A_{<\min B'}\}$ is a regular sequence in $\phi_{B'}^*(BP_A)$ and we notice $\{v_i : i \in A_{<b}\}$ provides an initial segment of such sequence and the next element to be added is v_b, thus the map

$$v_b \colon \phi_{B'}^*(BP_A)/(v_i : i \in A_{<b}) \to \phi_{B'}^*(BP_A)/(v_i : i \in A_{<b})$$

is injective and we proved $\phi_{B'}^*(BP_A)/(v_i : i \in A_{<b}) \neq 0$. Thus v_b is not a zero-divisor in this BP_*-algebra, in particular it is not nilpotent and we conclude the localization $v_b^{-1}(\phi_{B'}^*(BP_A)/(v_i : i \in A_{<b}))$ is not zero.

The claim that $\{v_i : i \in A_{<b}\}$ constitutes a regular sequence in $\phi_B^*(BP_A)$ follows easily from the inductive assumption and Lemma 4.4.7 (c). □

Proof of Proposition 4.4.4. Take v_i with $i \in A$, then there are clearly maps of BP_*-algebras

$$v_i^{-1}BP_Q \to \phi_{A_{\leq i}}^*(BP_Q) \xrightarrow{\theta_{A_{\leq i},A}} \phi_A^*(BP_Q)$$

ensuring v_i is invertible in $\phi_A^*(BP_Q)$.

Now the inclusion $A \subseteq Q$ induces the quotient map

$$BP_Q \to BP_Q/(v_i : i \notin A) = BP_A$$

and we can consider its image under ϕ_A^*. If v_i for $i \notin A$ were invertible in $\phi_A^*(BP_Q)$ then we would get $\phi_A^*(BP_A) = 0$. But Lemma 4.4.8 for the case $B = A$ establishes that $\phi_A^*(BP_A)/I_{\min A} = \phi_A^*(BP_A)$ is not trivial.

We conclude v_i is invertible in $\phi_A^*(BP_Q)$ if and only if $i \in A$. $\square$

The connection between the algebraic functors ϕ_A^* and the iterated localizations ϕ_A of Example 2.1.10 is established by the next result. Recall that for any $i \in \mathbb{N}$ the spectrum $BP\langle i\rangle$ is given by $BP/(v_{i+1}, v_{i+2}, \dots)$, hence $BP\langle i\rangle_* = \mathbb{Z}_{(p)}[v_1, \dots, v_i]$.

Proposition 4.4.9. *Let $A \subseteq N$, then we have an isomorphism of graded BP_*-modules*

$$\pi_*(\phi_A BP\langle n-1\rangle) \cong \phi_A^*(BP_N).$$

Proof. This follows directly from [19, Lemma 2.3] and the surrounding discussion. The cited Lemma implies immediately that $\pi_*(\phi_m BP) = \phi_m^* BP_*$, but the proof can be adapted to any BP-algebra E so that the sequence $v_0, \dots, v_m$ is regular in $\pi_*(E)$ and $\pi_*(E)/I_m \neq 0$. This way $v_m^{-1}E$ results a v_m-periodic Landweber exact spectrum. This lets us prove the claim by induction on $|A|$.

Let $A = \{a\} \cup B$ with $a < \min B$ and $|B| \geq 1$. Since the localization functors ϕ_m have a clear lax monoidal structure it follows $\phi_B(BP\langle n-1\rangle)$ admits the structure of BP-algebra, by inductive assumption $\pi_*(\phi_B BP\langle n-1\rangle) = \phi_B^* BP_N$ and Lemma 4.4.8 guarantees $v_0, v_1, \dots, v_a$ is a regular sequence in this ring.

Thus $v_a^{-1}\phi_B BP\langle n-1\rangle$ is a v_a-periodic Landweber exact spectrum and [19, Cor. 1.12] implies it is Bousfield equivalent to $E(a)$. Since it is a ring spectrum, it must be local with respect to itself, therefore $v_a^{-1}\phi_B BP\langle n-1\rangle$ is $E(a)$-local.

We now claim that $\phi_B BP\langle n-1\rangle \to v_a^{-1}\phi_B BP\langle n-1\rangle$ is an $E(a)$-equivalence: the proof is exactly the same as the second part of [19, Lemma 2.3]. We have to verify that the map $v_a\colon \Sigma^{|v_a|}\phi_B BP\langle n-1\rangle \to \phi_B BP\langle n-1\rangle$ is an $E(a)$-equivalence, this is equivalent to proving that $\eta_R(v_a)$ is a unit in $E(a)_*\phi_B BP\langle n-1\rangle \cong E(a)_* \otimes_{BP_*} BP_*BP \otimes_{BP_*} \phi_B^*(BP_N)$. Again the formula

$$\eta_R(v_a) = v_a \bmod I_a$$

immediately shows the claim to be true. Notice that, differently from the starting case, here $\pi_*\phi_B BP\langle n-1\rangle$ is $I_{\min B}$-complete hence any element of $v_a + I_a$ is invertible.

At this point we proved $L_{E(a)}\phi_B BP\langle n-1\rangle = v_a^{-1}\phi_B BP\langle n-1\rangle$. Now [19, Cor. 2.2] states $L_{K(m)} = L_{F(m)}L_{E(m)}$, thus

$$\phi_A BP\langle n-1\rangle = \phi_a\phi_B BP\langle n-1\rangle = L_{F(a)}L_{E(a)}\phi_B BP\langle n-1\rangle = L_{F(a)}(v_a^{-1}\phi_B BP\langle n-1\rangle).$$

Finally [19, Thm. 2.1] implies

$$\pi_*(L_{F(a)}v_a^{-1}\phi_B BP\langle n-1\rangle) \cong (v_a^{-1}\phi_B^* BP_N)^\wedge_{I_a} = \phi_A^*(BP_N).$$

$\square$

Remark 4.4.10. The same reasoning would allow us to prove that $\pi_*(\phi_A BP) \cong \phi_A^* BP_*$ if we could show that the functor ϕ_a^* preserves the regularity of the sequence $v_0, v_1, \dots, v_i$ for $i < a$ from $\phi_B^* BP_*$ to $\phi_a^*\phi_B^* BP_*$. Unfortunately, since BP_* is not Noetherian, the arguments adopted in Lemma 4.4.7 and 4.4.8 do not work.

Corollary 4.4.11. *The iterated localizations ϕ_A of Example 2.1.10 cannot be naturally isomorphic for different subsets $A \subseteq N$.*

Proof. Immediate from Proposition 4.4.9 and Proposition 4.4.4. $\square$

The same computations of Proposition 4.4.9 also show the ϕ_A's are distinct in the case of Example 2.1.11.

Proposition 4.4.12. *Let E be a BP-module spectrum, then the canonical map*

$$L_{T(n)}E \to L_{K(n)}E$$

is an equivalence.

Proof. We have just to show $L_{T(n)}E$ is $K(n)$-local.

Fix $F(n)$ a finite spectrum of type n, it can be proved that we can choose a v_n-self map $v\colon \Sigma^k F(n) \to F(n)$ such that $1 \wedge v = v_n^{p^l} \wedge 1$ as endomorphism of $BP \wedge F(n)$ for some $l \in \mathbb{N}$, a reference for this assertion is [21, Thm. 4.6]. This implies that $BP \wedge T(n) = BP \wedge v^{-1}F(n) = v_n^{-1}BP \wedge F(n)$.

We claim that for any spectrum X the wedge product $BP \wedge T(n) \wedge X$ is $K(n)$-local. Consider W a $K(n)$-acyclic spectrum then

$$[W, BP \wedge T(n) \wedge X] \cong [W, v_n^{-1}BP \wedge F(n) \wedge X] \cong [W \wedge DF(n), v_n^{-1}BP \wedge X].$$

By Remark 1.2.10 the spectrum $DF(n)$ is of type n, hence $W \wedge DF(n)$ is $E(n)$-acyclic. Since $v_n^{-1}BP$ is a BP-algebra which is a v_n-periodic Landweber exact spectrum it is Bousfield equivalent to $E(n)$ by [19, Cor. 1.12] and being $v_n^{-1}BP \wedge X$ a module over $v_n^{-1}BP$ it must be $E(n)$-local. We conclude the above hom-group is zero hence $BP \wedge T(n) \wedge X$ is indeed $K(n)$-local.

Because E is a BP-module we have $T(n) \wedge E$ is a retract of $BP \wedge T(n) \wedge E$, hence even this is $K(n)$-local.

By [21, Prop. 4.22] there exists a tower of generalized Moore spectra of type n

$$\cdots \to M_n \to M_{n-1} \to \cdots \to M_1 \to M_0$$

with $BP_*M_k \cong BP_*/J_k$ for some ideal $J_k = (p^{i_0}, v_1^{i_1}, \dots v_{n-1}^{i_{n-1}})$ of BP_* such that the sequence of n-uples $(i_0, \dots, i_{n-1})$ is cofinal at the varying of k. Also, we can choose v_n-self maps $w_k\colon \Sigma^{m_k}M_k \to M_k$ which are compatible with the tower so that we can form the homotopy limit

$$\underset{\underset{k}{\longleftarrow}}{\operatorname{holim}}\, \operatorname{Tel}(M_k) \wedge E = \underset{\underset{k}{\longleftarrow}}{\operatorname{holim}}\, w_k^{-1}M_k \wedge E.$$

We showed each term in the inverse system is $K(n)$-local, so the limit has this property as well.

This concludes our argument once we realize the presented homotopy limit is a model for the $T(n)$-localization of E: for a reference of this fact see [25, Prop. 5.1], the proposed proof actually deals with the localization of the sphere spectrum but it can be immediately adapted to any other spectrum. $\square$

In the general case the fracture axiom immediately implies that if $\phi_a = \phi_b$ for $a < b$ then $\phi_a = \phi_b\phi_a = 0$. But we do not know if we can distinguish the compositions ϕ_A's. In the chromatic case we relied on the properties of the Morava K-theories and on the existence of a universal example (namely BP) which contains the information of all the involved homology theories.

In the situation of Definition 2.1.1 we could ask the homotopy category $\mathcal{B}$ to be monoidal and the homology theories $K(i)_*$ to be represented by objects $K(i)$ with good properties: e.g.

they are ring objects and they are skew fields. Then we could think of providing as universal discriminant object $U = \bigoplus_{i=0}^{n-1} K(i)$, but this will not do the trick.

Being $K(i)$ a ring object it is local with respect to itself, hence we have

$$\phi_a K(i) = \begin{cases} K(i) & \text{if } a = i \\ 0 & \text{else} \end{cases}$$

thus

$$\phi_A U = \begin{cases} K(i) & \text{if } A = \{i\} \\ 0 & \text{else.} \end{cases}$$

The fact is that that the direct sum of the objects $K(i)$ does not keep track of the gluing data between the homology theories. Thus while U contains information about every isolated object $K(i)$, it does not provide any data about how they interact.

In fact, it is known that $\langle BP \rangle \neq \bigvee_{i \in \mathbb{N}} \langle K(i) \rangle$, for reference see [34, Thm. 2.2].

This could be interpreted in the language of algebraic geometry as the fact that the Hopf algebroid (BP_*, BP_*BP) represents the moduli stack of formal group laws and the localization $L_{K(i)}$ corresponds to the completion along the locally closed substack of formal group laws of height exactly i. But to put together the information provided by these completions we need the underlying complete stack.

4.5 Fracture diagrams

We now define the anafunctor version of the iterated localisations with respect to the direct sums of the fixed homology theories $K(i)_*$.

Definition 4.5.1. Recall the poset $\mathbb{Q}$ and the order-embedding $u\colon \mathbb{P} \to \mathbb{Q}$ we presented in Definition 2.2.2. Consider an object $X \in \mathcal{C}(\mathbb{Q} \times R)$, and the restriction $(u \times 1)^*\colon \mathcal{C}(\mathbb{Q} \times R) \to \mathcal{C}(\mathbb{P} \times R)$ induced by $u\colon \mathbb{P} \to \mathbb{Q}$. We say that X is *u-cartesian* if the natural map

$$(u \times 1)^*\colon \mathcal{C}(\mathbb{Q} \times R)(W, X) \to \mathcal{C}(\mathbb{P} \times R)((u \times 1)^*(W), (u \times 1)^*(X))$$

is an isomorphism for all W, or equivalently X is in the essential image of the functor

$$(u \times 1)_*\colon \mathcal{C}(\mathbb{P} \times R) \to \mathcal{C}(\mathbb{Q} \times R).$$

This equivalence comes from Lemma 3.2.3 and the triangular equalities.

We say that X is a *fracture object* if it is u-cartesian, and $(u \times 1)^*(X)$ is fully localising. We write $\mathcal{F}(R)$ for the subcategory of fracture objects in $\mathcal{C}(\mathbb{Q} \times R)$. We also define $j\colon R \to \mathbb{Q} \times R$ by $j(r) = (u\emptyset, R)$.

Remark 4.5.2. Because we have ordered $\mathbb{Q}$ by reverse inclusion, for $U \in \mathbb{Q}$ we have $U \leq uA$ if and only if $uA \subseteq U$, which is equivalent to $A \in U$. Using this together with the Kan formula (D4), the u-cartesian condition becomes

$$X_U = \underset{\substack{\longleftarrow \\ A \in U}}{\operatorname{holim}} X_{uA}.$$

Example 4.5.3. Consider the case $N = \{0, 1\}$, and put

$$W = vN = u\{0\} \cup u\{1\} = \{A \subseteq N \mid A \neq \emptyset\} \in \mathbb{Q}.$$

We then have $\mathbb{Q} = \{uA \mid A \in \mathbb{P}\} \cup \{W, \emptyset\}$, with partial order as shown below.

$$\begin{tikzcd}
u\emptyset \arrow[rrr] \arrow[dddd] \arrow[dr] & & & u\{0\} \arrow[dd] \arrow[dddd] \\
& W \arrow[urr] \arrow[dr] \arrow[ddl] & & \\
& & u\{0,1\} \arrow[ddr] & \\
& & & \\
u\{1\} \arrow[rrr] \arrow[uur] & & & \emptyset
\end{tikzcd}$$

An associated fracture diagram is in the shape is

$$\begin{tikzcd}
X \arrow[rrr] \arrow[dddd] \arrow[dr] & & & \phi_0 X \arrow[dd] \arrow[dddd] \\
& \lambda_{01} X \arrow[urr] \arrow[dr] \arrow[ddl] & & \\
& & \phi_{01} X \arrow[ddr] & \\
& & & \\
\phi_1 X \arrow[rrr] \arrow[uur] & & & 0.
\end{tikzcd}$$

Observe we are extending the fully localizing square associated to $X \in \mathcal{C}(R)$ by adding 0 (the zero-localization) and $\lambda_{01}X$ by embedding the fracture square associated to $K(0)$ and $K(1)$.

Now consider the case $N = \{0, 1, 2\}$, where $|\mathbb{P}| = 8$. It turns out that $|\mathbb{Q}| = 20$: while it is possible to present a graphical representation of $\mathbb{Q}$ it is not rather pleasant or illuminating. We present instead a partial picture of a corresponding fracture diagram

$$\begin{tikzcd}
X \arrow[r] & \lambda_{012} X \arrow[rrrr] \arrow[dddddd] \arrow[ddddrr] & & & & \phi_0 X \arrow[dddddd] \arrow[ddddrr] & & \\
& & \lambda_{02} X \arrow[urrr] \arrow[dddddl] & & & & & \\
& & & & \lambda_{01} X \arrow[uur] \arrow[ddl] & & & \\
& & & & & & \phi_0 \lambda_{12} X \arrow[dr] \arrow[dddl] & \\
& & & \phi_1 X \arrow[rrrr] \arrow[dddd] & & & & \phi_{01} X \arrow[dddd] \\
& & \lambda_{12} X \arrow[ur] \arrow[dl] & & & & & \\
& \phi_2 X \arrow[rrrr] \arrow[ddrr] & & & & \phi_{02} X \arrow[ddrr] & & \\
& & & & \lambda_{01} \phi_2 X \arrow[ur] \arrow[dl] & & & \\
& & & \phi_{12} X \arrow[rrrr] & & & & \phi_{012} X.
\end{tikzcd}$$

Proposition 4.5.4. *$\mathcal{F}$ is a thick subderivator of $\mathcal{C}^{\mathbb{Q}}$, and $j^* \colon \mathcal{F} \to \mathcal{C}$ is an equivalence of derivators.*

Proof. Let $\mathcal{E}(R)$ be the subcategory of u-cartesian objects in $\mathcal{C}(\mathbb{Q} \times R)$, so the functor $(u \times 1)^* \colon \mathcal{E}(R) \to \mathcal{C}(\mathbb{P} \times R)$ is an equivalence. An object is u-cartesian if and only if the unit map $X \to (u \times 1)_*(u \times 1)^*(X)$ is an isomorphism, and from this we see that $\mathcal{E}(R)$ is a thick subcategory of $\mathcal{C}(\mathbb{Q} \times R)$.

Now consider a morphism $f\colon R \to T$ of finite posets. We know from [10, Prop. 2.6] that the resulting morphism $f^*\colon \mathcal{C}^T \to \mathcal{C}^R$ preserves homotopy Kan extensions (both left and right). Thus it commutes up to a canonical isomorphism with the functors $(u \times 1)_*$, so it restricts to give a functor $f^*\colon \mathcal{E}(T) \to \mathcal{E}(R)$. It is straightforward that the functors $(1 \times f)_*$ commute with $(u \times 1)_*$, so they restrict to give $f_*\colon \mathcal{E}(R) \to \mathcal{E}(T)$. By the dual of Lemma 3.8.11, we deduce that $\mathcal{E}$ is a thick subderivator of $\mathcal{C}^{\mathbb{Q}}$.

We can see that $\mathcal{E}$ is actually the essential image of $u_*\colon \mathcal{C}^{\mathbb{P}} \to \mathcal{C}^{\mathbb{Q}}$, by Proposition 3.4.1 this morphims must be fully faithful. Thus the restriction $u_*\colon \mathcal{C}^{\mathbb{P}} \to \mathcal{E}$ is an equivalence of derivators.

By the fully faithfulness of u_* we see that an object $X \in \mathcal{C}(\mathbb{Q} \times R)$ lies in $\mathcal{F}(R)$ only if it is the right Kan extension along u of an object of $\mathcal{P}(R)$, hence we obtain the diagram of derivators

$$\begin{tikzcd} \mathcal{C}^{\mathbb{P}} \arrow[r, "u_*"] & \mathcal{E} \\ \mathcal{P} \arrow[u, hook] \arrow[r, "u_*"'] & \mathcal{F} \arrow[u, hook] \end{tikzcd}$$

where the vertical arrows are inclusions and the horizontal ones equivalences. Therefore, composing the inverse equivalence $u^*\colon \mathcal{F} \to \mathcal{P}$ with $\emptyset^*$ of Proposition 4.3.7 we get the second claim. □

Definition 4.5.5. For $U \in \mathbb{Q}$, we define θ_U to be the anafunctor

$$\mathcal{C} \xleftarrow{j^*} \mathcal{F} \xrightarrow{U^*} \mathcal{C}.$$

We also note that an inequality $U \leq V$ gives a natural transformation $U^* \Rightarrow V^*$ and thus a 2-cell $\theta_U \Rightarrow \theta_V$ between anafunctors, as discussed in Remark 4.2.3.

Remark 4.5.6. Consider an object $X \in \mathcal{C}(R)$, and a fracture object $Y \in \mathcal{F}(R)$ with $j^*Y \simeq X$. Then the object $Y_U = U^*Y \in \mathcal{C}(R)$ is a choice of $\theta_U(X)$. As Y is u-cartesian we have

$$Y_U = \underset{\substack{\longleftarrow \\ U \leq uA}}{\operatorname{holim}} Y_{uA} = \underset{\substack{\longleftarrow \\ A \in U}}{\operatorname{holim}} Y_{uA}.$$

As $u^*Y \in \mathcal{P}(R)$ we know that Y_{uA} is a choice of $\phi_A(X)$. Thus, the basic idea is that $\theta_U(X) = \underset{\longleftarrow\ A \in U}{\operatorname{holim}} \phi_A(X)$.

Remark 4.5.7. Consider the original chromatic context where the homology theory $K(i)_*$ is represented by a spectrum, so we can apply ϕ_A or θ_U to that spectrum. Using [34, Thm. 2.1 (i)] it is easy to see that $\phi_A(K(i)) \cong K(i)$ if $A \subseteq \{i\}$, and $\phi_A(K(i)) = 0$ in all other cases. From this we find that $\theta_U(K(i)) \cong K(i)$ if $\{i\} \in U$, and $\theta_U(K(i)) = 0$ in all other cases.

Thus, for any $U, V \in \mathbb{Q}$ if there exists $0 \leq i \leq n-1$ such that $\{i\}$ belongs to just one of these two elements then the corresponding θ_U and θ_V are different.

However, since the occurrence $\{i\} \in U$ is very rare this statement is not particularly strong.

Lemma 4.5.8. *If $A = \{a_1, \dots, a_r\}$ with $a_1 < \cdots < a_r$ then there are equivalences of anafunctors*

$$\theta_{uA} \simeq \phi_A \simeq L_{K(a_1)} \cdots L_{K(a_r)}.$$

Proof. We have a diagram as follows, which commutes on the nose:

$$\begin{tikzcd} \mathcal{F} \arrow[r, "uA^*"] \arrow[d, "j^*"', "\simeq"] \arrow[dr, "u^*", "\simeq"'] & \mathcal{C} \\ \mathcal{C} & \mathcal{P} \arrow[l, "\simeq", "\emptyset^*"'] \arrow[u, "A^*"'] \end{tikzcd}$$

This gives an equivalence $uA^*(j^*)^{-1} \simeq A^*(\emptyset^*)^{-1}$ of anafunctors, or in other words $\theta_{uA} \simeq \phi_A$. Moreover, Corollary 4.3.18 gives $\phi_A \simeq L_{K(a_1)} \cdots L_{K(a_r)}$. □

Lemma 4.5.9. *The functor $\theta_\emptyset$ is zero.*

Proof. Fix $X \in \mathcal{C}(R)$ and choose $Y \in \mathcal{F}(R)$ together with an isomorphism $X \simeq j^*(Y)$. It will suffice to prove that $Y_\emptyset = 0$. By Remark 4.5.6 we have

$$Y_\emptyset = \underset{\substack{\longleftarrow \\ A\in\emptyset}}{\operatorname{holim}} Y_{uA}.$$

Since $(\emptyset/u) = \{A \in \mathbb{P} : A \in \emptyset\} = \emptyset$ clearly $Y_\emptyset = 0$. □

Proposition 4.5.10. *For $vA = \{T \subseteq N \mid T \cap A \neq \emptyset\}$ we have $\theta_{vA} \simeq \lambda_A$.*

Proof. Let Y be any object of $\mathcal{F}(R)$. We claim that the morphism $Y_{u\emptyset} \to Y_{vA}$ is a localisation with respect to $K(A)_*$. In order to simplify notation, we replace $\mathcal{C}$ by $\mathcal{C}^R$ and thus reduce to the case $R = e$. As Y is a u-cartesian object, we see that Y_{vA} is the homotopy inverse limit of $(u^*Y)|_{vA}$. We define the poset

$$P' = \mathcal{P}'(A) = \{B \subset A : B \neq \emptyset\}.$$

Note that the inclusion $i\colon P' \to vA$ is left adjoint to the map

$$r\colon vA \to P' \qquad B \mapsto B \cap A.$$

It follows from Proposition 3.6.15 that i is homotopy cofinal, so

$$Y_{vA} = \underset{\substack{\longleftarrow \\ B\in vA}}{\operatorname{holim}} Y_{uB} \cong \underset{\substack{\longleftarrow \\ B\in P'}}{\operatorname{holim}} Y_{uB}.$$

We can now apply Proposition 4.3.12 (with N replaced by A) to see that this homotopy limit is a $K(A)$-localisation, as required.

Now define $k\colon [1] \to \mathbb{Q}$ by $k(0) = u\emptyset$ and $k(1) = vA$. This gives a morphism $k^*\colon \mathcal{C}^{\mathbb{Q}} \to \mathcal{C}^{[1]}$, and the previous paragraph shows that this restricts to give a morphism $\mathcal{F} \to \mathcal{L}$ (where $\mathcal{L}$ is as in Definition 4.3.1, for localisation with respect to $K(A)$). We now have a diagram as follows, which commutes on the nose:

$$\begin{tikzcd}
\mathcal{F} \arrow[r, "vA^*"] \arrow[d, "j^*"', "\simeq"] \arrow[dr, "k^*", "\simeq"'] & \mathcal{C} \\
\mathcal{C} & \mathcal{L} \arrow[l, "\simeq", "0^*"'] \arrow[u, "1^*"']
\end{tikzcd}$$

As 0^* and j^* are equivalences, we see that k^* is also an equivalence. This gives an isomorphism $vA^*(j^*)^{-1} \simeq 1^*(0^*)^{-1}$ of anafunctors, or in other words $\theta_{vA} \simeq \lambda_A$. □

The crucial point of our proof will be the following result which reduces the composition of the anafunctors θ_U to the operation $*$ on $\mathbb{Q}$ of Lemma 2.2.4.

Theorem 4.5.11. *The composite $\theta_U\theta_V$ is naturally isomorphic to θ_{U*V}.*

The proof will be given after some preliminaries.

We start with the following result, which will be needed in the proof of Theorem 4.5.11, and which also shows that Theorem 4.5.11 is consistent with Proposition 4.3.17.

Lemma 4.5.12. *For $A, B \in \mathbb{P}$ we have*

$$uA * uB = \begin{cases} u(A \cup B) & \text{if } A \angle B \\ \emptyset & \text{otherwise.} \end{cases}$$

Proof. By definition, we have

$$uA * uB = \{C \cup D \mid A \subseteq C, B \subseteq D, C \angle D\} \subseteq u(A \cup B).$$

If $A \angle B$ then we can choose k with $a \leq k$ for all $a \in A$, and $k \leq b$ for all $b \in B$. Then any $E \in u(A \cup B)$ can be written as $C \cup D$ with $C = \{j \in E \mid j \leq k\} \supseteq A$ and $D = \{j \in E \mid j \geq k\} \supseteq B$, so $E \in uA * uB$. We therefore have $uA * uB = u(A \cup B)$ in this case. On the other hand, if it is not true that $A \angle B$ then we can choose $a \in A$ and $b \in B$ with $a > b$. If C and D are as in the definition then $a \in C$ and $b \in D$ so it cannot be that $C \angle D$. From this it follows that $uA * uB = \emptyset$. $\square$

Definition 4.5.13. We say that an object $X \in \mathcal{C}(\mathbb{Q} \times \mathbb{Q} \times R)$ is a *double fracture object* if

(a) X is a fracture object relative to $\mathbb{Q} \times R$, so it lies in $\mathcal{F}(\mathbb{Q} \times R)$.

(b) The restriction to $\{u\emptyset\} \times \mathbb{Q} \times R \simeq \mathbb{Q} \times R$ lies in $\mathcal{F}(R)$.

We write $\mathcal{F}_2(R)$ for the full subcategory of double fracture objects in $\mathcal{C}(\mathbb{Q} \times \mathbb{Q} \times R)$. We also define $k, l \colon \mathbb{Q} \to \mathbb{Q} \times \mathbb{Q}$ by $k(V) = (u\emptyset, V)$ and $l(U) = (U, u\emptyset)$. This gives functors $k^*, l^* \colon \mathcal{F}_2(R) \to \mathcal{C}(\mathbb{Q} \times R)$. Finally, we define $j_2 \colon e \to \mathbb{Q} \times \mathbb{Q}$ to be the map with image $(u\emptyset, u\emptyset)$.

Proposition 4.5.14. *$\mathcal{F}_2$ is a thick subderivator of $\mathcal{C}^{\mathbb{Q} \times \mathbb{Q}}$, and the maps k and l give equivalences as shown:*

$$\begin{tikzcd} \mathcal{F}_2 \arrow[r, "k^*", "\simeq"'] \arrow[d, "l^*"', "\simeq"] \arrow[dr, "j_2^*", "\simeq"'] & \mathcal{F} \arrow[d, "\simeq"', "j^*"] \\ \mathcal{F} \arrow[r, "\simeq", "j^*"'] & \mathcal{C}. \end{tikzcd}$$

Proof. Put $\mathcal{E}(R) = \mathcal{F}(\mathbb{Q} \times R) \subset \mathcal{C}(\mathbb{Q} \times \mathbb{Q} \times R)$ (so this is the subcategory of objects satisfying condition (a)). From Proposition 4.3.12 we see that $\mathcal{E}$ is a thick subderivator of $\mathcal{C}^{\mathbb{Q} \times \mathbb{Q}}$ and that $k^* \colon \mathcal{E} \to \mathcal{C}^{\mathbb{Q}}$ is an equivalence of derivators. Moreover, it is clear k^* restricts to an equivalence $k^* \colon \mathcal{F}_2 \to \mathcal{F}$, since $\mathcal{F}$ is a thick subderivator of $\mathcal{C}^{\mathbb{Q}}$ we deduce $\mathcal{F}_2$ must be a thick subderivator of $\mathcal{C}^{\mathbb{Q} \times \mathbb{Q}}$. We have seen that $j^* \colon \mathcal{F} \to \mathcal{C}$ is also an equivalence.

Next, recall again that $\mathcal{F}$ is a thick subderivator. Any monotone map $f \colon R \to T$ gives a functor $(1 \times f)^* \colon \mathcal{C}(\mathbb{Q} \times T) \to \mathcal{C}(\mathbb{Q} \times R)$, and the subderivator property implies that $(1 \times f)^*(\mathcal{F}(T)) \subseteq \mathcal{F}(R)$. Take $T = \mathbb{Q} \times R$ and $f(r) = (u\emptyset, r)$; the conclusion is then that $l^*(\mathcal{E}(R)) \subseteq \mathcal{F}(R)$, and so $l^*(\mathcal{F}_2(R)) \subseteq \mathcal{F}(R)$. This means that we have a diagram of functors as claimed. The upper right triangle commutes on the nose, while the lower left triangle commutes up to a natural isomorphism interchanging the two copies of $\mathbb{Q} \times \mathbb{Q}$ to identify the subderivator of $\mathcal{C}^{\mathbb{Q} \times \{u\emptyset\}}$ with $\mathcal{F} \subset \mathcal{C}^{\{u\emptyset\} \times \mathbb{Q}}$.

As j^* and k^* are equivalences, we can chase the diagram to see that l^* and $(u\emptyset, u\emptyset)^*$ are equivalences as well. $\square$

Example 4.5.15. We give an example of a diagram in $\mathcal{F}_2$ to make sense of the above Proposition 4.5.14. We consider the case $n = 2$, since the cardinality of $\mathbb{Q} \times \mathbb{Q}$ increases quickly this is the only non-trivial case we would be able to present concretely (e.g. if $n = 3$ then $|\mathbb{Q}| = 20$ and $|\mathbb{Q} \times \mathbb{Q}| = 400$). We also remove the element $\emptyset$ from $\mathbb{Q}$ to simplify the picture further, since in this position the value of the diagram will be zero. Thus $\mathbb{Q} \times \mathbb{Q}$ has shape

$$
\begin{tikzcd}[column sep=small, row sep=small]
(u\emptyset,u\emptyset) \arrow[rr] \arrow[dd] \arrow[dr] & & (u\emptyset,u\{0\}) \arrow[dd] & & & & (u\{0\},u\emptyset) \arrow[rr] \arrow[dd] \arrow[dr] & & (u\{0\},u\{0\}) \arrow[dd] \\
& (u\emptyset,W) \arrow[ur] \arrow[dl] \arrow[rrrrrr] \arrow[dddrrr] \arrow[dddddd] & & & & & & (u\{0\},W) \arrow[ur] \arrow[dl] \arrow[dddddd] & \\
(u\emptyset,u\{1\}) \arrow[rr] & & (u\emptyset,uN) & & & & (u\{0\},u\{1\}) \arrow[rr] & & (u\{0\},uN) \\
& & & (W,u\emptyset) \arrow[rr] \arrow[dd] \arrow[dr] & & (W,u\{0\}) \arrow[dd] & & & \\
& & & & (W,W) \arrow[ur] \arrow[dl] \arrow[uuurrr] \arrow[dddlll] & & & & \\
& & & (W,u\{1\}) \arrow[rr] & & (W,uN) & & & \\
(u\{1\},u\emptyset) \arrow[rr] \arrow[dd] \arrow[dr] & & (u\{1\},u\{0\}) \arrow[dd] & & & & (uN,u\emptyset) \arrow[rr] \arrow[dd] \arrow[dr] & & (uN,u\{0\}) \arrow[dd] \\
& (u\{1\},W) \arrow[ur] \arrow[dl] \arrow[rrrrrr] & & & & & & (uN,W) \arrow[ur] \arrow[dl] & \\
(u\{1\},u\{1\}) \arrow[rr] & & (u\{1\},uN) & & & & (uN,u\{1\}) \arrow[rr] & & (uN,uN)
\end{tikzcd}
$$

and a corresponding double fracture diagram is

$$
\begin{tikzcd}[column sep=small, row sep=small]
X \arrow[rr] \arrow[dd] \arrow[dr] & & \phi_0X \arrow[dd] & & & & \phi_0X \arrow[rr] \arrow[dd] \arrow[dr] & & \phi_0\phi_0X \arrow[dd] \\
& \lambda_{01}X \arrow[ur] \arrow[dl] \arrow[rrrrrr] \arrow[dddrrr] \arrow[dddddd] & & & & & & \phi_0\lambda_{01}X \arrow[ur] \arrow[dl] \arrow[dddddd] & \\
\phi_1X \arrow[rr] & & \phi_{01}X & & & & \phi_0\phi_1X \arrow[rr] & & \phi_0\phi_{01}X \\
& & & \lambda_{01}X \arrow[rr] \arrow[dd] \arrow[dr] & & \lambda_{01}\phi_0X \arrow[dd] & & & \\
& & & & \lambda_{01}\lambda_{01}X \arrow[ur] \arrow[dl] \arrow[uuurrr] \arrow[dddlll] & & & & \\
& & & \lambda_{01}\phi_1X \arrow[rr] & & \lambda_{01}\phi_{01}X & & & \\
\phi_1X \arrow[rr] \arrow[dd] \arrow[dr] & & \phi_1\phi_0X \arrow[dd] & & & & \phi_{01}X \arrow[rr] \arrow[dd] \arrow[dr] & & \phi_{01}\phi_0X \arrow[dd] \\
& \phi_1\lambda_{01}X \arrow[ur] \arrow[dl] \arrow[rrrrrr] & & & & & & \phi_{01}\lambda_{01}X \arrow[ur] \arrow[dl] & \\
\phi_1\phi_1X \arrow[rr] & & \phi_1\phi_{01}X & & & & \phi_{01}\phi_1X \arrow[rr] & & \phi_{01}\phi_{01}X.
\end{tikzcd}
$$

The smaller squares are indexed on the second copy of $\mathbb{Q}$, while the bigger square which contains these is indexed on the first copy of $\mathbb{Q}$. Thus the subdiagram on the upper left corner starting with X occupies the positions $(u\emptyset, V)$ for $V \in \mathbb{Q}$ and by condition (b) of Definition 4.5.13 it must be a fracture diagram. Condition (a) means that we have to obtain the other subdiagrams by applying θ_U to get the one placed in position U. For example in the lower left corner we have the positions $(u\{1\}, V)$ so the associated diagram is produced just by applying ϕ_1 to the starting square.

It is clear that at the position (U, V) we have just $\theta_U\theta_V$ of the value in $(u\emptyset, u\emptyset)$.

Both l^* and k^* being equivalences corresponds to the fact that restricting along l and k we obtain two fracture diagrams starting with X, and since fracture diagrams are determined by their initial value this means we did not lose any information.

Corollary 4.5.16. *For any $U, V \in \mathbb{Q}$, the composite anafunctor $\theta_U\theta_V$ is isomorphic to the fraction*

$$\mathcal{C} \xleftarrow{j_2^*} \mathcal{F}_2 \xrightarrow{(U,V)^*} \mathcal{C}.$$

Proof. Note that if $X \in \mathcal{F}_2(R)$ then $X \in \mathcal{F}(\mathbb{Q} \times R)$ and $\mathcal{F}$ is a subderivator so we have $V^*X \in \mathcal{F}(R)$. We can thus interpret V^* as a morphism from $\mathcal{F}_2$ to $\mathcal{F}$. It fits into a diagram as follows, which commutes on the nose:

$$\begin{tikzcd}
 & & \mathcal{C} \\
 & \mathcal{F}_2 \arrow[ur, "(U,V)^*"] \arrow[r, "V^*"'] \arrow[d, "k^*", "\simeq"'] \arrow[dl, "j_2^*"', "\simeq"] & \mathcal{F} \arrow[u, "U^*"'] \arrow[d, "j^*", "\simeq"'] \\
\mathcal{C} & \mathcal{F} \arrow[l, "\simeq"', "j^*"] \arrow[r, "V^*"'] & \mathcal{C}.
\end{tikzcd}$$

The bottom edge represents the anafunctor θ_V, whereas the right hand edge represents θ_U. The claim is clear from this. □

Proposition 4.5.17. *The morphism $u^2_*\colon \mathcal{C}^{\mathbb{P}\times\mathbb{P}} \to \mathcal{C}^{\mathbb{Q}\times\mathbb{Q}}$ restricts to give an equivalence $\mathcal{P}_2 \to \mathcal{F}_2$, with inverse $(u^2)^*$.*

Proof. In this proof all the restrictions will be with respect to the base derivator $\mathcal{C}$, even if we will apply them to elements we will prove are in the derivator of (doubly) localizing or fracture objects. This lets us avoid awkward notation and it is not confusing since the above derivators are subderivators of appropriate shifts of $\mathcal{C}$.

We must show that $\mathcal{P}_2(R) \simeq \mathcal{F}_2(R)$ for all R, but we can reduce to the case $R = e$ by replacing $\mathcal{C}$ with $\mathcal{C}^R$.

We will factor the map $u^2\colon \mathbb{P}\times\mathbb{P} \to \mathbb{Q}\times\mathbb{Q}$ as $u^2 = u_1 \circ u_2$, where $u_1 = u \times 1\colon \mathbb{P}\times\mathbb{Q} \to \mathbb{Q}\times\mathbb{Q}$ and $u_2 = 1 \times u\colon \mathbb{P}\times\mathbb{P} \to \mathbb{P}\times\mathbb{Q}$. We also use the maps $i_\emptyset\colon \mathbb{P} \to \mathbb{P}\times\mathbb{P}$ and $i_{u\emptyset}\colon \mathbb{Q} \to \mathbb{Q}\times\mathbb{Q}$ given by $i_\emptyset(B) = (\emptyset, B)$ and $i_{u\emptyset}(V) = (u\emptyset, V)$. These fit in a commutative diagram

$$\begin{tikzcd}
\mathbb{P} \arrow[r, "i_\emptyset"] \arrow[d, "u"'] \arrow[dr, phantom, "\overset{Id}{\nearrow}"] & \mathbb{P}\times\mathbb{P} \arrow[d, "u^2"] \\
\mathbb{Q} \arrow[r, "i_{u\emptyset}"'] & \mathbb{Q}\times\mathbb{Q}.
\end{tikzcd}$$

Note that an object $X \in \mathcal{C}(\mathbb{P}\times\mathbb{P})$ lies in $\mathcal{P}_2(e)$ if and only if it satisfies the following conditions:

(a) $X \in \mathcal{P}(\mathbb{P})$

(b) $i_\emptyset^* X \in \mathcal{P}(e)$.

Similarly, by unwinding the definitions a little we see that an object $Y \in \mathcal{C}(\mathbb{Q}\times\mathbb{Q})$ lies in $\mathcal{F}_2(e)$ if and only if the following hold:

(c) $u_1^* Y \in \mathcal{P}(\mathbb{Q})$

(d) $Y = (u_1)_*(u_1^* Y)$

(e) $i_{u\emptyset}^* Y \in \mathcal{F}(e)$.

Suppose that $Y \in \mathcal{F}_2(e)$, so that (c), (d) and (e) are satisfied. Put $X = (u^2)^* Y \in \mathcal{C}(\mathbb{P}\times\mathbb{P})$; we must show that $X \in \mathcal{P}_2(e)$, or in other words that (a) and (b) are satisfied. Note that $X = u_2^*(u_1^* Y)$ and $u_1^* Y \in \mathcal{P}(\mathbb{Q})$ by (c) and $\mathcal{P}$ is a subderivator so $u_2^*(u_1^* Y) \in \mathcal{P}(\mathbb{P})$ so (a) is satisfied. Moreover, the diagram shows that $i_\emptyset^* X = i_\emptyset^*(u^2)^* Y = u^* i_{u\emptyset}^* Y$, and $i_{u\emptyset}^* Y \in \mathcal{F}(e)$ by (e), so $u^* i_{u\emptyset}^* Y \in \mathcal{P}(e)$ and (b) holds.

Suppose instead that we start with $X \in \mathcal{P}_2(e)$, so that (a) and (b) hold. Put $Y = u_*^2 X \in \mathcal{C}(\mathbb{Q}\times\mathbb{Q})$; we must then prove (c), (d) and (e). We first note that $Y = (u_1)_*(u_2)_* X$ and u_1 is fully faithful so $\varepsilon\colon u_1^*(u_1)_* \cong Id$ and we deduce $u_1^* Y \cong (u_2)_* X$. Moreover, we have $X \in \mathcal{P}(\mathbb{P})$ by (a) and $\mathcal{P}$ is a subderivator so $(u_2)_* X \in \mathcal{P}(\mathbb{Q})$ and this proves (c). Condition (d) is also clear from this discussion. For condition (e), we note that the diagram gives a canonical mate

$$(Id_*)_X \colon i_{u\emptyset}^* Y = i_{u\emptyset}^* u_*^2 X \to u_* i_\emptyset^* X \in \mathcal{C}(\mathbb{Q}).$$

We know that $i_\emptyset^* X \in \mathcal{P}(e)$ by (b), and it follows that $u_* i_\emptyset^* X \in \mathcal{F}(e)$. It will therefore suffice to show that $(Id_*)_X$ is an isomorphism. Using (D2) this is equivalent to to checking that $V^*(Id_*)_X$ is an isomorphism in $\mathcal{C}(e)$ for all $V \in \mathbb{Q}$. Put

$$\mathbb{B} = \{B \in \mathbb{P} \mid V \leq uB\}$$
$$\mathbb{C} = \{(A,B) \in \mathbb{P}\times\mathbb{P} \mid (u\emptyset, V) \leq (uA, uB)\}.$$

(These can in fact be simplified to $\mathbb{B} = V$ and $\mathbb{C} = \mathbb{P}\times V$.) The map $i_\emptyset$ restricts to give a map $\mathbb{B} \to \mathbb{C}$. The Kan formula tells us that the domain of $V^*(Id_*)_X$ is $\underleftarrow{\operatorname{holim}}_{\mathbb{C}} X$, whereas the codomain is $\underleftarrow{\operatorname{holim}}_{\mathbb{B}} i_\emptyset^* X$. The evident projection $\mathbb{C} \to \mathbb{B}$ is right adjoint to $i_\emptyset$, so $i_\emptyset \colon \mathbb{B} \to \mathbb{C}$ is homotopy cofinal by Proposition 3.6.15, so $(Id_*)_X$ is an isomorphism as required.

The discussion above shows that the adjunction of morphisms

$$(u^2)^* \colon \mathcal{C}^{\mathbb{Q}\times\mathbb{Q}} \rightleftarrows \mathcal{C}^{\mathbb{P}\times\mathbb{P}} : u_*^2$$

restricts to an adjunction

$$(u^2)^* \colon \mathcal{F}_2 \rightleftarrows \mathcal{P}_2 : u_*^2$$

and by Proposition 3.4.1 the counit $\varepsilon\colon (u^2)^* u_*^2 \Rightarrow Id$ is invertible. All that is left is to prove is that when $Y \in \mathcal{F}_2(e)$, the unit map $Y \to u_*^2(u^2)^* Y$ is also an isomorphism. Put $Z = u_1^* Y$, so condition (d) gives $Y \cong (u_1)_* Z$ (via the unit associated to $u_1^* \dashv (u_1)_*$). It will suffice to show that $Z \cong (u_2)_* u_2^* Z$ (via the unit for $u_2^* \dashv (u_2)_*$). Note that $Z \in \mathcal{P}(\mathbb{Q})$ by condition (c), and $\mathcal{P}$ is a subderivator, so $(u_2)_* u_2^* Z$ also lies in $\mathcal{P}(\mathbb{Q})$. As $i_\emptyset^* \colon \mathcal{P} \to \mathcal{C}$ is an equivalence, it will suffice to check that the map $i_\emptyset^* Z \to i_\emptyset^*(u_2)_* u_2^* Z$ is an isomorphism. For this, we claim that $i_\emptyset^*(u_2)_* u_2^* Z \cong u_* u^* i_\emptyset^* Z$. This follows from the fact that $i_\emptyset^* \colon \mathcal{C}^{\mathbb{P}} \to \mathcal{C}$ is a morphism of derivators and so it is compatible with u_* and u^*. We must therefore check that the map $i_\emptyset^* Z \to u_* u^* i_\emptyset^* Z$ is an isomorphism. Here $i_\emptyset^* Z = i_\emptyset^* u_1^* Y = i_{u\emptyset}^* Y$, and this lies in $\mathcal{F}(e)$ by condition (e), so the claim follows from the definition of $\mathcal{F}$. $\square$

Definition 4.5.18. Given $U, V \in \mathbb{Q}$ we put

$$U \boxtimes V = (U \times V) \cap \mathbb{M} = \{(A, B) \in \mathbb{P} \times \mathbb{P} \mid A \in U,\ B \in V,\ A \angle B\}.$$

The definition of $U * V$ can then be written as

$$U * V = \{A \cup B \mid (A, B) \in U \boxtimes V\}.$$

Note that $U \boxtimes V$ and $U * V$ can be seen as subposets of $\mathbb{M}$ and $\mathbb{P}$ respectively. We define $\sigma \colon U \boxtimes V \to U * V$ by $\sigma(A, B) = A \cup B$, and note that this is a morphism of posets.

Proposition 4.5.19. *The map $\sigma \colon U \boxtimes V \to U * V$ is homotopy cofinal.*

Proof. If $U \boxtimes V = \emptyset$ the claim becomes trivial, hence we examine the case $U \boxtimes V \neq \emptyset$.

We apply Proposition 3.6.15 after showing that for every $C \in U * V$ the slice category (σ/C) is $\mathcal{D}$-contractible.

If $C = \emptyset$ then $(\sigma/C) = \{(\emptyset, \emptyset)\}$ thus we can assume C is not empty. For any integer i we set $C_{\le i} = \{c \in C : c \le i\}$ and $C_{\ge i} = \{c \in C : c \ge i\}$.

We start by defining the map

$$\phi \colon (\sigma/C) \to (\sigma/C) \quad (A, B) \mapsto (C_{\le \max A}, C_{\ge \min B})$$

where we assume the convention $\min \emptyset = n$ and $\max \emptyset = -1$. This ensures $C_{\le \max A} \angle C_{\ge \min B}$ even in the case one among A and B is empty. It is easy to see that ϕ is well-defined: since $A \subset C$ we have $A \subset C_{\le \max A}$ and being U upward closed we deduce $C_{\le \max A} \in U$, similarly $C_{\ge \min B} \in V$. Also ϕ is a morphism of posets: if $(A, B) \le (A', B')$ then $A \subset A'$ implies $\max A \le \max A'$ which means $C_{\le \max A} \subset C_{\le \max A'}$, similarly $B \subset B'$ implicates $C_{\ge \min B} \subset C_{\ge \min B'}$. Moreover, we have $Id \le \phi$.

Now we observe there exists $k \in \{0, 1, \ldots, n-1\}$ such that $(C_{\le k}, C_{\ge k}) \in U \boxtimes V$: since by the definitions (σ/C) is not empty we can take an element $(A, B) \in (\sigma/C)$, then if $A \neq \emptyset$ we set $k = \max A$. Clearly $A \subset C_{\le k}$ and $B \subset C_{\ge k}$ thus $(C_{\le k}, C_{\ge k}) \in U \boxtimes V$ again because both U and V are upward closed. If $A = \emptyset$ take $k = \min C$ and it is trivial to see this does the trick.

We define the map

$$\begin{aligned} &\lambda_k \colon (\sigma/C) \to (\sigma/C) \\ &\qquad (A, B) \mapsto (\tilde{A}, B) \text{ where } \tilde{A} = \begin{cases} A & \text{if } \max A \le k, \\ C_{\le k} & \text{if } \max A > k. \end{cases} \end{aligned}$$

The idea is to truncate A at k if its maximum is bigger that this value, but since U is not downward closed in general $A_{\le k}$ will not be an element of U, thus we substitute A with $C_{\le k}$ which is in U by assumption.

It is easy to see λ_k is well defined: if $\max A > k$ then clearly $C_{\le k} \angle B$ hence $(C_{\le k}, B) \in (\sigma/C)$. We verify λ_k is a map of posets, that is $A \subset A'$ implies $\tilde{A} \subset \tilde{A}'$. As before we notice that $\max A \le \max A'$, if $\max A' \le k$ then $\tilde{A} = A \subset A' = \tilde{A}'$, if $k < \max A$ then $\tilde{A} = C_{\le k} = \tilde{A}'$. For the case $\max A \le k < \max A'$ we observe that the condition $(A, B) \in (\sigma/C)$ implies $A \subset C$ thus $\tilde{A} = A \subset C_{\le k} = \tilde{A}'$. Moreover we have the inequality $\phi \ge \lambda_k$.

Similarly we can define a map

$$\begin{aligned} &\varrho_k \colon (\sigma/C) \to (\sigma/C) \\ &\qquad (A, B) \mapsto (A, \tilde{B}) \text{ where } \tilde{B} = \begin{cases} B & \text{if } k \le \min B, \\ C_{\ge k} & \text{if } \min B < k. \end{cases} \end{aligned}$$

We can use the same arguments to show this is well defined and it is non-decreasing, also $\phi \geq \varrho_k$ and $\varrho_k \lambda_k = \lambda_k \varrho_k$.

If we apply λ_k to the inequalities $Id \leq \phi \geq \varrho_k$ we get $\lambda_k \leq \lambda_k \phi \geq \lambda_k \varrho_k$, thus in the end we have a chain of inequalities

$$Id \leq \phi \geq \lambda_k \leq \lambda_k \phi \geq \lambda_k \varrho_k.$$

But $(C_{\leq k}, C_{\geq k})$ is the maximal element in the image of the last map, thus by Remark 3.6.6 we conclude the slice is contractible. □

Proposition 4.5.20. *Consider the map $\mu\colon \mathbb{Q} \times \mathbb{Q} \to \mathbb{Q}$ (given by $(U, V) \mapsto U * V$) and the induced morphism $\mu^*\colon \mathcal{C}^{\mathbb{Q}} \to \mathcal{C}^{\mathbb{Q}\times\mathbb{Q}}$. This restricts to give an equivalence $\mu^*\colon \mathcal{F} \to \mathcal{F}_2$, making the following diagram commute up to natural isomorphism:*

$$\begin{tikzcd}
\mathcal{P} \arrow[r, "u_*", "\simeq"'] \arrow[d, "\mathrm{inc}_*\,\sigma^*"', "\simeq"] & \mathcal{F} \arrow[d, "\mu^*", "\simeq"'] \\
\mathcal{P}_2 \arrow[r, "\simeq", "u^2_*"'] & \mathcal{F}_2.
\end{tikzcd}$$

Proof. By the usual reduction via shifts, it will suffice to work with $\mathcal{P}(e)$, $\mathcal{F}(e)$ and so on. Proposition 4.3.16 gives the left hand equivalence, the top and bottom equivalences are given by Proposition 4.5.4 and Proposition 4.5.17 respectively. We claim that for $X \in \mathcal{P}(e)$, there is a natural isomorphism $\mu^* u_* X \simeq u^2_* \,\mathrm{inc}_*\, \sigma^* X$. Assuming this, everything else follows easily by chasing the diagram.

To prove the claim, we apply Proposition 3.6.17 to the square

$$\begin{tikzcd}
\mathbb{M} \arrow[r, "\sigma"] \arrow[d, "u^2 \circ \mathrm{inc}"'] & \mathbb{P} \arrow[d, "u"] \\
\mathbb{Q} \times \mathbb{Q} \arrow[r, "\mu"'] \arrow[ur, phantom, "\overset{Id}{\nearrow}"] & \mathbb{Q},
\end{tikzcd}$$

which commutes by Lemma 4.5.12. This square induces the canonical mate $(Id)_*\colon \mu^* u_* \Rightarrow u^2_* \,\mathrm{inc}_*\, \sigma^*$, and the Proposition tells us that this is an isomorphism provided that the map

$$\sigma_{(U,V)}\colon ((U, V)/(u^2 \circ \mathrm{inc})) \to (\mu(U, V)/u)$$

is homotopy cofinal for all $U, V \in \mathbb{Q}$. By unwinding the definitions, we see that this is just the map $U \boxtimes V \to U * V$ whose cofinality was proved in Proposition 4.5.19. □

Proof of Theorem 4.5.11. By Proposition 4.5.20 we obtain the following commutative diagram:

$$\begin{tikzcd}
\mathcal{F} \arrow[r, "(U*V)^*"] \arrow[d, "j^*", "\simeq"'] \arrow[dr, "\mu^*", "\simeq"'] & \mathcal{C} \\
\mathcal{C} & \mathcal{F}_2. \arrow[l, "\simeq"', "j_2^*"] \arrow[u, "(U,V)^*"']
\end{tikzcd}$$

The anafunctor provided by the upper left corner is θ_{U*V} by definition, and the other given on the lower right corner is $\theta_U \theta_V$ by Corollary 4.5.16. □

Proposition 4.5.10 established the identification $\theta_{vA} \cong \lambda_A$, thus by Theorem 4.5.11 we have for any k-uple of subsets $\mathbb{A} = (A_1, \dots, A_k)$

$$\lambda_{\mathbb{A}} = \lambda_{A_1} \circ \cdots \circ \lambda_{A_k} \cong \theta_{vA_1 * vA_2 * \cdots * vA_k}.$$

But it would be better to give a more intrinsic characterization of the element of $\mathbb{Q}$ presenting such iterated localization.

Definition 4.5.21. Let $\mathbb{A} = (A_1, \dots, A_k)$ an uple of elements of $\mathbb{P}$, a subset $T \in \mathbb{P}$ is called a *thread set* of $\mathbb{A}$ if and only if there exists an increasing sequence $a_1 \leq a_2 \leq \cdots \leq a_k$ such that $a_i \in T \cap A_i$ for all $1 \leq i \leq k$.

We define $T(\mathbb{A})$ to be the collection of all thread sets of $\mathbb{A}$.

Lemma 4.5.22. *For any $\mathbb{A}$ we have $T(\mathbb{A}) \in \mathbb{Q}$. Moreover, we have the equality*

$$T(\mathbb{A}) = vA_1 * \cdots * vA_k.$$

Proof. The claim $T(\mathbb{A}) \in \mathbb{Q}$ is trivial: by definition $T(\mathbb{A}) \subset \mathbb{P}$ and if T is a thread set of $\mathbb{A}$ then clearly any $B \in \mathbb{P}$ containing it contains also an increasing sequence presenting B as a thread set.

The second assertion can be easily be proven by induction. If $k = 1$ clearly $T(A_1) = vA_1$. For $k > 1$ we have to apply the induction hypothesis after showing

$$T(A_1, \dots, A_k) = T(A_1, \dots, A_{k-1}) * vA_k.$$

For $T \in T(\mathbb{A})$ there exits a sequence $a_1 \leq \cdots \leq a_k$ with $a_i \in A_i \cap T$, so we can give a decomposition $T = T_{\leq a_k} \cup T_{\geq a_k}$ which presents T as an element of the set on the right hand side.

Conversely take $T = B \cup C$ with $B \in T(A_1, \dots, A_{k-1})$, $C \in vA_k$ and $B \angle C$. By assumption there exists a sequence $a_1 \leq \cdots \leq a_{k-1}$ with $a_i \in B \cap A_i$, now $B \angle C$ implies $a_{k-1} \leq \max B \leq c$ for any $c \in C \cap A_k$. Therefore $a_1 \leq \cdots \leq a_{k-1} \leq c$ is a sequence establishing $T \in T(\mathbb{A})$. $\square$

Corollary 4.5.23. *Let $\mathbb{A}$ and $\mathbb{B}$ be two uples of subsets, even with a different number of elements. If $T(\mathbb{A}) = T(\mathbb{B})$ then $\lambda_{\mathbb{A}} \cong \lambda_{\mathbb{B}}$.*

Proof. Immediate from Theorem 4.5.11 and the above discussion. $\square$

Lemma 4.5.24. *Let $A, B \in \mathbb{P}$ with $A \subseteq B$, then $vA * vB = vB * vA = vA$.*

Proof. We prove only $vA * vB = vA$, the other equality can be showed with similar arguments.

We first establish $vA \subseteq vA * vA$: if $C \in vA$ by definition there exists $x \in C \cap A$, so $C = C_{\leq x} \cup C_{\geq x}$ giving the inclusion.

It is immediate that if $U, V, W \in \mathbb{Q}$ and $V \subseteq W$ then $U * V \subseteq U * W$ and $V * U \subseteq W * U$, thus $vA \subseteq vA * vA \subseteq vA * vB$ (because $A \subseteq B$ implies $vA \subseteq vB$) and $vA * vB \subseteq vA * u\emptyset = vA$ (because $u\emptyset$ is the minimal element of $\mathbb{Q}$). $\square$

Corollary 4.5.25. *Let $\mathbb{A} = (A_1, \dots, A_k)$ as above and suppose for some l we have $A_l \subseteq A_i$ for all $1 \leq i \leq k$. Then $\lambda_{\mathbb{A}} \cong \lambda_{A_l}$.*

Proof. This is a straightforward consequence of Lemma 4.5.22 and Lemma 4.5.24. $\square$

Remark 4.5.26. Lemma 4.5.24 is just the combinatorial version of the first part of Lemma 1.1.7. For two localizations L_E and L_F the inequality $\langle E \rangle \leq \langle F \rangle$ between the associated Bousfield classes means that being L_F-acyclic implies being L_E-acyclic, therefore $L_E L_F \cong L_F L_E \cong L_E$.

By the notation fixed in Definition 2.1.1, it is clear that for $A \subseteq B$ we have $\langle K(A) \rangle \leq \langle K(B) \rangle$.

Example 4.5.27. We will use Corollary 4.5.23 to compute all the iterated compositions of $\lambda_{\mathbb{A}}$ for $\mathbb{A} = (\{0,1,3\}, \{0,2,3\})$. It is easy to verify that

$$v\{0,2,3\} * v\{0,1,3\} = v\{0,3\},$$

this and Lemma 4.5.24 imply that

$$T(\mathbb{A})^{*k} = v\{0,3\} \quad \forall k \geq 2$$

thus $\lambda_{\mathbb{A}}^k = \lambda_{\{0,3\}}$.

Another application of Corollary 4.5.23 lets us provide an explicit formula for λ_i in the situation of Example 2.1.12.

Proposition 4.5.28. *In the setting of Example 2.1.12 we fix $i \in N$ and define $A = N_{\geq i}$, $B = N_{\leq i}$. Then for any G-equivariant spectrum X we have*

$$\lambda_A(X) = F(\Sigma_+^\infty E\mathcal{F}_i, X) \qquad \lambda_B(X) = \Sigma^\infty \tilde{E}\mathcal{F}_{i+1} \wedge X$$

where $F(-,-)$ denotes the internal hom of the G-equivariant stable homotopy category.

Proof. Recall we showed that for any $0 \leq j \leq n-1$ a G-spectrum Y is $K(j)_*$-acyclic if and only if $\phi^{H_j} Y = 0$.

For the first claim we observe that [32, Prop. 3.3.10] implies a G-spectrum X is $K(A)_*$-acyclic if and only if $\pi_*^{H_j}(X) = 0$ for all $j \geq i$. The localization corresponding to this class of acyclics has been studies extensively in the literature and it is known to have such expression: for a reference see [27, Ch. IV, Prop. 6.4].

We now pass to the second identification. For $i = n-1$ we observe the $K(B)_*$-acyclics are just 0, hence λ_B is trivially the identity functor. So we can assume $i < n-1$, we will be proving directly that $X \to \Sigma^\infty \tilde{E}\mathcal{F}_{i+1} \wedge X$ is a $K(B)_*$-localization.

The fiber of this map is $\Sigma_+^\infty E\mathcal{F}_{i+1} \wedge X$, since $(E\mathcal{F}_{i+1})^{H_j} = \emptyset$ for all $j \leq i$ it is immediate to see this G-spectrum is $K(B)_*$-acyclic.

To conclude we prove $\Sigma^\infty \tilde{E}\mathcal{F}_{i+1} \wedge X$ is $K(B)_*$-local: fix Z a $K(B)_*$-acyclic G-spectrum, we have to verify that

$$[Z, \Sigma^\infty \tilde{E}\mathcal{F}_{i+1} \wedge X] = 0.$$

Since $\Phi^{H_j} Z = 0$ for $0 \leq j \leq i$ the map $\Sigma_+^\infty E\mathcal{F}_{i+1} \wedge Z \to Z$ must be an equivalence by [32, Prop. 3.3.10], moreover $\Sigma_+^\infty E\mathcal{F}_{i+1}$ belongs to the localizing subcategory of $\mathcal{B}$ generated by $\{\Sigma_+^\infty G/H_j : j \geq i+1\}$ (see [27, Ch. IV, §6] and surrounding discussion) so we can reduce to prove

$$[Z \wedge \Sigma_+^\infty G/H_j, \Sigma^\infty \tilde{E}\mathcal{F}_{i+1} \wedge X] = 0$$

for any $j \geq i+1$. Using the self-duality of the suspension spectra of the orbits we have

$$[Z \wedge \Sigma_+^\infty G/H_j, \Sigma^\infty \tilde{E}\mathcal{F}_{i+1} \wedge X] = [Z, \Sigma_+^\infty G/H_j \wedge \Sigma^\infty \tilde{E}\mathcal{F}_{i+1} \wedge X]$$

and we notice that the G-spectrum $\Sigma_+^\infty G/H_j \wedge \Sigma^\infty \tilde{E}\mathcal{F}_{i+1}$ is contractible since all its geometric fixed points are trivial. $\square$

Corollary 4.5.29. *In the setting of Example 2.1.12, for any $i \in N$ and any G-spectrum X we have*

$$\lambda_i(X) = F(\Sigma_+^\infty E\mathcal{F}_i, \Sigma^\infty \tilde{E}\mathcal{F}_{i+1} \wedge X).$$

Proof. Taken A and B as in the notation of Proposition 4.5.28 it is trivial to verify the equality $T(A, B) = u\{i\}$, then Corollary 4.5.23 implies $\lambda_i(X) = \lambda_A\lambda_B(X)$. We have only to use the formulas showed in Proposition 4.5.28 to conclude. □

A question which immediately arises from the above discussion is that if we can express every element of $\mathbb{Q}$ as $T(\mathbb{A})$ for some uple of subsets. In fact we proved that the all the iterated localizations $\lambda_{\mathbb{A}}$ can be expressed as anafunctors θ_U for particular elements of $\mathbb{Q}$, so it comes natural asking if also the converse holds. Unfortunately this is false: we can easily present an explicit element U which is not a collection of thread sets.

Example 4.5.30. Fix $n = 3$ and take $U = \{\{0, 1\}, \{1, 2\}, \{0, 1, 2\}\} \in \mathbb{Q}$. We claim that $U \neq T(\mathbb{A})$ for any $\mathbb{A}$. Indeed, suppose that $U = T(\mathbb{A})$ with $\mathbb{A} = (A_1, \dots, A_k)$ as usual. Then $\{1\} \notin T(\mathbb{A})$, so we can choose m with $1 \notin A_m$. Similarly $\{0\} \notin T(\mathbb{A})$, so $0 \notin A_l$ for some l. On the other hand, we have $\{0, 1\} \in T(\mathbb{A})$, which means that there exists p with $1 < p \leq k$ and $0 \in A_i$ for $i < p$ and $1 \in A_i$ for $i \geq p$. From this it is clear that $p > m$. Similarly, as $\{1, 2\} \in T(A)$ there must exist q with $1 \leq q < k$ and $1 \in A_i$ for $i \leq q$ and $2 \in A_i$ for $i > q$. From this we see that $q < m$ and so $q \leq p - 2$. It follows in turn that $\{0, 2\} \in T(\mathbb{A})$, contradicting our assumption that $T(\mathbb{A}) = U$.

Remark 4.5.31. If we consider Example 4.5.3 we see that in such diagram the object $\theta_U(X)$, for the element U defined in Example 4.5.30, is presented as the homotopy pullback of the span

$$\begin{tikzcd} & \phi_{01}X \arrow[d] \\ \phi_{12}X \arrow[r] & \phi_{012}X. \end{tikzcd}$$

This cannot be reduced to a chromatic fracture square. We currently do not know whether it can be actually be written as iterated localization of X at all.

To conclude this section we discuss if the above formulation provides a concrete upper bound to the number of iterated localizations for a fixed n. That is, can we compute the cardinality of $\mathbb{Q}$?

Since $\mathbb{Q} \subset \mathcal{P}(\mathbb{P})$ we have $|\mathbb{Q}| \leq 2^{2^n}$ but this number obviously amply overestimates such cardinality.

It is known that an upward closed subset U of a finite poset P is uniquely identifiable with an antichain, which is a collection of non-comparable elements of P. The bijection is provided just by taking the minimal elements in U.

In the case of $\mathbb{P}$, the power set of a set with n elements, the question of counting the number of such antichains is a famous problem: this number is usually referred as the *Dedekind number* $M(n)$. This can also be characterized as the cardinality of a free distributive lattice in n generators, or the number of monotone Boolean functions in n variables.

Unfortunately, even this better estimate has exponential growth. Moreover, $M(n)$ has been computed explicitly only for $n \leq 8$. In [23] it is proved that

$$2^{\binom{n}{\lfloor n/2 \rfloor}} \leq M(n) \leq 2^{\left(1+O\left(\frac{\log n}{n}\right)\right)\binom{n}{\lfloor n/2 \rfloor}},$$

where $O\left(\frac{\log n}{n}\right)$ indicates a function in n with the same asymptotic behaviour as $\frac{\log n}{n}$.

Other more accurate bounds exist, but they are more complicated to present. The bottom line is that we cannot expect to determine $|\mathbb{Q}|$ exactly and our approach does not offer any further insight on this problem than the results already existing in the literature.

4.6 Monoidal structures

In the setting of Definition 2.1.1, if the homotopy category $\mathcal{B}$ has a symmetric monoidal structure and the kernels of the homology theories $K(i)_*$ are tensor ideals it is know that the localizations λ_A admit the structure of lax monoidal functors.

We want to show that also our generalized localizations θ_U can be endowed with a lax monoidal structure, spelled out in the appropriate sense for anafunctors.

We refer to [9, §2] for the complete account on the notion of monoidal derivator.

Lemma 4.6.1. *Let $\mathcal{C}, \mathcal{D}$ be two prederivators, then we can define a new prederivator $\mathcal{C} \times \mathcal{D}$ by the following composition of 2-functors*

$$\mathrm{PoSet}^{op} \xrightarrow{\Delta} \mathrm{PoSet}^{op} \times \mathrm{PoSet}^{op} \xrightarrow{\mathcal{C}\times\mathcal{D}} \mathrm{CAT} \times \mathrm{CAT} \xrightarrow{\times} \mathrm{CAT},$$

where $\Delta\colon \mathrm{PoSet}^{op} \to \mathrm{PoSet}^{op} \times \mathrm{PoSet}^{op}$ is the diagonal and $\times\colon \mathrm{CAT} \times \mathrm{CAT} \to \mathrm{CAT}$ the cartesian product of categories.

Moreover, if both $\mathcal{C}$ and $\mathcal{D}$ are derivators so is $\mathcal{C} \times \mathcal{D}$. If $\mathcal{C}$ and $\mathcal{D}$ are both strong or stable so is $\mathcal{C} \times \mathcal{D}$.

Proof. See [9, Lemma 1.2]. □

This means that the usual cartesian product of categories endows PDER, the 2-category of pre-derivators, with the structure of symmetric monoidal 2-category. Its unit consists of $y(e)$, the derivator represented by the category e, and the braiding isomorphism τ is induced by the one on CAT exchanging positions in the product of two categories.

This allows us to provide the following easy definition

Definition 4.6.2. A (symmetric) monoidal prederivator is a (symmetric) monoidal object of the monoidal 2-category $(\mathrm{PDER}, \times, y(e))$.

While conceptually neat, it is better to unravel this definition to have a better understanding of the notion.

A monoidal prederivator consists in a 6-uple of objects $(\mathcal{C}, \otimes, \mathbb{1}, a, l, r)$ where $\mathcal{C}$ is a prederivator, $\otimes\colon \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ and $\mathbb{1}\colon y(e) \to \mathcal{C}$ are two morphisms of derivators, and finally a, l, r are three invertible modifications as in the following diagrams

$$\begin{tikzcd} y(e)\times\mathcal{C} \arrow[r, "\mathbb{1}\times\mathcal{C}"] \arrow[dr, bend right, "\cong"'] \arrow[r, phantom, shift right=6, "\swarrow l"] & \mathcal{C}\times\mathcal{C} \arrow[d, "\otimes"] \arrow[r, phantom, shift right=6, "\searrow r"] & \mathcal{C}\times y(e) \arrow[l, "\mathcal{C}\times\mathbb{1}"'] \arrow[dl, bend left, "\cong"] \\ & \mathcal{C} & \end{tikzcd} \qquad \begin{tikzcd} \mathcal{C}\times\mathcal{C}\times\mathcal{C} \arrow[r, "\otimes\times\mathcal{C}"] \arrow[d, "\mathcal{C}\times\otimes"'] \arrow[dr, phantom, "\swarrow a"] & \mathcal{C}\times\mathcal{C} \arrow[d, "\otimes"] \\ \mathcal{C}\times\mathcal{C} \arrow[r, "\otimes"] & \mathcal{C}. \end{tikzcd}$$

These modifications are required to satisfy the usual pentagon and triangle identity. Moreover, $\mathcal{C}$ is symmetric if also we have an invertible 2-cell b fitting in the diagram

$$\begin{tikzcd} \mathcal{C}\times\mathcal{C} \arrow[rr, "\tau"] \arrow[dr, "\otimes"'] \arrow[rr, phantom, shift right=5, "\xRightarrow{b}"] & & \mathcal{C}\times\mathcal{C} \arrow[dl, "\otimes"] \\ & \mathcal{C} & \end{tikzcd}$$

and satisfying the appropriate coherence conditions, we refer to [6, Def. 13] for the explicit spelling.

By specifying these data at any finite poset R, we easily deduce $\mathcal{C}(R)$ is a (symmetric) monoidal category with tensor product $\otimes_R$, unit $\mathbb{1}_R$ and the required associator and units given by a_R, l_R and r_R.

Since $\otimes$ and $\mathbb{1}$ are morphism of prederivators, for every $u\colon R \to T$ we have associated compatibility isomorphism $\gamma_\otimes^u$ and $\gamma_{\mathbb{1}}^u$ which endow u^* with the structure of strong (symmetric) monoidal functor. The coherence conditions required by monoidal functors are obtained after unravelling the fact that a, l, r are modifications.

Therefore, $\mathcal{C}$ factors as $\mathcal{C}\colon \mathrm{PoSet}^{op} \to \mathrm{MonCAT}$, where the target is the 2-category of monoidal categories. Similarly in the symmetric case $\mathcal{C}$ lands in sMonCAT, the 2-category of symmetric monoidal categories.

In the case of derivators, we want also to ensure the left Kan extensions to be (strong) monoidal functors if possible. This is why in [9, Def. 2.4] we require $\otimes$ to preserve homotopy left Kan extensions in both variables: this implies $u_!$ admits a canonical (i.e. arising from the appropriate mates) structure of strong monoidal functor.

We do not require $\otimes$ to commute with right Kan extensions simply because usually this is not the case. In a lot of situations we have that the tensor product on a category $\otimes\colon \mathcal{M} \times \mathcal{M} \to \mathcal{M}$ is a left adjoint of two variables: that is we have two functors

$$\mathrm{hom}_l, \mathrm{hom}_r\colon \mathcal{M}^{op} \times \mathcal{M} \to \mathcal{M}$$

called respectively the left and right internal hom functors, which come with compatible families of isomorphisms

$$\mathrm{Hom}_{\mathcal{M}}(X \otimes Y, Z) \cong \mathrm{Hom}_{\mathcal{M}}(Y, \mathrm{hom}_l(X, Z)) \cong \mathrm{Hom}_{\mathcal{M}}(X, \mathrm{hom}_r(Y, Z))$$

for any $X, Y, Z \in \mathcal{M}$, providing us with adjunctions $X \otimes - \dashv \mathrm{hom}_l(X, -)$ and $- \otimes Y \dashv \mathrm{hom}_r(Y, -)$. Thus, we should expect the tensor product to preserve left Kan extensions in both variables, but we have no indication it should do the same for right Kan extensions in general.

In the case of a derivator $\mathcal{C}$, if for every R the tensor product $\otimes_R$ on $\mathcal{C}(R)$ is a left adjoint of two variables there is a canonical way to collect the two right adjoints $\mathrm{hom}_l(-,-)$ and $\mathrm{hom}_r(-,-)$ into lax bimorphisms of derivators. In this manner $\otimes$ preserving left Kan extensions corresponds to some of the natural transformations $\gamma^{\mathrm{hom}_l}, \gamma^{\mathrm{hom}_r}$ to be invertible (see [9, Lemma 1.9], which is an adaptation of Lemma 3.9.4 to bimorphisms). This justifies the following definition.

Definition 4.6.3. A monoidal derivator $\mathcal{C}$ is *biclosed* if its tensor product $\otimes$ is a left adjoint of two variables. If additionally $\mathcal{C}$ is symmetric then the braiding isomorphism let us identify the two right adjoints of the tensor product $\otimes$, so we call $\mathcal{C}$ simply *closed.*

As we could expect, (symmetric) monoidal model categories induce a monoidal structure on the associated homotopy derivator.

Theorem 4.6.4. *Let $\mathcal{M}$ be a combinatorial monoidal model category, then the associated homotopy derivator $\mathcal{H}_{\mathcal{M}}$ admits the structure of biclosed derivator. Furthermore, if $\mathcal{M}$ is symmetric then $\mathcal{H}_{\mathcal{M}}$ is closed.*

Therefore, for the remainder of this section we suppose that the geometric model $\mathcal{B}_0$ invoked in Definition 2.1.1 is a combinatorial (symmetric) monoidal model category.

This is not actually too restricting: we observe that in the first three examples presented (Examples 2.1.9, 2.1.10 and 2.1.11) the chosen model $\mathcal{B}_0$ is in this form. For Example 2.1.12 the model category of G-orthogonal spectra we proposed is not combinatorial, nevertheless it is Quillen equivalent to the flat model structure on the category of symmetric G-spectra ([15, Thm. 7.5]) which has this property.

We can finally start with the preliminary results.

Lemma 4.6.5. *Let R be a finite poset, take $X, Y \in \mathcal{C}(\mathbb{Q} \times R)$ such that Y is u-cartesian and the associated restrictions u^*X, u^*Y satisfy the assumptions of Lemma 4.3.8. Then we have a bijection*

$$j^*\colon \mathcal{C}(\mathbb{Q} \times R)(X, Y) \to \mathcal{C}(R)(j^*X, j^*Y).$$

Proof. Since Y is u-cartesian by Lemma 3.4.1 the unit morphism $Y \to u_*u^*Y$ is invertible. Thus composing along it induces an isomorphism

$$\mathcal{C}(\mathbb{Q} \times R)(X, Y) \cong \mathcal{C}(\mathbb{Q} \times R)(X, u_*u^*Y)$$

now the target is isomorphic to $\mathcal{C}(\mathbb{P} \times R)(u^*X, u^*Y)$ and the composition these two bijections coincides with

$$u^*\colon \mathcal{C}(\mathbb{Q} \times R)(X, Y) \to \mathcal{C}(\mathbb{P} \times R)(u^*X, u^*Y).$$

Finally apply Lemma 4.3.8 to get the desired isomorphism

$$\emptyset^*u^* = j^*\colon \mathcal{C}(\mathbb{Q} \times R)(X, Y) \to \mathcal{C}(R)(j^*X, j^*Y).$$

□

Lemma 4.6.6. *Suppose that $X, Y, Z \in \mathcal{F}(R)$. Then the natural map*

$$\alpha_R\colon \mathcal{C}(\mathbb{Q} \times R)(X \otimes Y, Z) \to \mathcal{C}(R)(j^*X \otimes j^*Y, j^*Z)$$

is a bijection. Moreover, this is compatible with the restrictions.

Proof. We start by observing that in general $X \otimes Y$ will not be an object of $\mathcal{F}(R)$ any more (since the tensor product of two $K(i)_*$-local objects is not necessarily $K(i)_*$-local), but it still retains the property that the induced maps $u^*(X \otimes Y)_A \to u^*(X \otimes Y)_{tA}$ are $K(t)_*$-equivalences (where $A \in \mathbb{P}$ with $t < a$ for any $a \in A$) by our assumption that the localizing categories of $K(t)_*$-acyclics are tensor ideals.

Lemma 4.6.5 gives a bijection

$$j^*\colon \mathcal{C}(\mathbb{Q} \times R)(X \otimes Y, Z) \to \mathcal{C}(R)(j^*(X \otimes Y), j^*Z)$$

and precomposing with the isomorphism

$$(\gamma_j^{\otimes})^{-1}\colon j^*X \otimes j^*Y \cong j^*(X \otimes Y)$$

coming from the fact that j^* is a strong monoidal functor we obtain the desired α_R.

We now spell out explicitly the compatibility with restrictions along $u\colon T \to R$: we claim the following diagram commutes

$$\begin{tikzcd}
\mathcal{C}(\mathbb{Q} \times R)(X \otimes Y, Z) \arrow[r, "j^*"] \arrow[d, "u^*"'] & \mathcal{C}(R)(j^*(X \otimes Y), j^*Z) \arrow[r, "-\circ(\gamma_j^{\otimes})^{-1}"] & \mathcal{C}(R)(j^*X \otimes j^*Y, j^*Z) \arrow[d, "u^*"] \\
\mathcal{C}(\mathbb{Q} \times T)(u^*(X \otimes Y), u^*Z) \arrow[d, "-\circ(\gamma_u^{\otimes})^{-1}"'] & & \mathcal{C}(T)(u^*(j^*X \otimes j^*Y), u^*j^*Z) \arrow[d, "-\circ(\gamma_u^{\otimes})^{-1}"] \\
\mathcal{C}(\mathbb{Q} \times T)(u^*X \otimes u^*Y, u^*Z) \arrow[r, "j^*"] & \mathcal{C}(T)(j^*(u^*X \otimes u^*Y), j^*u^*Z) \arrow[r, "-\circ(\gamma_j^{\otimes})^{-1}"] & \mathcal{C}(T)(j^*u^*X \otimes j^*u^*Y, j^*u^*Z)
\end{tikzcd}$$

where the two horizontal compositions coincide with α_R and α_T. Recall that for any morphism of derivators F the compatibility conditions on the natural transformations γ^F require for any two composable morphisms of posets $I \xrightarrow{a} J \xrightarrow{b} K$ the equality

$$\gamma^F_{ba} = (a^*b^*F_K \xrightarrow{\gamma^F_b} a^*F_Jb^* \xrightarrow{\gamma^F_a} F_Ia^*b^*)$$

must hold. Specifying this for $F = \otimes$ and j, u and considering these two maps commute appropriately the claim follows trivially. $\square$

Proposition 4.6.7. *Define $\mathcal{E}$ to be the bipullback derivator of the span*

$$\begin{tikzcd} & & \mathcal{F} \arrow[d, "j^*"] \\ \mathcal{F}\times\mathcal{F} \arrow[r, "j^*\times j^*"] & \mathcal{C}\times\mathcal{C} \arrow[r, "\otimes"] & \mathcal{C}. \end{tikzcd}$$

Let $\mathcal{G}$ be the prederivator determined by

$$\mathcal{G}(R) = \{(X,Y,Z,\psi) : X,Y,Z \in \mathcal{F}(R),\ \psi\colon X\otimes Y \to Z \text{ s.t. } \alpha_R(\psi) \text{ is invertible}\}$$

for any R finite poset and for any $u\colon T\to R$ the associated restriction is given by

$$u^*\colon \mathcal{G}(R)\to\mathcal{G}(T) \qquad (X,Y,Z,\psi)\mapsto(u^*X,u^*Y,u^*Z,\psi_u)$$

where the morphism ψ_u is the composition

$$u^*X\otimes u^*Y \xrightarrow{(\gamma^{\otimes}_u)^{-1}} u^*(X\otimes Y) \xrightarrow{u^*\psi} u^*Z.$$

Then the bijection α_R induces an equivalence of derivators $F\colon\mathcal{G}\to\mathcal{E}$

$$F_R\colon \mathcal{G}(R)\to\mathcal{E}(R) \qquad (X,Y,Z,\psi)\mapsto(X,Y,Z,\alpha_R(\psi)).$$

Proof. Observe the compatibility of α with restrictions proved in Lemma 4.6.6 implies for any $u\colon T\to R$ we have $\alpha_T(\psi_u) = (\alpha_R\psi)_u$, ensuring that the restriction for $\mathcal{G}$ is well defined: that is $\alpha_T(\psi_u)$ is a bijection.

We now claim F is a strict morphisms of prederivators, i.e. it commutes with restrictions. This is equivalent to the statement $\alpha_R(\psi_u) = \alpha_T(\psi)_u$ (for the restrictions with respect to $\mathcal{G}$ and $\mathcal{E}$ respectively) which can be reduced to $\gamma^{\otimes}_{ju} = \gamma^{\otimes}_{uj}$. More explicitly, recalling the description of u^* for the bipullback $\mathcal{E}$ we have $(\alpha_T\psi)_u$ and $\alpha_R(\psi_u)$ are given by the two rows in the diagram

$$\begin{tikzcd} u^*j^*X\otimes u^*j^*Y \arrow[r, "(\gamma^{\otimes}_u)^{-1}"] \arrow[d, equal] & u^*(j^*X\otimes j^*Y) \arrow[r, "u^*(\gamma^{\otimes}_j)^{-1}"] & u^*j^*(X\otimes Y) \arrow[r, "u^*j^*\psi"] \arrow[d, equal] & u^*j^*Z \arrow[d, equal] \\ j^*u^*X\otimes j^*u^*Y \arrow[r, "(\gamma^{\otimes}_j)^{-1}"] & j^*(u^*X\otimes u^*Y) \arrow[r, "j^*(\gamma^{\otimes}_u)^{-1}"] & j^*u^*(X\otimes Y) \arrow[r, "j^*u^*\psi"] & j^*u^*Z \end{tikzcd}$$

and the left square commutes by $\gamma^{\otimes}_{ju} = \gamma^{\otimes}_{uj}$.

It is trivial to see F is an equivalence, hence by Lemma 3.9.8 also $\mathcal{G}$ is a derivator. $\square$

Proposition 4.6.8. *For any $U \in \mathbb{Q}$ there exist 2-cells $m_U\colon \otimes\circ\theta_U\times\theta_U \Rightarrow \theta_U\circ\otimes$ and $u_U\colon \mathbb{1}\Rightarrow\theta_U\circ\mathbb{1}$ endowing θ_U with the structure of lax monoidal functor. Moreover, these natural transformations are compatible in the sense that for $U\le V$ in $\mathbb{Q}$ the appropriate diagrams involving m_U, u_U, m_V, u_V and $\theta_U\Rightarrow\theta_V$ commute.*

Proof. We first provide explicitly m_U: recall this must be a 2-cell between the anafunctors

$$F\colon \mathcal{C}\times\mathcal{C} \xleftarrow{j^*\times j^*} \mathcal{F}\times\mathcal{F} \xrightarrow{U^*\times U^*} \mathcal{C}\times\mathcal{C} \xrightarrow{\otimes} \mathcal{C}$$
$$G\colon \mathcal{C}\times\mathcal{C} \xrightarrow{\otimes} \mathcal{C} \xleftarrow{j^*} \mathcal{F} \xrightarrow{U^*} \mathcal{C}.$$

We have an invertible natural transformation $\gamma_U^{\otimes}\colon U^*\circ\otimes \Rightarrow \otimes\circ U^*\times U^*\colon \mathcal{C}^{\mathbb{Q}}\times\mathcal{C}^{\mathbb{Q}} \to \mathcal{C}$ which restricted to $\mathcal{F}\times\mathcal{F}\subset\mathcal{C}^{\mathbb{Q}}\times\mathcal{C}^{\mathbb{Q}}$ allows us to rewrite F as

$$F\colon \mathcal{C}\times\mathcal{C} \xleftarrow{j^*\times j^*} \mathcal{F}\times\mathcal{F} \xrightarrow{\otimes} \mathcal{C}^{\mathbb{Q}} \xrightarrow{U^*} \mathcal{C}.$$

By the definition of the composition of anafunctors, G coincides with the fraction induced by the following bipullback

$$\begin{tikzcd}
\mathcal{P} \arrow[r] \arrow[dd] & \mathcal{F} \arrow[d, hook] & \\
 & \mathcal{C}^{\mathbb{Q}} \arrow[r, "U^*"] \arrow[d, "j^*"] \arrow[ul, phantom, "\cong"] & \mathcal{C} \\
\mathcal{C}\times\mathcal{C} \arrow[r, "\otimes"] & \mathcal{C} &
\end{tikzcd}$$

if we precompose $\otimes\colon\mathcal{C}\times\mathcal{C}\to\mathcal{C}$ along $j^*\times j^*$ we obtain

$$\begin{tikzcd}
\mathcal{E} \arrow[r] \arrow[dd] & \mathcal{P} \arrow[r] \arrow[dd] & \mathcal{F} \arrow[d, hook] & \\
 & & \mathcal{C}^{\mathbb{Q}} \arrow[r, "U^*"] \arrow[d, "j^*"] & \mathcal{C}. \\
\mathcal{F}\times\mathcal{F} \arrow[r, "j^*\times j^*"] \arrow[ur, phantom, "\Nearrow"] & \mathcal{C}\times\mathcal{C} \arrow[r, "\otimes"] \arrow[ur, phantom, "\cong"] & \mathcal{C} &
\end{tikzcd}$$

By the pasting law for bipullbacks the left square is also a bipullback. Since $j^*\times j^*$ is an equivalence the induced morphism $\mathcal{E}\to\mathcal{P}$ is an equivalence of derivators as well.

Now Proposition 4.6.7 implies we can substitute $\mathcal{E}$ with $\mathcal{G}$. Moreover, we can extend this diagram of derivators as follows

$$\begin{tikzcd}
\mathcal{G} \arrow[r, "b"] \arrow[d, "a"'] & \mathcal{F} \arrow[r, "U^*"] \arrow[d, hook] & \mathcal{C} \arrow[d, equal] \\
\mathcal{F}\times\mathcal{F} \arrow[r, "\otimes"] \arrow[d, "j^*\times j^*"'] \arrow[ur, phantom, "\mu"] & \mathcal{C}^{\mathbb{Q}} \arrow[r, "U^*"] \arrow[d, "j^*"] & \mathcal{C} \\
\mathcal{C}\times\mathcal{C} \arrow[r, "\otimes"'] \arrow[ur, phantom, "\cong\ (\gamma_j^{\otimes})^{-1}"] & \mathcal{C} &
\end{tikzcd}$$

where for a generic poset R the functors a_R and b_R are given by the formulas

$$a_R\colon \mathcal{G}(R)\to\mathcal{F}(R)\times\mathcal{F}(R) \qquad (X,Y,Z,\psi)\mapsto(X,Y)$$
$$b_R\colon \mathcal{G}(R)\to\mathcal{F}(R) \qquad (X,Y,Z,\psi)\mapsto Z$$

and the natural transformation μ_R is provided by $(X,Y,Z,\psi)\mapsto\psi$.

This induces the 2-cell $m_U\colon F\Rightarrow G$: we write down it explicitly via the diagram

$$
\begin{tikzcd}
& \mathcal{F}\times\mathcal{F} \arrow[dl, "j^*\times j^*"'] \arrow[drr, bend left, "\otimes"] & & & \\
\mathcal{C}\times\mathcal{C} & \mathcal{G} \arrow[l] \arrow[u, "a"'] \arrow[d, equal] \arrow[r, phantom, "\Downarrow\mu"] & & \mathcal{C}^{\mathbb{Q}} \arrow[r, "U^*"] & \mathcal{C}. \\
& \mathcal{G} \arrow[ul, "j^*\times j^*\circ a"] \arrow[r, "b"] & \mathcal{F} \arrow[ur, hook] & &
\end{tikzcd}
$$

We can form the 2-cell u_U in a similar fashion: first we observe that the composite $\theta_U \circ \mathbb{1}$ consists in the fraction obtained from the pibullback

$$
\begin{tikzcd}
\mathcal{I} \arrow[r] \arrow[d] \arrow[dr, phantom, "\cong\Nearrow"] & \mathcal{F} \arrow[r, "U^*"] \arrow[d, "j^*"] & \mathcal{C} \\
y(e) \arrow[r, "\mathbb{1}"] & \mathcal{C} &
\end{tikzcd}
$$

where for any R finite poset

$$\mathcal{I}(R) = \{(*, Y, \varrho) : * \in y(e)(R),\ Y \in \mathcal{F}(R),\ \varrho\colon \mathbb{1}_R(*) \xrightarrow{\cong} j^*Y\}.$$

Lemma 4.6.5 provides a bijection

$$\beta_R\colon \mathcal{C}(\mathbb{Q}\times R)(\mathbb{1}_{\mathbb{Q}\times R}(*), Y) \xrightarrow{j^*_R} \mathcal{C}(R)(j^*\mathbb{1}_{\mathbb{Q}\times R}(*), j^*Y) \cong \mathcal{C}(R)(\mathbb{1}_R(*), j^*Y)$$

which we can use as in Proposition 4.6.7 to define an equivalence between $\mathcal{I}$ and another derivator $\mathcal{J}$ given by

$$\mathcal{J}(R) = \{(*, Y, \sigma) : * \in y(e)(R),\ Y \in \mathcal{F}(R),\ \sigma\colon \mathbb{1}_{\mathbb{Q}\times R}(*) \to Y \text{ s.t. } \beta_R(\sigma) \text{ is invertible}\}.$$

This new model for the bipullback lets us provide a new diagram as follows

$$
\begin{tikzcd}
\mathcal{J} \arrow[r, "d"] \arrow[d, "c"'] \arrow[dr, phantom, "\varepsilon\Nearrow"] & \mathcal{F} \arrow[d, hook] & \\
y(e)^{\mathbb{Q}} \arrow[r, "\mathbb{1}^{\mathbb{Q}}"] \arrow[d, "j^*"'] \arrow[dr, phantom, "\cong\Nearrow"] & \mathcal{C}^{\mathbb{Q}} \arrow[r, "U^*"] \arrow[d, "j^*"] & \mathcal{C} \arrow[dl, equal] \\
y(e) \arrow[r, "\mathbb{1}"] & \mathcal{C} &
\end{tikzcd}
$$

with $(\varepsilon_R)_{(*,Y,\sigma)} = \sigma$, furthermore we added the transformation $j^* \Rightarrow U^*$ arising from the inequality $u\emptyset \leq U$ in $\mathbb{Q}$.

From this we see ε induces a 2-cell $u_U\colon \mathbb{1} \Rightarrow \theta_U \circ \mathbb{1}$: explicitly we write $\mathbb{1}$ as the fraction

$$y(e) \xleftarrow{j^*\circ c} \mathcal{J} \xrightarrow{j^*\circ c} y(e) \xrightarrow{\mathbb{1}} \mathcal{C}$$

which is isomorphic via $\gamma_j^{\mathbb{1}}$ to

$$y(e) \xleftarrow{j^*\circ c} \mathcal{J} \xrightarrow{c} y(e)^{\mathbb{Q}} \xrightarrow{\mathbb{1}^{\mathbb{Q}}} \mathcal{C}^{\mathbb{Q}} \xrightarrow{j^*} \mathcal{C}$$

and now form the diagram

$$
\begin{tikzcd}
 & \mathcal{J} \arrow[dl, "j^*c"'] \arrow[dd, equal] \arrow[r, "c"] & y(e)^{\mathbb{Q}} \arrow[dr, "\mathbb{1}^{\mathbb{Q}}"] \arrow[dd, phantom, "\varepsilon\Downarrow"] & & \\
y(e) & & & \mathcal{C}^{\mathbb{Q}} \arrow[r, bend left, "j^*"] \arrow[r, bend right, "U^*"'] \arrow[r, phantom, "\Downarrow"] & \mathcal{C}. \\
 & \mathcal{J} \arrow[ul, "j^*c"] \arrow[r, "d"] & \mathcal{F} \arrow[ur, hook] & &
\end{tikzcd}
$$

Compatibility of m_U and u_U with the inequalities $U \leq V$ is immediate from the construction: we saw these two natural transformations are induced respectively by μ and ε which are modifications between morphisms of derivators landing in $\mathcal{C}^{\mathbb{Q}}$, before even taking the evaluation at $U \in \mathbb{Q}$.

From the construction of μ and ε it is also clear that compatibility of m_U and u_U with the associator a and the two unitors l, r can be checked after applying the isomorphisms induced from j^* by Lemma 4.3.8. At this point the desired equalities between morphisms in the appropriate categories follow from the monoidality of the derivator $\mathcal{C}$. $\square$

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