Value and Nash Equilibrium in Games of Optimal Stopping

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Abstract

We study games of optimal stopping (Dynkin games). A Dynkin game is a mathematical model involving several competing players, each of them interested in capturing the moments when certain stochastic processes are at an extremum. Actions of players are referred to as "stopping" (of an underlying process), and the outcome for every player depends on the stopping decisions of the other players.

Our focus is on Dynkin games with asymmetric information. Asymmetry of information refers to the situation in which different players have different (possibly incomplete) knowledge of the underlying world. Observations of the underlying processes (or of a more general information flow) and of the actions of competitors allow the players to make optimal stopping choices. An important aspect of our framework is a possibility of randomising these choices: for example, in order to avoid revealing private information to competitors.

We develop a general stochastic framework for studying Dynkin games with asymmetric information. In particular, we provide conditions for the existence of the value in such games. Separately, we study issues arising in games with mixed first-mover advantage, in which sometimes it is beneficial for the players to act as soon as possible, and sometimes to wait for another player to act.

Abbreviations

Below we present an incomplete list of the mathematical notation used in the thesis.

$\mathbb{R}$	the set of real numbers $(-\infty, \infty)$
$\mathbb{N}$	the set of natural numbers $\{1, 2, \ldots\}$
$\mathbb{P}$	a probability measure
$\mathbb{E}$	the mathematical expectation (with respect to a probability measure)
$\mathcal{F}$, $\mathcal{G}$	a sigma-algebra
$\sigma(X)$	the sigma-algebra generated by a random variable X
$\mathcal{B}(A)$	the Borel sigma-algebra on a set A
$\mathcal{T}$	a set of stopping times (with respect to a sigma-algebra)
λ	the Lebesgue measure
I_A	the indicator function of a set (event) A
$a \vee b$	maximum of a and b
$a \wedge b$	minimum of a and b
$\langle a, b \rangle$	scalar product of vectors a and b
a^T	transpose of a matrix a

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Chapter 1

Introduction

Optimal stopping is a stochastic problem of the following kind. Suppose an actor observes a stochastic process and wishes to capture the moment when the process is at its maximum (or minimum). Possible tasks include characterising this time in terms of parameters of the process, or estimating the expected value of the process at that time. This results in a problem of a form

$$\sup_{\tau \in \mathcal{T}} \mathbb{E} X_\tau,$$

where $\mathcal{T}$ is a set of *stopping times*, (X_t) is a stochastic process, and $\mathbb{E}$ denotes the mathematical expectation.

A *Dynkin game* (or a *game of optimal stopping*, or an *optimal stopping game*) is an extension of an optimal stopping problem that allows for multiple actors (or *players*). Such games originate from Dynkin (1969). A variation of the Dynkin game appeared shortly after in the textbook Neveu (1975) and in the following years received attention of researchers worldwide Bismut (1977b), Stettner (1982a), Stettner (1982b), Lepeltier & Maingueneau (1984), Yasuda (1985). More recently, Dynkin games have gained popularity due to their financial applications, particularly *game options* Kifer (2000) and *real options* Steg & Thijssen (2015), De Angelis & Ekström (2020). We provide a review of the most relevant literature in Chapter 3.

Dynkin games appear in the literature in numerous variations — as discrete-time games or continuous-time games, in a zero-sum or a nonzero-sum framework, as two-player or multiple-player games, under different assumptions on the set of stopping times and on the underlying processes. We now outline a framework that is close to our research. Consider two stochastic processes (f_t), (g_t). Let there be two players, and denote by τ and σ the stopping times used by the first and the second player, respectively. The game ends at time $\tau \wedge \sigma := \min\{\tau, \sigma\}$ with the first player delivering to the second player the random *payoff*

$$\mathcal{P}(\tau,\sigma) = f_\tau I_{\{\tau\le\sigma\}} + g_\sigma I_{\{\sigma<\tau\}},$$

where $I_{\{\tau\le\sigma\}}$ and $I_{\{\sigma<\tau\}}$ denote the indicators of the events $\{\tau\le\sigma\}$ and $\{\sigma<\tau\}$. Since the first player (the τ-*player*) is the one who pays the amount $\mathcal{P}$, she is referred to as the *minimiser* in the game. On the other hand, the second player (the σ-*player*) is the one who receives $\mathcal{P}$, and hence is the *maximiser*.

Let us mention a fundamental concept from game theory. A couple (τ^*,σ^*) is called a *Nash equilibrium*, if for any other couple (τ,σ) the following holds:

$$\mathbb{E}[\mathcal{P}(\tau^*,\sigma)] \le \mathbb{E}[\mathcal{P}(\tau^*,\sigma^*)] \le \mathbb{E}[\mathcal{P}(\tau,\sigma^*)]. \tag{1.1}$$

One way to interpret these inequalities is that neither minimiser nor maximiser can obtain a more desirable expected payoff by deviating from the Nash equilibrium.

Another important concept is the *value* of the game, which is defined as

$$V = \sup_\sigma \inf_\tau \mathbb{E}[\mathcal{P}(\tau,\sigma)] = \inf_\tau \sup_\sigma \mathbb{E}[\mathcal{P}(\tau,\sigma)], \tag{1.2}$$

provided that the second equality holds.

The reader might have noticed that we did not specify over which sets the supremum and infimum are taken in (1.2), and what set τ^*, σ^*, τ, σ in (1.1) belong to. Traditionally, the set of stopping times is the one that appears in these definitions. However, one can also consider the set of *randomised* or *mixed* stopping times (rigorously introduced in Section 2.4). This set is larger than the set of usual (*pure*) stopping times, and therefore, the existence of value and/or of Nash equilibrium can be ensured for a broader class of games. Such relaxation is particularly useful for *games with incomplete/asymmetric information* that we proceed to introduce.

So far, we implicitly assumed that both of the players observe both of the *payoff processes* (f_t), (g_t), fully and without any noise, and their actions τ, σ are based on this observation. In such situations, we say that the game is a *game with full and symmetric information*. However, it is easy to think of situations when the information is incomplete and/or there is no symmetry of information available to the players. For example, the payoff processes may depend on an exogenous random variable which is only known to one of the players (the *informed* player) and could be viewed as a *scenario* in which the game is played, or as *insider information*.

A possible way for randomised stopping times to appear in this framework is the following: they are used by the informed player to gradually reveal the information to the uninformed player. To give an intuition, imagine a game with two possible scenarios: in one of them, it is optimal for the informed player to stop at time 0, in the other one — at time 1. If the game is played in the latter scenario, a "naive" informed player would wait until time 1 to stop. But then, immediately after time 0, the uninformed player learns the true scenario (from the fact that the informed player did not stop at 0). To avoid the information being revealed in such a way, the informed player may want to include randomness in her choice of the stopping time, and to manipulate the beliefs of the uninformed player in an optimal way.

Dynkin games with asymmetric information are the main object of our study. Chapter 4 is devoted to our main results in this area. Particularly, the question of existence of the value in such games is our focus. We specify conditions under which the supremum and infimum in (1.2) can be interchanged. In general, both players in our framework use randomised stopping times.

Traditionally in the study of Dynkin games with asymmetric information Grün (2013), Lempa & Matomäki (2013), Gensbittel & Grün (2019), De Angelis *et al.* (2021b), De Angelis *et al.* (2021a), certain specific assumptions on the structure of information available to the players are imposed. Moreover, the payoff processes (f_t), (g_t) are assumed to be Markovian. In many cases, they are specific deterministic functions of a single underlying Markov process. The advantages of our approach in Chapter 4, compared to results in the literature, are that we allow for a general structure of information in our game, and we do not impose any Markovian assumptions. Chapter 4 is based on the joint paper with my PhD supervisors De Angelis *et al.* (2021c).

Initially, our research was focused on classical Dynkin games (with full and symmetric information). For such games, one of the important conditions for existence of the value used to be the Mokobodzki condition Bismut (1977b), Mokobodzki (1978), Stettner (1982a) on the payoff processes (f_t), (g_t). Mokobodzki condition states that there exist two supermartingales whose difference lies between g and f. However, in Lepeltier & Maingueneau (1984) it was proven that Dynkin games have a value under a weaker condition:

$$f \geq g. \tag{1.3}$$

This condition (that has no scientist's name attached to it, so we call it the *order condition*) is, as we will see, quite natural, and it was adopted in most of the literature.

To the best of our knowledge, the order condition was first challenged in Stettner (1982b), and later studied in a more general set-up in Touzi & Vieille (2002) and Laraki & Solan (2005). In the latter two papers, a relaxed Dynkin game (with randomised stopping times) is considered, and the value is proven to exist *without* the order condition. In Chapter 5, we unify certain results and ideas from the literature in order to show that the value exists under a condition weaker than (1.3), in a setting when players only use pure stopping times.

For the reader's convenience, we clarify the content of the chapters we did not mention so far — Chapter 2, Appendix A, and Appendix B. In the former two, we discuss several general concepts and preliminary facts that are used throughout the thesis. Chapter 2 can be viewed as a mathematical introduction to the main chapters of the thesis, while Appendix A collects the textbook definitions and theorems. Finally, Appendix B is devoted to auxiliary proofs that may be of interest at various stages of reading the thesis.

We conclude the introduction with a discussion of financial applications of Dynkin games. The game options introduced by Kifer (2000) are option contracts which enable *both* the buyer and the seller to stop them at any time before (and including) the time of expiration of the contract. Thus, the game options extend the notion of American options (which only allow the buyer to exercise the option). We also see that this financial situation is, mathematically, precisely a Dynkin game.

A popular real-world instrument embodying the concept of a game option is a *convertible bond*. Such contracts are issued by a *firm*, held by a *bondholder* and typically prescribe the following: the bondholder is entitled to receiving the coupons while she holds the bond, and, further, has the option to exchange (*convert*) the bond for the firm's stocks. The firm, on the other hand, has the right to *call* the bond, in which case the bondholder must surrender the bond for a pre-specified price. Finally, at the maturity of the contract, if neither the conversion nor the call have happened, the bondholder must sell the bond back to the firm, for a pre-specified price or in exchange for the firm's stocks. We see that such a convertible bond can be viewed as a game option, or a Dynkin game between the firm and the bondholder, with specific payoff processes tied to the value of the firm's stock, to the coupon rate, and to the fixed prices agreed in the contract.

The mathematical study of convertible bonds dates back to Brennan & Schwartz (1977), Ingersoll (1977), but it was not until more recently Sirbu *et al.* (2004), Sirbu & Shreve (2006) that researchers began to explore the connection with Dynkin games. For various extensions of the model and properties of the value and the Nash equilibrium in this setting, we refer to Bielecki *et al.* (2008), Crépey & Rahal (2011/2012), Chen *et al.* (2013), Yan *et al.* (2015), Liang & Sun (2019).

Moving on to financial interpretations of the informational features of Dynkin games, certain parameters of the underlying assets may be unknown to market players, giving rise to games with incomplete and/or asymmetric information. For example, in Lempa & Matomäki (2013), the Dynkin game is set on a random time horizon, and the occurence of the expiring event is only observable by one of the two players. From the financial standpoint, this situation corresponds to one of the players having inside information about a default taking place. In De Angelis *et al.* (2021b), the two players are trading a call option on an asset with the drift which is random and only partially observable. De Angelis & Ekström (2020) introduce a Dynkin game with both players not being certain about the existence of the competitor. This may be interpreted as investors not wanting to publicly reveal their interest in a certain business opportunity. Another interesting view on the information is provided in Ekström *et al.* (2017), where the two players have heterogeneous beliefs, i.e. disagree about how to estimate the drift of the underlying asset. This idea is justified by the following example: for the buyer of a call option, it is natural to estimate the drift of the underlying asset higher than the seller's estimation.

Chapter 2

Preliminaries

In this section, we agree on several general concepts that are used throughout the thesis, introduce notation and give a number of preliminary facts we will refer to. Note that the textbook definitions (e.g. of a stopping time) and theorems (e.g. Lebesgue's dominated convergence theorem) are given in Appendix A.

2.1 Probabilistic setting

Unless specified otherwise, random variables and stochastic processes we consider are defined on the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and take values in $\mathbb{R}$. We always assume that the probability space is *complete*, that is, for any set $A \in \mathcal{F}$ of $\mathbb{P}$-measure zero, the sigma-algebra $\mathcal{F}$ includes also all the subsets of A.

When we say "with probability one" or "almost surely", it is implied that the probability measure $\mathbb{P}$ is the one in question. Another synonym to describe the same concept is $\mathbb{P}$-a.s. We write $\mathbb{E}$ for the expectation $\mathbb{E}^{\mathbb{P}}$ with respect to measure $\mathbb{P}$. We work in *continuous time*, and $T \in (0, \infty]$ denotes the *time horizon*.

We call a stochastic process $(X_t)_{t\in[0,T]}$ *càdlàg* (from the French "continu à droite, limites à gauche"), if its trajectories $t \mapsto X_t$ are right-continuous with left limits everywhere, with probability one.

Let $(\mathcal{F}_t)_{t\in[0,T]}$ be a filtration (Definition A.1.1) on $(\Omega, \mathcal{F}, \mathbb{P})$. The set of $(\mathcal{F}_t)$-stopping times (Definition A.1.2) is denoted $\mathcal{T}(\mathcal{F}_t)$. We omit the mention of the filtration and simply write $\mathcal{T}$ whenever it causes no ambiguity.

We consider situations when the time horizon T is infinite, as well as the situations when it is finite. These two types of horizon can be treated similarly, as explained in the following remark.

Remark 2.1.1. *If $T = \infty$, then ∞ is a one-point compactification of $[0,\infty)$ (c.f. (Dellacherie & Meyer, 1982, Remark VI.53e)). There are bijections between any finite interval and $[0,\infty]$, and, intuitively, the properties of stochastic processes should be preserved if such a bijection is applied to the time. In particular, càdlàg processes on $[0,\infty]$ are understood as follows: a process $(X_t)_{t\in[0,\infty]}$ is càdlàg if it is càdlàg on $[0,\infty)$ and the limit $X_{\infty-} := \lim_{t\to\infty} X_t$ exists. When we say that the process $(X_t)_{t\in[0,\infty]}$ is $(\mathcal{F}_t)_{t\in[0,\infty]}$-adapted, it is also assumed that the random variable X_∞ is $\mathcal{F}_\infty$-measurable, where $\mathcal{F}_\infty$ is potentially larger than $\mathcal{F}_{\infty-} := \sigma(\cup_{t\in[0,\infty)}\mathcal{F}_t)$.*

For simplicity of notation, we often denote filtrations as $(\mathcal{F}_t)$ instead of $(\mathcal{F}_t)_{t\in[0,T]}$, and similarly for processes: (X_t) instead of $(X_t)_{t\in[0,T]}$.

2.2 Lebesgue–Stieltjes measure and integral

We will employ integrals with respect to processes with paths of finite variation (Definition A.4.13). Such integrals will be understood in pathwise Lebesgue–Stieltjes sense. We recall the relevant theory in this section.

Let F be a measurable function on a measure space (S,Σ,μ). We denote by $\int_S F(x)d\mu(x)$ the Lebesgue integral of the function F over the set S with respect to the measure μ. We sometimes omit the variable of integration x and write $\int_S F d\mu$.

Let us now focus on functions on the real line. As we mentioned, the results of this section will be applied to paths of stochastic processes. It will be convenient to treat the value of these paths at time zero as a jump from an "initial" value, and to think that the process takes this "initial" value at time "just before zero". This is why some of the functions below are defined on $[0-,\infty)$.

Let $F : [0-,\infty) \mapsto \mathbb{R}$ be a right-continuous function of finite variation. The corresponding *Lebesgue–Stieltjes measure* is denoted by μ^F. Note that, in general, it is a signed measure; it is only a positive measure if the function F is non-decreasing. If the function F has values between 0 and 1, the positive measure μ^F is a probability measure.

Let $G : [0,\infty) \mapsto \mathbb{R}$ be a measurable function, $F : [0-,\infty) \mapsto \mathbb{R}$ a function of finite variation. It is known (Proposition A.4.14) that $F = F^+ - F^-$, where F^+, F^- are non-negative and non-decreasing. For $t \in [0,\infty)$, we define the *Lebesgue–Stieltjes integral* of G on $[0,t]$ as

$$\int_{[0,t]} G(s)dF(s) := \int_{[0,t]} G(s)d\mu^{F^+}(s) - \int_{[0,t]} G(s)d\mu^{F^-}(s),$$

where we assume that the terms on the right-hand side are finite. The Lebesgue–Stieltjes integral on an open or a semi-open interval is defined in the analogous way. We remark that the value of the integral does not depend on the choice of F^+ and F^-.

In our proofs, there are several important building blocks related to Lebesgue–Stieltjes integrals. We state the corresponding propositions here.

Proposition 2.2.1 (Integration by parts)**.** *If $F, G : [0-, \infty) \mapsto \mathbb{R}$ are two right-continuous functions of finite variation such that $F(0-) = G(0-) = 0$, then for every $t \in [0, \infty)$ we have*

$$F(t)G(t) = \int_{[0,t]} F(s)dG(s) + \int_{[0,t]} G(s-)dF(s).$$

Proof. Denote $\Delta F(0) = F(0) - F(0-)$, $\Delta G(0) = G(0) - G(0-)$. Observe that in our case

$$\begin{aligned}
&F(0)G(0) = \big(F(0-) + \Delta F(0)\big)\big(G(0-) + \Delta G(0)\big) = \Delta F(0)\Delta G(0); \\
&\int_{(0,t]} F(s)dG(s) = \int_{[0,t]} F(s)dG(s) - \Delta F(0)\Delta G(0); \\
&\int_{(0,t]} G(s-)dF(s) = \int_{[0,t]} G(s-)dF(s).
\end{aligned} \tag{2.1}$$

Thus,

$$F(t)G(t) = F(0)G(0) + \int_{(0,t]} F(s)dG(s) + \int_{(0,t]} G(s-)dF(s) = \int_{[0,t]} F(s)dG(s) + \int_{[0,t]} G(s-)dF(s),$$

where the first equality is due to the integration by parts formula on the semi-open interval (Revuz & Yor, 1999, Prop. 0.4.5), and the second one is due to (2.1). □

For a non-decreasing right-continuous function $F : [0-, \infty) \mapsto \mathbb{R}$, let us define a function $F^{\leftarrow}$ for $s \in [0, \infty)$ as

$$F^{\leftarrow}(s) = \inf\{t \in [0, \infty) : F(t) > s\},$$

with the convention $\inf \emptyset = \infty$, and $F^{\leftarrow}(0-) = 0$. The function $F^{\leftarrow}$ is called a *generalised inverse* of the function F.

Proposition 2.2.2 (Change of variables)**.** *Let $F : [0-, \infty) \mapsto \mathbb{R}$ be a non-decreasing right-continuous function, and let $F^{\leftarrow}$ be its generalised inverse. Let additionally $F(t) = 1$ for some $t \in [0, \infty)$. Let $H : [0, \infty) \mapsto \mathbb{R}$ be a measurable function. Then,*

$$\int_0^1 H(F^{\leftarrow}(u)) I_{\{F^{\leftarrow}(u) < t\}} du = \int_{[0,t)} H(s)dF(s).$$

Proof. For $s \in [0,\infty)$, define $\widehat{H}(s) = H(s)I_{\{s<t\}}$. Due to the right-continuity of the function F at t, for $u \in [0,1)$, we have $F^{\leftarrow}(u) < t$, and for $u \in [1,\infty)$, we have $F^{\leftarrow}(u) \geq t$. Then,

$$\int_0^\infty \hat{H}(F^{\leftarrow}(u))I_{\{F^{\leftarrow}(u)<\infty\}}du = \int_0^\infty H(F^{\leftarrow}(u))I_{\{F^{\leftarrow}(u)<t\}} = \int_0^1 H(F^{\leftarrow}(u))I_{\{F^{\leftarrow}(u)<t\}},$$

and

$$\int_{[0,\infty)} \hat{H}(s)dF(s) = \int_{[0,\infty)} H(s)I_{\{s<t\}}dF(s) = \int_{[0,t)} H(s)dF(s).$$

Applying (Revuz & Yor, 1999, Prop. 0.4.9) to the function $\widehat{H}$, we obtain the desired equality. □

Remark 2.2.3. *The condition that $F(t) = 1$ for some $t \in [0,\infty)$ is convenient for our purposes below, but it is not a key condition in the change of variables formula. Without it, we could make a similar claim for integrals limiting at infinity:*

$$\int_0^\infty H(F^{\leftarrow}(u))I_{\{F^{\leftarrow}(u)<\infty\}}du = \int_{[0,\infty)} H(s)dF(s),$$

which follows directly from (Revuz & Yor, 1999, Prop. 0.4.9).

2.3 Regular processes and projections

In this section, we focus on properties of stochastic processes in continuous time that play the key role in Chapters 4 and 5, as well as in most literature on optimal stopping (Chapter 3). On our complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, consider a filtration $(\mathcal{F}_t)_{t\in[0,T]}$ (recall that $T \in (0,\infty]$ denotes the time horizon).

We say that the filtration $(\mathcal{F}_t)$ is *right-continuous*, if $\mathcal{F}_t = \mathcal{F}_{t+}$ for all t, where $\mathcal{F}_{t+} := \cap_{s>t}\mathcal{F}_s$. We say that it is *complete* if the probability space is complete (which we always assume) and if the sigma-algebra $\mathcal{F}_0$ contains all sets of $\mathbb{P}$-measure zero. When a filtration is both complete and right-continuous, we say that it satisfies the *usual conditions*. The completeness condition is not restrictive, since, given a filtration $(\mathcal{F}_t)$, one can add to $\mathcal{F}_t$ (for every t) all the $\mathbb{P}$-measure zero sets from $\mathcal{F}$ in order to obtain a complete filtration (this process is called the *usual augmentation*).

By a *measurable process* we mean a stochastic process (X_t) such that the mapping $(t,\omega) \mapsto X_t(\omega)$ is $\mathcal{B}([0,T]) \times \mathcal{F}$-measurable. For a measurable process (X_t), we denote by $(\mathcal{F}_t^X)$ its natural filtration (i.e. the filtration such that $\mathcal{F}_t^X = \sigma(X_t)$ for all $t \in [0,T]$). We assume that $(\mathcal{F}_t^X)$ is augmented in a usual way and therefore complete.

Let $\mathcal{L}_b$ be a Banach space of càdlàg measurable processes (X_t) with the norm

$$\|X\|_{\mathcal{L}_b} := \mathbb{E}\Big[\sup_{t\in[0,T]} |X_t|\Big] < \infty. \tag{2.2}$$

Remark 2.3.1. *In the general literature on stochastic processes (e.g. (Rogers & Williams, 2000, Definition VI.29.5)), a class of processes larger than $\mathcal{L}_b$ is frequently considered: the class (D). A measurable process (X_t) is said to belong to class (D), if the family of random variables*

$$\{X_\eta : \eta \text{ is a finite stopping time}\}$$

is uniformly integrable (i.e. $\lim_{K\to\infty}(\sup_\eta \mathbb{E}[|X_\eta| I_{\{|X_\eta|\geq K\}}]) = 0$). We see that the class (D) is indeed larger than $\mathcal{L}_b$, because for any process $(X_t) \in \mathcal{L}_b$ we have

$$\lim_{K\to\infty}(\sup_\eta \mathbb{E}[I_{\{|X_\eta|\geq K\}}]) \leq \lim_{K\to\infty}(\mathbb{E}(I_{\{\sup_{t\in[0,T]}|X_t|\geq K\}}]) = 0,$$

and therefore (X_t) belongs to class (D). In line with most literature on optimal stopping (see Chapter 3), we choose to focus only on the processes from $\mathcal{L}_b$.

The space $\mathcal{L}_b$ will be extensively used in Chapter 4, as well as the definition below that we take from Meyer (1978). To avoid a terminological confusion, we remark that *previsible* stopping times in the definition below are also known as *predictable* or *announceable* stopping times (see Definition A.1.3).

Definition 2.3.2. *A process $(X_t)_{t\in[0,T]} \in \mathcal{L}_b$ is called* regular, *if*

$$\mathbb{E}[X_\eta - X_{\eta-}|\mathcal{F}_{\eta-}] = 0 \quad \mathbb{P}\text{-a.s. for all previsible } (\mathcal{F}_t)\text{-stopping times } \eta.$$

Example 2.3.3. *It is immediate from Definition 2.3.2 that a continuous process is regular, since for such processes a stronger property $\mathbb{P}(\{\omega : X_t(\omega) - X_{t-}(\omega) = 0 \text{ for all } t \in [0,T]\}) = 1$ holds.*

Example 2.3.4. *A process (X_t) is said to be* quasi left-continuous *(or* left-continuous over stopping times*), if for all stopping times η_n, η such that $\eta_n \nearrow \eta$ as $n \to \infty$ we have*

$$X_{\eta_n} \to X_\eta \ \mathbb{P}\text{-a.s.}$$

Let (X_t) be a quasi left-continuous process and η a previsible stopping time. By the definition above,

$$X_\eta = \lim_{\eta_n \nearrow \eta} X_{\eta_n} = X_{\eta-}, \ \mathbb{P}\text{-a.s.},$$

and therefore $\mathbb{E}[X_\eta - X_{\eta-}|\mathcal{F}_{\eta-}] = 0$, $\mathbb{P}$-a.s. Thus, quasi left-continuous processes are regular.

Example 2.3.5. *Let θ be a random variable with continuous distribution on $[0,\infty)$. Let $\Lambda_t := I_{\{t\geq\theta\}}$, and let $\mathcal{F}_t := \sigma(\Lambda_s, 0 \leq s \leq t)$. By (Rogers & Williams, 2000, Example VI.14.4), θ is a totally inaccessible $(\mathcal{F}_t)$-stopping time.*

From Definition 2.3.2 and Lemma A.1.6 we see that regular processes can only jump in totally inaccessible times. Therefore, processes that only jump at such (continuously distributed) θ are regular.

We will use the concepts of $\mathcal{F}_t$-*optional* and $\mathcal{F}_t$-*previsible projection* of a measurable stochastic process (see Section A.5.1 for the formal definitions). Here, we state the non-standard results that will play an important role in our proofs.

Theorem 2.3.6. *(Bismut, 1978, Theorem 3) Let $(X_t) \in \mathcal{L}_b$ be a càdlàg, $(\mathcal{F}_t)$-adapted, and regular process. Then there exists $(\tilde{X}_t) \in \mathcal{L}_b$ with continuous trajectories (not necessarily $\mathcal{F}_t$-adapted) such that (X_t) is an $(\mathcal{F}_t)$-optional projection of $(\tilde{X}_t)$.*

Theorem 2.3.7. *(Dellacherie & Meyer, 1982, Thm VI.57, Remark VI.58.d) Let $(X_t) \in \mathcal{L}_b$, and let (Y_t) be its $(\mathcal{F}_t)$-optional projection. Let (ρ_t) be a non-decreasing $(\mathcal{F}_t)$-adapted process. Then*

$$\mathbb{E}\left[\int_{[0,T]} X_t d\rho_t\right] = \mathbb{E}\left[\int_{[0,T]} Y_t d\rho_t\right]. \tag{2.3}$$

Following (Dellacherie & Meyer, 1982, Remark VI.50.d), we note that there is an equivalent definition of regularity: a process (X_t) is regular if its previsible projection is indistinguishable from the left-limit process (X_{t-}). In the original (Bismut, 1978, Theorem 3), this equivalent definition of regularity is used. It is also formulated for class (D) processes and thus is applicable, in particular, in our case (see Remark 2.3.1).

Remark 2.3.8. *Throughout, integrals of processes in $\mathcal{L}_b$ that appear in (2.3) are understood in pathwise Lebesgue–Stieltjes sense. Formally, for integrals in (2.3) to be well-defined, we consider $\Omega_X := \{\omega \in \Omega : \int_{[0,T]} |X_t(\omega)| d\rho_t(\omega) < \infty\}$, and similarly for Ω_Y, and take $\omega \in \Omega_X \cap \Omega_Y$. However, by definition (2.2) of the space $\mathcal{L}_b$, we have $\mathbb{P}(\Omega_X) = \mathbb{P}(\Omega_Y) = 1$. Therefore, we usually omit a mention of ω and assume that integrals are finite in calculations (e.g. when applying results of Section 2.2 to integrals of processes).*

2.4 Randomised, mixed, distribution stopping times

In this section, we introduce several closely related concepts that extend the notion of a stopping time. They will be important for studying optimal stopping problems and games, particularly the games with asymmetric information (see Section 3.4 for the literature review, and Chapter 4 for the main asymmetric information problem of the thesis).

Fix a filtration $(\mathcal{F}_t)_{t \in [0,T]}$ that satisfies the usual conditions. Recall that $\mathcal{T}$ denotes the set of $(\mathcal{F}_t)$-stopping times, and λ denotes the Lebesgue measure. By a *mixed stopping time*, we mean a measurable function $\mu : \Omega \times [0,1] \mapsto [0,T]$ such that for λ-almost every $r \in [0,1]$, the mapping $\omega \mapsto \mu(\omega, r)$ belongs to $\mathcal{T}$.

It is clear that any "usual" (*pure*) stopping time $\tau \in \mathcal{T}$ is a mixed stopping time. To be more precise, to any $\tau \in \mathcal{T}$, there corresponds a mixed stopping time defined as $\mu(\omega, r) := \tau(\omega)$ for any $r \in [0,1]$.

Now we introduce another concept that, as we will see below, is equivalent to the concept of a mixed stopping time in a suitable sense. For a filtration $(\mathcal{G}_t) \subseteq (\mathcal{F}_t)$, define the following set of processes:

$$\begin{aligned} \mathcal{A}^\circ(\mathcal{G}_t) := \{(\rho_t) \,:\, (\rho_t) \text{ is } (\mathcal{G}_t)\text{-adapted with } t \mapsto \rho_t(\omega) \text{ càdlàg}, \\ \text{non-decreasing, } \rho_{0-}(\omega) = 0 \text{ and } \rho_T(\omega) = 1 \text{ for all } \omega \in \Omega\}, \end{aligned} \tag{2.4}$$

where we adopt a convention that $\mathcal{G}_{0-}$ is the trivial sigma-algebra.

Remark 2.4.1. *In the definition of the set $\mathcal{A}^\circ(\mathcal{G}_t)$, we take the opportunity to require that the stated properties hold for all $\omega \in \Omega$. This leads to no loss of generality, if $\mathcal{G}_0$ contains all $\mathbb{P}$-null sets of Ω (in particular, if $(\mathcal{G}_t)$ satisfies the usual conditions). In this case, for any $\omega \in \mathcal{N} \subset \Omega$ with $\mathbb{P}(\mathcal{N}) = 0$, we can simply set $\rho_t(\omega) = 0$ for $t \in [0,T)$ and $\rho_T(\omega) = 1$.*

Let Z be a random variable with uniform distribution $U([0,1])$, independent of $\mathcal{F}_T$, and let the process $(\rho_t) \in \mathcal{A}^\circ(\mathcal{G}_t)$. A random variable η defined as

$$\eta = \eta(\rho, Z) = \inf\{t \in [0,T] : \rho_t > Z\}, \quad \mathbb{P}\text{-a.s.}$$

is called a $(\mathcal{G}_t)$-*randomised stopping time*. We call the variable Z the *randomisation device* for the randomised stopping time η, and the process (ρ_t) the *generating process*. The set of $(\mathcal{G}_t)$-randomised stopping times will be denoted $\mathcal{T}^R(\mathcal{G}_t)$. In case $(\mathcal{G}_t) = (\mathcal{F}_t)$, we will use a shorter notation $\mathcal{T}^R$. Whenever we consider multiple randomised stopping times, we additionally assume that their randomisation devices are independent.

Any pure stopping time is a randomised stopping time, in a sense that, to any $\tau \in \mathcal{T}$, there corresponds the generating process defined as $\rho_t(\omega) := I_{\{t \geq \tau(\omega)\}}$ for any $t \in [0,T]$. Indeed, note that $\mathbb{P}(\{\omega : Z(\omega) = 0\}) = \mathbb{P}(\{\omega : Z(\omega) = 1\}) = 0$. In other words, $Z \in (0,1)$ with probability 1. Therefore, with probability 1, we have $\eta = \inf\{t \in [0,T] : I_{\{t \geq \tau\}} > Z\} = \tau$. We see that in this case, with probability 1 the randomisation device Z does not affect the realised value of the randomised stopping time η.

Note that in Shmaya & Solan (2014), the terminology is different: the generating process (ρ_t) is called a randomised stopping time. However, given a randomisation device Z, the correspondence between randomised stopping times and the generating processes is one-to-one. Therefore, there is no conceptual difference between our approach and the one in Shmaya & Solan (2014).

In a similar way one can identify our definition of a randomised stopping time with the ones used in Meyer (1978) and Touzi & Vieille (2002). We clarify this further in Sections 3.1.3 and 3.5.

Shmaya & Solan (2014) introduce a concept that unifies the randomised and mixed stopping times. Denote by $\mathcal{M}$ the set of probability measures on $\Omega \times [0,T]$. For $\delta \in \mathcal{M}$, $t \in [0,T]$, $A \in \mathcal{F}$, denote

$$\delta^t(A) := \delta(A \times [0,t]).$$

We call a measure $\delta \in \mathcal{M}$ a *distribution stopping time*, if it satisfies the following properties:

- The marginal distribution of δ on Ω is $\mathbb{P}$.
- For every $t \in [0,T]$, the Radon–Nikodym derivative of δ^t with respect to $\mathbb{P}$ is $\mathcal{F}_t$-measurable.

Observe that any mixed stopping time μ naturally defines a measure $\delta_\mu \in \mathcal{M}$ via

$$\delta_\mu(A \times [0,t]) := (\mathbb{P} \otimes \lambda)(\{(\omega, r) : \omega \in A,\ \mu(\omega, r) \leq t\}) \tag{2.5}$$

for $A \in \mathcal{F}$, $t \in [0,T]$. Similarly, a randomised stopping time with the generating process (ρ_t) defines a measure $\delta_\rho \in \mathcal{M}$ via

$$\delta_\rho(A \times [0,t]) := \int_A \rho_t(\omega) d\mathbb{P}(\omega) = \mathbb{E}^{\mathbb{P}}[I_A \rho_t] \tag{2.6}$$

for $A \in \mathcal{F}$, $t \in [0,T]$.

In (Shmaya & Solan, 2014, Section 2.3), it is proven that the measures δ_μ, δ_ρ defined above are distribution stopping times. On the other hand, by (Shmaya & Solan, 2014, Corollary 2), for any distribution stopping time δ, there exists a mixed stopping time μ and a randomised stopping time with the generating process (ρ_t) such that $\delta = \delta_\mu = \delta_\rho$. Moreover, the generating process (ρ_t) is unique up to indistinguishability. In particular, we see that there exist a mapping $\mathcal{H}$ from the set of mixed stopping times onto $\mathcal{A}^\circ(\mathcal{F}_t)$. We clarify the importance of this correspondence in Section 2.4.1.

2.4.1 Equivalence of functionals

We now clarify the relation between mixed and randomised stopping times using equivalence of certain functionals. Let the filtration $(\mathcal{F}_t)$ satisfy the usual conditions. To a pure stopping time $\tau \in \mathcal{T}$ and a process $(X_t) \in \mathcal{L}_b$, we associate the *expected payoff* as follows:

$$N^P(\tau, X) := \mathbb{E}^{\mathbb{P}}[X_{\tau(\omega)}(\omega)].$$

To a mixed stopping time μ, we associate the expected payoff

$$N^m(\mu, X) := \mathbb{E}^{\mathbb{P}\times\lambda}[X_{\mu(\omega,r)}(\omega)],$$

and to a randomised stopping time with the generating process (ρ_t) the expected payoff

$$N^r(\rho, X) := \mathbb{E}^{\mathbb{P}}\left[\int_{[0,T]} X_t(\omega) d\rho_t(\omega)\right]. \tag{2.7}$$

All these functionals agree on pure stopping times in the following sense.

Proposition 2.4.2. *Let $\tau \in \mathcal{T}$ and $(X_t) \in \mathcal{L}_b$. Let $\mu(\omega, r) = \tau(\omega)$ for all $r \in [0,1]$. Let $\rho_t(\omega) = I_{\{t \geq \tau(\omega)\}}$ for all $t \in [0,T]$. Then,*

$$N^p(\tau, X) = N^m(\mu, X) = N^r(\rho, X).$$

Proof. For the first equality, we by definition of N^m and μ have

$$N^m(\mu, X) = \mathbb{E}^{\mathbb{P}\times\lambda}[X_{\mu(\omega,r)}(\omega)] = \mathbb{E}^{\mathbb{P}}[X_{\tau(\omega)}](\omega) = N^p(\tau, X).$$

Further, by definition of N^r and ρ we have

$$N^r(\rho, X) = \mathbb{E}^{\mathbb{P}}\left[\int_{[0,T]} X_t(\omega) d\rho_t(\omega)\right] = \mathbb{E}^{\mathbb{P}}\left[\int_{[0,T]} X_t(\omega) d(I_{\{t\geq\tau(\omega)\}})\right] = \mathbb{E}^{\mathbb{P}}[X_{\tau(\omega)}](\omega) = N^p(\tau, X).$$

□

Moreover, there is the following correspondence between an arbitrary mixed stopping time and a generating process of a randomised stopping time.

Theorem 2.4.3. *(Shmaya & Solan, 2014, Theorem 3) Let μ be a mixed stopping time. Let $(X_t) \in \mathcal{L}_b$. Let $\mathcal{H}$ be the mapping from the set of mixed stopping times onto $\mathcal{A}^\circ(\mathcal{F}_t)$. Then,*

$$N^m(\mu, X) = N^r(\mathcal{H}(\mu), X).$$

This result can be extended to a payoff functional that depends on two random stopping times. Fix processes $(f_t), (g_t), (h_t) \in \mathcal{L}_b$. For $\tau, \sigma \in \mathcal{T}$, define

$$\tilde{N}^p(\tau, \sigma) = \mathbb{E}^{\mathbb{P}}[f_{\tau(\omega)}(\omega) I_{\{\tau(\omega)<\sigma(\omega)\}} + g_{\sigma(\omega)}(\omega) I_{\{\sigma(\omega)<\tau(\omega)\}} + h_{\tau(\omega)}(\omega) I_{\{\sigma(\omega)=\tau(\omega)\}}].$$

For mixed stopping times μ, ν, consider

$$\begin{aligned}\tilde{N}^m(\mu, \nu) = \mathbb{E}^{\mathbb{P}\otimes\lambda\otimes\lambda}\big[&f_{\mu(\omega,r_1)}(\omega) I_{\{\mu(\omega,r_1)<\nu(\omega,r_2)\}} + g_{\nu(\omega,r_2)}(\omega) I_{\{\nu(\omega,r_2)<\mu(\omega,r_1)\}} \\ &+ h_{\mu(\omega,r_1)}(\omega) I_{\{\mu(\omega,r_1)=\nu(\omega,r_2)\}}\big],\end{aligned}$$

and for randomised stopping times with generating processes $(\rho_t), (\chi_t)$, define

$$\tilde{N}^r(\rho, \chi) = \mathbb{E}^{\mathbb{P}}\left[\int_{[0,T]} f_t(\omega)(1-\chi_t) d\rho_t(\omega) + \int_{[0,T]} g_t(\omega)(1-\rho_t) d\chi_t(\omega) + \sum_{t\in[0,T]} h_t \Delta\rho_t \Delta\chi_t\right].$$

Proposition 2.4.4. *(Touzi & Vieille, 2002, p. 1075) Let* $\tau, \sigma \in \mathcal{T}$. *Let* $\rho_t = I_{\{t \geq \tau\}}$, $\chi_t = I_{\{t \geq \sigma\}}$ *for all* $t \in [0,T]$. *Then,*

$$\tilde{N}^p(\tau,\sigma) = \tilde{N}^r(\rho,\chi).$$

The functional $\tilde{N}^r$ plays the central role in Touzi & Vieille (2002) (see Section 3.5). The proposition below provides a mapping from the set of mixed stopping times onto the set of generating processes $\mathcal{A}^\circ(\mathcal{F}_t)$. This mapping preserves the functional $\tilde{N}^r$, therefore proving that the concepts of mixed and randomised stopping time are equivalent in the setting of the paper.

Proposition 2.4.5. *(Touzi & Vieille, 2002, Proposition 7.1) There exists a mapping* $\tilde{\mathcal{H}}$ *from the set of mixed stopping times onto* $\mathcal{A}^\circ(\mathcal{F}_t)$ *such that, for every pair of mixed stopping times* μ, ν, *we have*

$$\tilde{N}^m(\mu,\nu) = \tilde{N}^r(\tilde{\mathcal{H}}(\mu),\tilde{\mathcal{H}}(\nu)).$$

Chapter 3

Background and literature review

3.1 Optimal stopping theory

Before we can start the review of literature on optimal stopping games, we need to recall certain facts and methods from the optimal stopping theory. We follow Karatzas & Shreve (1998) and Peskir & Shiryaev (2006) to outline the martingale and the Markovian aspects of the theory. We refer to Definition A.1.9 for the concept of a martingale (and also of a super- and submartingale), and to Definition A.1.10 for the concepts of Markov and strong Markov process.

3.1.1 Martingale approach

Let a filtration $(\mathcal{F}_t)_{[0,T]}$ satisfy the usual conditions, and let (X_t) be an $(\mathcal{F}_t)$-adapted stochastic process. Assume that $(X_t) \in \mathcal{L}_b$. In order to avoid some technical difficulties, we additionally take (X_t) to be non-negative (and explain in Remark 3.1.8 how to treat the general case).

Consider an *optimal stopping problem*

$$V = \sup_{\tau \in \mathcal{T}} \mathbb{E} X_\tau. \tag{3.1}$$

The problem involves two tasks: to characterise the *value* V, and to present an *optimal* stopping time τ^* at which the supremum is attained. In order to do this, we need the notion of essential supremum/infimum of a family of random variables.

Definition 3.1.1. *Let $\{Z_\alpha\}_{\alpha\in\mathbb{I}}$ be a non-empty family of random variables. The random variable Z^* is called the* essential supremum *of the family $\{Z_\alpha\}_{\alpha\in\mathbb{I}}$ (relative to the probability measure $\mathbb{P}$), if the following conditions are satisfied:*

$$\mathbb{P}(Z_\alpha \le Z^*) = 1 \text{ for each } \alpha \in \mathbb{I};$$

if $\tilde{Z}$ is another random variable satisfying $\mathbb{P}(Z_\alpha \le \tilde{Z}) = 1$ for each $\alpha \in \mathbb{I}$, then

$$\mathbb{P}(Z^* \le \tilde{Z}) = 1. \tag{3.2}$$

The random variable Z^* is called the essential infimum *of the family $\{Z_\alpha\}_{\alpha\in\mathbb{I}}$ (relative to the probability measure $\mathbb{P}$), if the following conditions are satisfied:*

$$\mathbb{P}(Z_\alpha \ge Z^*) = 1 \text{ for each } \alpha \in \mathbb{I};$$

if $\tilde{Z}$ is another random variable satisfying $\mathbb{P}(Z_\alpha \ge \tilde{Z}) = 1$ for each $\alpha \in \mathbb{I}$, then

$$\mathbb{P}(Z^* \ge \tilde{Z}) = 1. \tag{3.3}$$

The essential supremum and infimum are denoted $\operatorname{ess\,sup}_{\alpha\in\mathbb{I}} Z_\alpha$ *and* $\operatorname{ess\,inf}_{\alpha\in\mathbb{I}} Z_\alpha$*.*

Remark 3.1.2. *It is clear from* (3.2), (3.3) *that the essential supremum and infimum, when exist, are unique up to $\mathbb{P}$-a.s.*

Definition 3.1.3. *A family of random variables $\{Z_\alpha\}_{\alpha\in\mathbb{I}}$ is called* upwards directed *if it is closed under pairwise maximisation, that is, if for any $\alpha, \beta \in \mathbb{I}$ there exists $\gamma \in \mathbb{I}$ such that $Z_\alpha \vee Z_\beta \le Z_\gamma$, $\mathbb{P}$-a.s. A family of random variables is called* downwards directed *if it is closed under pairwise minimisation.*

Lemma 3.1.4. *(Karatzas & Shreve, 1998, Theorem A.3) Let $\{Z_\alpha\}_{\alpha\in\mathbb{I}}$ be a non-empty family of non-negative random variables. Then* $\operatorname{ess\,sup}_{\alpha\in\mathbb{I}} Z_\alpha$ *exists. Moreover, if the family $\{Z_\alpha\}_{\alpha\in\mathbb{I}}$ is upwards directed, then there exists a countable set $\{\alpha_n\}_{n\ge1}$ such that*

$$\operatorname*{ess\,sup}_{\alpha\in\mathbb{I}} Z_\alpha = \lim_{n\to\infty} Z_{\alpha_n} \ \mathbb{P}\text{-a.s.},$$

where $Z_{\alpha_1} \le Z_{\alpha_2} \le \dots$ $\mathbb{P}$-a.s.

Remark 3.1.5. *For the essential infimum, the symmetric property holds: if the family $\{Z_\alpha\}_{\alpha\in\mathbb{I}}$ is downwards directed, then there exists a countable set $\{\alpha_n\}_{n\ge1}$ such that*

$$\operatorname*{ess\,inf}_{\alpha\in\mathbb{I}} Z_\alpha = \lim_{n\to\infty} Z_{\alpha_n} \ \mathbb{P}\text{-a.s.},$$

where $Z_{\alpha_1} \ge Z_{\alpha_2} \ge \dots$ $\mathbb{P}$-a.s.

Let us return to the optimal stopping problem (3.1). We will think of it in an extended sense and will allow it to "start" not necessarily from time zero, but from an arbitrary stopping time. More precisely, for $\theta \in \mathcal{T}(\mathcal{F}_t)$, we define

$$\mathcal{T}_\theta = \{\tau \in \mathcal{T}(\mathcal{F}_t) : \ \tau \ge \theta, \ \mathbb{P}\text{-a.s.}\}, \tag{3.4}$$

so that $\mathcal{T}_0 = \mathcal{T}(\mathcal{F}_t) = \mathcal{T}$, and

$$Z(\theta) = \operatorname*{ess\,sup}_{\tau \in \mathcal{T}_\theta} \mathbb{E}[X_\tau | \mathcal{F}_\theta].$$

The family of random variables $\{Z(\theta)\}_{\theta \in \mathcal{T}}$ gives rise to the so-called *Snell envelope* of (X_t) — the process (Z_t^0) from the following theorem.

Theorem 3.1.6. *(Karatzas & Shreve, 1998, (D.6)), (Karatzas & Shreve, 1998, Theorem D.7) There exists a càdlàg $(\mathcal{F}_t)$-supermartingale (Z_t^0) such that*

$$Z_\theta^0 = Z(\theta), \ \mathbb{P}\text{-a.s. for every } \theta \in \mathcal{T};$$
$$(Z_t^0) \text{ dominates } (X_t) \text{ (Definition A.1.8).}$$

Moreover, (Z_t^0) is the smallest càdlàg supermartingale that dominates (X_t), i.e. if $(\tilde{Z}_t)$ is another càdlàg $(\mathcal{F}_t)$-supermartingale that dominates (X_t), then $(\tilde{Z}_t)$ dominates (Z_t^0).

Note that to say that a stopping time τ^* is optimal in (3.1) is equivalent to saying that

$$\mathbb{E}X_{\tau^*} = Z_0^0 = \sup_{\tau \in \mathcal{T}} \mathbb{E}X_\tau.$$

The following theorem further explains the link between the concept of the Snell envelope and the optimal stopping problem (3.1).

Theorem 3.1.7. *(Karatzas & Shreve, 1998, Theorem D.9) The stopping time τ^* is optimal in (3.1), if and only if*

- $Z_{\tau^*}^0 = X_{\tau^*}$, $\mathbb{P}$-*a.s.,*
- $(Z_{t \wedge \tau^*}^0)_{t \in [0,T]}$ *is an $(\mathcal{F}_t)$-martingale.*

Due to the latter claim of Theorem 3.1.7 and to the supermartingale property of the Snell envelope, the approach to optimal stopping based on Snell envelopes is often referred as *martingale approach*.

Remark 3.1.8. *The results of this section can be extended beyond non-negative processes in the following way. For $(X_t) \in \mathcal{L}_b$ and any $s \in [0,T]$, note that*

$$\mathbb{E}\Big[\inf_{t \in [0,T]} X_t | \mathcal{F}_s\Big] > -\infty.$$

Define for $s \in [0,T]$

$$\tilde{X}_s = X_s - \mathbb{E}\Big[\inf_{t \in [0,T]} X_t | \mathcal{F}_s\Big].$$

Observe that $\tilde{X}_s \geq 0$, because

$$\mathbb{E}\Big[\inf_{t\in[0,T]} X_t|\mathcal{F}_s\Big] = X_s \wedge \mathbb{E}\Big[\inf_{t\in[0,T]\setminus\{s\}} X_t|\mathcal{F}_s\Big].$$

Then we have

$$\begin{aligned} Z(\theta) &= \operatorname*{ess\,sup}_{\tau\in\mathcal{T}_\theta} \mathbb{E}[X_\tau|\mathcal{F}_\theta] = \operatorname*{ess\,sup}_{\tau\in\mathcal{T}_\theta} \left(\mathbb{E}[\tilde{X}_\tau|\mathcal{F}_\theta] + \mathbb{E}\Big[\mathbb{E}[\inf_{t\in[0,T]} X_t|\mathcal{F}_\tau]|\mathcal{F}_\theta\Big]\right) \\ &= \operatorname*{ess\,sup}_{\tau\in\mathcal{T}_\theta} \mathbb{E}[\tilde{X}_\tau|\mathcal{F}_\theta] + \mathbb{E}\Big[\inf_{t\in[0,T]} X_t|\mathcal{F}_\theta\Big]. \end{aligned} \tag{3.5}$$

Since the process $(\tilde{X}_t)$ is non-negative, for the family $\{\tilde{Z}(\theta)\}_{\theta\in\mathcal{T}}$ defined as

$$\tilde{Z}(\theta) := \operatorname*{ess\,sup}_{\tau\in\mathcal{T}_\theta} \mathbb{E}[\tilde{X}_\tau|\mathcal{F}_\theta],$$

we can construct the corresponding Snell envelope (Theorem 3.1.6), and apply Theorem 3.1.7 to characterise the optimal stopping time τ^*. Since the second term in (3.5) does not depend on τ, the stopping time τ^* would be optimal in the original problem (3.1) as well.

3.1.2 Markovian approach

The martingale approach of Section 3.1.1 is applicable for a large class of processes (X_t), but it has a disadvantage of not providing an *explicit* solution to the optimal stopping problem (3.1). One way to make the approach more explicit is to additionally assume that the Markov property holds.

Let $(X_t)_{t\in[0,T]}$ be a strong Markov process defined on a filtered probability space $(\Omega,\mathcal{F},(\mathcal{F}_t),\mathbb{P}_x)$, where $X_0 = x$ under the measure $\mathbb{P}_x$, and $(\mathcal{F}_t)$ satisfies the usual conditions. We assume that (X_t) takes values in a measurable space $(E,\mathcal{B})$. We assume that (X_t) is right-continuous and quasi left-continuous. For the ease of presentation, we also assume that (X_t) is time-homogeneous.

Let a measurable function $F : E \mapsto \mathbb{R}$ belong to the space $\mathcal{L}_b^X$ defined as

$$\mathcal{L}_b^X := \{R : E \mapsto \mathbb{R} \text{ such that } \mathbb{E}_x\big[\sup_{t\in[0,T]} |R(X_t)|\big] < \infty \text{ for all } x \in E\}, \tag{3.6}$$

where $\mathbb{E}_x$ denotes the expectation with respect to the measure $\mathbb{P}_x$.

Let $T = \infty$. Consider the optimal stopping problem

$$V(x) = \sup_{\tau\in\mathcal{T}} \mathbb{E}_x F(X_\tau). \tag{3.7}$$

The case of finite horizon $T < \infty$ can be treated similarly. Note that the process (t, X_t) is an $\mathbb{R}^{d+1}$-valued Markov process. In this case, we consider the optimal stopping problem

$$V(t,x) = \sup_{\tau \in \mathcal{T}: 0 \le \tau \le T-t} \mathbb{E}_x F(t+\tau, X_{t+\tau}).$$

For simplicity of notation, in the sequel we assume that $T = \infty$, and additionally that $F(X_\infty) = 0$.

As with (3.1), the goal is to characterise the *value function* $V(x)$ and the *optimal stopping time* τ^* that delivers the supremum in (3.7). For a subset $D \subseteq E$, denote

$$\tau_D = \inf\{t \in [0,\infty] : X_t \in D\}, \tag{3.8}$$

where we adopt the convention $\inf \emptyset = \infty$. We will need the following definition.

Definition 3.1.9. *Let $C \subseteq E$ be a measurable set, and let $D = E \setminus C$. A measurable function $R : E \mapsto \mathbb{R}$ is said to be* superharmonic *in C if*

$$\mathbb{E}_x R(X_{\tau \wedge \tau_D}) \le R(x)$$

for all $\tau \in \mathcal{T}$ and all $x \in E$.

R is said to be subharmonic *in C if*

$$\mathbb{E}_x R(X_{\tau \wedge \tau_D}) \ge R(x)$$

for all $\tau \in \mathcal{T}$ and all $x \in E$.

R is said to be harmonic *in C if*

$$\mathbb{E}_x R(X_{\tau \wedge \tau_D}) = R(x)$$

for all $\tau \in \mathcal{T}$ and all $x \in E$.

Remark 3.1.10. *A measurable function $R : E \mapsto \mathbb{R}$ is superharmonic in E if and only if*

$$\mathbb{E}_x R(X_\tau) \le R(x)$$

for all $\tau \in \mathcal{T}$ and all $x \in E$. Similarly for subharmonic and harmonic functions in E, the variable $\tau_D \equiv \infty$ disappears from the definition.

Proposition 3.1.11. *(Peskir & Shiryaev, 2006, (2.2.8)) R is superharmonic in E if and only if $(R(X_t))$ is a right-continuous supermartingale under $\mathbb{P}_x$ for every $x \in E$.*

Introduce the *continuation set*

$$C = \{x \in E : V(x) > F(x)\}$$

and the *stopping set*

$$D = \{x \in E : V(x) = F(x)\}.$$

The sets C and D have such names due to (Peskir & Shiryaev, 2006, Theorem 2.4), which states that, if there exists an optimal stopping time τ^* in (3.7), then (under some additional continuity assumptions) the stopping time τ_D is optimal in (3.7).

Remark 3.1.12. *The algorithm of finding the value function of the Markovian optimal stopping problem* (3.7) *can be summarised as follows. By (Peskir & Shiryaev, 2006, Theorem 2.7), (Peskir & Shiryaev, 2006, Corollary 2.9), the problem reduces to* finding the smallest superharmonic function $\widehat{V}$ that dominates the function F on E. *The latter problem is equivalent to solving the following obstacle problem:*

$$\mathbb{L}_X \widehat{V} \leq 0,$$
$$\widehat{V} \geq F,$$

where $\mathbb{L}_X$ is the infinitesimal operator of (X_t) — an integro-differential operator acting on functions $G : E \mapsto \mathbb{R}$ given by (Peskir & Shiryaev, 2006, (7.0.11)). This connection between the optimal stopping in Markovian framework and partial integro-differential equations will be further explored in Sections 3.3 *and* 3.4 *in the context of optimal stopping games.*

3.1.3 Approach based on functional analysis

Recall the functional N^r defined in Section 2.4.1. We will follow Meyer (1978), which extended the earlier results of Baxter & Chacon (1977), and study certain compactness and continuity properties of this functional. Our goal is to use these properties to show that the supremum of this functional is attained, and then deduce the existence of an optimal stopping time in (3.1).

Recall that, for a filtration $(\mathcal{F}_t)_{t\in[0,T]}$, by $\mathcal{A}^\circ(\mathcal{F}_t)$ (2.4) we denote the set of generating processes of $(\mathcal{F}_t)$-randomised stopping times. Consider

$$\mathcal{A}^- := \{(\rho_t) : t \mapsto \rho_t(\omega) \text{ càdlàg, non-decreasing, } \rho_{0-}(\omega) = 0 \text{ and } \rho_T(\omega) = 1 \text{ for all } \omega \in \Omega\},$$

which is the same set of generating processes but without the adaptivity condition. Note that the functional N^r (2.7) can be extended to $(\rho_t) \in \mathcal{A}^-$: let

$$\widehat{N}^r(\rho, X) = \mathbb{E}^{\mathbb{P}}\left[\int_{[0,T]} X_s d\rho_s\right].$$

With an abuse of notation, we denote this extended functional $N^r(\rho, X)$. There is the following result on convergence of such functionals.

Theorem 3.1.13. *(Meyer, 1978, Theorem 8') Assume that, for some $(\rho^n_t), (\rho_t) \in \mathcal{A}^-$, we have $N^r(\rho^n, X) \overset{n\to\infty}{\to} N^r(\rho, X)$ for all continuous bounded processes (X_t). Then, $N^r(\rho^n, X) \overset{n\to\infty}{\to} N^r(\rho, X)$ for all càdlàg regular (Definition 2.3.2) processes $(X_t) \in \mathcal{L}_b$.*

In Chapter 4, the reader will see an extension of Theorem 3.1.13 to a functional depending on two randomised stopping times. Due to an additional adaptivity assumption therein, we were able to prove this in a more straightforward way. Results of this kind emphasise the importance of regular processes in our study.

The proof of Theorem 3.1.13 relies on several ideas from topology, functional analysis, and theory of projections of stochastic processes (Section A.5.1) which we proceed to overview.

Let D denote the space of bounded càdlàg processes, and $C \subset D$ the space of bounded continuous processes. The following theorem provides a correspondence between linear functionals on C and elements of $\mathcal{A}^-$.

Theorem 3.1.14. *(Meyer, 1978, Theorem 2) For any linear functional $\mathcal{H}$ on C such that $|\mathcal{H}(X)| \leq \mathbb{E}[\sup_t |X_t|]$, $\mathcal{H}(1) = 1$, there exists a unique (up to indistinguishability) $(\rho_t) \in \mathcal{A}^-$ such that*

$$N^r(\rho, X) = \mathcal{H}(X).$$

Remark 3.1.15. *Let us draw a parallel between the set $\mathcal{A}^-$ and a certain set of probability measures. Meyer (1978) calls a probability measure ν on $\Omega \times [0,T]$ a* randomised random variable *(variable aléatoire floue), if the projection of ν on Ω is $\mathbb{P}$. For a randomised random variable ν and a bounded measurable process (X_t), denote*

$$\nu(X) = \int_{\Omega\times[0,T]} X_s(\omega) d\nu(\omega, s).$$

Disintegration theorem (Dellacherie & Meyer, 1978, pp. 78-80) provides a correspondence between a randomised random variable ν and a unique (up to indistinguishability) càdlàg non-decreasing process (ρ_t) with $\rho_{0-} = 0$ via equality

$$\nu(X) = N^r(\rho, X) \tag{3.9}$$

that holds for all non-negative processes (X_t). Since the projection of ν on Ω is $\mathbb{P}$, we see that $\rho_T = 1$, and therefore $(\rho_t) \in \mathcal{A}^-$.

The terminology of Meyer (1978) evolves around the set of randomised random variables, but we will use the identification described above and proceed to study the set $\mathcal{A}^-$.

Note that, according to (3.9), for any $(\rho_t) \in \mathcal{A}^-$ there exists a linear functional on C satisfying the properties listed in Theorem 3.1.14. Therefore, the correspondence between $(\rho_t) \in \mathcal{A}^-$ and such functionals $\mathcal{H}$ on C is one-to-one.

The weak topology (see Section A.3.1) on the set $\mathcal{A}^-$ is introduced as the coarsest topology such that the mapping $\rho \mapsto N^r(\rho, X)$ is continuous for any $(X_t) \in C$. Meyer (1978) establishes weak compactness of the set $\mathcal{A}^-$, and further proves that the set $\mathcal{A}^\circ(\mathcal{F}_t)$ is weakly closed. We see that the set $\mathcal{A}^\circ(\mathcal{F}_t)$ is a weakly closed subset of the weakly compact set $\mathcal{A}^-$. Therefore (Theorem A.2.3), the following holds.

Theorem 3.1.16. *(Meyer, 1978, Theorem 3) The set $\mathcal{A}^\circ(\mathcal{F}_t)$ is weakly compact.*

By definition, the weak convergence of $(\rho_t^n) \in \mathcal{A}^-$ to $(\rho_t) \in \mathcal{A}^-$ means that $N^r(\rho^n, X) \to N^r(\rho, X)$ for processes $(X_t) \in C$. It turns out that the convergence holds for a more general class of processes. The following theorem is a step in the proof of Theorem 3.1.13.

Theorem 3.1.17. *(Meyer, 1978, Theorem 5) Let $(\rho_t^n) \subset \mathcal{A}^\circ(\mathcal{F}_t)$ be a sequence that converges weakly to $(\rho_t) \in \mathcal{A}^-$. Then, $N^r(\rho^n, X) \to N^r(\rho, X)$ for any regular process $(X_t) \in D$.*

The result of Theorem 3.1.17 can be applied as follows. The mapping $\rho \mapsto N^r(\rho, X)$ is continuous on $\mathcal{A}^\circ(\mathcal{F}_t)$ with respect to the weak topology on $\mathcal{A}^-$. We also know (Theorem 3.1.16) that the set $\mathcal{A}^\circ(\mathcal{F}_t)$ is compact with respect to the same topology. This allows to apply the extreme value theorem (Theorem A.2.4) to deduce that, for a regular process $(X_t) \in D$, the supremum

$$\sup_{\rho \in \mathcal{A}^\circ(\mathcal{F}_t)} N^r(\rho, X) \tag{3.10}$$

is attained. This is equivalent to existence of an optimal randomised stopping time in

$$V = \sup_{\tau \in \mathcal{T}^R(\mathcal{F}_t)} \mathbb{E} X_\tau.$$

Moreover, it can be shown that the supremum is attained at a pure stopping time. To clarify this, we need a concept of extreme point. A point x of a vector space S is called an *extreme point*, if it cannot be expressed as a convex combination of two points of S different from x. It turns out that the generating processes of pure stopping times are extreme points of the set of generating processes (Pennanen & Perkkiö, 2018, Lemma 2). Then, to prove that the supremum in (3.10) is attained at a generating process of a pure stopping time, one needs to verify (Pennanen & Perkkiö, 2018, Theorem 1) the assumptions of Bauer's maximum principle (Choquet, 1969, Theorem 25.9), which are the conditions for a function to attain its maximum at an extreme point.

In Bismut (1977a), the ideas described in this section are used to prove the existence of a solution to a certain mixed problem of optimal stopping and control. We also refer to Pennanen & Perkkiö (2018) for a more recent generalisation of this approach. In the optimal stopping problem of Pennanen & Perkkiö (2018), the process (X_t) is required to be neither càdlàg nor regular for a solution to exist. However, in Pennanen & Perkkiö (2018), the notion of solution is different: instead of optimal stopping times, the so-called optimal quasi-stopping times (or split stopping times (Dellacherie & Meyer, 1982, Appendix I.14)) are studied.

We note that the approach of Meyer (1978) is close to the one we take in Chapter 4. Albeit the core functional studied therein $(\rho,\chi) \mapsto N(\rho,\chi)$ is bilinear, while the functional $\rho \mapsto N^r(\rho,X)$ is linear, and despite the fact that in Chapter 4 we work in a different topology on $\mathcal{A}^\circ(\mathcal{F}_t)$, the main ideas are still applicable.

3.2 Martingale approach to games

In Section 3.1.1, we introduced an optimal stopping problem and outlined the martingale approach to finding an optimal stopping time that delivers the supremum therein. The main object of our study is optimal stopping *games*, where there are several (in this thesis, usually two) players and each of them chooses a stopping time in order to maximise or minimise a certain functional. Some methods used in order to find "optimal" stopping strategies in this context have a parallel in optimal stopping theory. In this section, we describe the martingale approach to optimal stopping games.

3.2.1 Zero-sum game

We start by reviewing the set-up and results of Lepeltier & Maingueneau (1984).

Consider the continuous-time setting of Section 2.3 with the time horizon $T = \infty$. Let a filtration $(\mathcal{F}_t)_{t\in[0,\infty]}$ satisfy the usual conditions. Let $(f_t)_{t\in[0,\infty]}, (g_t)_{t\in[0,\infty]}$ be right-continuous processes optional with respect to $(\mathcal{F}_t)$. Lepeltier & Maingueneau (1984) additionally assume that $f_\infty = g_\infty = 0$, and that the following *order condition* holds:

$$f_t \geq g_t \text{ for all } t \in [0,\infty) \ \mathbb{P}\text{-a.s.} \tag{3.11}$$

Let τ and σ be $(\mathcal{F}_t)$-stopping times chosen by, respectively, the first and the second player. We assume that at time $\tau \wedge \sigma$, the first player delivers to the second player the random *payoff*

$$\mathcal{P}(\tau,\sigma) = f_\tau I_{\{\tau<\sigma\}} + g_\sigma I_{\{\sigma\leq\tau\}}. \tag{3.12}$$

Since the first player (or τ-player) is the one who pays this amount, she is the *minimiser* in the game, while the second player (or σ-player) is the *maximiser*. The combined wealth of the two players does not change, and for this reason such games are known as *zero-sum* games.

We denote the *expected payoff* as

$$N(\tau,\sigma) = \mathbb{E}[\mathcal{P}(\tau,\sigma)].$$

Remark 3.2.1. *The game described above is a* war-of-attrition *due to the condition* (3.11)*. War-of-attrition refers to the fact that each player at each moment of time would benefit more from the other player stopping the game rather than from stopping herself. This canonical class of Dynkin games will appear multiple times below. Its complement are the so-called* pre-emption games, *in which players have an incentive to stop first (see e.g. De Angelis & Ekström (2020), (Fudenberg & Tirole, 1991, Section 4.5.3), Boyarchenko & Levendorskii (2014)). In Chapter 5, we study a "mixed" optimal stopping game that is neither a war-of-attrition nor a pre-emption game.*

Remark 3.2.2. *In the literature, a more general payoff than* (3.12) *is sometimes considered (see e.g.* (3.18) *below): one of a form*

$$\mathcal{P}(\tau,\sigma) = f_\tau I_{\{\tau<\sigma\}} + g_\sigma I_{\{\sigma<\tau\}} + h_\tau I_{\{\tau=\sigma\}}, \tag{3.13}$$

where the "middle" payoff process (h_t) is commonly assumed to satisfy

$$f_t \geq h_t \geq g_t \text{ for all } t \in [0,\infty) \ \mathbb{P}\text{-a.s.} \tag{3.14}$$

In Section 3.3, we consider a game with such "middle" payoff (that additionally has a specific Markovian structure).

Remark 3.2.3. *In papers concerned with financial applications of Dynkin games (see e.g. Sections 3.4.2-3.4.4 below), one can often encounter the payoff of a form*

$$\mathcal{P}(\tau,\sigma) = e^{-r\tau} f_\tau I_{\{\tau<\sigma\}} + e^{-r\sigma} g_\sigma I_{\{\sigma\leq\tau\}},$$

for some fixed constant $r \in (0,\infty)$. The exponential discounting *term has a financial meaning of money being worth more the sooner it is received, due to its capacity to earn interest. This term does not cause any mathematical difficulties, as one can consider payoff processes $(e^{-rt} f_t)$, $(e^{-rt} g_t)$.*

The two concepts defined below play the key role in studying zero-sum optimal stopping games.

Definition 3.2.4. *Define the* lower value *and* upper value *of the game as*

$$V_* := \sup_{\sigma\in\mathcal{T}(\mathcal{F}_t)} \inf_{\tau\in\mathcal{T}(\mathcal{F}_t)} N(\tau,\sigma) \quad \textit{and} \quad V^* := \inf_{\tau\in\mathcal{T}(\mathcal{F}_t)} \sup_{\sigma\in\mathcal{T}(\mathcal{F}_t)} N(\tau,\sigma).$$

If they coincide, the game is said to have a value $V = V_* = V^*$. *Existence of a value is also sometimes called* Stackelberg equilibrium.

Remark 3.2.5. *It is clear from the definition of supremum and infimum that* $V_* \leq V^*$.

Definition 3.2.6. *A pair* $(\tau^*,\sigma^*) \in \mathcal{T}(\mathcal{F}_t)\times\mathcal{T}(\mathcal{F}_t)$ *is called a* Nash equilibrium point *(*NEP*, or simply a* Nash equilibrium*, or sometimes a* saddle point*), if the following holds*

$$N(\tau^*,\sigma) \leq N(\tau^*,\sigma^*) \leq N(\tau,\sigma^*)$$

for any $\tau,\sigma \in \mathcal{T}(\mathcal{F}_t)$.

Remark 3.2.7. *Existence of a Nash equilibrium implies existence of a value. Indeed, if* (τ^*,σ^*) *is a Nash equilibrium, then for arbitrary* $\tau,\sigma\in\mathcal{T}(\mathcal{F}_t)$ *we have*

$$\begin{aligned} V^* = \inf_{\tau\in\mathcal{T}(\mathcal{F}_t)} \sup_{\sigma\in\mathcal{T}(\mathcal{F}_t)} N(\tau,\sigma) &\leq \sup_{\sigma\in\mathcal{T}(\mathcal{F}_t)} N(\tau,\sigma^*) \leq N(\tau^*,\sigma^*) \\ \leq \inf_{\tau\in\mathcal{T}(\mathcal{F}_t)} N(\tau^*,\sigma) &\leq \sup_{\sigma\in\mathcal{T}(\mathcal{F}_t)} \inf_{\tau\in\mathcal{T}(\mathcal{F}_t)} N(\tau,\sigma) = V_*, \end{aligned}$$

therefore $V^* \leq V_*$. *The opposite equality always holds (Remark 3.2.5), so the upper and the lower values coincide, and the value* $V = V_* = V^* = N(\tau^*,\sigma^*)$ *exists.*

Lepeltier & Maingueneau (1984) study the existence of the value and of the Nash equilibrium in the game using an extended Snell envelope approach described in Section 3.1.1 in the context of optimal stopping *problems*. More precisely, they define two families of random variables indexed by stopping times θ (c.f. Theorem A.5.2):

$$S_*(\theta) = \operatorname*{ess\,sup}_{\sigma\in\mathcal{T}_\theta} \operatorname*{ess\,inf}_{\tau\in\mathcal{T}_\theta} \mathbb{E}[\mathcal{P}(\tau,\sigma)|\mathcal{F}_\theta] \quad \text{and} \quad S^*(\theta) = \operatorname*{ess\,inf}_{\tau\in\mathcal{T}_\theta} \operatorname*{ess\,sup}_{\sigma\in\mathcal{T}_\theta} \mathbb{E}[\mathcal{P}(\tau,\sigma)|\mathcal{F}_\theta], \tag{3.15}$$

where $\mathcal{T}_\theta$ is the set of stopping times that exceed θ as in (3.4).

The authors prove (Lepeltier & Maingueneau, 1984, Theorem 7) that there exist measurable optional processes $(\hat{S}_*(t))_{t\in[0,\infty]}$ and $(\hat{S}^*(t))_{t\in[0,\infty]}$ such that

$$S_*(\theta) = \hat{S}_*(\theta),\ S^*(\theta) = \hat{S}^*(\theta),\ \mathbb{P}\text{-a.s. for all } \theta\in\mathcal{T}.$$

Moreover, if we define, for $\varepsilon > 0$, the random times

$$D_*(\varepsilon) = \inf\{t\in[0,\infty] : \hat{S}_*(t) \geq f_t - \varepsilon\}, \quad D^*(\varepsilon) = \inf\{t\in[0,\infty] : \hat{S}^*(t) \leq g_t + \varepsilon\},$$

then it turns out (Lepeltier & Maingueneau, 1984, Theorem 11) that the stopped process $(\hat{S}_*(t \wedge D_*(\varepsilon)))_{t\in[0,\infty]}$ is a submartingale, while the stopped process $(S_*(t \wedge D^*(\varepsilon)))_{t\in[0,\infty]}$ is a supermartingale (c.f. the supermartingale property of the Snell envelope of Theorem 3.1.6). This allows to deduce (Lepeltier & Maingueneau, 1984, Corollary 12) that, for all stopping times θ,

$$\hat{S}_*(\theta) = \hat{S}^*(\theta), \ \mathbb{P}\text{-a.s.}, \tag{3.16}$$

and, in particular (for $\theta = 0$), that $V_* = V^*$, and hence the game has a value.

The equality (3.16) holds for stopping times θ, and, in particular, for any constant time $t \in [0,\infty]$ we have $\hat{S}_*(t) = \hat{S}^*(t) := S_t$. We already mentioned that $(S_t)_{t\in[0,\infty]}$ is a measurable process. Under an additional condition

$$\lim_{s\nearrow t} g_s \leq {}^p g_t, \quad \limsup_{s\nearrow t} f_s \leq {}^p f_t,$$

where the $({}^p f_t),({}^p g_t)$ denote the previsible projections of $(f_t),(g_t)$, the process (S_t) yields the saddle point of the game (Lepeltier & Maingueneau, 1984, Theorem 15) via

$$\tau^* = \inf\{t \in [0,\infty] : S_t = f_t\}; \quad \sigma^* = \inf\{t \in [0,\infty] : S_t = g_t\}.$$

3.2.2 Nonzero-sum game

Ideas of Lepeltier & Maingueneau (1984) are extended in Hamadène & Zhang (2010) for a study of a *nonzero-sum* game.

Let the time horizon $T < \infty$, and let $(\mathcal{F}_t)_{t\in[0,T]}$ be a filtration that satisfies the usual conditions. Consider four càdlàg $(\mathcal{F}_t)$-adapted processes $(f_t^1),(f_t^2),(g_t^1),(g_t^2)$ of class (D) (recall the definition from Remark 2.3.1). As above, let there be the first and the second player who choose $(\mathcal{F}_t)$-stopping times τ and σ, respectively. Define the random payoffs

$$\begin{aligned} \mathcal{P}_1(\tau,\sigma) &= g_\tau^1 I_{\{\tau\leq\sigma\}} + f_\sigma^1 I_{\{\sigma<\tau\}}, \\ \mathcal{P}_2(\tau,\sigma) &= g_\sigma^2 I_{\{\sigma<\tau\}} + f_\tau^2 I_{\{\tau\leq\sigma\}}, \end{aligned} \tag{3.17}$$

and denote

$$N_i(\tau,\sigma) = \mathbb{E}[\mathcal{P}_i(\tau,\sigma)]$$

for $i = 1,2$. We assume that at time $\tau\wedge\sigma$, the first player obtains the amount $\mathcal{P}_1$ and the second player obtains $\mathcal{P}_2$. Therefore, unlike in the zero-sum of Lepeltier & Maingueneau (1984), both players in this nonzero-sum set-up are maximisers. We also assume that $f_t^i \geq g_t^i$ $(i = 1,2)$ for all $t \in [0,T]$ $\mathbb{P}$-a.s. In other words, if the player i is the *leader* (i.e. stops the game first), she receives the (smaller) payoff g^i. If the player i is the *follower* (i.e. does not stop first), she receives the (larger) amount f^i. Thus, this game is a war-of-attrition (recall Remark 3.2.1).

Hamadène & Zhang (2010) impose additional assumptions on the processes $(f^i_t),(g^i_t)$ $(i = 1,2)$: namely, the processes (g^i_t) have only positive jumps, and for any $\eta \in \mathcal{T}(\mathcal{F}_t)$, there holds $\mathbb{P}(\{g^1_\eta < f^1_\eta\} \setminus \{g^2_\eta < f^2_\eta\}) = 0$. Under these and the above assumptions, the authors study the Nash equilibrium of the game, which in this setting is defined as follows.

Definition 3.2.8. *A pair* $(\tau^*,\sigma^*) \in \mathcal{T}(\mathcal{F}_t) \times \mathcal{T}(\mathcal{F}_t)$ *is called a* Nash equilibrium point, *if*

$$N_1(\tau,\sigma^*) \leq N_1(\tau^*,\sigma^*), \quad N_2(\tau^*,\sigma) \leq N_2(\tau^*,\sigma^*)$$

for any $\tau,\sigma \in \mathcal{T}(\mathcal{F}_t)$.

The main result of Hamadène & Zhang (2010) is that a Nash equilibrium exists. The proof relies on constructing a decreasing sequence of pairs of stopping times (τ_n,σ_n) whose limit is a Nash equilibrium. The construction uses a sequence of Snell envelopes and iterative application of Theorem 3.1.7 to the resulting sequence of optimal stopping problems.

More precisely, let $\tau_1 = \sigma_1 = T$. Assume that for some $n \geq 1$, τ_n and σ_n are defined. Set for every $t \in [0,T]$

$$S^{1,n}_t = \operatorname*{ess\,sup}_{\eta \in \mathcal{T}_t} \mathbb{E}[g^1_\eta I_{\{\eta<\sigma_n\}} + f^1_{\sigma_n} I_{\{\eta\geq\sigma_n\}} | \mathcal{F}_t],$$

and

$$\tilde{\tau}_{n+1} = \inf\{t \in [0,T] : S^{1,n}_t = g^1_t\} \wedge \sigma_n; \quad \tau_{n+1} = \begin{cases} \tilde{\tau}_{n+1} & \text{if } \tilde{\tau}_{n+1} < \sigma_n; \\ \tau_n & \text{if } \tilde{\tau}_{n+1} = \sigma_n. \end{cases}$$

Similarly, let

$$S^{2,n}_t = \operatorname*{ess\,sup}_{\eta \in \mathcal{T}_t} \mathbb{E}[g^2_\eta I_{\{\eta<\tau_{n+1}\}} + f^2_{\tau_{n+1}} I_{\{\eta\geq\tau_{n+1}\}} | \mathcal{F}_t],$$

and

$$\tilde{\sigma}_{n+1} = \inf\{t \in [0,T] : S^{2,n}_t = g^2_t\} \wedge \tau_{n+1}; \quad \sigma_{n+1} = \begin{cases} \tilde{\sigma}_{n+1} & \text{if } \tilde{\sigma}_{n+1} < \tau_{n+1}; \\ \sigma_n & \text{if } \tilde{\sigma}_{n+1} = \tau_{n+1}. \end{cases}$$

This iterative procedure yields a sequence of "best responses" for each player, in a sense that (Hamadène & Zhang, 2010, Lemma 3.3), for any $\eta \in \mathcal{T}$ and any $n \geq 1$,

$$N_1(\eta,\sigma_n) \leq N_1(\tau_{n+1},\sigma_n), \quad N_2(\tau_{n+1},\eta) \leq N_2(\tau_{n+1},\sigma_{n+1}).$$

This, together with a few auxiliary lemmata, proves (Hamadène & Zhang, 2010, Theorem 2.2), that is, that the couple $\tau^* := \lim_{n\to\infty} \tau_n, \sigma^* := \lim_{n\to\infty} \sigma_n$ forms a Nash equilibrium of the game.

3.2.3 Backward stochastic differential equations

In this section, we again consider a zero-sum game. Cvitanic & Karatzas (1996) draw a connection between its value and a solution of a certain system of stochastic differential equations (Section A.7). First, we need to give a rigorous definition of such a solution.

Let $T < \infty$, and let $(\mathcal{F}_t)_{t\in[0,T]}$ be a filtration that satisfies the usual conditions. Let $(f_t)_{t\in[0,T]}$, $(g_t)_{t\in[0,T]}$ be continuous processes adapted to $(\mathcal{F}_t)$. Assume that

$$\mathbb{E}\Big[\sup_{t\in[0,T]}(f_t)^2\Big] < \infty, \quad \mathbb{E}\Big[\sup_{t\in[0,T]}(g_t)^2\Big] < \infty,$$

and

$$f_t \geq g_t \text{ for all } t \in [0,T]\ \mathbb{P}\text{-a.s.}$$

Let h be an $\mathcal{F}_T$-measurable square-integrable random variable such that

$$g_T \leq h \leq f_T,\ \mathbb{P}\text{-a.s.}$$

Let $R : [0,T]\times\Omega\times\mathbb{R}\times\mathbb{R}^d \mapsto \mathbb{R}$ be a $\mathfrak{P}\times\mathcal{B}(\mathbb{R})\times\mathcal{B}(\mathbb{R}^d)$-measurable function, where $\mathfrak{P}$ denotes the previsible sigma-algebra. Assume that

$$\begin{aligned}
&\mathbb{E}\int_0^T (R(t,\omega,0,0))^2 dt < \infty,\\
&|R(t,\omega,x,y) - R(t,\omega,x',y')| \leq k(|x-x'| + \|y-y'\|),
\end{aligned}$$

for all $t \in [0,T]$; $\omega \in \Omega$; $x, x' \in \mathbb{R}$; $y, y' \in \mathbb{R}^d$ and for some $k \in (0,\infty)$.

Definition 3.2.9. *Let the processes $(f_t), (g_t)$, the random variable h, and the function R be as above. Let $(X_t), (K_t)$ be $\mathbb{R}$-valued and (Y_t) an $\mathbb{R}^d$-valued stochastic processes. Let $(X_t), (Y_t), (K_t)$ be $(\mathcal{F}_t)$-adapted. We say that the triple $((X_t), (Y_t), (K_t))$ is the* solution of the backward stochastic differential equation (BSDE) with reflecting barriers $(f_t), (g_t)$, terminal condition h and driver R, *if*

$$\begin{aligned}
&K_t = K_t^+ - K_t^-,\\
&X_t = h + \int_t^T R(s,X_s,Y_s)ds + K_T^+ - K_t^+ - (K_T^- - K_t^-) - \int_t^T \langle Y_s, dW_s\rangle,\\
&g_t \leq X_t \leq f_t,\\
&\int_0^T (X_s - g_s)dK_s^+ = \int_0^T (f_s - X_s)dK_s^- = 0,
\end{aligned}$$

for all $t \in [0,T]$ $\mathbb{P}$-a.s., where (W_t) is the standard d-dimensional $(\mathcal{F}_t)$-Wiener process, and where the following additional properties hold:

- $(K_t^+), (K_t^-)$ *are continuous, non-decreasing,* $(\mathcal{F}_t)$*-adapted processes with* $K_0^+ = K_0^- = 0$, $\mathbb{E}[(K_T^+)^2], \mathbb{E}[(K_T^-)^2] < \infty$,
- (Y_t) *is an* $(\mathcal{F}_t)$*-previsible process with* $\int_0^T \mathbb{E}\|Y_s\|^2 ds < \infty$.

The process (X_t) *is called the* state process *and* (Y_t) *the* noise process *of the solution.*

Assume that there exists a solution $((X_t), (Y_t), (K_t))$ of the BSDE above. Define for $t \in [0, T]$, $\omega \in \Omega$,

$$r_t(\omega) = R(t, \omega, X_t(\omega), Y_t(\omega)).$$

Finally, fix $t \in [0, T]$, and for $\tau, \sigma \in \mathcal{T}_t$, define the payoff of a game at time t via

$$\mathcal{P}_t(\tau, \sigma) = f_\tau I_{\{\tau<\sigma\}} + g_\sigma I_{\{\sigma \le \tau\} \cap \{\sigma < T\}} + h I_{\{\sigma = T\} \cap \{\tau = T\}} + \int_t^{\tau \wedge \sigma} r_u du.$$

Here, similarly to the classical set-up of Lepeltier & Maingueneau (1984), τ is the stopping time chosen by the first player, σ is the stopping time chosen by the second player, and at time $\tau \wedge \sigma \wedge T$, the first player delivers to the second player the random amount $\mathcal{P}_0(\tau, \sigma)$.

Consider

$$S_*(t) = \operatorname*{ess\,sup}_{\sigma \in \mathcal{T}_t} \operatorname*{ess\,inf}_{\tau \in \mathcal{T}_t} \mathbb{E}[\mathcal{P}_t(\tau, \sigma) | \mathcal{F}_t] \quad \text{and} \quad S^*(t) = \operatorname*{ess\,inf}_{\tau \in \mathcal{T}_t} \operatorname*{ess\,sup}_{\sigma \in \mathcal{T}_t} \mathbb{E}[\mathcal{P}_t(\tau, \sigma) | \mathcal{F}_t].$$

According to (Cvitanic & Karatzas, 1996, Theorem 4.1),

$$S_*(t) = S^*(t) = X_t, \ \mathbb{P}\text{-a.s. for all } t \in [0, T],$$

where (X_t) is the state process (Definition 3.2.9) of the BSDE above.

3.3 Markovian approach to games

As with optimal stopping problems (see Section 3.1.2), optimal stopping games that have Markovian structure are of special interest. Following Ekström & Peskir (2008), we provide the conditions for existence of the value function and of Nash equilibrium in a Markovian optimal stopping game.

Consider a filtration $(\mathcal{F}_t)_{t \in [0,\infty]}$ that satisfies the usual conditions. Consider a strong Markov process (X_t), defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P}_x)$, with values in a measurable space $(E, \mathcal{B})$. Here, for $x \in E$, we assume that $X_0 = x$ under $\mathbb{P}_x$.

Consider a zero-sum game with the payoff

$$\mathcal{P}(\tau,\sigma) = F(X_\tau)I_{\{\tau<\sigma\}} + G(X_\sigma)I_{\{\sigma<\tau\}} + H(X_\tau)I_{\{\tau=\sigma\}}, \tag{3.18}$$

where $F,G,H \in \mathcal{L}_b^X$ (recall the integrability condition (3.6)). As before, we denote

$$N_x(\tau,\sigma) = \mathbb{E}_x[\mathcal{P}(\tau,\sigma)].$$

We assume that F,G,H are continuous functions that satisfy (c.f. the order condition (3.14))

$$G(x) \le H(x) \le F(x) \quad \forall x \in E, \tag{3.19}$$

and that

$$\lim_{t\to\infty} G(X_t) = \lim_{t\to\infty} F(X_t), \ \mathbb{P}_x\text{-a.s.} \tag{3.20}$$

Recall the non-Markovian Definition 3.2.4 of the lower and upper value. Since in the current set-up, the probability measure depends on the starting point x of the underlying process, the lower and upper value of the game become a function of x, as in the following.

Definition 3.3.1. *For the game with the payoff* (3.18)*, the lower and upper value (functions) are*

$$V_*(x) = \sup_{\sigma\in\mathcal{T}} \inf_{\tau\in\mathcal{T}} N_x(\tau,\sigma); \quad V^*(x) = \inf_{\tau\in\mathcal{T}} \sup_{\sigma\in\mathcal{T}} N_x(\tau,\sigma)$$

If

$$V_*(x) = V^*(x) \text{ for all } x \in E,$$

we say that there exists a value (function) $V(x) := V_*(x) = V^*(x)$ *for* $x \in E$.

Definition 3.3.2. *A pair* $(\tau^*,\sigma^*) \in \mathcal{T}\times\mathcal{T}$ *is called a* Nash equilibrium*, if*

$$N_x(\tau^*,\sigma) \le N_x(\tau^*,\sigma^*) \le N_x(\tau,\sigma^*)$$

for all $\tau,\sigma \in \mathcal{T}$ *and for all* $x \in E$.

Let us quote the main result of Ekström & Peskir (2008).

Theorem 3.3.3. *Consider the optimal stopping game (3.18). Let* $F,G,H \in \mathcal{L}_b^X$ *be continuous functions satisfying* (3.19) *and* (3.20)*. If the strong Markov process* (X_t) *is right-continuous, then there exists a measurable value function* V*. If* (X_t) *is right-continuous and quasi left-continuous (recall Example 2.3.4), then the Nash equilibrium holds with*

$$\tau^* = \inf\{t : X_t \in D_1\}; \ \sigma^* = \inf\{t : X_t \in D_2\}, \tag{3.21}$$

where $D_1 = \{x \in E : V(x) = F(x)\}$ *and* $D_2 = \{x \in E : V(x) = G(x)\}$.

The proof is closely related to the theory of Snell envelopes and the martingale approach to optimal stopping. We provide a brief overview.

Fix arbitrary $\tau \in \mathcal{T}$, and consider the optimal stopping problem of the σ-player

$$\hat{V}^*_\tau(x) = \sup_{\sigma \in \mathcal{T}} \hat{N}_x(\tau, \sigma),$$

where we set

$$G^\tau_s = F(X_\tau) I_{\{\tau \le s\}} + G(X_s) I_{\{s < \tau\}}, \; \hat{N}_x(\tau, \sigma) = \mathbb{E}_x[G^\tau_\sigma].$$

Let $(\hat{S}^\tau_t)$ be the Snell envelope of the process (G^τ_t) (recall Theorem 3.1.6). Using the Markov property, the authors obtain a result stronger than in Section 3.1.1:

$$\hat{V}^*_\tau(X_\rho) = \mathbb{E}_x \hat{S}^\tau_\rho, \; \mathbb{P}_x\text{-a.s.},$$

for any $\rho \le \tau$. The authors also prove that the family of random variables

$$\{\sup_\sigma \hat{M}_{X_\rho}(\tau, \sigma)\}_{\tau \in \mathcal{T}}$$

is downwards directed for any $\rho \le \tau$. Roughly speaking, this allows to replace the infimum/supremum with the the limit and swap them with the expectation in

$$\hat{V}^*(x) = \inf_{\tau \in \mathcal{T}} \sup_{\sigma \in \mathcal{T}} \hat{M}_x(\tau, \sigma).$$

We note the parallel with (Karatzas & Shreve, 1998, Proposition D.2), and with our approach in Section 4.5.3. Ultimately, this results in the equality $\hat{V}^* = V^* = V_*$, which in particular implies existence of the value. We omit the details on the proof of existence of Nash equilibrium and its characterisation (5.23), and only mention that it is as well related to Snell envelopes and Theorems 3.1.6 and 3.1.7.

One advantage of the Markovian framework is that it enables to characterise the value function of the stopping game as a solution of a certain system of variational inequalities. This approach was pioneered by Bensoussan & Friedman (1974). Let $\mathbb{L}$ be the infinitesimal generator (see (Dynkin, 1965, Chapter III)) of the process (X_t). The value function of a Markovian optimal stopping game is the solution w of the equation of the following kind:

$$\max\left\{\min\left\{(-\frac{\partial}{\partial t} - \mathbb{L})[w], w - G(x)\right\}, w - F(x)\right\}\right\} = 0. \tag{3.22}$$

Depending on the regularity assumptions on F and G, the exact definition of the solution of (3.22) relies either on theory of solutions in Sobolev spaces or on theory of viscosity solutions to partial differential equations. We refer to (Brezis, 2010, Chapters 8-9), (Fleming & Soner, 2006, Chapter II), respectively, for the details of these theories.

To finish this section, we mention that the superharmonic characterisation of the value function of an optimal stopping *problem* (Remark 3.1.12) is generalised in Peskir (2008) for the stopping *game* described in this section. It turns out that the value function V, when it exists, admits a so-called semiharmonic characterisation. Roughly speaking, it is the smallest superharmonic function and the largest subharmonic function between G and F.

Our main results (Chapter 4) do not rely on the Markov property, and our approach is closer to the martingale approach described in Section 3.2. For this reason, we omit further details on the semi-harmonic characterisation of Peskir (2008), as well as a review of other important results Ekström & Villeneuve (2006), Ekström (2006), Alvarez (2008) related to *classical* Markovian optimal stopping games. On the other hand, in Chapter 4 we study a (non-Markovian) optimal stopping game *with asymmetric information*, and the existing literature on such games traditionally follows the Markovian approach. Therefore, we illustrate the other aspects of this approach below in Section 3.4, in the framework of asymmetric information games.

3.4 Asymmetric information games

So far, we have only considered games with *full* information, where both players observe the underlying filtration $(\mathcal{F}_t)$, and their strategies are $(\mathcal{F}_t)$-stopping times. This section is devoted to games with one or both players having *incomplete* information. The game itself in such a situation is referred as a game with *asymmetric* information. Incompleteness of information can be formalised via player(s) only having access to a subfiltration $(\mathcal{G}_t)$ of the full information filtration $(\mathcal{F}_t)$. As we will see, the set of players' strategies also changes in this situation.

Recall Section 2.4 and the concepts of mixed/randomised stopping times. As we will see, these concepts are crucial for studying asymmetric information games. The value and Nash equilibrium in such games may not exist in the classical sense of Definitions 3.2.4 and 3.2.6 (see, for instance, an example in (Grün, 2013, Section 2.1)). However, enlarging the set of players' strategies to include mixed/randomised stopping times allows to prove existence results and study properties of the value/Nash equilibrium. This idea is not specific to Dynkin games with asymmetric information and appears in more general game-theoretical frameworks - see e.g. Cardaliaguet & Rainer (2009), Cardaliaguet *et al.* (2016), Gensbittel (2019).

In Section 2.4, we showed that equivalent stopping times induce the equivalent game payoffs. Therefore, in the sequel, while reviewing papers on asymmetric information games, we omit the specifics of the definition of a mixed/randomised stopping time the authors use.

3.4.1 Game with partially observed scenarios

In this section, we review the results of Grün (2013). We demonstrate the methodology the author uses to prove that the value of a certain asymmetric information game exists, and outline the link between the value and the solution of a certain system of partial differential equations (PDEs).

Fix $d \in \mathbb{N}$ and $t \in [0,T]$. Let $(W_u)_{u\in[0,T]}$ be the standard Wiener process in $\mathbb{R}^d$. For $s \in [t,T]$, let $\mathcal{H}_{t,s}$ be the sigma-algebra generated by paths of (W_u) on $[t,s]$. Consider the diffusion process that starts at time t from the point $x \in \mathbb{R}^d$, i.e. the process $(X_s)_{s\in[t,T]}$ with dynamics

$$dX_s^{t,x} = b(s,X_s^{t,x}) + a(s,X_s^{t,x})dW_s, \quad X_t^{t,x} = x,$$

where $a = (a_{k,l})_{1\le k,l\le d}$ and $b : [0,T]\times\mathbb{R}^d \mapsto \mathbb{R}^d$, $a_{k,l} : [0,T]\times\mathbb{R}^d \mapsto \mathbb{R}$ $(1 \le k,l \le d)$ are bounded and Lipschitz continuous functions (see Section A.7 for a justification of the latter assumption).

Let $I \in \mathbb{N}$, and let the *scenario* random variable $\mathcal{I}$ take values $\{1,\dots,I\}$ with probabilities $\{p_1,\dots,p_I\}$. We require $\mathcal{I}$ to be independent of $\mathcal{H}_{0,T}$. The idea behind the asymmetric information game is that $\mathcal{I}$ is observed by one of the players but not by her opponent. For $i \in \{1,\dots,I\}$, let $F_i, G_i, H_i : [0,T]\times\mathbb{R}^d \mapsto \mathbb{R}$ be bounded and Lipschitz continuous functions.

Let us denote by $\mathcal{T}(t)$ the set of $(\mathcal{H}_{t,s})_{s\in[t,T]}$-stopping times, and by $\mathcal{T}^R(t)$ the set of $(\mathcal{H}_{t,s})$-randomised stopping times. For $i \in \{1,\dots,I\}$ and $\tau_i, \sigma \in \mathcal{T}^R(t)$, define the payoff

$$N_i(t,x,\tau_i,\sigma) = \mathbb{E}[F_i(\tau_i, X_{\tau_i}^{t,x})I_{\{\tau_i\le\sigma\}\cap\{\tau_i<T\}} + G_i(\sigma, X_\sigma^{t,x})I_{\{\sigma<\tau_i\}} + H_i(X_T^{t,x})I_{\{\sigma=\tau_i=T\}}].$$

Denote by $\Delta(I)$ the simplex of $\mathbb{R}^I$. For vectors $p := (p_1,\dots,p_I) \in \Delta(I)$ (satisfying $p_i \ge 0$ for all i and $\sum_{i=1}^I p_i = 1$ by definition of the simplex) and $\tau := (\tau_1,\dots,\tau_I) \in (\mathcal{T}^R(t))^I$, let

$$N(t,x,p,\tau,\sigma) = \sum_{i=1}^{I} p_i N_i(t,x,\tau_i,\sigma).$$

The lower and upper value of the game are defined as

$$V_*(t,x,p) = \sup_{\sigma\in\mathcal{T}^R(t)} \inf_{\tau\in(\mathcal{T}^R(t))^I} N(t,x,p,\tau,\sigma) \quad \text{and} \quad V^*(t,x,p) = \inf_{\tau\in(\mathcal{T}^R(t))^I} \sup_{\sigma\in\mathcal{T}^R(t)} N(t,x,p,\tau,\sigma).$$

To prove that the value of the game exists means to prove that, for every $(t,x,p) \in [0,T]\times\mathbb{R}^d\times\Delta(I)$,

$$V_*(t,x,p) = V^*(t,x,p).$$

In order to prove this equality, Grün (2013) uses the theory of viscosity solutions to PDEs. Define the differential operator

$$\mathbb{L}[w](t,x,p) = \frac{1}{2}\mathrm{trace}\left(a(t,x)a^T(t,x)\frac{\partial^2(w(t,x,p))}{\partial x^2}\right) + \langle b(t,x), \frac{\partial(w(t,x,p))}{\partial x}\rangle$$

(recall that the trace of a matrix is the sum of the elements on its main diagonal). Define, for $p \in \Delta(I)$ and a symmetric $I \times I$ matrix A,

$$\lambda_{\min}(p,A) = \min_{z \in C\setminus\{0\}} \frac{\langle Az, z\rangle}{|z|^2},$$

where $C = \overline{\cup_{\lambda>0}(\Delta(I)-p)/\lambda}$ is the tangent cone to $\Delta(I)$ at p.

Finally, consider the following equation on w:

$$\max\left\{\max\left\{\min\left\{(-\frac{\partial}{\partial t} - \mathbb{L})[w], w - \langle G(t,x),p\rangle\right\}, w - \langle F(t,x),p\rangle\right\}, \lambda_{\min}(p, \frac{\partial^2 w}{\partial p^2})\right\} = 0,$$
$$w(T,x,p) = \langle p, H(T,x)\rangle, \tag{3.23}$$

where, similarly to the above, $F = (F_1,\dots,F_I)$, $G = (G_1,\dots,G_I)$, $H = (H_1,\dots,H_I)$.

Roughly speaking, λ_{min} appears in (3.23) as a condition of concavity of w with respect to p. Let us clarify this using the example $I = 2$. If there are only two possible scenarios, then their probabilities satisfy $p_2 = 1 - p_1$. Then, (p_1, p_2) is a point of an interval on the plane, and C is either a line or (if $p_1 = 0$ or 1) a half-line. In this case, it can be proven that the directional derivative $\lambda_{\min}(p, \frac{\partial^2 w}{\partial p^2})$ reduces to the partial derivative $\frac{\partial^2 \widetilde{w}}{\partial p_1^2}$, where $\widetilde{w}(p_1) := w(p_1, 1-p_1)$.

By (Grün, 2013, Theorem 3.4), the value function of the game exists and is a solution of (3.23) in a suitable viscosity sense (Grün, 2013, Definitions 3.1 and 3.2). This result is obtained by first proving certain continuity and convexity properties of the functions V_* and V^* (Grün, 2013, Propositions 5.1 and 5.2), and then by deriving the so-called dynamic programming principles (see (Fleming & Soner, 2006, Section II.3)) for V_* and V^* (Grün, 2013, Theorems 5.8 and 5.3). Finally, the comparison principle (see (Fleming & Soner, 2006, Section V.8)) (Grün, 2013, Theorem 3.3) allows to combine the results for V_* and V^* into (Grün, 2013, Theorem 3.4).

3.4.2 Game with two partially observed dynamics

In the set-up of Grün (2013), as well as in the papers considered below in Sections 3.4.3 and 3.4.4, the information parameters of the game do not evolve over time. In Gensbittel & Grün (2019), they do: the information available to a player consists of observations of a Markov process, and the two players observe two different processes. The payoff of the game then depends on both of the Markov processes. We specify the set-up and overview the results below.

Consider two independent time-homogeneous Markov processes (X_t), (Y_t) with state spaces $\{1,\ldots,I\}$, $\{1,\ldots,J\}$, initial laws $p \in \Delta(I)$, $q \in \Delta(J)$ and infinitesimal generators $R = (R_{i,i'})_{i,i' \in \{1,\ldots,I\}}$, $Q = (Q_{j,j'})_{j,j' \in \{1,\ldots,J\}}$, respectively.

The payoff processes are discounted functions $F \geq G$ of the underlying processes (X_t), (Y_t). That is, the expected payoff of the game, given a choice of random times τ and σ for the first and the second player, reads

$$N_{p,q}(\tau,\sigma) = \mathbb{E}[e^{-r\tau}F(X_t,Y_t)I_{\{\tau<\sigma\}} + e^{-r\sigma}G(X_t,Y_t)I_{\{\sigma\leq\tau\}}].$$

The minimiser is assumed to have access to the information $(\mathcal{F}^X_t)$ (recall Section 2.3) and the maximiser is assumed to have access to the information $(\mathcal{F}^Y_t)$. The upper and lower value functions of the game are defined as

$$V_*(p,q) = \sup_{\sigma\in\mathcal{T}^R(\mathcal{F}^Y_t)} \inf_{\tau\in\mathcal{T}^R(\mathcal{F}^X_t)} N_{p,q}(\tau,\sigma); \quad V^*(p,q) = \inf_{\tau\in\mathcal{T}^R(\mathcal{F}^X_t)} \sup_{\sigma\in\mathcal{T}^R(\mathcal{F}^Y_t)} N_{p,q}(\tau,\sigma).$$

The authors prove (Gensbittel & Grün, 2019, Theorem 3.3) that the game has a value $V(p,q)$ for all $(p,q) \in \Delta(I)\times\Delta(J)$, and provide the following variational characterisation of V:

$$\begin{aligned}\max\Big\{ \min\Big\{ rV(p,q) - \langle\nabla_p V(p,q), R^T p\rangle - \langle\nabla_q V(p,q), Q^T q\rangle, V(p,q) - G(p,q)\Big\},\\ V(p,q) - F(p,q)\Big\} = 0,\end{aligned} \tag{3.24}$$

where $\nabla_p V(p,q)$ and $\nabla_q V(p,q)$ are the components of the gradient of V corresponding to vectors p and q, respectively. As with the characterisation (3.23) of the value function in Grün (2013), we omit the rigorous definition of the solution to (3.24) and the related constraints. Similarly to (Grün, 2013, Theorem 3.4), the proof of (Gensbittel & Grün, 2019, Theorem 3.3) relies on certain continuity and convexity properties of the function V^* (Gensbittel & Grün, 2019, Lemma 3.5) and on the dynamic programming principle (Gensbittel & Grün, 2019, Proposition 3.7). The corresponding properties of V_* follow by symmetry of the model, and the comparison principle (Gensbittel & Grün, 2019, Theorem 3.12) finishes the proof.

3.4.3 Games with a single partially observed dynamics

In De Angelis *et al.* (2021b) and De Angelis *et al.* (2021a), the incomplete/asymmetric information feature is modelled differently from Grün (2013) and Gensbittel & Grün (2019). The underlying dynamic (X_t) itself is not fully observable by one (De Angelis *et al.* (2021a)) or both (De Angelis *et al.* (2021b)) of the players.

Let us start by describing the set-up of De Angelis *et al.* (2021b). Let the underlying process be a geometric Brownian motion with a random drift that is unobservable by both players. More precisely, let

$$dX_t = (r - \delta_0 \mathcal{I})X_t dt + \sigma X_t dW_t, \quad X_0 = x > 0,$$

where (W_t) is the standard Wiener process in $\mathbb{R}$, r, δ_0, σ are positive constants, and $\mathcal{I}$ is a random variable that takes values 0 or 1 with $\mathbb{P}(\mathcal{I} = 1) = \pi$ and is independent of (W_t). The constant π is assumed to be known to both players, but none of them observes $\mathcal{I}$.

The payoff functions of the game are specified as follows: fix $K, \varepsilon_0 > 0$ and consider

$$G(x) = (x - K)^+, \; F(x) = (x - K)^+ + \varepsilon_0.$$

Note that the integrability condition (3.6) is not satisfied for the functions G, F (De Angelis *et al.*, 2021b, Remark 2.1).

Let $\mathcal{T} = \mathcal{T}(\hat{\mathcal{F}}^X_t)$, where $(\hat{\mathcal{F}}^X_t)$ is the filtration generated by (X_t) augmented in a certain way (we omit further details as they are not essential to describe the problem). For $\tau, \sigma \in \mathcal{T}$, the expected payoff of the game is defined as

$$N_{x,\pi}(\tau, \sigma) = \mathbb{E}[e^{-r\tau} F(X_\tau) I_{\{\tau < \sigma\}} + e^{-r\sigma} G(X_\sigma) I_{\{\sigma \le \tau\}}].$$

Note that, since $\delta_0 > 0$, the speed of growth of trajectories of $F(X_t)$ and $G(X_t)$ is $\mathbb{P}$-a.s. at most exponential with the rate smaller than r, and therefore we have

$$\limsup_{t \to \infty} e^{-rt} F(X_t) = \limsup_{t \to \infty} e^{-rt} G(X_t) = 0, \; \mathbb{P}\text{-a.s.}$$

(observe the similarity with the condition at infinity for the full information game without discounting (3.20)). The upper and lower value functions are

$$V_*(x, \pi) = \sup_{\sigma \in \mathcal{T}} \inf_{\tau \in \mathcal{T}} N_{x,\pi}(\tau, \sigma); \quad V^*(x, \pi) = \inf_{\tau \in \mathcal{T}} \sup_{\sigma \in \mathcal{T}} N_{x,\pi}(\tau, \sigma).$$

Note that, due to the dependence on $\mathcal{I}$, the process (X_t) is not Markovian. In (De Angelis *et al.*, 2021b, Section 2), the authors apply the filtering theory (Lipster & Shiryaev, 2001, Chapter 9) in order to increase the dimension of the state space and formulate an equivalent Markovian problem. Informally speaking, filtering is used to progressively update the players' estimate on $\mathcal{I}$ based on their observation of (X_t). The existence of the value and Nash equilibrium (De Angelis *et al.*, 2021b, Theorem 3.2) are then closely related to the classical results described in Section 3.3.

We emphasise that no randomisation is needed in this set-up: the strategies τ,σ are pure stopping times. An intuitive reason is that the information available to players is incomplete but symmetric (they both cannot observe $\mathcal{J}$). On the contrary, in De Angelis *et al.* (2021a), one of the players learns the true value of $\mathcal{J}$ as soon as the game starts. We proceed to describe the resulting asymmetric information game.

The underlying dynamics of the game in De Angelis *et al.* (2021a) is a diffusion whose drift depends on the realisation of the random variable $\mathcal{J}$ ($\mathcal{J}$ as above) in a general way, i.e. a process (X_t) in $\mathbb{R}$ with the dynamics

$$dX_t = \sum_{i=0}^{1} I_{\{\mathcal{J}=i\}}\mu_i(X_t)dt + \sigma(X_t)dW_t, \quad X_0 = x,$$

where (W_t) is the standard Wiener process in $\mathbb{R}$, and μ_0, μ_1, σ are positive continuous functions.

The payoff processes are continuous functions F,G of the underlying process. That is, the expected payoff of the game, given a choice of random times τ and σ for the first and the second player, reads

$$N_{x,\pi}(\tau,\sigma) = \mathbb{E}[F(X_\tau)I_{\{\tau\leq\sigma\}} + G(X_\sigma)I_{\{\sigma<\tau\}}].$$

We assume that $F \geq G \geq 0$ (c.f. the order condition in the classical set-up (3.19)). Note that the functions F,G used in De Angelis *et al.* (2021b) by definition satisfy this property.

As we already mentioned, one of the players (the minimiser) is assumed to have access to the information $\mathcal{J}$. Her information flow is therefore modelled as $\mathcal{F}_t^{X,\mathcal{J}} := \mathcal{F}_t^X \vee \sigma(\mathcal{J})$ (recall Section 2.3 for the definition of the filtration $(\mathcal{F}_t^X)$). The upper and lower value functions of the game are defined as

$$V_*(x,\pi) = \sup_{\sigma\in\mathcal{T}(\mathcal{F}_t^X)} \inf_{\tau\in\mathcal{T}^R(\mathcal{F}_t^{X,\mathcal{J}})} N_{x,\pi}(\tau,\sigma); \quad V^*(x,\pi) = \inf_{\tau\in\mathcal{T}^R(\mathcal{F}_t^{X,\mathcal{J}})} \sup_{\sigma\in\mathcal{T}(\mathcal{F}_t^X)} N_{x,\pi}(\tau,\sigma).$$

The intuition behind τ being randomised is that the informed player uses randomisation in order to "gradually reveal" the information to their opponent when optimal. We also note the parallel with the set-up of Section 3.4.1, where the informed player uses a vector of randomised stopping times as her strategy. This parallel is clear upon noticing that $\tau \in \mathcal{T}^R(\mathcal{F}_t^{X,\mathcal{J}})$ can be decomposed as in Lemma B.1.2:

$$\tau = \tau_0 I_{\{\mathcal{J}=0\}} + \tau_1 I_{\{\mathcal{J}=1\}},$$

where $\tau_0,\tau_1 \in \mathcal{T}^R(\mathcal{F}_t^X)$.

An important step in solving the problem is to rewrite the game in terms of singular controls - namely, the generating processes of the randomised stopping times τ_0, τ_1 (De Angelis *et al.*, 2021a, Proposition 3.1). We will heavily rely on this idea in Section 4.3. For the set-up of the current section, it allows to characterise the value function of the game as a solution of a certain quasi-variational inequality (De Angelis *et al.*, 2021a, (42)-(43)) that comes with a set of non-standard constraints related to the informational asymmetry (De Angelis *et al.*, 2021a, Theorem 5.1).

3.4.4 Game with a random horizon

In Lempa & Matomäki (2013), the information asymmetry is introduced in yet another way. Consider a classical Markovian stopping game (3.18). Assume additionally that it stops at an exogenous random time θ (more formally, that its payoff is zero after the time θ). Assume that only one of the players observes the occurrence of θ, and that only she is able to make a stopping decision at the time θ. We formalise this description below.

The underlying dynamics of the game is a one-dimensional diffusion

$$dX_t = \mu(X_t)dt + \sigma(X_t)dW_t, \quad X_0 = x.$$

The payoff functions F, G, H are assumed to be continuous, non-decreasing, and to satisfy $c \leq G \leq H \leq F$ for some constant $c \in \mathbb{R}$ (c.f. (3.19)). Further, they are assumed to satisfy

$$\mathbb{E}_x\left[\int_0^\infty e^{-rt}|F(X_t)+G(X_t)+H(X_t)|\right] < \infty \text{ for all } x,$$

where the discounting rate $r \in \mathbb{R}$ is fixed (c.f. the integrability condition for Markovian problems without discounting (3.6)).

Let θ be an exponentially distributed random variable independent from (X_t). Let $\Lambda_t := I_{\{t\geq\theta\}}$, and let $\mathcal{F}_t^{X,\theta} := \mathcal{F}_t^X \vee \sigma(\Lambda_s, 0 \leq s \leq t)$. The occurrence of θ indicates the end of the game that only one of the players observes. That is, the expected game payoff is defined as

$$N_x(\tau,\sigma) = \mathbb{E}_x\left[\left(e^{-r\tau}F(X_\tau)I_{\{\tau<\sigma\}} + e^{-r\sigma}G(X_\sigma)I_{\{\sigma<\tau\}} + e^{-r\tau}H(X_\tau)I_{\{\tau=\sigma\}}\right)I_{\{\tau\wedge\sigma\leq\theta\}}\right].$$

Lempa & Matomäki (2013) do not fix which player is informed (has access to the filtration $(\mathcal{F}_t^{X,\theta})$), but instead consider two symmetric definitions of the lower/upper value (and thus two symmetric games):

$$V_*(x) = \sup_{\sigma\in\mathcal{T}(\mathcal{F}_t^X)} \inf_{\tau\in\mathcal{T}(\mathcal{F}_t^{X,\theta})} N_x(\tau,\sigma); \quad V^*(x) = \inf_{\tau\in\mathcal{T}(\mathcal{F}_t^{X,\theta})} \sup_{\sigma\in\mathcal{T}(\mathcal{F}_t^X)} N_x(\tau,\sigma)$$

and

$$\widehat{V}_*(x) = \sup_{\sigma \in \mathcal{T}(\mathcal{F}_t^{X,\theta})} \inf_{\tau \in \mathcal{T}(\mathcal{F}_t^X)} N_x(\tau,\sigma); \quad \widehat{V}^*(x) = \inf_{\tau \in \mathcal{T}(\mathcal{F}_t^X)} \sup_{\sigma \in \mathcal{T}(\mathcal{F}_t^{X,\theta})} N_x(\tau,\sigma).$$

The analysis of the two games is analogous. It relies on the specific structure of $(\mathcal{F}_t^{X,\theta})$ thanks to which, for any $\eta \in \mathcal{T}(\mathcal{F}_t^{X,\theta})$, there exists $\widetilde{\eta} \in \mathcal{T}(\mathcal{F}_t^X)$ such that $\eta \wedge \theta = \widetilde{\eta} \wedge \theta$ $\mathbb{P}$-a.s. (Lempa & Matomäki, 2013, Lemma 3.1). This allows to rewrite the lower/upper value of the games in a way that only involves $(\mathcal{F}_t^X)$-stopping times (Lempa & Matomäki, 2013, Proposition 3.2, Proposition 4.1). These auxiliary expressions involve the different payoffs that are obtained, roughly speaking, by integrating out the information θ from the payoff N_x. The existence of the value then follows from the classical results of Section 3.3. Using the diffusion structure of the problem, the authors provide explicit expressions for the value (Lempa & Matomäki, 2013, (3.9), (4.6)), and study its asymptotic behaviour with respect to the parameter of the distribution of θ.

3.5 Approach based on functional analysis

In this section, we provide a review of Touzi & Vieille (2002) which largely influenced the approach we take in Chapter 4. Touzi & Vieille (2002) apply the ideas discussed in Section 3.1.3 to Dynkin games (with full information).

3.5.1 Setting

Consider $(f_t)_{t\in[0,T]}, (g_t)_{t\in[0,T]}, (h_t)_{t\in[0,T]} \in \mathcal{L}_b$. Let $\mathcal{F}_t = \mathcal{F}_t^f \vee \mathcal{F}_t^g \vee \mathcal{F}_t^h$ (recall Section 2.3). As in (3.13), the payoff of the game is set to be

$$\mathcal{P}(\tau,\sigma) = f_\tau I_{\{\tau<\sigma\}} + g_\sigma I_{\{\sigma<\tau\}} + h_\tau I_{\{\sigma=\tau\}}. \tag{3.25}$$

Recall that in Lepeltier & Maingueneau (1984), the value of the game was proven to exist for the infinite horizon game without the "middle" payoff (h_t) (i.e. therein $h_t = f_t$ for all t) and under the order assumption

$$f_t \geq g_t \text{ for all } t \in [0,\infty) \ \mathbb{P}\text{-a.s.}$$

Touzi & Vieille (2002) relax the order assumption and prove that the value of the game exists if the players are allowed to use randomised stopping time as their strategies.

Recall the set $\mathcal{A}^\circ(\mathcal{F}_t)$ of generating processes of randomised stopping times defined in (2.4). Instead of studying the expected payoff functional $(\tau,\sigma) \mapsto \mathbb{E}[\mathcal{P}(\tau,\sigma)]$, consider a functional acting on this set: for $(\xi_t), (\zeta_t) \in \mathcal{A}^\circ(\mathcal{F}_t)$, let

$$N(\xi,\zeta) = \mathbb{E}\left[\int_{[0,T]} f_t(1-\zeta_t)d\xi_t + \int_{[0,T]} g_t(1-\xi_t)d\zeta_t + \sum_{t\in[0,T]} h_t \Delta\xi_t \Delta\zeta_t\right], \tag{3.26}$$

where $\Delta\xi_t = \xi_t - \xi_{t-}$ and $\Delta\zeta_t = \zeta_t - \zeta_{t-}$. From Section 2.4.1 we know that, for $\tau,\sigma \in \mathcal{T}^R(\mathcal{F}_t)$ with generating processes $(\xi_t), (\zeta_t)$, we have

$$N(\xi,\zeta) = \mathbb{E}[\mathcal{P}(\tau,\sigma)],$$

so the notation N that we commonly use for the expected payoff of a game comes at no surprise.

Remark 3.5.1. *In the sequel, we will introduce a payoff* (4.9)

$$N(\xi,\zeta) = \mathbb{E}\left[\int_{[0,T)} f_t(1-\zeta_{t-})d\xi_t + \int_{[0,T)} g_t(1-\xi_t)d\zeta_t + \tilde{h}\Delta\xi_T\Delta\zeta_T\right], \tag{3.27}$$

where $\tilde{h}$ is an $\mathcal{F}_T$-measurable random variable. This payoff is a particular case of (3.26)*, to which* (3.26) *reduces if $h_t = f_t$ on $[0,T)$. Indeed, in this case,*

$$\int_{[0,T)} f_t(1-\zeta_t)d\xi_t + \sum_{t\in[0,T]} h_t\Delta\xi_t\Delta\zeta_t = \int_{[0,T)} f_t(1-\zeta_{t-})d\xi_t + h_T\Delta\xi_T\Delta\zeta_T$$

by definition of $\Delta\zeta_t$ and $\Delta\xi_t$, and the random variable h_T becomes the same as the random variable $\tilde{h}$ in (3.27)*. See Section 4.5.4 for a further connection between the payoffs* (3.26) *and* (3.27)*.*

The lower and upper value of the game are then defined via the functional (3.26) as

$$V_* = \sup_{\zeta\in\mathcal{A}^\circ(\mathcal{F}_t)} \inf_{\xi\in\mathcal{A}^\circ(\mathcal{F}_t)} N(\xi,\zeta), \qquad V^* = \inf_{\xi\in\mathcal{A}^\circ(\mathcal{F}_t)} \sup_{\zeta\in\mathcal{A}^\circ(\mathcal{F}_t)} N(\xi,\zeta).$$

The main result of Touzi & Vieille (2002) is the that the game has a value in this extended sense under the assumptions of the theorem below.

Theorem 3.5.2. *(Touzi & Vieille, 2002, Theorem 3.1) Let $(f_t), (g_t), (h_t) \in \mathcal{L}_b$. Let (f_t), (g_t) be semimartingales with trajectories continuous at time T, $\mathbb{P}$-a.s. Let $f_t \geq h_t$ for all $t \in [0,T]$ $\mathbb{P}$-a.s. Then,*

$$V_* = V^*.$$

The proof of Theorem 3.5.2 involves an application of a general min-max theorem known as Sion's theorem.

Theorem 3.5.3 (Sion's theorem). *(Sion, 1958, Corollary 3.3) Let A and B be convex subsets of a linear topological space one of which is compact. Let $\varphi(\mu,\nu)$ be a function $A\times B\mapsto\mathbb{R}$ that is quasi-concave and upper semicontinuous in μ for each $\nu\in B$, and quasi-convex and lower semicontinuous in ν for each $\mu\in A$. Then,*

$$\sup_{\mu\in A}\inf_{\nu\in B}\varphi(\mu,\nu)=\inf_{\nu\in B}\sup_{\mu\in A}\varphi(\mu,\nu).$$

Definitions of terms used in the theorem are listed in Section A.4.1. For its proof and related examples, we refer to Section 3.6.

The difficulty arises from the fact that the set $\mathcal{A}^\circ(\mathcal{F}_t)$ does not satisfy the assumptions of Sion's theorem (this will become clear in Section 3.5.3, after introducing the topology we work in). In order to overcome this, we will consider auxiliary subsets of the set of players' controls (and quantities that can be viewed as lower/upper values of auxiliary games). We then prove that these auxiliary games have the same lower and upper value V_* and V^*. This is achieved by constructing a sequence of controls that approximate a general control from $\mathcal{A}^\circ(\mathcal{F}_t)$ in a suitable sense. The final step is to verify the conditions of Sion's theorem and apply it to show that the auxiliary games have a value.

More precisely, define

$$\mathcal{A}_1=\{(\xi_t)\in\mathcal{A}^\circ(\mathcal{F}_t): f_T\Delta\xi_T\le 0\ \mathbb{P}\text{-a.s.}\};\quad \mathcal{A}_2=\{(\zeta_t)\in\mathcal{A}^\circ(\mathcal{F}_t):(\zeta_t)\text{ is continuous }\mathbb{P}\text{-a.s.}\}.$$

These sets will act as the sets of players' strategies in the auxiliary games for which, as we will see, an application of Sion's theorem yields existence of the value. The following are the main steps in the proof of Theorem 3.5.2.

Proposition 3.5.4. *(Touzi & Vieille, 2002, Proposition 4.1) For $(f_t),(g_t),(h_t)\in\mathcal{L}_b$,*

$$\sup_{\zeta\in\mathcal{A}_2}\inf_{\xi\in\mathcal{A}_1}N(\xi,\zeta)=\sup_{\zeta\in\mathcal{A}_2}\inf_{\xi\in\mathcal{A}^\circ(\mathcal{F}_t)}N(\xi,\zeta).$$

Proposition 3.5.5. *(Touzi & Vieille, 2002, Proposition 4.2) Under the assumptions of Theorem 3.5.2,*

$$\inf_{\xi\in\mathcal{A}_1}\sup_{\zeta\in\mathcal{A}_2}N(\xi,\zeta)=\inf_{\xi\in\mathcal{A}_1}\sup_{\zeta\in\mathcal{A}^\circ(\mathcal{F}_t)}N(\xi,\zeta).$$

Proposition 3.5.6. *(Touzi & Vieille, 2002, Proposition 4.3) Let $(f_t),(g_t),(h_t)\in\mathcal{L}_b$. Let $(f_t),(g_t)$ be semimartingales. Then,*

$$\sup_{\zeta\in\mathcal{A}_2}\inf_{\xi\in\mathcal{A}_1}N(\xi,\zeta)=\inf_{\xi\in\mathcal{A}_1}\sup_{\zeta\in\mathcal{A}_2}N(\xi,\zeta).$$

With these results in place, recalling that $\mathcal{A}_2, \mathcal{A}_1 \subset \mathcal{A}^\circ(\mathcal{F}_t)$, we deduce

$$\begin{aligned} V_* = & \sup_{\zeta \in \mathcal{A}^\circ(\mathcal{F}_t)} \inf_{\xi \in \mathcal{A}^\circ(\mathcal{F}_t)} N(\xi,\zeta) \geq \sup_{\zeta \in \mathcal{A}_2} \inf_{\xi \in \mathcal{A}^\circ(\mathcal{F}_t)} N(\xi,\zeta) = \sup_{\zeta \in \mathcal{A}_2} \inf_{\xi \in \mathcal{A}_1} N(\xi,\zeta) \\ = & \inf_{\xi \in \mathcal{A}_1} \sup_{\zeta \in \mathcal{A}_2} N(\xi,\zeta) = \inf_{\xi \in \mathcal{A}_1} \sup_{\zeta \in \mathcal{A}^\circ(\mathcal{F}_t)} N(\xi,\zeta) \geq \inf_{\xi \in \mathcal{A}^\circ(\mathcal{F}_t)} \sup_{\zeta \in \mathcal{A}^\circ(\mathcal{F}_t)} N(\xi,\zeta) = V^*. \end{aligned}$$

The reverse inequality is a general property of the lower/upper value of a game (Remark 3.2.5), and the statement of Theorem 3.5.2 follows.

3.5.2 Approximation with auxiliary controls

The proof of Proposition 3.5.4 relies on constructing, for arbitrary $(\xi_t) \in \mathcal{A}^\circ(\mathcal{F}_t)$, a strategy $(\widehat{\xi}_t) \in \mathcal{A}_1$ as

$$\begin{aligned} &\widehat{\xi}_T = \xi_{T-} \text{ on the event } \{f_T > 0\}; \\ &\widehat{\xi} = \xi \text{ otherwise.} \end{aligned}$$

It is then verified that, for every $(\zeta_t) \in \mathcal{A}_2$,

$$N(\widehat{\xi},\zeta) \leq N(\xi,\zeta),$$

which implies that

$$\sup_{\zeta \in \mathcal{A}_2} \inf_{\xi \in \mathcal{A}_1} N(\xi,\zeta) \leq \sup_{\zeta \in \mathcal{A}_2} \inf_{\xi \in \mathcal{A}^\circ(\mathcal{F}_t)} N(\xi,\zeta).$$

The opposite inequality follows from the fact that $\mathcal{A}_1 \subset \mathcal{A}^\circ(\mathcal{F}_t)$, and the statement of Proposition 3.5.4 follows.

The proof of Proposition 3.5.5 employs a similar idea but requires extra technical steps. Define another subset of the set of players' controls as

$$\mathcal{A}_3 = \{(\zeta_t) \in \mathcal{A}^\circ(\mathcal{F}_t) : \Delta\zeta_T = 0 \text{ on } \{f_T \geq 0, g_T > 0\}\}.$$

By (Touzi & Vieille, 2002, Lemma 5.1), for any $(\xi_t) \in \mathcal{A}_1$ and $(\zeta_t) \in \mathcal{A}^\circ(\mathcal{F}_t)$, there exists a sequence $(\zeta_t^n)_{n\geq 1} \subset \mathcal{A}_3$ such that

$$\limsup_{n\to\infty} N(\xi,\zeta^n) \geq N(\xi,\zeta).$$

The construction of the sequence (ζ_t^n) utilises Snell envelopes of the processes (f_t), (g_t) and relies on their continuity at the terminal time T.

Further, in (Touzi & Vieille, 2002, Lemma 5.2), for arbitrary strategy $(\zeta_t) \in \mathcal{A}_3$, a sequence $(\zeta_t^n)_{n\geq 1} \subset \mathcal{A}_2$ is defined as

$$\zeta_t^n = \zeta_t - \sum_{s<t} \Delta\zeta_s I_{\{\Delta\zeta_s \leq \frac{1}{n}\}}. \tag{3.28}$$

It is then proved that, for any $(\xi_t) \in \mathcal{A}_1$,

$$\limsup_{n\to\infty} N(\xi, \zeta^n) \geq N(\xi, \zeta).$$

The proof uses integration by parts of the integral terms in (3.26), and therefore relies on the semimartingale assumption on (f_t), (g_t) that ensures that the stochastic integrals with respect to these processes are well-defined.

Applying (Touzi & Vieille, 2002, Lemma 5.1, 5.2), we see that

$$\inf_{\xi\in\mathcal{A}_1} \sup_{\zeta\in\mathcal{A}_2} N(\xi,\zeta) \geq \inf_{\xi\in\mathcal{A}_1} \sup_{\zeta\in\mathcal{A}^\circ(\mathcal{F}_t)} N(\xi,\zeta).$$

The opposite inequality is due to $\mathcal{A}_2 \subset \mathcal{A}^\circ(\mathcal{F}_t)$, and the statement of Proposition 3.5.5 follows.

3.5.3 Verification of the conditions of Sion's theorem

Consider the Banach space $\mathcal{S}$ of $(\mathcal{F}_t)$-adapted processes (ρ_t) with $\rho_{0-} = 0$ and

$$||\rho||^2 := \mathbb{E}\left[\int_0^T (\rho_t)^2 dt + (\Delta\rho_T)^2\right] < \infty, \quad \Delta\rho_T := \rho_T - \liminf_{t\uparrow T} \rho_t, \tag{3.29}$$

and consider the weak topology on $\mathcal{S}$ (see Section A.3.1). The weak topology is used to prove the compactness and continuity results required to apply Sion's theorem. The equivalence of strong and weak closedness for convex sets (Theorem A.3.3) allows to work mainly in the strong topology on $\mathcal{S}$ (induced by the norm (3.29)).

As we already mentioned, the set $\mathcal{A}^\circ(\mathcal{F}_t)$ does not satisfy the assumptions of Sion's theorem (Theorem 3.5.3) — in particular, this set is not compact in the weak topology on $\mathcal{S}$, and the continuity properties of the functional N are violated on this set. However, both conditions can be verified for sets $\mathcal{A}_2$, $\mathcal{A}_1$ in place of A, B in Sion's theorem.

The first condition of Sion's theorem is verified by (Touzi & Vieille, 2002, Lemma 6.1) that proves compactness of the set $\mathcal{A}_1$ in the weak topology. The proof goes through showing the convexity of $\mathcal{A}_1$ and the sequential closedness of $\mathcal{A}_1$ in the strong topology on $\mathcal{S}$, and applying Theorem A.3.3.

In order to verify the continuity conditions of Sion's theorem, (Touzi & Vieille, 2002, Lemma 6.3) shows continuity of the mapping $\zeta \mapsto N(\xi, \zeta)$ for all $(\xi_t) \in \mathcal{A}_1$ in the strong topology on

$\mathcal{S}$. The proof is sequential: for arbitrary $(\xi_t) \in \mathcal{A}_1$, take an arbitrary sequence $(\zeta_t^n) \subset \mathcal{A}_2$ that converges to $(\zeta_t) \in \mathcal{A}_2$, and prove the convergence

$$\lim_{n\to\infty} N(\xi, \zeta^n) = N(\xi, \zeta).$$

The proof again utilises integration by parts and hence relies on the semimartingale assumption. After several technical steps, the convergence result is obtained by the dominated convergence theorem.

Similarly, (Touzi & Vieille, 2002, Lemma 6.4) shows continuity of the mapping $(\xi_t) \mapsto N(\xi, \zeta)$ for all $(\zeta_t) \in \mathcal{A}_2$ in the strong topology on $\mathcal{S}$. The proof is analogous to the proof of (Touzi & Vieille, 2002, Lemma 6.3).

The convexity requirements of Sion's theorem are fulfilled by bilinearity of the functional N. They also allow to apply Theorem A.3.3 again to deduce from the continuity results of (Touzi & Vieille, 2002, Lemma 6.3, 6.4) that hold in the strong topology the continuity properties of the functional N in the weak topology. This, together with the weak compactness established in (Touzi & Vieille, 2002, Lemma 6.1), finishes the verification of conditions of Sion's theorem, and its application finishes the proof of Proposition 3.5.6.

3.6 Sion's theorem: proof and examples

Sion's theorem (Theorem 3.5.3) is a key tool for proving the existence of the value not only in Section 3.5, but also in the main chapter of the thesis — Chapter 4. The theorem was originally proved in Sion (1958) using a topological result known as the Knaster–Kuratowski–Mazurkiewicz theorem Knaster *et al.* (1929). In this section, we review a more straightforward proof of Sion's theorem Komiya (1988). In fact, below we work under assumptions symmetric to Komiya (1988). Recall that in Sion's theorem, one of the spaces in question is compact. In Komiya (1988), the infimum is taken over a compact space, while for our purposes (see Theorem 4.4.5) it will be convenient to have the supremum taken over a compact space instead.

Lemma 3.6.1. *(Komiya, 1988, Lemma 1) Under the assumptions of Theorem 3.5.3, for any* $\nu_1, \nu_2 \in B$ *and any real number* $\alpha > \sup_{\mu\in A} \min\{\phi(\mu,\nu_1), \phi(\mu,\nu_2)\}$*, there exists* $\nu \in B$ *such that* $\alpha > \sup_{\mu\in A} \phi(\mu, \nu)$.

It is worth emphasising that the proof of Lemma 3.6.1 is elementary, in a sense that it only uses definitions and basic properties of the conditions in Sion's theorem (see Section A.4.1), and does not use non-standard results, unlike the proofs in Sion (1958). From Lemma 3.6.1, one can by induction deduce the following.

Lemma 3.6.2. *(Komiya, 1988, Lemma 2) Under the assumptions of Theorem 3.5.3, for any* $\nu_1, \dots, \nu_n \in B$ *and any real number* $\alpha > \sup_{\mu \in A} \min_{1 \le i \le n} \{\phi(\mu, \nu_i)\}$*, there exists* $\nu \in B$ *such that* $\alpha > \sup_{\mu \in A} \phi(\mu, \nu)$.

We now reproduce a proof from Komiya (1988) in order to show how Theorem 3.5.3 follows from Lemma 3.6.2, as well as to introduce some notation that will be useful in the sequel. Recall that we work under assumption that the space A in Theorem 3.5.3 is the one that is compact.

Proof of Theorem 3.5.3. Fix $\alpha \in \mathbb{R}$ such that $\alpha > \sup_{\mu \in A} \inf_{\nu \in B} \phi(\mu, \nu)$. For every $\nu \in B$, the level set

$$\mathcal{Z}^{\alpha}(\nu) := \{\mu \in A : \phi(\mu, \nu) \ge \alpha\}$$

is closed by the upper semicontinuity of $\phi(\cdot, \nu)$ (Theorem A.4.5). By the choice of α, we have $\cap_{\nu \in B} \mathcal{Z}^{\alpha}(\nu) = \emptyset$. In other words, sets $\{A \setminus \mathcal{Z}^{\alpha}(\nu) : \nu \in B\}$ form an open cover of the compact A. Hence, by Definition A.2.2, there exist $\nu_1, \dots, \nu_n \in B$ such that $\cap_{i=1}^{n} \mathcal{Z}^{\alpha}(\nu_n) = \emptyset$, i.e. $\alpha > \sup_{\mu \in A} \min_{1 \le i \le n} \phi(\mu, \nu_i)$. By Lemma 3.6.2, there exists $\nu \in B$ such that $\alpha > \sup_{\mu \in A} \phi(\mu, \nu)$, and hence $\alpha > \inf_{\nu \in B} \sup_{\mu \in A} \phi(\mu, \nu)$. Therefore,

$$\sup_{\mu \in A} \inf_{\nu \in B} \phi(\mu, \nu) \ge \inf_{\nu \in B} \sup_{\mu \in A} \phi(\mu, \nu).$$

The reverse inequality is always true (Remark 3.2.5), which finishes the proof of Theorem 3.5.3. □

Let us provide a lemma that allows to write max instead of sup in Theorem 3.5.3.

Lemma 3.6.3. *Under the assumptions of Theorem 3.5.3 with the space A compact, there exists* $\mu^* \in A$ *such that*

$$\sup_{\mu \in A} \inf_{\nu \in B} \phi(\mu, \nu) = \inf_{\nu \in B} \phi(\mu^*, \nu).$$

Proof. Consider the mapping $\hat{\phi} : A \mapsto \mathbb{R}$ defined as $\hat{\phi}(\mu) = \inf_{\nu \in B} \phi(\mu, \nu)$. Take arbitrary $\alpha \in \mathbb{R}$. Consider the level set for $\hat{\phi}$:

$$\hat{\mathcal{Z}}^{\alpha} = \{\mu \in A : \inf_{\nu \in B} \phi(\mu, \nu) \ge \alpha\}.$$

Then, $\hat{\mathcal{Z}}^{\alpha} = \cap_{\nu \in B} \mathcal{Z}^{\alpha}(\nu)$. For any $\nu \in B$, by the upper semicontinuity of $\phi(\cdot, \nu)$, the set $\mathcal{Z}^{\alpha}(\nu)$ is closed (Theorem A.4.5). Therefore, the set $\hat{\mathcal{Z}}^{\alpha}$ is closed for arbitrary $\alpha \in \mathbb{R}$, i.e. the function $\hat{\phi}$ is upper semicontinuous. Since the space A is compact, by Theorem A.4.6 $\hat{\phi}$ attains its supremum on A, which finishes the proof. □

We conclude with the examples illustrating the necessity of conditions in Theorem 3.5.3.

Example 3.6.4. *Consider* $\phi : \mathbb{R} \times \mathbb{R} \mapsto \mathbb{R}$ *such that* $\phi(\mu, \nu) = \mu + \nu$. *Then*

$$\sup_{\mu \in \mathbb{R}} \inf_{\nu \in \mathbb{R}} \phi(\mu, \nu) = -\infty,$$

but

$$\inf_{\nu \in \mathbb{R}} \sup_{\mu \in \mathbb{R}} \phi(\mu, \nu) = +\infty,$$

so the infimum and the supremum cannot be swapped. This is due to $\mathbb{R}$ *not being a compact space, so Theorem 3.5.3 does not apply.*

Example 3.6.5. *Consider* $\phi : [0,1] \times [0,1] \mapsto \mathbb{R}$ *such that* $\phi(\mu, \nu) = \mu^2 - 2\mu\nu + \nu^2$. *Consider, for a fixed* $\hat{\nu} \in [0,1]$*, the task of maximising* $\phi(\mu, \hat{\nu})$ *over* μ*. The maximum*

$$\max_{\mu \in [0,1]} \phi(\mu, \hat{\nu}) = \max_{\mu \in [0,1]} (\mu^2 - 2\mu\hat{\nu} + \hat{\nu}^2)$$

is attained at the boundary $\mu = 0$ *or* $\mu = 1$*, and equals* $\hat{\nu}^2 \vee (\hat{\nu}^2 - 2\hat{\nu} + 1)$*. Then it is easy to see that*

$$\min_{\nu \in [0,1]} \max_{\mu \in [0,1]} \phi(\mu, \nu) = \frac{1}{4}.$$

On the other hand, consider, for a fixed $\hat{\mu} \in [0,1]$*, the task of minimising* $\phi(\hat{\mu}, \nu)$ *over* ν*. The minimum*

$$\min_{\nu \in [0,1]} \phi(\hat{\mu}, \nu) = \min_{\nu \in [0,1]} (\hat{\mu}^2 - 2\hat{\mu}\nu + \nu^2)$$

is attained at $\nu = \hat{\mu}$*, and equals* 0*, therefore*

$$\max_{\mu \in [0,1]} \min_{\nu \in [0,1]} \phi(\mu, \nu) = 0.$$

Thus, the infimum and the supremum (which are the minimum and the maximum in this set-up) cannot be swapped. This is due to ϕ *not being quasi-concave (Definition A.4.2) in* μ *for each* ν*, so Theorem 3.5.3 does not apply. Indeed, the quasi-concavity is violated because, for example, for* $\hat{\nu} = \frac{1}{2}$ *the level set* $\{\mu \in [0,1] : \mu^2 - 2\mu\hat{\nu} + \hat{\nu}^2 \geq \frac{1}{4}\} = \{0,1\}$*, which is not connected and therefore is not convex.*

Chapter 4

Value of asymmetric information games

In this chapter, we develop a framework for studying the existence of the value in zero-sum Dynkin games with partial/asymmetric information. In contrast with most literature on such games (see Section 3.4), our set-up is non-Markovian. The games are considered on both the finite and infinite-time horizon and, as always in the thesis, the horizon is denoted by T. We assume that the payoff processes are the sum of a *regular* process (in the sense of Meyer (1978); recall Definition 2.3.2) and a pure jump process with mild restrictions on the direction of jumps for one of the two players. The rigorous description of our set-up can be found in Section 4.1.

We allow for a general structure of the information available to the players. All processes are adapted to an overarching filtration $(\mathcal{F}_t)$, whereas each player makes decisions based on her own filtration, representing her access to information. Letting $(\mathcal{F}_t^i)$ be filtration of the i-th player, with $i = 1, 2$, we only assume that $\mathcal{F}_t^i \subseteq \mathcal{F}_t$ for all $t \in [0, T]$. In particular, we cover the case in which the players are equally (partially) informed, i.e. $\mathcal{F}_t^1 = \mathcal{F}_t^2$, and the case in which they have asymmetric (partial) information, i.e. $\mathcal{F}_t^1 \neq \mathcal{F}_t^2$.

Under this generality we prove that Dynkin games in the form of war-of-attrition (recall Remark 3.2.1) admit a value in randomised stopping times. We emphasise that, due to the asymmetry of information, the value may not exist in pure stopping times (i.e. in the classical sense of Definition 3.2.4). Our Definition 4.2.2 of the value is, instead, similar to the one used in the literature on asymmetric information games (see, in particular, Sections 3.4.1 and 3.4.2).

Our framework encompasses all the examples of zero-sum Dynkin games (in continuous time) with partial/asymmetric information that we could find in the literature. We will give a detailed account of this fact in Section 4.6. As explained in Section 3.4 where we review this literature, the methods traditionally employed to study asymmetric information games hinge on variational inequalities and partial differential equations. The classical assumptions share two key features: (i) a specific structure of the information flow in the game and (ii) the Markovian assumption. In our work, instead, we are able to analyse the games at a more abstract level that allows us to drop the Markovian assumption and to avoid specifying an information structure. We will also show by several counterexamples in Section 4.7 that our main assumptions cannot be further relaxed, as otherwise a value for the game may no longer exist.

Our methodology draws on the idea presented in Touzi & Vieille (2002) (recall Section 3.5) of using Sion's min-max theorem Sion (1958). In Touzi & Vieille (2002), the authors study non-Markovian zero-sum Dynkin games with *full* information, in which the first- and second-mover advantage may occur at different points in time, depending on the stochastic dynamics of the underlying payoff processes. This "order" feature is studied in Chapter 5. Since our set-up is different, due to the partial/asymmetric information features and to relaxed assumptions on the payoff processes, we encounter some non-trivial technical difficulties in following the arguments from Touzi & Vieille (2002): for example, our class of randomised stopping times is not closed with respect to the topology used in Touzi & Vieille (2002) (see Remark 4.4.17). For this reason, we develop an alternative approach based on the general theory of stochastic processes (see Sections 2.3 and A.5) combined with ideas from functional analysis (Sections A.3 and A.4).

4.1 Setting

We consider two-player zero-sum Dynkin games on the horizon $T \in (0,\infty]$. Basic probabilistic features of our problem are outlined in Section 2.1, and we will extensively use definitions of regularity, projections, and of space $\mathcal{L}_b$ from Section 2.3. For notation, recall that we work on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$. Actions of the first player are based on the information contained in a filtration $(\mathcal{F}^1_t) \subseteq (\mathcal{F}_t)$ (the rigorous meaning of this will be clarified later). Actions of the second player are based on the information contained in a filtration $(\mathcal{F}^2_t) \subseteq (\mathcal{F}_t)$. Each player selects a random time based on the information she acquires via her filtration; the first player's random time is denoted by τ while the second player's random time is σ. The game terminates at time $\tau \wedge \sigma \wedge T$ with the first player delivering to the second player the random *payoff*

$$\mathcal{P}(\tau,\sigma) = f_\tau I_{\{\tau\le\sigma\}\cap\{\tau<T\}} + g_\sigma I_{\{\sigma<\tau\}\cap\{\sigma<T\}} + h I_{\{\sigma=T\}\cap\{\tau=T\}}. \tag{4.1}$$

The first player (or τ-player) is the minimiser in the game whereas the second player (or σ-player) is the maximiser.

The *payoff processes* (f_t) and (g_t), and the *terminal payoff* h satisfy the following conditions:

(A1) $(f_t), (g_t) \in \mathcal{L}_b$,

(A2) $(f_t), (g_t)$ are $(\mathcal{F}_t)$-adapted regular processes,

(A3) $f_t \geq g_t$ for all $t \in [0,T]$ $\mathbb{P}$-a.s.,

(A4) the random variable h is $\mathcal{F}_T$-adapted and satisfies

$$g_T \leq h \leq f_T, \quad \mathbb{P}\text{-a.s.},$$

(A5) the filtrations $(\mathcal{F}_t)$ and $(\mathcal{F}_t^i)$, $i = 1,2$, satisfy the usual conditions, i.e. they are right-continuous and $\mathcal{F}_0^i$, $i = 1,2$, contain all sets of $\mathbb{P}$-measure zero (see Section 2.3).

We elaborate on our assumptions (and relax one of them) in Section 4.1.1.

Players assess the game by looking at the *expected payoff*

$$N(\tau,\sigma) = \mathbb{E}\big[\mathcal{P}(\tau,\sigma)\big]. \tag{4.2}$$

Recall that the game is said to have a *value* if

$$\sup_{\sigma}\inf_{\tau} N(\tau,\sigma) = \inf_{\tau}\sup_{\sigma} N(\tau,\sigma), \tag{4.3}$$

where for now we do not specify the nature of players' strategies (τ,σ). The mathematical difficulty with establishing existence of a value lies in the possibility to swap the order of 'inf' and 'sup', and this is closely linked to the choice of the set of strategies that the players are allowed to use (recall the discussion at the start of Section 3.4).

Remark 4.1.1. *By (4.1), if the players stop simultaneously, they exchange the larger payoff (f_t). This choice causes no loss of generality, since we do not make assumptions on the sign of (f_t), (g_t), h. Indeed, if the value exists for the game with payoff $\mathcal{P}(\tau,\sigma)$, the same is true for the game with payoff $\mathcal{P}'(\tau,\sigma) = -\mathcal{P}(\tau,\sigma)$. However, in the latter game the τ-player is a maximiser and the σ-player is a minimiser, since*

$$\sup_{\sigma}\inf_{\tau} \mathcal{P}(\tau,\sigma) = -\inf_{\sigma}\sup_{\tau} \mathcal{P}'(\tau,\sigma).$$

Defining $f_t' := -f_t$, $g_t' := -g_t'$, we have $f_t' \leq g_t'$. So, in this case, if the players stop simultaneously, they exchange the smaller payoff (f_t').

It has been indicated in the literature that games with asymmetric information may not have a value if players' strategies are stopping times for their respective filtrations. Indeed, in Section 4.7.2 we demonstrate that the game studied in this paper may not, in general, have a value if the i-th player uses $(\mathcal{F}_t^i)$-stopping times, $i = 1,2$ (that is, if infimum and supremum in (4.3) are taken over the sets $\mathcal{T}(\mathcal{F}_t^1)$ and $\mathcal{T}(\mathcal{F}_t^2)$). It has been proven in certain Markovian set-ups that the relaxation of player controls to randomised stopping times may be sufficient for the existence of the value (see Sections 3.4.1 and 3.4.2). Our goal is to show that this is indeed true in the generality of our non-Markovian set-up for the game with payoff (4.1).

4.1.1 Overview of the assumptions

The integrability Assumption (A1) is natural in the framework of optimal stopping problems and Dynkin games, as we saw in Section 3. Most of our proofs in Sections 4.3-4.5 rely on finiteness of integrals in question, and the latter is ensured by Remark 2.3.8.

By the regularity Assumption (A2) we replace semimartingale assumptions on (f_t) and (g_t) from Touzi & Vieille (2002) (recall Section 3.5). In the optimal stopping framework, it dates back to Meyer (1978) (Section 3.1.3). Regular processes encompass a large family of stochastic processes encountered in applications. It is straightforward to see (Example 2.3.4) that quasi left-continuous processes are regular. In the Markovian framework, strong and weak solutions of stochastic differential equations (Section A.7) are continuous and therefore regular.

We subsequently relax Assumption (A2) by allowing the payoff processes to have previsible jumps with nonzero (conditional) mean. In particular, in (A2') we allow either jumps of (f_t) in any direction and upward jumps of (g_t) or, vice versa, jumps of (g_t) in any direction and downward jumps of (f_t). This ensures a certain closedness property (see Section 4.5.1).

(A2') Processes (f_t) and (g_t) have the decomposition $f = \tilde{f} + \hat{f}$, $g = \tilde{g} + \hat{g}$ with

1. $(\tilde{f}_t), (\tilde{g}_t) \in \mathcal{L}_b$,
2. $(\tilde{f}_t), (\tilde{g}_t)$ are $(\mathcal{F}_t)$-adapted regular processes,
3. $(\hat{f}_t), (\hat{g}_t)$ are $(\mathcal{F}_t)$-adapted (right-continuous) piecewise-constant processes of integrable variation with $\hat{f}_0 = \hat{g}_0 = 0$, $\Delta\hat{f}_T = \hat{f}_T - \hat{f}_{T-} = 0$ and $\Delta\hat{g}_T = \hat{g}_T - \hat{g}_{T-} = 0$,
4. either $(\hat{f}_t)$ is non-increasing or $(\hat{g}_t)$ is non-decreasing.

Note that there are non-decreasing processes $(\hat{f}_t^+), (\hat{f}_t^-), (\hat{g}_t^+), (\hat{g}_t^-) \in \mathcal{L}_b$ starting from 0 such that $\hat{f} = \hat{f}^+ - \hat{f}^-$ and $\hat{g} = \hat{g}^+ - \hat{g}^-$ (Definition A.5.5).

Observe the similarity between the restriction on the direction of previsible jumps of one the payoff processes (item (4) in (A2')) and the assumption in the *non*zero-sum game of Hamadène & Zhang (2010) (recall Section 3.2.2). In zero-sum setting, this restriction is a new feature introduced by the asymmetry of information, i.e. for classical zero-sum Dynkin games it is not necessary, see Sections 3.2.1 and 3.3. It is worth emphasising that further relaxation of Assumption (A2') is not possible in the generality of our setting, as demonstrated in Remark 4.5.7 and in Section 4.7.3.

The order conditions similar to (or stricter than) (A3)-(A4), on the contrary, do appear in the full information setting as well (recall (3.11), (3.14), (3.19)). Although their necessity is challenged in Touzi & Vieille (2002) and Chapter 5, we require it to hold in the current setting, in order to focus on the specifics of the problem caused by the asymmetry of information.

Finally, Assumption (A5) on filtrations is technical and goes beyond the game applications of theory of stochastic processes (c.f. Section 2.3).

4.2 Main definitions and results

Recall the notation from Section 2.4: given a filtration $(\mathcal{G}_t) \subseteq (\mathcal{F}_t)$, we denote

$$\begin{aligned}\mathcal{A}^\circ(\mathcal{G}_t) := \{(\rho_t) : (\rho_t) \text{ is } (\mathcal{G}_t)\text{-adapted with } t \mapsto \rho_t(\omega) \text{ càdlàg},\\ \text{non-decreasing, } \rho_{0-}(\omega) = 0 \text{ and } \rho_T(\omega) = 1 \text{ for all } \omega \in \Omega\}.\end{aligned}$$

We emphasise that these properties are required to hold for all $\omega \in \Omega$; recall Remark 2.4.1 for why this requirement is not restrictive as long as $(\mathcal{G}_t)$ satisfies the usual conditions. In the infinite-time horizon case $T = \infty$ (recall Remark 2.1.1), we understand ρ_T as an $\mathcal{F}_\infty$-measurable random variable, while $\rho_{T-} := \lim_{t\to\infty} \rho_t$ (which exists by the assumption that (ρ_t) is a càdlàg process).

In Section 2.4, we encountered the concepts of a mixed/randomised/distribution stopping time and discussed their equivalence. In our work, we choose to use randomised stopping times. The main reason for this choice will be clear in Sections 4.3 and 4.4: randomised stopping times provide a way of working in a space of processes with a convenient tolopogical structure. Since randomised stopping times play the key role in our analysis, let us formalise the definition we gave in Section 2.4.

Definition 4.2.1. *Given a filtration $(\mathcal{G}_t) \subseteq (\mathcal{F}_t)$, a random variable η is called a $(\mathcal{G}_t)$-*randomised stopping time *if there exists a random variable Z with uniform distribution $U([0,1])$, independent of $\mathcal{F}_T$, and a process $(\rho_t) \in \mathcal{A}^\circ(\mathcal{G}_t)$ such that*

$$\eta = \eta(\rho, Z) = \inf\{t \in [0,T] : \rho_t > Z\}, \quad \mathbb{P}\text{-a.s.} \tag{4.4}$$

The variable Z is called a randomisation device *for the randomised stopping time η, and the process (ρ_t) is called the* generating process. *The set of $(\mathcal{G}_t)$-randomised stopping times is denoted by $\mathcal{T}^R(\mathcal{G}_t)$. It is assumed that randomisation devices of different randomised stopping times are independent.*

To avoid unnecessary complication of notation, we assume that the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ supports two independent random variables Z_τ and Z_σ which are also independent of $\mathcal{F}_T$ and are the randomisation devices for the randomised stopping times τ and σ of the two players.

Now we finally give the formal definition of the value of the game described in Section 4.1.

Definition 4.2.2. *Define*

$$V_* := \sup_{\sigma \in \mathcal{T}^R(\mathcal{F}_t^2)} \inf_{\tau \in \mathcal{T}^R(\mathcal{F}_t^1)} N(\tau, \sigma) \quad \textit{and} \quad V^* := \inf_{\tau \in \mathcal{T}^R(\mathcal{F}_t^1)} \sup_{\sigma \in \mathcal{T}^R(\mathcal{F}_t^2)} N(\tau, \sigma).$$

The lower value *and* upper value of the game in randomised strategies *are given by V_* and V^*, respectively. If they coincide, the game is said to have a* value in randomised strategies $V = V_* = V^*$.

The following theorem states the main result of this chapter.

Theorem 4.2.3. *Under Assumptions (A1), (A2'), (A3)-(A5), the game has a value in randomised strategies.*

For the clarity of presentation of our methodology, we first prove a theorem under more restrictive regularity properties of the payoff processes and then show how to extend the proof to the general case of Theorem 4.2.3.

Theorem 4.2.4. *Under Assumptions (A1)-(A5), the game has a value in randomised strategies.*

The proofs of the above theorems are given in Section 4.4. They rely on two key results: an approximation procedure (Propositions 4.4.6 and 4.5.2) and an auxiliary game with 'nice' regularity properties (Theorem 4.4.5 and 4.5.1) which enables the use of a known min-max theorem (Theorem 3.5.3).

The sigma-algebra $\mathcal{F}_0$ is not assumed to be trivial. It is therefore natural to consider a game in which players assess their strategies ex-post, i.e. after the observation available to them at time 0 when their first action may take place. Allowing for more generality, let $\mathcal{G}$ be a sigma-algebra contained in $\mathcal{F}_0^1$ and in $\mathcal{F}_0^2$, i.e. containing information available to both players at time 0. The expected payoff of the game in this case takes the form:

$$\mathbb{E}\big[\mathcal{P}(\tau, \sigma)\big|\mathcal{G}\big] = \mathbb{E}\big[f_\tau I_{\{\tau \le \sigma\} \cap \{\tau < T\}} + g_\sigma I_{\{\sigma < \tau\} \cap \{\sigma < T\}} + h I_{\{\sigma = T\} \cap \{\tau = T\}}\big|\mathcal{G}\big]. \tag{4.5}$$

The proof of the following theorem is in Section 4.5.3.

Theorem 4.2.5. *Under Assumptions (A1), (A2'), (A3)-(A5) and for any $\mathcal{G} \subseteq \mathcal{F}_0^1 \cap \mathcal{F}_0^2$, the $\mathcal{G}$-conditioned game has a value, i.e.*

$$\operatorname*{ess\,sup}_{\sigma\in\mathcal{T}^R(\mathcal{F}_t^2)} \operatorname*{ess\,inf}_{\tau\in\mathcal{T}^R(\mathcal{F}_t^1)} \mathbb{E}\big[\mathcal{P}(\tau,\sigma)\big|\mathcal{G}\big] = \operatorname*{ess\,inf}_{\tau\in\mathcal{T}^R(\mathcal{F}_t^1)} \operatorname*{ess\,sup}_{\sigma\in\mathcal{T}^R(\mathcal{F}_t^2)} \mathbb{E}\big[\mathcal{P}(\tau,\sigma)\big|\mathcal{G}\big]. \tag{4.6}$$

4.3 Reformulation as a game of (singular) controls

In order to integrate out the randomisation devices for τ and σ and obtain a reformulation of the payoff functional $N(\tau,\sigma)$ in terms of generating processes for randomised stopping times τ and σ, we need the two auxiliary lemmata below.

Remark 4.3.1. *If η is an $(\mathcal{G}_t)$-randomised stopping time for $(\mathcal{G}_t) \subseteq (\mathcal{F}_t)$, then η is also an $(\mathcal{F}_t)$-randomised stopping time. Indeed, by definition of a randomised stopping time, the generating process (ρ_t) of η belongs to $\mathcal{A}^\circ(\mathcal{G}_t)$. Then (ρ_t) is $(\mathcal{G}_t)$-adapted by definition of this set, therefore it is $(\mathcal{F}_t)$-adapted, and thus belongs to $\mathcal{A}^\circ(\mathcal{F}_t)$. Applying the definition of a randomised stopping time again, we see that η is an $(\mathcal{F}_t)$-randomised stopping time. Therefore, the results below are formulated for $(\mathcal{F}_t)$-randomised stopping times.*

Lemma 4.3.2. *Let $\eta \in \mathcal{T}^R(\mathcal{F}_t)$ with the generating process (ρ_t). Then, for any $\mathcal{F}_T$-measurable random variable κ with values in $[0,T]$,*

$$\mathbb{E}[I_{\{\eta\le\kappa\}}|\mathcal{F}_T] = \rho_\kappa, \qquad \mathbb{E}[I_{\{\eta>\kappa\}}|\mathcal{F}_T] = 1-\rho_\kappa, \tag{4.7}$$

$$\mathbb{E}[I_{\{\eta<\kappa\}}|\mathcal{F}_T] = \rho_{\kappa-}, \qquad \mathbb{E}[I_{\{\eta\ge\kappa\}}|\mathcal{F}_T] = 1-\rho_{\kappa-}. \tag{4.8}$$

Proof. The proof of (4.7) follows the lines of (De Angelis *et al.*, 2021a, Proposition 3.1). Let Z be the randomisation device for η. Since (ρ_t) is right-continuous, non-decreasing, and (4.4) holds, we have the following inclusion of events:

$$\{\rho_\kappa > Z\} \subseteq \{\eta \le \kappa\} \subseteq \{\rho_\kappa \ge Z\}.$$

Using that ρ_κ is $\mathcal{F}_T$-measurable and Z is uniformly distributed and independent of $\mathcal{F}_T$, we compute

$$\mathbb{E}[I_{\{\eta\le\kappa\}}|\mathcal{F}_T] \ge \mathbb{E}[I_{\{\rho_\kappa>Z\}}|\mathcal{F}_T] = \int_0^1 I_{\{\rho_\kappa>y\}}dy = \rho_\kappa,$$

and

$$\mathbb{E}[I_{\{\eta\le\kappa\}}|\mathcal{F}_T] \le \mathbb{E}[I_{\{\rho_\kappa\ge Z\}}|\mathcal{F}_T] = \int_0^1 I_{\{\rho_\kappa\ge y\}}dy = \rho_\kappa.$$

This completes the proof of the first equality in (4.7). Since the events $\{\eta \le \kappa\}$ and $\{\eta > \kappa\}$ are complements of each other, the other equality is a direct consequence.

To prove (4.8), we observe that, by (4.7), for any $\varepsilon > 0$ we have

$$I_{\{\kappa>0\}}\mathbb{E}[I_{\{\eta\le(\kappa-\varepsilon)\vee(\kappa/2)\}}|\mathcal{F}_T] = I_{\{\kappa>0\}}\rho_{(\kappa-\varepsilon)\vee(\kappa/2)}.$$

Dominated convergence theorem implies

$$\begin{aligned}\mathbb{E}[I_{\{\eta<\kappa\}}|\mathcal{F}_T] = I_{\{\kappa>0\}}\,\mathbb{E}[I_{\{\eta<\kappa\}}|\mathcal{F}_T] &= \lim_{\varepsilon\downarrow 0} I_{\{\kappa>0\}}\,\mathbb{E}[I_{\{\eta\le(\kappa-\varepsilon)\vee(\kappa/2)\}}|\mathcal{F}_T]\\ &= \lim_{\varepsilon\downarrow 0} I_{\{\kappa>0\}}\,\rho_{(\kappa-\varepsilon)\vee(\kappa/2)} = I_{\{\kappa>0\}}\,\rho_{\kappa-} = \rho_{\kappa-},\end{aligned}$$

where in the last equality we used that $\rho_{0-}=0$. This proves the first equality in (4.8). The other one is again a direct consequence. □

Lemma 4.3.3. *Let $\eta,\theta\in\mathcal{T}^R(\mathcal{F}_t)$ with generating processes (ρ_t), (χ_t) and independent randomisation devices Z_η, Z_θ. For $(X_t)\in\mathcal{L}_b$, we have*

$$\begin{aligned}\mathbb{E}\left[X_\eta I_{\{\eta\le\theta\}\cap\{\eta<T\}}\right] &= \mathbb{E}\left[\int_{[0,T)} X_t(1-\chi_{t-})d\rho_t\right],\\ \mathbb{E}\left[X_\eta I_{\{\eta<\theta\}}\right] &= \mathbb{E}\left[\int_{[0,T)} X_t(1-\chi_t)d\rho_t\right],\end{aligned}$$

where we use the notation $\int_{[0,T)}$ for the (pathwise) Lebesgue–Stieltjes integral (recall Section 2.2).

Proof. For $y\in[0,1)$, define a family of random variables

$$q(y) = \inf\{t\in[0,T] : \rho_t > y\}.$$

Then, $\eta = q(Z_\eta)$. Using that $Z_\eta\sim U(0,1)$ and Fubini's theorem, we see that

$$\begin{aligned}\mathbb{E}\left[X_\eta I_{\{\eta\le\theta\}\cap\{\eta<T\}}\right] &= \mathbb{E}\left[\int_0^1 X_{q(y)} I_{\{q(y)\le\theta\}\cap\{q(y)<T\}}dy\right]\\ &= \int_0^1 \mathbb{E}\left[\mathbb{E}\left[X_{q(y)} I_{\{q(y)\le\theta\}\cap\{q(y)<T\}}|\mathcal{F}_T\right]\right]dy.\end{aligned}$$

Since $X_{q(y)}I_{\{q(y)<T\}}$ is $\mathcal{F}_T$-measurable and the randomisation device Z_θ is independent of $\mathcal{F}_T$, we continue as follows:

$$\begin{aligned}\int_0^1 \mathbb{E}\left[\mathbb{E}\left[X_{q(y)} I_{\{q(y)\le\theta\}\cap\{q(y)<T\}}|\mathcal{F}_T\right]\right]dy &= \int_0^1 \mathbb{E}\left[X_{q(y)} I_{\{q(y)<T\}}\mathbb{E}[I_{\{q(y)\le\theta\}}|\mathcal{F}_T]\right]dy\\ &= \mathbb{E}\left[\int_0^1 X_{q(y)} I_{\{q(y)<T\}}(1-\chi_{q(y)-})dy\right]\\ &= \mathbb{E}\left[\int_{[0,T)} X_t(1-\chi_{t-})d\rho_t\right],\end{aligned}$$

where in the second equality we apply Lemma 4.3.2 with $\kappa = q(y)$, and in the third equality we change the variable of integration applying Proposition 2.2.2 ω-wise and using the fact that the function $y \mapsto q(y)(\omega)$ is the generalised inverse of $t \mapsto \rho_t(\omega)$. The first statement of the lemma is now proved.

For the second statement, we adapt the arguments above to write

$$\begin{aligned}\mathbb{E}\left[X_\eta I_{\{\eta<\theta\}}\right] &= \int_0^1 \mathbb{E}\left[X_{q(y)}\mathbb{E}[I_{\{q(y)<\theta\}}|\mathcal{F}_T]\right]dy = \mathbb{E}\left[\int_0^1 X_{q(y)}(1-\chi_{q(y)})dy\right]\\ &= \mathbb{E}\left[\int_{[0,T]} X_t(1-\chi_t)d\rho_t\right] = \mathbb{E}\left[\int_{[0,T)} X_t(1-\chi_t)d\rho_t\right],\end{aligned}$$

where in the last equality we used that $\chi_T = 1$. □

Applying Lemma 4.3.2 and 4.3.3 to (4.1) and (4.2) and noticing that

$$\mathbb{E}[I_{\{\eta=\kappa\}}|\mathcal{F}_T] = \mathbb{E}[I_{\{\eta\le\kappa\}} - I_{\{\eta<\kappa\}}|\mathcal{F}_T] = \rho_\kappa - \rho_{\kappa-}$$

we obtain the following reformulation of the game.

Proposition 4.3.4. *For $\tau \in \mathcal{T}^R(\mathcal{F}_t^1)$, $\sigma \in \mathcal{T}^R(\mathcal{F}_t^2)$,*

$$N(\tau,\sigma) = \mathbb{E}\left[\int_{[0,T)} f_t(1-\zeta_{t-})d\xi_t + \int_{[0,T)} g_t(1-\xi_t)d\zeta_t + h\Delta\xi_T\Delta\zeta_T\right], \tag{4.9}$$

where (ξ_t) and (ζ_t) are the generating processes for τ and σ, respectively, and $\Delta\xi_T = \xi_T - \xi_{T-} = 1-\xi_{T-}$ denotes the jump of (ξ_t) at T, and $\Delta\zeta_T = 1-\zeta_{T-}$.

With a slight abuse of notation, we will denote the right-hand side of (4.9) by $N(\xi,\zeta)$.

Remark 4.3.5. *In the Definition 4.2.2 of the lower value, the infimum can always be replaced by infimum over* pure *stopping times (c.f. Laraki & Solan (2005)). Same holds for the supremum in the definition of the upper value.*

Let us look at the upper value: take arbitrary $\tau \in \mathcal{T}^R(\mathcal{F}_t^1)$, $\sigma \in \mathcal{T}^R(\mathcal{F}_t^2)$, and define the family of stopping times

$$q(y) = \inf\{t \in [0,T] : \zeta_t > y\}, \qquad y \in [0,1),$$

similarly to the proof of Lemma 4.3.3 and with (ζ_t) the generating process of σ. Then,

$$N(\tau,\sigma) = \int_0^1 N(\tau,q(y))dy \le \sup_{y\in[0,1)} N(\tau,q(y)) \le \sup_{\sigma\in\mathcal{T}(\mathcal{F}_t^2)} N(\tau,\sigma)$$

(recall that $\mathcal{T}(\mathcal{F}_t^2)$ denotes the set of pure $(\mathcal{F}_t^2)$-stopping times). Since $\mathcal{T}(\mathcal{F}_t^2) \subset \mathcal{T}^R(\mathcal{F}_t^2)$, we have

$$\sup_{\sigma\in\mathcal{T}^R(\mathcal{F}_t^2)} N(\tau,\sigma) = \sup_{\sigma\in\mathcal{T}(\mathcal{F}_t^2)} N(\tau,\sigma),$$

and, consequently, the 'inner' *optimisation can be done over pure stopping times:*

$$\inf_{\tau\in\mathcal{T}^R(\mathcal{F}^1_t)}\sup_{\sigma\in\mathcal{T}^R(\mathcal{F}^2_t)} N(\tau,\sigma)=\inf_{\tau\in\mathcal{T}^R(\mathcal{F}^1_t)}\sup_{\sigma\in\mathcal{T}(\mathcal{F}^2_t)} N(\tau,\sigma).$$

By the same argument one can show that

$$\sup_{\sigma\in\mathcal{T}^R(\mathcal{F}^2_t)}\inf_{\tau\in\mathcal{T}^R(\mathcal{F}^1_t)} N(\tau,\sigma)=\sup_{\sigma\in\mathcal{T}^R(\mathcal{F}^2_t)}\inf_{\tau\in\mathcal{T}(\mathcal{F}^1_t)} N(\tau,\sigma).$$

However, in general an analogue result for the 'outer' *optimisation does not hold, i.e.*

$$\sup_{\sigma\in\mathcal{T}^R(\mathcal{F}^2_t)}\inf_{\tau\in\mathcal{T}^R(\mathcal{F}^1_t)} N(\tau,\sigma)\neq\sup_{\sigma\in\mathcal{T}(\mathcal{F}^2_t)}\inf_{\tau\in\mathcal{T}^R(\mathcal{F}^1_t)} N(\tau,\sigma)$$

as shown by an example in Section 4.7.

4.4 Sion's theorem and existence of value

The proofs of Theorems 4.2.3 and 4.2.4, i.e. that the game with payoff (4.2) has a value in randomised strategies, utilise Sion's min-max theorem (Theorem 3.5.3) originally proved in Sion (1958). We also refer to Komiya (1988) for a proof that is reviewed in Section 3.6. The idea of relying on Sion's theorem comes from Touzi & Vieille (2002) where the authors study zero-sum Dynkin games with full and symmetric information (see Section 3.5). Here, however, we need different key technical arguments as explained in e.g. Remark 4.4.17 below.

An important step in applying Sion's theorem is to find a topology on the set of randomised stopping times, or, equivalently, on the set of corresponding generating processes so that the functional $N(\cdot,\cdot)$ satisfies the assumptions. We will use the weak topology (c.f. Section A.3) of

$$\mathcal{S}:=L^2\big([0,T]\times\Omega,\mathcal{B}([0,T])\times\mathcal{F},\lambda\times\mathbb{P}\big),$$

where λ denotes the Lebesgue measure on $[0,T]$.

Given a filtration $(\mathcal{G}_t)\subseteq(\mathcal{F}_t)$, in addition to the class of increasing processes $\mathcal{A}^\circ(\mathcal{G}_t)$ introduced earlier, here we also need

$$\mathcal{A}^\circ_{ac}(\mathcal{G}_t):=\{(\rho_t)\in\mathcal{A}^\circ(\mathcal{G}_t):t\mapsto\rho_t(\omega)\text{ is absolutely continuous on }[0,T)\text{ for all }\omega\in\Omega\}.$$

We refer to Section A.4.5 for the definition and basic properties of absolutely continuous functions and measures. It is important to notice that $(\rho_t)\in\mathcal{A}^\circ_{ac}(\mathcal{G}_t)$ may have a jump at time T if

$$\rho_{T-}(\omega):=\lim_{t\uparrow T}\int_0^t\big(\tfrac{d}{dt}\rho_s\big)(\omega)ds<1=\rho_T(\omega).$$

As with $\mathcal{A}^\circ(\mathcal{G}_t)$, in the definition of $\mathcal{A}^\circ_{ac}(\mathcal{G}_t)$ we require that the stated properties hold for all $\omega \in \Omega$, which causes no loss of generality if $\mathcal{G}_0$ contains all $\mathbb{P}$-null sets of Ω. It is clear that $\mathcal{A}^\circ_{ac}(\mathcal{G}_t) \subset \mathcal{A}^\circ(\mathcal{G}_t) \subset \mathcal{S}$.

For reasons that will become clear later (e.g. see Lemma 4.4.16), we prefer to work with slightly more general processes than those in $\mathcal{A}^\circ(\mathcal{G}_t)$ and $\mathcal{A}^\circ_{ac}(\mathcal{G}_t)$. Let us denote

$$\mathcal{A}(\mathcal{G}_t) := \{(\rho_t) \in \mathcal{S} : \exists\, (\hat{\rho}_t) \in \mathcal{A}^\circ(\mathcal{G}_t) \text{ such that } \rho = \hat{\rho} \text{ for } (\lambda \times \mathbb{P})\text{-a.e. } (t,\omega) \in [0,T] \times \Omega\},$$
$$\mathcal{A}_{ac}(\mathcal{G}_t) := \{(\rho_t) \in \mathcal{S} : \exists\, (\hat{\rho}_t) \in \mathcal{A}^\circ_{ac}(\mathcal{G}_t) \text{ such that } \rho = \hat{\rho} \text{ for } (\lambda \times \mathbb{P})\text{-a.e. } (t,\omega) \in [0,T] \times \Omega\}.$$

Definition 4.4.1. *We call the process $(\hat{\rho}_t)$ in the definition of the set $\mathcal{A}$ (and $\mathcal{A}_{ac}$) the* càdlàg *(and* absolutely continuous*)* representative *of (ρ_t).*

Although not unique, all càdlàg representatives are indistinguishable (Definition A.1.7), as Lemma 4.4.7 below shows. Hence, all càdlàg representatives of $(\rho_t) \in \mathcal{A}$ define the same positive Lebesgue–Stieltjes measure $t \mapsto \hat{\rho}_t(\omega)$ on $[0,T]$ for $\mathbb{P}$-a.e. $\omega \in \Omega$. Then, given any bounded measurable process (X_t) the stochastic process (Lebesgue–Stieltjes integral)

$$t \mapsto \int_{[0,t]} X_s \, d\hat{\rho}_s, \qquad t \in [0,T],$$

does not depend on the choice of the càdlàg representative $(\hat{\rho}_t)$ in the sense that it is defined up to indistinguishability.

The next definition connects the randomised stopping times that we use in the construction of the game's payoff (Proposition 4.3.4) with processes from the classes $\mathcal{A}(\mathcal{F}^1_t)$ and $\mathcal{A}(\mathcal{F}^2_t)$. Note that $\mathcal{A}(\mathcal{G}_t) \subseteq \mathcal{A}(\mathcal{F}_t)$ whenever $(\mathcal{G}_t) \subseteq (\mathcal{F}_t)$, so the definition can be stated for $\mathcal{A}(\mathcal{F}_t)$ without any loss of generality.

Definition 4.4.2. *For $(X_t) \in \mathcal{L}_b$ and $(\chi_t), (\rho_t) \in \mathcal{A}(\mathcal{F}_t)$, we define the Lebesgue–Stieltjes integral processes*

$$t \mapsto \int_{[0,t]} X_s \, d\rho_s, \quad t \mapsto \int_{[0,t]} X_s (1-\chi_s) d\rho_s \quad \textit{and} \quad t \mapsto \int_{[0,t]} X_s (1-\chi_{s-}) d\rho_s \qquad t \in [0,T],$$

by

$$t \mapsto \int_{[0,t]} X_s \, d\hat{\rho}_s, \quad t \mapsto \int_{[0,t]} X_s (1-\hat{\chi}_s) d\hat{\rho}_s \quad \textit{and} \quad t \mapsto \int_{[0,t]} X_s (1-\hat{\chi}_{s-}) d\hat{\rho}_s \qquad t \in [0,T],$$

for any choice of the càdlàg representatives $(\hat{\rho}_t)$ and $(\hat{\chi}_t)$, uniquely up to indistinguishability.

With a slight abuse of notation, we define a functional $N : \mathcal{A}(\mathcal{F}^1_t) \times \mathcal{A}(\mathcal{F}^2_t) \to \mathbb{R}$ by the right-hand side of (4.9).

Proposition 4.4.3. *The lower and the upper value of our game satisfy*

$$V_* = \sup_{\zeta \in \mathcal{A}(\mathcal{F}_t^2)} \inf_{\xi \in \mathcal{A}(\mathcal{F}_t^1)} N(\xi,\zeta), \qquad V^* = \inf_{\xi \in \mathcal{A}(\mathcal{F}_t^1)} \sup_{\zeta \in \mathcal{A}(\mathcal{F}_t^2)} N(\xi,\zeta). \tag{4.10}$$

Proof. By Proposition 4.3.4, for $\tau \in \mathcal{T}^R(\mathcal{F}_t^1)$, $\sigma \in \mathcal{T}^R(\mathcal{F}_t^2)$, the functional $N(\tau,\sigma)$ equals the sum of integrals (4.9) involving the generating processes $(\xi_t) \in \mathcal{A}^\circ(\mathcal{F}_t^1)$ and $(\zeta_t) \in \mathcal{A}^\circ(\mathcal{F}_t^2)$ of τ and σ, respectively. Therefore, taking supremum and infimum over the sets $\mathcal{T}^R(\mathcal{F}_t^1)$ and $\mathcal{T}^R(\mathcal{F}_t^2)$ in Definition 4.2.2 of the lower and upper value is equivalent to taking them over the sets $\mathcal{A}^\circ(\mathcal{F}_t^1)$ and $\mathcal{A}^\circ(\mathcal{F}_t^2)$. And the latter is, for the sum of integrals on the right-hand side of (4.9), equivalent to taking supremum and infimum over the sets $\mathcal{A}(\mathcal{F}_t^1)$ and $\mathcal{A}(\mathcal{F}_t^2)$, thanks to Definition 4.4.2. □

Remark 4.4.4. *The mapping $\mathcal{A}(\mathcal{F}_t^1) \times \mathcal{A}(\mathcal{F}_t^2) \ni (\xi,\zeta) \mapsto N(\xi,\zeta)$ does not satisfy the conditions of Sion's theorem. Indeed, taking $\xi_t^n = I_{\{t \geq \frac{T}{2}+\frac{1}{n}\}}$, we have $\xi_t^n \to I_{\{t \geq \frac{T}{2}\}} =: \xi_t$ for λ-a.e. $t \in [0,T]$, so that by dominated convergence (ξ^n) also converges to (ξ_t) in $\mathcal{S}$. Then, fixing $\zeta_t = I_{\{t \geq \frac{T}{2}\}}$ in $\mathcal{A}(\mathcal{F}_t^2)$ we have $N(\xi^n,\zeta) = \mathbb{E}[g_{\frac{T}{2}}]$ for all $n \geq 1$ whereas $N(\xi,\zeta) = \mathbb{E}[f_{\frac{T}{2}}]$. So the lower semicontinuity of $\xi \mapsto N(\xi,\zeta)$ cannot be ensured if, for example, $\mathbb{P}(f_{\frac{T}{2}} > g_{\frac{T}{2}}) > 0$.*

Due to the issues indicated in the above remark, as in Touzi & Vieille (2002) (Section 3.5), we "smoothen" the control strategy of one player in order to introduce additional regularity in the payoff. We will show that this procedure does not change the value of the game (Proposition 4.4.6). We choose to consider an auxiliary game in which the first player can only use controls from $\mathcal{A}_{ac}(\mathcal{F}_t^1)$. Let us define the associated lower/upper values:

$$W_* = \sup_{\zeta \in \mathcal{A}(\mathcal{F}_t^2)} \inf_{\xi \in \mathcal{A}_{ac}(\mathcal{F}_t^1)} N(\xi,\zeta) \quad \text{and} \quad W^* = \inf_{\xi \in \mathcal{A}_{ac}(\mathcal{F}_t^1)} \sup_{\zeta \in \mathcal{A}(\mathcal{F}_t^2)} N(\xi,\zeta). \tag{4.11}$$

Here, the first player is chosen arbitrarily and with no loss of generality. In Section 4.5.2, we explain why we could instead consider a game in which the second player can only use controls from $\mathcal{A}_{ac}(\mathcal{F}_t^2)$.

Note that we work under the regularity assumption on the payoff processes (A2). Relaxation of this assumption is conducted in Section 4.5.1.

The main steps in the proof of our main Theorem 4.2.4 are the following:

Theorem 4.4.5. *Under Assumptions (A1)-(A5), the game (4.11) has a value, i.e.*

$$W_* = W^* := W.$$

Moreover, the ζ-player (maximiser) has an optimal strategy, i.e. there exists $(\zeta_t^*) \in \mathcal{A}(\mathcal{F}_t^2)$ such that

$$\inf_{\xi \in \mathcal{A}_{ac}(\mathcal{F}_t^1)} N(\xi,\zeta^*) = W.$$

Proposition 4.4.6. *Under Assumptions (A1)-(A5), for any* $(\zeta_t) \in \mathcal{A}(\mathcal{F}_t^2)$ *and* $(\xi_t) \in \mathcal{A}(\mathcal{F}_t^1)$*, there is a sequence* $(\xi_t^n) \subset \mathcal{A}_{ac}(\mathcal{F}_t^1)$ *such that*

$$\limsup_{n\to\infty} N(\xi^n, \zeta) \le N(\xi, \zeta).$$

The proofs will be conducted in the following subsections: Section 4.4.1 contains a number of technical results which we then use to prove Theorem 4.4.5 (in Section 4.4.2) and Proposition 4.4.6 (in Section 4.4.3). With the results from Theorem 4.4.5 and Proposition 4.4.6 in place we can provide the proof of Theorem 4.2.4.

Proof of Theorem 4.2.4. Recall that $\mathcal{A}_{ac}(\mathcal{F}_t^1) \subseteq \mathcal{A}(\mathcal{F}_t^1)$. Since the infimum in the defintions of W_* and W^* is taken over a smaller set than the infimum in the definitions of V_* and V^*, we have $V_* \le W_*$ and $V^* \le W^*$. However, Proposition 4.4.6 implies that

$$\inf_{\xi \in \mathcal{A}_{ac}(\mathcal{F}_t^1)} N(\xi, \zeta) \le \inf_{\xi \in \mathcal{A}(\mathcal{F}_t^1)} N(\xi, \zeta)$$

for any $(\zeta_t) \in \mathcal{A}(\mathcal{F}_t^2)$, so $V_* \ge W_*$. Therefore, $V_* = W_*$. Then, thanks to Theorem 4.4.5, we have a sequence of inequalities which complete the proof

$$W = W_* = V_* \le V^* \le W^* = W.$$

□

4.4.1 Technical results

In this section we give a series of results concerning the convergence of integrals when either the integrand or the integrator converges in a suitable sense. We start with a technical lemma.

Lemma 4.4.7. *Let* (X_t) *and* (Y_t) *be càdlàg measurable processes such that* $X_t = Y_t$*,* $\mathbb{P}$*-a.s. for all* $t \in D$*, where* D *is a countable and dense subset of* $[0,T)$*, and* $X_{0-} = Y_{0-}$*,* $X_T = Y_T$*,* $\mathbb{P}$*-a.s. Then* (X_t) *is indistinguishable from* (Y_t)*.*

Proof. Define

$$\Omega_0 = \{\omega \in \Omega : (X_t(\omega)), (Y_t(\omega)) \text{ are càdlàg and } X_t(\omega) = Y_t(\omega) \text{ for all } t \in D\}.$$

We have $\mathbb{P}(\Omega_0) = 1$. Fix $\omega \in \Omega_0$. Since the set D is dense in $[0,T)$, and the processes (X_t) and (Y_t) are càdlàg, for any $t_0 \in (0,T)$, there exists a sequence $\{t_n\}_{n\in\mathbb{N}}$ in D such that

$$X_{t_0}(\omega) = \lim_{n\to\infty} X_{t_n}(\omega), \; Y_{t_0}(\omega) = \lim_{n\to\infty} Y_{t_n}(\omega).$$

Further, for any $n \in \mathbb{N}$ we have $X_{t_n}(\omega) = Y_{t_n}(\omega)$ by assumption. Therefore, $X_{t_0}(\omega) = Y_{t_0}(\omega)$. Since t_0 and ω are arbitrary, we obtain

$$\mathbb{P}\{\omega \in \Omega : X_t(\omega) = Y_t(\omega) \text{ for all } t \in (0,T)\} = 1.$$

This, together with the assumption $X_{0-} = Y_{0-}$, $X_T = Y_T$, $\mathbb{P}$-a.s., proves the indistinguishability of (X_t) and (Y_t). □

Definition 4.4.8. *Given a càdlàg measurable process (X_t), for each $\omega \in \Omega$ we denote*

$$C_X(\omega) := \{t \in [0,T] : X_{t-}(\omega) = X_t(\omega)\}.$$

Our next result tells us that the convergence $(\lambda \times \mathbb{P})$-a.e. of processes in $\mathcal{A}(\mathcal{G}_t)$ can be lifted to $\mathbb{P}$-a.s. convergence at all points of continuity of the corresponding càdlàg representatives (recall Definition 4.4.1).

Lemma 4.4.9. *For a filtration $(\mathcal{G}_t) \subseteq (\mathcal{F}_t)$, let $(\rho_t^n)_{n\geq 1} \subset \mathcal{A}(\mathcal{G}_t)$ and $(\rho_t) \in \mathcal{A}(\mathcal{G}_t)$ with $\rho^n \to \rho$ $(\lambda \times \mathbb{P})$-a.e. as $n \to \infty$. Then for any càdlàg representatives $(\hat{\rho}_t^n)$ and $(\hat{\rho}_t)$ we have*

$$\mathbb{P}\Big(\big\{\omega \in \Omega : \lim_{n\to\infty} \hat{\rho}_t^n(\omega) = \hat{\rho}_t(\omega) \text{ for all } t \in C_{\hat{\rho}}(\omega)\big\}\Big) = 1. \tag{4.12}$$

Proof. The $(\lambda \times \mathbb{P})$-a.e. convergence of (ρ_t^n) to (ρ_t) means that the càdlàg representatives $(\hat{\rho}_t^n)$ converge to $(\hat{\rho}_t)$ also $(\lambda \times \mathbb{P})$-a.e. Hence, there is a set $D \subset [0,T]$ with $\lambda([0,T] \setminus D) = 0$ such that $\hat{\rho}_t^n \to \hat{\rho}_t$ $\mathbb{P}$-a.s. for $t \in D$. Since $\lambda([0,T] \setminus D) = 0$, there is a countable subset $D_0 \subset D$ that is dense in $[0,T]$. Define

$$\Omega_0 := \{\omega \in \Omega : \ \hat{\rho}_t^n(\omega) \to \hat{\rho}_t(\omega) \text{ for all } t \in D_0\}.$$

Then $\mathbb{P}(\Omega_0) = 1$. □

Now, fix $\omega \in \Omega_0$ and let $t \in C_{\hat{\rho}}(\omega) \cap (0,T)$. Take an increasing sequence $(t_k^1)_{k\geq 1} \subset D_0$ and a decreasing one $(t_k^2)_{k\geq 1} \subset D_0$, both converging to t as $k \to \infty$. For each $k \geq 1$, we have

$$\hat{\rho}_t(\omega) = \lim_{k\to\infty} \hat{\rho}_{t_k^2}(\omega) = \lim_{k\to\infty} \lim_{n\to\infty} \hat{\rho}_{t_k^2}^n(\omega) \geq \limsup_{n\to\infty} \hat{\rho}_t^n(\omega), \tag{4.13}$$

where the first equality holds because $t \in C_{\hat{\rho}}(\omega)$, and in the final inequality we use that $\hat{\rho}_{t_k^2}^n(\omega) \geq \hat{\rho}_t^n(\omega)$ by monotonicity. By the same argument, we also obtain

$$\hat{\rho}_t(\omega) = \lim_{k\to\infty} \hat{\rho}_{t_k^1}(\omega) = \lim_{k\to\infty} \lim_{n\to\infty} \hat{\rho}_{t_k^1}^n(\omega) \leq \liminf_{n\to\infty} \hat{\rho}_t^n(\omega).$$

Combining the above, we obtain (4.12) (apart from $t \in \{0,T\}$) by recalling that $\omega \in \Omega_0$ and $\mathbb{P}(\Omega_0) = 1$. The convergence at $t = T$, irrespective of whether it belongs to $C_{\hat{\rho}}(\omega)$, is trivial as $\hat{\rho}_T^n(\omega) = \hat{\rho}_T(\omega) = 1$. If $0 \in C_{\hat{\rho}}(\omega)$, then $\hat{\rho}_0(\omega) = \hat{\rho}_{0-}(\omega) = 0$. Inequality (4.13) reads $0 = \hat{\rho}_0(\omega) \geq \limsup_{n\to\infty} \hat{\rho}_0^n(\omega)$. Since $\hat{\rho}_0^n(\omega) \geq 0$, this proves that $\hat{\rho}_0^n(\omega) \to \hat{\rho}_0(\omega) = 0$.

Lemma 4.4.10. *For a filtration $(\mathcal{G}_t) \subseteq (\mathcal{F}_t)$, let $(\rho^n_t)_{n\geq 1} \subset \mathcal{A}^\circ(\mathcal{G}_t)$ and $(\rho_t) \in \mathcal{A}^\circ(\mathcal{G}_t)$ with $\rho^n \to \rho$ $(\lambda \times \mathbb{P})$-a.e. as $n \to \infty$. For any $t \in [0,T]$ and any random variable $X \geq 0$ with $\mathbb{E}[X] < \infty$, we have*

$$\limsup_{n\to\infty} \mathbb{E}[X\Delta\rho^n_t] \leq \mathbb{E}[X\Delta\rho_t].$$

Proof. Fix $t \in (0,T)$. Arguing by contradiction, assume there exists a subsequence n_k such that

$$\lim_{k\to\infty} \mathbb{E}[X\Delta\rho^{n_k}_t] > \mathbb{E}[X\Delta\rho_t].$$

Therefore, using $(\lambda\times\mathbb{P})$-a.e. convergence of (ρ^n_t) to (ρ_t), i.e. that $\int_0^T \mathbb{P}\big(\lim_{k\to\infty}\rho^{n_k}_t = \rho_t\big)dt = T$, there is a decreasing sequence $\delta_m \to 0$ such that

$$\lim_{k\to\infty} \rho^{n_k}_{t-\delta_m} = \rho_{t-\delta_m}, \qquad \lim_{k\to\infty} \rho^{n_k}_{t+\delta_m} = \rho_{t+\delta_m}, \qquad \mathbb{P}\text{-a.s.}$$

Then, by the dominated convergence theorem,

$$\begin{aligned}
\mathbb{E}[X\Delta\rho_t] &= \lim_{m\to\infty} \mathbb{E}[X(\rho_{t+\delta_m} - \rho_{t-\delta_m})] \\
&= \lim_{m\to\infty}\lim_{k\to\infty} \mathbb{E}[X(\rho^{n_k}_{t+\delta_m} - \rho^{n_k}_{t-\delta_m})] \\
&= \lim_{m\to\infty}\limsup_{k\to\infty} \mathbb{E}[X(\rho^{n_k}_{t+\delta_m} - \rho^{n_k}_{t-\delta_m})] \\
&= \lim_{m\to\infty}\limsup_{k\to\infty} \mathbb{E}[X(\rho^{n_k}_{t+\delta_m} - \rho^{n_k}_{t} + \rho^{n_k}_{t-} - \rho^{n_k}_{t-\delta_m} + \Delta\rho^{n_k}_t)] \geq \limsup_{k\to\infty} \mathbb{E}[X\Delta\rho^{n_k}_t],
\end{aligned}$$

where the last inequality is due to $t \mapsto \rho^{n_k}_t$ being non-decreasing. This contradiction finishes the proof for $t \in (0,T)$. The proof for $t \in \{0,T\}$ is a simplified version of the argument above. Indeed, for $t = T$ we have

$$\begin{aligned}
\mathbb{E}[X\Delta\rho_T] &= \lim_{m\to\infty} \mathbb{E}[X(\rho_T - \rho_{T-\delta_m})] \\
&= \lim_{m\to\infty}\lim_{k\to\infty} \mathbb{E}[X(\rho^{n_k}_T - \rho^{n_k}_{T-\delta_m})] \\
&= \lim_{m\to\infty}\limsup_{k\to\infty} \mathbb{E}[X(\rho^{n_k}_T - \rho^{n_k}_{T-\delta_m})] \\
&= \lim_{m\to\infty}\limsup_{k\to\infty} \mathbb{E}[X(\rho^{n_k}_{T-} - \rho^{n_k}_{T-\delta_m} + \Delta\rho^{n_k}_T)] \geq \limsup_{k\to\infty} \mathbb{E}[X\Delta\rho^{n_k}_T],
\end{aligned}$$

and for $t = 0$ we have

$$\begin{aligned}
\mathbb{E}[X\Delta\rho_0] &= \lim_{m\to\infty} \mathbb{E}[X(\rho_{\delta_m} - \rho_{0-})] \\
&= \lim_{m\to\infty}\lim_{k\to\infty} \mathbb{E}[X(\rho^{n_k}_{\delta_m} - \rho^{n_k}_{0-})] \\
&= \lim_{m\to\infty}\limsup_{k\to\infty} \mathbb{E}[X(\rho^{n_k}_{\delta_m} - \rho^{n_k}_{0-})] \\
&= \lim_{m\to\infty}\limsup_{k\to\infty} \mathbb{E}[X(\rho^{n_k}_{\delta_m} - \rho^{n_k}_0 + \Delta\rho^{n_k}_0)] \geq \limsup_{k\to\infty} \mathbb{E}[X\Delta\rho^{n_k}_0].
\end{aligned}$$

□

We need to consider a slightly larger class of processes $\tilde{\mathcal{A}}^\circ(\mathcal{G}_t) \supset \mathcal{A}^\circ(\mathcal{G}_t)$ defined by

$$\tilde{\mathcal{A}}^\circ(\mathcal{G}_t) := \{(\rho_t) : (\rho_t) \text{ is } (\mathcal{G}_t)\text{-adapted with } t \mapsto \rho_t(\omega) \text{ càdlàg}, \\ \text{non-decreasing, } \rho_{0-}(\omega) = 0 \text{ and } \rho_T(\omega) \leq 1 \text{ for all } \omega \in \Omega\}.$$

Proposition 4.4.11. *For a filtration $(\mathcal{G}_t) \subseteq (\mathcal{F}_t)$, let $(\rho^n_t)_{n\geq 1} \subset \tilde{\mathcal{A}}^\circ(\mathcal{G}_t)$ and $(\rho_t) \in \tilde{\mathcal{A}}^\circ(\mathcal{G}_t)$. Assume*

$$\mathbb{P}\Big(\big\{\omega \in \Omega : \lim_{n\to\infty} \rho^n_t(\omega) = \rho_t(\omega) \text{ for all } t \in C_\rho(\omega) \cup \{T\}\big\}\Big) = 1.$$

Then for any $(X_t) \in \mathcal{L}_b$ that is also $(\mathcal{F}_t)$-adapted and regular, we have

$$\lim_{n\to\infty} \mathbb{E}\left[\int_{[0,T]} X_t d\rho^n_t\right] = \mathbb{E}\left[\int_{[0,T]} X_t d\rho_t\right]. \tag{4.14}$$

Proof. Let us first assume (with a slight abuse of notation compared to the statement of the proposition) that $(X_t) \in \mathcal{L}_b$ has continuous trajectories but is *not* necessarily adapted. If we prove that

$$\lim_{n\to\infty} \int_{[0,T]} X_t(\omega) d\rho^n_t(\omega) = \int_{[0,T]} X_t(\omega) d\rho_t(\omega), \quad \text{for } \mathbb{P}\text{-a.e. } \omega \in \Omega, \tag{4.15}$$

then the result in (4.14) will follow by the dominated convergence theorem. By assumption there is $\Omega_0 \subset \Omega$ with $\mathbb{P}(\Omega_0) = 1$ and such that $\rho^n_t(\omega) \to \rho_t(\omega)$ at all points of continuity of $t \mapsto \rho_t(\omega)$ and at the terminal time T for all $\omega \in \Omega_0$. Let us also assume that $\sup_{t\in[0,T]} |X_t(\omega)| < \infty$ for all $\omega \in \Omega_0$, which is justified by the assumption $(X_t) \in \mathcal{L}_b$. Since $d\rho^n_t(\omega)$ and $d\rho_t(\omega)$ define positive measures on $[0,T]$ for each $\omega \in \Omega_0$, the convergence of integrals in (4.15) can be deduced from the weak convergence of finite measures (see Section A.6). Indeed, if $\omega \in \Omega_0$ is such that $\rho_T(\omega) = 0$, the right-hand side of (4.15) is zero and we have

$$\limsup_{n\to\infty} \left| \int_{[0,T]} X_t(\omega) d\rho^n_t(\omega) \right| \leq \limsup_{n\to\infty} \sup_{t\in[0,T]} |X_t(\omega)| \rho^n_T(\omega) = 0,$$

where we used that $\sup_{t\in[0,T]} |X_t(\omega)| < \infty$. If, instead, $\omega \in \Omega_0$ is such that $\rho_T(\omega) > 0$, then for all sufficiently large n's, we have $\rho^n_T(\omega) > 0$ and $t \mapsto \rho^n_t(\omega)/\rho^n_T(\omega)$ define cumulative distribution functions converging pointwise to $\rho_t(\omega)/\rho_T(\omega)$ at the points of continuity of $\rho_t(\omega)$. Since $t \mapsto X_t(\omega)$ is continuous, Theorem A.6.3 justifies

$$\begin{aligned} \lim_{n\to\infty} \int_{[0,T]} X_t(\omega) d\rho^n_t(\omega) &= \lim_{n\to\infty} \rho^n_T(\omega) \int_{[0,T]} X_t(\omega) d\left(\frac{\rho^n_t(\omega)}{\rho^n_T(\omega)}\right) \\ &= \rho_T(\omega) \int_{[0,T]} X_t(\omega) d\left(\frac{\rho_t(\omega)}{\rho_T(\omega)}\right) = \int_{[0,T]} X_t(\omega) d\rho_t(\omega). \end{aligned} \tag{4.16}$$

Now we drop the continuity assumption on (X_t). We turn our attention to càdlàg, $(\mathcal{F}_t)$-adapted and regular $(X_t) \in \mathcal{L}_b$. By Theorem 2.3.6, there is $(\tilde{X}_t) \in \mathcal{L}_b$ with continuous trajectories such that (X_t) is an $(\mathcal{F}_t)$-optional projection of $(\tilde{X}_t)$ (Definition A.5.3). From the first part of the proof we know that (4.14) holds for $(\tilde{X}_t)$. To show that it holds for (X_t) it is sufficient to notice that (ρ_t^n) and (ρ_t) are $(\mathcal{F}_t)$-adapted processes and apply Theorem 2.3.7 to obtain

$$\mathbb{E}\left[\int_{[0,T]} X_t d\rho_t^n\right] = \mathbb{E}\left[\int_{[0,T]} \tilde{X}_t d\rho_t^n\right] \quad \text{and} \quad \mathbb{E}\left[\int_{[0,T]} X_t d\rho_t\right] = \mathbb{E}\left[\int_{[0,T]} \tilde{X}_t d\rho_t\right].$$

□

Remark 4.4.12. *Theorems 2.3.6 and 2.3.7, even though they may seem to only have appeared in the proof of Proposition 4.4.11 for technical reasons, enabled us to generalise one of our main results — Theorem 4.2.4. Originally, we assumed that the payoff processes (f_t) and (g_t) are continuous. The current setting of the chapter assumes no continuity from the beginning — only regularity (A2). This is only possible because in the proof of Proposition 4.4.11 we are able to extend the convergence result from continuous processes to regular and adapted ones.*

Proposition 4.4.13. *For a filtration $(\mathcal{G}_t) \subseteq (\mathcal{F}_t)$, let $(\chi_t) \in \mathcal{A}^\circ(\mathcal{G}_t)$ and $(\rho_t) \in \mathcal{A}_{ac}(\mathcal{G}_t)$, and consider $(X_t) \in \mathcal{L}_b$ which is $(\mathcal{F}_t)$-adapted and regular. If $(\rho_t^n)_{n\geq 1} \subset \mathcal{A}_{ac}(\mathcal{G}_t)$ converges $(\lambda \times \mathbb{P})$-a.e. to (ρ_t) as $n \to \infty$, then*

$$\lim_{n\to\infty} \mathbb{E}\left[\int_{[0,T]} X_t(1-\chi_{t-})d\rho_t^n\right] = \mathbb{E}\left[\int_{[0,T]} X_t(1-\chi_{t-})d\rho_t\right]. \tag{4.17}$$

Proof. Define absolutely continuous adapted processes

$$R_t^n = \int_{[0,t]} (1-\chi_{s-})d\rho_s^n \quad \text{and} \quad R_t = \int_{[0,t]} (1-\chi_{s-})d\rho_s,$$

so that (see Propostition A.4.16)

$$\int_{[0,T]} X_t(1-\chi_{t-})d\rho_t^n = \int_{[0,T]} X_t dR_t^n \quad \text{and} \quad \int_{[0,T]} X_t(1-\chi_{t-})d\rho_t = \int_{[0,T]} X_t dR_t. \tag{4.18}$$

With no loss of generality (thanks to the Definition 4.4.2) we can consider the absolutely continuous representatives of (ρ_t) and (ρ_t^n) from the class $\mathcal{A}_{ac}^\circ(\mathcal{G}_t)$ in the definition of all the integrals above (which we still denote by (ρ_t) and (ρ_t^n) for simplicity). In light of this observation we see that $(R_t^n)_{n\geq 1} \subset \tilde{\mathcal{A}}^\circ(\mathcal{G}_t)$ and $(R_t) \in \tilde{\mathcal{A}}^\circ(\mathcal{G}_t)$, since $1-\chi_{s-} \leq 1$ for any $s \in [0,T]$. We indend to apply Proposition 4.4.11 to the integrals with (R_t^n) and (R_t) in (4.18). Let us verify the conditions of Proposition 4.4.11.

Thanks to Lemma 4.4.9 and recalling that $\rho_T^n = \rho_T = 1$, the set

$$\Omega_0 = \left\{\omega \in \Omega : \lim_{n\to\infty} \rho_t^n(\omega) = \rho_t(\omega) \text{ for all } t \in [0,T]\right\}$$

has full measure, i.e. $\mathbb{P}(\Omega_0)=1$. For any $\omega\in\Omega_0$ and $t\in[0,T]$, integrating by parts as in Proposition 2.2.1, using the dominated convergence theorem and then again integrating by parts gives

$$\lim_{n\to\infty}R^n_t=\lim_{n\to\infty}\left[(1-\chi_t)\rho^n_t-\int_{[0,t]}\rho^n_s d(1-\chi_s)\right]=(1-\chi_t)\rho_t-\int_{[0,t]}\rho_s d(1-\chi_s)=R_t. \tag{4.19}$$

Hence (R^n_t) and (R_t) satisfy the assumptions of Proposition 4.4.11 and we can conclude that (4.17) holds. □

We close this technical section with a similar result to the above but for approximations which are needed for the proof of Proposition 4.4.6. The next proposition is tailored for our specific type of regularisation of processes in $\mathcal{A}(\mathcal{F}^1_t)$. Notice that the left hand side of (4.21) features χ_{t-} while the right hand side has χ_t.

Proposition 4.4.14. *For a filtration $(\mathcal{G}_t)\subseteq(\mathcal{F}_t)$, let $(\chi_t),(\rho_t)\in\mathcal{A}^\circ(\mathcal{G}_t)$, $(\rho^n_t)_{n\geq1}\subset\mathcal{A}^\circ(\mathcal{G}_t)$ and consider $(X_t)\in\mathcal{L}_b$ which is $(\mathcal{F}_t)$-adapted and regular. Assume the sequence $(\rho^n_t)_{n\geq1}$ is non-decreasing and for $\mathbb{P}$-a.e. $\omega\in\Omega$*

$$\lim_{n\to\infty}\rho^n_t(\omega)=\rho_{t-}(\omega)\ \textit{for all}\ t\in[0,T). \tag{4.20}$$

Then

$$\lim_{n\to\infty}\mathbb{E}\left[\int_{[0,T)}X_t(1-\chi_{t-})d\rho^n_t\right]=\mathbb{E}\left[\int_{[0,T)}X_t(1-\chi_t)d\rho_t\right] \tag{4.21}$$

and for $\mathbb{P}$-a.e. $\omega\in\Omega$

$$\lim_{n\to\infty}\rho^n_{t-}(\omega)=\rho_{t-}(\omega)\quad \textit{for all}\ t\in[0,T]. \tag{4.22}$$

Proof. Denote by Ω_0 the set on which the convergence (4.20) holds. The first observation is that for all $\omega\in\Omega_0$ and $t\in(0,T]$

$$\lim_{n\to\infty}\rho^n_{t-}(\omega)=\lim_{n\to\infty}\lim_{u\uparrow t}\rho^n_u(\omega)=\lim_{u\uparrow t}\lim_{n\to\infty}\rho^n_u(\omega)=\lim_{u\uparrow t}\rho_{u-}(\omega)=\rho_{t-}(\omega), \tag{4.23}$$

where the order of limits can be swapped by monotonicity of the process and of the sequence (see Lemma B.2.1). The convergence at $t=0$ is obvious as $\rho^n_{0-}=\rho_{0-}=0$. This proves (4.22).

Define for $t\in[0,T)$,

$$R^n_t=\int_{[0,t]}(1-\chi_{s-})d\rho^n_s,\qquad R_t=\int_{[0,t]}(1-\chi_s)d\rho_s, \tag{4.24}$$

and extend both processes to $t=T$ in a continuous way by taking $R^n_T:=R^n_{T-}$ and $R_T:=R_{T-}$. Similarly to the proof of Proposition 4.4.13, by construction we have $(R^n_t)_{n\geq1}\subset\tilde{\mathcal{A}}^\circ(\mathcal{G}_t)$ and

$(R_t) \in \tilde{\mathcal{A}}^\circ(\mathcal{G}_t)$, and the idea is to apply Proposition 4.4.11. First we notice that for all $\omega \in \Omega$ and any $t \in [0,T)$ we have

$$\Delta R_t(\omega) = (1-\chi_t(\omega))\Delta\rho_t(\omega)$$

so that we can write the set of points of continuity of (R_t) as (recall Definition 4.4.8)

$$C_R(\omega) = C_\rho(\omega) \cup \{t \in [0,T] : \chi_t(\omega) = 1\}.$$

For any $t \in [0,T)$ and all $\omega \in \Omega_0$, integrating $R^n_t(\omega)$ by parts (Proposition 2.2.1) and then taking limits as $n \to \infty$, we obtain

$$\begin{aligned}\lim_{n\to\infty} R^n_t(\omega) &= \lim_{n\to\infty}\Big[(1-\chi_t(\omega))\rho^n_t(\omega) - \int_{[0,t]}\rho^n_s(\omega)d(1-\chi_s(\omega))\Big] \\ &= (1-\chi_t(\omega))\rho_{t-}(\omega) - \int_{[0,t]}\rho_{s-}(\omega)d(1-\chi_s(\omega)) \\ &= R_t(\omega) - (1-\chi_t(\omega))\Delta\rho_t(\omega) = R_{t-}(\omega),\end{aligned} \tag{4.25}$$

where the second equality uses dominated convergence and (4.20), and the third equality is integration by parts. We can therefore conclude that

$$\lim_{n\to\infty} R^n_t(\omega) = R_t(\omega), \quad \text{for all } t \in C_R(\omega) \cap [0,T) \text{ and all } \omega \in \Omega_0.$$

It remains to show the convergence at T (which belongs to $C_R(\omega)$ by our construction of (R_t)). Since the function $t \mapsto \rho_t(\omega)$ is non-decreasing and the sequence $(\rho^n_t(\omega))_n$ is non-decreasing, the sequence $(R^n_t(\omega))_n$ is as well non-decreasing (a proof of this fact is contained in Lemma B.2.2). As in (4.23), we show that $\lim_{n\to\infty} R^n_{T-}(\omega) = R_{T-}(\omega)$ for $\omega \in \Omega_0$. By construction of (R^n_t) and (R_t), this proves convergence of R^n_T to R_T.

Then, the processes (R^n_t) and (R_t) fulfil all the assumptions of Proposition 4.4.11 whose application allows us to obtain (4.21). □

From the convergence (4.25), an identical argument as in (4.23) (with (R^n_t), (R_t) in place of (ρ^n_t), (ρ_t)) proves convergence of left-limits of processes (R^n) at any $t \in [0,T]$. The following corollary formalises this observation. It will be used in Section 4.5.1.

Corollary 4.4.15. *Consider the processes (R^n_t) and (R_t) defined in (4.24). For $\mathbb{P}$-a.e. $\omega \in \Omega$ we have*

$$\lim_{n\to\infty} R^n_{t-}(\omega) = R_{t-}(\omega) \quad \textit{for all } t \in [0,T].$$

4.4.2 Verification of the conditions of Sion's theorem

For the application of Sion's theorem, we will consider the weak topology on $\mathcal{A}_{ac}(\mathcal{F}_t^1)$ and $\mathcal{A}(\mathcal{F}_t^2)$ inherited from the space $\mathcal{S}$. In our arguments, we will often use notions and theorems from Section A.3. In particular, we will use that for convex sets the weak and strong closedness are equivalent (Theorem A.3.3), and that any $\mathcal{S}$-converging sequence admits a $(\lambda \times \mathbb{P})$-a.e. -converging subsequence (Theorem A.3.7).

Lemma 4.4.16. *For any filtration $(\mathcal{G}_t) \subseteq (\mathcal{F}_t)$ satisfying the usual conditions, the set $\mathcal{A}(\mathcal{G}_t)$ is weakly compact in $\mathcal{S}$.*

Proof. We write $\mathcal{A}$ for $\mathcal{A}(\mathcal{G}_t)$ and $\mathcal{A}^\circ$ for $\mathcal{A}^\circ(\mathcal{G}_t)$. The set $\mathcal{A}$ is a subset of the unit ball in $\mathcal{S}$. Since $\mathcal{S}$ is a reflexive Banach space (Corollary A.3.9), this ball is weakly compact (Kakutani's Theorem A.3.5). Therefore, we only need to show that $\mathcal{A}$ is weakly closed (then the weak compactness is implied by Theorem A.2.3). Since $\mathcal{A}$ is convex, it is enough to show that $\mathcal{A}$ is strongly closed (Theorem A.3.3).

Take a sequence $(\rho_t^n)_{n\geq 1} \subset \mathcal{A}$ that converges strongly in $\mathcal{S}$ to (ρ_t). We will prove that $(\rho_t) \in \mathcal{A}$ by constructing a càdlàg non-decreasing adapted process $(\hat{\rho}_t)$ such that $\hat{\rho}_{0-} = 0$, $\hat{\rho}_T = 1$, and $\hat{\rho} = \rho$ $(\lambda \times \mathbb{P})$-a.e. With no loss of generality we can pass to the càdlàg representatives $(\hat{\rho}_t^n)_{n\geq 1} \subset \mathcal{A}^\circ$ which also converge to (ρ_t) in $\mathcal{S}$. Then, there is a subsequence $(n_k)_{k\geq 1}$ such that $\hat{\rho}_t^{n_k} \to \rho_t$ $(\lambda \times \mathbb{P})$-a.e. (Theorem A.3.7).

Since
$$\int_0^t \mathbb{P}\big(\lim_{k\to\infty} \hat{\rho}_s^{n_k} = \rho_s\big) ds = t, \qquad \text{for all } t \in [0,T],$$
we can find $\hat{D} \subset [0,T]$ with $\lambda([0,T] \setminus \hat{D}) = 0$ such that $\mathbb{P}(\Omega_t) = 1$ for all $t \in \hat{D}$, where

$$\Omega_t := \{\omega \in \Omega : \lim_{k\to\infty} \hat{\rho}_t^{n_k}(\omega) = \rho_t(\omega)\}.$$

Then we can take a dense countable subset $D \subset \hat{D}$ and define $\Omega_0 := \cap_{t\in D} \Omega_t$ so that $\mathbb{P}(\Omega_0) = 1$ and
$$\lim_{k\to\infty} \hat{\rho}_t^{n_k}(\omega) = \rho_t(\omega), \qquad \text{for all } (t,\omega) \in D \times \Omega_0.$$
Since $\hat{\rho}^{n_k}$ are non-decreasing, so is the mapping $D \ni t \mapsto \rho_t(\omega)$ for all $\omega \in \Omega_0$. Let us extend this mapping to $[0,T]$ by defining $\hat{\rho}_t(\omega) := \rho_t(\omega)$ for $t \in D$ and

$$\hat{\rho}_t(\omega) := \lim_{s\in D: s\downarrow t} \rho_s(\omega), \quad \hat{\rho}_{0-}(\omega) := 0, \quad \hat{\rho}_T(\omega) := 1, \quad \text{for all } \omega \in \Omega_0,$$

where the limit exists due to monotonicity. For $\omega \in \mathcal{N} := \Omega \setminus \Omega_0$, we set $\hat{\rho}_t(\omega) = 0$ for $t < T$ and $\hat{\rho}_T(\omega) = 1$. Notice that $\mathcal{N} \in \mathcal{G}_0$ since $\mathbb{P}(\mathcal{N}) = 0$ so that $\hat{\rho}_t$ is $\mathcal{G}_t$-measurable for $t \in D$. Moreover,

$(\hat{\rho}_t)$ is càdlàg by construction and $\hat{\rho}_t$ is measurable with respect to $\cap_{s\in D, s>t}\mathcal{G}_s = \mathcal{G}_{t+} = \mathcal{G}_t$ for each $t\in[0,T]$ by the right-continuity of the filtration (Assumption (A5)). Hence, $(\hat{\rho}_t)$ is $(\mathcal{G}_t)$-adapted and $(\hat{\rho}_t)\in\mathcal{A}^\circ$.

It remains to show that $\hat{\rho}^{n_k}\to\hat{\rho}$ in $\mathcal{S}$ so that $\hat{\rho}=\rho$ $(\lambda\times\mathbb{P})$-a.e. and therefore $(\rho_t)\in\mathcal{A}$. It suffices to show that $\hat{\rho}^{n_k}\to\hat{\rho}$ $(\lambda\times\mathbb{P})$-a.e. and then conclude by dominated convergence that $\hat{\rho}^{n_k}\to\hat{\rho}$ in $\mathcal{S}$. For each $\omega\in\Omega_0$ the process $t\mapsto\hat{\rho}(\omega)$ has at most countably many jumps (on any bounded interval) by monotonicity (Proposition A.4.15), i.e. $\lambda([0,T]\setminus C_{\hat{\rho}}(\omega))=0$ (recall Definition 4.4.8). Moreover, arguing as in the proof of Lemma 4.4.9, we conclude

$$\lim_{k\to\infty}\hat{\rho}_t^{n_k}(\omega)=\hat{\rho}_t(\omega),\quad \text{for all } t\in C_{\hat{\rho}}(\omega) \text{ and all } \omega\in\Omega_0.$$

Since $(\lambda\times\mathbb{P})(\{(t,\omega):t\in C_{\hat{\rho}}(\omega)\cap B,\omega\in\Omega_0\})=\lambda(B)$ for any bounded interval $B\subseteq[0,T]$, then $\hat{\rho}^{n_k}\to\hat{\rho}$ in $\mathcal{S}$. Thus, $\mathcal{A}$ is strongly closed in $\mathcal{S}$. □

Remark 4.4.17. *Our space $\mathcal{A}(\mathcal{G}_t)$ is the space of processes that generate randomised stopping times, and for any $(\rho_t)\in\mathcal{A}(\mathcal{G}_t)$ we require that $\rho_T(\omega)=1$ for all $\omega\in\Omega$. In the finite horizon problem, i.e. $T<\infty$, such specification imposes a constraint that prevents a direct use of the topology induced by the norm considered in Touzi & Vieille (2002). Indeed, in Touzi & Vieille (2002) the space $\mathcal{S}$ is that of $(\mathcal{G}_t)$-adapted processes (ρ_t) with*

$$||\rho||^2:=\mathbb{E}\left[\int_0^T(\rho_t)^2dt+(\Delta\rho_T)^2\right]<\infty,\quad \Delta\rho_T:=\rho_T-\liminf_{t\uparrow T}\rho_t.$$

The space of generating processes $\mathcal{A}(\mathcal{G}_t)$ is not closed in the topology induced by $\|\cdot\|$ above: for example, define a sequence $(\rho_t^n)_{n\geq1}\subset\mathcal{A}(\mathcal{G}_t)$ by

$$\rho_t^n=n\left(t-T+\frac{1}{n}\right)^+,\qquad t\in[0,T].$$

Then $||\rho^n||\to0$ as $n\to\infty$ but $\rho\equiv0\notin\mathcal{A}(\mathcal{G}_t)$ since it fails to be equal to one at T (and it is not possible to select a representative from $\mathcal{A}(\mathcal{G}_t)$ with the equivalence relation induced by $\|\cdot\|$).

Lemma 4.4.18. *Given any $(\xi,\zeta)\in\mathcal{A}_{ac}(\mathcal{F}_t^1)\times\mathcal{A}(\mathcal{F}_t^2)$, the functionals $N(\xi,\cdot):\mathcal{A}(\mathcal{F}_t^2)\to\mathbb{R}$ and $N(\cdot,\zeta):\mathcal{A}_{ac}(\mathcal{F}_t^1)\to\mathbb{R}$ are, respectively, upper semicontinuous and lower semicontinuous in the strong topology of $\mathcal{S}$.*

Proof. Recall from (4.9) that

$$N(\xi,\zeta)=\mathbb{E}\left[\int_{[0,T)}f_t(1-\zeta_{t-})d\xi_t+\int_{[0,T)}g_t(1-\xi_t)d\zeta_t+h\Delta\xi_T\Delta\zeta_T\right].\tag{4.26}$$

Upper semicontinuity of $N(\xi,\cdot)$. Fix $(\xi_t) \in \mathcal{A}_{ac}(\mathcal{F}_t^1)$ and consider a sequence $(\zeta_t^n)_{n\geq 1} \subset \mathcal{A}(\mathcal{F}_t^2)$ converging to $(\zeta_t) \in \mathcal{A}(\mathcal{F}_t^2)$ strongly in $\mathcal{S}$. We have to show that

$$\limsup_{n\to\infty} N(\xi,\zeta^n) \leq N(\xi,\zeta).$$

Assume, by contradiction, that $\limsup_{n\to\infty} N(\xi,\zeta^n) > N(\xi,\zeta)$. There is a subsequence (n_k) over which the limit on the left-hand side is attained. Along a further subsequence we have $(\mathbb{P}\times\lambda)$-a.e. convergence of (ζ_t^n) to (ζ_t) (Theorem A.3.7). With an abuse of notation we will assume that the original sequence posesses those two properties, i.e. the limit $\lim_{n\to\infty} N(\xi,\zeta^n)$ exists and it strictly dominates $N(\xi,\zeta)$, and there is $(\mathbb{P}\times\lambda)$-a.e. convergence of (ζ_t^n) to (ζ_t).

Since (ξ_t) is absolutely continuous on $[0,T)$,

$$\lim_{n\to\infty} \mathbb{E}\left[\int_{[0,T)} f_t(1-\zeta_{t-}^n)d\xi_t\right] = \mathbb{E}\left[\int_{[0,T)} f_t(1-\zeta_{t-})d\xi_t\right]$$

by the dominated convergence theorem. For the last two terms of $N(\xi,\zeta^n)$ in (4.26) we have

$$\begin{aligned}\mathbb{E}\left[\int_{[0,T)} g_t(1-\xi_t)d\zeta_t^n + h\Delta\xi_T\Delta\zeta_T^n\right] &= \mathbb{E}\left[\int_{[0,T)} g_t(1-\xi_{t-})d\zeta_t^n + h\Delta\xi_T\Delta\zeta_T^n\right]\\ &= \mathbb{E}\left[\int_{[0,T]} g_t(1-\xi_{t-})d\zeta_t^n + (h-g_T)\Delta\xi_T\Delta\zeta_T^n\right],\end{aligned}$$

where the first equality is by (absolute) continuity of (ξ_t) and for the second one we used that $1-\xi_{T-} = \Delta\xi_T$. From Lemma 4.4.9 and the boundedness and continuity of (ξ_t) we verify the assumptions of Proposition 4.4.11 (with $X_t = g_t(1-\xi_{t-})$ therein since (ξ_{t-}) is continuous on $[0,T]$), so

$$\lim_{n\to\infty} \mathbb{E}\left[\int_{[0,T]} g_t(1-\xi_{t-})d\zeta_t^n\right] = \mathbb{E}\left[\int_{[0,T]} g_t(1-\xi_{t-})d\zeta_t\right].$$

Recalling that $g_T \leq h$, we obtain from Lemma 4.4.10

$$\limsup_{n\to\infty} \mathbb{E}\big[(h-g_T)\Delta\xi_T\Delta\zeta_T^n\big] \leq \mathbb{E}\big[(h-g_T)\Delta\xi_T\Delta\zeta_T\big].$$

Combining above convergence results contradicts $\lim_{n\to\infty} N(\xi,\zeta^n) > N(\xi,\zeta)$, hence, proves the upper semicontinuity.

Lower semicontinuity of $N(\cdot,\zeta)$. Fix $(\zeta_t) \in \mathcal{A}(\mathcal{F}_t^2)$ and consider a sequence $(\xi_t^n)_{n\geq 1} \subset \mathcal{A}_{ac}(\mathcal{F}_t^1)$ converging to $(\xi_t) \in \mathcal{A}_{ac}(\mathcal{F}_t^1)$ strongly in $\mathcal{S}$. Arguing by contradiction as above, we assume that there is a subsequence of (ξ_t^n) which we denote the same, such that $\xi^n \to \xi$ $(\mathbb{P}\times\lambda)$-a.e. and

$$\lim_{n\to\infty} N(\xi^n,\zeta) < N(\xi,\zeta). \tag{4.27}$$

By Lemma 4.4.9 and the continuity of (ξ_t) we have for $\mathbb{P}$-a.e. $\omega \in \Omega$

$$\lim_{n\to\infty} \xi_t^n(\omega) = \xi_t(\omega) \quad \text{for all } t \in [0,T).$$

Then by dominated convergence for the second term of $N(\xi^n,\zeta)$ in (4.26) we get

$$\lim_{n\to\infty} \mathbb{E}\left[\int_{[0,T)} g_t(1-\xi_t^n)d\zeta_t\right] = \mathbb{E}\left[\int_{[0,T)} g_t(1-\xi_t)d\zeta_t\right].$$

For the remaining terms of $N(\xi^n,\zeta)$, we have

$$\mathbb{E}\left[\int_{[0,T)} f_t(1-\zeta_{t-})d\xi_t^n + h\Delta\xi_T^n\Delta\zeta_T\right] = \mathbb{E}\left[\int_{[0,T]} f_t(1-\zeta_{t-})d\xi_t^n + (h-f_T)\Delta\xi_T^n\Delta\zeta_T\right].$$

Observe that, by Lemma 4.4.10,

$$\liminf_{n\to\infty} \mathbb{E}\big[(h-f_T)\Delta\xi_T^n\Delta\zeta_T\big] \geq \mathbb{E}\big[(h-f_T)\Delta\xi_T\Delta\zeta_T\big],$$

because $h - f_T \leq 0$. Further,

$$\lim_{n\to\infty} \mathbb{E}\left[\int_{[0,T]} f_t(1-\zeta_{t-})d\xi_t^n\right] = \mathbb{E}\left[\int_{[0,T]} f_t(1-\zeta_{t-})d\xi_t\right]$$

by Proposition 4.4.13. The above results contradict (4.27), therefore, proving the lower semicontinuity. □

We are now ready to prove that the game with continuous randomisation for the first player (τ-player) has a value.

Proof of Theorem 4.4.5. We will show that the conditions of Sion's theorem hold (recall the notation in Theorem 3.5.3) with $(A,B) = (\mathcal{A}(\mathcal{F}_t^2), \mathcal{A}_{ac}(\mathcal{F}_t^1))$ on the space $\mathcal{S}\times\mathcal{S}$ equipped with its weak topology. For the sake of compactness of notation, we will write $\mathcal{A}$ for $\mathcal{A}(\mathcal{F}_t^2)$ and $\mathcal{A}_{ac}$ for $\mathcal{A}_{ac}(\mathcal{F}_t^1)$. A straightforward proof of the fact that the sets $\mathcal{A}$ and $\mathcal{A}_{ac}$ are convex is contained in Lemma B.2.3. Compactness of $\mathcal{A}$ in the weak topology of $\mathcal{S}$ follows from Lemma 4.4.16. It remains to prove the convexity and semicontinuity properties of the functional N with respect to the weak topology of $\mathcal{S}$. This is equivalent to showing (see Theorem A.4.5) that for any $a \in \mathbb{R}$, $(\hat{\xi}_t) \in \mathcal{A}_{ac}$ and $(\hat{\zeta}_t) \in \mathcal{A}$ the level sets

$$\mathcal{K}^a(\hat{\zeta}) = \{(\xi_t) \in \mathcal{A}_{ac} : N(\xi,\hat{\zeta}) \leq a\} \qquad \text{and} \qquad \mathcal{Z}^a(\hat{\xi}) = \{(\zeta_t) \in \mathcal{A} : N(\hat{\xi},\zeta) \geq a\}$$

are convex and closed in $\mathcal{A}_{ac}$ and $\mathcal{A}$, respectively, with respect to the weak topology of $\mathcal{S}$. For any $\lambda \in [0,1]$ and $(\xi_t^1),(\xi_t^2) \in \mathcal{A}_{ac}$, $(\zeta_t^1),(\zeta_t^2) \in \mathcal{A}$, using the expression in (4.9) it is immediate (by linearity) that

$$N(\lambda\xi^1 + (1-\lambda)\xi^2, \hat{\zeta}) = \lambda N(\xi^1,\hat{\zeta}) + (1-\lambda)N(\xi^2,\hat{\zeta}),$$
$$N(\hat{\xi}, \lambda\zeta^1 + (1-\lambda)\zeta^2) = \lambda N(\hat{\xi},\zeta^1) + (1-\lambda)N(\hat{\xi},\zeta^2).$$

This proves the convexity of the level sets. Their closedness in the strong topology of $\mathcal{S}$ follows from Lemma 4.4.18 (an attentive reader may have noticed that only sequential closedness was proven, but then an application of Lemma A.2.5 proves the topological closedness). The latter two properties imply, by Theorem A.3.3, that the level sets are closed in the weak topology of $\mathcal{S}$. Sion's theorem (Theorem 3.5.3) yields the existence of the value of the game: $W_* = W^*$.

The second part of the statement results from using Lemma 3.6.3 which allows to write max instead of sup in (4.11), i.e.

$$\sup_{\zeta\in\mathcal{A}} \inf_{\xi\in\mathcal{A}_{ac}} N(\xi,\zeta) = \max_{\zeta\in\mathcal{A}} \inf_{\xi\in\mathcal{A}_{ac}} N(\xi,\zeta) = \inf_{\xi\in\mathcal{A}_{ac}} N(\xi,\zeta^*),$$

where $(\zeta_t^*) \in \mathcal{A}$ delivers the maximum. □

4.4.3 Approximation with continuous controls

We now prove Proposition 4.4.6 by constructing a sequence $(\xi_t^n)_n$ of Lipschitz continuous processes with the Lipschitz constant for each process bounded by n for all ω. This uniform bound on the Lipschitz constant is not used below, as we only need that each of the processes (ξ_t^n) has absolutely continuous trajectories with respect to the Lebesgue measure on $[0,T)$ so that it belongs to $\mathcal{A}_{ac}(\mathcal{F}_t^1)$.

Proof of Proposition 4.4.6. Fix $(\zeta_t) \in \mathcal{A}(\mathcal{F}_t^2)$. We need to show that for any $(\xi_t) \in \mathcal{A}(\mathcal{F}_t^1)$, there exists a sequence $(\xi_t^n)_{n\geq 1} \subset \mathcal{A}_{ac}(\mathcal{F}_t^1)$ such that

$$\limsup_{n\to\infty} N(\xi^n,\zeta) \leq N(\xi,\zeta). \tag{4.28}$$

We will explicitly construct absolutely continuous (ξ_t^n) that approximate (ξ_t) in a suitable sense. As $N(\xi,\zeta)$ does not depend on the choice of càdlàg representatives, by Definition 4.4.2, without loss of generality we assume that $(\xi_t) \in \mathcal{A}^\circ(\mathcal{F}_t^1)$ and $(\zeta_t) \in \mathcal{A}^\circ(\mathcal{F}_t^2)$. Define a function $\phi_t^n = (nt) \wedge 1 \vee 0$. Let $\xi_t^n := \int_{[0,t]} \phi_{t-s}^n d\xi_s$ for $t \in [0,T)$, and $\xi_T^n := 1$. We shall show that (ξ_t^n) are n-Lipschitz, hence (Lemma A.4.21) absolutely continuous on $[0,T)$. Note that $\phi_t^n \equiv 0$ for $t \leq 0$, and therefore $\xi_t^n = \int_{[0,T]} \phi_{t-s}^n d\xi_s$ for $t \in [0,T)$. For arbitrary $t_1, t_2 \in [0,T)$ we have

$$\begin{aligned}|\xi_{t_1}^n - \xi_{t_2}^n| &= \left|\int_{[0,T]} (\phi_{t_1-s}^n - \phi_{t_2-s}^n) d\xi_s\right| \leq \int_{[0,T]} |\phi_{t_1-s}^n - \phi_{t_2-s}^n| d\xi_s \\ &\leq \int_{[0,T]} n|(t_1-s)-(t_2-s)| d\xi_s = \int_{[0,T]} n|t_1-t_2| d\xi_s = n|t_1-t_2|,\end{aligned}$$

where the first inequality is a generic estimate of the absolute value of an integral, and the second inequality follows by the definition of ϕ^n.

We will verify the assumptions of Proposition 4.4.14. Clearly the sequence (ξ^n_t) is non-decreasing in n, as the measure $d\xi(\omega)$ is positive for each $\omega \in \Omega$ and the sequence ϕ^n is non-decreasing. By the construction of (ξ^n_t) we have $\xi^n_0 = 0 \to \xi_{0-}$ as $n \to \infty$. Moreover, for any $t \in (0,T)$ and $n > \frac{1}{t}$

$$\xi^n_t = \int_{[0,t)} \phi^n_{t-s} d\xi_s = \xi_{t-\frac{1}{n}} + \int_{(t-\frac{1}{n},t)} n(t-s) d\xi_s$$

where the first equality uses that $\phi^n_0 = 0$, so that jumps of (ξ_t) at time t give zero contribution, and the second one uses the definition of ϕ^n. Letting $n \to \infty$, we see that the second term above vanishes since

$$0 \leq \int_{(t-\frac{1}{n},t)} n(t-s) d\xi_s \leq \xi_{t-} - \xi_{t-\frac{1}{n}} \to 0,$$

and therefore obtain $\xi^n_t \to \xi_{t-}$ for all $t \in [0,T)$. Proposition 4.4.14 implies that

$$\lim_{n\to\infty} \mathbb{E}\left[\int_{[0,T)} f_t(1-\zeta_{t-}) d\xi^n_t\right] = \mathbb{E}\left[\int_{[0,T)} f_t(1-\zeta_t) d\xi_t\right], \tag{4.29}$$

and $\lim_{n\to\infty} \xi^n_{T-} = \xi_{T-}$ so that

$$\lim_{n\to\infty} \Delta\xi^n_T = \Delta\xi_T, \tag{4.30}$$

since $\xi^n_T = 1$ for all $n \geq 1$. The dominated convergence theorem (applied to the second integral below) and (4.29), (4.30) yield

$$\begin{aligned} \lim_{n\to\infty} N(\xi^n,\zeta) &= \lim_{n\to\infty} \mathbb{E}\left[\int_{[0,T)} f_t(1-\zeta_{t-}) d\xi^n_t + \int_{[0,T)} g_t(1-\xi^n_t) d\zeta_t + h\Delta\xi^n_T\Delta\zeta_T\right] \\ &= \mathbb{E}\left[\int_{[0,T)} f_t(1-\zeta_t) d\xi_t + \int_{[0,T)} g_t(1-\xi_{t-}) d\zeta_t + h\Delta\xi_T\Delta\zeta_T\right]. \end{aligned} \tag{4.31}$$

Note that

$$\begin{aligned} N(\xi,\zeta) &= \mathbb{E}\left[\int_{[0,T)} f_t(1-\zeta_{t-}) d\xi_t + \int_{[0,T)} g_t(1-\xi_t) d\zeta_t + h\Delta\xi_T\Delta\zeta_T\right] \\ &= \mathbb{E}\left[\int_{[0,T)} f_t(1-\zeta_t) d\xi_t + \int_{[0,T)} g_t(1-\xi_{t-}) d\zeta_t + \sum_{t\in[0,T)} (f_t - g_t)\Delta\xi_t\Delta\zeta_t + h\Delta\xi_T\Delta\zeta_T\right] \\ &\geq \mathbb{E}\left[\int_{[0,T)} f_t(1-\zeta_t) d\xi_t + \int_{[0,T)} g_t(1-\xi_{t-}) d\zeta_t + h\Delta\xi_T\Delta\zeta_T\right], \end{aligned} \tag{4.32}$$

where the last inequality is due to Assumption (A3). Combining this with (4.31) completes the proof of (4.28). □

Remark 4.4.19. *Recall the similar approximating construction* (3.28) *of (Touzi & Vieille, 2002, Lemma 5.2). The advantage of our construction in the proof of Proposition 4.4.6 is that it allows to avoid integrating with respect to the payoff processes* (f_t), (g_t). *Hence, unlike Touzi & Vieille (2002), we do not have to impose a semimartingality assumption on the payoff processes.*

4.5 Extensions and complimentary results

4.5.1 Relaxation of Assumption (A2)

Assumption (A2), which requires that the payoff processes are regular, can be relaxed to allow for previsible jumps. In this section, we extend Theorem 4.4.5 and Proposition 4.4.6 to the case of Assumption (A2') with the payoff process (g_t) having non-negative previsible jumps, i.e. $(\hat{g}_t)$ from the decomposition in (A2') being non-decreasing. Arguments when $(\hat{f}_t)$ is non-increasing (the payoff process (f_t) has non-positive previsible jumps) are analogous. However, in that case, we define the analogue of problem (4.11) where the second player (maximiser) uses absolutely continuous generating processes $(\zeta_t) \in \mathcal{A}_{ac}(\mathcal{F}_t^2)$ and the first player (minimiser) picks $(\xi_t) \in \mathcal{A}(\mathcal{F}_t^1)$. We give more details on this symmetric problem below in this section and in Section 4.5.2.

Theorem 4.5.1. *Under Assumptions (A1), (A2'), (A3)-(A5) (with* $(\hat{g}_t)$ *non-decreasing), the game* (4.11) *has a value, i.e.*

$$W_* = W^* := W.$$

Moreover, the ζ*-player (maximiser) has an optimal strategy, i.e. there exists* $(\zeta_t^*) \in \mathcal{A}(\mathcal{F}_t^2)$ *such that*

$$\inf_{\xi \in \mathcal{A}_{ac}(\mathcal{F}_t^1)} N(\xi, \zeta^*) = W.$$

Proposition 4.5.2. *Under assumptions (A1), (A2'), (A3)-(A5) (with* $(\hat{g}_t)$ *non-decreasing), for any* $(\zeta_t) \in \mathcal{A}(\mathcal{F}_t^2)$ *and* $(\xi_t) \in \mathcal{A}(\mathcal{F}_t^1)$*, there is a sequence* $(\xi_t^n) \subset \mathcal{A}_{ac}(\mathcal{F}_t^1)$ *such that*

$$\limsup_{n\to\infty} N(\xi^n, \zeta) \le N(\xi, \zeta).$$

We remark that, in order to treat the case of Assumption (A2') with the payoff process $(\hat{f}_t)$ non-increasing, we need to define

$$\widetilde{W}_* = \sup_{\zeta \in \mathcal{A}_{ac}(\mathcal{F}_t^2)} \inf_{\xi \in \mathcal{A}(\mathcal{F}_t^1)} N(\xi, \zeta) \quad \text{and} \quad \widetilde{W}^* = \inf_{\xi \in \mathcal{A}(\mathcal{F}_t^1)} \sup_{\zeta \in \mathcal{A}_{ac}(\mathcal{F}_t^2)} N(\xi, \zeta), \tag{4.33}$$

and to state the following theorem analogous to Theorem 4.5.1.

Theorem 4.5.3. *Under Assumptions (A1), (A2'), (A3)-(A5) (with $(\hat{f}_t)$ non-increasing), the game (4.33) has a value, i.e.*

$$\widetilde{W}_* = \widetilde{W}^* := \widetilde{W}.$$

Moreover, the ξ-player (minimiser) has an optimal strategy, i.e. there exists $(\xi^*_t) \in \mathcal{A}(\mathcal{F}^2_t)$ such that

$$\sup_{\zeta \in \mathcal{A}_{ac}(\mathcal{F}^2_t)} N(\xi^*, \zeta) = \widetilde{W}.$$

Similarly, Proposition 4.5.2 has the following counterpart.

Proposition 4.5.4. *Under assumptions (A1), (A2'), (A3)-(A5) (with $(\hat{f}_t)$ non-increasing), for any $(\xi_t) \in \mathcal{A}(\mathcal{F}^1_t)$ and $(\zeta_t) \in \mathcal{A}(\mathcal{F}^2_t)$, there is a sequence $(\zeta^n_t) \subset \mathcal{A}_{ac}(\mathcal{F}^2_t)$ such that*

$$\liminf_{n\to\infty} N(\xi, \zeta^n) \geq N(\xi, \zeta).$$

Proof of Theorem 4.2.3. The proof is identical to the proof Theorem 4.2.4 but with references to Theorem 4.4.5 and Proposition 4.4.6 replaced by the above results: more precisely, by Theorem 4.5.1 and Proposition 4.5.2 in case when $(\hat{g}_t)$ is non-decreasing, and by Theorem 4.5.3 and Proposition 4.5.4 in case when $(\hat{f}_t)$ non-increasing. □

Section 4.5.2 is devoted to proving Theorem 4.5.3 and Proposition 4.5.4. In the rest of this section we prove Theorem 4.5.1 and Proposition 4.5.2.

By Assumption (A2'), the processes $(\hat{f}_t), (\hat{g}_t)$ have integrable variation (in the sense of Definition A.5.5), and, in particular, finite variation (Definition A.5.4). By Corollary A.5.7, we then have the following decomposition: there exist $(\mathcal{F}_t)$-stopping times $(\eta^f_k)_{k\geq 1}$ and $(\eta^g_k)_{k\geq 1}$, non-negative $\mathcal{F}_{\eta^f_k}$-measurable random variables X^f_k, $k \geq 1$, and non-negative $\mathcal{F}_{\eta^g_k}$-measurable random variables X^g_k, $k \geq 1$, such that

$$\hat{f}_t = \sum_{k=1}^{\infty} (-1)^k X^f_k I_{\{t \geq \eta^f_k\}}, \qquad \hat{g}_t = \sum_{k=1}^{\infty} X^g_k I_{\{t \geq \eta^g_k\}}. \tag{4.34}$$

The condition $\hat{f}_0 = \hat{g}_0 = 0$ implies that $\eta^f_k, \eta^g_k > 0$ for all $k \geq 1$. Since $(\hat{f}_t), (\hat{g}_t)$ have integrable variation, the infinite sequences in (4.34) are dominated by integrable random variables X^f and X^g: for any $t \in [0, T]$

$$|\hat{f}_t| \leq X^f := \sum_{k=1}^{\infty} X^f_k, \qquad \text{and} \qquad \hat{g}_t \leq X^g := \sum_{k=1}^{\infty} X^g_k. \tag{4.35}$$

To handle convergence of integrals of piecewise-constant processes, we need to extend the results of Proposition 4.4.11.

Proposition 4.5.5. *For a filtration $(\mathcal{G}_t) \subseteq (\mathcal{F}_t)$, consider $(\rho^n_t)_{n\geq 1} \subset \tilde{\mathcal{A}}^\circ(\mathcal{G}_t)$ and $(\rho_t) \in \tilde{\mathcal{A}}^\circ(\mathcal{G}_t)$ with*

$$\mathbb{P}\Big(\Big\{\omega \in \Omega : \lim_{n\to\infty} \rho^n_t(\omega) = \rho_t(\omega), \quad \textit{for all } t \in C_\rho(\omega) \cup \{T\}\Big\}\Big) = 1.$$

Then for any random variables $\theta \in (0,T]$ and $(X_t) \in [0,\infty)$ with $\mathbb{E}[X] < \infty$ we have

$$\limsup_{n\to\infty} \mathbb{E}\Big[\int_{[0,T]} I_{\{t\geq\theta\}} X d\rho^n_t\Big] \leq \mathbb{E}\Big[\int_{[0,T]} I_{\{t\geq\theta\}} X d\rho_t\Big]. \tag{4.36}$$

Furthermore, if $\mathbb{P}(\{\omega : \theta(\omega) \in C_\rho(\omega) \text{ or } X(\omega) = 0\}) = 1$, then

$$\lim_{n\to\infty} \mathbb{E}\Big[\int_{[0,T]} I_{\{t\geq\theta\}} X d\rho^n_t\Big] = \mathbb{E}\Big[\int_{[0,T]} I_{\{t\geq\theta\}} X d\rho_t\Big]. \tag{4.37}$$

Proof. Let Ω_0 be the set of $\omega \in \Omega$ for which $\rho^n_t(\omega) \to \rho_t(\omega)$ for all $t \in C_\rho(\omega) \cup \{T\}$. We have $\mathbb{P}(\Omega_0) = 1$ by assumption. Fix $\omega \in \Omega_0$. Take an arbitrary $t \in C_\rho(\omega)$ such that $t < \theta(\omega)$ (such t always exists, since $\theta(\omega) > 0$ and since $t \mapsto \rho_t(\omega)$ has at most countably many jumps on any bounded interval by Proposition A.4.15). We have $\rho^n_t(\omega) \leq \rho^n_{\theta(\omega)-}(\omega)$, therefore by assumption

$$\liminf_{n\to\infty} \rho^n_{\theta(\omega)-}(\omega) \geq \rho_t(\omega).$$

Since $C_\rho(\omega)$ is dense in $(0,T)$, by arbitrariness of $t < \theta(\omega)$ we have

$$\liminf_{n\to\infty} \rho^n_{\theta(\omega)-}(\omega) \geq \rho_{\theta(\omega)-}(\omega). \tag{4.38}$$

We rewrite the integral as follows: $\int_{[0,T]} I_{\{t\geq\theta\}} X d\rho^n_t = X(\rho^n_T - \rho^n_{\theta-})$. Therefore,

$$\limsup_{n\to\infty} \mathbb{E}\Big[\int_{[0,T]} I_{\{t\geq\theta\}} X d\rho^n_t\Big] = \limsup_{n\to\infty} \mathbb{E}\big[X(\rho^n_T - \rho^n_{\theta-})\big] = \limsup_{n\to\infty} \mathbb{E}[X\rho^n_T] - \liminf_{n\to\infty} \mathbb{E}[X\rho^n_{\theta-}].$$

The dominated convergence theorem yields $\lim_{n\to\infty} \mathbb{E}[X\rho^n_T] = \mathbb{E}[X\rho_T]$, while applying Fubini's theorem gives

$$\liminf_{n\to\infty} \mathbb{E}[X\rho^n_{\theta-}] \geq \mathbb{E}[\liminf_{n\to\infty} X\rho^n_{\theta-}] \geq \mathbb{E}[X\rho_{\theta-}],$$

where the last inequality is by (4.38). Combining the above completes the proof of (4.36).

Assume now that $\theta(\omega) \in C_\rho(\omega)$ or $X(\omega) = 0$ for $\mathbb{P}$-a.e. $\omega \in \Omega_0$. This and the dominated convergence theorem yield

$$\mathbb{E}[X(\rho_T - \rho_{\theta-})] = \mathbb{E}[X(\rho_T - \rho_\theta)] = \lim_{n\to\infty} \mathbb{E}[X(\rho^n_T - \rho^n_\theta)] \leq \limsup_{n\to\infty} \mathbb{E}[X(\rho^n_T - \rho^n_{\theta-})],$$

where the last inequality follows from the monotonicity of (ρ^n_t). This estimate and (4.36) prove (4.37). □

Remark 4.5.6. *The inequality* (4.36) *in Proposition 4.5.5 can be strict even if $\rho^n_t \to \rho_t$ for all $t \in [0,T]$, because this condition does not imply that $\rho^n_{t-} \to \rho_{t-}$. One needs further continuity assumptions on (ρ_t) to establish equality* (4.37).

Proof of Theorem 4.5.1. Compared to the proof of the analogue result under the more stringent condition (A2) (i.e. Theorem 4.4.5), we only need to establish lower and upper semicontinuity of the functional N, while all the remaining arguments stay valid.

Indeed, in the proof of Theorem 4.4.5, we verify that the sets $\mathcal{A}$ and $\mathcal{A}_{ac}$ are convex; compactness of $\mathcal{A}$ in the weak topology of $\mathcal{S}$ is proven in Lemma 4.4.16. In the proof of Theorem 4.4.5, we also show that for any $a \in \mathbb{R}$, $(\hat{\xi}_t) \in \mathcal{A}_{ac}$ and $(\hat{\zeta}_t) \in \mathcal{A}$ the level sets

$$\mathcal{K}^a(\hat{\zeta}) = \{(\xi_t) \in \mathcal{A}_{ac} : N(\xi, \hat{\zeta}) \leq a\} \qquad \text{and} \qquad \mathcal{Z}^a(\hat{\xi}) = \{(\zeta_t) \in \mathcal{A} : N(\hat{\xi}, \zeta) \geq a\}$$

are convex in $\mathcal{A}_{ac}$ and $\mathcal{A}$, respectively. What remains to prove before applying Sion's theorem is that these level sets are closed with respect to the weak topology of $\mathcal{S}$. This, due to the convexity and Theorem A.3.3, follows from their closedness in the strong topology of $\mathcal{S}$, which we establish below by extending arguments of Lemma 4.4.18.

Upper semicontinuity of $N(\xi, \cdot)$. Fix $(\xi_t) \in \mathcal{A}_{ac}(\mathcal{F}^1_t)$ and consider a sequence $(\zeta^n_t)_{n\geq 1} \subset \mathcal{A}(\mathcal{F}^2_t)$ converging to $(\zeta_t) \in \mathcal{A}(\mathcal{F}^2_t)$ strongly in $\mathcal{S}$. Arguing by contradiction, we assume that there is a subsequence of $(\zeta^n_t)_{n\geq 1}$, denoted the same with an abuse of notation, that converges $(\mathbb{P} \times \lambda)$-a.e. to (ζ_t) and such that

$$\lim_{n\to\infty} N(\xi, \zeta^n) > N(\xi, \zeta).$$

Without loss of generality, we can further require that $(\zeta^n_t)_{n\geq 1} \subset \mathcal{A}^\circ(\mathcal{F}^2_t)$ and $(\zeta_t) \in \mathcal{A}^\circ(\mathcal{F}^2_t)$. Since (ξ_t) is absolutely continuous on $[0,T)$, we have

$$\lim_{n\to\infty} \mathbb{E}\left[\int_{[0,T)} f_t(1-\zeta^n_{t-})d\xi_t\right] = \mathbb{E}\left[\int_{[0,T)} f_t(1-\zeta_{t-})d\xi_t\right] \tag{4.39}$$

by the dominated convergence theorem. For the last two terms of $N(\xi, \zeta^n)$ (recall (4.9)) we have

$$\mathbb{E}\left[\int_{[0,T)} g_t(1-\xi_t)d\zeta^n_t + h\Delta\xi_T\Delta\zeta^n_T\right] = \mathbb{E}\left[\int_{[0,T]} g_t(1-\xi_{t-})d\zeta^n_t + (h-g_T)\Delta\xi_T\Delta\zeta^n_T\right].$$

By the proof of Lemma 4.4.18, for the regular part $(\tilde{g}_t)$ of the process (g_t) we have

$$\lim_{n\to\infty} \mathbb{E}\left[\int_{[0,T]} \tilde{g}_t(1-\xi_{t-})d\zeta^n_t\right] = \mathbb{E}\left[\int_{[0,T]} \tilde{g}_t(1-\xi_{t-})d\zeta_t\right]. \tag{4.40}$$

For the pure jump part $(\hat{g}_t)$ of the process (g_t), we will prove that

$$\limsup_{n\to\infty} \mathbb{E}\left[\int_{[0,T]} \hat{g}_t(1-\xi_{t-})d\zeta^n_t\right] \leq \mathbb{E}\left[\int_{[0,T]} \hat{g}_t(1-\xi_{t-})d\zeta_t\right]. \tag{4.41}$$

To this end, let us define

$$R_t^n = \int_{[0,t]} (1-\xi_{s-}) d\zeta_s^n, \qquad R_t = \int_{[0,t]} (1-\xi_{s-}) d\zeta_s, \qquad \text{for } t \in [0,T], \tag{4.42}$$

with $R_{0-}^n = R_{0-} = 0$; then we are going to apply Proposition 4.5.5 with (R_t^n) and (R_t) in place of (ρ_t^n) and (ρ_t). In order to do that, we need to check that $R_t^n(\omega) \to R_t(\omega)$ as $n \to \infty$ for $t \in C_R(\omega) = C_\zeta(\omega) \cup \{t \in [0,T] : \xi_t(\omega) = 1\}$, for $\mathbb{P}$-a.e. $\omega \in \Omega$. The latter is indeed true. Setting $\Omega_0 = \{\omega \in \Omega : \lim_{n\to\infty} \zeta_t^n(\omega) = \zeta_t(\omega)\ \forall t \in C_\zeta(\omega)\}$, we have $\mathbb{P}(\Omega_0) = 1$ by Lemma 4.4.9. For any $\omega \in \Omega_0$ and $t \in C_\zeta(\omega)$, invoking the absolute continuity of (ξ_t), we obtain (omitting the dependence on ω)

$$\lim_{n\to\infty} R_t^n = \lim_{n\to\infty} \left[(1-\xi_t)\zeta_t^n + \int_{[0,t]} \zeta_s^n d\xi_s \right] = (1-\xi_t)\zeta_t + \int_{[0,t]} \zeta_s d\xi_s = R_t,$$

where the convergence of the second term is due to the dominated convergence theorem and the fact that $\lambda([0,T] \setminus C_\zeta(\omega)) = 0$ and $\zeta_T^n = \zeta_T = 1$. On the other hand, for t such that $\xi_t = 1$ we obtain

$$\lim_{n\to\infty} R_t^n = \lim_{n\to\infty} \left[\int_{[0,t]} \zeta_s^n d\xi_s \right] = \int_{[0,t]} \zeta_s d\xi_s = R_t,$$

where we again applied the dominated convergence.

For any $k \geq 1$, since $X_k^g \geq 0$, Proposition 4.5.5 gives (recall (4.34))

$$\limsup_{n\to\infty} \mathbb{E}\left[\int_{[0,T]} X_k^g I_{\{t \geq \eta_k^g\}} dR_t^n \right] \leq \mathbb{E}\left[\int_{[0,T]} X_k^g I_{\{t \geq \eta_k^g\}} dR_t \right]. \tag{4.43}$$

We first apply the decomposition of $(\hat{g}_t)$ and then the monotone convergence theorem:

$$\mathbb{E}\left[\int_{[0,T]} \hat{g}_t (1-\xi_{t-}) d\zeta_t^n \right] = \mathbb{E}\left[\sum_{k=1}^\infty \int_{[0,T]} X_k^g I_{\{t \geq \eta_k^g\}} dR_t^n \right] = \sum_{k=1}^\infty \mathbb{E}\left[\int_{[0,T]} X_k^g I_{\{t \geq \eta_k^g\}} dR_t^n \right].$$

Since $(\hat{g}_t) \in \mathcal{L}_b$, we have the bound (recall (4.35))

$$\sum_{k=1}^\infty \sup_n \mathbb{E}\left[\int_{[0,T]} X_k^g I_{\{t \geq \eta_k^g\}} dR_t^n \right] \leq \sum_{k=1}^\infty \mathbb{E}[X_k^g] < \infty.$$

This allows us to apply reverse Fatou's lemma (with respect to the counting measure on $\mathbb{N}$)

$$\begin{aligned} \limsup_{n\to\infty} \sum_{k=1}^\infty \mathbb{E}\left[\int_{[0,T]} X_k^g I_{\{t \geq \eta_k^g\}} dR_t^n \right] &\leq \sum_{k=1}^\infty \limsup_{n\to\infty} \mathbb{E}\left[\int_{[0,T]} X_k^g I_{\{t \geq \eta_k^g\}} dR_t^n \right] \\ &\leq \sum_{k=1}^\infty \mathbb{E}\left[\int_{[0,T]} X_k^g I_{\{t \geq \eta_k^g\}} dR_t \right] = \mathbb{E}\left[\int_{[0,T]} \hat{g}_t (1-\xi_{t-}) d\zeta_t \right], \end{aligned} \tag{4.44}$$

where the last inequality is due to (4.43) and the final equality follows by monotone convergence and the decomposition of $(\hat{g}_t)$. This completes the proof of (4.41).

Recalling that $g_T \leq h$, we obtain from Lemma 4.4.10

$$\limsup_{n\to\infty} \mathbb{E}\big[(h-g_T)\Delta\xi_T\Delta\zeta_T^n\big] \leq \mathbb{E}\big[(h-g_T)\Delta\xi_T\Delta\zeta_T\big].$$

Combining the latter with (4.39), (4.40), and (4.41) shows that

$$\limsup_{n\to\infty} N(\xi,\zeta^n) \leq N(\xi,\zeta). \tag{4.45}$$

Hence we have a contradiction with $\lim_{n\to\infty} N(\xi,\zeta^n) > N(\xi,\zeta)$, which proves the upper semi-continuity.

Lower semicontinuity of $N(\cdot,\zeta)$. The proof follows closely the argument of the proof of Lemma 4.4.18: we fix $(\zeta_t) \in \mathcal{A}(\mathcal{F}_t^2)$, consider a sequence $(\xi_t^n)_{n\geq 1} \subset \mathcal{A}_{ac}(\mathcal{F}_t^1)$ converging to $(\xi_t) \in \mathcal{A}_{ac}(\mathcal{F}_t^1)$ strongly in $\mathcal{S}$, assume that

$$\lim_{n\to\infty} N(\xi^n,\zeta) < N(\xi,\zeta) \tag{4.46}$$

and reach a contradiction. We focus on how to handle the convergence for $(\hat{f}_t)$ as all other terms are handled by the proof of Lemma 4.4.18.

By Lemma 4.4.9 and the continuity of (ξ_t) we have $\mathbb{P}\big(\lim_{n\to\infty}\xi_t^n(\omega) = \xi_t(\omega)\ \forall t \in [0,T)\big) = 1$. Let

$$R_t^n = \int_{[0,t]} (1-\zeta_{t-})d\xi_t^n, \qquad R_t = \int_{[0,t]} (1-\zeta_{t-})d\xi_t,$$

with $R_{0-}^n = R_{0-} = 0$. Due to the continuity of (ξ_t^n) and (ξ_t) for $t \in [0,T)$, processes (R_t^n) and (R_t) are continuous on $[0,T)$ with a possible jump at T. From (4.19) in the proof of Proposition 4.4.13 we conclude that for $\mathbb{P}$-a.e. $\omega \in \Omega$

$$\lim_{n\to\infty} R_t^n(\omega) = R_t(\omega) \quad \text{for all } t \in [0,T].$$

Since $\Delta\hat{f}_T = 0$ (see Assumption (A2')), there is a decomposition such that $X_k^f I_{\{\eta_k^f = T\}} = 0$ $\mathbb{P}$-a.s. for all k. Recalling that (R_t) is continuous on $[0,T)$, we can apply (4.37) in Proposition 4.5.5: for any $k \geq 1$,

$$\lim_{n\to\infty} \mathbb{E}\bigg[\int_{[0,T]} X_k^f I_{\{t\geq\eta_k^f\}} dR_t^n\bigg] = \mathbb{E}\bigg[\int_{[0,T]} X_k^f I_{\{t\geq\eta_k^f\}} dR_t\bigg].$$

Combining the latter with decomposition (4.34) and the dominated convergence theorem (with the bound X^f) we obtain

$$\lim_{n\to\infty} \mathbb{E}\bigg[\int_{[0,T]} \hat{f}_t dR_t^n\bigg] = \mathbb{E}\bigg[\int_{[0,T]} \hat{f}_t dR_t\bigg].$$

This, together with an identical result for $(\tilde{f}_t)$ that follows from the proof of Lemma 4.4.18, shows that

$$\lim_{n\to\infty} \mathbb{E}\left[\int_{[0,T]} f_t dR^n_t\right] = \mathbb{E}\left[\int_{[0,T]} f_t dR_t\right], \tag{4.47}$$

and the rest of the proof of lower semicontinuity from Lemma 4.4.18 applies. □

Remark 4.5.7. *Item (4) in Assumption (A2') implies in particular that the payoff process* (g_t) *does not have previsible jumps that are* $\mathbb{P}$*-a.s. negative. This assumption cannot be further relaxed as this may cause the proof of the upper semicontinuity in Theorem 4.5.1 to fail. Recall that the process* (g_t) *corresponds to the payoff of the second player and her strategy* (ζ_t) *is not required to be absolutely continuous. For example, fix* $t_0 \in (0,T)$ *and take* $g_t = 1 - I_{\{t\geq t_0\}}$, $\zeta_t = I_{\{t\geq t_0\}}$ *and* $\xi_t = I_{\{t=T\}}$*. Let us consider the sequence* $\zeta^n_t = I_{\{t\geq t_0-\frac{1}{n}\}}$*, which converges to* (ζ_t) *pointwise and also strongly in* $\mathcal{S}$*. We have*

$$\int_{[0,T]} g_t(1-\xi_{t-})d\zeta^n_t \equiv 1, \text{ for all } n\text{'s, but } \int_{[0,T]} g_t(1-\xi_{t-})d\zeta_t \equiv 0,$$

hence (4.41) *fails and so does* (4.45).

Proof of Proposition 4.5.2. Here, we show how to extend the proof of Proposition 4.4.6 to the more general setting. Fix $(\zeta_t) \in \mathcal{A}^\circ(\mathcal{F}^2_t)$ and $(\xi_t) \in \mathcal{A}^\circ(\mathcal{F}^1_t)$. Construct a sequence $(\xi^n_t) \subset \mathcal{A}^\circ_{ac}(\mathcal{F}^1_t)$ as in the proof of Proposition 4.4.6. We need to show that

$$\limsup_{n\to\infty} N(\xi^n,\zeta) \leq N(\xi,\zeta). \tag{4.48}$$

From the proof of Proposition 4.4.6 we have

$$\begin{aligned} &\lim_{n\to\infty} \mathbb{E}\left[\int_{[0,T)} \tilde{f}_t(1-\zeta_{t-})d\xi^n_t + \int_{[0,T)} \tilde{g}_t(1-\xi^n_t)d\zeta_t + h\Delta\xi^n_T\Delta\zeta_T\right] \\ &= \mathbb{E}\left[\int_{[0,T)} \tilde{f}_t(1-\zeta_t)d\xi_t + \int_{[0,T)} \tilde{g}_t(1-\xi_{t-})d\zeta_t + h\Delta\xi_T\Delta\zeta_T\right]. \end{aligned} \tag{4.49}$$

For $t \in [0,T]$, define

$$R^n_t = \int_{[0,t]} (1-\zeta_{s-})d\xi^n_s, \qquad R_t = \int_{[0,t]} (1-\zeta_s)d\xi_s,$$

with $R^n_{0-} = R_{0-} = 0$. Corollary 4.4.15 implies that for $\mathbb{P}$-a.e. $\omega \in \Omega$

$$\lim_{n\to\infty} R^n_{t-}(\omega) = R_{t-}(\omega) \quad \text{for all } t \in [0,T]. \tag{4.50}$$

By the decomposition of $(\hat{f}_t)$ in (4.34) and the dominated convergence theorem for the infinite sum (recalling (4.35)) we obtain

$$\mathbb{E}\left[\int_{[0,T)} \hat{f}_t(1-\zeta_{t-})d\xi_t^n\right] = \sum_{k=1}^{\infty}\mathbb{E}\left[(-1)^k\int_{[0,T)} X_k^f I_{\{t\geq\eta_k^f\}}dR_t^n\right] = \sum_{k=1}^{\infty}\mathbb{E}\left[(-1)^k X_k^f(R_{T-}^n - R_{\eta_k^f-}^n)\right].$$

We further apply dominated convergence (with respect to the product of the counting measure on $\mathbb{N}$ and the measure $\mathbb{P}$) to obtain

$$\begin{aligned}\lim_{n\to\infty}\mathbb{E}\left[\int_{[0,T)} \hat{f}_t(1-\zeta_{t-})d\xi_t^n\right] &= \sum_{k=1}^{\infty}\mathbb{E}\left[(-1)^k\lim_{n\to\infty} X_k^f(R_{T-}^n - R_{\eta_k^f-}^n)\right]\\ &= \sum_{k=1}^{\infty}\mathbb{E}\left[(-1)^k X_k^f(R_{T-} - R_{\eta_k^f-})\right] = \mathbb{E}\left[\int_{[0,T)} \hat{f}_t(1-\zeta_t)d\xi_t\right],\end{aligned} \tag{4.51}$$

where the second equality uses (4.50) and the final one the decomposition of $(\hat{f}_t)$. Recalling that $\xi_t^n \to \xi_{t-}$ as $n\to\infty$ by construction, dominated convergence yields

$$\lim_{n\to\infty}\mathbb{E}\left[\int_{[0,T)} \hat{g}_t(1-\xi_t^n)d\zeta_t\right] = \mathbb{E}\left[\int_{[0,T)} \hat{g}_t(1-\xi_{t-})d\zeta_t\right]. \tag{4.52}$$

Putting together (4.49), (4.51) and (4.52) shows that

$$\lim_{n\to\infty} N(\xi^n,\zeta) = \mathbb{E}\left[\int_{[0,T)} f_t(1-\zeta_t)d\xi_t + \int_{[0,T)} g_t(1-\xi_{t-})d\zeta_t + h\Delta\xi_T\Delta\zeta_T\right].$$

It remains to notice that, by (4.32), the right-hand side is dominated by $N(\xi,\zeta)$, which completes the proof of (4.48). □

4.5.2 Regularising the control of the other player

Recall that initially in Section 4.4, we regularised the control (ξ_t) of the first player to belong to $\mathcal{A}_{ac}(\mathcal{F}_t^1)$ (see (4.11)). In this subsection, we show that the choice of the player caused no loss of generality, by adjusting our proofs to the situation when, instead, the control (ζ_t) of the second player belongs to $\mathcal{A}_{ac}(\mathcal{F}_t^2)$.

We note that the ideas of the proofs are similar to Section 4.5.1. Again, we will have to verify semicontinuity of the functional N in order to prove Theorem 4.5.3, and we will approximate a control by a sequence of Lipschitz-continuous ones to prove Proposition 4.5.4. Some arguments are the same as in Theorem 4.5.1 and Proposition 4.5.2, up to swapping the roles of (ξ_t) and (ζ_t). Note, however, that the decomposition (4.54) below is different from (4.35), and this causes certain differences in the proofs. In particular, in (4.65) we apply the usual Fatou's lemma and not its reverse counterpart as we did in (4.44). Another example is that the inequality (4.32) is written for the functional $N(\xi,\zeta)$, while the symmetric version (4.68) bounds the value of its approximation $N(\xi,\zeta^n)$.

Semicontinuity

Recall Assumption (A2) and its generalisation (A2') — in particular, the last bullet point of the latter. In Section 4.5.1, we covered the case when Assumption (A2') holds with the payoff process (g_t) having only non-negative previsible jumps, i.e. $(\hat{g}_t)$ from the decomposition in (A2') being non-decreasing. In the current symmetric set-up, we assume that, instead, $(\hat{f}_t)$ is non-increasing (the process (f_t) has only non-positive previsible jumps).

The processes $(\hat{f}_t),(\hat{g}_t)$ have the following decomposition (see Section 4.5.1 for the explaination and related references in the symmetric case): there are $(\mathcal{F}_t)$-stopping times $(\eta^f_k)_{k\geq 1}$ and $(\eta^g_k)_{k\geq 1}$ and $\mathcal{F}$-measurable, non-negative random variables $(X^f_k)_{k\geq 1}$ and $(X^g_k)_{k\geq 1}$ such that

$$\hat{f}_t = -\sum_{k=1}^{\infty} X^f_k I_{\{t\geq \eta^f_k\}}, \qquad \hat{g}_t = \sum_{k=1}^{\infty} (-1)^k X^g_k I_{\{t\geq \eta^g_k\}}, \tag{4.53}$$

and the infinite sequences in (4.53) are dominated by integrable random variables X^f and X^g: for any $t\in[0,T]$

$$|\hat{f}_t| \leq X^f := \sum_{k=1}^{\infty} X^f_k, \qquad \text{and} \qquad |\hat{g}_t| \leq X^g := \sum_{k=1}^{\infty} X^g_k. \tag{4.54}$$

Proof of Theorem 4.5.3. As explained in the proof of Theorem 4.5.1, in order to prove Theorem 4.5.3 we only need to establish lower and upper semicontinuity of the functional N. We do so by extending arguments of Lemma 4.4.18.

Upper semicontinuity of $N(\xi,\cdot)$. Fix $(\xi_t)\in\mathcal{A}(\mathcal{F}^1_t)$ and consider a sequence $(\zeta^n_t)_{n\geq 1}\subset\mathcal{A}_{ac}(\mathcal{F}^2_t)$ converging to $(\zeta_t)\in\mathcal{A}_{ac}(\mathcal{F}^2_t)$ strongly in $\mathcal{S}$. Arguing by contradiction, we assume that there is a subsequence of $(\zeta^n_t)_{n\geq 1}$, denoted the same with an abuse of notation, that converges $(\mathbb{P}\times\lambda)$-a.e. to (ζ_t) and such that

$$\lim_{n\to\infty} N(\xi,\zeta^n) > N(\xi,\zeta). \tag{4.55}$$

Without loss of generality, we can further require that $(\zeta^n_t)_{n\geq 1}\subset\mathcal{A}^\circ_{ac}(\mathcal{F}^2_t)$ and $(\zeta_t)\in\mathcal{A}^\circ_{ac}(\mathcal{F}^2_t)$. By the dominated convergence,

$$\lim_{n\to\infty}\mathbb{E}\left[\int_{[0,T)} f_t(1-\zeta^n_{t-})d\xi_t\right] = \mathbb{E}\left[\int_{[0,T)} f_t(1-\zeta_{t-})d\xi_t\right]. \tag{4.56}$$

For the last two terms of $N(\xi,\zeta^n)$ (recall (4.9)) we have, using continuity of (ζ_t^n) on $[0,T)$,

$$\mathbb{E}\Big[\int_{[0,T)} g_t(1-\xi_t)d\zeta_t^n + h\Delta\xi_T\Delta\zeta_T^n\Big] = \mathbb{E}\Big[\int_{[0,T]} g_t(1-\xi_{t-})d\zeta_t^n + (h-g_T)\Delta\xi_T\Delta\zeta_T^n\Big].$$

By Lemma 4.4.9 and the continuity of (ζ_t) we have $\mathbb{P}\big(\lim_{n\to\infty}\zeta_t^n(\omega)=\zeta_t(\omega)\ \forall t\in[0,T)\big)=1$. Then, in particular, we can apply Proposition 4.4.11 to the regular part $(\tilde{g}_t)$ of the process (g_t) and obtain

$$\lim_{n\to\infty}\mathbb{E}\Big[\int_{[0,T]}\tilde{g}_t(1-\xi_{t-})d\zeta_t^n\Big] = \mathbb{E}\Big[\int_{[0,T]}\tilde{g}_t(1-\xi_{t-})d\zeta_t\Big]. \tag{4.57}$$

For the pure jump part $(\hat{g}_t)$ of the process (g_t), we will prove that

$$\lim_{n\to\infty}\mathbb{E}\Big[\int_{[0,T]}\hat{g}_t(1-\xi_{t-})d\zeta_t^n\Big] = \mathbb{E}\Big[\int_{[0,T]}\hat{g}_t(1-\xi_{t-})d\zeta_t\Big]. \tag{4.58}$$

Let us define

$$R_t^n = \int_{[0,t]}(1-\xi_{s-})d\zeta_s^n, \qquad R_t = \int_{[0,t]}(1-\xi_{s-})d\zeta_s, \qquad \text{for } t\in[0,T],$$

with $R_{0-}^n = R_{0-} = 0$.

Due to the continuity of (ζ_t^n) and (ζ_t) for $t\in[0,T)$, processes (R_t^n) and (R_t) are continuous on $[0,T)$ with a possible jump at T. It follows from (4.19) from the proof of Proposition 4.4.13 that for $\mathbb{P}$-a.e. $\omega\in\Omega$

$$\lim_{n\to\infty}R_t^n(\omega) = R_t(\omega) \quad \text{for all } t\in[0,T].$$

Since $\Delta\hat{g}_T = 0$ (see Assumption (A2')), there is a decomposition such that $X_k^g I_{\{\eta_k^g=T\}} = 0$ $\mathbb{P}$-a.s. for all k. Recalling that (R_t) is continuous on $[0,T)$, we can apply (4.37) in Proposition 4.5.5: for any $k\geq 1$,

$$\lim_{n\to\infty}\mathbb{E}\Big[\int_{[0,T]}X_k^g I_{\{t\geq\eta_k^g\}}dR_t^n\Big] = \mathbb{E}\Big[\int_{[0,T]}X_k^g I_{\{t\geq\eta_k^g\}}dR_t\Big].$$

The dominated convergence theorem (with the bound X^g) implies that

$$\lim_{n\to\infty}\mathbb{E}\Big[\int_{[0,T]}\hat{g}_t dR_t^n\Big] = \mathbb{E}\Big[\int_{[0,T]}\hat{g}_t dR_t\Big],$$

which proves (4.58).

Finally, recalling that $g_T\leq h$, we obtain from Lemma 4.4.10

$$\limsup_{n\to\infty}\mathbb{E}\big[(h-g_T)\Delta\xi_T\Delta\zeta_T^n\big] \leq \mathbb{E}\big[(h-g_T)\Delta\xi_T\Delta\zeta_T\big].$$

Combining the latter with (4.56), (4.57) and (4.58) shows that

$$\limsup_{n\to\infty} N(\xi,\zeta^n) \le N(\xi,\zeta). \tag{4.59}$$

Hence we have a contradiction with (4.55), which proves the upper semicontinuity.

Lower semicontinuity of $N(\cdot,\zeta)$. Fix $(\zeta_t) \in \mathcal{A}_{ac}(\mathcal{F}_t^2)$ and consider a sequence $(\xi_t^n)_{n\ge1} \subset \mathcal{A}(\mathcal{F}_t^1)$ converging to $(\xi_t) \in \mathcal{A}(\mathcal{F}_t^1)$ strongly in $\mathcal{S}$. Arguing by contradiction as above, we assume that there is a subsequence of (ξ_t^n) which we denote the same, with an abuse of notation, such that $\xi^n \to \xi$ $(\mathbb{P}\times\lambda)$-a.e. and

$$\lim_{n\to\infty} N(\xi^n,\zeta) < N(\xi,\zeta). \tag{4.60}$$

Since (ζ_t) is absolutely continuous on $[0,T)$,

$$\lim_{n\to\infty} \mathbb{E}\left[\int_{[0,T)} g_t(1-\xi_t^n)d\zeta_t\right] = \mathbb{E}\left[\int_{[0,T)} g_t(1-\xi_t)d\zeta_t\right] \tag{4.61}$$

by the dominated convergence theorem. For the other two terms of $N(\xi^n,\zeta)$ we have

$$\mathbb{E}\left[\int_{[0,T)} f_t(1-\zeta_{t-})d\xi_t^n + h\Delta\xi_T^n\Delta\zeta_T\right] = \mathbb{E}\left[\int_{[0,T]} f_t(1-\zeta_{t-})d\xi_t^n + (h-f_T)\Delta\xi_T^n\Delta\zeta_T\right],$$

where we used that $1-\zeta_{T-} = \Delta\zeta_T$. Let us first focus on the regular part $(\tilde{f}_t)$ of the process (f_t). From Lemma 4.4.9 and the boundedness and continuity of (ζ_t) we verify the assumptions of Proposition 4.4.11, so

$$\lim_{n\to\infty} \mathbb{E}\left[\int_{[0,T]} \tilde{f}_t(1-\zeta_{t-})d\xi_t^n\right] = \mathbb{E}\left[\int_{[0,T]} \tilde{f}_t(1-\zeta_{t-})d\xi_t\right]. \tag{4.62}$$

We now show how to handle the convergence for $(\hat{f}_t)$. To be precise, we will prove that

$$\liminf_{n\to\infty} \mathbb{E}\left[\int_{[0,T]} \hat{f}_t(1-\zeta_{t-})d\xi_t^n\right] \ge \mathbb{E}\left[\int_{[0,T]} \hat{f}_t(1-\zeta_{t-})d\xi_t\right]. \tag{4.63}$$

Let

$$R_t^n = \int_{[0,t]} (1-\zeta_{t-})d\xi_t^n, \qquad R_t = \int_{[0,t]} (1-\zeta_{t-})d\xi_t,$$

with $R_{0-}^n = R_{0-} = 0$.

In the proof of Theorem 4.5.1, we verify that (R_t^n), (R_t) defined as (4.42) satisfy the conditions of Proposition 4.5.5. Note that in the current proof, (R_t^n) and (R_t) are the same up to notation. Applying Proposition 4.5.5 for all $k\ge1$ and using that $X_k^f \ge 0$, we obtain

$$\limsup_{n\to\infty} \mathbb{E}\left[\int_{[0,T]} X_k^f I_{\{t\ge\eta_k^f\}} dR_t^n\right] \le \mathbb{E}\left[\int_{[0,T]} X_k^f I_{\{t\ge\eta_k^f\}} dR_t\right],$$

and therefore

$$\liminf_{n\to\infty}\mathbb{E}\left[\int_{[0,T]}-X_k^f I_{\{t\ge\eta_k^f\}}dR_t^n\right]\ge\mathbb{E}\left[\int_{[0,T]}-X_k^f I_{\{t\ge\eta_k^f\}}dR_t\right]. \tag{4.64}$$

We first apply the decomposition of $(\hat{f}_t)$ and then the monotone convergence theorem:

$$\mathbb{E}\left[\int_{[0,T]}\hat{f}_t(1-\zeta_{t-})d\xi_t^n\right]=\mathbb{E}\left[\sum_{k=1}^{\infty}\int_{[0,T]}-X_k^f I_{\{t\ge\eta_k^f\}}dR_t^n\right]=\sum_{k=1}^{\infty}\mathbb{E}\left[\int_{[0,T]}-X_k^f I_{\{t\ge\eta_k^f\}}dR_t^n\right].$$

Since $(\hat{f}_t)\in\mathcal{L}_b$ we have the bound (recall (4.54))

$$\sum_{k=1}^{\infty}\sup_n\mathbb{E}\left[\int_{[0,T]}X_k^f I_{\{t\ge\eta_k^f\}}dR_t^n\right]\le\sum_{k=1}^{\infty}\mathbb{E}[X_k^f]<\infty.$$

Then we can apply Fatou's lemma (with respect to the counting measure on $\mathbb{N}$) to obtain

$$\begin{aligned}\liminf_{n\to\infty}\sum_{k=1}^{\infty}\mathbb{E}\left[\int_{[0,T]}-X_k^f I_{\{t\ge\eta_k^f\}}dR_t^n\right]&\ge\sum_{k=1}^{\infty}\liminf_{n\to\infty}\mathbb{E}\left[\int_{[0,T]}-X_k^f I_{\{t\ge\eta_k^f\}}dR_t^n\right]\\&\ge\sum_{k=1}^{\infty}\mathbb{E}\left[\int_{[0,T]}-X_k^f I_{\{t\ge\eta_k^f\}}dR_t\right]\\&=\mathbb{E}\left[\int_{[0,T]}\hat{f}_t(1-\zeta_{t-})d\xi_t\right],\end{aligned} \tag{4.65}$$

where the last inequality is due to (4.64) and the final equality follows by monotone convergence and the decomposition of $(\hat{f}_t)$. This completes the proof of (4.63).

The final step is to recall that $f_T\ge h$, and thus we obtain from Lemma 4.4.10

$$\liminf_{n\to\infty}\mathbb{E}\left[(h-f_T)\Delta\xi_T^n\Delta\zeta_T\right]\ge\mathbb{E}\left[(h-f_T)\Delta\xi_T\Delta\zeta_T\right]. \tag{4.66}$$

Combining the above convergence results (4.61), (4.62), (4.63), (4.66) contradicts (4.60), hence, proves the lower semicontinuity. □

Approximation with continuous controls

Proof of Proposition 4.5.4. Fix $(\xi_t)\in\mathcal{A}(\mathcal{F}_t^1)$. We need to show that for any $(\zeta_t)\in\mathcal{A}(\mathcal{F}_t^2)$, there exists a sequence $(\zeta_t^n)_{n\ge1}\subset\mathcal{A}_{ac}(\mathcal{F}_t^2)$ such that

$$\liminf_{n\to\infty}N(\xi,\zeta^n)\ge N(\xi,\zeta). \tag{4.67}$$

Without loss of generality we assume that $(\zeta_t)\in\mathcal{A}^\circ(\mathcal{F}_t^2)$ and $(\xi_t)\in\mathcal{A}^\circ(\mathcal{F}_t^1)$. Recall the function $\phi_t^n=(nt)\wedge1\vee0$ from the proof of Proposition 4.4.6. Let $\zeta_t^n=\int_{[0,t]}\phi_{t-s}^n d\zeta_s$ for $t\in[0,T)$, and

$\zeta_T^n = 1$. By the argument identical to the proof of Proposition 4.4.6 (for (ξ_t^n) therein), we see that (ζ_t^n) are absolutely continuous on $[0,T)$, and that $\zeta_t^n \to \zeta_{t-}$ for all $t \in [0,T)$.

Note that

$$\begin{aligned}
N(\xi,\zeta^n) &= \mathbb{E}\left[\int_{[0,T)} f_t(1-\zeta_{t-}^n)d\xi_t + \int_{[0,T)} g_t(1-\xi_t)d\zeta_t^n + h\Delta\xi_T\Delta\zeta_T^n\right] \\
&= \mathbb{E}\left[\int_{[0,T)} f_t(1-\zeta_t^n)d\xi_t + \int_{[0,T)} g_t(1-\xi_{t-})d\zeta_t^n + \sum_{t\in[0,T)}(f_t-g_t)\Delta\zeta_t^n\Delta\xi_t + h\Delta\xi_T\Delta\zeta_T^n\right] \\
&\geq \mathbb{E}\left[\int_{[0,T)} f_t(1-\zeta_t^n)d\xi_t + \int_{[0,T)} g_t(1-\xi_{t-})d\zeta_t^n + h\Delta\xi_T\Delta\zeta_T^n\right],
\end{aligned} \tag{4.68}$$

where the last inequality is due to Assumption (A3). In order to prove (4.67), we will use the decomposition (A2') of (f_t) and (g_t) and show that each of the terms on the right-hand side of (4.68) converges to a corresponding term of $N(\xi,\zeta)$.

Let us consider the regular parts $(\tilde{f}_t)$, $(\tilde{g}_t)$ or the processes (f_t), (g_t). An argument identical to the one that was used in the proof of Proposition 4.4.6 (for (ξ_t^n) therein) to verify the assumptions of Proposition 4.4.14 proves that it is applicable in the current case as well. Proposition 4.4.14 implies that

$$\lim_{n\to\infty}\mathbb{E}\left[\int_{[0,T)} \tilde{g}_t(1-\xi_{t-})d\zeta_t^n\right] = \mathbb{E}\left[\int_{[0,T)} \tilde{g}_t(1-\xi_t)d\zeta_t\right], \tag{4.69}$$

and $\lim_{n\to\infty}\zeta_{T-}^n = \zeta_{T-}$ so that

$$\lim_{n\to\infty}\Delta\zeta_T^n = \Delta\zeta_T \tag{4.70}$$

since $\zeta_T^n = 1$ for all $n \geq 1$. Recalling that $\zeta_t^n \to \zeta_{t-}$ by construction, the dominated convergence theorem (applied to the first integral below) and (4.69), (4.70) yield

$$\begin{aligned}
&\lim_{n\to\infty}\mathbb{E}\left[\int_{[0,T)} \tilde{f}_t(1-\zeta_t^n)d\xi_t + \int_{[0,T)} \tilde{g}_t(1-\xi_{t-})d\zeta_t^n + h\Delta\xi_T\Delta\zeta_T^n\right] \\
&= \mathbb{E}\left[\int_{[0,T)} \tilde{f}_t(1-\zeta_{t-})d\xi_t + \int_{[0,T)} \tilde{g}_t(1-\xi_t)d\zeta_t + h\Delta\xi_T\Delta\zeta_T\right].
\end{aligned} \tag{4.71}$$

Now we turn our attention to the jump parts $(\hat{f}_t)$, $(\hat{g}_t)$ or the processes (f_t), (g_t). For $t \in [0,T]$, define

$$R_t^n = \int_{[0,t]}(1-\xi_{s-})d\zeta_s^n, \qquad R_t = \int_{[0,t]}(1-\xi_s)d\zeta_s,$$

with $R_{0-}^n = R_{0-} = 0$. Corollary 4.4.15 implies that for $\mathbb{P}$-a.e. $\omega \in \Omega$

$$\lim_{n\to\infty} R_{t-}^n(\omega) = R_{t-}(\omega) \quad \text{for all } t \in [0,T]. \tag{4.72}$$

By the decomposition of $(\hat{g}_t)$ and the dominated convergence theorem for the infinite sum (recalling (4.54)) we obtain

$$\mathbb{E}\left[\int_{[0,T)} \hat{g}_t(1-\xi_{t-})d\zeta_t^n\right] = \sum_{k=1}^{\infty} \mathbb{E}\left[(-1)^k \int_{[0,T)} X_k^g I_{\{t\geq\eta_k^g\}} dR_t^n\right] = \sum_{k=1}^{\infty} \mathbb{E}\big[(-1)^k X_k^g (R_{T-}^n - R_{\eta_k^g-}^n)\big].$$

We further apply dominated convergence with respect to the product of the counting measure on $\mathbb{N}$ and the measure $\mathbb{P}$ to obtain

$$\begin{aligned}\lim_{n\to\infty} \mathbb{E}\left[\int_{[0,T)} \hat{g}_t(1-\xi_{t-})d\zeta_t^n\right] &= \sum_{k=1}^{\infty} \mathbb{E}\big[(-1)^k \lim_{n\to\infty} X_k^g (R_{T-}^n - R_{\eta_k^g-}^n)\big] \\ &= \sum_{k=1}^{\infty} \mathbb{E}\big[(-1)^k X_k^g (R_{T-} - R_{\eta_k^g-})\big] = \mathbb{E}\left[\int_{[0,T)} \hat{g}_t(1-\xi_t)d\zeta_t\right],\end{aligned} \tag{4.73}$$

where the second equality uses (4.72) and the final one the decomposition of $(\hat{g}_t)$.

On the other hand, recalling that $\zeta_t^n \to \zeta_{t-}$ by construction, we can apply the dominated convergence to see that

$$\lim_{n\to\infty} \mathbb{E}\left[\int_{[0,T)} \hat{f}_t(1-\zeta_t^n)d\xi_t\right] = \mathbb{E}\left[\int_{[0,T)} \hat{f}_t(1-\zeta_{t-})d\xi_t\right]. \tag{4.74}$$

Putting together (4.68), (4.71), (4.73) and (4.74) shows that

$$\liminf_{n\to\infty} N(\xi^n,\zeta) \geq \mathbb{E}\left[\int_{[0,T)} f_t(1-\zeta_{t-})d\xi_t + \int_{[0,T)} g_t(1-\xi_t)d\zeta_t + h\Delta\xi_T\Delta\zeta_T\right] = N(\xi,\zeta),$$

which completes the proof of (4.67). □

4.5.3 Value of $\mathcal{G}$-conditioned game

This section is devoted to the proof of Theorem 4.2.5. Let us copy the statement for the ease of reference.

Theorem. *Under Assumptions (A1), (A2'), (A3)-(A5) and for any $\mathcal{G} \subseteq \mathcal{F}_0^1 \cap \mathcal{F}_0^2$, the $\mathcal{G}$-conditioned game has a value, i.e.*

$$\operatorname*{ess\,sup}_{\sigma\in\mathcal{T}^R(\mathcal{F}_t^2)} \operatorname*{ess\,inf}_{\tau\in\mathcal{T}^R(\mathcal{F}_t^1)} \mathbb{E}\big[\mathcal{P}(\tau,\sigma)\big|\mathcal{G}\big] = \operatorname*{ess\,inf}_{\tau\in\mathcal{T}^R(\mathcal{F}_t^1)} \operatorname*{ess\,sup}_{\sigma\in\mathcal{T}^R(\mathcal{F}_t^2)} \mathbb{E}\big[\mathcal{P}(\tau,\sigma)\big|\mathcal{G}\big]. \tag{4.75}$$

By definition, randomisation devices Z_τ and Z_σ associated to a pair $(\tau,\sigma) \in \mathcal{T}^R(\mathcal{F}_t^1) \times \mathcal{T}^R(\mathcal{F}_t^2)$ are independent of $\mathcal{F}_T$ and, in particular, of $\mathcal{G}$. Denoting by $(\xi_t) \in \mathcal{A}^\circ(\mathcal{F}_t^1)$ and $(\zeta_t) \in \mathcal{A}^\circ(\mathcal{F}_t^2)$ the generating processes for τ and σ, respectively, the statement of Proposition 4.3.4 (the alternative expression for the functional N in terms of players' controls) can be extended to encompass the conditional functional (4.5):

$$\mathbb{E}\big[\mathcal{P}(\tau,\sigma)\big|\mathcal{G}\big] = \mathbb{E}\bigg[\int_{[0,T)} f_t(1-\zeta_{t-})d\xi_t + \int_{[0,T)} g_t(1-\xi_t)d\zeta_t + h\Delta\xi_T\Delta\zeta_T\bigg|\mathcal{G}\bigg]. \tag{4.76}$$

Remark 4.5.8. *Similarly to Remark 4.3.5, the "internal" essential infimum/supremum can be taken over pure stopping times, i.e. the following holds for the conditioned lower and upper value:*

$$\underline{V} := \operatorname*{ess\,sup}_{\sigma\in\mathcal{T}^R(\mathcal{F}^2_t)}\ \operatorname*{ess\,inf}_{\tau\in\mathcal{T}^R(\mathcal{F}^1_t)} \mathbb{E}\big[\mathcal{P}(\tau,\sigma)\big|\mathcal{G}\big] = \operatorname*{ess\,sup}_{\sigma\in\mathcal{T}^R(\mathcal{F}^2_t)}\ \operatorname*{ess\,inf}_{\tau\in\mathcal{T}(\mathcal{F}^1_t)} \mathbb{E}\big[\mathcal{P}(\tau,\sigma)\big|\mathcal{G}\big] \tag{4.77}$$

and

$$\overline{V} := \operatorname*{ess\,inf}_{\tau\in\mathcal{T}^R(\mathcal{F}^1_t)}\ \operatorname*{ess\,sup}_{\sigma\in\mathcal{T}^R(\mathcal{F}^2_t)} \mathbb{E}\big[\mathcal{P}(\tau,\sigma)\big|\mathcal{G}\big] = \operatorname*{ess\,inf}_{\tau\in\mathcal{T}^R(\mathcal{F}^1_t)}\ \operatorname*{ess\,sup}_{\sigma\in\mathcal{T}(\mathcal{F}^2_t)} \mathbb{E}\big[\mathcal{P}(\tau,\sigma)\big|\mathcal{G}\big]. \tag{4.78}$$

Indeed, take arbitrary $\tau\in\mathcal{T}^R(\mathcal{F}^1_t)$, $\sigma\in\mathcal{T}^R(\mathcal{F}^2_t)$, and define the family of stopping times

$$q(y) = \inf\{t\in[0,T] : \zeta_t > y\}, \qquad y\in[0,1).$$

Then,

$$\mathbb{E}\big[\mathcal{P}(\tau,\sigma)\big|\mathcal{G}\big] = \int_0^1 \mathbb{E}\big[\mathcal{P}(\tau,q(y))\big|\mathcal{G}\big]dy \le \operatorname*{ess\,sup}_{y\in[0,1]}\mathbb{E}\big[\mathcal{P}(\tau,q(y))\big|\mathcal{G}\big] \le \operatorname*{ess\,sup}_{\sigma\in\mathcal{T}(\mathcal{F}^2_t)}\mathbb{E}\big[\mathcal{P}(\tau,\sigma)\big|\mathcal{G}\big],$$

where the first inequality holds $\mathbb{P}$-a.s. by definition of essential supremum. Therefore,

$$\operatorname*{ess\,sup}_{\sigma\in\mathcal{T}^R(\mathcal{F}^2_t)} \mathbb{E}\big[\mathcal{P}(\tau,\sigma)\big|\mathcal{G}\big] \le \operatorname*{ess\,sup}_{\sigma\in\mathcal{T}(\mathcal{F}^2_t)}\mathbb{E}\big[\mathcal{P}(\tau,\sigma)\big|\mathcal{G}\big],\ \mathbb{P}\text{-a.s.}$$

while the reverse inequality is obvious since $\mathcal{T}(\mathcal{F}^2_t)\subset\mathcal{T}^R(\mathcal{F}^2_t)$. This proves (4.77), and the proof of (4.78) is analogous.

Notice that $\overline{V}\ge\underline{V}$, $\mathbb{P}$-a.s. We will show that

$$\mathbb{E}[\underline{V}] = \mathbb{E}[\overline{V}] \tag{4.79}$$

so that $\overline{V}=\underline{V}$, $\mathbb{P}$-a.s.

In order to prove (4.79), let us define

$$\overline{M}(\tau) := \operatorname*{ess\,sup}_{\sigma\in\mathcal{T}(\mathcal{F}^2_t)}\mathbb{E}\big[\mathcal{P}(\tau,\sigma)\big|\mathcal{G}\big], \quad \text{for } \tau\in\mathcal{T}^R(\mathcal{F}^1_t)$$

and

$$\underline{M}(\sigma) := \operatorname*{ess\,inf}_{\tau\in\mathcal{T}(\mathcal{F}^1_t)}\mathbb{E}\big[\mathcal{P}(\tau,\sigma)\big|\mathcal{G}\big], \quad \text{for } \sigma\in\mathcal{T}^R(\mathcal{F}^2_t).$$

These are two standard optimal stopping problems and the theory of Snell envelope applies (see Section 3.1). We adapt some results from that theory to suit our needs in the game setting. Recall Definition 3.1.3 of a downwards/upwards directed family of random variables.

Lemma 4.5.9. *The family* $\{\overline{M}(\tau), \tau \in \mathcal{T}^R(\mathcal{F}^1_t)\}$ *is downwards directed and the family* $\{\underline{M}(\sigma), \sigma \in \mathcal{T}^R(\mathcal{F}^2_t)\}$ *is upwards directed.*

Proof. Let $\tau^{(1)}, \tau^{(2)} \in \mathcal{T}^R(\mathcal{F}^1_t)$ and let $(\xi^{(1)}_t), (\xi^{(2)}_t) \in \mathcal{A}^\circ(\mathcal{F}^1_t)$ be the corresponding generating processes. Fix the $\mathcal{G}$-measurable event $B = \{\overline{M}(\tau^{(1)}) \le \overline{M}(\tau^{(2)})\}$ and define another randomised stopping time as $\hat{\tau} = \tau^{(1)} I_B + \tau^{(2)} I_{B^c}$ (the fact $\mathcal{G} \subset \mathcal{F}^1_0$ ensures that $\hat{\tau} \in \mathcal{T}^R(\mathcal{F}^1_t)$). The generating process of $\hat{\tau}$ reads $\hat{\xi}_t = \xi^{(1)}_t I_B + \xi^{(2)}_t I_{B^c}$ for $t \in [0,T]$. Using the linear structure of $(\hat{\xi}_t)$ and recalling (4.76), for any $\sigma \in \mathcal{T}(\mathcal{F}^2_t)$ we have

$$\begin{aligned}\mathbb{E}\big[\mathcal{P}(\hat{\tau},\sigma)|\mathcal{G}\big] =& I_B \mathbb{E}\Big[\int_{[0,T)} I_{\{u\le\sigma\}} f_u d\xi^{(1)}_u + g_\sigma(1-\xi^{(1)}_\sigma)) + h\Delta\xi^{(1)}_T I_{\{\sigma=T\}}\Big|\mathcal{G}\Big]\\ &+ I_{B^c}\mathbb{E}\Big[\int_{[0,T)} I_{\{u\le\sigma\}} f_u d\xi^{(2)}_u + g_\sigma(1-\xi^{(2)}_\sigma)) + h\Delta\xi^{(2)}_T I_{\{\sigma=T\}}\Big|\mathcal{G}\Big]\\ =& I_B\mathbb{E}\big[\mathcal{P}(\tau^{(1)},\sigma)|\mathcal{G}\big] + I_{B^c}\mathbb{E}\big[\mathcal{P}(\tau^{(2)},\sigma)|\mathcal{G}\big]\\ \le& I_B\overline{M}(\tau^{(1)}) + I_{B^c}\overline{M}(\tau^{(2)}) = \overline{M}(\tau^{(1)}) \wedge \overline{M}(\tau^{(2)}),\end{aligned}$$

where the inequality is by definition of essential supremum and the final equality by definition of the event B. Thus, taking essential supremum over $\sigma \in \mathcal{T}(\mathcal{F}^2_t)$ we get

$$\overline{M}(\hat{\tau}) \le \overline{M}(\tau^{(1)}) \wedge \overline{M}(\tau^{(2)}),$$

hence the family $\{\overline{M}(\tau), \tau \in \mathcal{T}^R(\mathcal{F}^1_t)\}$ is downwards directed. A symmetric argument proves that the family $\{\underline{M}(\sigma), \sigma \in \mathcal{T}^R(\mathcal{F}^2_t)\}$ is upwards directed. □

By of Lemma 4.5.9 and Lemma 3.1.4, there exist sequences $(\sigma_n)_{n\ge1} \subset \mathcal{T}^R(\mathcal{F}^2_t)$ and $(\tau_n)_{n\ge1} \subset \mathcal{T}^R(\mathcal{F}^1_t)$ such that $\mathbb{P}$-a.s.

$$\overline{V} = \lim_{n\to\infty} \overline{M}(\tau_n) \quad \text{and} \quad \underline{V} = \lim_{n\to\infty} \underline{M}(\sigma_n), \tag{4.80}$$

where the convergence is monotone in both cases.

Analogous results hold for the optimisation problems defining $\overline{M}(\tau)$ and $\underline{M}(\sigma)$.

Lemma 4.5.10. *The family* $\{\mathbb{E}\big[\mathcal{P}(\tau,\sigma)|\mathcal{G}\big], \tau \in \mathcal{T}(\mathcal{F}^1_t)\}$ *is downwards directed for each* $\sigma \in \mathcal{T}^R(\mathcal{F}^2_t)$. *The family* $\{\mathbb{E}\big[\mathcal{P}(\tau,\sigma)|\mathcal{G}\big], \sigma \in \mathcal{T}(\mathcal{F}^2_t)\}$ *is upwards directed for each* $\tau \in \mathcal{T}^R(\mathcal{F}^1_t)$.

Proof. Let $\tau^{(1)},\tau^{(2)} \in \mathcal{T}^R(\mathcal{F}^1_t)$ and let $(\xi^{(1)}_t),(\xi^{(2)}_t) \in \mathcal{A}^\circ(\mathcal{F}^1_t)$ be the corresponding generating processes. Fix the $\mathcal{G}$-measurable event $B = \{\mathbb{E}\big[\mathcal{P}(\tau^{(1)},\sigma)|\mathcal{G}\big] \le \mathbb{E}\big[\mathcal{P}(\tau^{(2)},\sigma)|\mathcal{G}\big]\}$ and define another $(\mathcal{F}^1_t)$-randomised stopping time as $\hat\tau = \tau^{(1)}I_B + \tau^{(2)}I_{B^c}$. The generating process of $\hat\tau$ reads $\hat\xi_t = \xi^{(1)}_t I_B + \xi^{(2)}_t I_{B^c}$ for $t \in [0,T]$. Using the linear structure of $(\hat\xi_t)$ and recalling (4.76), for any $\sigma \in \mathcal{T}(\mathcal{F}^2_t)$ we have

$$\begin{aligned}\mathbb{E}\big[\mathcal{P}(\hat\tau,\sigma)|\mathcal{G}\big] =&I_B\mathbb{E}\Big[\int_{[0,T)} I_{\{u\le\sigma\}}f_u d\xi^{(1)}_u + g_\sigma(1-\xi^{(1)}_\sigma)) + h\Delta\xi^{(1)}_T I_{\{\sigma=T\}}\Big|\mathcal{G}\Big]\\ &+I_{B^c}\mathbb{E}\Big[\int_{[0,T)} I_{\{u\le\sigma\}}f_u d\xi^{(2)}_u + g_\sigma(1-\xi^{(2)}_\sigma)) + h\Delta\xi^{(2)}_T I_{\{\sigma=T\}}\Big|\mathcal{G}\Big]\\ =&I_B\mathbb{E}\big[\mathcal{P}(\tau^{(1)},\sigma)|\mathcal{G}\big] + I_{B^c}\mathbb{E}\big[\mathcal{P}(\tau^{(2)},\sigma)|\mathcal{G}\big]\\ =&\mathbb{E}\big[\mathcal{P}(\tau^{(1)},\sigma)|\mathcal{G}\big]\wedge\mathbb{E}\big[\mathcal{P}(\tau^{(2)},\sigma)|\mathcal{G}\big],\end{aligned} \tag{4.81}$$

hence the family $\{\mathbb{E}\big[\mathcal{P}(\tau,\sigma)|\mathcal{G}\big], \tau \in \mathcal{T}^R(\mathcal{F}^1_t)\}$ is downwards directed. A symmetric argument proves that the family $\{\mathbb{E}\big[\mathcal{P}(\tau,\sigma)|\mathcal{G}\big], \sigma \in \mathcal{T}(\mathcal{F}^2_t)\}$ is upwards directed. □

Remark 4.5.11. *There is no inequality in (4.81), so, considering $\hat\tau = \tau^{(1)}I_{B^c} + \tau^{(2)}I_B$, the same argument shows that the family $\{\mathbb{E}\big[\mathcal{P}(\tau,\sigma)|\mathcal{G}\big], \tau \in \mathcal{T}^R(\mathcal{F}^1_t)\}$ is upwards directed. One can compare this result to a similar one for usual stopping tmies (Karatzas & Shreve, 1998, Lemma D.1).*

As with (4.80), Lemma 4.5.10 implies that for each $\tau \in \mathcal{T}^R(\mathcal{F}^1_t)$ and $\sigma \in \mathcal{T}^R(\mathcal{F}^2_t)$, there are sequences $(\sigma^\tau_n)_{n\ge1} \subset \mathcal{T}(\mathcal{F}^2_t)$ and $(\tau^\sigma_n)_{n\ge1} \subset \mathcal{T}(\mathcal{F}^1_t)$ such that

$$\overline{M}(\tau) = \lim_{n\to\infty}\mathbb{E}\big[\mathcal{P}(\tau,\sigma^\tau_n)|\mathcal{G}\big] \quad \text{and} \quad \underline{M}(\sigma) = \lim_{n\to\infty}\mathbb{E}\big[\mathcal{P}(\tau^\sigma_n,\sigma)|\mathcal{G}\big], \tag{4.82}$$

where the convergence is monotone in both cases. Equipped with these results we can prove the following lemma which will quickly lead to (4.79). Recall V_* and V^* from Definition 4.2.2.

Lemma 4.5.12. *We have*

$$\mathbb{E}[\overline{V}] = V^*, \qquad \textit{and} \qquad \mathbb{E}[\underline{V}] = V_*. \tag{4.83}$$

Proof. Fix $\tau \in \mathcal{T}^R(\mathcal{F}^1_t)$. By (4.82) and the monotone convergence theorem

$$\mathbb{E}[\overline{M}(\tau)] = \lim_{n\to\infty}\mathbb{E}[\mathcal{P}(\tau,\sigma^\tau_n)] \le \sup_{\sigma\in\mathcal{T}(\mathcal{F}^2_t)}\mathbb{E}[\mathcal{P}(\tau,\sigma)].$$

The opposite inequality follows from the fact that $\overline{M}(\tau) \ge \mathbb{E}[\mathcal{P}(\tau,\sigma)|\mathcal{G}]$ for any $\sigma \in \mathcal{T}(\mathcal{F}^2_t)$ by the definition of the essential supremum. Therefore, we have

$$\mathbb{E}[\overline{M}(\tau)] = \sup_{\sigma\in\mathcal{T}(\mathcal{F}^2_t)}\mathbb{E}[\mathcal{P}(\tau,\sigma)]. \tag{4.84}$$

From (4.80), an analogous argument proves that

$$\mathbb{E}[\overline{V}] = \inf_{\tau \in \mathcal{T}^R(\mathcal{F}^1_t)} \mathbb{E}[\overline{M}(\tau)]. \tag{4.85}$$

Combining (4.84) and (4.85) completes the proof that $\mathbb{E}[\overline{V}] = V^*$. The second part of the statement requires symmetric arguments. □

With Lemma 4.5.12 proved, Theorem 4.2.5 follows almost immediately.

Proof of Theorem 4.2.5. By (4.83) and Theorem 4.2.3, we obtain (4.79), which is equivalent to the statement of Theorem 4.2.5. □

4.5.4 Generalised payoff and Nash equilibrium

In the joint paper with my PhD supervisors De Angelis *et al.* (2021c), we extend the results of the present chapter to the case of a payoff more general than (4.1). Consider

$$\widetilde{\mathcal{P}}(\tau,\sigma) = f_\tau I_{\{\tau<\sigma\}} + g_\sigma I_{\{\sigma<\tau\}} + \tilde{h}_\tau I_{\{\tau=\sigma\}}, \tag{4.86}$$

where $\tau \in \mathcal{T}^R(\mathcal{F}^1_t)$, $\sigma \in \mathcal{T}^R(\mathcal{F}^2_t)$. Recall that the same payoff (3.25) was studied in Section 3.5, but the game therein is a full-information one, which in the notation of this chapter corresponds to the case $(\mathcal{F}^1_t) = (\mathcal{F}^2_t) = (\mathcal{F}_t)$.

The payoff processes (f_t), (g_t) are assumed to be as above, i.e. to satisfy (A1), (A2'). Further, the "middle" payoff process $(\tilde{h}_t)$ in De Angelis *et al.* (2021c) satisfies the following:

(A3') $f_t \geq \tilde{h}_t \geq g_t$ for all $t \in [0,T]$, $\mathbb{P}$-a.s.,

(A4') $(\tilde{h}_t)$ is an $(\mathcal{F}_t)$-adapted, measurable process.

Note that we do not assume that $(\tilde{h}_t)$ is càdlàg. The filtrations $(\mathcal{F}^1_t), (\mathcal{F}^2_t) \subseteq (\mathcal{F}_t)$ are assumed to be as above, i.e. to satisfy (A5). Let us also define the expected payoff functional

$$\widetilde{N}(\tau,\sigma) = \mathbb{E}[\widetilde{\mathcal{P}}(\tau,\sigma)].$$

Note that the current setting reduces to the setting of Section 4.1 if $\tilde{h}_t(\omega) = f_t(\omega)$ for all $t \in [0,T)$ and all $\omega \in \Omega$. The random variable $\tilde{h}_T$ in this case plays the same role as the random variable h in Section 4.1.

The value in randomised strategies of the asymmetric information game with the payoff (4.86) is defined as in Definition 4.2.2. The classical definition of a Nash equilibrium 3.2.6

in the current framework needs to be adjusted as follows: a pair $(\tau_*,\sigma_*) \in \mathcal{T}^R(\mathcal{F}_1)\times\mathcal{T}^R(\mathcal{F}_2)$ is said to be a Nash equilibrium, if

$$\widetilde{N}(\tau_*,\sigma) \le \widetilde{N}(\tau_*,\sigma_*) \le \widetilde{N}(\tau,\sigma_*),$$

for all pairs $(\tau,\sigma) \in \mathcal{T}^R(\mathcal{F}_1)\times\mathcal{T}^R(\mathcal{F}_2)$.

The main result of De Angelis *et al.* (2021c) extends Theorem 4.2.3.

Theorem 4.5.13 (De Angelis *et al.* (2021c), Theorem 2.4)**.** *Under Assumptions (A1), (A2'), (A3'), (A4'), (A5), the game with the payoff* (4.86) *has a value in randomised strategies. Moreover, if $\hat{f}$ and $\hat{g}$ in (A2') are non-increasing and non-decreasing, respectively, there exists a Nash equilibrium* (τ_*,σ_*).

To avoid a terminological confusion, we mention that in De Angelis *et al.* (2021c) a Nash equilibrium is called a pair of optimal strategies (c.f. the term used in Theorem 4.4.5 to describe a strategy of *one* of the players).

There is also the following extension of Theorem 4.2.5 on the value of the $\mathcal{G}$-conditioned game.

Theorem 4.5.14 (De Angelis *et al.* (2021c), Theorem 2.6)**.** *Under the assumptions of Theorem 4.5.13 and for any σ-algebra $\mathcal{G} \subseteq \mathcal{F}^1_0 \cap \mathcal{F}^2_0$, the $\mathcal{G}$-conditioned game has a value, i.e.*

$$\operatorname*{ess\,sup}_{\sigma\in\mathcal{T}^R(\mathcal{F}^2_t)}\ \operatorname*{ess\,inf}_{\tau\in\mathcal{T}^R(\mathcal{F}^1_t)} \mathbb{E}\big[\widetilde{\mathcal{P}}(\tau,\sigma)\big|\mathcal{G}\big] = \operatorname*{ess\,inf}_{\tau\in\mathcal{T}^R(\mathcal{F}^1_t)}\ \operatorname*{ess\,sup}_{\sigma\in\mathcal{T}^R(\mathcal{F}^2_t)} \mathbb{E}\big[\widetilde{\mathcal{P}}(\tau,\sigma)\big|\mathcal{G}\big], \qquad \mathbb{P}\text{-}a.s.$$

Moreover, if $\hat{f}$ and $\hat{g}$ in (A2') are non-increasing and non-decreasing, respectively, there exists a pair (τ_*,σ_*) of optimal strategies in the sense that

$$\mathbb{E}\big[\widetilde{\mathcal{P}}(\tau_*,\sigma)\big|\mathcal{G}\big] \le \mathbb{E}\big[\widetilde{\mathcal{P}}(\tau_*,\sigma_*)\big|\mathcal{G}\big] \le \mathbb{E}\big[\widetilde{\mathcal{P}}(\tau,\sigma_*)\big|\mathcal{G}\big], \qquad \mathbb{P}\text{-}a.s.$$

for all pairs $(\tau,\sigma) \in \mathcal{T}^R(\mathcal{F}^1_t)\times\mathcal{T}^R(\mathcal{F}^2_t)$.

The proofs of Theorems 4.5.13 and 4.5.14 go through deriving, similarly to Proposition 4.3.4, an alternative representation for the functional $\widetilde{N}$ involving the generating processes (ξ_t), (ζ_t) of the randomised stopping times τ,σ (De Angelis *et al.*, 2021c, Proposition 4.4):

$$\widetilde{N}(\tau,\sigma) = \mathbb{E}\bigg[\int_{[0,T]} f_t(1-\zeta_t)d\xi_t + \int_{[0,T]} g_t(1-\xi_t)d\zeta_t + \sum_{t\in[0,T]} \tilde{h}_t\Delta\xi_t\Delta\zeta_t\bigg]. \tag{4.87}$$

Note that the right-hand side is the same as in (3.26) in Section 3.5, i.e. it does not depend on the information features of the game but only on the form of the payoff. In De Angelis *et al.*

(2021c), (4.87) is written with integrals over $[0,T)$, which is equivalent since $\xi_T = \zeta_T = 1$. The rest of the argument in De Angelis *et al.* (2021c) is similar to the current chapter, and the main steps of the proof of Theorem 4.5.13 (namely, (De Angelis *et al.*, 2021c, Theorem 5.21) and (De Angelis *et al.*, 2021c, Proposition 5.22)) are up to notation the same as Theorem 4.5.1 and Proposition 4.5.2. However, we decided to present the results of the current chapter for the less general payoff (4.1), because the majority of the work on the extension to the payoff (4.86) was done by my PhD supervisors.

4.6 Examples

In this section, we illustrate some of the specific games for which our general results apply. We draw from the existing literature on two-player zero-sum Dynkin games in continuous time (Chapter 3) and show that a broad class of these fits within our framework. Indeed, when $(\mathcal{F}^1_t) = (\mathcal{F}^2_t) = (\mathcal{F}_t)$, the game (4.2) is the classical Dynkin game with full information for both players (see Sections 3.2, 3.3, 3.5). The case $(\mathcal{F}^1_t) = (\mathcal{F}^2_t) \subsetneq (\mathcal{F}_t)$ corresponds to a game with partial but symmetric information about the payoff processes (Section 3.4.3, De Angelis *et al.* (2021b)), whereas $(\mathcal{F}^1_t) \neq (\mathcal{F}^2_t)$ is the game with asymmetric information. One can have $(\mathcal{F}^1_t) = (\mathcal{F}_t)$, i.e. only the second player is uninformed (Section 3.4.1, Grün (2013)), or $(\mathcal{F}^1_t) \neq (\mathcal{F}_t)$ and $(\mathcal{F}^2_t) \neq (\mathcal{F}_t)$, i.e. both players access different information flows and neither of them has full knowledge of the underlying world (Section 3.4.2, Gensbittel & Grün (2019)).

Remark 4.6.1. *Recall that in the classical non-Markovian full information framework of Section 3.2, the value of the game exists in* pure *strategies. On the other hand, our main Theorem 4.2.4 proves the existence of the value in* randomised *strategies. However, recall Section 3.1.3 where we mention that the generating processes of pure stopping times are extreme points of the set of generating processes (Pennanen & Perkkiö, 2018, Lemma 2). In our notation, we see that the set $\mathcal{T}(\mathcal{F}_t)$ consists of extreme points of the set $\mathcal{T}^R(\mathcal{F}_t)$. In the full information framework $(\mathcal{F}^1_t) = (\mathcal{F}^2_t) = (\mathcal{F}_t)$, this fact could be possible to apply to deduce the existence of the value in pure strategies. However, it is technically more difficult than the similar result for an optimal stopping problem (Pennanen & Perkkiö, 2018, Theorem 1), since, to the best of our knowledge, there is no min-max analogue of Bauer's maximum principle. Since our contribution is mainly to the theory of games with partial/asymmetric information, we leave further details on the full information case to the future research, and below focus on the examples corresponding to the papers reviewed in Section 3.4.*

4.6.1 Game with partially observed scenarios

Our first example extends the setting of Section 3.4.1. On a discrete probability space $(\Omega^s, \mathcal{F}^s, \mathbb{P}^s)$, consider two random variables $\mathcal{I}$ and $\mathcal{J}$ taking values in $\{1,\dots,I\}$ and in $\{1,\dots,J\}$, respectively. Denote their joint distribution by $(\pi_{i,j})_{i=1,\dots,I,j=1,\dots,J}$ so that $\pi_{i,j} = \mathbb{P}^s(\mathcal{I}=i, \mathcal{J}=j)$. The indices (i,j) are used to identify the *scenario* in which the game is played and are the key ingredient to model the asymmetric information feature. Consider another probability space $(\Omega^p, \mathcal{F}^p, \mathbb{P}^p)$ with a filtration $(\mathcal{F}^p_t)$ satisfying the usual conditions, $(\mathcal{F}^p_t)$-adapted payoff processes $(f^{i,j}_t)$ and $(g^{i,j}_t)$, and $\mathcal{F}_T$-measurable terminal payoffs $h^{i,j}$, with (i,j) taking values in $\{1,\dots,I\}\times\{1,\dots,J\}$. For all i,j, we assume that $(f^{i,j}_t)$, $(g^{i,j}_t)$, $h^{i,j}$ satisfy conditions (A1)-(A4).

The game is set on the probability space $(\Omega, \mathcal{F}, \mathbb{P}) := (\Omega^p\times\Omega^s, \mathcal{F}^p\vee\mathcal{F}^s, \mathbb{P}^p\otimes\mathbb{P}^s)$. The first player is informed about the outcome of $\mathcal{I}$ before the game starts but never directly observes $\mathcal{J}$. Hence, her actions are adapted to the filtration $\mathcal{F}^1_t = \mathcal{F}^p_t\vee\sigma(\mathcal{I})$. Conversely, the second player knows $\mathcal{J}$ but not $\mathcal{I}$, so her actions are adapted to the filtration $\mathcal{F}^2_t = \mathcal{F}^p_t\vee\sigma(\mathcal{J})$. Given a choice of random times τ,σ for the first and the second player the payoff is

$$\mathcal{P}(\tau,\sigma) = f^{\mathcal{I},\mathcal{J}}_\tau I_{\{\tau\le\sigma\}\cap\{\tau<T\}} + g^{\mathcal{I},\mathcal{J}}_\sigma I_{\{\sigma<\tau\}\cap\{\sigma<T\}} + h^{\mathcal{I},\mathcal{J}} I_{\{\sigma=T\}\cap\{\tau=T\}}.$$

Players assess the game by looking at the expected payoff as in (4.2). The structure of the game is common knowledge, i.e. both players know all processes $(f^{i,j}_t)$, $(g^{i,j}_t)$ and all random variables $h^{i,j}$ involved; however, they have partial and asymmetric knowledge on the couple (i,j) which is drawn at the start of the game from the distribution of $(\mathcal{I},\mathcal{J})$.

In the framework of Section 4.1, the above setting corresponds to $f_t = f^{\mathcal{I},\mathcal{J}}_t$, $g_t = g^{\mathcal{I},\mathcal{J}}_t$, and $h = h^{\mathcal{I},\mathcal{J}}$ with the filtration $\mathcal{F}_t = \mathcal{F}^p_t\vee\sigma(\mathcal{I},\mathcal{J})$. The observation flows for the players are given by $(\mathcal{F}^1_t)$ and $(\mathcal{F}^2_t)$, respectively.

We note that the set-up of Section 3.4.1 corresponds to the case $J=1$, so the observation flow of the second player contains no scenario information, i.e. $(\mathcal{F}^2_t) = (\mathcal{F}^p_t)$. Moreover, the payoff processes (f_t), (g_t) and h in Section 3.4.1 are deterministic functions of a diffusion process (X_t) on $\mathbb{R}^d$ (Section A.7), i.e. $f_t = F(t,X_t)$, $g_t = G(t,X_t)$ and $h = H(X_T)$.

The particular structure of players' filtrations $(\mathcal{F}^1_t)$ and $(\mathcal{F}^2_t)$ allows for the following decomposition of randomised stopping times, see Lemma B.1.2.

Lemma 4.6.2. *Any $\tau\in\mathcal{T}^R(\mathcal{F}^1_t)$ has a representation*

$$\tau = \sum_{i=1}^{I} I_{\{\mathcal{I}=i\}}\tau_i, \tag{4.88}$$

where $\tau_1,\dots,\tau_I \in \mathcal{T}^R(\mathcal{F}^p_t)$*, with generating processes* $(\xi^1_t),\dots,(\xi^I_t) \in \mathcal{A}^\circ(\mathcal{F}^p_t)$ *and a common randomisation device* Z_τ*. An analogous representation holds for* $\sigma \in \mathcal{T}^R(\mathcal{F}^2_t)$ *with* $\sigma_1,\dots,\sigma_J \in \mathcal{T}^R(\mathcal{F}^p_t)$*, generating processes* $(\zeta^1_t),\dots,(\zeta^J_t) \in \mathcal{A}^\circ(\mathcal{F}^p_t)$*, and a common randomisation device* Z_σ*.*

Corollary 4.6.3. *Any* $(\mathcal{F}^1_t)$*-stopping time* τ *has a decomposition* (4.88) *with* $\tau_1,\dots,\tau_I$ *being* $(\mathcal{F}^p_t)$*-stopping times (and analogously for* $(\mathcal{F}^2_t)$*-stopping times).*

Hence, given a realisation of the scenario variable $\mathcal{I}$ (resp. $\mathcal{J}$), the first (second) player chooses a randomised stopping time whose generating process is adapted to the common filtration $(\mathcal{F}^p_t)$. The resulting expected payoff can be written as

$$N(\tau,\sigma) = \sum_{i=1}^{I}\sum_{j=1}^{J} \pi_{i,j}\mathbb{E}\Big[f^{i,j}_{\tau_i} I_{\{\tau_i\le\sigma_j\}\cap\{\tau_i<T\}} + g^{i,j}_{\sigma_j} I_{\{\sigma_j<\tau_i\}\cap\{\sigma_j<T\}} + h^{i,j} I_{\{\sigma_j=T\}\cap\{\tau_i=T\}}\Big].$$

4.6.2 Game with two partially observed dynamics

Here we show how the setting of Section 3.4.2 also fits in our framework. On a probability space $(\Omega,\mathcal{F},\mathbb{P})$, consider two strong Markov processes (X_t) and (Y_t) (in Section 3.4.2 these are time-homogenous continuous-time Markov chains). The first player only observes the process (X_t) while the second player only observes the process (Y_t). In the notation of Section 4.1, we have $(\mathcal{F}^1_t) = (\mathcal{F}^X_t)$, $(\mathcal{F}^2_t) = (\mathcal{F}^Y_t)$ and $(\mathcal{F}_t) = (\mathcal{F}^X_t \vee \mathcal{F}^Y_t)$ (recall that $(\mathcal{F}^X_t)$ and $(\mathcal{F}^Y_t)$ denote the filtrations generated by sample paths of (X_t) and (Y_t), respectively, and augmented with $\mathbb{P}$-null sets, see Section 2.3). The payoff processes are deterministic functions of the underlying dynamics, i.e. $f_t = F(t,X_t,Y_t)$, $g_t = G(t,X_t,Y_t)$ and $h = H(X_T,Y_T)$, and they satisfy conditions (A1)-(A4). Given a choice of random times $\tau \in \mathcal{T}^R(\mathcal{F}^1_t)$ and $\sigma \in \mathcal{T}^R(\mathcal{F}^2_t)$ for the first and the second player, the payoff of the game reads

$$\mathcal{P}(\tau,\sigma) = F(\tau,X_\tau,Y_\tau)I_{\{\tau\le\sigma\}\cap\{\tau<T\}} + G(\sigma,X_\sigma,Y_\sigma)I_{\{\sigma<\tau\}\cap\{\sigma<T\}} + H(X_T,Y_T)I_{\{\sigma=T\}\cap\{\tau=T\}}.$$

Players assess the game by looking at the expected payoff as in (4.2).

We recall that the proofs of existence of the value in Sections 3.4.1 and 3.4.2 are based on variational inequalities. Further, the “scenario” interpretation in Section 3.4.2 is only possible due to the finiteness of the state spaces of both underlying processes. Therefore, the existence result cannot be extended to our general non-Markovian framework.

4.6.3 Game with a single partially observed dynamics

Our next example generalises the set-up of Section 3.4.3. Recall that therein the underlying dynamics of the game is a diffusion whose drift depends on the realisation of an independent random variable $\mathcal{J} \in \{1,\dots,I\}$. Formally, on a probability space $(\Omega,\mathcal{F},\mathbb{P})$ we have a Wiener process (W_t) on $\mathbb{R}^d$, an independent random variable $\mathcal{J} \in \{1,\dots,I\}$ with distribution $\pi_i = \mathbb{P}(\mathcal{J}=i)$, and a process (X_t) on $\mathbb{R}^d$ with the dynamics

$$dX_t = \sum_{i=1}^{I} I_{\{\mathcal{J}=i\}}\mu_i(X_t)dt + \sigma(X_t)dW_t, \quad X_0 = x,$$

where σ, $(\mu_i)_{i=1,\dots I}$ are given functions (known to both players) that guarantee existence of a unique strong solution of the SDE (see Section A.7). The payoff processes are deterministic functions of the underlying process, i.e. $f_t = F(t,X_t)$, $g_t = G(t,X_t)$ and $h = h(X_T)$, and they are known to both players. We assume that the payoff processes satisfy conditions (A1)-(A4).

To draw a parallel with the notation from Section 4.1, here we take $\mathcal{F}_t = \mathcal{F}^W_t \vee \sigma(\mathcal{J})$ (recall also Section 2.3). Both players observe the dynamics of (X_t), however they have partial/asymmetric information on the value of $\mathcal{J}$. If neither of the two players knows the true value of $\mathcal{J}$, we have $(\mathcal{F}^1_t) = (\mathcal{F}^2_t) = (\mathcal{F}^X_t)$ (notice that $\mathcal{F}^X_t \subsetneq \mathcal{F}_t$). If, instead, the first player (minimiser) observes the true value of $\mathcal{J}$, then $(\mathcal{F}^1_t) = (\mathcal{F}_t)$ and $(\mathcal{F}^2_t) = (\mathcal{F}^X_t)$, so that $\mathcal{F}^2_t \subsetneq \mathcal{F}^1_t$. Recall that both situations are considered in Section 3.4.3. However, therein only two scenarios are possible ($I = 2$), and the payoff processes are particular time-homogeneous functions of a particular one-dimensional diffusion.

Using the notation $X^{\mathcal{J}}$ to emphasise the dependence of the underlying dynamics on $\mathcal{J}$, and given a choice of random times τ and σ for the first and the second player, the payoff of the game reads

$$\mathcal{P}(\tau,\sigma) = F(\tau,X^{\mathcal{J}}_\tau)I_{\{\tau\le\sigma\}\cap\{\tau<T\}} + G(\sigma,X^{\mathcal{J}}_\sigma)I_{\{\sigma<\tau\}\cap\{\sigma<T\}} + H(X^{\mathcal{J}}_T)I_{\{\sigma=T\}\cap\{\tau=T\}}.$$

Players assess the game by looking at the expected payoff as in (4.2).

Recall that in Section 3.4.3, the value exists in a smaller class of strategies. If neither of the players is informed, they both use $(\mathcal{F}^X_t)$-stopping times, with no need for additional randomisation. If one of the players is informed, she uses $(\mathcal{F}_t)$-randomised stopping times, but the uninformed player uses $(\mathcal{F}^X_t)$-stopping times. We emphasise that in our general setting the randomisation is necessary for both players, as shown in Section 4.7.2.

4.6.4 Game with a random horizon

Here we consider a non-Markovian extension of the framework of Section 3.4.4, where the time horizon of the game is random and independent of the payoff processes. On a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ we have a filtration $(\mathcal{G}_t)_{t\in[0,T]}$ satisfying the usual conditions and a positive random variable θ which is independent of $\mathcal{G}_T$ and has a continuous distribution. Let $\Lambda_t := I_{\{t\geq\theta\}}$ and take $\mathcal{F}_t = \mathcal{G}_t \vee \sigma(\Lambda_s, 0 \leq s \leq t)$.

The players have asymmetric knowledge of the random variable θ. The first player observes the occurrence of θ, whereas the second player does not. We have $(\mathcal{F}^1_t) = (\mathcal{F}_t)$ and $(\mathcal{F}^2_t) = (\mathcal{G}_t) \subsetneq (\mathcal{F}_t)$. Given a choice of random times $\tau \in \mathcal{T}^R(\mathcal{F}^1_t)$ and $\sigma \in \mathcal{T}^R(\mathcal{F}^2_t)$ for the first and the second player, the game's payoff reads

$$\mathcal{P}(\tau,\sigma) = I_{\{\tau\wedge\sigma\leq\theta\}}\big(f^0_\tau I_{\{\tau\leq\sigma\}\cap\{\tau<T\}} + g^0_\sigma I_{\{\sigma<\tau\}\cap\{\sigma<T\}} + h^0 I_{\{\sigma=T\}\cap\{\tau=T\}}\big),$$

where we assume that (f^0_t), (g^0_t), h^0 satisfy the conditions (A1)-(A4), and that $f^0_t \geq 0$ for all $t \in [0,T]$.

Note that the problem above does not fit directly into the framework of Section 4.1: Assumption (A1) is indeed violated, because the processes $(I_{\{t\leq\theta\}}f^0_t), (I_{\{t\leq\theta\}}g^0_t)$ are not càdlàg. However, we now show that the game can be equivalently formulated as a game satisfying conditions of our framework. The expected payoff can be rewritten as follows

$$\begin{aligned} N^0(\tau,\sigma) &:= \mathbb{E}\big[\mathcal{P}(\tau,\sigma)\big] \\ &= \mathbb{E}\big[f^0_\tau I_{\{\tau\leq\theta\}} I_{\{\tau\leq\sigma\}\cap\{\tau<T\}} + g^0_\sigma I_{\{\sigma\leq\theta\}} I_{\{\sigma<\tau\}\cap\{\sigma<T\}} + h^0 I_{\{\sigma\leq\theta\}} I_{\{\tau=T\}\cap\{\sigma=T\}}\big] \\ &= \mathbb{E}\big[f^0_\tau I_{\{\tau\leq\theta\}} I_{\{\tau\leq\sigma\}\cap\{\tau<T\}} + g^0_\sigma I_{\{\sigma<\theta\}} I_{\{\sigma<\tau\}\cap\{\sigma<T\}} + h^0 I_{\{\sigma<\theta\}} I_{\{\tau=T\}\cap\{\sigma=T\}}\big], \end{aligned} \tag{4.89}$$

where the second equality holds because θ is continuously distributed and independent of $\mathcal{F}^2_T$, so $\mathbb{P}(\sigma = \theta) = 0$ for any $\sigma \in \mathcal{T}^R(\mathcal{F}^2_t)$. Fix $\varepsilon > 0$ and set

$$f^\varepsilon_t := f^0_t I_{\{t<\theta+\varepsilon\}}, \quad g_t := g^0_t I_{\{t<\theta\}}, \quad h := h^0 I_{\{T<\theta\}}, \qquad t \in [0,T]. \tag{4.90}$$

We see that conditions (A1), (A3), (A4) hold for (f_t), (g_t), h (for conditions (A3), (A4) we use that $f^0 \geq 0$). Condition (A2) (regularity of payoffs (f_t) and (g_t)) is also satisfied, because (f^0_t), (g^0_t) are regular and θ has a continuous distribution, so it is a totally inaccessible stopping time for the filtration $(\mathcal{F}_t)$ (recall Example 2.3.5). Therefore, by Theorem 4.2.4, the game with expected payoff

$$N^\varepsilon(\tau,\sigma) = \mathbb{E}\big[\mathcal{P}^\varepsilon(\tau,\sigma)\big] := \mathbb{E}\big[f^\varepsilon_\tau I_{\{\tau\leq\sigma\}\cap\{\tau<T\}} + g_\sigma I_{\{\sigma<\tau\}\cap\{\sigma<T\}} + h I_{\{\tau=T\}\cap\{\sigma=T\}}\big]$$

has a value.

We now show that the game with expected payoff N^0 has the same value as the one with expected payoff N^ε, for any $\varepsilon > 0$. First observe that

$$N^\varepsilon(\tau,\sigma) - N^0(\tau,\sigma) = \mathbb{E}\big[f^0_\tau I_{\{\tau\le\sigma\}} I_{\{\theta<\tau<\theta+\varepsilon\}} I_{\{\tau<T\}}\big] \ge 0 \tag{4.91}$$

by the assumption that $f^0 \ge 0$. Hence,

$$\inf_{\tau\in\mathcal{T}^R(\mathcal{F}^1_t)} \sup_{\sigma\in\mathcal{T}^R(\mathcal{F}^2_t)} N^\varepsilon(\tau,\sigma) \ge \inf_{\tau\in\mathcal{T}^R(\mathcal{F}^1_t)} \sup_{\sigma\in\mathcal{T}^R(\mathcal{F}^2_t)} N^0(\tau,\sigma). \tag{4.92}$$

To derive the opposite inequality for the lower values, fix $\sigma \in \mathcal{T}^R(\mathcal{F}^2_t)$. For $\tau \in \mathcal{T}^R(\mathcal{F}^1_t)$, define

$$\hat\tau = \begin{cases} \tau, & \tau \le \theta, \\ T, & \tau > \theta. \end{cases}$$

Then, using that $\mathcal{P}^\varepsilon(\tau,\sigma) = \mathcal{P}(\tau,\sigma)$ on $\{\tau\le\theta\}$ and $\mathcal{P}^\varepsilon(T,\sigma) = g^0_\sigma I_{\{\sigma<\theta\}} = \mathcal{P}(\tau,\sigma)$ on $\{\tau > \theta\}$, we have $N^\varepsilon(\hat\tau,\sigma) = N^0(\tau,\sigma)$. It then follows that

$$\inf_{\tau\in\mathcal{T}^R(\mathcal{F}^1_t)} N^\varepsilon(\tau,\sigma) \le \inf_{\tau\in\mathcal{T}^R(\mathcal{F}^1_t)} N(\tau,\sigma),$$

which implies

$$\sup_{\sigma\in\mathcal{T}^R(\mathcal{F}^2_t)} \inf_{\tau\in\mathcal{T}^R(\mathcal{F}^1_t)} N^\varepsilon(\tau,\sigma) \le \sup_{\sigma\in\mathcal{T}^R(\mathcal{F}^2_t)} \inf_{\tau\in\mathcal{T}^R(\mathcal{F}^1_t)} N(\tau,\sigma). \tag{4.93}$$

Since the value of the game with expected payoff N^ε exists, combining (4.92) and (4.93) we see that the value of the game with expected payoff N also exists.

Recall that in Section 3.4.4 the setting is Markovian with $T = \infty$, $f^0_t = e^{-rt}F(X_t)$, $g^0_t = e^{-rt}G(X_t)$ (F, G continuous deterministic functions and $r \ge 0$), $h^0 = 0$, θ is exponentially distributed, and (X_t) is a one-dimensional linear diffusion. Under specific requirements on the functions F and G, Lempa & Matomäki (2013) find that a Nash equilibrium for the game exists when the first player uses $(\mathcal{F}^1_t)$-stopping times and the second player uses $(\mathcal{F}^2_t)$-stopping times, with no need for randomisation. Their methods rely on the theory of one-dimensional linear diffusions, hence do not admit an extension to a non-Markovian case.

4.7 Necessity of assumptions

In the subsections below we show that: (a) relaxing condition (A4) may lead to a game without a value, (b) in situations where one player has all the informational advantage, the use of randomised stopping times may still be beneficial also for the uninformed player, and (c) Assumption (A2') is tight in requiring that either $(\hat{f}_t)$ is non-increasing or $(\hat{g}_t)$ is non-decreasing.

In order to keep the exposition simple we consider the framework of Section 4.6.1 with $I = 2$, $J = 1$, and impose that $(\mathcal{F}^p_t)$ be the trivial filtration (hence all payoff processes are deterministic, since they are $(\mathcal{F}^p_t)$-adapted). Furthermore we restrict to the case in which $f^{1,1} = f^{2,1} = f$ and $g^{1,1} = g^{2,1} = g$. Only the terminal payoff depends on the scenario, i.e. $h^{1,1} \neq h^{2,1}$ (both deterministic since $(\mathcal{F}^p_T)$-measurable). For notational simplicity we set $h^1 := h^{1,1}$ and $h^2 := h^{2,1}$.

Note that only the first player (minimiser) observes the true value of $\mathcal{I}$, so she has a strict informational advantage over the second player (maximiser). The second player will be referred to as the *uninformed player* while the first player as the *informed player*.

We denote by $\mathcal{T}^R = \mathcal{T}^R(\mathcal{F}^p_t)$ the set of $(\mathcal{F}^p_t)$-randomised stopping times. The informed player chooses two randomised stopping times τ_1, τ_2 (one for each scenario, recall Lemma 4.6.2) with the generating processes $(\xi^1_t), (\xi^2_t)$ which, due to the triviality of the filtration $(\mathcal{F}^p_t)$, are deterministic functions. Pure $(\mathcal{F}^p_t)$-stopping times are constants in $[0, T]$. Similarly, the uninformed player's randomised stopping time σ has a deterministic function (ζ_t) as the generating process.

4.7.1 Necessity of Assumption (A4)

Let us consider specific payoff functions

$$f \equiv 1, \quad g_t = \frac{1}{2}t, \quad h^1 = 2, \quad h^2 = 0,$$

and let us also set $T = 1$, $\pi_1 = \pi_2 = \frac{1}{2}$.

Proposition 4.7.1. *In the example of this subsection we have*

$$V_* \leq \frac{1}{2}, \qquad \text{and} \qquad V^* > \frac{1}{2},$$

so the game does not have a value.

Proof. First we show that $V_* \leq \frac{1}{2}$, i.e. that for any $\sigma \in \mathcal{T}^R$, there exist $\tau_1, \tau_2 \in \mathcal{T}^R$ such that $N((\tau_1, \tau_2), \sigma) \leq \frac{1}{2}$. Recall that (see Remark 4.3.5)

$$V_* = \sup_{\sigma \in \mathcal{T}^R} \inf_{\tau_1, \tau_2 \in \mathcal{T}^R} N((\tau_1, \tau_2), \sigma) = \sup_{\sigma \in \mathcal{T}^R} \inf_{\tau_1, \tau_2 \in [0,1]} N((\tau_1, \tau_2), \sigma),$$

so we can take $\tau_1, \tau_2 \in [0, 1]$ deterministic in the arguments below. Take any $\sigma \in \mathcal{T}^R$ and the corresponding generating process (ζ_t) which is, due to the triviality of the filtration $(\mathcal{F}^p_t)$, a deterministic function. For $\tau_1 \in [0,1)$, $\tau_2 = 1$ we obtain

$$N((\tau_1,\tau_2),\sigma) = \mathbb{E}\big[(\frac{1}{2}\sigma I_{\{\sigma<\tau_1\}} + 1\cdot I_{\{\sigma\geq\tau_1\}})I_{\{\mathcal{J}=1\}} + (\frac{1}{2}\sigma I_{\{\sigma<1\}} + 0\cdot I_{\{\sigma=1\}})I_{\{\mathcal{J}=2\}}\big]$$
$$\leq \frac{1}{2}(\frac{1}{2}\zeta_{\tau_1-} + (1-\zeta_{\tau_1-})) + \frac{1}{4}\zeta_{1-} = \frac{1}{2} - \frac{1}{4}\zeta_{\tau_1-} + \frac{1}{4}\zeta_{1-},$$

where we used Lemma 4.3.2, the fact that σ is bounded above by 1, and that $\mathcal{J}$ is independent of σ with $\mathbb{P}(\mathcal{J}=1) = \mathbb{P}(\mathcal{J}=2) = \frac{1}{2}$. In particular,

$$\inf_{\tau_1,\tau_2\in[0,1]} N((\tau_1,\tau_2),\sigma) \leq \lim_{\tau_1\to 1-} N((\tau_1,1),\sigma) = \frac{1}{2}.$$

This proves that $V_* \leq \frac{1}{2}$.

Now we turn our attention to demonstrating that $V^* > \frac{1}{2}$, i.e. that for any $\tau_1,\tau_2 \in \mathcal{T}^R$, there exists $\sigma \in \mathcal{T}^R$ such that $N((\tau_1,\tau_2),\sigma) > \frac{1}{2}$. Noting again that

$$V^* = \inf_{\tau_1,\tau_2\in\mathcal{T}^R}\sup_{\sigma\in\mathcal{T}^R} N((\tau_1,\tau_2),\sigma) = \inf_{\tau_1,\tau_2\in\mathcal{T}^R}\sup_{\sigma\in[0,1]} N((\tau_1,\tau_2),\sigma),$$

we can restrict our attention to constant $\sigma \in [0,1]$. Take any $\tau_1,\tau_2 \in \mathcal{T}^R$ and the corresponding generating processes $(\xi_t^1),(\xi_t^2)$ which are also deterministic functions.

Take any $\delta \in (0,\frac{1}{2})$. If $\xi_{1-}^1 > \delta$, then for any $\sigma < 1$ we have, using Lemma 4.3.2,

$$N((\tau_1,\tau_2),\sigma) \geq \mathbb{E}\big[\big(1\cdot I_{\{\tau_1\leq\sigma\}} + \frac{1}{2}\sigma I_{\{\sigma<\tau_1\}}\big)I_{\{\mathcal{J}=1\}} + \frac{1}{2}\sigma I_{\{\mathcal{J}=2\}}\big]$$
$$= \mathbb{E}\big[\big(\xi_\sigma^1 + \frac{1}{2}\sigma(1-\xi_\sigma^1)\big)I_{\{\mathcal{J}=1\}} + \frac{1}{2}\sigma I_{\{\mathcal{J}=2\}}\big]$$
$$= \frac{1}{2}\xi_\sigma^1 - \frac{1}{4}\sigma\xi_\sigma^1 + \frac{1}{2}\sigma = \frac{1}{2}\xi_\sigma^1(1-\frac{1}{2}\sigma) + \frac{1}{2}\sigma$$

and, in particular,

$$\sup_{\sigma\in[0,1]} N((\tau_1,\tau_2),\sigma) \geq \lim_{\sigma\to 1-} N((\tau_1,\tau_2),\sigma) \geq \frac{1}{4}\xi_{1-}^1 + \frac{1}{2} \geq \frac{1}{2} + \frac{1}{4}\delta > \frac{1}{2}.$$

On the other hand, if $\xi_{1-}^1 \leq \delta$, taking $\sigma = 1$ yields

$$\sup_{\sigma\in[0,1]} N((\tau_1,\tau_2),\sigma) \geq N((\tau_1,\tau_2),1) \geq \mathbb{E}[2\cdot I_{\{\tau_1=1\}}I_{\{\mathcal{J}=1\}}] = 1 - \xi_{1-}^1 \geq 1-\delta > \frac{1}{2},$$

where the equality is due to Lemma 4.3.2. This completes the proof that $V^* > \frac{1}{2}$. □

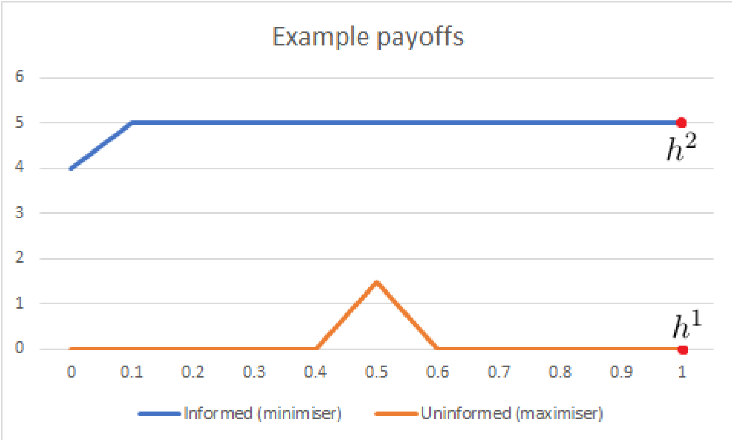

Figure 4.1: Payoff functions: (f_t) in blue, (g_t) in orange.

4.7.2 Necessity of randomisation

Here we argue that randomisation is not only sufficient in order to find the value in Dynkin games with asymmetric information but, in many cases, it is also necessary. In De Angelis *et al.* (2021a) (see Sections 3.4.3 and 4.6.3), there is a rare example of explicit construction of equilibrium strategies for a zero-sum Dynkin game with asymmetric information in a diffusive set-up. The peculiarity of the solution in De Angelis *et al.* (2021a) lies in the fact that the informed player uses a randomised stopping time whereas the uninformed player sticks to a pure stopping time. An interpretation of that result suggests that the informed player uses randomisation to "gradually reveal information" about the scenario in which the game is being played, in order to induce the uninformed player to act in a certain desirable way. Since the uninformed player has "no information to reveal" one may be tempted to draw a general conclusion that she should never use randomised stopping rules. However, Proposition 4.7.2 below shows that such conclusion would be wrong in general and even the *uninformed* player may benefit from randomisation of stopping times.

We consider specific payoff functions (f_t) and (g_t) plotted on Figure 4.1. Their analytic formulae read

$$f_t = (10t+4)I_{\{t\in[0,\frac{1}{10})\}} + 5I_{\{t\in[\frac{1}{10},1]\}}, \qquad g_t = (15t-6)I_{\{t\in[\frac{2}{5},\frac{1}{2})\}} + (9-15t)I_{\{t\in[\frac{1}{2},\frac{3}{5})\}}$$

with

$$h^1 = 0 = g_{1-}, \quad h^2 = 5 = f_{1-}.$$

We also set $T = 1$, $\pi_1 = \pi_2 = \frac{1}{2}$. As always, we identify randomised strategies with their generating processes. In particular, we denote by (ζ_t) the generating process for $\sigma \in \mathcal{T}^R$.

By Theorem 4.2.4, the game has a value in randomised strategies, i.e. $V^* = V_*$. Restriction of the uninformed player's (player 2) strategies to pure stopping times affects only the lower value, see Remark 4.3.5. The lower value of the game in which player 2 is restricted to using pure stopping times reads

$$\widehat{V}_* := \sup_{\sigma \in [0,1]} \inf_{\tau_1, \tau_2 \in \mathcal{T}^R} N((\tau_1, \tau_2), \sigma) = \sup_{\sigma \in [0,1]} \inf_{\tau_1, \tau_2 \in [0,1]} N((\tau_1, \tau_2), \sigma),$$

where the equality is again due to Remark 4.3.5 (we are using that $(\mathcal{F}_t^p)$-stopping times are deterministic, because $(\mathcal{F}_t^p)$ is trivial). As the following proposition shows, $\widehat{V}_* < V_*$, so the game in which the uninformed player does not randomise does not have a value. This confirms that the randomisation can play a strategic role beyond manipulating/revealing information.

Proposition 4.7.2. *In the example of this subsection, we have*

$$V_* > \widehat{V}_*.$$

Proof. First, notice that

$$\widehat{V}_* \le \sup_{\sigma \in [0,1]} N(\hat{\tau}(\sigma), \sigma),$$

where we take

$$\hat{\tau}(\sigma) = (\tau_1(\sigma), \tau_2(\sigma)) = \begin{cases} (1,1), & \text{for } \sigma \in [0,1), \\ (1,0), & \text{for } \sigma = 1. \end{cases}$$

We see that $\sup_{\sigma \in [0,1]} N(\hat{\tau}(\sigma), \sigma) = N(\hat{\tau}(1), 1) \vee \sup_{\sigma \in [0,1)} N(\hat{\tau}(\sigma), \sigma) = 2$.

We will show that the σ-player can ensure a strictly larger payoff by using a randomised strategy. Define $\zeta_t = a I_{\{t \ge \frac{1}{2}\}} + (1-a) I_{\{t=1\}}$, i.e. the corresponding $\sigma \in \mathcal{T}^R$ prescribes to 'stop at time $\frac{1}{2}$ with probability a and at time 1 with probability $1-a$'. The value of the parameter $a \in [0,1]$ will be determined below. We claim that

$$\inf_{\tau_1, \tau_2 \in [0,1]} N((\tau_1, \tau_2), \zeta) = N((1,0), \zeta) \wedge N((1,1), \zeta). \tag{4.94}$$

Assuming that the above is true, we calculate

$$N((1,0), \zeta) = 2 + \frac{3}{4} a, \qquad N((1,1), \zeta) = \frac{5}{2} - a.$$

Picking $a = \frac{2}{7}$, the above quantities are equal to $\frac{31}{14}$. Hence $V_* \ge \frac{31}{14} > 2$. □

It remains to prove (4.94). Recall that $\zeta_t = aI_{\{t\geq\frac{1}{2}\}} + (1-a)I_{\{t=1\}}$ is the generating process of σ and the expected payoff reads

$$N((\tau_1,\tau_2),\zeta) = \sum_{i=1}^{2} \mathbb{E}\left[I_{\{\mathcal{I}=i\}}\left(f_{\tau_i} I_{\{\tau_i\leq\sigma\}\cap\{\tau_i<1\}} + g_\sigma I_{\{\sigma<\tau_i\}\cap\{\sigma<1\}} + h^i I_{\{\tau_i=\sigma=1\}}\right)\right].$$

It is clear that on the event $\{\mathcal{I}=1\}$ the infimum is attained for $\tau_1 = 1$, irrespective of the choice of (ζ_t). On the event $\{\mathcal{I}=2\}$ the informed player would only stop either at time zero, where the function (f_t) attains the minimum cost $f_0 = 4$, or at time $t > \frac{1}{2}$, since (ζ_t) only puts mass at $t = \frac{1}{2}$ and at $t = 1$ (the informed player knows her opponent may stop at $t = \frac{1}{2}$ with probability a). The latter strategy corresponds to a payoff $5 - \frac{7}{2}a$ and can also be achieved by picking $\tau_2 = 1$. Then, the informed player needs only to consider the expected payoff associated to the strategies $(\tau_1,\tau_2) = (1,0)$ and $(\tau_1,\tau_2) = (1,1)$, so that (4.94) holds.

4.7.3 Necessity of Assumption (A2')

Our final counterexample shows that violating Assumption (A2') by allowing both previsible upward jumps of (f_t) *and* previsible downward jumps of (g_t) may also lead to a game without a value.

Consider the payoffs

$$f_t = 1 + 2I_{\{t\geq\frac{1}{2}\}}, \quad g_t = -I_{\{t\geq\frac{1}{2}\}}, \quad h^1 = 3, \quad h^2 = -1,$$

so that $h^1 = f_{1-}$ and $h^2 = g_{1-}$, and let us also set $T = 1$, $\pi_1 = \pi_2 = \frac{1}{2}$. Assumption (A2') is violated as (g_t) has a previsible downward jump and (f_t) has a previsible upward jump at time $t = \frac{1}{2}$.

Proposition 4.7.3. *In the example of this subsection we have*

$$V_* \leq 0, \quad \textit{and} \quad V^* > 0,$$

so the game does not have a value.

Proof. First we show that $V_* \leq 0$. For this step, it is sufficient to restrict our attention to pure stopping times $\tau_1,\tau_2 \in [0,1]$ for the informed player (c.f. Remark 4.3.5). Let $\sigma \in \mathcal{T}^R$ with a (deterministic) generating process (ζ_t) and fix $\varepsilon \in (0,\frac{1}{2})$. For $\tau_1 = \frac{1}{2} - \varepsilon$ and $\tau_2 = 1$ we obtain

$$\begin{aligned} N((\tau_1,\tau_2),\sigma) &= \mathbb{E}\left[I_{\{\mathcal{I}=1\}}(0\cdot I_{\{\sigma<\tau_1\}} + 1\cdot I_{\{\sigma\geq\tau_1\}}) + I_{\{\mathcal{I}=2\}}(0\cdot I_{\{\sigma<\frac{1}{2}\}} - 1\cdot I_{\{\sigma\geq\frac{1}{2}\}})\right] \\ &= \frac{1}{2}(1-\zeta_{(\frac{1}{2}-\varepsilon)-}) - \frac{1}{2}(1-\zeta_{\frac{1}{2}-}), \end{aligned}$$

where the final equality is due to Lemma 4.3.2. Therefore, using that (ζ_t) is càdlàg, we have

$$\inf_{\tau_1,\tau_2\in[0,1]} N((\tau_1,\tau_2),\sigma) \le \lim_{\varepsilon\to 0}\frac{1}{2}\cdot(\zeta_{\frac{1}{2}-}-\zeta_{(\frac{1}{2}-\varepsilon)-}) = 0.$$

Since the result holds for all $\sigma \in \mathcal{T}^R$, we have $V_* \le 0$.

Next, we demonstrate that $V^* > 0$. For this step it is again sufficient to consider pure stopping times $\sigma \in [0,1]$ for the uninformed player (Remark 4.3.5). Let $\tau_1,\tau_2 \in \mathcal{T}^R$ and let ξ^1,ξ^2 be the associated (deterministic) generating processes. Consider first the case in which $\xi^1_{\frac{1}{2}-}+\xi^2_{\frac{1}{2}-} > \delta$ for some $\delta \in (0,1)$, and fix $\varepsilon \in (0,\frac{1}{2})$. For $\sigma = \frac{1}{2}-\varepsilon$ we have

$$\begin{aligned} N((\tau_1,\tau_2),\sigma) &= \mathbb{E}\big[I_{\{\mathcal{J}=1\}}(1\cdot I_{\{\tau_1\le\sigma\}}+0\cdot I_{\{\sigma<\tau_1\}})+I_{\{\mathcal{J}=2\}}(1\cdot I_{\{\tau_2\le\sigma\}}+0\cdot I_{\{\sigma<\tau_2\}})\big] \\ &= \frac{1}{2}(\xi^1_{\frac{1}{2}-\varepsilon}+\xi^2_{\frac{1}{2}-\varepsilon}), \end{aligned}$$

where the final equality is due to Lemma 4.3.2. Thus,

$$\sup_{\sigma\in[0,1]} N((\tau_1,\tau_2),\sigma) \ge \lim_{\sigma\to\frac{1}{2}-} N((\tau_1,\tau_2),\sigma) = \frac{1}{2}(\xi^1_{\frac{1}{2}-}+\xi^2_{\frac{1}{2}-}) > \delta > 0. \tag{4.95}$$

If, instead, $\xi^1_{\frac{1}{2}-}+\xi^2_{\frac{1}{2}-} \le \delta$, so that in particular $\xi^1_{\frac{1}{2}-}\vee\xi^2_{\frac{1}{2}-} \le \delta$, then, again applying Lemma 4.3.2, we obtain

$$\begin{aligned} \sup_{\sigma\in[0,1]} N((\tau_1,\tau_2),\sigma) &\ge N((\tau_1,\tau_2),1) \\ &\ge \mathbb{E}\big[I_{\{\mathcal{J}=1\}}(1\cdot I_{\{\tau_1<\frac{1}{2}\}}+3\cdot I_{\{\tau_1\ge\frac{1}{2}\}})+I_{\{\mathcal{J}=2\}}(-1)\big] \\ &\ge \frac{1}{2}\left(\xi^1_{\frac{1}{2}-}+3\big(1-\xi^1_{\frac{1}{2}-}\big)\right)-\frac{1}{2} = 1-\xi^1_{\frac{1}{2}-} \ge 1-\delta > 0. \end{aligned} \tag{4.96}$$

Combining (4.95) and (4.96) we have $V^* > 0$. □

Chapter 5

Ordering of payoffs

5.1 Setting

In this chapter, we study a Markovian zero-sum game with full information. Our framework is similar to Section 3.3. Our focus is on the *order* condition (3.19), which is commonly imposed in order to prove that the game has a value. Recall that a similar condition appears also in non-Markovian and nonzero-sum settings (see Sections 3.2.1 and 3.2.2). We unify certain ideas from the literature in order to relax the order condition. The exact framework we work in is described below.

Consider a filtration $(\mathcal{F}_t)_{t\in[0,\infty]}$ that satisfies the usual conditions. Let (X_t) be a Markov process defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P}_x)$, with values in a measurable space $(E, \mathcal{B})$. Here, for $x \in E$, we assume that $X_0 = x$ under $\mathbb{P}_x$. We will assume for simplicity that $E = \mathbb{R}^d$ for some $d \in \mathbb{N}$.

We consider an optimal stopping game with the following payoff:

$$\mathcal{P}(\tau, \sigma) = F(X_\tau) I_{\{\tau<\sigma\}} + G(X_\sigma) I_{\{\sigma<\tau\}} + H(X_\tau) I_{\{\tau=\sigma\}}, \tag{5.1}$$

where $F, G, H : E \mapsto \mathbb{R}$. As usual, we assume that the τ-player is the minimiser in the game and the σ-player is the maximiser.

Refer to Section 3.3 for results in the literature specific to the payoffs of such form (see in particular Ekström & Peskir (2008) and Theorem 3.3.3). Recall that one of the commonly imposed conditions for existence of the value and Nash equilibrium (Definitions 3.3.1 and 3.3.2) in the game (5.1) is the order condition

$$G(x) \le H(x) \le F(x) \quad \forall x \in E \tag{5.2}$$

(c.f. the order condition in a non-Markovian setting (3.14)). In Section 5.2, we show that the game (5.1) has the value and Nash equilibrium under a condition weaker than (5.2).

Recall that Touzi & Vieille (2002) (Section 3.5) relax the assumption (5.2) by extending the set of strategies to randomised stopping times (Section 2.4). In our Markovian set-up, such extension is not required, and we show the existence of the value and Nash equilibrium in *pure* (non-randomised) strategies, i.e. with $\tau, \sigma \in \mathcal{T}(\mathcal{F}_t) =: \mathcal{T}$.

We will also consider a special case of the payoff (5.1):

$$\mathcal{P}'(\tau,\sigma) = F(X_\tau)I_{\{\tau\leq\sigma\}} + G(X_\sigma)I_{\{\sigma<\tau\}}. \tag{5.3}$$

This is indeed a special case because $\mathcal{P}(\tau,\sigma) \equiv \mathcal{P}'(\tau,\sigma)$ if in (5.1) we have $H \equiv F$. For the game (5.3), the order condition (5.2) translates into

$$G(x) \leq F(x) \quad \forall x \in E. \tag{5.4}$$

The reason behind (5.4) becomes intuitively clear if one considers a situation when it is violated. Let, at some $t \geq 0$, the payoffs be ordered differently: $G(X_t) > F(X_t)$. If the minimiser decides to stop the game at time t, she obtains the smaller payoff F, and if the maximiser stops the game at time t, she obtains the larger payoff G. Thus, both players "like their payoffs" and would prefer to stop the game immediately. We will show this more rigorously below.

Recall also one of the financial applications of the theory of optimal stopping games — the game options Kifer (2000). A game option is an option contract which enables both its buyer and seller to stop it at any time. If the seller is the one to cancel the contract, it is natural that they will pay to the buyer more than if the buyer exercises the contract (recall e.g. De Angelis *et al.* (2021b) in Section 3.4.3 where such "penalty" is a constant $\varepsilon_0 > 0$). Hence, for game options, the order condition (5.4) is not restrictive.

In Section 5.3, we show that the order condition (5.4) is, in fact, *not* necessary for the existence of the value and Nash equilibrium in the game (5.3).

We conclude this section by formally listing our assumptions on the underlying Markov process (X_t) and on the payoff functions F, G, H in (5.1).

(A1) (X_t) is a càdlàg quasi left-continuous strong Markov process,

(A2) $F, G, H : E \mapsto \mathbb{R}$ are continuous and bounded functions,

(A3) the following condition at infinity holds:

$$\lim_{t\to\infty} (G \wedge H)(X_t) = \lim_{t\to\infty} (F \vee H)(X_t), \; \mathbb{P}_x\text{-a.s.} \; \forall x \in E.$$

For the game (5.3) without the "middle" payoff H, the assumption (A3) reads

(A3')

$$\lim_{t\to\infty} G(X_t) = \lim_{t\to\infty} F(X_t), \ \mathbb{P}_x\text{-a.s. } \forall x \in E.$$

5.2 Existence of value and Nash equilibrium

We will use the notation

$$N_x(\tau,\sigma) = \mathbb{E}_x\big[\mathcal{P}(\tau,\sigma)\big],$$

where $\mathbb{E}_x$ is the expectation with respect to the measure $\mathbb{P}_x$, and $\tau,\sigma \in \mathcal{T}$.

The following lemma allows us to bound the lower and upper value of the game by the value of the payoff functions.

Lemma 5.2.1. *For every $x \in E$ we have*

$$G(x) \wedge H(x) \le V_*(x) \le V^*(x) \le F(x) \vee H(x). \tag{5.5}$$

Proof. Let us substitute $\tau = 0$ and $\sigma = 0$ in the expressions for the upper/lower value in Definition 3.3.1. More precisely, observe that

$$\begin{aligned} V^*(x) &\le \sup_{\sigma\in\mathcal{T}} N_x(0,\sigma) \\ &= \sup_{\sigma\in\mathcal{T}} \mathbb{E}_x\big[F(X_0)I_{\{0<\sigma\}} + H(X_\sigma)I_{\{\sigma=0\}}\big] \\ &= F(x) \vee H(x). \end{aligned} \tag{5.6}$$

On the other hand,

$$\begin{aligned} V_*(x) &\ge \inf_{\tau\in\mathcal{T}} N_x(\tau,0) \\ &= \inf_{\tau\in\mathcal{T}} \mathbb{E}_x\big[G(X_0)I_{\{\tau>0\}} + H(X_\tau)I_{\{\tau=0\}}\big] \\ &= G(x) \wedge H(x). \end{aligned} \tag{5.7}$$

Recall that $V_*(x) \le V^*(x)$ by definition (Remark 3.2.5). Thus, (5.6) and (5.7) imply (5.5). □

Define the set

$$A = \{x \in E : G(x) \ge H(x) \ge F(x)\}. \tag{5.8}$$

The following corollary of Lemma 5.2.1 enables us to determine the value on A.

Corollary 5.2.2. *For every $x \in A$, we have*

$$V_*(x) = V^*(x) = H(x). \tag{5.9}$$

Consider an auxiliary game with the payoff

$$\widetilde{\mathcal{P}}_x(\tau,\sigma) = \widetilde{F}(X_\tau) I_{\{\tau<\sigma\}} + \widetilde{G}(X_\sigma) I_{\{\sigma<\tau\}} + \widetilde{H}(X_\tau) I_{\{\tau=\sigma\}}, \tag{5.10}$$

where $\widetilde{F} = F \vee H$, $\widetilde{G} = G \wedge H$, $\widetilde{H} = H$. Denote

$$\widetilde{N}_x(\tau,\sigma) = \mathbb{E}_x\big[\widetilde{\mathcal{P}}(\tau,\sigma)\big],$$

and let $\widetilde{V}_*$ and $\widetilde{V}^*$ be the lower and upper value of the game (5.10).

Remark 5.2.3. *For $x \in A$, we have $\widetilde{F}(x) = \widetilde{G}(x) = H(x) = \widetilde{V}_*(x) = \widetilde{V}^*(x)$, where the first two equalities are by definition of $\widetilde{F}$, $\widetilde{G}$ and the last two are due to Corollary 5.2.2.*

The auxiliary game (5.10) is linked to the original game (5.1) as follows.

Theorem 5.2.4. *Suppose that $H(x)$ always lies between $G(x)$ and $F(x)$, i.e.*

$$G(x) \wedge F(x) \le H(x) \le G(x) \vee F(x) \quad \forall x \in E. \tag{5.11}$$

Suppose there exists a Nash equilibrium $(\widetilde{\tau}^*, \widetilde{\sigma}^*)$ of the game with the payoff (5.10) such that

$$\widetilde{\tau}^* \le \eta_A,\ \widetilde{\sigma}^* \le \eta_A,\ \mathbb{P}\text{-a.s.}, \tag{5.12}$$

where $\eta_A = \inf\{t : X_t \in A\}$ for the set $A \subset E$ defined in (5.8). Then, there exists the value function V of the game with the payoff (5.1), and $V \equiv \widetilde{V}$, where $\widetilde{V}$ is the value of the game with the payoff (5.10). Further, the couple $(\widetilde{\tau}^*, \widetilde{\sigma}^*)$ is a Nash equilibrium in the game (5.1).

Proof. Fix $x \in E$. For arbitrary $\sigma \in \mathcal{T}$, define $\hat{\sigma} := \sigma \wedge \eta_A$. For the future use, observe that

$$\begin{aligned} I_{\{\widetilde{\tau}^*<\eta_A\}} I_{\{\widetilde{\tau}^*<\sigma\}} &= I_{\{\widetilde{\tau}^*<\eta_A\}} I_{\{\widetilde{\tau}^*<\hat{\sigma}\}}, \\ I_{\{\widetilde{\tau}^*<\eta_A\}} I_{\{\sigma<\widetilde{\tau}^*\}} &= I_{\{\widetilde{\tau}^*<\eta_A\}} I_{\{\hat{\sigma}<\widetilde{\tau}^*\}}, \\ I_{\{\widetilde{\tau}^*<\eta_A\}} I_{\{\widetilde{\tau}^*=\sigma\}} &= I_{\{\widetilde{\tau}^*<\eta_A\}} I_{\{\widetilde{\tau}^*=\hat{\sigma}\}}, \end{aligned} \tag{5.13}$$

and that

$$I_{\{\sigma<\eta_A\}} = I_{\{\hat{\sigma}<\eta_A\}},\ I_{\{\eta_A\le\sigma\}} = I_{\{\eta_A=\hat{\sigma}\}}. \tag{5.14}$$

We have

$$\begin{aligned} N_x(\widetilde{\tau}^*,\sigma) &= \mathbb{E}_x\big[F(X_{\widetilde{\tau}^*}) I_{\{\widetilde{\tau}^*<\sigma\}} + G(X_\sigma) I_{\{\sigma<\widetilde{\tau}^*\}} + H(X_{\widetilde{\tau}^*}) I_{\{\widetilde{\tau}^*=\sigma\}}\big] \\ &= \mathbb{E}_x\big[I_{\{\widetilde{\tau}^*<\eta_A\}}\big(F(X_{\widetilde{\tau}^*}) I_{\{\widetilde{\tau}^*<\sigma\}} + G(X_\sigma) I_{\{\sigma<\widetilde{\tau}^*\}} + H(X_{\widetilde{\tau}^*}) I_{\{\widetilde{\tau}^*=\sigma\}}\big) \\ &\quad + I_{\{\widetilde{\tau}^*=\eta_A\}}\big(F(X_{\eta_A}) I_{\{\eta_A<\sigma\}} + G(X_\sigma) I_{\{\sigma<\eta_A\}} + H(X_{\eta_A}) I_{\{\eta_A=\sigma\}}\big)\big]. \end{aligned}$$

Observe that, by (5.11), on the event $\{\widetilde{\tau}^* < \eta_A\}$ we have $F(X_{\widetilde{\tau}^*}) \geq H(X_{\widetilde{\tau}^*})$, and, since $\widetilde{\tau}^* \leq \eta_A$, $\mathbb{P}$-a.s. by assumption, on the event $\{\sigma < \widetilde{\tau}^*\}$ we have $G(X_\sigma) \leq H(X_\sigma)$. Therefore,

$$\begin{aligned}
&\mathbb{E}_x\big[I_{\{\widetilde{\tau}^*<\eta_A\}}\big(F(X_{\widetilde{\tau}^*})I_{\{\widetilde{\tau}^*<\sigma\}} + G(X_\sigma)I_{\{\sigma<\widetilde{\tau}^*\}} + H(X_{\widetilde{\tau}^*})I_{\{\widetilde{\tau}^*=\sigma\}}\big)\big] \\
&= \mathbb{E}_x\big[I_{\{\widetilde{\tau}^*<\eta_A\}}\big((F\vee H)(X_{\widetilde{\tau}^*})I_{\{\widetilde{\tau}^*<\sigma\}} + (G\wedge H)(X_\sigma)I_{\{\sigma<\widetilde{\tau}^*\}} + H(X_{\widetilde{\tau}^*})I_{\{\widetilde{\tau}^*=\sigma\}}\big)\big] \\
&= \mathbb{E}_x\big[I_{\{\widetilde{\tau}^*<\eta_A\}}\big((F\vee H)(X_{\widetilde{\tau}^*})I_{\{\widetilde{\tau}^*<\hat{\sigma}\}} + (G\wedge H)(X_{\hat{\sigma}})I_{\{\hat{\sigma}<\widetilde{\tau}^*\}} + H(X_{\widetilde{\tau}^*})I_{\{\widetilde{\tau}^*=\hat{\sigma}\}}\big)\big],
\end{aligned}$$

where in the last inequality we used (5.13). On the event $\{\sigma > \eta_A\}$, again using (5.11), we have $F(X_\sigma) \leq H(X_\sigma)$, and on the event $\{\sigma < \eta_A\}$, we have $G(X_\sigma) \leq H(X_\sigma)$. Thus,

$$\begin{aligned}
&\mathbb{E}_x\big[I_{\{\widetilde{\tau}^*=\eta_A\}}\big(F(X_{\eta_A})I_{\{\eta_A<\sigma\}} + G(X_\sigma)I_{\{\sigma<\eta_A\}} + H(X_{\eta_A})I_{\{\eta_A=\sigma\}}\big)\big] \\
&\leq \mathbb{E}_x\big[I_{\{\widetilde{\tau}^*=\eta_A\}}\big(H(X_{\eta_A})I_{\{\eta_A<\sigma\}} + (G\wedge H)(X_\sigma)I_{\{\sigma<\eta_A\}} + H(X_{\eta_A})I_{\{\eta_A=\sigma\}}\big)\big] \\
&= \mathbb{E}_x\big[I_{\{\widetilde{\tau}^*=\eta_A\}}\big((G\wedge H)(X_{\hat{\sigma}})I_{\{\hat{\sigma}<\eta_A\}} + H(X_{\eta_A})I_{\{\eta_A=\hat{\sigma}\}}\big)\big],
\end{aligned}$$

where in the last inequality we used (5.14). Combining the above, we see that

$$N_x(\widetilde{\tau}^*, \sigma) \leq \widetilde{N}_x(\widetilde{\tau}^*, \hat{\sigma}). \tag{5.15}$$

For the upper value V^* of the *initial* game (5.1), we then have

$$V^*(x) \leq \sup_{\sigma\in\mathcal{T}} N_x(\widetilde{\tau}^*, \sigma) \leq \sup_{\sigma\in\mathcal{T}} \widetilde{N}_x(\widetilde{\tau}^*, \hat{\sigma}) \leq \sup_{\sigma\in\mathcal{T}} \widetilde{N}_x(\widetilde{\tau}^*, \sigma) = \widetilde{V}(x), \tag{5.16}$$

where the third inequality is due to the supremum on the right-hand side of it being taken over a larger set, and the final equality is the general relation between a Nash equilibrium and the value (Remark 3.2.7).

We now provide a symmetric argument for the lower value of the game (5.1). For arbitrary $\tau \in \mathcal{T}$, let us define $\hat{\tau} := \tau \wedge \eta_A$. We have

$$\begin{aligned}
N_x(\tau, \widetilde{\sigma}^*) = \mathbb{E}_x\big[&I_{\{\widetilde{\sigma}^*<\eta_A\}}\big(F(X_{\widetilde{\tau}^*})I_{\{\widetilde{\tau}^*<\sigma\}} + G(X_\sigma)I_{\{\sigma<\widetilde{\tau}^*\}} + H(X_{\widetilde{\tau}^*})I_{\{\widetilde{\tau}^*=\sigma\}}\big) \\
&+ I_{\{\widetilde{\sigma}^*=\eta_A\}}\big(F(X_\tau)I_{\{\tau<\eta_A\}} + G(X_{\eta_A})I_{\{\eta_A<\tau\}} + H(X_{\eta_A})I_{\{\tau=\eta_A\}}\big)\big].
\end{aligned}$$

On the event $\{\widetilde{\sigma}^* < \eta_A\}$ we have $G(X_{\widetilde{\sigma}^*}) \leq H(X_{\widetilde{\sigma}^*})$, and on the event $\{\tau < \widetilde{\sigma}^*\}$ we have $F(X_\tau) \geq H(X_\tau)$. Therefore,

$$\begin{aligned}
&\mathbb{E}_x\big[I_{\{\widetilde{\sigma}^*<\eta_A\}}\big(F(X_\tau)I_{\{\tau<\widetilde{\sigma}^*\}} + G(X_{\widetilde{\sigma}^*})I_{\{\widetilde{\sigma}^*<\tau\}} + H(X_\tau)I_{\{\tau=\widetilde{\sigma}^*\}}\big)\big] \\
&= \mathbb{E}_x\big[I_{\{\widetilde{\sigma}^*<\eta_A\}}\big((F\vee H)(X_\tau)I_{\{\tau<\widetilde{\sigma}^*\}} + (G\wedge H)(X_{\widetilde{\sigma}^*})I_{\{\widetilde{\sigma}^*<\tau\}} + H(X_\tau)I_{\{\tau=\widetilde{\sigma}^*\}}\big)\big] \\
&= \mathbb{E}_x\big[I_{\{\widetilde{\sigma}^*<\eta_A\}}\big((F\vee H)(X_{\hat{\tau}})I_{\{\hat{\tau}<\widetilde{\sigma}^*\}} + (G\wedge H)(X_{\widetilde{\sigma}^*})I_{\{\widetilde{\sigma}^*<\hat{\tau}\}} + H(X_{\hat{\tau}})I_{\{\hat{\tau}=\widetilde{\sigma}^*\}}\big)\big].
\end{aligned}$$

On the event $\{\tau > \eta_A\}$, we have $G(X_\tau) \geq H(X_\tau)$, and on the event $\{\tau < \eta_A\}$, we have $F(X_\tau) \geq H(X_\tau)$. Thus,

$$\begin{aligned}
&\mathbb{E}_x\big[I_{\{\widetilde{\sigma}^*=\eta_A\}}\big(F(X_\tau)I_{\{\tau<\eta_A\}} + G(X_{\eta_A})I_{\{\eta_A<\tau\}} + H(X_{\eta_A})I_{\{\tau=\eta_A\}}\big)\big] \\
&\geq \mathbb{E}_x\big[I_{\{\widetilde{\sigma}^*=\eta_A\}}\big((F\vee H)(X_\tau)I_{\{\tau<\eta_A\}} + H(X_{\eta_A})I_{\{\eta_A<\tau\}} + H(X_{\eta_A})I_{\{\tau=\eta_A\}}\big)\big] \\
&= \mathbb{E}_x\big[I_{\{\widetilde{\sigma}^*=\eta_A\}}\big((F\vee H)(X_{\hat{\tau}})I_{\{\hat{\tau}<\eta_A\}} + H(X_{\eta_A})I_{\{\hat{\tau}=\eta_A\}}\big)\big].
\end{aligned}$$

Combining the above, we see that

$$N_x(\tau, \widetilde{\sigma}^*) \geq \widetilde{N}_x(\hat{\tau}, \widetilde{\sigma}^*), \tag{5.17}$$

which, for the lower value V_* of the initial game (5.1), implies

$$V_*(x) \geq \inf_{\tau\in\mathcal{T}} N_x(\tau, \widetilde{\sigma}^*) \geq \inf_{\tau\in\mathcal{T}} \widetilde{N}_x(\hat{\tau}, \widetilde{\sigma}^*) \geq \widetilde{V}(x). \tag{5.18}$$

From (5.16) and (5.18), we obtain

$$V_*(x) = V^*(x) = \widetilde{V}(x) \quad \forall x \in E, \tag{5.19}$$

i.e. the initial game has a value $V \equiv \widetilde{V}$.

For the second claim of the theorem, using the inequalities (5.15), (5.17), and the fact that $(\widetilde{\tau}^*, \widetilde{\sigma}^*)$ is a Nash equilibrium of the auxiliary game (5.10), we obtain, for arbitrary $\tau, \sigma \in \mathcal{T}$,

$$N_x(\widetilde{\tau}^*, \sigma) \leq \widetilde{N}_x(\widetilde{\tau}^*, \hat{\sigma}) \leq \widetilde{N}_x(\widetilde{\tau}^*, \widetilde{\sigma}^*) \leq \widetilde{N}_x(\hat{\tau}, \widetilde{\sigma}^*) \leq N_x(\tau, \widetilde{\sigma}^*). \tag{5.20}$$

Recall that $\widetilde{\sigma}^* \wedge \eta_A = \widetilde{\sigma}^*$, $\mathbb{P}$-a.s. by assumption. Using this and (5.15) we obtain

$$N_x(\widetilde{\tau}^*, \widetilde{\sigma}^*) \leq \widetilde{N}_x(\widetilde{\tau}^*, \widetilde{\sigma}^*).$$

The opposite inequality follows from the fact that $\widetilde{\tau}^* \wedge \eta_A = \widetilde{\tau}^*$ and (5.17). Using (5.20), we conclude that the couple $(\widetilde{\tau}^*, \widetilde{\sigma}^*)$ is a Nash equilibrium in the game (5.1).

□

Remark 5.2.5. *In the proof of Theorem 5.2.4, we do not use the Markov property of the process (X_t). An analogous result could be established in a non-Markovian framework of e.g. Section 3.2.1 or Chapter 4. For further details, see Section 5.4.*

The following theorem gives the sufficient conditions for existence of the value and the Nash equilibrium of the game (5.1).

Theorem 5.2.6. *Let the assumptions (A1), (A2), (A3) hold. Suppose further that*

$$G(x) \wedge F(x) \leq H(x) \leq G(x) \vee F(x) \quad \forall x \in E. \tag{5.21}$$

Then, there exists the value function V of the optimal stopping game with the payoff (5.1). Moreover, the Nash equilibrium holds with

$$\tau^* = \inf\{t : X_t \in D_1 \cup A\}; \; \sigma^* = \inf\{t : X_t \in D_2 \cup A\}, \tag{5.22}$$

where $D_1 = \{x \in E : V(x) = F(x)\}$, $D_2 = \{x \in E : V(x) = G(x)\}$, $A = \{x \in E : G(x) \geq H(x) \geq F(x)\}$.

Remark 5.2.7. *Under condition (5.21) we have*

$$E = A \cup \{x \in E : G(x) < H(x) < F(x)\}.$$

In other words, the state space E is partitioned into the region A (where the lower and upper value are known to coincide, see Corollary 5.2.2) and the region in which the "classical" order condition $G < H < F$ holds (and where, under additional conditions, the value exists due to Theorem 3.3.3).

Proof of Theorem 5.2.6. For the auxiliary game (5.10), we have $\widetilde{G}(x) \leq \widetilde{H}(x) \leq \widetilde{F}(x)$ $\forall x \in E$ due to (5.21), i.e. the order condition (5.2) holds. Let us show that the other conditions of Theorem 3.3.3 are satisfied for the game (5.10). Indeed, the strong Markov process (X_t) is càdlàg and quasi left-continuous by (A1); the functions $F, G, H \in \mathcal{L}_b^X$ as in (3.6) since they are bounded, and they are continuous by (A2); (3.20) holds due to (A3). Applying Theorem 3.3.3, we see that the game (5.10) has the value $\widetilde{V}$ and a Nash equilibrium $(\widetilde{\tau}^*, \widetilde{\sigma}^*)$ such that

$$\widetilde{\tau}^* = \inf\{t : X_t \in \widetilde{D}_1\}; \; \widetilde{\sigma}^* = \inf\{t : X_t \in \widetilde{D}_2\}, \tag{5.23}$$

where $\widetilde{D}_1 = \{x \in E : \widetilde{V}(x) = \widetilde{G}(x)\}$ and $\widetilde{D}_2 = \{x \in E : \widetilde{V}(x) = \widetilde{F}(x)\}$.

By Remark 5.2.3, we have

$$A \subset \widetilde{D}_i, \; i = 1, 2, \tag{5.24}$$

and thus $\widetilde{\tau}^* \leq \eta_A$, $\widetilde{\sigma}^* \leq \eta_A$. In other words, the condition (5.12) is satisfied, and we can apply Theorem 5.2.4 to deduce that the initial game (5.1) has the value $V \equiv \widetilde{V}$, and the couple $(\widetilde{\tau}^*, \widetilde{\sigma}^*)$ is a Nash equilibrium in game (5.1). It remains to note that, by definitions of functions $\widetilde{F}$, $\widetilde{G}$ and of the set A, we have

$$\widetilde{D}_i = D_i \cup A, \; i = 1, 2.$$

Thus, $\tau^* = \widetilde{\tau}^*$, $\sigma^* = \widetilde{\sigma}^*$, and the couple (τ^*, σ^*) is a Nash equilibrium in the game (5.1). □

Remark 5.2.8. *The condition (5.21) is only a condition on H, unlike the classical order condition (5.2) which bounds all the three payoff functions.*

Example 5.2.9. *If the condition (5.21) is violated, the value of the game (5.1) may not exist. Suppose, for instance, that*

$$\sup_{x\in E} H(x) < \inf_{x\in E} \big(G(x) \wedge F(x)\big).$$

In this situation, H is the smallest payoff, so the minimiser "would like to stop together with the maximiser". Hence, for the lower value of the game (5.1), we have

$$V_* = \sup_{\sigma\in\mathcal{T}} \inf_{\tau\in\mathcal{T}} N_x(\tau,\sigma) \le \sup_{\sigma\in\mathcal{T}} N_x(\sigma,\sigma) = \sup_{\sigma\in\mathcal{T}} H(X_\sigma) \le \sup_{x\in E} H(x),$$

and for the upper value we have

$$V^* = \inf_{\tau\in\mathcal{T}} \sup_{\sigma\in\mathcal{T}} N_x(\tau,\sigma) \ge \inf_{\tau\in\mathcal{T}} \sup_{\sigma\neq\tau} N_x(\tau,\sigma) > \sup_{x\in E} H(x).$$

Therefore, $V^* > V_*$ and there is no value.

5.3 The case $H \equiv F$

Consider now a game with $H \equiv F$, i.e. with the payoff (5.3). With a slight abuse of notation of Section 5.2, we denote

$$N_x(\tau,\sigma) = \mathbb{E}_x\big[\mathcal{P}'(\tau,\sigma)\big].$$

Further, the lower and upper value of the game (5.3) are denoted as V_* and V^*. We emphasise that this is only for notational convenience, and V_* and V^* are *not* necessarily equal to the lower and upper value of the general game considered in Section 5.2.

Similarly to Section 5.2, the order condition (5.4) can be relaxed (in fact, removed). The following lemma is a direct corollary of Lemma 5.2.1.

Lemma 5.3.1. *For the game* (5.3)*, for every $x \in E$ we have*

$$F(x) \wedge G(x) \le V_*(x) \le V^*(x) \le F(x). \tag{5.25}$$

The set A from (5.8) reads

$$A = \{x \in E : G(x) \ge F(x)\}.$$

Corollary 5.3.2. *For the game* (5.3)*, for every $x \in A$, we have*

$$V_*(x) = V^*(x) = F(x). \tag{5.26}$$

Remark 5.3.3. *Consider the game (5.3) in case when $A = E$, i.e. $G(x) \geq F(x)$ $\forall x \in E$. In line with Laraki & Solan (2005), such game stops at time 0, and $\tau^* = \sigma^* \equiv 0$ form a Nash equilibrium. Indeed, for arbitrary $x \in E$, $\tau, \sigma \in \mathcal{T}$, we have*

$$N_x(0,\sigma) = F(x) = N_x(0,0) \leq N_x(\tau,0), \tag{5.27}$$

so the couple $(0,0)$ satisfies the Definition 3.3.2 of a Nash equilibrium.

The following theorem for the game (5.3) holds true. Note that there are no additional requirements on the relations between F and G.

Theorem 5.3.4. *Under assumptions (A1), (A2), (A3'), there exists the value function V of the optimal stopping game with the payoff (5.3), and the Nash equilibrium holds*

$$\tau^* = \inf\{t : X_t \in D_1 \cup A\}; \ \sigma^* = \inf\{t : X_t \in D_2 \cup A\}, \tag{5.28}$$

where $D_1 = \{x \in E : V(x) = F(x)\}$, $D_2 = \{x \in E : V(x) = G(x)\}$, $A = \{x \in E : G(x) \geq F(x)\}$.

Since $V(x) = F(x)$ $\forall x \in A$ (Corollary 5.3.2), we have $A \subset D_1$. Hence, the stopping time τ^* in (5.28) has a shorter representation $\tau^* = \inf\{t : X_t \in D_1\}$.

Remark 5.3.5. *In the game with the payoff (5.3), the "middle" payoff H equals the payoff F of the τ-player: that is, on the event $\{\tau = \sigma\}$, the payoff is $F(X_\tau)$. Consider a similar game, but with the opposite behaviour on the event $\{\tau = \sigma\}$, i.e. with the payoff*

$$\hat{\mathcal{P}}(\tau,\sigma) = F(X_\tau) I_{\{\tau<\sigma\}} + G(X_\sigma) I_{\{\sigma \leq \tau\}}. \tag{5.29}$$

Such game fits the framework of (5.1) with $H \equiv G$, and therefore a result symmetric to Lemma 5.3.1 holds:

$$G(x) \leq \hat{V}_*(x) \leq \hat{V}^*(x) \leq F(x) \vee G(x), \tag{5.30}$$

where $\hat{V}_*$ and $\hat{V}^*$ are the lower and the upper value of the game (5.29). In particular, similarly to Corollary 5.3.2, for the game (5.29) we have $\hat{V}_*(x) = \hat{V}^*(x) = G(x)$ $\forall x \in A$.

In the case $H \equiv G$, the following theorem holds true.

Theorem 5.3.6. *Under assumptions (A1), (A2), (A3'), there exists the value function V of the optimal stopping game with the payoff (5.29), and the Nash equilibrium holds with (τ^*, σ^*) as in Theorem 5.3.4.*

Theorems 5.3.4 and 5.3.6 are a special case of the Theorem 5.2.6. Indeed, (A3') is a special case of (A3), and the condition (5.21) is automatically satisfied if $H \equiv F$ (as in the payoff (5.3)) or if $H \equiv G$ (as in the payoff (5.29)). We also remark the similarity between Theorem 5.3.6 and the following fact from the literature.

Theorem 5.3.7. *(Stettner, 1982b, Theorem 3) The game (5.29) has the same value and Nash equilibrium as the auxiliary game with the function $F \vee G$ instead of F in (5.29).*

However, in Theorem 5.3.7, the assumptions on the process (X_t) ensuring the existence of the value in the auxiliary game are different from ours.

Remark 5.3.8. *So far in this section, we worked under assumption that there is no "middle" payoff H, and in the infinite-horizon case. It is possible to consider a similar framework but with finite horizon T, and with the "middle" payoff reducing to the terminal-time payoff — a Markovian analogue of the payoff studied in Chapter 4. That is, consider*

$$\mathring{\mathcal{P}}(\tau,\sigma) = F(X_\tau)I_{\{\tau\leq\sigma\}\cap\{\tau<T\}} + G(X_\sigma)I_{\{\sigma<\tau\}\cap\{\sigma<T\}} + \mathring{H}(X_T)I_{\{\tau=T\}\cap\{\sigma<T\}}.$$

The order condition (5.21) in this case is no longer automatically satisfied, and we would need its weaker analogue

$$G(X_T) \wedge F(X_T) \leq \mathring{H}(X_T) \leq G(X_T) \vee F(X_T), \quad \mathbb{P}\text{-a.s.}$$

in order to claim, as in Theorem 5.3.4, that the value and the Nash equilibrium exist.

5.4 Possible non-Markovian extension

In Remark 5.2.5, we mentioned that the proof of Theorem 5.2.4 does not use the Markovian properties of the game. In fact, the main reason for assuming Markovianity was to fit the auxiliary game (5.10) into the framework of Theorem 3.3.3 and use the latter to prove Theorem 5.2.6.

Instead, one could consider the non-Markovian payoff

$$\check{\mathcal{P}}(\tau,\sigma) = f_\tau I_{\{\tau<\sigma\}} + g_\sigma I_{\{\sigma<\tau\}} + h_\tau I_{\{\tau=\sigma\}},$$

as in (3.13) in Section 3.2.1. Using the non-Markovian Definition 3.2.4 of the lower and upper value $\check{V}_*$ and $\check{V}^*$ of the game with the payoff $\check{\mathcal{P}}$, we can establish a non-Markovian counterpart of Lemma 5.2.1:

$$\mathbb{E}[g_0 \wedge h_0] \leq \check{V}_* \leq \check{V}^* \leq \mathbb{E}[f_0 \vee h_0]. \tag{5.31}$$

The proof of this chain of inequalities is analogous to the proof of Lemma 5.2.1 and goes through substituting $\tau = 0$ and $\sigma = 0$ in Definition 3.2.4. Note that (5.31) only addresses the behaviour of the payoff processes at time 0, while Lemma 5.2.1 holds on the whole state space E. Let us introduce a counterpart of the set A (5.8), i.e. the set on which the payoff processes are in the "wrong" order:

$$\check{A} = \{t \in [0, \infty] : g_t \geq h_t \geq f_t\}.$$

In order to study the game on the (random) set $\check{A}$, one may need to extend the ideas related to Snell envelopes in optimal stopping theory (see Section 3.1.1) to the game framework and introduce lower/upper value *processes* of the game, similarly to Section 3.2.1 (see in particular (3.15) and references thereafter). However, we leave the details of this extension to future research.

Chapter 6

Conclusions

In the main chapter of the thesis (Chapter 4), we study Dynkin games with asymmetric information. In Theorem 4.2.3, we prove that the value of games exists under general assumptions on payoff processes, and without specific assumptions on the information structure. This result generalises previous results from the literature on Dynkin games with asymmetric information (see Section 4.6). In Section 4.5.4, we explain that in fact the value exists for a more general payoff, and under a further condition on the jumps of the payoff processes the Nash equilibrium also exists.

Our methodology enables us to work in a general non-Markovian framework, but this generality also limits us to the existence results. A possible next step is to introduce Markovian assumptions in the framework in order to provide more explicit results. Indeed, recall Theorem 3.3.3 that characterises the Nash equilibrium of a classical full-information Markovian Dynkin games as a couple of stopping times that are hitting times. In that setting, it is optimal for the minimiser (resp. the maximiser) to stop as soon as the underlying process enters the set D_1 (resp. D_2). However, in the setting of Chapter 4, the stopping decisions of the players are randomised. If the Nash equilibrium (τ_*, σ_*) exists for the asymmetric information game of Chapter 4, the natural question is the behaviour of the corresponding generating processes (ξ^*, ζ^*). It is of interest to try and deduce a similarly explicit "rule", to specify when it is optimal for the minimiser (resp. the maximiser) to increase the value of her strategy ξ^* (resp. ζ^*).

Another possible direction of future research is developing financial applications of our model. In Chapter 1 we briefly introduced convertible bonds, which are real-world instruments that can be viewed as a Dynkin game (with specific payoff processes) between a firm issuing the bond and a bondholder. The payoff (4.1) studied in Chapter 4 can be reduced to the payoff of a convertible bond, if the minimiser is interpreted as the firm, the maximiser is interpreted as the bondholder, the time horizon T is the time of maturity of the bond, the payoff process (f_t) is a constant, and the payoff process (g_t) and the terminal payoff h are functions of the value of the firm's stock (S_t). Indeed, the process (f_t) is the payoff delivered by the minimiser (i.e. the firm) to the maximiser (i.e. the bondholder) on the event $\{\tau \leq \sigma\} \cap \{\tau < T\}$ (i.e. if the bond is *called* before the maturity T). Similarly, the process (g_t) is the payoff delivered by the firm to the bondholder if the bond is *converted* before the maturity T. Finally, the terminal payoff h (i.e. the payoff at the maturity T of the bond) is delivered by the firm to the bondholder if none of the players have stopped before the time T (i.e. if neither the conversion nor the call have happened before T), and thus the bondholder chooses between receiving a pre-specified price or a function of the value of the firm's stock S_T. With such an interpretation, the framework of Chapter 4 can be used to study a convertible bond traded in a situation of asymmetry of information between the firm and the bondholder. A natural example would be the bondholder not being able to accurately estimate the value of the firm's stock.

In Chapter 5, we study a different Dynkin game — the Markovian game with full information. We focus on the order condition on the payoff processes, and, drawing on the existing literature, provide a way to relax this condition. Possible future steps in studying this problem include the extension to non-Markovian set-up (see Section 5.4). Another direction is to apply the ideas from Chapter 5 to relax the order conditions (A3)-(A4) in the non-Markovian asymmetric information framework of Chapter 4.

Appendix A

Auxiliary facts

A.1 Probability and stochastic processes

In this section, we follow Rogers & Williams (2000) and Blumental & Getoor (1968) to give some important definitions used in the thesis and discuss relations between them. Recall that $T \in (0, \infty]$ denotes the time horizon.

Definition A.1.1. *A family of sigma-algebras* $(\mathcal{F}_t)_{t \in [0,T]}$ *is called a* filtration *on the probability space* $(\Omega, \mathcal{F}, \mathbb{P})$*, if for every* $t_1, t_2 \in [0,T]$ *such that* $t_1 \leq t_2$*, we have* $\mathcal{F}_{t_1} \subseteq \mathcal{F}_{t_2} \subseteq \mathcal{F}$.

Definition A.1.2. *Given a filtration* $(\mathcal{F}_t)_{t \in [0,T]}$*, a random variable* $\eta : \Omega \mapsto [0,T]$ *is called an* $(\mathcal{F}_t)$-stopping time*, if the event* $\{\eta \leq t\}$ *belongs to the sigma-algebra* $\mathcal{F}_t$ *for all* $t \in [0,T]$.

Definition A.1.3. *An* $(\mathcal{F}_t)$*-stopping time* η *is called* previsible *(or sometimes* predictable *or* announceable*), if there exists a sequence of* $(\mathcal{F}_t)$*-stopping times* $\{\eta_n\}_{n \in \mathbb{N}}$ *that* announces η*, i.e. such that* $\eta_n < \eta$ *on the event* $\{\eta > 0\}$*, and* $\eta_n \to \eta$.

Definition A.1.4. *An* $(\mathcal{F}_t)$*-stopping time* η *is called* accessible*, if there exists a sequence of* previsible $(\mathcal{F}_t)$*-stopping times* $\{\eta_n\}_{n \in \mathbb{N}}$ *such that* $\mathbb{P}(\{\text{exists } n \in \mathbb{N} \text{ such that } \eta = \eta_n\}) = 1$.

Every previsible stopping time is accessible — to see that one needs to take $\eta_n = \eta$ for all n in the above definition.

Definition A.1.5. *An* $(\mathcal{F}_t)$*-stopping time* η *is called* totally inaccessible*, if for every* previsible *stopping time* χ *there holds* $\mathbb{P}(\{\eta = \chi\} \cap \{\chi < \infty\}) = 0$.

Every stopping time can be decomposed into an accessible and a totally inaccessible "parts", as the following lemma formalises.

Lemma A.1.6. *(Rogers & Williams, 2000, Lemma VI.13.4) For any finite stopping time η, there exist an accessible stopping time η_1, a totally inaccessible stopping time η_2, and two disjoint events A_1, A_2 such that $A_1 \cup A_2 = \Omega$, for which the following holds:*

$$\eta = \eta_1 I_A + \eta_2 I_B.$$

Properties of random variables usually do not hold for all $\omega \in \Omega$, but rather with probability one. For example, when we say that two random variables are equal, we almost always mean that they are equal $\mathbb{P}$-a.s. (and usually we do specify this to avoid confusion). For stochastic processes, there is a similar notion of being "morally equal" — indistinguishability.

Definition A.1.7. *Two stochastic processes $(X_t)_{t\in[0,T]}$ and $(Y_t)_{t\in[0,T]}$ are called* indistinguishable*, if*

$$\mathbb{P}(\{\omega \in \Omega : X_t(\omega) = Y_t(\omega) \text{ for all } t \in [0,T]\}) = 1.$$

A similar definition can be given to address the issue of ordering of stochastic processes.

Definition A.1.8. *Let $(X_t^1)_{t\in[0,T]}$, $(X_t^2)_{t\in[0,T]}$ be measurable processes. We say that the process (X_t^1)* dominates *the process (X_t^2), if*

$$\mathbb{P}(\{\omega \in \Omega : X_t^1(\omega) \geq X_t^2(\omega) \text{ for all } t \in [0,T]\}) = 1.$$

Let us also define some fundamental types of stochastic processes.

Definition A.1.9. *For a filtration $(\mathcal{F}_t)_{t\in[0,T]}$, a process $(X_t)_{t\in[0,T]}$ is called an $(\mathcal{F}_t)$-*martingale*, if*

(1) (X_t) is $(\mathcal{F}_t)$-adapted,

(2) $\mathbb{E}[|X_t|] < \infty$ for every $t \in [0,T]$,

(3) for every $t_1, t_2 \in [0,T]$ such that $t_1 \leq t_2$ we have $\mathbb{E}[X_{t_2}|\mathcal{F}_{t_1}] = X_{t_1}$, $\mathbb{P}$-a.s.

*A process $(X_t)_{t\in[0,T]}$ is called an $(\mathcal{F}_t)$-*supermartingale *(resp. $(\mathcal{F}_t)$-*submartingale*), if it satisfies (1) and (2) above, and in (3), the equality $=$ is replaced by the inequality $\leq$ (resp. $\geq$).*

Definition A.1.10. *Let $(E, \mathcal{B})$ be a measurable space, $(\Omega, \mathcal{F}, \mathbb{P}_x)$ a probability space for all $x \in E$, and $(\mathcal{F}_t)_{t\in[0,T]}$ a filtration of sub-sigma-algebras of $\mathcal{F}$. A stochastic process $(X_t)_{t\in[0,T]}$ with values in E is called* Markov *(with respect to the filtration $(\mathcal{F}_t)$), if*

(1) (X_t) is $(\mathcal{F}_t)$-adapted,

(2) the map $x \mapsto \mathbb{P}_x(X_t \in B)$ is in $\mathcal{B}$ for every $t \in [0,T]$ and $B \in \mathcal{B}$,

(3) $\mathbb{P}_x(X_{t+s} \in B|\mathcal{F}_t) = \mathbb{P}_{X_t}(X_s \in B)$ *for all* $x \in E$, $B \in \mathcal{B}$, $0 \leq t \leq s \leq T$.

A Markov process (X_t) *is called* strong Markov, *if for every* $(\mathcal{F}_t)$*-stopping time* η *and every bounded measurable function* $F : E \mapsto \mathbb{R}$*, the following holds:*

(4) X_η *is* $\mathcal{F}_\eta$*-measurable,*

(5) $\mathbb{E}_x(F(X_{t+\eta})|\mathcal{F}_\eta) = \mathbb{E}_{X_\eta}(F(X_t))$ *for all* $x \in E$, $t \in [0,T]$,

where $\mathbb{E}_x$ *is the expectation with respect to the measure* $\mathbb{P}_x$.

Remark A.1.11. *In the literature, definitions related to continuous-time Markov processes often differ from Definition A.1.10. For example, it is possible to define a Markov process without introducing the measures* $\mathbb{P}_x$*, by stating that "given the present, the past and the future are independent" (for the rigorous definition, see (Blumental & Getoor, 1968, Definition I.1.1)). Another example is adding a "cemetery state" to the space E and postulating that the process* (X_t) *never leaves this state (for details, see (Blumental & Getoor, 1968, Definition I.3.1)). There are also concept related to time-homogeneity of Markov processes, of translation operators and infinitesimal generators corresponding to a Markov process. Since our main results in Chapter 4 do not rely on Markovianity and do not benefit from this theory, we omit further details and refer to (Dynkin, 1965, Chapters II-III).*

A.2 General topology

The fact that a continuous function on a compact set attains its maximum and minimum (sometimes referred as *extreme value theorem*) holds true for functions on a general topological space. We provide the background necessary to prove the extreme value theorem. The definitions of topological space, topology, open sets and neighbourhoods, convergence, continuity are omitted; we refer to (Armstrong, 1983, Chapters 1-3).

A (topologically) *closed* set is defined as a complement of an open set. To define topological compactness, we need a concept of cover.

Definition A.2.1. *Let B be a set in a topological space, and* $\mathcal{O}$ *a collection of open sets. We say that* $\mathcal{O}$ *is a* cover *of B, if* $B = \cup_{A \in \mathcal{O}} A$.

Definition A.2.2. *A set B in a topological space is called* compact, *if every open cover of B has a finite subcover.*

There is the following general relation between topological closedness and compactness.

Theorem A.2.3. *(Armstrong, 1983, Theorem 3.5) A closed subset of a compact set is compact.*

We now can formally state the extreme value theorem.

Theorem A.2.4 (Extreme value theorem)**.** *(Armstrong, 1983, Theorem 3.10) A continuous real-valued function defined on a compact space is bounded and attains its bounds.*

An alternative way of introducing closedness and compactness is via converging sequences. A set is called *sequentially closed*, if every converging sequence of its elements converges to an element of this set. A set is called *sequentially compact*, if every sequence of its elements has a converging subsequence.

In the proof of Theorem 4.4.5, we claim that closedness of certain sets follows from Lemma 4.4.18. In fact, in Lemma 4.4.18 only sequential closedness of these sets is proven. But, as the following theorem explains, these concepts are equivalent in metric spaces.

Theorem A.2.5. *(Willard, 1970, Theorem 10.4) In a metric space $\mathcal{M}$, sequential closedness is equivalent to topological closedness.*

Proof. First, take a closed set $C \subset \mathcal{M}$, and take a sequence $\{x_n\}$ in C such that $x_n \to x \in \mathcal{M}$. We need to show that $x \in C$. Assume otherwise, i.e. $x \in \mathcal{M} \setminus C$. Then, since the set $\mathcal{M} \setminus C$ is open, there exists a neighbourhood O of x such that $O \subset \mathcal{M} \setminus C$. But, by the definition of convergence, $\exists N : \forall n > N \; x_n \in O \subset \mathcal{M} \setminus C$, which contradicts the fact that $x_n \in C$. This contradiction proves that the set C is sequentially closed.

For the other direction, take a sequentially closed set C. Consider its *closure* $\bar{C}$ (the smallest closed set containing C). Take an arbitrary $x \in \bar{C}$. Consider the open balls $B(x, \frac{1}{n})$ for $n \in \mathbb{N}$. We claim that the sets $B(x, \frac{1}{n}) \cap C$ are non-empty. Indeed, assume otherwise: that there exists n such that $B(x, \frac{1}{n}) \cap C = \emptyset$. But, by the definition of closure, the set $\mathcal{M} \setminus \bar{C}$ is the largest open set that does not intersect with C. Then, $B(x, \frac{1}{n}) \subset \mathcal{M} \setminus \bar{C}$, which contradicts the fact that $x \in \bar{C}$.

Let us for each n select an arbitrary point $x_n \in B(x, \frac{1}{n}) \cap C$. By construction, $x_n \to x$, and, by sequential closedness of C, it implies $x \in C$. Since this was proven for arbitrary $x \in \bar{C}$, we see that $C = \bar{C}$, hence C is closed. □

Note that, in general, closedness and sequential closedness are not equivalent. However, we can see from the proof of Theorem A.2.5 that a closed set is always sequentially closed (indeed, we did not use the metric to conduct the proof in this direction). Spaces in which the opposite implication holds true (i.e. any sequentially closed set is closed) are called *sequential* spaces. Thus, Theorem A.2.5 proves that metric spaces are sequential.

In the proof we have used another property — metric spaces are *first-countable*, that is, each point $x \in \mathcal{M}$ has a *countable neighbourhood basis*: there exists a sequence of neighbourhoods of x such that any neighbourhood of x contains an element of this sequence. From the proof we see that, in a metric space, the balls $B(x, \frac{1}{n})$ form a countable neighbourhood basis of a point x. One can use a similar argument to show that any first-countable space is sequential.

We conclude with some definitions that will help us to avoid confusion in Section A.3.1.

Definition A.2.6. *Let $\mathcal{O}_1$, $\mathcal{O}_2$ be two topologies on a general topological space S. We say that $\mathcal{O}_1$ is* coarser *(or* weaker*) than $\mathcal{O}_2$, if $\mathcal{O}_1 \subseteq \mathcal{O}_2$.*

Definition A.2.7. *A set $A \subset S$ is said to be* dense *in S, if for any point $x \in S$ and for any neighbourhood O of x, $O \cap A \neq \emptyset$.*

A.3 Properties of Banach spaces

The content of this section is used in Section 4.4; in particular, many facts we outline below are necessary to prove Lemma 4.4.16.

A.3.1 Weak topologies

Let E be a Banach space with a norm $||\cdot||$. Denote by E^* the *dual space* of E, that is, the space of all continuous linear functionals on E. For $F \in E^*$, the *dual norm* is defined by

$$||F||_{E^*} = \sup_{x \in E : ||x|| \leq 1} F(x).$$

The space E^* with this norm is Banach.

The topology on E generated by the norm $||\cdot||$ is often referred as *strong topology*. There is another topology on E defined as follows.

Definition A.3.1. *The topology on E is called the* weak topology $\sigma(E, E^*)$*, if it is the coarsest topology on E such that all the maps $\{F : E \mapsto \mathbb{R}\}_{F \in E^*}$ are continuous in this topology.*

The weak topology exists and is unique for any Banach space E by (Brezis, 2010, Lemma 3.1). In fact, for any collection of maps from a set to a general topological space, the coarsest topology such that these maps are continuous exists and is unique (recall e.g. Section 3.1.3 and the weak topology on the set $\mathcal{A}^\circ$ defined via the functional N^r). However, in Chapter 4, we use specifically the weak topology $\sigma(E, E^*)$, and therefore we focus on this topology in the sequel.

Remark A.3.2. *By definition of the space E^*, all its elements are continuous in the strong topology, and therefore the weak topology is weaker than the strong topology. In other words, sets that are open in the weak topology $\sigma(E, E^*)$ are always open in the strong topology. It implies, for example, that if (x_n) in E converges to x in the strong topology, then (x_n) converges to x in the weak topology (Brezis, 2010, Proposition 3.5.ii).*

For brevity, we often say "weakly open/closed/compact" instead of "open/closed/compact in the weak topology", and use the terms "strongly open/closed/compact" in a similar way.

In general, strongly open/closed sets are not necessarily weakly open/closed (Brezis, 2010, Section 3.2, Remark 2). For convex sets, however, there is the equivalence.

Theorem A.3.3. *(Brezis, 2010, Theorem 3.7) Let C be a convex subset of E. Then C is weakly closed if and only if it is strongly closed.*

Finally, we say a few words about reflexive spaces. Note that, given $x \in E$, the mapping $\mathfrak{J}_x : F \mapsto F(x)$ is a continuous (with respect to the norm $||\cdot||_{E^*}$) linear functional on E^*, i.e. an element of $(E^*)^*$.

Definition A.3.4. *Let $\mathfrak{J}_x$ be as above. The mapping $J : E \mapsto (E^*)^*$ defined as $J(x) = \mathfrak{J}_x$ is called a* canonical injection. *If, additionally, $J(E) = (E^*)^*$, the space E is called* reflexive.

We will use the following property of reflexive spaces.

Theorem A.3.5 (Kakutani)**.** *(Brezis, 2010, Theorem 3.17) Let E be a Banach space. Then E is reflexive if and only if the* unit ball

$$B_E = \{x \in E : ||x|| \leq 1\}$$

is weakly compact.

A.3.2 L^p-spaces

Let (S, Σ, μ) be a measure space. For $p \in [1, \infty)$, by $L^p(S, \Sigma, \mu)$ (or simply L^p when no ambiguity arises) we denote the space

$$\left\{ F : S \to \mathbb{R} \text{ such that } ||F||_{L^p} := \left(\int_S |F|^p d\mu \right)^{\frac{1}{p}} < \infty \right\}.$$

Note that the functions that coincide μ-a.e. are indistinguishable in this space: indeed, the $||\cdot||_{L^p}$-norm of their difference is zero by definition. Therefore, elements of an L^p-space are actually *equivalence classes* of functions. In this sense, one can speak of a function-*representative* of an element of L^p-space.

Theorem A.3.6. *(Brezis, 2010, Theorems 4.7-4.8) L^p with the norm $||\cdot||_{L^p}$ is a Banach space.*

By L^p*-convergence* we mean convergence in the $||\cdot||_{L^p}$-norm. More precisely, for $(F_n), F \in L^p$, we say that $F_n \to F$ in L^p if $||F_n - F||_{L^p} \to 0$ as $n \to \infty$.

Sometimes, it is more convenient to deal with μ-a.e. convergence than with the L^p-convergence.

Theorem A.3.7. *(Brezis, 2010, Theorem 4.9.a) Let $(F_n), F \in L^p$ be such that $F_n \to F$ in L^p. Then, there exist a subsequence (F_{n_k}) such that $F_{n_k} \to F$ μ-a.e. on S.*

Finally, let us draw a link between this section and Section A.3.1. A corollary of so-called *Riesz representation theorem* (Brezis, 2010, Theorem 4.11) is that the dual of an L^p space can be identified with $L^{p'}$ for a certain p'.

Theorem A.3.8. *Let $p \in (1, \infty)$, and let p' be such that $\frac{1}{p} + \frac{1}{p'} = 1$. Then the space $(L^p)^*$ is isometric to $L^{p'}$.*

In particular, the dual of L^2 space is (isometric to) itself.

Corollary A.3.9. *L^2 is a reflexive space.*

A.4 Functional analysis and measure theory

A.4.1 Functions on a linear topological space

In this section, we give definitions from Sion (1958) necessary to understand conditions of Sion's theorem (Theorem 3.5.3), which is a very important tool we use to obtain results of Chapter 4. An overview of its proof is carried out in Section 3.6.

Definition A.4.1. *A linear space S (in this thesis, over $\mathbb{R}$) is called a* linear topological space, *if it is endowed with a topology such that the vector addition and the scalar multiplication are continuous functions (in the product topologies on $S \times S$ and $\mathbb{R} \times S$).*

For example, every Banach space with the strong topology is a topological vector space, and therefore (see Remark A.3.2) every Banach space with the weak topology is.

Let A be a subset of a linear topological space S.

Definition A.4.2. *A function $F : A \to \mathbb{R}$ is called* quasi-concave, *if $\{x \in A : F(x) \geq a\}$ is a convex set for any $a \in \mathbb{R}$. A function $F : A \to \mathbb{R}$ is called* quasi-convex, *if $\{x \in A : F(x) \leq a\}$ is a convex set for any $a \in \mathbb{R}$.*

Definition A.4.3. *A function $F : A \to \mathbb{R}$ is called* upper semicontinuous, *if for any sequence $(x_n) \subset A$ converging to $x \in A$ there holds*

$$\limsup_{n\to\infty} F(x_n) \le F(x).$$

A function $F : A \to \mathbb{R}$ is called lower semicontinuous, *if the function $-F$ is upper semicontinuous.*

Remark A.4.4. *Definition A.4.2 can be given for any linear space S. Definition A.4.3, on the other hand, can be given for any topological space S. Thus, the concept of linear topological space allows to study functions that are quasi-convex/quasi-concave* and *upper/lower semicontinuous — in particular, to state Sion's theorem (Theorem 3.5.3).*

There is an equivalent definition of upper and lower semicontinuity involving the *level sets*.

Theorem A.4.5. *(Bourbaki, 1998, Propositions IV.6.2.1 and IV.6.2.3) A function $F : A \to \mathbb{R}$ is upper semicontinuous if and only if the sets $\{x \in A : F(x) \ge a\}$ are closed for any $a \in \mathbb{R}$. A function $F : A \to \mathbb{R}$ is lower semicontinuous if and only if the sets $\{x \in A : F(x) \le a\}$ are closed for any $a \in \mathbb{R}$.*

Finally, we state an extension of the extreme value theorem (Theorem A.2.4).

Theorem A.4.6. *(Bourbaki, 1998, Theorem IV.6.2.3) An upper semicontinuous (lower semicontinuous) real-valued function defined on a compact space is bounded from above (from below) and attains its supremum (infimum).*

A.4.2 Convergence theorems

For convergence of integrals, we often use Lebesgue's dominated convergence theorem. We refer to it as "dominated convergence theorem", or we say "by dominated convergence".

Theorem A.4.7 (Dominated convergence)**.** *Let $(F_n)_{n\in\mathbb{N}}$ be a sequence of real-valued measurable functions on a measure space (S, Σ, μ). Suppose that the sequence converges μ-a.e. to a measurable function F, and is dominated by some integrable function G, i.e. for all $n \in \mathbb{N}$ and μ-almost all $x \in S$*

$$|F_n(x)| \le G(x),$$

where

$$\int_S G d\mu < \infty.$$

Then F is integrable and

$$\lim_{n\to\infty} \int_S F_n d\mu = \int_S F d\mu.$$

Sometimes, we use the following theorem instead.

Theorem A.4.8 (Monotone convergence)**.** *Let $(F_n)_{n\in\mathbb{N}}$ be a non-decreasing sequence of real-valued integrable functions on a measure space (S,Σ,μ). Suppose that the sequence converges μ-a.e. to a measurable function F. Then F is integrable and*

$$\lim_{n\to\infty}\int_S F_n d\mu = \int_S F d\mu.$$

Another imporant fact about convergence of integrals is Fatou's lemma.

Lemma A.4.9 (Fatou's lemma)**.** *Let $(F_n)_{n\in\mathbb{N}}$ be a sequence of real-valued non-negative integrable functions on a measure space (S,Σ,μ). Define for all $x\in S$*

$$F(x) := \liminf_{n\to\infty} F_n(x).$$

Then F is integrable and

$$\int_S F d\mu \le \liminf_{n\to\infty}\int_S F_n d\mu.$$

The following corollary is a version of Fatou's lemma and is often called the same.

Corollary A.4.10. *Let $(F_n)_{n\in\mathbb{N}}$ be a sequence of real-valued integrable functions on a measure space (S,Σ,μ), and let G be an integrable function on the same space such that $F_n \ge G$ μ-a.e. for all $n\in\mathbb{N}$. Then F defined in Fatou's lemma is integrable and*

$$\int_S F d\mu \le \liminf_{n\to\infty}\int_S F_n d\mu.$$

Proof. Follows immedeately by applying Fatou's lemma to non-negative functions $\hat{F}_n := F_n - G$ and to $\hat{F} := F - G$. □

Corollary A.4.11 (Reverse Fatou's lemma)**.** *Let $(F_n)_{n\in\mathbb{N}}$ be a sequence of real-valued integrable functions on a measure space (S,Σ,μ), and let G be an integrable function on the same space such that $F_n \le G$ μ-a.e. for all $n\in\mathbb{N}$. Define for all $x\in S$*

$$F(x) := \limsup_{n\to\infty} F_n(x).$$

Then F is integrable and

$$\limsup_{n\to\infty}\int_S F_n d\mu \le \int_S F d\mu.$$

Proof. Follows by applying Fatou's lemma to non-negative functions $\hat{F}_n := G - F_n$ and to $\hat{F} := F - G$, and upon noticing that

$$\liminf_{n\to\infty}\int_S -\hat{F}_n = -\limsup_{n\to\infty}\int_S \hat{F}_n.$$

□

Finally, we state an important theorem concerning integrals of functions of two variables.

Theorem A.4.12 (Fubini–Tonelli theorem)**.** *Let F be a real-valued measurable function on a measure space $(S_1 \times S_2, \Sigma_1 \times \Sigma_2, \mu_1 \times \mu_2)$. Then,*

$$\int_{S_1} \left(\int_{S_2} |F(x,y)| d\mu_2(y) \right) d\mu_1(x) = \int_{S_2} \left(\int_{S_1} |F(x,y)| d\mu_1(x) \right) d\mu_2(y) = \int_{S_1 \times S_2} |F(x,y)| d(\mu_1 \times \mu_2)(x,y).$$

If, additionally, any of the above integrals is finite, then the same equality holds for integrals of F (without the absolute value).

The theorem often goes by a shorter name of Fubini's theorem.

A.4.3 Functions of finite variation

In the sequel, we focus on real-valued functions on a real interval $[t_1, t_2]$, rather than on an arbitrary measure space. Most of the presentation is due to Revuz & Yor (1999).

Let $F : [t_1, t_2] \mapsto \mathbb{R}$ be a right-continuous function. For $s \in [t_1, t_2]$ and a subdivision Ξ of the interval $[t_1, s]$ into $t_1 = u_0 < u_1 < \ldots < u_n = s$, let

$$S^{\Xi}(s) = \sum_{i=1}^{n} |F(u_{i+1}) - F(u_i)|.$$

Definition A.4.13. *We say that the function F has* finite variation*, if for every $s \in [t_1, t_2]$ we have*

$$S(s) = \sup_{\Xi} S^{\Xi}(s) < \infty.$$

The finite function S is sometimes called the total variation of F on $[t_1, s]$.

Proposition A.4.14. *(Revuz & Yor, 1999, Proposition 0.4.2) Any function of finite variation is the difference of two non-negative non-decreasing finite functions.*

The converse is also true, since, if we are given two non-negative non-decreasing finite functions, the total variation of their difference is bounded by the sum of total variations of the two functions, and is therefore finite. Hence, Proposition A.4.14 can be used as an equivalent definition of a function of finite variation.

Let $F : [t_1, t_2] \mapsto \mathbb{R}$ be a finite non-decreasing right-continuous function. Recall that we associate to it the (positive) *Lebesgue–Stieltjes measure* μ^F by putting

$$\mu^F((u,v]) = F(v) - F(u) \text{ for } t_1 \leq u < v \leq t_2$$

and extending μ^F to the whole σ-algebra $\mathcal{B}([t_1,t_2])$ by Caratheodory's extension theorem (Athreya & Lahiri, 2006, Theorem 1.3.3). To a right-continuous function of finite variation $F:[t_1,t_2]\mapsto\mathbb{R}$, we associate the (signed) Lebesgue–Stieltjes measure in the same way. This correspondence is one-to-one (Revuz & Yor, 1999, Theorem 0.4.3).

Finally, we state a property of the set of discontinuities (jumps) of a function of finite variation.

Proposition A.4.15. *(Rudin, 1976, Theorem 4.30) Let $F:[t_1,t_2]\mapsto\mathbb{R}$ be a monotone function. Then the set of its discontinuities is at most countable.*

Note that Proposition A.4.15 implies, in particular, that the set of jumps of a monotone function has Lebesgue measure zero. Same holds true for a function of finite variation due to Proposition A.4.14.

A.4.4 Measure induced by Lebesgue–Stieltjes integral

Let $F:[t_1,t_2]\mapsto\mathbb{R}$ be a finite non-decreasing function, and let $H:[t_1,t_2]\mapsto\mathbb{R}$ be a bounded function. Define, for $s\in[t_1,t_2]$,

$$G(s)=\int_{[t_1,s]}H(t)dF(t).$$

Then, G inherits the properties of F: G is finite, and, since μ^F is a positive measure, G is non-decreasing. It is also right-continuous. Indeed, consider the decomposition $F(t)=F^c(t)+\sum_{u\in[t_1,s]}\Delta F(u)$, where F^c is the continuous part of F. Then,

$$G(s)=\int_{[t_1,s]}H(t)dF^c(t)+\sum_{u\in[t_1,s]}H(u)\Delta F(u),$$

where the first term is continuous, and the second term is right-continuous. We additionally see that the set of discontinuities of G is contained in the set of discontinuities of F.

Since G is non-decreasing and right-continuous, there exists the Lebesgue–Stieltjes measure μ^G. By its definition, for any $u,v\in[t_1,t_2]$ we have

$$\int_{(u,v]}H(s)dF(s)=G(v)-G(u)=\mu^G((u,v]).$$

The Lebesgue–Stieltjes integral with respect to G is then linked to the Lebesgue–Stieltjes integral with respect to F as follows.

Proposition A.4.16. *(Obłój, 2017, Proposition 6.1.8) For F and G as above and for any bounded function $X:[t_1,t_2]\to\mathbb{R}$, we have*

$$\int_{[t_1,t_2]}X(t)dG(t)=\int_{[t_1,t_2]}X(t)H(t)dF(t).$$

A.4.5 Absolute continuity

In this section, we follow Athreya & Lahiri (2006) to introduce absolute continuity of functions and measures.

Definition A.4.17. *A function $F : [t_1, t_2] \mapsto \mathbb{R}$ is called* absolutely continuous*, if*

$$\forall \varepsilon > 0\ \exists \delta > 0 : \forall \{u_i, v_i\}_{i=1}^N : t_1 \le u_1 \le v_1 \le u_2 \le v_2 \le \ldots \le u_N \le v_N \le t_2$$
$$\textit{with } \sum_i (v_i - u_i) < \delta \textit{ we have } \sum_i |F(u_i) - F(v_i)| < \varepsilon.$$

The definition can be straightforwardly extended to functions on an open or a semi-open interval.

There is an important equivalent definition of absolute continuity.

Theorem A.4.18. *(Athreya & Lahiri, 2006, Theorem 4.4.1) A function $F : [t_1, t_2] \mapsto \mathbb{R}$ is absolutely continuous, if and only if there exists a Borel measurable function $\alpha : [t_1, t_2] \mapsto \mathbb{R}$, called the* density *of F, such that*

$$F(s) = F(t_1) + \int_{[t_1, s]} \alpha(t) d\lambda(t)$$

for every $s \in [t_1, t_2]$, where λ is the Lebesgue measure on $[t_1, t_2]$.

Definition A.4.19. *A measure μ on the interval $[t_1, t_2]$ equipped with the Borel σ-algebra $\mathcal{B}([t_1, t_2])$ is called* absolutely continuous*, if it is absolutely continuous with respect to the Lebesgue measure λ, i.e.*

$$\forall A \in \mathcal{B}([t_1, t_2]) \textit{ such that } \lambda(A) = 0 \textit{ we have } \mu(A) = 0.$$

Theorem A.4.20. *(Athreya & Lahiri, 2006, Theorem 4.4.3) The absolute continuity of a non-decreasing function F is equivalent to the absolute continuity of the associated Lebesgue–Stieltjes measure μ^F.*

Lemma A.4.21. *Let $F : [t_1, t_2] \to \mathbb{R}$ be a Lipschitz continuous function. Then F is absolutely continuous.*

Proof. Let L be the Lipschitz constant of F. For arbitrary $\varepsilon > 0$, take $\delta = \frac{\varepsilon}{L}$. Then, for any $\{u_i, v_i\}_{i=1}^N$ from Definition A.4.17, we have

$$\sum_i |F(u_i) - F(v_i)| \le \sum_i L|u_i - v_i| \le L\delta = \varepsilon.$$

Thus, F is absolutely continuous. □

A.5 Projections and processes of finite variation

In this section, we follow (Dellacherie & Meyer, 1982, Section VI.2) to introduce optional and previsible processes and projections, and provide a decomposition for a piecewise-constant process of finite variation.

A.5.1 Optional and previsible processes and projections

Definition A.5.1. *Given a filtration $(\mathcal{F}_t)_{t\in[0,T]}$, the sigma-algebra on $[0,T]\times\Omega$ generated by càdlàg $(\mathcal{F}_t)$-adapted processes is called the* optional *sigma-algebra. The sigma-algebra on $(0,T]\times\Omega$ generated by left-continuous $(\mathcal{F}_{t-})$-adapted processes, where $\mathcal{F}_{t-} := \cup_{s<t}\mathcal{F}_s$, is called the* previsible *sigma-algebra. A process $(X_t)_{t\in[0,T]}$ is called $\mathcal{F}_t$-optional ($\mathcal{F}_t$-previsible), if the mapping $(t,\omega)\to X_t(\omega)$ is measurable with respect to the optional (previsible) sigma-algebra.*

It is immediate from the definition above that càdlàg $\mathcal{F}_t$-adapted processes are $\mathcal{F}_t$-optional. It is used in Chapter 4, where effectively all the processes we study are càdlàg and adapted.

Theorem A.5.2. *(Dellacherie & Meyer, 1982, Thm VI.43) Let (X_t) be a bounded (or unbounded but positive) measurable process. Then, there exist a unique optional process (Y_t) and a unique previsible process (Z_t) such that*

$$\mathbb{E}[X_\eta I_{\{\eta<\infty\}}|\mathcal{F}_\eta] = Y_\eta I_{\{\eta<\infty\}}\ \mathbb{P}\text{-a.s. for all stopping times } \eta;$$
$$\mathbb{E}[X_\eta I_{\{\eta<\infty\}}|\mathcal{F}_{\eta-}] = Z_\eta I_{\{\eta<\infty\}}\ \mathbb{P}\text{-a.s. for all previsible stopping times } \eta.$$

Definition A.5.3. *Processes (Y_t) and (Z_t) from Theorem A.5.2 are called, respectively, $\mathcal{F}_t$-*optional *and $\mathcal{F}_t$-*previsible projection *of (X_t).*

A.5.2 Decomposition of a piecewise-constant process

In the literature (e.g. (Rogers & Williams, 2000, Definition IV.7.2)), a stochastic process of finite variation is often defined as a stochastic process (Y_t) whose trajectories $t\mapsto Y(t,\omega)$ are functions of finite variation (in the sense of Definition A.4.13) for all $\omega\in\Omega$. Further, it is a standard convention to only consider $(\mathcal{F}_t)$-adapted and right-continuous finite variation processes. We will use the following definition from (Dellacherie & Meyer, 1982, p.115), which is equivalent due to Proposition A.4.14.

Definition A.5.4. *We say that a process (Y_t) is of* finite variation*, if it is a difference of two $(\mathcal{F}_t)$-adapted, non-decreasing, finite, right-continuous processes.*

Definition A.5.5. *We say that a non-decreasing process* $(Y_t)_{t\in[0,T]}$ *is* integrable, *if* $\mathbb{E}Y_T < \infty$. *We say that a process* (Y_t) *is of* integrable variation, *if it is a difference of two* $(\mathcal{F}_t)$*-adapted, integrable, non-decreasing, finite, right-continuous processes.*

Clearly, processes of integrable variation are a subclass of processes of finite variation.

Theorem A.5.6. *(Dellacherie & Meyer, 1982, Theorem VI.52 and Remark VI.53.a) Let* (Y_t) *be a finite* $(\mathcal{F}_t)$*-adapted non-decreasing process with* $Y_{0-} = 0$*. Then, there exist a continuous non-decreasing process* (Y_t^c)*,* $(\mathcal{F}_t)$*-stopping times* $(\eta_k)_{k\geq 1}$*, and non-negative* $\mathcal{F}_{\eta_k}$*-measurable random variables* X_k*,* $k \geq 1$*, such that*

$$Y_t = Y_t^c + \sum_{k=1}^{\infty} X_k I_{\{t\geq\eta_k\}}.$$

We apply this result in a specific set-up as follows.

Corollary A.5.7. *Let* (Y_t) *be a bounded* $(\mathcal{F}_t)$*-adapted piecewise-constant process of finite variation with* $Y_{0-} = 0$*. Then, there exist* $(\mathcal{F}_t)$*-stopping times* $(\eta_k)_{k\geq 1}$*, and non-negative* $\mathcal{F}_{\eta_k}$*-measurable random variables* X_k*,* $k \geq 1$*, such that*

$$Y_t = \sum_{k=1}^{\infty} (-1)^k X_k I_{\{t\geq\eta_k\}}. \tag{A.1}$$

The alternating terms in (A.1) come from interweaving sequences for the two non-decreasing processes (Y_t^+) and (Y_t^-) from the decomposition $Y_t = Y_t^+ - Y_t^-$ (Definition A.5.4). This is for notational convenience and resulting in no mathematical complications as the infinite sum is absolutely convergent.

A.6 Weak convergence of non-decreasing functions

The convergence (4.15) from the proof of Proposition 4.4.11 is similar to weak convergence of random variables and of the corresponding cumulative distribution functions (CDFs) (Billingsley, 1995, Theorem 25.8), the biggest difference being that the integrators $\rho^n(\omega)$, $\rho(\omega)$ in (4.15) may not be CDFs.

Instead of normalisation (4.16) that enabled us to prove (4.15), we could refer to the extended notion of weak convergence of Shiryaev (1996). Indeed, one can define the weak convergence for arbitrary bounded right-continuous non-decreasing functions (not necessarily CDFs), and for arbitrary finite measures (not necessarily probability measures).

Definition A.6.1. *A sequence of bounded right-continuous non-decreasing functions* $F^n : \mathbb{R} \mapsto \mathbb{R}$ converges weakly *to a function* $F : \mathbb{R} \mapsto \mathbb{R}$ *(denoted* $F^n \xrightarrow{w} F$*), if, for any continuous and bounded function* $H : \mathbb{R} \mapsto \mathbb{R}$*,*

$$\int_{\mathbb{R}} H(t)dF^n(t) \to \int_{\mathbb{R}} H(t)dF(t).$$

Similarly, a sequence of finite measures μ^n *on* $\mathbb{R}$ *converges weakly to a measure* μ *on* $\mathbb{R}$ *(denoted* $\mu^n \xrightarrow{w} \mu$*), if, for any continuous and bounded function* $H : \mathbb{R} \mapsto \mathbb{R}$*,*

$$\int_{\mathbb{R}} H(t)d\mu^n(t) \to \int_{\mathbb{R}} H(t)d\mu(t).$$

Note that, by definition of Lebesgue–Stieltjes measure, the convergence $F^n \xrightarrow{w} F$ is equivalent to $\mu^{F^n} \xrightarrow{w} \mu^F$ of the corresponding measures.

For convergence of functions, another (equivalent, as we will see shortly) definition is often more convenient.

Definition A.6.2. *A sequence of bounded right-continuous non-decreasing functions* $F^n : \mathbb{R} \mapsto \mathbb{R}$ converges in general *to a function* $F : \mathbb{R} \mapsto \mathbb{R}$ *(denoted* $F^n \Rightarrow F$*), if, for any point* t *of continuity of* F*,*

$$F^n(t) \to F(t).$$

A sequence of finite measures μ^n *on* $\mathbb{R}$ *converges in general to a measure* μ *on* $\mathbb{R}$ *(denoted* $\mu^n \Rightarrow \mu$*), if, for any set* A *with* $\mu(\partial A) = 0$ *(where* ∂A *denotes the boundary of* A*),*

$$\mu^n(A) \to \mu(A).$$

Note that, unlike with Definition A.6.1, the equivalence between convergence in general of functions and of corresponding Lebesgue–Stieltjes measures is not immediate. But the equivalence does hold by the following theorem.

Theorem A.6.3. *(Shiryaev, 1996, Theorem III.1.2 and Remark III.1.2) Let* $F^n, F : \mathbb{R} \mapsto \mathbb{R}$ *be bounded right-continuous non-decreasing functions. If* $\mu^{F^n}(\mathbb{R}) \to \mu^F(\mathbb{R})$*, then following conditions are equivalent:*

a) $F^n \Rightarrow F$

b) $\mu^{F^n} \Rightarrow \mu^F$

c) $\mu^{F^n} \xrightarrow{w} \mu^F$

d) $F^n \xrightarrow{w} F$

Remark A.6.4. *In Propositions 4.4.11, we assume that (omitting the dependence on the fixed* $\omega \in \Omega_0$, $\mathbb{P}(\Omega_0) = 1$*)* $\rho_T^n \to \rho_T$*, i.e. that* $\mu^{\rho^n}([0,T]) = \mu^{\rho^n}(\mathbb{R}) \to \mu^{\rho}(\mathbb{R})$*. We also assume that* $\rho^n \Rightarrow \rho$*. Therefore, we could use Theorem A.6.3 to see that* $\rho^n \xrightarrow{w} \rho$*, and then the convergence* (4.15) *for continuous* (X_t) *follows from Definition A.6.1 of weak convergence.*

A.7 Stochastic differential equations

In this section, we follow (Karatzas & Shreve, 1991, Chapter 5) to outline certain aspects of theory of stochastic differential equations and their solutions. The theory relies on a process known as *Wiener process* (or *Brownian motion*). We refer to (Karatzas & Shreve, 1991, Chapter 2) (particularly (Karatzas & Shreve, 1991, Definitions 2.1.1 and 2.5.1)) for the properties and the construction of a Wiener process.

Fix $d, r \geq 1$. Consider Borel-measurable functions $b_i(t,x)$, $\sigma_{ij}(t,x)$ from $[0,\infty) \times \mathbb{R}^d$ to $\mathbb{R}$, where $1 \leq i \leq d$, $1 \leq j \leq r$. Define the $(d \times 1)$ *drift vector* $b(t,x) = \{b_i(t,x)\}_{1 \leq i \leq d}$ and the $(d \times r)$ *dispersion matrix* $\sigma(t,x) = \{\sigma_{ij}(t,x)\}_{1 \leq i \leq d, 1 \leq j \leq r}$. Fix a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0,\infty)}, \mathbb{P})$, where the filtration $(\mathcal{F}_t)$ satisfies the usual conditions. Let (W_t) be an r-dimensional $(\mathcal{F}_t)$-Wiener process. Consider the *stochastic differential equation (SDE)*

$$dX_t = b(t,X_t)dt + \sigma(t,X_t)dW_t. \tag{A.2}$$

Definition A.7.1. *Let* $(X_t)_{t \in [0,\infty)}$ *be a continuous* $\mathbb{R}^d$*-valued* $(\mathcal{F}_t)$*-adapted stochastic process. We say that the process* (X_t) *is a* strong solution *to the SDE* (A.2) *relative to* (W_t) *with the initial condition* $x_0 \in \mathbb{R}^d$*, if*

$$X_t = X_0 + \int_0^t b(s,X_s)ds + \int_0^t \sigma(s,X_s)dW_s \tag{A.3}$$

for all $t \in [0,\infty)$ $\mathbb{P}$*-a.s.,* $X_0 = x_0$*,* $\mathbb{P}$*-a.s., and*

$$\int_0^t |b_i(s,X_s)| + \sigma_{ij}^2(s,X_s)ds < \infty \tag{A.4}$$

for all $t \in [0,\infty)$*,* $1 \leq i \leq d$*,* $1 \leq j \leq r$*,* $\mathbb{P}$*-a.s.*

Definition A.7.2. *Let the coefficients* b*,* σ *be fixed. Assume that, for any filtered probability space* $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$*, any* r*-dimensional* $(\mathcal{F}_t)$*-Wiener process, any* $x_0 \in \mathbb{R}^d$*, and any two strong solutions* (X_t)*,* $(\hat{X}_t)$ *of the SDE* (A.2)*,* (X_t) *and* $(\hat{X}_t)$ *are indistinguishable. Then we say that* strong uniqueness *holds for the pair* (b,σ)*.*

The following condition guarantees existence of a strong solution and strong uniqueness.

Theorem A.7.3. *(Karatzas & Shreve, 1991, Theorems 5.2.5 and 5.2.9) Let $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ be a filtered probability space with $(\mathcal{F}_t)$ satisfying the usual conditions, and let (W_t) be an r-dimensional $(\mathcal{F}_t)$-Wiener process. Let $x_0 \in \mathbb{R}^d$. Suppose that the coefficients b, σ satisfy the (global) Lipschitz and linear growth conditions*

$$||b(t,x) - b(t,y)|| + ||\sigma(t,x) - \sigma(t,y)|| \leq K||x-y||,$$
$$||b(t,x)||^2 + ||\sigma(t,x)||^2 \leq K^2(1 + ||x||^2),$$

for some $K > 0$ and every $t \in [0,\infty)$, $x, y \in \mathbb{R}^d$. Then, there exists a continuous $(\mathcal{F}_t)$-adapted process (X_t) which is a strong solution to the SDE (A.2) with the initial condition x_0. Moreover, the strong uniqueness holds for the pair (b, σ).

We note that existence and strong uniqueness hold, in particular, in case when the coefficients b, σ are bounded and Lipschitz-continuous, as in e.g. Section 3.4.1.

The notion of strong solution of an SDE assumes that we fix a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ and a Wiener process (W_t). We now introduce a different notion that requires to fix neither.

Definition A.7.4. *A* weak solution *of the SDE (A.2) is a triple $((X_t), (W_t), (\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P}))$ such that*

- *$(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ is a filtered probability space, and $(\mathcal{F}_t)$ satisfies the usual conditions,*
- *(X_t) is a continuous $(\mathcal{F}_t)$-adapted $\mathbb{R}^d$-valued process, (W_t) an r-dimensional $(\mathcal{F}_t)$-Wiener process,*
- *(A.3), (A.4) are satisfied.*

Clearly, if a strong solution exists, then a weak solution exists with the same process (X_t). A weak solution also exists under weaker regularity conditions on the coefficients b, σ; we omit further details and refer to (Karatzas & Shreve, 1991, Proposition 5.3.6, Theorem 5.4.22).

Appendix B

Auxiliary proofs

B.1 Decomposition of a randomised stopping time

In this section, we study a decomposition of a randomised stopping time with respect to a filtration enlarged by a sigma-algebra generated by a random variable. A similar decomposition is provided in (Esmaeeli & Imkeller, 2018, Proposition 3.3) for pure stopping times.

On a discrete probability space $(\Omega^s, \mathcal{F}^s, \mathbb{P}^s)$, consider a random variable $\mathcal{I}$ taking values in $\{1, \ldots, I\}$. Consider another probability space $(\Omega^p, \mathcal{F}^p, \mathbb{P}^p)$ with a filtration $(\mathcal{F}^p_t)$ satisfying the usual conditions. On the probability space $(\Omega, \mathcal{F}, \mathbb{P}) := (\Omega^p \times \Omega^s, \mathcal{F}^p \vee \mathcal{F}^s, \mathbb{P}^p \otimes \mathbb{P}^s)$, consider another filtration $\mathcal{F}^{\mathcal{I}}_t := \mathcal{F}^p_t \vee \sigma(\mathcal{I})$.

We start with a decomposition for generating processes: it turns out that elements of $\mathcal{A}^\circ(\mathcal{F}^{\mathcal{I}}_t)$ (recall (2.4)) can be decomposed into a sum of elements of $\mathcal{A}^\circ(\mathcal{F}^p_t)$.

Lemma B.1.1. *Any $(\xi_t) \in \mathcal{A}^\circ(\mathcal{F}^{\mathcal{I}}_t)$ has a representation*

$$\xi_t = \sum_{i=1}^{I} I_{\{\mathcal{I}=i\}} \xi^i_t, \tag{B.1}$$

for every $t \in [0,T]$, where $(\xi^1_t), \ldots, (\xi^I_t) \in \mathcal{A}^\circ(\mathcal{F}^p_t)$.

Proof. For $i \in \{1, \ldots, I\}$, let $\omega_i \in \Omega^s$ be such that $\mathcal{I}(\omega_i) = i$. For $\omega^p \in \Omega^p$, $t \in [0,T]$, let us define $\xi^i_t(\omega^p) := \xi_t((\omega^p, \omega_i))$. Then, for every $\omega^p \in \Omega^p$, $\omega^s \in \Omega^s$, $t \in [0,T]$, (B.1) holds. Further, for every $i \in \{1, \ldots, I\}$, the process (ξ^i_t) is càdlàg, non-decreasing with $\xi^i_{0-}(\omega^p) = 0$, $\xi^i_T(\omega^p) = 1$ for all $\omega^p \in \Omega^p$, since these properties hold for the process $(\xi_t(\cdot, \omega_i))_{t \in [0,T]}$. Finally, for every $t \in [0,T]$, the random variable $\xi_t(\cdot, \omega_i)$ is $\mathcal{F}^p_t$-measurable. Thus, $(\xi^i_t) \in \mathcal{A}^\circ(\mathcal{F}^p_t)$ for every i, which finishes the proof. □

Now we show that $(\mathcal{F}_t^{\mathcal{J}})$-randomised stopping times can be decomposed into a sum of $(\mathcal{F}_t^p)$-randomised stopping times.

Lemma B.1.2. *Any* $\tau \in \mathcal{T}^R(\mathcal{F}_t^{\mathcal{J}})$ *has a representation*

$$\tau = \sum_{i=1}^{I} I_{\{\mathcal{J}=i\}} \tau_i, \tag{B.2}$$

where $\tau_1, \ldots, \tau_I \in \mathcal{T}^R(\mathcal{F}_t^p)$, *with generating processes* $(\xi_t^1), \ldots, (\xi_t^I) \in \mathcal{A}^\circ(\mathcal{F}_t^p)$ *and a common randomisation device* Z.

Proof. Let $(\xi_t) \in \mathcal{A}^\circ(\mathcal{F}_t^{\mathcal{J}})$ be the generating process and Z the randomisation device of τ. For $i \in \{1, \ldots, I\}$, let $(\xi_t^i) \in \mathcal{A}^\circ(\mathcal{F}_t^p)$ be as in Lemma B.1.1. Define $\tau_i = \inf\{t \in [0,T] : \xi_t^i > Z\}$. By Lemma B.1.1, $\tau_i \in \mathcal{T}^R(\mathcal{F}_t^p)$. Further, for every $\omega^p \in \Omega^p$, $\omega^s \in \Omega^s$, $t \in [0,T]$, setting i_0 to be such that $\mathcal{J}(\omega^s) = i_0$, we have

$$\sum_{i=1}^{I} I_{\{\mathcal{J}(\omega^s)=i\}} \tau_i(\omega^p) = \tau_{i_0}(\omega^p) = \inf\{t \in [0,T] : \xi_t^{i_0}(\omega^p) > Z\} = \tau(\omega^p, \omega^s),$$

which proves (B.2). □

B.2 Various technical lemmata

Lemma B.2.1. *Let* $(\rho^n)_{n\geq 1}$ *be a sequence of non-decreasing functions* $[0,T] \mapsto [0,1]$ *that is non-decreasing in n. Assume that for some* $t \in \mathbb{R}$ *the limit* $\lim_{n\to\infty} \lim_{u\uparrow t} \rho^n(u)$ *exists. Then the limit* $\lim_{u\uparrow t} \lim_{n\to\infty} \rho^n(u)$ *exists, and*

$$\lim_{n\to\infty} \lim_{u\uparrow t} \rho^n(u) = \lim_{u\uparrow t} \lim_{n\to\infty} \rho^n(u).$$

Proof. For any $n \geq 1$, we have

$$\lim_{u\uparrow t} \rho^n(u) = \sup_{u<t} \rho^n(u), \tag{B.3}$$

since ρ^n is a non-decreasing function. Similarly,

$$\lim_{n\to\infty} \sup_{u\leq t} \rho^n(u) = \sup_{n\geq 1} \sup_{u<t} \rho^n(u),$$

since the sequence $(\sup_{u<t} \rho^n)_{n\geq 1}$ is non-decreasing, which follows from the fact that $(\rho^n(u))_{n\geq 1}$ is non-decreasing for any $u < t$. Thus,

$$\lim_{n\to\infty} \lim_{u\uparrow t} \rho^n(u) = \sup_{n\geq 1} \sup_{u<t} \rho^n(u). \tag{B.4}$$

Further, observe that for any $n \geq 1$ the mapping $t \mapsto \sup_{u<t} \rho^n(u)$ is non-decreasing, since the mapping $t \mapsto \rho^n(u)$ is non-decreasing for any $u < t$. This and (B.3) allow us to deduce

$$\lim_{u \uparrow t} \lim_{n \to \infty} \rho^n(u) = \sup_{u<t} \sup_{n \geq 1} \rho^n(u). \tag{B.5}$$

On the other hand, by definition of supremum, for any $n \geq 1$, $u \leq t$, we have

$$\rho^n(u) \leq \sup_{n \geq 1} \sup_{u<t} \rho^n(u).$$

Taking double supremum on the left-hand side, we obtain

$$\sup_{u<t} \sup_{n \geq 1} \rho^n(u) \leq \sup_{n \geq 1} \sup_{u<t} \rho^n(u).$$

Note that the reverse inequality holds true by a symmetric argument, and therefore

$$\sup_{u<t} \sup_{n \geq 1} \rho^n(u) = \sup_{n \geq 1} \sup_{u<t} \rho^n(u),$$

which, combined with (B.4), (B.5), finishes the proof. □

Lemma B.2.2. *Let* $\chi : [0,T] \mapsto [0,1]$ *be a right-continuous non-decreasing function, and* $(\rho^n)_{n \geq 1}$ *be a sequence of non-decreasing right-continuous functions* $[0,T] \mapsto [0,1]$ *that is non-decreasing in n. Assume* $\rho^n(0-) = \chi(0-) = 0$ *and define*

$$R^n(t) = \int_{[0,t]} (1 - \chi(s-)) d\rho^n(s), \; n \geq 1. \tag{B.6}$$

Then the sequence $(R^n)_{n \geq 1}$ *is non-decreasing in n.*

Proof. Fix $t \in [0,T]$. Integrating by parts (Proposition 2.2.1) and using that $\rho^n(0-) = 0$, we obtain

$$R^n(t) = (1 - \chi(t))\rho^n(t) - \int_{[0,t]} \rho^n(s) d(1 - \chi(s)).$$

Observe that the mapping $t \mapsto (1 - \chi(t))$ defines a negative measure, and recall that $1 - \chi(t) \geq 0$. Since for any $n \geq 1$, $s \in [0,t]$ we have $\rho^{n+1}(s) \geq \rho^n(s)$ by assumption, it follows that

$$(1 - \chi(t))\rho^{n+1}(t) \geq (1 - \chi(t))\rho^n(t),$$

and

$$\int_{[0,t]} \rho^n(s) d(1 - \chi(s)) \geq \int_{[0,t]} \rho^{n+1}(s) d(1 - \chi(s)).$$

Thus, $R^{n+1}(t) \geq R^n(t)$, which finishes the proof. □

Lemma B.2.3. *Let $(\mathcal{F}_t)$ be a filtration satisfying the usual conditions. For any filtration $(\mathcal{G}_t) \subseteq (\mathcal{F}_t)$, the sets $\mathcal{A}(\mathcal{G}_t)$ and $\mathcal{A}_{ac}(\mathcal{G}_t)$ defined in Section 4.4 are convex in $\mathcal{S}$.*

Proof. Let us denote $\mathcal{A}(\mathcal{G}_t) = \mathcal{A}$ for brevity. Take $\rho, \chi \in \mathcal{A}$ with càdlàg representatives $\hat{\rho}, \hat{\chi}$, and let $\alpha, \beta \in [0,1] : \alpha + \beta = 1$. For $0 \leq t \leq s \leq T$ we have

$$\begin{aligned} \alpha\hat{\rho}_t + \beta\hat{\chi}_t &\leq \alpha\hat{\rho}_s + \beta\hat{\chi}_s, \\ 0 \leq \alpha\hat{\rho}_t + \beta\hat{\chi}_t &\leq \alpha + \beta = 1, \\ \alpha\hat{\rho}_T + \beta\hat{\chi}_T &= \alpha + \beta = 1. \end{aligned} \tag{B.7}$$

Moreover, the process $\alpha\hat{\rho} + \beta\hat{\chi}$ is càdlàg, because the processes $\hat{\rho}$, $\hat{\chi}$ are.

Let $\tilde{\rho}, \tilde{\chi}$ be some (not necessarily càdlàg) representatives of ρ, χ. Observe that the $\mathcal{S}$-norm of the difference $\alpha\tilde{\rho} + \beta\tilde{\chi} - (\alpha\hat{\rho} + \beta\hat{\chi})$ equals zero, because the $\mathcal{S}$-norms of the differences $\tilde{\rho} - \hat{\rho}$ and $\tilde{\chi} - \hat{\chi}$ both equal zero by definition of a representative of an element of $\mathcal{S}$. Thus, the process $\alpha\hat{\rho} + \beta\hat{\chi}$ is a càdlàg representative of $\alpha\rho + \beta\chi$. Then, by (B.7), we have $\alpha\rho + \beta\chi \in \mathcal{A}$, which proves the convexity of $\mathcal{A}$.

The proof of convexity of the set $\mathcal{A}_{ac}(\mathcal{G}_t)$ is analogous, the only difference is that it additionally uses the fact that for absolutely continuous $\hat{\rho}, \hat{\chi}$ their convex combination $\alpha\hat{\rho} + \beta\hat{\chi}$ is absolutely continuous. □

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