Lattices of congruence relations for inverse semigroups

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Abstract

The study of congruence relations is acknowledged as fundamental to the study of algebras. Inverse semigroups are a widely studied class for which congruences are well understood. We study one sided congruences on inverse semigroups. We develop the notion of an inverse kernel and show that a left congruence is determined by its trace and inverse kernel. Our strategy identifies the lattice of left congruences as a subset of the direct product of the lattice of congruences on the idempotents and the lattice of full inverse subsemigroups. This is a natural way to describe one sided congruences with many desirable properties, including that a pair is the inverse kernel and trace of a left congruence precisely when it is the inverse kernel and trace of a right congruence. We classify inverse semigroups for which every Rees left congruence is finitely generated, and provide alternative proofs to classical results, including classifications of left Noetherian inverse semigroups, and Clifford semigroups for which the lattice of left congruences is modular or distributive.

In the second half of this thesis we study the partial automorphism monoid of a finite rank free group action, which we denote by $G \wr \mathcal{I}_n$, where G is the group in question and n the rank. Congruences on $G \wr \mathcal{I}_n$ are described in terms a Rees congruence, subgroups of G^i for $1 \leq i \leq n$ and a subgroup of $G \wr \mathcal{S}_m$ for some $1 \leq m \leq n$. Via analysis of the subgroups arising in this description we show that, for finite G, the number of congruences on $G \wr \mathcal{I}_n$ grows polynomially in n with an exponent related to the chief length of G. We consider in detail $G \wr \mathcal{I}_n$ for finite simple groups; in this case we exactly describe the lattice of congruences. Finally, we describe one sided congruences on $G \wr \mathcal{I}_n$.

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Introduction

Inverse semigroups are one of the most important and widely studied classes of semigroups. The story of inverse semigroups begins in the Erlanger Programm; the opinion, advocated by Felix Klein in the nineteenth century, that every geometry may be regarded as the theory of invariants of a group of transformations. The existence of geometries with symmetries (or invariants) which do not form groups led to the search for a wider class of algebras which might fill the void. Via a series of abstractions of the concept of a group, the axiomatisation of inverse semigroups was reached by Wagner in 1952 ([78] in Russian, [29] in English), and independently by Preston in 1953 ([65]). Much has been written about the early development of the theory of inverse semigroups and we refer the reader to [28, Chapter 10] for a comprehensive review of their history.

The study of congruences is acknowledged as fundamental to understanding the structure of algebras since congruences determine homomorphic images, and one of the cornerstones of the approach to congruences on semigroups is the study of the lattice of congruences. There are well trodden paths for the hardy semigroup theorist to follow in order to get hold of congruences on inverse semigroups. In broad generality there are two routes to describe a congruence. The first uses the fact that a congruence on an inverse semigroup is completely determined by the equivalence classes of idempotents, and describes such collections of sets. The second makes use of the fact that a congruence is determined by its trace (the restriction to the idempotents), and its kernel (the union of the congruence classes containing idempotents). From a lattice perspective this second approach is advantageous, as the ordering of left congruences is induced by the natural orderings on the sets of kernels and traces.

Strongly related to the idea of a congruence is the notion of a one sided congruence. Whereas congruences determine homomorphic images, one sided congruences determine semigroup actions. In the first half of this thesis we study one sided congruences on inverse semigroups. Both approaches to describing congruences on inverse semigroups have been successfully applied to the case of one sided congruences. Following the first philosophy, in 1974 Meakin [46] developed left-kernel systems for inverse semigroups. These are exactly the collections of subsets that are the equivalence classes of a left congruence which contain idempotents. Following the second philosophy, in 1992 Petrich and Rankin [61] developed the kernel trace approach for one sided congruences on inverse semigroups.

Our approach is motivated by and builds upon these foundations. We introduce and characterise the *inverse kernel* for a left congruence on an inverse semigroup. This is a full inverse subsemigroup, and may be realised as the largest inverse subsemigroup contained in the kernel of the left congruence. We show that a left congruence can be recovered from its trace and inverse kernel, and we give necessary and sufficient conditions for an inverse subsemigroup and a congruence on the idempotents to be the inverse kernel and trace of a left congruence. For inverse semigroups there is a natural isomorphism between the lattices of left and right congruences. Our description of one sided congruences, which we term the inverse kernel approach, is closely connected with this isomorphism; a pair is the inverse kernel and trace of a left congruence if and only if it is the inverse kernel and trace of a right congruence. Using the inverse kernel approach the lattice of left congruences may be realised as a subset of the direct product of the lattice of congruences on the idempotents and the lattice of full inverse subsemigroups. This is a natural way to view left congruences, as the ordering on the left congruences coincides with the ordering in the direct product.

Of central importance in inverse semigroup theory is the symmetric inverse monoid $\mathcal{I}_X$ (or $\mathcal{I}_n$ when X is finite). Accepting that a group is the set of symmetries of a geometry, possibly the most important group theoretic result is the Cayley theorem, that every group is isomorphic to a subgroup of a symmetric group. The symmetric inverse monoid plays the role for inverse semigroups which the symmetric group takes within group theory. The analogous result is the *Wagner-Preston representation theorem* [31, Theorem 5.1.7], which states that every inverse semigroup may be embedded into a symmetric inverse monoid.

Monoids and semigroups similar to symmetric inverse monoids in derivation or structure are valuable and interesting objects of study. There are many directions in which it is possible to expand from symmetric inverse monoids, one productive and well plumbed vein is to regard $\mathcal{I}_n$ as a diagram monoid. Congruences on such monoids are well understood [13], [14], [16]. Natural generalisations of $\mathcal{I}_X$ arise from the partial automorphism monoids of independence algebras, a concept introduced as v^*-algebras in [53] and formulated in its modern style in [23] and [19]. Independence algebras are defined using generalised notions of linear independence and spanning sets, and include the classes of sets, vector spaces and free group actions. Of course, we can regard a set X as a universal algebra with no (basic) operations; viewed in this way it is an independence algebra and $\mathcal{I}_X$ is the partial automorphism monoid.

In the second half of this thesis we study the partial automorphism monoid of a finite rank free group action. Such monoids have a wreath product like structure and so we denote then by $G \wr \mathcal{I}_n$, where G is the group in question and n the rank. We build on general results concerning congruences on partial automorphism monoids of independence algebras due to Lima [42] and decompose congruences on $G \wr \mathcal{I}_n$ in terms a Rees congruence, an idempotent separating congruence and a congruence on a principal factor. We further describe idempotent separating congruences in terms a set of subgroups of G^i for $1 \leq i \leq n$, and the congruence on a principal factor in terms of a subgroup of $G \wr \mathcal{S}_m$ (the usual group theoretic wreath product of a group G with the symmetric group $\mathcal{S}_m$) for some $1 \leq m \leq n$. Our decomposition is compatible in a natural way with the ordering of congruences and allows us to efficiently describe the lattice of congruences.

Chapter 1 comprises an account of introductory ideas for inverse semigroups. Our particular focus is on congruences on inverse semigroups and how the kernel trace description may be utilised to give correspondence theorems between intervals in the lattice of congruences and suitable sets of subsemigroups. We describe the analogous results for one sided congruences and highlight the areas in which the descriptions of one sided congruences lack the strength of those for two sided congruences. The chapter is largely a survey of the concepts and ideas which we shall use throughout the thesis, particularly in Chapters 2, 3 and 4. We include examples throughout, and in particular we finish the chapter by introducing, and describing congruences on, four families of inverse semigroups upon which we shall regularly call. These are: Clifford semigroups, the bicyclic monoid, Brandt semigroups, and finite symmetric inverse monoids.

In Chapter 2 we develop the notion of an inverse kernel for a left congruence and show that a left congruence is determined by its trace and its inverse kernel. We argue that this is a natural way to describe left congruences and has many desirable properties, including that both the trace and inverse kernel maps are onto $\cap$-homomorphisms and that the sets of left congruences that share the same trace or share the same inverse kernel have minimum elements. It is known [46] that the set of left congruences which have the same trace is an interval in the lattice of left congruences; we prove a correspondence theorem between such an interval and the lattice of full inverse subsemigroups of a particular semigroup defined by the trace. Turning our attention more broadly to the lattice of left congruences we describe the meet and the join of left congruences in terms of the inverse kernel and trace. We consider the relationship between the lattice of left congruences and the direct product of the lattice of congruences on the idempotents and the lattice of full inverse subsemigroups. The former is a join-homomorphic image of the latter, and there is a meet-homomorphism from the latter to the former.

Chapter 3 is focused on examples. We describe left congruences on each of the examples introduced in the first chapter. Descriptions of one sided congruences on Clifford semigroups [61], the bicyclic monoid [54], [10], and Brandt semigroups [60] are known. In the first case the inverse kernel description for left congruences coincides with the kernel trace description, however in the following two cases the inverse kernel description offers a new method to describe one sided congruences. There was no description, of which I am aware, of one sided congruences on (finite) symmetric inverse monoids. Our results are technical, and the fundamental complexities result in our description perhaps not being "user-friendly." However, we are able to use it to give bounds for the growth of the number of left congruences on $\mathcal{I}_n$ in terms of n.

Concluding the first part of the thesis, Chapter 4 is concerned with applications of the inverse kernel approach to general inverse semigroups. We show that the usual kernel trace description of two sided congruences on inverse semigroups is an elementary application of the inverse kernel approach. We use our methodology to provide alternate proofs of classical results about the lattice of left congruences of an inverse semigroup, including when an inverse semigroup is left Noetherian [38], or when the lattice of left congruences on a Clifford semigroup is modular or distributive [18]. The first of these in particular is an immediate consequence of the identification of the lattice of left congruences as a subset of the direct product of the lattice of congruences on the idempotents and the lattice of full inverse subsemigroups. In an original contribution we also use our description to classify when a Rees left congruence is finitely generated and so classify those inverse semigroups for which every Rees left congruence is finitely generated.

Moving into the second half of the thesis, we again begin with preliminaries in Chapter 5. Here we introduce independence algebras and results from [42] concerning congruences on their partial automorphism monoids. We discuss in detail the partial automorphism monoid of a finite rank free group action and show that it has a wreath product like structure. Subgroups of direct and semidirect products of groups play an important role in our description of congruences on $G \wr \mathcal{I}_n$ and we introduce the approaches which have been used to describe such subgroups [2], [77], on which we later build.

Chapter 6 is the meat of the second part of the thesis. We decompose a congruence on $G \wr \mathcal{I}_n$ in terms of a Rees congruence, a subgroup of $G \wr \mathcal{S}_m$ for some $1 \leq m \leq n$ and a set of normal subgroups of G^i for $1 \leq i \leq n$ which are invariant under the action of $\mathcal{S}_i$ on their coordinates. We describe such invariant normal subgroups via a Goursat's Lemma style result, that is, in terms of a set of subgroups of G and a homomorphism between quotients of these subgroups. We obtain normal subgroups of $G \wr \mathcal{S}_m$ in terms of an invariant normal subgroup $K \trianglelefteq G^m$, a normal subgroup $Q \trianglelefteq \mathcal{S}_m$ and a homomorphism $Q \to G^m/K$. Using these descriptions we show that, for a finite group G, the number of congruences on $G \wr \mathcal{I}_n$ grows polynomially in n with an exponent related to the maximum length of a chain of normal subgroups of G.

In many ways Chapter 7 is a continuation of the previous chapter; considering two specialisations of the results from Chapter 6. First we examine the lattice of congruences on $G \wr \mathcal{I}_n$ when G is a finite simple group. The cases when G is abelian or non-abelian are distinct, so we consider these separately. In particular for a fixed n, for all non-abelian finite simple groups G the lattices of congruences on $G \wr \mathcal{I}_n$ are isomorphic. However, for each abelian finite simple group G the lattice of congruences on $G \wr \mathcal{I}_n$ is distinct. We provide diagrams of lattices of congruences on $G \wr \mathcal{I}_n$ for finite simple G and for small n. The second specialisation we consider is that of order preserving automorphisms; to do this we must endow a free G-act with a partial order. We show that, in a natural way, we may choose a partial order under which the monoid of order preserving partial automorphisms is isomorphic to $G \wr \mathcal{O}_n$, where $\mathcal{O}_n$ is the submonoid of $\mathcal{I}_n$ consisting of order preserving partial bijections. Congruences on $G \wr \mathcal{O}_n$ decompose in a similar way to congruences on $G \wr \mathcal{I}_n$, in terms of a Rees congruence and an idempotent separating congruence. Furthermore, we may again describe idempotent separating congruences in terms of a set of subgroups of G^i for $1 \leq i \leq n$.

Finally, in Chapter 8, we draw upon both parts of the thesis and utilise the inverse kernel approach to describe one sided congruences on $G \wr \mathcal{I}_n$. We relate left congruences on $G \wr \mathcal{I}_n$ to left congruences on $\mathcal{I}_n$ and on a particular Clifford submonoid of $G \wr \mathcal{I}_n$. Writing $G \wr \mathcal{P}_n$ for this submonoid, we show that full inverse subsemigroups of $G \wr \mathcal{I}_n$ may be described in terms of a full inverse subsemigroup $U \subseteq \mathcal{I}_n$, a full inverse subsemigroup $V \subseteq G \wr \mathcal{P}_n$ and a function $U \to V$.

Throughout the thesis we shall assume that any reader has a fairly strong background in algebra and in particular is familiar with semigroup theory.

However, the subject matter of this thesis is elementary in the sense that it allows us to start at the very beginning, which we do by defining a semigroup. We will endeavour to define all terms that are used, though a few may be missed. In general we follow standard algebraic and set theoretic notation and customs.

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Author's declaration

I declare that the work presented in this thesis, except where otherwise stated, is based on my own research carried out at the University of York and has not been submitted previously for any degree at this or any other university. Sources are acknowledged by explicit references.

The work presented in Chapter 2 and Sections 4.2 and 4.3 in Chapter 4 is available as a preprint:

M. Brookes. "The Lattice of One-Sided Congruences on an Inverse Semigroup". *preprint: arXiv:1901.11455* (2019).

The work presented in Chapter 6 is awaiting publication in the International Journal of Algebra and Computation:

M. Brookes. "Congruences on the Partial Automorphism Monoid of a Free Group Action". *International Journal of Algebra and Computation (to appear)* (2020).

Law of the Thesis

NOW this is the Law of the Thesis - as old and as true as the sky;
And the Student that keeps it may prosper, but the student that breaks it must die.

As the creeper that girdles the tree-trunk the viva brings forth the attack -
To find cracks in what you have written, so ensure what you've written won't crack.

Wash daily from nose-tip to tail-tip; drink deeply, but never too deep;
And remember the night is for research, and forget not the day is for sleep.

The Advisor may make a suggestion, but Cub, when thy whiskers are grown,
Remember that you are a Scholar - go forth with ideas of thine own.

Keep peace withe Lords of the Jungle - the Fellows, Profess'rs, and Dons.
And trouble not the Head of department, offend not academic liaisons.

When thought meets with thought in discussion, and neither withdraws from debate,
Quiet down, take a break to consider - is now a chance to collaborate.

Because of their age and their cunning, because of their gripe and their paw,
In all that the Law leaveth open, the word of your Advisor is Law.

Now these are the Laws of the Thesis, and many and mighty are they;
But the head and the hoof of the Law and the haunch and the hump is - OBEY!

— 1 —

Preliminaries

The first question that one has to ask when starting out into algebraic study is: "What are the rules that govern the system in which we find ourselves?" The formal posing of this question is the basis for the field of universal algebra, which at its heart seeks the most general formulations and theorems and describes commonalities between seemingly disparate subject matter.

Definition 1.0.1. Let $F = \{f_i \mid i \in I\}$ be a set of function symbols (or *operations*) where f_i has degree (or *arity*) $n_i \geq 0$. An *algebra* $\mathbf{A} = (A, F)$ is a set A together with a set of functions $\{f_i^A \colon A^{n_i} \to A \mid i \in I\}$. Algebras $\mathbf{A}$ and $\mathbf{B}$ are said to be of the same *type* if they arise from the same set of function symbols.

The second question that one asks is: "Given a structure how do we form new structures of the same type?" The answer to this question is usually threefold: first we may take a substructure, second we take multiple copies of the structure (a direct product) and third we define a function to a new set such that the operations are preserved. Very loosely the third of these options is what we touch upon in this thesis; the functions are of course homomorphisms.

Definition 1.0.2. Let $\mathbf{A} = (A, F)$ and $\mathbf{B} = (B, F)$ be algebras of the same type, and let $\theta \colon \mathbf{A} \to \mathbf{B}$ be a function. Then θ is a *homomorphism* if

$$(f^A(a_1, \ldots, a_n))\theta = f^B(a_1\theta, \ldots, a_n\theta)$$

for each $f \in F$ (with degree n say) and all $a_1, \ldots, a_n \in A$. The *image of* θ, written $\mathrm{Im}(\theta)$, is the set $\{a\theta \mid a \in A\} \subseteq B$. A homomorphism is an *isomorphism* if there exists an inverse, by which we mean a homomorphism $\theta^{-1} \colon \mathbf{B} \to \mathbf{A}$ such that $\theta\theta^{-1}$ is the identity function on A and $\theta^{-1}\theta$ is the identity function on B.

In universal algebra theory it is irresponsible to introduce the idea of a homomorphism without introducing the notion of the kernel of the homomorphism.

Definition 1.0.3. Let $\mathbf{A} = (A, F)$ and $\mathbf{B} = (B, F)$ be algebras of the same type, and let $\theta\colon \mathbf{A} \to \mathbf{B}$ be a homomorphism. The *kernel* of θ, written $\mathrm{Ker}(\theta)$, is defined by

$$\mathrm{Ker}(\theta) = \{(a, b) \in A \times A \mid a\theta = b\theta\}.$$

In the first half of this thesis at least we are not overly invested in or concerned with general universal algebra theory and the algebraic structures we are concerned with are semigroups.

Definition 1.0.4. A *semigroup* is a universal algebra (S, m) with just one binary operation $m\colon S \times S \to S$ where m is *associative*, by which we mean: for all $a, b, c \in S$

$$m(m(a, b), c) = m(a, m(b, c)).$$

As is usual we shall drop the notation for m and write ab for $m(a, b)$, we also simply say that S is a semigroup instead of (S, m) is a semigroup. We remark regarding functions of degree 0 that $S^0 = \{\emptyset\}$ so a function $f\colon S^0 \to S$ may be identified with the unique element in its image. We usually refer to degree 0 functions as *constants*. A *monoid* is a universal algebra $(S, m, 1)$ where (S, m) is a semigroup and $1\colon S^0 \to S$ is a constant such that for all $a \in S$

$$1a = a = a1.$$

A *group* is a universal algebra $(G, m, 1, ^{-1})$ where $(G, m, 1)$ is a monoid and $^{-1}\colon S \to S$ is a unary function such that for all $a \in G$

$$aa^{-1} = 1 = a^{-1}a.$$

In particular, when S, T are semigroups then a function $\theta\colon S \to T$ is a *(semigroup) homomorphism* if for all $a, b \in S$

$$(a\theta)(b\theta) = (ab)\theta.$$

If S, T are monoids and also $1_S\theta = 1_T$ then we say that θ is then a *(monoid) homomorphism*. If S, T are groups and $s^{-1}\theta = (s\theta)^{-1}$ for all $s \in S$ then we say θ is *(group) homomorphism*. We remark that if S, T are both groups then a semigroup homomorphism $S \to T$ is a group homomorphism; the analogue does not hold for monoids, though it is true that if S is a monoid then a semigroup homomorphism $\theta \colon S \to T$ is a monoid homomorphism from S to the image of θ. This justifies us not usually specifying to which type of homomorphism we refer.

On occasion we reach points when we want to pretend that all semigroups are monoids, usually because we want a property such as: for all $a \in S$ there is some $t \in S$ such that $ta = a$. Often we seek this for no other reason than it makes the statements of results easier, we can say "for all $s \in aS$" rather than "for $s = a$ or for $s \in aS$". To facilitate our pretence, a common construction is to adjoin an identity to a semigroup S. We write S^1 for S *with an identity adjoined* which we define as: if S is a monoid then $S^1 = S$, and if S is not a monoid then $S^1 = S \cup \{1\}$ (where $1 \notin S$) and $1s = s = s1$ for all $s \in S^1$. A similar operation is *adjoining a zero* to S, for which we write S^0, where S^0 is the set $S \cup \{0\}$ and $s0 = 0 = 0s$ for all $s \in S^0$. We note that when adjoining a zero to S we do not consider whether S already has a zero.

Homomorphisms naturally form a fundamental part of our understanding of universal algebras and semigroups and there is a huge body of work concerned with their study. Every introductory algebra text, undergraduate or graduate course spends a great deal of time in this area, thus there is little value in a general discussion in this work, there is nowt that I can say that ain't been said better before. The reader's attention is directed to Chapter 1 of [31] for a detailed introduction to the area in the field of semigroups and to [45] for a wide introduction to the study of algebras in general. Our introduction here by its nature bears some resemblance to both these, a fact we deem unavoidable. We shall be heavily invested in relations and relational structures, so we remind ourselves of the common terminology.

Definition 1.0.5. Let A be a set. A *binary relation* on S is a subset $\kappa \subseteq A \times A$. Standard properties which κ might satisfy include:

(i) *reflexivity*: $(a, a) \in \kappa$ for each $a \in A$;

(ii) *symmetry*: if $(a, b) \in \kappa$ then $(b, a) \in \kappa$;

(iii) *antisymmetry*: if $(a, b) \in \kappa$ and $(b, a) \in \kappa$ then $a = b$;

(iv) *transitivity*: if $(a, b) \in \kappa$ and $(b, c) \in \kappa$ then $(a, c) \in \kappa$.

For notation we shall write $a\ \kappa\ b$ and $(a, b) \in \kappa$ interchangeably as they have different strengths and flavours. If $B \subseteq A$ is a subset then we write $\kappa|_B$ for $\kappa \cap (B \times B)$, the *restriction* of κ to B.

Particularly important combinations of properties of relations are given combined names.

Definition 1.0.6. Let κ be a binary relation on a set A. Then κ is an *equivalence relation* if κ is reflexive, symmetric and transitive. The κ-class of a is

$$[a]_\kappa = \{b \in A \mid b\ \kappa\ a\}.$$

and on occasion we may swap to writing $a\kappa$ instead. We drop the subscript when this does not cause confusion. A subset $B \subseteq A$ is said to be *saturated* by κ if B is a union of κ-classes.

Definition 1.0.7. Let P be a set and let $\leq$ be a binary relation on P. Then $\leq$ is a *preorder on* P if it is reflexive and transitive. If in addition $\leq$ is antisymmetric then $\leq$ is a partial order and we say that P is a *partially ordered set* or *poset*.

A strongly related notion (one could say equivalent) to that of an equivalence relation is that of a partition. A *partition* of a set X is a set $\{A_i \mid i \in I\}$ of non-empty subsets of X such that $\bigcup_{i \in I} A_i = X$, and $A_i \cap A_j \neq \emptyset$ implies that $i = j$. Equivalence relations define partitions via taking the set of equivalence classes as our collection of subsets and partitions define equivalence relations by letting elements be related if they lie in the same part.

Definition 1.0.8. Let $\mathbf{A} = (A, F)$ be an algebra, and let κ be an equivalence relation on A. Then κ is a *congruence* on $\mathbf{A}$ if the following property holds. Let $f \in F$ have degree n, and let $a_1, \ldots, a_n, b_1, \ldots, b_n \in A$. If $a_i \ \kappa \ b_i$ for all $1 \leq i \leq n$ then $f(a_1, \ldots, a_n) \ \kappa \ f(b_1, \ldots, b_n)$.

Translated into the language of semigroup theory this has the following formulation.

Definition 1.0.9. Let S be a semigroup and $\rho \subseteq S \times S$ be an equivalence relation. Then ρ is a congruence on S if

$$(a, b),\ (c, d) \in \rho \implies (ac, bd) \in \rho.$$

Two congruences of particular import that deserve special mention and their own notation are the *identity relation* $\iota = \{(a, a) \mid a \in S\}$ and the *universal relation* $\omega = \{(a, b) \mid a, b \in S\}$.

Congruences on universal algebras are fundamental to the study of homomorphisms. Let $\mathbf{A} = (A, F)$ be an algebra and let ρ be a congruence on $\mathbf{A}$. Then we define the *quotient algebra* $\mathbf{A}/\rho$ as the set $\{[a]_\rho \mid a \in A\}$ with operations $F_\rho = \{f_\rho \mid f \in F\}$ where $f_\rho([a_1]_\rho, \ldots, [a_n]_\rho) = [f(a_1, \ldots, a_n)]_\rho$, which we note is well defined as ρ is a congruence. We call the function $\mathbf{A} \to \mathbf{A}/\rho$ defined by $a \mapsto [a]_\rho$ the *quotient map.*

Theorem 1.0.10 (The fundamental theorem of homomorphisms for universal algebras)**.** *Let* $\mathbf{A} = (A, F)$ *be an algebra and let* ρ *be a congruence on* $\mathbf{A}$. *Then the quotient algebra* $\mathbf{A}/\rho$ *is an algebra of the same type as* $\mathbf{A}$ *and the quotient map is a surjective homomorphism.*

Let $\mathbf{B}$ *be an algebra of the same type as* $\mathbf{A}$, *and let* $\theta \colon \mathbf{A} \to \mathbf{B}$ *be a homomorphism. Then* $\operatorname{Ker}(\theta)$ *is a congruence on* $\mathbf{A}$ *and* $\operatorname{Im}(\theta)$ *is a subalgebra of* $\mathbf{B}$. *Furthermore, the function* $\mathbf{A}/\rho \to \operatorname{Im}(\theta)$ *defined by* $[a]_\rho = a\theta$ *is an isomorphism between* $\mathbf{A}/\rho$ *and* $\operatorname{Im}(\theta)$.

As has become usual, we transfer the result to the language of semigroups. Given a semigroup S and a congruence ρ on S the *quotient semigroup* S/ρ is the set of ρ-classes $\{[a]_\rho \mid a \in S\}$ and multiplication is defined by $[a]_\rho[b]_\rho = [ab]_\rho$.

Theorem 1.0.11 (The fundamental theorem of homomorphisms for semigroups)**.** *Let S be a semigroup and let ρ be a congruence on S. Then the quotient semigroup S/ρ is a semigroup and the map $S \to S/\rho$ defined by $a \mapsto [a]_\rho$ is a surjective homomorphism.*

Let T be a semigroup and let $\theta\colon S \to T$ be a homomorphism. Then $\mathrm{Im}(\theta)$ *is a subsemigroup of T and* $\mathrm{Ker}(\theta)$ *is a congruence on S. Furthermore the function $S/\rho \to \mathrm{Im}(\theta)$ defined $[a]_{\mathrm{Ker}(\theta)} \mapsto a\theta$ is an isomorphism between S/ρ and* $\mathrm{Im}(\theta)$.

At this time we say goodbye to general universal algebra, we shall return briefly later in this chapter and again in Chapter 5. We turn to semigroup theory, with a smattering of lattice theory for good measure. The study of congruences on semigroups is crucial to the study of homomorphisms. Unlike in other widely studied areas in algebra (for instance group or ring theory) congruences on semigroups do not correspond with subsemigroups, so considering congruences is unavoidable when investigating homomorphisms. One significant theme throughout the history of research in semigroup theory has been to consider what is known as the lattice of congruences on a semigroup.

Let P be a partially ordered set under the ordering $\leq$ and let $Q \subseteq P$. We say that Q is *convex* if $x, y \in Q$ and $p \in P$ with $x \leq p \leq y$ implies $p \in Q$. For $a \leq b$ in P we define the *(closed) interval* as the set

$$[a, b] = \{p \in P \mid a \leq p \leq b\}.$$

The *open interval* is defined with the strict ordering $<$ (which is defined as $a < b$ when $a \leq b$ and $a \neq b$). We note that intervals are certainly convex.

Still with P a poset and $Q \subseteq P$, we say that $q \in Q$ is *minimal in Q* if for $x \in Q$

$$x \leq q \implies x = q,$$

in other words there is no element of Q that is strictly less than q. If in addition $q \leq x$ for all $x \in Q$ then q is said to be the *minimum* in Q. The notions of being *maximal* and *maximum* are defined dually. We say that $p \in P$ is a *lower bound* for Q if $p \leq q$ for all $q \in Q$, and if the set of lower bounds for Q has a maximum element then we say this the *greatest lower bound* or *meet* of Q, and we write $\bigwedge_{q\in Q} q$ for this element. Again the notion of a *least upper bound* or *join* for Q is defined dually and is written $\bigvee_{q\in Q} q$. If they exist then the meet or join are unique, and when $Q = \{a, b\}$ then we write $a \wedge b$ for the meet and $a \vee b$ for the join.

Definition 1.0.12. Let L be a partially ordered set under the ordering $\leq$. Then L is a *lattice* if $a \wedge b$ and $a \vee b$ exist for all $a, b \in L$. Sometimes we shall conform to universal algebra type notation and write $L = (L, \vee, \wedge)$ for the lattice and its operations. L is called *complete* if $\bigwedge_{i\in I} a_i$ and $\bigvee_{i\in I} a_i$ exist for arbitrary subsets $\{a_i \mid i \in I\} \subseteq L$.

The study of lattices is a rich and interesting area and shall play a significant supporting role throughout this work. For a comprehensive guide to all the lattice theory on which we shall call (and much more) the reader is directed to [45]. We shall introduce lattice ideas and notions throughout this thesis as and when they are relevant and needed, it seems somehow improper to devote significant time to the area at this point. There are however a few remarks that do warrant being made here.

If L is a lattice then both $\vee$ and $\wedge$ are binary operations on S and, as (for example) both $a \vee (b \vee c)$ and $(a \vee b) \vee c$ are least upper bounds for the set $\{a, b, c\}$, both $\vee$ and $\wedge$ are associative. Thus L is a semigroup under both $\vee$ and $\wedge$. A quick terminology note: when talking about algebras that have multiple operations (e.g. lattices) we may want to ignore some operations, this is usually called a *reduct*. In particular we may use functions that preserve (are homomorphisms) some operations but not others, in this case we specify which operations are preserved, for instance saying that $f\colon L \to L'$ is a $\wedge$-homomorphism if $(a \wedge b)f = af \wedge bf$ for all $a, b \in L$.

Definition 1.0.13. Let S be a semigroup. An element $a \in S$ is called *idempotent* if $a^2 = a$. The set of idempotents of a semigroup S is written $E(S)$.

We continue to let L be a lattice, it is immediate from the definitions that $a \vee a = a$ and $a \wedge a = a$, so under both $\vee$ and $\wedge$ all elements of L

are idempotent, so $E(L) = L$. Furthermore, it is clear for any $a, b \in L$ that (for example) both $a \vee b$ and $b \vee a$ are joins for $\{a, b\}$, in other words the semigroups $(L, \vee)$ and $(L, \wedge)$ are *commutative* (which, as usual, means that $ab = ba$ for all a, b). The following definition shall be of immense importance to us.

Definition 1.0.14. Let S be a semigroup such that $E(S) = E$ (every element is idempotent) and S is commutative. Then S is called a *semilattice*.

If E is a semilattice then the relation on E defined by $e \leq f$ if $ef = e$ for $e, f \in E$ is a partial ordering on E, and all pairs of elements have a meet in this partial order, the product of the elements. Furthermore, the standard result in this area is that semilattice may refer to either a commutative semigroup of idempotents or a poset such that all pairs of elements have a greatest lower bound, and there is a natural correspondence between the two definitions. If E is a semigroup semilattice then we know that the relation $e \leq f$ if $ef = e$ is a partial order on E, and under this partial order, with $e \wedge f = ef$, E is a poset semilattice. Conversely, if E is a poset semilattice then, with the operation $ef = e \wedge f$, E is a semigroup semilattice. Furthermore, if you move from a semigroup or poset semilattice to the alternate and then back again then you obtain the semilattice that you started with. Both viewpoints have benefits and are more useful at different times, and we shall use 'semilattice' to mean either viewpoint interchangeably. Sometimes people differentiate the semigroups $(L, \vee)$ and $(L, \wedge)$ calling the former an *upper (or join) semilattice* and the latter a *lower (or meet) semilattice*. Unless otherwise stated semilattice shall refer to a lower semilattice.

Given any collection of subsets $\mathcal{A} = \{A_i \mid i \in I\}$ of a set we may impose a partial order on $\mathcal{A}$ by $A_i \leq A_j$ whenever $A_i \subseteq A_j$. This is the *subset* or *inclusion ordering* on $\mathcal{A}$.

Definition 1.0.15. We define the *lattice of binary relations* $\mathfrak{BR}(S)$ on a semigroup S to be the set of all binary relations ordered by inclusion.

Since a binary relation is a subset $\kappa \subseteq S \times S$ and any such subset is a binary relation this lattice may simply be regarded as $\mathbb{P}(S \times S)$, where we use $\mathbb{P}(X)$ to mean the power set (set of subsets) of a set X. The operations $\vee$ and $\wedge$ for $\mathfrak{BR}(S)$ are then union $\cup$ and intersection $\cap$ respectively in $\mathbb{P}(S \times S)$. As is becoming a common remark we should mention that there is a heavy body of literature concerning the algebraic structures associated with the set of binary relations on a set. There are several definitions for composition of binary relations and each endows $\mathfrak{BR}(X)$ with an interesting set of properties (see [32]). This is outside the scope of this thesis, and if I wandered off on every tangent that appears then there would be no hope of finishing this within a readable length. Therefore we proceed with the important matters at hand. The only composition of relations which is relevant to us is the following. If $\kappa, \sigma \in \mathfrak{BR}(S)$ then we define the *composition of* κ *and* σ as

$$\kappa \circ \sigma = \{(a, b) \in S \times S \mid \exists c \in S,\ (a, c) \in \kappa,\ (c, b) \in \sigma\}.$$

Definition 1.0.16. Let S be a semigroup. Define $\mathfrak{ER}(S)$, the *lattice of equivalence relations* on S, as the set of equivalence relations on S ordered by inclusion. The *lattice of congruences* $\mathfrak{C}(S)$ is defined similarly, as the set of congruences on S ordered by inclusion.

The first thing to notice about the lattices we have just met is that as posets

$$\mathfrak{C}(S) \subseteq \mathfrak{ER}(S) \subseteq \mathfrak{BR}(S)$$

where the subset notation also means that the orderings are suborderings. The relationship between the three lattices is somewhat complicated (and as usual a full account can be found in [45]).

Given a binary relation κ on a semigroup S there is a minimum equivalence relation that contains κ. We construct this relation as follows. The first step is to add in all relations (a, a) for $a \in S$ to κ, the second step is to add in all relations (b, a) where $(a, b) \in \kappa$ and finally we take what is known as the *transitive closure* of a relation. For a (symmetric) relation κ the transitive closure of κ is the relation $\overline{\kappa}$ defined by $a\ \overline{\kappa}\ b$ if $a = b$ or there is a sequence $x_1, x_2, \ldots, x_n \in S$ (with $n \in \mathbb{N}$) such that $(a, x_1), (x_n, b) \in \kappa$, and $(x_i, x_{i+1}) \in \kappa$ for each $i = 1, \ldots, n-1$.

This is not limited to the lattices $\mathfrak{BR}(S)$ and $\mathfrak{ER}(S)$, given an equivalence relation κ on S there is also a smallest congruence on S which contains κ. Combining these ideas it is clear that for a binary relation on S there is a smallest congruence on S that contains this binary relation. If κ is our initial binary relation then this congruence is the *congruence on S generated by κ*. We construct this congruence now.

Definition 1.0.17. Let κ be a binary relation on S. Say that there is a *κ-sequence from a to b* in S if there is a sequence $x_1, \ldots, x_n, y_1, \ldots, y_n$ (for some $n \in \mathbb{N}$) such that $(x_i, y_i) \in \kappa$ or $(y_i, x_i) \in \kappa$ for each $1 \leq i \leq n$ and there are $u_1, \ldots, u_n, v_1, \ldots, v_n \in S^1$ (where S^1 is the semigroup S with an identity adjoined) such that

$$a = u_1 x_1 v_1,\ u_1 y_1 v_1 = u_2 x_2 v_2,\ u_2 y_2 v_2 = u_2 x_3 v_3,\ \ldots$$
$$\ldots, u_{n-1} y_{n-1} v_{n-1} = u_n x_n v_n,\ u_n y_n v_n = b.$$

The relation ρ defined by $a\ \rho\ b$ if: $a = b$ or there is a κ-sequence from a to b is a congruence on S, and is the congruence generated by κ. We write $\langle \kappa \rangle$ for this congruence, later (Section 1.4) we shall focus on one sided congruences and (for instance) write $\langle \kappa \rangle$ for the smallest left congruence containing κ. We endeavour to ensure that what we mean is clear from the context and, when there is confusion, we shall use subscripts to differentiate between the constructions to which we refer, for example $\langle \kappa \rangle_{ER}$ will be the equivalence relation generated by κ and $\langle \kappa \rangle_C$ the congruence generated by κ.

Now return to considering the set of three lattices of relations on S: $\mathfrak{C}(S) \subseteq \mathfrak{ER}(S) \subseteq \mathfrak{BR}(S)$. It is an exercise from a first course on algebra to show that if two binary relations κ and γ are equivalence relations then $\kappa \cap \gamma$ is also an equivalence relation. The same holds true for congruences, if $\kappa, \gamma \in \mathfrak{C}(S)$ then $\kappa \cap \gamma \in \mathfrak{C}(S)$. Thus the meet operation in $\mathfrak{C}(S)$ and $\mathfrak{ER}(S)$ is intersection just as is the case for $\mathfrak{BR}(S)$. In other words $\mathfrak{C}(S) \subseteq \mathfrak{ER}(S) \subseteq \mathfrak{BR}(S)$ as meet subsemilattices. On the other hand, in general, the union of two equivalence relations is not an equivalence relation and similarly the union of two congruences may not be a congruence. To realise the join in $\mathfrak{ER}(S)$ or $\mathfrak{C}(S)$ we instead take the equivalence relation or congruence generated by the union. However we do have the following relationship between the lattices $\mathfrak{C}(S)$ and $\mathfrak{ER}(S)$.

Theorem 1.0.18. *Let S be a semigroup and let ρ, σ be congruences on S. Then*

$$\langle \rho \cup \sigma \rangle_{ER} = \langle \rho \cup \sigma \rangle_{C}.$$

In particular, $\mathfrak{C}(S)$ is a sublattice of $\mathfrak{ER}(S)$ so the join $\rho \vee \sigma$ in $\mathfrak{C}(S)$ is the transitive closure of $\rho \cup \sigma$.

1.1 Green's Relations

It is impossible to go far into the world of structural semigroup theory without meeting Green's relations. We shall eschew the usual strategy to introducing Green's relations via ideals and take a roundabout approach. Just as one meets in a first course on group theory semigroups are often studied by how they 'act' on a set.

Definition 1.1.1. Let S be a semigroup and A a set. Then a function $\bullet \colon S \times A \to A$ is a *left S-action* or *S-act* if for all $s, t \in S$ and $a \in A$

$$s \bullet (t \bullet a) = (st) \bullet a.$$

We say that S *acts on the left on* A, and we usually drop the $\bullet$ notation. If S is a monoid then as usual we impose the additional restriction that $1a = a$ for all $a \in A$. Equivalently we can define actions on the right.

One possible motivation for the study of semigroups is the that they may be regarded as the most general algebraic structure that that can act in a "meaningful way". The condition of associativity is equivalent to the condition that the multiplication function defines an action. As our definition of semigroup action relies on the multiplication in the semigroup, to formalise the prior assertion we need a self referential definition of set acting on itself. This may be done by saying $f\colon A \times A \to A$ is an action if $((a,b)f,c)f = (a,(b,c)f)f$. This is merely an aside, we return to the normal definition of action and resume the main narrative.

Now if we were in the world of group theory we would define the orbits of the action, however this requires that if there is s such that $sx = y$ then there is t such that $ty = x$ which is not true for an arbitrary semigroup action. Instead for semigroups we can define an ordering, which is defined by the action. Let $\bullet\colon S \times A \to A$ be an S-act, then for $a, b \in A$ say that $a \lesssim_\bullet b$ if $a = b$ or there is $s \in S$ such that $sb = a$. It is straightforward that this is a preorder (which we recall means reflexive and transitive). From a preorder it is a standard operation to construct a equivalence relation, we say that $a \sim_\bullet b$ if $a \lesssim_\bullet b$ and $b \lesssim_\bullet a$. Further, we may define a partial order on the set of $\sim_\bullet$-classes: $[a]_{\sim_\bullet} \leq_\bullet [b]_{\sim_\bullet}$ if $a \lesssim_\bullet b$. Green's $\mathcal{L}$ relation can now be defined.

Definition 1.1.2. Let S be a semigroup and let $S \times S \to S$ be the left S-act defined as $(s,t) \mapsto st$ (we note that this may be regarded as either a left or a right action). Then *Green's $\mathcal{L}$ relation* (simply called the $\mathcal{L}$-relation) is the $\sim$-relation for this (left) action, so $s \mathrel{\mathcal{L}} t$ if there are $u, v \in S^1$ such that $us = t$ and $vt = s$. The *$\mathcal{L}$-order* $\leq_\mathcal{L}$ is the preorder defined $s \leq_\mathcal{L} t$ if there is u such that $us = t$, though we sometimes abuse terminology and use $\mathcal{L}$-order to mean the associated partial order on the equivalence classes (which is defined $[a]_\mathcal{L} \leq [b]_\mathcal{L}$ if $a \leq_\mathcal{L} b$). For an element $a \in S$ we write L_a for the $\mathcal{L}$-class containing a.

The standard way to introduce Green's relations is to define ideals of semigroups.

Definition 1.1.3. Let S be a semigroup and let $A \subseteq S$ be a subset. Then A is a *left ideal* of S if $S^1A \subseteq A$, and A is a *right ideal* of S if $AS^1 \subseteq A$. Finally A is a *(two sided) ideal* of S if A is a left ideal and a right ideal, or equivalently if $S^1AS^1 \subseteq A$.

The set $S^1a = \{sa \mid s \in S^1\}$ is called the *principal left ideal generated by a*; the principal right ideal generated by a is defined dually. Green's relations can be defined in terms of ideals, $a \mathrel{\mathcal{L}} b$ if $S^1a = S^1b$, and it is a standard exercise to show that the two definitions are equivalent. The $\mathcal{L}$-order in terms of ideals becomes $a \leq_{\mathcal{L}} b$ if $S^1a \subseteq S^1b$.

Definition 1.1.4. Let S be a semigroup. Then Green's relations (except for $\mathcal{L}$) are defined as follows

(i) *Green's $\mathcal{R}$-relation* is defined dually to the $\mathcal{L}$-relation, so $a \mathrel{\mathcal{R}} b$ precisely when $aS^1 = bS^1$; or equivalently when there are $u, v \in S^1$ such that $au = b$ and $bv = a$.

(ii) *Green's $\mathcal{H}$-relation* is defined as $\mathcal{H} = \mathcal{L} \cap \mathcal{R}$, the meet of the $\mathcal{L}$ and $\mathcal{R}$ relations in $\mathfrak{ER}(S)$.

(iii) *Green's $\mathcal{D}$-relation* is defined as $\mathcal{D} = \langle \mathcal{L} \cup \mathcal{R} \rangle_{ER}$, the join of the $\mathcal{L}$ and $\mathcal{R}$ relations in $\mathfrak{ER}(S)$.

(iv) *Green's $\mathcal{J}$-relation* is defined as $a \mathrel{\mathcal{J}} b$ when $S^1aS^1 = S^1bS^1$, or equivalently when there are $u, v, x, y \in S^1$ such that $a = ubv$ and $b = xay$.

We write R_a, H_a, D_a, J_a for the $\mathcal{R}, \mathcal{H}, \mathcal{D}, \mathcal{J}$-class of a. Also when we need to specify in which semigroup we are taking Green's relation we use, for example, $\mathcal{H}(S)$ for $\mathcal{H}$ on S.

Correspondingly we may define partial orders associated with $\mathcal{R}$ and $\mathcal{J}$. We say that $a \leq_{\mathcal{R}} b$ if $aS^1 \subseteq bS^1$ and $a \leq_{\mathcal{J}} b$ if $S^1aS^1 \subseteq S^1bS^1$.

We remark that this is not the only formulation of the $\mathcal{D}$-relation. It can be easily shown that the relations $\mathcal{L}$ and $\mathcal{R}$ commute, by which we mean that, for $a, b \in S$, if there is $c \in S$ such that $a \mathrel{\mathcal{R}} c \mathrel{\mathcal{L}} b$ then there is $d \in S$ such that $a \mathrel{\mathcal{L}} d \mathrel{\mathcal{R}} b$. In such a scenario the join of the equivalence relations is precisely the composition of the two relations, which we recall is $\mathcal{R} \circ \mathcal{L}$, the set of pairs (a, b) such that there is c such that $a \mathrel{\mathcal{R}} c$ and $c \mathrel{\mathcal{L}} b$. This formulation of the $\mathcal{D}$-relation, that $\mathcal{D} = \mathcal{R} \circ \mathcal{L}$, is equivalent to the previous formulation, and we shall use both henceforth. We remark that on a commutative semigroup all Green's relations agree.

Also of great importance is the result that if $e \in E(S)$ (which we recall says that e is idempotent) then H_e, the $\mathcal{H}$-class containing e, is a group with identity e. We call such $\mathcal{H}$-classes *group $\mathcal{H}$-classes*.

All of Green's relations shall crop up here and there throughout this thesis so it behoves us to introduce them here. We refrain from further comment at this point and direct the reader to the usual place for a fuller introduction, Chapter 2 of [31]. We look further at semigroup actions in Section 1.4.

Before we move on there is one further construction associated with ideals of a semigroup which we shall use. We know that in general congruences are not determined by substructures, however given an ideal of a semigroup there is a congruence which we construct from that ideal. If $A \subseteq S$ is an ideal then we define the *Rees congruence* on S by

$$\rho_A = \iota \cup \{(a, b) \mid a, b \in I\}$$

where we recall that ι is the identity relation. The quotient semigroup S/ρ_A is called the *Rees quotient* or *Rees factor*.

1.2 Inverse Semigroups

Semigroup theory is a phenomenally broad topic, encompassing as many different flavours as a Ben 'n' Jerry's store, so we specialise and impose further structure onto the semigroups that we shall be concerned with in this thesis. The root down which we travel is a familiar one, itself having many branching paths for us to carefully navigate. We shall be concerned with the theory of *inverse semigroups*.

Definition 1.2.1. A semigroup is said to be *inverse* if for each $a \in S$ there is a unique x such that $axa = a$ and $x = xax$. This element x is the *inverse* of a.

The study of inverse semigroups is motivated by the axiomatisation of sets of functions that arise as partial transformations of geometrical objects. For a good introduction to the area see Chapter 5 of [31] and for a more comprehensive accounting the reader is directed to [39]. A semigroup S is said to be *regular* if for each a in S there is $x \in S$ such that $a = axa$. This is a weaker condition than being inverse, so every inverse semigroup is regular. Idempotents (which we recall are elements $a \in S$ such that $a^2 = a$) play important roles in the study of regular and inverse semigroups. We notice that if $axa = a$ then ax is an idempotent so regular and inverse semigroups 'contain lots' of idempotents. In fact we may classify which regular semigroups are inverse.

Theorem 1.2.2 ([62]). *Let S be a semigroup. Then S is inverse if and only if S is regular and the idempotents in S commute.*

An inverse semigroup is an example of a *unary* semigroup, which is a semigroup that has an additional operation $a \mapsto a^\star$ such that this operation satisfies: $(a^\star)^\star = a$ and $(ab)^\star = b^\star a^\star$ for all $a, b \in S$. As the inverse of an element is unique it is common to write the inverse of $a \in S$ as a^{-1}. In particular the inverse map $a \mapsto a^{-1}$ obeys these rules, or equivalently is an anti-homomorphic involution (where an *involution* is a function $\theta\colon X \to X$ such that θ^2 is the identity, and an *anti-homomorphism* is a function $\theta\colon S \to T$ such that $(ab)\theta = (b\theta)(a\theta)$).

Since the idempotents of an inverse semigroup S commute, $E(S)$ necessarily forms a semilattice; it is immediate that $E(S)$ is a subsemigroup of S. We recall that on a semilattice there is a partial order defined by $f \leq e$ if $ef = f$. We can extend this partial order to the whole of S by setting $a \leq b$ if there is $e \in E(S)$ such that $ae = b$; on $E(S)$ this agrees with the original partial order. This partial order is known as the *natural ordering on S*. In fact there are many equivalent definitions for when $a \leq b$, and since we shall use several interchangeably later on there is value in giving a range of examples here.

Definition 1.2.3 (see [31, Proposition 5.2.1]). If S is an inverse semigroup then the *natural order* on S may be defined in any of the following ways. If $a, b \in S$ then $a \leq b$ if

(i) $\exists e \in E$ with $a = be$,

(ii) $\exists e \in E$ with $a = eb$,

(iii) $aa^{-1} = ab^{-1}$,

(iv) $a^{-1}a = b^{-1}a$,

(v) $a = ab^{-1}b$

(vi) $a = aa^{-1}b$,

(vii) $a = bb^{-1}a$.

(viii) $a = ab^{-1}a$,

Our final comments on the natural order relation on an inverse semigroup are to mention that it is compatible with the operations of both multiplication and taking inverse, so for $a, b, c, d \in S$

$$a \leq b \implies a^{-1} \leq b^{-1}$$
$$a \leq b,\ c \leq d \implies ac \leq bd.$$

On inverse semigroups many of Green's relations are closely related to the idempotents, if S is inverse then

$$a\ \mathcal{R}\ b \iff aa^{-1} = bb^{-1} \qquad \text{and} \qquad a\ \mathcal{L}\ b \iff a^{-1}a = b^{-1}b.$$

As $\mathcal{H}$ and $\mathcal{D}$ are defined in terms of $\mathcal{L}$ and $\mathcal{R}$ it follows that

$$\begin{aligned} & a\ \mathcal{H}\ b \iff aa^{-1} = bb^{-1} \text{ and } a^{-1}a = b^{-1}b, \\ \text{and} \quad & a\ \mathcal{D}\ b \iff \exists c \in S \text{ with } aa^{-1} = cc^{-1} \text{ and } b^{-1}b = c^{-1}c. \end{aligned}$$

It follows that on $E(S)$ the relation $\mathcal{D}(S)$ may be further refined to the statement that $e\ \mathcal{D}(S)\ f$ if and only if there is $a \in S$ such that $aa^{-1} = e$ and $a^{-1}a = f$. On a related note it is easy to see that principal left and right ideals of an inverse semigroup S are each in bijection with $E(S)$; the principal left ideals are Se for $e \in E(S)$ and the principal right ideals are eS for $e \in E(S)$ (we note that $e = ee$ which means $e \in eS$ and $e \in Se$ so $eS = eS^1$ and $Se = S^1e$).

The natural order on S is also closely tied with Green's relations. It is easy to see from the description of $\mathcal{R}$ on an inverse semigroup that in each $\mathcal{R}$-class there is a unique idempotent; the idempotent in R_a is aa^{-1}. The same holds true for the $\mathcal{L}$-classes, the unique idempotent in L_a is $a^{-1}a$. This can be taken as an alternative definition of an inverse semigroup, that every $\mathcal{L}$ and $\mathcal{R}$ class contains a unique idempotent. Restricting attention to $E(S)$ we note that $e \leq f$ (in the natural order) precisely when $e \leq_{\mathcal{R}(S)} f$ so the natural order on $E(S)$ is exactly the $\mathcal{R}(S)$-order and equivalently is the $\mathcal{L}(S)$-order. Furthermore, the order on the $\mathcal{R}$-classes, which we recall is given by $R_a \leq_{\mathcal{R}} R_b$ if $aS^1 \subseteq bS^1$, reduces to the partial order on the idempotents in the sense that $R_a \leq_{\mathcal{R}} R_b$ exactly when $aa^{-1} \leq bb^{-1}$ in the natural order. Dual results hold for the $\mathcal{L}$-order.

Green's relations on inverse semigroups are particularly useful in describing the structure of the semigroup (the strength is actually due to the regularity of the semigroup). One method by which Green's relations are used to describe semigroups is via what are known as *egg-box diagrams*. This is a diagram in which a semigroup S is first partitioned into $\mathcal{D}$-classes, and then each $\mathcal{D}$-class is partitioned into the $\mathcal{H}$-classes. Remembering that a $\mathcal{D}$-class is a union of both $\mathcal{L}$- and $\mathcal{R}$-classes and that a $\mathcal{H}$-class is the intersection of an $\mathcal{L}$-class and an $\mathcal{R}$-class we see that each $\mathcal{D}$-class has a "rectangular" structure. Each row is an $\mathcal{R}$-class, and each column is an $\mathcal{L}$-class. For an example of an egg-box diagram for a $\mathcal{D}$-class see Fig. 1.1.

For an inverse semigroup S the $\mathcal{D}$-classes have a square structure, each $\mathcal{R}$- and $\mathcal{L}$-class is R_e or L_e for some idempotent e which lies in the D-class. This means that each $\mathcal{H}$-class is the intersection of an $\mathcal{L}$-class and an $\mathcal{R}$-class each indexed by an idempotent, so for $a \in S$

$$H_a = R_{aa^{-1}} \cap L_{a^{-1}a} = \{b \in D_e \mid bb^{-1} = aa^{-1},\ b^{-1}b = a^{-1}a\}$$

Further, on any semigroup S we may define left and right translation maps $\phi_a \colon S \to S$ and $\theta_a \colon S \to S$ defined by $s \mapsto sa$ and $s \mapsto as$ respectively. When S is inverse these combine to give isomorphisms between the group $\mathcal{H}$-classes (which we recall are $\mathcal{H}$-classes which contain idempotents). Explicitly for each $a \in S$ the composition $\theta_{a^{-1}}\phi_a$ is an isomorphism $H_{aa^{-1}} \to H_{a^{-1}a}$ upon restriction to $H_{aa^{-1}}$. The left and right translation maps are shown in Fig. 1.1.

There is a huge amount that one could say about motivation for, and elegant results and structure theorems that arise in the study of inverse semigroups. However very little is directly relevant to this thesis so we

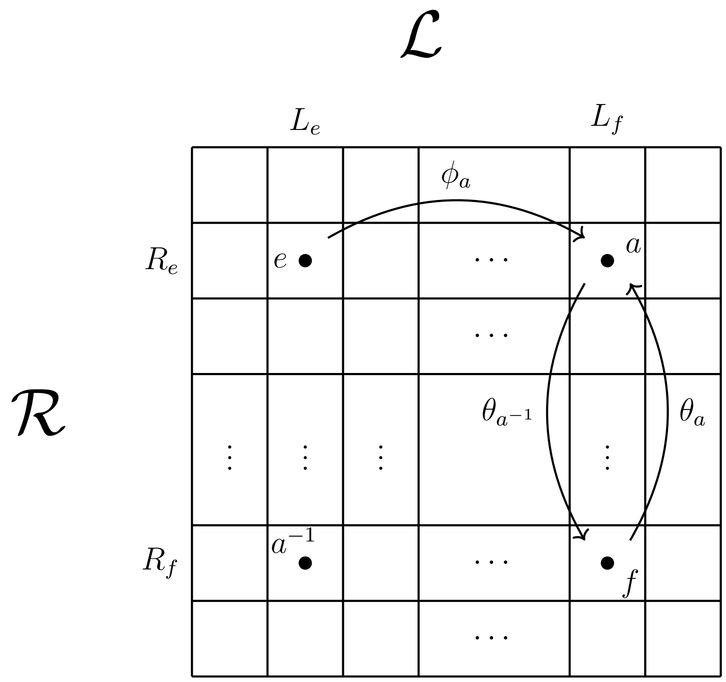


Figure 1.1: The egg-box diagram for a $\mathcal{D}$-class

restrict ourselves and halt after this very brief general introduction. For a longer introduction we recommend [31, Chapter 5], for a comprehensive survey we advise [58] and for a "big-picture" accounting we direct attention to [39].

1.3 Congruences on Inverse Semigroups

As this thesis is primarily focused on looking at (one and two sided) congruence lattices for inverse semigroups it is logical for a goodly portion of the preliminaries chapter to focus on introducing what is known about congruences on inverse semigroups and describing the approaches to getting hold of the congruence lattices. This we shall proceed with presently, and this is the first location where we do not direct the reader to [31] for a fuller accounting. While [31, Section 5.3] describes congruences on inverse semigroups and forms a useful background we shall go into more detail here. There is large overlap with this section and the content of [58, Chapter 3], which comprises a formal summary of the area. In this section we attempt to provide the motivation for the descriptions of congruences and to emphasise the connections between the lattice of congruences and lattices of substructures which these descriptions imply. Aside from the rare occasion where we see value in providing a proof we continue to direct the reader to original source material for the results we summarise. From this point on we let S be an inverse semigroup.

Inverse semigroups are - with a high degree of justification - often said to be the class of semigroups that is closest to groups and a significant theme in the study of inverse semigroups is the generalisation of group theoretic results to this wider class. Probably the first significant result any undergraduate meets in an algebra course is the fundamental theorem of homomorphisms for groups, which is a special case of the result for semigroup theory (Theorem 1.0.11), which in turn is a special case of the general result for universal algebra (Theorem 1.0.10). For group theory of course this result has a particularly elegant form. We recall that given a group G and a normal subgroup $N \trianglelefteq G$, the set of cosets $\{gN \mid g \in G\}$ forms a group with multiplication $(gN)(hN) = (gh)N$; this group is written G/N.

Theorem 1.3.1 (Fundamental theorem of homomorphisms for groups)**.** *Let G and H be groups and let $\theta\colon G \to H$ be a homomorphism. Let $N = \{g \in G \mid g\theta = 1\}$. Then N is normal in G and $G/N \cong \mathrm{Im}(\theta)$ via the function $gN \mapsto g\theta$.*

Since all groups are semigroups this must agree with Theorem 1.0.11 (the fundamental theorem of homomorphisms for semigroups) which says that $\mathrm{Im}(\theta) \cong G/\operatorname{Ker}(\theta)$, and of course the two do agree. If $\theta\colon G \to H$ is a homomorphism and $N = \{g \in G \mid g\theta = 1\}$ then the equivalence classes of $\operatorname{Ker}(\theta)$ are the cosets of N, in other words if $a\theta = b\theta$ then $aN = bN$. Thus for groups congruences are in bijection with normal subgroups. Recalling that $\mathfrak{C}(S)$ is the lattice of congruences on S and writing $\mathfrak{N}(G)$ for the lattice of normal subgroups of G, the following is the explicit correspondence between

$\mathfrak{C}(G)$ and $\mathfrak{N}(G)$:

$$\mathfrak{C}(G) \longrightarrow \mathfrak{N}(G); \qquad \rho \mapsto \{g \in G \mid (g, 1) \in \rho\},$$
$$\mathfrak{N}(G) \longrightarrow \mathfrak{G}(G); \qquad N \mapsto \{(g, h) \in G \times G \mid gh^{-1} \in N\}.$$

It is an easy question to ask as to whether this elegant form for the fundamental homomorphism theorem extends to the case of inverse semigroups. The short answer to this question is 'somewhat', the glib answer is 'what do you mean by extends?' and the long answer is what we devote our time to here. Actually the glib answer touches upon an important issue, "what do we actually mean by extends". We cannot say that the result holds for inverse semigroups as stated, but there are multiple ways in which we can consider an extension. We shall detail two approaches. The first describes a collection of sets of inverse subsemigroups that is in correspondence with the collection of congruences. The second describes congruences in terms of pairs (τ, K) where τ is a congruence on $E(S)$ and K is a subsemigroup of S.

Strategies to describe congruences on inverse semigroups make use of the fact that a congruence is determined by the classes that contain the idempotents, a fact observed in both of the foundational papers for inverse semigroups, Wagner's [78] (see [29] for an English translation) and Preston's [62]. In other words if ρ is a congruence on an inverse semigroup S then, from any $(a, b) \in \rho$ we can determine a set of elements of $S \times S$, each of which contain at least one idempotent, and from these we can again deduce the initial relation (a, b). Indeed, we suppose that $a \; \rho \; b$ then we note that

$$a^{-1}a \; \rho \; a^{-1}b, \; b^{-1}a \; \rho \; b^{-1}b, \;\text{ and }\; aa^{-1} \; \rho \; ba^{-1}.$$

We then observe

$$a^{-1}a \; \rho \; a^{-1}b = (a^{-1}b)(b^{-1}b) \; \rho \; (a^{-1}a)(b^{-1}b)$$

and

$$b^{-1}b \; \rho \; b^{-1}a = (b^{-1}a)(a^{-1}a) \; \rho \; (b^{-1}b)(a^{-1}a).$$

Thus we have that $a \; \rho \; b$ implies that

$$aa^{-1} \; \rho \; ba^{-1} \quad \text{and} \quad a^{-1}a \; \rho \; b^{-1}b.$$

Conversely, if $aa^{-1} \; \rho \; ba^{-1}$ and $a^{-1}a \; \rho \; b^{-1}b$ then

$$a = (aa^{-1})a \; \rho \; (ba^{-1})a = b(a^{-1}a) \; \rho \; bb^{-1}b = b.$$

The question of describing congruences on inverse semigroups becomes that of describing the classes of the idempotents. Before we start our in depth look at this field we make an important remark. We know that in the case of a congruence ρ on a group if $a \; \rho \; b$ then $a^{-1} \; \rho \; b^{-1}$, and this fact extends to inverse semigroups. Indeed, suppose that ρ is a congruence on an inverse semigroup S and that $a \; \rho \; b$. Then we have just seen that this implies that $a^{-1}a \; \rho \; b^{-1}b$ and that $a^{-1}b \; \rho \; a^{-1}a$, and similarly we may show that $aa^{-1} \; \rho \; bb^{-1}$. We then observe

$$a^{-1} = a^{-1}(aa^{-1}) \; \rho \; a^{-1}(bb^{-1}) = (a^{-1}b)b^{-1} \; \rho \; (a^{-1}a)b^{-1} \; \rho \; (b^{-1}b)b^{-1} = b^{-1}.$$

Further, if $\theta \colon S \to T$ is a semigroup homomorphism and T is inverse then it is easy to verify that $a^{-1}\theta$ is the inverse to $a\theta$, in other words $(a^{-1})\theta = (a\theta)^{-1}$. In addition, the idempotents in $\mathrm{Im}(\theta)$ are precisely the images of idempotents in S under θ. Indeed, suppose that $(a\theta)^2 = (a\theta)$. Then certainly $(a\theta)^{-1} = a\theta$, so

$$a^{-1}\theta = (a\theta)^{-1} = a\theta = (a\theta)^2 = (a^{-1}\theta)^2.$$

Then we observe that

$$a\theta = (aa^{-1}a)\theta = (a\theta)(a^{-1}\theta)(a\theta) = (a\theta)(a^{-1}\theta)^2(a\theta) = ((aa^{-1})(a^{-1}a))\theta$$

and $(aa^{-1})(a^{-1}a) \in E(S)$, so $a\theta$ is the image of an idempotent.

We may now remark that the homomorphic image of an inverse semigroup (under a semigroup homomorphism) is itself an inverse semigroup. It is natural to regard inverse semigroups as a separate type of universal algebra from semigroups, and so to consider inverse semigroup homomorphisms in the sense of homomorphism of universal algebras. However, the above argument implies that semigroup homomorphisms on inverse semigroups are inverse semigroup homomorphisms so from the point of view of homomorphisms it makes no difference whether we consider semigroup or inverse semigroup homomorphisms. We now turn our attention to our discussion of the varied approaches to congruences on inverse semigroups.

Definition 1.3.2. Let S be an inverse semigroup and let K be a subsemigroup of S. We say that K is *full* if $E(S) \subseteq K$, and that K is *self conjugate* if for each $a \in S$ we have $aKa^{-1} \subseteq K$. We write $\mathfrak{K}(S)$ for the lattice of full subsemigroups of S, and $\mathfrak{V}(S)$ for the lattice of full inverse subsemigroups of S.

Definition 1.3.3 (see [62])**.** Let $\mathcal{A} = \{A_i \mid i \in I\}$ be a set of disjoint inverse subsemigroups of an inverse semigroup S and let $E_i = E(A_i)$. Then $\mathcal{A}$ is *kernel normal system* for S if it satisfies the conditions:

(N1) $E(S) = \bigcup_{i \in I} E_i$;

(N2) for all $i, j \in I$ there is $l \in I$ such that $E_i E_j \subseteq E_l$;

(N3) for all $i \in I$ and $a \in S$ there is $j \in I$ such that $aE_i a^{-1} \subseteq E_j$;

(N4) if $aa^{-1}, bb^{-1}, a, ab^{-1} \in A_i$ then $b \in A_i$;

(N5) if $aa^{-1}, bb^{-1}, ab^{-1} \in A_i$ then for each $j \in I$, $aA_j b^{-1} \subseteq A_l$ where $aE_j a^{-1} \subseteq E_l$.

Kernel normal systems shall be the sets which are the congruence classes containing idempotents. We remark that the characterisation given in Definition 1.3.3 is not unique, the definition given here is the original due to Preston [62]. Later versions, such as that in [58, III.1.7], often use fewer conditions to describe such systems. We also remark that a similar result to that of Preston may be found in Wagner's original paper on inverse semigroups [78, Theorem 3.7], which states that a congruence is determined by the classes which contain idempotents.

Theorem 1.3.4 ([62, Theorem 1], see also [78, Theorem 3.7])**.** *Let S be an inverse semigroup and let $\mathcal{A} = \{A_i \mid i \in I\}$ be a kernel normal system for S. Then $\rho(\mathcal{A})$, which is defined by*

$$\rho(\mathcal{A}) = \{(a, b) \in S \times S \mid \exists i \in I \text{ such that } aa^{-1}, bb^{-1}, ab^{-1} \in A_i\}$$

is a congruence on S.

Conversely, if ρ is a congruence on S then $\mathcal{A}(\rho) = \{[e]_\rho \mid e \in E(S)\}$, the set of ρ-classes which contain idempotents, is a kernel normal system for S.

Furthermore $\mathcal{A}(\rho(\mathcal{A})) = \mathcal{A}$ and $\rho(\mathcal{A}(\rho)) = \rho$.

In other words congruences on inverse semigroups exactly correspond with kernel normal systems. Thus we may define congruences in terms of sets of substructures of inverse semigroups. In fact when S is a group then if $\mathcal{A}$ is a kernel normal system then $\mathcal{A}$ has exactly one element, and, as an inverse subsemigroup of a group is a subgroup, the kernel normal system is a subgroup of S. Moreover, in this case each of the conditions (N1)-(N4) is trivial and (N5) is equivalent to the normality of the subgroup. Therefore, on groups, the correspondence between congruences and kernel normal systems becomes the usual one between congruences and normal subgroups.

In many ways the second approach to congruences on inverse semigroups grows out of the kernel normal system description. To motivate this idea we make a series of observations about a kernel normal system $\{A_i \mid i \in I\}$. First we notice that (N1) and (N2) are equivalent to the statement that the set $\{E_i \mid i \in I\}$ is a partition of $E(S)$ and the equivalence relation that this defines is a congruence on $E(S)$.

The second observation that we make is that the union of the A_i is a subsemigroup of S. If we take as understood that the elements in A_i are precisely those that are ρ-related to an idempotent for a congruence ρ then this is immediate as, if $a \;\rho\; e$ and $b \;\rho\; f$ for idempotents e and f, then $ab \;\rho\; ef$ and as ef is idempotent it follows that $ab \in A_j$ for some j. Of course it must be possible to deduce that the union of the A_i is a subsemigroup directly from the abstract characterisation of kernel normal systems, however this is a slightly more involved calculation. We include it for completeness. Suppose that $x \in A_i$ and $y \in A_j$, then as A_i and A_j are inverse subsemigroups certainly $xx^{-1}, x^{-1}x \in A_i$ and $yy^{-1}, y^{-1}y \in A_j$. With $a = x = b$ in (N5) we have that there is $l \in I$ with $xA_jx^{-1} \subseteq A_l$ and l is such that $xE_jx^{-1} \subseteq E_l$. In particular, $xy^{-1}x^{-1} \in A_l$ and $xy^{-1}yx^{-1} \in E_l$. Similarly with $a = x$ and $b = x^{-1}x$ in (N5) we have that there is $k \in I$ with

$xA_jx^{-1}x \subseteq A_k$ where $xE_jx^{-1} \subseteq E_k$. However we know that $xE_jx^{-1} \subseteq E_l$ and as the E_i partition $E(S)$ it follows that $l = k$. Therefore $xA_jx^{-1}x \subseteq A_l$. In particular, $x(y^{-1}y)x^{-1}x = xy^{-1}y \in A_l$. We can then apply (N4) with $a = xy^{-1}y$ and $b = xy$, noting that

$$aa^{-1} = xy^{-1}yx^{-1} \in E_l, \quad bb^{-1} = xyy^{-1}x^{-1} \in E_l,$$
$$\text{and} \quad ab^{-1} = xy^{-1}yy^{-1}x^{-1} = xy^{-1}x^{-1} \in A_l.$$

Thus $b = xy \in A_l$ so in particular xy is in the union of the A_i so this union is a subsemigroup. As each A_i is inverse it follows that $\bigcup_{i\in I} A_i$ is inverse and from (N5) it is easily seen that $\bigcup_{i\in I} A_i$ is self conjugate.

The pair of observations we have just made motivates the following definitions.

Definition 1.3.5. Let ρ be a congruence on an inverse semigroup S. The *kernel* of ρ - written $\ker(\rho)$ - is the set of elements that are related to idempotents:

$$\ker(\rho) = \{a \in S \mid \exists e \in E(S) \text{ with } a \ \rho \ e\} = \bigcup_{e\in E(S)} [e]_\rho.$$

Definition 1.3.6. Let ρ be a congruence on an inverse semigroup S and let $E = E(S)$. The *trace* of ρ - written $\mathrm{trace}(\rho)$ - is the restriction of ρ to the idempotents:

$$\mathrm{trace}(\rho) = \rho \cap (E \times E).$$

The use of kernel to for the union of the congruence classes that contain idempotents is a little awkward as the congruence itself is the kernel of a homomorphism. To attempt to reduce confusion we differentiate our notation: $\mathrm{Ker}(\bullet)$ shall mean the kernel of a homomorphism, and $\ker(\bullet)$ shall be the kernel of a congruence - the union of the classes with idempotents.

The important observation to make regarding the trace of ρ, a congruence on S, is that $\mathrm{trace}(\rho)$ is a congruence on $E(S)$. Congruences on semilattices have the property that every congruence class is a convex subsemilattice, so in particular each equivalence class of $\mathrm{trace}(\rho)$ is a convex subsemilattice of $E(S)$.

The second type of description for congruences on inverse semigroups describes congruences in terms of their kernel and trace.

Definition 1.3.7. Let S be an inverse semigroup and let τ be a congruence on $E(S)$. Then τ is *normal* if for each $a \in S$, when $e \ \tau \ f$,

$$aea^{-1} \ \tau \ afa^{-1}.$$

We write $\mathfrak{C}_N(E)$ for the lattice of normal congruences on E, it is elementary that this is a sublattice of $\mathfrak{C}(E)$.

Definition 1.3.8 ([26])**.** Let S be an inverse semigroup and K a subsemigroup of S. We say that K is *normal* if K is full, inverse and for any $x, y \in S$ we have that $k \in K$ and $xy \in K$ implies that $xky \in K$. Let $\mathfrak{N}(S)$ be the lattice of normal subsemigroups of S.

We remark that if $K \subseteq S$ is a normal subsemigroup then K is certainly self conjugate. Indeed, as K is full, we notice that $aa^{-1} \in K$ for all $a \in S$, and it follows that $aKa^{-1} \subseteq K$ so K is self conjugate. On the other hand, a full self conjugate inverse subsemigroup is not necessarily normal. Indeed, the Brandt semigroup (see Section 1.5) $S = \mathcal{B}(\{1, 2, 3\}, 1)$ has every full inverse subsemigroup self conjugate but the only normal subsemigroups are $E(S)$ and S (see [58, Example III.4.10] for the details of this example).

There are several versions of the kernel trace description of congruences on inverse semigroups. We first give three of the potential definitions which characterise the trace and kernel of a congruence. Each of these definitions is equivalent, which justifies the same term being used in all three. Following this we state the theorem in which we may choose any of these options as the definition of a congruence pair.

Definition 1.3.9 ([57, Definition 4.2])**.** Let S be an inverse semigroup and let $K \subseteq S$ be a full self conjugate inverse subsemigroup. Let τ be a normal congruence on $E(S)$. Then (τ, K) is called a *congruence pair for S* if

(CP1) $ae \in K$ and $e \ \tau \ a^{-1}a$ implies that $a \in K$;

(CP2) $a \in K$ implies that $a^{-1}ea \ \tau \ a^{-1}ae$ for all $e \in E(S)$.

Definition 1.3.10 ([31, Section 5.3])**.** Let S be an inverse semigroup and let $K \subseteq S$ be a full self conjugate inverse subsemigroup. Let τ be a normal congruence on $E(S)$. Then (τ, K) is called a *congruence pair for S* if

(CP1) $ae \in K$ and $e \; \tau \; a^{-1}a$ implies that $a \in K$;

(CP3) $a \in K$ implies that $a^{-1}a \; \tau \; aa^{-1}$.

Definition 1.3.11 ([26, Proposition 3.9])**.** Let S be an inverse semigroup and let $K \subseteq S$ be a normal subsemigroup. Let τ be a normal congruence on $E(S)$. Then (τ, K) is called a *congruence pair for S* if

(CP3) $a \in K$ implies that $a^{-1}a \; \tau \; aa^{-1}$;

(CP4) for any $x, y \in S^1$ and $e, f \in E(S)$ if $xey \in K$ and $e \; \tau \; f$ then $xfy \in K$.

Theorem 1.3.12 ([57, Theorem 4.4],[31, Theorem 5.3.3],[26, Proposition 3.9])**.** *Let S be an inverse semigroup and let ρ be a congruence on S. Then* $(\mathrm{trace}(\rho), \ker(\rho))$ *is a congruence pair for S. Conversely, if* (τ, K) *is a congruence pair then*

$$\rho(\tau, K) = \{(a, b) \in S \times S \mid a^{-1}a \; \tau \; b^{-1}b, \; ab^{-1} \in K\}$$

is a congruence on S. Moreover,

$$\ker(\rho(\tau, K)) = K, \quad \mathrm{trace}(\rho(\tau, K)) = \tau \quad \textit{and} \quad \rho(\mathrm{trace}(\rho), \ker(\rho)) = \rho.$$

As previously remarked the definitions of a congruence pair are equivalent, so it is possible to move directly between the abstract characterisations in each of the definitions for a congruence pair. However this is perhaps more work than proving the theorem in all three cases, and as it involves nothing more than technical manipulation there is little benefit to its inclusion here. All three versions of this result may be proved in the same way, by directly showing that the kernel and trace of a congruence satisfy (whichever set of) the conditions, and then showing that the relation defined from a congruence pair is a congruence. The first direction is straightforward in every case, each of (CP1)-(CP4) are easily verified. The reverse direction is harder but still generally straightforward, the different sets of conditions come from slight variations in the strategies used to show that $\rho(\tau, K)$ is a congruence.

The method in which the kernel trace description is often applied, and how we shall commonly use these ideas, is that the usual ordering on the congruences - that of inclusion as subsets of $S \times S$ - coincides with the obvious ordering of the direct product $\mathfrak{C}_N(S) \times \mathfrak{N}(S)$ (recalling $\mathfrak{C}_N(E)$ is the lattice of normal congruences on E and $\mathfrak{N}(S)$ is the lattice of normal subsemigroups of S).

Corollary 1.3.13 ([58, Proposition III.2.3]). *Let S be an inverse semigroup and let ρ_1, ρ_2 be congruences on S. Then $\rho_1 \subseteq \rho_2$ if and only if*

$$\mathrm{trace}(\rho_1) \subseteq \mathrm{trace}(\rho_2) \qquad \textit{and} \qquad \ker(\rho_1) \subseteq \ker(\rho_2).$$

In particular, we have that $\rho_1 = \rho_2$ if and only if

$$\mathrm{trace}(\rho_1) = \mathrm{trace}(\rho_2) \qquad \textit{and} \qquad \ker(\rho_1) = \ker(\rho_2).$$

In fact, none of the previously mentioned results can be regarded as the first foray into the notion of the kernel trace approach. This honour goes to Scheiblich [72], whose brief note on the topic is a bridge between the kernel normal system and the kernel-trace approaches. Many of the later versions can be largely deduced from this work though are usually more refined and easier to use. We include the statement here for a more complete account of the development of the area, as well as to introduce a definition that we shall assume familiarity with henceforth.

Definition 1.3.14. Let S be an inverse semigroup with semilattice of idempotents E. The *closure* of a subset $X \subseteq S$ is the set

$$X\omega = \{s \in S \mid \exists e \in E \ \text{ with } \ se \in X\} = \{s \in S \mid \exists x \in X \ \text{ with } \ x \leq s\}.$$

A subset X is said to be *closed* if $X\omega = X$.

We proceed with the early version of the kernel trace approach from [72]. Let S be an inverse semigroup with semilattice of idempotents $E = E(S)$

and let $F \subseteq E$ be a subsemilattice. Then define

$$M_F = \{a \in S \mid aa^{-1}, a^{-1}a \in F, \text{ and } \forall e \in F\ aea^{-1}, a^{-1}ea \in F\}.$$

The following lemma is a specific application of a result for regular semigroups to the inverse case, we include a proof for completeness.

Lemma 1.3.15 (see [69, Theorem 1.5])**.** *Let S be an inverse semigroup with semilattice of idempotents $E = E(S)$, and let $F \subseteq E$ be a subsemilattice. Then M_F (as defined above) is the largest inverse subsemigroup of S which has semilattice of idempotents F.*

Proof. Suppose that $V \subseteq S$ is an inverse subsemigroup such that $E(V) = F$. If $a \in V$ then $aa^{-1}, a^{-1}a \in E(V)$ so $aa^{-1}, a^{-1}a \in F$. If also $e \in F$ then $e \in V$ so $aea^{-1}, a^{-1}ea \in V$ so $aea^{-1}, a^{-1}ea \in F$. Therefore $V \subseteq M_F$, so to complete the proof it suffices to show that M_F is an inverse subsemigroup of S and that $E(M_F) = F$.

First we observe that certainly $E(M_F) = F$. Indeed, if $e \in E(M_F)$ then $e = ee^{-1} \in F$, whence it follows that $E(M_F) \subseteq F$. For the reverse inclusion we observe that if $f \in F$ then, noting that as F is a subsemilattice we have $fe \in F$ for all $e \in F$, it is clear that $f \in M_F$. Thus we have that $E(M_F) = F$.

Now we show that M_F is an inverse subsemigroup. Suppose $a, b \in M_F$, so aa^{-1}, $a^{-1}a$, bb^{-1}, $b^{-1}b \in F$ and, if $e \in F$ then aea^{-1}, $a^{-1}ea$, beb^{-1}, $b^{-1}eb \in F$. Then, as $a^{-1}a, bb^{-1} \in F$, we have

$$(ab)(ab)^{-1} = a(bb^{-1})a^{-1} \in F \qquad \text{and} \qquad (ab)^{-1}(ab) = b^{-1}(a^{-1}a)b \in F.$$

Further, if $e \in F$ then $beb^{-1} \in F$ and then also

$$a(beb^{-1})a^{-1} = (ab)e(ab)^{-1} \in F.$$

Similarly $(ab)^{-1}e(ab) \in F$. Therefore $ab \in M_F$, so we have that M_F is a subsemigroup. That M_F is inverse is immediate from the definition. □

Proceeding with the results from [72] we let τ be a normal congruence on $E(S)$ and write $\{E_i \mid i \in I\}$ for the τ-classes. We recall that each τ-class is a subsemilattice of $E(S)$ so for $i \in I$ we may define $M_i = M_{E_i}$ (as specified before Lemma 1.3.15), and we also let

$$K_i = M_i \cap E_i\omega.$$

Since τ is a normal congruence on $E(S)$ for each $i, j \in I$ there is $l \in I$ such that $E_iE_j \subseteq E_l$ and, if $a \in S$ and $i \in I$ then there is $k \in I$ such that $aE_ia \subseteq E_k$. Let

$$U_i = \{a \in M_i \mid \forall j, l \in I,\ (E_iE_j \subseteq E_l \implies aE_ja^{-1} \subseteq E_l)\}$$

and define

$$K_\tau = \bigcup_{i \in I} K_i, \qquad \text{and} \qquad U_\tau = \bigcup_{i \in I} U_i.$$

Finally we define

$$\mathfrak{L}(\tau) = \{T \in \mathfrak{V}_{SC}(S) \mid K_\tau \subseteq T \subseteq U_\tau,\ T \cap M_i = (T \cap M_i)\omega \cap M_i\}.$$

where $\mathfrak{V}_{SC}(S)$ is the lattice of self conjugate full inverse subsemigroups of S.

Theorem 1.3.16 ([72, Theorem 2.1])**.** *Let S be an inverse semigroup and let τ be a normal congruence on $E(S)$. Then the map*

$$T \mapsto \{(a, b) \in S \times S \mid a^{-1}a\ \tau\ b^{-1}b,\ ab^{-1} \in T\}$$

is a bijective order preserving map of $\mathfrak{L}(\tau)$ onto the set of congruences on S with trace equal to τ. Furthermore, $K_\tau, U_\tau \in \mathfrak{L}(\tau)$.

This is an early example of what I shall term a *correspondence theorem*, a result which relates a set of congruences on a semigroup to a set of substructures of the semigroup. This terminology is common in the area. The classical correspondence theorem for groups is the result that the lattice of normal subgroups of G which contain a given normal subgroup N is isomorphic (as a lattice) to the lattice of normal subgroups of the quotient group G/N. We shall see shortly (see Corollary 1.3.30) that we may rewrite Theorem 1.3.16 as a correspondence between congruences on an inverse semigroup semigroup with a fixed trace and normal subsemigroups of a quotient semigroup.

We return to discussion of the kernel trace approach to describing $\mathfrak{C}(S)$. Just like the kernel normal system approach, the kernel trace approach directly extends the usual description of congruences on groups. When the initial inverse semigroup is a group the idempotent semilattice is just a singleton, the identity - in fact groups are exactly those inverse semigroups that have a unique idempotent - and so the set of congruences on the idempotents is just a singleton. Furthermore, the definition of normal subsemigroup and self conjugate inverse subsemigroup coincide and reduce to the definition of a normal subgroup; also, all of the conditions (CP1)-(CP4) reduce to triviality. Hence the set of congruence pairs is in bijection with the set of normal subgroups.

Many of the advantages that the kernel trace approach offers over that of kernel normal systems are in describing the lattice of congruences on an inverse semigroup. We recall that, via the kernel trace approach, the lattice $\mathfrak{C}(S)$ can be 'found' within the direct product $\mathfrak{C}_N(E) \times \mathfrak{N}(S)$. In particular Theorem 1.3.16 suggests that it may be illuminating to study the set of congruences which share the same trace. Motivated by this we make the following definitions.

Definition 1.3.17. Let S be an inverse semigroup. Then the *kernel map* is the function

$$\mathfrak{C}(S) \to \mathfrak{N}(S); \qquad \rho \mapsto \ker(\rho).$$

If $\rho \in \mathfrak{C}(S)$ then the *kernel class* of ρ - written $[\rho]_{\ker}$ - is

$$\{\kappa \in \mathfrak{C}(S) \mid \ker(\kappa) = \ker(\rho)\}$$

the set of congruences with the same kernel as ρ.

Definition 1.3.18. Let S be an inverse semigroup. Then the *trace map* is the function

$$\mathfrak{C}(S) \to \mathfrak{C}_N(E); \qquad \rho \mapsto \mathrm{trace}(\rho).$$

If $\rho \in \mathfrak{C}(S)$ then the *trace class* of ρ - written $[\rho]_{\mathrm{trace}}$ - is

$$\{\kappa \in \mathfrak{C}(S) \mid \mathrm{trace}(\kappa) = \mathrm{trace}(\rho)\}$$

the set of congruences with the same trace as ρ.

The first question which we address is one of the first one could ask in this scenario: 'Is the lattice of congruences a sublattice of $\mathfrak{C}_N(E) \times \mathfrak{N}(S)$?' This is a badly posed question, the answer could easily be yes if we allow ourselves arbitrary lattice embeddings (an *embedding* is an injective homomorphism), so we instead ask the question: 'Is the function

$$\mathfrak{C}(S) \to \mathfrak{C}_N(E) \times \mathfrak{N}(S); \qquad \rho \mapsto (\mathrm{trace}(\rho), \ker(\rho))$$

a lattice embedding?' The answer to this is 'nearly' - an answer as badly formulated as our initial question.

Theorem 1.3.19 ([26, Theorem 3.4], [69, Theorem 5.1])**.** *Let S be an inverse semigroup and let $\{\rho_i \mid i \in I\}$ be a family of congruences on S. Then*

$$\mathrm{trace}\left(\bigcap_{i\in I} \rho_i\right) = \bigcap_{i\in I} \mathrm{trace}(\rho_i); \qquad \mathrm{trace}\left(\bigvee_{i\in I} \rho_i\right) = \bigvee_{i\in I} \mathrm{trace}(\rho_i);$$
$$\textit{and} \qquad \ker\left(\bigcap_{i\in I} \rho_i\right) = \bigcap_{i\in I} \ker(\rho_i).$$

In particular, the trace map is a surjective complete lattice homomorphism $\mathfrak{C}(S) \to \mathfrak{C}_N(E)$ and the kernel map is a surjective (semilattice) $\cap$-homomorphism $\mathfrak{C}(S) \to \mathfrak{N}(S)$.

Example 1.3.20. To see that the kernel map does not in general preserve joins we consider the semigroup defined by the following multiplication table (this is the Clifford semigroup on two copies of $\mathbb{Z}_2$ with a connecting isomorphism; we define Clifford semigroups properly in Section 1.5).

	1	a	e	b
1	1	a	e	b
a	a	1	b	e
e	e	b	e	b
b	b	e	b	e

We consider the congruences defined by the partitions

$$\rho_1: \ \{1, e\}, \ \{a, b\}; \qquad \rho_2: \ \{1\}, \ \{a\}, \ \{e, b\}$$

which has $\ker(\rho_1) = \{1, e\}$ and $\ker(\rho_2) = \{1, e, b\}$. Then $\ker(\rho_1) \vee \ker(\rho_2) = \ker(\rho_2) = \{1, e, b\}$ however $\rho_1 \vee \rho_2 = \omega$ - the universal congruence - which has $\ker(\omega) = S = \{1, a, e, b\}$. Thus in this case the kernel map is not join preserving.

The fact that the kernel map is not join preserving is an issue when considering the lattice of congruences, in particular, when we want to consider lattice properties of $\mathfrak{C}(S)$ (which are often preserved under taking direct products and sublattices of lattices). It means that we cannot easily deduce properties of $\mathfrak{C}(S)$ from properties of $\mathfrak{N}(S)$ and $\mathfrak{C}_N(E)$, though we will later examine what deductions we may make. However, in many cases the kernel trace description does provide a powerful method to tackle problems related to congruences on inverse semigroups. In fact, less is lost by the lack of 'join-preserve-icity' of this failed embedding than we might assume. The following lemma may be deduced from stronger results which we shall see shortly and which are usually proved in a different fashion, skipping this preliminary result. However I believe that there is value in a direct proof.

Lemma 1.3.21. *Let S be an inverse semigroup and let K be a normal subsemigroup of S. Let $\{(\tau_i, K) \mid i \in I\}$ be a family of congruence pairs for S, and let $\rho_i = \rho(\tau_i, K)$ for $i \in I$. Then*

$$\ker\left(\bigvee_{i\in I} \rho_i\right) = K.$$

Proof. There are two methods to choose from to prove this claim. Option one is to show explicitly that the kernel of the join of a set of congruences with kernel K has kernel equal to K. Option two is to show that $(\bigvee_{i\in I} \tau_i, K)$ is a congruence pair. It is this second option that we pursue. We shall use the definition of a congruence pair from [31] given in Definition 1.3.10.

We initially note that K is certainly a full self conjugate inverse subsemigroup of S. Also, as $\bigvee_{i\in I} \tau_i$ is the transitive closure of the binary relation $\bigcup_{i\in I} \tau_i$, if $e\ (\bigvee_{i\in I} \tau_i)\ h$ then there is sequence $f_1, f_2, \ldots, f_n \in E(S)$ and $i_1, \ldots, i_{n+1} \in I$ such that

$$e\ \tau_{i_1}\ f_1\ \tau_{i_2}\ f_2\ \tau_{i_3}\ \ldots\ \tau_{i_n}\ f_n\ \tau_{i_{n+1}}\ h.$$

As each τ_i is normal, at each stage we may conjugate by a or a^{-1} and we obtain that $aea^{-1}\ (\bigvee_{i\in I}\tau_i)\ aha^{-1}$ and $a^{-1}ea\ (\bigvee_{i\in I}\tau_i)\ a^{-1}ha$, so $\bigvee_{i\in I}\tau_i$ is normal. It remains to verify that (CP1) and (CP3) hold.

Suppose $a \in K$. We first note that for each $i \in I$, as (τ_i, K) is a congruence pair, $aa^{-1}\ \tau_i\ a^{-1}a$ so certainly $aa^{-1}\ (\bigvee_{i\in I}\tau_i)\ a^{-1}a$. Hence (CP3) is verified so we must show (CP1). Suppose that $ae \in K$ and $e\ (\bigvee_{i\in I}\tau_i)\ a^{-1}a$. Again as the congruence $\bigvee_{i\in I}\tau_i$ is the transitive closure of the binary relation $\bigcup_{i\in I}\tau_i$, there is a sequence $f_1, f_2, \ldots, f_n \in E(S)$ and $i_1, \ldots, i_{n+1} \in I$ such that

$$e\ \tau_{i_1}\ f_1\ \tau_{i_2}\ f_2\ \tau_{i_3}\ \ldots\ \tau_{i_n}\ f_n\ \tau_{i_{n+1}}\ a^{-1}a.$$

We then observe that from $e\ \tau_{i_1}\ f_1$ we obtain

$$a^{-1}ae\ \tau_{i_1}\ a^{-1}af_1 = (af_1)^{-1}(af_1).$$

Also, as $ae \in K$, we notice that

$$(af_1)(a^{-1}ae) = (af_1a^{-1})(ae) = af_1e \in K.$$

Hence, as (τ_{i_1}, K) is a congruence pair, by (CP1) for (τ_{i_1}, K) we have that $af_1 \in K$. Proceeding inductively gives that $af_i \in K$ for each i, and in particular that $a(a^{-1}a) = a \in K$. Whence (CP1) holds for the pair $(\bigvee_{i\in I}\tau_i, K)$ so this is also a congruence pair. Therefore by Corollary 1.3.13 we have that $(\bigvee_{i\in I}\tau_i, K)$ is the congruence pair for $\bigvee_{i\in I}\rho_i$, in particular $\ker(\bigvee_{i\in I}\rho_i) = K$. $\square$

In particular Lemma 1.3.21 demonstrates that there is a maximum congruence in each kernel class. The usual strategy in proving results of this kind in the area is to give an expression for a congruence which has the given subsemigroup as the kernel and prove that every congruence with this kernel is contained in the given congruence. The advantage to this methodology is that it leads to explicit descriptions of the maximum and minimum congruences in the trace and kernel classes, which is the next result we mention.

Theorem 1.3.22 ([26, Theorem 3.3], [31, Proposition 5.3.4]). *Let S be an inverse semigroup. Let τ be a normal congruence on $E(S)$ and let K be a normal subsemigroup of S. Then the kernel class $\{\rho \in \mathfrak{C}(S) \mid \ker(\rho) = K\}$ and the trace class $\{\rho \in \mathfrak{C}(S) \mid \operatorname{trace}(\rho) = \tau\}$ are (non-empty) intervals in $\mathfrak{C}(S)$. In particular they have maximum and minimum elements, for the kernel class the maximum element is γ_K and the minimum is λ_K where*

$$\gamma_K = \{(a,b) \in S \times S \mid \forall x, y \in S^1,\ xay \in K \iff xby \in K\},$$
$$\lambda_K = \{(a,b) \in S \times S \mid \exists r, s \in K,\ \exists x, y \in S^1,\ rr^{-1} = ss^{-1},\ xry = a,\ xsy = b\}.$$

For the trace class the maximum element is μ_τ and the minimum is ν_τ where

$$\mu_\tau = \{(a,b) \in S \times S \mid \forall e \in E(S),\ a^{-1}ea\ \tau\ b^{-1}eb\},$$
$$\nu_\tau = \{(a,b) \in S \times S \mid aa^{-1}\ \tau\ bb^{-1},\ \exists e \in E(S)\ (e\ \tau\ aa^{-1} \text{ and } ea = eb)\}.$$

We shall use the notation established in Theorem 1.3.22 for the maximum and minimum congruences with a given kernel or trace. We remark that another way to think of γ_K is as the *principal congruence generated by* K, which is the largest congruence on S which saturates K.

It should come as no surprise that one of the cases most studied is when the trace or kernel is extremal, so when the trace is ι or ω or when the kernel is $E(S)$ or S.

Definition 1.3.23. Let S be an inverse semigroup and let ρ be a congruence on S. Then ρ is said to be

- *idempotent determined* if $\ker(\rho) = E(S)$;
- a *semilattice congruence* if $\ker(\rho) = S$ (so called as this is when S/ρ is a semilattice);
- a *group congruence* if $\operatorname{trace}(\rho) = \omega = E \times E$ (so called as this is when S/ρ is a group);
- *idempotent separating* when $\operatorname{trace}(\rho) = \iota$.

Probably the one of the properties in Definition 1.3.23 that lends itself most to further study and leads to the most elegant results is that of idempotent separation. We shall soon analyse the interplay between the kernel trace approach to congruences and Green's relations and this connection is perhaps strongest for idempotent separating congruences.

Theorem 1.3.24 (see [31, Proposition 5.3.7], [30, Theorem 2.5], [51])**.** *Let S be an inverse semigroup and let ρ be a congruence on S. Then ρ is idempotent separating if and only if $\rho \subseteq \mathcal{H}$.*

One thing that Theorem 1.3.24 implies is that there is a maximum congruence contained in $\mathcal{H}$; however this should not be surprising. For any equivalence relation there is a maximum congruence contained in the relation, a fact readily apparent from Theorem 1.0.18, the result that the lattice of congruences is a sublattice of the lattice of equivalence relations, which in particular implies that the join of congruences contained in an equivalence relation is still contained in this relation.

Definition 1.3.25. Let S be an inverse semigroup with semilattice of idempotents E. Then the *centraliser* of E is

$$E\zeta = \{a \in S \mid \forall e \in E,\ ea = ae\}.$$

If $E\zeta = S$ then the idempotents are said to be *central*.

Inverse semigroups for which the idempotents are central are called *Clifford semigroups*, and we shall introduce these properly later on (see Section 1.5). For now the important observation is that in this scenario the definition of a normal subsemigroup reduces to a full, self conjugate inverse subsemigroup. We know that a normal subsemigroup is always full, self conjugate and inverse so we must show that the reverse is true for a Clifford semigroup. Suppose that $K \subseteq S$ is a self conjugate full inverse subsemigroup of a Clifford semigroup S. Suppose also that $k \in K$ and there are $x, y \in S$ with $xy \in K$ then

$$xky = xk(yy^{-1})y = x(yy^{-1})ky = (xy)(y^{-1}ky)$$

which is an element of K as $xy \in K$ and, because $k \in K$ and K is self conjugate, $y^{-1}ky \in K$. Thus K is normal.

Theorem 1.3.26 ([57, Proposition 5.14]). *Let S be an inverse semigroup. The lattice of idempotent separating congruences on S is isomorphic to the lattice of normal subsemigroups of S contained in $E\zeta$.*

The following maps are inverse lattice isomorphisms:

$$K \mapsto \rho = \{(a,b) \mid a^{-1}a = b^{-1}b,\ ab^{-1} \in K\},$$
$$\rho \mapsto \ker(\rho).$$

The problem of describing the lattice of idempotent separating congruences on S then becomes describing the lattice of normal subsemigroups of S contained in $E\zeta$. By Theorem 1.3.24 we know that there is a maximum idempotent separating congruence, which, by Theorem 1.3.26, has kernel equal to the largest normal subsemigroup of S contained in $E\zeta$. It is easy to see that $E\zeta$ itself is a subsemigroup of S. Suppose that $e \in E(S)$, $a \in E\zeta$ and $x, y \in S$ with $xy \in E\zeta$, then

$$(xay)e = xa(yey^{-1})y = (xy)e(y^{-1}ay) = ex(yy^{-1})ay = e(xay),$$

so $xay \in E\zeta$. Thus we see that $E\zeta$ is normal. Hence the kernel of μ_ι (the maximum idempotent separating congruence) is $E\zeta$. Therefore, by applying the kernel trace description of congruences, we may give the following formulation for μ_ι

$$\mu_\iota = \rho(\iota, E\zeta) = \{(a,b) \in S \times S \mid a^{-1}a = b^{-1}b,\ ab^{-1} \in E\zeta\}.$$

It is an elementary verification to check that this formulation for μ_ι agrees with that given in Theorem 1.3.22.

In fact we may 'strengthen' the statement of Theorem 1.3.26, replacing normal subsemigroups with full self conjugate inverse subsemigroups. Indeed, suppose $K \subseteq E\zeta$ is a full self conjugate inverse subsemigroup of S, we show that K is normal. If $x, y \in S$ with $xy \in K$ and $k \in K$, then, using that $k \in E\zeta$ so $yy^{-1}k = kyy^{-1}$, we observe that

$$xky = xk(yy^{-1})y = x(yy^{-1})ky = (xy)(y^{-1}ky)$$

which is an element of K, as $xy \in K$ and K is a self conjugate subsemigroup so $y^{-1}ky \in K$. Thus K is normal. Therefore we may alter the statement of Theorem 1.3.26, changing "the lattice of normal subsemigroups of S contained in $E\zeta$" to "the lattice of full self conjugate inverse subsemigroups of S contained in $E\zeta$".

Theorem 1.3.26 is a correspondence theorem between the lattice of idempotent separating congruences and the lattice of normal subsemigroups. This may be extended to the trace class for an arbitrary normal congruence on $E(S)$. Suppose $\rho \in \mathfrak{C}(S)$, we know that if $a \;\rho\; e \in E(S)$ then $a^{-1} \;\rho\; e$ and for all $f \in E(S)$

$$afa^{-1} \;\rho\; efe = eef \;\rho\; aa^{-1}f.$$

This fact motivates the following definition.

Definition 1.3.27. Let S be an inverse semigroup and let τ be a normal congruence on $E(S)$. Then $Z(\tau)$, the *centre of* τ, is the set

$$Z(\tau) = \{a \in S \mid \forall e \in E(S),\ aea^{-1} \;\tau\; aa^{-1}e\}.$$

By the remark prior to Definition 1.3.27 we see that if ρ is a congruence with trace τ then $\ker(\rho) \subseteq Z(\tau)$. The next corollary follows from the description of μ_τ, the maximum congruence with trace τ, given in Theorem 1.3.22.

Corollary 1.3.28 ([57, Proposition 5.6])**.** *Let S be an inverse semigroup and let τ be a normal congruence on $E(S)$. Then*

$$\ker(\mu_\tau) = Z(\tau).$$

Therefore, in the terms of the kernel trace description for congruences, $\mu_\tau = \rho(\tau, Z(\tau))$.

We observe that when τ is the trivial relation, $Z(\tau)$ is exactly $E\zeta$, the centraliser of E. Indeed, $Z(\iota) = \{a \in S \mid \forall e \in E\ aea = aa^{-1}e\}$ and we note that $aea^{-1} = aa^{-1}e$ implies that

$$ae = ae(a^{-1}a) = (aea^{-1})a = (aa^{-1}e)a = ea.$$

Conversely, from $ae = ea$ it is immediate that $aea^{-1} = eaa^{-1}$. Theorem 1.3.26 is a correspondence theorem between the lattice of congruences with trivial trace and the lattice of normal subsemigroups of S contained in $E\zeta$. We may extend this to an arbitrary trace via the following lemma, which is the observation that (CP1) is equivalent to K being saturated by the relation ν_τ, the minimum congruence with trace τ.

Lemma 1.3.29. *Let S be an inverse semigroup, let τ be a normal congruence on $E(S)$ and let $K \subseteq S$ be a self conjugate full inverse subsemigroup. Then (K, τ) satisfies (CP1) if and only if K is saturated by ν_τ.*

Proof. We recall the definitions of (CP1) and $\nu = \nu_\tau$:

$$\text{(CP1)} \ : \text{ if } ae \in K \text{ and } e \ \tau \ a^{-1}a, \text{ then } a \in K;$$

$$\nu = \{(a, b) \in S \times S \mid aa^{-1} \ \tau \ bb^{-1}, \ \exists e \in E(S) \ (e \ \tau \ aa^{-1} \text{ and } ea = eb)\}.$$

Suppose first that K satisfies (CP1) and that $a \ \nu \ b$ for some $b \in K$. Then there is $e \in E(S)$ with $aa^{-1} \ \tau \ bb^{-1} \ \tau \ e$ and $ea = eb$. It follows that $a(a^{-1}ea) = ea = eb \in K$ and, as τ is normal, from conjugating $e \ \tau \ aa^{-1}$ by a we obtain that $a^{-1}ea \ \tau \ a^{-1}a$. By (CP1) we have that $a \in K$.

For the converse we suppose that K is saturated by ν and and suppose that there is $a \in S$ and $e \in E(S)$ with $e \ \tau \ a^{-1}a$ and $ae \in K$. Then, as τ is normal, we may conjugate $e \ \tau \ a^{-1}a$ by a to obtain $(ae)(ae)^{-1} = aea^{-1} \ \tau \ aa^{-1}$. Also we note that $(aea^{-1})a = (aea^{-1})ae$. Therefore $a \ \nu \ ae$, whence, as K is saturated by ν, we have $a \in K$, so (CP1) is satisfied. $\square$

We now extend Theorem 1.3.26 to an arbitrary trace class, using the observation that (CP2) exactly says that the kernel of a congruence with trace τ is contained in $Z(\tau)$. We note that, as $\ker(\mu_\tau)$ is equal to $Z(\tau)$, certainly $Z(\tau)$ is saturated by ν_τ. For $T \subseteq S$ a subsemigroup we write $\mathfrak{N}_S(T)$ for the lattice of normal subsemigroups of S which are contained in T and we recall that, for a congruence ρ on S, $\rho|_T$ is the restriction of ρ to T, which is a congruence on T.

Corollary 1.3.30. *Let S be an inverse semigroup and let τ be a normal congruence on $E(S)$. Then the lattice of congruences on S with trace τ is* *isomorphic to the lattice of normal subsemigroups of S which are contained in $Z = Z(\tau)$ and are saturated by ν_τ. The following maps are inverse lattice isomorphisms:*

$$K \mapsto \rho = \{(a,b) \mid aa^{-1} \ \tau \ bb^{-1}, \ a^{-1}b \in K\},$$
$$\rho \mapsto \ker(\rho).$$

Furthermore,

$$[\nu_\tau]_{\mathrm{trace}} \cong \mathfrak{N}_{S/\nu_\tau}(Z/\nu_\tau|_Z).$$

Another way to view Theorem 1.3.26 is that is provides a lattice embedding of the set of normal subsemigroups contained in $E\zeta$ into the lattice of congruences, and in doing so provides a partial inverse to the kernel map. It is possible to, in an admittedly weaker fashion, extend this embedding to all normal subsemigroups.

Proposition 1.3.31. *Let S be an inverse semigroup. For $K \subseteq S$ a normal subsemigroup let λ_K be the minimum congruence with kernel K. The function defined*

$$\Theta \colon \mathfrak{N}(S) \to \mathfrak{C}(S); \qquad K \mapsto \lambda_K$$

is a $\vee$-semilattice embedding.

In particular, Θ is a one sided inverse to the kernel map, in the sense that $\ker(K\Theta) = K$.

Proof. To show that Θ is a $\vee$-semilattice embedding we must show that $\lambda_K \vee \lambda_T = \lambda_{K\vee T}$ for $K, T \in \mathfrak{N}(S)$. We recall the expression for λ_K from Theorem 1.3.22:

$$\lambda_K = \{(a,b) \in S \times S \mid \exists r, s \in K, \ \exists x, y \in S^1, \ rr^{-1} = ss^{-1}, \ xry = a, \ xsy = b\}.$$

It is clear that if $K \subseteq V$ then $\lambda_K \subseteq \lambda_V$ (or in other words that Θ is order preserving). As $K, T \subseteq K \vee T$ we have $\lambda_K, \lambda_T \subseteq \lambda_{K\vee T}$ and it follows that $\lambda_K \vee \lambda_T \subseteq \lambda_{K\vee T}$.

For the reverse inclusion we observe that $K, T \subseteq \ker(\lambda_K \vee \lambda_T)$ so, as $\ker(\lambda_K \vee \Lambda_T)$ is a normal subsemigroup, $K \vee T \subseteq \ker(\lambda_K \vee \lambda_T)$. Then

$$K \vee T \subseteq \ker(\lambda_K \vee \lambda_T) \subseteq \ker(\lambda_{K\vee T}) = K \vee T,$$

where the second subset inclusion comes from applying Corollary 1.3.13 to the inclusion $\lambda_K \vee \lambda_T \subseteq \lambda_{K\vee T}$. Therefore we have $\ker(\lambda_K \vee \lambda_T) = K \vee T$, and as $\lambda_{K\vee T}$ is the minimum congruence with this kernel it follows that $\lambda_{K\vee T} \subseteq \lambda_K \vee \lambda_T$, so the two must be equal. $\square$

It is possible to prove corresponding results for the lattice $\mathfrak{C}_N(E)$ which are slightly stronger. Although, it is still not true that there is a "natural" embedding of $\mathfrak{C}_N(E)$ into $\mathfrak{C}(S)$.

Proposition 1.3.32 ([58, Lemma III.3.9])**.** *Let S be an inverse semigroup. For $\tau \in \mathfrak{C}_N(E)$ let ν_τ and μ_τ be the minimum and maximum congruences with trace τ. Let $\{\tau_i \mid i \in I\}$ be a family of normal congruences on E. Then*

$$\bigvee_{i\in I} \nu_{\tau_i} = \nu_{(\bigvee_{i\in I}\tau_i)} \qquad \text{and} \qquad \bigcap_{i\in I} \mu_{\tau_i} = \mu_{(\bigcap_{i\in I}\tau_i)}.$$

In particular, if we define the functions

$$\Theta\colon \mathfrak{C}_N(S) \to \mathfrak{C}(S); \quad \tau \mapsto \nu_\tau \qquad \text{and} \qquad \Phi\colon \mathfrak{C}_N(S) \to \mathfrak{C}(S); \quad \tau \mapsto \mu_\tau,$$

then Θ is a $\vee$-semilattice embedding, and Φ is a $\cap$-semilattice embedding.

At this point we refer back to when we claimed that it was partially possible to deduce lattice theoretic properties of $\mathfrak{C}(S)$ from properties of $\mathfrak{C}_N(E)$ and $\mathfrak{N}(S)$ (after Example 1.3.20). Since both these lattices embed as semilattices into $\mathfrak{C}(S)$ and can be realised as homomorphic images ($\cap$-homomorphism in the case of $\mathfrak{N}(S)$) we may deduce that if $\mathfrak{C}_N(E)$ or $\mathfrak{N}(S)$ fail to have some property that is preserved by (semilattice) homomorphism or by moving to a sub(semi)lattice then $\mathfrak{C}(S)$ also fails to have that property. More useful to us is the fact that the semilattice embeddings offer a strategy to determine when a pair $(\tau, K) \in \mathfrak{C}_N(E) \times \mathfrak{N}(S)$ is a congruence pair.

Theorem 1.3.33 (see [26, Theorem 3.8])**.** *Let S be an inverse semigroup, let $\tau \in \mathfrak{C}_N(E)$ and $K \in \mathfrak{N}(S)$ and let γ_K, λ_K be the maximum and minimum congruences with kernel K and let μ_τ, ν_τ be the maximum and minimum congruences with trace τ. Then (τ, K) is a congruence pair for S if and only if*

$$\nu_\tau \vee \lambda_K = \mu_\tau \cap \gamma_K.$$

In particular the following are equivalent

(i) (τ, K) *is a congruence pair for* S*;*

(ii) $\ker(\nu_\tau \vee \lambda_K) = K$ *and* $\operatorname{trace}(\nu_\tau \vee \lambda_K) = \tau$*;*

(iii) $\ker(\mu_\tau \cap \gamma_K) = K$ *and* $\operatorname{trace}(\mu_\tau \cap \gamma_K) = \tau$.

Proof. The forward implication (that if (τ, K) is a congruence pair then $\nu_\tau \vee \lambda_K = \mu_\tau \cap \gamma_K$) follows from [26, Theorem 3.8]. The reverse implication remains for us to prove. We suppose that $\nu_\tau \vee \lambda_K = \mu_\tau \cap \gamma_K$. Then certainly

$$K = \ker(\lambda_K) \subseteq \ker(\nu_\tau \vee \lambda_K) = \ker(\mu_\tau \cap \gamma_K) \subseteq \ker(\gamma_K) = K.$$

It follows that $\ker(\nu_\tau \vee \lambda_K) = K$. Replacing the kernel with the trace we obtain that $\operatorname{trace}(\nu_\tau \vee \lambda_K) = \tau$. Hence $\nu_\tau \vee \lambda_K$ is a congruence with kernel K and trace τ which precisely says that (τ, K) is a congruence pair. The equivalence of the three conditions follows immediately from this proof. $\square$

In many ways Theorem 1.3.33 exemplifies the kernel trace approach. As we know the kernel trace approach describes a congruence in terms of its trace and kernel, but another way of viewing this is that the kernel trace approach partitions the lattice of congruences in two ways, firstly into the trace classes and secondly into the kernel classes. The kernel trace description then tells us that if a trace class and a kernel class intersect then they intersect in a singleton, and that this intersection is both the join of the minimum elements and meet of the maximum elements in the trace and kernel classes. Visually this is represented in Fig. 1.2.

We now turn to considering the relationship between congruences (described with the kernel trace methodology) and Green's relations. We have seen that a congruence is idempotent separating exactly when it is contained within $\mathcal{H}$. In fact, we can weaken this condition. Suppose that ρ is a congruence on S such that $\rho \subseteq \mathcal{R}$ and suppose that $e \; \rho \; f$ for $e, f \in E(S)$. Then certainly $e \; \mathcal{R} \; f$ but then $e = ee^{-1} = ff^{-1} = f$, so we obtain that $\operatorname{trace}(\rho) = \iota$ so ρ is idempotent separating. Thus a congruence is idempotent separating if and only if it is contained in the $\mathcal{R}$ relation. Similarly ρ is

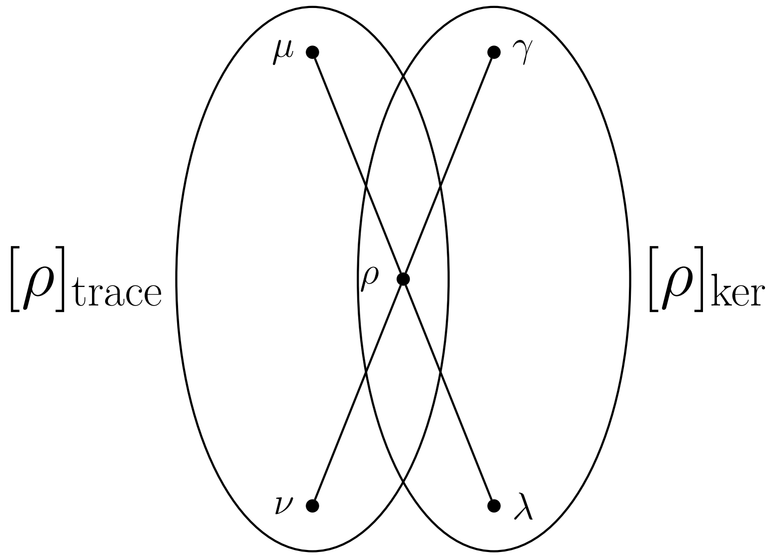


Figure 1.2: A congruence as the meet and join of maximal and minimal elements

idempotent separating if and only if $\rho \subseteq \mathcal{L}$. However, although $\mathcal{D} = \mathcal{R} \vee \mathcal{L}$ (with the join in $\mathfrak{ER}(S)$) it does not follow that if $\rho \subseteq \mathcal{D}$ then ρ is idempotent separating. Indeed, if S is a bisimple inverse semigroup with at least two idempotents (for instance the bicyclic monoid which we introduce Section 1.5) then the universal congruence is equal to the $\mathcal{D}$-relation, and this certainly is not idempotent separating.

Before we proceed further we shall need some general machinery. For any type of algebra (in the sense of universal algebra) the isomorphism theorems are of great importance; the third isomorphism theorem has the following formulation. If $\sigma \subseteq \rho$ are congruences on S then define

$$\rho/\sigma = \{([a]_\sigma, [b]_\sigma) \in S/\sigma \times S/\sigma \mid (a, b) \in \rho\}.$$

Then ρ/σ is a congruence on S/σ and $(S/\sigma)/(\rho/\sigma) \cong S/\rho$ via the obvious map. The fourth isomorphism theorem (sometimes known as the correspondence theorem) is valuable to us now and later, so we state it here.

Theorem 1.3.34 (The fourth isomorphism theorem for semigroups)**.** *Let S be a semigroup and let σ be a congruence on S. Let $[\sigma, \omega] \subseteq \mathfrak{C}(S)$ be the* *set of congruences on S that contain σ. Then the map*

$$\alpha\colon [\sigma,\omega] \to \mathfrak{C}(S/\sigma); \quad \rho \mapsto \rho/\sigma$$

is a lattice isomorphism.

This result is not restricted to congruences. We may replace $[\sigma,\omega]$ with the set of equivalence relations that contain the congruence σ, and then α is an isomorphism onto $\mathfrak{ER}(S/\sigma)$.

As idempotent separating congruences are precisely those contained in $\mathcal{H}$ we may make the following reformulation of Corollary 1.3.30, which we recall is the result that the lattice of congruences on S with trace τ is isomorphic to the lattice of normal subsemigroups of S which are contained in $Z(\tau)$ and are saturated by ν_τ.

Corollary 1.3.35. *Let S be an inverse semigroup and let τ be a congruence on $E(S)$. Then the trace class $\{\rho \in \mathfrak{C}(S) \mid \mathrm{trace}(\rho) = \tau\}$ is isomorphic to the set of idempotent separating congruences on S/ν_τ. The following are mutually inverse lattice isomorphisms*

$$\{\rho \in \mathfrak{C}(S) \mid \mathrm{trace}(\rho) = \tau\} \longrightarrow \{\sigma \in \mathfrak{C}(S/\nu_\tau) \mid \mathrm{trace}(\sigma) = \iota\};$$
$$\rho \mapsto \rho/\nu_\tau,$$
$$\{\sigma \in \mathfrak{C}(S/\nu_\tau) \mid \mathrm{trace}(\sigma) = \iota\} \longrightarrow \{\rho \in \mathfrak{C}(S) \mid \mathrm{trace}(\rho) = \tau\};$$
$$\sigma \mapsto \{(a,b) \in S \times S \mid [a]_{\nu_\tau}\ \sigma\ [b]_{\nu_\tau}\}.$$

Theorem 1.3.12 informs us that with (τ, K) a congruence pair and $\rho = \rho(\tau, K)$, if $a\ \rho\ b$ then $a^{-1}a\ \tau\ b^{-1}b$. On the other hand, since $a\ \rho\ b$ exactly when $a^{-1}\ \rho\ b^{-1}$ we have that also $aa^{-1}\ \tau\ bb^{-1}$. Bearing this in mind we define the relation on S

$$\mathcal{H} /\!\!/ \tau = \{(a,b) \in S \times S \mid aa^{-1}\ \tau\ bb^{-1},\ a^{-1}a\ \tau\ b^{-1}b\},$$

so that certainly $\rho \subseteq \mathcal{H} /\!\!/ \tau$. We observe that

$$\begin{aligned}
[a]_\rho\ \mathcal{H}(S/\rho)\ [b]_\rho &\iff [aa^{-1}]_\rho = [bb^{-1}]_\rho,\ [a^{-1}a]_\rho = [b^{-1}b]_\rho \\
&\iff aa^{-1}\ \tau\ bb^{-1},\ a^{-1}a\ \tau\ b^{-1}b \\
&\iff a\ \mathcal{H} /\!\!/ \tau\ b
\end{aligned}$$

So $\mathcal{H} /\!\!/ \tau$ is the preimage of $\mathcal{H}(S/\rho)$ under the map $S \to S/\rho$. In particular this is true when $\rho = \nu_\tau$ the minimum congruence with trace τ.

We recall that for inverse semigroups the egg-box diagram for each $\mathcal{D}$-class is a square. We can view $\mathcal{H} /\!\!/ \tau$ pictorially in just the same way we usually view the $\mathcal{H}$-relation on the egg-box diagram. Fig. 1.3 shows $\mathcal{H} /\!\!/ \tau$ for a $\mathcal{D}$-class containing 7 idempotents $\{e_1, \dots, e_7\}$ with a congruence τ defined by the partition which has parts $\{e_1, e_2\}$, $\{e_3, e_5, e_6\}$, $\{e_4\}$ and $\{e_7\}$.

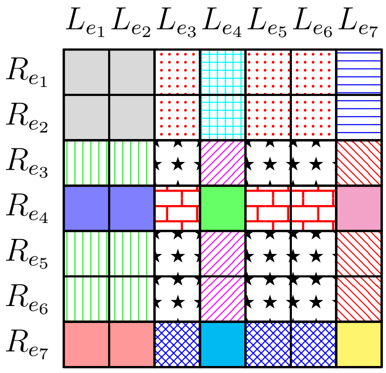


Figure 1.3: The relation $\mathcal{H} /\!\!/ \tau$

Corollary 1.3.30 informs us that the set of idempotent separating congruences is in correspondence with a subset of the set of normal subsemigroups contained in $Z(\tau)$. Getting hold of $Z(\tau)$ can be non trivial, so we make the following observation.

Lemma 1.3.36. *Let S be an inverse semigroup, let $\tau \in \mathfrak{C}(E)$ be normal and let $K \subseteq S$ be a full subsemigroup. Then $K \subseteq Z(\tau)$ if and only if $K \subseteq \{a \in S \mid aa^{-1} \; \tau \; a^{-1}a\}$.*

In particular, $Z(\tau) \subseteq \{a \in S \mid aa^{-1} \; \tau \; a^{-1}a\}$.

Proof. We first suppose that $K \subseteq Z(\tau)$, and take $a \in K$. Then $aea^{-1} \; \tau \; aa^{-1}e$ for all $e \in E(S)$, so, with $e = a^{-1}a$, we have that $aa^{-1} \; \tau \; (aa^{-1})(a^{-1}a)$. Also, with $e = aa^{-1}$, we have $aaa^{-1}a^{-1} \; \tau \; aa^{-1}$, which we conjugate by a to obtain that

$$(a^{-1}a)(aa^{-1}) = a^{-1}(aaa^{-1}a^{-1})a \; \tau \; a^{-1}(aa^{-1})a = a^{-1}a.$$

Therefore $aa^{-1} \mathrel{\tau} (a^{-1}a)(aa^{-1}) \mathrel{\tau} a^{-1}a$ so $a \in \{b \in S \mid bb^{-1} \mathrel{\tau} b^{-1}b\}$.

Conversely we suppose that $K \subseteq \{b \in S \mid bb^{-1} \mathrel{\tau} b^{-1}b\}$. Again we take $a \in K$, so $aa^{-1} \mathrel{\tau} a^{-1}a$, and we take $e \in E$. Then $ae \in K$ (since K is full) so

$$aea^{-1} = (ae)(ae)^{-1} \mathrel{\tau} (ae)^{-1}(ae) = ea^{-1}ae = a^{-1}ae \mathrel{\tau} aa^{-1}e.$$

Thus $a \in Z(\tau)$, and the proof of the first claim is complete.

The second claim follows from the first together with the recollection that $Z(\tau)$ is a full inverse subsemigroup of S. $\square$

We observe that when τ is the trivial congruence Lemma 1.3.36 states that normal subsemigroups of $E\zeta$ (which we recall is equal to $Z(\iota)$) are exactly normal subsemigroups contained in $\{a \in S \mid aa^{-1} = a^{-1}a\}$. We note that, by the description of $\mathcal{H}$ on an inverse semigroup, $aa^{-1} = a^{-1}a$ precisely says that $a \mathrel{\mathcal{H}} aa^{-1}$, so, in particular,

$$\{a \in S \mid aa^{-1} = a^{-1}a\} = \bigcup_{e \in E(S)} H_e.$$

This observation provides an opportunity to view Theorem 1.3.26 (the isomorphism between idempotent separating congruences and normal subsemigroups contained in $E\zeta$) visually. Fig. 1.4 shows a $\mathcal{D}$-class containing 7 idempotents $\{e_1, \ldots, e_7\}$, the $\mathcal{H}$-classes which contain idempotents lie along the diagonal. Theorem 1.3.26 says that the idempotent separating congruences correspond with normal subsemigroups that are contained in this diagonal.

Let τ be a non trivial normal congruence on the idempotents then, by Lemma 1.3.36

$$Z(\tau) \subseteq \{a \in S \mid aa^{-1} \mathrel{\tau} a^{-1}a\} = \bigcup_{e \in E} [e]_{\mathcal{H}/\!\!/\tau}.$$

We may think of $\bigcup_{e \in E} [e]_{\mathcal{H}/\!\!/\tau}$ as a τ-expanded diagonal in the egg-box diagram. A τ-expanded diagonal for a $\mathcal{D}$-class containing seven idempotents $\{e_1, e_2, \ldots, e_7\}$ with τ defined by the partition: $\{e_1, e_2\}$, $\{e_3, e_5, e_6\}$, $\{e_4\}$, $\{e_7\}$ is shown in Fig. 1.5. The kernels of congruences with trace τ are precisely those contained in the τ-expanded diagonal and saturated by ν_τ.

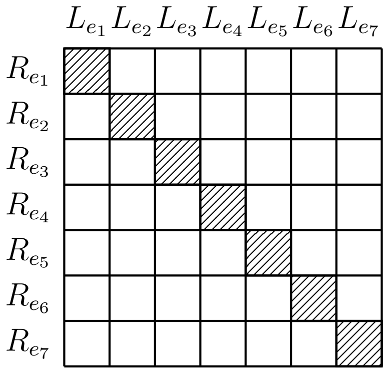


Figure 1.4: An egg-box diagram highlighting $\mathcal{H}$-classes containing idempotents

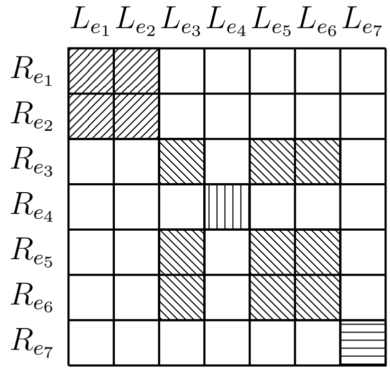


Figure 1.5: An egg-box diagram demonstrating a τ-expanded diagonal

Our discussion of congruences has so far been largely in the direction in which we take a normal congruence on the idempotents and then look at the appropriate normal subsemigroups that appear as kernels. Within the literature concerned with lattices of congruences on inverse semigroups this is the more common direction and does tend to be easier to get to grips with. The other direction starts with a normal subsemigroup and describes the set of congruences which have this as kernel. This latter direction is generally harder as kernel classes and the kernel map are less well behaved than the trace classes and the trace map. In Chapter 2, when we present our analysis of one sided congruences, it becomes even harder to go from a kernel to a congruence; we have focused in this preliminary chapter on presenting the results which we are able to mimic in the one sided case.

We shall not dwell too much on the kernel classes as there is much more we have to cover in this chapter and, interesting a field of study as it is, we do have to move on. Thus we simply state the corresponding result to Corollary 1.3.35.

Theorem 1.3.37 ([58, III.4.13]). *Let S be an inverse semigroup, let K be a normal subsemigroup of S and let $\lambda = \lambda_K$ be the minimum congruence with kernel K. Then the kernel class $\{\rho \in \mathfrak{C}(S) \mid \ker(\rho) = K\}$ is isomorphic to $\mathfrak{C}_{ID}(S/\lambda) = \{\sigma \in \mathfrak{C}(S/\lambda_K) \mid \ker(\sigma) = E(S/\lambda_K)\}$, the lattice of idempotent* *determined congruences on S/λ.*

The following are mutually inverse lattice isomorphisms

$$\{\rho \in \mathfrak{C}(S) \mid \ker(\rho) = K\} \longrightarrow \mathfrak{C}_{ID}(S/\lambda); \quad \rho \mapsto \rho/\lambda_K,$$
$$\mathfrak{C}_{ID}(S/\lambda) \longrightarrow \{\rho \in \mathfrak{C}(S) \mid \ker(\rho) = K\}; \quad \sigma \mapsto \{(a,b) \mid [a]_{\lambda_K}\ \sigma\ [b]_{\lambda_K}\}.$$

It would also be remiss of us to spend this long on the kernel trace approach without mentioning another of its motivations. It is a common occurrence in the study of algebra that one is in a situation where one is faced with a sequence

$$K \xrightarrow{\ \alpha\ } S \xrightarrow{\ \beta\ } T$$

where α and β are homomorphisms and the image of α is precisely the set of elements which become trivial under β. This is very commonly studied in the case of groups and is often known as a short exact sequence; S is said to be an extension of K by T (or an extension of T by K). To study extensions for inverse semigroups slightly more structure is needed. An inverse semigroup S is an *extension of* K *by* T if there is an embedding of $\alpha\colon K \hookrightarrow S$ and a surjective homomorphism $\beta\colon S \to T$ such that $(E(T))\beta^{-1} = K\alpha$. Then the kernel of a congruence ρ is precisely the subsemigroup K such that S is an extension of K by S/ρ. This subject has been widely studied, for instance presentations of inverse semigroups from presentations of their kernels are described in [6].

To conclude this discussion of the kernel trace approach to congruences on inverse semigroups we make a final observation to which we shall later refer back.

Lemma 1.3.38. *Let S be an inverse semigroup and let ρ be a congruence on S. Then the following are equivalent definitions for the kernel of ρ:*

(i) $\ker(\rho) = \{a \in S \mid \exists e \in E(S),\ a\ \rho\ e\}$

(ii) $\ker(\rho) = \{a \in S \mid \exists e, f \in E(S),\ a\ \rho\ e,\ a^{-1}\ \rho\ f\}$

(iii) $\ker(\rho) = \{a \in S \mid a\ \rho\ aa^{-1}\}$

(iv) $\ker(\rho) = \{a \in S \mid a\ \rho\ a^{-1}a\}$

Proof. All implications are immediate with the possibly exception of (ii) implies (iii) which we demonstrate now. Suppose $a \ \rho \ e$ and $a^{-1} \ \rho \ f$. Then

$$a = aa^{-1}a \ \rho \ efe = ef \ \rho \ aa^{-1}.$$

□

1.4 ONE SIDED CONGRUENCES

We turn now to what is probably the most relevant section in this chapter to the rest of the thesis - well at least the first half. This is the introduction and preliminary discussion of one sided congruences on inverse semigroups, with which we now proceed. This shall be somewhat shorter than the section for two sided congruences as we shall delay the introduction of some concepts and results to Chapter 2 - in which a new theory for one sided congruences on inverse semigroups is developed - in order for the narrative in Chapter 2 to have better flow.

Definition 1.4.1. Let S be a semigroup and let ρ be an equivalence relation on S. We say that ρ is a *left congruence* on S if for all $a, b, c, \in S$

$$a \ \rho \ b \implies ca \ \rho \ cb.$$

Dually, ρ is a *right congruence* if $a \ \rho \ b$ implies $ac \ \rho \ bc$. A *one sided congruence* refers to either a left or a right congruence. We write $\mathfrak{LC}(S)$ for the set of left congruences and $\mathfrak{RC}(S)$ for the set of right congruences.

Of particular importance is the remark that the $\mathcal{R}$-relation is a left congruence and $\mathcal{L}$ is a right congruence.

Being a one sided congruence is a weaker condition than being a congruence, a left congruence is not necessarily a right congruence or vice versa. For instance, in general, $\mathcal{R}$ is not a right congruence nor is $\mathcal{L}$ a left congruence. In fact, a congruence is an equivalence relation which is both a left and a right congruence.

Just as for congruences, left congruences or right congruences are partially ordered by inclusion as subsets of $S \times S$ and the set of left (or right) congruences is a complete lattice. Much of our preliminary discussion of the lattice of congruences carries over to one sided congruences. We may realise $\mathfrak{LC}(S)$ (or $\mathfrak{RC}(S)$) as a subset of $\mathfrak{ER}(S)$, so we have

$$\mathfrak{C}(S) \;\subseteq\; \mathfrak{LC}(S), \mathfrak{RC}(S) \;\subseteq\; \mathfrak{ER}(S).$$

Furthermore the same logic applies with regards to the meet and join of one sided congruences being the intersection and join of the relations regarded as equivalence relations. Thus there is a one sided analogue of Theorem 1.0.18.

Theorem 1.4.2. *Let S be a semigroup. Then $\mathfrak{LC}(S)$ is a sublattice of $\mathfrak{ER}(S)$, dually $\mathfrak{RC}(S)$ is a sublattice of $\mathfrak{ER}(S)$. Moreover, $\mathfrak{C}(S)$ is a sublattice of $\mathfrak{LC}(S)$ and of $\mathfrak{RC}(S)$. Furthermore,*

$$\mathfrak{C}(S) = \mathfrak{LC}(S) \cap \mathfrak{RC}(S)$$

regarded as sublattices of $\mathfrak{ER}(S)$.

We recall that if we start with a binary relation κ on S then the congruence generated by κ is the relation ρ defined by $a \; \rho \; b$ if $a = b$ or there is a κ-sequence from a to b. In a very similar fashion we may construct the smallest left congruence on S which contains κ, which we call the *left congruence generated by* κ.

Definition 1.4.3. Let κ be a binary relation on S. Say that there is a *κ-left-sequence from a to b* in S if there is a sequence $x_1, \ldots, x_n, y_1, \ldots, y_n$ such that $(x_i, y_i) \in \kappa$ or $(y_i, x_i) \in \kappa$ for each $1 \leq i \leq n$ and there are $u_1, \ldots, u_n \in S^1$ (where S^1 is the semigroup S with an identity adjoined) such that

$$a = u_1x_1, \; u_1y_1 = u_2x_2, \; u_2y_2 = u_2x_3, \; \ldots, \; u_{n-1}y_{n-1} = u_nx_n, \; u_ny_n = b.$$

Then the left congruence generated by κ is the relation ρ, defined by $a \; \rho \; b$ if $a = b$ or there is a κ-left-sequence from a to b. We shall drop the 'left' from κ-left-sequence when this is clear from the context.

Whereas congruences define homomorphic images, one sided congruences determine semigroup actions. A semigroup action $S \times A \to A$ is *monogenic* if there is $a \in A$ such that $S^1 a = A$ (where if we have added an identity in S^1 then this acts as expected: $1a = a$ for all $a \in A$). In this scenario we define a relation on S by

$$\{(s,t) \in S \times S \mid sa = ta\}$$

which is a left congruence. Conversely, if ρ is a left congruence then we define

$$S /\!\!/ \rho = \{[s]_\rho \mid s \in S\}$$

as the set of equivalence classes, then $S \times (S /\!\!/ \rho) \to S /\!\!/ \rho$ given by $(s, [t]_\rho) \mapsto [st]_\rho$ is a left action which we call the *quotient action*. If ρ is the trivial relation then $S /\!\!/ \rho$ is S and the action is that used in Definition 1.1.2, the action by left multiplication of S on itself. We remark that, in the case when S is not a monoid, S itself may not be a monogenic S-act. To overcome this minor niggle we consider the action $S \times S^1 \to S^1$; which we note is monogenic. The quotient action becomes $(S /\!\!/ \rho) \cup \{1\}$ which is again monogenic. This means we can (informally) pretend that S is a monogenic S-act.

We can define a semigroup action as a universal algebra, and the second half of the thesis is largely devoted to the consideration of the partial automorphism monoid of a group action. As a universal algebra an S-act is defined as $\mathbf{A} = (A, \{f_s \mid s \in S\})$ where for each $s \in S$ the function symbol f_s denotes the unary operation $x \mapsto sx$. An S-act homomorphism from $\mathbf{A} = (A, \{f_s \mid s \in S\})$ to $\mathbf{B} = (B, \{f_s \mid s \in S\})$ is a function $\theta \colon A \to B$, such that $(f_s(a))\theta = f_s(a\theta)$ for all $s \in S$ and $a \in A$, or equivalently, $(sa)\theta = s(a\theta)$. If $A = Sa$ is a monogenic S-act and $\theta \colon A \to B$ is a homomorphism of S-acts then the image of θ is also a monogenic S-act, indeed it is generated by $a\theta$. From this viewpoint left congruences on S are the kernels of S-act homomorphisms from S where S is regarded as a monogenic semigroup action. In fact, the correspondence between left congruences and monogenic S-acts is a bijection, up to the isomorphism class of left actions (in the sense isomorphic as universal algebras).

Theorem 1.4.4 (Homomorphism theorems for semigroup actions (see [45, Theorem 1.16]))**.** *Let S be a semigroup. Then the following hold.*

(1) Let ρ be a left congruence on S. Then the quotient action $S \mathbin{/\!\!/} \rho$ is an S-act and the map $S \to S \mathbin{/\!\!/} \rho$ defined by $a \mapsto [a]_\rho$ is an S-act homomorphism.

(2) Let A be an S-act and let $\theta \colon S \to A$ be an S-act homomorphism. Then $\mathrm{Im}(\theta)$ is a subact of A and $\mathrm{Ker}(\theta)$ is a left congruence on S. Furthermore the function $S \mathbin{/\!\!/} \rho \to A$ defined $[a]_{\mathrm{Ker}(\theta)} \mapsto a\theta$ is an S-act isomorphism between $S \mathbin{/\!\!/} \rho$ and $\mathrm{Im}(\theta)$.

(3) Let σ be a left congruence on S. Let $[\sigma, \omega] \subseteq \mathfrak{LC}(S)$ be the set of left congruences on S that contain σ. Then the map

$$\alpha \colon [\sigma, \omega] \to \mathfrak{C}(S \mathbin{/\!\!/} \sigma); \quad \rho \mapsto \rho/\sigma$$

is a lattice isomorphism, where $\mathfrak{C}(S \mathbin{/\!\!/} \sigma)$ is the lattice of congruences on $S \mathbin{/\!\!/} \sigma$ regarded as a left S-act.

The dual results hold for right congruences and right S-acts.

In particular in (3) if σ is a two sided congruence then this isomorphism between $[\sigma, \omega] \subseteq \mathfrak{LC}(S)$ and $\mathfrak{C}(S \mathbin{/\!\!/} \sigma)$ becomes a version of Theorem 1.3.34 and says that there is a lattice isomorphism $\alpha \colon [\sigma, \omega] \to \mathfrak{LC}(S/\sigma)$ given by $\rho \mapsto \rho/\sigma$.

When considering a group G, one sided congruences are determined by subgroups. For left congruences this is via the maps

$$\rho \mapsto \{a \in G \mid a \;\rho\; 1\},$$
$$H \mapsto \{(a, b) \in S \times S \mid a^{-1}b \in H\}.$$

For right congruences we take the dual. When we move from group theory to the realm of semigroups there are many scenarios when replacing the notion of subgroup with subsemigroup does not cut the mustard. Therefore, just as when we replace normal subgroups with congruences we often replace subgroups with one sided congruences.

At this point we focus on inverse semigroups, and from this point will assume that, unless otherwise stated, S is an inverse semigroup and $E = E(S)$. Methods used to describe one sided congruences on inverse semigroups largely mirror those used to describe two sided congruences, and, just as for two sided congruences, this relies on the observation that it is possible to recover arbitrary relations in a left congruence from relations in which at least one element is an idempotent. Indeed, we suppose $a \; \rho \; b$, from which it follows that

$$a^{-1}a \; \rho \; a^{-1}b, \quad b^{-1}a \; \rho \; b^{-1}b \quad \text{and} \quad a^{-1}b = a^{-1}bb^{-1}b \; \rho \; a^{-1}bb^{-1}a,$$

which is a trio of relations each containing at least one idempotent. Conversely, given this trio of relations we observe that

$$a = a(a^{-1}a) \; \rho \; a(a^{-1}b) \; \rho \; a(a^{-1}bb^{-1}a) = b(b^{-1}a) \; \rho \; bb^{-1}b = b.$$

We bear this in mind as we present the one sided analogues of the kernel normal system and kernel trace approaches which are found in the literature. First though, so we do not have to continue to repeat ourselves for both left and right congruences we should comment on the relationship between left and right congruences.

Proposition 1.4.5. *Let S be an inverse semigroup. Then there is a lattice isomorphism $\mathfrak{LC}(S) \to \mathfrak{RC}(S)$ given by the map*

$$\rho \mapsto \rho_{-1} = \{(a^{-1}, b^{-1}) \in S \times S \mid (a, b) \in \rho\}.$$

This is a result specific to inverse semigroups, in the sense that it is not true for arbitrary semigroups (a^{-1} is not even defined in general). While there is an isomorphism between left congruences on a semigroup S and right congruences on the dual semigroup S^d - which is defined as having the same set as S with multiplication, $\bullet$, defined as $a \bullet b = ba$ - in general it is not the case that left congruences on S are in bijection with left congruences on S^d. That the function defined in Proposition 1.4.5 is a bijection between left and right congruences on S is a consequence of the inverse map ($a \mapsto a^{-1}$) being an isomorphism between S and S^d.

From this point we shall not double up and state every result twice, once for left congruences and once for right sided congruences, we shall take as understood that an analogous result is true. This does necessitate choosing which sided congruences to focus on, and while there are solid arguments for each depending on the context, we shall dwell primarily in the realm of left congruences.

Next we proceed with a discussion of how left congruences on inverse semigroups are described. We begin with the notion of a left kernel system.

Definition 1.4.6 ([46, Definition 2.1])**.** Let $\mathcal{A} = \{A_i \mid i \in I\}$ be a set of disjoint subsets of an inverse semigroup S and let $A = \bigcup_{i \in I} A_i$. Then $\mathcal{A}$ is a *left kernel system* for S if it satisfies the conditions:

(L1) $E(S) = \bigcup_{i \in I} E(A_i)$;

(L2) for each $i \in I$, $E(S) \cap A_i \neq \emptyset$

(L3) for all $i \in I$ and $a \in A$ there is $j \in I$ such that $a^{-1}A_i \subseteq A_j$;

(L4) if $a \in A_i$ and $a^{-1}a \in A_j$ then $aA_i \subseteq A_j$;

(L5) if $a^{-1}b, a^{-1}a \in A_i$ for some $b \in A$ then $a \in A$.

Similar to in the two sided case, left kernel systems will define left congruences, and the subsets in the left kernel system will be the equivalence classes which contain idempotents. It is possible to show directly from the definition that a kernel normal system is a left kernel system.

Theorem 1.4.7 ([46, Theorem 2.1])**.** *Let S be an inverse semigroup and let $\mathcal{A} = \{A_i \mid i \in I\}$ be a left kernel system. Then the relation*

$$\rho(\mathcal{A}) = \{(a,b) \in S \times S \mid \exists i,j \in I,\ a^{-1}b, a^{-1}a \in A_i,\ b^{-1}a, b^{-1}b \in A_j\}$$

is a left congruence on S. Conversely, if ρ is a left congruence on S then $\mathcal{A}(\rho) = \{[e]_\rho \mid e \in E(S)\}$ is a left kernel system.

Furthermore, $\mathcal{A}(\rho(\mathcal{A})) = \mathcal{A}$ and $\rho(\mathcal{A}(\rho)) = \rho$.

Just as for kernel normal systems there are many ways to characterise left kernel systems, the properties given in Definition 1.4.6 are not unique in specifying that a set of disjoint subsets are a left kernel system. In [61, Proposition 7.2] the following alternate classification is given. A set $\mathcal{A} = \{A_i \mid i \in I\}$ of disjoint subsets of S (with $A = \bigcup_{i\in I} A_i$) is a left kernel system for S if it satisfies (L1), (L2) and (L6), where

(L6) for all $a \in S$ and $i, j \in I$ if $aA_i \cap A_j \neq \emptyset$ then $aA_i \subseteq A_j$.

In contrast to the two sided case, if $\mathcal{A} = \{A_i \mid i \in I\}$ is a left kernel system then each $A_i \in \mathcal{A}$ is not necessarily a subsemigroup of S. However A, the union of the A_i, is a subsemigroup. It therefore makes sense to extend the definition of the *kernel* to left congruences. Also, it is clear that if ρ is a left congruence on S then, as $E(S)$ is commutative, the trace of ρ is a congruence on $E(S)$. Therefore we also extend the definition of the *trace* to left congruences on S. The kernel-trace approach to one sided congruences is described in [61].

Definition 1.4.8 ([61, Definition 3.3])**.** Let S be an inverse semigroup. Let $K \subseteq S$ be a full subsemigroup and τ be a congruence on $E(S)$. Then (τ, K) is a *left congruence pair* if the following conditions are satisfied,

(i) for all $a \in S$ and $b \in K$, if $a \geq b$ and $a^{-1}a \;\tau\; b^{-1}b$ then $a \in K$;

(ii) for all $a \in K$ and $e, f \in E(S)$, if $e \;\tau\; f$ then $a^{-1}ea \;\tau\; a^{-1}fa$;

(iii) for all $a \in K$ there is $b \in S$ with $a \geq b$, $a^{-1}a \;\tau\; b^{-1}b$ and $b^{-1} \in K$.

If (i) and (ii) are satisfied then (τ, K) is a *pseudo left congruence pair.*

Theorem 1.4.9 ([61, Theorem 3.5])**.** *Let S be an inverse semigroup and let (τ, K) be a left congruence pair. Then the relation*

$$\rho(\tau, K) = \{(a, b) \in S \times S \mid a^{-1}b, b^{-1}a \in K,\ a^{-1}bb^{-1}a \;\tau\; a^{-1}a,\ b^{-1}aa^{-1}b \;\tau\; b^{-1}b\}$$

is a left congruence on S with $\operatorname{trace}(\rho(\tau, K)) = \tau$ *and* $\ker(\rho(\tau, K)) = K$.

Conversely, if ρ is a left congruence on S then $(\operatorname{trace}(\rho), \ker(\rho))$ *is a left congruence pair and* $\rho(\operatorname{trace}(\rho), \ker(\rho)) = \rho$.

We remark that we define a pseudo left congruence pair because $\rho(\tau, K)$ (as defined in Theorem 1.4.9) makes sense for any pair $(\tau, K) \in \mathfrak{C}(E) \times \mathfrak{K}(S)$ (we recall that $\mathfrak{K}(S)$ is the lattice of full subsemigroups of S), and $\rho(\tau, K)$ is a left congruence when (τ, K) is a pseudo left congruence pair. However, many pseudo left congruence pairs may give the same left congruence. If ρ is a left congruence and (τ, K) is a pseudo left congruence pair such that $\rho = \rho(\tau, K)$ then K is said to be an *pseudo kernel for* ρ. It can be shown (see [61]) that each pseudo kernel for ρ contains $\ker(\rho)$ and is saturated by ρ.

In the two sided case we observed that when the inverse semigroup is a group the two approaches to describing congruences reduce to the same thing - the definition of a normal subgroup. The good news is that, in a similar way, both descriptions of one sided congruences reduce to describing a subgroup. In a left kernel system there is just one subset which is a subgroup, and in the kernel trace approach the congruence on $E(S)$ is trivial and the full subsemigroup is again a subgroup. Thus both approaches are extensions of the usual description of one sided congruences for groups to the wider class of inverse semigroups.

As we have remarked the reason that describing congruences in terms of the kernel and trace in the realm of two sided congruences is useful stems in large part from the fact that the ordering of congruences is the same as the natural inclusion ordering on the set of congruence pairs. In one sided land the same is true, the set of left congruence pairs is a subset of the direct product $\mathfrak{C}(E) \times \mathfrak{K}(S)$ and we have the following analogue of Corollary 1.3.13.

Corollary 1.4.10. *Let S be an inverse semigroup and let ρ_1, ρ_2 be left congruences on S. Then $\rho_1 \subseteq \rho_2$ if and only if*

$$\operatorname{trace}(\rho_1) \subseteq \operatorname{trace}(\rho_2) \qquad \textit{and} \qquad \ker(\rho_1) \subseteq \ker(\rho_2).$$

In particular we have that $\rho_1 = \rho_2$ if and only if

$$\operatorname{trace}(\rho_1) = \operatorname{trace}(\rho_2) \qquad \textit{and} \qquad \ker(\rho_1) = \ker(\rho_2).$$

Following the format of the discussion for two sided congruences we extend the notion of the *kernel and trace maps* and that of *kernel and trace* *classes* to left congruences. We can then seek analogues of results that say that the kernel and trace classes are intervals in the lattice of congruences (Theorem 1.3.19 and Lemma 1.3.21) and that the trace and kernel maps are, respectively, lattice and semilattice homomorphisms (Theorem 1.3.19). Further, we could ask for analogous results to those that describe the kernel and trace classes for given subsemigroups and congruences on $E(S)$ (Corollary 1.3.30 and Theorem 1.3.37), or even of those results that express semilattice embeddings of the lattices $\mathfrak{N}(S)$ and $\mathfrak{C}_N(E)$ into $\mathfrak{C}(S)$ (Proposition 1.3.31 and Proposition 1.3.32). The approaches to one sided congruences which we have described give partial analogues to some of these results. The rest of this section is devoted to introducing these ideas. In Chapter 2 we obtain further results in this area.

Theorem 1.4.11 ([46, Theorem 3.1])**.** *Let S be an inverse semigroup and let τ be a congruence on $E(S)$. Then there is a left congruence on S with trace τ. Furthermore, the trace class $\{\rho \in \mathfrak{LC}(S) \mid \mathrm{trace}(\rho) = \tau\}$ is an interval in $\mathfrak{LC}(S)$ with maximum and minimum elements μ_τ and ν_τ, which may be realised as*

$$\begin{aligned}\mu_\tau = \{(a,b) \in S \times S \mid \ & a^{-1}a \ \tau \ a^{-1}bb^{-1}a, \ b^{-1}b \ \tau \ b^{-1}aa^{-1}b, \\ & e \ \tau \ f \implies a^{-1}beb^{-1}a \ \tau \ a^{-1}bfb^{-1}a, \ b^{-1}aea^{-1}b \ \tau \ b^{-1}afa^{-1}b\}.\end{aligned}$$

and

$$\nu_\tau = \{(a,b) \in S \times S \mid \ \exists e \in E(S), \ a^{-1}a \ \tau \ b^{-1}b \ \tau \ e, \ ae = be\}$$

We use the notation established in Theorem 1.4.11 for the maximum and minimum left congruences with a given trace, and shall endeavour to avoid confusion arising from overlap with notation for the two sided case.

Maximum and minimum one sided congruences with a given trace are also described in [61], making use of the notion of pseudo left congruence pair. We define $N_L(\tau)$, the *left normaliser* of $\tau \in \mathfrak{C}(E)$ by

$$N_L(\tau) = \{a \in S \mid e \ \tau \ f \implies a^{-1}ea \ \tau \ a^{-1}fa\}.$$

Further we define $C_L(\tau)$, the *left closure* of a congruence τ on E as

$$C_L(\tau) = \{a \in S \mid \exists e \in E \text{ such that } e \; \tau \; a^{-1}a \text{ and } ae = e\}.$$

Both $N_L(\tau)$ and $C_L(\tau)$ are easily seen to be full subsemigroups of S. Both are used in [61] to give the following description of μ_τ and ν_τ.

Theorem 1.4.12 ([61, Propositions 4.1 and 4.2])**.** *Let S be an inverse semigroup and let $\tau \in \mathfrak{C}(E)$. Then $(\tau, N_L(\tau))$ is a pseudo left congruence pair for S and $(\tau, C_L(\tau))$ is a left congruence pair for S. Moreover,*

$$\mu_\tau = \rho(\tau, N_L(\tau)) \quad \textit{and} \quad \nu_\tau = \rho(\tau, C_L(\tau)).$$

The kernel classes are less well behaved. As observed in [61], there is a maximum left congruence with a given kernel.

Proposition 1.4.13. *If $K \subseteq S$ is the kernel of a left congruence then there is a maximum left congruence γ_K with kernel K, and*

$$\gamma_K = \{(a,b) \in S \times S \mid \forall s \in S,\ sa \in K \iff sb \in K\}.$$

Similar to the two sided case, the maximum left congruence with kernel K may be viewed as the *principal left congruence* generated by K, which is the largest left congruence on S which saturates K. However, there is in general no minimum left congruence with a given kernel. We include a simple example of a semigroup for which there is a kernel class with no minimum element, as we are aware of no finite example in the literature.

Example 1.4.14. We consider the inverse semigroup $S \subseteq \mathcal{I}_6$ (and seek forgiveness not yet defining $\mathcal{I}_n$, we do this in Section 1.5)

$$\left\{ \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}, \begin{pmatrix} 4 & 5 & 6 \\ 1 & 2 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 4 & 5 \end{pmatrix}, \begin{pmatrix} 1 & 3 \\ 4 & 6 \end{pmatrix}, \begin{pmatrix} 4 & 6 \\ 1 & 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 4 \end{pmatrix}, \begin{pmatrix} 4 \\ 1 \end{pmatrix}, \right.$$
$$\left. e_{\{1,2,3\}}, e_{\{4,5,6\}}, e_{\{1,2\}}, e_{\{1,3\}}, e_{\{4,5\}} e_{\{4,6\}}, e_{\{1\}}, e_{\{4\}}, e_\emptyset \right\}$$

The semilattice $E(S)$ is shown in Fig. 1.6, with e_X labelled by X. We note that, with $a = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$ we may write the elements of S in the form

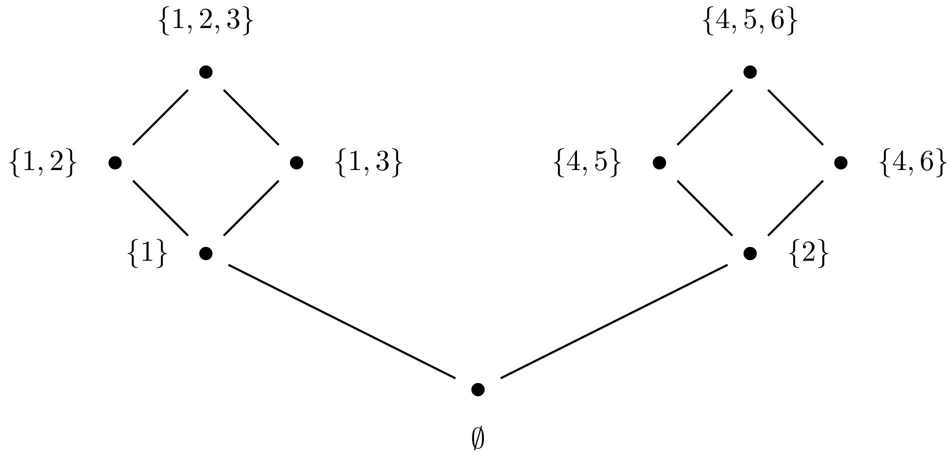


Figure 1.6: The semilattice $E(S)$ for Example 1.4.14

$e_X a$ or $e_X a^{-1}$ for some idempotent e_X. Let ρ_1 and ρ_2 be the partitions of S defined by:

$$\begin{aligned}
\rho_1 : \quad & \{e_{1,2,3}\},\ \{a^{-1}\},\ \{e_{4,5,6}, e_{4,5}, e_{4,5}a^{-1}\},\ \{a, e_{1,2}, e_{1,2}a\}, \\
& \qquad \{e_{4,6}, e_{4,6}a^{-1}, e_4, e_4a^{-1}\},\ \{e_{1,3}, e_{1,3}a, e_1, e_1a\},\ \{e_\emptyset\}; \\
\rho_2 : \quad & \{e_{1,2,3}\},\ \{a^{-1}\},\ \{e_{4,5,6}, e_{4,6}, e_{4,6}a^{-1}\},\ \{a, e_{1,3}, e_{1,3}a\}, \\
& \qquad \{e_{4,5}, e_{4,5}a^{-1}, e_4, e_4a^{-1}\},\ \{e_{1,2}, e_{1,2}a, e_1, e_1a\},\ \{e_\emptyset\}.
\end{aligned}$$

Both ρ_1 and ρ_2 are left congruences on S, and both have kernel $S\backslash\{a^{-1}\}$. In terms of the kernel trace description $\rho_i = \rho(\tau_i, S\backslash\{a^{-1}\})$ where τ_1 has non trivial congruence classes $\{e_{4,5,6}, e_{4,5}\}$, $\{e_{4,6}, e_4\}$ and $\{e_{1,3}, e_1\}$, and τ_2 has non trivial classes $\{e_{4,5,6}, e_{4,6}\}$, $\{e_{4,5}, e_4\}$ and $\{e_{1,2}, e_1\}$. Then we observe that $\rho_1 \cap \rho_2$ is defined by the partition

$$\begin{gathered}
\{e_{1,2,3}\},\ \{a^{-1}\},\ \{e_{4,5,6}\},\ \{a\},\ \{e_{1,2}, e_{1,2}a\},\ \{e_{4,5}, e_{4,5}a^{-1}\},\ \{e_{4,6}, e_{4,6}a^{-1}\}, \\
\{e_4, e_4a^{-1}\},\ \{e_{1,3}, e_{1,3}a\},\ \{e_1, e_1a\},\ \{e_\emptyset\}.
\end{gathered}$$

Thus $\ker(\rho_1 \cap \rho_2) = S\backslash\{a, a^{-1}\}$, so the kernel class corresponding to $S\backslash\{a^{-1}\}$ is not closed under intersection, so in particular, does not contain a minimum element.

A further difference to the two sided case is that given a full subsemigroup K it is not obvious if K is the kernel of left congruence. The following is a characterisation which states which full subsemigroups are the kernel of a left congruence.

Proposition 1.4.15 ([61, Proposition 4.4])**.** *Let S be an inverse semigroup, and let $K \subseteq S$ be a full subsemigroup. Then K is the kernel of some left congruence on S if and only if for $a \in K$ there is $b \leq a$ such that $b^{-1} \in K$ and $bx \in K$ implies $ax \in K$ for all $x \in S$.*

We recall Example 1.3.20, in which we considered a semigroup for which the kernel map for two sided congruences was not join preserving. Since congruences are left congruences, and the join as left congruences or as congruences is equal to the join as equivalence relations the same example suffices to demonstrate that the kernel map is not join preserving for left congruences.

We have shown that in general, for a family of left congruences $\{\rho_i \mid i \in I\}$ the kernel map is **not** either join or meet preserving, equivalently in symbols

$$\ker\left(\bigcap_{i\in I}\rho_i\right) \neq \bigcap_{i\in I}\ker(\rho_i), \qquad \text{and} \qquad \ker\left(\bigvee_{i\in I}\rho_i\right) \neq \bigvee_{i\in I}\ker(\rho_i).$$

In fact there is even more bad news. The trace map is not necessarily a lattice homomorphism, it need not preserve join, as the following example shows.

Example 1.4.16. We consider the combinatorial Brandt semigroup with two non zero idempotents, which has multiplication table

	e	f	a	a^{-1}	0
e	e	0	a	0	0
f	0	f	0	a^{-1}	0
a	0	a	0	e	0
a^{-1}	a^{-1}	0	f	0	0
0	0	0	0	0	0

We consider the left congruences ρ_1, the Rees left congruence for the left ideal $\{f, a, 0\}$, which is defined by the partition

$$\{f, a, 0\},\ \{e\},\ \{a^{-1}\}$$

and ρ_2, which we take to be $\mathcal{R}$, so has partition

$$\{f, a^{-1}\},\ \{e, a\},\ \{0\}.$$

Then ρ_2 has trivial trace, so $\operatorname{trace}(\rho_1) \vee \operatorname{trace}(\rho_2) = \operatorname{trace}(\rho_1)$ which is the relation $\{(f, 0), (0, f)\} \cup \iota$. However $\rho_1 \vee \rho_2 = \omega$ which has universal trace. Thus $\operatorname{trace}(\rho_1) \vee \operatorname{trace}(\rho_2) \neq \operatorname{trace}(\rho_1 \vee \rho_2)$.

We have shown that, in general, given $\{\rho_i \mid i \in I\}$ a family of left congruences on S

$$\operatorname{trace}\left(\bigvee_{i\in I} \rho_i\right) \neq \bigvee_{i\in I} \operatorname{trace} \rho_i.$$

However there is some good news.

Theorem 1.4.17 (see [61, Proposition 3.9 & Proposition 6.2])**.** *Let S be an inverse semigroup and let $\{\rho_i \mid i \in I\}$ be a family of left congruences on S. Then*

$$\operatorname{trace}\left(\bigcap_{i\in I} \rho_i\right) = \bigcap_{i\in I} \operatorname{trace}(\rho_i),$$

so the trace map is a complete surjective $\cap$-homomorphism.

Moreover, $(\bigcap_{i\in I} \operatorname{trace}(\rho_i), \bigcap_{i\in I} \ker \rho_i)$ is a pseudo left congruence pair and the corresponding left congruence is $\bigcap_{i\in I} \rho_i$.

We do need the "pseudo" in the final claim of Theorem 1.4.17; as we have seen in Example 1.4.14, the intersection of the kernels of left congruences is not necessarily the kernel of the intersection of the left congruences.

We recall from the discussion for two sided congruences that there are semilattice embeddings of $\mathfrak{C}_N(E)$ and $\mathfrak{N}(S)$ into $\mathfrak{C}(S)$. We recall ν_τ, the minimum (two sided) congruence with trace τ; μ_τ, the maximum congruence with trace τ and λ_K, the minimum congruence with kernel K. Then the maps $\mathfrak{C}_N(E) \hookrightarrow \mathfrak{C}(S)$ defined by $\tau \mapsto \nu_\tau$ and $\mathfrak{N}(S) \hookrightarrow \mathfrak{C}(S)$ defined by $K \mapsto \lambda_K$ are both $\vee$-homomorphisms, and the map $\mathfrak{C}_N(S) \hookrightarrow \mathfrak{C}(S)$ defined by $\tau \mapsto \mu_\tau$ is a $\cap$-homomorphism. For left congruences we do not even have an analogue of λ_K, so the map involving λ_K certainly has no analogue.

Proposition 1.4.18. *Let S be an inverse semigroup and let $E = E(S)$. Then the map $\tau \mapsto \nu_\tau$ is a $\vee$-semilattice embedding $\mathfrak{C}(E) \hookrightarrow \mathfrak{LC}(S)$.*

Proof. Let $\{\tau_i \mid i \in I\}$ be a family of congruences on E, write ν_i for ν_{τ_i} and let $\xi = \bigvee_{i\in I} \tau_i$. It is immediate that the map $\tau \mapsto \nu_\tau$ is order preserving, so certainly $\nu_i \subseteq \nu_\xi$ for each $i \in I$. Thus $\bigvee_{i\in I} \nu_i \subseteq \nu_\xi$ so, in particular, $\operatorname{trace}(\bigvee_{i\in I} \nu_i) \subseteq \xi$. On the other hand, it is clear that $\bigcup_{i\in I} \tau_i \subseteq \operatorname{trace}(\bigvee_{i\in I} \nu_i)$ and, as $\operatorname{trace}(\bigvee_{i\in I} \nu_i)$ is a congruence on E, we have that $\xi = \bigvee_{i\in I} \tau_i \subseteq \operatorname{trace}(\bigvee_{i\in I} \nu_i)$ and so the two are equal. As ν_ξ is the minimum left congruence with trace ξ we have that $\nu_\xi \subseteq \bigvee_{i\in I} \nu_i$ and again we have that the two are equal. $\square$

The map $\tau \mapsto \nu_\tau$ defined in Proposition 1.4.18 shall be important in Chapter 2. We call the set $\{\nu_\tau \mid \tau \in \mathfrak{C}(E)\}$ the set of *trace minimal left congruences.*

The remaining map we considered in the two sided case was $\tau \mapsto \mu_\tau$, which is (for two sided congruences) a $\cap$-homomorphism. We have already seen an example showing that this is not a $\cap$-homomorphism in the one sided case. The congruences ρ_1 and ρ_2 considered in Example 1.4.14 are equal to μ_{τ_1} and μ_{τ_2} where τ_1, τ_2 are as stated in the example. However the intersection $\tau_1 \cap \tau_2$ is the trivial relation, and we know that $\mu_\iota = \mathcal{R}$. However $\rho_1 \cap \rho_2 \neq \mathcal{R}$.

A further comment on a feature of the kernel trace approach to one sided congruences which we may wish were true is that a left congruence pair is also a right congruence pair, unfortunately this is not true in general, which may be seen from the fact that Definition 1.4.8 is not self dual under taking the inverse. The dual conditions to those in Definition 1.4.8 specify the pairs which are the kernel and trace of a right congruence, which we refer to as *right congruence pairs.* These conditions are

(i) for all $a \in S$ and $b \in K$, if $a \geq b$ and $aa^{-1} \; \tau \; bb^{-1}$ then $a \in K$;

(ii) for all $a \in K$ and $e, f \in E(S)$, if $e \; \tau \; f$ then $aea^{-1} \; \tau \; afa^{-1}$;

(iii) for all $a \in K$ there is $b \in S$ with $a \geq b$, $aa^{-1} \; \tau \; bb^{-1}$ and $b^{-1} \in K$.

An example of a left congruence pair which is not a right congruence pair again arises from Example 1.4.14. It is easy to see that the pair $(\tau_1, S \backslash \{a^{-1}\})$

from Example 1.4.14 fails to satisfy any of the three properties above. In Chapter 2 we present a new methodology to characterise left congruences such that more of these properties of the two sided approach have analogues in the one sided case, and in particular the concept we use to replace that of a left congruence pair is left/right dual.

One result for which there is a satisfactory one sided analogue is Theorem 1.3.26. Just as for two sided congruences we say that a left congruence ρ is *idempotent separating* if $\operatorname{trace}(\rho) = \iota$. We write $\mathfrak{LC}_{IS}(S)$ for the lattice of idempotent separating left congruences on S. Also recall that $\mathfrak{V}(S)$ is the lattice of full inverse subsemigroups of S.

Theorem 1.4.19 ([46, Theorem 4.2])**.** *Let S be an inverse semigroup. The lattice of idempotent separating left congruences is isomorphic to the lattice of full inverse subsemigroups.*

The following maps are mutually inverse lattice isomorphisms

$$\mathfrak{V}(S) \to \mathfrak{LC}_{IS}(S); \quad K \mapsto \chi_K = \{(a,b) \in S \times S \mid aa^{-1} = bb^{-1},\ a^{-1}b \in K\},$$
$$\mathfrak{LC}_{IS}(S) \to \mathfrak{V}(S); \quad \rho \mapsto \ker(\rho).$$

Idempotent separating left congruences will play an important role in Chapter 2, so we establish notation. For a full inverse subsemigroup $K \subseteq S$ we let χ_K be the idempotent separating left congruence with kernel K.

For two sided congruences being idempotent separating is equivalent to being contained in $\mathcal{H}$ and we have seen that it is also equivalent to being contained in $\mathcal{R}$. This weaker property is what carries forward to left congruences. For left congruences being idempotent separating is equivalent to being contained in $\mathcal{R}$ and dually for right congruences idempotent separating is equivalent to being contained in $\mathcal{L}$. Indeed, suppose ρ is an idempotent separating left congruence and $a \;\rho\; b$. Then

$$a^{-1}a \;\rho\; a^{-1}b = a^{-1}bb^{-1}b \;\rho\; a^{-1}bb^{-1}a,$$

and as ρ is idempotent separating this implies that $a^{-1}a = a^{-1}bb^{-1}a$. Similarly $b^{-1}b = b^{-1}aa^{-1}b$. Then

$$aa^{-1} = a(a^{-1}a)a^{-1} = a(a^{-1}bb^{-1}a)a^{-1} = b(b^{-1}aa^{-1}b)b^{-1} = b(b^{-1}b)b^{-1} = bb^{-1}.$$

Thus $a \mathcal{R} b$, so $\rho \subseteq \mathcal{R}$. Conversely if $\rho \subseteq \mathcal{R}$, then as each $\mathcal{R}$-class contains exactly one idempotent, each ρ-class contains at most one idempotent or, in other words, ρ is idempotent separating. We shall use idempotent separating left congruences as a cornerstone of our characterisation of left congruences in Chapter 2.

1.5 The inverse semigroups to think about

Everybody thinks about every area of mathematics differently and what is obvious to one person is completely opaque to another. One mechanism commonly used to attempt to bridge divides in thought processes is to provide examples. In this branch of abstract algebra the term examples is less explicit than in other areas, since we generally apply results to structures which are themselves abstract. Nonetheless, we must persevere and there are certain classes and examples of inverse semigroups that we shall refer to frequently through this thesis. We must introduce them somewhere, and so we do so here. We shall characterise left congruences on each of these families of semigroups in Chapter 3, here we describe two sided congruences for comparative purposes.

Clifford semigroups

We have mentioned Clifford semigroups previously. One motivation for inverse semigroup theory is to generalise and connect notions of groups and semilattices. With this in mind Clifford semigroups are perhaps the purest class of inverse semigroups. The explicit characterisation for Clifford semigroups makes this clear, describing Clifford semigroups as strong semilattices of groups. It is this version that we shall make most use of, as it seems to be the easiest for us to use when describing congruences.

We define a *strong semilattice of groups* as follows. Let Y be a semilattice and let $\{G_e \mid e \in Y\}$ be a pairwise disjoint set of groups indexed by Y. Suppose that for $f \leq e$ in Y we have a homomorphism $\phi_{e,f} \colon G_e \to G_f$ such that

(i) $\phi_{e,e}$ is the identity for each $e \in Y$,

(ii) if $f \leq e \leq h$ then $\phi_{h,e}\phi_{e,f} = \phi_{h,f}$.

The semigroup we consider is the set $S = \bigcup_{e \in Y} G_e$, with multiplication defined by

$$xy = (x\phi_{e,ef})(y\phi_{f,ef})$$

where $x \in G_e$ and $y \in G_f$. We write this semigroup $\mathcal{C}(Y, G_e, \phi_{e,f})$. Idempotents in S are the identities in the constituent groups, we write 1_e for the identity in G_e. It is clear that the semilattice of idempotents is isomorphic to Y via the map $1_e \mapsto e$, we shall blur the distinction between e and 1_e, sometimes taking e as an element of Y and sometimes as an element of S.

We remark that it is possible to "do away with" the $\phi_{e,f}$. A disjoint union $\{G_e \mid e \in Y\}$ of groups, with Y a semilattice, is called a *semilattice of groups* if $G_e G_f \subseteq G_{ef}$. If S is a strong semilattice of groups then it is immediate that S is a semilattice of groups. Conversely, if S is a semilattice of groups then $\phi_{e,f}$ (where $f \leq e$), defined by $g\phi_{e,f} = g1_f$, is a homomorphism and it easy to show that S is a strong semilattice of groups. We will use the strong semilattice formulation as it is often useful to know the "fine" structure of S.

There are many characterisations of Clifford semigroups and they pervade many branches of semigroup theory, not just the study of inverse semigroups. The following is a selection of classifications for Clifford semigroups, which we shall generally call upon as properties of Clifford semigroups.

Definition 1.5.1. A semigroup S is called a *Clifford semigroup* if one of the following equivalent conditions hold:

(i) S is a strong semilattice of groups;

(ii) S is a semilattice of groups;

(iii) S is regular and idempotents are central (commute with all elements);

(iv) S is inverse and $ss^{-1} = s^{-1}s$ for all $s \in S$;

(v) S is regular and $\mathcal{D} \cap (E(S) \times E(S))$ is the identity relation.

The kernel trace description of congruences is applied to Clifford semigroups in [59]; this approach is well suited to congruences on Clifford semigroups. A large reason for this is the observation that any congruence τ on the idempotents is normal, indeed if $e \; \tau \; f$ then

$$aea^{-1} = aa^{-1}e \; \tau \; aa^{-1}f \; = afa^{-1}$$

and similarly $a^{-1}ea \; \tau \; a^{-1}fa$. A further reason is that inverse subsemigroups of Clifford semigroups are also Clifford, and a final reason is that, as previously remarked, a normal subsemigroup of a Clifford semigroup is precisely one which is full, self conjugate and inverse. Bearing in mind these remarks we proceed to describe inverse subsemigroups of Clifford semigroups.

Lemma 1.5.2. *Let $S = \mathcal{C}(Y, G_e, \phi_{e,f})$ be a Clifford semigroup and let $T \subseteq S$ be an inverse subsemigroup. Then there is a subsemilattice $X \subseteq Y$ and there are subgroups $H_e \leq G_e$ for $e \in X$ such that $H_e\phi_{e,f} \subseteq H_f$ for $f \leq e \in X$, with*

$$T = \mathcal{C}(X, H_e, \phi_{e,f}|_{H_e}) = \bigcup_{e \in X} H_e.$$

Conversely, if X and H_e (for $e \in X$) satisfy these conditions then the union of the subgroups is an inverse subsemigroup of S.

Furthermore, T is full if and only if $X = Y$, and T is self conjugate if and only if each H_e is normal in G_e.

To describe congruences we want to classify the congruence pairs, we use Definition 1.3.10, which we recall says that a pair (τ, K), with $K \subseteq S$ a full self conjugate inverse subsemigroup and τ a normal congruence on $E(S)$, is a congruence pair if

(CP1) $ae \in K$ and $e \; \tau \; a^{-1}a$ implies that $a \in K$;

(CP3) $a \in K$ implies that $a^{-1}a \; \tau \; aa^{-1}$.

The computation of which pairs are congruence pairs is made easier by the fact that $a^{-1}a = aa^{-1}$ for all $a \in S$ so (CP3) is automatically true. Therefore it remains to apply the condition (CP1).

Lemma 1.5.3. *Let $S = \mathcal{C}(Y, G_e, \phi_{e,f})$ be a Clifford semigroup, let $T = \mathcal{C}(Y, H_e, \phi_{e,f}|_{H_e})$ be a self conjugate full inverse subsemigroup of S and let τ be a congruence on Y. Then the following are equivalent*

(i) $ae \in T$ and $e \ \tau \ a^{-1}a$ imply that $a \in T$;

(ii) $f \leq e$ and $e \ \tau \ f$ imply that $H_e = \{g \in G_e \mid g\phi_{e,f} \in H_f\}$.

Proof. Suppose first that (i) is satisfied. Take $f \leq e$ with $e \ \tau \ f$. Since T is a subsemigroup, $\{g\phi_{e,f} \mid g \in H_e\} \subseteq H_f$, which precisely says that

$$H_e \subseteq \{g \in G_e \mid g\phi_{e,f} \in H_f\}.$$

For the reverse inclusion suppose that $g \in G_e$ and $g\phi_{e,f} \in H_f$. Then

$$g1_f = (g\phi_{e,f})(1_f\phi_{f,f}) = (g\phi_{e,f})1_f = g\phi_{e,f},$$

so $g1_f \in T$. As $1_f \ \tau \ 1_e = g^{-1}g$, by applying (i) we obtain that $g \in T$. Thus $g \in H_e$, so we have that (ii) is satisfied.

For the converse we suppose that (ii) holds, and suppose that there is $a \in S$ for which $ae \in T$ and $a^{-1}a \ \tau \ e$. Then

$$ae = (a\phi_{a^{-1}a,a^{-1}ae})(e\phi_{e,a^{-1}ae}) = (a\phi_{a^{-1}a,a^{-1}ae}) \in H_{a^{-1}ae},$$

and also $a^{-1}ae \ \tau \ a^{-1}a$. By (ii)

$$H_{a^{-1}a} = \{g \in G_{a^{-1}a} \mid g\phi_{a^{-1}a,a^{-1}ae} \in H_{a^{-1}ae}\}.$$

As $ae = a\phi_{a^{-1}a,a^{-1}ae}$, we have that $a \in H_{a^{-1}a}$, so $a \in T$ and (i) is satisfied. $\square$

We can now state the kernel trace description of congruences on Clifford semigroups.

Theorem 1.5.4 ([59])**.** *Let S be the Clifford semigroup (G_e, Y) and let τ be a congruence on Y and $T = \mathcal{C}(Y, H_e, \phi_{e,f}|_{H_e})$ be a normal subsemigroup. Then (τ, T) is a congruence pair for S if and only if $f \leq e$ and $e \ \tau \ f$ imply that $H_e = \{g \in G_e \mid g\phi_{e,f} \in H_f\}$.*

The bicyclic monoid

The bicyclic monoid is an important inverse semigroup, it forms the basis of the study of *bisimple* ($\mathcal{D} = \omega$) inverse semigroups and crops up in all sorts of interesting places, and its inclusion here has the added advantage of it being my supervisor's favourite semigroup. We write $\mathbb{N}^0$ for the set $\mathbb{N} \cup \{0\}$.

Definition 1.5.5. The *bicyclic monoid* is defined as the set $\mathbb{N}^0 \times \mathbb{N}^0$ with multiplication

$$(a, b)(c, d) = (a - b + t, d - c + t)$$

where $t = \max\{b, c\}$.

For the rest of this section on the bicyclic monoid we let B be the bicyclic monoid. As mentioned $\mathcal{D} = \mathcal{J} = \omega$, which follows from the fact that the other Green's relations are given by

$$(a, b)\ \mathcal{R}\ (c, d) \iff a = c \qquad \text{and} \qquad (a, b)\ \mathcal{L}\ (c, d) \iff b = d$$

so that $\mathcal{H}$ is the trivial relation. Semigroups for which $\mathcal{H}$ is trivial are called *combinatorial*. Idempotents in B are the elements (n, n) for $n \in \mathbb{N}^0$, and it follows that the idempotent semilattice of B is a chain with $(n, n) \leq (m, m)$ when $n \geq m$.

Congruences on B are described by the kernel trace approach in the following way. As shall be explored later (in Chapter 3) congruences on $E(B)$ correspond to partitions of the set $\mathbb{N}^0$.

Lemma 1.5.6. *The only normal congruences on $E(B)$ are the trivial and universal congruences.*

Proof. We suppose that $\tau \in \mathfrak{C}_N(E)$ is not the trivial relation, we will show that τ is the universal relation. Since τ is non-trivial there are idempotents $(m, m) \neq (n, n)$ such that $(m, m)\ \tau\ (n, n)$. We assume without loss of generality that $n > m$. We then note that

$$(0, 0) = (0, m)(m, m)(m, 0)\ \tau\ (0, m)(n, n)(m, 0) = (n - m, n - m).$$

In particular, as each τ-class is a subsemilattice and $n > m$, we have that $(0,0)\ \tau\ (1,1)$. We proceed by induction to show that $(0,0)\ \tau\ (k,k)$ for all k. Indeed, if $(0,0)\ \tau\ (k,k)$ then

$$(0,0)\ \tau\ (1,1) = (1,0)(0,0)(0,1)\ \tau\ (1,0)(k,k)(0,1) = (k+1,k+1).$$

This completes the proof. $\square$

Lemma 1.5.7. *The normal subsemigroups of B are the sets*

$$T(d) = \{(a,b) \in B\ :\ d \mid b-a\}$$

for each $d \in \mathbb{N}^0$ (noting that $E(B) = T(0)$).

Proof. That $T(d)$ is a normal semigroup follows from an elementary direct computation, we show that all normal subsemigroups are of this form. We remark that as a full inverse subsemigroup $T(d)$ is generated by $(0,d)$, by which we mean every element in $T(d)$ is of the form $(0,d)^n(m,m)$ for $n \in \mathbb{Z}$ and $m \in \mathbb{N}^0$.

Suppose that $K \subseteq B$ is a normal subsemigroup with $K \neq E(B)$. Choose $(a,b) \in K \backslash E(B)$ with $|b-a|$ minimal, note that we may assume that $b > a$ (else take (b,a)), and let $d = b-a$. We will show that $K = T(d)$. By the remark in the previous paragraph, to show that $T(d) \subseteq K$ it suffices to show that $(0,d) \in K$. We note that $(0,0) = (0,a)(a,0) \in K$ so as K is normal we have

$$(0,a)(a,b)(a,0) = (0,b-a) = (0,d) \in K.$$

Thus we have that $T(d) \subseteq K$. For the reverse inclusion we note that, by a similar argument, if there is $(x,y) \in K$ (with $y > x$) then $(0,y-x) \in K$. Recall we chose (a,b) such that $b-a = d$ is minimal, so $d \leq y-x$. Let $n \geq 1$ be chosen such that $nd \leq y-x < (n+1)d$. Then

$$(0,y-x)(d,0)^n = (0,y-x)(nd,0) = (0,(y-x)-nd) \in K.$$

But $0 \leq (y-x)-nd < d$, which implies (as d was chosen minimally) that $(y-x)-nd = 0$, so $d \mid y-x$ or equivalently $(x,y) \in T(d)$. Thus $K = T(d)$. $\square$

We notice that $T(d) \subseteq T(c)$ if and only if $c \mid d$. We write $\mathbf{D}$ for the lattice on $\mathbb{N}$ with ordering $m \leq_D n$ if $n \mid m$, so that join is greatest common divisor and meet is least common multiple. Thus the lattice of normal subsemigroups is isomorphic to $\mathbf{D}^0$, which is $\mathbf{D}$ with a zero adjoined. Adjoining a zero to a lattice L is a common operation, and is similar to adjoining a zero to a semigroup; L^0 is the set $L \cup \{0\}$ with $0 \vee a = a$ and $0 \wedge a = 0$ for all $a \in L^0$.

Applying the definition of congruence pair, it is immediate that the only congruence pair containing ι as the trace is $(\iota, E(B))$, indeed this comes immediately from (CP3) as, if (m, n) is in the kernel of some congruence with trace ι then, by (CP3), $(m, m)\ \iota\ (n, n)$, so $m = n$. On the other hand every normal subsemigroup forms a congruence pair with ω.

Theorem 1.5.8 (see [52, Theorem 1.3]). *The congruence pairs for the bicyclic monoid B are the pairs $(\omega, T(d))$ for $d \in \mathbb{N}$, $(\omega, E(B))$ and $(\iota, E(B))$. The lattice $\mathfrak{C}(B)$ is isomorphic to $(\mathbf{D}^0)^0$ (where yes we adjoin a second zero to $\mathbf{D}^0$ even though one exists).*

Brandt semigroups

Brandt semigroups are the building blocks for many inverse semigroups, in particular, in the happy land where everything is finite. Explicitly Brandt semigroups are inverse semigroups which are also completely 0-simple.

Definition 1.5.9. A semigroup S is *completely* 0*-simple* if S has a zero (an element $0 \in S$ such that $0s = 0 = s0$ for all $s \in S$), $S^2 \neq 0$, the only $\mathcal{J}$-classes are $\{0\}$ and $S \backslash \{0\}$, and S has a *primitive idempotent*, by which we mean an idempotent which is minimal in the set of non-zero idempotents.

Definition 1.5.10. Let I be a set and let G be a group. Then the Brandt semigroup $\mathcal{B}(I, G)$ is defined as

$$\mathcal{B}(I, G) = (I \times G \times I) \cup \{0\}$$

with multiplication

$$(i,g,j)(k,h,l) = \begin{cases} (i,gh,l) & \text{if } \ j=k \\ 0 & \text{otherwise} \end{cases}$$

and $0(i,g,j) = 0 = (i,g,j)0 = 0^2$.

Green's relations on the Brandt semigroup $\mathcal{B}(I,G)$ are elementary:

$$(i,g,j) \ \mathcal{R} \ (k,h,l) \iff i=k, \quad \text{and} \quad (i,g,j) \ \mathcal{L} \ (k,h,l) \iff j=l.$$

Then

$$(i,g,j) \ \mathcal{H} \ (k,h,l) \iff i=k \text{ and } j=l,$$

and $\mathcal{D} = \mathcal{J}$ is the relation $((S\backslash 0)\times(S\backslash 0)) \cup \{(0,0)\}$. Further, each non-zero group $\mathcal{H}$-class is isomorphic to the group G. Brandt semigroups play an important role via principal factors, which we now define.

Definition 1.5.11. Let S be a semigroup and take $a \in S$. Define $I(a) = \{b \in S^1aS^1 \mid a \notin S^1bS^1\}$ the set of elements strictly below a in the $\mathcal{J}$-order. If $I(a)$ is non empty then it is an ideal and the *principal factor* generated by a is the Rees quotient

$$S^1aS^1/I(a).$$

If I_a is empty (a is $\mathcal{J}$-minimal) then the principal factor generated by a is J_a, where we recall that J_a is the $\mathcal{J}$-class of a.

The set $\{J_a/I(a) \mid a \in S\}$ is the set of *principal factors of* S.

The principal factors are 0-simple semigroups and if S is a finite inverse semigroup it follows that the principal factors are completely 0-simple, so are isomorphic to Brandt semigroups. While much of the work in this thesis is not confined to the finite world our results are often stronger when we do restrict ourselves. Actually it is the case that the principal factors of an inverse semigroup are Brandt semigroups in a large number of scenarios not just when S is finite.

Congruences on Brandt semigroups are straightforward to describe. Idempotents in $B = \mathcal{B}(I,G)$ are the elements of the form $(i,1,i)$ for $i \in I$ and

1 the identity in G, along with 0. It is easy to see that the idempotent semilattice of B is the "null" semilattice on $I \cup \{0\}$ where a *null* semigroup is one in which every product is zero, so the *null semilattice* has all products $ef = 0$ when $e \neq f$ and $ee = e$ for all e. For any non-zero idempotents e, f there is $a \in S$ such that $aea^{-1} = f$, indeed if $e = (i, 1, i)$ and $f = (j, 1, j)$ then $a = (j, 1, i)$ works.

It is now immediate that the only normal congruences are the trivial and universal congruences. Indeed, suppose τ is a normal congruence on $E(B)$ then, if there are $e \neq d \in E(S)$ and $e \ \tau \ d$, then $e \ \tau \ ed = 0$. For each $f \in E(B)$, by choosing $a \in B$ such that $aea^{-1} = f$, we have that $f = aea^{-1} \ \tau \ a0a^{-1} = 0$. The fact that the only normal congruences on the idempotents are the trivial and universal congruences is the same as the case for the bicyclic monoid, which suggests that the condition of normality for a congruence on the idempotents is quite strong.

It is also an elementary verification exercise to show that normal subsemigroups of $\mathcal{B}(I, G)$ are $\mathcal{B}(I, G)$ itself and the subsemigroups T_N for each $N \trianglelefteq G$, which is defined as

$$T_N = \{(i, g, i) \mid i \in I, \ g \in N\}.$$

Theorem 1.5.12 ([63, Theorem 2])**.** *Let $S = \mathcal{B}(I, G)$ be a Brandt semigroup. Then the congruence pairs for S are (ι, T_N) for $N \trianglelefteq G$ and (ω, S). In particular, the non-universal congruences form a sublattice of $\mathfrak{C}(S)$ which is isomorphic to the lattice of normal subgroups of G.*

The symmetric inverse monoid

This is probably the best known and possibly the most important inverse semigroup. It plays the role within inverse semigroup theory that the symmetric group does within group theory. There is lots that we might wish to say about symmetric inverse monoids, however as usual we cannot improve upon previous works. On this occasion we direct the reader to Lipscomb's comprehensive book [44].

Definition 1.5.13. Let X be a set. The *symmetric inverse monoid* $\mathcal{I}_X$ on X is defined as the set

$$\mathcal{I}_X = \{f\colon A \to B \mid A, B \subseteq X,\ f \text{ a bijection}\}$$

of bijective maps between subsets of X. Composition of $f, g \in \mathcal{I}_X$ is defined as composition of f, g as partial functions, that is, $x(fg) = y$ if there is $z \in X$ such that $xf = z$ and $zg = y$.

Usually the labels of the elements of X are unimportant, and when X is finite we assume that they are labelled $\{1, 2, \dots, n\}$. In this case, when $|X| = n$, we write $\mathcal{I}_n$ for $\mathcal{I}_X$, and call this the *rank n symmetric inverse monoid*. We remark that a symmetric inverse monoid is finite precisely when the ground set X is finite.

It is often helpful to think of $\mathcal{I}_n$ visually. Elements are partial matchings between two rows of n vertices, and composition is given by layering the graphs and then removing the middle set of vertices. For example see Fig. 1.7. Monoids which may be described graphically in this way, as graphs on two rows of vertices (subject to some conditions) are called diagram monoids, and form a natural generalisation of "transformation-type" monoids. Congruences on such monoids are described in [13].

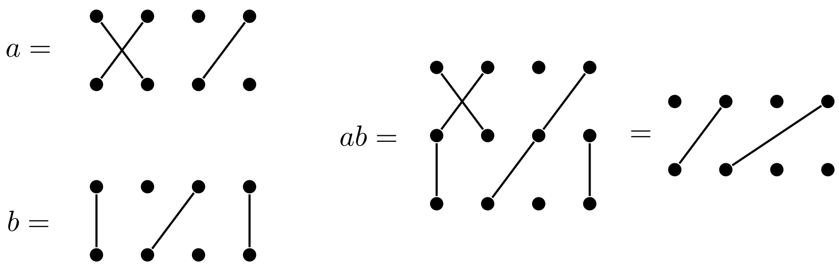


Figure 1.7: Multiplication in $\mathcal{I}_n$

The idempotents for $\mathcal{I}_X$ are the partial identity functions $e_A\colon A \to A$ and in particular $e_A e_B = e_{A\cap B}$. Therefore the semilattice of idempotents $E(\mathcal{I}_X)$ is isomorphic to the *intersection monoid* $\mathcal{P}_X$, that is the powerset of X under intersection. We usually use $\mathcal{P}_X$ when we refer to the semilattice of idempotents in $\mathcal{I}_X$, and we write $\mathcal{P}_n$ for the intersection monoid when

$X = \{1, 2, \ldots, n\}$. Inverses in $\mathcal{I}_X$ are inverses in the sense of partial functions, so f^{-1} has $xf^{-1} = y$ exactly when $yf = x$.

On any inverse semigroup 'range' and 'domain' functions $\mathbf{r}\colon S \to E(S)$ and $\mathbf{d}\colon S \to E(S)$ are defined by $\mathbf{r}(a) = a^{-1}a$ and $\mathbf{d}(a) = aa^{-1}$. This is an extension of the category theoretic notions of the domain and range of morphisms and comes from the viewing inverse semigroups as a particular type of category known as an inductive groupoid, an approach embodied by the Ehresmann-Schein-Nambooripad Theorem. On symmetric inverse monoids $\mathbf{d}$ and $\mathbf{r}$ are the domain and range functions regarding elements of $\mathcal{I}_X$ as partial functions. In the case of $\mathcal{I}_X$ we usually use the terms *domain* and *image* instead of domain and range, and we write $\operatorname{Dom}(a)$ and $\operatorname{Im}(a)$. Furthermore we define the *rank* of an element $a \in \mathcal{I}_X$ as $\operatorname{rank}(a) = |\operatorname{Im}(a)| = |\operatorname{Dom}(a)|$.

Green's relations on $\mathcal{I}_n$ are straightforward and elegant,

$$\begin{aligned}
a \mathrel{\mathcal{R}} b &\iff \operatorname{Dom}(a) = \operatorname{Dom} b), \\
a \mathrel{\mathcal{L}} b &\iff \operatorname{Im}(a) = \operatorname{Im}(b), \\
a \mathrel{\mathcal{H}} b &\iff \operatorname{Dom}(a) = \operatorname{Dom}(b) \quad \text{and} \quad \operatorname{Im}(a) = \operatorname{Im}(b), \\
a \mathrel{\mathcal{D}} b &\iff \operatorname{rank}(a) = \operatorname{rank} b), \\
a \mathrel{\mathcal{J}} b &\iff \operatorname{rank}(a) = \operatorname{rank}(b).
\end{aligned}$$

It is relevant to observe that this implies that there are $n+1$ $\mathcal{J}$-classes (and so $n+1$ $\mathcal{D}$-classes) and further that the ideal structure of $\mathcal{I}_n$ is a chain of length $n+1$

$$I_0 \subseteq I_1 \subseteq I_2 \subseteq \cdots \subseteq I_{n-1} \subseteq I_n = \mathcal{I}_n$$

where $I_k = \{a \in \mathcal{I}_n \mid \operatorname{rank}(a) \leq n\}$. The group $\mathcal{H}$-classes are symmetric groups, for $A \subseteq \{1, 2, \ldots, n\}$ with $|A| = k$ the $\mathcal{H}$-class $H_{e_A} \cong \mathcal{S}_k$ (we write $\mathcal{S}_k$ for the k^{th} symmetric group).

The reason that symmetric inverse semigroups play a central role in inverse semigroup theory is the fact that every inverse semigroup embeds into a symmetric inverse monoid.

Theorem 1.5.14 (The Vagner-Preston representation theorem)**.** *Let S be an inverse semigroup. Then there exists a symmetric inverse monoid $\mathcal{I}_X$*

and an embedding $\phi\colon S \hookrightarrow \mathcal{I}_X$. Furthermore, when S is finite, X can be chosen to be finite.

Congruences on $\mathcal{I}_X$ have been thoroughly described, first by Liber in [41] and then later the kernel normal system approach is applied by Scheiblich in [71]. We summarise here the kernel trace approach to congruences on symmetric inverse monoids.

Lemma 1.5.15. *Let τ be a congruence on $\mathcal{P}_n$. Then τ is normal in $\mathcal{I}_n$ if and only if there is $0 \leq k \leq n$ such that*

$$\tau = \tau_k = \{(e_A, e_B) \in \mathcal{P}_n \times \mathcal{P}_n \mid A = B \quad or \quad |A|, |B| \leq K\}.$$

For $A \subseteq [n]$ we define $\gamma_A\colon [|A|] \to A$ to be the unique order preserving bijection. We know that the $\mathcal{H}$-classes containing idempotents are isomorphic to symmetric groups, in particular, for $A \subseteq [n]$ the function $\theta_A\colon H_{e_A} \to \mathcal{S}_{|A|}$ defined by $a \mapsto \gamma_A a \gamma_A^{-1}$ is an isomorphism.

Lemma 1.5.16. *Let $K \subset \mathcal{I}_n$ be a normal inverse subsemigroup not equal to $\mathcal{I}_n$. Then K is equal to $K(k, N)$ for some $1 \leq k \leq n$ and a normal subgroup $N \trianglelefteq \mathcal{S}_k$, where*

$$K(k, N) = I_{k-1} \cup \{a \in \mathcal{I}_n \mid \operatorname{rank}(a) = k,\ a \mathrel{\mathcal{H}} e_A,\ a\theta_A \in N\} \cup E(\mathcal{I}_n)$$

where $\theta_A\colon H_{e_A} \to \mathcal{S}_k$ is as defined above.

Conversely, if $1 \leq k \leq n$ and $N \trianglelefteq \mathcal{S}_k$ a normal subgroup, then $K(k, N)$ is a normal inverse subsemigroup of $\mathcal{I}_n$ (and is not equal to $\mathcal{I}_n$).

It follows from elementary computation which pairs of a normal congruence on $\mathcal{P}_n$ and a normal subsemigroup of $\mathcal{I}_n$ are congruence pairs.

Theorem 1.5.17 ([71, Theorem 2.7])**.** *The congruence pairs for $\mathcal{I}_n$ are exactly $(\omega, \mathcal{I}_n)$ and $(\tau_k, K(k, N))$ for $1 \leq k \leq n$ and $N \trianglelefteq \mathcal{S}_k$. In particular, $\mathfrak{C}(\mathcal{I}_n)$ is a chain, and $|\mathfrak{C}(\mathcal{I}_n)| = 3n - 1$ (unless $n = 2, 3$ when $|\mathfrak{C}(\mathcal{I}_n)| = 4, 7$ respectively).*

We shall spend quite a while considering one sided congruences on $\mathcal{I}_n$ in Chapter 3, and will dive deeper into the structure of $\mathcal{I}_n$ at that point, this is partly why we are skimpy on the details here, along with the assumption that the reader is familiar with $\mathcal{I}_n$.

— 2 —

One sided congruences on inverse semigroups

As explained in Chapter 1, it is possible to reconstruct a left congruence on an inverse semigroup from the kernel and trace (Theorem 1.4.9). This chapter is devoted to developing and improving upon this theory. We introduce the notion of an inverse kernel for a left congruence on an inverse semigroup and show that a left congruence is determined by its trace and inverse kernel. We follow along the paths trodden when analysing two sided congruences and discuss various properties of the trace and inverse kernel, in particular that both the trace and inverse kernel maps are onto $\cap$-homomorphisms. We identify the lattice of left congruences as a subset of the direct product of the lattice of congruences on the idempotents and the lattice of full inverse subsemigroups, and we compute the meet and join of left congruences in the terms of this identification.

2.1 The Inverse Kernel

We follow on from Section 1.4, so will reuse our notation and assumptions, though where more precision is needed - now that we have moved from the preliminary section where the rules regarding rigour are somewhat more lax than in the meat of the thesis - we may repeat ourselves a little. One example of this is in the upcoming definition which builds upon the left normaliser introduced previously. Throughout this chapter we take S to be an inverse semigroup and let $E = E(S)$ be the semilattice of idempotents.

Definition 2.1.1. Let τ be a congruence on E. Define $N_L(\tau)$, the *left-normaliser* of τ by

$$N_L(\tau) = \{a \in S \mid e \;\tau\; f \implies a^{-1}ea \;\tau\; a^{-1}fa\},$$

and $N_R(\tau)$, the *right normaliser* by

$$N_R(\tau) = \{a \in S \mid e \;\tau\; f \implies aea^{-1} \;\tau\; afa^{-1}\}.$$

The *normaliser* $N(\tau)$ is then defined as

$$\begin{aligned} N(\tau) &= N_R(\tau) \cap N_L(\tau) \\ &= \{a \in S \mid e \ \tau \ f \implies aea^{-1} \ \tau \ afa^{-1} \text{ and } a^{-1}ea \ \tau \ a^{-1}fa\}. \end{aligned}$$

This is an important definition for us, and we shall use the normaliser with great frequency. Relevant observations to make at this point include that $N(\tau)$ is a full inverse subsemigroup, indeed it is the largest inverse subsemigroup contained in $N_L(\tau)$ (or indeed in $N_R(\tau)$). This follows from the observation $N_R(\tau) = \{a \in S \mid a^{-1} \in N_L(\tau)\}$, and in turn that

$$N(\tau) = \{a \in N_L(\tau) \mid a^{-1} \in N_L(\tau)\}.$$

It is also worth reminding ourselves that for a left congruence ρ on S with trace τ, we have that $\ker(\rho) \subseteq N_L(\tau)$. This follows immediately from Theorem 1.4.12 or can be simply shown directly. Indeed, suppose that $a \in \ker(\rho)$, so there is $g \in E$ with $a \ \rho \ g$. For $e, f \in E$ with $e \ \tau \ f$ we observe

$$a^{-1}ea \ \rho \ a^{-1}eg = a^{-1}ge \ \rho \ a^{-1}gf = a^{-1}fg \ \rho \ a^{-1}fa,$$

hence $a \in N_L(\tau)$.

In [60] it is noted that the primary issue with the kernel is that given $a \in K$ it is not possible to determine to which idempotent a is related. With this motivation we make the following definition.

Definition 2.1.2. For a left congruence ρ on S the *inverse kernel* of ρ is the set

$$\text{Inker}(\rho) = \{a \mid a \ \rho \ aa^{-1}\}.$$

We immediately note that the inverse kernel of a left congruence ρ is contained in the kernel, and that while given $a \in \text{Inker}(\rho)$ and $e \in E$ it is not possible to determine from $\text{Inker}(\rho)$ alone whether $a \ \rho \ e$, we are able to specify one idempotent to which a is ρ-related, namely aa^{-1}. We also recall Lemma 1.3.38, which says that for a two sided congruence κ the kernel is equal to the set $\{a \mid a \ \kappa \ aa^{-1}\}$. Thus the definition of inverse kernel extends the definition of the kernel for a two sided congruence to one sided congruences, albeit in a slightly less obvious way than the usual definition of kernel for a left congruence.

Proposition 2.1.3. *Let ρ be a left congruence on S, let $\tau = \operatorname{trace}(\rho)$, and let $K = \ker(\rho)$. Then the following hold:*

(i) $\operatorname{Inker}(\rho)$ *is a full inverse subsemigroup of S;*

(ii) $\operatorname{Inker}(\rho) = \ker(\rho \cap \mathcal{R})$;

(iii) $\operatorname{Inker}(\rho) = \{a \in K \mid a^{-1} \in K\}$;

(iv) $\operatorname{Inker}(\rho) = K \cap N(\tau)$.

Proof. (i) First we observe that $\operatorname{Inker}(\rho)$ is a subsemigroup. Indeed suppose that $a, b \in \operatorname{Inker}(\rho)$, so $a \;\rho\; aa^{-1}$, and $b \;\rho\; bb^{-1}$. Then

$$ab \;\rho\; abb^{-1} = abb^{-1}a^{-1}a \;\rho\; abb^{-1}a^{-1}aa^{-1} = abb^{-1}a^{-1} = (ab)(ab)^{-1}.$$

Further, as $e \;\rho\; e = ee^{-1}$ we have that $E \subseteq \operatorname{Inker}(\rho)$, so $\operatorname{Inker}(\rho)$ is full. Also if $a \;\rho\; aa^{-1}$ then by multiplying on the left by a^{-1} we observe that $a^{-1}a \;\rho\; a^{-1}$, hence $\operatorname{Inker}(\rho)$ is closed under taking inverses. Thus $\operatorname{Inker}(\rho)$ is a full inverse subsemigroup of S.

(ii) This is immediate from the definition of the inverse kernel, recalling that on an inverse semigroup $a \;\mathcal{R}\; e$ for $e \in E$ exactly when $aa^{-1} = e$.

(iii) We have already noted that $\operatorname{Inker}(\rho)$ is an inverse subsemigroup contained in the kernel, thus

$$\operatorname{Inker}(\rho) \subseteq \{a \in K \mid a^{-1} \in K\}.$$

For the reverse inclusion we suppose that $a, a^{-1} \in K$ so there are $e, f \in E$ with $a \;\rho\; e$ and $a^{-1} \;\rho\; f$. We note that since $a^{-1} \;\rho\; f$ we have $fa^{-1} \;\rho\; f \;\rho\; a^{-1}$. We then observe that

$$a = aa^{-1}a \;\rho\; aa^{-1}e = eaa^{-1} \;\rho\; eafa^{-1} = afa^{-1}e \;\rho\; afa^{-1}a = af \;\rho\; aa^{-1}.$$

Thus $a \in \operatorname{Inker}(\rho)$, so $\{a \in K \mid a^{-1} \in K\} \subseteq \operatorname{Inker}(\rho)$ and we have that the two are equal.

(iv) We recall that if $\mathrm{trace}(\rho) = \tau$ then $K = \ker(\rho) \subseteq N_L(\tau)$. We have observed that $N(\tau) = \{a \in N_L(\tau) \mid a^{-1} \in N_L(\tau)\}$. Then (iii) gives $\mathrm{Inker}(\rho) \subseteq N(\tau)$. Thus we have that $\mathrm{Inker}(\rho) \subseteq N(\tau) \cap K$. For the reverse inclusion suppose $a \in N(\tau) \cap K$. As $a \in K$ there is $e \in E$ with $a \; \rho \; e$ from which we obtain that $a^{-1}ea \; \rho \; a^{-1}e \; \rho \; a^{-1}a$. As $a^{-1}a, a^{-1}ea$ are idempotents this implies $a^{-1}a \; \tau \; a^{-1}ea$, which we may conjugate by $a \in N(\tau)$ to obtain that

$$aa^{-1} = a(a^{-1}a)a^{-1} \; \tau \; a(a^{-1}ea)a^{-1} = aa^{-1}e.$$

As $a \; \rho \; e$ it follows that $(aa^{-1})a \; \rho \; (aa^{-1})e$ and so

$$a = aa^{-1}a \; \rho \; aa^{-1}e \; \rho \; aa^{-1}.$$

Thus $a \in \mathrm{Inker}(\rho)$, and so we have that $\mathrm{Inker}(\rho) = K \cap N(\tau)$.

□

From Proposition 2.1.3(iii) we observe that when the kernel of a left congruence is closed under taking inverses then the inverse kernel is equal to the kernel. For instance for two sided congruences and idempotent separating left congruences the notion of kernel and inverse kernel coincide. We recall Theorem 1.4.19, which states that the lattice of idempotent separating left congruences is isomorphic to the lattice of full inverse subsemigroups.

Corollary 2.1.4. *Let $T \subseteq S$ be a full inverse subsemigroup. Then there is a unique idempotent separating congruence χ on S such that*

$$\mathrm{Inker}(\chi) = T = \ker(\chi).$$

For certain classes of inverse semigroups, including Clifford semigroups (see [61] or Chapter 3) the kernel of a left congruence is always an inverse subsemigroup. The description of one sided congruences on inverse semigroups given in this chapter coincides with the kernel-trace description from [61] on these classes of semigroups.

As we have remarked previously the normaliser of a congruence on E is the maximum inverse subsemigroup contained in the left normaliser. From

Proposition 2.1.3(iii) we have that the inverse kernel of a left congruence is the largest inverse subsemigroup contained in the kernel. Applying this to the maximum left congruence with a fixed trace (for which we recall we write μ_τ) we have the following corollary.

Corollary 2.1.5. *Let τ be a congruence on E, then*

$$N(\tau) = \text{Inker}(\mu_\tau) = \ker(\mu_\tau \cap \mathcal{R})$$

Proof. We recall the description of μ_τ from Theorem 1.4.11,

$$\begin{aligned} \mu_\tau = \{(a,b) \in S \times S \mid\ & a^{-1}a \ \tau \ a^{-1}bb^{-1}a,\ b^{-1}b \ \tau \ b^{-1}aa^{-1}b, \\ & e \ \tau \ f \implies a^{-1}beb^{-1}a \ \tau \ a^{-1}bfb^{-1}a,\ b^{-1}aea^{-1}b \ \tau \ b^{-1}afa^{-1}b\}. \end{aligned}$$

It is an elementary verification exercise that if $a \in N(\tau)$ then $a \ \mu_\tau \ aa^{-1}$. Hence $N(\tau) \subseteq \text{Inker}(\mu_\tau)$. However, by applying Proposition 2.1.3(iv) we have that $\text{Inker}(\mu_\tau) \subseteq N(\tau)$. $\square$

Furthermore from Proposition 2.1.3 we observe that the kernel determines the inverse kernel so the set of left congruences with the same inverse kernel is a union of kernel classes. Inspired by the kernel and trace maps defined in Chapter 1 we make the following natural definition, recalling that $\mathfrak{V}(S)$ is the lattice of full inverse subsemigroups of S.

Definition 2.1.6. The *inverse kernel map* is the function

$$\text{Inker}\colon \mathfrak{LC}(S) \to \mathfrak{V}(S); \quad \rho \mapsto \text{Inker}(\rho).$$

If ρ is a left congruence the *inverse kernel class* of ρ is

$$[\rho]_{\text{Inker}} = \{\kappa \in \mathfrak{LC}(S) \mid \text{Inker}(\kappa) = \text{Inker}(\rho)\}.$$

We know (Theorem 1.4.17) that the map $\rho \mapsto \text{trace}(\rho)$ is a complete surjective $\cap$-homomorphism from $\mathfrak{LC}(S)$ onto $\mathfrak{C}(E)$, so in particular:

$$\text{trace}(\rho_1 \cap \rho_2) = \text{trace}(\rho_1) \cap \text{trace}(\rho_2).$$

We have remarked that the kernel map is not in general a $\cap$-homomorphism. However it is elementary, following immediately from the definition, that the inverse kernel map is such. Recall for a full inverse subsemigroup T that we write χ_T for the idempotent separating congruence with kernel and inverse kernel equal to T.

Theorem 2.1.7. *The inverse kernel map, $\rho \mapsto \mathrm{Inker}(\rho)$, is a complete surjective $\cap$-homomorphism of $\mathfrak{LC}(S)$ onto $\mathfrak{V}(S)$.*

Moreover, the inverse kernel class $\{\rho \in \mathfrak{LC}(S) \mid \mathrm{Inker}(\rho) = T\}$ is closed under $\cap$ and has a minimum element, which is χ_T. In particular, if κ is a left congruence then the minimum left congruence in $[\kappa]_{\mathrm{Inker}}$ is $\kappa \cap \mathcal{R}$.

Proof. Let $\{\rho_i \mid i \in I\}$ be a family of left congruences. Then

$$\begin{aligned} a \in \mathrm{Inker}\left(\bigcap_{i\in I} \rho_i\right) &\iff a \ \left(\bigcap_{i\in I} \rho_i\right) \ aa^{-1} \\ &\iff a \ \rho_i \ aa^{-1} \quad \text{for all} \quad i \in I \\ &\iff a \in \mathrm{Inker}(\rho_i) \quad \text{for all} \quad i \in I \\ &\iff a \in \bigcap_{i\in I} \mathrm{Inker}(\rho_i) \end{aligned}$$

This proves that the inverse kernel map is a complete $\cap$-homomorphism. By Corollary 2.1.4, for each full inverse subsemigroup T there is a unique idempotent separating left congruence for which T is the inverse kernel. Since every inverse kernel is a full inverse subsemigroup (Proposition 2.1.3(i)) and every full inverse subsemigroup is the inverse kernel of an idempotent separating left congruence it follows that the inverse kernel map is surjective onto $\mathfrak{V}(S)$.

That each inverse kernel class is closed under $\cap$ is now immediate. Furthermore, if ρ is a left congruence then, as $\mathcal{R}$ is left congruence, $\rho \cap \mathcal{R}$ is a left congruence, and, as $\mathcal{R}$ is idempotent separating, $\rho \cap \mathcal{R}$ is idempotent separating. Then, by Proposition 2.1.3, $\mathrm{Inker}(\rho) = \ker(\rho \cap \mathcal{R}) = \mathrm{Inker}(\rho \cap \mathcal{R})$. Hence $\rho \cap \mathcal{R}$ is in the inverse kernel class of ρ. It follows from Corollary 2.1.4 that $\rho \cap \mathcal{R}$ is the unique idempotent separating left congruence with inverse kernel equal to $\mathrm{Inker}(\rho)$. The same argument implies that for any $\kappa \in [\rho]_{\mathrm{Inker}}$, $\kappa \cap \mathcal{R}$ is also an idempotent separating congruence with inverse kernel $\mathrm{Inker}(\kappa) = \mathrm{Inker}(\rho)$. Then $\kappa \cap \mathcal{R} = \rho \cap \mathcal{R}$, so in particular $\rho \cap \mathcal{R} \subseteq \kappa$. Thus we have that $\rho \cap \mathcal{R}$ is minimum in the inverse kernel class of ρ. □

The remainder of this section is devoted to showing that a left congruence is characterised by its trace and inverse kernel.

Definition 2.1.8. Let τ be a congruence on E, and let $T \subseteq S$ be a full inverse subsemigroup. We say that (τ, T) is an *inverse congruence pair* for S if (τ, T) satisfies the following conditions:

(ICP1) $T \subseteq N(\tau)$;

(ICP2) for $x \in S$, if there exist $e, f \in E$ such that $x^{-1}x \;\tau\; e,\; xx^{-1} \;\tau\; f$ and $xe, fx \in T$, then we have $x \in T$.

For an inverse congruence pair (τ, T), define the relation

$$\rho_{(\tau,T)} = \{(x, y) \mid x^{-1}y \in T,\; x^{-1}yy^{-1}x \;\tau\; x^{-1}x,\; y^{-1}xx^{-1}y \;\tau\; y^{-1}y\}.$$

Proposition 2.1.9. *If* (τ, T) *is an inverse congruence pair for* S*, then* $\rho_{(\tau,T)}$ *is a left congruence on* S *such that* $\operatorname{Inker}(\rho_{(\tau,T)}) = T$ *and* $\operatorname{trace}(\rho_{(\tau,T)}) = \tau$.

Proof. Let $\rho = \rho_{(\tau,T)}$. First we show that ρ is a left congruence. It is immediate that ρ is reflexive and symmetric. We next show left compatibility. Suppose that $a \;\rho\; b$, so $a^{-1}b \in T$, $a^{-1}bb^{-1}a \;\tau\; a^{-1}a$ and $b^{-1}aa^{-1}b \;\tau\; b^{-1}b$. We want to show that $ca \;\rho\; cb$. Since T is a full inverse subsemigroup we have

$$(ca)^{-1}(cb) = a^{-1}c^{-1}cb = (a^{-1}c^{-1}ca)a^{-1}b \in T.$$

We also note that

$$(a^{-1}c^{-1}ca)(a^{-1}bb^{-1}a) = a^{-1}c^{-1}cbb^{-1}c^{-1}ca = (ca)^{-1}(cb)(cb)^{-1}(ca).$$

Thus, as $a^{-1}bb^{-1}a \;\tau\; a^{-1}a$,

$$(ca)^{-1}(cb)(cb)^{-1}(ca) = a^{-1}c^{-1}ca(a^{-1}bb^{-1}a) \;\tau\; a^{-1}c^{-1}ca(a^{-1}a) = (ca)^{-1}(ca),$$

and similarly we obtain $(cb)^{-1}(ca)(ca)^{-1}(cb) \;\tau\; (cb)^{-1}(cb)$. Thus $ca \;\rho\; cb$ and we have shown that ρ is left compatible.

We now show that ρ is transitive, to which end suppose that $a \;\rho\; b$ and $b \;\rho\; c$. Thus $a^{-1}b,\; b^{-1}c \in T$, and $a^{-1}bb^{-1}a \;\tau\; a^{-1}a$, $b^{-1}aa^{-1}b \;\tau\; b^{-1}b$,

$b^{-1}cc^{-1}b \;\tau\; b^{-1}b$ and $c^{-1}bb^{-1}c \;\tau\; c^{-1}c$. We need to show that $c^{-1}a \in T$, and that $c^{-1}aa^{-1}c \;\tau\; c^{-1}c$ and $a^{-1}cc^{-1}a \;\tau\; a^{-1}a$. For the latter claim note that as τ is a congruence

$$(a^{-1}bb^{-1}a)(a^{-1}cc^{-1}a) \;\tau\; (a^{-1}a)(a^{-1}cc^{-1}a) = a^{-1}cc^{-1}a.$$

Also, as $T \subseteq N(\tau)$ (by (ICP1)) and $a^{-1}b \in T$, we conjugate $b^{-1}cc^{-1}b \;\tau\; b^{-1}b$ by $a^{-1}b$ to obtain

$$\begin{aligned}(a^{-1}bb^{-1}a)(a^{-1}cc^{-1}a) &= (a^{-1}b)(b^{-1}cc^{-1}b)(a^{-1}b)^{-1}\\ &\qquad \tau\; (a^{-1}b)(b^{-1}b)(a^{-1}b)^{-1} = a^{-1}bb^{-1}a.\end{aligned}$$

We now have that

$$a^{-1}cc^{-1}a \;\tau\; a^{-1}bb^{-1}a \;\tau\; a^{-1}a,$$

and the dual argument gives that $c^{-1}aa^{-1}c \;\tau\; c^{-1}c$.

To show ρ is a left congruence it remains to show that $c^{-1}a \in T$. As ρ is left compatible we have that $a^{-1}b \;\rho\; a^{-1}c$ and $c^{-1}a \;\rho\; c^{-1}b$, hence $c^{-1}aa^{-1}b \in T$ and $b^{-1}cc^{-1}a \in T$. From $a \;\rho\; b$ and $b \;\rho\; c$, we have $b^{-1}a,\ c^{-1}b \in T$, and, since T is a subsemigroup, $(c^{-1}aa^{-1}b)(b^{-1}a) \in T$ and $(c^{-1}b)(b^{-1}cc^{-1}a) \in T$. Also,

$$c^{-1}bb^{-1}c \;\tau\; c^{-1}c \;\tau\; c^{-1}aa^{-1}c = (c^{-1}a)(c^{-1}a)^{-1}$$

and

$$a^{-1}bb^{-1}a \;\tau\; a^{-1}a \;\tau\; a^{-1}cc^{-1}a = (c^{-1}a)^{-1}(c^{-1}a).$$

Thus by (ICP2) with $x = c^{-1}a$, $e = a^{-1}bb^{-1}a$, and $f = c^{-1}bb^{-1}c$ we have that $c^{-1}a \in T$. Hence ρ is transitive and thus a left congruence on S.

Finally we show that $\operatorname{trace}(\rho) = \tau$ and $\operatorname{Inker}(\rho) = T$. Suppose that we have $e, f \in E$ with $e \;\rho\; f$. Then, since $e^{-1} = e$, and $f^{-1} = f$, we have $e \;\tau\; ef \;\tau\; f$ and so $e \;\tau\; f$. Conversely, if $e \;\tau\; f$ then it is immediate that $e \;\rho\; f$, so $\operatorname{trace}(\rho) = \tau$. To see that the inverse kernel is T we note that if $a \in T$ then $a \;\rho\; aa^{-1}$, so $T \subseteq \operatorname{Inker}(\rho)$. Conversely if $a \;\rho\; aa^{-1}$, then $a^{-1}(aa^{-1}) = a^{-1} \in T$ and since T is inverse we get $a \in T$, thus $\operatorname{Inker}(\rho) = T$. $\square$

Shortly we shall see that every left congruence is of the form $\rho_{(\tau,T)}$ for an inverse congruence pair (τ, T). However, it is beneficial to first consider how it is possible to recover a left congruence from the minimum left congruence with the same trace and the minimum left congruence with the same inverse kernel. We recall that we write ν_τ for the minimum left congruence with trace τ and χ_T for the idempotent separating (and so minimum) left congruence with inverse kernel T.

Theorem 2.1.10. *For every left congruence ρ we have*

$$\rho = \nu_{\mathrm{trace}(\rho)} \vee \chi_{\mathrm{Inker}(\rho)}.$$

Moreover, for each $a \in \ker(\rho)$ there is $f \in E$ such that

$$f \ \chi_{\mathrm{Inker}(\rho)} \ fa \ \nu_{\mathrm{trace}(\rho)} \ a.$$

Proof. We write ν for $\nu_{\mathrm{trace}(\rho)}$ and χ for $\chi_{\mathrm{Inker}(\rho)}$ and recall that $\chi = \rho \cap \mathcal{R}$. We shall show that $\ker(\nu \vee \chi) = \ker(\rho)$ and $\mathrm{trace}(\nu \vee \chi) = \mathrm{trace}(\rho)$, whence, by Corollary 1.4.10, $\nu \vee \chi = \rho$. Certainly $\nu, \chi \subseteq \rho$, so $\nu \vee \chi \subseteq \rho$. As the kernel map is order preserving we have $\ker(\nu \vee \chi) \subseteq \ker(\rho)$. Also $\nu \subseteq \nu \vee \chi \subseteq \rho$, so since the trace map is order preserving it follows that $\mathrm{trace}(\nu \vee \chi) = \mathrm{trace}(\rho)$. To complete the proof it therefore suffices to prove the final claim of the theorem, from which it is immediate that $\ker(\nu \vee \chi) \supseteq \ker(\rho)$.

To this end suppose $a \in \ker(\rho)$, let e be any idempotent in the ρ-class of a and let $f = eaa^{-1}$. As $e \ \rho \ a$ we have

$$f = aa^{-1}e \ \rho \ aa^{-1}a = a,$$

and as ρ is a left congruence this implies $fa \ \rho \ f$. We also note that $(fa)(fa)^{-1} = faa^{-1}f = faa^{-1} = f$, so $fa \ \mathcal{R} \ f$ and therefore $f \ \chi \ fa$. As $fa \ \rho \ a$ we have that $a^{-1}fa \ \rho \ a^{-1}a$, and so $a^{-1}fa \ \mathrm{trace}(\rho) \ a^{-1}a$. Thus $a^{-1}fa \ \nu \ a^{-1}a$, and as ν is a left congruence we obtain $fa \ \nu \ a$, completing the proof. $\square$

Theorem 2.1.10 is reminiscent of Theorem 1.3.33 which states that a two sided congruence is the join of the minimum congruences with the same trace and the same kernel, as well as the meet of the maximum congruences with the same trace and the same kernel. While Theorem 2.1.10 states that a left congruence is the join of the minimum left congruences with the same trace and inverse kernel we shall see (Example 2.1.13) that in general there is no maximum left congruence with a given inverse kernel.

We now complete the fundamental result in the inverse kernel approach to left congruences, that a left congruence is uniquely determined by its trace and inverse kernel.

Theorem 2.1.11. *If* (τ, T) *is an inverse congruence pair for* S*, then* $\rho_{(\tau,T)}$ *is a left congruence on* S *with trace* τ *and inverse kernel* T*. Conversely, if* ρ *is a left congruence on* S *then* $(\mathrm{trace}(\rho), \mathrm{Inker}(\rho))$ *is an inverse congruence pair for* S *and* $\rho = \rho_{(\mathrm{trace}(\rho),\mathrm{Inker}(\rho))}$.

Proof. The first statement is precisely that of Proposition 2.1.9, so we want to prove the second statement. To this end suppose that ρ is a left congruence and let $\tau = \mathrm{trace}(\rho)$ and $T = \mathrm{Inker}(\rho)$. By Proposition 2.1.3(i), $\mathrm{Inker}(\rho)$ is a full inverse subsemigroup, and we know that $\mathrm{trace}(\rho)$ is a congruence on E. From Proposition 2.1.3(iv) we have $T = N(\tau) \cap \ker(\rho)$, so $T \subseteq N(\tau)$. Thus to show that (τ, T) is an inverse congruence pair we need to verify that (ICP2) holds.

Suppose that $a \in S$ and that there are $e, f \in E$ with $ae, fa \in T$ and $a^{-1}a \ \tau \ e$, $aa^{-1} \ \tau \ f$. Since $fa \in T$ we may conjugate $a^{-1}a \ \tau \ e$ by fa to observe that

$$aa^{-1}f = (fa)a^{-1}a(fa)^{-1} \ \tau \ (fa)e(fa)^{-1} = aea^{-1}f.$$

As $aa^{-1} \ \tau \ f$ we have $aa^{-1} \ \tau \ aa^{-1}f$ and as $a^{-1}a \ \tau \ e$ we have $ae \ \rho \ aa^{-1}a = a$. As $ae \in T$ we know that $ae \ \rho \ (ae)(ae)^{-1} = aea^{-1}$. We then obtain that

$$aa^{-1} \ \tau \ aa^{-1}f \ \tau \ aea^{-1}f \ \tau \ aea^{-1}(aa^{-1}) = aea^{-1} \ \rho \ ae \ \rho \ a.$$

Hence $a \in \mathrm{Inker}(\rho) = T$, so (ICP2) is satisfied and (τ, T) is an inverse congruence pair.

It remains to show that $\rho = \rho_{(\tau,T)}$. We know that $\mathrm{trace}(\rho_{(\tau,T)}) = \tau = \mathrm{trace}(\rho)$. From Theorem 2.1.7 we know that the minimum left congruence in $[\rho]_{\mathrm{Inker}}$ is $\rho \cap \mathcal{R}$ and the minimum left congruence in $[\rho_{(\tau,T)}]_{\mathrm{Inker}}$ is $\rho_{(\tau,T)} \cap \mathcal{R}$.

Then, by Theorem 2.1.10 we have that $\rho = (\rho \cap \mathcal{R}) \vee \nu_\tau$, and also $\rho_{(\tau,T)} = (\rho_{(\tau,T)} \cap \mathcal{R}) \vee \nu_\tau$. However $\operatorname{Inker}(\rho) = T = \operatorname{Inker}(\rho_{(\tau,T)})$ and since idempotent separating congruences are uniquely determined by their inverse kernel $\rho_{(\tau,T)} \cap \mathcal{R} = \rho \cap \mathcal{R}$. Hence $\rho = \rho_{(\tau,T)}$. □

We have shown that left congruences on inverse semigroups are determined by their trace and inverse kernel, and thus we may realise the lattice of left congruences as a subset of $\mathfrak{C}(E) \times \mathfrak{V}(S)$. We denote the set of inverse congruence pairs by $\mathfrak{ICP}(S)$. As in the case of the kernel trace description the ordering of left congruences coincides with the natural ordering in the lattice $\mathfrak{C}(E) \times \mathfrak{V}(S)$.

Corollary 2.1.12. *Let ρ_1, ρ_2 be left congruences on S, then*

$$\rho_1 \subseteq \rho_2 \iff \operatorname{trace}(\rho_1) \subseteq \operatorname{trace}(\rho_2) \text{ and } \operatorname{Inker}(\rho_1) \subseteq \operatorname{Inker}(\rho_2).$$

Consequently,

$$\rho_1 = \rho_2 \iff \operatorname{trace}(\rho_1) = \operatorname{trace}(\rho_2) \text{ and } \operatorname{Inker}(\rho_1) = \operatorname{Inker}(\rho_2).$$

As promised in the Preliminary chapter we shall liberally sprinkle examples to illuminate our results. Chapter 3 is devoted to demonstrating the usage of the inverse kernel approach to describing lattices of left congruences on various inverse semigroups. However it is useful to provide a very simple example at this stage to which we can refer and which can serve to demonstrate the failure of properties we might desire.

Example 2.1.13. We consider $\mathcal{I}_2$, the symmetric inverse monoid on a 2 element set. We label the elements of $\mathcal{I}_2$ by $e_{1,2}, e_1, e_2, e_\emptyset, \alpha, \beta, \beta^{-1}$ where e_X is the idempotent with domain $X \subseteq \{1,2\}$, α is the non-identity invertible element, and β has domain $\{1\}$ and image $\{2\}$. The semigroup $\mathcal{I}_2$ has 3 distinct full inverse subsemigroups: $E = E(\mathcal{I}_2)$, $\mathcal{I}_2$ and $T = E \cup \{\beta, \beta^{-1}\}$. The lattice of full inverse subsemigroups is displayed in Fig. 2.2. The semilattice of idempotents is isomorphic to the powerset of a 2 element set under intersection, for which we recall we write $\mathcal{P}_2$. The lattice of

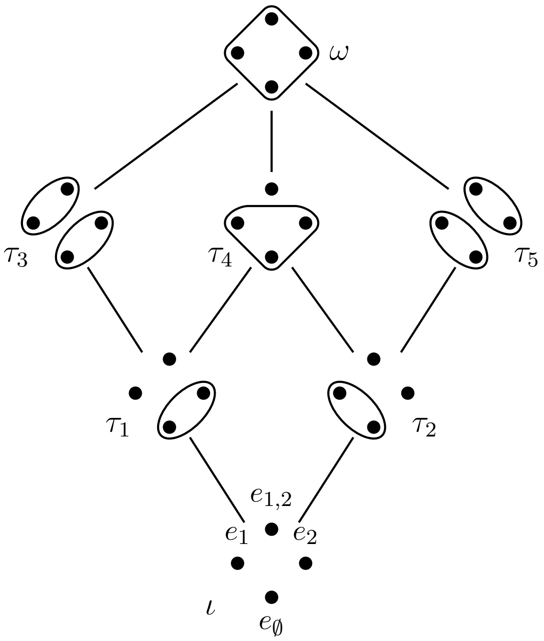


Figure 2.1: The lattice $\mathfrak{C}(E(\mathcal{I}_2))$

Figure 2.2: The lattice $\mathfrak{V}(\mathcal{I}_2)$

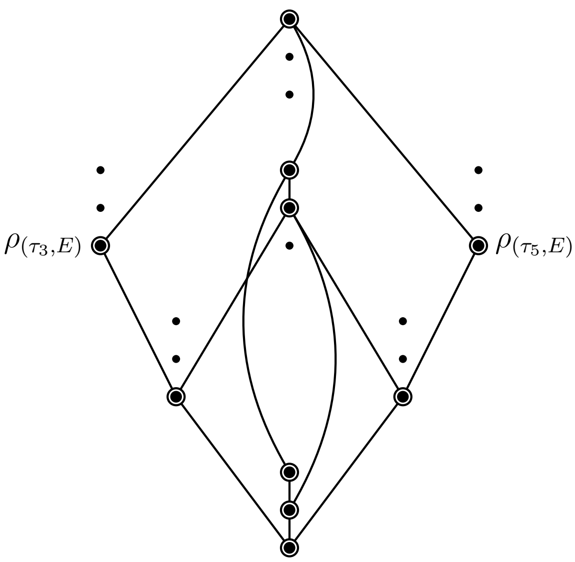


Figure 2.3: The lattice $\mathfrak{LC}(\mathcal{I}_2)$ as a subset of $\mathfrak{C}(E) \times \mathfrak{V}(S)$

congruences on the idempotents is illustrated in Fig. 2.1, in which the partitions of the idempotents are shown.

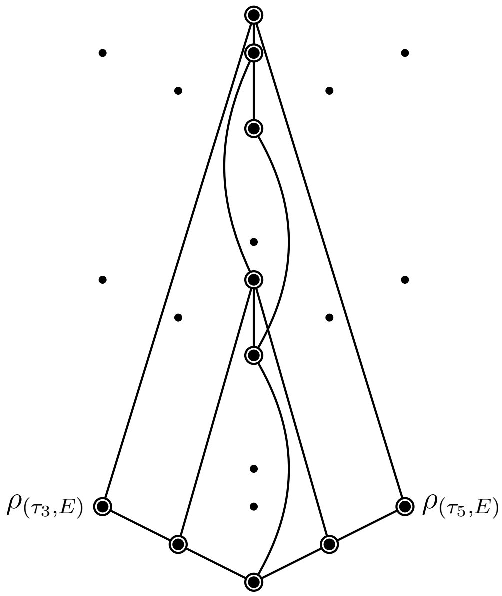


Figure 2.4: The lattice $\mathfrak{LC}(\mathcal{I}_2)$ as a subset of $\mathfrak{V}(S) \times \mathfrak{C}(E)$

The lattice of left congruences is then realised as a subset of the direct product of these two lattices. After elementary calculations to determine which elements of the direct product are inverse congruence pairs we obtain the lattice of left congruences as shown in Fig. 2.3 and Fig. 2.4. Both figures show the lattice of left congruences as a subset of a direct product, with the inverse congruence pairs indicated by the larger, circled vertices. Fig. 2.3 shows $\mathfrak{LC}(\mathcal{I}_2)$ as a subset of $\mathfrak{V}(\mathcal{I}_2) \times \mathfrak{C}(\mathcal{P}_2)$ the elements grouped into trace classes, whereas Fig. 2.4 shows $\mathfrak{LC}(\mathcal{I}_2)$ as a subset of $\mathfrak{C}(\mathcal{P}_2) \times \mathfrak{V}(\mathcal{I}_2)$, the elements grouped by inverse kernel. It is then easy to observe that the inverse kernel class of E contains no maximum elements, indeed the labelled inverse congruence pairs $\rho_{(\tau_3,E)}$ and $\rho_{(\tau_5,E)}$ both have inverse kernel E, however the join of these left congruences is ω, which has inverse kernel $\mathcal{I}_2$.

The following is an important corollary, and is the primary method with which the idea of the inverse kernel characterisation of left congruence will be applied in the future - and as one of the central themes of the thesis it shall be called upon a lot!

Corollary 2.1.14. *Let ρ be a left congruence on S. Let $T = \mathrm{Inker}(\rho)$, and $\tau = \mathrm{trace}(\rho)$. Then (τ, T) is the unique element in $\mathfrak{C}(E) \times \mathfrak{V}(S)$ such that (τ, T) is an inverse congruence pair, and*

$$\rho = \rho_{(\tau,T)} = \chi_T \vee \nu_\tau.$$

Proof. This is an immediate consequence of Theorem 2.1.11, which says that every left congruence on S is of the form $\rho_{(\tau,T)}$ for a unique inverse congruence pair, and Theorem 2.1.10, which says that $\rho_{(\tau,T)} = \chi_T \vee \nu_\tau$. $\square$

Corollary 2.1.14 suggests considering the function

$$\Theta\colon \mathfrak{C}(E) \times \mathfrak{V}(S) \to \mathfrak{LC}(S);\ (\tau, T) \mapsto \nu_\tau \vee \chi_T,$$

and we shall see that this is a fruitful endeavour, in particular when paired with the function

$$\Phi\colon \mathfrak{LC}(S) \to \mathfrak{C}(E) \times \mathfrak{V}(S);\ \rho \mapsto (\mathrm{trace}(\rho), \mathrm{Inker}(\rho)),$$

the natural embedding of the set of left congruences onto the set of inverse congruence pairs regarded as a subset of $\mathfrak{C}(E) \times \mathfrak{V}(S)$. Before we embark on a proper consideration of these maps we recall (from Proposition 1.4.18) that the map $\tau \mapsto \nu_\tau$ is a $\vee$-semilattice embedding $\mathfrak{C}(E) \hookrightarrow \mathfrak{LC}(S)$. While we have seen a proof, there is value in making the following observation which can easily be used to reprove the claim. Recall that given a congruence τ on E we may view τ as a binary relation on S and so we may consider $\langle \tau \rangle$, the left congruence on S generated by τ. It is clear that $\langle \tau \rangle$ is the minimum left congruence with trace τ, so we have that $\nu_\tau = \langle \tau \rangle$. Given traces τ_1, τ_2 it is clear that $\langle \tau_1 \rangle,\ \langle \tau_2 \rangle \subseteq \langle \tau_1 \vee \tau_2 \rangle$ so certainly $\langle \tau_1 \rangle \vee \langle \tau_2 \rangle \subseteq \langle \tau_1 \vee \tau_2 \rangle$. On the other hand, it is clear that regarded as binary relations

$$\tau_1 \cup \tau_2 \subseteq \langle \tau_1 \rangle \vee \langle \tau_2 \rangle.$$

Since $\langle \tau_1 \vee \tau_2 \rangle$ is generated as a congruence by $\tau_1 \cup \tau_2$ it follows that $\langle \tau_1 \vee \tau_2 \rangle \subseteq \langle \tau_1 \rangle \vee \langle \tau_2 \rangle$. Hence the two are equal. We recall that we call $\{\nu_\tau \mid \tau \in \mathfrak{C}(E)\}$ the set of trace minimal left congruences. We now proceed with a discussion of Φ and Θ.

Theorem 2.1.15. *The function* Φ *is a complete* $\cap$*-homomorphism, and* Θ *is an onto* $\vee$*-homomorphism. Moreover,* $\Phi\Theta\colon \mathfrak{LC}(S) \to \mathfrak{LC}(S)$ *is the identity map.*

Proof. That Φ is a complete $\cap$-homomorphism is immediate as the trace and inverse kernel maps are complete $\cap$-homomorphisms (by Theorem 1.4.17 and Theorem 2.1.7 respectively). Suppose $(\tau_1, T_1), (\tau_2, T_2) \in \mathfrak{C}(E) \times \mathfrak{V}(S)$. Then, utilising that the trace minimal elements and the idempotent separating left congruences form $\vee$-subsemilattices (by Proposition 1.4.18 and Theorem 1.4.19 respectively), we obtain

$$\begin{aligned}(\tau_1, T_1)\Theta \vee (\tau_2, T_2)\Theta &= (\nu_{\tau_1} \vee \chi_{T_1}) \vee (\nu_{\tau_2} \vee \chi_{T_2}) \\ &= (\nu_{\tau_1} \vee \nu_{\tau_2}) \vee (\chi_{T_1} \vee \chi_{T_2}) \\ &= \nu_{\tau_1 \vee \tau_2} \vee \chi_{T_1 \vee T_2} \\ &= (\tau_1 \vee \tau_2, T_1 \vee T_2)\Theta.\end{aligned}$$

Thus Θ is a $\vee$-homomorphism. From Corollary 2.1.14 we know that a left congruence $\rho_{(\tau,T)}$ is equal to $\nu_\tau \vee \chi_T$. Thus it is clear both that Φ is onto, and that the function $\Phi\Theta$ is the identity map. $\square$

Aside 2.1.16. While not important to us, it is of interest to remark upon the reverse composition $\Theta\Phi$. If L is a lattice then a function $f\colon L \to L$ is a *closure operator* if f is

- *extensive*: $a \leq f(a)$;
- *idempotent*: $f(f(a)) = f(a)$;
- *order preserving* (sometimes called *isotone*): if $a \leq b$ then $f(a) \leq f(b)$.

Note that we write closure operators on the left. Then (no surprises coming) $\Theta\Phi\colon \mathfrak{C}(E) \times \mathfrak{V}(S) \to \mathfrak{C}(E) \times \mathfrak{V}(S)$ is a closure operator, with associated closed sets (the image of the operator) the set of inverse congruence pairs.

Back to the main plot. In general neither is Φ a $\vee$-homomorphism, nor is Θ a $\cap$-homomorphism. To see that Φ does not preserve join recall that for $\mathcal{I}_2$ there are distinct left congruences ($\rho_{(\tau_3,E)}$ and $\rho_{(\tau_5,E)}$ in Fig. 2.3) with inverse kernel equal to E and join equal to the universal congruence, which has inverse kernel $\mathcal{I}_2$. Thus

$$(\rho_{(\tau_3,E)} \vee \rho_{(\tau_5,E)})\Phi = \omega\Phi = (\omega, \mathcal{I}_2),$$

however

$$\rho_{(\tau_3,E)}\Phi \vee \rho_{(\tau_3,E)}\Phi = (\tau_3, E) \vee (\tau_5, E) = (\omega, E).$$

Therefore Φ is not a $\vee$-homomorphism.

The example $\mathcal{I}_2$ also suffices to show that Θ is not a $\cap$-homomorphism. Indeed consider τ_3, τ_5 from Fig. 2.1, then $N(\tau_3) = E = N(\tau_5)$. We observe that $\tau_3 \cap \tau_5 = \iota$, the trivial congruence. We also note that $(\tau_3, \mathcal{I}_2)\Theta = \omega = (\tau_5, \mathcal{I}_2)\Theta$. Thus

$$(\tau_3, \mathcal{I}_2)\Theta \cap (\tau_5, \mathcal{I}_2)\Theta = \omega.$$

On the other hand

$$(\tau_3 \cap \tau_5, \mathcal{I}_2)\Theta = (\iota, \mathcal{I}_2)\Theta = \mathcal{R}.$$

Since $\omega \neq \mathcal{R}$ we have that Θ is not a $\cap$-homomorphism.

The map Θ defines an equivalence relation on $\mathfrak{C}(E) \times \mathfrak{V}(S)$ via taking the kernel of the function (in the sense of kernel of homomorphism). The following definition draws on this relation.

Definition 2.1.17. Let ρ be a left congruence on S. Then (τ, T) is a *pseudo inverse congruence pair* for ρ if $(\tau, T)\Theta = \rho$. A full inverse subsemigroup $T \subseteq S$ is a *pseudo inverse kernel* for ρ if $(\text{trace}(\rho), T)$ is a pseudo inverse congruence pair for ρ (so $(\text{trace}(\rho), T)\Theta = \rho$).

We make comparison with the definition of pseudo kernel from [61], which is given in Definition 1.4.8. As remarked after Theorem 1.4.9, the kernel of a left congruence is contained in every pseudo kernel for this left congruence. However, it is clear that every pseudo inverse kernel is contained in the inverse kernel. The notion of pseudo inverse kernel shall come into its own in Chapter 4; pseudo inverse kernels may not be used explicitly a large number of times but we shall lean heavily on the underlying idea.

2.2 Right Congruences

As described in Proposition 1.4.5, for inverse semigroups there is an isomorphism between the lattices of left and right congruences given by

$$\rho \mapsto \rho_{-1} = \{(a^{-1}, b^{-1}) \mid (a, b) \in \rho\}.$$

The inverse kernel approach to one sided congruences has a natural connection to this isomorphism.

All results and discussion thus far have analogues for right congruences. In particular, the inverse kernel of a right congruence ρ is defined as

$$\operatorname{Inker}(\rho) = \{a \in S \mid a \;\rho\; a^{-1}a\}.$$

We remark that by Proposition 2.1.3(iii) the inverse kernel of a left congruence ρ is equal to

$$\{a \in S \mid \exists e, f \in E,\; a \;\rho\; e,\; a^{-1} \;\rho\; f\}.$$

On the other hand, the right sided analogue of Proposition 2.1.3(iii) implies that $\{a \in S \mid \exists e, f \in E,\; a \;\rho\; e,\; a^{-1} \;\rho\; f\}$ is also an expression for the inverse kernel of a right congruence. Therefore, by taking this to be the definition of the inverse kernel it is possible to "unify" the definitions of inverse kernel for a left congruence and a right congruence. We remark that the usual definition of inverse kernel for a left congruence ρ (that is $\{a \in S \mid a \;\rho\; aa^{-1}\}$) was chosen as it is both easier to compute and use, and it more obviously "fixes" the problem of not knowing to which idempotent an element in the kernel is related.

When ρ is a left congruence it is straightforward to see that $\operatorname{trace}(\rho) = \operatorname{trace}(\rho_{-1})$ and noting that $\ker(\rho_{-1}) = \{a \in S \mid a^{-1} \in \ker(\rho)\}$ it follows from Proposition 2.1.3(iii) that

$$\operatorname{Inker}(\rho) = \ker(\rho) \cap \ker(\rho_{-1}) = \operatorname{Inker}(\rho_{-1}).$$

The next result then follows and makes clear the link between the inverse kernel approach and the isomorphism between $\mathfrak{LC}(S)$ and $\mathfrak{RC}(S)$.

Corollary 2.2.1. *The pair (τ, T) is the trace and inverse kernel of a left congruence if and only if it is the trace and inverse kernel of a right congruence. Moreover if ρ is a left congruence then the right congruence with the same trace and inverse kernel is ρ_{-1}.*

2.3 Trace classes

In the remaining sections of this chapter we focus on providing one sided analogues to results for two sided congruences concerning trace and kernel classes and trace and kernel maps. In this section we are motivated by the result describing the lattice of idempotent separating left congruences, and we describe the trace class for an arbitrary trace. We recall Theorem 1.4.11, that given a congruence τ on the idempotents the trace class $\{\rho \mid \text{trace}(\rho) = \tau\}$ is an interval with minimum and maximum left congruences ν_τ and μ_τ, respectively. Just as is done for the left kernel system and kernel trace descriptions of left congruences, we give the inverse kernel description of the maximum and minimum elements in each trace class.

We recall the definition of the left closure of $\tau \in \mathfrak{C}(E)$,

$$C_L(\tau) = \{a \in S \mid \exists e \in E \text{ such that } e \;\tau\; a^{-1}a \text{ and } ae = e\}.$$

Similarly we may define the *right closure*

$$C_R(\tau) = \{a \in S \mid \exists e \in E \text{ such that } e \;\tau\; aa^{-1} \text{ and } ea = e\}.$$

We then define the *closure* of a trace

$$C(\tau) = C_L(\tau) \cap C_R(\tau).$$

We recall, from Theorem 1.4.12, that $(\tau, C_L(\tau))$ is the left congruence pair for ν_τ. Therefore, as the kernel of a left congruence with trace τ is a subset of $N_L(\tau)$, we have that $C_L(\tau) \subseteq N_L(\tau)$. Similarly, $C_R(\tau) \subseteq N_R(\tau)$. It follows that $C(\tau) \subseteq N(\tau)$. In fact, it is easy to see from the definition of $C_L(\tau)$ and $C_R(\tau)$ that $C_L(\tau) = \{a \in S \mid a^{-1} \in C_R(\tau)\}$. It follows that

$$C(\tau) = \{a \in C_L(\tau) \mid a^{-1} \in C_L(\tau)\}.$$

Recalling that $N(\tau)$ may be obtained as $N(\tau) = \{a \in N_L(\tau) \mid a^{-1} \in N_L(\tau)\}$ we see that $C(\tau) = N(\tau) \cap C_L(\tau)$. Therefore we may realise $C(\tau)$ as

$$C(\tau) = \{a \in N(\tau) \mid \exists e \in E \text{ such that } e\ \tau\ a^{-1}a \text{ and } ae = e\}.$$

We use this expression for $C(\tau)$ in the future.

Lemma 2.3.1. *Let τ be a congruence on E. Then $C(\tau)$ is a full self conjugate inverse subsemigroup of $N(\tau)$.*

Proof. It is immediate that $C(\tau)$ is a full inverse subsemigroup. This can be seen directly, or deduced from the fact that $C(\tau)$ is the largest inverse subsemigroup contained in $C_L(\tau)$, which we know is a full subsemigroup as it is the kernel of a left congruence.

We need to show that $C(\tau)$ is self conjugate in $N(\tau)$. To this end suppose that $a \in C(\tau)$ and $b \in N(\tau)$, so there is $e \in E$ with $e\ \tau\ a^{-1}a$ and $ae = e$. Then $beb^{-1} \in E$ and

$$(bab^{-1})(beb^{-1}) = baeb^{-1} = beb^{-1}.$$

It then follows that

$$\begin{aligned} beb^{-1} = (beb^{-1})(beb^{-1}) &= (beb^{-1})(bab^{-1})^{-1}(bab^{-1})(beb^{-1}) \\ &= (bab^{-1})^{-1}(bab^{-1})(beb^{-1}). \end{aligned}$$

Also as $b \in N(\tau)$ we may conjugate $a^{-1}a\ \tau\ e$ by b to obtain $ba^{-1}ab^{-1}\ \tau\ beb^{-1}$. Thus

$$\begin{aligned} beb^{-1} = (bab^{-1})^{-1}(bab^{-1})(beb^{-1})\ \tau\ &(bab^{-1})^{-1}(bab^{-1})(ba^{-1}ab^{-1}) \\ &= (bab^{-1})^{-1}(bab^{-1}). \end{aligned}$$

Hence $bab^{-1} \in C(\tau)$ and thus $C(\tau)$ is a self conjugate full inverse subsemigroup of $N(\tau)$. □

The following result is the cornerstone of our extension of Theorem 1.4.19. It can be deduced from [61, Proposition 6.4] and the usual kernel-trace description of a two sided congruence on an inverse semigroup. However we shall later call on this result so it is worthwhile to include a direct proof.

Proposition 2.3.2 (see [61, Proposition 6.4])**.** *If τ is a congruence on E then $\nu_\tau|_{N(\tau)}$ is a two sided congruence on $N(\tau)$, and*

$$\nu_\tau|_{N(\tau)} = \{(a,b) \in N(\tau) \times N(\tau) \mid a^{-1}a \ \tau \ b^{-1}b,\ ab^{-1} \in C(\tau)\}.$$

Moreover, $\nu_\tau|_{N(\tau)}$ is the minimum congruence on $N(\tau)$ with trace τ.

Proof. We write N for $N(\tau)$ and ψ for the expression given on the right hand side of the displayed expression in the statement. We seek to prove that $\langle\tau\rangle \cap (N \times N) = \psi$, with $\langle\tau\rangle$ here the left congruence on S generated by τ. Suppose $(a,b) \in \psi$, so $a^{-1}a \ \tau \ b^{-1}b$ and, as $ab^{-1} \in C(\tau)$, there is $e \in E$ such that $ba^{-1}ab^{-1} \ \tau \ e$ and $ab^{-1}e = e$. As $b \in N$ we may conjugate $ba^{-1}ab^{-1} \ \tau \ e$ by b, so we have

$$a^{-1}ab^{-1}b = b^{-1}(ba^{-1}ab^{-1})b \ \tau \ b^{-1}eb.$$

Noting that from $a^{-1}a \ \tau \ b^{-1}b$ we have $a \ \langle\tau\rangle \ ab^{-1}b$ and $ba^{-1}a \ \langle\tau\rangle \ b$, we then observe that

$$a \ \langle\tau\rangle \ ab^{-1}b = a(a^{-1}ab^{-1}b) \ \langle\tau\rangle \ ab^{-1}eb = eb = bb^{-1}eb \ \langle\tau\rangle \ ba^{-1}ab^{-1}b = ba^{-1}a \ \langle\tau\rangle \ b.$$

Thus $(a,b) \in \langle\tau\rangle$, so $\psi \subseteq \langle\tau\rangle$.

Next we show that $\langle\tau\rangle \cap (N \times N) \subseteq \psi$. We shall show that if there is a τ-left-sequence from a to b then $(a,b) \in \psi$. To this end suppose that $a, b \in N$ and there is a τ-left-sequence from a to b. In other words we have $c_1, \dots, c_n \in S$ and $(e_1, f_1), \dots, (e_n, f_n) \in \tau$ such that

$$a = c_1e_1,\ c_if_i = c_{i+1}e_{i+1} \text{ for } 1 \leq i \leq n-1, \text{ and } c_nf_n = b.$$

We note that then $a^{-1}a = e_1c_1^{-1}c_1$, $f_ic_i^{-1}c_i = e_{i+1}c_{i+1}^{-1}c_{i+1}$ for $1 \leq i \leq n-1$ and $f_nc_n^{-1}c_n = b^{-1}b$. For each i, as $e_i \ \tau \ f_i$ we see that $e_ic_i^{-1}c_i \ \tau \ f_ic_i^{-1}c_i$, thus

$$a^{-1}a = e_1c_1^{-1}c_1 \ \tau \ f_1c_1^{-1}c_1 = e_2c_2^{-1}c_2 \ \tau \ \dots \ \tau \ f_nc_n^{-1}c_n = b^{-1}b.$$

Therefore $a^{-1}a \ \tau \ b^{-1}b$.

To show $ab^{-1} \in C(\tau)$ we will proceed by induction on the length of τ-left-sequence. Suppose that $a(c_ie_i)^{-1} \in C(\tau)$, which as $C(\tau)$ is inverse is equivalent to $c_ie_ia^{-1} \in C(\tau)$. Then there is $g \in E$ with $ae_ic_i^{-1}c_ia^{-1} \ \tau \ g$ and $c_ie_ia^{-1}g = g$. We will show that $a(c_if_i)^{-1} \in C(\tau)$. For the moment we drop the subscript i's. We know from the previous paragraph that $ec^{-1}c \ \tau \ fc^{-1}c$, so using that τ is a normal congruence in N and $a \in N$ we may conjugate $ec^{-1}c \ \tau \ fc^{-1}c$ by a to obtain

$$a(ec^{-1}c)a^{-1} \ \tau \ a(fc^{-1}c)a^{-1} = (cfa^{-1})^{-1}(cfa^{-1}).$$

As $e \ \tau \ f$ we have $e \ \tau \ f \ \tau \ ef$, which we may also conjugate by a to obtain that

$$aea^{-1} \ \tau \ afa^{-1} \ \tau \ aefa^{-1}.$$

We then observe

$$g(aefa^{-1}) \ \tau \ (aec^{-1}ca^{-1})(aea^{-1}) = aec^{-1}ca^{-1} \ \tau \ (cfa^{-1})^{-1}(cfa^{-1}).$$

Also,

$$(cfa^{-1})(gaefa^{-1}) = (cea^{-1})(gaefa^{-1}) = gaefa^{-1}.$$

Therefore $c(af)^{-1} \in C(\tau)$, and so also $a(cf)^{-1} \in C(\tau)$ as required. We add back in the subscripts. We have shown that if $a(c_ie_i)^{-1} \in C(\tau)$ then $a(c_if_i)^{-1} \in C(\tau)$ and as $a(c_if_i)^{-1} = a(c_{i+1}e_{i+1})^{-1}$ this completes the induction step and we have that $a(c_{i+1}e_{i+1})^{-1} \in C(\tau)$. It follows that $a(c_nf_n)^{-1} = ab^{-1} \in C(\tau)$. We have shown that $\langle\tau\rangle \cap (N \times N) \subseteq \psi$, whence the two are equal so we have that ψ is a left congruence on N.

We have shown that ψ is the minimum left congruence on N with trace τ, or equivalently is the left congruence on N generated by τ. We note that the congruence on N generated by τ is also generated as a congruence by the minimum left congruence with trace τ. Thus to complete the proof it suffices to show that ψ is also right congruence on N, whence ψ is a congruence so is the congruence on N generated by τ. Suppose that $a \ \psi \ b$, so $a^{-1}a \ \tau \ b^{-1}b$ and $ab^{-1} \in C(\tau)$ and let $c \in N$. Then we conjugate the relation $a^{-1}a \ \tau \ b^{-1}b$ by c to obtain

$$(ac)^{-1}(ac) = c^{-1}a^{-1}ac \ \tau \ c^{-1}b^{-1}bc = (bc)^{-1}(bc).$$

Since $C(\tau)$ is a full subsemigroup and $ab^{-1} \in C(\tau)$ we have that

$$(ac)(bc)^{-1} = acc^{-1}b^{-1} = ab^{-1}(bcc^{-1}b^{-1}) \in C(\tau).$$

Thus $ac\ \psi\ bc$ and so ψ is a right congruence. $\square$

It is worth noting that we have not used any prior knowledge about ν_τ in the proof of Proposition 2.3.2, and in fact we can deduce directly from Proposition 2.3.2 that $C(\tau) = \text{Inker}(\nu_\tau)$. Indeed, suppose τ is a congruence on E and let $\psi = \nu_\tau \cap (N \times N)$. Since $\text{Inker}(\nu_\tau) \subseteq N(\tau)$ certainly $\text{Inker}(\nu_\tau) = \text{Inker}(\psi)$. Thus we must show that $\text{Inker}(\psi) = C(\tau)$. We use the expression for ψ from Proposition 2.3.2. If $a\ \psi\ aa^{-1}$ then $a^{-1}a\ \tau\ aa^{-1}$ and $a(aa^{-1}) \in C(\tau)$. Hence there is some $e \in E$ such that $(aa^{-1})(a^{-1}a)\ \tau\ e$ and $a(aa^{-1})e = e$. Thus

$$(aa^{-1})e = (aa^{-1})a(aa^{-1})e = a(aa^{-1})e = e,$$

and we have $ae = a(aa^{-1})e = e$. We also note that $a^{-1}a\ \tau\ (aa^{-1})(a^{-1}a)\ \tau\ e$. Thus $a \in C(\tau)$. Thus we have that $\text{Inker}(\psi) \subseteq C(\tau)$.

Conversely we observe that if $a \in C(\tau)$ then certainly $a\ \psi\ a^{-1}a$, so $a \in \ker(\psi)$. However as previously noted for a two sided congruence the notion of kernel and inverse kernel coincide, hence $a \in \text{Inker}(\psi)$. Thus $C(\tau) = \text{Inker}(\psi) = \text{Inker}(\nu_\tau)$. We reinforce the remark that $C(\tau)$ is the inverse kernel of ν_τ and moreover is the kernel of $\nu_\tau|_{N(\tau)}$ (as this is a two sided congruence) as we shall assume familiarity with this fact in the rest of this section.

Descriptions of the maximum and minimum elements in a trace interval are available in both [46] and [61]. Let τ be a congruence on E, we have noted that the inverse kernel of the minimum left congruence with trace τ is $C(\tau)$. We also note that $(\tau, N(\tau))$ is certainly an inverse congruence pair. In terms of the inverse kernel description we get the following description for the maximum and minimum elements in a trace class.

Corollary 2.3.3. *Let τ be a congruence on E. The minimum and maximum left congruences with trace τ are respectively*

$$\nu_\tau = \rho_{(\tau, C(\tau))} = \{(x, y) \mid x^{-1}y \in C(\tau),\ x^{-1}yy^{-1}x\ \tau\ x^{-1}x,\ y^{-1}xx^{-1}y\ \tau\ y^{-1}y\},$$

and

$$\mu_\tau = \rho_{(\tau,N(\tau))} = \{(x,y) \mid x^{-1}y \in N(\tau),\ x^{-1}yy^{-1}x \ \tau\ x^{-1}x,\ y^{-1}xx^{-1}y \ \tau\ y^{-1}y\}.$$

Now we move to extending Theorem 1.4.11 to an arbitrary trace class.

Proposition 2.3.4. *Let τ be a congruence on E and let $T \subseteq N(\tau)$ be a full inverse subsemigroup. Then (τ, T) is an inverse congruence pair if and only if T is a union of $\nu_\tau|_{N(\tau)}$-classes.*

Proof. Let $\psi = \nu_\tau|_{N(\tau)}$ and recall (ICP2), which says that if $a \in S$ and there are $e, f \in E$ with $a^{-1}a \ \tau\ e$, $aa^{-1} \ \tau\ f$ and $ae, fa \in T$, then $a \in T$.

First suppose that (τ, T) is an inverse congruence pair. We want to show that T is a union of ψ-classes, to which end we suppose that $a \in N(\tau)$ and $a \ \psi\ b$ for some $b \in T$. From the description of ψ in Proposition 2.3.2 we have $a^{-1}a \ \tau\ b^{-1}b$ and $ab^{-1} \in C(\tau)$. Since $a \in N(\tau)$ we can conjugate $a^{-1}a \ \tau\ b^{-1}b$ by a to get $aa^{-1} \ \tau\ ab^{-1}ba^{-1}$. Then letting $e = b^{-1}b,\ f = ab^{-1}ba^{-1}$ we have $ae = ab^{-1}b = fa$. As $\rho_{(\tau,C(\tau))}$ is the minimum left congruence with trace τ we note that $\rho_{(\tau,C(\tau))} \subseteq \rho_{(\tau,T)}$. Recalling that the inverse kernel map is order preserving we see that $ab^{-1} \in C(\tau)$ implies that $ab^{-1} \in T$. We then note that as $b, ab^{-1} \in T$ we have that $ab^{-1}b = ae \in T$. Applying (ICP2) we obtain that $a \in T$. Thus T is a union of ψ-classes.

Conversely suppose that T is a union of ψ-classes. We have $T \subseteq N(\tau)$, so to show (τ, T) is an inverse congruence pair we need to verify that (ICP2) holds. Suppose that $a \in S$ and there exist $e, f \in E$ such that $a^{-1}a \ \tau\ e,\ aa^{-1} \ \tau\ f$, and $ae, fa \in T$. We first show that $a \in N(\tau)$. Suppose that $g, h \in E$ with $g \ \tau\ h$. Then as $fa \in N(\tau)$ we conjugate $g \ \tau\ h$ by fa to get $faga^{-1} \ \tau\ faha^{-1}$. Since $aa^{-1} \ \tau\ f$ we then have

$$aga^{-1} = aa^{-1}(aga^{-1}) \ \tau\ f(aga^{-1}) \ \tau\ f(aha^{-1}) \ \tau\ aa^{-1}(aha^{-1}) = aha^{-1}.$$

Conjugating $g \ \tau\ h$ by $(ae)^{-1}$, and using that $a^{-1}a \ \tau\ e$, a similar argument gives that $a^{-1}ga \ \tau\ a^{-1}ha$, hence $a \in N(\tau)$. To see that in fact $a \in T$ we note that $a^{-1}a \ \tau\ e$ implies that $a \ \psi\ ae$ since $\operatorname{trace}(\psi) = \tau$. As $ae \in T$ and T is a union of ψ-classes it follows that $a \in T$. $\square$

We can now formulate the main result of this section, the extension of Theorem 1.4.11 to an arbitrary trace class.

Theorem 2.3.5. *For every congruence τ on E the lattice of left congruences on S with trace τ is isomorphic to the lattice of full inverse subsemigroups of $N(\tau)/\nu_\tau|_{N(\tau)}$.*

Proof. We know (Proposition 2.3.4) that the set of full inverse subsemigroups of $N(\tau)$ that form an inverse congruence pair with τ consists precisely of those saturated by $\nu_\tau|_{N(\tau)}$. Therefore to complete the proof it suffices to note that by standard universal arguments this set is exactly the pre-image of the set of full inverse subsemigroups of $N(\tau)/\nu_\tau|_{N(\tau)}$ under the natural homomorphism $N(\tau) \mapsto N(\tau)/\nu_\tau|_{N(\tau)}$. □

We have shown that given a trace τ the trace class in $\mathfrak{LC}(S)$ is isomorphic to the lattice $\mathfrak{V}(N(\tau)/\nu_\tau|_{N(\tau)})$. For the discussion in the rest of this section take $\tau \in \mathfrak{C}(E)$ and write N for $N(\tau)$ and ν for $\nu_\tau|_N$. Then define the functions

$$\Pi : \mathfrak{V}(N) \to \mathfrak{V}(N); \qquad T \mapsto \bigcup_{t\in T} [t]_\nu$$

and

$$\pi \colon \mathfrak{V}(N) \to \mathfrak{V}(N/\nu); \qquad T \mapsto \left(\bigcup_{t\in T} [t]_\nu\right)/\nu.$$

We shall shortly show that both functions are well defined. We first remark that the two maps Π and π are obviously closely related, π is equal to Π followed by the function $\Psi\colon \operatorname{Im}(\Pi) \to \mathfrak{V}(N/\nu)$ defined by $T\Pi \mapsto T\Pi/\nu$. Noting that $\operatorname{Im}(\Pi)$ is the set of full inverse subsemigroups of N which are saturated by ν we see that Ψ is precisely the function considered in the proof of Theorem 2.3.5, and by standard universal arguments Ψ is a lattice isomorphism.

We now show that both π and Π are well defined. Let $T \in \mathfrak{V}(N)$. We first show that $T\Pi$ is a full inverse subsemigroup. Recall from Proposition 2.3.2 that ν is a two sided congruence on N. First we observe that if $a, b \in T\Pi$ then there exist $x, y \in T$ with $a \ \nu \ x,\ b \ \nu \ y$. As ν is a two sided congruence we obtain $ab \ \nu \ xy$, hence $ab \in T\Pi$. Again as ν is two sided, if $a \ \nu \ b$ then

$a^{-1}\ \nu\ b^{-1}$, so as T is inverse it follows that $T\Pi$ is inverse. We also note that as $E \subseteq T \subseteq T\Pi$ it is immediate that $T\Pi$ is full. Thus $T\Pi$ is a full inverse subsemigroup and so Π is well defined. That π is well defined then follows from the fact Ψ is a lattice homomorphism, so is certainly well defined.

We define both π and Π as there is value in each when considering different viewpoints. We will show that π is lattice homomorphism, but Π, in general, is not, since the join of two full inverse subsemigroups that are saturated by ν is not necessarily saturated by ν. In particular, we note that $C(\tau) = E\Pi$ because $C(\tau)$ is the kernel of ν (as remarked after Proposition 2.3.2).

Proposition 2.3.6. *Let τ be a congruence on E, let N, ν, π and Π be as defined above. Then π is a lattice homomorphism and Π is a $\cap$-homomorphism.*

Proof. Take $T_1, T_2 \in \mathfrak{V}(N)$. We first show that Π is a $\cap$-homomorphism, for which we need to prove that

$$\bigcup_{t \in T_1 \cap T_2} [t]_\nu = \bigcup_{t \in T_1} [t]_\nu \cap \bigcup_{t \in T_2} [t]_\nu.$$

As the term on the left hand side is a subset of both terms on the right hand side the inclusion left to right is immediate.

For the reverse inclusion we suppose that there is $s \in N$ such that we have $a \in T_1$ and $b \in T_2$ with $s\ \nu\ a$ and $s\ \nu\ b$ (which says that s is a element of the right hand side). We use the description of ν from Proposition 2.3.2, which is

$$\nu = \{(x,y) \in N \times N \mid x^{-1}x\ \tau\ y^{-1}y,\ xy^{-1} \in C(\tau)\}.$$

Then $s^{-1}s\ \tau\ a^{-1}a$ and $s^{-1}s\ \tau\ b^{-1}b$, and $sa^{-1}, sb^{-1} \in C(\tau)$. Then certainly $s^{-1}s\ \tau\ (a^{-1}a)(b^{-1}b)$. As ν is a congruence on N with trace τ this implies that $s\ \nu\ s(a^{-1}ab^{-1}b)$. Furthermore, since $C(\tau)$ is the kernel of ν so is equal to $\bigcup_{e \in E}[e]_\nu$ we have $sa^{-1} \in C(\tau) \subseteq T_1$. Then as $ab^{-1}ba^{-1} \in E(S) \subseteq T_1$ and $a \in T_1$, we have

$$s(a^{-1}ab^{-1}b) = (sa^{-1})(ab^{-1}ba^{-1})a \in T_1.$$

Similarly, $s(a^{-1}ab^{-1}b) = (sb^{-1})(ba^{-1}ab^{-1})b \in T_2$. Thus $s \in \bigcup_{t \in T_1 \cap T_2} [t]_\nu$, and we have shown that Π is a $\cap$-homomorphism. That π is also a $\cap$-homomorphism follows from the fact that the function $T\Pi \mapsto (T\Pi)\nu$ (previously called Ψ) is a lattice homomorphism.

We now turn our attention to the claim that π is a $\vee$-homomorphism. We must show that $(T_1 \vee T_2)\pi = T_1\pi \vee T_2\pi$. We note that, as $T_1, T_2 \subseteq T_1 \vee T_2$ it is immediate that $T_1\pi, T_2\pi \subseteq (T_1 \vee T_2)\pi$, and thus that $T_1\pi \vee T_2\pi \subseteq (T_1 \vee T_2)\pi$. For the reverse inclusion we observe that the “preimage” of $T_1\pi \vee T_2\pi$, by which we mean $a \in S$ such that $[a]_\nu \cap (T_1\pi \vee T_2\pi) \neq \emptyset$, is saturated by ν, and certainly contains T_1 and T_2. Also, by standard universal arguments, this preimage is a full inverse subsemigroup of N, and thus must contain $T_1 \vee T_2$. It follows that $(T_1 \vee T_2)\pi \subseteq T_1\pi \vee T_2\pi$, so the two are equal. Thus we have completed the proof of the proposition. $\square$

We now show that Π is not in general a $\vee$-homomorphism. Consider the inverse semigroup $S \subseteq \mathcal{I}_5$ where S is equal to

$$\left\{ \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}, \begin{pmatrix} 4 & 5 \\ 5 & 4 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 5 & 4 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 4 & 5 \end{pmatrix}, \begin{pmatrix} 4 & 5 \\ 1 & 2 \end{pmatrix}, \begin{pmatrix} 4 & 5 \\ 2 & 1 \end{pmatrix}, \right.$$
$$\left. e_{\{1,2,3\}}, e_{\{1,2\}}, e_{\{4,5\}}, e_\emptyset \right\}$$

Let τ be the relation

$$\tau = \{(e,e) \mid e \in E(S)\} \cup \{(e_{\{1,2\}}, e_{\{1,2,3\}}), (e_{\{1,2,3\}}, e_{\{1,2\}})\}.$$

Then τ is a normal congruence on $E(S)$, so $N(\tau) = S$, and we note that

$$\nu = \nu_\tau = \{(a,a) \mid a \in S\} \cup \{(e_{\{1,2\}}, e_{\{1,2,3\}}), (e_{\{1,2,3\}}, e_{\{1,2\}}), (b,c), (c,b)\},$$

where

$$b = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}, \qquad c = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}.$$

Let $T_1,\ T_2 \subseteq S$ be the following full inverse subsemigroups:

$$T_1 = \left\{ \begin{pmatrix} 1 & 2 \\ 4 & 5 \end{pmatrix}, \begin{pmatrix} 4 & 5 \\ 1 & 2 \end{pmatrix} \right\} \cup E(S),$$

$$T_2 = \left\{ \begin{pmatrix} 1 & 2 \\ 5 & 4 \end{pmatrix}, \begin{pmatrix} 4 & 5 \\ 2 & 1 \end{pmatrix} \right\} \cup E(S).$$

We notice that both T_1 and T_2 are saturated by ν. However we also observe that $T_1 \vee T_2 = S \backslash \{b\}$, which is not saturated by ν. In other words

$$T_1\Pi \vee T_2\Pi = T_1 \vee T_2 = S\backslash\{b\} \neq S = (S\backslash\{b\})\Pi = (T_1 \vee T_2)\Pi.$$

Hence the image of Π is not closed under taking join, and Π is not a $\vee$-homomorphism. Thus the set of full inverse subsemigroups that "form" a trace class in not a sublattice of the lattice of full inverse subsemigroups.

2.4 Inverse Kernel Classes

We now seek to give analogues for inverse kernel classes of results concerning trace classes. We know that given a full inverse subsemigroup there is an idempotent separating left congruence such that this subsemigroup is the inverse kernel and that this is the minimum element in the inverse kernel class. As we have seen in the case of $\mathcal{I}_2$, in general there is no maximum element in an inverse kernel class.

As a brief deviation from the story of the inverse kernel we make a small technical observation.

Lemma 2.4.1. *Let τ be a congruence on $E(S)$ and $T \subseteq N(\tau)$ be a full inverse subsemigroup of S. Let x be an element of S. Then the following are equivalent:*

(i) there are $e, f \in E$ such that $x^{-1}x \;\tau\; e,\; xx^{-1} \;\tau\; f$ and $xe, fx \in T$;

(ii) there is $e \in E$ such that $x^{-1}x \;\tau\; e,\; xx^{-1} \;\tau\; xex^{-1}$ and $xe \in T$;

(iii) x is an element of $N(\tau)$ and there is $e \in E$ such that $x^{-1}x \;\tau\; e$ and $xe \in T$.

Proof. We note that if (ii) holds then (i) holds with $f = xex^{-1}$ (as $xex^{-1}x = xe$). Suppose that (i) holds so there are $e, f \in E$ such that $x^{-1}x \;\tau\; e,\; xx^{-1} \;\tau\; f$

and $xe, fx \in T$. Then as $fx \in T \subseteq N(\tau)$ we can conjugate $x^{-1}x \ \tau \ e$ by fx to obtain $fxx^{-1} \ \tau \ xex^{-1}f$. We observe

$$xx^{-1} \ \tau \ fxx^{-1} \ \tau \ xex^{-1}f \ \tau \ xex^{-1}xx^{-1} = xex^{-1}.$$

Thus (ii) holds

If (iii) holds then certainly (ii) holds, since we can conjugate $x^{-1}x \ \tau \ e$ by x to obtain $xx^{-1} \ \tau \ xex^{-1}$. Suppose instead that (ii) holds, we must show that $x \in N(\tau)$. Suppose $f, g \in E$ are such that $f \ \tau \ g$. As $xe \in T \subseteq N(\tau)$, we may conjugate τ-relations by xe. Noting that $x^{-1}x \ \tau \ e$ implies

$$x^{-1}fx = (x^{-1}x)(x^{-1}fx) \ \tau \ e(x^{-1}fx)$$

and similarly $x^{-1}gx \ \tau \ e(x^{-1}gx)$, we obtain

$$x^{-1}fx \ \tau \ e(x^{-1}fx) = (xe)^{-1}f(xe) \ \tau \ (xe)^{-1}g(xe) = e(x^{-1}gx) \ \tau \ x^{-1}gx.$$

Also, we similarly obtain that

$$xfx^{-1} = (xx^{-1})(xfx^{-1}) \ \tau \ (xex^{-1})(xfx^{-1}) = (xe)(x^{-1}xf)(xe)^{-1}$$

and

$$xgx^{-1} = (xx^{-1})(xgx^{-1}) \ \tau \ (xex^{-1})(xgx^{-1}) = (xe)(x^{-1}xg)(xe)^{-1}.$$

As $x^{-1}xf \ \tau \ x^{-1}xg$, it follows that $xfx^{-1} \ \tau \ xgx^{-1}$. Thus $x \in N(\tau)$, so (iii) holds. $\square$

We remark that Lemma 2.4.1 implies that under the assumption that $T \subseteq N(\tau)$ we may rewrite (ICP2) as

(ICP2$'$) for $x \in S$, if there is $e \in E$ such that $x^{-1}x \ \tau \ e$, $xx^{-1} \ \tau \ xex^{-1}$ and $xe \in T$ then we have $x \in T$.

We shall sometimes use this as the definition for (ICP2) in the future, we shall make it clear when this is the case.

Our objective now is to describe the set of congruences on E that are the traces of left congruences in an inverse kernel class. Motivated by (ii) from Lemma 2.4.1 we make the following definition.

Definition 2.4.2. Let τ be a congruence on E and $T \subseteq S$ a subset. We define the τ-*closure* of T as

$$T\tau = \{a \in S \mid \exists e \in E \text{ such that } ae \in T,\ e \;\tau\; a^{-1}a,\ aea^{-1} \;\tau\; aa^{-1}\}.$$

If $T = T\tau$ then we say that T is τ-*closed.*

This extends the well known definition of the closure of a subset T of an inverse semigroup, which is $\{s \in S \mid \exists t \in T, t \leq s\}$. The closure is usually written $T\omega$ and coincides with the ω-closure for our definition of closure. In particular, for any $T \subseteq S$ and $\tau \in \mathfrak{C}(E)$ we have that $T \subseteq T\tau$, as if $a \in T$ then $e = a^{-1}a$ demonstrates that $a \in T\tau$.

We also note that the closure $C(\tau)$ of $\tau \in \mathfrak{C}(E)$ is precisely $E\tau$, the τ-closure of the idempotents. Indeed, the definition of $C(\tau)$ is

$$C(\tau) = \{a \in N(\tau) \mid \exists e \in E \text{ such that } e \;\tau\; a^{-1}a \text{ and } ae = e\}.$$

If $a \in C(\tau)$ then $a \in N(\tau)$, and there is $e \in E$ such that $ae = e$ and $e \;\tau\; a^{-1}a$. It follows that $aea^{-1} \;\tau\; a^{-1}a$, whence $a \in E\tau$. Conversely, if $b \in E\tau$ then there is $f \in E$ such that $bf \in E$, $b^{-1}b \;\tau\; f$ and $bb^{-1} \;\tau\; bfb^{-1}$. To show that $b \in C(\tau)$ we must show that $b \in N(\tau)$. To this end we suppose that $p, q \in E$ and $p \;\tau\; q$. We then note that as $bf \in E$ we have

$$\begin{aligned} b^{-1}pb = (b^{-1}b)b^{-1}pb(b^{-1}b) &\;\tau\; (fb^{-1})p(bf) \\ &\;\tau\; (fb^{-1})q(bf) \;\tau\; (b^{-1}b)b^{-1}qb(b^{-1}b) = b^{-1}qb. \end{aligned}$$

A similar argument implies that $bpb^{-1} \;\tau\; bqb^{-1}$, so we have that $b \in N(\tau)$, so $b \in C(\tau)$. Thus we have shown that $C(\tau) = E\tau$.

It is possible to describe traces of left congruences τ with a given inverse kernel T in terms of the τ-closure. We observe that for $T \in \mathfrak{V}(S)$ with $T \subseteq N(\tau)$ and $\tau \in \mathfrak{C}(E)$, T being τ-closed is equivalent to (ICP2$'$) being satisfied, which itself precisely says that (τ, T) is an inverse congruence pair.

Corollary 2.4.3. *Let T be a full inverse subsemigroup of S. A congruence τ on E is the trace of a left congruence with inverse kernel T if and only if τ is normal in T and $T = T\tau$.*

To obtain a more precise description of which $\tau \in \mathfrak{C}(E)$ are the trace of a left congruence with a given inverse kernel we need to restrict attention to inverse semigroups which have some additional structure.

A partial order is said to have the *descending chain condition* if it contains no infinite descending chains. Any partially ordered set with the descending chain condition has minimal elements; if a meet-semilattice has the descending chain condition then it contains a minimum element. Let τ be a congruence on a semilattice E. Since a congruence class is a subsemilattice we note that if the semilattice has the descending chain condition then each τ-class has a minimum element. We observe that when E has the descending chain condition then the usual partial order on S also has the descending chain condition, in this case we say that *S has the descending chain condition.*

If S has the descending chain condition and $T \subseteq S$ is a full inverse subsemigroup then it is obvious that for each $a \in S \backslash T$ there is $b \in S \backslash T$ such that $b \leq a$ and b is minimal in $S \backslash T$. We now give criteria for a congruence on E to form an inverse congruence pair with a given full inverse subsemigroup when S has the descending chain condition.

Proposition 2.4.4. *Let S be an inverse semigroup with the descending chain condition, and let $T \subseteq S$ be a full inverse subsemigroup. If τ is a congruence on $E(S)$ which is normal in T, then (τ, T) is an inverse congruence pair if and only if for each minimal element $a \in S \backslash T$ at least one of aa^{-1} and $a^{-1}a$ is the minimum in its τ-class.*

Proof. Suppose that $a \in S \backslash T$ is minimal, then as T is inverse it is clear that a^{-1} is also minimal. We note that if $e < a^{-1}a$ then $ae < a$. We assume initially that (τ, T) is an inverse congruence pair, and that both $a^{-1}a,\ aa^{-1}$ are not minimum in their τ-class. So there are $e < a^{-1}a,\ f < aa^{-1}$, with $e \ \tau \ a^{-1}a,\ f \ \tau \ aa^{-1}$. Then as a, a^{-1} are minimal in $S \backslash T$ we get $ae, a^{-1}f \in T$. But since (τ, T) is an inverse congruence pair (ICP2) gives that $a \in T$, which is a contradiction. Thus at least one of aa^{-1} and $a^{-1}a$ is minimum in its congruence class.

For the converse suppose that for any minimal $x \in S\backslash T$ at least one of $xx^{-1}, x^{-1}x$ is minimum in its congruence class. We need to verify (ICP2). Let $a \in S$ and suppose there are $e, f \in E$ such that $e \ \tau \ a^{-1}a, \ f \ \tau \ aa^{-1}$ and $ae, fa \in T$. Suppose that $a \in S\backslash T$. Then since S satisfies the descending chain condition we have that there is some $h \in E$ such that $b = ah$ and b is minimal in $S\backslash T$. By assumption at least one of bb^{-1} and $b^{-1}b$ is minimum in its τ-class. Suppose $b^{-1}b$ is a minimum, then $b^{-1}b = b^{-1}ba^{-1}a \ \tau \ b^{-1}be$, so $b^{-1}b = b^{-1}be$ and $b = be$. Then $b = be = ahe = (ae)h \in T$, a contradiction. Similarly, bb^{-1} cannot be minimum it its τ-class. It follows that $a \in T$ and (ICP2) holds. $\square$

Proposition 2.4.4 suggests the potential for computational application of the inverse kernel approach to describing left congruences. If the lattices $\mathfrak{V}(S)$ and $\mathfrak{C}(E)$ are known - both of which are much easier computational problems than computing $\mathfrak{LC}(S)$ - and we also know $N(\tau)$ for each $\tau \in \mathfrak{C}(E)$, then Proposition 2.4.4 provides a mechanism to calculate $\mathfrak{LC}(S)$ using the partial order structure as opposed to the multiplicative structure of S.

2.5 The lattice of left congruences

We now consider the lattice of left congruences on S from the perspective of the inverse kernel approach, regarding $\mathfrak{LC}(S)$ as a subset of $\mathfrak{C}(E) \times \mathfrak{V}(S)$. We shall describe the meets and joins of left congruences in terms of the trace and inverse kernel. We remark that we use the notation $\vee$ in multiple contexts, if $\tau_1, \tau_2 \in \mathfrak{C}(E)$ then $\tau_1 \vee \tau_2$ is the join in $\mathfrak{C}(E)$ and if $T_1, T_2 \in \mathfrak{V}(S)$ then $T_1 \vee T_2$ is the join as full inverse subsemigroups. It shall be clear from the context which lattice we are using.

We have seen that, like the trace map (Theorem 1.4.17) and unlike the kernel map, the inverse kernel map is a complete $\cap$-homomorphism (Theorem 2.1.7). Since we know that the restriction of the inverse kernel map to the set of idempotent separating left congruences is onto, and the restriction of the trace map to the set of trace minimal left congruences is onto we have shown the following result. We recall that $\mathfrak{ICP}(S)$ is the lattice of inverse congruence pairs.

Corollary 2.5.1. *The lattice $\mathfrak{ICP}(S)$ is a complete $\cap$-subsemilattice of the of direct product $\mathfrak{C}(E) \times \mathfrak{V}(S)$. Furthermore, it is a subdirect product (which just means the projection onto both coordinates is surjective).*

In particular, if $\{\rho_i \mid i \in I\}$ is a family of left congruences then

$$\operatorname{trace}\left(\bigcap_{i\in I} \rho_i\right) = \bigcap_{i\in I} \operatorname{trace}(\rho_i) \quad \textit{and} \quad \operatorname{Inker}\left(\bigcap_{i\in I} \rho_i\right) = \bigcap_{i\in I} \operatorname{Inker}(\rho_i).$$

On the other hand it is a non trivial question to determine the join of two left congruences on S. We now show that the inverse kernel approach provides a mechanism to handle this problem smoothly. Initially we make an observation, recalling that $(\tau, T) \in \mathfrak{C}(E) \times \mathfrak{V}(S)$ is a pseudo inverse congruence pair for $\rho \in \mathfrak{LC}(S)$ if $\nu_\tau \vee \chi_T = \rho$.

Lemma 2.5.2. *Let (τ_1, T_1) and (τ_2, T_2) be inverse congruence pairs. Then $(\tau_1 \vee \tau_2, T_1 \vee T_2)$ is a pseudo inverse congruence pair for $\rho_{(\tau_1,T_1)} \vee \rho_{(\tau_2,T_2)}$.*

Proof. This follows from Corollary 2.1.14, which states that for an inverse congruence pair (τ, T) we have $\rho_{(\tau,T)} = \nu_\tau \vee \chi_T$, and the fact that $\tau \mapsto \nu_\tau$ and $T \mapsto \chi_T$ are $\vee$-embeddings. Together these imply that

$$\begin{aligned}\nu_{\tau_1\vee\tau_2} \vee \chi_{T_1\vee T_2} &= (\nu_{\tau_1} \vee \nu_{\tau_2}) \vee (\chi_{T_1} \vee \chi_{T_2}) \\ &= (\nu_{\tau_1} \vee \chi_{T_1}) \vee (\nu_{\tau_2} \vee \chi_{T_2}) = \rho_{(\tau_1,T_1)} \vee \rho_{(\tau_2,T_2)}.\end{aligned}$$

Which says that $(\tau_1 \vee \tau_2, T_1 \vee T_2)$ is a pseudo inverse congruence pair for $\rho_{(\tau_1,T_1)} \vee \rho_{(\tau_2,T_2)}$. $\square$

We can do much better than a pseudo inverse congruence pair, as we shall now demonstrate. We begin by making a straightforward observation about how the normalisers of congruences on E are related to normalisers of the joins and meets of these congruences.

Lemma 2.5.3. *Let $\{\tau_i \mid i \in I\}$ be a set of congruences on E with normalisers $N_i = N(\tau_i)$ respectively. Then*

$$N\left(\bigcap_{i\in I} \tau_i\right) \supseteq \bigcap_{i\in I} N_i \quad \textit{and} \quad N\left(\bigvee_{i\in I} \tau_i\right) \supseteq \bigcap_{i\in I} N_i.$$

Proof. The first part is straightforward, suppose $a \in \bigcap_{i\in I} N_i$ and that $e\ (\bigcap_{i\in I}\tau_i)\ f$. Then for each i, $e\ \tau_i\ f$ and $a \in N_i$, so $aea^{-1}\ \tau_i\ afa^{-1}$. Therefore $aea^{-1}\ (\bigcap_{i\in I}\tau_i)\ afa^{-1}$. Similarly we obtain that $a^{-1}ea\ (\bigcap_{i\in I}\tau_i)\ a^{-1}fa$, so $a \in N(\bigcap_{i\in I}\tau_i)$.

For the second claim we suppose that $e\ (\bigvee_{i\in I}\tau_i)\ f$ and $a \in \bigcap_{i\in I} N_i$. As $\bigvee_{i\in I}\tau_i$ is the transitive closure of the union of $\{\tau_i \mid i \in I\}$ there are elements $g_1, \ldots, g_m \in E$ and $i_1, \ldots, i_m \in I$ such that

$$e\ \tau_{i_1}\ g_1\ \tau_{i_2}\ g_2\ \ldots\ g_{m-1}\ \tau_{i_{m-1}}\ g_m\ \tau_{i_m}\ f.$$

At each stage we can conjugate by a and thus obtain that

$$aea^{-1}\ (\bigvee_{i\in I}\tau_i)\ afa^{-1}.$$

By the same argument swapping a and a^{-1} we have $a^{-1}ea\ (\bigvee_{i\in I}\tau_i)\ a^{-1}fa$. Thus $a \in N(\bigvee_{i\in I}\tau_i)$. □

We note that these may both be strict inclusions. Consider again the symmetric inverse monoid $\mathcal{I}_2$, and recall the congruences τ_3 and τ_5 from Fig. 2.1, which are defined by the following partitions of $E(\mathcal{I}_2)$:

$$\tau_3\colon \{e_{1,2},\ e_1\},\ \{e_2,\ e_\emptyset\} \quad \text{and} \quad \tau_5\colon \{e_{1,2},\ e_2\},\ \{e_1,\ e_\emptyset\}.$$

We observe that $N(\tau_3) = E = N(\tau_5)$, but $N(\tau_3 \cap \tau_5) = N(\iota) = \mathcal{I}_2$, and also $N(\tau_3 \vee \tau_5) = N(\omega) = \mathcal{I}_2$. The following preliminary result we have already shown, we state it here in this form as it is directly relevant to the sequel.

Lemma 2.5.4. *Let τ be a congruence on E and let $T \subseteq N(\tau)$ be a full inverse subsemigroup. Then $V = \bigcup_{t\in T}[t]_{\nu_\tau|_{N(\tau)}}$ is a full inverse subsemigroup and (τ, V) is an inverse congruence pair.*

Proof. We have seen that V is a full inverse subsemigroup of $N(\tau)$ (see the remark preceding Proposition 2.3.6). By definition V is saturated by $\nu_\tau|_{N(\tau)}$, so applying Proposition 2.3.4 we have that (τ, V) is an inverse congruence pair. □

Theorem 2.5.5. *Let (τ_1, T_1), (τ_2, T_2) be inverse congruence pairs and let ξ be the least congruence on E such that $\tau_1 \vee \tau_2 \subseteq \xi$ and $T_1 \vee T_2 \subseteq N(\xi)$. Then*

$$\rho_{(\tau_1,T_1)} \vee \rho_{(\tau_2,T_2)} = \rho_{(\xi,V)}$$

where $V = \bigcup_{t \in T_1 \vee T_2} [t]_{\nu_\xi|_{N(\xi)}}$.

Proof. We first note that ξ is well defined. Indeed, let $\{\xi_i \mid i \in I\} \subseteq \mathcal{C}(E)$ be the set of all congruences on E such that $\tau_1 \vee \tau_2 \subseteq \xi_i$ and $T_1 \vee T_2 \subseteq N(\xi_i)$, and note that this set is non-empty as it contains the universal relation. Let $\xi = \bigcap_{i \in I} \xi_i$. Then it is immediate that $\tau_1 \vee \tau_2 \subseteq \xi$, and, by Lemma 2.5.3, $T_1 \vee T_2 \subseteq \bigcap_{i \in I} N(\xi_i) \subseteq N(\xi)$. Moreover, by definition ξ is the least congruence such that $\tau_1 \vee \tau_2 \subseteq \xi$ and $T_1 \vee T_2 \subseteq N(\xi)$. From Lemma 2.5.4 we observe that (ξ, V) is a inverse congruence pair, and by appeal to Corollary 2.1.12 we have that $\rho_{(\tau_1,T_1)} \vee \rho_{(\tau_2,T_2)} \subseteq \rho_{(\xi,V)}$.

We now show that $\rho_{(\xi,V)} \subseteq \rho_{(\tau_1,T_1)} \vee \rho_{(\tau_2,T_2)}$. Let (ζ, W) be the inverse congruence pair such that $\rho_{(\tau_1,T_1)} \vee \rho_{(\tau_2,T_2)} = \rho_{(\zeta,W)}$. We note that, by the previous paragraph, $\zeta \subseteq \xi$ and $W \subseteq V$. By Corollary 2.1.12 we have that $T_1, T_2 \subseteq W$ and $\tau_1, \tau_2 \subseteq \zeta$, so $T_1 \vee T_2 \subseteq W$ and $\tau_1 \vee \tau_2 \subseteq \zeta$. As (ζ, W) is an inverse congruence pair, $W \subseteq N(\zeta)$ so it follows that $T_1 \vee T_2 \subseteq N(\zeta)$. However, by definition ξ is the least congruence on E that has these properties. Therefore $\xi \subseteq \zeta$, so we have that $\xi = \zeta$. Finally we can note that $T_1 \vee T_2 \subseteq W$, and, as (ξ, W) is an inverse congruence pair, W is saturated by $\nu_\xi|_{N(\xi)}$ (by Proposition 2.3.4). It is then clear that $V \subseteq W$, so we have $V = W$. We have shown that $(\xi, V) = (\zeta, W)$, so $\rho_{(\xi,V)} = \rho_{(\tau_1,T_1)} \vee \rho_{(\tau,T_2)}$. □

2.6 Concluding remarks

To conclude this chapter I would like to make a few brief comments about how this inverse kernel approach which we have developed fits into the broader mathematical framework which has been built to describe one and two sided congruences on inverse semigroups.

First we comment on the relationship between the kernel and the inverse kernel. From Proposition 2.1.3 it follows that if two left congruences have the same kernel then they have the same inverse kernel. Thus the inverse kernel classes are unions of the kernel classes. In the other direction, starting from the inverse kernel of a left congruence it not possible to construct the kernel. However, by Theorem 2.1.10, we know that if $\rho = \rho_{(\tau,T)}$ is a left congruence then if $a \in \ker(\rho)$ there is $f \in E$ such that $f\ \chi_T\ fa\ \nu_\tau\ a$. Noting that $fa \in \ker(\chi_T) = \operatorname{Inker}(\rho)$ we have shown that if $a \in \ker(\rho)$ then $a\ \nu_\tau\ b$ for some $b \in \operatorname{Inker}(\rho)$. Conversely if $a\ \nu_\tau\ b$ for $b \in \operatorname{Inker}(\rho)$ then it is immediate that $a \in \ker(\rho)$. Thus we observe that for a left congruence ρ on S with trace τ the kernel is

$$\ker(\rho) = \bigcup_{a \in \operatorname{Inker}(\rho)} [a]_{\nu_\tau}.$$

In fact we can say slightly more.

Corollary 2.6.1. *Let $T \subseteq S$ be a full inverse subsemigroup and let τ be a congruence on E such that $\tau = \operatorname{trace}(\nu_\tau \vee \chi_T)$. Then*

$$\ker(\nu_\tau \vee \chi_T) = \bigcup_{a \in T} [a]_{\nu_\tau}.$$

Proof. Let $V = \bigcup_{t \in T} [t]_{\nu_\tau|_{N(\tau)}}$, then by Proposition 2.3.6 we have that V is a full inverse subsemigroup of $N(\tau)$, and so of S as well.

Using that $\tau = \operatorname{trace}(\nu_\tau \vee \chi_T)$ and applying Proposition 2.3.4 we have that the inverse kernel of $\nu_\tau \vee \chi_T$ is saturated by $\nu_\tau|_{N(\tau)}$. As $\chi_T \subseteq \nu_\tau \vee \chi_T$ and since the inverse kernel map is order preserving, we observe that $V \subseteq \operatorname{Inker}(\nu_\tau \vee \chi_T)$. It follows that $\chi_T \subseteq \chi_V \subseteq \nu_\tau \vee \chi_T$ and therefore that $\nu_\tau \vee \chi_T = \nu_\tau \vee \chi_V$. Furthermore, as $V \subseteq N(\tau)$ and V is saturated by $\nu_\tau|_{N(\tau)}$, by Proposition 2.3.4 we have that (τ, V) is an inverse congruence pair. From Corollary 2.3.3 we have that $\nu_\tau = \rho_{(\tau,C(\tau))}$. Since $\rho_{(\tau,C(\tau)} \subseteq \rho_{(\tau,V)}$ we know that $C(\tau) \subseteq V$.

As χ_T is idempotent separating $\chi_T = \rho_{(\iota,T)}$. Applying Theorem 2.5.5 we have that

$$\operatorname{Inker}(\nu_\tau \vee \chi_T) = \bigcup_{t \in T \vee C(\tau)} [t]_{\nu_\tau|_{N(\tau)}}.$$

By the same argument, also

$$\operatorname{Inker}(\nu_\tau \vee \chi_V) = \bigcup_{t \in V \vee C(\tau)} [t]_{\nu_\tau|_{N(\tau)}}.$$

Since $C(\tau) \subseteq V$ (from the previous paragraph) $V \vee C(\tau) = V$. We also know that $\nu_\tau \vee \chi_T = \nu_\tau \vee \chi_V$, so

$$\mathrm{Inker}(\nu_\tau \vee \chi_T) = \mathrm{Inker}(\nu_\tau \vee \chi_V) = \bigcup_{t \in V} [t]_{\nu_\tau|_{N(\tau)}}.$$

Since by definition V is saturated by $\nu_\tau|_{N(\tau)}$ it follows that

$$\mathrm{Inker}(\nu_\tau \vee \chi_T) = \bigcup_{t \in V} [t]_{\nu_\tau|_{N(\tau)}} = V = \bigcup_{t \in T} [t]_{\nu_\tau|_{N(\tau)}}$$

As in the remark preceding the result the kernel of $\nu_\tau \vee \chi_T$ may be realised as

$$\ker(\nu_\tau \vee \chi_T) = \bigcup_{a \in \mathrm{Inker}(\nu_\tau \vee \chi_T)} [a]_{\nu_\tau}.$$

The claim now follows immediately.

□

We have already seen that the inverse kernel approach is closely connected with natural map between left and right congruences on an inverse semigroup. The kernel trace description for two sided congruences on inverse semigroups is well known (and is described in Theorem 1.3.12) and states that $\mathfrak{C}(S)$ is realised as the the lattice of congruence pairs. In Chapter 4 we shall demonstrate that the two sided kernel trace description follows in an elementary fashion from the inverse kernel approach to one sided congruences.

In the case for two sided congruences, as discussed in Chapter 1, a pair $(K, \tau) \in \mathfrak{N}(S) \times \mathfrak{C}(E)$ is the kernel and trace of a congruence if and only if the kernel class $\{\kappa \mid \ \ker(\kappa) = K\}$ and the trace class $\{\kappa \mid \mathrm{trace}(\kappa) = \tau\}$ have non-empty intersection. Analogues are clearly true in the one sided case, both for the kernel trace and inverse kernel approaches. Where the one sided approaches fall down is that this intersection cannot be represented as both the join of the minimum and meet of maximum elements in the (inverse) kernel and trace classes. If K is the kernel of a left congruence then there is a maximum left congruence γ_K which has kernel K (Proposition 1.4.13) and it is straightforward to see that a left congruence ρ with trace τ and kernel K has

$$\rho = \gamma_K \cap \mu_\tau.$$

Indeed, since γ_K and μ_τ are the maximum left congruences with the same kernel and trace as ρ it is immediate that $\rho \subseteq \gamma_K \cap \mu_\tau$. We also note that $\gamma_K \cap \mu_\tau \subseteq \gamma_K$. Since the kernel map is order preserving this implies that

$$K = \ker(\rho) \subseteq \ker(\gamma_K \cap \mu_\tau) \subseteq \ker(\gamma_K) = K.$$

Hence $K = \ker(\gamma_K \cap \mu_\tau)$. Similarly we obtain that $\mathrm{trace}(\gamma_K \cap \mu_\tau) = \tau$ and as a left congruence is determined by its trace and kernel (Theorem 1.4.9) this implies that $\rho = \gamma_K \cap \mu_\tau$.

Combining this observation with the fact that $\rho_{(\tau,T)} = \nu_\tau \vee \chi_T$ is the join of the minimum elements in the inverse kernel and trace classes we may use both the kernel and the inverse kernel to represent every congruence as both a meet and join. This is shown in Fig. 2.5 in which μ and ν are the maximum and minimum elements in $[\rho]_{\text{trace}}$, γ is the maximum element in $[\rho]_{\ker}$ and χ is the minimum element in $[\rho]_{\text{Inker}}$.

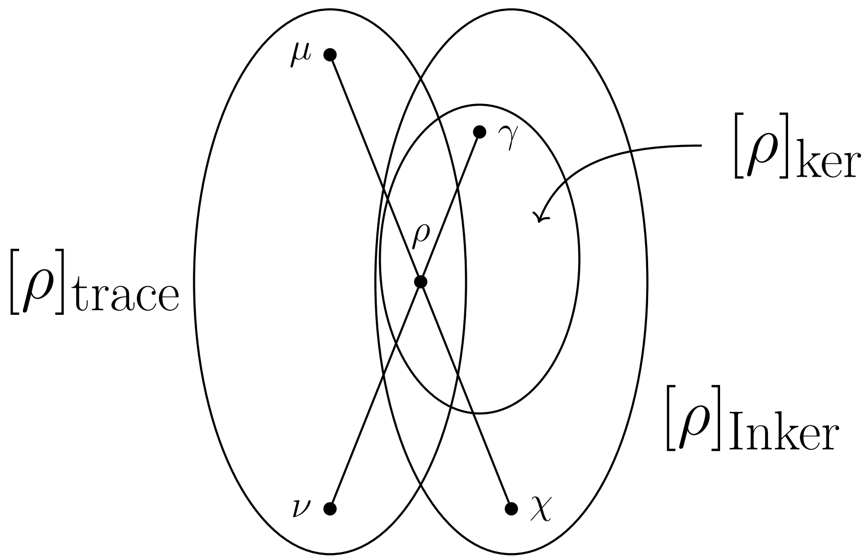


Figure 2.5: A left congruence as the meet and join of maximal and minimal elements

We recall that in the case of two sided congruences we write λ_K for the minimum congruence with kernel K and that $\rho \in \mathfrak{C}(S)$ is an idempotent determined congruence if $\ker(\rho) = E$. In Theorem 1.3.37 and Corollary 1.3.35 the kernel and trace classes of $\rho(\tau, K)$ were described via isomorphisms with the lattice of idempotent determined congruences on S/λ_K and the lattice of idempotent separating congruences on S/ν_τ (this ν_τ agrees with the ν_τ in this chapter by Proposition 2.3.2). For a one sided analogue for the trace class we can use Theorem 2.3.5 and Theorem 1.4.19, the results that the lattice of left congruences on S with trace τ is isomorphic to the lattice of full inverse subsemigroups of $N(\tau)/\nu_\tau|_{N(\tau)}$ and that the lattice of idempotent separating left congruences on an inverse semigroup is isomorphic to the lattice of full inverse subsemigroups.

Theorem 2.6.2. *Let S be an inverse semigroup and let τ be a congruence on $E(S)$. Then the trace class $\{\rho \mid \mathrm{trace}(\rho) = \tau\}$ is isomorphic to the set of idempotent separating left congruences on $N(\tau)/\nu_\tau|_{N(\tau)}$.*

As a final remark we consider the inverse congruence pair that we obtain when we restrict a left congruence on an inverse semigroup to an inverse subsemigroup.

Theorem 2.6.3. *Let S be an inverse semigroup and let $\rho = \rho_{(\tau,T)}$ be a left congruence on S. Let $V \subseteq S$ be an inverse subsemigroup. Then $\rho|_V = \rho \cap (V \times V)$ is a left congruence on V and $(\tau|_{E(V)}, T \cap V)$ is the inverse congruence pair for $\rho|_V$.*

Proof. It is immediate that $\rho|_V$ is a left congruence on V, we must show that $\mathrm{trace}(\rho|_V) = \tau|_{E(V)}$ and $\mathrm{Inker}(\rho|_V) = T \cap V$. We first consider the trace. It is clear that if $e, f \in E(V)$ with $e \;\tau\; f$ then $e \;\rho|_V\; f$, so $\tau|_{E(V)} \subseteq \mathrm{trace}(\rho|_V)$. For the reverse inclusion suppose that $e, f \in E(V)$ with $e \;\rho|_V\; f$, then $e \;\rho\; f$ so $e \;\tau\; f$. Thus we have $\mathrm{trace}(\rho|_V) = \tau|_{E(V)}$.

For the inverse kernel suppose $a \in T \cap V$, then $a \in T$ so $a \;\rho\; aa^{-1}$. As V is an inverse subsemigroup and $a \in V$ we have $a^{-1} \in V$ so $aa^{-1} \in V$. Therefore $a \;\rho|_V\; aa^{-1}$ and we have $a \in \mathrm{Inker}(\rho|_V)$. Thus $T \cap V \subseteq \mathrm{Inker}(\rho|_V)$. For the reverse inclusion we suppose that $a \in \mathrm{Inker}(\rho|_V)$, so $a \in V$ and $a \;\rho|_V\; aa^{-1}$. Then $a \;\rho\; aa^{-1}$ so $a \in \mathrm{Inker}(\rho) = T$. Thus we have $T \cap V = \mathrm{Inker}(\rho|_V)$. This completes the proof. $\square$

At this point we shall draw this chapter to a close and press pause on our evaluation of the theoretical uses of the inverse kernel approach. We shall return to look more deeply at how we apply our methodology to link properties of an inverse semigroup to properties of the lattice of left congruences in Chapter 4, after a brief interlude focusing on explicit examples of the usage of the inverse kernel description of left congruences in Chapter 3.

2.7 Appendix A: Inverse Kernel Systems

Thus far in this chapter we have focused almost exclusively on the inverse kernel approach to describing left congruences, which follows a similar philosophy to the kernel trace approach. As described in Chapter 1 there is an alternate approach, using that each left congruence on an inverse semigroup is completely determined by its idempotent classes and so specifying the sets that arise as classes of a one sided congruence which contain idempotents. Both Meakin ([46, Definition 2.1], see Theorem 1.4.7), and Petrich and Rankin ([61, Definition 7.1], stated after Theorem 1.4.7) give descriptions of these sets. This therefore begs the questions: is it possible to specify the sets that arise as idempotent containing classes in the inverse kernel, and if so is there any benefit to doing so? The first of these questions is easily and unsurprisingly answered affirmatively and, while it is difficult to answer in the negative for the second question, there is no obvious benefit to be obtained. First we shall describe those sets that arise as the partitions of the inverse kernel, though we note that, as in the description of left kernel systems, the conditions given here are not unique.

Definition 2.7.1. Let $\mathcal{A} = \{A_i \mid i \in I\}$ be a set of disjoint subsets of S, and let $A = \bigcup_{i \in I} A_i$, and for $a \in A$ write $A(a)$ for the A_i containing a. Then $\mathcal{A}$ is called an inverse kernel system if it satisfies the conditions:

(I1) $E(S) \subseteq A$;

(I2) for each $i \in I$, $E(S) \cap A_i \neq \emptyset$;

(I3) for $a, b \in A$, $aA(b) \subseteq A(abb^{-1}a^{-1})$;

(I4) for each $a \in A$, $a^{-1}A(aa^{-1}) \subseteq A(a^{-1}a)$;

(I5) if $a^{-1}b \in A(a^{-1}a)$, and $ab^{-1} \in A(aa^{-1})$ for some $b \in A$ then $a \in A$.

Definition 2.7.2. For a left congruence ρ on S, the *inverse system* of ρ is

$$\operatorname{Isys}(\rho) = \{e\rho \cap \operatorname{Inker}(\rho) \mid e \in E(S)\}.$$

Lemma 2.7.3. *If ρ is a left congruence on S then* $\operatorname{Isys}(\rho)$ *is an inverse kernel system.*

Proof. This is a verification exercise in which use the fact that ρ is a left congruence and that $\operatorname{Inker}(\rho) = \{a \in S \mid a \;\rho\; aa^{-1}\}$. For $a \in \operatorname{Inker}(\rho)$ let $A(a)$ be the set in $\operatorname{Isys}(\rho)$ containing a. From the definition of the inverse kernel we note that $A(a) = A(aa^{-1})$ for all $a \in \operatorname{Inker}(\rho)$. We observe that (I1) and (I2) are immediate, we prove the remainder.

Let $a, b \in \operatorname{Inker}(\rho)$ and suppose that $x \in A(b) = A(bb^{-1})$. That $x \in A(b)$ says that $x \;\rho\; b$ so, as ρ is a left congruence, $ax \;\rho\; ab$. Then, since $a, b \in \operatorname{Inker}(\rho)$ and $\operatorname{Inker}(\rho)$ is a subsemigroup, we have $ab \in \operatorname{Inker}(\rho)$, so $ab \;\rho\; abb^{-1}a^{-1}$. Also $ax \in \operatorname{Inker}(\rho)$, thus $ax \;\rho\; abb^{-1}a^{-1}$. Therefore $ax \in A(abb^{-1}a^{-1})$ and we have that (I3) holds.

For (I4) let $a \in \operatorname{Inker}(\rho)$ and suppose $x \in A(aa^{-1})$. Then $x \;\rho\; aa^{-1}$ so $a^{-1}x \;\rho\; a^{-1}$, and, as $\operatorname{Inker}(\rho)$ is inverse, we have $a^{-1} \in \operatorname{Inker}(\rho)$. Since $x, a^{-1} \in \operatorname{Inker}(\rho)$ we have $a^{-1}x \in \operatorname{Inker}(\rho)$. Thus $a^{-1}x \in A(a^{-1}) = A(a^{-1}a)$, so (I4) holds.

Finally, suppose $a^{-1}b \in A(a^{-1}a)$ and $ab^{-1} \in A(aa^{-1})$ for some $b \in \operatorname{Inker}(\rho)$, so $a^{-1}a \;\rho\; a^{-1}b$ and $aa^{-1} \;\rho\; ab^{-1}$. Then, as $b \in \operatorname{Inker}(\rho)$ and $\operatorname{Inker}(\rho)$ is inverse, $b \;\rho\; bb^{-1}$ and $b^{-1} \;\rho\; b^{-1}b$. Also, as $ab^{-1}, a^{-1}b \in \operatorname{Inker}(\rho)$, we have $a^{-1}b \;\rho\; a^{-1}bb^{-1}a$ and $ab^{-1} \;\rho\; ab^{-1}ba^{-1}$. We then observe that

$$\begin{aligned} a = aa^{-1}a \;\rho\; aa^{-1}b \;\rho\; aa^{-1}bb^{-1} = bb^{-1}aa^{-1} \;\rho\; bb^{-1}ab^{-1} \\ \rho\; bb^{-1}ab^{-1}b = aa^{-1}bb^{-1}ab^{-1}b = ab^{-1}ba^{-1}bb^{-1}a \\ \rho\; ab^{-1}ba^{-1}b \;\rho\; ab^{-1}ba^{-1}a = ab^{-1}b \;\rho\; ab^{-1} \;\rho\; aa^{-1}. \end{aligned}$$

Thus $a \in \mathrm{Inker}(\rho) = A$, so (I5) is satisfied and so $\mathrm{Isys}(\rho)$ is an inverse kernel system. $\square$

Theorem 2.7.4. *If $\mathcal{A} = \{A_i \mid i \in I\}$ is an inverse kernel system, then the relation*

$$\rho_{\mathcal{A}} = \{(a, b) \mid a^{-1}b \in A(a^{-1}a),\ b^{-1}a \in A(b^{-1}b)\}$$

is a left congruence on S with inverse system $\mathcal{A}$. Moreover, if ρ is a left congruence on S with inverse system $\mathcal{A}$, then $\rho = \rho_{\mathcal{A}}$.

Proof. Let $A = \bigcup_{i\in I} A_i$ and for $a \in A$ write $A(a)$ for the A_i with $a \in A_i$. Let τ be the relation on $E = E(S)$ defined by the partition of E given by $\mathcal{A}$. We will show that (τ, A) is an inverse congruence pair and that $\rho_{(\tau,A)} = \rho_{\mathcal{A}}$. Initially we note that (I3) implies that τ is a congruence on $E(S)$. Also, by (I1), $E(S) \subseteq A$, and, by (I3) again, A is a subsemigroup. From (I4) we have that, for $a \in A$, $a^{-1} = a^{-1}(aa^{-1}) \in A(a^{-1}a) \subseteq A$. Hence A is a full inverse subsemigroup of S. We also note that, by (I3), if $a \in A$ then $aA(a^{-1}a) \subseteq A(aa^{-1})$, thus $a \in A(aa^{-1})$.

Next we show that $A \subseteq N(\tau)$. Suppose that $a \in A$, $e, f \in E$ and $e \ \tau\ f$, so $f \in A(e)$. Then by (I3) we have that $af \in A(aea^{-1})$. As $af = afa^{-1}a$, applying (I3) again gives $afa^{-1}A(a) \subseteq A(afa^{-1})$. Thus $af \in A(aea^{-1})$ and $af \in A(afa^{-1})$. Since the A_i are disjoint this implies $A(aea^{-1}) = A(afa^{-1})$. Similarly can show that $A(a^{-1}ea) = A(a^{-1}fa)$. Hence $a \in N(\tau)$, so we have shown that $A \subseteq N(\tau)$.

Our next step is to verify (ICP2). Suppose that $a \in S$ and there are $e, f \in E$ such that $a^{-1}a\ \tau\ e$, $aa^{-1}\ \tau\ f$, with $ae, fa \in A$. We must show that $a \in A$. Since $A \subseteq N(\tau)$, from $a^{-1}a\ \tau\ e$ and $fa \in A$ we obtain

$$faa^{-1} = (fa)a^{-1}a(fa)^{-1}\ \tau\ (fa)e(fa)^{-1} = faea^{-1}.$$

Then we observe that

$$aa^{-1}\ \tau\ faa^{-1}\ \tau\ faea^{-1}\ \tau\ (aa^{-1})aea^{-1} = aea^{-1}.$$

Hence we may take $f = aea^{-1}$. We let $b = ae$, and note that we have $a^{-1}b = a^{-1}ae$, and since τ is a congruence we have $a^{-1}ae\ \tau\ a^{-1}a$, so $a^{-1}b =$

$a^{-1}ae \in A(a^{-1}a)$. We also have that $ab^{-1} = aea^{-1} \in A(aa^{-1})$. Hence by (I5) we have $a \in A$. Thus (ICP2) holds and (τ, A) is an inverse congruence pair.

Next we show that $\rho_{(\tau,A)} = \rho_{\mathcal{A}}$. We have previously remarked that if $a \in A$ then $a \in A(aa^{-1})$, it follows that for $a, b \in S$, $a^{-1}b \in A$ if and only if $a^{-1}b \in A(a^{-1}bb^{-1}a)$. We then observe that

$$\begin{aligned} a\ \rho_{(\tau,A)}\ b &\iff a^{-1}b \in A,\ a^{-1}bb^{-1}a\ \tau\ a^{-1}a,\ b^{-1}aa^{-1}b\ \tau\ b^{-1}b \\ &\iff a^{-1}b \in A(a^{-1}bb^{-1}a),\ b^{-1}a \in A(b^{-1}aa^{-1}b) \\ &\qquad \text{and } A(a^{-1}bb^{-1}a) = A(a^{-1}a),\ A(b^{-1}aa^{-1}b) = A(b^{-1}b) \\ &\iff a^{-1}b \in A(a^{-1}a),\ b^{-1}a \in A(b^{-1}b) \\ &\iff a\ \rho_{\mathcal{A}}\ b. \end{aligned}$$

Hence $\rho_{\mathcal{A}} = \rho_{(\tau,A)}$.

It remains to show that if $\mathcal{A}$ is a inverse system then $\mathrm{Isys}(\rho_{\mathcal{A}}) = \mathcal{A}$ and that if ρ is a left congruence with inverse system $\mathcal{A}$ then $\rho = \rho_{\mathcal{A}}$. Both these claims follow from the definitions. Let $\mathcal{A}$ be an inverse system, we know that if b is an element in some part of the inverse system then $b \in A(b)$. It is clear that $b \in A(b)$ if and only if $b\ \rho_{\mathcal{A}}\ bb^{-1}$ which in turn is equivalent to b being in the same subset bb^{-1} in the inverse system of $\rho_{\mathcal{A}}$. Thus $\mathcal{A} = \mathrm{Isys}(\rho_{\mathcal{A}})$.

For the second claim we suppose that ρ is a left congruence and let $\mathcal{A} = \mathrm{Isys}(\rho)$. Then $a\ \rho\ b$ implies $a^{-1}a\ \rho\ a^{-1}b$ and $b^{-1}a\ \rho\ b^{-1}b$, which is equivalent to the fact that $a^{-1}b \in A(a^{-1}a)$ and $b^{-1}a \in A(b^{-1}b)$, which in turn implies that $a\ \rho_{\mathcal{A}}\ b$. Conversely if $a\ \rho_{\mathcal{A}}\ b$ then $a^{-1}b \in A(a^{-1}a)$ and $b^{-1}a \in A(b^{-1}b)$, so $a^{-1}b\ \rho\ a^{-1}a$ and $b^{-1}a\ \rho\ b^{-1}b$. By (I4) we also know that $a^{-1}b \in A(a^{-1}bb^{-1}a)$, so $a^{-1}b\ \rho\ a^{-1}bb^{-1}a$. These three relations together imply that $a\ \rho\ b$. Thus we have shown that $\rho = \rho_{\mathcal{A}}$. $\square$

At this point we have shown that we may characterise the sets that arise as the partitions of the inverse kernel determined by a left congruence. In the two sided case this coincides with the notion of kernel normal system (this is clear from the fact that the inverse kernel is equal to the kernel in this case) and kernel normal systems have a number of nice properties, including that each part of the partition is an inverse subsemigroup. In the one sided case this falls apart completely, each part is not necessarily any of inverse, self conjugate or a subsemigroup, but can be any combination of them. The root cause of everything failing for the one sided case is that if $a \in \mathrm{Inker}(\rho)$ (or indeed $\ker(\rho)$ for that matter) then it does not follow that $a \ \rho \ a^{-1}$, a fact which is true in the two sided case.

— 3 —

Left congruences on inverse semigroups

In the preliminary chapter it was promised that we would consider examples to illuminate our work. In this chapter we describe left congruences on the selection of inverse semigroups that we introduced in Section 1.5. While there are descriptions of one sided congruences on many commonly studied families and examples of inverse semigroups, and much of that which we cover in the upcoming sections (the sections on Clifford semigroups, the bicyclic monoid and Brandt semigroups) can be found elsewhere in the literature there is value in a systematic approach and in seeing how the inverse kernel approach works in each case.

We recall that the inverse kernel approach describes left congruences in terms of congruences on idempotents and full inverse subsemigroups. Before we dive into the list of examples we first present a brief discussion about what we can say, in general, about the lattices $\mathfrak{C}(E)$ and $\mathfrak{V}(S)$.

3.1 The lattices $\mathfrak{C}(E)$ and $\mathfrak{V}(S)$

Describing congruence lattices for semilattices and their properties has attracted wide interest and study. We shall be brief and only mention that which is relevant to our use, or is of particular interest. Throughout, E shall refer to an arbitrary semilattice.

It is immediate that if τ is a congruence on E then each τ-class is a subsemilattice, and moreover, is *convex*, by which we mean a subset $F \subseteq E$ such that if $e, f \in F$ and $h \in E$ with $f \leq h \leq e$ then $h \in F$. In fact, as we shall see, each convex subsemilattice is a congruence class of some congruence. A subsemigroup $T \subseteq S$ is said to have the *congruence extension property* if for each congruence ρ on T there is a congruence κ on S such that $\kappa \cap (T \times T) = \rho$.

Theorem 3.1.1 ([76, Theorem 2.1])**.** *Every subsemilattice of a semilattice has the congruence extension property.*

Related to the congruence extension property is the following weaker result, which is useful to us, and which we shall use regularly in this and the following chapter, so we provide a proof.

Lemma 3.1.2. *Let E be a semilattice and $F \subseteq E$ be a subsemilattice. Let $\overline{F} = \{e \in E \mid \exists f_1, f_2 \in F$ with $f_1 \leq e \leq f_2\}$ be the convex closure of F. Then $\overline{F}$ is a congruence class of $\tau = \langle F \times F \rangle$ (the congruence generated by the binary relation $F \times F$).*

Proof. We first note that certainly for $k_1, k_2 \in \overline{F}$ we have $k_1 \; \tau \; k_2$, hence $\overline{F} \times \overline{F} \subseteq \tau$. For the reverse inclusion we suppose $h \; \tau \; g$ with $h \in \overline{F}$, so there is a τ-sequence from h to g. Thus there are $p_i, q_i \in F$, and $y_i \in E$ for $1 \leq i \leq n$ such that

$$h = y_1 p_1, \; y_1 q_1 = y_2 p_2, \ldots, y_n q_n = g.$$

We prove $g \in \overline{F}$ by induction. Suppose $y_i p_i \in \overline{F}$, then there is $f \in F$ such that $f \leq y_i p_i$. Then $f q_i \leq y_i p_i q_i \leq y_i q_i \leq q_i$. As $f, q_i \in F$ and F is a subsemilattice we have $f q_i \in F$. Therefore $y_i q_i \in \overline{F}$. Since $y_i q_i = y_{i+1} p_{i+1}$ this completes the inductive step, and as $y_1 p_1 = h \in \overline{F}$ it follows that $g = y_n q_n \in \overline{F}$. □

Congruence lattices of semilattices satisfy a range of properties, for a summary of the area see [49, Section 4] and [48, Section 3]. It is possible to abstractly characterise the lattices which arise as $\mathfrak{C}(E)$ for a semilattice E. This characterisation is due to Zhitomersky in [80], this is in Russian so the following statement is a translation from [48]. We say that elements a, b in a lattice (which has a 1 and a 0) are *Boolean* if $a \vee b = 1$ and $a \wedge b = 0$.

Theorem 3.1.3 ([48, remark after Theorem 3.5])**.** *A complete lattice L is isomorphic to the lattice of congruences of a semilattice if and only if*

(i) for all elements $x, y \in L$ there are Boolean elements $b, c \in L$ such that $(b \leq c \vee x \iff b \leq c \vee y)$ implies $x = y$;

(ii) for all elements $x, y \in L$ there is a maximal element $m \in L$ such that $(x \leq m \iff y \leq m)$ implies $x = y$.

Just like lattices of congruences on semilattices, lattices of full inverse subsemigroups of inverse semigroups are of broad interest and have been extensively studied. A sizeable quantity of this research can be attributed to Jones, who has a hefty back-catalogue of research into inverse subsemigroup lattices, for example [35], [37] and [36] to mention just three of many papers. Much that is known about $\mathfrak{V}(S)$ is concerned with the question: "If $\mathfrak{V}(S)$ has property P then what does S look like?" At this time this is not our focus, though we turn to a related question in Section 4.4. For the moment we are more interested in how we describe full inverse subsemigroups. The main result to which we shall appeal is the following, originally due to Jones in [37].

Theorem 3.1.4 ([37, Theorem 1.4])**.** *For any inverse semigroup S the lattice of full inverse subsemigroups is a subdirect product of the lattices of full inverse subsemigroups of the principal factors.*

We now turn our attention to examples of left congruences on inverse semigroups described by the inverse kernel approach. We remind ourselves of the definition of an inverse congruence pair which we shall be using. If τ is a congruence on E and $T \subseteq S$ is a full inverse subsemigroup then (τ, T) is an inverse congruence pair for S if (τ, T) satisfies the following conditions:

(ICP1) $T \subseteq N(\tau)$;

(ICP2) for $x \in S$, if there exist $e, f \in E$ such that $x^{-1}x \;\tau\; e,\; xx^{-1} \;\tau\; f$ and $xe, fx \in T$, then we have $x \in T$.

The left congruence corresponding to an inverse congruence pair (τ, T) is

$$\rho(\tau, T) = \{(x, y) \mid x^{-1}y \in T,\; x^{-1}yy^{-1}x \;\tau\; x^{-1}x,\; y^{-1}xx^{-1}y \;\tau\; y^{-1}y\}.$$

We note that to reduce the number of subscripts we have altered the notation for the congruence defined by an inverse congruence pair, we now write $\rho(\tau, T)$ for this relation. Now on with the examples.

3.2 Clifford semigroups

We start with Clifford semigroups, we recall that Clifford semigroups are strong semilattices of groups, and are written $\mathcal{C}(Y, G_e, \phi_{e,f})$ where Y is a semilattice, G_e is a group for each $e \in Y$ and $\phi_{e,f} \colon G_e \to G_f$ is a homomorphism for each $f \leq e$ in Y with $\phi_{e,e}$ the identity homomorphism. Then $\mathcal{C}(Y, G_e, \phi_{e,f})$ is the disjoint union $\bigcup_{e \in Y} G_e$ and multiplication is defined by $gh = (g\phi_{gg^{-1},hh^{-1}gg^{-1}})(h\phi_{hh^{-1},hh^{-1}gg^{-1}})$.

One sided congruences on Clifford semigroups are described by Petrich and Rankin in [61, Section 8]. As we have stated previously, on a Clifford semigroup the notion of kernel and inverse kernel coincide, so our description here coincides with the description in [61]. However, the description we give here is cleaner and more explicit. We first make formal the observation that the kernel and inverse kernel are equal for Clifford semigroups.

Lemma 3.2.1. *Let S be a Clifford semigroup and let ρ be a left congruence on S. Then* $\ker(\rho) = \operatorname{Inker}(\rho)$.

Proof. We recall, for a Clifford semigroup S and $s \in S$, that $ss^{-1} = s^{-1}s$. Suppose that $a \in S$ and $e \in E(S)$ with $a \mathrel{\rho} e$, then

$$a = aa^{-1}a \mathrel{\rho} aa^{-1}e = (ea)(ea)^{-1} = (ea)^{-1}(ea) = a^{-1}ea \mathrel{\rho} a^{-1}e \mathrel{\rho} a^{-1}a = aa^{-1}.$$

Therefore $\ker(\rho) \subseteq \operatorname{Inker}(\rho)$. It follows that the two are equal, since it is always the case that $\operatorname{Inker}(\rho) \subseteq \ker(\rho)$. □

The inverse kernel description of left congruences uses full inverse subsemigroups, which we have described for Clifford semigroups in Lemma 1.5.2. We recall that full inverse subsemigroups of $S = \mathcal{C}(Y, G_e, \phi_{e,f})$ are precisely the sets $\mathcal{C}(Y, H_e, \phi_{e,f}|_{H_e})$ where $H_e \leq G_e$ is a subgroup for each $e \in Y$ and $H_e\phi_{e,f} \leq H_f$ whenever $f \leq e$.

As previously remarked, for Clifford semigroups the idempotents are central, so all congruences on E are normal in S or equivalently if τ is a congruence on E then $N(\tau) = S$. This means that when using the inverse kernel approach to describe congruences on a Clifford semigroup the condition (ICP1) is trivial.

Theorem 3.2.2. *Let $S = \mathcal{C}(Y, G_e, \phi_{e,f})$ be a Clifford semigroup. Let τ be a congruence on Y and let $T = \mathcal{C}(Y, H_e, \phi_{e,f}|_{H_e})$ be a full inverse subsemigroup of S. Then (τ, T) is an inverse congruence pair for S if and only if, for $e, f \in Y$,*

$$f \leq e \text{ and } e \; \tau \; f \implies H_e = \{g \in G_e \mid g\phi_{e,f} \in H_f\} = H_f \phi_{e,f}^{-1}.$$

Proof. This follows identically to Lemma 1.5.3, the description of congruence pairs for two sided congruences on Clifford semigroups. We provide a reminder of the proof.

Suppose first that (τ, T) is an inverse congruence pair, and take $e, f \in Y$ with $f \leq e$ and $e \; \tau \; f$. Suppose $g \in G_e$ with $g\phi_{e,f} \in H_f$. Then $gf = g\phi_{e,f} \in T$ and $gg^{-1} = g^{-1}g = e \; \tau \; f$. Therefore, by applying (ICP2), we have $g \in T$, so $g \in H_e$. Thus we have that $\{g \in G_e \mid g\phi_{e,f} \in H_f\} \subseteq H_e$. As T is a subsemigroup it is immediate that $H_e \subseteq \{g \in G_e \mid g\phi_{e,f} \in H_f\}$. It follows that we have equality.

For the converse we suppose that $H_e = \{g \in G_e \mid g\phi_{e,f} \in H_f\}$ whenever $e \; \tau \; f$ and $f \leq e$. As (ICP1) is trivial for Clifford semigroups we need to show that (ICP2) holds. Let $a \in S$ and $e \in E$ be such that $aa^{-1} = a^{-1}a \; \tau \; e$ and $ae = ea \in T$. Then $ae \in H_{a^{-1}ae}$. Also, $a^{-1}ae \leq a^{-1}a$, so by assumption $H_{a^{-1}a} = \{g \in G_{a^{-1}a} \mid g\phi_{a^{-1}a, a^{-1}ae} \in H_{a^{-1}ae}\}$. As $ae = a\phi_{a^{-1}a, a^{-1}ae} \in H_{a^{-1}ae}$, we have that $a \in H_{a^{-1}a}$, whence $a \in T$ and (ICP2) holds. Thus (τ, T) is an inverse congruence pair, completing the proof. $\square$

We compare this with the kernel-trace description of two sided congruences on Clifford semigroups. We recall one definition of congruence pair (Definition 1.3.10): a pair (τ, K) where $K \subseteq S$ is a full self conjugate inverse subsemigroup, τ is a normal congruence on $E(S)$ and

(CP1) $ae \in K$ and $e \; \tau \; a^{-1}a$ implies that $a \in K$;

(CP3) $a \in K$ implies that $a^{-1}a \; \tau \; aa^{-1}$.

In general Item (CP1) is a stronger version of (ICP2), it is always the case that if (τ, T) satisfies (CP1) then it satisfies (ICP2). For Clifford semigroups the two conditions, (CP1) and (ICP2), are equivalent. Indeed, let S be a Clifford semigroup and take $\tau \in \mathfrak{C}(E)$ and $T \in \mathfrak{V}(S)$. If $ae \in T$ and $a^{-1}a \; \tau \; e$ (so the conditions in (CP1) are satisfied) then $ea = ae \in T$ and $aa^{-1} = a^{-1}a \; \tau \; e$, so the conditions for (ICP2) are satisfied. It follows that (CP1) and (ICP2) are the same for Clifford semigroups. We also recall that for Clifford semigroups (CP3) and (ICP1) are trivial. Therefore, for a Clifford semigroup, the only difference between a congruence pair and an inverse congruence pair is that the subsemigroup in the case of a congruence pair is self conjugate.

Corollary 3.2.3. *Let S be a Clifford semigroup and let ρ be a left congruence on S. Then ρ is a two sided congruence if and only if* $\mathrm{Inker}(\rho)$ *is self conjugate.*

3.3 The Bicyclic Monoid

Our second example is the bicyclic monoid. Descriptions of one sided congruences on the bicyclic monoid B are known ([54] and [10]). However, it is an illuminating illustration of our techniques to apply the inverse kernel approach to the lattice of left congruences on B. Recall that we use the following description of the bicyclic monoid: $B = \mathbb{N}^0 \times \mathbb{N}^0$ with multiplication

$$(a, b)(c, d) = (a - b + t, d - c + t)$$

where $t = \max\{b, c\}$. For the remainder of this section we shall use B to denote the bicyclic monoid.

The application of the inverse kernel approach to describing left congruences on the bicyclic monoid is fairly technical and involved. We describe full inverse subsemigroups and congruences on the idempotents separately, and then classify inverse congruence pairs.

Full inverse subsemigroups

The lattice of full inverse subsemigroups of the bicyclic monoid is discussed in Jones [35] and Descalço and Ruškuc [9]. However, both these approaches in the literature leave something to be desired considering our needs, the former focuses on describing the lattice of full inverse subsemigroups (with emphasis on the lattice), and the latter is an application of a general description of subsemigroups of B. It is not that difficult to give a rigorous description of full inverse subsemigroups, so we shall start from the beginning.

Definition 3.3.1. For $k \in \mathbb{N}^0$ and $d \in \mathbb{N}$ define:

$$T_{k,d} = \{(x, y) \mid x, y \geq k,\ d|x - y\} \cup E(B).$$

A pictorial representation of $T_{k,d}$ is shown in Fig. 3.1, in which elements that lie on the solid lines are elements in $T_{k,d}$.

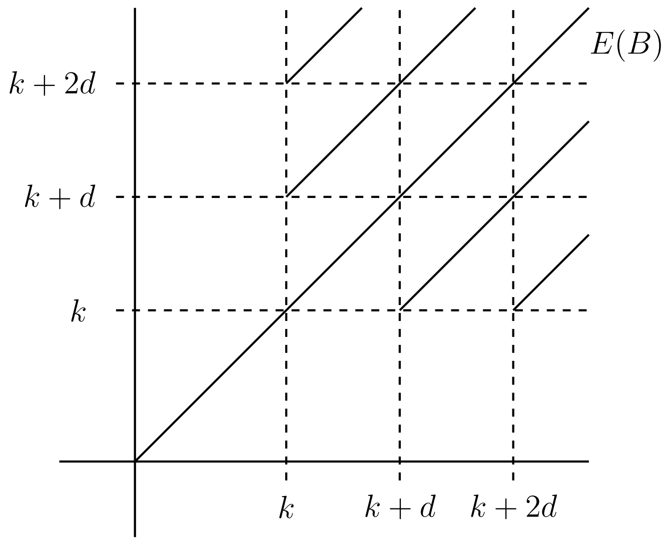


Figure 3.1: $T_{k,d}$, a full inverse subsemigroup of B

We note that each non-idempotent element in $T_{k,d}$ is of the form $(i, i+md)$ or $(i + md, i)$ for some $i \geq k$ and $m \geq 1$. We now show that together with $E(B)$, the $T_{k,d}$'s form a complete list of all full inverse subsemigroups of B.

Proposition 3.3.2. *Let $k \in \mathbb{N}^0$ and $d \in \mathbb{N}$. Then $T = T_{k,d}$ is a full inverse subsemigroup of B.*

Proof. It is immediate that T is full, and as $(x, y)^{-1} = (y, x)$ it is clear that T is inverse. It remains to show that T is a subsemigroup. Suppose that

$(x, y), (z, w) \in T$, then $(x, y)(z, w) = (x-y+t, w-z+t)$ with $t = \max\{y, z\}$. We suppose that neither (x, y) nor (z, w) are idempotents. Then we note that $x - y + t \geq x \geq k$, $w - z + t \geq w \geq k$ and $(x - y + t) - (w - z + t) = (x - y) - (w - z)$ which is divisible by d since both $x - y$ and $w - z$ are divisible by d. Hence $(x, y)(z, w) \in T$. Similar arguments demonstrate that $(x, y)(z, w) \in T$ when one or both of (x, y) and (z, w) are idempotents. Thus T is a subsemigroup. □

Theorem 3.3.3 (see [9, Theorem 7.1])**.** *For $k \geq 0$, $d \geq 1$, $T_{k,d}$ is a full inverse subsemigroup of B. Moreover if $T \neq E(B)$ is a full inverse subsemigroup of B, then $T = T_{k,d}$ for some $k \geq 0$, $d \geq 1$.*

Proof. From Proposition 3.3.2, for $k \geq 0$ and $d \geq 1$, we know that $T_{k,d}$ is a full inverse subsemigroup, so it remains to prove the converse, that every full inverse subsemigroup of B is of this form. Suppose that $T \neq E(S)$ is a full inverse subsemigroup. Let $k = \min\{a \in \mathbb{N}^0 \mid \exists (a, b) \in T \backslash E(S)\}$ so k is the smallest integer that appears in a non-idempotent element. Also let $d = \min\{a \in \mathbb{N} \mid (k, k + a) \in T\}$, noting that k, and so also d, exists since $T \neq E(B)$. We claim that $T = T_{k,d}$.

By the definition of k and d, we have that $(k, k + d) \in T$. We notice that $(k, k + d)^n = (k, k + nd)$ and that for $b \geq 0$

$$(k + b, k + b)(k, k + nd) = (k + b, k + b + nd).$$

Together these imply that $(k + b, k + b + nd) \in T$ for any $n \geq 1$ and $b \geq 0$. Similarly we obtain that $(k + b + nd, k + b) \in T$. Every non idempotent element of $T_{k,d}$ is of the form $(k + b + nd, k + b)$ or $(k + b, k + b + nd)$ for some $n \geq 1$ and some $b \in \mathbb{N}^0$, so we have that $T_{k,d} \subseteq T$.

For the reverse inclusion suppose that $(x, x + y) \in T \backslash T_{k,d}$ with $y \geq 1$, so $y \nmid d$. Certainly $x \geq k$ as k was chosen minimally such that it appeared in a non-idempotent element in T. Let $n \geq 0$ be such that $nd < y < (n + 1)d$ (we note strict inequality is possible as $d \nmid y$) so $0 < y - nd < d$. Then $(x + nd, x) \in T_{k,d} \subseteq T$ so

$$(x, x + y)(x + nd, x) = (x, x + (y - nd)) \in T.$$

Similarly we choose m such that $k+(m-1)d \leq x \leq k+md$ and we observe that we then have

$$(k, k+md)(x, x+(y-nd))(k+md, k) = (k, k+(y-nd)) \in T.$$

However $0 < y-nd < d$, and d was chosen to be minimum such that $(k, k+d) \in T$. This is a contradiction, so there are no x and y such that $(x, x+y) \in T \backslash T_{k,d}$. Thus $T = T_{k,d}$. $\square$

We observe that $T_{k,d} \subseteq T_{j,c}$ if and only if $j \leq k$ and $c \,|\, d$; we can use this observation to describe the lattice of full inverse subsemigroups of B, which has an interesting structure. Let $\mathbf{C}$ be the non-negative integers under $\leq_{\mathbf{C}}$, the reverse of the usual order, and recall that $\mathbf{D}$ is the lattice consisting of the natural numbers with $m \leq_{\mathbf{D}} n$ if $n \mid m$. For a lattice $\mathbf{L}$ by $\mathbf{L}^0$ we mean the lattice with a 0 adjoined.

Corollary 3.3.4. *The lattice of full inverse subsemigroups of the bicyclic monoid ($\mathfrak{V}(B)$) is isomorphic to* $(\mathbf{C} \times \mathbf{D})^0$, *where the ordering is the usual direct product ordering. The* 0 *of* $\mathfrak{V}(B)$ *corresponds to* $E(B)$.

Congruences on the idempotents

Turning our attention to the trace lattice we know that idempotents in B are of the form (x, x) for some $x \in \mathbb{N}^0$, and that

$$(x, x)(y, y) = (\max\{x, y\}, \max\{x, y\}).$$

Thus $E(B)$ is a lattice and is isomorphic to the lattice $\mathbf{C}$. We now analyse congruences on $\mathbf{C}$ which we regard as a semilattice. Explicitly, $\mathbf{C}$ is the set $\mathbb{N}^0$ with multiplication $ab = \max\{a, b\}$.

It is natural to view congruences as partitions, and this is particularly effective for congruences on a chain of idempotents, as each part is an interval in the chain. Viewed in this way a congruence on $\mathbf{C}$ corresponds to a partition of $\mathbb{N}^0$ in which each part is an interval. Such a partition of $\mathbb{N}^0$ is determined by the set of the minimum (under the usual order on $\mathbb{N}^0$) elements of each equivalence class; in the case that there is an infinite equivalence class this corresponds to a finite set of minimum elements. This gives a bijection between the set of congruences on $\mathbb{N}^0$ and $\mathbb{P}(\mathbb{N}^0)$, the powerset of $\mathbb{N}^0$. Under this correspondence the ordering on the congruences becomes the reverse of the usual subset inclusion ordering on $\mathbb{P}(\mathbb{N}^0)$. We write $\mathbf{P}$ for this lattice, and $\leq_P$ for the ordering on $\mathbf{P}$.

An alternate way to describe a partition of $\mathbb{N}^0$ arising from a congruence is to give the size of each part sequentially. We now establish notation for both methods of characterising a congruence on $\mathbf{C}$.

Definition 3.3.5. Let τ be a congruence on $\mathbf{C}$. Let $\Xi(\tau) = \{c_1, c_2, c_3 \dots\}$ be the set of integers a that are maximal with respect to $\leq_{\mathbf{C}}$ (so minimal with respect to $\leq$, the usual ordering on $\mathbb{N}^0$) in a congruence class. Also let $\Gamma(\tau) = (m_1, m_2, m_3, \dots)$ be the sequence of integers corresponding to sizes of the finite congruence classes.

Until otherwise stated we shall assume that τ is a congruence on $\mathbf{C}$, and we let $\Xi(\tau) = \{c_1, c_2, \dots\}$ and $\Gamma(\tau) = (m_1, m_2, m_3, \dots)$. We remark that $x \ \tau \ y$ if and only if there is $u \in \mathbb{N}$ such that

$$c_u \leq x, y < c_{u+1}$$

(under the usual ordering on $\mathbb{N}$) where if $\Xi(\tau) = \{c_1, \dots, c_q\}$ is finite we define $c_{q+1} = \infty$. We note that $\Xi(\tau)$ and $\Gamma(\tau)$ are both finite if and only if τ has an infinite congruence class. Furthermore, it is clear that $c_1 = 0$ and $c_u = \sum_{i=1}^{u-1} m_i$ for $u \geq 2$. Also, $m_1 = c_2$ and for $u \geq 2$ we have $m_u = c_{u+1} - c_u$.

Our next objective is to describe $N(\tau)$ for each congruence τ on $\mathbf{C}$. We shall see that the normaliser of τ is of one of three types, depending on a certain property which τ may satisfy. We now define this property.

Definition 3.3.6. Let τ be a congruence on $\mathbf{C}$ with no infinite congruence class. We say that τ is *eventually periodic* if there are $r, p \geq 1$ such that

$$s \geq r \implies m_s = m_{s+p}.$$

We note that r, p can certainly be chosen to both be minimum. Indeed, if there is a repeating sequence in $\Gamma(\tau)$ then there is a shortest repeating sequence, the length of which we set to p. There is then an earliest point this pattern starts (where we consider the pattern cyclically), which we call r. With r, p chosen to be minimum let $k = c_r$ and let $d = \sum_{j=r}^{r+p-1} m_j$. Then d is the *period* of τ and we say that τ is *d-periodic after k*.

Let τ be an eventually periodic congruence on $\mathbf{C}$, and let p, r be as in Definition 3.3.6 chosen to be minimum. Define $l(\tau)$ as

$$l(\tau) = c_r - \min\{m_{r-1}, m_{r+p-1}\}.$$

We note that we cannot have $m_{r-1} = m_{r+p-1}$ because we chose r as the earliest point from which there is a repeating pattern of the m_i. In particular, either $l(\tau) = c_{r-1}$ (if $m_{r-1} < m_{m+p-1}$) or $c_{r-1} < l(\tau) < c_r$ (if $m_{r+p-1} < m_r$). In Fig. 3.2 we have an example of an eventually periodic congruence in the case where $m_{r+p-1} < m_{r-1}$.

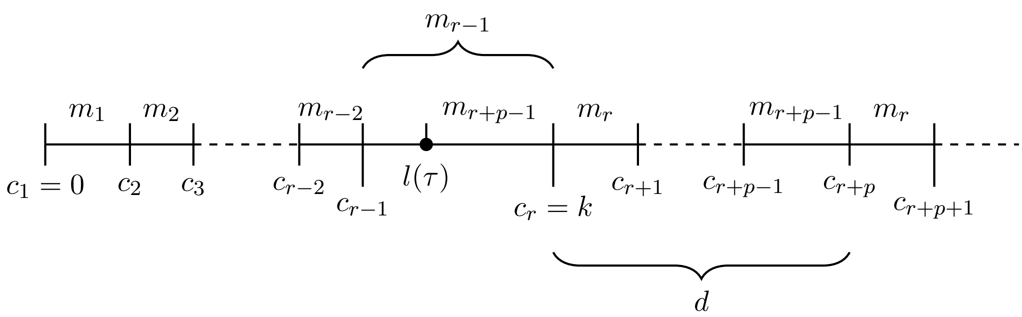


Figure 3.2: An eventually periodic congruence on $\mathbf{C}$

We now proceed with three technical lemmas in which we show that, for an eventually periodic congruence, $l(\tau)$ is the earliest point in from which we can deduce that $a \; \tau \; b$ if and only if $a + d \; \tau \; b + d$. This is the reason that $l(\tau)$ is important.

Lemma 3.3.7. *Let τ be an eventually periodic congruence on $\mathbf{C}$ and let $l = l(\tau)$. Then exactly one of $(l-1) \; \tau \; l$ or $(l+d-1) \; \tau \; (l+d)$.*

Proof. Let p, r be as in Definition 3.3.6 chosen to be minimum. Suppose $m_{r+p-1} < m_{r-1}$. Then $c_{r-1} < l = c_r - m_{r+p-1} < c_r$, so we have that $(l-1) \; \tau \; l$.

Noting that by definition $c_r + \sum_{j=r}^{r+p-1} m_j = c_{r+p}$, we have

$$l+d = c_r - m_{r+p-1} + d = c_r - m_{r+p-1} + \sum_{j=r}^{r+p-1} m_j = c_{r+p} - m_{r+p-1} = c_{r+p-1}.$$

Then, as each c_i is minimum in its congruence class, we have $(l+d-1) \not\tau (l+d)$.

Suppose instead that $m_{r-1} < m_{r+p-1}$, recalling that $m_{r-1} \neq m_{r+p-1}$ so we may use strict inequality here. Then $l = c_{r-1}$, so $(l-1) \not\tau\, l$. In this case

$$\begin{aligned} c_{r+p} = c_r + d > l + d = c_{r-1} + d = c_{r-1} + \sum_{j=r}^{r+p-1} m_j \\ > c_{r-1} + \sum_{j=r}^{r+p-1} m_j + (m_{r-1} - m_{r+p-1}) \\ = c_{r-1} + \sum_{j=r-1}^{r+p-2} m_j = c_{r+p-1}. \end{aligned}$$

Thus in this case $(l+d-1)\ \tau\ (l+d)$. $\square$

In other words, Lemma 3.3.7 states that exactly one of $l(\tau)$ or $l(\tau)+d$ is the minimum (with respect to the usual order) in a τ-class.

Lemma 3.3.8. *Let τ be a congruence on $\mathbf{C}$. Suppose there are k, d such that for $x, y \geq k$ we have $x\ \tau\ y$ if and only if $(x+d)\ \tau\ (y+d)$. Then either τ has an infinite congruence class, or τ is eventually periodic with period $d' \mid d$.*

Proof. Suppose τ has no infinite congruence class. Let r be the integer such that $c_{r-1} \leq k < c_r$. If $k + d < c_r$ then $k\ \tau\ (k+d)$. By the hypothesis it follows that $(k+(n-1)d)\ \tau\ (k+nd)$ for each $n \in \mathbb{N}$ and this implies that $k\ \tau\ (k+nd)$ for any $n \in \mathbb{N}$. This implies that there is an infinite congruence class, contradicting our assumption. Hence we must have $c_r \leq k + d$.

We will show by that there is p such that $c_r + d = c_{r+p}$, and, for each $0 \leq i \leq p-1$, that $m_{r+i} = m_{r+p+i}$. By definition of r, we have $k \leq c_r - 1$ and, since $(c_r - 1) \not\tau\, c_r$, it follows from the hypothesis that $(c_r + d - 1) \not\tau\, (c_r + d)$. Therefore $c_r + d$ is minimal in a τ-class, so $c_r + d = c_{r+p}$ for some $p \geq 1$.

We now prove by induction that $c_{r+i}+d=c_{r+p+i}$ for all $i\geq 0$. Let $i\geq 0$ and suppose that $c_{r+i}+d=c_{r+p+i}$, then for each $1\leq x<m_{r+i}$ we have that $c_{r+i}\ \tau\ (c_{r+i}+x)$. Therefore, by the hypothesis,

$$c_{r+p+i}=(c_{r+i}+d)\ \tau\ (c_{r+i}+d+x)=c_{r+p+i}+x.$$

This implies that $m_{r+p+i}\geq m_{r+i}$. As $c_{r+p+i+1}=c_{r+p+i}+m_{r+p+i}$ we have shown that $c_{r+p+i+1}\geq c_{r+p+i}+m_{r+i}$. Also $(c_{r+i}+m_{r+i}-1)\ \not\tau\ (c_{r+i}+m_{r+i})$, so again applying the hypothesis we have

$$c_{r+p+i}+m_{r+i}-1=(c_{r+i}+m_{r+i}-1+d)\ \not\tau\ (c_{r+i}+m_{r+i}+d)=c_{r+p+i}+m_{r+i}.$$

Therefore $c_{r+p+i}+m_{r+i}=c_j$ for some $j\geq r+p+i+1$. Combining this with the fact that $c_{r+p+i+1}\geq c_{r+p+i}+m_{r+i}$ we see that $c_{r+p+i+1}=c_{r+p+i}+m_{r+i}$. Therefore $c_{r+p+i+1}=c_{r+i}+m_{r+i}+d=c_{r+i+1}+d$. Thus we have completed the inductive step. Since $m_j=c_{j+1}-c_j$ it follows that for $0\leq i\leq p-1$

$$m_{r+p+i}=c_{r+p+i+1}-c_{r+p+i}=c_{r+i+1}-c_{r+i}=m_{r+i}.$$

Hence $m_s=m_{s+p}$ for all $s\geq r$, thus τ is eventually periodic.

It remains to prove that the period of τ divides d. Suppose that $p'>0$ is chosen minimally such that for $s\geq r$ we also have $m_{s+p'}=m_s$. Suppose that $p'\neq p$. We claim that for $q=p-p'$ we also have $m_{s+q}=m_s$ for $s\geq r$. Indeed, we have

$$m_{s+q}=m_{(s+p)-p'}=m_{s+p}=m_s.$$

Let n be such that $np'\leq p<(n+1)p'$. Let $q=p-np'$ then $0\leq q<p'$. Then by the previous argument we have that $m_{s+q}=m_s$ for $s\geq r$. However $p'>0$ was chosen to the be the minimum with this property. Hence $q=0$ so $p'\mid p$. As the sequence $m_r,m_{r+1},\dots m_{r+p}$ is p/p' copies of the sequence $m_r,\dots,m_{r+p'-1}$ it is immediate that the period of τ, $d'=\sum_{i=0}^{p'-1}m_{r+i}$ has $d'\mid d=\sum_{i=0}^{p-1}m_{r+i}$. $\square$

Lemma 3.3.9. *Let τ be a congruence on $\mathbf{C}$ which is d-periodic after k, and let $x,y\geq l(\tau)$. Then $x\ \tau\ y$ if and only if $x+d\ \tau\ y+d$.*

Proof. Choose r, p minimally as in Definition 3.3.6. We observe, that as the sequence of m_u repeats, for any $q \geq r$ we have $\sum_{i=q}^{q+p-1} m_i = d$. Hence for any $u \geq r$ we have that

$$c_{u+p} = c_u + \sum_{i=u}^{u+p-1} m_i = c_u + d.$$

Suppose $x\ \tau\ y$, where, without loss of generality, we assume that $x \leq y$. First consider the case when $c_r = k \leq x \leq y$. There is some $u \geq r$ such that

$$c_u \leq x \leq y < c_{u+1}.$$

Then we note that

$$c_{u+p} = c_u + d \leq x + d \leq y + d < c_{u+1} + d = c_{u+1+p}.$$

Thus $x + d\ \tau\ y + d$.

We now suppose that $l(\tau) \leq x < k$ so, since $x\ \tau\ y$, we have $l(\tau) \leq y < c_r = k$. By definition $c_{r+p-1} \leq l(\tau) + d$, so

$$c_{r+p-1} \leq l(\tau) + d \leq x + d \leq y + d < k + d = c_r + d = c_{r+p}.$$

Thus $x + d\ \tau\ y + d$.

Conversely, suppose that $x + d\ \tau\ y + d$ (and continue to assume that $x \leq y$). Then there is some v such that $c_v \leq x + d \leq y + d < c_{v+1}$. Suppose $x, y \geq k = c_r$, then $x + d, y + d \geq c_{r+p}$. Thus $v \geq r + p$, so we have that $r \leq v - p$. Then $c_{v-p} = c_v - d$ and we observe

$$c_{v-p} = c_v - d \leq (x + d) - d \leq (y + d) - d < c_{v+1} - d = c_{v-p+1}.$$

Hence $x\ \tau\ y$. If $l(\tau) \leq x < k$ then, recalling that $c_{r+p-1} \leq l(\tau) + d$,

$$c_{r+p-1} \leq l(\tau) + d \leq x + d < k + d = c_r + d = c_{r+p}.$$

As $x + d\ \tau\ y + d$, also $c_{r+p-1} \leq y + d < c_{r+p}$. Then, recalling that $c_{r-1} \leq l(\tau)$,

$$c_{r-1} \leq l(\tau) \leq x \leq y < c_{r+p} - d = c_r.$$

Thus $x\ \tau\ y$ and we have completed the proof. $\square$

We are now going to use the previous lemmas and determine the normaliser of a trace. We return to the usual notation for elements in $E(B)$. We first compute the normaliser for an eventually periodic trace.

Proposition 3.3.10. *Let τ be a congruence on $E(B)$ which is d-periodic after k, and let $l = l(\tau)$. Then*

$$N(\tau) = T_{l,d}.$$

Proof. We first show that $T_{l,d} \subseteq N(\tau)$. Take $a \geq l$ and $b \in \mathbb{N}^0$ so that $(a, a+bd) \in T_{l,d}$; note every element in $T_{l,d}$ is either of this form or has its inverse of this form. Also suppose that $(s,s) \; \tau \; (t,t)$ (with $s \leq t$) then consider

$$(a, a+bd)(s,s)(a+bd, a) = (x-bd, x-bd) \quad \text{ for } x = \max\{a+bd, s\}$$
$$(a, a+bd)(t,t)(a+bd, a) = (y-bd, y-bd) \quad \text{ for } y = \max\{a+bd, t\}$$

We claim that $(x-bd, x-bd) \; \tau \; (y-bd, y-bd)$. We consider 3 cases.

(i) If $s \leq t \leq a+bd$ (so $x = a+bd = y$) then this is immediate as $x - bd = a = y - bd$.

(ii) If $a+bd \leq s \leq t$ (so $x = s$ and $y = t$) then by repeated application of Lemma 3.3.9, as $l \leq a \leq s-bd \leq t-bd$, we have that

$$(s-bd, s-bd) \; \tau \; (t-bd, t-bd) \iff (s,s) \; \tau \; (t,t).$$

Hence $(s-bd, s-bd) \; \tau \; (t-bd, t-bd)$.

(iii) If $s \leq a+bd \leq t$ (so $x = a+bd$ and $y = t$) then using that each τ class is an interval in $E(B)$ we certainly have that $(a+bd, a+bd) \; \tau \; (t,t)$. We apply the same argument as in *(ii)* to obtain that $(a,a) \; \tau \; (t-bd, t-bd)$. Noting that in this case $(x-bd, x-bd) = (a,a)$ we have shown the claim is true.

Therefore we have that $(x-bd, x-bd) \; \tau \; (y-bd, y-bd)$ in all cases. It is very similar to show that if $(s,s) \; \tau \; (t,t)$ then

$$(a+bd, a)(s,s)(a, a+bd) \; \tau \; (a+bd, a)(t,t)(a, a+bd),$$

and thus we obtain $(a, a+bd) \in N(\tau)$. We have shown that $T_{l,d} \subseteq N(\tau)$.

We now prove the reverse inclusion, that $N(\tau) \subseteq T_{l,d}$. As $N(\tau)$ is a full inverse subsemigroup of B and by the previous paragraph is not equal to $E(B)$ there are j, c such that $N(\tau) = T_{j,c}$. As $T_{l,d} \subseteq T_{j,c}$ by the ordering on the full inverse subsemigroups we must have that $j \leq l$ and $c \mid d$.

We observe for $s \geq a$ that $(a+b, a)(s, s)(a, a+b) = (s+b, s+b)$, and for $s \geq a+b$ we have $(a, a+b)(s, s)(a+b, a) = (s-b, s-b)$. Suppose $s, t \geq j$, then, by the previous observation with $a = j$ and $b = c$, as $(j, j+c) \in N(\tau)$ we have

$$(s, s) \ \tau \ (t, t) \iff (s+c, s+c) \ \tau \ (t+c, t+c).$$

Applying Lemma 3.3.8 we obtain that τ is d'-periodic after k' with $d' \mid c$. By assumption τ has period d, so $d' = d$. Furthermore, $d \mid c$ but we also have $c \mid d$. Hence we obtain that $c = d$.

Suppose that $j < l(\tau) = l$ so we have $(l-1, l-1+d) \in N(\tau)$. By Lemma 3.3.7 exactly one of

$$(l-1, l-1) \ \tau \ (l, l) \quad \text{or} \quad (l-1+d, l-1+d) \ \tau \ (l+d, l+d).$$

Suppose that $(l-1+d, l-1+d) \ \tau \ (l+d, l+d)$ then

$$\begin{aligned}(l-1, l-1) &= (l-1, l-1+d)(l-1+d, l-1+d)(l-1+d, l-1),\\ (l, l) &= (l-1, l-1+d)(l+d, l+d)(l-1+d, l-1)\end{aligned}$$

so we have that $(l-1, l-1) \ \tau \ (l, l)$, a contradiction. Suppose instead that $(l-1, l-1) \ \tau \ (l, l)$, then

$$\begin{aligned}(l-1+d, l-1+d) &= (l-1+d, l-1)(l-1, l-1)(l-1, l-1+d),\\ (l+d, l+d) &= (l-1+d, l-1)(l, l)(l-1, l-1+d).\end{aligned}$$

This implies that $(l-1+d, l-1+d) \ \tau \ (l+d, l+d)$, which is again a contradiction. Hence we must have $j = l$ and so $N(\tau) = T_{l,d}$ $\square$

We now compute the normaliser for a trace with an infinite congruence class.

Proposition 3.3.11. *Let τ be a congruence on $E(B)$ such that τ has the infinite congruence class $\{(j,j) \mid j \geq n\}$. Then*

$$N(\tau) = T_{n,1}.$$

Proof. We first suppose $x, y \geq n$. Then for any s we have $(x,y)(s,s)(y,x) = (z_s, z_s)$, where $z_s = \max\{x - y + s, x\} \geq x$. Hence if $(s,s)\ \tau\ (t,t)$ we have $z_s, z_t \geq n$, so $(z_s, z_s)\ \tau\ (z_t, z_t)$. Thus $(x,y) \in N(\tau)$ so $T_{n,1} \subseteq N(\tau)$.

We proceed with the reverse inclusion. First suppose that $x < n \leq y$. Then $(y,y)\ \tau\ (y+n, y+n)$. However, we observe that $(x,y)(y,y)(y,x) = (x,x)$, and $(x,y)(y+n,y+n)(y,x) = (x+n, x+n)$. Since $x < n$ we have that $(x,x)\ \not\tau\ (x+n, x+n)$, thus $(x,y) \notin N(\tau)$.

Suppose finally that $x < y < n$. We note that $(n,n)\ \tau\ (y+n, y+n)$. Then $(x,y)(n,n)(y,x) = (x-y+n, x-y+n)$, and $(x,y)(y+n,y+n)(y,x) = (x+n, x+n)$. As $x-y+n < n \leq n+x$, we have $(x-y+n, x-y+n)\ \not\tau\ (x+n, x+n)$. Hence $(x,y) \notin N(\tau)$. Thus we have that $N(\tau) = T_{n,1}$. □

Proposition 3.3.12. *Let τ be a congruence on $E = E(B)$, with no infinite congruence class. Then τ is eventually periodic if and only if $N(\tau) \neq E(B)$.*

Consequently, if $\tau \in \mathfrak{C}(E(B))$ has no infinite congruence class and is not eventually periodic then $N(\tau) = E(B)$.

Proof. We first note that if τ is d-periodic after k then, by Proposition 3.3.10, we have $T_{l(\tau),d} = N(\tau)$. In particular, $N(\tau) \neq E(B)$.

For the converse suppose that $N(\tau) \neq E(B)$. Choose $a \geq 0, b \geq 1$ such that $(a, a+b) \in N(\tau)$. If $v \geq a+b$ then $(a+b, a)(v,v)(a, a+b) = (v+b, v+b)$, and $(a, a+b)(v,v)(a+b, a) = (v-b, v-b)$. Then for $s, t \geq a+b$, we have that

$$(s,s)\ \tau\ (t,t) \iff (s+b, s+b)\ \tau\ (t+b, t+b).$$

Thus by Lemma 3.3.8, since τ has no infinite congruence class we have that τ is eventually periodic. □

We have then described the normaliser of every congruence on $E(B)$, we summarise in the following corollary.

Corollary 3.3.13. *Let τ be a congruence on $E(B)$. Then*

(i) if τ has an infinite congruence class - which we say is $\{(x,x) \mid x \geq n\}$ - then $N(\tau) = T_{n,1}$;

(ii) if τ is eventually periodic - suppose it is d-periodic after k - then $N(\tau) = T_{l(\tau),d}$;

(iii) if τ is not eventually periodic and has no infinite congruence class then $N(\tau) = E(B)$.

Inverse congruence pairs

To determine inverse congruence pairs for B, for each congruence τ on $E(B)$, we must calculate which inverse subsemigroups of B which are contained in $N(\tau)$ satisfy (ICP2). For the bicyclic monoid (ICP2) has the following formulation.

(ICP2) For $(x,y) \in B$ and $(w,w),(z,z) \in E(B)$, if

$$\begin{aligned} (y,y)\ \tau\ (w,w), &\qquad (x,x)\ \tau\ (z,z) \\ (x,y)(w,w) &= (x-y+\max\{y,w\}, \max\{y,w\}) \in T \\ \text{and} \quad (z,z)(x,y) &= (\max\{x,z\}, y-x+\max\{x,z\}) \in T, \end{aligned}$$

then $(x,y) \in T$.

The following is a special case of Proposition 2.3.4, which says that (τ, T) is an inverse congruence pair exactly when $T \subseteq N(\tau)$ and T is saturated by $\nu_\tau|_{N(\tau)}$, giving an alternate characterisation of (ICP2). We include a proof here for completion.

Lemma 3.3.14. *Let τ be a congruence on $E(B)$ and $T \subseteq N(\tau)$ a full inverse subsemigroup. Let $(x,y) \in N(\tau)$. Then the following are equivalent:*

(i) if $(w,w),(z,z) \in E(B)$ are such that $(y,y)\ \tau\ (w,w)$, $(x,x)\ \tau\ (z,z)$ and $(x-y+\max\{y,w\}, \max\{y,w\}), (\max\{x,z\}, y-x+\max\{x,z\}) \in T$ then $(x,y) \in T$ (in other words, (ICP2) holds);

(ii) if $(w,z) \in T$, $(w,w)\ \tau\ (y,y)$ *and* $w - z = y - x$ *then* $(x,y) \in T$.

Proof. We first suppose that (i) holds and that we have $(w,z) \in T$, $(w,w)\ \tau\ (y,y)$ and $w - z = y - x$. If $y \geq w$ then

$$(z,w)(y,y) = (z - w + y, y) = (x - y + y, y) = (x,y),$$

so $(x,y) \in T$. We suppose that $w > y$. We claim that $(z,z)\ \tau\ (x,x)$. Indeed, as $(x,y) \in N(\tau)$ we conjugate $(y,y)\ \tau\ (w,w)$ by (x,y) to obtain

$$(x,x) = (x,y)(y,y)(y,x)\ \tau\ (x,y)(w,w)(y,x) = (z,z)$$

where we note that $x - y + w = z - w + w = z$. Further,

$$(x - y + \max\{y,w\}, \max\{y,w\}) = (z,w) = (\max\{x,z\}, y - x + \max\{x,z\})$$

so the conditions for (i) are satisfied so $(x,y) \in T$. Hence (ii) holds.

For the converse we suppose (ii) holds and that there are $(w,w),(z,z) \in E(B)$ such that $(y,y)\ \tau\ (w,w)$, $(x,x)\ \tau\ (z,z)$ and

$$(x - y + \max\{y,w\}, \max\{y,w\}),\ (\max\{x,z\}, y - x + \max\{x,z\}) \in T.$$

Then certainly $(y,y)\ \tau\ (\max\{y,w\}, \max\{y,w\})$. Further, as T is inverse, $(\max\{y,w\}, x - y + \max\{y,w\}) \in T$ and

$$\max\{y,w\} - (x - y + \max\{y,w\}) = y - x.$$

Thus the conditions for (ii) are satisfied with $z \mapsto x - y + \max\{y,w\}$ and $w \mapsto \max\{y,w\}$. Therefore $(x,y) \in T$. It follows that (i) is satisfied so we have that (i) and (ii) are equivalent. $\square$

The alternative method to see that Lemma 3.3.14 holds uses Proposition 2.3.4. We observe that, for B, the minimum left congruence with a given trace, which we recall is

$$\nu_\tau = \{(a,b) \in S \times S \mid\ \exists e \in E(S),\ a^{-1}a\ \tau\ b^{-1}b\ \tau\ e,\ ae = be\}$$

becomes

$$(m,n)\ \nu_\tau\ (p,q) \iff (n,n)\ \tau\ (q,q) \text{ and } n - m = q - p.$$

Indeed, let $a = (m, n)$ and $b = (p, q)$ then $a^{-1}a = (n, n)$ and $b^{-1}b = (q, q)$. We note that for the condition $ae = be$ we may assume that $e \leq a^{-1}a,\ b^{-1}b$. For any $e = (x, x)$ with $x \geq \max\{n, q\}$ we observe that $ae = (m, n)(x, x) = (m-n+x, x)$ and $be = (p, q)(x, x) = (p-q+x, x)$. Thus $ae = be$ is equivalent to $m - n = p - q$.

Lemma 3.3.14 then follows as subsemigroups contained in $N(\tau)$ satisfying (ICP2) are those saturated by $\nu_\tau|_{N(\tau)}$ (by Proposition 2.3.4), and saturation by $\nu_\tau|_{N(\tau)}$ is precisely the statement of (ii) in Lemma 3.3.14.

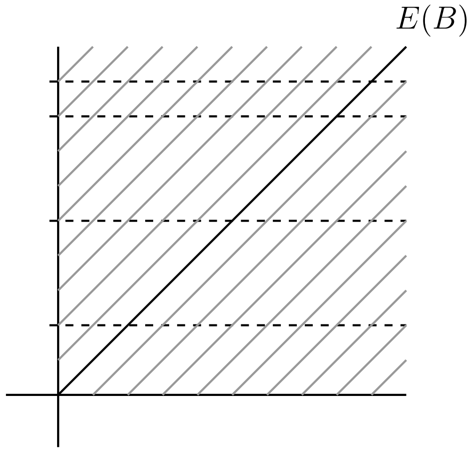


Figure 3.3: ν_τ (left) on B

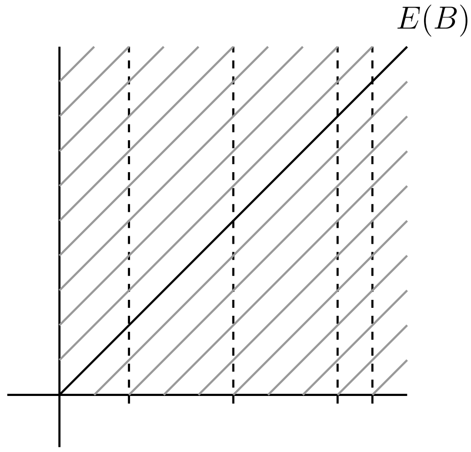


Figure 3.4: ν_τ (right) on B

We can view this pictorially. Fig. 3.3 depicts B, with diagonal lines representing the equivalence classes of the relation defined by $n - m = p - q$ and the dashed horizontal lines representing the τ relation (so there are dashed lines at $c_2, c_3, \ldots$). Elements are ν_τ related if they lie on the same diagonal and are both between the same two dashed lines (where between means equal to or greater than the lower and strictly less than the higher).

The corresponding trace minimal right congruences can be viewed similarly, Fig. 3.4 represents equivalent formulation for the right sided version. Here the dashed lines representing τ are vertical. In this case elements (m, n) and (p, q) are related if $(m, m)\ \tau\ (p, p)$ and $n - m = q - p$, which can again be thought of as being on the same diagonal and between two dashed lines.

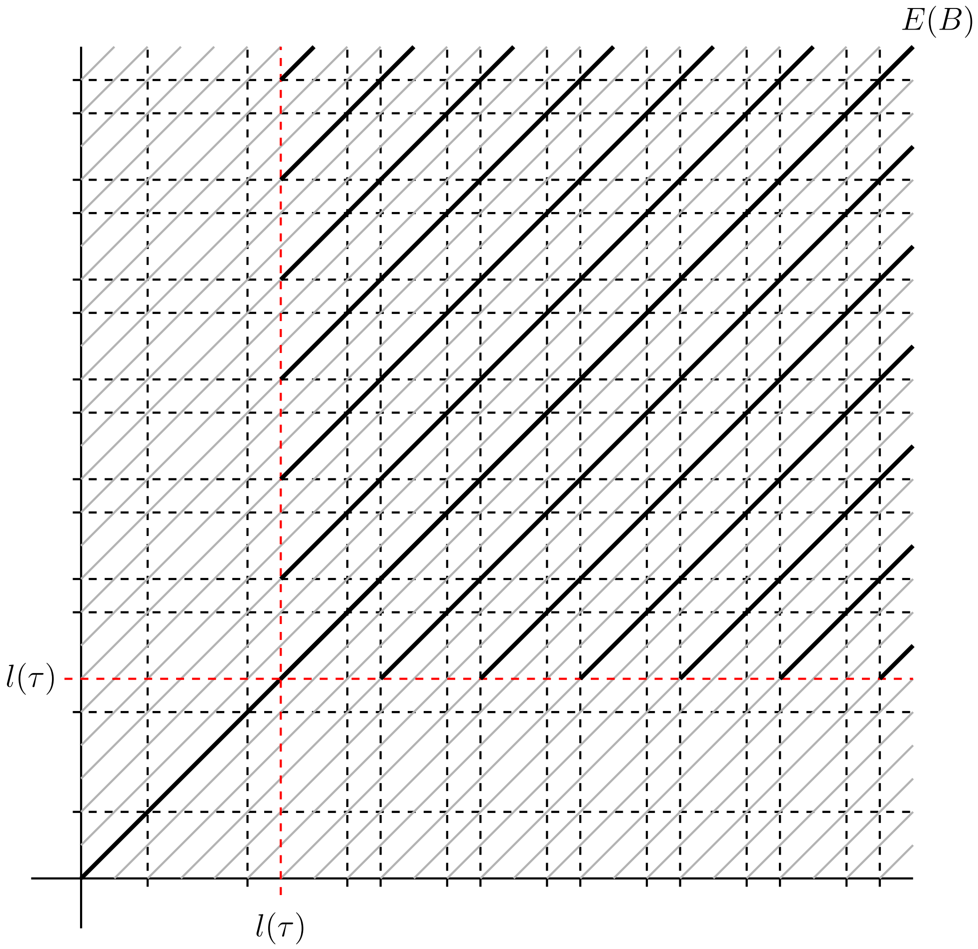


Figure 3.5: $N(\tau)$ on B for an eventually periodic trace

Since an inverse congruence pair defines both a left and a right congruence the inverse kernel of a left congruence ρ must be saturated by both ν_L and ν_R, the minimum left and right congruence with trace equal to trace(ρ). In particular, if τ is a congruence on $E(B)$ then $N(\tau)$ is saturated by both ν_L and ν_R. The normaliser of τ can be thought of as elements on lines that go "corner to corner" in the grid created by adding both horizontal and vertical dashed lines corresponding to the trace, in the sense that they never cross a dashed line in any point which isn't a corner. Fig. 3.5 shows the normaliser for an eventually periodic trace, τ is represented by the black dashed lines (in this example the line representing $l(\tau)$ does not represent a τ relation) and elements that lie on the bolder diagonal lines are in the normaliser.

We use Corollary 3.3.13 and apply Lemma 3.3.14 for each type of congruence on $E(B)$ to determine inverse congruence pairs for the bicyclic monoid.

Corollary 3.3.15. *Let τ be a congruence on $E(B)$ with the infinite congruence class $\{(x, x) \mid x \geq n\}$ and let $T \in \mathfrak{V}(B)$. Then (τ, T) is an inverse congruence pair if and only if only if $T = E(B)$ or $T = T_{n,d}$ for some d.*

Proof. By Corollary 3.3.13, since τ has an infinite class, $N(\tau) = T_{n,1}$. Suppose $(\tau, T_{k,d})$ is an inverse congruence pair. As $T_{k,d} \subseteq T_{n,1}$ we have $k \geq n$ so $(k, k)\ \tau\ (n, n)$. Also, $(k + d, k) \in T_{k,d}$ and $(k + d) - k = d = (n + d) - n$. As $(\tau, T_{k,d})$ is an inverse congruence pair it satisfies (ICP2) so we may apply Lemma 3.3.14 to obtain that $(n+d, n) \in T_{k,d}$. Thus $n \geq k$ so we have $k = n$.

The converse is straightforward, applying Corollary 3.3.13 we have that $T_{n,d} \subseteq N(\tau)$ and it is clear that $T_{n,d}$ is saturated by ν_τ, so by Lemma 3.3.14 we have that (ICP2) holds. Thus $(\tau, T_{n,d})$ is an inverse congruence pair for all $d \geq 1$. Noting that $(\tau, E(B))$ is also an inverse congruence pair completes the proof. □

Corollary 3.3.16. *Let τ be a congruence on $E(B)$ which has no infinite class and is not eventually periodic. Then the only inverse congruence pair containing τ is $(\tau, E(B))$.*

Proof. By Corollary 3.3.13, $N(\tau) = E(B)$, the claim follows immediately. □

Corollary 3.3.17. *Let τ be a congruence on $E(B)$ which is eventually periodic - say τ is d-periodic after k. Let $\Xi(\tau) = \{c_1, c_2, \dots\}$ (the integers minimal in each τ-class) and let r be such that $c_r = k$. Then (τ, T) is an inverse congruence pair if and only if $T = E(B)$ or $T = T_{j,b}$ where $d \mid b$ and either $j = l(\tau)$ or there is $v \geq r$ such that $j = c_v$.*

Proof. We first note that $(\tau, E(B))$ is an inverse congruence pair for B. Therefore we must prove that $(\tau, T_{j,b})$ is an inverse congruence pair if and only if $d \mid b$ and either $j = l(\tau)$ or there is $v \geq r$ such that $j = c_v$.

We suppose that $(\tau, T_{j,b})$ is an inverse congruence pair. By Corollary 3.3.13, $N(\tau) = T_{l(\tau),d}$, so we have that $T_{j,b} \subseteq T_{l(\tau),d}$. Therefore we have that $d \mid b$ and $j \geq l(\tau)$. Let u be such that $c_u \leq j < c_{u+1}$, so $(c_u, c_u)\ \tau\ (j, j)$. If $(c_u+b, c_u) \in N(\tau)$, then, as $(j+b, j) \in T_{j,b}$ and $(\tau, T_{j,b})$ satisfies (ICP2) we may apply Lemma 3.3.14 to obtain that also $(c_u+b, c_u) \in T_{j,b}$. This implies $j = c_u$ with $u \geq r$ (as $j \geq l(\tau)$). If $(c_u + b, c_u) \notin N(\tau)$ then $c_u < l(\tau) \leq j < c_{u+1}$. As $(j + b, j) \in T_{j,b}$, $(l(\tau) + b, l(\tau)) \in N(\tau)$ and $(\tau, T_{j,b})$ satisfies (ICP2) we again apply Lemma 3.3.14 to obtain that $(l(\tau) + b, l(\tau)) \in T_{j,b}$, so in particular $l(\tau) \geq j$. It follows that $j = l(\tau)$. Thus we have that j, b are as claimed.

For the converse we suppose that $d \mid b$ and either $j = l(\tau)$ or there is $v \geq r$ such that $j = c_v$. By Corollary 3.3.13 we have $T_{j,b} \subseteq N(\tau)$. We shall show that $T_{j,b}$ is saturated by $\nu_\tau|_{N(\tau)}$ whence, by Lemma 3.3.14, $(\tau, T_{j,b})$ satisfies (ICP2) so is an inverse congruence pair. Suppose $(x+mb, x)\ \nu_\tau\ (p, q)$ with $x \geq j$ and $m \geq 0$ (so $(x + mb, x) \in T_{j,b}$) and $(p, q) \in N(\tau)$. This says that $(x, x)\ \tau\ (q, q)$ and $mb = p-q$ so $(p, q) = (q+mb, q)$. Since $(x, x)\ \tau\ (q, q)$ there is $u \geq r$ such $c_u \leq x, q < c_{u+1}$. If $j = c_v$ then, as $(x + mb, x) \in T_{j,b}$ we have $u \geq v$, so $j \leq q$ and thus $(q + mb, q) = (p, q) \in T_{j,b}$. If $j = l(\tau)$ and $u > r$ then $j < c_u \leq q$ so $(p, q) \in T_{j,b}$. Finally, if $j = l(\tau)$ and $u = r$ then, as $(p, q) \in N(\tau)$ we have $q \geq l(\tau)$ so $(p, q) \in T_{j,b}$. The argument showing that if $(x, x + mb)\ \nu_\tau\ (p, q)$ then $(p, q) \in T_{j,b}$ is very similar. This completes the proof. $\square$

We may now summarise our description of inverse congruence pairs for the bicyclic monoid in the following theorem.

Theorem 3.3.18. *Let τ be a congruence on $E(B)$ with $\Xi(\tau) = \{c_1, c_2, \dots\}$. Then (τ, T) is an inverse congruence pair for B if and only if at least one of the following holds:*

(i) $T = E(B)$;

(ii) τ *has an infinite congruence class,* $\{(x,x) \mid x \geq n\}$*, and there is* $c \geq 1$ *with* $T = T_{n,c}$*;*

(iii) τ *is* d*-periodic after* k *and* $T = T_{j,c}$ *with* $d \mid c$ *and either* $j = l(\tau)$ *or* $j = c_u$ *for some* u *with* $c_u \geq k$*.*

We recall that $T_{m,c} \subseteq T_{l,d}$ if $l \leq m$ and $d \mid c$. Therefore when τ has an infinite congruence class the trace class of τ is isomorphic to $\mathbf{D}^0$. For the trace class of an eventually periodic trace we have the following.

Corollary 3.3.19. *Let* $\tau \in \mathfrak{C}(E(B))$ *be eventually periodic. Then the trace class* $\{\rho \in \mathfrak{LC}(B) \mid \mathrm{trace}(\rho) = \tau\}$ *is isomorphic to* $\mathfrak{V}(B)$*.*

If τ *is* d*-periodic after* k*,* $\Xi(\tau) = \{c_1, c_2, \dots\}$ *and* $c_r = k$ *then the map*

$$\Theta \colon \mathfrak{V}(B) \to \{\rho \in \mathfrak{LC}(B) \mid \mathrm{trace}(\rho) = \tau\}$$

defined by $E(B) \mapsto \rho(\tau, E(B))$*,* $T_{0,b} \mapsto \rho(\tau, T_{l(\tau),bd})$ *and* $T_{k,b} \mapsto \rho(\tau, T_{c_{r-1+k},bd})$ *for* $k \geq 1$ *is an isomorphism.*

Proof. We note that the ordering on the trace class $\{\rho \in \mathfrak{LC}(B) \mid \mathrm{trace}(\rho) = \tau\}$ is given by the ordering on the inverse kernels. It is elementary that Θ is a lattice homomorphism. Indeed, it follows from the ordering of the full inverse subsemigroups of B that in $\mathfrak{V}(B)$

$$T_{j,b} \vee T_{l,c} = T_{\min\{l,k\},\gcd\{b,c\}} \quad \text{and} \quad T_{j,b} \cap T_{l,c} = T_{\max\{j,l\},\mathrm{lcm}\{b,c\}}.$$

By Theorem 3.3.18 we know that every left congruence with trace τ has inverse kernel $E(B)$ or $T_{j,bd}$ where $j = l(\tau)$ or $j = c_v$ for $v \geq r$ and $b \geq 1$. Using that the ordering of left congruences in a trace class coincides with the ordering on the inverse kernels we see that in $\mathfrak{LC}(B)$

$$\rho(\tau, T_{c_v,bd}) \vee \ \rho(\tau, T_{c_u,ad}) = \rho(\tau, T_{c_{\min\{v,u\}},(\gcd\{b,a\})d})$$

and

$$\rho(\tau, T_{c_v,bd}) \cap \ \rho(\tau, T_{c_u,ad}) = \rho(\tau, T_{c_{\max\{u,v\}},(\mathrm{lcm}\{b,a\})d}).$$

In these expressions we use c_0 to denote $l(\tau)$. That Θ is a lattice homomorphism is now clear.

That Θ is injective is immediate and we notice that Theorem 3.3.18 implies that Θ is surjective. Thus Θ is a lattice isomorphism. This proves the second claim, the first claim follows immediately. $\square$

When considering the lattice of left congruences on B it is interesting to observe that the lattice of full inverse subsemigroups - isomorphic to $(\mathbf{C}\times\mathbf{D})^0$ - is countable, whereas the lattice of congruences on $E(B)$ - isomorphic to $\mathbb{P}(\mathbb{N})$ (the powerset of $\mathbb{N}$) - is uncountable. It follows that the lattice of left congruences is uncountable; indeed for each congruence on $E(B)$ there is a left congruence on B with this a trace. Furthermore, the set of congruences on $E(B)$ with an infinite congruence class corresponds to finite subsets of $\mathbb{N}$ under our description of congruences by the minimum elements of each congruence class. It is also easy to see that the set of eventually periodic traces is countable. Indeed, let $\Xi(\tau) = \{c_1, c_2, \dots\}$ then if τ is eventually periodic it is determined by $(d, \{c_1, \dots, c_{r+p}\})$ where d is the period. This claim follows from the fact that $c_{r+p+i} = c_{r+i} + d$ for all $i \geq 0$. Thus the map $\phi\colon \tau \mapsto (d, \{c_1, \dots, c_{r+p}\})$ is injective. As the set of finite subsets of $\mathbb{N}$ is countable it follows that the image of ϕ is countable. Thus the set of eventually periodic congruences on $E(B)$ is countable. Furthermore, each trace class is countable, as it is in bijection with a subset of the set of full inverse subsemigroups, which is countable. Therefore, there are countably many left congruences on B with an eventually periodic trace or a trace that has an infinite class.

Since $\mathfrak{C}(E(B))$ is uncountable we see that the set of left congruences on B with a trace which has no infinite class and is not eventually periodic is uncountable. Such left congruences have inverse kernel $E(B)$, so are totally determined by their trace. Thus "most" of the structure of the lattice of left congruences on B comes from the lattice of congruences on $E(B)$.

3.4 BRANDT SEMIGROUPS

We recall that Brandt semigroups are completely 0-simple inverse semigroups and we describe them with the structure $B = \mathcal{B}(I, G)$ where

$$\mathcal{B}(I, G) = (I \times G \times I) \cup \{0\}$$

with multiplication

$$(i, g, j)(k, h, l) = \begin{cases} (i, gh, l) & \text{if } \ j = k \\ 0 & \text{otherwise.} \end{cases}$$

and $0(i, g, j) = 0 = (i, g, j)0 = 0^2$.

One sided congruences on Brandt semigroups have a particularly nice structure and are classified by Petrich and Rankin in [60] using the kernel trace approach to one sided congruences on inverse semigroups. It is shown that right congruences (and so also left congruences) are in bijective correspondence with the set of inverse subsemigroups that contain the zero. In fact, the left congruences on a Brandt semigroup B are precisely the principal left congruences P_{T^*} where $T \subseteq S$ is an inverse subsemigroup containing 0 and $T^* = T\backslash\{0\}$. We recall that the principal left congruence P_X on S for a subset $X \subseteq S$ is defined as $a \; P_X \; b$ if and only if for all $u \in S$, $au \in X \iff bu \in X$. Here we reproduce this result via the inverse kernel approach, and demonstrate that by imposing an arbitrary structure (by which we mean imaginary and irrelevant to the structure of the semigroup) to I we may give a slicker description (in my opinion) of the lattice of left congruences.

Throughout this section we take B to be the Brandt semigroup $\mathcal{B}(I, G)$. In this section we swap to the notation $i\kappa$, from $[i]_\kappa$, for the κ-class containing i. We do this to avoid having too many subscripts.

We now present the inverse kernel approach to left congruences. First we analyse full inverse subsemigroups and then we describe inverse congruence pairs.

Full inverse subsemigroups of Brandt semigroups

The following definition is from [60], and is used to describe the set of inverse subsemigroups of B.

Definition 3.4.1 ([60, Definition 4.1])**.** Let κ be an equivalence relation on $I \cup \{0\}$ and let $\Gamma\colon i \mapsto \Gamma_i$ be a mapping from I into the set of all left cosets of subgroups of G. We say that (κ, Γ) is an *inverse subsemigroup pair* for B if $i\kappa = j\kappa$ implies that $\Gamma_i^{-1}\Gamma_i = \Gamma_j^{-1}\Gamma_j$.

For an inverse subsemigroup pair (κ, Γ) define

$$T_{\kappa,\Gamma} = \{0\} \cup \bigcup_{i\kappa = j\kappa \neq 0\kappa} \{i\} \times \Gamma_i\Gamma_j^{-1} \times \{j\}.$$

Proposition 3.4.2 ([60, Proposition 4.2])**.** *Let (κ, Γ) be an inverse subsemigroup pair for B. Then $T = T_{\kappa,\Gamma}$ is an inverse subsemigroup of B containing 0. Conversely, every inverse subsemigroup of B that contains 0 is of the form $T_{\kappa,\Gamma}$ for some inverse subsemigroup pair (κ, Γ).*

Moreover, $T_{\kappa,\Gamma} = T_{\kappa',\Gamma'}$ if and only if $\kappa = \kappa'$ and for any $i, j \in I \backslash 0\kappa$ we have $\Gamma_i\Gamma_j^{-1} = \Gamma_i'\Gamma_j'^{-1}$

The proof of Proposition 3.4.2 is elementary and there is little value in repeating it here. We are primarily interested in full inverse subsemigroups, so we specialise Proposition 3.4.2. We notice first that each full inverse subsemigroup of B contains 0 so Proposition 3.4.2 applies. Furthermore, the idempotents of B that are contained in $T_{\kappa,\Gamma}$ are exactly the elements $\{(i, 1, i) \mid i \notin 0\kappa\}$. This implies the following result.

Corollary 3.4.3. *Let (κ, Γ) be an inverse subsemigroup pair for B. Then $T_{\kappa,\Gamma}$ is full if and only $0\kappa = \{0\}$.*

While we do not include a proof of Proposition 3.4.2 it is worth explicitly giving the relationship between the inverse subsemigroup and the inverse subsemigroup pair. In particular, it is worth emphasising the connection between the full inverse subsemigroup $T = T_{\kappa,\Gamma}$ and the relation κ. For $i, j \in I$, we observe that $i \; \kappa \; j$ precisely when there is some $g \in G$ with $(i, g, j) \in T$, equivalently this says that $(i, 1, i) \; \mathcal{D}(T) \; (j, 1, j)$.

The mapping Γ is somewhat harder to examine, largely because it is not uniquely determined by the inverse subsemigroup. The additional structure which we later impose on I resolves this uniqueness issue. Here we explain why Γ is not unique. For each $i \in I$ say that $\Gamma_i = x_i H_i$ for a subgroup $H_i \leq G$ and $x_i \in G$. Choose $y \in G$ and for each $i \in I$ let $\Gamma'_i = x_i H_i y = x_i y(y^{-1} H_i y)$. Then, as $y^{-1} H_i y$ is a subgroup of G, we see that Γ'_i is a mapping from I to the set of left cosets of subgroups of G. Furthermore,

$$\Gamma_i \Gamma_j^{-1} = x_i H_i H_j x_j^{-1} = x_i H_i y y^{-1} H_j x_j^{-1} = \Gamma'_i {\Gamma'_j}^{-1}.$$

Therefore, by Proposition 3.4.2, we have that $T_{\kappa,\Gamma} = T_{\kappa,\Gamma'}$, however, unless $y \in H_i$ for every i, $\Gamma \neq \Gamma'$ thus we see that the mapping Γ is not necessarily uniquely determined by the subsemigroup.

Inverse congruence pairs for Brandt semigroups

We recall that the idempotents of the Brandt semigroup $B = \mathcal{B}(I, G)$ are the elements $\{(i, 1, i) \mid i \in I\}$, so we may identify the idempotent semilattice $E(B)$ with the null semilattice $I \cup \{0\}$. With this identification, for a congruence on $E(B)$ we write $i\ \tau\ j$ to mean $(i, 1, i)\ \tau\ (j, 1, j)$. It is clear that congruences on $E(B)$ are partitions of $I \cup \{0\}$ such that if there is a non-trivial part then it is the part containing 0.

Corollary 3.4.4. *Congruences on $E(B)$ are the Rees congruences. Consequently, $\mathfrak{C}(E(B))$ is isomorphic to $\mathcal{P}_I$, the powerset of I with lattice operations intersection and union. Moreover, the functions*

$$\tau \mapsto 0\tau \backslash \{0\}; \qquad J \mapsto \tau_J = \iota \cup \{(i, j) \mid i, j \in J \cup \{0\}\}$$

are mutually inverse lattice isomorphisms.

Proof. It is straightforward that all the congruences on $E(B)$ are Rees congruences and that the ideals of $E(B)$ are exactly the sets $J \cup \{0\}$ for $J \subseteq I$. As in the statement of the claim we write τ_J for the congruence $\iota \cup \{(i, j) \mid i, j \in J \cup \{0\}\}$. Then the function $\tau \mapsto 0\tau \backslash \{0\}$ may be written as $\tau_J \mapsto J$. That this is a lattice isomorphism is immediate from the observation that for any Rees congruences ρ_A and ρ_B on any semigroup (where ρ_A is the Rees congruence corresponding to the ideal A),

$$\rho_A \cap \rho_B = \rho_{A \cap B} \quad \text{and} \quad \rho_A \vee \rho_B = \rho_{A \cup B}.$$

□

We recall that computing inverse congruence pairs requires us to know the normaliser of a trace, thus this is our first step.

Lemma 3.4.5. *Let τ be a congruence on $E(B)$. Then*

$$N(\tau) = \{0\} \cup \bigcup_{i,j \in 0\tau \backslash \{0\}} \{(i,g,j) \mid g \in G\} \cup \bigcup_{i,j \notin 0\tau} \{(i,g,j) \mid g \in G\}.$$

In terms of inverse subsemigroup pairs, $N(\tau) = T_{\kappa,\Gamma}$ where κ has three parts: $\{0\}$, $0\tau\backslash\{0\}$ and $I\backslash 0\tau$, and $\Gamma_i = G$ for all i.

Proof. Let M be the set on the RHS of the above claim. First we observe that

$$(i,g,j)(k,1,k)(j,g^{-1},i) = \begin{cases} 0 & \text{if } j \neq k \\ (i,1,i) & \text{if } j = k. \end{cases}$$

Also, by the nature of congruences on $E(B)$, if $k \;\tau\; l$ then either $k = l$ or $k \;\tau\; 0$.

Suppose that $(i,g,j) \in M$ and that $k \;\tau\; l$. If $k = l$ then we have nothing to prove so suppose $k \neq l$ (so $k \;\tau\; 0$). If $i,j \in 0\tau$ then certainly

$$(i,g,j)(k,1,k)(j,g^{-1},i) \;\tau\; (i,g,j)(l,1,l)(j,g^{-1},i)$$

as the two idempotents are elements of the set $\{0,(i,1,i)\} \subseteq 0\tau$. If $i,j \notin 0\tau$ then conjugating any idempotent in 0τ gives 0. Thus certainly $M \subseteq N(\tau)$.

For the reverse inclusion suppose that $(i,g,j) \in N(\tau)\backslash M$. We suppose (without loss of generality) that $j \;\tau\; 0$ and $i \not\!\tau\; 0$. We then observe that

$$(i,1,i) = (i,g,j)(j,1,j)(j,g^{-1},i) \;\tau\; (i,g,j)0(j,g^{-1},i) = 0.$$

This is a contradiction, so $(i,g,j) \notin N(\tau)$, and we have that $N(\tau) = M$. □

To compute (ICP2) for Brandt semigroups we appeal to Proposition 2.3.4 which says that, if $\tau \in \mathfrak{C}(E)$, subsemigroups that satisfy (ICP2) are precisely those saturated by $\nu_\tau|_{N(\tau)}$. With this in mind we describe ν_τ.

Lemma 3.4.6. *Let τ be a congruence on $E(B)$. Then*

$$(i,g,j)\ \nu_\tau\ (k,h,l) \iff j,l \in 0\tau \quad or \quad i=k,\ j=l,\ g=h$$

and $0\ \nu_\tau\ (i,g,j)$ *when* $j \in 0\tau$.

Proof. This is a straightforward application of the description of the minimum left congruence with trace τ, which we recall from Theorem 1.4.11 is

$$\nu_\tau = \{(a,b) \in S \times S \mid\ \exists e \in E(S),\ a^{-1}a\ \tau\ b^{-1}b\ \tau\ e,\ ae = be\}.$$

Using that $j\ \tau\ l$ exactly when $j = l$ or $j,l \in 0\tau$ gives the result. □

Lemma 3.4.6 has the following immediate corollary.

Corollary 3.4.7. *Let τ be a congruence on $E(B)$ and let $T \subseteq N(\tau)$ be a full inverse subsemigroup of B. Then T is saturated by $\nu_\tau|_{N(\tau)}$ if and only if*

$$\bigcup_{i,j\in 0\tau\setminus\{0\}} \{(i,g,j) \mid g \in G\} \subseteq T.$$

Proof. Suppose T is saturated by $\nu_\tau|_{N(\tau)}$. As T is full, $0 \in T$ so, by Lemma 3.4.6, for all $i,j \in 0\tau\setminus\{0\}$ and for all $g \in G$, we have $(i,g,j)\ \nu_\tau\ 0$, thus $(i,g,j) \in T$. Hence T contains the set claimed.

For the converse suppose that $\bigcup_{i,j\in 0\tau\setminus\{0\}}\{(i,g,j) \mid g \in G\} \subseteq T$. From Lemma 3.4.5 we have that

$$N(\tau) = \{0\} \cup \bigcup_{i,j\in 0\tau\setminus\{0\}} \{(i,g,j) \mid g \in G\} \cup \bigcup_{i,j\notin 0\tau} \{(i,g,j) \mid g \in G\}.$$

Further, by applying Lemma 3.4.6 we obtain that the only non trivial $\nu_\tau|_{N(\tau)}$ class is $\bigcup_{i,j\in 0\tau\setminus\{0\}}\{(i,g,j) \mid g \in G\} \cup \{0\}$ and all other classes are singletons. Thus as $\bigcup_{i,j\in 0\tau\setminus\{0\}}\{(i,g,j) \mid g \in G\} \subseteq T$ we certainly have that T is saturated by $\nu_\tau|_{N(\tau)}$. □

If (κ, Γ) is an inverse subsemigroup pair with $0\kappa = \{0\}$ (in other words $T_{\kappa,\Gamma}$ is full) and τ is a congruence on $E(B)$, then the condition $\bigcup_{i,j\in 0\tau\setminus\{0\}}\{(i,g,j) \mid g \in G\} \subseteq T_{\kappa,\Gamma}$ precisely says that

$$\{(i,j) \mid i,j \in 0\tau\setminus\{0\}\} \subseteq \kappa$$

and that $\Gamma_i = G$ for each $i \in 0\tau\setminus\{0\}$. Also, that $T_{\kappa,\Gamma} \subseteq N(\tau)$ says that

$$\kappa \subseteq \{(i,j) \mid i,j \in 0\tau\setminus\{0\}\} \cup \{(i,j) \mid i,j \notin 0\tau\} \cup \{(0,0)\}.$$

Combining these conditions we see that $0\tau\setminus\{0\}$ is an equivalence class of κ.

From the reverse direction it follows from Lemma 3.4.5 that if $0\tau\setminus\{0\}$ is an equivalence class of κ then $T_{\kappa,\Gamma} \subseteq N(\tau)$. If in addition $\Gamma_i = G$ for $i \in 0\tau\setminus\{0\}$ then $\bigcup_{i,j\in 0\tau\setminus\{0\}}\{(i,g,j) \mid g \in G\} \subseteq T_{\kappa,\Gamma}$, then, by Lemma 3.4.6, $T_{\kappa,\Gamma}$ is saturated by $\nu_\tau|_{N(\tau)}$. We have now computed the inverse congruence pairs for B, which we summarise in the following result.

Theorem 3.4.8. *Let $B = \mathcal{B}(I,G)$ be a Brandt semigroup, let τ be a congruence on $E(B)$ and let $T_{\kappa,\Gamma}$ be a full inverse subsemigroup (so $0\kappa = \{0\}$). Then $(\tau, T_{\kappa,\Gamma})$ is an inverse congruence pair for B if and only if $0\tau\setminus\{0\}$ is an equivalence class of κ and $\Gamma_i = G$ for $i \in 0\tau\setminus\{0\}$.*

In [60] it is shown that a left congruence may be totally determined by an inverse subsemigroup. We now illustrate how this result follows from the inverse kernel approach. Some details are left to the reader. Let $(\tau, T_{\kappa,\Gamma})$ be an inverse congruence pair. We make the observation that there is overlap in the information that κ and τ provide, and there is an equivalence relation on $I \cup \{0\}$ from which we can recover τ and κ. We consider the relation $\kappa \cup \tau$. We know that $0\tau\setminus\{0\}$ is a κ-class, and all τ-classes apart from 0τ are singletons. Therefore

$$\kappa \cup \tau = \kappa \cup \{(i,0) \mid i \in 0\tau\setminus\{0\}\} \cup \{(0,i) \mid i \in 0\tau\setminus\{0\}\}.$$

It is straightforward that $\kappa \cup \tau$ is an equivalence relation and we observe that

$$\kappa = \Big(\kappa \cup \tau\Big) \cap \Big((I \times I) \cup \{(0,0)\}\Big)$$

and, noting that $0(\kappa \cup \tau) = 0\tau$,

$$\tau = \iota \cup \Big(0(\kappa \cup \tau) \times 0(\kappa \cup \tau)\Big).$$

It is easy to check that $(\kappa \cup \tau, \Gamma)$ is an inverse subsemigroup pair, and that

$$T_{\kappa\cup\tau,\Gamma} = T_{\kappa,\Gamma} \cap \{(i,g,j) \mid g \in G,\ i,j \notin 0\tau\}.$$

We have shown that can recover the inverse congruence pair $(\tau, T_{\kappa,\Gamma})$ from the inverse subsemigroup pair $(\kappa \cup \tau, \Gamma)$. It follows that we may define a left congruence from an inverse subsemigroup pair. The result from [60] follows.

Corollary 3.4.9 (cf [60, Theorem 3.2])**.** *Let $T = T_{\kappa,\Gamma}$ be an inverse subsemigroup of $B = \mathcal{B}(I,G)$ containing 0, and let $K_T = T \cup \{(i,g,j) \mid g \in G,\ i,j \in 0\kappa\backslash\{0\}\}$. Let $\tau_T = \iota \cup (0\kappa \times 0\kappa)$. Then (τ_T, K_T) is an inverse congruence pair for B. Moreover, every inverse congruence pair is of this form. Consequently, left congruences on B are in bijection with inverse subsemigroups of B which contain 0.*

The lattice of left congruences

As explained the most significant issue with the approach presented thus far is that an inverse subsemigroup does not specify a unique inverse subsemigroup pair, in fact arbitrarily many inverse subsemigroup pairs may correspond to the same inverse subsemigroup. The reason that this ambiguity is a problem for us is that it makes it more complicated to give the ordering on inverse subsemigroup pairs that corresponds to the ordering on inverse subsemigroups, and we are interested in the ordering so that we may describe the lattice of full inverse subsemigroups and then the lattice of left congruences. We would like to able to say "$T_{\kappa',\Gamma'} \subseteq T_{\kappa,\Gamma}$ if and only $\kappa' \leq \kappa$ and $\Gamma'_i \subseteq \Gamma_i$ for each $i \in I$". This would imply a unique correspondence between inverse subsemigroups and inverse subsemigroup pairs, which we do not have. The best we can currently say for the ordering on inverse subsemigroup pairs is that is implied by Proposition 3.4.2, which is: "$T' \subseteq T$

if and only $\kappa' \leq \kappa$ and $\Gamma'_i {\Gamma'_j}^{-1} \subseteq \Gamma_i \Gamma_j^{-1}$ for each $i, j \in I$ with $i \;\kappa\; j$." This is not significantly easier to check than directly verifying whether $T_{\kappa',\Gamma'} \subseteq T_{\kappa,\Gamma}$.

In turn this difficulty describing the ordering on full inverse subsemigroups makes it difficult to describe the ordering in the lattice of left congruences, and even more difficult to describe the meet and join of left congruences. In [61] this difficulty is tackled via the kernel trace approach, computing the kernel of the join of pairs of congruences. However, this approach does not connect well with the description of inverse subsemigroups by inverse subsemigroup pairs. We improve upon this aspect in this section.

We start by describing how we specify a unique inverse subsemigroup pair to define a subsemigroup. Let (κ, Γ) be an inverse subsemigroup pair. Our solution is to fix an element of each κ-class - by which we mean having some deterministic algorithm that outputs an element of I given an input of a subset of I - and insist that each identified element i has Γ_i a subgroup. As temporary notation we shall write $\bar{i}$ to be the identified element of $i\kappa$. Since $i \;\kappa\; \bar{i}$ and (κ, Γ) is an inverse subsemigroup pair, $\Gamma_i^{-1}\Gamma_i = \Gamma_{\bar{i}}^{-1}\Gamma_{\bar{i}}$, which says that Γ_i and $\Gamma_{\bar{i}}$ are left cosets of the same group. As we insist that $\Gamma_{\bar{i}}$ is a subgroup it follows that Γ_i must be a left coset of $\Gamma_{\bar{i}}$ and consequently $\Gamma_i \Gamma_{\bar{i}} = \Gamma_i$. When $i \;\kappa\; j$ we know that $\Gamma_i \Gamma_j^{-1} = \{g \in G \mid (i, g, j) \in T_{\kappa,\Gamma}\}$, so

$$\Gamma_i = \Gamma_i \Gamma_{\bar{i}} = \Gamma_i \Gamma_{\bar{i}}^{-1} = \{g \in G \mid (i, g, \bar{i}) \in T_{\kappa,\Gamma}\}.$$

We shall see that this provides a way to give a unique inverse subsemigroup pair.

We comment on our identified element $\bar{i}$. As we know any set may be well ordered so we assume from this point that I is well ordered and for $J \subseteq I$ we write "$\min J$" for the minimum element of J. We shall use the minimum element in a κ-class as our identified element. We reinforce the message that this is not the natural partial order on the idempotent semilattice, it is an arbitrary well order, which is unrelated to the multiplicative structure of the semigroup.

Definition 3.4.10. Let (κ, Γ) be an inverse subsemigroup pair. Then (κ, Γ) is *special* if for each $i \in I$ with $i = \min i\kappa$ the coset Γ_i is a subgroup.

We remark that when (κ, Γ) is a special inverse subsemigroup pair Γ_i is a left coset of $\Gamma_{\min i\kappa}$ for each $i \in I$.

Proposition 3.4.11. *Let $T \subseteq B$ be a full inverse subsemigroup. Then there is a unique special inverse subsemigroup pair (κ, Γ) such that $T = T_{\kappa,\Gamma}$.*

Proof. Let κ be $\{(i,j) \mid \exists (i,g,j) \in T\} \cup \{(0,0)\}$ and define

$$\Gamma_i = \{g \in G \mid (i, g, \min i\kappa) \in T\}.$$

We claim that this is a special inverse subsemigroup pair. We observe that κ is certainly an equivalence relation as T is a full inverse subsemigroup. We show that $\Gamma_{\min i\kappa}$ is a subgroup of G and that $\Gamma_i^{-1}\Gamma_i = \Gamma_{\min i\kappa}$ for each i. Indeed, if $i \in I$ and $g, h \in \Gamma_i$ then $(i, g, \min i\kappa)$ and $(i, h, \min i\kappa)$ are elements of T. Then

$$\begin{aligned}(i, g, \min i\kappa)^{-1}(i, h, \min i\kappa) &= (\min i\kappa, g^{-1}, i)(i, h, \min i\kappa)\\ &= (\min i\kappa, g^{-1}h, \min i\kappa) \in T,\end{aligned}$$

so $g^{-1}h \in \Gamma_{\min i\kappa}$. It follows that if $i = \min i\kappa$ then Γ_i is a subgroup, and also for any i that $\Gamma_i^{-1}\Gamma_i = \Gamma_{\min i\kappa}$. Thus we have that (κ, Γ) is a special inverse subsemigroup pair.

We recall that $T_{\kappa,\Gamma}$ is defined as

$$T_{\kappa,\Gamma} = \{0\} \cup \{(i,g,j) \mid g \in \Gamma_i\Gamma_j^{-1}\}.$$

If $(i,g,j) \in T_{\kappa,\Gamma}$ then $g = hk^{-1}$ for $h \in \Gamma_i$ and $k \in \Gamma_j$. It follows that $(i, h, \min i\kappa)$ and $(\min j\kappa, k^{-1}, j)$ are elements of T, whence $(i,g,j) \in T$. Therefore $T_{\kappa,\Gamma} \subseteq T$. For the reverse inclusion, suppose $(i,g,j) \in T$, so in particular $i \;\kappa\; j$, and choose $h \in \Gamma_j$ so that $(j, h, \min j\kappa) \in T$. Then

$$(i,g,j)(j,h,\min j\kappa) = (i, gh, \min j\kappa) = (i, gh, \min i\kappa) \in T.$$

We then have that $gh \in \Gamma_i$, so $g \in \Gamma_i\Gamma_j^{-1}$, whence $(i,g,j) \in T_{\kappa,\Gamma}$. Thus we have shown that $T = T_{\kappa,\Gamma}$.

It remains to show that (κ, Γ) is the unique special inverse subsemigroup pair for $T = T_{\kappa,\Gamma}$. Suppose that (δ, Δ) is also a special inverse subsemigroup pair such that $T_{\delta,\Delta} = T_{\kappa,\Gamma}$. Then, as both (κ, Γ) and (δ, Δ) are inverse subsemigroup pairs, by Proposition 3.4.2 we have that $\kappa = \delta$. From the description of $T_{\kappa,\Delta}$ and $T_{\kappa,\Gamma}$ we have that, since $\Delta_{\min i\kappa}$ and $\Gamma_{\min i\kappa}$ are both subgroups,

$$\begin{aligned}\Delta_{\min i\kappa} = \Delta_{\min i\kappa}\Delta^{-1}_{\min i\kappa} &= \{g \in G \mid (\min i\kappa, g, \min i\kappa) \in T\}\\ &= \Gamma_{\min i\kappa}\Gamma^{-1}_{\min i\kappa} = \Gamma_{\min i\kappa}.\end{aligned}$$

Furthermore, applying Proposition 3.4.2 we have that

$$\Delta_i\Delta^{-1}_{\min i\kappa} = \{g \in G \mid (i, g, \min i\kappa) \in T\} = \Gamma_i\Gamma^{-1}_{\min i\kappa}.$$

Since both (κ, Γ) and (δ, Δ) are special inverse subsemigroup pairs we know that Δ_i and Γ_i are left cosets of $\Delta_{\min i\kappa} = \Gamma_{\min i\kappa}$. Thus

$$\Delta_i = \Delta_i\Delta^{-1}_{\min i\kappa} \quad \text{and} \quad \Gamma_i\Gamma^{-1}_{\min i\kappa} = \Gamma_i.$$

Therefore $\Delta_i = \Gamma_i$. We now have that $\kappa = \delta$ and $\Gamma = \Delta$, whence (κ, Γ) is the unique special inverse subsemigroup pair for which $T_{\kappa,\Gamma} = T$. □

In particular, Proposition 3.4.11 implies that if $T \subseteq \mathcal{B}(I, G)$ is a full inverse subsemigroup then the special inverse subsemigroup pair for T is (κ, Γ) where $\kappa = \{(i, j) \mid \exists (i, g, j) \in T\} \cup \{(0, 0)\}$ and

$$\Gamma_i = \{g \in G \mid (i, g, \min i\kappa) \in T\}.$$

On special inverse subsemigroup pairs we can now describe the ordering induced by the inclusion of subgroups.

Theorem 3.4.12. *Let (κ, Γ) and (δ, Δ) be special inverse subsemigroup pairs. Then $T_{\kappa,\Gamma} \subseteq T_{\delta,\Delta}$ if and only if $\kappa \subseteq \delta$ and for each $i \in I$*

$$\Gamma_i \subseteq \Delta_i\Delta^{-1}_{\min i\kappa}.$$

Proof. Initially we suppose that $\kappa \subseteq \delta$ and that $\Gamma_i \subseteq \Delta_i\Delta^{-1}_{\min i\kappa}$ for each $i \in I$. From the definition of $T_{\kappa,\Gamma}$, we know that $(i, g, j) \in T_{\kappa,\Gamma}$ precisely when $i \ \kappa \ j$ and $g \in \Gamma_i\Gamma_j^{-1}$. We suppose that $(i,g,j) \in T_{\kappa,\Gamma}$. Then, as $\kappa \subseteq \delta$, $i \ \delta \ j \ \delta \ \min i\kappa$. Also, as $\Gamma_i \subseteq \Delta_i\Delta_{\min i\kappa}^{-1}$,

$$\Gamma_i\Gamma_j^{-1} \subseteq (\Delta_i\Delta_{\min i\kappa}^{-1})(\Delta_{\min i\kappa}\Delta_j^{-1}).$$

As (δ, Δ) is an inverse subsemigroup pair and $i \ \delta \ j \ \delta \ \min i\kappa$, we have

$$\Delta_i^{-1}\Delta_i = \Delta_j^{-1}\Delta_j = \Delta_{\min i\kappa}^{-1}\Delta_{\min i\kappa}$$

and so

$$\Gamma_i\Gamma_j^{-1} \subseteq \Delta_i\Delta_{\min i\kappa}^{-1}\Delta_{\min i\kappa}\Delta_j^{-1} = \Delta_i\Delta_i^{-1}\Delta_i\Delta_j^{-1} = \Delta_i\Delta_j^{-1}.$$

Thus $(i,g,j) \in T_{\delta,\Delta}$, whence $T_{\kappa,\Gamma} \subseteq T_{\delta,\Delta}$.

For the converse we suppose that $T_{\kappa,\Gamma} \subseteq T_{\delta,\Delta}$. It is immediate that $\kappa \subseteq \delta$. Furthermore, since (κ, Γ) is a special inverse subsemigroup pair,

$$\Gamma_i = \Gamma_i\Gamma_{\min i\kappa} = \{g \in G \mid (i, g, \min i\kappa) \in T_{\kappa,\Gamma}\}.$$

On the other hand, noting that as $\kappa \subseteq \delta$ we have $i \ \delta \ \min i\kappa$, by the definition of $T_{\delta,\Delta}$,

$$\{g \in G \mid (i, g, \min i\kappa) \in T_{\delta,\Delta}\} = \Delta_i\Delta_{\min i\kappa}^{-1}.$$

Therefore, as $T_{\kappa,\Gamma} \subseteq T_{\delta,\Delta}$, we have that $\Gamma_i \subseteq \Delta_i\Delta_{\min i\kappa}^{-1}$. □

Theorem 3.4.12 is a significant step forward in describing the ordering of full inverse subsemigroups, before, to determine if $T_{\kappa,\Gamma} \subseteq T_{\delta,\Delta}$ we needed to check that $\kappa \subseteq \delta$ and then consider all pairs i, j such that $i \ \kappa \ j$ and check if $\Gamma_i\Gamma_j^{-1} \subseteq \Delta_i\Delta_j^{-1}$. Now we only have to perform one check for each i, which is far more computationally efficient. We can also use special inverse semigroup pairs to compute the intersection and join of full inverse subsemigroups.

Proposition 3.4.13. *Let $B = \mathcal{B}(I, G)$ and let (κ, Γ) and (κ', Γ') be special inverse subsemigroup pairs. Then the intersection and join of $T_{\kappa,\Gamma}$ and $T_{\kappa',\Gamma'}$ are as follows:*

(i)

$$T_{\kappa,\Gamma} \cap T_{\kappa',\Gamma'} = T_{\delta,\Delta}$$

where $0\delta = \{0\}$ *and, for* $i, j \in I$, $i\ \delta\ j$ *precisely when* $i\ \kappa\ j$, $i\ \kappa'\ j$ *and* $\Gamma_i\Gamma_j^{-1} \cap \Gamma'_i\Gamma'^{-1}_j \neq \emptyset$, *and*

$$\Delta_i = \Gamma_i\Gamma^{-1}_{\min i\delta} \cap \Gamma'_i\Gamma'^{-1}_{\min i\delta};$$

(ii)

$$T_{\kappa,\Gamma} \vee T_{\kappa',\Gamma'} = T_{\delta,\Delta}$$

where $\delta = \kappa \vee \kappa'$ *and, with* $j = \min j\delta$ *and* L_m *the set of* m*-tuples* $(i_1, i_2, \ldots, i_m)$ *such that*

$$j\ \kappa\ i_1\ \kappa'\ i_2\ \kappa\ i_3\ \kappa'\ \ldots\ \kappa'\ i_{m-1}\ \kappa\ i_m\ \kappa'\ j,$$

we have

$$\Delta_j = \bigcup_{m\geq 1}\ \bigcup_{(i_1\ldots,i_m)\in L_m} \Gamma^{-1}_{i_1}(\Gamma'_{i_1}\Gamma'^{\,-1}_{i_2})(\Gamma_{i_2}\Gamma^{-1}_{i_3})(\Gamma'_{i_3}\Gamma'^{\,-1}_{i_4})\ldots(\Gamma_{i_{m-1}}\Gamma^{-1}_{i_m})\Gamma'_{i_m},$$

and for $j \neq \min j\delta$ *choose some sequence* $i_1, \ldots, i_m$ *such that*

$$j\ \kappa\ i_1\ \kappa'\ i_2\ \kappa\ \ldots\ \kappa\ i_m\ \kappa'\ \min j\delta$$

then

$$\Delta_j = (\Gamma_j\Gamma^{-1}_{i_1})(\Gamma'_{i_1}\Gamma'^{\,-1}_{i_2})(\Gamma_{i_2}\Gamma^{-1}_{i_3})\ldots(\Gamma_{i_{m-1}}\Gamma^{-1}_{i_m})\Gamma'_{i_m}\Delta_{\min j\delta}.$$

Proof. The intersection is almost immediate. We construct (δ, Δ) the special inverse subsemigroup pair for $T_{\kappa,\Gamma} \cap T_{\kappa',\Gamma'}$. We observe that $(i, g, j) \in T_{\kappa,\Gamma} \cap T_{\kappa',\Gamma'}$ if and only if $i\ \kappa\ j$, $i\ \kappa'\ j$ and $g \in \Gamma_i\Gamma_j^{-1} \cap \Gamma'_i\Gamma'^{-1}_j$, so this is certainly non-empty. Therefore we have δ is as claimed. Furthermore,

$$\begin{aligned}
\Delta_i &= \{g \in G \mid (i, g, \min i\delta) \in T_{\kappa,\Gamma} \cap T_{\kappa',\Gamma'}\} \\
&= \{g \in G \mid (i, g, \min i\delta) \in T_{\kappa,\Gamma}\} \cap \{g \in G \mid (i, g, \min i\delta) \in T_{\kappa',\Gamma'}\} \\
&= \Gamma_i\Gamma^{-1}_{\min i\delta} \cap \Gamma'_i\Gamma'^{-1}_{\min i\delta}.
\end{aligned}$$

It follows that $\Delta_i = \Gamma_i\Gamma^{-1}_{\min i\delta} \cap \Gamma'_i\Gamma'_{\min i\delta}$ so Δ is as claimed.

The join is more complicated to prove, which is unsurprising looking at the statement of the claim. The crux of the proof is the observation that the join of $T_{\kappa,\Gamma}$ and $T_{\kappa',\Gamma'}$ as subsemigroups is equal to the set

$$\{s_1 s_2 s_3 \dots s_m \mid m \geq 1,\ s_1, s_3, \dots \in T_{\kappa,\Gamma},\ s_2, s_4, \dots \in T_{\kappa',\Gamma'}\},$$

the products of elements alternating between $T_{\kappa,\Gamma}$ and $T_{\kappa',\Gamma'}$. We note that we can assume the first element in in $T_{\kappa,\Gamma}$ and the last element is in $T_{\kappa',\Gamma'}$ since both subsemigroups are full so we may append idempotents on the start or end of the sequence if necessary. Let (δ, Δ) be the special inverse subsemigroup pair for $T_{\kappa,\Gamma} \vee T_{\kappa',\Gamma'}$. We shall show that form claimed for (δ, Δ) is accurate. That δ is the transitive closure of κ and κ', which is as claimed, is immediate.

For Δ we will show that the expressions claimed are precisely

$$\{g \in G \mid (j, g, \min j\delta) \in T_{\delta,\Delta}\}.$$

We first consider j such that $j = \min j\delta$. We know that $(j, g, j) \in T_{\delta,\Delta}$ precisely when there is a sequence $(j, h_1, i_1) \in T_{\kappa,\Gamma}$, $(i_1, h_2, i_2) \in T_{\kappa',\Gamma'}$, $\dots$, $(i_{m-1}, h_m, i_m) \in T_{\kappa,\Gamma}$, $(i_m, h_{m+1}, j) \in T_{\kappa',\Gamma'}$. We note that $(j, h_1, i_1) \in T_{\kappa,\Gamma}$ says that $h_1 \in \Gamma_{i_1}^{-1}$ (using that $j = \min j\delta$ so certainly $j = \min j\kappa$), $(i_1, h_2, i_2) \in T_{\kappa',\Gamma'}$ says $h_2 \in \Gamma'_{i_1}{\Gamma'_{i_2}}^{-1}$ and so on till $(i_m, h_{m+1}, \min j\delta) \in T_{\kappa',\Gamma'}$ which says that $h_{m+1} \in \Gamma'_{i_m}$ (again using that $j = \min j\delta$ so certainly $j = \min j\kappa'$). Therefore

$$g \in \Gamma_{i_1}^{-1}\Gamma'_{i_1}{\Gamma'}_{i_2}^{-1}\Gamma_{i_2}\Gamma_{i_3}^{-1}\Gamma'_{i_3} \dots \Gamma_{i_{m-1}}\Gamma_{i_m}^{-1}\Gamma'_{i_m}.$$

Thus we have that

$$\Delta_j \subseteq \bigcup_{m \geq 1} \bigcup_{(i_1, \dots, i_m) \in L_m} \Gamma_{i_1}^{-1}\Gamma'_{i_1}{\Gamma'}_{i_2}^{-1}\Gamma_{i_2}\Gamma_{i_3}^{-1}\Gamma'_{i_3} \dots \Gamma_{i_{m-1}}^{-1}\Gamma_{i_m}\Gamma'_{i_m}.$$

The reverse inclusion follows from the reverse of the above argument, that from any $g \in \Gamma_{i_1}^{-1}\Gamma'_{i_1} \dots \Gamma_{i_{m-1}}\Gamma_{i_m}^{-1}\Gamma'_{i_m}$ we may construct a sequence of elements in either $T_{\kappa,\Gamma}$ or $T_{\kappa',\Gamma'}$ such that their product is (j, g, j).

We now consider Δ_j for those j with $j \neq \min j\delta$, take $i_1, \dots, i_m$ as in the statement. By the same argument as in the previous paragraph, we see that if $g \in \Gamma_j\Gamma_{i_1}^{-1} \ldots \Gamma_{i_{m-1}}\Gamma_{i_m}^{-1}\Gamma'_{i_m}\Delta_{\min j\delta}$ then we obtain an element $(j, g, \min j\delta) \in T_{\delta,\Delta}$. For the reverse inclusion we suppose that we have some $(j, g, \min j\delta) \in T_{\delta,\Delta}$. We must show that $g \in \Gamma_j\Gamma_{i_1}^{-1} \ldots \Gamma_{i_{m-1}}\Gamma_{i_m}^{-1}\Gamma'_{i_m}\Delta_{\min j\delta}$. Choose $h \in \Gamma_j\Gamma_{i_1}^{-1} \ldots \Gamma_{i_{m-1}}\Gamma_{i_m}^{-1}\Gamma'_{i_m}\Delta_{\min j\delta}$, then $(j, h, \min j\delta) \in T_{\delta,\Delta}$ so as $T_{\delta,\Delta}$ is inverse we know that $(\min j\delta, h^{-1}, j) \in T_{\delta,\Delta}$. Then

$$(\min j\delta, h^{-1}, j)(j, g, \min j\delta) = (\min j\delta, h^{-1}g, \min j\delta) \in T_{\delta,\Delta}.$$

Thus we have that $h^{-1}g \in \Delta_{\min j\delta}$ and so

$$g = h(h^{-1}g) \in \Gamma_j\Gamma_{i_1}^{-1} \ldots \Gamma_{i_{m-1}}\Gamma_{i_m}^{-1}\Gamma'_{i_m}\Delta_{\min j\delta}$$

completing the proof. □

It is possible to combine Proposition 3.4.13 with the description of the intersection and join of left congruences in terms of the kernel trace approach from Theorem 2.5.5. The combination is straightforward, but the statement is even more technical that the statement of Proposition 3.4.13, so we refrain from giving it here.

3.5 Symmetric inverse monoids

The the main objective of this section is to describe one sided congruences on $\mathcal{I}_n$ via the inverse kernel approach; this is what we proceed with from this point. That this entirely straightforward question has no elegant solution in the literature indicates that it is a question far more easily posed than solved. In this section we continue use $i\kappa$ to denote the κ-class containing i, the notation $[n]$ is now used to denote the set of integers $\{1, 2, \ldots, n\}$. Also, E will denote $E(\mathcal{I}_n)$.

This section grew from a desire to compute $\mathfrak{LC}(\mathcal{I}_n)$ efficiently and produce diagrams of $\mathfrak{LC}(\mathcal{I}_n)$ for small n. This lends a slight technical slant to what follows, and it is probably not the easiest to read section of the thesis. We postpone much of the material which is only relevant to computational efficiency till Section 3.6. As has become common we describe full inverse subsemigroups first, following this we explain how we think of congruences on the idempotents of $\mathcal{I}_n$ and then we conclude by drawing both together to describe inverse congruence pairs.

Full inverse subsemigroups

Various types of subsemigroups of symmetric inverse semigroups have been studied, including maximal subsemigroups [79] and self-conjugate inverse semigroups [43], however I am aware of no description of arbitrary of full inverse subsemigroups. We do not claim that the description given here is elegant or user friendly however it does provide illumination as to the shape of the lattice of full inverse subsemigroups and the complexity of the problem.

An important piece of notation for this section will be the following, which was introduced when $\mathcal{I}_n$ was first defined in Chapter 1. For $A \subseteq [n]$ let

$$\gamma_A \colon [|A|] \to A$$

be the unique order preserving map. For $i \in A$ we see that $i\gamma_A^{-1}$ is the number of j less that i in A, so $i\gamma_A^{-1} = |\{j \in A \mid j \leq i\}|$, equivalently $i\gamma_A^{-1}$ is the position (first, second, third, etc.) of i in A. We note that for $a \in \mathcal{I}_n$, the composition $\gamma_{\mathrm{Dom}(a)} a \gamma_{\mathrm{Im}(a)}^{-1}$ is a bijective function $[\mathrm{rank}(a)] \to [\mathrm{rank}(a)]$, so can be thought of as an element of $\mathcal{S}_{\mathrm{rank}(a)}$. In fact, the set $\{\gamma_A \mid A \subseteq [n]\}$ is a submonoid of $\mathcal{I}_n$ [11].

To get hold of the lattice $\mathfrak{V}(\mathcal{I}_n)$ we shall appeal to Theorem 3.1.4, which we recall states that for any inverse semigroup S the lattice of full inverse subsemigroups is a subdirect product of the lattices of full inverse subsemigroups of the principal factors. For $\mathcal{I}_n$ the principal factors are isomorphic to the Brandt semigroups $B_k = \mathcal{B}^0(P_k, \mathcal{S}_k)$ for $i = 1, \ldots, k$ where $P_k = \{A \subseteq [n] \mid |A| = k\}$, the set of subsets of $[n]$ of size k. We think of B_k "living inside" $\mathcal{I}_n$ as $D_k \cup \{0\}$ where D_k is the $\mathcal{D}$-class $\{a \in \mathcal{I}_n \mid \mathrm{rank}(a) = k\}$. With this in mind we define a function

$$\theta \colon \mathcal{I}_n \to \{0\} \cup \bigcup_{1 \leq k \leq n} B_k \backslash \{0\}; \quad a \mapsto (\mathrm{Dom}(a), \gamma_{\mathrm{Dom}(a)} a \gamma_{\mathrm{Im}(a)}^{-1}, \mathrm{Im}(a))$$

if $a \neq 0$ and $0 \mapsto 0$. We call this map the *Brandt decomposition map*, and say that $(\mathrm{Dom}(a), \gamma_{\mathrm{Dom}(a)} a \gamma_{\mathrm{Im}(a)}^{-1}, \mathrm{Im}(a))$ is the *Brandt decomposition* for $a \in \mathcal{I}_n$.

It is easy to see that θ is well defined, and moreover, since

$$\theta^{-1}\colon (A, g, B) \mapsto \gamma_A^{-1} g \gamma_B$$

is the inverse function to θ, that θ is a bijection. Also, we can view each B_k as a subset of $\mathrm{Im}(\theta)$, indeed $B_k \approx D_k\theta \cup \{0\}$. In this way, as a set, we view B_k living inside $\mathcal{I}_n$ as $(D_k\theta \cup \{0\})\theta^{-1}$. Following this train of thought we view $\mathcal{I}_n$ as the union $\bigcup_{1\leq k\leq n} B_k$ where we identify the zeros of each B_k and in particular we think of θ as a function $\mathcal{I}_n \to \bigcup_{1\leq k\leq n} B_k$.

We have already produced a description of full inverse subsemigroups of Brandt semigroups in Proposition 3.4.11 and we shall build on this foundation and determine how subsemigroups of the Brandt semigroups fit together to give a full inverse subsemigroup of $\mathcal{I}_n$. Via the Brandt decomposition map θ, a full inverse subsemigroup $T \subseteq \mathcal{I}_n$ defines a set $\{T_k \subseteq B_k \mid 1 \leq k \leq n\}$ of subsets of B_k. Explicitly $T_k = T\theta \cap B_k$, regarding B_k as a subset of $\mathrm{Im}(\theta)$ as in the previous paragraph. It is easy to see that $T_k \subseteq B_k$ is a full inverse subsemigroup. Further, given a set $\{T_k \subseteq B_k \mid 1 \leq k \leq n\}$ of full inverse subsemigroups which arose in this way from a full inverse subsemigroup $T \subseteq \mathcal{I}_n$ we may recover T as

$$T = \{0\} \cup \bigcup_{1\leq k\leq n} (T_k \backslash \{0\})\theta^{-1} = \bigcup_{1\leq k\leq n} T_k\theta^{-1}.$$

To make our notation easier we blur the distinction between an element $a \in \mathcal{I}_n$ and the Brandt decomposition for a. We compose Brandt decompositions as elements in $\mathcal{I}_n$. Formally, for $a, b \in \bigcup_{1\leq i\leq n} B_k$, we set the product in $\mathcal{I}_n$ as $ab = ((a\theta^{-1})(b\theta^{-1}))\theta$. We note that if $a, b \in B_k$ and $ab \neq 0$ when multiplied in B_k then the product in $\mathcal{I}_n$ is the same as the product in B_k. It shall be clear from context which multiplication we use.

We now demonstrate that for a set $\{T_k \subseteq B_k \mid 1 \leq k \leq n\}$ of full inverse subsemigroups to combine to give a full inverse subsemigroup of $\mathcal{I}_n$ it suffices to be able to multiply across the different factors by idempotents. For a set $V \subseteq D_k$ we observe that

$$EV \cap D_{k-1} = \{ev \mid e \in E(\mathcal{I}_n),\ v \in V,\ e \leq vv^{-1},\ \mathrm{rank}(e) = k-1\}$$

is the set of elements in D_{k-1} that are lower covers - which means are directly below in the partial order - of elements of V in the natural partial order. This extends in the obvious way, for $i < k$,

$$EV \cap D_i = \{ev \mid e \in E(\mathcal{I}_n),\ v \in V,\ e \leq vv^{-1},\ \text{rank}(e) = i\}.$$

Proposition 3.5.1. *Let $\{T_k \subseteq B_k \mid 1 \leq k \leq n\}$ be full inverse subsemigroups, and let*

$$T = \bigcup_{1 \leq k \leq n} T_k\theta^{-1} \subseteq \mathcal{I}_n.$$

Then T is a full inverse subsemigroup of $\mathcal{I}_n$ if and only if for each $1 < k \leq n$

$$E(T \cap D_k) \cap D_{k-1} \subseteq T$$

where $E(T \cap D_k)$ refers to the set product of E with $T \cap D_k$.

Proof. For each $1 \leq i \leq n$ write $\overline{T_i}$ for $T \cap D_i$. Then as a set $T_i = \overline{T}_i$ (using our blurring of notation between subsets of B_k and subsets of D_k), in this proof we regard $\overline{T_i}$ as a subset of $\mathcal{I}_n$ and $T_i \cup \{0\}$ as a full inverse subsemigroup of B_i. We first remark that if $E\overline{T_k} \cap D_{k-1} \subseteq \overline{T_{k-1}}$ for each k then $E\overline{T_k} \cap D_i \subseteq \overline{T_i}$ for each $i < k$ and so $E\overline{T_k} \subseteq T$. Indeed, if $a \in E\overline{T}_k \cap D_{k-2}$ then $a = et$ for $e \in E(\mathcal{I}_n)$ and $t \in \overline{T}_k$ with $e \leq tt^{-1}$ and $\text{rank}(e) = k-2$. Then there is $f \in E(\mathcal{I}_n)$ such that $e < f < tt^{-1}$ and $\text{rank}(f) = k-1$. Then $ft \in E\overline{T}_k \cap D_{k-1}$, and by assumption $E\overline{T_k} \cap D_{k-1} \subseteq \overline{T_{k-1}}$ so $ft \in \overline{T_{k-1}}$. Also by assumption $E\overline{T_{k-1}} \cap D_{k-2} \subseteq \overline{T_{k-2}}$. We then note that $et = e(ft) \in E\overline{T_{k-1}} \cap D_{k-2}$, so $et \in \overline{T_{k-2}}$. By induction it follows that $E\overline{T_k} \cap D_i \subseteq \overline{T_i}$ for each $i < k$.

We next observe that if $a, b \in \mathcal{I}_n$ then

$$ab = (abb^{-1}a^{-1})a(a^{-1}abb^{-1})b,$$

and both $(abb^{-1}a^{-1})a$ and $(a^{-1}abb^{-1})b$ are elements of rank equal to $\text{rank}(ab)$.

We may now proceed with the proof of the result. Suppose first that $E\overline{T_k} \cap D_{k-1} \subseteq \overline{T_{k-1}}$ for each k and take $a, b \in T$, with $a \in \overline{T_k}$ and $b \in \overline{T_j}$. Then

$$(abb^{-1}a^{-1})a \in E\overline{T_k} \cap D_{\text{rank}(ab)} \subseteq \overline{T_{\text{rank}(ab)}}$$

and similarly

$$(a^{-1}abb^{-1})b \in \overline{T_{\mathrm{rank}(ab)}}.$$

Therefore $(abb^{-1}a^{-1})a,\ (a^{-1}abb^{-1})b \in T_{\mathrm{rank}(ab)}$, and, as $T_{\mathrm{rank}(ab)}$ is a full inverse subsemigroup of $B_{\mathrm{rank}(ab)}$ and $\mathrm{Im}((abb^{-1}a^{-1})a) = \mathrm{Dom}((a^{-1}abb^{-1})b)$, we have $ab \in T_{\mathrm{rank}(ab)}$. Thus $ab \in \overline{T_{\mathrm{rank}(ab)}}$, so T is a subsemigroup. As each T_k is full and inverse it is clear that T is full and inverse.

The converse is straightforward, we suppose T is a full inverse subsemigroup. If $a \in E(T \cap D_k) \cap D_{k-1}$ then $a = et$ for $t \in T \cap D_k$ and $e \in E$, and $\mathrm{rank}(a) = k - 1$. Then, as $\mathrm{rank}(a) = k - 1$, $a \in D_{k-1}$ and, as T is a full subsemigroup $a = et \in T$. Thus $a \in T \cap D_{k-1}$. $\square$

Proposition 3.5.1 tells us that when we seek conditions for a “chain” of subsemigroups of the Brandt semigroups $\mathcal{B}^0(\mathcal{S}_k, P_k)$ to combine to give a subsemigroup of $\mathcal{I}_n$ we need only to consider closure under the projections down the “chain” that correspond to multiplication by idempotents.

With this in mind we make a digression and discuss multiplication by idempotents in terms of the Brandt decomposition. Each $g \in \mathcal{S}_k$ defines a set of elements of $\mathcal{S}_{k-1}$. We recall that $\gamma_A \colon [|A|] \to A$ is the unique order preserving function and we consider $\mathcal{S}_k$ as the set of bijective functions $[k] \to [k]$. Define the function

$$\Lambda_k \colon \mathcal{S}_k \times [k] \to \mathcal{S}_{k-1}; \qquad (g, i) \mapsto \gamma_{[k]\setminus\{i\}} g \gamma^{-1}_{[k]\setminus\{ig\}}.$$

For an example of the Λ_k function see Fig. 3.6.

We note that

$$\begin{aligned}(gh, i)\Lambda_k &= \gamma_{[k]\setminus\{i\}} gh \gamma^{-1}_{[k]\setminus\{i(gh)\}} \\ &= (\gamma_{[k]\setminus\{i\}} g \gamma^{-1}_{[k]\setminus\{ig\}})(\gamma_{[k]\setminus\{ig\}} h \gamma^{-1}_{[k]\setminus\{i(gh)\}}) = (g, i)\Lambda_k (h, ig)\Lambda_k,\end{aligned}$$

and also that $((g, i)\Lambda_k)^{-1} = (g^{-1}, ig)\Lambda_k$. The following lemma is elementary, though the proof is technical. It is much easier to convince oneself of its validity if one draws pictures along the lines of Fig. 3.6.

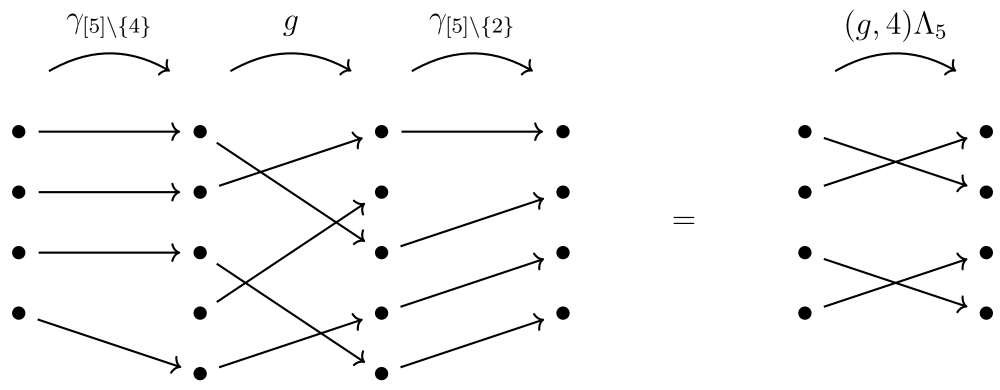


Figure 3.6: The function Λ_k, computing $(g,4)\Lambda_5$

Lemma 3.5.2. *Let $a \in \mathcal{I}_n$ have Brandt decomposition (A, g, B) and let $e \in E(\mathcal{I}_n)$ have* $\mathrm{Dom}(e) = A\backslash i$ *for some i. Then ea has Brandt decomposition*

$$(\ A\backslash\{i\},\ (g, i\gamma_A^{-1})\Lambda_k,\ B\backslash\{i(\gamma_A^{-1}g\gamma_B)\}\ )$$

where $k =$ $\mathrm{rank}(a)$.

Proof. It is immediate that $\mathrm{Dom}(ea) = A\backslash\{i\}$. Also, we know $a = \gamma_A^{-1}g\gamma_B$ so

$$ia = i(\gamma_A^{-1}g\gamma_B).$$

Therefore

$$\mathrm{Im}(ea) = \mathrm{Im}(a)\backslash\{i(\gamma_A^{-1}g\gamma_B)\} = B\backslash\{i(\gamma_A^{-1}g\gamma_B)\}.$$

For the group component we note that $e = \gamma_{A\backslash\{i\}}^{-1}1\gamma_{A\backslash\{i\}}$. Thus

$$ea = \gamma_{A\backslash\{i\}}^{-1}1\gamma_{A\backslash\{i\}}\gamma_A^{-1}g\gamma_B.$$

Also, as $\mathrm{Im}(ea) = B\backslash\{i(\gamma_A^{-1}g\gamma_B)\}$ we see that

$$\begin{aligned} ea &= ea\gamma_{B\backslash\{i(\gamma_A^{-1}g\gamma_B)\}}^{-1}\gamma_{B\backslash\{i(\gamma_A^{-1}g\gamma_B)\}} \\ &= \gamma_{A\backslash\{i\}}^{-1}(\gamma_{A\backslash\{i\}}\gamma_A^{-1})g(\gamma_B\gamma_{B\backslash\{i(\gamma_A^{-1}g\gamma_B)\}}^{-1})\gamma_{B\backslash\{i(\gamma_A^{-1}g\gamma_B)\}}. \end{aligned}$$

We then observe that $\gamma_{A\backslash\{i\}}\gamma_A^{-1}\colon [|A|-1] \to [|A|]$ has image $[|A|]\backslash i\gamma_A^{-1}$. It follows that $\gamma_{A\backslash\{i\}}\gamma_A^{-1} = \gamma_{[|A|]\backslash i\gamma_A^{-1}}$. Similarly, the partial function

$$\gamma_B\gamma^{-1}_{B\backslash\{i(\gamma_A^{-1}g\gamma_B)\}}\colon [|B|] \to [|B|-1]$$

has domain $[|B|]\backslash\{i(\gamma_A^{-1}g)\}$ so is equal to $\gamma^{-1}_{[|B|]\backslash\{i(\gamma_A^{-1}g)\}}$. Therefore we have

$$\begin{aligned} ea &= \gamma^{-1}_{A\backslash\{i\}}(\gamma_{[|A|]\backslash i\gamma_A^{-1}}g\gamma^{-1}_{[|B|]\backslash\{i(\gamma_A^{-1}g)\}})\gamma_{B\backslash\{i(\gamma_A^{-1}g\gamma_B)\}} \\ &= \gamma^{-1}_{A\backslash\{i\}}((g, i\gamma_A^{-1})\Lambda_k)\gamma_{B\backslash\{i(\gamma_A^{-1}g\gamma_B)\}}. \end{aligned}$$

Thus ea has the Brandt decomposition claimed in the statement of the lemma. □

For Brandt semigroups we defined subsemigroups in terms of inverse subsemigroup pairs. If we have a set $\{T_k \subseteq B_k \mid 1 \le k \le n\}$ of subsemigroups which combine to define a full inverse subsemigroup of $\mathcal{I}_n$ then we can specify each of the T_k in terms of an inverse subsemigroup pair. Each of these inverse subsemigroup pairs consists of an equivalence relation κ_k on $P_k \cup \{0\}$ and a mapping Γ_k from P_k to the set of left cosets of subgroups of $\mathcal{S}_k$. Since every T_k is full, for each k we have $0\kappa_k = \{0\}$. We equate the zero of each B_k with the empty set in $\mathcal{P}_n$ and we define

$$\kappa = \bigcup_{1\le k\le n} \kappa_k.$$

Then it is clear that κ is an equivalence relation on $\mathcal{P}_n$. Also, the union of the Γ_k defines a mapping Γ from $\mathcal{P}_n\backslash\{\emptyset\}$ to the set of left cosets of subgroups of the symmetric groups $\mathcal{S}_1, \dots, \mathcal{S}_n$. Further, it is clear that we can recover κ_k and Γ_k from κ and Γ so the pair (κ, Γ) specifies each of the subsemigroups $T_k \subseteq B_k$. Our next step is to characterise the equivalence relations and mappings which arise in this way, which will give us a description of full inverse subsemigroups of $\mathcal{I}_n$ in terms of "inverse subsemigroup pairs". In order to reduce notational clutter, for $A \subseteq [n]$ and $i \in A$ we write $A\backslash i$ for $A\backslash\{i\}$.

Definition 3.5.3. Let κ be an equivalence relation on $\mathcal{P}_n$ and let $\Gamma\colon A \mapsto \Gamma_A$ be a mapping from $\mathcal{P}_n\backslash\{0\}$ to the set of left cosets of subgroups of $\mathcal{S}_k$ where

$1 \leq k \leq n$. Then (κ, Γ) is an *inverse subsemigroup pair for $\mathcal{I}_n$* if all the following hold

(i) if $A\ \kappa\ B$ then that $|A| = |B|$;

(ii) Γ_A is a left coset of a subgroup of $\mathcal{S}_{|A|}$;

(iii) $A\ \kappa\ B$ implies that $\Gamma_A^{-1}\Gamma_A = \Gamma_B^{-1}\Gamma_B$;

(iv) for each A, B in $\mathcal{P}_n$ such that $A\ \kappa\ B$, and for each $i \in A$

$$\{(A\backslash i, B\backslash\{i(\gamma_A^{-1}g\gamma_B)\}) \mid g \in \Gamma_A\Gamma_B^{-1}\} \subseteq \kappa;$$

(v) for $A, B \in \mathcal{P}_n$ with $A\ \kappa\ B$ and for each $i \in A$

$$\{(g, i\gamma_A^{-1})\Lambda_k \mid g \in \Gamma_A\Gamma_B^{-1}\} \subseteq \Gamma_{A\backslash i}\Gamma^{-1}_{B\backslash(i\gamma_A^{-1}g\gamma_B)}.$$

If (κ, Γ) is an inverse subsemigroup pair then we define

$$T_{\kappa,\Gamma} = \left(\{0\} \cup \bigcup_{\substack{A\kappa = B\kappa,\\ |A|\neq 0}} \{A\} \times \Gamma_A\Gamma_B^{-1} \times \{B\}\right)\theta^{-1}.$$

We remark that (i) and (ii) of Definition 3.5.3 are required for an element (A, g, B) to be the Brandt decomposition of some element in $\mathcal{I}_n$.

Lemma 3.5.4. *Let (κ, Γ) be an inverse subsemigroup pair for $\mathcal{I}_n$, let $1 \leq k \leq n$ and let (κ_k, Γ_k) be the restriction of κ to $P_k \cup \{0\}$ and Γ to P_k respectively. Then (κ_k, Γ_k) is an inverse subsemigroup pair for $B_k = \mathcal{B}(P_k, \mathcal{S}_k)$. Furthermore, $T_{\kappa_k,\Gamma_k} \subseteq B_k$ is a full inverse subsemigroup, and*

$$T_{\kappa_k\Gamma_k} = (T_{\kappa,\Gamma} \cap D_k)\theta \cup \{0\}.$$

Proof. Actually, this follows from (i), (ii) and (iii) of the definition. It is clear from (i) and (ii) that κ_k is an equivalence relation on $P_k \cup \{0\}$ and Γ_k is a mapping from P_k to the set of left cosets of subgroups of $\mathcal{S}_k$. That (κ_k, Γ_k) is an inverse congruence pair is then exactly the statement of (iii).

The final parts of the claim are immediate. That T_{κ_k,Γ_k} is full follows from (i) as $0\kappa = \{0\}$ so $0\kappa_k = \{0\}$. Finally $T_{\kappa_k\Gamma_k} = (T_{\kappa,\Gamma} \cap D_k)\theta \cup \{0\}$ follows from the definitions of $T_{\kappa,\Gamma}$ and T_{κ_k,Γ_k}. □

We now use inverse subsemigroup pairs to determine full inverse subsemigroups of $\mathcal{I}_n$.

Theorem 3.5.5. *Let (κ, Γ) be an inverse subsemigroup pair for $\mathcal{I}_n$. Then $T_{\kappa,\Gamma}$ is a full inverse subsemigroup of $\mathcal{I}_n$. Conversely, all full inverse subsemigroups $\mathcal{I}_n$ are equal to $T_{\kappa,\Gamma}$ for an inverse subsemigroup pair (κ, Γ).*

Proof. Throughout this proof we abuse notation and use the Brandt decomposition for elements in $\mathcal{I}_n$ and multiply Brandt decompositions "through" $\mathcal{I}_n$.

Let (κ, Γ) be an inverse subsemigroup pair. By (i) and (ii) of the definition every element of $T_{\kappa,\Gamma}$ is an element of $\mathcal{I}_n$. Also, by Lemma 3.5.4 defining κ_k and Γ_k as the restrictions of κ and Γ to P_k we have that (κ_k, Γ_k) is an inverse subsemigroup pair for B_k and for each k that $T_k = T_{\kappa_k,\Gamma_k}$ is full. Therefore the pair (κ, Γ) defines a set of full inverse subsemigroups $T_k \subseteq B_k$. We apply Proposition 3.5.1 to obtain that $T_{\kappa,\Gamma}$ is a full inverse subsemigroup of $\mathcal{I}_n$ if and only if $ET_k \cap D_{k-1} \subseteq T_{k-1}$ for each $1 < k \leq n$ (here we abuse notation and regard the subsemigroup of B_k as a set of elements in $\mathcal{I}_n$).

If $a \in T_k$ and $e \in E$ such that $ea \in T_{k-1}$ then we may assume that $\mathrm{Dom}(e) = \mathrm{Dom}(a)\backslash\{i\}$ for some $i \in \mathrm{Dom}(a)$. Next we apply Lemma 3.5.2 to obtain that if $a = (A, g, B)$ then

$$ea = \left(A\backslash\{i\},\ (g, i\gamma_A^{-1})\Lambda_k,\ B\backslash\{i(\gamma_A^{-1}g\gamma_B)\}\right)$$

It follows that

$$ET_k \cap D_{k-1} = \{(A\backslash\{i\}, (g, i\gamma_A^{-1})\Lambda_k, B\backslash\{i(\gamma_A^{-1}g\gamma_B)\}) \mid (A, g, B) \in T_k,\ i \in A\}.$$

We suppose $(A, g, B) \in T_{\kappa,\Gamma}$, and that this says that $A\ \kappa\ B$ and $g \in \Gamma_A\Gamma_B^{-1}$. Then (iv) states that $(A\backslash\{i\}, B\backslash\{i(\gamma_A^{-1}g\gamma_B)\}) \in \kappa$, and (v) states that $(g, i\gamma_A^{-1})\Lambda_k \in \Gamma_{A\backslash i}\Gamma^{-1}_{B\backslash(i\gamma_A^{-1}g\gamma_B)}$. Together these imply that

$$(\ A\backslash\{i\},\ (g, i\gamma_A^{-1})\Lambda_k,\ B\backslash\{i(\gamma_A^{-1}g\gamma_B)\}\ ) \in T_{\kappa,\Gamma}.$$

Thus we have that $ET_k \cap D_{k-1} \subseteq T_{k-1}$, and it follows from Proposition 3.5.1 that $T_{\kappa,\Gamma}$ is a full inverse subsemigroup.

We now show that all full inverse subsemigroups are of the form $T_{\kappa,\Gamma}$ for an inverse subsemigroup pair (κ,Γ). Let $T \subseteq \mathcal{I}_n$ be a full inverse subsemigroup. Then $T_k = (T \cap D_k) \cup \{0\}$ may be regarded as a full inverse subsemigroup of B_k for each k. Therefore there is an inverse subsemigroup pair for B_k, say (κ_k, Γ_k) for which $T_k = T_{\kappa_k,\Gamma_k}$. We let κ and Γ be the union of the κ_k and Γ_k respectively. That (i),(ii) of Definition 3.5.3 hold is immediate by construction and that (iii) holds follows from the fact that (κ_k, Γ_k) is an inverse subsemigroup pair for B_k for each k. That (iv) and (v) hold follows by the reverse of the argument in the previous paragraph. If $A \ \kappa_k \ B$ and $g \in \Gamma_A\Gamma_B^{-1}$ then $(A, g, B) \in T_{\kappa_k,\Gamma_k}$. Since T is a full inverse subsemigroup by Proposition 3.5.1 $ET_{\kappa_k,\Gamma_k} \cap D_{k-1} \subseteq T_{\kappa_{k-1}\Gamma_{k-1}}$. Thus for each $i \in A$, by Lemma 3.5.2 we have that

$$(\ A\backslash\{i\}, \ (g, i\gamma_A^{-1})\Lambda_k, \ B\backslash\{i(\gamma_A^{-1}g\gamma_B)\} \ ) \in T_{\kappa_{k-1},\Gamma_{k-1}}.$$

Whence $(A\backslash\{i\}, B\backslash\{i(\gamma_A^{-1}g\gamma_B)\}) \in \kappa_{k-1}$ and $(g, i\gamma_A^{-1})\Lambda_k \in \Gamma_{A\backslash i}\Gamma^{-1}_{B\backslash(i\gamma_A^{-1}g\gamma_B)}$. Thus (iv) and (v) hold, so (κ,Γ) is an inverse subsemigroup pair for $\mathcal{I}_n$.

It is clear from the definition of $T_{\kappa,\Gamma}$ that $(A, g, B) \in T_{\kappa_k,\Gamma_k}$ for some k exactly when $(A, g, B) \in T_{\kappa,\Gamma}$. As $T = \{0\} \cup \bigcup_{1\leq i\leq n} T_{\kappa_k,\Gamma_k}$ we have that $T = T_{\kappa,\Gamma}$. This completes the proof. $\square$

We can describe the ordering on full inverse subsemigroups in terms of the inverse subsemigroup pairs, we have $T_{\kappa,\Gamma} \subseteq T_{\kappa',\Gamma'}$ if and only if $\kappa \subseteq \kappa'$ and $\Gamma_A\Gamma_B^{-1} \subseteq \Gamma'_A(\Gamma'_B)^{-1}$ for all $A, B \in \mathcal{P}_n$.

We remark that for $\mathcal{I}_n$ the notion of inverse subsemigroup pairs we have produced only captures full inverse subsemigroups, it is possible to use a similar formulation to describe other inverse subsemigroups, however the notation very quickly becomes even less manageable. Further, we also notice that the description of full inverse subsemigroups via inverse subsemigroup pairs has the same drawbacks as the corresponding version for Brandt semigroups, that a subsemigroup does not uniquely specify an inverse subsemigroup pair. This issue can be overcome in the same way as for Brandt semigroups, the details of which are included in Section 3.6, though the details become increasing technical and opaque, which is why we present the argument in this format, without these details.

Congruences on $\mathcal{P}_n$

It is worth spending some time dwelling on congruences on $\mathcal{P}_n$, even though it is not strictly necessary for our description of inverse congruence pairs. As this is a finite semilattice it is often helpful to think of a congruence as a partition. We say that a partition of $\mathcal{P}_n$ defined by a congruence is a *congruence partition*.

We recall that each congruence class of $\tau \in \mathfrak{C}(\mathcal{P}_n)$ is a convex subsemilattice, and that each convex subsemilattice $\mathcal{B} \subseteq \mathcal{P}_n$ is a class in some congruence (Lemma 3.1.2). In Section 3.6 we describe the minimum congruence which has $\mathcal{B}$ as a congruence class, and describe one way in which it is possible to describe a congruence on $\mathcal{P}_n$ by a "small" unique set of convex subsemilattices. For our current purposes we view a congruence in terms of the set of convex subsemilattices which are the non-trivial congruence classes. For example, in the congruence shown in Fig. 3.7 we think of the congruence as defined by $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}$.

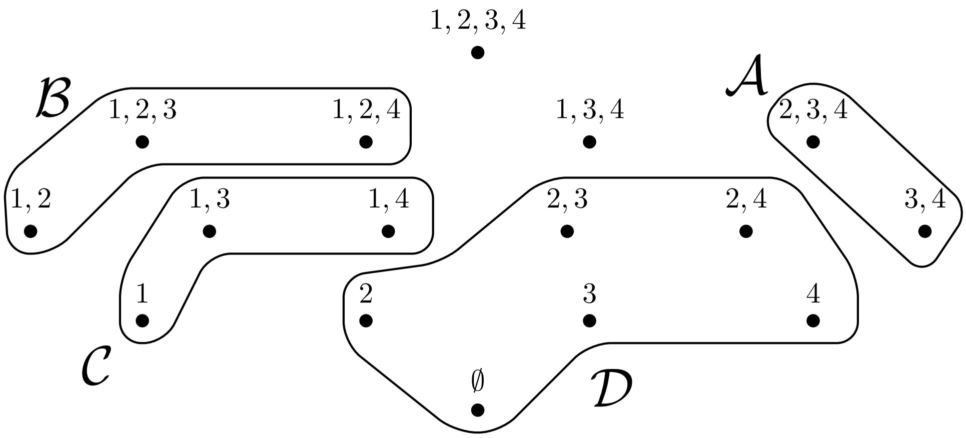


Figure 3.7: A congruence on $\mathcal{P}_n$ as a set of convex subsemilattices

Let $\mathcal{B} \subseteq \mathcal{P}_n$ be a convex subsemilattice. As $\mathcal{P}_n$ is finite, certainly $\mathcal{B}$ is finite, so, in particular, $\mathcal{B}$ has a minimum element, say X. Then each $B \in \mathcal{B}$ has $B = X \cup B\backslash X$, and the set $\{B\backslash X \mid B \in \mathcal{B}\}$ is a convex subsemilattice. It is this viewpoint of convex subsemilattices which shall be useful (particularly in Section 3.6), a minimum element X and then a set of subsets $\{C_i \subseteq [n] \mid i \in I\}$ such that $C_i \cap X = \emptyset$ for each i, and the convex subsemilattice is then $\{X \cup C_i \mid i \in I\}$.

The lattice of congruences on $\mathcal{P}_n$ grows rapidly as n increases. Computation of values for small n is possible (using GAP) and the sizes of $\mathfrak{C}(\mathcal{P}_n)$ for $n = 0, 1, 2, 3, 4$ are $1, 2, 7, 61, 2480$. We have seen the lattice of congruences on $\mathcal{P}_2$ in Fig. 2.1. We include the corresponding picture for $\mathcal{P}_3$. The semilattice $\mathcal{P}_3$ is shown in Fig. 3.8, and the lattice $\mathfrak{C}(\mathcal{P}_3)$ is shown in Fig. 3.9.

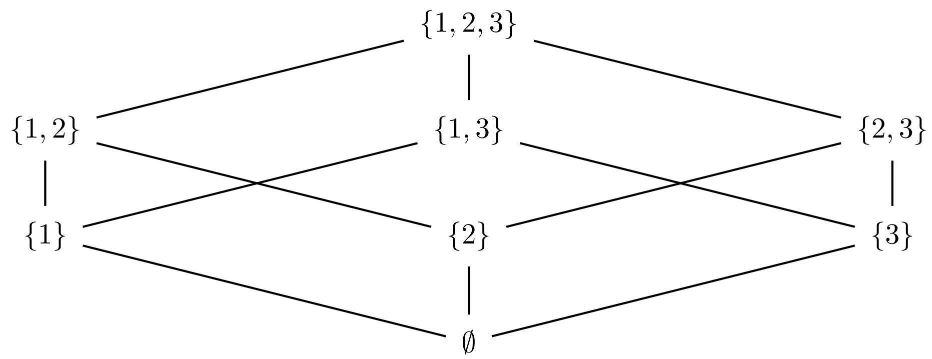


Figure 3.8: The intersection monoid $\mathcal{P}_3$

With a view to describing left congruences using inverse congruence pairs we now compute the normaliser of a congruence τ on $\mathcal{P}_n$. We write e_X for the idempotent in $\mathcal{I}_n$ with domain X. In this section we shall equivalently write $e_X \ \tau \ e_Y$ and $X \ \tau \ Y$.

Definition 3.5.6. Let τ be a congruence on $\mathcal{P}_A$ and σ a congruence on $\mathcal{P}_B$. Then we say that τ is *isomorphic* to σ - written $\tau \cong \sigma$ - if there is a bijection $f\colon A \to B$ such that $\tau f = \sigma$ (where $\tau f = \{(Xf, Yf) \mid (X, Y) \in \tau\}$). Such a function f is said to an *isomorphism* between τ and σ. We write $\mathrm{Isom}(\tau, \sigma)$ for the set of isomorphisms between τ and σ.

We observe that for $\tau \in \mathfrak{C}(\mathcal{P}_A)$ and $\sigma \in \mathfrak{C}(\mathcal{P}_B)$, whilst the condition $\tau \cong \sigma$ certainly implies that $\mathcal{P}_A/\tau \cong \mathcal{P}_B/\sigma$ this is not sufficient as the pair of congruences on $\mathcal{P}_2$ shown in Fig. 3.10 demonstrates. We also remark that

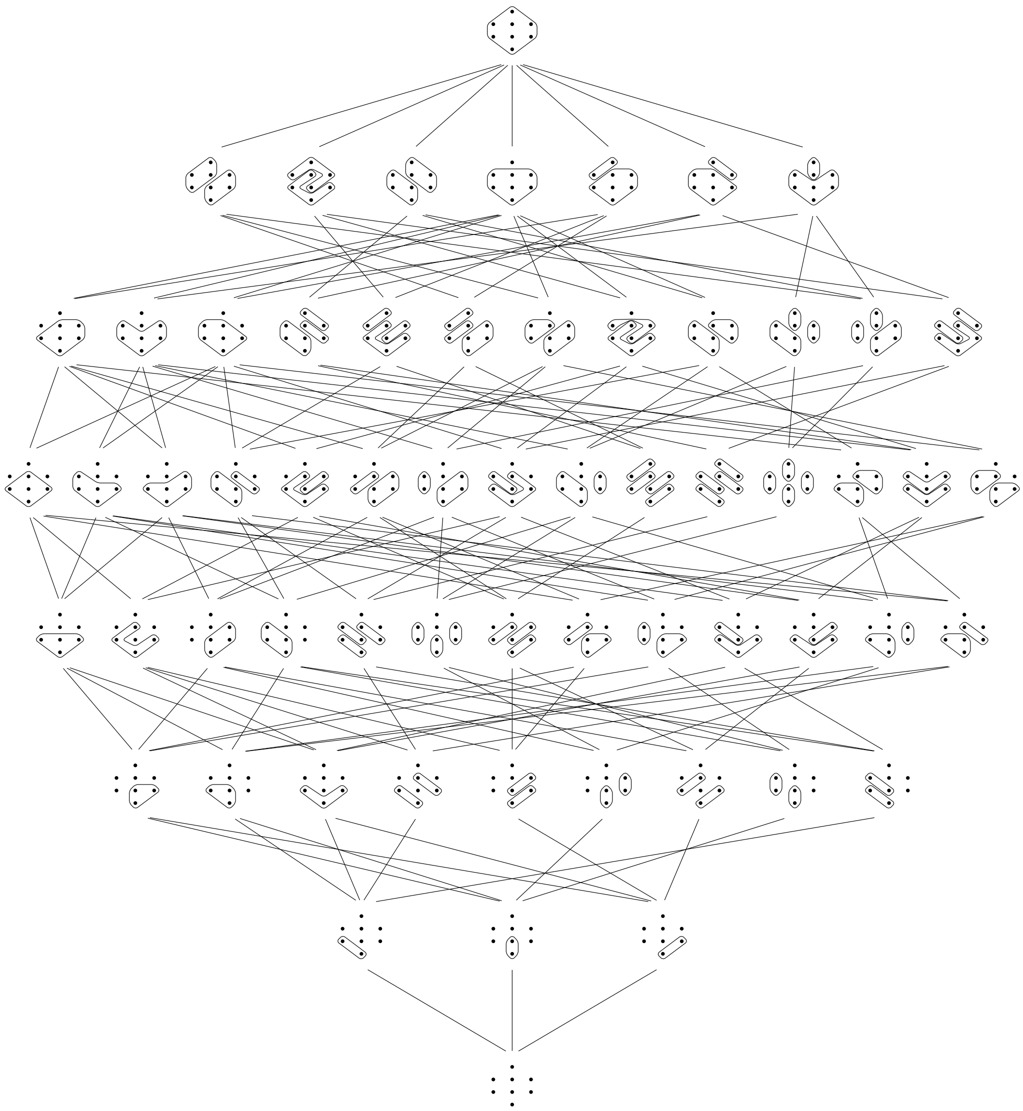


Figure 3.9: Congruences on $\mathcal{P}_3$

if $f \in \operatorname{Isom}(\tau, \sigma)$, then $X \subseteq A$ is minimum in its τ-class if and only if Xf is minimum in its σ-class.

Proposition 3.5.7. *Let τ be a congruence on $\mathcal{P}_n$. Then there is an element $a\colon A \to B$ with $a \in N(\tau)$ if and only if $\tau \cap (A \times A) \cong \tau \cap (B \times B)$. Moreover, $a\colon A \to B$ is an element of $N(\tau)$ if and only if a is an isomorphism between*

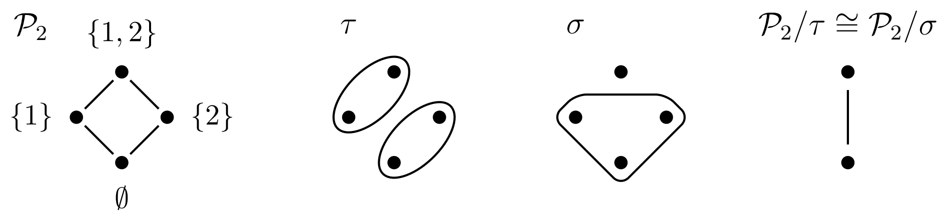


Figure 3.10: Non-isomorphic congruences on $\mathcal{P}_2$

$\tau \cap (A \times A)$ *and* $\tau \cap (B \times B)$. *Consequently,*

$$N(\tau) = \{a \in \mathcal{I}_n \mid a \in \operatorname{Isom}(\tau|_{\operatorname{Dom}(a)}, \tau|_{\operatorname{Im}(a)}).$$

Proof. By definition, the normaliser is

$$N(\tau) = \{a \in \mathcal{I}_n \mid e_X \;\tau\; e_Y \implies ae_Xa^{-1} \;\tau\; ae_Ya^{-1} \text{ and } a^{-1}e_Xa \;\tau\; a^{-1}e_Ya\}.$$

We observe that $ae_Xa^{-1} = e_{Xa^{-1}}$, and $a^{-1}e_Xa = e_{Xa}$.

Suppose that $a \in N(\tau)$ with $\operatorname{Dom}(a) = A$ and $\operatorname{Im}(a) = B$ and that $X, Y \subseteq A$ have $X \;\tau\; Y$. Then from $a^{-1}e_Xa \;\tau\; a^{-1}e_Ya$ we have that $Xa \;\tau\; Ya$. If $X, Y \subseteq B$ have $X \;\tau\; Y$ then $ae_Xa^{-1} \;\tau\; ae_Ya^{-1}$ gives us that $Xa^{-1} \;\tau\; Ya^{-1}$. Thus, for $X, Y \subseteq A$, we have $X \;\tau\; Y$ if and only if $Xa \;\tau\; Ya$, which implies that a is an isomorphism from $\tau \cap (A \times A)$ to $\tau \cap (B \times B)$.

For the converse we suppose that $a \colon A \to B$ is an isomorphism from $\tau \cap (A \times A)$ to $\tau \cap (B \times B)$. We must show that $a \in N(\tau)$. To this end suppose that $X \;\tau\; Y$, so we need to show that $Xa \;\tau\; Ya$ and $Xa^{-1} \;\tau\; Ya^{-1}$. We first notice that as τ is a congruence $X \;\tau\; Y$ forces $X \cap \operatorname{Dom}(a) \;\tau\; Y \cap \operatorname{Dom}(a)$, and $Xa = (X \cap \operatorname{Dom}(a))a$ and $Ya = (Y \cap \operatorname{Dom}(a))a$. As a is an isomorphism $X \cap \operatorname{Dom}(a) \;\tau\; Y \cap \operatorname{Dom}(a)$ precisely when $(X \cap \operatorname{Dom}(a))a \;\tau\; (Y \cap \operatorname{Dom}(a))a$ so we have that $Xa \;\tau\; Ya$. The argument for $Xa^{-1} \;\tau\; Ya^{-1}$ is very similar.

This completes the proof of the second claim, the first follows immediately. $\square$

We can then give a description of the normaliser of $\tau \in \mathfrak{C}(\mathcal{P}_n)$ in the terms of inverse subsemigroup pairs. To enable us to do this we suppose that there is a total ordering on each set $P_k = \{A \in \mathcal{P}_n \mid |A| = k\}$, and given a set $X \subseteq P_k$ we write $\min X$ for the minimum element of X with respect to this ordering. We also recall our usual definition of $\gamma_A \colon [|A|] \to A$ as the unique order preserving map. For $\tau \in \mathfrak{C}(\mathcal{P}_n)$ we define ξ_τ as the relation

$$\xi_\tau = \{(A, B) \in \mathcal{P}_n \times \mathcal{P}_n \mid \tau \cap (A \times A) \ \cong \ \tau \cap (B \times B)\}$$

and we define a mapping Ω^τ from $\mathcal{P}_n$ to the set of subsets of symmetric groups by

$$\Omega^\tau_A = \{\gamma_A a \gamma^{-1}_{\min A\xi_\tau} \mid a \in \operatorname{Isom}(\tau \cap (A \times A), \tau \cap (\min A\xi_\tau \times \min A\xi_\tau)\}.$$

Corollary 3.5.8. *Let τ be a congruence on $\mathcal{P}_n$. Then (ξ_τ, Ω^τ) is an inverse subsemigroup pair for $N(\tau)$.*

Proof. It is clear that ξ_τ is an equivalence relation on $\mathcal{P}_n$. Also,

$$\operatorname{Isom}(\tau \cap (B \times B), \tau \cap (B \times B))$$

is clearly a subgroup of $\mathcal{S}_B$, so if $B = \min B\xi_\tau$ we see that $\Omega^\tau_B \leq \mathcal{S}_{|B|}$ is a subgroup. Further, for any A, we see that if $g, h \in \Omega^\tau_A$ then $g^{-1}h \in \Omega^\tau_{\min A\xi_\tau}$, so Ω^τ_A is a left coset of $\Omega^\tau_{\min A\xi_\tau}$. To show that (ξ_τ, Ω^τ) is an inverse subsemigroup pair we must verify conditions (iii)-(v) from Definition 3.5.3. The first of these conditions, (iii), follows immediately from the definition of Ω^τ and ξ_τ. For the latter two conditions we suppose that $A, B \in \mathcal{P}_n$ with $A \ \xi_\tau \ B$ and that $i \in A$. Then $g \in \Omega^\tau_A {\Omega^\tau_B}^{-1}$ says that (A, g, B) is the Brandt decomposition for an element, which we call a, in $\operatorname{Isom}(\tau|_A, \tau|_B)$. Furthermore, if $e \in E(\mathcal{I}_n)$ is the idempotent with domain $A \backslash i$ then, by Lemma 3.5.2, ea has Brandt decomposition $(A\backslash i,\ (g, i\gamma_A^{-1})\Lambda_k,\ B\backslash(i\gamma_A^{-1} g\gamma_B))$. Also, as $a \in N(\tau)$ and $N(\tau)$ is full we have that $ea \in N(\tau)$ which implies that ea is an isomorphism from $\tau|_{A\backslash i}$ to $\tau|_{B\backslash(ia)}$. Conditions (iv) and (v) now follow, so we have that (ξ_τ, Ω^τ) is an inverse subsemigroup pair

We now show that $N(\tau) = T_{\xi_\tau, \Omega^\tau}$. We recall the definition of $T_{\xi_\tau, \Omega^\tau}$ which is

$$T_{\xi_\tau, \Omega^\tau} = \{(A, g, B) \mid A \ \xi_\tau \ B,\ g \in \Omega^\tau_A (\Omega^\tau_B)^{-1}\} \cup \{0\}.$$

We note that $(A, B) \in \xi_\tau$ precisely says that $\tau \cap (A \times A) \cong \tau \cap (B \times B)$ and $g \in \Omega^\tau_A (\Omega^\tau_B)^{-1}$ is equivalent to $\gamma_A^{-1} g \gamma_B$ being an isomorphism between $\tau \cap (A \times A)$ and $\tau \cap (B \times B)$. However, $\gamma_A^{-1} g \gamma_B$ is precisely the element of $\mathcal{I}_n$ with Brandt decomposition (A, g, B). Thus by Proposition 3.5.7 we have that $N(\tau) = T_{\xi_\tau, \Omega^\tau}$ □

For our purposes we are most interested in specifying which full inverse subsemigroups are contained in the normaliser of a trace. The following is immediate from Corollary 3.5.8 and the remark following Theorem 3.5.5 on the ordering of full inverse subsemigroups.

Corollary 3.5.9. *Let τ be a congruence on $\mathcal{P}_n$ and let (κ, Γ) be an inverse subsemigroup pair for $\mathcal{I}_n$. Then $T_{\kappa,\Gamma} \subseteq N(\tau)$ if and only if*

(i) $\kappa \subseteq \xi_\tau$;

(ii) for all A, B with $A \; \kappa \; B$,

$$\Gamma_A \Gamma_B^{-1} \subseteq \Omega^\tau_A {\Omega^\tau_B}^{-1}.$$

Inverse congruence pairs

We now turn our attention to describing (ICP2), which we recall from Lemma 2.4.1 has one possible formulation

(ICP2) for $x \in S$, if there $e \in E$ such that $x^{-1}x \; \tau \; e$, $xx^{-1} \; \tau \; xex^{-1}$ and $xe \in T$ then we have $x \in T$.

We apply this to $\mathcal{I}_n$ via the Brandt decomposition. Let e have Brandt decomposition $(Y, 1, Y)$, we may assume that $e \leq x^{-1}x$ so that xe has Brandt decomposition (X, h, Y) where xex^{-1} has Brandt decomposition $(X, 1, X)$. We may then say that x has Brandt decomposition $(X \cup A, g, Y \cup B)$. To describe the relationship between g and h we need to extend the Λ_k functions to the function:

$$\Lambda\colon \bigcup_{1 \leq k \leq n} \mathcal{S}_k \times \mathcal{P}_k \to \bigcup_{1 \leq k \leq n} \mathcal{S}_k; \qquad (g, Z) \mapsto \gamma_{[k] \setminus Z}\, g \gamma^{-1}_{[k] \setminus Z}.$$

where $\mathcal{P}_k = \{A \subseteq [k]\}$ is the powerset of $[k]$. Then Λ tells us the relationship of h to g.

Lemma 3.5.10. *Take $X \subseteq [n]$ and $A \subseteq X$. If $e \in E(\mathcal{I}_n)$ has Brandt decomposition $(X\backslash A, 1, X\backslash A)$, and $a \in \mathcal{I}_n$ has Brandt decomposition (X, g, Y), then ea has Brandt decomposition*

$$(\ X\backslash A,\ (g, A\gamma_X^{-1})\Lambda,\ Y\backslash(Aa)\ ).$$

Proof. This result follows either by repeated application of Lemma 3.5.2 or using the same argument as in the proof of Lemma 3.5.2. The details are left to the reader. $\square$

In essence (ICP2) tells us when an element higher up in the natural order is forced to be in T by an element lower down the partial order. To apply this we shall need an "inverse" of Λ which tells us which group elements g are such that $(X \cup A, g, Y \cup B)$ is above (X, h, Y) in $\mathcal{I}_n$. To this end we define $Q_{k,j} = \{A \subseteq [k+j] \mid |A| = k\}$, then define the function $\Delta_{k,j} \colon \mathcal{S}_k \times Q_{k,j} \times Q_{k,j} \to \mathcal{P}(\mathcal{S}_{k+j})$ by

$$(h, U, V) \mapsto \{g \in \mathcal{S}_{k+j} \mid h = (g, [k+j]\backslash U)\Lambda \text{ and } Ug = V\}$$

where we write $\mathcal{P}(\mathcal{S}_k)$ for the powerset of $\mathcal{S}_k$. We shall also drop the subscripts k, j from $\Delta_{k,j}$ when this is convenient and does not cause confusion. The function Δ is the function we require.

Lemma 3.5.11. *Let $X, Y, A, B \in \mathcal{P}_n$ with $|X| = |Y|$, $|A| = |B|$ and $X \cap A = \emptyset = Y \cap B$ and let $h \in \mathcal{S}_{|X|}$. Then $g \in (h, X\gamma_{X\cup A}^{-1}, Y\gamma_{Y\cup B}^{-1})\Delta_{|X|,|A|}$ if and only if*

$$(X, h, Y) = (X, 1, X)(X \cup A, g, Y \cup B).$$

Proof. Suppose first that $g \in (h, X\gamma_{X\cup A}^{-1}, Y\gamma_{Y\cup B}^{-1})\Delta_{|X|,|A|}$. By Lemma 3.5.10 we have that

$$(X, 1, X)(X\cup A, g, Y\cup B) = (\ X,\ (g, A\gamma_{X\cup A}^{-1})\Lambda,\ (Y\cup B)\backslash(A(\gamma_{X\cup A}^{-1} g \gamma_{Y\cup B}))\ ).$$

Then we note that $g \in (h, X\gamma_{X\cup A}^{-1}, Y\gamma_{Y\cup B}^{-1})\Delta_{|X|,|A|}$ says that $(g, A\gamma_{X\cup A}^{-1})\Lambda = h$ and $X\gamma_{X\cup A}^{-1}g = Y\gamma_{Y\cup B}^{-1}$, and this final condition may be rewritten as $X\gamma_{X\cup A}^{-1}g\gamma_{Y\cup B} = Y$, or equivalently $A\gamma_{X\cup A}^{-1}g\gamma_{Y\cup B} = B$. Thus

$$(\ X, \ (g, A\gamma_{X\cup A}^{-1})\Lambda, \ (Y \cup B)\backslash(A(\gamma_{X\cup A}^{-1}g\gamma_{Y\cup B})) \ ) = (X, h, Y).$$

For the converse we suppose that $(X, h, Y) = (X, 1, X)(X \cup A, g, Y \cup B)$. Again Lemma 3.5.10 tells us that

$$(X, 1, X)(X \cup A, g, Y \cup B) = (\ X, \ (g, A\gamma_{X\cup A}^{-1})\Lambda, \ (Y \cup B)\backslash(A(\gamma_{X\cup A}^{-1}g\gamma_{Y\cup B})) \ )$$

so we have $(g, A\gamma_{X\cup A}^{-1})\Lambda = h$ and $(Y \cup B)\backslash(A(\gamma_{X\cup A}^{-1}g\gamma_{Y\cup B})) = Y$. This latter condition precisely says that $A(\gamma_{X\cup A}^{-1}g\gamma_{Y\cup B}) = B$ so we have that $g \in (h, X\gamma_{X\cup A}^{-1}, Y\gamma_{Y\cup B}^{-1})\Delta_{|X|,|A|}$. This completes the proof. $\square$

We remark that it is clear for any choice $(h, A, B) \in \mathcal{S}_k \times Q_{k,j} \times Q_{k,j}$ that $(h, A, B)\Delta_{k,j}$ is a "copy" of S_j, the elements of $[k + j]\backslash A$ are mapped bijectively to $[k + j]\backslash B$ without restriction. It must be noted that it is not necessarily a subgroup of $\mathcal{S}_k$, it is in fact a coset of a subgroup isomorphic to S_j inside $\mathcal{S}_k$.

Using this language for (ICP2) we classify inverse congruence pairs in the following way, recalling the notation ξ_τ, and Ω^τ from Corollary 3.5.9.

Theorem 3.5.12. *Let τ be a congruence on $\mathcal{P}_n$ and let (κ, Γ) be an inverse subsemigroup pair for $\mathcal{I}_n$. Then $(\tau, T_{\kappa,\Gamma})$ is an inverse congruence pair for $\mathcal{I}_n$ if and only if the following hold.*

(i) $\kappa \subseteq \xi_\tau$;

(ii) for all A, B with $A \ \kappa \ B$,

$$\Gamma_A\Gamma_B^{-1} \subseteq \Omega_A^\tau {\Omega_B^\tau}^{-1};$$

(iii) for $X, Y, A, B \in \mathcal{P}_n$ with $|X| = |Y|$, $|A| = |B|$, $X \cap A = \emptyset = Y \cap B$, $X \ \tau \ X \cup A$ and $Y \ \tau \ Y \cup B$, if $X \ \kappa \ Y$ then $X \cup A \ \kappa \ Y \cup B$;

(iv) for $X, Y, A, B \in \mathcal{P}_n$ with $|X| = |Y|$, $|A| = |B|$, $X \cap A = \emptyset = Y \cap B$, $X \; \tau \; X \cup A$ and $Y \; \tau \; Y \cup B$,

$$\{(h, X\gamma_{X\cup A}^{-1}, Y\gamma_{Y\cup B}^{-1})\Delta \mid h \in \Gamma_X\Gamma_Y^{-1}\} \subseteq \Gamma_{X\cup A}\Gamma_{Y\cup B}^{-1}.$$

Proof. By Corollary 3.5.9, we know that (i) and (ii) are equivalent to $T_{\kappa,\Gamma}$ being contained in $N(\tau)$. Thus to complete the proof if remains to check that (ICP2) is equivalent to (iii) and (iv). This is a largely straightforward verification exercise, however it is very heavy on notation (even after the abuse of using the Brandt decomposition informally) and becomes an technical definition chasing exercise very quickly.

First we suppose that (iii) and (iv) hold and that there for $x \in S$ there is $e \in E$ with $x^{-1}x \; \tau \; e$, $xx^{-1} \; \tau \; xex^{-1}$ and $xe \in T_{\kappa,\Gamma}$. Then suppose that in terms of Brandt decomposition $xe = (X, h, Y)$ and $x = (X \cup A, g, Y \cup B)$. Then we have $X, Y, A, B \in \mathcal{P}_n$ with $|X| = |Y|$, $|A| = |B|$, $X \cap A = \emptyset = Y \cap B$, $X \; \tau \; X \cup A$, $Y \; \tau \; Y \cup B$ and $X \; \kappa \; Y$, so, by (iii), $X \cup A \; \kappa \; Y \cup B$. Also, by (iv),

$$\{(h, X\gamma_{X\cup A}^{-1}, Y\gamma_{Y\cup B}^{-1})\Delta \mid h \in \Gamma_X\Gamma_Y^{-1}\} \subseteq \Gamma_{X\cup A}\Gamma_{Y\cup B}^{-1}.$$

By Lemma 3.5.11, $(X, 1, X)(X \cup A, g, Y \cup B) = (X, h, Y)$ implies that

$$g \in (h, X\gamma_{X\cup A}^{-1}, Y\gamma_{Y\cup B}^{-1})\Delta.$$

Therefore, $g \in \Gamma_{X\cup A}\Gamma_{Y\cup B}^{-1}$ and, as $X \cup A \; \kappa \; Y \cup B$, we have that

$$(X \cup A, g, Y \cup B) \in T_{\kappa,\Gamma},$$

so (ICP2) is satisfied.

For the converse we assume that (ICP2) is satisfied. Suppose that we have $X, Y, A, B \in \mathcal{P}_n$ with $|X| = |Y|$, $|A| = |B|$, $X \cap A = \emptyset = Y \cap B$, $X \; \tau \; X \cup A$, $Y \; \tau \; Y \cup B$ and $X \; \kappa \; Y$. Since $X \; \kappa \; Y$ we have that

$$\Gamma_X\Gamma_Y^{-1} = \{h \in \mathcal{S}_{|X|} \mid (X, h, Y) \in T_{\kappa,\Gamma}\},$$

and, in particular, this is non-empty. We take $h \in \Gamma_X\Gamma_Y^{-1}$, and choose $g \in (h, X\gamma_{X\cup A}^{-1}, Y\gamma_{Y\cup B}^{-1})\Delta$. Then, by Lemma 3.5.11,

$$(X, 1, X)(X \cup A, g, Y \cup B) = (X, h, Y).$$

Then the condition for (ICP2) are satisfied with $x = (X \cup A, g, Y \cup B)$ and $e = (Y, 1, Y)$, as this means that $xex^{-1} = (X, 1, X)$ and $xe = (xex^{-1})x = (X, h, Y) \in T_{\kappa,\Gamma}$. To see that $xex^{-1} = (X, 1, X)$ we note that

$$xex^{-1} = (xe)(xe)^{-1} = (X, h, Y)(X, h, Y)^{-1} = (X, 1, X).$$

Then by (ICP2) we have that $x = (X \cup A, g, Y \cup B) \in T_{\kappa,\Gamma}$. In particular this means that $X \cup A \ \kappa \ Y \cup B$ so (iii) is satisfied. Furthermore, the argument holds for any $h \in \Gamma_X \Gamma_Y^{-1}$ and any $g \in (h, X\gamma_{X\cup A}^{-1}, Y\gamma_{Y\cup B}^{-1})\Delta$. Thus we have that

$$\{(h, X\gamma_{X\cup A}^{-1}, Y\gamma_{Y\cup B}^{-1})\Delta \mid h \in \Gamma_X\Gamma_Y^{-1}\} \subseteq \Gamma_{X\cup A}\Gamma_{Y\cup B}^{-1},$$

so (iv) is satisfied and the proof is complete. □

The size of $\mathfrak{LC}(\mathcal{I}_n)$

We briefly comment on the number of left congruences on $\mathcal{I}_n$. We will give asymptotic bounds for $|\mathfrak{LC}(\mathcal{I}_n)|$, but the main objective of this section is to try to indicate why $|\mathfrak{LC}(\mathcal{I}_n)|$ grows so rapidly and why it is hard to compute. We lift general results from the literature to aid us in this endeavour.

The main strategy is to use the fact that $\mathfrak{V}(\mathcal{I}_n)$ and $\mathfrak{C}(\mathcal{P}_n)$ embed as semilattices into $\mathfrak{LC}(\mathcal{I}_n)$ which in turn embeds (as a semilattice) into $\mathfrak{C}(\mathcal{P}_n) \times \mathfrak{V}(\mathcal{I}_n)$. This implies that

$$\max\{|\mathfrak{C}(\mathcal{P}_n)|, |\mathfrak{V}(S)|\} \leq |\mathfrak{LC}(\mathcal{I}_n)| \leq |\mathfrak{C}(\mathcal{P}_n)||\mathfrak{V}(\mathcal{I}_n)|.$$

We first give an upper bound for $|\mathfrak{C}(\mathcal{P}_n)|$. We do not give a lower bound as we shall see that the $|\mathfrak{V}(\mathcal{I}_n)|$ dwarfs $|\mathfrak{C}(\mathcal{P}_n)|$.

Theorem 3.5.13 (see [7, Theorem 2.3])**.** *For all n the congruence lattice for $\mathcal{P}_n$ has*

$$|\mathfrak{C}(\mathcal{P}_n)| \leq 25(2^{2^n-6}).$$

We now consider $\mathfrak{V}(\mathcal{I}_n)$. For an upper bound we use the description of full inverse subsemigroups in terms of inverse semigroup pairs. An inverse subsemigroup pair (κ, Γ) consists of an equivalence relation on $\mathcal{P}_n$ and a mapping from $\mathcal{P}_n$ to the set of cosets of subgroups of symmetric groups. The number of equivalence relations on (or the number of partitions of) the set $\{1, \ldots, m\}$, for which we write Q_m, is known as the m^{th} *Bell number* and is an important combinatorial object. We let W_n be the number of mappings from $\mathcal{P}_n$ to the set of cosets of subgroups of symmetric groups. Then we have that

$$|\mathfrak{V}(\mathcal{I}_n)| \leq Q_{|\mathcal{P}_n|} W_n.$$

The best bounds for Bell numbers that I am aware of are as follows.

Theorem 3.5.14 ([3])**.** *The m^{th} Bell number Q_m satisfies*

$$\left(\frac{m}{e \ln m}\right)^m \leq Q_m \leq \left(\frac{0.792m}{\ln(m+1)}\right)^m.$$

For our purposes we may simplify the bounds for Bell numbers.

Corollary 3.5.15. *The m^{th} Bell number Q_m satisfies*

$$Q_m \leq m^m.$$

In particular, $Q_{|\mathcal{P}_n|}$, the number of equivalence relations on $\mathcal{P}_n$, satisfies

$$Q_{|\mathcal{P}_n|} \leq 2^{n2^n}.$$

To bound the number of mappings from $\mathcal{P}_n$ to cosets of subgroups of symmetric groups we need to know the number of cosets.

Theorem 3.5.16 ([68, Corollary 3.3])**.** *Let R_n be the number of subgroups of the symmetric group $\mathcal{S}_n$. Then*

$$2^{\left(\frac{1}{16}+o(1)\right)n^2} \leq R_n \leq 24^{\left(\frac{1}{6}+o(1)\right)n^2}.$$

In particular, there are $A, B > 1$ such that for all $n > 1$

$$A^{n^2} \leq R_n \leq B^{n^2}.$$

If H is a subgroup of a group G then the number of cosets of H is $|G|/|H|$. The number of cosets of subgroups of G is then bounded above by

$|G|$ multiplied by the number of subgroups of G. Consequently we have that the number of cosets of subgroups of $\mathcal{S}_n$ is bounded above by $n!R_n$, where R_n is the number of subgroups of $\mathcal{S}_n$.

Furthermore, when $n \geq m$ we may view $\mathcal{S}_m$ as a subgroup of $\mathcal{S}_n$, and therefore may view a coset of a subgroup of $\mathcal{S}_m$ as a coset of a subgroup of $\mathcal{S}_n$.

Lemma 3.5.17. *Let W_n be the number of mappings from $\mathcal{P}_n$ to the set of cosets of subgroups of the symmetric groups $\{\mathcal{S}_m \mid 1 \leq i \leq n\}$. Then there is $C > 1$ such that for all $n > 1$*

$$W_n \leq C^{n^2 2^n}.$$

Proof. As remarked prior to the result we may consider a coset of subgroup of $\mathcal{S}_m$ for $m \leq n$ as a coset of a subgroup of $\mathcal{S}_n$. Theorem 3.5.16 informs us that there is $B > 1$ such that there are at most B^{n^2} subgroups of $\mathcal{S}_n$. We then note that

$$n! \leq n^n \leq (2^n)^n = 2^{n^2},$$

therefore, taking $C = 2B$, there are at most

$$n!B^{n^2} \leq (2^{n^2})(B^{n^2}) = (2B)^{n^2} = C^{n^2}$$

cosets of subgroups of $\mathcal{S}_m$. Consequently there at most

$$(C^{n^2})^{|\mathcal{P}_n|} = \left(C^{n^2}\right)^{2^n} = C^{n^2 2^n}$$

mappings from $\mathcal{P}_n$ to the set of cosets of subgroups of $\mathcal{S}_n$, so $W_n \leq C^{n^2 2^n}$. $\square$

We can now state our upper bound for $|\mathfrak{V}(\mathcal{I}_n)|$.

Corollary 3.5.18. *There is $D > 1$ such that for all $n > 1$*

$$|\mathfrak{V}(\mathcal{I}_n)| \leq D^{n^2 2^n}.$$

Proof. We use the fact that $|\mathfrak{V}(\mathcal{I}_n)|$ is bounded by the product $Q_{|\mathcal{P}_n|}W_n$ where $Q_{|\mathcal{P}_n|}$ is the number of equivalence relations on $\mathcal{P}_n$ and W_n is the number of mappings from $\mathcal{P}_n$ to the set of cosets of subgroups of symmetric groups. Applying the bounds on $Q_{|\mathcal{P}_n|}$ and W_n from Corollary 3.5.15 and Lemma 3.5.17 we have that there is $C > 1$ such that

$$|\mathfrak{V}(\mathcal{I}_n)| \leq Q_{|\mathcal{P}_n|}W_n \leq 2^{n2^n}C^{n^22^n} \leq (2C)^{n^22^n}.$$

Taking $D = 2C$ gives the result. $\square$

At this stage we have an upper bound for $|\mathfrak{LC}(\mathcal{I}_n)|$.

Proposition 3.5.19. *There is $B > 1$ such that for all $n > 1$*

$$|\mathfrak{LC}(\mathcal{I}_n)| \leq B^{n^22^n}.$$

Proof. We apply Theorem 3.5.13 and Corollary 3.5.18 to obtain upper bounds for $|\mathfrak{C}(\mathcal{P}_n)|$ and $|\mathfrak{V}(\mathcal{I}_n)|$. Then there is $D > 1$ such that

$$|\mathfrak{LC}(\mathcal{I}_n)| \leq |\mathfrak{C}(\mathcal{P}_n)||\mathfrak{V}(\mathcal{I}_n)| \leq 25(2^{2^n-6})(D^{n^22^n}) \leq 2^{2^n}(D^{n^22^n}) \leq (2D)^{n^22^n}.$$

Taking $B = 2D$ completes the proof. $\square$

We now turn our attention to a lower bound for $|\mathfrak{LC}(\mathcal{I}_n)|$. We shall give a lower bound for $|\mathfrak{V}(\mathcal{I}_n)|$, which is then also a lower bound for $|\mathfrak{LC}(\mathcal{I}_n)|$. We recall that the lattice $\mathfrak{V}(\mathcal{I}_n)$ is a subdirect product of the lattices of full inverse subsemigroups of the principal factors (Theorem 3.1.4), which we know are the semigroups $B_k = \mathcal{B}(P_k, \mathcal{S}_k)$. It is obvious therefore, that

$$\max_{1\leq k\leq n}\{|\mathfrak{V}(B_k)|\} \leq |\mathfrak{V}(\mathcal{I}_n)|.$$

With this is mind it makes sense for us to obtain a lower bound for $|\mathfrak{V}(B_k)|$. We again use inverse subsemigroup pairs, this time for the Brandt semigroup B_k. Thus we have a partition of P_k and a mapping from P_k to the set of cosets of subgroups of $\mathcal{S}_k$. If κ is a partition of P_k, then by setting $\Gamma_A = \{1\}$ for each $A \in P_k$ we see that (κ, Γ) is an inverse subsemigroup pair. Also, if Γ is a mapping from P_k into the set of subgroups (yes subgroups) of $\mathcal{S}_k$ then (ι, Γ) is an inverse subsemigroup pair. It follows that $|\mathfrak{V}(B_k)|$ bounded below by the maximum of the number of partitions and the number of mappings into subgroups.

Lemma 3.5.20. *There is $A > 1$ such that for all $n > 1$ and for all $1 < k \leq n$*

$$\left(A^{k^2}\right)^{\binom{n}{k}} \leq |\mathfrak{V}(B_k)|.$$

Proof. By Theorem 3.5.16 we know that there is $A > 1$ such that $A^{k^2} \leq R^k$, where R_k is the number of subgroups of $\mathcal{S}_k$. As remarked prior to the result, if Γ is any mapping from P_k into the set of subgroups of $\mathcal{S}_k$ then (ι, Γ) is an inverse subsemigroup pair. Therefore

$$\left(A^{k^2}\right)^{\binom{n}{k}} \leq |\mathfrak{V}(B_k)|.$$

□

Our next step is to choose a suitable value of k such that we may use Lemma 3.5.20 to bound $|\mathfrak{V}(\mathcal{I}_n)|$. We shall use $k = \lceil n/2 \rceil$, where $\lceil x \rceil$ is the smallest integer m such that $m \geq x$. The binomial coefficient $\binom{n}{\lceil n/2 \rceil}$ is known as the *central binomial coefficient* and it is a standard combinatorics exercise to show that there are $a, b > 1$ such that

$$a\frac{2^n}{\sqrt{n}} \leq \binom{n}{\lceil n/2 \rceil} \leq b\frac{2^n}{\sqrt{n}}.$$

For bounds on binomial coefficients, including these, see [75].

Proposition 3.5.21. *There is $A > 1$ such that for all $n > 1$*

$$A^{\frac{n^2 2^n}{\sqrt{n}}} \leq |\mathfrak{V}(\mathcal{I}_n)|.$$

Proof. We use the fact that $|\mathfrak{V}(B_k)| \leq |\mathfrak{V}(\mathcal{I}_n)|$ for each k. We apply Lemma 3.5.20 with $k = \lceil n/2 \rceil$ to obtain an $A' > 1$ and we apply the bounds on the central binomial coefficient given before the lemma to obtain an $a > 1$, such that

$$\left(A'^{(n/2)^2}\right)^{a\frac{2^n}{\sqrt{n}}} \leq \left(A'^{(\lceil n/2 \rceil)^2}\right)^{\binom{n}{\lceil n/2 \rceil}} \leq |\mathfrak{V}(B_k)| \leq |\mathfrak{V}(\mathcal{I}_n)|.$$

We note that

$$\left(A'^{(n/2)^2}\right)^{a\frac{2^n}{\sqrt{n}}} = \left(A'^{(a/4)}\right)^{\frac{n^2 2^n}{\sqrt{n}}}.$$

Taking $A = A'^{(a/4)}$ completes the proof. □

We may now conclude this section with the main result, asymptotic bounds for $|\mathfrak{LC}(\mathcal{I}_n)|$.

Theorem 3.5.22. *There are $A, B > 1$ such that for all n*

$$A^{\frac{n^2 2^n}{\sqrt{n}}} \leq |\mathfrak{LC}(\mathcal{I}_n)| \leq B^{n^2 2^n}.$$

Proof. The upper bound is precisely that from Proposition 3.5.19. For the lower bound we use Proposition 3.5.21 and the fact that $|\mathfrak{V}(\mathcal{I}_n)| \leq |\mathfrak{LC}(\mathcal{I}_n)|$. □

3.6 Appendix B: Further analysis of $\mathfrak{LC}(\mathcal{I}_n)$

As promised, a highly technical and difficult to read section is approaching. Also as stated earlier, we reiterate that this grew out of a desire to draw pictures, or to get a computer to draw pictures. The lack of pictures should indicate that this was not a successful endeavour. This section is motivated by the fact that the lattice of left congruences on $\mathcal{I}_n$ becomes very large very quickly. It is therefore hard to compute and to do so via the inverse kernel approach requires efficient descriptions of the lattices $\mathfrak{V}(\mathcal{I}_n)$ and $\mathfrak{C}(\mathcal{P}_n)$. Thus far we have described $\mathfrak{V}(S)$ in terms of inverse subsemigroup pairs and mentioned that we think of a congruence on $\mathcal{P}_n$ in terms of convex subsemilattices. In this section we go further down these paths. First we define "special" inverse subsemigroup pairs and use these to further refine our description of $\mathfrak{V}(S)$. Second we go into much more detail regarding congruences and describe a provide a method to efficiently describe a congruence in terms of convex subsemilattices. This section is informal, we do not include complete proofs. Most results are similar to those we have seen, and the proof method would be similar; however, there is usually an extra level of technical detail.

We remind ourselves that a significant issue with describing the lattice of full inverse subsemigroups of $\mathcal{I}_n$ via inverse subsemigroup pairs is that more than one pair may correspond to more than one inverse subsemigroup pair, and this makes determining the ordering on $\mathfrak{V}(\mathcal{I}_n)$ difficult.

We recall that in the approach to describing inverse subsemigroups of Brandt semigroups in order to overcome the same issue we impose a total order on the indexing set (the I for the Brandt semigroup $\mathcal{B}^0(I, G)$) and then use this ordering to give a method to identify "special" elements in any subset of I. In the case of $\mathcal{I}_n$, we view the $\mathcal{D}$-class $D_k = \{a \in \mathcal{I}_n \mid \operatorname{rank}(a) = k\}$ through the "Brandt semigroup lens" as $\mathcal{B}(P_k, \mathcal{S}_k)$, so our indexing set is $P_k = \{A \subseteq [n] \mid |A| = k\}$. There are two common orderings for P_k, the lexicographic and colexicographic orderings. If $A = \{a_1, a_2, \ldots, a_k\}$ and $B = \{b_1, b_2, \ldots, b_k\}$ with $a_1 < a_2 < \cdots < a_k$ and $b_1 < b_2 < \cdots < b_k$ then the lexicographic ordering is defined as $A < B$ if $a_i < b_i$ for the first i such that $a_i \neq b_i$ and the colexicographic ordering is defined by $A < B$ if $a_i < b_i$ for the last i such that $a_i \neq b_i$. Both orderings may be extended to orderings on $\mathcal{P}_n$ by setting $A < B$ when $|A| < |B|$ and using the (co)lexicographic ordering when $|A| = |B|$. There are advantages to each ordering in various settings, however in this setting the existence of the ordering is the relevant part so we do not specify and the reader may just choose their favourite ordering on $\mathcal{P}_n$ (though it should have the property that $|A| \leq |B|$ implies $A \leq B$). We shall use $\min A\kappa$ to mean the minimum element in $A\kappa$ under whichever ordering is being used.

Just as in the case for Brandt semigroups we may use an ordering on $\mathcal{P}_n$ to describe unique pairs that correspond to a full inverse subsemigroup.

Definition 3.6.1. Let (κ, Γ) be an inverse semigroup pair for $\mathcal{I}_n$. Then we say (κ, Γ) is special if Γ_A is a subgroup whenever $A = \min A\kappa$.

It is fairly straightforward to show that there is a unique special inverse subsemigroup pair that corresponds to each full inverse subsemigroup. It is simply the "union" of the special inverse subsemigroup pairs for the intersection of the subsemigroup with each principal factor. The proof is left to the reader.

Corollary 3.6.2. *The full inverse subsemigroups of $\mathcal{I}_n$ are precisely $T_{\kappa,\Gamma}$ for (κ, Γ) a special inverse subsemigroup pair.*

The benefit of using special inverse subsemigroup pairs is just the same as for Brandt semigroups. It is far easier to describe the ordering on subsemigroups, which we use to determine whether a full inverse subsemigroup is contained in the normaliser of a congruence on $\mathcal{P}_n$ (which we recall is (ICP1)). The description of the ordering on special inverse subsemigroup pairs is as follows. If (κ, Γ) and (δ, Δ) are special inverse subsemigroup pairs, then $T_{\kappa,\Gamma} \subseteq T_{\delta,\Delta}$ if and only if $\kappa \subseteq \delta$ and for each $A \in \mathcal{P}_n$

$$\Gamma_A \subseteq \Delta_A \Delta^{-1}_{\min A\kappa}.$$

The proof for this is very similar to the proof of Theorem 3.4.12, the analogous result for the ordering of special inverse subsemigroup pairs for Brandt semigroups.

We now turn our attention to the lattice $\mathfrak{C}(\mathcal{P}_n)$. In the previous description of $\mathfrak{LC}(\mathcal{I}_n)$ we also promised to elaborate on how one might minimally specify a congruence on $\mathcal{P}_n$. As $\mathfrak{C}(S)$ is a complete lattice for any semigroup, it follows that for each convex subsemilattice $\mathcal{B} \subseteq \mathcal{P}_n$ there is a minimum congruence on $\mathcal{P}_n$ for which $\mathcal{B}$ is a congruence class. The *congruence generated by* $\mathcal{B}$ is the smallest congruence for which $\mathcal{B}$ is a congruence class and the *congruence partition generated by* $\mathcal{B}$ is the congruence partition for this minimum congruence (we recall that a congruence partition is a partition corresponding to a congruence). We shall describe congruences generated by a convex subsemilattice and provide a mechanism by which congruences generated by subsemilattices may be used as building blocks for $\mathfrak{C}(\mathcal{P}_n)$.

One observation we ought to make is that a principal congruence is certainly a congruence generated by a convex subsemilattice. If $\tau = \langle (A, B) \rangle$ then τ is the congruence partition generated by

$$\mathcal{B} = \{C \in \mathcal{P}_n \mid A \cap B \subseteq C \subseteq A \text{ or } A \cap B \subseteq C \subseteq B\}.$$

We note that in this section we use set notation to refer to elements of $\mathcal{P}_n$, using $\cap$ for the multiplication. Conversely, a congruence generated by a convex subsemilattice need not be principal.

We shall now give an explicit formulation for the congruence generated by a convex subsemilattice $\mathcal{B} \subseteq \mathcal{P}_n$. Let $X = \bigcap_{B\in\mathcal{B}} B$ be the minimum element in $\mathcal{B}$. For $Z \subseteq X$ define the set

$$\mathcal{B}_Z = \{Z \cup (B\backslash X) \mid B \in \mathcal{B}\}.$$

Then we define

$$\tilde{\mathcal{B}} = \{\mathcal{B}_Z \mid Z \subseteq X\} \cup \{\{C\} \mid C \not\subseteq B \text{ for any } B \in \mathcal{B}\}.$$

As $\mathcal{B}$ is a convex semilattice it follows that, for each $Z \subseteq X$, the set $\mathcal{B}_Z$ is a convex subsemilattice. An example of such a partition (which we shall prove this is in just a moment) is given in Fig. 3.11. We will show that $\tilde{\mathcal{B}}$ is a congruence partition of $\mathcal{P}_n$, and is in fact the congruence partition generated by $\mathcal{B}$.

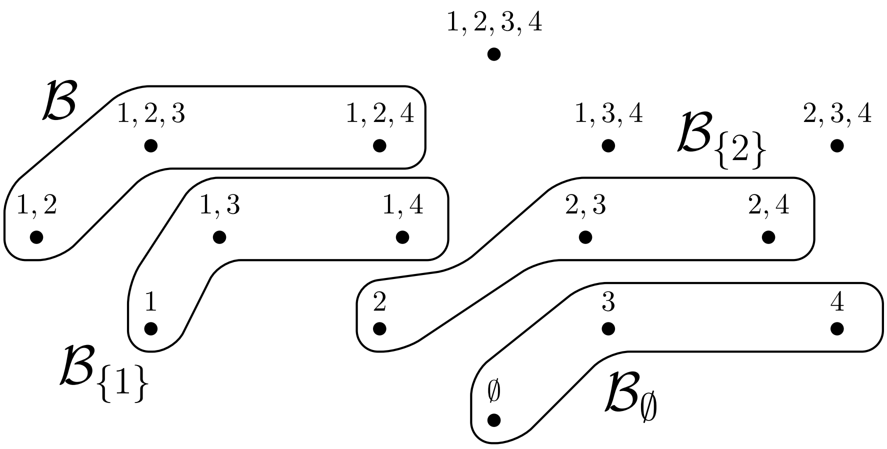


Figure 3.11: $\tilde{\mathcal{B}}$ for $\mathcal{B} = \{\{1,2\},\{1,2,3\},\{1,2,4\}\}$

Proposition 3.6.3. *Let $\mathcal{B} \subseteq \mathcal{P}_n$ be a non-singleton convex semilattice and let $\tilde{\mathcal{B}}$ be defined as above. Then $\tilde{\mathcal{B}}$ is the congruence partition generated by $\mathcal{B}$.*

Proof. This proof is in three parts, first we show that $\tilde{\mathcal{B}}$ is a partition of $\mathcal{P}_n$, then we show it is a congruence partition. Finally we prove that it is generated by $\mathcal{B}$. Let X be the minimum element of $\mathcal{B}$ and note that $\mathcal{B}_X = \mathcal{B}$.

It is clear from the definition of $\tilde{\mathcal{B}}$ that every element of $\mathcal{P}_n$ is an element in some element of $\tilde{\mathcal{B}}$. To show that $\tilde{\mathcal{B}}$ is a partition it remains to show that for $\mathcal{C}, \mathcal{D} \in \tilde{\mathcal{B}}$ either $\mathcal{C} \cap \mathcal{D} = \emptyset$ or $\mathcal{C} = \mathcal{D}$. The only thing to check is that if $Y, Z \subseteq X$ then $\mathcal{B}_Y \cap \mathcal{B}_Z \neq \emptyset$ then $Y = Z$. If $C \in \mathcal{B}_Y \cap \mathcal{B}_Z$ then there are $C_Y, C_Z \in \mathcal{B}$ such that $C = Y \cup (C_Y \backslash X) = Z \cup (C_Z \backslash X)$. For each $B \in \mathcal{B}$, $(B \backslash X) \cap X = \emptyset$. Therefore

$$(Y \cup (C_Y \backslash X)) \cap X = (Y \cap X) \cup ((C_Y \backslash X) \cap X) = Y \cap X = Y$$

and similarly $(Z \cup (C_Z \backslash X)) \cap X = Z$. It follows that $Y = Z$.

We now show that $\tilde{\mathcal{B}}$ defines a congruence on $\mathcal{P}_n$. As the only non-singleton classes in $\tilde{\mathcal{B}}$ are the $\mathcal{B}_Z$ it is sufficient to show that if $A, C \in \mathcal{B}_Z$ for $Z \subseteq X$ and $D \in \mathcal{P}_n$ then there is $Y \subseteq X$ such that $A \cap D,\ C \cap D \in \mathcal{B}_Y$. Write $\overline{A} = A \backslash Z$ and $\overline{C} = C \backslash Z$ so that $X \cup \overline{A},\ X \cup \overline{C} \in \mathcal{B}$. Then $A \cap D = (Z \cap D) \cup (\overline{A} \cap D)$ and $C \cap D = (Z \cap D) \cup (\overline{C} \cap D)$. As $\mathcal{B}$ is convex, we know that $X \cup (\overline{A} \cap D),\ X \cup (\overline{C} \cap D) \in \mathcal{B}$. It follows that $A \cap D, C \cap D \in \mathcal{B}_{Z \cap D}$. So we have that $\tilde{\mathcal{B}}$ is a congruence partition.

Finally we show that $\tilde{\mathcal{B}}$ is generated by $\mathcal{B}$. Suppose that $A, C \in \mathcal{B}_Z$, so that, with $\overline{A} = A \backslash Z$ and $\overline{C} = C \backslash Z$ we have $X \cup \overline{A},\ X \cup \overline{C} \in \mathcal{B}$. Let

$$Y = \bigcup_{y \in B \in \mathcal{B},\ y \notin X} y$$

so Y is the set of elements of $[n]$ that appear in elements of $\mathcal{B}$ but are not in X. Then observe that

$$\begin{aligned}(X \cup \overline{A}) \cap (Z \cup Y) &= (X \cap Z) \cup (\overline{A} \cap Z) \cup (X \cap Y) \cup (\overline{A} \cap Y) \\ &= Z \cup \emptyset \cup \emptyset \cup \overline{A} = Z \cup \overline{A} = A.\end{aligned}$$

Similarly $(X \cup \overline{C}) \cap (Z \cup Y) = Z \cup \overline{C} = C$. This implies that A, C must be in the same part of any congruence partition containing $\mathcal{B}$, and it follows that $\tilde{\mathcal{B}}$ is the minimum congruence partition containing $\mathcal{B}$. □

We have said that these congruences generated by convex subsemilattices shall be our building blocks as we look at the set of all congruences on $\mathcal{P}_n$. We must show that all congruences can be written as a "combination" of these building blocks. We have noted that principal congruences are congruences generated by convex subsemilattices, so, as all congruences can be written as the join of principal congruences, the set of congruences generated by convex subsemilattices generates $\mathfrak{C}(E)$ under the usual join of congruences. In this next segment we shall see that given $\tau \in \mathfrak{C}(E)$ we may choose "unique" convex subsemilattices which combine to give τ.

We shall temporarily ignore any distinction between partitions and congruences and shall write $\tilde{\mathcal{B}}$ for the congruence defined by the partition $\tilde{\mathcal{B}}$. We observe that if $\mathcal{B} = \{X \cup B_i \mid i \in I\}$ (with each B_i distinct and $B_i \cap X = \emptyset$) then

$$\tilde{\mathcal{B}} = \iota \cup \{(Z \cup B_i, Z \cup B_j) \mid Z \subseteq X,\ i, j \in I\}.$$

We remark that viewed as congruences it makes sense to define unions of $\tilde{\mathcal{B}}$.

Definition 3.6.4. We define a partial ordering on convex subsemilattices of $\mathcal{P}_n$. Let $\mathcal{B}$, $\mathcal{C}$ be convex subsemilattices then $\mathcal{C} \lesssim \mathcal{B}$ if $\tilde{\mathcal{C}} \subseteq \tilde{\mathcal{B}}$ as congruences.

We leave it to the reader to check that $\lesssim$ is a partial order. We notice that if $\mathcal{C} \subseteq \mathcal{B}$ as subsets of $\mathcal{P}_n$ then certainly $\mathcal{C} \lesssim \mathcal{B}$. We can give an explicit description of exactly when $\mathcal{C} \lesssim \mathcal{B}$. Let X_B be the minimum element of $\mathcal{B}$ and for $Z \subseteq X_B$ define $\mathcal{B}_Z$ as usual. Then $\mathcal{C} \lesssim \mathcal{B}$ if and only if $\mathcal{C} \subseteq \mathcal{B}_Z$ for some $Z \subseteq X_B$. In fact - with X_C the minimum element of $\mathcal{C}$ - we must have $\mathcal{C} \subseteq \mathcal{B}_{X_B \cap X_C}$. Equivalently, if $\{B_1 \ldots, B_r\}$ and $\{C_1, \ldots, C_m\}$ are the sets such that $B_i \cap X_B = \emptyset = C_j \cap X_C$ for each $1 \leq i \leq r$ and $1 \leq j \leq m$, and

$$\mathcal{B} = \{X_B \cup B_1, \ldots, X_B \cup B_r\}, \qquad \mathcal{C} = \{X_C \cup C_1, \ldots, X_C \cup C_m\}.$$

Then $\mathcal{C} \lesssim \mathcal{B}$ if and only if both

(i) there exists $1 \leq i \leq r$ such that $X_C = (X_B \cap X_C) \cup B_i$, and

(ii) $\{C_1, \ldots, C_m\} \subseteq \{B_1, \ldots, B_r\}$.

We now explain how, given a congruence τ we get a hold of an appropriate set of subsemilattices.

(1) Start by taking the set $\mathcal{B}_1, \dots \mathcal{B}_m$ of all the congruence classes of τ.

(2) Consider each $\mathcal{B}_i$ in turn, if there is $\mathcal{B}_j$ (for some $j \neq i$ with $\mathcal{B}_j$ left in the list) such that $\tilde{\mathcal{B}}_i \lesssim \tilde{\mathcal{B}}_j$ then remove $\mathcal{B}_i$ from the list.

We shall see that it doesn't matter in what order you consider the convex semilattices, the final list is the same. The technical detail follows but is not overly illuminating. It is easier to think in terms of examples, such as that shown in Fig. 3.12, in which the initial set of convex subsemilattices is $\{\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}\}$ and the only convex subsemilattice removed is $\mathcal{C}$ because $\mathcal{C} = \mathcal{B}_{\{1\}}$, so $\mathcal{C} \lesssim \mathcal{B}$. Note that $\mathcal{D}$ is not removed even though the relations in $\mathcal{D}$ are implied by the combination of the relations in $\mathcal{B}$ and $\mathcal{A}$.

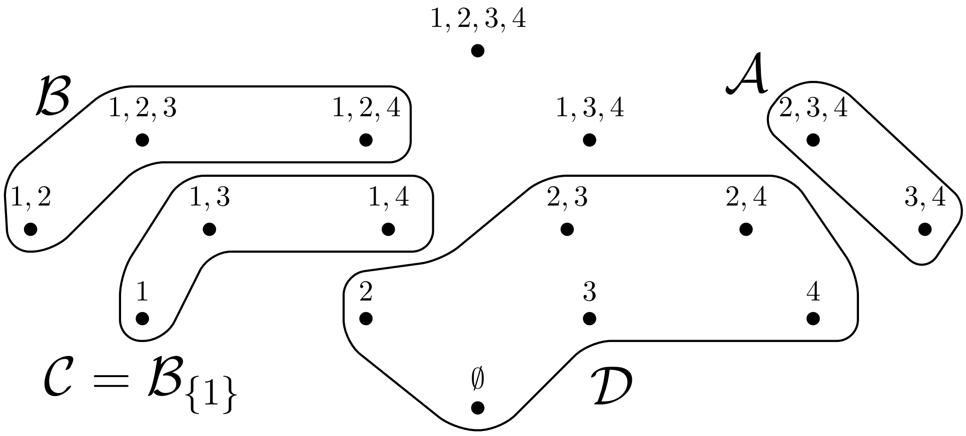


Figure 3.12: Compatible convex subsemilattices

Definition 3.6.5. Let $\{\mathcal{B}_i \subseteq \mathcal{P}_n \mid i \in I\}$ be convex subsemilattices. We say that $\{\mathcal{B}_i \mid i \in I\}$ are *compatible* if the following hold

(i) $\{\mathcal{B}_i \mid i \in I\}$ is an antichain with respect to $\lesssim$;

(ii) $\bigcup_{i \in I} \tilde{\mathcal{B}}_i$ is a congruence;

(iii) the congruence classes of $\bigcup_{i \in I} \tilde{\mathcal{B}}_i$ are of the form $(\mathcal{B}_i)_Z$ for some $i \in I$ (in particular, each $\mathcal{B}_i$ is a congruence class).

The following lemma is slightly informal, and tells us that the process described above produces compatible convex subsemilattices.

Lemma 3.6.6. *If τ is a congruence on $\mathcal{P}_n$ and we produce $\{\mathcal{B}_i \mid i \in I\}$ via the algorithm above then $\{\mathcal{B}_i \mid i \in I\}$ are compatible. In particular, every congruence τ on $\mathcal{P}_n$ is the union of congruences generated by compatible convex subsemilattices.*

Proof. It is immediate that $\{\mathcal{B}_i \mid i \in I\}$ is an antichain, as during the construction we remove the smaller of any comparable elements. Also, if the initial list is $\{\mathcal{B}_1, \ldots, \mathcal{B}_m\}$ (so we assume $I \subseteq [m]$) then, as the $\mathcal{B}_j$ are the congruence classes of τ, we see that $\tau = \bigcup_{1 \leq j \leq m} \tilde{\mathcal{B}}_i$. On the other hand, we claim that each $\mathcal{B}_j$ that we remove has $\tilde{\mathcal{B}}_j \subseteq \tilde{\mathcal{B}}_i$ for some $i \in I$. Indeed, if we remove some $\mathcal{B}_j$ because $\mathcal{B}_j \lesssim \mathcal{B}_k$ and later remove $\mathcal{B}_k$ because $\mathcal{B}_k \lesssim \mathcal{B}_l$ then as $\lesssim$ is a partial order we have that $\mathcal{B}_j \lesssim \mathcal{B}_l$. As we make finitely many removals the claim follows. To see that the union is a congruence we note that, as each $\mathcal{B}_j$ which we remove has $\tilde{\mathcal{B}}_j \subseteq \tilde{\mathcal{B}}_i$ for some $i \in I$,

$$\bigcup_{i \in I.} \tilde{\mathcal{B}}_i = \bigcup_{1 \leq j \leq m} \tilde{\mathcal{B}}_j = \tau$$

To complete the proof we note that by definition each τ-class is equal to $\mathcal{B}_j$ for some $1 \leq j \leq m$ and if we remove $\mathcal{B}_j$ from the list then $\mathcal{B}_j \subseteq (\mathcal{B}_l)_Z$ for some $l \in I$. However, as the union $\bigcup_{i \in I} \tilde{B}_i$ is a congruence, we have that $(\mathcal{B}_l)_Z$ is contained in some τ-class. It follows that $(\mathcal{B}_l)_Z = \mathcal{B}_j$. Therefore each τ-class is of the required form. ☐

We now show that compatible convex subsemilattices uniquely determine congruences on $\mathcal{P}_n$.

Theorem 3.6.7. *Let τ be a congruence on $\mathcal{P}_n$ then there convex subsemilattices $\{\mathcal{B}_i \mid i \in I\}$ which are compatible, and*

$$\tau = \bigcup_{i \in I} \tilde{\mathcal{B}}_i.$$

Moreover, these $\mathcal{B}_i$ are unique subject to these restrictions.

Proof. We have just seen, in Lemma 3.6.6, that every congruence is the union of a set of congruences generated by compatible convex subsemilattices. It remains to prove uniqueness. Suppose that $\{\mathcal{B}_i \mid i \in I\}$ and $\{\mathcal{C}_j \mid j \in J\}$ are sets of compatible convex subsemilattices such that

$$\bigcup_{i \in I} \tilde{\mathcal{B}}_i = \tau = \bigcup_{j \in J} \tilde{\mathcal{C}}_j.$$

We claim that for $i \in I$ there is $j \in J$ such that $\mathcal{B}_i = \mathcal{C}_j$. Combining this with the dual argument will imply that $\{\mathcal{B}_i \mid i \in I\} = \{\mathcal{C}_j \mid j \in J\}$, which will complete the proof, so it suffices to prove our claim.

Take $i \in I$. By definition, $\mathcal{B}_i$ is a τ-class. On the other hand each τ-class is of the form $(\mathcal{C}_j)_Z$ for some $j \in J$ and some $Z \in \mathcal{P}_n$. Therefore $\mathcal{B}_i = (\mathcal{C}_j)_Z$ and so $\mathcal{B}_i \lesssim \mathcal{C}_j$. Conversely, applying the same argument to $\mathcal{C}_j$ there is some $k \in I$ and $Y \in \mathcal{P}_n$ with $\mathcal{C}_j = (\mathcal{B}_k)_Y$, so $\mathcal{C}_j \lesssim \mathcal{B}_k$. As $\lesssim$ is a partial order this implies that $\mathcal{B}_i \lesssim \mathcal{B}_k$ and as $\{\mathcal{B}_i \mid i \in I\}$ is an antichain we have $\mathcal{B}_i = \mathcal{B}_k$. Whence also, $\mathcal{B}_i = \mathcal{C}_j$. Hence we have completed the proof. $\square$

At this point we conclude our discussion of how to efficiently describe congruences on $\mathcal{P}_n$.

To use what we have talked about in this section to describe inverse congruence pairs the route is very similar to using (non-special) inverse subsemigroup pairs and congruences defined as congruences not in terms of convex subsemilattices. It is an exercise in definition chasing to write down the normaliser of a congruence in terms of the set of compatible convex subsemilattices which define the congruence. We do not do that here. It is then also an exercise in definition chasing to write down the normaliser of a congruence defined by a set of compatible convex subsemilattices in the terms of a special inverse subsemigroup pair. Then (brace yourself) it is an even longer exercise in definition chasing to write down which other special inverse subsemigroup pairs correspond to full inverse subsemigroups contained in the normaliser of a congruence defined by a set of compatible convex subsemilattices - actually that one is easy as the benefit to using special inverse subsemigroup pairs is that it is easier to describe the ordering of subsemigroups. It is a hair-pulling, eye-gouging and gut-wrenching definition chasing exercise to write down what it means for a special inverse subsemigroup pair which corresponds to a full inverse subsemigroup which is contained in the normaliser of a congruence which is in turn defined by a set of compatible convex subsemilattices to satisfy (ICP2) from the definition of inverse congruence pair. It is probably only slightly more trying and infuriating than trying to parse the previous sentence. This chapter is already far too long and technical, and none of the above definition chasing exercises is particularly enlightening. Thus we finish this chapter on examples of the use of the inverse kernel approach at this point.

— 4 —

Applications of the inverse kernel approach

In this chapter we focus on applying the inverse kernel approach for left congruences on inverse semigroups to relate properties of the lattice of left congruences with properties of the inverse semigroup. The majority of our discussion in this chapter may be found elsewhere in the literature however our methods provide new proof mechanisms and we shall often use only simple and elementary steps to prove otherwise complicated results. Through this chapter, unless otherwise stated, we assume that S is an inverse semigroup and that E is the semilattice of idempotents.

We shall assume familiarity with the inverse kernel approach on many occasions throughout this Chapter. We recall the main definition (Definition 2.1.8) and result (Theorem 2.1.11). Let τ be a congruence on E, and let $T \subseteq S$ be a full inverse subsemigroup. We say that (τ, T) is an *inverse congruence pair* for S if (τ, T) satisfies the following conditions:

(ICP1) $T \subseteq N(\tau)$;

(ICP2) for $x \in S$, if there exist $e, f \in E$ such that $x^{-1}x \;\tau\; e,\; xx^{-1} \;\tau\; f$ and $xe, fx \in T$, then we have $x \in T$.

For an inverse congruence pair (τ, T), define the relation

$$\rho(\tau, T) = \{(x, y) \mid x^{-1}y \in T,\; x^{-1}yy^{-1}x \;\tau\; x^{-1}x,\; y^{-1}xx^{-1}y \;\tau\; y^{-1}y\}.$$

This is slightly different notation than in Chapter 2 in order to reduce subscript splurge. Then left congruences on S are exactly $\rho(\tau, T)$ where (τ, T) is an inverse congruence pair. Also, if ρ is a left congruence on S then the associated inverse congruence pair is $(\mathrm{trace}(\rho), \mathrm{Inker}(\rho))$, where $\mathrm{Inker}(\rho) = \{a \in S \mid a \;\rho\; aa^{-1}\}$.

4.1 Two sided Congruences

Our first foray was previously promised, we discuss the lattice of two sided congruences, which we regard as a subset of the lattice of left congruences and shall show that we may deduce the two sided kernel trace description of congruences from the inverse kernel approach to left congruences. Given a family of two sided congruences we recall that their join as congruences is equal to their join as equivalence relations, which is also equal to their join as left congruences. Hence $\mathfrak{C}(S)$ is a sublattice of $\mathfrak{LC}(S)$. When ρ is a two sided congruence the kernel is an inverse subsemigroup so we have that

$$\ker(\rho) = \ker(\rho \cap \mathcal{R}) = \mathrm{Inker}(\rho).$$

The kernel trace approach to two sided congruences is discussed at some length in Chapter 1, for reference we recall one of the possible definitions for a congruence pair, Definition 1.3.10. A pair (τ, T), where $T \subseteq S$ is a self conjugate full inverse subsemigroup and τ is a congruence on E with $N(\tau) = S$, is a *congruence pair* if it satisfies:

(CP1) for $x \in S$ and $e \in E$, if $xe \in T$ and $e \ \tau \ x^{-1}x$ then $x \in T$;

(CP3) for each $x \in T$ we have $xx^{-1} \ \tau \ x^{-1}x$.

The relevant result is Theorem 1.3.12, that congruence pairs are in bijection with congruences. If (τ, T) is a congruence pair for S then the associated congruence is

$$P_{(\tau,T)} = \{(a, b) \mid a^{-1}a \ \tau \ b^{-1}b, \ ab^{-1} \in T\}$$

and if ρ is a congruence on S then the associated congruence pair is $(\mathrm{trace}(\rho), \ker(\rho))$. We will show that this follows in straightforward fashion from the description of one sided congruences in terms of the inverse kernel and trace.

We have seen that inverse congruence pairs give rise to both left and right congruences and the corresponding congruences are related via the usual isomorphism between the lattices of left and right congruences (Proposition 1.4.5) which has the form

$$\rho \mapsto \rho_{-1} = \{(a^{-1}, b^{-1}) \mid (a, b) \in \rho\}.$$

We recall Corollary 2.2.1 which states that if (τ, T) is an inverse congruence pair then it defines both a left and a right congruence and if ρ is the left congruence then ρ_{-1} is the right congruence. In this section we use subscripts L and R to differentiate between left and right congruences. For an inverse congruence pair (τ, T) the corresponding left and right congruences are, respectively,

$$\rho_L(\tau, T) = \{(a, b) \mid a^{-1}b \in T,\ a^{-1}bb^{-1}a \ \tau \ a^{-1}a,\ b^{-1}aa^{-1}b \ \tau \ b^{-1}b\},$$

and

$$\rho_R(\tau, T) = \{(a, b) \mid ab^{-1} \in T,\ ab^{-1}ba^{-1} \ \tau \ aa^{-1},\ ba^{-1}ab^{-1} \ \tau \ bb^{-1}\}.$$

We recall that the definition which we use for the inverse kernel of a left congruence ρ_L is

$$\operatorname{Inker}(\rho_L) = \{a \in S \mid a \ \rho_L \ aa^{-1}\}$$

and the inverse kernel for a right congruence ρ_R is

$$\operatorname{Inker}(\rho_R) = \{a \in S \mid a \ \rho_R \ a^{-1}a\}.$$

The crux of the deduction of the kernel trace description for two sided congruences from the inverse kernel approach to one sided congruences is the observation that a left congruence $\rho_L(\tau, T)$ is a two sided congruence if and only if $\rho_L(\tau, T) = \rho_R(\tau, T)$.

Lemma 4.1.1. *Let (τ, T) be an inverse congruence pair. Then $\rho_L(\tau, T)$ is a two sided congruence if and only if $\rho_L(\tau, T) = \rho_R(\tau, T)$.*

Proof. Write ρ_L and ρ_R for $\rho_L(\tau, T)$ and $\rho_R(\tau, T)$. If $\rho_L = \rho_R$ then certainly ρ_L is a two sided congruence. For the converse we suppose that ρ_L is a two sided congruence. Then $(a, b) \in \rho_L$ if and only if $(a^{-1}, b^{-1}) \in \rho_L$. However, this exactly says that $\rho_L = \rho_R$. $\square$

The remainder of the proof that congruence pairs determine congruences is an elementary verification exercise in which we check that $\rho_L(\tau, T) = \rho_R(\tau, T)$ if and only if (τ, T) is a congruence pair. We include a proof for completeness.

Proposition 4.1.2. *Let (τ, T) be an inverse congruence pair and let ρ_L, ρ_R be the corresponding left and right congruences. Then (τ, T) is a congruence pair if and only if $\rho_L = \rho_R$.*

Proof. Initially we assume that $\rho_L = \rho_R = \rho$, so we need to show that (τ, T) is a congruence pair. We shall do this by repeatedly bashing the definition of inverse congruence pair against ρ until the all the correct bits fall off and we are left with a congruence pair. Since (τ, T) is an inverse congruence pair we have that T is a full inverse subsemigroup, and, as ρ is two sided, we have that $\mathrm{Inker}(\rho) = T = \ker(\rho)$.

We first establish that T is self conjugate. Suppose that $b \in T$ so $b \; \rho \; bb^{-1}$. As ρ is a two sided congruence we obtain $aba^{-1} \; \rho \; abb^{-1}a^{-1}$. As $abb^{-1}a^{-1} \in E$, we have $aba^{-1} \in \ker(\rho) = T$, so T is self conjugate.

Next we show that $N(\tau) = S$. Suppose that $e \; \tau \; f$ and $a \in S$. As ρ is a two sided congruence it is immediate that $aea^{-1}\rho \; afa^{-1}$, so $aea^{-1} \; \tau \; afa^{-1}$. Similarly $a^{-1}ea \; \tau \; a^{-1}fa$, and thus $a \in N(\tau)$.

We now establish (CP1). Suppose that $ae \in T$, and $e \; \tau \; a^{-1}a$; we need that $a \in T$. As $N(\tau) = S$ we have that $a \in N(\tau)$ so we conjugate $e \; \tau \; a^{-1}a$ by a to obtain $aea^{-1} \; \tau \; aa^{-1}$, and we note that $ae = (aea^{-1})a$. Then by applying (ICP2) (from the definition of inverse congruence pair) we have that $a \in T$.

To establish (CP3) we apply the left and right definitions of the inverse kernel to obtain that

$$\mathrm{Inker}(\rho_R) = \{a \mid a \; \rho \; a^{-1}a\} = T = \{a \mid a \; \rho \; aa^{-1}\} = \mathrm{Inker}(\rho_L).$$

Therefore, if $a \in T$ then $a^{-1}a \; \rho \; a \; \rho \; aa^{-1}$. Thus we have that (τ, T) is a congruence pair.

For the converse we suppose that (τ, T) is a congruence pair. We first note that this implies that (τ, T) is an inverse congruence pair, as $T \subseteq S = N(\tau)$, and (CP1) is a strengthening of (ICP2).

We show that $\rho_L \subseteq \rho_R$. Suppose $a \; \rho_L \; b$, so $a^{-1}b \in T$ and $a^{-1}bb^{-1}a \; \tau \; a^{-1}a$, $b^{-1}aa^{-1}b \; \tau \; b^{-1}b$. As $a^{-1}b \in T$, by (CP3) we have that $b^{-1}aa^{-1}b \; \tau \; a^{-1}bb^{-1}a$.

Then we have that

$$a^{-1}a \ \tau \ a^{-1}bb^{-1}a \ \tau \ b^{-1}aa^{-1}b \ \tau \ b^{-1}b.$$

Since $N(\tau) = S$, we conjugate the relation $a^{-1}a \ \tau \ b^{-1}b$ by a and b to obtain

$$aa^{-1} \ \tau \ ab^{-1}ba^{-1} \quad \text{and} \quad bb^{-1} \ \tau \ ba^{-1}ab^{-1}.$$

We also conjugate $a^{-1}a \ \tau \ a^{-1}bb^{-1}a$ by a, and we conjugate $b^{-1}b \ \tau \ b^{-1}aa^{-1}b$ by b from which we see

$$aa^{-1} \ \tau \ aa^{-1}bb^{-1} \ \tau \ bb^{-1}.$$

Since $b^{-1}a \in T$, and T is self conjugate, we have $ab^{-1}aa^{-1} \in T$. Also

$$aa^{-1} \ \tau \ bb^{-1} \ \tau \ ba^{-1}ab^{-1} = (ab^{-1})^{-1}(ab^{-1}),$$

so (CP1) with $x = ab^{-1}$ and $e = aa^{-1}$ gives that $ab^{-1} \in T$. Whence we shown that $a \ \rho_R \ b$. The dual argument gives that $\rho_R \subseteq \rho_L$, hence the two are equal. □

To complete a proof of Theorem 1.3.12 (the kernel trace description for two sided congruences) it then suffices to show that when (τ, T) is a congruence pair the left congruence

$$\rho_L(\tau, T) = \{(a, b) \mid a^{-1}b \in T, \ a^{-1}bb^{-1}a \ \tau \ a^{-1}a, \ b^{-1}aa^{-1}b \ \tau \ b^{-1}b\}$$

reduces to the form for $P_{(\tau,T)}$. We note that in the proof of Proposition 4.1.2 we saw that when (τ, T) is a congruence pair and $a \ \rho_L \ b$ we have that $a^{-1}a \ \tau \ b^{-1}b$. Thus if (τ, T) is a congruence pair then

$$\rho_L(\tau, T) \subseteq \{(a, b) \mid a^{-1}a \ \tau \ b^{-1}b, \ ab^{-1} \in T\} = P_{(\tau,T)}.$$

The reverse inclusion is also straightforward, suppose that $a^{-1}a \ \tau \ b^{-1}b$ and $ab^{-1} \in T$. Then conjugating $a^{-1}a \ \tau \ b^{-1}b$ by a and b gives $aa^{-1} \ \tau \ ab^{-1}ba^{-1}$ and $bb^{-1} \ \tau \ ba^{-1}ab^{-1}$. Hence $P_{(\tau,T)} \subseteq \rho_R(\tau, T)$. Since (τ, T) being a congruence pair implies that $\rho_L(\tau, T) = \rho_R(\tau, T)$ we have that in this case $\rho_L(\tau, T) = P_{(\tau,T)}$.

4.2 Finitely generated left congruences

In this section and the one following we shift our focus and consider left congruences as defined by a generating set. With this in mind we make a slight change of notation. For Sections 4.2 and 4.3 we use subscript notation to define in which lattice we are working, so, for instance, we write $\langle R\rangle_{\mathrm{LC}(S)}$ and $\langle R\rangle_{\mathrm{C}(E)}$ for, respectively, the left congruence generated on S and the congruence generated on E by the binary relation R (usually we shall refer to congruences on E and left congruences on the whole semigroup). This change enables us to write $\langle Z\rangle_{\mathrm{IS}}$ for the inverse subsemigroup generated by a set $Z \subseteq S$.

Several properties of semigroups are related to (or defined in terms of) which one sided congruences are finitely generated, for example whether a semigroup is left Noetherian [38], or is left coherent [22]. We will see that for inverse semigroups finite generation of left congruences is closely tied to finite generation of the trace and the inverse kernel. We write $\mathfrak{LC}_{\mathrm{FG}}(S)$ and $\mathfrak{C}_{\mathrm{FG}}(S)$ for the lattices of finitely generated left congruences and congruences, respectively, on S.

We recall that we often assume that the generating set R for a (left) congruence is symmetric, by which we mean that $(a,b) \in R$ whenever $(b,a) \in R$, and this assumption does not affect whether R is finite. Similarly, without affecting finiteness, we commonly assume that the generating set Z for an inverse subsemigroup is closed under taking inverses, in other words that $a^{-1} \in Z$ whenever $a \in Z$. Initially we prove a technical lemma regarding generating sets for one sided congruences, which is the basis for much of the discussion in this section.

Lemma 4.2.1. *Let $H \subseteq S \times S$ be a binary relation, and let $\rho = \langle H\rangle_{\mathrm{LC}(S)}$. Then there exists a (symmetric) binary relation $H' \subseteq S \times S$ such that $\rho = \langle H'\rangle_{\mathrm{LC}(S)}$ and*

$$H' \subseteq (E \times E) \cup \{(e,a) \mid a \mathrel{\mathcal{R}} e\} \cup \{(a,e) \mid a \mathrel{\mathcal{R}} e\}.$$

Moreover, if H is finite then H' is also finite.

Proof. Suppose that $(a,b) \in H$, so certainly $a \; \rho \; b$. Then, as we have seen previously, $a^{-1}a \; \rho \; a^{-1}b$, $b^{-1}a \; \rho \; b^{-1}b$ and $a^{-1}b = a^{-1}bb^{-1}b \; \rho \; a^{-1}bb^{-1}a$. Consequently, $a^{-1}a \; \rho \; a^{-1}bb^{-1}a$. Dually we obtain that $b^{-1}aa^{-1}b \; \rho \; b^{-1}b$.

Conversely, we note that the three relations:

$$a^{-1}b \; \rho \; a^{-1}bb^{-1}a, \quad a^{-1}a \; \rho \; a^{-1}bb^{-1}a \quad \text{and} \quad b^{-1}aa^{-1}b \; \rho \; b^{-1}b,$$

together imply that $a \; \rho \; b$. Indeed, from $a^{-1}b \; \rho \; a^{-1}bb^{-1}a$ we have

$$b^{-1}a(a^{-1}b) \; \rho \; b^{-1}a(a^{-1}bb^{-1}a) = b^{-1}a.$$

We then observe that

$$a = a(a^{-1}a) \; \rho \; a(a^{-1}bb^{-1}a) = bb^{-1}a \; \rho \; b(b^{-1}aa^{-1}b) \; \rho \; b(b^{-1}b) = b.$$

Hence we may replace each pair $(a,b) \in H$ with the 3 pairs $(a^{-1}a, a^{-1}bb^{-1}a)$, $(b^{-1}b, b^{-1}aa^{-1}b)$ and $(a^{-1}b, a^{-1}bb^{-1}a)$ which have the required form, and the left congruence generated by this new set is the same as the original. Explicitly

$$\begin{aligned} H' = \{(a^{-1}a, a^{-1}bb^{-1}a) \mid (a,b) \in \rho\} \cup \{(b^{-1}b, b^{-1}aa^{-1}b) \mid (a,b) \in \rho\} \\ \cup \{(a^{-1}b, a^{-1}bb^{-1}a) \mid (a,b) \in \rho\}, \end{aligned}$$

and it is clear that $\langle H' \rangle_{\mathrm{LC}(S)} = \langle H \rangle_{\mathrm{LC}(S)}$. Moreover, if H is finite then $|H'| \leq 3|H|$ so H' is also finite.

To complete the proof it suffices to remark that if required we can "symmetrise" H' by adding in (b,a) for each $(a,b) \in H'$ and the resulting symmetric set is finite if H (and so H') is finite. $\square$

The reason that Lemma 4.2.1 is valuable to us is that it relates a generating set to the inverse kernel approach, via the following corollary. We recall that ν_τ is the minimum left congruence on S with trace τ.

Corollary 4.2.2. *Every finitely generated left congruence ρ on S can be written as the join $\nu_\tau \vee \chi$, where τ is a finitely generated congruence on E and χ is a finitely generated idempotent separating left congruence on S.*

Proof. Let ρ be a left congruence on S generated by a finite set H. By Lemma 4.2.1 we may assume that

$$H \subseteq (E \times E) \cup \{(aa^{-1}, a) \mid a \in S\} \cup \{(a, aa^{-1}) \mid a \in S\}.$$

Let $\tau = \langle H \cap (E \times E)\rangle_{\mathrm{C}(E)}$ and let

$$\chi = \langle H \cap (\{(aa^{-1}, a) \mid a \in S\} \cup \{(a, aa^{-1}) \mid a \in S\})\rangle_{\mathrm{LC}(S)}.$$

Then, as H is finite, τ and χ are finitely generated. Also, χ is idempotent separating as it is contained in $\mathcal{R}$. Further, it is clear that $\rho = \nu_\tau \vee \chi$, so the proof is complete. □

The inverse kernel approach describes left congruences via congruences on E and full inverse subsemigroups of S. Corollary 4.2.2 relates finite generation of left congruences to finite generation of congruences on E and finite generation of idempotent separating congruences. This suggests the question: "How does finite generation of idempotent separating left congruences relate to finite generation of full inverse subsemigroups?" We answer this question presently, but first we explain how the inverse kernel of an idempotent separating left congruence may be used to generate the left congruence.

Lemma 4.2.3. *Let χ be an idempotent separating left congruence and let $T = \mathrm{Inker}(\chi)$. Then $\chi = \langle\{(a, aa^{-1}) \mid a \in T\}\rangle_{\mathrm{LC}(S)}$. Furthermore if $T = \langle X\rangle_{\mathrm{IS}}$ then $\chi = \langle\{(a, aa^{-1}) \mid a \in X\}\rangle_{\mathrm{LC}(S)}$.*

Proof. First let $R = \{(a, aa^{-1}) \mid a \in T\}$; we claim that $\chi = \langle R\rangle_{\mathrm{LC}(S)}$. We initially note that $\langle R\rangle_{\mathrm{LC}(S)}$ is certainly idempotent separating (as, for example, $R \subseteq \mathcal{R}$). Since $T = \mathrm{Inker}(\chi)$, we have that $x \;\chi\; xx^{-1}$ for each $x \in T$, so certainly $R \subseteq \chi$ and thus $\langle R\rangle_{\mathrm{LC}(S)} \subseteq \chi$.

For the reverse inclusion we note $T \subseteq \mathrm{Inker}(\langle R\rangle_{\mathrm{LC}(S)})$. As the lattice of idempotent separating left congruences is isomorphic to the lattice of full inverse subsemigroups (Theorem 1.4.19) and both χ and $\langle R\rangle_{\mathrm{LC}(S)}$ are idempotent separating it follows that $\chi \subseteq \langle R\rangle_{\mathrm{LC}(S)}$. Hence the two are equal.

For the final claim we note that $X \subseteq \mathrm{Inker}(\langle\{(a, aa^{-1}) \mid a \in X\}\rangle_{\mathrm{LC}(S)})$ and as the inverse kernel is a full inverse subsemigroup it follows that $T \subseteq \mathrm{Inker}(\langle\{(a, aa^{-1}) \mid a \in X\}\rangle_{\mathrm{LC}(S)})$. On the other hand $X \subseteq T$ so certainly $\{(a, aa^{-1}) \mid a \in X\} \subseteq \{(a, aa^{-1}) \mid a \in T\}$. Then

$$\begin{aligned} T \subseteq \mathrm{Inker}(\langle\{(a, aa^{-1}) \mid a \in X\}\rangle_{\mathrm{LC}(S)}) &\subseteq \mathrm{Inker}(\langle\{(a, aa^{-1}) \mid a \in T\}\rangle_{\mathrm{LC}(S)}) \\ &= \mathrm{Inker}(\chi) = T. \end{aligned}$$

Again using that idempotent separating congruences are determined by their inverse kernel implies that $\chi = \langle\{(a, aa^{-1}) \mid a \in X\}\rangle_{\mathrm{LC}(S)}$ □

We now answer the question of how finite generation of idempotent separating left congruences is related to generating sets for the inverse kernel.

Definition 4.2.4. A full inverse subsemigroup $T \subseteq S$ is said to be *almost finitely generated* if there exists a finite set $X \subseteq S$ such that $T = \langle X \cup E\rangle_{\mathrm{IS}}$. We write $\mathfrak{V}_{\mathrm{AFG}}(S)$ for the lattice of almost finitely generated full inverse subsemigroups.

The notion of almost finitely generated exactly captures which full inverse subsemigroups are the inverse kernel of a finitely generated idempotent separating congruence.

Proposition 4.2.5. *Let χ be an idempotent separating left congruence on S. Then $T = \mathrm{Inker}(\chi)$ is almost finitely generated if and only if χ is finitely generated.*

Proof. First suppose that T is almost finitely generated, say $T = \langle X \cup E\rangle_{\mathrm{IS}}$, where X is a finite set. Let $R = \{(x, xx^{-1}) \mid x \in X\}$; we claim that $\chi = \langle R\rangle_{\mathrm{LC}(S)}$. By Lemma 4.2.3 we have that

$$\chi = \langle\{(a, aa^{-1}) \mid a \in X \cup E\}\rangle_{\mathrm{LC}(S)}.$$

However it is clear that

$$\langle\{(a, aa^{-1}) \mid a \in X \cup E\}\rangle_{\mathrm{LC}(S)} = \langle\{(a, aa^{-1}) \mid a \in X\}\rangle_{\mathrm{LC}(S)},$$

as, for instance, $(e, e) \in \langle \emptyset \rangle_{\mathrm{LC}(S)}$. Thus we have that $\chi = \langle R \rangle_{\mathrm{LC}(S)}$, so χ is finitely generated.

For the converse we suppose that χ is finitely generated. By Lemma 4.2.1, using that χ is idempotent separating, we can choose a finite generating set Q for χ such that $Q = \{(p, pp^{-1}) \mid p \in P\}$ for some finite set $P \subseteq S$. We claim that $\mathrm{Inker}(\chi) = \langle P \cup E \rangle_{\mathrm{IS}}$. It is immediate that $P \subseteq \mathrm{Inker}(\chi)$, so, as the inverse kernel is a full inverse subsemigroup, we have $\langle P \cup E \rangle_{\mathrm{IS}} \subseteq \mathrm{Inker}(\chi)$. Let ζ be the idempotent separating left congruence with inverse kernel equal to $\langle P \cup E \rangle_{\mathrm{IS}}$. As $\mathfrak{V}(S) \cong \mathfrak{LC}_{IS}(S)$ (Theorem 1.4.19) it suffices to show that $\chi \subseteq \zeta$. We note that $(p, pp^{-1}) \in \zeta$ for each $p \in P$. Thus $Q \subseteq \zeta$, and hence $\langle Q \rangle_{\mathrm{LC}(S)} = \chi \subseteq \zeta$. Thus we have that $\mathrm{Inker}(\chi)$ is almost finitely generated. $\square$

For ease of notation if $Y \subseteq S$ is an inverse subsemigroup then we write χ_Y for the idempotent separating left congruence with inverse kernel $\langle Y \cup E \rangle_{\mathrm{IS}}$. We remark that if Y is an inverse subsemigroup of S then $\langle Y \cup E \rangle_{\mathrm{IS}} = YE \cup E$. Indeed, this follows from the observation that if $a \in S$ and $e \in E$ then $ae = (aea^{-1})a$, and $aea^{-1} \in E$, so $YE = EY$.

We note that $\mathfrak{C}_{\mathrm{FG}}(E)$ and $\mathfrak{V}_{\mathrm{AFG}}(S)$ are $\vee$-subsemilattices of $\mathfrak{C}(E)$ and $\mathfrak{V}(S)$ respectively. In general neither is a $\cap$-subsemilattice, there are principal congruences on countable semilattices that have non finitely generated intersection, and it is possible for finitely generated subgroups of a group to have non finitely generated intersection. The following examples illustrate this occurrence.

Example 4.2.6. We define the *pendulum semilattice* as

$$E = \{e_i \mid i \in \mathbb{N}\} \cup \{f,\ g,\ 0\}$$

with multiplication defined by: 0 acts as zero,

$$e_i e_j = e_{\max\{i,j\}}, \qquad fe_i = e_i f = f, \qquad ge_i = e_i g = g \ \text{ and } \ fg = 0$$

where the indices are ordered in the obvious way. The semilattice is shown in Fig. 4.1.

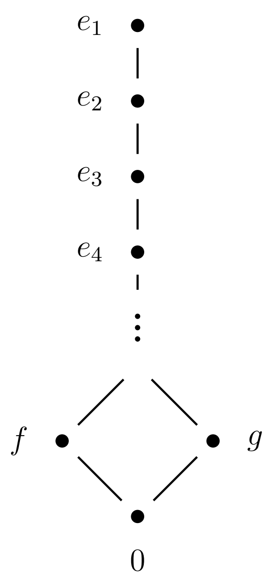


Figure 4.1: The pendulum semilattice

The principal congruences we shall consider are ρ_1 generated by (e_1, f) and ρ_2 generated by (e_1, g). It is straightforward that ρ_1 has partition

$$\{e_i \mid 1 \in \mathbb{N}\} \cup \{f\}, \quad \{g, 0\}.$$

Also the partition defined by ρ_2 is

$$\{e_i \mid 1 \in \mathbb{N}\} \cup \{g\}, \quad \{f, 0\}.$$

It is then clear that $\rho_1 \cap \rho_2$ has only one non trivial part which is

$$\{e_i \mid i \in \mathbb{N}\}.$$

The universal congruence on an infinite descending chain is not finitely generated so $\rho_1 \cap \rho_2$ is not finitely generated. Thus the pendulum semilattice provides an example of when two principal congruences have infinite intersection. The congruences ρ_1, ρ_2 and $\rho_1 \cap \rho_2$ are shown in Fig. 4.2.

Example 4.2.7. For a group that has finitely generated subgroups with non-finitely generated intersection our example is from [50]. We consider the group $G = F_2 \times \mathbb{Z}$, where F_2 is the free group on 2 generators. We note

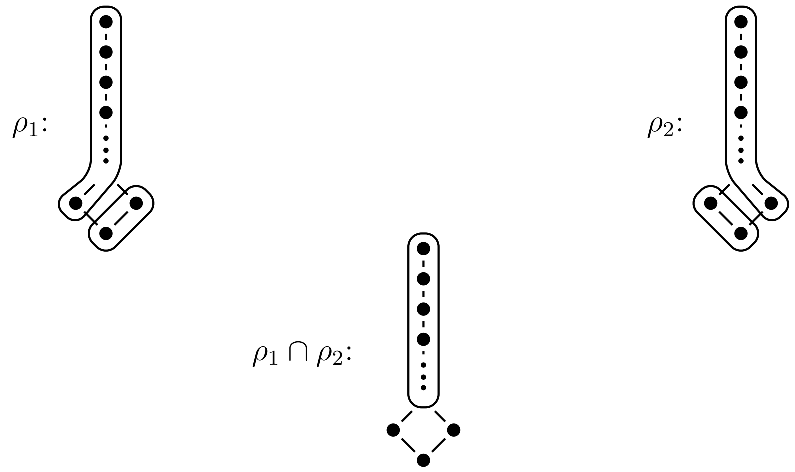


Figure 4.2: The congruences ρ_1, ρ_2 and $\rho_1 \cap \rho_2$ on the pendulum semilattice

that G has presentation $\langle a, b, c \mid ac = ca,\ bc = cb\rangle$. The relevant subgroups are $P = \langle a, bc\rangle$ and $Q = \langle a, b\rangle$. Then $P \cap Q = \langle \{b^i ab^{-i} \mid i \in I\}\rangle$, which is a free group with countably infinitely many generators.

Returning to our consideration of finitely generated left congruences, we recall from Chapter 2 the function

$$\Theta : \mathfrak{C}(E) \times \mathfrak{V}(S) \to \mathfrak{LC}(S);\ (\tau, T) \mapsto \nu_\tau \vee \chi_T$$

where we recall that χ_T is the idempotent separating left congruence with inverse kernel T. Theorem 2.1.15 informs us that $\mathfrak{LC}(S)$ is the image of Θ, and that Θ is a $\vee$-homomorphism. Combining Corollary 4.2.2 and Proposition 4.2.5 gives us the following result describing finitely generated left congruences on an inverse semigroup.

Theorem 4.2.8. *Let ρ be a left congruence on S. Then ρ is finitely generated if and only if there are $T \in \mathfrak{V}_{\mathrm{AFG}}(S)$ and $\tau \in \mathfrak{C}_{\mathrm{FG}}(E)$ such that*

$$\rho = \chi_T \vee \nu_\tau.$$

In particular, $\mathfrak{LC}_{\mathrm{FG}}(S)$ is the image $(\mathfrak{C}_{\mathrm{FG}}(E) \times \mathfrak{V}_{\mathrm{AFG}}(S))\Theta$.

One avenue of interest is to consider when every left congruence is finitely generated. A partial order P is said to have the *ascending chain condition* if every increasing sequence is eventually constant.

Definition 4.2.9. A semigroup S is called *left Noetherian* if every left congruence on S is finitely generated, or equivalently, if the lattice of left congruences has the ascending chain condition.

Left Noetherian inverse semigroups have been classified [38], we reproduce this classification in an elementary fashion. Theorem 4.2.8 implies that S is left Noetherian if and only if every left congruence is the join of a finitely generated trace minimal left congruence and a finitely generated idempotent separating left congruence.

The following is a straightforward observation about the ascending chain condition on partial orders.

Lemma 4.2.10. *Let P, Q be partial orders that have the ascending chain condition, and let $R \subseteq P$ be a suborder. Then R and $P \times Q$ both have the ascending chain condition.*

We can now classify left Noetherian inverse semigroups.

Theorem 4.2.11. *Let S be an inverse semigroup. The lattice $\mathfrak{LC}(S)$ has the ascending chain condition if and only if $\mathfrak{V}(S)$ and $\mathfrak{C}(E)$ have the ascending chain condition.*

Proof. This follows directly from Lemma 4.2.10. We have $\vee$-semilattice embeddings $\mathfrak{V}(S) \hookrightarrow \mathfrak{LC}(S)$ and $\mathfrak{C}(E) \hookrightarrow \mathfrak{LC}(S)$ onto the sets on idempotent separating and trace minimal left congruences respectively. These are certainly embeddings as partial orders, so Lemma 4.2.10 implies that if $\mathfrak{LC}(S)$ has the ascending chain condition then so do both $\mathfrak{V}(S)$ and $\mathfrak{C}(E)$. For the converse, the inverse kernel approach describes $\mathfrak{LC}(S)$ as a subset of $\mathfrak{V}(S) \times \mathfrak{C}(E)$, thus, by Lemma 4.2.10, if $\mathfrak{V}(S)$ and $\mathfrak{C}(E)$ have the ascending chain condition then so does $\mathfrak{LC}(S)$. $\square$

It is easily seen that the lattice $\mathfrak{V}(S)$ has the ascending chain condition if and only if every full inverse subsemigroup is almost finitely generated; the proof for this is almost identical to the group theoretic analogue. We include an outline for completeness.

Lemma 4.2.12. *Let S be an inverse semigroup. Then $\mathfrak{V}(S)$ has the ascending chain condition if and only if every full inverse subsemigroup is almost finitely generated.*

Proof. Suppose first that there is some not almost finitely generated full inverse subsemigroup V. We may suppose that V is generated by $\{a_1, a_2, \ldots\}$ together with E, and $a_i \notin \langle E \cup \{a_1, \ldots, a_{i-1}\}\rangle_{\mathrm{IS}}$ for each i. Then, letting $T_i = \langle E \cup \{a_1, \ldots, a_i\}\rangle_{\mathrm{IS}}$, we have an infinite ascending chain $T_1 \subset T_2 \subset \ldots$, so $\mathfrak{V}(S)$ does not have the ascending chain condition.

Conversely, if all full inverse subsemigroups are almost finitely generated and $T_1 \subseteq T_2 \subseteq \ldots$ is an ascending chain in $\mathfrak{V}(S)$ then we let $V = \bigcup T_i$, which is easily seen to be a full inverse subsemigroup. Then V is almost finitely generated, by $\{a_1, \ldots, a_n\}$ say, so for each $1 \leq j \leq n$ there is some i_j such that $a_j \in T_{i_j}$. Letting $m = \max\{i_j \mid 1 \leq j \leq n\}$ we obtain that $T_m = V$, thus the initial sequence of subsemigroups is eventually constant, so $\mathfrak{V}(S)$ has the ascending chain condition. $\square$

For the lattice $\mathfrak{C}(E)$ we have the following result from [38]. We include the outline of a proof using Lemma 3.1.2, which we recall states that if E is a semilattice and $F \subseteq E$ is a subsemilattice then the convex closure of F is a congruence class of $\langle F \times F\rangle_{\mathrm{C}(E)}$.

Lemma 4.2.13 ([38, Proposition 3.4])**.** *If E is a semilattice then $\mathfrak{C}(E)$ has the ascending chain condition if and only if E is finite.*

Proof. We show that if $\mathfrak{C}(E)$ contains no infinite ascending chains then E cannot contain either an infinite chain or an infinite antichain from which it follows that E is finite.

First suppose that there is an infinite descending (or ascending) chain C which we write $e_1, e_2, \ldots$ (with the appropriate ordering). We define $\tau_i = \langle (e_1, e_i)\rangle_{\mathrm{C}(E)}$. Then, by Lemma 3.1.2, the set $\{f \in E \mid e_i \leq f \leq e_1\}$ (or the set with $e_1 \leq f \leq e_i$ if C is ascending) is a congruence class of τ_i. It follows that $\tau_i \subsetneq \tau_{i+1}$, and so we have an infinite ascending chain of congruences on E.

Suppose there is an infinite antichain A for which we write $\{f_1, f_2, \dots\}$. Define $\tau_i = \langle\{(f_1 \dots f_i, f_k) \mid 1 \leq k \leq i\}\rangle_{\mathrm{C}(E)}$ for $1 \leq i$. Again by Lemma 3.1.2 the set $\{e \in E \mid \exists 1 \leq k \leq i,\ f_1 \dots f_i \leq e \leq f_k\}$ is a congruence class of τ_i and it again follows that $\tau_i \subsetneq \tau_{i+1}$. Therefore we have an infinite ascending chain of left congruences.

That a lack of infinite chains and infinite antichains implies that the semilattice is finite is a a result known as the *chain-antichain theorem*, for details see [27]. □

In the case that E is finite the notions of finitely generated and almost finitely generated full inverse subsemigroups coincide. The usual formulation for the classification of left Noetherian inverse semigroups is now a consequence of combining Theorem 4.2.11, Lemma 4.2.12 and Lemma 4.2.13.

Theorem 4.2.14 ([38, Proposition 4.3])**.** *An inverse semigroup is left Noetherian if and only if every full inverse subsemigroup is finitely generated (as an inverse semigroup).*

Proof. We have shown that S is left Noetherian if and only if E is finite and every full inverse subsemigroup is almost finitely generated. To complete the proof it remains to note that E itself is a full inverse subsemigroup and is finitely generated (as a semigroup) if and only if E is finite. □

4.3 Rees Congruences

An important family of congruences on any semigroup are the Rees congruences. In this section we discuss the left analogue of Rees congruences, with particular focus on when these are finitely generated. Given a left ideal $A \subseteq S$ we define a relation ρ_A on S by

$$a\ \rho_A\ b \iff a = b, \text{ or } a, b \in A.$$

Since A is a left ideal it is immediate that ρ_A is a left congruence which we call the *Rees left congruence*; if A is an ideal then ρ_A is the two sided Rees congruence with which we are familiar. For the remainder of this section we shall assume that A is a left ideal of S.

We recall that the principal left ideals of S are $\{Se \mid e \in E\}$ and that a left ideal is finitely generated if it is the union of finitely many principal left ideals. The following is an elementary alternate characterisation of finitely generated left ideals for inverse semigroups.

Lemma 4.3.1. *Let A be a left ideal of S. Then A is finitely generated if and only if $E(A)$ has finitely many maximal idempotents and every idempotent in A is below a maximal idempotent in $E(A)$.*

Proof. We know that A is finitely generated if and only if $A = Se_1 \cup \cdots \cup Se_n$ for some $e_1, \ldots, e_n \in E$. If $f \in E(A)$ then there is some $1 \leq i \leq n$ and $s \in S$ with $f = se_i$, whence $fe_i = (se_i)e_i = se_i = f$ so $f \leq e_i$. It follows that all idempotents in $E(A)$ are below some e_i. Also, the same argument implies that any maximal idempotent is equal to some e_i, of which there are finitely many, so there are certainly finitely many maximal idempotents.

Conversely we assume that there are finitely many maximal idempotents in $E(A)$ and every idempotent in $E(A)$ is below a maximal idempotent. Let $m_1, \ldots, m_n$ be the maximal idempotents. Then we claim $A = Sm_1 \cup \cdots \cup Sm_n$. It is clear, as $m_i \in A$ for each i and A is a left ideal, that $Sm_1 \cup \cdots \cup Sm_n \subseteq A$. For the reverse inclusion we suppose $a \in A$. Then $a^{-1}a \in E(A)$ so $a^{-1}a$ is below some maximal idempotent, m_i say. Then $a = a(a^{-1}a) = a(a^{-1}a)m_i$, so $a \in Sm_i$. This completes the proof. $\square$

We are interested in computing which left ideals A correspond to finitely generated Rees left congruences. We note that

$$\begin{aligned}
\operatorname{trace}(\rho_A) &= (E(A) \times E(A)) \cup \{(e,e) \mid e \in E(S)\}, \\
\operatorname{Inker}(\rho_A) &= \{a \in A \mid a^{-1} \in A\} \cup E(S), \\
\ker(\rho_A) &= A \cup E(S).
\end{aligned}$$

We first note that if ρ_A is finitely generated then it follows in a straightforward fashion that A is finitely generated. Indeed, suppose ρ_A is generated by a finite set H, which we assume is symmetric and contains no pairs of the form (a, a). If $a \neq b \in A$ there is a H-sequence from a to b, so there are

$s_1, \ldots, s_n \in S^1$ and $(h_1, k_1) \in H$ such that

$$a = s_1 h_1,\ s_1 k_1 = s_2 h_2, \ldots, s_n k_n = b.$$

In particular, $a = s_1 h_1$ so $a \in S h_1$. Thus

$$A \subseteq \bigcup_{(h,k) \in H} Sh.$$

On the other hand, if $(h, k) \in H$ then the ρ_A-class of h is non trivial, as it contains $h \neq k$. By definition of ρ_A, the only non-trivial ρ_A-class is A so we have $h \in A$. Thus $A = \bigcup_{(h,k)\in H} Sh$. Therefore, as H is finite we have that A is finitely generated.

However, the converse is not true, there are finitely generated ideals A for which ρ_A is not finitely generated. Indeed, consider an infinite descending chain of idempotents indexed by $\mathbb{N}$. This has all ideals principal, but no Rees congruence is finitely generated.

In this section we shall classify those left ideals $A \subseteq S$ for which ρ_A is finitely generated. Our first step is a technical lemma which may be considered a partial refinement of Theorem 2.5.5 in terms of generating sets for a left congruence.

Lemma 4.3.2. *Let $Z \subseteq E \times E$ be a binary relation, let $Y \subseteq S$ be an inverse subsemigroup, and let $\rho = \chi_Y \vee \nu_{\langle Z \rangle_{\mathrm{C}(E)}}$. Then*

$$\operatorname{trace}(\rho) = \langle Z \cup \{(aea^{-1}, afa^{-1}) \mid (e, f) \in Z,\ a \in Y\} \rangle_{\mathrm{C}(E)}.$$

Proof. Let $\tau = \langle Z \rangle_{\mathrm{C}(E)}$ and let

$$X = Z \cup \{(aea^{-1}, afa^{-1}) \mid (e, f) \in Z,\ a \in Y\}.$$

It is immediate that $\tau \subseteq \langle X \rangle_{\mathrm{C}(E)}$. Further, if $e\ \tau\ f$ then there is a Z-sequence from e to f. Thus there are $(p_i, q_i) \in Z$ (we assume without loss of generality that Z is symmetric) and $u_i \in E$ such that

$$e = u_1 p_1, u_1 q_1 = u_2 p_2, \ldots, u_n q_n = f.$$

If $a \in Y$ then taking $u_i' = au_ia^{-1}$ and $(p_i', q_i') = (ap_ia^{-1}, aq_ia^{-1})$ we obtain an X-sequence from aea^{-1} to afa^{-1}. Thus $(aea^{-1}, afa^{-1}) \in \langle X \rangle_{\mathrm{C}(E)}$. It follows that

$$\langle X \rangle_{\mathrm{C}(E)} = \langle \tau \cup \{(aea^{-1}, afa^{-1}) \mid (e, f) \in \tau,\ a \in Y\} \rangle_{\mathrm{C}(E)}.$$

This demonstrates that without loss of generality we may assume that $Z = \tau$ in other words that Z is a congruence on E. For the remainder of the proof we use τ as our initial relation on E, including changing our definition of X to include τ instead of Z.

Let $\xi = \operatorname{trace}(\rho)$, we need to show that $\xi = \langle X \rangle_{\mathrm{C}(E)}$. As $(\xi, \operatorname{Inker}(\rho))$ is an inverse congruence pair we know $\operatorname{Inker}(\rho) \subseteq N(\xi)$ and, by the definition of ρ, we have that $Y \subseteq \operatorname{Inker}(\rho)$. It follows that $Y \subseteq N(\xi)$. Since ξ is normal in Y we have that if $a \in Y$ and $e \mathrel{\xi} f$ then $aea^{-1} \mathrel{\xi} afa^{-1}$. Also, it is clear from the definition of ρ that $\tau \subseteq \xi$. This implies that $X \subseteq \xi$. Therefore $\langle X \rangle_{\mathrm{C}(E)} \subseteq \xi$.

For the reverse inclusion we first show that $Y \subseteq N(\langle X \rangle_{\mathrm{C}(E)})$. Suppose that $a \in Y$ and $e \mathrel{\langle X \rangle_{\mathrm{C}(E)}} f$ so there are $(p_i, q_i) \in X$ and $h_i \in E$ such that there is an X-sequence

$$e = h_1p_1,\ h_1q_1 = h_2p_2,\ \ldots,\ h_nq_n = f$$

from e to f. As Y is a subsemigroup and $a \in Y$, if $(p, q) \in X$ then $(apa^{-1}, aqa^{-1}) \in X$. Further, if $x, y \in E$ then

$$axya^{-1} = (axa^{-1})(aya^{-1}).$$

We thus obtain an X-sequence from aea^{-1} to afa^{-1} by taking $p_i' = ap_ia^{-1}$ and $q_i' = aq_ia^{-1}$ and $h_i' = ah_ia^{-1}$. Indeed, we know $(p_i', q_i') \in X$ and for each $1 \leq i \leq n-1$ we have

$$h_i'q_i' = ah_iq_ia^{-1} = ah_{i+1}p_{i+1}a^{-1} = h_{i+1}'p_{i+1}'$$

so that

$$aea^{-1} = ah_1p_1a^{-1} = h_1'p_1',\ h_1'q_1' = h_2'p_2',\ \ldots,\ h_n'q_n' = ah_nq_na^{-1} = afa^{-1}.$$

Thus $(aea^{-1}, afa^{-1}) \in \langle X\rangle_{\mathrm{C}(E)}$ so we have shown that $Y \subseteq N(\langle X\rangle_{\mathrm{C}(E)})$.

For any congruence $\tau \in \mathfrak{C}(E)$, the pair $(\tau, N(\tau))$ is an inverse congruence pair, so we have that $(\langle X\rangle_{\mathrm{C}(E)}, N(\langle X\rangle_{\mathrm{C}(E)}))$ is an inverse congruence pair. By definition $\tau \subseteq \langle X\rangle_{\mathrm{C}(E)}$ and we have shown that $Y \subseteq N(\langle X\rangle_{\mathrm{C}(E)})$. It follows that $\nu_\tau \subseteq \nu_{\langle X\rangle_{\mathrm{C}(E)}}$ and $\chi_Y \subseteq \chi_{N(\langle X\rangle_{\mathrm{C}(E)})}$. Thus

$$\rho = \nu_\tau \vee \chi_Y \subseteq \nu_{\langle X\rangle_{\mathrm{C}(E)}} \vee \chi_{N(\langle X\rangle_{\mathrm{C}(E)})} = \rho(\langle X\rangle_{\mathrm{C}(E)}, N(\langle X\rangle_{\mathrm{C}(E)})).$$

By Corollary 2.1.12 (the result that says the ordering on left congruences agrees with the inclusion ordering on inverse congruences pairs), we have that $\mathrm{trace}(\rho) = \xi \subseteq \langle X\rangle_{\mathrm{C}(E)}$. This completes the proof. □

We may now proceed with the main result of this section. We recall Lemma 3.1.2, which states that given a semilattice E, and a subsemilattice F the convex closure of F, which is $\overline{F} = \{e \in E \mid \exists f_1, f_2 \in F \mid f_1 \leq e \leq f_2\}$ is a congruence class of the congruence $\langle F \times F\rangle_{\mathrm{C}(E)}$. We remark that this one of the results in this section which is new and cannot be found in the literature.

Theorem 4.3.3. *Let S be an inverse semigroup, and let $A \subseteq S$ be a left ideal. Then ρ_A is a finitely generated left congruence on S if and only if A is finitely generated, and there is a finitely generated subsemigroup $W \subseteq A$ such that for each $a \in A$ there is some $f \in E(W)$ with $af \in W$.*

Proof. Suppose first that ρ_A is generated by a finite set, so (by Theorem 4.2.8) there are finite sets $Y \subseteq S$ and $Q \subseteq E \times E$ such that $\rho_A = \nu_{\langle Q\rangle_{\mathrm{C}(E)}} \vee \chi_{\langle Y\rangle_{\mathrm{IS}}}$. We note that since all non-trivial ρ_A relations are within A we may assume that $Y \subseteq A$ and $Q \subseteq E(A) \times E(A)$.

Let X be the set of all idempotents which appear in Q and note that X is finite. Certainly $\langle Q\rangle_{\mathrm{C}(E)} \subseteq \langle X \times X\rangle_{\mathrm{C}(E)}$, so, since

$$X \times X \subseteq E(A) \times E(A) \subseteq \mathrm{trace}(\rho_A)$$

we have that

$$\rho_A = \nu_{\langle X\times X\rangle_{\mathrm{C}(E)}} \vee \chi_{\langle Y\rangle_{\mathrm{IS}}}.$$

We now assume that Q is of the form $X \times X$.

We have previously observed that if ρ_A is finitely generated then A is finitely generated. By Lemma 4.3.1 there are then finitely many maximal idempotents in A and each idempotent in A is below some maximal idempotent. If necessary we add any idempotents which are maximal in A to X and note that we still have that X is finite and that $\rho_A = \nu_{\langle X\times X\rangle_{\mathrm{C}(E)}} \vee \chi_{\langle Y\rangle_{\mathrm{IS}}}$. Therefore, we assume that all maximal idempotents in A are elements of X.

We now show that there is a finitely generated subsemigroup W such that for $a \in A$ there is some $f \in E(W)$ with $af \in W$. Let $W = \langle X \cup Y\rangle_{\mathrm{IS}}$ and note that, as Y, X are finite, W is finitely generated. Also, as $X, Y \subseteq A$ and A is a subsemigroup, we have that $W \subseteq A$. As $\rho_A = \nu_{\langle X\times X\rangle_{\mathrm{C}(E)}} \vee \chi_{\langle Y\rangle_{\mathrm{IS}}}$ we apply Lemma 4.3.2 to obtain

$$\mathrm{trace}(\rho_A) = \langle (X \times X) \cup \{(aea^{-1}, afa^{-1}) \mid e, f \in X,\ a \in \langle Y\rangle_{\mathrm{IS}}\}\rangle_{\mathrm{C}(E)}.$$

We remark that

$$(X \times X) \cup \{(aea^{-1}, afa^{-1}) \mid e, f \in X,\ a \in \langle Y\rangle_{\mathrm{IS}}\} \subseteq E(W) \times E(W).$$

As $E(W) \times E(W) \subseteq E(A) \times E(A) \subseteq \mathrm{trace}(\rho_A)$ we have that

$$\mathrm{trace}(\rho_A) = \langle E(W) \times E(W)\rangle_{\mathrm{C}(E)}.$$

By applying Lemma 3.1.2 we obtain that $\overline{E(W)} = \{e \in E(S) \mid \exists f_1, f_2 \in E(W) \text{ with } f_1 \leq e \leq f_2\}$ (the convex closure of $E(W)$) is a congruence class of $\mathrm{trace}(\rho_A)$, so $\overline{E(W)} = E(A)$. Thus given $e \in E(A)$ there is some $f \in E(W)$ such that $f \leq e$.

We have that $\chi_{\langle Y\rangle_{\mathrm{IS}}} \vee \nu_{\mathrm{trace}(\rho_A)} = \rho_A$ and $\ker(\rho_A) = A \cup E$. We apply Corollary 2.6.1 to obtain that

$$A \cup E = \ker(\rho_A) = \ker(\chi_{\langle Y\rangle_{\mathrm{IS}}} \vee \nu_{\mathrm{trace}(\rho_A)}) = \bigcup_{t \in \langle Y \cup E\rangle_{\mathrm{IS}}} [t]_{\nu_{\mathrm{trace}(\rho_A)}}.$$

We recall the definition of ν_ξ, the minimum left congruence with trace ξ, from Theorem 1.4.11

$$\nu_\xi = \{(a, b) \in S \times S \mid\ \exists e \in E(S),\ a^{-1}a\ \xi\ b^{-1}b\ \xi\ e,\ ae = be\}.$$

Thus for each $a \in A$ there is $t \in \langle Y \cup E \rangle_{\mathrm{IS}}$ and $e \in E$ such that $ae = te$ and

$$a^{-1}a \ \operatorname{trace}(\rho_A) \ t^{-1}t \ \operatorname{trace}(\rho_A) \ e.$$

Since $a^{-1}a \in A$ we obtain, from the definition of trace(ρ_A), that $e \in A$, and we also note that $ae = te \in \langle Y \cup E \rangle_{\mathrm{IS}}$.

If $ae \in E(A)$ then as $E(A) = \overline{E(W)}$ there is $f \in E(W)$ such that $f \leq ae$. Then certainly $f \leq e$, so $af = aef = f$, thus $af \in W$. Suppose now that $ae \notin E(A)$. As $\langle Y \cup E \rangle_{\mathrm{IS}} = E \cup \langle Y \rangle_{\mathrm{IS}} E$ this implies that $ae \in \langle Y \rangle_{\mathrm{IS}} E$, so there are $b \in \langle Y \rangle_{\mathrm{IS}}$ and $e' \in E$ such that $ae = be'$. Note that we may choose $e' \in E(A)$ as, for instance, we may assume $e' = e'b^{-1}b$ and $b \in Y \subseteq A$ and A is a left ideal. Then as $ee' \in E(A)$ and $E(A) = \overline{E(W)}$ we may take $f \in E(W)$ with $f \leq ee'$. Then $af = aef = be'f = bf \in W$. This completes the proof that A satisfies the properties claimed.

Conversely, assume that A is as claimed, with $W = \langle V \rangle_{\mathrm{IS}}$ for some finite set $V \subseteq A$ closed under taking inverses. We assume, without loss of generality, that all maximal idempotents of A are in V. Let $T = \langle V \cup E \rangle_{\mathrm{IS}}$, and $X = \{vv^{-1} \mid v \in V\}$. Let $\tau = \langle X \times X \rangle_{\mathrm{C}(E)}$ and let $\rho = \chi_T \vee \nu_\tau$, we shall show that $\rho = \rho_A$. We note that this will imply that ρ_A is finitely generated as τ is certainly finitely generated and we see that χ_T is finitely generated by applying Proposition 4.2.5, since T is almost finitely generated.

Since V is closed under taking inverses, for any $v, u \in V$ we have that $v^{-1}v, uu^{-1} \in X$ so that $vv^{-1} \ \nu_\tau \ u^{-1}u$. Also for any $v \in V$ we have that $v \ \chi_T \ vv^{-1}$. For any $a \in W$ we may write a as a product of $v_i \in V$, say $a = v_1 \dots v_n$. Then we observe that

$$a = v_1 \dots v_{n-1}v_n \ \chi_T \ v_1 \dots v_{n-1}v_nv_n^{-1} \ \nu_\tau \ v_1 \dots v_{n-1}v_{n-1}^{-1}v_{n-1} = v_1 \dots v_{n-1}.$$

Proceeding in the same fashion we obtain $a \ \rho \ v_1$, so we have $a \ \rho \ v_1v_1^{-1}$. This implies that a is ρ-related to every element of X. Since our choice of $a \in W$ was arbitrary it follows that for every pair $a, b \in W$ we have $a \ \rho \ b$. In particular, for $e, f \in E(W)$ we have $e \ \rho \ f$. By assumption for $a \in A$ we have some $f \in E(W)$ with $af \in W$. We also assume that there are finitely many maximal idempotents in A and every idempotent is below some idempotent (this is equivalent to A being finitely generated by Lemma 4.3.1). Thus we may choose a maximal idempotent m in A with $m \geq a^{-1}a$, and we recall that we assume all maximal idempotents are in $V \subseteq W$. As $m, f \in E(W)$ we have $m \ \rho \ f$, and thus $a = am \ \rho \ af \in W$. Thus every element in A is ρ-related to an element of W. As all elements of W are ρ-related we have shown that $\rho_A \subseteq \rho$. The reverse inclusion follows immediately from the observation that that if $Z \subseteq A$ for a left ideal A then $\langle Z \times Z \rangle_{\mathrm{LC}(S)} \subseteq \rho_A$ and the recollection that $X, T \subseteq A$. Thus we have $\rho = \rho_A$ as claimed. $\square$

It is of interest to consider semigroups for which the universal relation ω is finitely generated as a left congruence. For monoids this is equivalent to the monoid being of type left FP_1 (see, for example, [8]). This has been studied for several classes of semigroups, and a classification for inverse semigroups is available [8]. Since $\omega = \rho_S$ for the ideal S we may deduce the classification of those inverse semigroups for which the universal relation is finitely generated.

Corollary 4.3.4 ([8, Theorem 5.1])**.** *Let S be an inverse semigroup. Then ω is finitely generated as a left congruence on S if and only if E contains finitely many maximal elements and every idempotent is below a maximal idempotent, and there is a finitely generated subsemigroup $W \subseteq S$ such that for each $a \in S$ there is some $f \in E(W)$ with $af \in W$.*

To conclude this section we will classify those subsemigroups such that for every left ideal A the left congruence ρ_A is finitely generated. Before we state this result we provide a pair of elementary lemmata from which the result follows. The first is immediate from the fact that the join of two finitely generated left congruences is finitely generated and the observation that if A, B are left ideals then $\rho_A \vee \rho_B = \rho_{A \cup B}$.

Lemma 4.3.5. *Let S be an inverse semigroup. Then the following statements are equivalent.*

(i) For all finitely generated left ideals $A \subseteq S$ the left congruence ρ_A is finitely generated.

(ii) For all principal left ideal $B \subseteq S$ the left congruence ρ_B is finitely generated.

The second lemma is a standard result, which is essentially folklore, about which inverse semigroups have every left ideal finitely generated.

Lemma 4.3.6 (see [24, Proposition 3.1])**.** *Let S be an inverse semigroup. Then the following statements are equivalent.*

(i) Every left ideal of S is finitely generated.

(ii) Every ideal of E is finitely generated.

(iii) The semilattice E contains no infinite antichains and no infinite ascending chains.

Theorem 4.3.7. *Let S be an inverse semigroup. Every Rees left congruence is finitely generated if and only if E contains no infinite antichains and no infinite ascending chains, and for all principal left ideals B there is a finitely generated subsemigroup $W \subseteq B$ such that for all $b \in B$ there is $f \in E(W)$ such that $bf \in W$.*

4.4 LATTICE PROPERTIES OF $\mathfrak{LC}(S)$

It is a broad and interesting question to ask "how do properties of the lattice of congruences on a semigroup relate to properties of the semigroup?" A detailed analysis of this area is outside the scope of the thesis and so we direct the reader to the excellent pair of surveys by Mitsch [49] and [48].

In a similarly well plumbed vein there is a significant quantity of research into lattice properties of lattices of subsemigroups, a comprehensive survey [73] (and later book [74]) of results in this area is available. As mentioned, Jones has a hefty back-catalogue of research in this area (for example [35], [37] and [36]).

At this time we are dwelling in the world of one sided congruences so would like to ask the corresponding question: "how do properties of $\mathfrak{LC}(S)$ relate to properties of S?" This is a far less well studied area, and little is known in general about lattice properties of the lattice of one sided congruences for inverse semigroups. Our inverse kernel approach has been effective thus far in this area, for example providing a simple proof as to when the lattice of one sided congruences contains no infinite chains. In this section, while we do not attain general results relating properties of S to common properties of $\mathfrak{LC}(S)$, we can utilise the inverse kernel approach, along with general results about $\mathfrak{C}(E)$ and $\mathfrak{V}(S)$, to demonstrate that certain properties are highly restrictive.

We start with the definitions of the two lattice properties that we shall focus on in this section.

Definition 4.4.1. Let L be a lattice. Then L is said to be *modular* if $a \leq b \in L$ implies that $a \vee (x \wedge b) = (a \vee x) \wedge b$ for all $x \in L$.

Definition 4.4.2. Let L be a lattice. Then L is said to be *distributive* if $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ for all $x, y, z \in L$.

We remark that the definition of distributive is $\vee, \wedge$ dual, and that a distributive lattice is modular however the converse is not true.

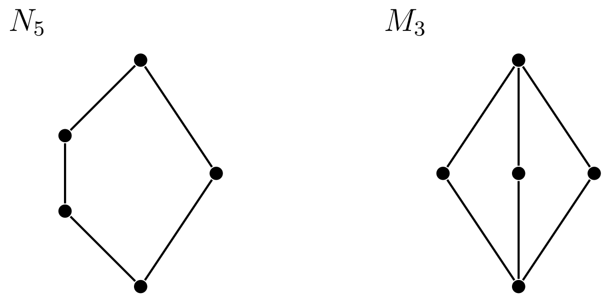


Figure 4.3: The pentagon lattice N_5 and the diamond lattice M_3

Equivalently (via standard results, see, for example, [45]) a lattice is *modular* if it contains no sublattice isomorphic to the pentagon lattice N_5 and *distributive* if it contains no sublattice isomorphic to either N_5 or the diamond lattice M_3. The lattices N_5 and M_3 are shown in Fig. 4.3.

Since the lattice $\mathfrak{V}(S)$ embeds into $\mathfrak{LC}(S)$ as the set of idempotent separating congruences the following is immediate.

Proposition 4.4.3. *Let P be any lattice property that is preserved under moving to a sublattice (e.g. modularity, distributivity). If $\mathfrak{LC}(S)$ has property P then $\mathfrak{V}(S)$ has property P.*

In general the analogue for converse is not true, though we make one remark in that direction.

Corollary 4.4.4. *Let S be an inverse semigroup and let P be any lattice property that is preserved under moving to a sublattice and taking a homomorphic image. If $\mathfrak{V}(S)$ has property P then every trace class in $\mathfrak{LC}(S)$ satisfies has property P.*

Proof. This follows immediately from Proposition 2.3.6 which says that each trace class is isomorphic to a lattice homomorphic image of a sublattice of $\mathfrak{V}(S)$. □

For the lattice $\mathfrak{C}(E)$ we do not have a natural lattice embedding into $\mathfrak{LC}(S)$. It follows that we cannot make a similar statement about modularity/distributivity of $\mathfrak{LC}(S)$ passing to $\mathfrak{C}(E)$. We recall Theorem 1.3.19, which states that the trace map restricted to $\mathfrak{C}(S)$ is a surjective lattice homomorphism onto $\mathfrak{C}_N(E)$, the lattice of normal congruences on E. As $\mathfrak{C}(S) \subseteq \mathfrak{LC}(S)$ is a sublattice, it follows that if $\mathfrak{LC}(S)$ is modular/distributive then so is $\mathfrak{C}(S)$ and further, so is $\mathfrak{C}_N(E)$. In particular, if every congruence is normal (for instance, for Clifford semigroups) then if $\mathfrak{LC}(S)$ is modular then so is $\mathfrak{C}(E)$.

Corollary 4.4.5. *Let S be a Clifford semigroup. If $\mathfrak{LC}(S)$ is modular (distributive) then $\mathfrak{C}(E)$ is modular (distributive).*

We remark that it is possible to define modularity/distributivity for semilattices (see, for instance [70]) which might lend hope to the prospect of deducing (semi)lattice properties of $\mathfrak{LC}(S)$ from properties of $\mathfrak{C}(E)$, since we recall that we have a semilattice embedding of $\mathfrak{C}(E)$ into $\mathfrak{LC}(S)$. However, the conditions on the functions between semilattices required for these properties to pass via the function are stronger than the properties satisfied by our embedding so this is a dead end. Similarly, the $\cap$-homomorphism from $\mathfrak{LC}(S) \to \mathfrak{C}(E) \times \mathfrak{V}(S)$ is too "weak" a function for us to deduce many properties of $\mathfrak{LC}(S)$ from properties of $\mathfrak{C}(E) \times \mathfrak{V}(S)$.

We now turn attention to positive results which we can provide for modularity and distributivity of $\mathfrak{LC}(S)$. When S is itself a semilattice $\mathfrak{LC}(S)$ is $\mathfrak{C}(E)$, in this case the conditions for modularity and distributivity agree and are known.

Theorem 4.4.6 ([49, Theorem 4.4])**.** *Let E be a semilattice. Then the following are equivalent:*

(i) $\mathfrak{C}(E)$ *is modular;*

(ii) $\mathfrak{C}(E)$ *is distributive;*

(iii) the natural partial order on E is a tree.

Next we summarise the relevant results that describe inverse semigroups for which $\mathfrak{V}(S)$ is modular or distributive. We recall Theorem 3.1.4, which says that the lattice of full inverse subsemigroups of an inverse semigroup may be regarded as a subdirect product of the lattices of full inverse subsemigroups of the principal factors. This means that the lattice of full inverse subsemigroups of an inverse semigroup has lattice properties (for example modularity or distributivity) if and only if the lattice of full inverse subsemigroups of each principal factor does. This is because the properties we are interested in are preserved under homomorphism, taking direct product and moving to a sublattice. The lattice of full inverse subsemigroups of each principal factor is a homomorphic image of $\mathfrak{V}(S)$, the homomorphism is projection onto that coordinate of the product. Thus $\mathfrak{V}(S)$ having a lattice property implies that the lattice of full inverse subsemigroups of each principal factor has this property. Conversely, if the lattice of full inverse subsemigroups of every principal factor has a lattice property then the direct product of these lattices has that property. Consequently so does $\mathfrak{V}(S)$ which is a sublattice of this direct product. Using this fact most results concerning lattice properties are given for simple inverse semigroups.

For an inverse semigroup S an element a is said to be *(strictly) right regular* (see [35]) if $a^{-1}a \leq aa^{-1}$ ($a^{-1}a < aa^{-1}$); the definition for (strict) left regular elements is dual. An equivalent (for strictly right regular) condition is that $a^2 \mathcal{R} a$ and $a^2 \not{\mathcal{H}} a$. It is worth noting that if a is strictly left or right regular then $\langle a\rangle_{\mathrm{IS}}$ is a bicyclic semigroup. If S is a simple inverse semigroup then E is *Archimedean* in S if for any $g \in E$ and any strictly right regular $a \in S$ we have $a^{-n}a^n < g$ for some $n > 0$. We recall that a semigroup is said to be *combinatorial* if $\mathcal{H}$ is trivial on S.

Theorem 4.4.7 ([35])**.** *Let S be a simple inverse semigroup which is not a group. Then $\mathfrak{V}(S)$ is distributive if and only if:*

(i) S is combinatorial,

(ii) $E(S)$ is Archimedean in S,

(iii) the maximum group homomorphic image of S is locally cyclic,

(iv) for each $\mathcal{D}$-class the idempotents form a chain.

Theorem 4.4.8 ([34])**.** *Let S be a simple inverse semigroup which is not a group. Then $\mathfrak{V}(S)$ is modular if and only if:*

(i) S is combinatorial,

(ii) $E(S)$ is Archimedean in S,

(iii) the maximum group homomorphic image of S is locally cyclic,

(iv) for each $\mathcal{D}$-class the idempotents form a chain or contain exactly one pair of incomparable elements each of which is maximal.

Notice that while the classification for which inverse semigroups have $\mathfrak{V}(S)$ modular or distributive is not quite like that for $\mathfrak{C}(E)$, in that the conditions for modularity and distributivity do not coincide, they are very similar.

Example 4.4.9. We provide an example to show that it does not follow that if both $\mathfrak{V}(S)$ and $\mathfrak{C}(S)$ are modular/distributive that $\mathfrak{LC}(S)$ is as well. In fact we notice that that the bicyclic monoid B has both $\mathfrak{V}(B)$ and $\mathfrak{C}(E(B))$ distributive, indeed $E(B)$ is a chain, so is certainly a tree, and B is simple and satisfies the conditions of Theorem 4.4.7. We shall demonstrate however, that $\mathfrak{LC}(B)$ is not even modular. We recall that congruences on $E(B)$ are given by partitions of $\mathbb{N}^0$ and full inverse subsemigroups of B are of the form

$$T_{k,d} = \{(x, y) \in B \mid x, y \geq n,\ d \mid x - y\} \cup E(B).$$

Let τ_1, τ_2 be the congruences on $E(B)$ defined by the partitions of $\mathbb{N}^0$:

$$\tau_1\colon \{0, 1\}, \{2, 3\}, \{4, 5\}, \ldots$$
$$\tau_2\colon \{0\}, \{1, 2\}, \{3, 4\}, \{5, 6\}, \ldots.$$

We recall our description of inverse congruence pairs for the bicyclic monoid Theorem 3.3.18, which implies that $T_{k,d}$ forms an inverse congruence pair with τ_1 when $2 \mid d$ and k is even, and $T_{k,d}$ forms an inverse congruence pair with τ_2 when $2 \mid d$, and $k = 0$ or k is odd. We also note that $\tau_1 \vee \tau_2 = \omega$ and $\tau_1 \cap \tau_2 = \iota$. We recall (again from Theorem 3.3.18) that $(\omega, T_{k,d})$ is an inverse congruence pair precisely when $k = 0$ and that $(\iota, T_{k,d})$ is always an inverse congruence pair. This implies that

$$\rho(\tau_1, T_{6,2}) \vee \rho(\tau_2, T_{7,2}) = \rho(\omega, T_{0,2}) = \rho(\tau_1, T_{4,2}) \vee \rho(\tau_2, T_{7,2})$$

and that

$$\rho(\tau_1, T_{6,2}) \cap \rho(\tau_2, T_{7,2}) = \rho(\iota, T_{7,2}) = \rho(\tau_1, T_{4,2}) \cap \rho(\tau_2, T_{7,2}).$$

We then consider the subsemilattice of $\mathfrak{LC}(B)$ consisting of these 5 left congruences, which is shown in Fig. 4.4 with vertices labelled with the inverse congruence pairs. As this sublattice is isomorphic to the pentagon lattice N_5 it follows that $\mathfrak{LC}(B)$ is not modular.

We now state our one original and general result about inverse congruence pairs when $\mathfrak{LC}(S)$ is modular.

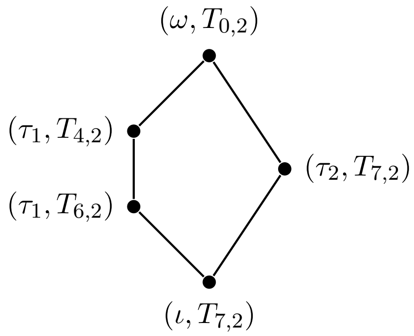


Figure 4.4: A sublattice of $\mathfrak{LC}(B)$ isomorphic to N_5

Proposition 4.4.10. *Let S be an inverse semigroup such that $\mathfrak{LC}(S)$ is modular. Then for each congruence τ on $E = E(S)$ the trace class is*

$$[\tau]_{\text{trace}} = \{\rho(\tau, T) \mid T \in \mathfrak{V}(S),\ C(\tau) \subseteq T \subseteq N(\tau)\}.$$

Consequently, each trace class is isomorphic to an interval in the lattice $\mathfrak{V}(S)$.

Proof. Let τ be a congruence on E and suppose for a contradiction that there is some T with $C(\tau) \subsetneq T \subsetneq N(\tau)$ with (τ, T) not an inverse congruence pair for S. Let $V = \text{Inker}(\nu_\tau \vee \chi_T)$. Then certainly $\chi_V \subseteq \nu_\tau \vee \chi_T$ so

$$\nu_\tau \vee \chi_T = \rho(\tau, V) = \nu_\tau \vee \chi_V.$$

Further, since the inverse kernel and trace of the intersection of left congruences is the intersection of the inverse kernels and traces respectively, and $\nu_\tau = \rho(\tau, C(\tau))$, we have

$$\nu_\tau \cap \chi_T = \rho(\iota, C(\tau)) = \chi_{C(\tau)} = \nu_\tau \cap \chi_V.$$

We consider the set of left congruences: $\chi_{C(\tau)}$, χ_T, χ_V, ν_τ and $\rho(\tau, V)$. In particular these are all distinct and form a sublattice of $\mathfrak{LC}(S)$, shown in Fig. 4.5, which is isomorphic to N_5.

Therefore $\mathfrak{LC}(S)$ is not modular, contradicting our assumption. Thus there can be no T with $C(\tau) \subsetneq T \subsetneq N(\tau)$ such that (τ, T) is not an inverse congruence pair. The final claim of the proposition is immediate. $\square$

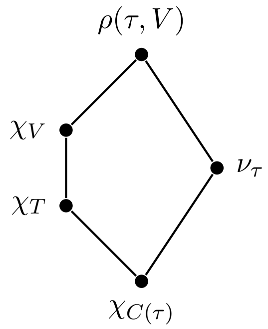


Figure 4.5: A sublattice of $\mathfrak{LC}(S)$ isomorphic to N_5 from Proposition 4.4.10

Proposition 4.4.10 appears to be a very restrictive condition on the structure of inverse semigroups for which $\mathfrak{LC}(S)$ is modular. At this stage we move from looking at general inverse semigroups to Clifford semigroups. In the rest of this section on lattice properties we reprove the conditions required for $\mathfrak{LC}(S)$ to be modular or distributive for a Clifford semigroup; a phenomenon which has been previously studied in [18].

We recall that a Clifford semigroup S is written $S = \mathcal{C}(Y, G_e, \phi_{e,f})$ where $\{G_e \mid e \in Y\}$ are groups, Y is a semilattice, and $\phi_{e,f} \colon G_e \to G_f$ for $f \leq e$ is a homomorphism. The semigroup S is then the disjoint union of the G_e as e ranges over Y. Further, full inverse subsemigroups of S are defined by specifying a subgroup $H_e \leq G_e$ for each $e \in Y$ such that $H_e\phi_{e,f} \subseteq H_f$ whenever $f \leq e$, and the subsemigroup is then $\mathcal{C}(Y, H_e, \phi_{e,f}|_{H_e})$. As described in Theorem 3.2.2 inverse congruence pairs for S are pairs $(\tau, \mathcal{C}(Y, H_e, \phi_{e,f}|_{H_e}))$ such that if $e \mathrel{\tau} f$ and $f \leq e$ then $H_e = \{g \in G_e \mid g\phi_{e,f} \in H_f\}$.

We have mentioned that modularity and ditributivity of $\mathfrak{LC}(S)$ for Clifford semigroups has been studied in [18], and as often happens when reproducing known results it is impossible to depart entirely from the previous method of proof. We shall need one lemma from [18] but the rest of the proof follows from the description of the intersection and join of left congruences in terms of the kernel and trace.

Lemma 4.4.11 ([18, Lemma 1])**.** *Let $S = \mathcal{C}(Y, G_e, \phi_{e,f})$ be a Clifford semigroup. If $\mathfrak{LC}(S)$ is modular then $\phi_{e,f}$ is the trivial homomorphism for* *all e, f with $f < e$.*

Proof. We provide an indication of how to proceed with the proof, for a complete proof see [18].

First we consider a simple case and suppose that the semilattice Y is the two element semilattice consisting of e and f with $f < e$. We assume that $\phi_{e,f}$ is not the trivial homomorphism so $\ker(\phi_{e,f}) = K \neq G_e$. Then let

$$A_e = K, \quad A_f = G_f \quad \text{and} \quad A = \mathcal{C}(Y, A_y, \phi_{y,x}),$$
$$B_e = G_e, \quad B_f = G_f \quad \text{and} \quad B = \mathcal{C}(Y, B_y, \phi_{y,x}),$$

and

$$D_e = K, \quad D_f = \{1_f\} \quad \text{and} \quad D = \mathcal{D}(Y, A_y, \phi_{y,x}).$$

It is en elementary check using the description of inverse congruence pairs on Clifford semigroups (Theorem 3.2.2) that (ι, A), (ι, B) and (ω, D) are inverse congruence pairs. Further, we note that

$$\rho(\iota, A) \vee \rho(\omega, D) = \omega = \rho(\iota, B) \vee \rho(\omega, D)$$

and

$$\rho(\iota, A) \cap \rho(\omega, D) = \rho(\iota, D) = \rho(\iota, B) \cap \rho(\omega, D).$$

These 5 left congruences: $\rho(\iota, A)$, $\rho(\iota, B)$, $\rho(\omega, D)$, $\rho(\iota, D)$ and ω are a sublattice of $\mathfrak{LC}(S)$ which is isomorphic to N_5. This is shown in Fig. 4.6. Thus $\mathfrak{LC}(S)$ is not modular.

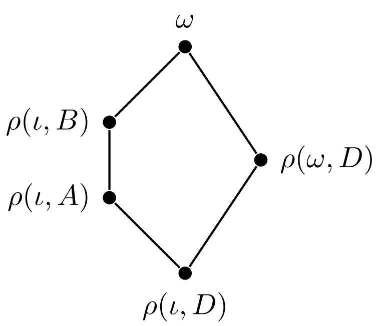


Figure 4.6: A sublattice of $\mathfrak{LC}(S)$ isomorphic to N_5 from Lemma 4.4.11

For the general case, for a Clifford semigroup S with an arbitrary semilattice Y the proof proceeds by showing that given a pair $e, f \in Y$ with $f < e$ there is a homomorphism $\lambda\colon S \to (G_e \cup G_f \cup \{0\})$ defined by

$$s\lambda = \begin{cases} se & \text{if } se \in G_e \\ sf & \text{if } sf \in G_f \text{ but } se \notin G_e \\ 0 & \text{if } sf \notin G_f \end{cases},$$

and that this is either surjective or has image $G_e \cup G_f$. Furthermore, a homomorphism $\theta\colon S \to T$ of semigroups induces a homomorphism $\mathfrak{LC}(S) \to \mathfrak{LC}(\mathrm{Im}(\theta))$. Note that the obvious inclusion $G_e \cup G_f \hookrightarrow G_e \cup G_f \cup \{0\}$ is also a homomorphism. It follows that if $\mathfrak{LC}(S)$ is modular then $\mathfrak{LC}(G_e \cup G_f)$ is modular (where we regard $G_e \cup G_f$ as a subsemigroup of S). To prove the general result we suppose $f < e$ in Y has $\phi_{e,f}$ non trivial, and apply the previous argument from the special case (with a two element semilattice) to obtain that $\mathfrak{LC}(G_e \cup G_f)$ is not modular, whence $\mathfrak{LC}(S)$ is not modular. $\square$

We remark that it is possible to directly prove Lemma 4.4.11 by describing inverse congruence pairs in an arbitrary Clifford semigroup, however this quickly becomes difficult to follow. Thus we chose to proceed with the method used in the proof.

If every linking homomorphism in a Clifford semigroup S is trivial then S is said to have *trivial multiplication.* It follows immediately from Lemma 4.4.11 that if a Clifford semigroup S has $\mathfrak{LC}(S)$ distributive then S has trivial multiplication. We note in particular that when the multiplication is trivial any collection of subgroups $\{H_e \leq G_e \mid e \in Y\}$ defines a full inverse subsemigroup of $\mathcal{C}(Y, G_e, \phi_{e,f})$. We specialise our description of inverse congruence pairs on Clifford semigroups (Theorem 3.2.2) to Clifford semigroups with trivial multiplication.

Corollary 4.4.12. *Let $S = \mathcal{C}(Y, G_e, \phi_{e,f})$ be a Clifford semigroup with trivial multiplication, let τ be a congruence on Y and let $T = \mathcal{C}(Y, H_e, \phi_{e,f})$ be a full inverse subsemigroup of S. Then (τ, T) is an inverse congruence pair if and only if $H_e = G_e$ for any e which is not minimal in $[e]_\tau$.*

Proof. We recall Theorem 3.2.2, which states that (T, τ) is an inverse congruence pair if and only if, for $e, f \in Y$, $e \;\tau\; f$ and $f < e$ implies that $H_e = \{g \in G_e \mid g\phi_{e,f} \in H_f\}$.

We first assume that $H_e = G_e$ for any e which is not minimal in $[e]_\tau$. We suppose that $e \;\tau\; f$ and $f < e$, and note that this implies that e is not minimal in $[e]_\tau$, so, by assumption, we have $H_e = G_e$. On the other hand, we observe that $g\phi_{e,f} = 1_f$ for any $g \in G_e$ so $\{g \in G_e \mid g\phi_{e,f} \in H_f\} = G_e = H_e$. Thus we have that (τ, T) is an inverse congruence pair.

For the converse we suppose that (τ, T) is an inverse congruence pair, and take e which is not minimal in $[e]_\tau$. Choose $f \in [e]_\tau$ such that $f < e$, then, as (τ, T) is an inverse congruence pair we have that $H_e = \{g \in G_e \mid g\phi_{e,f} \in H_f\}$. However, $g\phi_{e,f} = 1_f$ for any $g \in G_e$, so we have that $H_e = G_e$. This completes the proof. □

We may now give the meets and joins of congruences on Clifford semigroups with trivial multiplication.

Proposition 4.4.13. *Let $S = \mathcal{C}(Y, G_e, \phi_{e,f})$ be a Clifford semigroup with trivial multiplication and let*

$$\rho = \rho(\tau, \{H_e \mid e \in Y\}) \quad \textit{and} \quad \rho' = \rho(\tau', \{H'_e \mid e \in Y\})$$

be left congruences on S. Then

$$\rho \cap \rho' = \rho(\tau \cap \tau', \{H_y \cap H'_y \mid y \in Y\}),$$

and

$$\rho \vee \rho' = \rho(\tau \vee \tau', \{K_y \mid y \in Y\})$$

where

$$K_y = \begin{cases} H_y \vee H'_y & \textit{if } y \textit{ is minimal in } [y]_{\tau \vee \tau'} \\ G_y & \textit{otherwise} \end{cases}.$$

Proof. The first half of the claim - the part for the intersection - is immediate as the trace and kernel maps are $\cap$-homomorphisms (by Corollary 2.5.1). We prove the second half. Let $T = \mathcal{C}(Y, H_e, \phi_{e,f})$ and $T' = \mathcal{C}(Y, H'_e, \phi_{e,f})$.

We apply Theorem 2.5.5, which we recall states that the trace of $\rho \vee \rho'$ is ξ, the least congruence on E such that $\tau \vee \tau' \subseteq \xi$ and $T_1 \vee T_2 \subseteq N(\xi)$, and that the inverse kernel of $\rho \vee \rho'$ is the least subsemigroup containing T and T' which forms an inverse congruence pair with ξ.

We know that every congruence on the idempotents of a Clifford semigroup is a normal congruence, so it is clear that $\xi = \tau \vee \tau'$. We then note that certainly $U = \mathcal{C}(Y, \{K_y \mid y \in Y\})$ contains both T and T', and is a full inverse subsemigroup. Also, by Corollary 4.4.12, (ξ, U) is an inverse congruence pair.

We must show that U is the smallest inverse full inverse subsemigroup containing T and T' which forms an inverse congruence pair with ξ. Say that $W = \mathcal{C}(Y, \{J'_y \mid y \in Y\})$ is the smallest such subsemigroup, so $W \subseteq U$. We first note that when y is minimal in $[y]_\xi$ then, as $T, T' \subseteq W$, certainly $H_y \vee H'_y \subseteq J_y$. By Corollary 4.4.12, when y is not minimal in $[y]_\xi$ we have $J_y = G_y$. Thus we have $U \subseteq W$, which, by minimality of W, implies $U = W$. This completes the proof. □

Utilising the description of the intersection and join of left congruences on a Clifford semigroup with trivial multiplication it is then very straightforward to classify those Clifford semigroups for which $\mathfrak{LC}(S)$ is modular or distributive.

Theorem 4.4.14 (see [18, Theorem 4]). *Let $S = \mathcal{C}(Y, G_e, \phi_{e,f})$ be a Clifford semigroup. Then $\mathfrak{LC}(S)$ is modular (distributive) if and only if the following conditions hold:*

(i) the partial order on E is a tree;

(ii) S has trivial multiplication;

(iii) G_e has modular (distributive) lattice of subgroups for each $e \in Y$.

Proof. We begin with the only if direction (that if $\mathfrak{LC}(S)$ is modular then the conditions hold). That S has trivial multiplication is exactly Lemma 4.4.11. That the partial order on Y is a tree is Corollary 4.4.5 and Theorem 4.4.6. Finally that each G_f has modular (distributive) lattice of subgroups for each $e \in Y$ follows as the lattice of subgroups of G_e embeds into $\mathfrak{V}(S)$ for each $e \in Y$. To see this either appeal to Theorem 3.1.4, or consider the map $\mathfrak{V}(G_e) \to \mathfrak{V}(S)$ defined by $H \mapsto \mathcal{C}(Y, K_y, \phi_{y,f}|_{K_y})$ where $K_e = H$ and $K_y = \{y\}$ otherwise.

We now turn to the other direction. Suppose that the partial order on E is a tree, that S has trivial multiplication and G_e has modular lattice of subgroups for each $e \in Y$. Suppose also that $\rho_a = \rho(\tau_a, \{A_e \mid e \in Y\})$, $\rho_b = \rho(\tau_b, \{B_e \mid e \in Y\})$ and $\rho_c = \rho(\tau_c, \{C_e \mid e \in Y\})$ are left congruences on S with $\rho_a \leq \rho_b$ (so $\tau_a \subseteq \tau_b$ and $A_e \subseteq B_e$ for each $e \in Y$). Then, by applying Proposition 4.4.13, we observe that

$$\rho_a \vee (\rho_c \cap \rho_b) = \rho(\tau_a \vee (\tau_c \cap \tau_b), \{K_e \mid e \in Y\})$$

where

$$K_e = \begin{cases} A_e \vee (C_e \cap B_e) & \text{if } e \text{ is minimal in } [e]_{\tau_a \vee (\tau_c \cap \tau_b)} \\ G_e & \text{otherwise} \end{cases}.$$

As the partial order on Y is a tree by Theorem 4.4.6 the lattice of congruences on Y is modular, so, as $\tau_a \subseteq \tau_b$,

$$\tau_a \vee (\tau_c \cap \tau_b) = (\tau_a \vee \tau_c) \cap \tau_b.$$

Similarly, as each G_e has modular subgroup lattice and $A_e \leq B_e$

$$A_e \vee (C_e \cap B_e) = (A_e \vee C_e) \cap B_e.$$

It follows that

$$K_e = \begin{cases} (A_e \vee C_e) \cap B_e & \text{if } e \text{ is minimal in } [e]_{(\tau_a \vee \tau_c) \cap \tau_b} \\ G_e & \text{otherwise} \end{cases}$$

and therefore that

$$\begin{aligned} \rho_a \vee (\rho_c \cap \rho_b) &= \rho(\tau_a \vee (\tau_c \cap \tau_b), \{K_e \mid e \in Y\}) \\ &= \rho((\tau_a \vee \tau_c) \cap \tau_b, \{K_e \mid e \in Y\}) = (\rho_a \vee \rho_c) \cap \rho_b. \end{aligned}$$

The proof in the distributive case is very similar. $\square$

We remark that is a standard result that a group G has distributive subgroup lattice if and only if G is *locally cyclic*, by which we mean that every finitely generated subgroup of G is cyclic; for details see [55].

We also comment that classifications for modularity and distributivity of lattices of left congruences are known for other classes of inverse semigroups. Similar strategies may be employed to use the inverse kernel approach to reproduce these results, but we see little value doing so here. We state the result for Brandt semigroups as they form another class of the examples we have considered.

Theorem 4.4.15 ([60, Proposition 6.1])**.** *Let $S = \mathcal{B}(I, G)$ be a Brandt semigroup. Then $\mathfrak{LC}(S)$ is modular if and only if either $|I| = 1$ and G has modular lattice of subgroups, or $|I| = 2$ and G is trivial. Furthermore, $\mathfrak{LC}(S)$ is distributive if and only if $|I| = 1$ and G has distributive lattice of subgroups.*

— 5 —

Preliminaries II

I would like to reassure the reader that this second preliminary section is shorter (though I am typing this before I have written this chapter, so I may yet be proven wrong). I will endeavour to be more concise and only lay enough ground work to set the scene, and then only build up that which we actually need. Without further ado let us begin on our next journey, into the wondrous world of independence algebras, their partial automorphism monoids, and the lattices of one and two sided congruences on these latter beasts.

This section builds upon the work of Lima, whose thesis [42], entitled "The Local Automorphism Monoid of an Independence Alegbra", describes partial automorphism monoids of independence algebras, and constructs a framework to describe congruences on these monoids. This second preliminary chapter is largely devoted to summarising the relevant results from [42]; we lean upon this foundation in the remaining chapters,

In the introduction to Lima's thesis may be found the following quotation, regarding the description of the trace classes in the lattice of congruences.

> *We show, by considering the example of a free G-set for an arbitrary group G, that in general there is no hope of giving a precise description of the congruences in these intervals [the trace classes].*

To me that reads like a challenge. While in this half of the thesis we are not solely devoted to proving that it is possible to give a precise description of the trace classes in the lattice of congruences for the partial automorphism monoid of a free G-act, from a certain viewpoint Chapter 6 is essentially giving such a description.

5.1 Independence algebras

We return to the language of universal algebra, which we introduced in the first preliminary chapter. In the rest of the thesis we are largely concerned with partial automorphism monoids of a specific type of independence algebra, for which we do not need too much in the way of explicit universal algebra theory. With this in mind it is not really helpful or necessary to approach every concept entirely formally or even rigorously here; instead we aim to provide an introduction and overview.

The concept of independence algebras was introduced as v^*-algebras in [53] and formulated in its modern style in [23] and [19]. The idea is to generalise the notion of "independence", in the sense of vector spaces to a wider class of algebras. The following is a bit "definition splurgy", but this is unavoidable. If $\mathbf{A} = (A, F)$ is an algebra and $B \subseteq A$ then we write $\langle B \rangle$ for the subalgebra of $\mathbf{A}$ generated by B.

Definition 5.1.1. Let $\mathbf{A} = (A, F)$ be an algebra and let X be a subset of A. Then we say that X is *independent* if $x \notin \langle X \backslash \{x\} \rangle$ for all $x \in X$.

It is an elementary application of Zorn's lemma that given an independent set X in an algebra $\mathbf{A}$ there is a maximal independent set Y containing X.

Definition 5.1.2 (see [40, 2.3.1])**.** An algebra $\mathbf{A} = (A, F)$ is said to have the *exchange property* if $\mathbf{A}$ satisfies the following equivalent conditions:

(i) for every subset $X \subseteq A$ and all $u, v \in A$, if $u \in \langle X \cup \{v\} \rangle$ and $u \notin \langle X \rangle$ then $v \in \langle X \cup \{u\} \rangle$;

(ii) for every subset $X \subseteq A$ and all $u \in A$, if X is independent and $u \notin \langle X \rangle$ then $X \cup \{u\}$ is independent;

(iii) for every subset $X \subseteq A$ if Y is a maximal independent subset of X then $\langle Y \rangle = \langle X \rangle$;

(iv) for every subset $X \subseteq A$ and every independent subset Y of X there is an independent subset Z such that $Y \subseteq Z \subseteq X$ and $\langle Z \rangle = \langle X \rangle$.

Definition 5.1.3 (see [23, Corollary 3.1])**.** Let $\mathbf{A} = (A, F)$ be an algebra that has the exchange property, let $\mathbf{B}$ be a subalgebra of $\mathbf{A}$ and let X be a subset of B. Then X is a *basis* for $\mathbf{B}$ if the X satisfies the following equivalent conditions:

(i) X is a maximal independent subset of $\mathbf{B}$;

(ii) X is independent and $\langle X \rangle = B$;

(iii) X is minimal with respect to $\langle X \rangle = B$.

It follows from Definition 5.1.2 and the observation that any independent subset is contained in a maximal independent subset, that if an algebra $\mathbf{A} = (A, F)$ has the exchange property then it has a basis. In fact, if $Y \subseteq A$ is an independent set then there is a basis Z for $\mathbf{A}$ such that $Y \subseteq Z$. It is often said that Z *extends* Y *to a basis*.

Essentially, the exchange property says that bases for the algebra exist and every independent subset is a subset of some basis. However, it is possible for X and Y to be independent with $\langle X \rangle \cap \langle Y \rangle = \langle \emptyset \rangle$ and for $X \cup Y$ to not be independent. In terms of our intuition from vector spaces this would say that X and Y are bases for "orthogonal" subspaces and $X \cup Y$ is not independent, which is impossible. Therefore to fully capture the essence of the linear algebra notions we seek to generalise we strengthen the exchange property in the following way.

Definition 5.1.4. Let $\mathbf{A} = (A, F)$ be an algebra. We say that $\mathbf{A}$ has the *strong exchange property* if $\mathbf{A}$ satisfies the following conditions:

(i) for any independent subsets X and Y of A, if $\langle X \rangle \cap \langle Y \rangle = \langle \emptyset \rangle$ then $X \cup Y$ is independent;

(ii) for any $x, y \in A$, if $x \in \langle y \rangle$ and $x \notin \langle \emptyset \rangle$, then $y \in \langle x \rangle$.

We stated that the strong exchange property would be a strengthening of the exchange property, and we remark that this is indeed the case. Indeed, suppose $\mathbf{A}$ satisfies the strong exchange property, $X \subseteq A$ is an independent subset and $u \in A$ is such that $u \notin \langle X \rangle$. Then, by (ii) of the strong exchange property, $\langle u\rangle \cap \langle X\rangle = \langle\emptyset\rangle$, indeed, if $y \in \langle u\rangle \cap \langle X\rangle$ then $y \in \langle u\rangle$ so if $y \notin \langle\emptyset\rangle$ then $u \in \langle y\rangle \subseteq \langle X\rangle$, a contradiction. Then, by (i) of strong exchange property, $X \cup \{u\}$ is independent. Thus $\mathbf{A}$ has the exchange property. The converse is not true, there are examples of algebras which satisfy the exchange property but not the strong exchange property, though we do not meet any examples in this work. An example of a "non-strong independence algebra", originally attributed to Bardelang in an unpublished set of notes may be found in [42, Examples 1.3.9].

Definition 5.1.5. An algebra $\mathbf{A}$ is said to have the *free basis property* if for any basis X for $\mathbf{A}$ and any function $\alpha : X \to A$, there is an endomorphism $\bar{\alpha}$ of $\mathbf{A}$ that extends α.

We may think of the free basis property as saying that homomorphisms from an algebra are determined by how they behave on a basis. At this point we may define an independence algebra.

Definition 5.1.6. Let $\mathbf{A}$ be an algebra. We say that $\mathbf{A}$ is an *independence algebra* if $\mathbf{A}$ has the exchange property and the free basis property. In addition, we say that $\mathbf{A}$ is a *strong independence algebra* if $\mathbf{A}$ has the strong exchange property and the free basis property.

The final general notion which we shall need is the idea of the rank of an independence algebra. Using a very similar argument to that in a first course in linear algebra, is it possible to show that any basis for an independence algebra has the same cardinality.

Definition 5.1.7. Let $\mathbf{A}$ be an algebra and let X be a basis for $\mathbf{A}$. The *rank* of $\mathbf{A}$ is $|X|$, the cardinality of X. The definition of rank extends to subalgebras, the rank of a subalgebra is the cardinality of a basis for the subalgebra.

Let us have some examples to illustrate these concepts.

Example 5.1.8. A set X is a universal algebra, $\mathbf{X} = (X, \emptyset)$, with no operations. For any subset $Y \subseteq X$ we have $\langle Y\rangle = Y$. In this case the strong exchange property is trivially satisfied, as $Y \subseteq X$ is independent for all Y and $\langle Y \rangle = Y$. The only basis for $\mathbf{X}$ is X itself, as if Y is a basis for $\mathbf{X}$ then $\langle Y \rangle = X$, so $Y = X$. The free basis property is then also trivially true, as any function from a basis to X is an endomorphism of $\mathbf{X}$. Therefore $\mathbf{X}$ is a strong independence algebra.

Example 5.1.9. The canonical example of a strong independence algebra is that of a vector space. As a universal algebra we view a vector space as $\mathbf{V} = (V; +, \{f_\lambda \mid \lambda \in F\}, 0)$ where F is a field, f_λ is the unary operation defined $f_\lambda(u) = \lambda u$ for each $\lambda \in F$ (scalar multiplication), $+$ is the obvious binary operation and 0 is the degree 0 (constant) operation taking the value 0.

It is easy to see that the notion of independence defined here agrees with the usual notion of linear independence. The strong exchange property and the free basis property are standard properties of vector spaces, so $\mathbf{V}$ is a strong independence algebra.

In fact we do not need to restrict ourselves to vector spaces over fields. Instead we allow ourselves to consider a module over a *division ring* (a ring such that every element has a multiplicative inverse). It is straightforward that we still obtain a strong independence algebra.

The final example of a strong independence algebra which we meet is the most important to us. We spend the vast majority of the rest of the thesis considering it. We have previously defined a semigroup action, and the following definition of a group action should be familiar to all readers.

Definition 5.1.10. Let G be a group and A a set. Then we say a function $\bullet \colon G \times A \to A$ is a *left group action* or *G-act* if for all $g, h \in G$ and $a \in A$

$$g \bullet (h \bullet a) = (gh) \bullet a \qquad \text{and} \qquad 1 \bullet a = a.$$

We then say that *G acts (on the left) on A*, or that *A is a (left) G-act* and as usual we drop the $\bullet$ notation. Equivalently one can define actions on the right, though as is the theme of the thesis we choose the left. As is common, we will often assume that the function is implicit, and say that A is a G-act.

Definition 5.1.11. Let G be a group and let $G \times A \to A$ be a G-act. We say that A is *free* if for all $a \in A$ and $g \in G$

$$ga = a \implies g = 1.$$

As a universal algebra we may view a G-act as $\mathbf{A} = (A, \{f_g \mid g \in G\})$, where for each $g \in G$, f_g is the unary function $a \mapsto ga$. We then notice that for $a \in A$ the subalgebra generated by a is

$$\langle a \rangle = \{ga \mid g \in G\},$$

the orbit of a under the action of G.

We recall what constitutes a G-act homomorphism, a specialisation of the usual definition of morphism between universal algebras.

Definition 5.1.12. Let $\mathbf{A} = (A, \{f_g \mid g \in G\})$ and $\mathbf{B} = (B, \{f_g \mid g \in G\})$ be G-acts and let $\theta\colon A \to B$ be a function. We say that θ is a *G-act homomorphism* if for all $g \in G$ and $a \in A$

$$(f_g(a))\theta = f_g(a\theta).$$

Equivalently, in more familiar language, θ is a G-act homomorphism if $(ga)\theta = g(a\theta)$ for all $g \in G$ and all $a \in A$.

Let A be a free G-act and define an equivalence relation $\sim$ on A by setting $a \sim b$ if there is $g \in G$ such that $ag = b$, in other words, if a and b are in the same orbit under the action of G. Let X be a subset of A such that X contains exactly one element of each $\sim$-class. Then

$$A = \bigcup_{x \in X} \{gx \mid g \in G\},$$

and it is clear that this is a disjoint union. As A is free, the map $G \times X \to A$ defined $(g, x) \mapsto gx$ is a bijection. Furthermore, the function $G \times (G \times X) \to G \times X$ defined by $g(h, x) = (gh, x)$ is a free group action, and it is easy to see that the function $G \times X \to A$; $(g, x) \mapsto gx$ is a homomorphism of group actions. Thus it is an isomorphism. This motivates the following construction for a free group action.

Definition 5.1.13. Let G be a group and let X be a set. Then the *free G-act over X* is the set $G \times X$ together with the action

$$G \times (G \times X) \to G \times X; \qquad g(h, x) = (gh, x).$$

We usually drop the brackets and write hx for (h, x), and we identify x with $1x$. We write $\mathbf{A}_X$ for this G-act. When $|X| = n \in \mathbb{N}$ (when X is finite) we write $\mathbf{A}_n$ for $\mathbf{A}_X$.

We may think of $\mathbf{A}_X$ as a set of disjoint copies of the free monogenic G-act indexed by elements of X, where we recall that the free monogenic G-act is the action of G on itself via multiplication on the left.

Having introduced $\mathbf{A}_X$ after promising an example of a strong independence algebra we are somewhat obligated to demonstrate that $\mathbf{A}_X$ is indeed a strong independence algebra. We note that for $hx \in \mathbf{A}_X$

$$\langle hx \rangle = \{gx \mid g \in G\}.$$

It is then clear that a subset $\{h_i x_i \mid i \in I\} \subseteq G \times X$ is independent if and only if $x_i \neq x_j$ when $i \neq j$. The strong exchange property is then immediate.

We now verify the free basis property. If $Y = \{h_i x_i \mid i \in I\} \subseteq G \times X$ is a basis for $\mathbf{A}_X$ then this says that for each $x \in X$ there is exactly one $i \in I$ with $x_i = x$, therefore we may write $Y = \{h_x x \mid x \in X\}$. If $\alpha\colon Y \to G \times X$ is a function then we define

$$\overline{\alpha}\colon G \times X \to G \times X; \qquad gx \mapsto (gh_x^{-1})((h_x x)\alpha).$$

Then $\overline{\alpha}$ is a G-act endomorphism of $\mathbf{A}_X$, and for $h_x x \in Y$ we note that $(h_x x)\overline{\alpha} = (h_x x)\alpha$. Thus $\mathbf{A}_X$ satisfies the free basis property so is a strong independence algebra.

5.2 Partial automorphism monoids

Until further notice we assume that $\mathbf{A}$ is a strong independence algebra. In this section we present an introduction to the partial automorphism monoid for a strong independence algebra. It is perfectly natural to construct independence algebras with infinite rank, and, where simple to do so, we do not restrict definitions and results to the finite rank case. Since in subsequent sections we shall be focused on the finite rank case, we do not go to significant effort to deal with complications that arise when dealing with infinite rank independence algebras.

We know that a subalgebra is a subset which is an algebra (of the same type) upon the restriction of the operations; we denote the set of subalgebras of $\mathbf{A}$ by $\mathrm{Sub}(\mathbf{A})$.

Definition 5.2.1. Let $\mathbf{A}$ be an algebra. Then the *partial automorphism monoid*, $\mathrm{PAut}(\mathbf{A})$ of $\mathbf{A}$ is the set of isomorphisms between two (not necessarily distinct) subalgebras, that is

$$\mathrm{PAut}(\mathbf{A}) = \{a\colon \mathbf{B} \to \mathbf{A} \mid \mathbf{B} \in \mathrm{Sub}(\mathbf{A}),\ a \text{ an injective homomorphism}\},$$

under composition of partial functions.

Associated to $\mathrm{PAut}(\mathbf{A})$ are the domain function $\mathrm{Dom} : \mathrm{PAut}(\mathbf{A}) \to \mathrm{Sub}(\mathbf{A})$ and image function $\mathrm{Im} : \mathrm{PAut}(\mathbf{A}) \to \mathrm{Sub}(\mathbf{A})$. Concretely, the composition of $a, b \in \mathrm{PAut}(\mathbf{A})$ has $\mathrm{Dom}(ab) = (\mathrm{Im}(a) \cap \mathrm{Dom}(b))a^{-1}$ and $\mathrm{Im}(ab) = (\mathrm{Im}(a) \cap \mathrm{Dom}(b))b$, with a^{-1} the inverse of a as a partial function, and $x(ab) = (xa)b$ for all $x \in \mathrm{Dom}\, ab$. With this operation the set of partial automorphisms is an inverse monoid such that the inverse of a is a^{-1}. The group of units of $\mathrm{PAut}(\mathbf{A})$ is the automorphism group of $\mathbf{A}$ and the semilattice of idempotents is actually a lattice, which is isomorphic to the lattice of subalgebras of $\mathbf{A}$. In addition, $\mathrm{PAut}(\mathbf{A})$ has a zero, it is the identity map on $\langle \emptyset \rangle$, where $\langle \emptyset \rangle$ is the subalgebra of $\mathbf{A}$ generated by $\emptyset$.

The following result is important to the study of partial automorphism monoids of independence algebras.

Lemma 5.2.2 ([23, Lemma 3.8])**.** *Let $\mathbf{A}$ be an independence algebra and let $a \in \mathrm{PAut}(\mathbf{A})$. Then $X \subseteq \mathrm{Dom}(a)$ is independent if and only if Xa is independent.*

Definition 5.2.3. Let $a \in \mathrm{PAut}(\mathbf{A})$ and let $Y \subseteq \mathrm{Im}(a)$ be independent such that $\langle Y \rangle = \mathrm{Im}(a)$. The *rank* of is $\mathrm{rank}(a) = |Y|$.

We remark that the rank of a partial automorphism is well defined since the rank of a subalgebra is well defined. It follows from Lemma 5.2.2 that the rank may also be defined in terms of the cardinality of a basis for the domain. It is easy to see that if $a, b \in \mathrm{PAut}(\mathbf{A})$ then $\mathrm{rank}(ab) \leq \mathrm{rank}(a)$ and $\mathrm{rank}(ab) \leq \mathrm{rank}(b)$. This implies that for each cardinal λ the set

$$I_\lambda = \{a \in \mathrm{PAut}(\mathbf{A}) \mid \mathrm{rank}(a) < \lambda\}$$

is an ideal of $\mathrm{PAut}(\mathbf{A})$ (noting that if $\lambda > \mathrm{rank}(\mathbf{A})$ then $I_\lambda = \mathrm{PAut}(\mathbf{A})$). In fact, these are all the ideals of $\mathrm{PAut}(\mathbf{A})$. If λ is a cardinal then λ^+ is the successor cardinal to λ.

Proposition 5.2.4 ([42, Proposition 2.1.1])**.** *Every ideal of* $\mathrm{PAut}(\mathbf{A})$ *is of the form* I_λ *for some cardinal* λ*, with* $1 \leq \lambda \leq (\mathrm{rank}(\mathbf{A}))^+$.

Our next result describes Green's relations on $\mathrm{PAut}(\mathbf{A})$.

Proposition 5.2.5 ([42])**.** *Green's relations on* $\mathrm{PAut}(\mathbf{A})$ *are as follows*

$$\begin{aligned} a \mathrel{\mathcal{R}} b &\iff \mathrm{Dom}(a) = \mathrm{Dom}(b), \\ a \mathrel{\mathcal{L}} b &\iff \mathrm{Im}(a) = \mathrm{Im}(b), \\ a \mathrel{\mathcal{H}} b &\iff \mathrm{Dom}(a) = \mathrm{Dom}(b) \quad \textit{and} \quad \mathrm{Im}(a) = \mathrm{Im}(b), \\ a \mathrel{\mathcal{D}} b &\iff \mathrm{rank}(a) = \mathrm{rank}(b) \iff a \mathrel{\mathcal{J}} b. \end{aligned}$$

It will also be important to understand the principal factors.

Proposition 5.2.6 ([42, Proposition 2.1.3])**.** *Let* $\mathbf{A}$ *be a strong independence algebra and let* n *be an natural number with* $1 \leq n \leq \mathrm{rank}\,\mathbf{A}$*. Then each principal factor* I_n/I_{n-1} *of* $\mathrm{PAut}(\mathbf{A})$ *is a completely* 0*-simple semigroup.*

We return to consider our examples: a set, a module over a division ring and a free G-act.

Example 5.2.7. First we consider a set $\mathbf{X}$. A partial automorphism is simply a partial bijection, and $\mathrm{PAut}(\mathbf{X})$ is the symmetric inverse monoid $\mathcal{I}_X$. The principal factor I_n/I_{n-1} is the Brandt semigroup $\mathcal{B}(P_n, \mathcal{S}_n)$, where P_n is the set of subsets of size n.

Example 5.2.8. Next we let $\mathbf{V} = (V; +, \{f_\alpha \mid \alpha \in F\}, 0)$ be a vector space over a field F. Then the domain and image of $a \in \mathrm{PAut}(\mathbf{V})$ are subspaces of V and a is an isomorphic linear map. In particular, a group $\mathcal{H}$-class is the set of automorphisms of a subspace, so the group $\mathcal{H}$-classes are general linear groups. The principal factor I_n/I_{n-1} is the Brandt semigroup $\mathcal{B}(W_n, GL_n(F))$, where W_n is the set of subspaces of dimension n and $GL_n(F)$ is the general linear group of an n-dimensional vector space over F.

Again we spend much more time considering a free G-act. First, we consider the subalgebras of a free G-act. We recall that we write $\mathbf{A}_X$ for the free G-act over X and we also write Gx for the set $\{gx \mid g \in G\}$.

Lemma 5.2.9. *Let $\mathbf{A}_X$ be a free G-act. If $Y \subseteq X$ then $\mathbf{A}_Y$ is a subact of $\mathbf{A}_X$. Conversely, if $\mathbf{B} \subseteq \mathbf{A}_X$ is a subact then there is $Y \subseteq X$ such that $\mathbf{B} = \mathbf{A}_Y$. Consequently, if $\mathbf{B} \subseteq \mathbf{A}$ is a subact then $\mathbf{B}$ is also a free G-act.*

Proof. This is immediate, noting that for $x \in X$ if there is $g \in G$ such that $gx \in B$ then $Gx \subseteq B$. $\square$

Lemma 5.2.9 implies that for the free G-act $\mathbf{A}_X$ the lattice of subacts is isomorphic to the lattice of subsets of X. Moreover, for $\theta \in \mathrm{PAut}(\mathbf{A}_X)$ and $x \in X$ with $x \in \mathrm{Dom}(\theta)$, if $x\theta = ky$ (where $y \in X$) then we observe that $\theta|_{Gx}$ is a G-act isomorphism from Gx to Gy. Thus θ defines an element $a_\theta \in \mathcal{I}_X$ where $x \in \mathrm{Dom}(a_\theta)$ if and only if $x \in \mathrm{Dom}(\theta)$, and $xa_\theta = y$ where $x\theta \in Gy$. Furthermore, it is clear that the map $\theta \mapsto a_\theta$ defines a homomorphism $\mathrm{PAut}(\mathbf{A}_X) \to \mathcal{I}_X$.

For a set X we write $G^{PX} = \{f \colon Y \to G \mid Y \subseteq X\}$ for the set of partial functions from X to G. We define the product of $f, g \in G^{PX}$ by:

$$\mathrm{Dom}(fg) = \mathrm{Dom}(f) \cap \mathrm{Dom}(g), \quad x(fg) = (xf)(xg) \text{ for } x \in \mathrm{Dom}(fg).$$

Given $\theta \in \mathrm{PAut}(\mathbf{A}_X)$ we define $f_\theta \in G^{PX}$ as the group label of the restriction of θ to the set $\mathrm{Dom}(\theta) \cap X$, so if $x\theta = gy$ then $xf_\theta = g$. We notice that θ is fully characterised by f_θ and a_θ as $x\theta = (xf_\theta)(xa_\theta)$ for each x with $Gx \subseteq \mathrm{Dom}(\theta)$, and we can then uniquely extend θ to the rest of $\mathrm{Dom}(\theta)$ by the freeness of $\mathbf{A}_X$. Explicitly,

$$(gx)\theta = g(x\theta) = g(xf_\theta)(xa_\theta).$$

This leads us to consider a partial wreath product as defined in [47]. For $a \in \mathcal{I}_X$ and $f \in G^{PX}$ define $f^a \in G^{PX}$ as:

$$\mathrm{Dom}(f^a) = \{x \in \mathrm{Dom}(a) \mid xa \in \mathrm{Dom}(f)\}, \quad x(f^a) = (xa)f.$$

Then we define the *partial wreath product* $G \wr \mathcal{I}_n$ to be:

$$G \wr \mathcal{I}_X = \{(f; a) \in G^{PX} \times \mathcal{I}_X \mid \mathrm{Dom}(f) = \mathrm{Dom}(a)\},$$

with multiplication

$$(f; a)(g; b) = (fg^a; ab).$$

The following result is essentially folklore. We present an outline proof here for completeness.

Theorem 5.2.10. *The function*

$$\Phi : \mathrm{PAut}(\mathbf{A}_X) \to G \wr \mathcal{I}_X; \quad \theta \mapsto (f_\theta; a_\theta)$$

is an isomorphism.

Proof. As we have noted a partial automorphism θ is determined by f_θ and a_θ, thus Φ is injective. It is also straightforward that given $f \in G^{PX}$ and $a \in \mathcal{I}_X$ with $\mathrm{Dom}(a) = \mathrm{Dom}(f)$ the function $x \mapsto (xf)(xa)$ (defined for x with $x \in \mathrm{Dom}(a)$) extends uniquely to a partial automorphism $\theta \in \mathrm{PAut}(\mathbf{A}_X)$ with $f = f_\theta$ and $a = a_\theta$, so Φ is surjective. Thus it remains to show that Φ is a homomorphism.

To this end suppose that $\theta, \gamma \in \mathrm{PAut}(\mathbf{A}_X)$. As remarked previously the function $\theta \mapsto a_\theta$ is a homomorphism so $a_{\theta\gamma} = a_\theta a_\gamma$, and the definitions give that $\mathrm{Dom}(f_{\theta\gamma}) = \mathrm{Dom}(a_{\theta\gamma})$. Also, for $x \in \mathrm{Dom}(a_{\theta\gamma})$, by definition $xf_{\theta\gamma}$ is the group label of

$$x(\theta\gamma) = (x\theta)\gamma = ((xf_\theta)(xa_\theta))\gamma = (xf_\theta)((xa_\theta)\gamma),$$

and the group label of $(xa_\theta)\gamma$ is $(xa_\theta)f_\gamma = xf_\gamma^{a_\theta}$. Thus

$$xf_{\theta\gamma} = (xf_\theta)(xf_\gamma^{a_\theta}) = x(f_\theta f_\gamma^{a_\theta}),$$

so $f_{\theta\gamma} = f_\theta f_\gamma^{a_\theta}$. It follows that Φ is a homomorphism. $\square$

While this is the morally correct way to construct $G \wr \mathcal{I}_X$, and justifies us referring to it as a partial wreath product, it is generally unwieldy, and sends one down endless rabbit-holes of notational difficulty. As mentioned, we are interested in the world where X is finite, and in this case, fortunately, it is possible to pin down a less aesthetically pleasing but more 'user-friendly' version of $G \wr \mathcal{I}_n$. For a group G we write G_0 for the group with a zero adjoined, so $G_0 = G \cup \{0\}$ with multiplication extended by declaring $g0 = 0 = 0g$ for all $g \in G_0$. We remark that this is different from our usual notation for adjoining a 0 to a semigroup, we choose this alternate notation as we shall want to use to the 0^{th} power of G_0 on several occasions and do not want to write $(G^0)^0$.

To establish our new formulation for our partial wreath product, let $a \in \mathcal{I}_n$, and $g \in (G_0)^n$. Write $g_a = (g_{1a}, \ldots, g_{na})$ where we take $g_{ia} = 0$ if ia is undefined. In particular for an idempotent $e \in E(\mathcal{I}_n)$ we write 1_e for $(1, 1, \ldots, 1)_e$ so that 1_e has a 1 in position i if $i \in \mathrm{Dom}(e)$ and 0's elsewhere. For each $a \in \mathcal{I}_n$ the function $g \mapsto g_a$ is an endomorphism of $(G_0)^n$ and is, in general, neither injective nor surjective.

Lemma 5.2.11. *The function*

$$\Psi \colon \mathcal{I}_n \to \mathrm{End}((G_0)^n); \qquad a \mapsto [g \mapsto g_a]$$

is an antihomomorphism (by which we mean that $(ab)\Psi = (b\Psi)(a\Psi)$).

Proof. Let $a, b \in \mathcal{I}_n$ and $(g_1, \ldots, g_n) \in G^0$, and for each $1 \leq i \leq n$ let $h_i = g_{ib}$ if ib is defined and 0 otherwise and let $k_i = h_{ia}$ if ia is defined and 0 otherwise. Then

$$(g_1, \ldots, g_n)(b\Psi)(a\Psi) = (h_1, \ldots, h_n)(a\Psi) = (k_1, \ldots, k_n).$$

We observe that if $h_i \neq 0$ then $i \in \mathrm{Dom}(b)$ and $h_i = g_j$ where $j = ib$, or equivalently $i = jb^{-1}$. Also for $1 \leq m \leq n$ we have that if $k_m \neq 0$, then $m \in \mathrm{Dom}(a)$ and $k_m = h_i$ where $ma = i$, and $h_i \neq 0$. Combining these we have $h_i = g_j$ where $i = jb^{-1}$, and so we have that $k_m = g_j$ where $ma = i = jb^{-1}$, or equivalently $j = m(ab)$. Thus $(k_1, \ldots, k_n) = (g_{1(ab)}, \ldots, g_{n(ab)})$ and we see that $(ab)\Psi = (b\Psi)(a\Psi)$. $\square$

We define

$$S = \{(g; a) \in (G_0)^n \times \mathcal{I}_n \mid g_i \neq 0 \iff i \in \mathrm{Dom}(a)\},$$

with multiplication

$$(g; a)(h; b) = (g_1, \ldots, g_n; a)(h_1, \ldots, h_n; b) = (g_1 h_{1a}, \ldots, g_n h_{na}; ab) = (gh_a; ab).$$

It is elementary that $S \cong G \wr \mathcal{I}_n$ via the map $(g; a) \mapsto (f; a)$ where $\mathrm{Dom}(f) = \mathrm{Dom}(a) = \{i \in [n] \mid g_i \neq 0\}$ and $if = g_i$. In the rest of the thesis we shall refer to this second formulation when we write $G \wr \mathcal{I}_n$.

We notice that the condition for $(g; a)$ to be in $G \wr \mathcal{I}_n$, that is $g_i \neq 0$ if and only if $i \in \mathrm{Dom}(a)$, can be reformulated as $g = g_{aa^{-1}}$ and $g \neq g_e$ for all $e \in E(\mathcal{I}_n)$ with $e < aa^{-1}$. We shall use this implicitly on occasion, particularly the consequence that if $(g; a) \in G \wr \mathcal{I}_n$ then $g_{aa^{-1}} = g$. We also define, for $e \in E(\mathcal{I}_n)$

$$G^e = \{g \in (G_0)^n \mid g_i \neq 0 \iff i \in \mathrm{Dom}(e)\} = \{g \in (G_0)^n \mid (g; e) \in G \wr \mathcal{I}_n\}.$$

There are some calculations in $G \wr \mathcal{I}_n$ which we shall do frequently, and it is helpful to gather them in one place so that we may refer to them. We do that here. The proof of each is elementary and is left to the reader.

Lemma 5.2.12. *The following hold in $G \wr \mathcal{I}_n$.*

1. *Let $(g; a)$ be an element in $G \wr \mathcal{I}_n$. Then*

$$(g; a)^{-1} = (g_1, \ldots, g_n; a)^{-1} = (g^{-1}_{1a^{-1}}, \ldots, g^{-1}_{na^{-1}}; a^{-1}) = (g^{-1}_{a^{-1}}; a^{-1})$$

where we write $0^{-1} = 0$.

2. *For any* $(g;a) \in G \wr \mathcal{I}_n$,

$$(g;a) = (g;aa^{-1})(1_{aa^{-1}};a) = (1_{aa^{-1}};a)(g_{a^{-1}};a^{-1}a).$$

3. *Let* $e \in E(\mathcal{I}_n)$, $g \in G^e$, *and* $a \in \mathcal{I}_n$, *then*

$$(1_{aa^{-1}};a)(g;e)(1_{aa^{-1}};a)^{-1} = (1_{aa^{-1}};a)(g;e)(1_{a^{-1}a};a^{-1}) = (g_a;aea^{-1}).$$

4. *Let* $e \in E(\mathcal{I}_n)$ *and* $(g;e),(h;a) \in G \wr \mathcal{I}_n$, *then*

$$(g;e)(h;a)(g;e)^{-1} = (g;e)(h;a)(g^{-1};e) = (gh_e g_{ea}^{-1};eae) = (ghg_a^{-1};eae).$$

In particular, if $a = f \in E(\mathcal{I}_n)$ *then*

$$(g;e)(h;f)(g;e)^{-1} = (g;e)(h;f)(g^{-1};e) = (ghg^{-1};ef).$$

5. *Let* $(g;e),(h;f) \in G \wr \mathcal{I}_n$ *with* $e,f \in E(\mathcal{I}_n)$, *then*

$$(g;e)(h;f) = (g_{ef}h_{ef};ef) = (gh;ef).$$

6. *Let* $e \in E(\mathcal{I}_n)$, $g \in G^e$ *and* $a \in \mathcal{I}_n$, *then* $g_a \in G^{aea^{-1}}$. *In particular,* $1_a \in G^{aa^{-1}}$ *and so* $1_a = 1_{aa^{-1}}$ *(note this implies that* $(1_{aa^{-1}};a)^{-1} = (1_{a^{-1}a};a^{-1})$*).*

We can view the elements of $G \wr \mathcal{I}_n$ pictorially. As we have seen before we consider $a \in \mathcal{I}_n$ as a graph with two rows each of n vertices - indexed as $1, 2, \ldots, n$ for the upper row and $1', 2', \ldots, n'$ for the lower row - with edges (i, j') if $ia = j$. We then consider $(g;a) \in G \wr \mathcal{I}_n$ as the graph of a with the top row labelled with the coordinates of G. We compose the graphs as elements of $\mathcal{I}_n$, and then "slide" the labels up adjacent edges. For an example refer to Fig. 5.1.

We now compute the principal factors of the form I_{m+1}/I_m (see [42, Examples 2.1.4]). The set of subacts of $\mathbf{A}_n$ of rank m is, by Lemma 5.2.9, in bijection with subsets of $[n]$ of size m. The $\mathcal{H}$-class for an idempotent $(1_e;e)$ of rank m is easily seen to be the set of elements $(g;a) \in G \wr \mathcal{I}_n$ such that $a \;\mathcal{H}(\mathcal{I}_n)\; e$. Further, we recall the usual isomorphism $\theta\colon H_e(\mathcal{I}_n) \to \mathcal{S}_m$, and

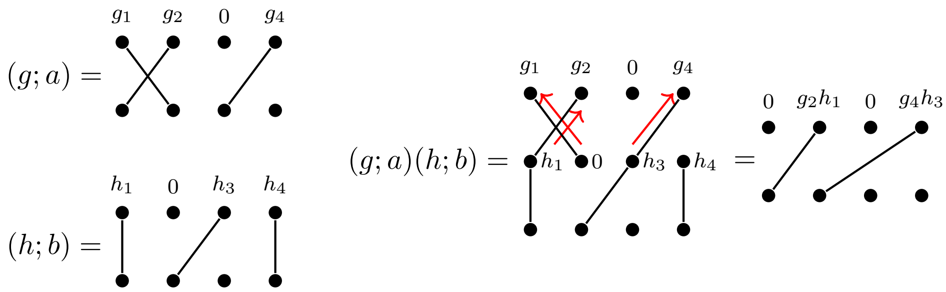


Figure 5.1: Multiplication in $G \wr \mathcal{I}_n$

we define the function $\omega\colon (G_0)^n \to \bigcup_{0\leq k\leq n} G^k$ as the function that ignores zero entries. We then observe that we may define a function

$$\Phi\colon H_{(1_e;e)} \to G^m \times \mathcal{S}_m; \qquad (g;a) \to (g\omega; a\theta),$$

and this is a bijection. Furthermore, there is a multiplication on $G^m \times \mathcal{S}_m$ induced by this mapping, which is given by

$$(g_1,\ldots,g_m;a)(h_1,\ldots,h_m;b) = (g_1h_{1a},\ldots,g_mh_{ma};ab).$$

This is easily seen to be a group, and is in fact the usual construction of the wreath product of G with the symmetric group $\mathcal{S}_m$. We write $G \wr \mathcal{S}_m$ for this group. Therefore the group $\mathcal{H}$-classes for idempotents of rank m in $G \wr \mathcal{I}_n$ are isomorphic to $G \wr \mathcal{S}_m$, and the principal factor I_{m+1}/I_m is isomorphic to the Brandt semigroup

$$\mathcal{B}(P_m, G \wr \mathcal{S}_m)$$

where $P_m = \{X \subseteq [n] \mid |X| = m\}$.

5.3 Congruences on the partial automorphism monoid of an independence algebra

One motivation for the study of partial automorphism monoids for independence algebras is to study inverse monoids that are similar in structure and derivation to the symmetric inverse monoid. We recall that the lattice of congruences on $\mathcal{I}_n$ has an elementary structure, it is a chain. One might expect therefore, that the study of congruences on $\mathrm{PAut}(\mathbf{A})$ is limited and the lattice of congruences is simple. While there are parallels to and inspiration drawn from the approaches to describing congruences on $\mathcal{I}_X$, and $\mathfrak{C}(\mathcal{I}_X)$ is a special case of the general description of $\mathfrak{C}(\mathrm{PAut}(\mathbf{A}))$ for a strong independence algebra, we shall see that these lattices may have a rich structure. Just as in the case for $\mathcal{I}_X$ the structure of the lattice of congruences looks very different when X is finite or when X is infinite. In later chapters we focus on finite rank independence algebras, which correspond to finite X. For this reason, and the fact that dealing with the infinite rank case requires a fair amount of set up and is definition heavy, we do not cover it in this thesis, but direct any interested party to [42]. In fact, although in the following discussion we talk about congruences on partial automorphism monoids for finite rank independence algebras, the results also hold in the infinite rank case, describing those congruences with finite *primary cardinal*, which for a congruence ρ is the cardinal λ which has $0\rho = I_\lambda$. For simplicity and clarity, until and unless otherwise stated we take $\mathbf{A}$ to be a finite rank independence algebra; we say its rank is n.

We know that the ideals of $\mathrm{PAut}(\mathbf{A})$ are the sets

$$I_m = \{a \in \mathrm{PAut}(\mathbf{A}) \mid \mathrm{rank}(a) < m\}$$

for $1 \leq m \leq n+1$. As 0ρ is an ideal of $\mathrm{PAut}(\mathbf{A})$ (this is the case for all congruences on semigroups with a 0) there is some m with $0\rho = I_m$, and ρ is the universal congruence precisely when $m = n+1$. If ρ is non-universal then ρ induces a non-universal congruence on the principal factor I_{m+1}/I_m. Furthermore, since I_{m+1}/I_m is a Brandt semigroup (Proposition 5.2.6), we may apply Theorem 1.5.12 which says that every non-universal congruence is contained in $\mathcal{H}$, and the lattice of non-universal congruences is isomorphic to the lattice of normal subgroups of a group $\mathcal{H}$-class.

It follows from Proposition 5.2.5 that the $\mathcal{D}$-classes in $\mathrm{PAut}(\mathbf{A})$ are the sets $D_m = \{a \in \mathrm{PAut}(\mathbf{A} \mid \mathrm{rank}(a) = m\}$ for $0 \leq m \leq n$. Therefore the ideal I_{m+1} is equal to $D_0 \cup D_1 \cup \cdots \cup D_m$. Therefore, if we are given a relation σ on I_{m+1}/I_m we can define a relation $\overline{\sigma}$ on D_m by

$$\overline{\sigma} = \{(a,b) \in D_m \times D_m \mid (a/I_m, b/I_m) \in \sigma\},$$

where we denote by a/I_m the equivalence class containing a of the Rees congruence defined by the ideal I_m. When we want to refer to this Rees congruence explicitly as a binary relation on $\mathrm{PAut}(\mathbf{A})$ we shall write $I_m^\star$. We may now state the decomposition for congruences on $\mathrm{PAut}(\mathbf{A})$ given in [42].

Theorem 5.3.1 ([42, Lemma 3.2.5 & Theorem 3.2.6])**.** *Let χ be an idempotent separating congruence on* $\mathrm{PAut}(\mathbf{A})$, *let $1 \leq m \leq n$ and let σ be a non-universal congruence on I_{m+1}/I_m such that $\chi \cap (D_m \times D_m) \subseteq \overline{\sigma}$. Then*

$$\rho(m, \sigma, \chi) = I_m^\star \cup \overline{\sigma} \cup \chi$$

is a non-universal congruence on $\mathrm{PAut}(\mathbf{A})$.

Conversely, if ρ is a non-universal congruence on $\mathrm{PAut}(\mathbf{A})$ *then with $\chi = \rho \cap \mu$ (where μ is the maximum idempotent separating congruence), m such that $0\rho = I_m$ and σ chosen such that $\overline{\sigma} = \rho \cap (D_m \times D_m)$, we have $\rho = \rho(m, \sigma, \chi)$.*

Consequently, the problem of describing congruences on $\mathrm{PAut}(\mathbf{A})$ reduces to describing idempotent separating congruences and to describing congruences on the principal factors.

Remark 5.3.2. Since the principal factors of $\mathrm{PAut}(\mathbf{A})$ are Brandt semigroups all non-universal congruences are idempotent separating congruences (see Theorem 1.5.12 and [63]) so it is possible to formulate Theorem 5.3.1 as the decomposition

$$\rho = I_m^\star \cup \tilde{\zeta}$$

where $\tilde{\zeta}$ is the lift of ζ - an idempotent separating congruence on the Rees quotient $(\mathrm{PAut}(\mathbf{A}))/I_m$ - to $\mathrm{PAut}(\mathbf{A})$. However, for each m an idempotent separating congruence on $(\mathrm{PAut}(\mathbf{A}))/I_m$ can be decomposed into a non-universal congruence on I_{m+1}/I_m and the projection of an idempotent separating congruence on $\mathrm{PAut}(\mathbf{A})$ onto $(\mathrm{PAut}(\mathbf{A}))/I_m$. Thus it is better to go straight for the decomposition given in Theorem 5.3.1.

We understand congruences on Brandt semigroups (Theorem 1.5.12), so the remaining obscure aspect of Theorem 5.3.1 is idempotent separating congruences. We recall Theorem 1.3.26, which says that the lattice of idempotent separating congruences is isomorphic to the lattice of normal subsemigroups contained in $E\zeta$, the centraliser of the idempotents. We reconsider our examples. If $\mathbf{A}$ is a set then the centraliser of the idempotents is just the set of idempotents, so the lattice of idempotent separating congruences is a singleton. If $\mathbf{A}$ is a vector space then in [42, Theorem 3.1.14] it is shown that a normal subsemigroup of $E\zeta$ is determined by a chain of normal subgroups of the multiplicative group of the field. The description of normal subsemigroups of $E\zeta$ in the case of a free group action is the subject of Chapter 6.

5.4 Subgroups of direct and semidirect products

It is probably not difficult to see that there ought to be some relation between the lattices $\mathfrak{C}(G \wr \mathcal{I}_n)$ and $\mathfrak{N}(G)$ (the lattice of normal subgroups of G). There is a strong relationship: in [42] it is shown that there are "many" embeddings of $\mathfrak{N}(G)$ into $\mathfrak{C}(G \wr \mathcal{I}_n)$. Part of the decomposition of congruences on $G \wr \mathcal{I}_n$ given in Theorem 5.3.1 is a congruence on a principal factor, and we know (Proposition 5.2.6) the principal factors for $G \wr \mathcal{I}_n$ are the Brandt semigroups $\mathcal{B}(P_k, G \wr \mathcal{S}_k)$. As non-universal congruences on Brandt semigroups correspond to normal subgroups of the group we need to understand normal subgroups of $G \wr \mathcal{S}_k$. In Chapter 6 we describe the set of normal subgroups of $G \wr \mathcal{S}_k$, so in this section we introduce the material on which we call.

In Chapter 8 we describe one sided congruences on $G \wr \mathcal{I}_n$ via the inverse kernel approach. This requires us to understand the lattice of full inverse subsemigroups of $G \wr \mathcal{I}_n$. As we know (Theorem 3.1.4) this may be realised as a subdirect product of the lattices of full inverse subsemigroups of the principal factors. It is clear that we can embed the lattice of subgroups of G^k into the lattice of subgroups of $G \wr \mathcal{S}_k$, and in turn embed the lattice of subgroups of $G \wr \mathcal{S}_k$ into the lattice of full inverse subsemigroups of $\mathcal{B}(P_k, G \wr \mathcal{S}_k)$. We shall see that in fact subgroups of G^k play an important role in describing left congruences on $G \wr \mathcal{I}_n$.

Getting hold of the sets of subgroups of G^k is remarkably difficult (when k is at least 3 anyway), in this section we introduce one way in which these sets of subgroups may be described.

The standard starting point in the consideration of subgroups of direct products of groups is Goursat's lemma.

Theorem 5.4.1 (Goursat's Lemma [25]). *Let G, H be groups. Then the subgroups $X \leq G \times H$ are exactly the sets*

$$X(A, B, C, D, \theta) = \{(a, b) \in A \times B \mid (aC)\theta = bD\},$$

where $C \trianglelefteq A \leq G$, $D \trianglelefteq B \leq H$ and $\theta\colon A/C \to B/D$ is an isomorphism.

The relationship between the quintuples and the subgroups in Goursat's Lemma is as follows. The subgroups A and B are the projections of $X = X(A, B, C, D, \theta) \leq G \times H$ onto the first and second coordinates respectively, the subgroups C and D are the kernels of the these projections so

$$C = \{c \in G \mid (c, 1) \in X\} \quad \text{and} \quad D = \{d \in H \mid (1, d) \in X\}$$

and finally the isomorphism θ is the function $aC \mapsto bD$ if $(a, b) \in X$. It is straightforward to see that the inclusion ordering on subgroups of $G \times H$ implies the following relationship between the quintuples to which the subgroups correspond.

Corollary 5.4.2. *Let G, H be groups and let $X = X(A, B, C, D, \theta)$ and $X' = X(A', B', C', D', \theta')$ be subgroups of $G \times H$. Then $X \leq X'$ if and only if: $A \leq A'$, $B \leq B'$, $C \leq C'$, $D \leq D'$ and if $(aC)\theta = bD$ then $(aC')\theta' = bD'$.*

We remark that we may rewrite the condition on the homomorphisms in Corollary 5.4.2 as $\theta'|_{A/C'} = \theta\pi$ where $\pi\colon B/D \to B'/D'$ is the obvious quotient map $bD \mapsto bD'$. In other words θ' is an extension of $\theta\pi$.

Subgroups of higher order direct products are harder to describe. A first attempt to generalise Goursat's lemma to subgroups of $G \times H \times K$ might be to consider the set

$$\{(a, b, e) \in A \times B \times E \mid aC\theta = bD,\ aC\phi = eF\}$$

where $A \trianglelefteq C \leq G$, $D \trianglelefteq B \leq H$, $F \trianglelefteq E \leq K$, and $\theta\colon A/C \to B/D$ and $\phi\colon A/C \to E/F$ are isomorphisms. It is straightforward to verify that this is a subgroup of $G \times H \times K$ and one might hope that subgroups of $G \times H \times K$ are in bijective correspondence with octuples which have the same properties as $(A, B, C, D, E, F, \theta, \phi)$. However this is not the case; the subgroups fail to be uniquely determined by the octuple, meaning that there are multiple subgroups of $G \times H \times K$ that would give the same octuple. For an example of when this approach fails to work we direct the reader to [2, Section 5]. The subject of the aforementioned paper ([2]) is to extend (or generalise) Goursat's lemma to higher order direct products. This is done with the following fashion.

Given groups A and B, a normal subgroup $D \trianglelefteq B$ and an onto homomorphism $\theta : A \to B/D$ define the set

$$\Gamma(A, B, D, \theta) = p^{-1}(\mathcal{G}_\theta) \leq A \times B$$

where $\mathcal{G}_\theta \subseteq A \times (B/D)$ is the graph of θ and $p\colon A \times B \to A \times (B/D)$ is the natural surjection. Equivalently:

$$\Gamma(A, B, D, \theta) = \{(a, b) \in A \times B \mid a\theta = bD\}$$

Theorem 5.4.3 (Generalised Goursat's Lemma, [2, Theorem 3.2])**.** *There is a bijective correspondence between the subgroups $G \leq A_1 \times \cdots \times A_n$ and $(3n-2)$-tuples*

$$(H_1; H_2, K_2, \theta_1; \ldots; H_n, K_n, \theta_{n-1})$$

Where $H_i \leq A_i$, $K_i \trianglelefteq H_i$ and $\theta_i : \Lambda_i \twoheadrightarrow H_{i+1}/K_{i+1}$ is a surjective homomorphism, and $\Lambda_i \leq A_1 \times \cdots \times A_i$ is defined inductively, with $\Lambda_1 = H_1$ and for $1 \leq i \leq n-1$

$$\Lambda_{i+1} = \Gamma(\Lambda_i, H_{i+1}, K_{i+1}, \theta_i) \leq (A_1 \times \cdots \times A_i) \times A_{i+1}.$$

We call the decomposition in Theorem 5.4.3 the *Goursat's decomposition* for a subgroup of a direct product. The construction is essentially an inductive one, constructing subgroups of $G \times H \times K$ from subgroups of $G \times H$ and K, and so on. We briefly explain the relationship between a subgroup and its Goursat's decomposition. Let $G \leq A_1 \times \cdots \times A_n$ be a subgroup and let $(H_1; H_2, K_2, \theta_1; \ldots; H_n, K_n, \theta_{n-1})$ be the Goursat's decomposition. Then H_i is the projection of G onto the i^{th} coordinate. We obtain K_i as

$$K_i = \{a \in A_i \mid \exists x_{j+1} \in A_{j+1}, \ldots, x_n \in A_n, \quad (1, \ldots, 1, a, x_{j+1}, \ldots, x_n) \in G\},$$

in other words, K_i is the projection onto the i^{th} coordinate of the kernel of the projection onto the first $i-1$ coordinates. Finally, we recover θ_i in the following way. Let Λ_i be the projection of G onto the first i coordinates. Then θ_i is defined by

$$\theta_i \colon \Lambda_i \to H_{i+1}/K_{i+1}; \quad (a_1, \ldots, a_i) \mapsto hK_{i+1} \text{ where } (a_1, \ldots, a_i, h) \in \Lambda_{i+1}.$$

Conversely, if we start with a Goursat's decomposition then the associated subgroup is Λ_n, as constructed in Theorem 5.4.3. Further, the two notions of Λ_i agree, in other words, the inductive construction in Theorem 5.4.3 is the projection of the associated subgroup onto the first i coordinates.

As we are interested in lattices it is relevant to remark on the inclusion ordering of the subgroups in terms of the Goursat's decomposition. Suppose that $A, B \leq G_1 \times \cdots \times G_n$ with $A \leq B$ and that A has Goursat's decomposition $(H_1^a; H_2^a, K_2^a, \theta_1^a; \ldots; H_n^a, K_n^a, \theta_{n-1}^a)$, and B has Goursat's decomposition $(H_1^b; H_2^b, K_2^b, \theta_1^b; \ldots; H_n^b, K_n^b, \theta_{n-1}^b)$. It is clear from the explanation of how $H_i^a, H_i^b, K_i^a, K_i^b$ are related to A, B that $H_i^a \leq H_i^b$, and $K_i^a \leq K_i^b$. Also, $(a_1, \ldots, a_n) \in A$ says that $(a_1, \ldots, a_i)\theta_i^a = a_{i+1}K_{i+1}^a$ for each i. Since $A \leq B$ we have $(a_1, \ldots, a_n) \in B$, which says that $(a_1, \ldots, a_i)\theta_i^b = a_{i+1}K_{i+1}^b$ for each i. We then note that with $\sigma_i : H_{i+1}^a/K_{i+1}^a \to H_{i+1}^a/K_{i+1}^b$ the usual quotient map we have for each $i = 1, \ldots, n-1$ that

$$(a_1, \ldots, a_i)\theta_i^a\sigma_i = a_{i+1}K_{i+1}^b.$$

On the other hand as $(a_1, \ldots, a_n) \in B$ we also observe that $(a_1, \ldots, a_i)\theta_i^b = a_{i+1}K_{i+1}^b$. Hence we have that for each i

$$\theta_i^b|_{\Lambda_i^a} = \theta_i^a\sigma_i.$$

Equivalently this says that θ_i^b is an extension of $\theta_i^a\sigma_i$.

Conversely, if $H_i^a \leq H_i^b$, $K_i^a \leq K_i^b$ and θ_i^b is an extension of $\theta_i^a\sigma_i$ for all i then the reverse of the above argument implies that $A \leq B$. Hence we have the following.

Theorem 5.4.4. *Let G be a group and let $A, B \leq G^n$ be subgroups with $(H_1^a; H_2^a, K_2^a, \theta_1^a; \ldots; H_n^a, K_n^a, \theta_{n-1}^a)$ and $(H_1^b; H_2^b, K_2^b, \theta_1^b; \ldots; H_n^b, K_n^b, \theta_{n-1}^b)$ the associated Goursat's decompositions. Then $A \leq B$ if and only if for each i we have $H_i^a \leq H_i^b$, $K_i^a \leq K_i^b$, and θ_i^b is a extension of $\theta_i^a\sigma_i$ (where σ_i is as defined before the theorem).*

We now introduce subgroups of semidirect products. We are motivated by the fact that for a finite rank free G-act the principal factors are Brandt semigroups over the group $G \wr \mathcal{S}_m$ for some m. This wreath product is a semidirect product, it is the product $G^m \rtimes \mathcal{S}_m$ under the action of $\mathcal{S}_m$ on the coordinates of G^m.

For this general discussion concerning semidirect products of groups we use the convention that P and H are groups and $\phi\colon P \to \operatorname{Aut} H$ is an antihomomorphism. For $p \in P$ and $h \in H$ we write $p\phi = \phi_p$ and $h\phi_p = h^p$. The semidirect product of P and H is then the set of all ordered pairs $\{(h,p) \mid h \in H,\ p \in P\}$, with the operation

$$(h,p)(g,q) = (hg^p, pq).$$

We denote this group by $H \rtimes_\phi P$. We remark that inverses in $H \rtimes_\phi P$ work as

$$(h,q)^{-1} = ((h^{-1})^{q^{-1}}, q^{-1}).$$

A subgroup $J \leq H$ is *ϕ-invariant* if for all $j \in J$ and $p \in P$, $j^p \in J$. When $J \trianglelefteq H$ is ϕ-invariant then

$$(hJ)^p = \{k^p \mid k \in hJ\} = h^p J.$$

In this case ϕ induces an antihomomorphism $\phi'\colon P \to \operatorname{Aut}(H/J)$ defined by $p \mapsto [hJ \mapsto h^p J]$ and, with $J' = \{(j,1) \mid j \in J\}$, we have

$$(H \rtimes_\phi P)/J' \cong (H/J) \rtimes_{\phi'} P.$$

The following discussion is included to show how the results we use in Chapter 6 evolve from previous work. In Chapter 6 we directly prove the results that we use. The first half of the following definition is taken from [77], the second half is "new" and we shall use it for an elementary refinement of results from [77].

Definition 5.4.5 (see [77])**.** Let $H \rtimes_\phi P$ be a semidirect product and let $Q \leq P$ and $J \leq H$ be subgroups. We say that a function $\psi : Q \to H$ is a *normal crossed* $\mathfrak{R}_\phi^J$ *(NCR) homomorphism* and the triple (J, Q, ψ) a *normal crossed* $\mathfrak{R}_\phi^J$ *(NCR) triple* if the following are satisfied:

(i) for all $r, q \in Q$ there is $j \in J$ such that $(rq)\psi = j(r\psi)(q\psi)^r$;

(ii) for all $q \in Q$ and $j \in J$ we have $(q\psi)j^q(q\psi)^{-1} \in J$.

Furthermore we say ψ is a *strongly normal crossed* $\mathfrak{R}_\phi^J$ *(SNCR) homomorphism* and (J, Q, ψ) a *strongly normal crossed* $\mathfrak{R}_\phi^J$ *(SNCR) triple* if $Q \trianglelefteq P$ and $J \trianglelefteq H$ is ϕ-invariant, and in addition to (i) and (ii) the following are satisfied:

(iii) for all $q \in Q$ and $p \in P$ we have that $(q\psi)^p J = ((pqp^{-1})\psi)J$;

(iv) for all $q \in Q$, and $h \in H$ we have that $(q\psi)h^q(q\psi)^{-1} \in hJ$.

For an NCR or SNCR triple (J, Q, ψ) define the set

$$\mathbf{L}(J, Q, \psi) = \{(j(q\psi), q) \mid j \in J,\ q \in Q\}.$$

Usenko provides the following description of subgroups of $H \rtimes_\phi P$.

Theorem 5.4.6 (see [77])**.** *Let $H \rtimes_\phi P$ be a semidirect product and let (J, Q, ψ) be an NCR triple. Then $\mathbf{L}(J, Q, \psi)$ is a subgroup of $H \rtimes_\phi P$.*

Moreover, given $L \leq H \rtimes_\phi P$ a subgroup, let $J = \{h \in H \mid (h, 1) \in L\}$ and $Q = \{q \in Q \mid \exists h \in H,\ (h, q) \in L\}$. For each $q \in Q$ choose $(h, q) \in L$ and define $q\psi = h$. Then (J, Q, ψ) is an NCR triple and $L = \mathbf{L}(J, Q, \psi)$.

In this description of subgroups of $H \rtimes_\phi P$ the group Q can be viewed as the projection of the subgroup $L \leq H \rtimes_\phi P$ onto the P coordinate, and J is the kernel of that projection, so $J = \{h \in H \mid (h,1) \in L\}$. One particular point that is important to note is that while $\mathbf{L}(J_1, Q_1, \psi_1) = \mathbf{L}(J_2, Q_2, \psi_2)$ does give that $J_1 = J_2$ and $Q_1 = Q_2$ it does not imply that $\psi_1 = \psi_2$. This is due to ψ coming from a "choice" of h for each q, where h is chosen such that $(h,q) \in L$. In general this is not a unique choice, so there are potentially many functions which may be "chosen". We specialise Theorem 5.4.6 to normal subgroups. We provide an outline proof at this stage, we shall directly prove a refinement of the following result in Chapter 6 (Theorem 6.4.2).

Corollary 5.4.7. *Let $H \rtimes_\phi P$ be a semidirect product and (J, Q, ψ) an NCR triple. Then $L = \mathbf{L}(J, Q, \theta)$ is normal in $H \rtimes_\phi P$ if and only if (J, Q, ψ) is an SNCR triple.*

Proof. This is straightforward; it is immediate that Q and J must be normal. That J must be ϕ-invariant follows from the observation that $(1,p)(j,1)(1,p^{-1}) = (j^p, 1)$. Also, (iii) from Definition 5.4.5 is equivalent to L being closed under conjugation by elements of the form $(1,p)$, and (iv) to L being closed under conjugation by elements of the form $(h,1)$. Elements of the form $(1,p)$ and $(h,1)$ generate $H \rtimes_\phi P$, hence L is normal if and only if (J, Q, ψ) is an SNCR triple. $\square$

The next step is to resolve the issue of having multiple SNCR triples corresponding to the same normal subgroup. Let (J, Q, ψ) be an SNCR triple for $H \rtimes_\phi P$. As $J \trianglelefteq H$ we can consider the quotient group H/J. As J is ϕ invariant the antihomomorphism $\phi\colon P \to \operatorname{Aut} H$ induces an antihomomorphism

$$\tilde{\phi}\colon P \to \operatorname{Aut} H/J; \qquad p \mapsto [hJ \mapsto h^p J].$$

We write $(hJ)^p$ for $h^p J$. Define $\overline{\psi} : Q \to H/J$ by $q\overline{\psi} = (q\psi)J$. As ψ is an SNCR homomorphism, by (i), we have that for all $p, q \in Q$ there is $j \in J$ such that $(qp)\psi = j(q\psi)(p\psi)^q$, and, by (iv), we have that for $q \in Q$ and $h \in H$ that $(q\psi)h^q(q\psi)^{-1} \in hJ$. Thus

$$\begin{aligned}(qp)\overline{\psi} = (qp)\psi J = j(q\psi)(p\psi)^q J &= (q\psi)(p\psi)^q J \\ &= (q\psi)(q\psi)^{-1}(p\psi)(q\psi)J = (p\psi)(q\psi)J = (p\overline{\psi})(q\overline{\psi}).\end{aligned}$$

Thus $\overline{\psi}$ is an anti-homomorphism. Conversely, using antihomomorphisms $Q \to H/J$ (along with the Q and the J) allows us to define unique triples to each normal subgroup of $H \rtimes_\phi P$. For the result that follows from this discussion see Theorem 6.4.2.

— 6 —

Congruences on $G \wr \mathcal{I}_n$

In Chapter 5 we described the partial automorphism monoid for the free group action of rank n as the partial wreath product $G \wr \mathcal{I}_n$, which is the set

$$G \wr \mathcal{I}_n = \{(g; a) \in (G_0)^n \times \mathcal{I}_n \mid g_i \neq 0 \iff i \in \mathrm{Dom}(a)\},$$

with multiplication

$$(g; a)(h; b) = (g_1, \ldots, g_n; a)(h_1, \ldots, h_n; b) = (g_1 h_{1a}, \ldots, g_n h_{na}; ab) = (gh_a; ab),$$

where we recall that for $(g_1, \ldots, g_n) \in (G_0)^n$ and $a \in \mathcal{I}_n$ we write g_a for $(g_{1a}, \ldots, g_{na})$ and $g_{ia} = 0$ where ia is undefined.

6.1 Preliminary results concerning $G \wr \mathcal{I}_n$

We first make some initial remarks and comments about $G \wr \mathcal{I}_n$, translating the previous general results for partial automorphism monoids of independence algebras to the setting and language of $G \wr \mathcal{I}_n$. We provide direct proofs for several results which it is possible to deduce from the general results concerning partial automorphism of an independence algebra contained in Chapter 5.

By Lemma 5.2.9 the subacts of a rank n free G-act are in bijective correspondence with subsets of $[n] = \{1, \ldots, n\}$; further, as previously remarked, the set of subacts is also in bijection with set of idempotents of $G \wr \mathcal{I}_n$.

Corollary 6.1.1. *The idempotents of $G \wr \mathcal{I}_n$ are precisely the elements $(1_e; e)$ for $e \in E(\mathcal{I}_n)$. Consequently, $E(G \wr \mathcal{I}_n)$ forms a lattice isomorphic to the subsets of $[n]$ under intersection, which is isomorphic to the lattice of idempotents $E(\mathcal{I}_n)$.*

Proof. Clearly $(1_e; e) \in E(G \wr \mathcal{I}_n)$ for any $e \in E(\mathcal{I}_n)$. Conversely, suppose that $(g; a)(g; a) = (gg_a; a^2) = (g; a)$. Then certainly $a^2 = a$, hence $a = e \in E(\mathcal{I}_n)$. We then note that $g_a = g_e = g$, so $gg_a = g^2 = g$; whence g is an idempotent in $(G_0)^n$, with $g_i = 0$ exactly when $i \notin \mathrm{Dom}\, e$, so $g = 1_e$. $\square$

We recall that if $(g; a) \in G \wr \mathcal{I}_n$ then $g_{aa^{-1}} = g$. We then observe that for $(g; a) \in G \wr \mathcal{I}_n$

$$(g^{-1}; aa^{-1})(g; a) = (1_{aa^{-1}}; a) = (g; a)(g^{-1}_{a^{-1}}; a^{-1}a),$$

and

$$(g; aa^{-1})(1_{aa^{-1}}; a) = (g; a) = (1_{aa^{-1}}; a)(g_{a^{-1}}; a^{-1}a).$$

From this pair of observations it is clear that $(g; a)\ \mathcal{H}\ (1_{aa^{-1}}; a)$ and it follows that the Green's relations for $G \wr \mathcal{I}_n$ are induced by those for $\mathcal{I}_n$, as we now show.

Lemma 6.1.2. *Let $\mathcal{K} \in \{\mathcal{H}, \mathcal{L}, \mathcal{R}, \mathcal{D}, \mathcal{J}\}$ be a Green's relation. Then*

$$(g; a)\ \mathcal{K}(G \wr \mathcal{I}_n)\ (h; b) \iff a\ \mathcal{K}(\mathcal{I}_n)\ b.$$

Proof. The proof in each case follows a similar strategy; we give the proof for $\mathcal{R}$. Let $x = (g; a)$ and $y = (h; b)$ be elements in $G \wr \mathcal{I}_n$. Then

$$\begin{aligned} x\ \mathcal{R}(G \wr \mathcal{I}_n)\ y &\iff xx^{-1} = yy^{-1} \iff (1_{aa^{-1}}; aa^{-1}) = (1_{bb^{-1}}; bb^{-1}) \\ &\iff aa^{-1} = bb^{-1} \iff a\ \mathcal{R}(\mathcal{I}_n)\ b. \end{aligned}$$

The proof for $\mathcal{L}$ is almost identical, and from $\mathcal{L}$ and $\mathcal{R}$ the claim for $\mathcal{H}$ follows. For $\mathcal{D}$, the result follows by a similar argument using the classification of $\mathcal{D}$ on an inverse semigroup which says $s\ \mathcal{D}\ t$ if there is u such that $ss^{-1} = uu^{-1}$ and $t^{-1}t = u^{-1}u$.

The proof for $\mathcal{J}$ uses the results for $\mathcal{R}$ and $\mathcal{L}$. We continue to let $(g; a)$ and $(h; b)$ be elements in $G \wr \mathcal{I}_n$. It is clear that if $(g; a)\ \mathcal{J}(G \wr \mathcal{I}_n)\ (h; b)$ then $a\ \mathcal{J}(\mathcal{I}_n)\ b$, we prove the converse. Suppose $a\ \mathcal{J}(\mathcal{I}_n)\ b$, and recall that in $\mathcal{I}_n$ we have $\mathcal{D} = \mathcal{J}$. Then $a\ \mathcal{D}(\mathcal{I}_n)\ b$ so there is $c \in \mathcal{I}_n$ such that $a\ \mathcal{R}(\mathcal{I}_n)\ c\ \mathcal{L}(\mathcal{I}_n)\ b$. Using the results for $\mathcal{R}$ and $\mathcal{L}$ then have that

$$(g; a)\ \mathcal{R}(G \wr \mathcal{I}_n)\ (1_{cc^{-1}}; c)\ \mathcal{L}(G \wr \mathcal{I}_n)\ (h; b).$$

Thus $(g; a)\ \mathcal{D}(G \wr \mathcal{I}_n)\ (h; b)$ so certainly $(g; a)\ \mathcal{J}(G \wr \mathcal{I}_n)\ (h; b)$. This completes the proof. $\square$

In particular, the $\mathcal{D}$-classes are the sets

$$D_k = \{(g; a) \in G \wr \mathcal{I}_n \mid \operatorname{rank}(a) = k\}$$

for $0 \le k \le n$. As Green's relations for $G \wr \mathcal{I}_n$ are inherited from those for $\mathcal{I}_n$ it follows that the $\mathcal{L}$, $\mathcal{R}$ and $\mathcal{J}$ partial orders on $G \wr \mathcal{I}_n$ are also inherited from the corresponding partial orders on $\mathcal{I}_n$. Therefore the two sided ideals of $G \wr \mathcal{I}_n$ are inherited from those in $\mathcal{I}_n$. Thus for each $0 \le m \le n$ the set

$$J_m = \{(g; a) \in G \wr \mathcal{I}_n \mid \operatorname{rank}(a) \le m\}$$

is an ideal of $G \wr \mathcal{I}_n$, and these are the only ideals of $G \wr \mathcal{I}_n$. Note that we have slightly switched notation for ideals from that in Chapter 5, where we took $I_m = \{a \mid \operatorname{rank}(a) < m\}$ and now we use $J_m = \{a \mid \operatorname{rank}(a) \le m\}$. This makes no real difference, we make the change to reduce subscript length and so that we index the ideals of $G \wr \mathcal{I}_n$ by 0 to n not 1 to $n + 1$. The reason that one needs to use the version used in Chapter 5 is to allow for infinite rank independence algebras, when we need strict inequality for ideals corresponding to limit cardinals. We also recall that the group $\mathcal{H}$-classes in D_k are isomorphic to the group $G \wr \mathcal{S}_k$, the wreath product of G with the symmetric group $\mathcal{S}_k$. This is the semidirect product of G^k with $\mathcal{S}_k$ under the action of $\mathcal{S}_k$ on the coordinates in G^k.

We recall that there is a notion of rank for partial automorphisms of an independence algebra, the cardinality of a basis for the image. In the setting $G \wr \mathcal{I}_n$ we notice that $(g; a)$ has rank equal to $|\operatorname{Im}(a)|$. This is equal to the rank of a as an element of $\mathcal{I}_n$ so there is no confusion using the notation $\operatorname{rank}(g; a)$ for the rank of an element in $G \wr \mathcal{I}_n$. We also have a notion of rank for a congruence on $G \wr \mathcal{I}_n$.

Definition 6.1.3. Let ρ be a congruence on $G \wr \mathcal{I}_n$. The *rank* of ρ is m where $0\rho = J_m$.

We consider idempotent separating congruences on $G \wr \mathcal{I}_n$. We recall that the lattice of idempotent separating congruences is isomorphic to the lattice of normal subsemigroups which are contained in $E\zeta$ (Theorem 1.3.26), where $E\zeta$ is the set of elements which commute with all idempotents. For $G \wr \mathcal{I}_n$ we know that the idempotents are elements of the form $(1_e; e)$ for some idempotent $e \in \mathcal{I}_n$. An elementary calculation gives that for $G \wr \mathcal{I}_n$ the centraliser of the idempotents is the set

$$E\zeta = \{(g; e) \in G \wr \mathcal{I}_n \mid e \in E(\mathcal{I}_n)\}.$$

For the foreseeable future when we write $E\zeta$ we refer to the centraliser of $E(G \wr \mathcal{I}_n)$. We use the description of the maximum idempotent separating congruence from Theorem 1.3.26, which is

$$\mu_\iota = \{(x, y) \mid x^{-1}x = y^{-1}y,\ xy^{-1} \in E\zeta\}.$$

Applied to $G \wr \mathcal{I}_n$ with $x = (g; a)$ and $y = (h; b)$ we see that $x^{-1}x = y^{-1}y$ says that $a^{-1}a = b^{-1}b$ and $xy^{-1} \in E\zeta$ says that $(g; a)(h; b)^{-1} = (k; e)$ for some $e \in E(\mathcal{I}_n)$. Indeed, we recall that $(h; b)^{-1} = (h^{-1}_{b^{-1}}; b^{-1})$ and also Lemma 5.2.11, which states that $(k_c)_d = k_{dc}$, where $k \in (G_0)^n$ and $c, d \in \mathcal{I}_n$. Thus

$$(g; a)(h; b)^{-1} = (g; a)(h^{-1}_{b^{-1}}; b^{-1}) = (g(h^{-1}_{b^{-1}})_a; ab^{-1}) = (gh^{-1}_{ab^{-1}}; ab^{-1}).$$

From the definition of $E\zeta$ we have that $ab^{-1} \in E(\mathcal{I}_n)$. Coupled with $a^{-1}a = b^{-1}b$ this implies that $a = b$. Indeed, we note that

$$\begin{aligned} a = a(a^{-1}a) = a(b^{-1}b) = (ab^{-1})b = (ab^{-1})(ab^{-1})b = (ab^{-1}a)(b^{-1}b) \\ = (ab^{-1}a)(a^{-1}a) = ab^{-1}a, \end{aligned}$$

and dually, as $ba^{-1} \in E(\mathcal{I}_n)$

$$b^{-1} = b^{-1}ab^{-1}.$$

This exactly says that b^{-1} is an inverse of a, in other words, $a^{-1} = b^{-1}$, so certainly $a = b$. We have shown that $(g; a)\ \mu_\iota\ (h; b)$ exactly when $a = b$, therefore on $G \wr \mathcal{I}_n$ we have the following expression for the maximum idempotent separating congruence,

$$\mu_\iota = \{((g; a), (h; b)) \mid a = b\}.$$

6.2 Congruences on $G \wr \mathcal{I}_n$

We now study general congruences on $G \wr \mathcal{I}_n$. We remark that as $J_m = D_0 \cup D_1 \cup \cdots \cup D_m$ we may view J_m/J_{m-1} as $D_m \cup \{0\}$. Recall the description of congruences on the partial automorphism monoid of an independence algebra from Theorem 5.3.1. Applied to $G \wr \mathcal{I}_n$ this states that the non universal congruences are of the form

$$\rho(m, \sigma, \chi) = J^{\star}_{m-1} \cup \overline{\sigma} \cup \chi$$

where $1 \leq m \leq n$, $J^{\star}_{m-1}$ is the Rees congruence with respect to the ideal J_{m-1}, χ is an idempotent separating congruence on $G \wr \mathcal{I}_n$ and $\overline{\sigma}$ is the restriction of σ, a congruence on J_m/J_{m-1}, to D_m (where we think of J_m/J_{m-1} as $D_m \cup \{0\}$).

For $G \wr \mathcal{I}_n$ the decomposition of congruences may be deduced directly by adapting the approach in [21, Chapter 6] from $\mathcal{I}_n$ to $G \wr \mathcal{I}_n$. We shall indicate how this is done, though the following skimps on the details and skips most proofs. The crux of this idea is the observation that we may 'project' congruences on $G \wr \mathcal{I}_n$ onto congruences on $\mathcal{I}_n$. We define the obvious map $\Psi \colon G \wr \mathcal{I}_n \to \mathcal{I}_n$ by $(g; a) \mapsto a$, which we know is a homomorphism. If ρ is a congruence on $G \wr \mathcal{I}_n$ then

$$\rho\Psi = \{(a, b) \in \mathcal{I}_n \times \mathcal{I}_n \mid \exists g, h \in (G_0)^n,\ (g; a)\ \rho\ (h; b)\}$$

is a congruence on $\mathcal{I}_n$.

Our first two results regarding congruences on $G \wr \mathcal{I}_n$ may be proved identically to the $\mathcal{I}_n$ case given in [21].

Lemma 6.2.1 (see [21, Lemma 6.3.5])**.** *Let ρ be congruence on $G \wr \mathcal{I}_n$, and $(g; a), (h; b) \in G \wr \mathcal{I}_n$ such that* $\operatorname{rank}(g; a) = k$, $\operatorname{rank}(h; b) = m$, $k > m$ *and $(g; a)\ \rho\ (h; b)$. Then $J_k \subseteq 0\rho$.*

Lemma 6.2.2 (see [21, Lemma 6.3.6])**.** *Let ρ be a non identity congruence on $G \wr \mathcal{I}_n$. If $(g; a)\ \rho\ (h; b)$ and $(g; a) \notin 0\rho$ then $(g; a)\ \mathcal{H}\ (h; b)$.*

Proof. While this can be proved directly following the same strategy as in [21, Lemma 6.3.6], which is the corresponding claim for $\mathcal{I}_n$, our result can also be deduced from the $\mathcal{I}_n$ result and the knowledge of Green's relations on $G \wr \mathcal{I}_n$. We know that if ρ is a congruence on $G \wr \mathcal{I}_n$ then $\rho\Psi$ (as defined above) is a congruence on $\mathcal{I}_n$. Further the rank of $\rho\Psi$ (defined analogously to the rank of ρ as the 'rank' of the ideal $0(\rho\Psi)$) is equal to the rank of ρ. Therefore if $(g; a)\ \rho\ (h; b)$ then $a\ \rho\Psi\ b$, so, from the result for $\mathcal{I}_n$ ([21, Lemma 6.3.6]), $a\ \mathcal{H}\ b$. This says that $(g; a)\ \mathcal{H}\ (h; b)$, thus our result is proven. □

The next result is the first slight deviation from the $\mathcal{I}_n$ case. The second result in the upcoming pair is again slightly different to the $\mathcal{I}_n$ case, but just as in the $\mathcal{I}_n$ case it is a direct consequence of the first result in the pair. Even though the statement of our results deviate from the results in [21], the direct proof for both is identical to the corresponding result in [21]. This similarity is due to our result only talking about the $\mathcal{I}_n$ coordinate. Just as in our proof of Lemma 6.2.2, both the following results may also be deduced from the corresponding result for $\mathcal{I}_n$ via consideration of $\rho\Psi$.

Lemma 6.2.3 (see [21, Lemma 6.3.7]). *Let ρ be congruence on $G \wr \mathcal{I}_n$ and $(g; a) \in G \wr \mathcal{I}_n$ with* $\operatorname{rank}(g; a) = k$ *such that there is $(h; b) \in G \wr \mathcal{I}_n$ with $(g; a)\ \rho\ (h; b)$ and $a \neq b$. Then $J_{k-1} \subseteq 0\rho$.*

Lemma 6.2.4 (see [21, Lemma 6.3.8]). *Let ρ be congruence on $G \wr \mathcal{I}_n$ with rank k (so $0\rho = J_k$). If $(g; a), (h; b) \in G \wr \mathcal{I}_n$ with* $\operatorname{rank}(g; a), \operatorname{rank}(h; b) > k+1$ *and $(g; a)\ \rho\ (h; b)$ then $a = b$.*

The direct deduction of the decomposition for congruences on $G \wr \mathcal{I}_n$ (previously given in Theorem 5.3.1) follows from the previous four lemmas. We give a brief indication of how the proof runs.

Theorem 6.2.5 (see [21, Lemma 6.3.9]). *Let ρ be a non-universal congruence on $G \wr \mathcal{I}_n$ with rank $k - 1$. Then $\rho = \rho(k, \sigma, \chi)$ for some congruence σ on J_k/J_{k-1} and some idempotent separating congruence χ on $G \wr \mathcal{I}_n$.*

Proof. We first recall the maximum idempotent separating congruence on $G \wr \mathcal{I}_n$

$$\mu_\iota = \{((g; a), (h; b)) \in G \wr \mathcal{I}_n \times G \wr \mathcal{I}_n \mid a = b\}.$$

We let $\chi = \rho \cap \mu_\iota$. We observe that Lemma 6.2.4 says that on restriction to elements of rank strictly greater than k the congruence ρ is contained in the congruence μ_ι. In other words, if $\operatorname{rank}(a) > k$ then $a\rho = a\chi$.

We define σ in the obvious way; viewing J_m/J_{m-1} as $D_m \cup \{0\}$ we take

$$\sigma = \{(x, y) \in D_m \times D_m \mid x \ \rho \ y\} \cup \{(0, 0)\}.$$

We also note that $\operatorname{rank}(\rho) = k-1$ implies $J^\star_{k-1} \subseteq \rho$. It is immediate from this construction that $\rho = \rho(k, \sigma, \chi)$. □

An alternative method to get hold of congruences on $G \wr \mathcal{I}_n$ is to appeal to results of East and Ruškuc in [15] describing how congruences on ideals of semigroups extend to congruences on the whole semigroup. To indicate how this is done we give the most relevant result. A $\mathcal{J}$-class J of a semigroup S is said to be stable if for all $x \in J$ and $a \in S$

$$x \ \mathcal{J} \ xa \implies x \ \mathcal{R} \ xa \quad \text{and} \quad x \ \mathcal{J} \ ax \implies x \ \mathcal{L} \ ax.$$

Let J be a stable regular $\mathcal{J}$-class with group $\mathcal{H}$-class G in a semigroup S and let $N \trianglelefteq G$ be a normal subgroup. Then define

$$\upsilon_N = \{(axb, ayb) \mid x, y \in N, \ a, b \in S^1, \ axb, ayb \in J\}.$$

If $T \subseteq S$ is a subsemigroup then we say a congruence ρ on T is *strongly liftable* to S if $\rho \cup \iota$ is a congruence on S, we write $\mathfrak{C}^S(T)$ for the set of congruences on T which are strongly liftable to S.

Theorem 6.2.6 ([15, Theorem 3.14])**.** *Let S be a semigroup with a maximum $\mathcal{J}$-class J which is regular and stable and suppose that for any $x \in J$ and any $y \in S \backslash H_x$ the congruence generated by (x, y) is the universal congruence ($\langle (x, y) \rangle = \omega$). Let $T = S \backslash J$ (so T is a subsemigroup) and let G be a group $\mathcal{H}$-class of J. For $N \trianglelefteq G$ define $\gamma_N = \langle \upsilon_N \rangle |_T$ (the restriction to T of the congruence on S generated by the relation υ_N). Then*

$$\mathfrak{C}(S) = \{\kappa \cup \upsilon_N \mid \kappa \in \mathfrak{C}^S(T), \ N \trianglelefteq G, \ \gamma_N \subseteq \kappa\} \cup \{\omega\}.$$

It is easy to see that $G \wr \mathcal{I}_n$ satisfies the conditions in Theorem 6.2.6, thus it may be applied in this case. It is possible to show that strongly liftable congruences on $J_{n-1} \subseteq G \wr \mathcal{I}_n$ precisely correspond to congruences on $G \wr \mathcal{I}_{n-1}$, thus Theorem 6.2.6 provides an induction-type method to produce the lattice of congruences on $G \wr \mathcal{I}_n$.

Our next objective is to refine the description of congruences on $G \wr \mathcal{I}_n$ from Theorem 5.3.1. To do this we shall appeal to the correspondence between idempotent separating congruences and normal subsemigroups of $G \wr \mathcal{I}_n$ contained in $E\zeta$ and so shall describe normal subsemigroups contained in $E\zeta$. We recall (from a remark after Theorem 1.3.26) that a subsemigroup T of an inverse semigroup S which is contained in $E\zeta$ is normal if and only if T is a full self-conjugate inverse subsemigroup of S. We shall show that normal subsemigroups contained in $E\zeta$ are determined by a set of subgroups of G^i, one for each $1 \leq i \leq n$. Combining this with the usual description of congruences on Brandt semigroups will allow us to describe a congruence on $G \wr \mathcal{I}_n$ in terms of set of subgroups of various groups associated with G.

First we consider normal subsemigroups of $E\zeta$. Define the function

$$\Omega \colon E\zeta \to \bigcup_{0 \leq m \leq n} G^m$$

to be the map that ignores zero entries in the $(G_0)^n$ component and ignores the final $(\mathcal{I}_n)$ coordinate. We know $E\zeta = \{(g; e) \in G \wr \mathcal{I}_n \mid e \in E(\mathcal{I}_n)\}$ and it follows that, for $(1_e; e) \in E(G \wr \mathcal{I}_n)$ of rank m, the restriction of Ω to $E\zeta \cap H_{(1_e;e)}$ is an isomorphism onto G^m, so $E\zeta \cap H_{(1_e;e)} \cong G^m$.

Given $h \in G^m$ and $e \in E(\mathcal{I}_n)$ with $\operatorname{rank}(e) = m$ write $\overline{{}^e h}$ for the element of $(G_0)^n$ that has $\overline{{}^e h}_i = 0$ for $i \notin \operatorname{Dom}(e)$ and $(\overline{{}^e h}; e)\Omega = h$. Equivalently, $\overline{{}^e h}$ is the unique element of $(G_0)^n$ that has $(\overline{{}^e h}; e) \in G \wr \mathcal{I}_n$ and $(\overline{{}^e h}; e)\Omega = h$.

Definition 6.2.7. A subgroup $K \leq G^m$ is (permutation) invariant if for all $\sigma \in \mathcal{S}_m$ we have that

$$(g_1, g_2, \ldots, g_m) \in K \iff (g_{1\sigma}, g_{2\sigma}, \ldots, g_{m\sigma}) \in K.$$

Write $PI(G, m)$ for the lattice of invariant subgroups of G^m.

Lemma 6.2.8. *Let $T \subseteq E\zeta$ be a normal subsemigroup of $G \wr \mathcal{I}_n$ and for $e \in E(\mathcal{I}_n)$ let $T_e = T \cap H_{(1_e;e)}$. If $e, f \in \mathcal{I}_n$ have* $\operatorname{rank}(e) = \operatorname{rank}(f) = m$ *then $T_e \cong T_f$. Moreover, the group $T_e\Omega \trianglelefteq G^m$ is normal and invariant, and $T_e\Omega = T_f\Omega$.*

Proof. For $a \in \mathcal{I}_n$ if $(g; a^{-1}a) \in T$ then as T is normal it follows that

$$(1_{aa^{-1}}; a)(g; a^{-1}a)(1_{a^{-1}a}; a^{-1}) = (g_a; aa^{-1}) \in T$$

(noting that this is one of our computations from Lemma 5.2.12). With this in mind for each $a \in \mathcal{I}_n$ we define the function

$$\Psi_a \colon T_{a^{-1}a} \to T_{aa^{-1}}; \quad (g; a^{-1}a) \mapsto (g_a; aa^{-1})$$

and it is easily seen that this is an isomorphism. Thus $T_{a^{-1}a} \cong T_{aa^{-1}}$ for each $a \in \mathcal{I}_n$. If $e, f \in E(\mathcal{I}_n)$ with $\operatorname{rank}(e) = \operatorname{rank}(f)$ then $e \mathrel{\mathcal{D}(\mathcal{I}_n)} f$ so, as $\mathcal{I}_n$ is inverse, we may choose $a \in \mathcal{I}_n$ with $a^{-1}a = e$ and $aa^{-1} = f$. It follows that if $\operatorname{rank}(e) = \operatorname{rank}(f)$ then $T_e \cong T_f$.

Let $e \in E(\mathcal{I}_n)$ with $\operatorname{rank}(e) = m$. If $a \mathrel{\mathcal{H}} e$ then $aa^{-1} = e = a^{-1}a$, and so Ψ_a is an automorphism of T_e. Moreover, Ψ_a acts as an element $\sigma_a \in \mathcal{S}_m$ permuting the coordinates of the non-zero entries in the $(G_0)^n$ component. Furthermore, the map $H_e(\mathcal{I}_n) \to \mathcal{S}_{\operatorname{rank}(e)}$ defined by $a \mapsto \sigma_a$ is surjective. This exactly says that $T_e\Omega$ is invariant.

Now let $e, f \in E(\mathcal{I}_n)$ with $\operatorname{rank}(e) = \operatorname{rank}(f)$ and again choose $a \in \mathcal{I}_n$ with $aa^{-1} = e$ and $a^{-1}a = f$. If $g \in T_e\Omega$ then $(\overline{{}^e g}; e)\Psi_a\Omega \in T_f\Omega$ and it is clear that $(\overline{{}^e g}; e)\Psi_a\Omega$ is equal to g under a permutation of the coordinates. As $T_f\Omega$ is invariant this implies that $g \in T_f\Omega$. Thus $T_e\Omega \subseteq T_f\Omega$. Symmetry in e, f in the argument implies $T_e\Omega = T_f\Omega$.

Let $e \in E(\mathcal{I}_n)$ with rank m. To see $T_e\Omega$ is normal in G^m suppose that $g \in T_e\Omega$ so $(\overline{{}^e g}; e) \in T_e$ (recalling that $(\overline{{}^e g}; e)\Omega = g$) and let $h \in G^m$. Note that $(\overline{{}^e(hgh^{-1})}; e) = (\overline{{}^e h}; e)(\overline{{}^e g}; e)(\overline{{}^e h^{-1}}; e)$. As T is normal this implies that $(\overline{{}^e(hgh^{-1})}; e) \in T_e$, so $(\overline{{}^e(hgh^{-1})}; e)\Omega = hgh^{-1} \in T_e\Omega$. Thus $T_e\Omega$ is normal. □

We have shown that to define a normal subsemigroup $T \subseteq G \wr \mathcal{I}_n$ contained in $E\zeta$ it suffices to describe a set of invariant normal subgroups

$\{T_i \trianglelefteq G^i \mid 0 \leq i \leq n\}$. Here T_0 is trivial since it is a subgroup of G^0 (the trivial group that is the 0^{th} power of G). The normal subsemigroup is then

$$T = \bigcup_{e \in E(\mathcal{I}_n)} \left\{ ({}^{\overline{e}}g; e) \mid g \in T_{\mathrm{rank}(e)} \right\}.$$

We call $\{T_i \trianglelefteq G^i \mid 0 \leq i \leq n\}$ the *defining groups for* T.

Our next step is to describe the collections of subgroups that will yield a suitable normal subsemigroup. For $1 \leq m \leq n$ we write

$$\pi_m \colon \bigcup_{m \leq i \leq n} G^i \to G^{m-1}$$

for the projection onto the first $m-1$ coordinates. We say a set $\{T_i \leq G^i \mid 0 \leq i \leq n\}$ is *closed* if $T_i \pi_i \subseteq T_{i-1}$ for each $1 \leq i \leq n$. Notice that when T_i is invariant the projection onto any equally sized subset of the coordinates has the same image.

Proposition 6.2.9. *Let $T \subseteq E\zeta$ be a normal subsemigroup of $G \wr \mathcal{I}_n$ and let $\{T_i \leq G^i \mid 0 \leq i \leq n\}$ be the defining groups for T. Then each T_i is an invariant normal subgroup and $\{T_i \mid 0 \leq i \leq n\}$ is closed.*

Moreover if $\{T_i \trianglelefteq G^i \mid 0 \leq i \leq n\}$ is a closed set of invariant normal subgroups then

$$T = \bigcup_{e \in E(\mathcal{I}_n)} \left\{ ({}^{\overline{e}}g; e) \mid g \in T_{\mathrm{rank}(e)} \right\}$$

is a normal subsemigroup, $T \subseteq E\zeta$ and $\{T_i \leq G^i \mid 0 \leq i \leq n\}$ are the defining groups for T.

Proof. Recall $\Omega \colon E\zeta \to \bigcup_{0 \leq m \leq n} G^m$, the function that ignores zero entries in the $(G_0)^n$ component, and ignores the $\mathcal{I}_n$ component.

Suppose that $T \subseteq E\zeta$ is a normal subsemigroup with defining groups $\{T_i \mid 1 \leq i \leq n\}$. By Lemma 6.2.8 each T_i is an invariant normal subgroup. Suppose $e \in E(\mathcal{I}_n)$ with rank m has domain $\{x_1 < x_2 < \cdots < x_m\}$ and let $f \in E(\mathcal{I}_n)$ be the idempotent with domain $\{x_1 < \cdots < x_{m-1}\}$. We notice that for $g \in G^m$ that

$${}^{\overline{e}}g 1_f = {}^{\overline{f}}(g\pi_{m-1}).$$

If $g \in T_m$ so $(\overline{^e g}; e) \in T$, then $(\overline{^e g}; e)(1_f; f) = (\overline{^f(g\pi_{m-1})}; f) \in T$. Also $g\pi_{m-1} = (\overline{^f(g\pi_{m-1})}; f)\Omega$, so $g\pi_{m-1} \in T_{m-1}$. Thus $\{T_i \mid 0 \leq i \leq n\}$ is closed.

For the converse, suppose that $\{T_i \trianglelefteq G^i \mid 0 \leq i \leq n\}$ is a closed set of invariant normal subgroups. To see that T (as defined in the statement) is a subsemigroup let $(\overline{^e g}; e), (\overline{^f h}; f) \in T$ and observe that

$$\begin{aligned}(\overline{^e g}; e)(\overline{^f h}; f) = ((\overline{^e g})(\overline{^f h})_e; ef) &= ((\overline{^e g})1_{ef}(\overline{^f h})1_{ef}; ef) \\ &= \Big((\overline{^e g})1_{ef}; ef\Big)\Big((\overline{^f h})1_{ef}; ef\Big).\end{aligned}$$

As each T_i is invariant (so the projection onto equally sized subsets has the same image) it is clear that $((\overline{^e g})1_{ef}; ef)\Omega \in T_{\mathrm{rank}(e)}\pi_{\mathrm{rank}(ef)}$, and that $((\overline{^f h})1_{ef}; ef)\Omega \in T_{\mathrm{rank}(f)}\pi_{\mathrm{rank}(ef)}$. As the set of subgroups is closed, both $T_{\mathrm{rank}(e)}\pi_{\mathrm{rank}(ef)} \subseteq T_{\mathrm{rank}(ef)}$ and $T_{\mathrm{rank}(f)}\pi_{\mathrm{rank}(ef)} \subseteq T_{\mathrm{rank}(ef)}$. Therefore

$$((\overline{^e g})1_{ef}; ef)\Omega((\overline{^f h})1_{ef}; ef)\Omega \in T_{\mathrm{rank}(ef)}.$$

As Ω is an isomorphism when restricted to $H_{(1_{ef}; ef)}$ we have that

$$((\overline{^e g})1_{ef}; ef)\Omega((\overline{^f h})1_{ef}; ef)\Omega = \Big(((\overline{^e g})1_{ef}; ef)((\overline{^f h})1_{ef}; ef)\Big)\Omega.$$

Therefore we have that

$$\Big((\overline{^e g}; e)(\overline{^f h}; f)\Big)\Omega = \Big(((\overline{^e g})1_{ef}; ef)((\overline{^f h})1_{ef}; ef)\Big)\Omega \in T_{\mathrm{rank}(ef)}.$$

This implies that $(\overline{^e g}; e)(\overline{^f h}; f) \in T$

To see that T is full we note that each T_i is a subgroup, so $1^i \in T_i$ and as $\overline{^e 1^i} = 1_e$ (where $\mathrm{rank}(e) = i$) we have that $(1_e, e) \in T$ for all $e \in \mathcal{I}_n$. We show that T is inverse in a similar fashion. As each T_i is a subgroup, if $g \in T_i$ then $g^{-1} \in T_i$ and we observe that $(\overline{^e g})^{-1} = \overline{^e(g^{-1})}$. For any $(h; e) \in E\zeta$ we know that $(h; e)^{-1} = (h^{-1}; e)$, thus if $(\overline{^e g}; e) \in T$ then

$$(\overline{^e g}; e)^{-1} = (\overline{^e(g^{-1})}; e) \in T,$$

so T is inverse. To see that T is self conjugate we recall (from Lemma 5.2.12) that $(g; a) \in G \wr \mathcal{I}_n$ decomposes as $(g; a) = (g; aa^{-1})(1_{aa^{-1}}; a)$. Then

$$(g; a)(\overline{^e h}; e)(g; a)^{-1} = (g; aa^{-1})(1_{aa^{-1}}; a)(\overline{^e h}; e)(1_{a^{-1}a}; a^{-1})(g^{-1}; aa^{-1}).$$

Thus T being closed under conjugation by any element in $G \wr \mathcal{I}_n$ is equivalent to T being closed under conjugation by elements of the form either $(1_{aa^{-1}}; a)$ or $(g; aa^{-1})$. Under conjugation by $(1_{aa^{-1}}; a)$ we have that $(\overline{{}^e h}; e)$ is mapped to $((\overline{{}^e h})_a; aea^{-1})$ (see Lemma 5.2.12). If $h \in T_{\mathrm{rank}(e)}$ then $(\overline{{}^e h})_a$ is equal to $\overline{{}^e h}$ after permuting indices and replacing some h_i by 0. As each T_i is invariant and the set of subgroups is closed it follows that $((\overline{{}^e h})_a; aea^{-1})\Omega \in T_{\mathrm{rank}(aea^{-1})}$, thus

$$((\overline{{}^e h})_a; aea^{-1}) \in T.$$

Closure under conjugation by $(g; aa^{-1})$ follows from each T_i being normal. From Lemma 5.2.12 we know that such conjugation maps $(\overline{{}^e h}; e)$ to $(g(\overline{{}^e h})_{aa^{-1}} g^{-1}; eaa^{-1})$. Then $g(\overline{{}^e h})_{aa^{-1}} g^{-1}$ is equal to $\overline{{}^e h}$ after replacing some h_i by 0 and then conjugating some non-zero h_i by elements in G. Using that the set of subgroups is closed and each T_i is normal this implies that if $h \in T_{\mathrm{rank}(e)}$ then $(g(\overline{{}^e h})_{aa^{-1}} g^{-1}; aa^{-1}e)\Omega \in T_{\mathrm{rank}(aa^{-1}e)}$, so that

$$(g(\overline{{}^e h})_{aa^{-1}} g^{-1}; aa^{-1}e) \in T.$$

We have now shown that T is normal, which completes the proof □

Similarly to the way in which we define Ω, we define the function $\omega\colon (G_0)^n \to \bigcup_{0 \le m \le n} G^m$, which ignores zero entries. We have shown that to define a normal subsemigroup of $E\zeta$ it is sufficient to provide a closed set of invariant normal subgroups $\{T_i \trianglelefteq G^i \mid 0 \le i \le n\}$. We recall that our objective in describing normal subsemigroups was to obtain idempotent separating congruences, and that if $T \subseteq E\zeta$ is a normal subsemigroup the associated idempotent separating congruence is

$$\chi_T = \{(x, y) \mid x^{-1}x = y^{-1}y,\ xy^{-1} \in T\}.$$

We apply this to $G \wr \mathcal{I}_n$. We suppose that we have $T \subseteq E\zeta$ and that $\{T_i \trianglelefteq G^i \mid 0 \le i \le n\}$ are the defining groups for T. Similar to our previous discussion of the maximum idempotent separating congruence, we notice that if $x = (g; a)$ and $y = (h; b)$, then $x^{-1}x = y^{-1}y$ and $xy^{-1} \in T$ implies that $a = b$ and

$$(g; a)(h; a)^{-1} = (g; a)(h^{-1}_{a^{-1}}; a^{-1}) = (gh^{-1}; aa^{-1}) \in T,$$

which in turn says that $(gh^{-1})\omega \in T_{\mathrm{rank}(aa^{-1})}$. The idempotent separating congruence χ_T can then be expressed explicitly as

$$\chi_T = \chi(T_0, T_1, \ldots, T_n) = \{((g;a),(h;a)) \mid a \in \mathcal{I}_n,\ (gh^{-1})\omega \in T_{\mathrm{rank}(a)}\}.$$

Furthermore, the ordering on idempotent separating congruences coincides with the ordering on closed sets of invariant normal subgroups induced by subgroup inclusion in each degree: that is, $\chi(T_0, \ldots, T_n) \subseteq \chi(K_0, \ldots, K_n)$ if and only if $T_i \subseteq K_i$ for each $0 \leq i \leq n$. In this fashion we have again deduced the maximum idempotent separating congruence on $G \wr \mathcal{I}_n$.

Corollary 6.2.10. *The maximum idempotent separating congruence on $G \wr \mathcal{I}_n$ is*

$$\chi(G^0, G, G^2, \ldots, G^n) = \{((g;a),(h;b)) \mid a = b\}.$$

The next stage is to describe non-universal congruences on J_m/J_{m-1}. The principal factors are the Brandt semigroups $\mathcal{B}(P_m, G \wr \mathcal{S}_m)$ (Proposition 5.2.6). We recall that for Brandt semigroups the lattice of non-universal congruences is isomorphic to the lattice of normal subgroups of the associated group (Theorem 1.5.12).

Corollary 6.2.11. *Let $1 \leq m \leq n$. Then the lattice of non-universal congruences on J_m/J_{m-1} is isomorphic to the lattice of normal subgroups of $G \wr \mathcal{S}_m$.*

For each $e \in E(\mathcal{I}_n)$ with $\mathrm{rank}(e) = m$ we know that $H_e(\mathcal{I}_n) \cong \mathcal{S}_m$, let $\theta\colon H_e \to \mathcal{S}_m$ be the usual isomorphism. Define $\Psi_e : H_{(1_e;e)} \to G \wr \mathcal{S}_m$ by $(g;a) \mapsto (g\omega; a\theta)$. Then Ψ_e is an isomorphism. For a normal subgroup $L \trianglelefteq G \wr \mathcal{S}_m$ write σ_L for the corresponding congruence on J_m/J_{m-1}. As a set we think of J_m/J_{m-1} as $D_m \cup \{0\}$, and from this viewpoint we may realise σ_L as the relation

$$\{((g;a),(h;b)) \in D_m \times D_m \mid (g;a)\ \mathcal{H}\ (h;b),\ ((g^{-1}h)_{a^{-1}}; a^{-1}b))\Psi_{a^{-1}a} \in L\} \\ \cup \{(0,0)\}.$$

It is clear that we may regard G^m as a subgroup of $G \wr \mathcal{S}_m$ via the embedding $g \mapsto (g;1)$, in the remainder of this section we shall use this identification liberally. We recall the requirement from Theorem 5.3.1, that $\chi \cap (D_m \times D_m) \subseteq \overline{\sigma}$. The maximum idempotent separating congruence $\mu = \mu_\iota$ has

$$\mu \cap (D_m \times D_m) = \{((g;a),(h;b)) \in D_m \times D_m \mid a = b\}$$

and we notice that this is equal to $\sigma_L \cap (D_m \times D_m)$ for $L = G^m$. Thus the requirement $\chi \cap (D_m \times D_m) \subseteq \overline{\sigma}$ is equivalent to $T_m \subseteq L$.

We can now give the promised refinement of the description of two sided congruences on $G \wr \mathcal{I}_n$, recalling that π_i is the projection onto the first i coordinates.

Theorem 6.2.12. *Let $1 \leq m \leq n$, let $\{T_i \trianglelefteq G^i \mid m+1 \leq i \leq n\}$ be a closed set of invariant normal subgroups, and let $L \trianglelefteq G \wr \mathcal{S}_m$ be such that $T_{m+1}\pi_{m+1} \leq L$. Then*

$$\rho(m, \{T_i\}, L) = J^\star_{m-1} \cup \overline{\sigma_L} \cup \chi(G^0, G, G^2, \ldots, G^{m-1}, T_{m+1}\pi_{m+1}, T_{m+1}, \ldots, T_n)$$

is a non-universal congruence on $G \wr \mathcal{I}_n$.

Moreover, all non-universal congruences on $G \wr \mathcal{I}_n$ are of this form.

The explicit form for $\rho = \rho(m, \{T_i\}, L)$ is: $(g;a)\ \rho\ (h;b)$ if one of the following:

- $\text{rank}(a) < m$ and $\text{rank}(b) < m$,
- $\text{rank}(a) > m$, $a = b$ and $(g^{-1}h)\omega \in T_{\text{rank}(a)}$,
- $\text{rank}(a) = m = \text{rank}(b)$, $a\ \mathcal{H}(\mathcal{I}_n)\ b$ and $((g^{-1}h)_{a^{-1}}; a^{-1}b)\Psi_{a^{-1}a} \in L$.

It is also worth remarking on the relation between the ordering on congruences and the description in Theorem 6.2.12.

Proposition 6.2.13. *Let $\rho = \rho(m, \{T_i \mid m+1 \leq i \leq n\}, L)$ and $\rho' = \rho(m', \{T'_i \mid m'+1 \leq i \leq n\}, L')$ be non universal congruences on $G \wr \mathcal{I}_n$. Then $\rho \subseteq \rho'$ if and only if $m \leq m'$, $T_i \leq T'_i$ for each $m'+1 \leq i \leq n$ and, if $m = m'$, then $L \leq L'$ or, if $m < m'$, then $T_{m'} \subseteq L'$.*

This ordering allows us to easily compute the intersection and join of congruences.

Corollary 6.2.14. *Let $m \geq m'$, and let $\rho = \rho(m, \{T_i \mid m+1 \leq i \leq n\}, L)$ and $\rho' = \rho(m', \{T_i' \mid m'+1 \leq i \leq n\}, L')$ be congruences on $G \wr \mathcal{I}_n$.*

(i) The join of ρ and ρ' is

$$\rho \vee \rho' = \rho(m, \{U_i \mid m+1 \leq i \leq n\}, A),$$

where if $m = m'$ then $A = L \vee L'$ and if $m > m'$ then $A = L \vee T_m'$ with either join taken in the lattice of normal subgroups of $G \wr \mathcal{S}_m$, and $U_i = T_i \vee T_i'$ for $m < i \leq n$ with this join in $PI(G, i)$ (the set of permutation invariant subgroups of G^i).

(ii) The intersection of ρ and ρ' is

$$\rho \cap \rho' = \rho(m', \{V_i \mid m'+1 \leq i \leq n\}, B),$$

where if $m = m'$ then $B = L \cap L'$ and if $m > m'$ then $B = L'$ and V_i for $i > m'$ is defined by

$$V_i = \begin{cases} T_i' & \text{for } m' < i < m \\ T_m' \cap L & \text{for } i = m \\ T_i \cap T_i' & \text{for } i > m. \end{cases}$$

We have shown that a congruence on $G \wr \mathcal{I}_n$ is determined by a set of subgroups of G^i for a range of i and a subgroup of $G \wr \mathcal{S}_m$, so it is a sensible next step to develop a theory describing the sets of these subgroups.

6.3 Invariant normal subgroups of G^m

In this section we present an analysis of the subgroups of G^m that arise as components in the prior description of congruences on $G \wr \mathcal{I}_n$. The standard starting point in the consideration of subgroups of direct products of groups is Goursat's lemma, which we met in Theorem 5.4.1. We recall that the subgroups $X \leq G \times H$ are exactly the sets

$$X(A, B, C, D, \theta) = \{(a, b) \in A \times B \mid (aC)\theta = bD\},$$

where $C \trianglelefteq A \leq G$, $D \trianglelefteq B \leq H$ and $\theta\colon A/C \to B/D$ is an isomorphism. We also recall that the subgroups A and B are the projections of $X = X(A,B,C,D,\theta) \leq G \times H$ onto the first and second coordinates respectively, the subgroups C and D are as follows: $C = \{c \in G \mid (c,1) \in X\}$ and $D = \{d \in H \mid (1,d) \in X\}$, and finally the isomorphism θ is the function $aC \mapsto bD$ if $(a,b) \in X$.

We shall be primarily interested in invariant normal subgroups, so we specialise Goursat's Lemma to accommodate our focus. If $X(A,B,C,D,\theta) \leq G^2$ is invariant then it follows that $A = B$ and $C = D$ and also that if $(aC)\theta = bC$ then by definition $(a,b) \in X$, so that $(b,a) \in X$ and $(bC)\theta = aC$, so θ is an automorphic involution of A/C.

If $X(A,B,C,D,\theta) \trianglelefteq G^2$ is normal then it follows that A and B are normal in G. Further, if $(a,b) \in X$ then $(gag^{-1},b) \in X$ for all $g \in G$, thus $(gag^{-1}a^{-1},1) \in X$ so $gag^{-1}a^{-1} \in C$. Equivalently $[G,A] \subseteq C$ (for $Z,Y \subseteq G$, the *commutator* of Z and Y is $[Z,Y] = \{zyz^{-1}y^{-1} \mid z \in Z,\ y \in Y\}$), and similarly $[G,B] \subseteq D$. In particular this implies that C and D are normal in G and that A/C and B/D are abelian. This discussion leads to the following elementary extension of Goursat's lemma which is more applicable in our case.

Corollary 6.3.1. *There is a bijective correspondence between invariant normal subgroups of G^2 and triples (A,C,θ) where $A, C \trianglelefteq G$, $C \trianglelefteq A$ such that $[G,A] \subseteq C$ and θ is an automorphic involution of A/C. The invariant normal subgroups of G^2 are*

$$X(A,A,C,C,\theta) = \{(a,b) \in A \times A \mid (aC)\theta = bC\}$$

for these triples.

For larger m or for semidirect as opposed to direct products the picture grows much more complicated. In Chapter 5 we discussed a general extension of Goursat's lemma (Theorem 5.4.3). However, for our purposes, this is too broad, so we shall directly produce a description tailored to our requirements. We shall demonstrate that an invariant normal subgroup of G^m is determined by certain normal subgroups $N \leq M \leq L$ of G and a map

$\phi\colon L \to L/N$. Initially we show how, given an invariant normal subgroup $K \trianglelefteq G^m$, we define these subgroups and following this we establish a collection of properties that the subgroups satisfy. Next we shall introduce the map ϕ and again establish the important properties it satisfies. Finally, we show how an invariant normal subgroup can be recovered from the quadruple (L, M, N, ϕ) and demonstrate that the properties previously established characterise the quadruples that arise from invariant normal subgroups.

For the remainder of this section unless otherwise stated we suppose $m \geq 3$ and let $K \trianglelefteq G^m$ be an invariant normal subgroup. Define

$$\begin{aligned} \mathbf{H}(K) &= \{(g,h) \in G^2 \mid (g,h,1,\ldots,1) \in K\} \\ \text{and} \quad \mathbf{N}(K) &= \{n \in G \mid (n,1,\ldots,1) \in K\}, \end{aligned}$$

and note that $\mathbf{N}(K) = \{n \in G \mid (n,1) \in \mathbf{H}(K)\}$. With π the projection onto the first coordinate define

$$\mathbf{L}(K) = K\pi \qquad \text{and} \qquad \mathbf{M}(K) = \mathbf{H}(K)\pi.$$

Since K is normal in G^m it is clear that $\mathbf{N}(K)$, $\mathbf{M}(K)$ and $\mathbf{L}(K)$ are normal subgroups of G, and, as K is also invariant, that $\mathbf{H}(K)$ is an invariant normal subgroup of G^2. Furthermore $\mathbf{N}(K) \leq \mathbf{M}(K) \leq \mathbf{L}(K)$. Until further notice we let $N = \mathbf{N}(K)$, $M = \mathbf{M}(K)$, $L = \mathbf{L}(K)$ and $H = \mathbf{H}(K)$.

Lemma 6.3.2. *The commutator $[G, L]$ is such that $[G, L] \subseteq N$. In particular the quotient group L/N is abelian.*

Proof. The proof is similar to the exposition earlier for a normal subgroup of G^2. For $l \in L$ there is some $k \in K$ with $k = (l, k_2, \ldots, k_m)$. As K is normal in G^m, if $g \in G$ then $(glg^{-1}, k_2, \ldots, k_m) \in K$. It follows that $(glg^{-1}l^{-1}, 1, \ldots, 1) \in K$ and therefore $glg^{-1}l^{-1} \in N$. Thus N contains the commutator $[G, L]$. □

As H is an invariant normal subgroup of G^2 we may apply Corollary 6.3.1 to deduce that there is an automorphic involution θ of M/N such that

$$H = \{(g,h) \in M^2 \mid (gN)\theta = hN\}.$$

Suppose $(g,h,1,\ldots,1) \in K$. As K is invariant, $(1,h^{-1},g^{-1},1,\ldots,1) \in K$ (we here use that $m \geq 3$) so $(g,1,g^{-1},1,\ldots,1) \in K$. As K is invariant, $(g,g^{-1},1,\ldots,1) \in K$, so $(g,g^{-1}) \in H$. It follows that $(gN)\theta = g^{-1}N$ and we note that the inverse map is an automorphism since L/N (and so also M/N) is abelian. Therefore

$$H = \{(g,h) \in M^2 |\ hg \in N\}.$$

In particular for $g \in M$ we have that $(g,g^{-1},1,\ldots,1) \in K$.

Lemma 6.3.3. *If* $(g_1,\ldots,g_m) \in K$, *then*

$$g_1M = g_2M = \cdots = g_mM.$$

Proof. Suppose that $(g_1,\ldots,g_m) \in K$. Then, since K is invariant, we have $(g_2,g_1,g_3,\ldots,g_m) \in K$, and thus

$$(g_2,g_1,g_3,\ldots,g_m)^{-1}(g_1,g_2,g_3,\ldots,g_m) = (g_2^{-1}g_1, g_1^{-1}g_2,1,\ldots,1) \in K.$$

Hence $(g_2^{-1}g_1, g_1^{-1}g_2) \in H$, so $g_1^{-1}g_2 \in M$, or equivalently $g_1M = g_2M$. Since K is invariant we may permute the g_i to obtain $g_1M = \cdots = g_mM$. □

We define the function

$$\phi_K : L \to L/N; \quad g \mapsto yN \ \text{ where } \ (y,g,g,\ldots,g) \in K.$$

We now show that ϕ_K is well defined. First recall that for $x \in M$ we have that $(x,x^{-1}) \in H$, thus for $x_2,\ldots,x_m \in M$ we have that $(x_2,x_2^{-1},1,\ldots,1)$, $(x_3,1,x_3^{-1},1,\ldots,1),\ldots,(x_m,1,\ldots,1,x_m^{-1})$ are elements of K. Hence their product

$$(x_2x_3\ldots x_m, x_2^{-1},\ldots,x_m^{-1}) \in K.$$

Suppose $g \in L$ so there is $k = (g,k_2,\ldots,k_m) \in K$. By Lemma 6.3.3 we have that $gM = k_2M = \cdots = k_mM$ so there are $x_2,\ldots,x_m \in M$ such that $k_i = gx_i$. Then

$$(g,k_2,\ldots,k_m)(x_2\ldots x_m, x_2^{-1},\ldots,x_m^{-1}) = (gx_2\ldots x_m, g,\ldots,g) \in K.$$

Thus $g\phi_K = gx_2\ldots x_mN$ so ϕ_K is certainly defined for each $g \in L$. Also if $(y,g,\ldots,g) \in K$ and $(x,g,\ldots,g) \in K$ then it is immediate that $y^{-1}x \in N$, so ϕ_K is a well defined function.

Lemma 6.3.4. *(i) The function ϕ_K defined previously is a homomorphism.*

(ii) The restriction of ϕ_K to M has the form $g \mapsto g^{1-m}N$.

(iii) For $g \in L$, if $g\phi_K = hN$ then $gM = hM$.

(iv) For $k_1, \ldots, k_m \in L$ we have that $(k_1, \ldots, k_m) \in K$ if and only if $k_1M = \cdots = k_mM$ and $k_1\phi_K = k_1^{2-m}k_2 \ldots k_mN$.

Proof. For convenience we write ϕ for ϕ_K.

(i) Suppose $(y, g, \ldots, g)$, $(x, h, \ldots, h) \in K$. Then $(yx, gh, \ldots, gh) \in K$. It is immediate that ϕ is a homomorphism.

(ii) Suppose $x \in M$. Then $(x^{-1}, x) \in H$, which implies that for each $1 \leq i \leq m$, $(x^{-1}, 1, \ldots, 1, x, 1, \ldots, 1) \in K$ where the x is in the i^{th} coordinate. Then

$$(x^{-1}, x, 1, \ldots, 1) \ldots (x^{-1}, 1, \ldots, 1, x) = (x^{1-m}, x, \ldots, x) \in K.$$

Therefore for $x \in M$ we have $x\phi = x^{1-m}N$.

(iii) This follows immediately from Lemma 6.3.3.

(iv) Suppose that $(k_1, \ldots, k_m) \in K$. By Lemma 6.3.3, $k_1M = \cdots = k_mM$. In the discussion prior to the lemma, in which it is shown that ϕ is well defined, we saw that $k_1\phi = k_1x_2 \ldots x_mN$ where x_i is such that $k_i = k_1x_i$ for $2 \leq i \leq m$. Then $x_i = k_1^{-1}k_i$, so $k_1\phi = k_1(k_1^{-1}k_2) \ldots (k_1^{-1}k_m)N$. Further, as L/N is abelian (by Lemma 6.3.2),

$$k_1(k_1^{-1}k_2) \ldots (k_1^{-1}k_m)N = k_1^{2-m}k_2 \ldots k_mN.$$

Therefore $k_1\phi = k_1^{2-m}k_2 \ldots k_mN$.

For the converse let $k_1, \ldots, k_m \in L$ and suppose $k_1M = k_2M = \cdots = k_mM$ and $k_1\phi = k_1^{2-m}k_2 \ldots k_mN$ which, as L/N is abelian, is equal to $k_1(k_1^{-1}k_2) \ldots (k_1^{-1}k_m)N$. Then, by the definition of ϕ, $(k_1(k_1^{-1}k_2) \ldots (k_1^{-1}k_m), k_1, \ldots, k_1) \in K$. For each $2 \leq i \leq m$ we have $k_i^{-1}k_1 \in M$ so that $(k_i^{-1}k_1, k_1^{-1}k_i) \in M$. This implies that $(k_i^{-1}k_1, 1, \ldots, 1, k_1^{-1}k_i, 1, \ldots, 1) \in K$. Taking the product of these we have

$$((k_m^{-1}k_1)\ldots(k_2^{-1}k_1), k_1^{-1}k_2, \ldots, k_1^{-1}k_m) \in K.$$

We then observe that

$$(k_1(k_1^{-1}k_2)\ldots(k_1^{-1}k_m), k_1, \ldots, k_1)((k_m^{-1}k_1)\ldots(k_2^{-1}k_1), k_1^{-1}k_2, \ldots, k_1^{-1}k_m) \\ = (k_1, k_2, \ldots, k_m) \in K.$$

We have shown that $(k_1, \ldots, k_m) \in K$ if and only if $k_1M = \cdots = k_mM$ and $k_1\phi = k_1^{2-m}k_2 \ldots k_mN$, so the proof is complete.

□

At this stage we have defined a set of subgroups and a homomorphism associated to an invariant normal subgroup $K \trianglelefteq G^m$ and have established some of the important properties. Next we aim to show the converse, namely that any suitable collection of subgroups and a homomorphism leads to such a K. At this time we drop the notational assumption that $K \trianglelefteq G^m$ is an invariant normal subgroup and that $L = \mathbf{L}(K)$, $M = \mathbf{M}(K)$, $N = \mathbf{N}(K)$ and $H = \mathbf{H}(K)$.

Definition 6.3.5. Let G be a group, let $m \geq 3$ be an integer, and let $N \leq M \leq L$ be normal subgroups of G. We say that a homomorphism $\phi \colon L \to L/N$ is an (L, M, N, m)-*homomorphism* if

(i) the restriction of ϕ to M has the form $g \mapsto g^{1-m}N$;

(ii) for $g \in L$, if $g\phi = hN$ then $gM = hM$.

We remark that if ϕ is an (L, M, N, m)-homomorphism then, as $N \leq M$, for any $g \in N$ we have $g\phi = g^{1-m}N = N$, so $N \leq \ker(\phi)$. Next we define the sets of subgroups and homomorphisms which will specify invariant normal subgroups.

Definition 6.3.6. Let G be a group and $m \geq 3$ an integer. Let $N \leq M \leq L$ be normal subgroups of G and $\phi\colon L \to L/N$ a homomorphism. Then (L, M, N, ϕ) is an *m-invariant quadruple for G* if $[G, L] \subseteq N$ and ϕ is an (L, M, N, m)-homomorphism.

Given an m-invariant quadruple (L, M, N, ϕ) we define the following subset of G^m

$$\mathbf{K}_m(L, M, N, \phi) = \{(g_1, \ldots, g_m) \in L^m \mid g_1M = \ldots = g_mM,$$
$$g_1\phi = g_1^{2-m}g_2 \ldots g_mN\}.$$

Proposition 6.3.7. *Let $K \trianglelefteq G^m$ be an invariant normal subgroup, and let $L = \mathbf{L}(K)$, $M = \mathbf{M}(K)$ and $N = \mathbf{N}(K)$. Then (L, M, N, ϕ_K) is an m-invariant quadruple. Furthermore, $K = \mathbf{K}_m(L, M, N, \phi_K)$.*

Proof. As noted previously $L, M, N \trianglelefteq G$ and $N \leq M \leq L$. Also, that $[G, L] \subseteq N$ is Lemma 6.3.2, and ϕ_K is an (L, M, N, m)-homomorphism by Lemma 6.3.4(i), (ii) and (iii).

It remains to show that $K = \mathbf{K}_m(L, M, N, \phi_K)$. By Lemma 6.3.4(iv), for $k_1, \ldots, k_m \in L$, we have $(k_1, \ldots, k_m) \in K$ if and only if $k_1M = \cdots = k_mM$ and $k_1\phi_K = k_1^{2-m}k_2 \ldots k_mN$. This says exactly that $K = \mathbf{K}_m(L, M, N, \phi_K)$. □

Theorem 6.3.8. *Let G be a group and let $m \geq 3$. Let (L, M, N, ϕ) be an m-invariant quadruple for G. Then $K = \mathbf{K}_m(L, M, N, \phi)$ is an invariant normal subgroup of G^m. Moreover, $\mathbf{L}(K) = L$, $\mathbf{M}(K) = M$, $\mathbf{N}(K) = N$ and $\phi_K = \phi$.*

Conversely, if K is an invariant normal subgroup of G^m then, writing $L = \mathbf{L}(K)$, $M = \mathbf{M}(K)$, $N = \mathbf{N}(K)$ and $\phi = \phi_K$, (L, M, N, ϕ) is an m-invariant quadruple and $K = \mathbf{K}_m(L, M, N, \phi)$.

Proof. The latter paragraph in the statement has been proven in Proposition 6.3.7. We proceed to prove the first paragraph.

Let (L, M, N, ϕ) be an m-invariant quadruple and $K = \mathbf{K}_m(L, M, N, \phi)$. We initially show that K is a subgroup of G^m. We first note that K is non-empty as $(1, \ldots, 1) \in K$. Suppose that $(g_1, \ldots, g_m)$, $(h_1, \ldots, h_m) \in K$, so that $g_1, \ldots, g_m, h_1, \ldots, h_m \in L$ and

$$g_1 M = \cdots = g_m M, \quad h_1 M = \cdots = h_m M,$$
$$g_1 \phi = g_1^{2-m} g_2 \ldots g_m N, \quad h_1 \phi = h_1^{2-m} h_2 \ldots h_m N.$$

As $M \trianglelefteq G$ is normal, it is immediate that $g_1 h_1 M = \cdots = g_m h_m M$, and, as ϕ is a homomorphism and L/N is commutative (since $[G, L] \subseteq N$), we have

$$\begin{aligned}(g_1 h_1)\phi = (g_1\phi)(h_1\phi) &= (g_1^{2-m} g_2 \ldots g_m N)(h_1^{2-m} h_2 \ldots h_m N) \\ &= (g_1 h_1)^{2-m} (g_2 h_2) \ldots (g_m h_m) N.\end{aligned}$$

Hence $(g_1 h_1, \ldots, g_m h_m) \in K$. Furthermore, again using that $M \trianglelefteq G$ is normal, $g_1^{-1} M = \cdots = g_m^{-1} M$, and, again as ϕ is a homomorphism,

$$g_1^{-1}\phi = (g_1\phi)^{-1} = (g_1^{2-m} g_2 \ldots g_m)^{-1} N = (g_1^{-1})^{2-m} (g_2^{-1} \ldots g_m^{-1}) N.$$

Thus $(g_1^{-1}, \ldots, g_m^{-1}) \in K$ and so K is a subgroup of G^m.

For the invariant property it is sufficient to show that K is closed under transposition of any two coordinates. Let $(g_1, \ldots, g_m) \in K$. Since L/N is commutative it is immediate that

$$g_1\phi = g_1^{2-m}(g_2 \ldots g_i \ldots g_j \ldots g_m) N = g_1^{2-m}(g_2 \ldots g_j \ldots g_i \ldots g_m) N,$$

hence K is closed under any transposition within the final $(m-1)$ coordinates. Therefore it remains to show that K is closed under swapping the first two coordinates. Suppose that $(g, h, k_3, \ldots, k_m) \in K$ so $g\phi = (g^{2-m} h k_3 \ldots k_m) N$ and $gM = hM$. Then $g^{-1}h \in M$ and as ϕ is an (L, M, N, m)-homomorphism we have $(g^{-1}h)\phi = (g^{-1}h)^{1-m} N$. Therefore

$$\begin{aligned}h\phi = (gg^{-1}h)\phi &= (g\phi)(g^{-1}h\phi) \\ &= (g^{2-m} h k_3 \ldots k_m)(g^{-1}h)^{1-m} N = h^{2-m} g k_3 \ldots k_m N.\end{aligned}$$

Hence $(h, g, k_3, \ldots, k_m) \in K$, and so K is invariant.

We next show that K is normal. As K is invariant it is sufficient to show that we can conjugate in the second coordinate. Suppose $(k_1, \ldots, k_m) \in K$

and $g \in G$. As $[G, L] \subseteq N$ we have that $gk_2g^{-1}N = k_2N$. Since $N \subseteq M$ it follows that $gk_2g^{-1}M = k_2M$ so $k_1M = gk_2g^{-1}M = k_3M = \cdots = k_mM$. Also, again using that $gk_2g^{-1}N = k_2N$,

$$k_1\phi = (k_1^{2-m}k_2 \dots k_m)N = (k_1^{2-m}(gk_2g^{-1})k_3 \dots k_m)N.$$

Thus $(k_1, gk_2g^{-1}, k_3, \dots, k_m) \in K$, and it follows that K is normal.

It remains to show that $\mathbf{L}(K) = L$, $\mathbf{M}(K) = M$, $\mathbf{N}(K) = N$ and $\phi_K = \phi$. As previously remarked, if ϕ is an (L, M, N, m)-homomorphism then $N \leq \ker(\phi)$. Therefore, for $x \in N$, we have $xM = M$ and $x\phi = N = x^{2-m}N$, so $(x, 1, \dots, 1) \in K$, hence $N \subseteq \mathbf{N}(K)$. Suppose that $x \in \mathbf{N}(K)$, so $(x, 1, \dots, 1) \in K$. Then $xM = M$ and $x\phi = x^{2-m}N$. However as $x \in M$ and ϕ is an (L, M, N, m)-homomorphism we also have that $x\phi = x^{1-m}N$. Thus $x^{1-m}N = x^{2-m}N$, so $xN = N$ or equivalently $x \in N$. Hence $\mathbf{N}(K) = N$.

For $y \in M$, as ϕ is an (L, M, N, m)-homomorphism, we have that $y\phi = y^{1-m}N = y^{2-m}y^{-1}N$. Thus $(y, y^{-1}, 1, \dots, 1) \in K$ so $M \subseteq \mathbf{M}(K)$. Suppose that $(x, y, 1, \dots, 1) \in K$. Then certainly $xM = yM = M$, so $\mathbf{M}(K) \subseteq M$ and the two are equal.

By definition $K \subseteq L^m$, so $\mathbf{L}(K) \subseteq L$. Conversely for $l \in L$ choose $x \in l\phi$. Then as ϕ is an (L, M, N, m)-homomorphism $lM = xM$. Also, $l\phi = l^{2-m}l^{m-2}xN$ so we have that $(l, \dots, l, x) \in K$. Thus $\mathbf{L}(K) = L$.

Let $g \in L$ and suppose that $g\phi_K = yN$ so $(y, g, \dots, g) \in K$. Then by the definition of K we have that $y\phi = y^{2-m}g^{m-1}N$ and $gM = yM$. Therefore $y^{-1}g \in M$ and, because ϕ is an (L, M, N, m)-homomorphism, $(y^{-1}g)\phi = (y^{-1}g)^{1-m}N = y^{m-1}g^{1-m}N$. Then

$$g\phi = (y\phi)((y^{-1}g)\phi) = (y^{2-m}g^{m-1}N)(y^{m-1}g^{1-m}N) = yN.$$

Thus $\phi_K = \phi$. $\square$

The ordering on invariant normal subgroups induces the ordering on $\mathfrak{C}_{IS}(G \wr \mathcal{I}_n)$ so it is worthwhile to remark upon the ordering of these groups. The following is an immediate consequence of Theorem 6.3.8.

Corollary 6.3.9. *Let G be a group, let $m \geq 3$ and let (L_1, M_1, N_1, ϕ_1) and (L_2, M_2, N_2, ϕ_2) be m-invariant quadruples. Then*

$$\mathbf{K}_m(L_1, M_1, N_1, \phi_1) \subseteq \mathbf{K}_m(L_2, M_2, N_2, \phi_2)$$

if and only if $L_1 \subseteq L_2$, $M_1 \subseteq M_2$, $N_1 \subseteq N_2$ and for all $l \in L_1$, if $l\phi_1 = xN_1$ then $l\phi_2 = xN_2$.

It is possible to use this ordering to compute the m-invariant quadruple for the joins and intersections of invariant normal subgroups, which can then be combined with Corollary 6.2.14 to give a method to compute the intersections and joins of congruences.

6.4 Normal subgroups of semidirect products

The next part of the description of congruences on $G \wr \mathcal{I}_n$ (Theorem 6.2.12) uses normal subgroups of $G \wr \mathcal{S}_m$. As we have seen, $G \wr \mathcal{S}_m$ is a semidirect product and subgroups of semidirect products are described by Usenko in [77] (see Theorem 5.4.6). As explained in Chapter 5 it is possible to reach the description given here starting from the description in [77]. However it is more straightforward to directly prove the result that we require. We recall that for semidirect products of groups we use the convention that P and H are groups and $\phi\colon P \to \operatorname{Aut} H$ is an antihomomorphism. For $p \in P$ and $h \in H$ we write $p\phi = \phi_p$ and $h\phi_p = h^p$. The semidirect product of P and H is then the set of all ordered pairs $\{(h, p) \mid h \in H,\ p \in P\}$, with the operation

$$(h, p)(g, q) = (hg^p, pq).$$

We denote this group by $H \rtimes_\phi P$. We also recall that a subgroup $J \leq H$ is *ϕ-invariant* if for all $j \in J$ and $p \in P$, $j^p \in J$. We have also seen that, when $J \trianglelefteq H$ is ϕ-invariant, $(hJ)^p = \{k^p \mid k \in hJ\}$ is equal to h^pJ for all $p \in P$. In this case ϕ induces an antihomomorphism $\phi'\colon P \to \operatorname{Aut}(H/J)$ defined by $p \mapsto [hJ \mapsto h^pJ]$ and, with $J' = \{(j, 1) \mid j \in J\}$, we have $(H \rtimes_\phi P)/J' \cong (H/J) \rtimes_{\phi'} P$.

Definition 6.4.1. Let $H \rtimes_\phi P$ be a semidirect product, let $Q \trianglelefteq P$ and let $J \trianglelefteq H$ be ϕ-invariant. Let $\xi\colon Q \to H/J$ be an antihomomorphism. We say that (J, Q, ξ) is a *normal subgroup triple for $H \rtimes_\phi P$* if:

(W1) for all $q \in Q$ and $p \in P$ we have that $(q\xi)^p = (pqp^{-1})\xi$;

(W2) for all $q \in Q$ and $h \in H$ we have that $h^q J = (q\xi)^{-1} hJ(q\xi)$.

For a normal subgroup triple (J, Q, ξ) we define the set

$$\mathbf{W}(J, Q, \xi) = \{(h, q) \mid h \in H,\ q \in Q,\ q\xi = hJ\}.$$

Theorem 6.4.2. *Let $H \rtimes_\phi P$ be a semidirect product and let (J, Q, ξ) be a normal subgroup triple. Then $W = \mathbf{W}(J, Q, \xi)$ is a normal subgroup of $H \rtimes_\phi P$. Moreover,*

$$J = \{h \in H \mid (h, 1) \in W\}, \quad Q = \{p \in P \mid \exists h \in H,\ (h, p) \in W\}$$

and, for $q \in Q$, $q\xi = hJ$ if and only if $(h, q) \in W$.

Conversely, let $W \trianglelefteq H \rtimes_\phi P$ and define $J = \{h \in H \mid (h, 1) \in W\}$ and $Q = \{p \in P \mid \exists h \in H,\ (h, p) \in W\}$. Also define $\xi\colon Q \to H/J$ by $q \mapsto hJ$ where $(h, q) \in W$. Then (J, Q, ξ) is a normal subgroup triple and $W = \mathbf{W}(J, Q, \xi)$.

Proof. Let (J, Q, ξ) be a normal subgroup triple and let $W = \mathbf{W}(J, Q, \xi)$. First we show that W is a subgroup. Suppose that $(h, q), (g, p) \in W$, so $q\xi = hJ$ and $p\xi = gJ$. Then $(h, q)(g, p) = (hg^q, qp)$ and observe that by applying (W2) we obtain

$$hg^q J = hJg^q J = hJ(q\xi)^{-1} gJ(q\xi) = hJ(hJ)^{-1} gJ(hJ) = gJhJ = (p\xi)(q\xi).$$

As ξ is an antihomomorphism from this we have $hg^q J = (qp)\xi$, which implies that $(hg^q, qp) \in W$. We note that $(h, q)^{-1} = ((h^{-1})^{q^{-1}}, q^{-1})$ and, as ξ is an antihomomorphism (so $(q\xi)^{-1} = q^{-1}\xi$), by applying (W2) we have that

$$(h^{-1})^{q^{-1}} J = (q^{-1}\xi)^{-1} h^{-1} J(q^{-1}\xi) = (hJ)h^{-1}J(q^{-1}\xi) = q^{-1}\xi.$$

Therefore $((h^{-1})^{q^{-1}}, q^{-1}) \in W$, so W is a subgroup.

To see that W is normal we suppose that $(h, q) \in W$ so $q\xi = hJ$. Let $p \in P$ and note that $(1, p)(h, q)(1, p)^{-1} = (h^p, pqp^{-1})$. We apply (W1) to obtain that $h^p J = (q\xi)^p = (pqp^{-1})\xi$ so $(h^p, pqp^{-1}) \in W$. Let $k \in H$ then $(k, 1)^{-1}(h, q)(k, 1) = (k^{-1}hk^q, q)$. By (W2),

$$k^q J = (q\xi)^{-1} kJ(q\xi) = (hJ)^{-1} kJ(hJ) = h^{-1}khJ.$$

Thus $k^{-1}hk^q J = hJ = q\xi$ and it follows that $(k^{-1}hk^q, q) \in W$. As $H \rtimes_\phi P$ is generated by the elements of the form $(1, p)$ and $(h, 1)$ we have shown that W is normal. The final claim in the first paragraph of the statement is immediate from the definition of $\mathbf{W}(J, Q, \xi)$.

For the converse we suppose that W is a normal subgroup of $H \rtimes_\phi P$ and let J, Q, ξ be defined as in the statement of the theorem. First we note that ξ is well defined. Indeed, suppose that $(h, q), (g, q) \in W$. Then as W is a subgroup

$$(h, q)(g, q)^{-1} = (h, q)((g^{-1})^{q^{-1}}, q^{-1}) = (hg^{-1}, 1) \in W$$

so $hg^{-1} \in J$. It follows that $hJ = gJ$ and thus ξ is well defined. Further, as W is normal, it is clear that Q is normal in P and that $J \trianglelefteq H$ is normal and ϕ-invariant.

If $(h, q), (g, p) \in W$ then $(h, q)(g, p) = (hg^q, qp) \in W$. It follows that $(qp)\xi = (q\xi)(p\xi)^q$. We next prove (W2). Suppose that $q \in Q$ and $h \in H$ and choose $g \in q\xi$, so $(g, q) \in W$. Then

$$(h^{-1}, 1)(g, q)(h, 1) = (h^{-1}gh^q, q) \in W$$

so $q\xi = h^{-1}gh^q J$. Also $g \in q\xi$ says that $q\xi = gJ$, so $q\xi = h^{-1}J(q\xi)h^q J$ or, equivalently, $h^q J = (q\xi)^{-1}hJ(q\xi)$. Thus (W2) holds. It follows that ξ is an antihomomorphism, indeed if $p, q \in Q$ then, using $(qp)\xi = (q\xi)(p\xi)^q$,

$$(qp)\xi = (q\xi)(p\xi)^q = (q\xi)(q\xi)^{-1}(p\xi)(q\xi) = (p\xi)(q\xi).$$

To see that (W1) holds we let $p \in P$, $q \in Q$ and choose $h \in q\xi$, so $(h, q) \in W$. Then $(1, p)(h, q)(1, p^{-1}) = (h^p, pqp^{-1}) \in W$ and it follows that $(pqp^{-1})\xi = h^p J = (q\xi)^p$.

It is immediate that $W = \mathbf{W}(J, Q, \xi)$, indeed, $(h, p) \in W$ exactly says that $p \in Q$ and $p\xi = hJ$, which in turn is equivalent to $(h, p) \in \mathbf{W}(J, Q, \xi)$. Thus the proof is complete. □

Next we apply Theorem 6.4.2 to $G \wr \mathcal{S}_m$, which we recall is the semidirect product of G^m and $\mathcal{S}_m$ under the action of $\mathcal{S}_m$ on the coordinates in G^m. Since our description of normal subgroups depends on invariant normal subgroups we split this application into two parts, first we describe normal subgroups of $G \wr \mathcal{S}_2$ and then move to $G \wr \mathcal{S}_m$ for larger m. We have to split the $m = 2$ case as our description of invariant normal subgroups of G^m (Theorem 6.3.8) only holds for $m \geq 3$, for the $m = 2$ case we rely on Corollary 6.3.1.

Proposition 6.4.3. *Let G be a group. The following is a complete list of all normal subgroups of $G \wr \mathcal{S}_2$.*

(i) For each triple (A, C, θ) where $A, C \trianglelefteq G$ with $C \leq A$ and $[G, A] \subseteq C$, and where $\theta\colon A/C \to A/C$ is an automorphic involution,

$$\{(x; 1) \mid x \in X\} \trianglelefteq G \wr \mathcal{S}_2,$$

where $X = X(A, A, C, C, \theta) = \{(g, h) \in A^2 \mid (gC)\theta = hC\}$.

(ii) For each pair (C, ζ) where $C \trianglelefteq G$ with G/C abelian, and $\zeta\colon \mathcal{S}_2 \to G/C$ is a homomorphism,

$$\{(g, h; s) \mid g, h \in G,\ s \in \mathcal{S}_2,\ ghC = s\zeta\}.$$

Proof. First we show that all normal subgroups of $G \wr \mathcal{S}_2$ are of the form claimed. Suppose that $W \trianglelefteq G \wr \mathcal{S}_2$ $(= G^2 \rtimes \mathcal{S}_2)$ is a normal subgroup. By Theorem 6.4.2 $W = \mathbf{W}(X, Q, \xi)$ for a normal subgroup triple (X, Q, ξ). Then X is an invariant normal subgroup of G^2, and so is described by Corollary 6.3.1 in terms of a triple (A, C, θ) where $A, C \trianglelefteq G$ with $C \leq A$

and $[G, A] \subseteq C$, and $\theta\colon A/C \to A/C$ is an automorphic involution. The associated subgroup is

$$X = X(A, A, C, C, \theta) = \{(g, h) \in A^2 \mid (gC)\theta = hC\}.$$

As $Q \trianglelefteq \mathcal{S}_2$ there are two options for Q: either Q is trivial or $Q = \mathcal{S}_2$. If Q is trivial then $W = \{(x; 1) \mid x \in X\}$, so W is of the type described in (i). Thus we suppose that $Q = \mathcal{S}_2$ and we show that in this case A must be equal to G. Suppose not, so there is $g \in G$ such that $gA \neq A$. Choose $(a, b) \in G^2$ such that $(1\ 2)\xi = (a, b)X$ (with $(1\ 2)$ the non identity element in $\mathcal{S}_2$). As (X, Q, ξ) is the normal subgroup triple for W it satisfies (W2), so

$$(1, g)X = (g, 1)^{(1\ 2)}X = (a^{-1}, b^{-1})(g, 1)(a, b)X = (a^{-1}ga, 1)X.$$

This implies that $(a^{-1}g^{-1}a, g) \in X$ which is a contradiction as $X \subseteq A^2$ and $g \notin A$. Therefore $A = G$ so $X = X(G, G, C, C, \theta)$, and in particular G/C is abelian. We also observe that θ is the inverse map on G/C. Indeed, suppose that $(g, h; (1\ 2)) \in W$ and take $a \in G$. Then

$$(a, 1; 1)(g, h; (1\ 2))(a^{-1}, 1; 1) = (ag, ha^{-1}; (1\ 2)) \in W.$$

Further,

$$(ag, ha^{-1}; (1\ 2))(g, h; (1\ 2))^{-1} = (a, ha^{-1}h^{-1}; 1) \in W$$

so $(a, ha^{-1}h^{-1}) \in X$ for all $a \in G$. Therefore, as G/C is abelian, $(aC)\theta = ha^{-1}h^{-1}C = a^{-1}C$, so θ is indeed the inverse map. Thus we can specify X exactly as

$$X = X(G, G, C, C, gC \mapsto g^{-1}C) = \{(g, h) \in G \times G \mid gh \in C\},$$

and we note that X is totally determined by C. We define $\Xi\colon G^2/X \to G/C$ by $(g, h)X \mapsto ghC$. It is easily seen that Ξ is an isomorphism, so in particular G^2/X is abelian. Further, we define $\zeta\colon \mathcal{S}_2 \to G/C$ by $\zeta = \xi\Xi$, and note that this is a homomorphism. Finally, we observe that by the definition of $\mathbf{W}(X, \mathcal{S}_2, \xi)$,

$$W = \mathbf{W}(X, \mathcal{S}_2, \xi) = \{(g, h; s) \mid g, h \in G,\ s \in \mathcal{S}_2,\ ghC = s\zeta\}.$$

Therefore W is of the form claimed in (ii), so every normal subgroup of $G \wr \mathcal{S}_2$ is of one of the types given in the theorem.

To complete the proof it remains to show that all the subsets of $G \wr \mathcal{S}_2$ listed are normal subgroups. That those in (i) are normal subgroups follows immediately from Corollary 6.3.1 as the conditions on (A, C, θ) are exactly those which imply that $X = X(A, A, C, C, \theta)$ is an invariant normal subgroup of G^2. For the subsets specified in (ii) we show directly that the subset is a normal subgroup. To this end let W be specified as in (ii). It is straightforward that W is a subgroup; we show that it is normal. Suppose that $(g_1, g_2; s) \in W$ and $(h_1, h_2; t) \in G \wr \mathcal{S}_2$. We note that

$$(h_1, h_2; t)(g_1, g_2; s)(h_1, h_2; t)^{-1} = (h_1 g_{1t} h_{1s}^{-1}, h_2 g_{2t} h_{2s}^{-1}; s).$$

Since G/C is abelian, we have that $h_1 g_{1t} h_{1s}^{-1} h_2 g_{2t} h_{2s}^{-1} C = g_1 g_2 C$. Therefore $(h_1 g_{1t} h_{1s}^{-1}, h_2 g_{2t} h_{2s}^{-1}; s) \in W$, so W is normal in $G \wr \mathcal{S}_2$. □

We now extend Proposition 6.4.3 to $G \wr \mathcal{S}_m$ for $m \geq 3$. First we prove a preliminary lemma.

Lemma 6.4.4. *Let $m \geq 3$ and let (L, M, N, ϕ) be an m-invariant quadruple for G. Let $K = \mathbf{K}_m(L, M, N, \phi)$, let $Q \neq \{1\}$ be a normal subgroup of $\mathcal{S}_m$, and let $\xi \colon Q \to G^m/K$ be an anti-homomorphism such that (K, Q, ξ) is a normal subgroup triple for $G \wr \mathcal{S}_m$. Then $L = M = G$, and consequently $K = \mathbf{K}_m(G, G, N, g \mapsto g^{1-m}N)$.*

Proof. Suppose for a contradiction that $M \neq G$, so there is $x \in G$ with $xM \neq M$. Take $a \in Q$ such that $1a = 2$, note that this is possible as Q is non-trivial and all non-trivial normal subgroups of $\mathcal{S}_m$ for $m \geq 3$ contain such elements. As (K, Q, ξ) is a normal subgroup triple, for $g \in G^m$ we have $(g^a)K = (a\xi)^{-1}(gK)(a\xi)$. Choose $(a_1, \ldots, a_m) \in a\xi$. This implies that

$$\begin{aligned}(1, x, 1, \ldots, 1)K &= (x, 1, \ldots, 1)^a K \\ &= (a_1, \ldots, a_m)^{-1}(x, 1, \ldots, 1)(a_1, \ldots, a_m)K \\ &= (a_1^{-1} x a_1, 1, \ldots, 1)K.\end{aligned}$$

In turn this implies that

$$(1, x^{-1}, 1, \dots, 1)(a_1^{-1}xa_1, 1 \dots, 1) = (a_1^{-1}xa_1, x^{-1}, 1, \dots, 1) \in K.$$

We recall the definition of $\mathbf{K}_m(L, M, N, \phi)$, that

$$K = \{(g_1, \dots, g_m) \in L^m \mid g_1M = \cdots = g_mM,\ g_1\phi = g_1^{2-m}g_2g_3 \dots g_mN\}.$$

Thus we have that $a_1^{-1}xa_1M = x^{-1}M = M$, a contradiction, so we must have that $M = G$. That $L = G$ follows as $M \leq L$, and the final claim is then immediate. $\square$

Theorem 6.4.5. *Let G be a group and $m \geq 2$. The following is a complete list of all normal subgroups of $G \wr \mathcal{S}_m$.*

(i) For each invariant normal subgroup $K \trianglelefteq G^m$,

$$\{(k; 1) \mid k \in K\} \trianglelefteq G \wr \mathcal{S}_m.$$

(ii) For each triple (N, Q, ζ) with $N \trianglelefteq G$ and G/N abelian, $Q \trianglelefteq \mathcal{S}_m$ non trivial and $\zeta\colon Q \to G/N$ a homomorphism such that $[\mathcal{S}_m, Q] \subseteq \ker(\zeta)$,

$$\{(g_1, \dots, g_m; q) \mid g_1, \dots, g_m \in G,\ q \in Q,\ q\zeta = g_1 \dots g_mN\} \trianglelefteq G \wr \mathcal{S}_m.$$

Proof. As the only non trivial subgroup of $\mathcal{S}_2$ is $\mathcal{S}_2$ itself the case for $m = 2$ is precisely that stated in Proposition 6.4.3. Therefore we suppose that $m \geq 3$. First we show that all normal subgroups of $G \wr \mathcal{S}_m$ are of the form specified. Suppose that $W \trianglelefteq G \wr \mathcal{S}_m$ is a normal subgroup. By Theorem 6.4.2, $W = \mathbf{W}(K, Q, \xi)$ for a normal subgroup triple (K, Q, ξ). If Q is trivial then $\xi\colon Q \to G^m/K$ is the trivial (anti)homomorphism and $W = \{(k; 1) \mid k \in K\}$. As K is an invariant normal subgroup of G^m it follows that W is one of the sets listed in (i). We now suppose Q is non trivial. By Lemma 6.4.4 $K = \mathbf{K}_m(G, G, N, g \mapsto g^{1-m}N)$ and we remark that $(G, G, N, g \mapsto g^{1-m}N)$ is an m-invariant quadruple precisely when G/N is abelian. We observe that the function

$$\Xi : G^m/K \to G/N; \quad (g_1, \dots, g_m)K \mapsto g_1 \dots g_mN$$

is an isomorphism, and we define $\zeta\colon Q \to G/N$ by $\zeta = \xi\Xi$. Then ζ is a homomorphism; it the composition of an antihomomorphism and a homomorphism with abelian image (as G/N is abelian). Further, as Ξ is an isomorphism, $\ker(\zeta) = \ker(\xi)$. We observe that

$$\begin{aligned}\mathbf{W}(K, Q, \xi) &= \{(g, q) \mid g \in G^m,\ q \in Q,\ q\xi = gK\}\\ &= \{(g_1, \ldots, g_m; q) \mid g_i \in G,\ q \in Q,\ q\zeta = g_1 \ldots g_m N\},\end{aligned}$$

so to complete the proof that W is one of sets listed in (ii) it remains to show that $[\mathcal{S}_m, Q] \subseteq \ker(\zeta)$ $(= \ker(\xi))$. Let $q \in Q$ and $p \in \mathcal{S}_m$ and say that $q\xi = (g_1, \ldots, g_m)K$. Then

$$\begin{aligned}q\xi &= (g_1, \ldots, g_m)K = (g_1 \ldots g_m N)\Xi^{-1}\\ (q\xi)^p &= (g_1, \ldots, g_m)^p K = (g_{1p}, \ldots, g_{mp})K = (g_{1p} \ldots g_{pm} N)\Xi^{-1}.\end{aligned}$$

As G/N is abelian it follows that $q\xi = (q\xi)^p$, so (W1) is equivalent to $q\xi = (pqp^{-1})\xi$ which in turn is equivalent to $[\mathcal{S}_m, Q] \subseteq \ker(\xi)$. Hence W is one the sets listed in (ii).

To complete the proof it remains to show that all the sets listed in the statement of the theorem are normal subgroups of $G \wr \mathcal{S}_m$. It is immediate that the sets listed in (i) are normal subgroups. Suppose W is a set as specified in (ii) for the triple (N, Q, ζ). As in the proof of Proposition 6.4.3 we prove directly that W is a normal subgroup. It is straightforward that W is a subgroup, so we show that it is normal. Suppose $(g_1, \ldots, g_m; q) \in W$. It follows from the definition of W that conjugation by elements of the form $(h_1, \ldots, h_m; 1)$ leaves the element in W. Indeed, if $h_1, \ldots, h_n \in G$ then

$$q\zeta = g_1 \ldots g_m N = (h_1 g_1 h_{1q}^{-1}) \ldots (h_m g_m h_{mq}^{-1})N,$$

so $(h_1 g_1 h_{1q}^{-1}, \ldots, h_m g_m h_{mq}^{-1}; q) \in W$. For $p \in \mathcal{S}_m$ we consider

$$(1, \ldots, 1; p)(g_1, \ldots, g_m; q)(1, \ldots, 1; p^{-1}) = (g_{1p}, \ldots, g_{mp}; pqp^{-1}).$$

We note that, as G/N is abelian, $g_{1p} \ldots g_{mp} N = g_1 \ldots g_m N = q\zeta$. As $[\mathcal{S}_m, q] \subseteq \ker(\zeta)$ we have that $(pqp^{-1}q^{-1})\zeta = N$ so $(pqp^{-1})\zeta = q\zeta$, thus $(g_{1p}, \ldots, g_{mp}; pqp^{-1}) \in W$. As all elements of $G \wr \mathcal{S}_m$ are products of elements of the form $(h_1, \dots, h_m; 1)$ and $(1, \dots, 1; p)$ it follows that W is normal in $G \wr \mathcal{S}_m$. $\square$

Aside 6.4.6. We make a brief comment on an alternate strategy to describe normal subgroups of $G \wr \mathcal{S}_m$. Let $K = \mathbf{K}_m(G, G, N, g \mapsto g^{1-m}N) \trianglelefteq G^m$ and, as in the proof of Theorem 6.4.5, notice that

$$K = \{(g_1, \dots, g_m) \in G^m \mid g_1 g_2 \dots g_m \in N\}.$$

Again as in the proof of Theorem 6.4.5 (remembering that G/N is abelian by Lemma 6.3.2), we notice that the function

$$\Xi : G^m/K \to G/N; \quad (g_1, \dots, g_m)K \mapsto g_1 \dots g_m N$$

is an isomorphism. The action of $\mathcal{S}_m$ on G^m (permuting the coordinates) carries forward to the quotient group G^m/K and this induces an action of $\mathcal{S}_m$ on G/N via the isomorphism Ξ. As G/N is abelian this induced action is trivial, so with $K' = \{(k; 1) \in G \wr \mathcal{S}_m \mid k \in K\}$ we obtain that $(G \wr \mathcal{S}_m)/K' \cong G/N \times \mathcal{S}_m$. By the correspondence theorem, normal subgroups of $G \wr \mathcal{S}_m$ that correspond to a normal subgroup triple (K, Q, ξ) for $K = \mathbf{K}_m(G, G, N, g \mapsto g^{1-m}N)$ (we vary the Q and ξ) are the lifts of normal subgroups of $G/N \times \mathcal{S}_m$, via the isomorphism Ξ to $G \wr \mathcal{S}_m$ such that the projection of the subgroup onto the second coordinate has trivial kernel. By the lift of a subgroup $C \leq A/B$ to A, we mean the set of all $a \in A$ such that $aB \in C$. Explicitly if $L \trianglelefteq G/N \times \mathcal{S}_m$ then the corresponding normal subgroup of $G \wr \mathcal{S}_m$ is

$$\{(g; a) \in G \wr \mathcal{S}_m \mid (gK\Xi; a) \in L\}.$$

We can use Goursat's lemma (Theorem 5.4.1) to obtain a normal subgroup $L \trianglelefteq G/N \times \mathcal{S}_m$ in terms of subgroups $A, B \trianglelefteq G/N$ and $Q, V \trianglelefteq \mathcal{S}_m$ such that $[G/N, A] \subseteq B$ and $[\mathcal{S}_m, Q] \subseteq V$, and an isomorphism $\psi \colon Q/V \to A/B$. As G/N is abelian the condition $[G/N, A] \subseteq B$ is trivially true. Further, B is the kernel of the projection of L onto the $\mathcal{S}_m$ coordinate, and we recall we are interested in subgroups such that this is trivial, thus we may assume that $B = \{N\}$.

We observe that V is the kernel of a homomorphism $Q \to A/B$ (defined by $q \mapsto (uV)\psi$). Therefore we may simplify the collection A, B, Q, V, ψ to a pair (Q, ζ) where $Q \trianglelefteq \mathcal{S}_m$ and $\zeta \colon Q \to G/N$ is a homomorphism such that $[\mathcal{S}_m, Q] \subseteq \ker(\zeta)$. We remark that, due to G/N being abelian, the condition $[\mathcal{S}_m, Q] \subseteq \ker(\zeta)$ is trivial for all choices of $Q \trianglelefteq \mathcal{S}_m$ and ζ when $m \geq 5$. The subgroup of $G/N \times \mathcal{S}_m$ is then

$$\{(q\zeta, q) \mid q \in Q\} \trianglelefteq G/N \times \mathcal{S}_n.$$

From this point it is straightforward to recover the description of normal subgroups of $G \wr \mathcal{S}_m$ given in Theorem 6.4.5.

6.5 THE SIZE OF $\mathfrak{C}(G \wr \mathcal{I}_n)$

We shall now delve deeper into the consideration of the set of congruences on $G \wr \mathcal{I}_n$ and will provide an answer to the question: what is the asymptotic growth of $|\mathfrak{C}(G \wr \mathcal{I}_n)|$? To allow us to answer this as precisely as possible in this section we shall assume that G is finite and non-trivial. We recall that for $\mathcal{I}_n$ the number of congruences grows linearly in n, and it is straightforward that when $|G| \neq 1$ the number of normal subgroups of G^n grows exponentially in n. We shall show that for a given group G the growth of the number of congruences on $G \wr \mathcal{I}_n$ is polynomial in n.

Proposition 6.5.1. *Let G be a finite group. Then there is an integer $\lambda_1(G)$ such that for all $m \in \mathbb{N}$ the number of permutation invariant subgroups of G^m is at most $\lambda_1(G)$.*

Proof. Notice that, as for each m the group G^m is finite, there are only finitely many subgroups of G^m. Therefore it suffices to prove the claim for m sufficiently large, which in this case is at least 3. Let

$$Z = \{(L, M, N, \phi) \mid N \leq M \leq L,\ N, M, L \trianglelefteq G,\ \phi : L \to L/N\}.$$

By Theorem 6.3.8 we have that invariant normal subgroups of G^m are determined by m-invariant quadruples. For each m we have that $\mathbf{Q}_m$ (the set of m-invariant quadruples) is a subset of Z. Since Z is obviously finite (as G is), this completes the proof. $\square$

In general for a finite group G it is difficult to calculate precise values or even efficient bounds for $\lambda_1(G)$, and it is similarly hard to compute precise values for $|\mathbf{Q}_m|$. We next extend Proposition 6.5.1 to normal subgroups of $G \wr \mathcal{S}_m$.

Corollary 6.5.2. *Let G be a finite group. Then there is an integer $\lambda_2(G)$ such that for all m the number of normal subgroups of $G \wr \mathcal{S}_m$ is at most $\lambda_2(G)$.*

Proof. As $G \wr \mathcal{S}_m$ is finite for each m it again suffices to prove the result for m sufficiently large, this time we take m at least 5. By Theorem 6.4.5 all subgroups of $G \wr \mathcal{S}_m$ are of one of two types, so it suffices to show that the number of each type is bounded. Proposition 6.5.1 precisely says that the number of subgroups of the type described in (i) of Theorem 6.4.5 is bounded by $\lambda_1(G)$, so it remains to show that the number of subgroups described in (ii) of Theorem 6.4.5 is also bounded. To do this we show that the number of triples (N, Q, ζ) is bounded above by a number that depends only on G and not m. It is clear that the number of $N \trianglelefteq G$ with G/N abelian depends only on G. Also the Q can only be $\mathcal{S}_m$ or $\mathcal{A}_m$ (as $Q \trianglelefteq \mathcal{S}_m$ is non trivial and $m \geq 5$). Furthermore, if $Q = \mathcal{S}_m$ then a homomorphism $\zeta \colon \mathcal{S}_m \to G/N$ is totally determined by $(1\ 2)\zeta$, since G/N is abelian so certainly $\mathcal{A}_m \subseteq \ker(\zeta)$. Also, if $Q = \mathcal{A}_m$, as G/N is abelian, the only homomorphism $\zeta \colon \mathcal{A}_m \to G/N$ is the trivial homomorphism. Therefore, for either option for Q, the number of homomorphisms $\zeta \colon Q \to G/N$ is bounded by $|G/N|$. Therefore the number of triples (N, Q, ζ) is bounded by $2(2^{|G|})|G/N|$ (as the number of subgroups of G is at most $2^{|G|}$). Thus we may take

$$\lambda_2(G) = \lambda_1(G) + 2(2^{|G|})|G/N|.$$

□

Similar to the case for $\lambda_1(G)$ it is difficult in general to compute efficient bounds for $\lambda_2(G)$. The following is a standard elementary combinatorial result, we state it here as we shall refer to it frequently.

Lemma 6.5.3. *Let C be a chain of length c, and let v_k be the number of sequences $t_1 \leq t_2 \leq \cdots \leq t_k$ of length k where each $t_i \in C$. Then*

$$v_k = \binom{k+c-1}{c-1}.$$

Moreover, for fixed c, there are $A, B > 0$ such that for all k

$$Ak^{c-1} \leq v_k \leq Bk^{c-1}.$$

Proof. Let $C = y_1 < y_2 < \cdots < y_c$. To prove the claim we define a bijection from the set of subsets of $[k+c-1]$ of size $c-1$, of which there are $\binom{k+c-1}{c-1}$, to the set of sequences $t_1 \leq t_2 \leq \cdots \leq t_k$ of length k. If $A = \{a_1, \ldots, a_{c-1}\} \subseteq [k+c-1]$ then we consider the sequence consisting of $a_1 - 1$ copies of y_1, then $a_i - a_{i-1} - 1$ copies of y_i for each $2 \leq i \leq c-1$, and finally $k+c-1-a_{c-1}$ copies of y_c. It is left to the reader to show that this gives a sequence of length k and that this identification defines a bijection from the set of subsets of $[k+c-1]$ of size $c-1$ to the set of sequences of length k. □

By Theorem 1.3.26 and Proposition 6.2.9, idempotent separating congruences on $G \wr \mathcal{I}_n$ correspond to closed sets of invariant normal subgroups $\{T_i \leq G^i \mid 0 \leq i \leq n\}$. Since T_0 is trivial, without loss of generality we may drop the T_0, and this does not make a difference to the set of closed sets of subgroups which arise from idempotent separating congruences. By Theorem 6.3.8 each invariant normal subgroup $K \leq G^m$ for $m \geq 3$ is of the form $K = \mathbf{K}_m(L, M, N, \phi)$ which we recall is equal to

$$\{(g_1, \ldots, g_m) \in L^m \mid g_1 M = \ldots = g_m M,\ g_1\phi = g_1^{2-m} g_2 \ldots g_m N\}$$

for an m-invariant quadruple (L, M, N, ϕ). For $m \geq 4$, $K\pi$, the projection onto the first $(m-1)$ coordinates (though any choice of $m-1$ coordinates is equivalent), is the set

$$\{(g_1, \ldots, g_{m-1}) \in L^{m-1} \mid g_1 M = \cdots = g_{m-1} M\} = \mathbf{K}_{m-1}(L, M, M, x \mapsto xM).$$

Indeed, if $(g_1, \ldots, g_{m-1}) \in K\pi$ then there is $g_m \in L$ such that $(g_1, \ldots, g_m) \in K$, so from the definition of $\mathbf{K}_m(L, M, N, \phi)$, $g_1 M = \cdots = g_{m-1} M$ so

$K\pi \subseteq \mathbf{K}_{m-1}(L, M, M, x \mapsto xM)$. Conversely, if $g_1M = \cdots = g_{m-1}M$ then, choosing g_m such that $(g_1^{2-m}g_2 \ldots g_{m-1}N)^{-1}(g_1\phi) = g_mN$, we observe that $g_mM = g_1M$ and so $(g_1, \ldots, g_{m-1}, g_m) \in K$. Thus we have that, as claimed, $K\pi = \mathbf{K}_{m-1}(L, M, M, x \mapsto xM)$.

When $m = 2$ an invariant normal subgroup of G^2 is of the form

$$X(L, L, N, N, \theta) = \{(g, h) \in L^2 \mid (gN)\theta = hN\}$$

(by Corollary 6.3.1). The projection of $\mathbf{K}_3(L, M, N, \phi)$ onto the first 2 coordinates is

$$\{(g, h) \in L^2 \mid gM = hM\} = X(L, L, M, M, gM \mapsto gM).$$

Also the projection of $X(L, L, N, N, \theta)$ on the first coordinate is $L \trianglelefteq G$. In this way, to each idempotent separating congruence on $G \wr \mathcal{I}_n$ we associate a set of normal subgroups $\{L_1, L_2, N_2\} \cup \{L_i, M_i, N_i \mid 3 \leq i \leq n\}$ of G and this set of subgroups is partially ordered as detailed in the following lemma.

Lemma 6.5.4. *Let G be a group. Let*

$$\begin{gathered} L_1 \trianglelefteq G, \\ X(L_2, L_2, N_2, N_2, \phi_2) \trianglelefteq G^2 \\ \textit{and} \quad \{\mathbf{K}_i(L_i, M_i, N_i, \phi_i) \trianglelefteq G^i \mid 3 \leq i \leq n\} \end{gathered}$$

form a closed set of invariant normal subgroups. Then $L_i \subseteq L_{i-1}$ for $2 \leq i \leq n$ and $M_i \subseteq N_{i-1}$ for $3 \leq i \leq n$.

Proof. This follows from the discussion prior to the lemma. We know that for $4 \geq i$

$$\mathbf{K}_i(L_i, M_i, N_i, \phi_i)\pi = \mathbf{K}_{i-1}(L_i, M_i, M_i, x \mapsto xM).$$

The set of subgroups being closed implies that

$$\mathbf{K}_{i-1}(L_i, M_i, M_i, x \mapsto xM) \subseteq \mathbf{K}_{i-1}(L_{i-1}, M_{i-1}, N_{i-1}, \phi_{i-1}).$$

Then Corollary 6.3.9, the description of the ordering of invariant normal subgroups, implies that $L_i \subseteq L_{i-1}$ and $M_i \subseteq N_{i-1}$. Similar arguments give the result when $i = 2, 3$. $\square$

The ordering on the set of subgroups is shown in Figure 6.1, where the arrows denote subset inclusion. We will refer to a lattice arising from a congruence in this way as a *(congruence) induced lattice*, and say that the congruence *induces* the lattice.

$$\begin{tikzcd}
L_1 & L_2 \arrow[l] & L_3 \arrow[l] & \cdots \arrow[l, dotted] & L_{n-2} \arrow[l, dotted] & L_{n-1} \arrow[l] & L_n \arrow[l] \\
& & M_3 \arrow[u] \arrow[dl] & \cdots \arrow[u, dotted] \arrow[dl, dotted] & M_{n-2} \arrow[u] \arrow[dl, dotted] & M_{n-1} \arrow[u] \arrow[dl] & M_n \arrow[u] \arrow[dl] \\
& N_2 \arrow[uu] & N_3 \arrow[u] & \cdots \arrow[u, dotted] & N_{n-2} \arrow[u] & N_{n-1} \arrow[u] & N_n \arrow[u]
\end{tikzcd}$$

Figure 6.1: Lattice of normal subgroups of G induced by a congruence on $G \wr \mathcal{I}_n$

When G is finite, by Lemma 6.5.1, for each m there are at most $\lambda_1(G)$ invariant normal subgroups of G^m. As an idempotent separating congruence is determined by n invariant normal subgroups (one for each $1 \leq i \leq n$) it follows that $|\mathfrak{C}_{IS}(G \wr \mathcal{I}_n)| \leq \lambda_1(G)^n$. However we can significantly improve on this bound for large n. For a group G, a maximal strictly increasing chain of normal subgroups is called a *chief series,* and the maximum length of a chief series is the *chief length*, for which we write $c(G)$.

Proposition 6.5.5. *Let G be a finite group with $c(G) = c$. Then there are $A, B > 0$ such that for all n*

$$An^{c-1} \leq |\mathfrak{C}_{IS}(G \wr \mathcal{I}_n)| \leq Bn^{2(c-1)}.$$

Proof. This result concerns asymptotic behaviour of $|\mathfrak{C}(G \wr \mathcal{I}_n)|$, so if we can prove that it holds for sufficiently large n then it follows for all n via adjusting the values of A and B. We assume that n is much larger than c.

First we demonstrate the lower bound. Notice that for $L \trianglelefteq G$ the group $\mathbf{K}_i(L, L, L, l \mapsto L) = L^i$ is an invariant normal subgroup of G^i for each i. Thus for each chain of normal subgroups $L_n \leq L_{n-1} \leq \cdots \leq L_1$ the set $\{L_i^i \mid 1 \leq i \leq n\}$ is a closed set of invariant normal subgroups, so $\chi(\{1\}, L_1, L_2^2, \ldots, L_n^n)$ is an idempotent separating congruence. Different chains of normal subgroups of G give different congruences. By Lemma

6.5.3 there is some $A > 0$ such there are at least An^{c-1} chains of normal subgroups of length n, so we have that $An^{c-1} \leq |\mathfrak{C}_{IS}(G \wr \mathcal{I}_n)|$.

In order to prove the upper bound we first show that for an induced lattice Y, there is an upper bound to the number of idempotent separating congruences which induce Y. Let $\{L_i, M_i, N_i \mid 3 \leq i \leq n\} \cup \{N_2, L_2, L_1\}$ be the labels of the vertices in Y, so, in particular, by Lemma 6.5.4

$$N_n \leq M_n \leq N_{n-1} \leq \cdots \leq N_3 \leq M_3$$

is a sequence of normal subgroups. As $c(G) = c$ there are at most $c - 1$ values of i between 3 and n for which $N_i \neq M_i$. If $\mathbf{K}_i(L_i, N_i, N_i, \phi)$ is an invariant normal subgroup then $\phi : L_i \to L_i/N_i$ is the standard quotient homomorphism, so when $M_i = N_i$ there is precisely one invariant normal subgroup $K \trianglelefteq G^i$ with $\mathbf{L}(K) = L_i$, $\mathbf{M}(K) = M_i$ and $\mathbf{N}(K) = N_i$.

Let q be the largest number of homomorphisms $L \to L/N$ where we vary L and N over normal subgroups of G. By Proposition 6.5.1 there are fewer than $\lambda_1(G)$ invariant normal subgroups $K \trianglelefteq G^2$, so at most $\lambda_1(G)$ invariant normal subgroups that have $N_2 = \{g \in G \mid (g, 1) \in K\}$. Hence there are at most $\lambda_1(G)q^{c-1}$ idempotent separating congruences which induce Y. Thus it suffices to show that there are at most $B'n^{2(c-1)}$ induced lattices.

To this end notice that a congruence induced lattice (with ordering as shown in Figure 6.1) can be decomposed into two sequences. The first is $L_n \leq L_{n-1} \leq \cdots \leq L_2 \leq L_1$ and is of length n, the second is $N_n \leq M_n \leq N_{n-1} \leq M_{n-1} \leq \cdots \leq M_3 \leq N_2$ and is of length $2n - 3$. When we ignore repeats in these sequences the resulting chains are each subchains of chief series.

Since G is a finite group there are finitely many chief series; say that there are r chief series. Then there are r^2 pairs of chief series. On the other hand, by Lemma 6.5.3, there is $D \in \mathbb{N}$ such that the number of sequences of length k arising from a chain of length x is bounded above by Dk^{x-1}. Thus given a pair chief series (which each have maximum length c), there are fewer than $(D(2n-3)^{c-1})(Dn^{c-1})$ distinct pairs of sequences, of lengths $2n - 3$ and n respectively, which reduce to subchains of this pair of chief series when repeats are ignored. It follows that there are at most

$r^2(D(2n-3)^{c-1})(Dn^{c-1})$ congruence induced lattices. Hence we have that

$$\begin{aligned}|\mathfrak{C}_{IS}(G \wr \mathcal{I}_n)| &\leq \lambda_1(G)q^{c-1}r^2(D(2n-3)^{c-1})(Dn^{c-1})\\ &\leq (\lambda_1(G)q^{c-1}D^2r^2 2^{c-1})n^{2(c-1)},\end{aligned}$$

and $(\lambda_1(G)q^{c-1}D^2r^2 2^{c-1}) = B$ is a constant determined by G. This completes the proof of the result. □

So far in this section we have demonstrated that the number of idempotent separating congruences on $G \wr \mathcal{I}_n$ for a finite group G is related to the chief length of G. Next we show that up to order of the polynomial, these bounds are the best possible in the sense that there are groups with arbitrarily large chief length that attain either the maximum or the minimum order growth for $|\mathfrak{C}_{IS}(G \wr \mathcal{I}_n)|$. We shall utilise the notation $\lfloor x \rfloor$ for the greatest integer at most x and $\lceil x \rceil$ for the least integer at least x.

Example 6.5.6. For the maximum growth of $|\mathfrak{C}_{IS}(G \wr \mathcal{I}_n)|$ we consider the group $\mathbb{Z}_2^c$, which has chief length $c(\mathbb{Z}_2^c) = c$. If C is a chief series of length c for $\mathbb{Z}_2^c$ then by applying Lemma 6.5.3 we obtain that there is $A' > 0$ such that for all k there are at least $A'k^{c-1}$ sequences of subgroups of $\mathbb{Z}_2^c$ of length k which reduce to a subchain of C when repeats are ignored. It is elementary that there is $A > 0$ such that $A'(\lfloor n/2 \rfloor)^{c-1} \geq An^{c-1}$ for all $n \geq 2$.

It is straightforward that for each pair $X, Y \trianglelefteq \mathbb{Z}_2^c$ with $X \leq Y$, and for each $3 \leq i \leq n$, $\mathbf{K}_i(X, Y, Y, x \mapsto xY) \trianglelefteq (\mathbb{Z}_2^c)^i$ is an invariant normal subgroup. Let

$$\{W_i \trianglelefteq \mathbb{Z}_2^c \mid 1 \leq i \leq \lfloor n/2 \rfloor\} \quad \text{and} \quad \{Y_i \trianglelefteq \mathbb{Z}_2^c \mid 1 \leq i \leq \lceil n/2 \rceil\}$$

be decreasing sequences of subgroups. Then let $K_1 = W_1$ and $K_2 = W_2^2$, and define $K_i = \mathbf{K}_i(\mathbb{Z}_2^c, W_i, W_i, g \mapsto gW_i)$ for $3 \leq i \leq \lfloor n/2 \rfloor$ and $K_{\lfloor n/2 \rfloor + i} = \mathbf{K}_{\lfloor n/2 \rfloor + i}(Y_i, \{1\}, \{1\}, y \mapsto y)$ for $1 \leq i \leq \lceil n/2 \rceil$. It is straightforward to see that $\{K_i \trianglelefteq (\mathbb{Z}_2^c)^i \mid 1 \leq i \leq n\}$ is a closed set of invariant normal subgroups, so defines an idempotent separating congruence. Moreover different choices of $\{W_i \mid 1 \leq i \leq \lfloor n/2 \rfloor\}$ and $\{Y_i \mid 1 \leq i \leq \lceil n/2 \rceil\}$ give distinct congruences. Thus

$$|\mathfrak{C}_{IS}(\mathbb{Z}_2^c \wr \mathcal{I}_n)| \geq (An^{c-1})(A'(\lceil n/2 \rceil)^{c-1}) \geq (AA'(1/2)^{c-1})n^{2(c-1)}.$$

Therefore the growth of $|\mathfrak{C}_{IS}(\mathbb{Z}_2^c \wr \mathcal{I}_n)|$ is polynomial in n of order $2(c-1)$.

Example 6.5.7. For a group that attains the minimum order growth for $|\mathfrak{C}_{IS}(G \wr \mathcal{I}_n)|$ we consider the group $\mathcal{A}_5^c$. We note that if $J \trianglelefteq \mathcal{A}_5^c$ then $J = J_1 \times \cdots \times J_c$ for $J_i \trianglelefteq \mathcal{A}_5$. Indeed, let $J\pi_i$ be the projection of J onto the i^{th} coordinate and let

$$N_i = \{a \in \mathcal{A}_5 \mid (1, \ldots, 1, a, 1, \ldots, 1) \in J\}$$

where the (possibly) non-identity entry is in the i^{th}-coordinate. Then $N_i \trianglelefteq \mathcal{A}_5$ and it is straightforward to see that $J\pi_i/N_i$ is abelian. As $\mathcal{A}_5$ is simple $J\pi_i$ is either trivial or $\mathcal{A}_5$. Since $\mathcal{A}_5$ is not abelian it follows that $N_i = J\pi_i$, and this then implies that

$$J = J\pi_1 \times \ldots J\pi_c.$$

It is then apparent that $c(\mathcal{A}_5^c) = c$. Moreover if also $N \trianglelefteq \mathcal{A}_5^c$ with J/N abelian then $J = N$. Thus the only invariant normal subgroups of $(\mathcal{A}_5^c)^m$ for $3 \leq m \leq n$ are of the form $\mathbf{K}_m(L, L, L, l \mapsto L) = L^m$ for $L \trianglelefteq \mathcal{A}_5^c$. Also the invariant normal subgroups of $(\mathcal{A}_5^c)^2$ are of the form L^2 for $L \trianglelefteq \mathcal{A}_5^c$. It follows that idempotent separating congruences on $\mathcal{A}_5^c \wr \mathcal{I}_n$ exactly correspond to chains of normal subgroups of $\mathcal{A}_5^c$ of length n, of which, by Lemma 6.5.3, there are at most Bn^{c-1}.

Since the chief length plays an important role in the size of $|\mathfrak{C}(G \wr \mathcal{I}_n)|$ it is worth noting that in general it is not possible to do better than the trivial bound on the chief length; that is that $c(G)$ is at most the number of prime factors (counted with multiplicity) of $|G|$.

Theorem 6.5.8. *Let G be a finite group with $c(G) = c$. Then there are $A, B > 0$ such that for all n*

$$An^c \leq |\mathfrak{C}(G \wr \mathcal{I}_n)| \leq Bn^{2c-1}.$$

Proof. As in Proposition 6.5.5 we may prove this for sufficiently large n and then the result holds for all n. The upper bound is straightforward: we note that by Theorem 6.2.12 each congruence ρ decomposes in terms of a Rees congruence, a normal subgroup of $G \wr \mathcal{S}_m$ and an idempotent separating congruence. There are $n+1$ ideals, by Corollary 6.5.2 at most $\lambda_2(G)$ normal subgroups of $G \wr \mathcal{S}_m$ and, by Proposition 6.5.5, at most $B'n^{2(c-1)}$ idempotent separating congruences. Thus letting $B = 2\lambda_2(G)B'$ we have

$$|\mathfrak{C}(G \wr \mathcal{I}_n)| \leq \lambda_2(G)B'(n+1)n^{2(c-1)} \leq Bn^{2c-1}.$$

We now prove the lower bound. Let $\{C_i \mid i \in I\}$ be the set of decreasing sequences of normal subgroups of G of length $\lfloor n/2 \rfloor$. We have seen (in Example 6.5.6) that there is $A' > 0$ such that $|I| \geq A'n^{c-1}$. We take $C_i = \{L_1 \supseteq L_2 \supseteq \cdots \supseteq L_{\lfloor n/2 \rfloor}\}$ and consider the set

$$X_i = \{G, G^2, \ldots, G^{\lceil n/2 \rceil}, K_1, K_2 \ldots, K_{\lfloor n/2 \rfloor}\}$$

where $K_j = \mathbf{K}_{\lceil n/2 \rceil + j}(L_j, L_j, L_j, l \mapsto L_j) = L_j^{\lceil n/2 \rceil + j}$ for $1 \leq j \leq \lfloor n/2 \rfloor$. Then X_i is a closed set of invariant normal subgroups, so defines an idempotent separating congruence, and for distinct sequences of normal subgroups of G the corresponding idempotent separating congruences are different. Then $\rho_{X_i,m} = \rho(m, X_i, \{(g; 1) \mid g \in G^m\})$ is a distinct congruence for each $i \in I$ and $1 \leq m \leq \lceil n/2 \rceil$. Thus with $A = A'/2$

$$|\mathfrak{C}(G \wr \mathcal{I}_n)| \geq A'n^{c-1}(\lceil n/2 \rceil) \geq An^c.$$

$\square$

As is the case for $\mathfrak{C}_{IS}(G \wr \mathcal{I}_n)$ in general these are the best possible bounds. There are groups that attain the maximum and minimum polynomial growth for the size of the congruence lattice; again the groups considered in Examples 6.5.6 and 6.5.7 attain the maximum and minimum respectively.

6.6 FURTHER REMARKS

To conclude this chapter we make a series of remarks primarily about embeddings of lattices. There are two natural questions that we consider. First, recalling that one motivation for the study of $G \wr \mathcal{I}_n$ is that it is a generalisation of $\mathcal{I}_n$, we ask how congruences on $G \wr \mathcal{I}_n$ relate to congruences on $\mathcal{I}_n$. Second, we ask about the relationship between $\mathfrak{C}(G\wr\mathcal{I}_n)$ and $\mathfrak{C}(G\wr\mathcal{I}_{n+1})$

We tackle the first of these questions. We make a direct observation without relying on the description of congruences we have produced thus far. We notice that we can embed $\mathcal{I}_n$ into $G \wr \mathcal{I}_n$ via the map $\eta : \mathcal{I}_n \hookrightarrow G \wr \mathcal{I}_n$; where $a \mapsto (1_{aa^{-1}}; a)$. If we have $\kappa \subseteq \mathcal{I}_n \times \mathcal{I}_n$ then we write

$$\kappa\eta = \{(a\eta, b\eta) \mid (a, b) \in \kappa\}.$$

Lemma 6.6.1. *Let κ be an equivalence relation on $\mathcal{I}_n$, and let ζ_κ be the equivalence relation on $G \wr \mathcal{I}_n$ generated by $\kappa\eta$. Then the restriction of ζ_κ to $\operatorname{Im}(\eta)$ is $\kappa\eta$. Moreover, if κ is a congruence on $\mathcal{I}_n$ and ζ_κ is the congruence on $G \wr \mathcal{I}_n$ generated by $\kappa\eta$ then the restriction of ζ_κ to $\operatorname{Im}(\eta)$ is $\kappa\eta$.*

Proof. We notice that the relation ξ_κ on $G \wr \mathcal{I}_n$ defined by

$$\xi_\kappa = \{((g; a), (h; b)) \mid (a, b) \in \kappa\}$$

is an equivalence relation on $G \wr \mathcal{I}_n$. Also, $\kappa\eta \subseteq \zeta_\kappa \subseteq \xi_\kappa$ and $\xi_\kappa|_{\operatorname{Im}(\eta)} = \kappa\eta$. This completes the proof of the first claim. Also when κ is a congruence then ξ_κ is a congruence, thus enabling the completion of the second claim. $\square$

Conversely, if ρ is a congruence on $G \wr \mathcal{I}_n$ then we restrict this to a congruence on $\mathcal{I}_n$ in two different ways:

$$\rho_1 = \{(a, b) \mid (1_{aa^{-1}}; a) \ \rho \ (1_{bb^{-1}}; b)\},$$
$$\rho_2 = \{(a, b) \mid \exists g, h \in (G^0)^n, \ (g; a) \ \rho \ (h; b)\}.$$

Though both are congruences on $\mathcal{I}_n$, in general these are not equal, although it is immediate that $\rho_1 \subseteq \rho_2$. Utilising our knowledge about the structure of congruences on $G \wr \mathcal{I}_n$ we can give explicit descriptions of ρ_1 and ρ_2. We remark that ρ_1 is the universal congruence on $\mathcal{I}_n$ if and only if ρ is the universal congruence on $G \wr \mathcal{I}_n$, and consequently the same is true for ρ_2. We suppose that our initial congruence on $G \wr \mathcal{I}_n$ is non-universal, so is of the form $\rho = \rho(m, \{T_i\}, L)$. We recall Theorem 1.5.17, which states that a non-universal congruence on $\mathcal{I}_n$ may be described via an integer k with $1 \leq k \leq n$ and a subgroup $N \trianglelefteq \mathcal{S}_k$. We take k_1, k_2 and N_1, N_2 to be the corresponding integers and subgroups associated with ρ_1, ρ_2 respectively, so $\rho_1 = \rho(k_1, N_1)$ and $\rho_2 = \rho(k_2, N_2)$. It is immediate that $k_1 = k_2 = m$. Further, considering the relation ρ on restriction to elements of rank m we recall that $(g; a)\ \rho\ (h; b)$ exactly when $a\ \mathcal{H}(\mathcal{I}_n)\ b$ and $((g^{-1}h)_{a^{-1}}; a^{-1}b)\Psi_{a^{-1}a} \in L$ where $\Psi_{a^{-1}a}\colon H_{(1_{a^{-1}a}; a^{-1}a)} \to G \wr \mathcal{S}_m$; $(h; c) \mapsto (h\omega; c\theta)$ is the isomorphism from $H_{a^{-1}a}$ to $G \wr \mathcal{S}_m$. Let $L \trianglelefteq G \wr \mathcal{S}_m$ have corresponding normal subgroup triple (J, Q, ξ). Then $a\ \rho_2\ b$ exactly when there is some h such that $(h; a^{-1}b)\Psi_{a^{-1}a} \in L$, which says that $(a^{-1}b)\theta \in Q$. Thus we have that $N_2 = Q$. Also, $a\ \rho_1\ b$ if $(1_{a^{-1}a}; a^{-1}b)\Psi_{a^{-1}a} \in L$ which is equivalent to $((a^{-1}b)\theta)\xi = J$. In turn this says that $a\ \rho_1\ b$ when $(a^{-1}b)\theta \in \ker(\xi)$, so we have that $N_1 = \ker(\xi)$. We have shown that ρ_1 and ρ_2 are determined by m and L; explicitly:

$$\rho_1 = \rho(m, \ker(\xi)) \quad \text{and} \quad \rho_2 = \rho(m, Q).$$

To conclude this chapter we briefly consider the relationship between $\mathfrak{C}(G \wr \mathcal{I}_n)$ and $\mathfrak{C}(G \wr \mathcal{I}_{n+1})$. We define

$$\Theta : G \wr \mathcal{I}_n \to G \wr \mathcal{I}_{n+1}; \quad (g_1, \ldots, g_n; a) \mapsto (g_1, \ldots, g_n, 0; a)$$

where in the image we regard a as an element of $\mathcal{I}_{n+1}$. It is clear that this is an embedding. For a relation $\kappa \subseteq G \wr \mathcal{I}_n \times G \wr \mathcal{I}_n$ we write

$$\kappa\Theta_2 = \{((g; a)\Theta, (h; b)\Theta) \mid ((g; a), (h; b)) \in \kappa\} \subseteq G \wr \mathcal{I}_{n+1} \times G \wr \mathcal{I}_{n+1}.$$

Proposition 6.6.2. *Let $\rho = \rho(m, \{T_i \mid m+1 \leq i \leq n\}, L)$ be a non universal congruence on $G \wr \mathcal{I}_n$. Then $\langle \rho\Theta_2 \rangle = \rho(m, \{T_i \mid m+1 \leq i \leq n+1\}, L)$, where T_{n+1} is the trivial group. Also, $\langle \omega\Theta_2 \rangle = \rho(n+1, \emptyset, \{(1; 1)\})$.*

Moreover, if ρ is any congruence $G \wr \mathcal{I}_n$, then

$$\langle \rho\Theta_2 \rangle \cap ((G \wr \mathcal{I}_n)\Theta \times (G \wr \mathcal{I}_n)\Theta) = \rho\Theta_2.$$

Proof. We recall how we recover the triple $(m, \{T_i\}, L)$ from ρ. We obtain m as one more than the maximum rank of $x \in G \wr \mathcal{I}_n$ such that $x\ \rho\ (0; 0)$.

With $e \in E(\mathcal{I}_n)$ an idempotent of rank m, the group $L \trianglelefteq G \wr \mathcal{S}_m$ is recovered as

$$\{(g;a)\Psi_e \mid (g;a)\ \rho\ (1_e;e)\},$$

where $\Psi_e \colon H_{(1_e;e)} \to G \wr \mathcal{S}_m$ is the usual isomorphism. Finally, with $f \in E(\mathcal{I}_n)$ an idempotent of rank $i \geq m+1$, we have that $T_i \leq G^i$ is equal to

$$\{h\omega \mid (h;f)\ \rho\ (1_f;f)\}.$$

where $\omega \colon G^f \to G^i$ is the function that ignores 0 entries.

Let $(m', \{T'_i \mid m'+1 \leq i \leq n+1\}, L')$ be such that

$$\langle \rho\Theta_2 \rangle = \rho(m', \{T'_i \mid m'+1 \leq i \leq n+1\}, L').$$

It is clear that $\rho\Theta_2 \subseteq \rho(m, \{T_i \mid m+1 \leq i \leq n+1\}, L)$, where T_{n+1} is the trivial group, so, as $\langle \rho\Theta_2 \rangle$ is a congruence on $G \wr \mathcal{I}_{n+1}$, we have that

$$\rho(m', \{T'_i \mid m'+1 \leq i \leq n+1\}, L') \subseteq \rho(m, \{T_i \mid m+1 \leq i \leq n+1\}, L).$$

We remark that this implies (Proposition 6.2.13) that $m' \leq m$, however, we observe that for $x \in G \wr \mathcal{I}_n$ with $\operatorname{rank}(x) < m$ that $x\ \rho\ (0;0)$. Then $x\Theta$ has $\operatorname{rank}(x\Theta) = \operatorname{rank}(x) < m$ and $x\Theta\ \langle \rho\Theta_2 \rangle\ (0;0)\Theta = (0;0)$. Therefore $m \leq m'$ so the two are equal.

Proposition 6.2.13 also implies that $L' \leq L$ and $T'_i \leq T_i$ for $m+1 \leq i \leq n+1$. On the other hand, choosing $e \in E(\mathcal{I}_n)$ of rank m and $a \in H_e$ we note that $((g;a)\Theta)\Psi_e = (g;a)\Psi_e$ so, if $(h;b) \in L$ then there is $(g;a) \in H_{(1_e;e)}$ such that $(g;a)\Psi_e = (h;b)$ and $(g;a)\ \rho\ (1;e)$. Then

$$(g;a)\Theta\ \langle \rho\Theta_2 \rangle\ (1_e;e)\Theta = (1_e;e),$$

so $((g;a)\Theta)\Psi_e = (g;a)\Psi_e = (h;b) \in L'$. Thus $L \leq L'$, so the two are equal. Similarly we obtain that $T'_i = T_i$ for each $m+1 \leq i \leq n$. That T'_{n+1} is the trivial group is clear from the fact that elements in the image of Θ are of rank at most n, so generate only trivial relations in D_{n+1}. $\square$

Proposition 6.6.2 demonstrates that the map $\mathfrak{C}(G \wr \mathcal{I}_n) \to \mathfrak{C}(G \wr \mathcal{I}_{n+1})$ defined by $\rho \mapsto \rho\Theta_2$ is an embedding so we may regard the lattice of congruences on $G \wr \mathcal{I}_{n+1}$ as an extension of the lattice of congruences on $G \wr \mathcal{I}_n$. We are reminded of Theorem 6.2.6, the general result regarding extending congruences on an ideal to congruences on the whole semigroup. As we stated earlier, applied to $G \wr \mathcal{I}_n$ this also provides a method to construct $\mathfrak{C}(G \wr \mathcal{I}_{n+1})$ from $\mathfrak{C}(G \wr \mathcal{I}_n)$ and the lattice of normal subgroups of a group $\mathcal{H}$-class in the topmost $\mathcal{J}$-class, which we know is isomorphic to the group $G \wr \mathcal{S}_{n+1}$. We indicate how this is done. Theorem 6.2.6 states that

$$\mathfrak{C}(G \wr \mathcal{I}_{n+1}) = \{\kappa \cup \sigma_N \mid \kappa \in \mathfrak{C}^{G \wr \mathcal{I}_{n+1}}(J_n),\ N \trianglelefteq G \wr \mathcal{S}_{n+1},\ \gamma_N \subseteq \kappa\} \cup \{\omega\},$$

where σ_N is our usual relation on D_{n+1} corresponding to the normal subgroup $N \trianglelefteq G \wr \mathcal{S}_{n+1}$ and $\gamma_N = \langle \sigma_N \rangle|_{J_n}$. As remarked previously, liftable congruences on J_n correspond with congruences on $G \wr \mathcal{I}_n$ so are the universal relation, or described in terms of triple $(m, \{T_i \mid m+1 \leq i \leq n\}, L)$. If κ is non-universal this implies that a congruence on $G \wr \mathcal{I}_{n+1}$ may be described in terms of a quadruple $(m, \{T_i \mid m+1 \leq i \leq n\}, L, N)$. However, the condition $\gamma_N \subseteq \kappa$ will imply that, since κ is not the universal relation on J_n, the subgroup $N \trianglelefteq G \wr \mathcal{S}_{n+1}$ is a subset of $\{(g; 1) \mid g \in G^{n+1}\}$, so we incorporate N as T_{i+1} into a triple for the congruence and have that

$$\kappa \cup \sigma_N = \rho(m, \{T_i \mid m+1 \leq i \leq n+1\}, L).$$

When κ is the universal relation we obtain that $\kappa \cup \sigma_N = \rho(n+1, \emptyset, N)$.

— 7 —

Congruences on $G \wr \mathcal{I}_n$ II

This chapter is largely a continuation of the previous, split off due to considerations of length and a change in tack. We follow two paths here. First we consider a specific case, when G is a finite simple group. Second we consider a particular submonoid of $G \wr \mathcal{I}_n$, motivated by the monoid of order preserving partial automorphisms on a set, which is usually written $\mathcal{O}_n$.

Our objective in the first half of this chapter is to the describe the lattice of congruences on $G \wr \mathcal{I}_n$ where G is a finite simple group. We shall consider as separate cases abelian and non-abelian groups. As a general overview, first we consider invariant normal subgroups of G^m, then describe idempotent separating congruences on $G \wr \mathcal{I}_n$ and subgroups of $G \wr \mathcal{S}_m$. Finally we conclude by describing the lattice of congruences on $G \wr \mathcal{I}_n$. We take as understood the classification of finite simple groups, in particular we use the fact that if G is abelian then $G \cong \mathbb{Z}_p$ for some prime p, the cyclic group of order p.

7.1 Subgroups of direct products of finite simple groups

We recall Theorem 6.3.8, our description of invariant normal subgroups of G^m (for $m \geq 3$) via m-invariant quadruples. These are quadruples (L, M, N, ϕ) where $N \leq M \leq L$ are normal subgroups of G and $\phi\colon L \to L/N$ is a homomorphism, such that the following hold:

(i) $[G, L] \subseteq N$;

(ii) the restriction of ϕ to M has the form $g \mapsto g^{1-m}N$;

(iii) for $g \in L$, if $g\phi = hN$ then $gM = hM$.

Given an m-invariant quadruple (L, M, N, ϕ) the associated invariant normal subgroup of G^m is $\mathbf{K}_m(L, M, N, \phi)$ which we recall is

$$\{(g_1, \ldots, g_m) \in L^m \mid g_1M = \ldots = g_mM, \quad g_1\phi = g_1^{2-m}g_2 \ldots g_mN\}.$$

For subgroups of G^2 we appeal to Corollary 6.3.1, which says invariant normal subgroups of G^2 are of the form

$$X(A, A, C, C, \theta) = \{(a, b) \in A \times A \mid (aC)\theta = bC\}$$

where $A, C \trianglelefteq G$, $C \trianglelefteq A$ such that $[G, A] \subseteq C$, and θ is an automorphic involution of A/C.

For finite simple groups we can refine our descriptions of invariant normal subgroups further.

Corollary 7.1.1. *Let G be a finite simple non-abelian group. Then for each $m \in \mathbb{N}$ there are precisely two invariant normal subgroups of G^m,*

$$\{(1, \ldots, 1)\} \qquad \textit{and} \qquad G^m.$$

Proof. If $m = 1$ then this is trivially true as G is simple. For $m \geq 3$ we apply Theorem 6.3.8 to describe subgroups in terms of m-invariant quadruples. Suppose (L, M, N, ϕ) is an m-invariant quadruple. Since G is simple either $L = \{1\}$ or $L = G$. If $L = \{1\}$, then the subgroup is $\{(1, 1, \ldots, 1)\}$. If $L = G$ then, as $[G, G] \subseteq N$ and G is non-abelian (so $[G, G] \neq \{1\}$) and simple we have that $N = G$, so $M = G$, L/N is again the trivial group and ϕ is the map $g \mapsto G$, so the subgroup is G^m. If $m = 2$ then the result follows from Corollary 6.3.1 via a very similar argument to the case for $m \geq 3$. □

Corollary 7.1.2. *Let G be a finite simple abelian group and let $m \geq 2$. If $G \cong \mathbb{Z}_2$ (say $G = \{1, x\}$) and $m = 2$ there are 3 invariant normal subgroups of G^m*

$$\{(1, 1)\}, \qquad \{(1, 1), (x, x)\} \qquad \textit{and} \qquad G^2.$$

Otherwise there are 4 invariant normal subgroups of G^m

$$\{(1, \ldots, 1)\}, \qquad \{(g, g, \ldots, g) \mid g \in G\},$$
$$\{(g_1, \ldots, g_m) \mid g_1 g_2 \ldots g_m = 1\} \qquad \textit{and} \qquad G^m.$$

Proof. Again this is entirely elementary. Let $m \geq 3$ and let (L, M, N, ϕ) be an m-invariant quadruple. As G is simple, L, M and N are each either $\{1\}$ or G. If $L = \{1\}$ then $M = N = \{1\}$ and ϕ is the map $1 \mapsto \{1\}$, the subgroup is then $\{(1, \ldots, 1)\}$. Thus we suppose that $L = G$. If $N = M = \{1\}$ then, by (iii) of the conditions for (L, M, N, ϕ) to be an m-invariant quadruple, ϕ is the identity map, so the subgroup is

$$\mathbf{K}_m(G, 1, 1, g \mapsto g) = \{(g, \ldots, g) \mid g \in G\}.$$

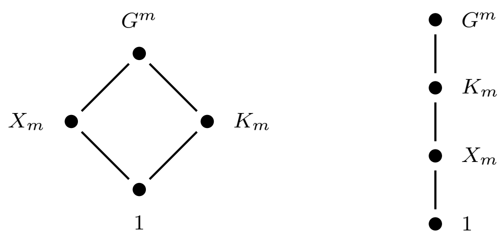
If $N = \{1\}$ and $M = G$ then, by (ii) of the conditions for (L, M, N, ϕ) to be an m-invariant quadruple, ϕ is the map $g \mapsto g^{1-m}$ so the subgroup is

$$\mathbf{K}_m(G, G, 1, g \mapsto g^{1-m}) = \{(g_1, \ldots, g_m) \mid g_1 \ldots g_m = 1\}.$$

Finally, if $N = M = G$ then ϕ is the map $g \mapsto G$ and the subgroup is G^m.

The case for $G \cong \mathbb{Z}_2$ and $m = 2$ follows from Corollary 6.3.1; the same result implies all other cases for $m = 2$. □

In Corollary 7.1.2 we see the first indication that the case for $G \cong \mathbb{Z}_2$ is an outlier. This crops up on several occasions and we shall indicate where this occurs. Our methods can be used, as we have done here, to describe the situation for $\mathbb{Z}_2$ but on future occasions this case deviates more substantially - in the details not in principle - and thus would require a large investment of effort. Extra considerations also arise when considering the group $\mathbb{Z}_3$. Bearing this in mind we shall not spend time dwelling on the case for $G \cong \mathbb{Z}_2$ or $G \cong \mathbb{Z}_3$.



$p \nmid m$ $\qquad$ $p \mid m$

Figure 7.1: Invariant normal subgroups of $\mathbb{Z}_p^m$

A slight complication arises when we consider the ordering of invariant normal subgroups of G^m. When G is non-abelian the issue does not arise, since the lattice of invariant normal subgroups of G^m is the two element lattice for every m. On the other hand, when G is abelian the structure of the lattice depends on m. Indeed, we notice that, if $G \cong \mathbb{Z}_p$ and $p \mid m$, then in G^m,

$$\{(g, g, \ldots, g) \mid g \in G\} \subseteq \{(g_1, \ldots, g_m) \mid g_1 g_2 \ldots g_m = 1\}$$

However, this inclusion does not occur when $p \nmid m$. Therefore the lattice of invariant subgroups of G^m has two forms, depending on whether $p \mid m$. When $p \mid m$ the lattice is a chain of length four, and when $p \nmid m$ the lattice is the four element diamond lattice. These options are shown in Fig. 7.1, in which we write $X_m = \mathbf{K}_m(G, 1, 1, g \mapsto g) = \{(g, \ldots, g) \mid g \in G\} \trianglelefteq G^m$ and $K_m = \mathbf{K}_m(G, G, 1, g \mapsto g^{1-m}) = \{(g_1, \ldots, g_m) \mid g_1 \ldots g_m = 1\} \trianglelefteq G^m$. Of course, when $p = 2$ and $m = 2$ the lattice is a chain of length 3.

We now consider normal subgroups of $G \wr \mathcal{S}_m$ (for $m \geq 2$), which we recall are described in Theorem 6.4.5. The following is the complete list of normal subgroups.

(i) For each invariant normal subgroup $K \trianglelefteq G^m$,

$$\{(k; 1) \mid k \in K\} \trianglelefteq G \wr \mathcal{S}_m.$$

(ii) For each triple (N, Q, ζ) with $N \trianglelefteq G$ and G/N abelian, $Q \trianglelefteq \mathcal{S}_m$ non trivial and $\zeta \colon Q \to G/N$ a homomorphism such that $[\mathcal{S}_m, Q] \subseteq \ker(\zeta)$,

$$\{(g_1, \ldots, g_m; q) \mid g_1, \ldots, g_m \in G,\ q \in Q,\ q\zeta = g_1 \ldots g_m N\} \trianglelefteq G \wr \mathcal{S}_m.$$

As we know the invariant normal subgroups of G^m we may write down a more explicit list of the normal subgroups of $G \wr \mathcal{S}_m$ and, more importantly, produce a diagram of the lattice of normal subgroups. We denote by V the normal Klein 4 subgroup of $\mathcal{S}_4$, explicitly

$$V = \Big\{1,\ (1\ 2)(3\ 4),\ (1\ 3)(2\ 4),\ (1\ 4)(2\ 3)\Big\}.$$

Corollary 7.1.3. *Let G be a non-abelian finite simple group and let $m \geq 2$ be an integer. Then the non trivial normal subgroups of $G \wr \mathcal{S}_m$ are $G \wr Q$ for $Q \trianglelefteq \mathcal{S}_m$. In particular:*

(i) if $m = 3$ or $m \geq 5$ then there are 4 normal subgroups of $G \wr \mathcal{S}_m$ which are: the trivial group $\{(1, \ldots, 1; 1)\}$, $G^m \times \{1\}$, $G \wr \mathcal{A}_m$ and the group itself $G \wr \mathcal{S}_m$;

(ii) if $m = 2$ then there are 3 normal subgroups of $G \wr \mathcal{S}_2$ which are: the trivial group $\{(1, 1; 1)\}$, $G^2 \times \{1\}$ and the group itself $G \wr \mathcal{S}_2$;

(iii) if $m = 4$ then there are 5 normal subgroups of $G \wr \mathcal{S}_4$ which are: the trivial group $\{(1, 1, 1, 1; 1)\}$, $G^4 \times \{1\}$, $G \wr V$, $G \wr \mathcal{A}_4$ and the group itself $G \wr \mathcal{S}_4$;

Furthermore, for each m the lattice of normal subgroups of $G \wr \mathcal{S}_m$ is a chain.

Proof. This is straightforward. The only observation needed is regarding normal subgroups of $G \wr \mathcal{S}_m$ with non-trivial projection onto the $\mathcal{S}_m$ component (type (ii) in the description of normal subgroups). The only subgroup $N \trianglelefteq G$ with G/N abelian is G itself, and then there is precisely one homomorphism $\zeta\colon Q \to G/G$ for each non trivial $Q \trianglelefteq \mathcal{S}_m$. □

Turning our attention to finite simple abelian groups we may produce a similar result.

Corollary 7.1.4. *Let G be an abelian finite simple group which is not isomorphic to $\mathbb{Z}_2$ or to $\mathbb{Z}_3$. Let $m \geq 2$, and let $K = \{(g_1, \ldots, g_m) \in G^m \mid g_1 \ldots g_m = 1\}$ (which we recall is equal to $\mathbf{K}_m(G, G, 1, g \mapsto g^{1-m})$ for $m \geq 3$). Then the subgroups of $G \wr \mathcal{S}_m$ are*

(i) the trivial group $\{(1, \ldots, 1; 1)\}$;

(ii) the subgroup $\{(g, g, \ldots, g; 1) \mid g \in G\}$;

(iii) the subgroups $K \wr Q$ for $Q \trianglelefteq \mathcal{S}_m$;

(iv) the subgroups $G \wr Q$ for $Q \trianglelefteq \mathcal{S}_m$.

In particular, there are 6 *normal subgroups of* $G \wr \mathcal{S}_2$*,* 10 *normal subgroups of* $G \wr \mathcal{S}_4$ *and* 8 *normal subgroups of* $G \wr \mathcal{S}_m$ *for* $m = 3$ *and* $m \geq 5$*.*

Proof. This is similar to the case for a non-abelian group though we have an additional case for subgroups of type (ii) from Theorem 6.4.5, described by a triple (N, Q, ζ). As we are now working with abelian groups we may take $\{1\} \trianglelefteq G$ as our normal subgroup N. In this case we note that, for each $Q \trianglelefteq \mathcal{S}_m$, any homomorphism $\zeta\colon Q \to G$ must be trivial. Indeed, if $m \geq 5$ then, by considering the possibilities for Q ($\mathcal{A}_m$ or $\mathcal{S}_m$) and $\ker(\zeta) \trianglelefteq Q$, we see that the image of ζ (i.e. $Q/\ker(\zeta)$) is trivial or isomorphic to $\mathbb{Z}_2$, and $\mathbb{Z}_2$ is not a subgroup of G. Thus ζ is determined by Q. The result for $m \geq 5$ now follows by applying the description of the normal subgroups of $G \wr \mathcal{S}_m$ from Theorem 6.4.5.

If $m = 4$ then the same logic implies that the image of ζ is isomorphic to one of $\{1\}$, $\mathbb{Z}_2$, $\mathbb{Z}_3$, V, $\mathcal{A}_4$, $\mathcal{S}_3$ and $\mathcal{S}_4$. The only one of these which is a subgroup of G is the trivial group $\{1\}$, so there is one isomorphism for each $Q \trianglelefteq \mathcal{S}_4$. Note that here we use that G is not isomorphic to $\mathbb{Z}_2$ or $\mathbb{Z}_3$.

Similar arguments give the claim for $m = 2$ and $m = 3$. □

The important conclusion to draw from the descriptions of normal subgroups of $G \wr \mathcal{S}_m$ for finite simple G is that, as a set, each normal subgroup is the direct product of an invariant normal subgroup of G^m and a normal subgroup of $\mathcal{S}_m$.

We remark that it is in the proof of Corollary 7.1.4 that it becomes clear where the problems mentioned previously for the groups $\mathbb{Z}_2$ and $\mathbb{Z}_3$ originate. For normal subgroups of $G \wr \mathcal{S}_m$ of type (ii), described by the triple $(\{1\}, Q, \zeta)$ we have a homomorphism $\zeta\colon Q \to G$ where $Q \trianglelefteq \mathcal{S}_m$. For $m = 3$ and $m = 4$ it is possible, for appropriately chosen Q, for a homomorphism from Q to have image isomorphic to $\mathbb{Z}_3$. Thus we do not obtain that ζ is determined by Q, so there are other normal subgroups of $\mathbb{Z}_3 \wr \mathcal{S}_3$ and $\mathbb{Z}_3 \wr \mathcal{S}_4$. We note that the $\mathbb{Z}_3$ case affects only normal subgroups of $G \wr \mathcal{S}_m$ for $m = 3$ and $m = 4$.

The $\mathbb{Z}_2$ case has a more systematic difference. For any $m \geq 2$ there is a surjective homomorphism $\zeta\colon \mathcal{S}_m \to \mathbb{Z}_2$, which has $[\mathcal{S}_m, \mathcal{S}_m] = \mathcal{A}_m = \ker(\zeta)$, so the triple $(\{1\}, \mathcal{S}_m, \zeta)$ gives a normal subgroup of $G \wr \mathcal{S}_m$ of type (ii). This is a more "global" issue, recalling that normal subgroups of $G \wr \mathcal{S}_m$ play an important role in the structure of $\mathfrak{C}(G \wr \mathcal{I}_n)$, when $G = \mathbb{Z}_2$ extra normal subgroups of $G \wr \mathcal{S}_m$ are added in for every m, which causes significant differences to the structure of $\mathfrak{C}(G \wr \mathcal{I}_n)$.

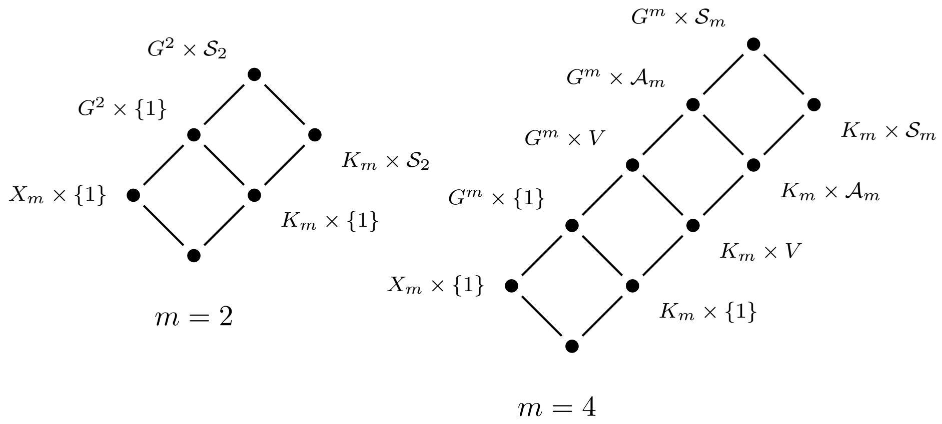


Figure 7.2: Normal subgroups of $\mathbb{Z}_p \wr \mathcal{S}_m$ for $p \geq 5$ and $m = 2, 4$

For non-abelian finite simple groups we observe that Corollary 7.1.3 implies that the lattice of normal subgroups is a chain. The lattice of normal subgroups of $G \wr \mathcal{S}_m$ has a slightly more complicated structure for abelian finite simple groups G than in the non-abelian case. This is largely due to the structure of the lattice of invariant normal subgroups of G^m depending on whether p (the order of G) divides m or not. We take $G = \mathbb{Z}_p$ for $p \geq 5$ a prime. As we have remarked, by Corollary 7.1.4, every normal subgroup of $G \wr \mathcal{S}_m$ is, as a set, a direct product of an invariant normal subgroup of G^m and a normal subgroup of $\mathcal{S}_m$. This makes it easy to compute the ordering on the normal subgroups of $G \wr \mathcal{S}_m$. This is shown in Fig. 7.2 for $m = 2, 4$ and Fig. 7.3 for $m = 3$ and $m \geq 5$. In these figures we write X_m for the subgroup $\mathbf{K}_m(G, 1, 1, g \mapsto g) = \{(g, \ldots, g) \mid g \in G\} \trianglelefteq G^m$ and K_m for the subgroup $\mathbf{K}_m(G, G, 1, g \mapsto g^{1-m}) = \{(g_1, \ldots, g_m) \mid g_1 \ldots g_m = 1\} \trianglelefteq G^m$. In Fig. 7.3 we see how the structure of $\mathfrak{N}(G \wr \mathcal{S}_m)$, the lattice of normal subgroups of $G \wr \mathcal{S}_m$, is related to the structure of $PI(G, m)$, the lattice of invariant normal subgroups of G^m (which is shown in Fig. 7.1). The lattice $PI(G,m)$ embeds into to lattice of $\mathfrak{N}(G \wr \mathcal{S}_m)$ via the map $H \mapsto H \times \{1\}$, and the lattice $\mathfrak{N}(G \wr \mathcal{S}_m)$ is "obtained" from $PI(G,m)$ by attaching a copy of the 4 element diamond lattice to the edge between K_m and G^m in $PI(G,m)$.

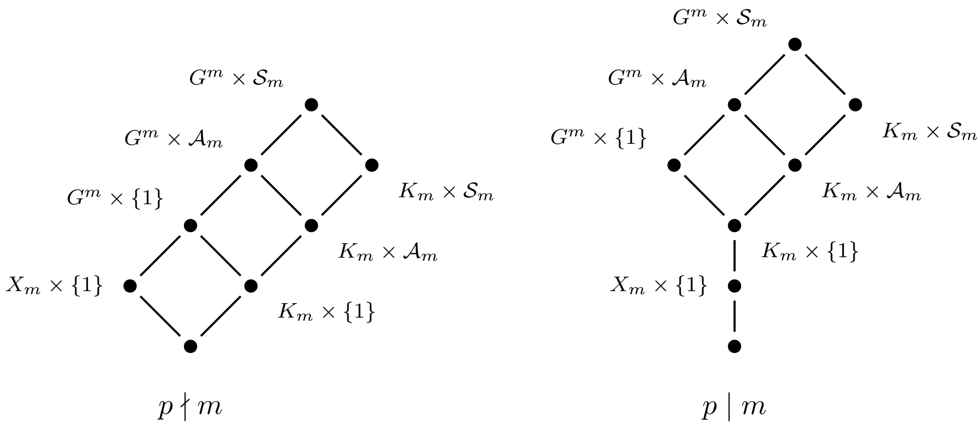


Figure 7.3: Normal subgroups of $\mathbb{Z}_p \wr \mathcal{S}_m$ for $p \geq 5$ and $m = 3$ or $m \geq 5$

7.2 Idempotent separating congruences

As we know (from Proposition 6.2.9) idempotent separating congruences on $G \wr \mathcal{I}_n$ are determined by a closed set of invariant normal subgroups $\{K_i \trianglelefteq G^i \mid 1 \leq i \leq n\}$, where closed means that, under the map $\pi_i \colon G^i \to G^{i-1}$ which ignores the i^{th} coordinate, $K_i\pi_i \subseteq K_{i-1}$ for each $2 \leq i \leq n$. We write $\chi(K_1, \ldots, K_n)$ for the corresponding idempotent separating congruence. Our analysis in Chapter 6 of idempotent separating congruences applies. However we can say more in the case of finite simple groups.

Proposition 7.2.1. *Let G be a non-abelian finite simple group. Then the lattice of idempotent separating congruences on $G \wr \mathcal{I}_n$ is a chain of length $n+1$. The idempotent separating congruences are $\chi_0 = \iota$ and χ_i for $1 \leq i \leq n$ where*

$$\chi_i = \chi(G, G^2, \ldots, G^i, 1^{i+1}, \ldots, 1^n),$$

and $\chi_0 \subseteq \chi_1 \subseteq \cdots \subseteq \chi_{n-1} \subseteq \chi_n$. This chain is shown in Fig. 7.4.

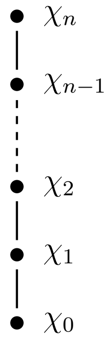


Figure 7.4: Idempotent separating congruences on $G \wr \mathcal{I}_n$ for non-abelian finite simple G

Proof. We recall that $\chi(K_1, \ldots, K_n) \subseteq \chi(L_1, \ldots, L_n)$ exactly when $K_i \subseteq L_i$. From Corollary 7.1.1 we know that for each i the only invariant normal subgroups of G^i are the trivial group and G^i. The result follows from the observation that $G^i \pi_i = G^{i-1}$. $\square$

The picture is significantly harder in the case of abelian finite simple groups. We seek to construct the lattice of idempotent separating congruences on $G \wr \mathcal{I}_n$ in an inductive fashion. We shall use Corollary 7.1.2, and the fact that if $\{K_1, \ldots, K_n\}$ is a closed set of invariant normal subgroups then so is $\{K_1, \ldots, K_{n-1}\}$. Let $m \geq 2$ and let $p \geq 3$ be a prime and consider the group $G = \mathbb{Z}_p$. We know (Corollary 7.1.2) that there are 4 invariant normal subgroups of G^m: the trivial group 1^m, the whole group G^m and two intermediate groups

$$A_m = \{(g, g, \ldots, g) \mid g \in G\} \quad \text{and} \quad B_m = \{(g_1, \ldots, g_m) \mid g_1 g_2 \ldots g_m = 1\}.$$

We notice that, for $m \geq 3$, $A_m \pi_m = A_{m-1}$ and $B_m \pi_m = G^{m-1}$. Suppose we have a closed set of invariant normal subgroups $\{K_1, \ldots, K_{n-1}\}$, and let $K_n \trianglelefteq G^n$ be an invariant normal subgroup. Then $\{K_1, \ldots, K_{n-1}, K_n\}$ is closed if one of the following occurs:

(i) $K_n = 1^n$;

(ii) $K_{n-1} = A_{n-1}$ and $K_n = A_n$;

(iii) $K_{n-1} = B_{n-1}$, $p \mid n-1$ and $K_n = A_n$;

(iv) $K_{n-1} = G^{n-1}$.

With this in mind we can describe the closed sets of invariant normal subgroups.

Proposition 7.2.2. *Let G be an abelian finite simple group not isomorphic to $\mathbb{Z}_2$. The following is a list of all the closed sets of invariant normal subgroups $\{K_i \trianglelefteq G^i \mid 1 \leq i \leq n\}$.*

(i) The "trivial" closed set, with $K_m = 1^m$ for each $1 \leq m \leq n$.

(ii) For any i, j with $1 \leq i \leq j \leq n$, we set $K_m = G^m$ for $1 \leq m \leq i$, $K_m = A_m$ for $i < m \leq j$ and $K_m = 1^m$ for $j < m \leq n$.

(iii) For any i with $2 \leq i \leq n$, we set $K_m = G^m$ for $1 \leq m < i$, $K_i = B_i$ and $K_m = 1^m$ for $i < m \leq n$.

(iv) For any i, j with $2 \leq i < j \leq n$ and $p \mid i$, we set $K_m = G^m$ for $m < i$, $K_i = B_i$, $K_m = A_m$ for $i < m \leq j$ and $K_m = 1^m$ for $j < m \leq n$.

Proof. Before we give a formal proof we note that we can think of each set of invariant normal subgroups as a string of letters (or a word) of length n with each letter being $1, A, B$ or G. The result claims that the permissible strings are of the form

(i) $111 \dots 1$;

(ii) $G \dots G \; \underline{A \dots A} \; \underline{1 \dots 1}$;

(iii) $G \dots G \; B \; \underline{1 \dots 1}$;

(iv) $G \dots G \; B \; A \dots A \; \underline{1 \dots 1}$ where if there are q letters G then $p \mid q + 1$.

Where the dashed underline indicates potentially empty parts of the string.

The formal proof is by an inductive argument using the observation prior to the proposition. The closed sets of length 1 are $\{1\}$ and $\{G\}$, which is as claimed in the proposition. For the inductive step, we know that if

$X_n = \{K_i \trianglelefteq G^i \mid 1 \leq i \leq n\}$ is a closed set of invariant normal subgroups then so is $X_{n-1} = \{K_i \trianglelefteq G^i \mid 1 \leq i \leq n-1\}$. Thus by assumption X_{n-1} is of a form claimed. If X_{n-1} is the trivial closed set, then $K_{n-1} = 1^{n-1}$, so we must have $K_n = 1^n$ so X_n is the trivial closed set. Suppose X_{n-1} is of type (ii), then $K_{n-1} = 1^{n-1}, A_{n-1}$ or G^{n-1}. If $K_{n-1} = 1^{n-1}$ then $K_n = 1^n$ and X_n is of type (ii). If $K_{n-1} = A_{n-1}$ then $K_n = 1^n$ or A_n, and X_n is again of type (ii). If $K_{n-1} = G^{n-1}$ then $K_n = 1^n, A_n, B_n$ or G^n, If $K_n = 1^n, A_n$ or G^n then X_n is of type (ii), if $K_n = B_n$ then X_n is of type (iii).

Similar arguments starting from X_{n-1} being of type (iii) or (iv) give that X_n is of a form claimed in the statement of the proposition. To conclude the proof we note that it is a straightforward check that any of the sets listed in the proposition are closed sets of invariant normal subgroups. $\square$

It is possible to use Proposition 7.2.2 to precisely count the number of closed sets of invariant normal subgroups, which is the same as counting the number of idempotent separating congruences.

Corollary 7.2.3. *Let G be an abelian finite simple group not isomorphic to $\mathbb{Z}_2$. Then there are*

$$\frac{n(n+1)}{2} \;+\; n \;+\; n\left\lfloor\frac{n-1}{p}\right\rfloor - \frac{p}{2}\left\lfloor\frac{n-1}{p}\right\rfloor\left(\left\lfloor\frac{n-1}{p}\right\rfloor + 1\right)$$

idempotent separating congruences on $G \wr \mathcal{I}_n$.

Proof. We remark that the closed sets listed in Proposition 7.2.2 are all distinct. Thus the result follows by counting the number of closed sets of invariant normal subgroups of each type. The number of closed sets of type (i) or (iii) are easiest to count, there are 1 and $n-1$ of them. Second easiest to count and are those of type (ii). For each $1 \leq i \leq n$ there are $n+1-i$ possibilities for j, thus there are

$$\sum_{i=1}^{n+1}(n+1-i) = \sum_{i=1}^{n+1} i = \frac{n(n+1)}{2}$$

sets of subgroups of type (ii). Type (iv) is the hardest to count, for each i such that $i \leq n-1$ with $p \mid i$ there are $n-i$ possibilities for j. Written as a sum this says that there are

$$\sum_{j=1}^{\lfloor\frac{n-1}{p}\rfloor}(n-jp) = n\left\lfloor\frac{n-1}{p}\right\rfloor - \frac{p}{2}\left\lfloor\frac{n-1}{p}\right\rfloor\left(\left\lfloor\frac{n-1}{p}\right\rfloor+1\right)$$

sets of subgroups of this type. □

At this stage we may draw the lattice of idempotent separating congruences on $G \wr \mathcal{I}_n$ for a finite abelian simple group G. First we describe diagrammatically how we move from the $\mathfrak{C}_{IS}(G \wr \mathcal{I}_n)$ to $\mathfrak{C}_{IS}(G \wr \mathcal{I}_{n+1})$ and then we include a diagram for $\mathfrak{C}_{IS}(\mathbb{Z}_5 \wr \mathcal{I}_{11})$.

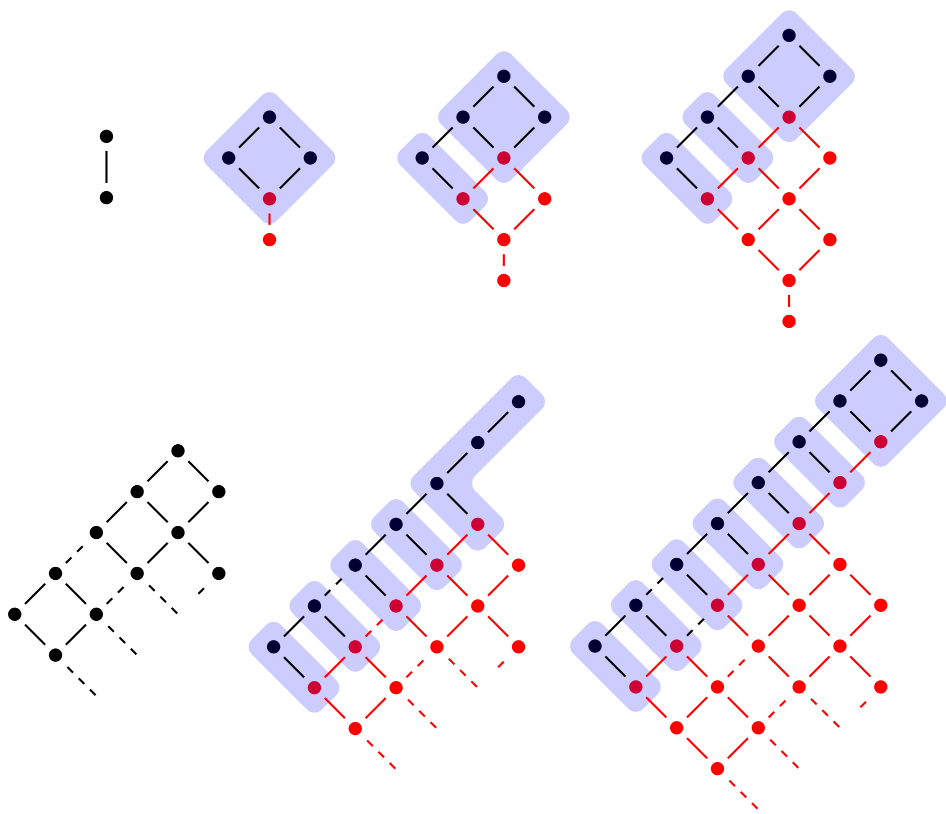


Figure 7.5: Idempotent separating congruences on $G \wr \mathcal{I}_n$ for a finite simple abelian group

The first of these is shown in Fig. 7.5. We draw $\mathfrak{C}_{IS}(G \wr \mathcal{I}_n)$ as a diamond with a chunk cut out of the right hand side. In the top row of the lattices in Fig. 7.5 we see the lattices of idempotent separating congruences for $G \wr \mathcal{I}_1$,

$G \wr \mathcal{I}_2$, $G \wr \mathcal{I}_3$ and $G \wr \mathcal{I}_4$; the group G is irrelevant provided we assume it is not $\mathbb{Z}_2$ or $\mathbb{Z}_3$ In this sequence of lattices the embedding of the m^{th} lattice into the $(m+1)^{th}$ lattice is shown via colouring the relevant sublattice of the $(m+1)^{th}$ lattice in red. The top row of lattices in Fig. 7.5 also demonstrates how we move from $\mathfrak{C}_{IS}(G \wr \mathcal{I}_n)$ to $\mathfrak{C}_{IS}(G \wr \mathcal{I}_{n+1})$ when $p \nmid n$. For each vertex along the top left side of the smaller lattice, except the topmost vertex, we add a vertex above and to the left. For the topmost vertex we add a 4-element diamond. The bottom row demonstrates what happens when $p \mid n$ (shown in the middle lattice of the 3). Again for each vertex along the top left side of the smaller lattice except the topmost vertex, we add a vertex above and to the left. This time to the topmost vertex we add a chain of length 4. In moving from the middle lattice on the bottom row to the right hand lattice we again see the extension of $\mathfrak{C}_{IS}(G \wr \mathcal{I}_n)$ to $\mathfrak{C}_{IS}(G \wr \mathcal{I}_{n+1})$, noting that as $p \mid n$ certainly $p \nmid n+1$.

To help explain what is happening in Fig. 7.5 we remark that the closed sets of invariant normal subgroups which correspond to nodes which are not along the top left side of the lattice have final subgroup 1^n or B_n. As remarked previously, to extend $\{K_1, \ldots, K_{n-1}, 1^n\}$ a closed set of invariant normal subgroups, to $\{K_1, \ldots, 1^n, K_{n+1}\}$ (again a closed set of invariant normal subgroups) we must have $K_{n+1} = 1^{n+1}$. Also if a set $\{K_1, \ldots, K_{n-1}, B_n\}$ does not correspond to a node along the top left side then $p \nmid n$ so extending this set by K_{n+1} to a closed set of invariant normal subgroups again forces $K_{n+1} = 1^{n+1}$. Therefore the nodes not along the top left side do not "expand" when moving from $\mathfrak{C}_{IS}(G \wr \mathcal{I}_n)$ to $\mathfrak{C}_{IS}(G \wr \mathcal{I}_{n+1})$. On the top left side of the lattice, with the exception of the topmost (and when $p \mid n$ the second topmost) node, the closed set of invariant normal subgroups has K_n (the subgroup of G^n) equal to A_n (which we recall is $\{(g, g, \ldots, g) \mid g \in G\}$). To extend $\{K_1, \ldots, K_{n-1}, A_n\}$ to $\{K_1, \ldots, A_n, K_{n+1}\}$ we must have $K_{n+1} = A_{n+1}$ or $K_n = 1^n$. Thus these nodes "expand" to a chain of length 2. Similarly, if $p \mid n$, then the second topmost node corresponds to a set of subgroups with final subgroup $K_n = B_n$ and we may extend the set of subgroups with $K_{n+1} = A_{n+1}$ or $K_{n+1} = 1^{n+1}$. Finally the topmost node corresponds to the set of subgroups $\{G, G^2, \ldots, G^m\}$, which we may expand with K_{n+1} as any of the 4 invariant normal subgroups of G^{n+1}. The difference between the case for $p \mid n+1$ and $p \nmid n+1$ corresponds to the difference between the lattices on invariant normal subgroups of G^n when $p \mid n$ or not, which is shown in Fig. 7.1.

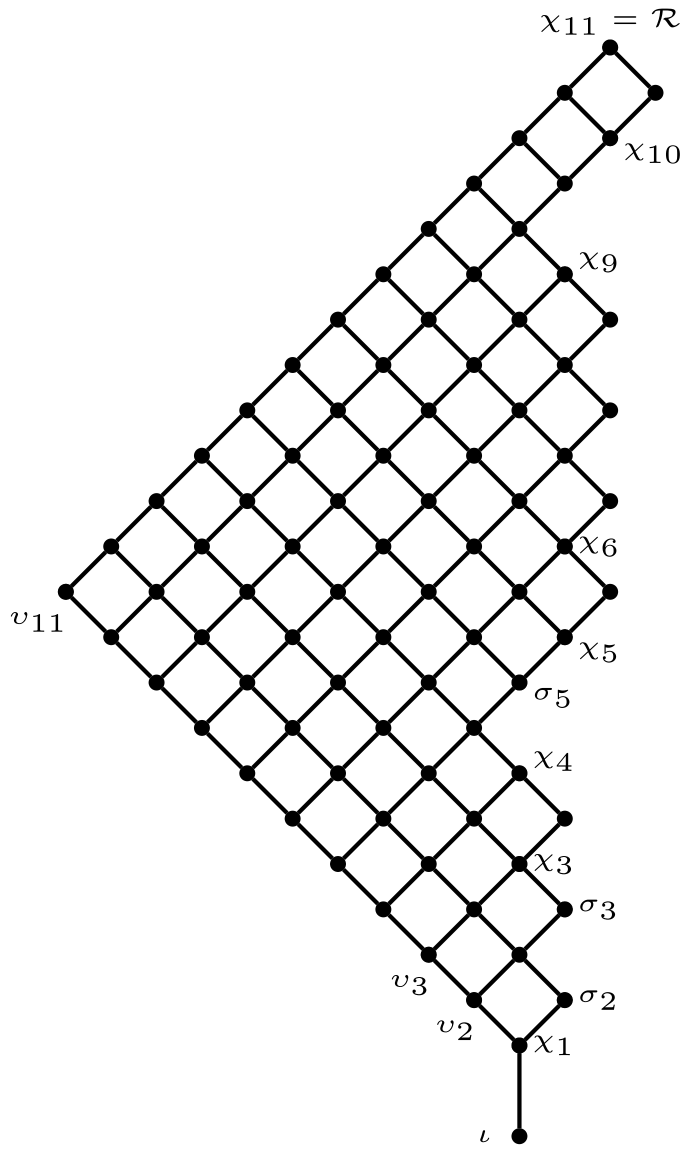


Figure 7.6: The lattice $\mathfrak{C}_{IS}(\mathbb{Z}_5 \wr \mathcal{I}_{11})$

The second of our diagrams is $\mathfrak{C}_{IS}(\mathbb{Z}_5 \wr \mathcal{I}_{11})$ the lattice of idempotent separating congruences on $\mathbb{Z}_5 \wr \mathcal{I}_{11}$. This is shown in Fig. 7.6, in which we write χ_m for $\chi(G, \dots, G^m, 1^{m+1}, \dots, 1^n)$, υ_m for $\chi(G, A_2, \dots, A_m, 1^{m+1}, \dots, 1^n)$ and σ_i for $\chi(G, \dots, G^{i-1}, B_i, 1^{i+1}, \dots, 1^n)$.

7.3 General congruences

We now build congruences on $G \wr \mathcal{I}_n$ for finite simple groups using the refinements to the general theory considered thus far in this chapter. We recall that the ideals of $G \wr \mathcal{I}_n$ are the sets

$$I_m = \{(g; a) \in G \wr \mathcal{I}_n \mid \operatorname{rank}(g; a) \leq m\},$$

and we note we use I_m instead of J_m as notation in this chapter. We then recall Theorem 6.2.12, which says that the congruences on $G \wr \mathcal{I}_n$ may be described as

$$\rho(m, \{T_i\}, L) = I^\star_{m-1} \cup \sigma_L \cup \chi(G^0, G, G^2, \dots, G^{m-1}, T_{m+1}\pi_{m+1}, T_{m+1}, \dots, T_n),$$

where $1 \leq m \leq n$, $\{T_i \trianglelefteq G^i \mid m+1 \leq i \leq n\}$ is a closed set of invariant normal subgroups, and $L \trianglelefteq G \wr \mathcal{S}_m$ is such that $T_{m+1}\pi_{m+1} \leq L$ (where π_k is the map that ignores the k^{th} coordinate). Here σ_L is the relation on D_m that corresponds to the normal subgroup $L \trianglelefteq G \wr \mathcal{S}_m$. By corresponds we formally mean the lift of the non universal congruence on I_m/I_{m-1} which is defined by the normal subgroup $L \trianglelefteq G \wr \mathcal{S}_m$.

We begin with non-abelian finite simple groups. For $Q \trianglelefteq \mathcal{S}_m$ a non trivial subgroup let σ_Q be the relation on D_m that corresponds to the normal subgroup $G \wr Q \trianglelefteq G \wr \mathcal{S}_m$.

Theorem 7.3.1. *Let G be a finite simple non-abelian group. Then the congruences on $G \wr \mathcal{I}_n$ are the following*

$$\rho(m, j, Q) = I^\star_m \cup \chi(G, \dots, G^j, 1^{j+1}, \dots, 1^n) \cup \sigma_Q$$

where $0 \leq m \leq j \leq n$, Q is a normal subgroup of $\mathcal{S}_{m+1}$ and if $m = j$ then $Q = \{1\}$, and σ_Q is the relation defined prior to the statement of the theorem if $Q \neq \{1\}$ and $\sigma_Q = \emptyset$ if $Q = \{1\}$.

Proof. This is an immediate consequence of our general result for congruences on $G \wr \mathcal{I}_n$ (Theorem 6.2.12). Indeed, let $\rho(m, \{T_i\}, L)$ be a congruence as described in Theorem 6.2.12. Then $\{T_i \mid m+1 \leq i \leq n\}$ is a closed set of invariant normal subgroups. By Proposition 7.2.1, there is $m \leq j \leq n$ such that

$$\{T_i \mid m+1 \leq i \leq n\} = \{G^{m+1}, \ldots, G^j, 1^{j+1}, \ldots, 1^n\}.$$

Furthermore, by Corollary 7.1.3, a normal subgroup $L \trianglelefteq G \wr \mathcal{S}_m$ is either trivial or of the form $G^m \times Q$ for $Q \trianglelefteq \mathcal{S}_m$. If L is trivial then (with an abuse of notation) $\rho(m, \{T_i\}, L) = \rho(m-1, m-1, \{1\})$, noting that the condition $T_{m+1}\pi_{m+1} \leq L$ forces $T_{m+1} = 1^{m+1}$ so that $j = m$. If L is non-trivial then it is determined by Q and $\rho(m, \{T_i\}, L) = \rho(m, j, Q)$. This completes the proof, though we remark upon the fact that when $L = G^m \times \{1\}$ we have $\rho(m, \{T_i\}, L) = I^\star_{m-1} \cup \chi(G, \ldots, G^m, T_{m+1}, \ldots, T_n) = \rho(m-1, m, \{1\})$ which is why we set $\sigma_Q = \emptyset$ when $Q = \{1\}$. $\square$

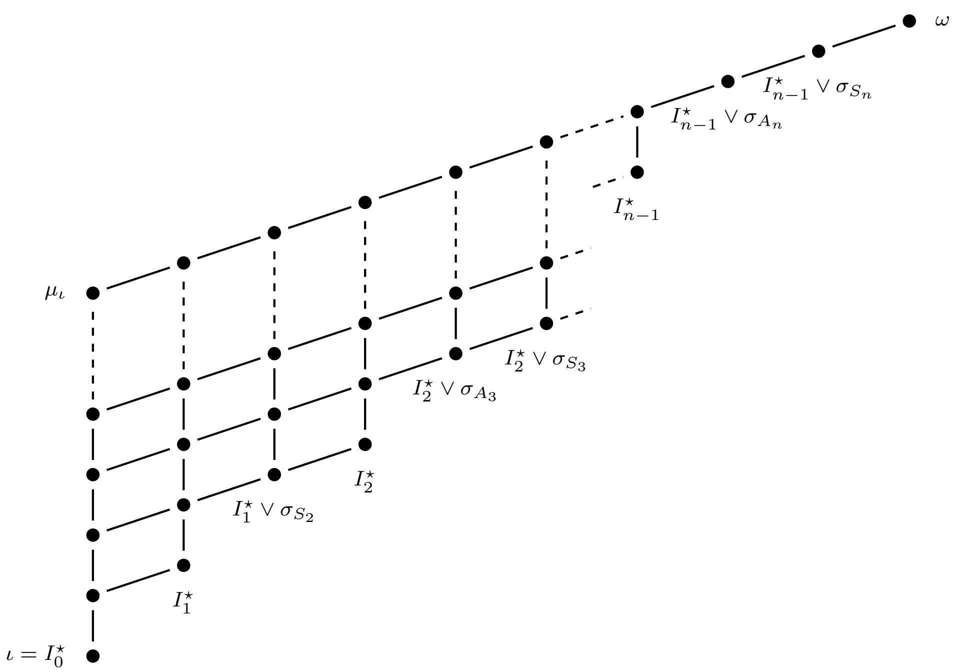


Figure 7.7: Congruences on $G \wr \mathcal{I}_n$ for a finite simple non-abelian group

We may now produce a diagram of the lattice of congruences on $G \wr \mathcal{I}_n$ for a finite non-abelian simple group. This is shown in Fig. 7.7. We note that the group in question does not impact the lattice.

Next we turn to considering $\mathfrak{C}(G \wr \mathcal{I}_n)$ for a finite simple abelian group G. The structure is more complex, and the lattice is substantially larger. As we have explained there are extra considerations when G is $\mathbb{Z}_2$ or $\mathbb{Z}_3$ so we shall exclude these groups from our results. We reinforce the message that there is no further complexity in the methods to describe $\mathfrak{C}(G \wr \mathcal{I}_n)$ in these cases, just the details of the results change. We recall the subgroup $B_m = \{(g_1, \ldots, g_m) \mid g_1 g_2 \ldots g_m = 1\}$, which is invariant and normal in G^m. We now write σ_L for the relation on D_m that corresponds to the normal subgroup $L \trianglelefteq G \wr \mathcal{S}_m$.

Theorem 7.3.2. *Let G be a finite abelian simple group not isomorphic to $\mathbb{Z}_2$ or $\mathbb{Z}_3$. Then the congruences on $G \wr \mathcal{I}_n$ are*

$$\rho(m, \{T_i\}, Q) = I^{\star}_{m-1} \cup \sigma_{T_m \rtimes Q} \cup \chi(G^0, G, G^2, \ldots, G^{m-1}, T_m, \ldots, T_n),$$

where $1 \leq m \leq n$, $\{T_i \trianglelefteq G^i \mid m \leq i \leq n\}$ is a closed set of invariant normal subgroups, $Q \trianglelefteq S_m$ is an normal subgroup and, if Q is non-trivial then $B^m \subseteq T_m$.

Proof. Again this is an exercise in applying Theorem 6.2.12. We leave the details to the reader. □

Now we reach what is - in my opinion - the highlight of this chapter, the pictures of the lattice $\mathfrak{C}(G \wr \mathcal{I}_n)$. We denote by σ_Q the the relation on D_m that corresponds to the normal subgroup $B_m \times Q \trianglelefteq G \wr \mathcal{S}_m$. We choose the group $\mathbb{Z}_5$ as this is the smallest "well-behaved" simple abelian group. Across Fig. 7.8, Fig. 7.9, Fig. 7.10, Fig. 7.11 and Fig. 7.12 we see how $\mathfrak{C}(\mathbb{Z}_5 \wr \mathcal{I}_n)$ grows.

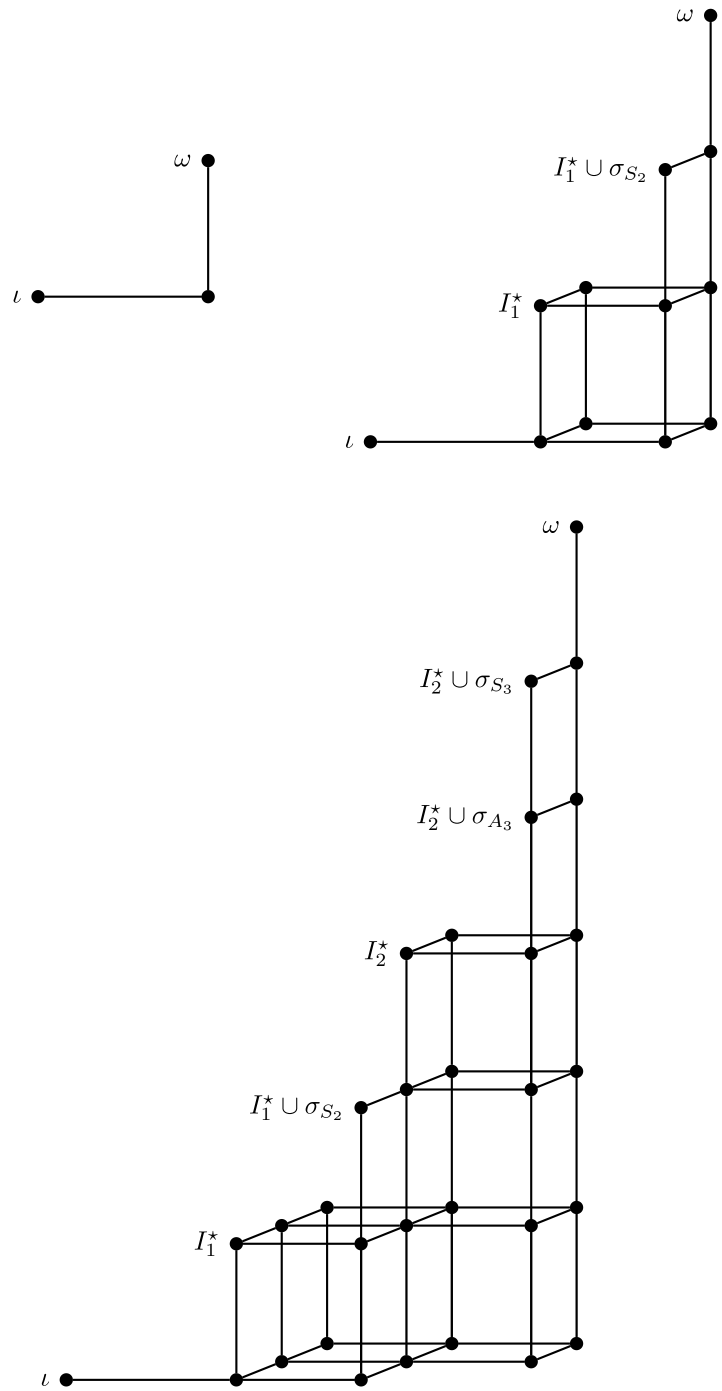


Figure 7.8: The lattices $\mathfrak{C}(\mathbb{Z}_5 \wr \mathcal{I}_n)$ for $n = 1, 2, 3$

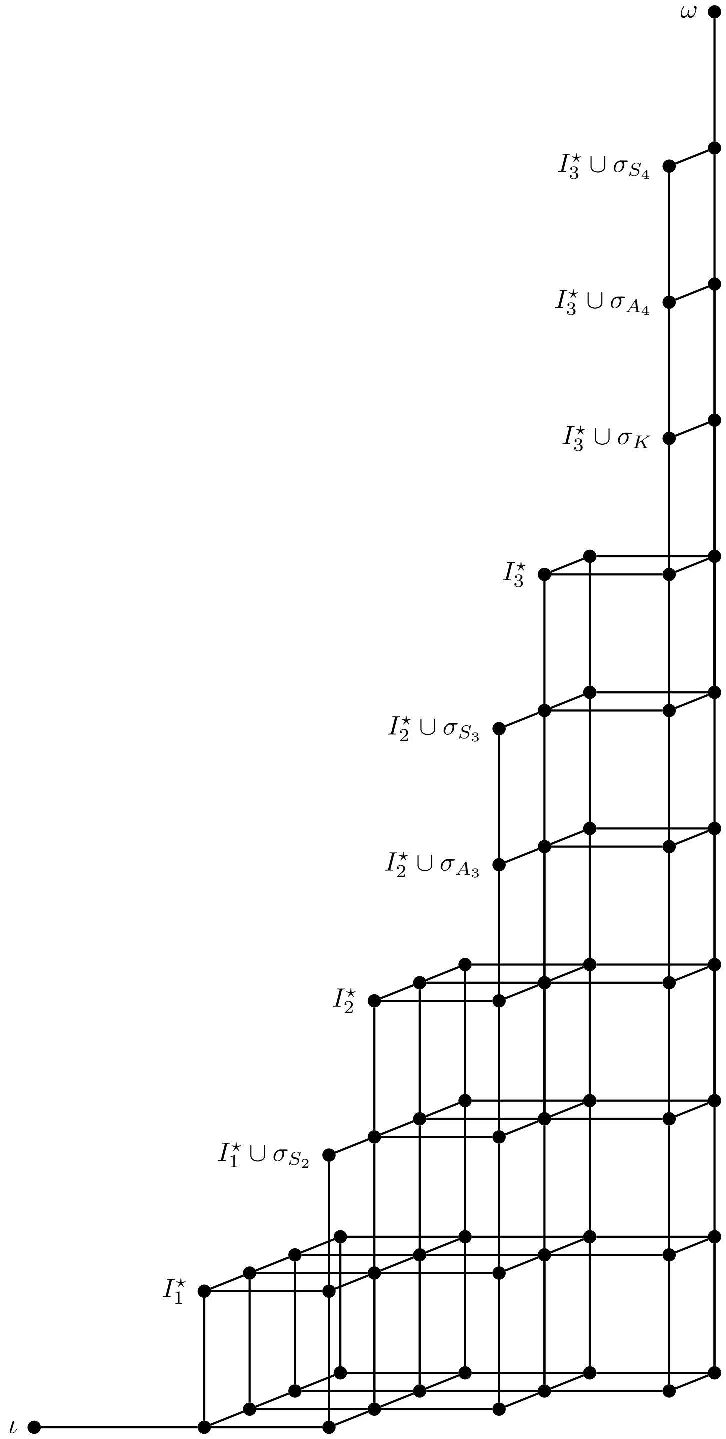


Figure 7.9: The lattice $\mathfrak{C}(\mathbb{Z}_5 \wr \mathcal{I}_4)$

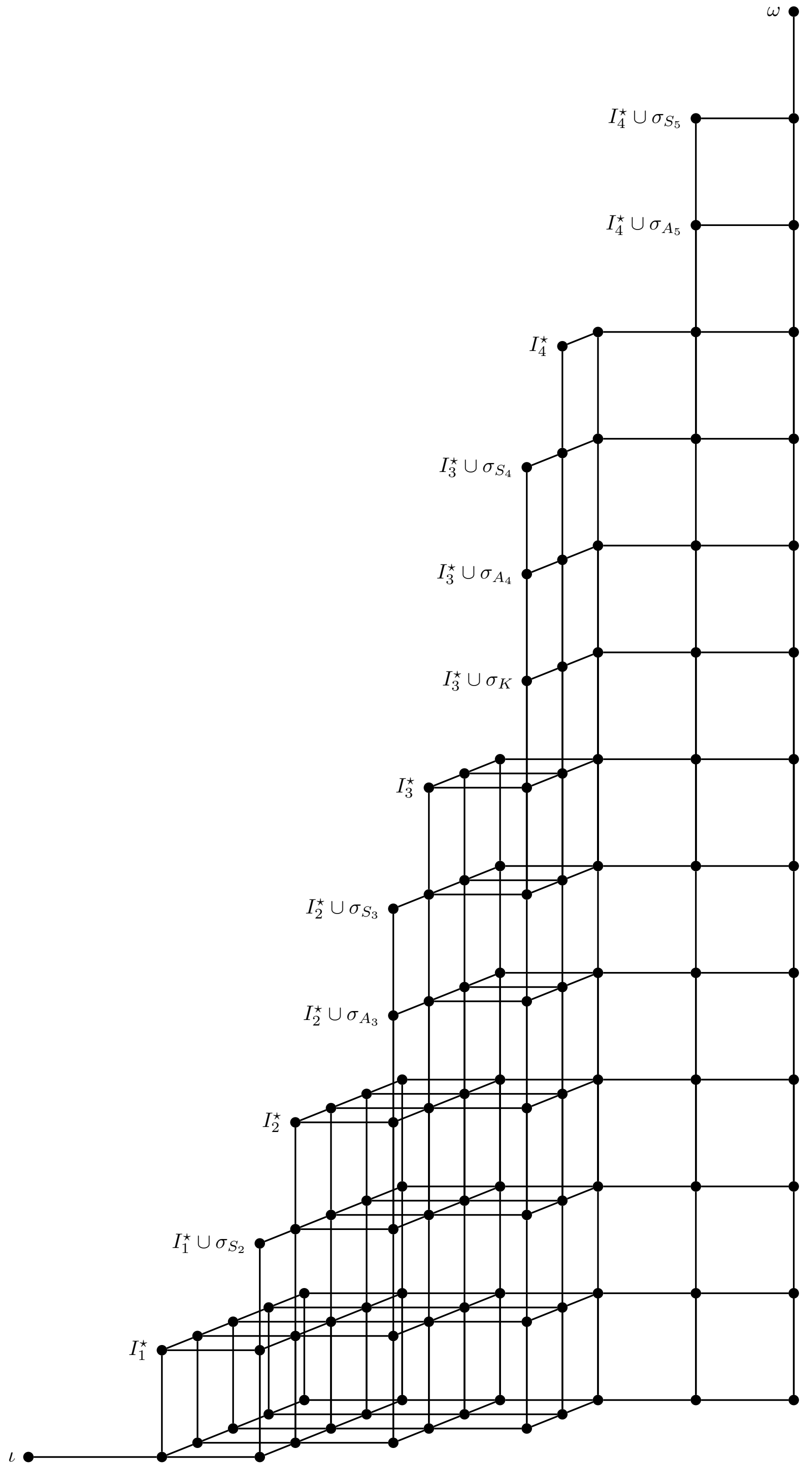


Figure 7.10: The lattice $\mathfrak{C}(\mathbb{Z}_5 \wr \mathcal{I}_5)$

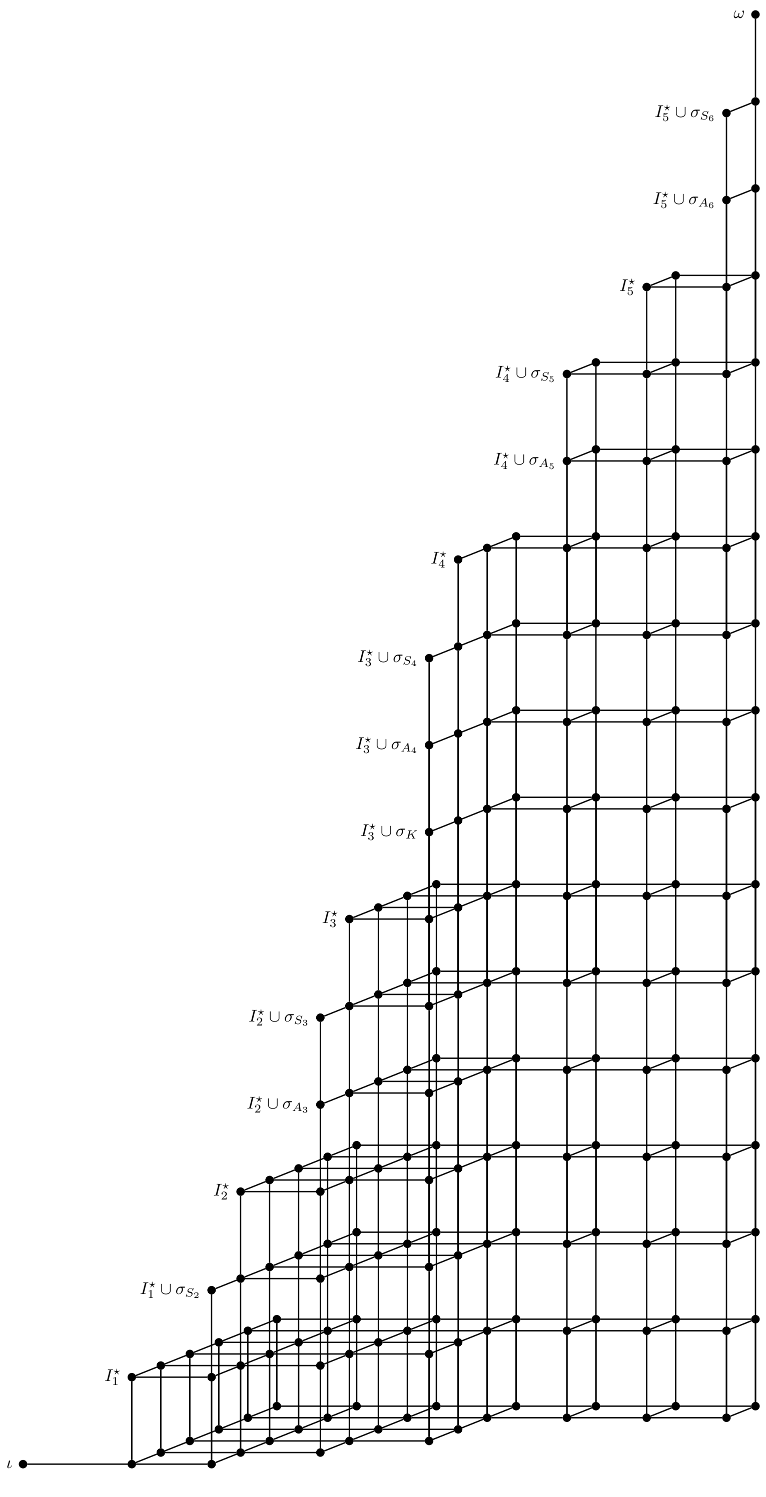


Figure 7.11: The lattice $\mathfrak{C}(\mathbb{Z}_5 \wr \mathcal{I}_6)$

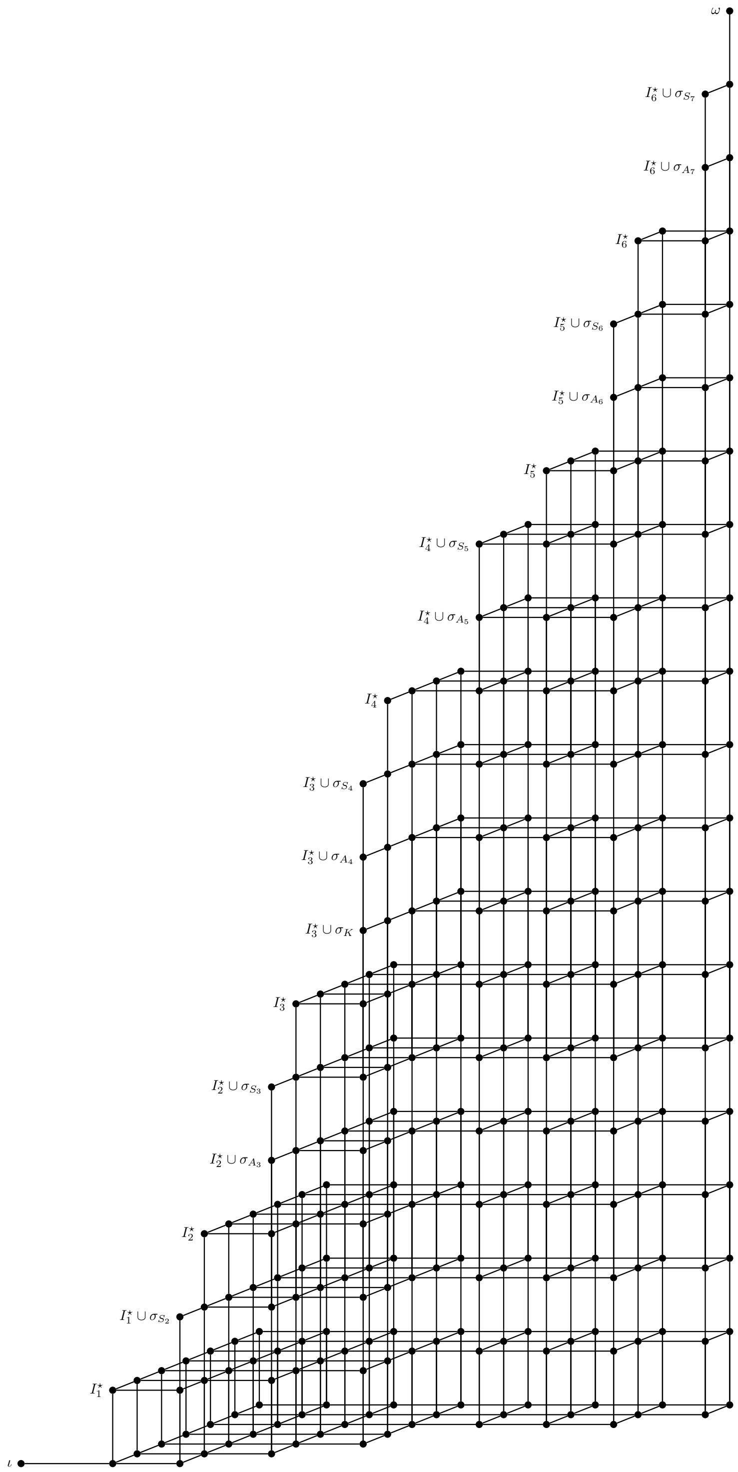


Figure 7.12: The lattice $\mathfrak{C}(\mathbb{Z}_5 \wr \mathcal{I}_7)$

7.4 Order preserving automorphisms

An important submonoid of $\mathcal{I}_n$ is $\mathcal{O}_n$, the submonoid of order preserving injective partial functions. For a detailed examination of $\mathcal{O}_n$ we recommend [17]. This is a full inverse submonoid of $\mathcal{I}_n$ and inherits much of the structure of $\mathcal{I}_n$. For our purposes the relevant properties are the descriptions of Green's relations and the congruence structure. Green's relations for $\mathcal{O}_n$ are exactly as in $\mathcal{I}_n$, determined by the domain and image. If $a, b \in \mathcal{O}_n$ then

$$a \mathcal{R} b \iff \mathrm{Dom}(a) = \mathrm{Dom}(b) \quad \text{and} \quad a \mathcal{L} b \iff \mathrm{Im}(a) = \mathrm{Im}(b).$$

The rest of the Green's relations follow. Of significant importance is the fact that $\mathcal{O}_n$ is combinatorial (which we recall means $\mathcal{H}$ is trivial); there is a unique order preserving bijection between subsets of $[n]$ of the same size. The notion of the *rank* of an element in $\mathcal{O}_n$ is inherited from the rank in $\mathcal{I}_n$, and the ideals of $\mathcal{O}_n$ are the sets

$$I_m = \{a \in \mathcal{O}_n \mid \mathrm{rank}(a) \leq m\}$$

for $0 \leq m \leq n$. The congruence lattice has an elementary structure.

Theorem 7.4.1 ([17, Proposition 2.6])**.** *The congruences on $\mathcal{O}_n$ are exactly the Rees congruences. Thus the congruence lattice is a chain of length $n + 1$.*

In the same way we can consider $\mathcal{O}\mathrm{PAut}(\mathbf{A})$, the order preserving partial automorphisms of a group action $\mathbf{A}$. Again this is a full inverse submonoid of $\mathrm{PAut}(\mathbf{A})$ and thus inherits much of its structure. Of course, to have order preserving (partial) automorphisms we need a group action A with an ordering, so we must assume that A is a poset. Different partial orders will produce different submonoids of $\mathrm{PAut}(\mathbf{A})$.

Definition 7.4.2. A group G is said to be *partially ordered* if there is a partial order on G. Say G is *properly* partially ordered if the partial order is compatible with right multiplication by G, in other words if $a, b \in G$, $g \in G$ and $a \leq b$ then $ag \leq bg$. We say that $\leq$ is a *proper partial order* on G.

For a partially ordered group G let

$$\mathrm{OP}\, G = \{h \in G \mid a \leq b \implies ah \leq bh\}.$$

We remark that in the definition of a properly partially ordered group we need compatibility with right multiplication not left multiplication in order to describe order preserving automorphisms of a G as a left group action. We explain why this is. We recall that every automorphism of G (as a group action) is of the form θ_k which is defined by $g \mapsto gk$. Let G be a partially ordered group and suppose θ_k is an order preserving automorphism of G (as a group action). If $g, h \in G$ with $g \leq h$ then $gk = g\theta_k \leq h\theta_k = hk$, so $k \in \mathrm{OP}\, G$. In fact, it is easy to see that θ_k is order preserving if and only if $k \in \mathrm{OP}\, G$.

We consider partially ordered free G-acts. Given a partial order $\leq_G$ on G we may extend this to a partial order on a chain of disjoint copies of G. If I is a chain ordered by $\leq_I$ then the free G-act $\mathbf{A}_I$, which as a set is $A = \{gx_i \mid g \in G,\ i \in I\}$ may be ordered by setting

$$gx_i \leq hx_j \iff i <_I j \ \text{ or } \ i = j \text{ and } g \leq_G h.$$

This is a partial order on A, and we say that this is the partial order on $\mathbf{A}_I$ *induced* by $\leq_G$.

On a free G-act there is a notion of multiplication on the right, $(gx_i)h \mapsto (gh)x_i$. Therefore we may extend the definition of proper partial order to free G-acts; we say that a partial order $\leq$ on $\mathbf{A}_X$ is *proper* if for all $g \in G$ and for all $a, b \in G$ and $x, y \in X$, if $ax \leq by$ then $agx \leq bgy$. Further, if $\leq_G$ is a proper partial order on G and I is a chain ordered by $\leq_I$ then, defining $\leq$ on A as in the previous paragraph, $\leq$ is a proper partial order. The study of order preserving partial automorphism monoids for free group actions with arbitrary partial orders is beyond the scope of this thesis; we restrict attention to partial orders of the type described above. In fact we shall restrict our attention to one particular partial order on $\mathbf{A}_n$ which has the partial order coming from the trivial partial order on G, and the obvious order on $[n]$.

Definition 7.4.3. Let G be a group and $n \in \mathbb{N}$. The *ordered free G-act of rank n* is $\{gx_i \mid g \in G,\ i \in [n]\}$ (the free G-act of rank n) with the partial order

$$gx_i \leq hx_j \iff i < j \ \text{ or } \ i = j \text{ and } g = h.$$

In Chapter 5 (Theorem 5.2.10) we demonstrated that the partial automorphism monoid of a free G-act was isomorphic to $G \wr \mathcal{I}_n$, a partial wreath product. We seek an analogue of this fact for the partial automorphism monoid of the ordered free G-act. It is possible to approach this formally in the same manner in which we dealt with the non-ordered case, however we shall again want to produce a "user-friendly" version of our monoid so we skip this stage. For a subsemigroup $T \subseteq \mathcal{I}_n$ the set

$$\{(g, a) \in G \wr \mathcal{I}_n \mid a \in T\}$$

is a subsemigroup of $G \wr \mathcal{I}_n$ and is full or inverse precisely when T is full or inverse. We write $G \wr T$ for this subsemigroup. In the rest of this chapter we shall consider $G \wr \mathcal{O}_n$ and in Chapter 8 we consider $G \wr \mathcal{P}_n$, where we write $\mathcal{P}_n$ for the set of idempotents in $\mathcal{I}_n$.

Theorem 7.4.4. *Let G be a group and let $\mathbf{A}_n$ be the ordered free G-act of rank n. Then*

$$\mathcal{O}\mathrm{PAut}(\mathbf{A}_n) \cong G \wr \mathcal{O}_n.$$

Proof. This is almost identical to the proof that the partial automorphism monoid of $\mathbf{A}_n$ is isomorphic to $G \wr \mathcal{I}_n$ (Theorem 5.2.10). We consider the function

$$\mathcal{O}\mathrm{PAut}(\mathbf{A}_n) \to G \wr \mathcal{I}_n; \quad \theta \mapsto (g_\theta; a_\theta),$$

where, if $x_i \in \mathrm{Dom}(\theta)$ and $x_i\theta = hx_j$ then $g_\theta \in (G_0)^n$ has $(g_\theta)_i = h$ and $a \in \mathcal{I}_n$ has $ia = j$. Thus (g_θ, a_θ) is the pair such that $x_i\theta = (g_\theta)_i x_{ia_\theta}$. This is the restriction of the isomorphism $\mathrm{PAut}(\mathbf{A}_n) \to G \wr \mathcal{I}_n$ to $\mathcal{O}\mathrm{PAut}(\mathbf{A}_n)$ so it suffices to show that $\theta \in \mathrm{PAut}(\mathbf{A}_n)$ is order preserving if and only if $a_\theta \in \mathcal{O}_n$.

Suppose first that θ is order preserving. If $i, j \in \mathrm{Dom}(a_\theta)$ and $i \leq j$ then $x_i, x_j \in \mathrm{Dom}(\theta)$ and $x_i < x_j$. We then note that, as θ is order preserving and injective, $x_i\theta < x_j\theta$, or equivalently $(g_\theta)_i x_{ia_\theta} < (g_\theta)_j x_{ja_\theta}$. By the definition of the partial order on $\mathbf{A}_n$ this says that $ia_\theta < ja_\theta$. Thus $a_\theta \in \mathcal{O}_n$.

For the converse we suppose that $a_\theta \in \mathcal{O}_n$ and we take $hx_i < kx_j$ in the G-act. Then $i < j$ and $(hx_i)\theta = h(g_\theta)_i x_{ia_\theta}$ and $(kx_j)\theta = k(g_\theta)_j x_{ja_\theta}$. Since

$a_\theta \in \mathcal{O}_n$ we have $ia_\theta < ja_\theta$ and thus

$$(hx_i)\theta = h(g_\theta)_i x_{ia_\theta} < k(g_\theta)_j x_{ja_\theta} = (kx_j)\theta.$$

We have shown that θ is order preserving, so have completed the proof. $\square$

For the rest of the chapter we focus on the monoid $G \wr \mathcal{O}_n$. Just as in the case when we move from $\mathcal{I}_n$ to $\mathcal{O}_n$, we see that $G \wr \mathcal{O}_n$ is a full inverse submonoid of $G \wr \mathcal{I}_n$ and inherits much of the structure of $G \wr \mathcal{I}_n$. Furthermore, $G \wr \mathcal{I}_n$ inherits much of its structure from $\mathcal{I}_n$ so in actuality $G \wr \mathcal{O}_n$ inherits much of its structure from $\mathcal{I}_n$. In particular, Green's relations in $G \wr \mathcal{O}_n$ are determined by Green's relations in $\mathcal{O}_n$ and the ideals in $G \wr \mathcal{I}_n$ are the sets

$$I_m = \{a \in \mathcal{O}_n \mid \operatorname{rank}(a) \leq m\}$$

for $0 \leq m \leq n$. We remark that the group $\mathcal{H}$-classes in $G \wr \mathcal{O}_n$ are isomorphic to direct products of G, if $e \in E(G \wr \mathcal{O}_n)$ has $\operatorname{rank}(e) = m$ then $H_e \cong G^m$.

7.5 CONGRUENCES ON $G \wr \mathcal{O}_n$

We describe congruences on $G \wr \mathcal{O}_n$. There are no major surprises, the picture is largely as one expects. Moving from congruences on $\mathcal{O}_n$ to congruences on $G \wr \mathcal{O}_n$ mirrors the pattern that we see moving from congruences on $\mathcal{I}_n$ to congruences on $G \wr \mathcal{I}_n$. We write $I_m^\star(\mathcal{O}_n)$ for the Rees congruence on $\mathcal{O}_n$ and $I_m^\star(G \wr \mathcal{O}_n)$ for the corresponding Rees congruence on $G \wr \mathcal{O}_n$.

Theorem 7.5.1. *The congruences on $G \wr \mathcal{O}_n$ are precisely the relations*

$$\rho(m, \chi) = I_m^\star(G \wr \mathcal{O}_n) \cup \chi$$

where $0 \leq m \leq n$ and χ is an idempotent separating congruence on $G \wr \mathcal{O}_n$.

Proof. We consider the projection map $\Theta \colon G \wr \mathcal{O}_n \to \mathcal{O}_n$ defined by $(g; a) \mapsto a$ and the corresponding map on congruences, which we also denote by Θ, so, if ρ is a congruence on $G \wr \mathcal{O}_n$, then

$$\rho\Theta = \{(a, b) \in \mathcal{O}_n \times \mathcal{O}_n \mid \exists g, h \in G_0^n,\ (g; a)\ \rho\ (h; b)\}.$$

It is elementary that $\rho\Theta$ is a congruence on $\mathcal{O}_n$. That $\rho\Theta$ is symmetric, reflexive and compatible is inherited directly from ρ. To see that $\rho\Theta$ is transitive we suppose $a\ \rho\Theta\ b$ and $b\ \rho\Theta\ c$. Then there are $(g;a),(h;b),(k,b),(l,c) \in G \wr \mathcal{O}_n$ such that

$$(g;a)\ \rho\ (h;b) \quad \text{and} \quad (k,b)\ \rho\ (l,c).$$

Let $\overline{h}$ and $\overline{k}$ be the elements in G^n such that $\overline{x}_i = x_i$ when $x_i \neq 0$ and $\overline{x}_i = 1$ when $x_i = 0$, where we take $x = h, k$. Then

$$(\overline{k}(\overline{h})^{-1}g;a) = (\overline{k}(\overline{h})^{-1};1)(g;a)\ \rho\ (\overline{k}(\overline{h})^{-1};1)(h;b) = (k;b).$$

Then, as ρ is transitive, $(\overline{k}(\overline{h})^{-1}g;a)\ \rho\ (l;c)$, so $a\ \rho\Theta\ c$. Therefore $\rho\Theta$ is transitive and so a congruence. By Theorem 7.4.1, $\rho\Theta$ is a Rees congruence, say $\rho\Theta = I_m^\star(\mathcal{O}_n)$. Furthermore, it follows that if $\operatorname{rank}(a) \leq m$ then there is g such that $(g;a)\ \rho\ (0;0)$. It is then clear that $I_m^\star(G \wr \mathcal{O}_n) \subseteq \rho$.

It remains to show that if $\operatorname{rank}(a) > m$ and $(g;a)\ \rho\ (h;b)$ then $a = b$. This is immediate as if $(g;a)\ \rho\ (h;b)$ then $a\ \rho\Theta\ b$. Therefore, if $\operatorname{rank}(a) > m$, we must have $a = b$. This completes the proof. $\square$

When studying congruences on $G \wr \mathcal{I}_n$ idempotent separating congruences were of great importance. For $G \wr \mathcal{O}_n$ they are arguably of even greater import. Just as for $G \wr \mathcal{I}_n$ we describe idempotent separating congruences via the isomorphism with normal subsemigroups contained in the centraliser of the idempotents. We regard $\mathcal{O}_n$ as a subsemigroup of $\mathcal{I}_n$, and we recall that the centraliser of $E(\mathcal{I}_n)$ is $E(\mathcal{I}_n)$. It follows that $E(\mathcal{O}_n)\zeta = E(\mathcal{O}_n) = E(\mathcal{I}_n)$. We regard $G \wr \mathcal{O}_n$ as a subsemigroup of $G \wr \mathcal{I}_n$ and we recall that the centraliser of the idempotents in $G \wr \mathcal{I}_n$ is

$$E(G \wr \mathcal{I}_n)\zeta = \{(g;e) \in G \wr \mathcal{I}_n \mid e \in E(\mathcal{I}_n)\}.$$

We notice that $E(G \wr \mathcal{I}_n) \subseteq G \wr \mathcal{O}_n$, which implies that $E(G \wr \mathcal{O}_n)\zeta = E(G \wr \mathcal{I}_n)\zeta$. For the remainder of this chapter we write $E\zeta$ for $E(G \wr \mathcal{O}_n)\zeta$. We shall see that while normal subsemigroups of $G \wr \mathcal{I}_n$ contained in $E\zeta$ are normal in $G \wr \mathcal{O}_n$, the converse is not true. Thus there shall be additional normal subsemigroups of $G \wr \mathcal{O}_n$ and so, additional idempotent separating congruences.

The following is similar to the discussion for $G \wr \mathcal{I}_n$. As when we described normal subsemigroups in the case of $G \wr \mathcal{I}_n$ we define the function

$$\Omega \colon E\zeta \to \bigcup_{0 \le m \le n} G^m$$

as the map that ignores zero entries and the final - now $\mathcal{O}_n$ - coordinate. It is clear from the fact that the centraliser of the idempotents is the "same" for both $G \wr \mathcal{I}_n$ and $G \wr \mathcal{O}_n$ that, for $e \in E(\mathcal{O}_n)$ of rank m, the restriction of Ω to $E\zeta \cap H_{(1_e;e)}$ is an isomorphism onto G^m, so $E\zeta \cap H_{(1_e;e)} \cong G^m$. As previously, given $h \in G^m$ and $e \in E(\mathcal{O}_n)$ with $\operatorname{rank}(e) = m$ write $\overline{{}^e h}$ for the unique element of G_0^n that has $\overline{{}^e h}_i = 0$ for $i \notin \operatorname{Dom}(e)$ and $(\overline{{}^e h}; e)\Omega = h$. We note that this says $(\overline{{}^e h}; e) \in G \wr \mathcal{O}_n$. For $T \subseteq E\zeta$ a normal subsemigroup of $G \wr \mathcal{O}_n$ and $e \in E(\mathcal{O}_n)$ we write T_e for $T \cap H_{(1_e;e)}$.

Lemma 7.5.2. *Let $T \subseteq E\zeta$ be a normal subsemigroup of $G \wr \mathcal{O}_n$ and let $e, f \in E(\mathcal{O}_n)$ with $\operatorname{rank}(e) = \operatorname{rank}(f) = m$. Then $T_e \cong T_f$. Moreover, the group $T_e\Omega \trianglelefteq G^m$ is normal and $T_e\Omega = T_f\Omega$.*

Proof. The proof is similar to that for Lemma 6.2.8.

For $a \in \mathcal{O}_n$ if $(g; a^{-1}a) \in T$ then as T is normal it follows that

$$(1_{aa^{-1}}; a)(g; a^{-1}a)(1_{a^{-1}a}; a^{-1}) = (g_a; aa^{-1}) \in T.$$

With this in mind for each $a \in \mathcal{O}_n$ we define the function

$$\Psi_a \colon T_{a^{-1}a} \to T_{aa^{-1}}; \quad (g; a^{-1}a) \mapsto (g_a; aa^{-1})$$

and it is easily seen that this is an isomorphism. Thus $T_{a^{-1}a} \cong T_{aa^{-1}}$ for each $a \in \mathcal{O}_n$.

Let $e, f \in E(\mathcal{O}_n)$ with $\operatorname{rank}(e) = \operatorname{rank}(f)$ then $e \; \mathcal{D}(\mathcal{O}_n) \; f$ and, as $\mathcal{O}_n$ is combinatorial, there is a unique $a \in \mathcal{O}_n$ with $a^{-1}a = e$ and $aa^{-1} = f$. It follows that if $\operatorname{rank}(e) = \operatorname{rank}(f)$ then $T_e \cong T_f$. Further we note that if $g \in T_e\Omega$ then $(\overline{{}^e g}; e) \in T$ and $(\overline{{}^e g}; e)\Psi_a = (\overline{{}^f g}; f)$. Thus $(\overline{{}^e g}; e)\Psi_a\Omega \in T_f\Omega$. However, $(\overline{{}^e g}; e)\Psi_a\Omega = g$ as, since a is order preserving, Ψ_a does not change the order of the non-zero entries in the $(G_0)^n$ component, just where these non-zero entries occur. Therefore we have that $T_e\Omega = T_f\Omega$.

Let $e \in E(\mathcal{I}_n)$ with rank m. To see $T_e\Omega$ is normal in G^m suppose that $g \in T_e\Omega$ so $(\overline{^e g}; e) \in T_e$, and let $h \in G^m$. Note that $(\overline{^e(hgh^{-1})}; e) = (\overline{^e h}; e)(\overline{^e g}; e)(\overline{^e h^{-1}}; e)$. As T is normal this implies that $(\overline{^e(hgh^{-1})}; e) \in T_e$, so $(\overline{^e(hgh^{-1})}; e)\Omega = hgh^{-1} \in T_e\Omega$. Thus $T_e\Omega$ is normal. $\square$

Just as in the case for $G \wr \mathcal{I}_n$ we have shown that to define a normal subsemigroup of $G \wr \mathcal{O}_n$ contained in $E\zeta$ it suffices to describe a set of normal subgroups $\{T_i \trianglelefteq G^i \mid 0 \leq i \leq n\}$. Again we note that T_0 is trivial since it is a subgroup of G^0. The normal subsemigroup is then

$$T = \bigcup_{e \in E(\mathcal{O}_n)} \left\{(\overline{^e g}; e) \mid g \in T_{\mathrm{rank}(e)}\right\}.$$

We also again call $\{T_i \trianglelefteq G^i \mid 0 \leq i \leq n\}$ the *defining groups for T*. We remark that the difference from the $G \wr \mathcal{I}_n$ case - given in Lemma 6.2.8 - is that subgroups of G^m are no longer invariant. Why this is the case can be seen by considering that the group $\mathcal{H}$-classes in $\mathcal{I}_n$ are symmetric groups, and there is an action of a symmetric group (given by a suitable conjugation in $G \wr \mathcal{I}_n$) upon the group G^m via permutation of the coordinates. The subgroups we need are invariant under this action, so are invariant. In the $G \wr \mathcal{O}_n$ case, the group $\mathcal{H}$-classes in $\mathcal{O}_n$ are trivial, so the "action" is also trivial and the subgroups of G^m are only "invariant" under the trivial action, which is to say, are not necessarily invariant.

Definition 7.5.3. For $1 \leq m \leq n$ let

$$\pi_m \colon \bigcup_{m \leq i \leq n} G^i \to \bigcup_{m-1 \leq i \leq n-1} G^i$$

be the projection onto all but the m^{th} coordinate. We say that a set $\{T_i \leq G^i \mid 0 \leq i \leq n\}$ is *closed* if $T_i\pi_m \subseteq T_{i-1}$ for each $1 \leq i \leq n$ and each $1 \leq m \leq i$.

Proposition 7.5.4. *Let $T \subseteq E\zeta$ be a normal subsemigroup of $G \wr \mathcal{O}_n$ and let $\{T_i \leq G^i \mid 0 \leq i \leq n\}$ be the defining groups for T. Then each T_i is an normal subgroup of G^i and $\{T_i \mid 0 \leq i \leq n\}$ is closed.*

Moreover if $\{T_i \trianglelefteq G^i \mid 0 \le i \le n\}$ *is a closed set of normal subgroups then*

$$T = \bigcup_{e \in E(\mathcal{O}_n)} \left\{ (\overline{{}^{e}g}; e) \mid g \in T_{\mathrm{rank}(e)} \right\}$$

is a normal subsemigroup, $T \subseteq E\zeta$ *and* $\{T_i \le G^i \mid 0 \le i \le n\}$ *are the defining groups for* T.

Proof. Again this is similar to the proof of the analogous result for $G \wr \mathcal{I}_n$ - Proposition 6.2.9. Recall $\Omega\colon E\zeta \to \bigcup_{0 \le m \le n} G^m$, the function that ignores zero entries in the $(G_0)^n$ component, and ignores the $\mathcal{O}_n$ component.

Suppose that $T \subseteq E\zeta$ is a normal subsemigroup. By Lemma 7.5.2 we know that each T_i is a normal subgroup. Suppose $e \in E(\mathcal{I}_n)$ with rank i has domain $\{x_1 < x_2 < \cdots < x_i\}$ and let $f \in E(\mathcal{I}_n)$ be an idempotent with domain $\{x_1 < \cdots < x_{m-1} < x_{m+1} < \cdots < x_i\}$. If $g \in T_i$ then $(\overline{{}^{e}g}; e) \in T$ and $g\pi_m = ((\overline{{}^{e}g}; e)(1_f; f))\Omega$, so $g\pi_m \in T_{i-1}$. Thus $\{T_i \mid 0 \le i \le n\}$ is closed.

For the converse, suppose that $\{T_i \trianglelefteq G^i \mid 0 \le i \le n\}$ is a closed set of normal subgroups. To see that T is a subsemigroup let $(\overline{{}^{e}g}; e), (\overline{{}^{f}h}; f) \in T$ and observe that (just as in the proof of Proposition 6.2.9)

$$(\overline{{}^{e}g}; e)(\overline{{}^{f}h}; f) = ((\overline{{}^{e}g})1_{ef}(\overline{{}^{f}h})1_{ef}; ef) = ((\overline{{}^{e}g})1_{ef}; ef)((\overline{{}^{f}h})1_{ef}; ef).$$

As the set of subgroups is closed it clear that for any $g \in T_m$ the projection of g onto a subset of the coordinates of size j is an element of T_j. Therefore $((\overline{{}^{e}g})1_{ef}; ef)\Omega \in T_{\mathrm{rank}(ef)}$ and $((\overline{{}^{f}h})1_{ef}; ef)\Omega \in T_{\mathrm{rank}(ef)}$. It follows that $((\overline{{}^{e}g})1_{ef}(\overline{{}^{f}h})1_{ef}; ef)\Omega \in T_{\mathrm{rank}(ef)}$. Also

$$(\overline{{}^{e}g})1_{ef}(\overline{{}^{f}h})1_{ef} = \overline{{}^{ef}(((\overline{{}^{e}g})1_{ef}(\overline{{}^{f}h})1_{ef}; ef)\Omega)},$$

hence $(\overline{{}^{e}g}; e)(\overline{{}^{f}h}; f) \in T$.

It is immediate that T is both full and inverse. To see that T is self conjugate we note that $(g; a) \in G \wr \mathcal{O}_n$ decomposes as $(g; a) = (g; aa^{-1})(1_{aa^{-1}}; a)$. Then, for $(\overline{{}^{f}h}; f) \in T$

$$(g; a)(\overline{{}^{f}h}; f)(g; a)^{-1} = (g; aa^{-1})(1_{aa^{-1}}; a)(\overline{{}^{f}h}; f)(1_{a^{-1}a}; a^{-1})(g^{-1}; aa^{-1}).$$

That T is closed under conjugation by elements of the form $(1_{aa^{-1}}; a)$ follows from this conjugation acting on the $(G_0)^n$ component by moving around the non-zero entries but preserving their order and possibly replacing some h_i by 0. Closure under conjugation by $(g; aa^{-1})$ follows from each T_i being normal. Since $G \wr \mathcal{O}_n$ is generated by the elements of the form $(1_{aa^{-1}}; a)$ and $(g; aa^{-1})$ we have that T is normal. $\square$

We remark that one reason that this proof is almost identical to the proof for Proposition 6.2.9 is that the definition of closed set of subgroups in this latter case agrees with the former when each of the subgroups of G^m is invariant.

Just as in the $G \wr \mathcal{I}_n$ case we define the function $\omega\colon (G_0)^n \to \bigcup_{0\leq m\leq n} G^m$, which ignores zero entries. We have shown that to define an idempotent separating congruence on $G \wr \mathcal{O}_n$ it is sufficient to provide a closed set of normal subgroups $\{T_i \trianglelefteq G^i \mid 0 \leq i \leq n\}$, and the idempotent separating congruence can then be expressed explicitly as

$$\chi(T_0, T_1, \ldots, T_n) = \{((g; a), (h; a)) \mid a \in \mathcal{O}_n,\ (h^{-1}g)\omega \in T_{\mathrm{rank}(a)}\}.$$

Furthermore the ordering on idempotent separating congruences coincides with the ordering on closed sets of normal subgroups induced by subgroup inclusion in each degree; that is $\chi(T_0, \ldots, T_n) \subseteq \chi(K_0, \ldots, K_n)$ if and only if $T_i \subseteq K_i$ for each $0 \leq i \leq n$.

Corollary 7.5.5. *The maximum idempotent separating congruence on $G \wr \mathcal{O}_n$ is*

$$\chi(G^0, G, G^2, \ldots, G^n) = \{((g; a), (h; b)) \mid a = b\}.$$

We may now state our main result for this section, an analogue of Theorem 6.2.12, describing congruences on $G \wr \mathcal{O}_n$. This follows directly from Theorem 7.5.1 and our description of the idempotent separating congruences in terms of closed sets of normal subgroups.

Theorem 7.5.6. *Let $0 \leq m \leq n$, let $\{T_i \trianglelefteq G^i \mid m+1 \leq i \leq n\}$ be a closed set of normal subgroups. Then*

$$\rho(m, \{T_i\}) = I_m^\star \cup \chi(G^0, G, G^2, \ldots, G^m, T_{m+1}, \ldots, T_n)$$

is a congruence on $G \wr \mathcal{O}_n$.

Moreover, all congruences on $G \wr \mathcal{O}_n$ are of this form.

We remark that whereas for invariant normal subgroups of G^m we were able to produce an elegant description, for normal subgroups of G^m in general we cannot do the same. It is possible to specialise the description of subgroups of direct products given in Theorem 5.4.3 to normal subgroups. One would do this by applying the same argument as when one specialises Goursat's lemma to normal subgroups, and then using the same inductive strategy that one uses to move from the usual Goursat's lemma to the generalised Goursat's lemma (Theorem 5.4.3). This is not a particularly informative exercise, and we refrain from producing it here; it tells us very little about the structure of the lattice of congruences.

— 8 —

One sided congruences on $G \wr \mathcal{I}_n$

In this chapter we draw together ideas from across this thesis and use the inverse kernel approach to consider the lattice of one sided congruences on $G \wr \mathcal{I}_n$. We remind ourselves of a couple of the relevant definitions and results from the previous chapters. The inverse semigroup $G \wr \mathcal{I}_n$ is

$$\{(g; a) \in (G_0)^n \times \mathcal{I}_n \mid g_i \neq 0 \iff i \in \mathrm{Dom}(a)\},$$

with multiplication

$$(g; a)(h; b) = (gh_a; ab)$$

where for $h = (h_1, \ldots, h_n) \in (G_0)^n$ and $a \in \mathcal{I}_n$

$$h_a = (h_{1a}, \ldots, h_{na}).$$

The inverse operation on $G \wr \mathcal{I}_n$ is

$$(g; a)^{-1} = (g^{-1}_{a^{-1}}; a^{-1}).$$

We recall the definition of inverse congruence pairs from Chapter 2. Let S be an inverse semigroup and let $E = E(S)$. If τ is a congruence on E then the *normaliser* of τ is

$$N(\tau) = \{a \in S \mid e \ \tau \ f \implies aea^{-1} \ \tau \ afa^{-1} \text{ and } a^{-1}ea \ \tau \ a^{-1}fa\}.$$

Let τ be a congruence on E, and let $T \subseteq S$ be a full inverse subsemigroup. Then (τ, T) is an *inverse congruence pair* if

(ICP1) $T \subseteq N(\tau)$;

(ICP2) for $x \in S$, if $e, f \in E$ are such that $x^{-1}x \ \tau \ e$, $xx^{-1} \ \tau \ f$ and $xe, fx \in T$ then we have $x \in T$.

One-sided congruences are determined by inverse congruence pairs. If (τ, T) is an inverse congruence pair then the associated left congruence is

$$\rho(\tau, T) = \{(a, b) \in S \times S \mid a^{-1}b \in T, \ a^{-1}bb^{-1}a \ \tau \ a^{-1}a, \ b^{-1}aa^{-1}b \ \tau \ b^{-1}b\}.$$

Conversely, given a left congruence ρ the inverse kernel of ρ is

$$\operatorname{Inker}(\rho) = \{a \in S \mid a \ \rho \ aa^{-1}\}$$

and the inverse congruence pair associated with ρ is $(\operatorname{trace}(\rho), \operatorname{Inker}(\rho))$.

Our objective is to describe one sided congruences on $G \wr \mathcal{I}_n$ in terms of one sided congruences on $\mathcal{I}_n$ and one sided congruences on $G \wr \mathcal{P}_n$. In this chapter there are many occasions where we shall talk about left congruences on multiple different inverse semigroups. Where necessary we use subscripts to provide clarity. For instance $\rho_S(\tau, T)$ is the left congruence on S with inverse congruence pair (τ, T).

We recall that the idempotents in $G \wr \mathcal{I}_n$ are the elements of the form $(1_e; e)$ for $e \in E(\mathcal{I}_n)$. When we are discussing idempotents it will usually make little difference whether we are concerned with $E(G \wr \mathcal{I}_n)$, $E(\mathcal{I}_n)$, $E(G \wr \mathcal{P}_n)$ or $\mathcal{P}_n$ (we recall that $\mathcal{P}_n$ is the set of subsets of $\{1, \ldots, n\}$ under intersection and is isomorphic to $E(\mathcal{I}_n)$). For notational simplicity, where it does not cause confusion, we shall henceforth blur the distinction between these four. In particular, we shall say τ is a congruence on $\mathcal{P}_n$ to mean that τ is a congruence on $E(G \wr \mathcal{I}_n)$ and on $E(\mathcal{I}_n)$ and shall write $e \ \tau \ f$ to also mean $(1_e; e) \ \tau \ (1_f; f)$. Further we write $N_S(\tau)$ for the normaliser of $\tau \in \mathfrak{C}(\mathcal{P}_n)$ in S where S is $G \wr \mathcal{I}_n$, $\mathcal{I}_n$ or $G \wr \mathcal{P}_n$.

8.1 Inducing a left congruence on $\mathcal{I}_n$

Our first step is to consider how a left congruence on $G \wr \mathcal{I}_n$ induces a left congruence on $\mathcal{I}_n$. In the two sided case (in Section 6.6) we observed that a congruence ρ on $G \wr \mathcal{I}_n$ induces a congruence on $\mathcal{I}_n$ in two ways. This carries forward to the one sided case: we can either consider the left congruences induced by the restriction to $\mathcal{I}_n$ or by the projection onto $\mathcal{I}_n$. For the first of these options we recall that $\mathcal{I}_n$ "lives inside" $G \wr \mathcal{I}_n$ as $\{(1_{aa^{-1}}; a) \mid a \in \mathcal{I}_n\}$, so the left congruence induced by $\rho \in \mathfrak{LC}(G \wr \mathcal{I}_n)$ on $\mathcal{I}_n$ is

$$\{(a, b) \in \mathcal{I}_n \times \mathcal{I}_n \mid (1_{aa^{-1}}; a) \ \rho \ (1_{bb^{-1}}; b)\}.$$

The second left congruence induced on $\mathcal{I}_n$ is of more interest to us. Define

$$\Psi \colon G \wr \mathcal{I}_n \to \mathcal{I}_n; \quad (g; a) \mapsto a,$$

that is the homomorphism that ignores the group component. For a left congruence ρ on $G \wr \mathcal{I}_n$ write $\rho\Psi$ for the relation

$$\rho\Psi = \{((g;a)\Psi, (h;b)\Psi) \mid ((g;a),(h;b)) \in \rho\}$$

which we note is equal to

$$\{(a,b) \in \mathcal{I}_n \times \mathcal{I}_n \mid \exists g, h \in (G_0)^n, \ (g;a) \ \rho \ (h;b)\}.$$

This is what was meant by the "projection" of ρ onto $\mathcal{I}_n$. We shall shortly show that this a left congruence, but first we comment on the projection of full inverse subsemigroups of $G \wr \mathcal{I}_n$ onto $\mathcal{I}_n$. Let $T \subseteq G \wr \mathcal{I}_n$ be a full inverse subsemigroup and consider the projection of T onto $\mathcal{I}_n$, namely

$$T\Psi = \{a \in \mathcal{I}_n \mid \exists g \in (G_0)^n \text{ with } (g;a) \in T\}.$$

It is immediate that $T\Psi$ is a full inverse subsemigroup of $\mathcal{I}_n$, moreover every full inverse subsemigroup of $\mathcal{I}_n$ arises in this way. Indeed, given a full inverse subsemigroup $V \subseteq \mathcal{I}_n$ define

$$T = \{(g;a) \in G \wr \mathcal{I}_n \mid a \in V\}.$$

As V is a full inverse subsemigroup of $\mathcal{I}_n$ we observe that T is a full inverse subsemigroup of $G \wr \mathcal{I}_n$ and it is clear that $T\Psi = V$.

We now show, given $\rho \in \mathfrak{LC}(G \wr \mathcal{I}_n)$, that $\rho\Psi$ is a left congruence on $\mathcal{I}_n$ and determine the inverse congruence pair for $\rho\Psi$.

Lemma 8.1.1. *Let τ be a congruence on $\mathcal{P}_n$ and let $T \subseteq G \wr \mathcal{I}_n$ be a full inverse subsemigroup such that (τ, T) is an inverse congruence pair for $G \wr \mathcal{I}_n$, and let $\rho = \rho_{G\wr\mathcal{I}_n}(\tau, T)$. Then $\rho\Psi$ is a left congruence on $\mathcal{I}_n$ and*

$$\rho\Psi = \rho_{\mathcal{I}_n}(\tau, T\Psi).$$

Proof. We first show that $\rho\Psi$ is a left congruence. It is clear that $\rho\Psi$ is symmetric, reflexive and left compatible as it inherits these properties from ρ. Thus it remains to show that $\rho\Psi$ is transitive. To this end we suppose that (a,b), $(b,c) \in \rho\Psi$. Then there exist $g, h, k, l \in (G_0)^n$ such that $(g;a) \ \rho \ (h;b)$

and $(k;b)\ \rho\ (l;c)$. For $g \in (G_0)^n$ let $\bar{g}$ be the element of G^n such that $\bar{g}_i = g_i$ whenever $g_i \neq 0$ and $\bar{g}_i = 1$ when $g_i = 0$. Then left multiplying $(g;a)\ \rho\ (h;b)$ by $(\bar{h}^{-1};1)$ gives $(\bar{h}^{-1}g;a)\ \rho\ (1_{bb^{-1}};b)$, and similarly we obtain that $(1_{bb^{-1}};b)\ \rho\ (\bar{k}^{-1}l;c)$. Then $(\bar{h}^{-1}g;a)\ \rho\ (\bar{k}^{-1}l;c)$, so $a\ \rho\Psi\ c$ and $\rho\Psi$ is transitive.

Now we show that $\rho\Psi = \rho_{\mathcal{I}_n}(\tau, T\Psi)$. In order to do this we shall show that $\mathrm{trace}(\rho\Psi) = \tau$ and $\mathrm{Inker}(\rho\Psi) = T\Psi$. We first handle the trace. It is clear that $\mathrm{trace}(\rho) \subseteq \mathrm{trace}(\rho\Psi)$. For the reverse inclusion suppose that $e\ \rho\Psi\ f$, so there are $(g;e),(h;f) \in G \wr \mathcal{I}_n$ such that $(g;e)\ \rho\ (h;f)$. Then (with reference to Lemma 5.2.12, our elementary computations in $G \wr \mathcal{I}_n$),

$$(1_e;e) = (g^{-1};e)(g;e)\ \rho\ (g^{-1};e)(h;f) = (g^{-1}h_e;ef) = (g_{ef}^{-1}h_{ef};ef).$$

Multiplying the relation $(1_e;e)\ \rho\ (g_{ef}^{-1}h_{ef};ef)$ on the left by $(1_f;f)$ gives that

$$(1_{ef};ef) = (1_f;f)(1_e;e)\ \rho\ (1_f;f)(g_{ef}^{-1}h_{ef};ef) = (g_{ef}^{-1}h_{ef};ef).$$

Thus $(1_e;e)\ \rho\ (1_{ef};ef)$. Similarly we obtain that $(1_f;f)\ \rho\ (1_{ef};ef)$, so we have that $(1_e;e)\ \rho\ (1_f;f)$. Therefore $\mathrm{trace}(\rho\Psi) \subseteq \mathrm{trace}(\rho)$, so we have shown that

$$\mathrm{trace}(\rho\Psi) = \mathrm{trace}(\rho) = \tau.$$

We now move to show that the inverse kernel of $\rho\Psi$ is $T\Psi$. Suppose that $a \in \mathrm{Inker}(\rho\Psi)$, so $a\ \rho\Psi\ aa^{-1}$. Thus there are $g,h \in (G_0)^n$ such that $(g;a)\ \rho\ (h;aa^{-1})$. Multiplying on the left by $(h^{-1};aa^{-1})$ we obtain that

$$(h^{-1}g;a)\ \rho\ (1_{aa^{-1}};aa^{-1}) = (h^{-1}g;a)(h^{-1}g;a)^{-1},$$

so $(h^{-1}g;a) \in \mathrm{Inker}(\rho) = T$ and thus $a \in T\Psi$. Thus $\mathrm{Inker}(\rho\Psi) \subseteq T\Psi$. On the other hand, if $a \in T\Psi$ then there is $g \in (G_0)^n$ such that $(g;a) \in T$ so as $T = \mathrm{Inker}(\rho)$ we have that $(g;a)\ \rho\ (1_{aa^{-1}};aa^{-1})$. Hence $a\ \rho\Psi\ aa^{-1}$, so $a \in \mathrm{Inker}(\rho\Psi)$. Thus $\mathrm{Inker}(\rho\Psi) = T\Psi$, so $(\tau, T\Psi)$ is an inverse congruence pair for $\mathcal{I}_n$ and $\rho\Psi = \rho_{\mathcal{I}_n}(\tau, T\Psi)$. $\square$

8.2 ONE SIDED CONGRUENCES ON $G \wr \mathcal{P}_n$

The next step on our journey to get hold of left congruences on $G \wr \mathcal{I}_n$ requires us to take a slight detour and describe left congruences on $G \wr \mathcal{P}_n$. We recall that $E\zeta$, the centraliser of the idempotents for $G \wr \mathcal{I}_n$, is the set $\{(g; e) \in G \wr \mathcal{I}_n \mid e \in E(\mathcal{I}_n)\}$. As in Chapter 7, for $T \subseteq \mathcal{I}_n$ we write $G \wr T$ for $\{(g; a) \in G \wr \mathcal{I}_n \mid a \in T\}$. Therefore

$$E\zeta = G \wr \mathcal{P}_n.$$

We define the homomorphism

$$\Omega\colon E\zeta \to (G_0)^n; \quad (g; e) \mapsto g$$

that ignores the $\mathcal{I}_n$ entry. Also, we recall the notation: for $e \in E(\mathcal{I}_n)$

$$G^e = \{g \in (G_0)^n \mid g_i \neq 0 \iff i \in \mathrm{Dom}(e)\} = \{g \in (G_0)^n \mid (g; e) \in G \wr \mathcal{I}_n\},$$

which is a group under the multiplication in $(G_0)^n$ and $G^e \cong G^k$ where $k = \mathrm{rank}(e)$.

We remark that the $\mathcal{P}_n$ component is obsolete in the expression $G \wr \mathcal{P}_n$, since the $\mathcal{P}_n$ coordinate is determined by the non-zero entries in the $(G_0)^n$ coordinate. In fact, Ω is an isomorphism, so we could consider the formulation $(G_0)^n$ for $E\zeta$. However, we stick to the notation established for $G \wr \mathcal{P}_n$ as we shall be regarding $E\zeta$ a subsemigroup of $G \wr \mathcal{I}_n$.

We recall that $E\zeta$ is a Clifford semigroup; it is the Clifford semigroup $\mathcal{C}(\mathcal{P}_n, G^e, \phi_{e,f})$, where, for $e \geq f$, $\phi_{e,f}\colon G^e \to G^f$ is defined by $g \mapsto g_f$. We fix the notation $\phi_{e,f}$ for the next few pages. We have already seen the descriptions of full inverse subsemigroups and left congruences on Clifford semigroups in Lemma 1.5.2 and Theorem 3.2.2 respectively. We state these descriptions here for $G \wr \mathcal{P}_n$ for completeness.

Corollary 8.2.1 (cf. Lemma 1.5.2). *The full inverse subsemigroups of $G \wr \mathcal{P}_n$ are the Clifford semigroups*

$$\mathcal{C}(\mathcal{P}_n, A_e, \phi_{e,f}|_{A_e}) = \bigcup_{e \in \mathcal{P}_n} \{(g; e) \mid g \in A_e\}$$

where $\{A_e \leq G^e \mid e \in \mathcal{P}_n\}$ *is a set of subgroups such that for* $f \leq e$ *we have* $A_e\phi_{e,f} \subseteq A_f$.

Motivated by Corollary 8.2.1 we make the following definition.

Definition 8.2.2. Let $\mathcal{A} = \{A_e \leq G^e \mid e \in E(\mathcal{I}_n)\}$ be a set of subgroups. We say that $\mathcal{A}$ is $\mathcal{P}_n$*-closed* if for each $e, f \in P_n$ with $f \leq e$ we have $A_e\phi_{e,f} \subseteq A_f$.

It is immediate that full inverse subsemigroups of $G \wr \mathcal{P}_n$ are in bijection with $\mathcal{P}_n$-closed sets of subgroups. We now give the description of inverse congruence pairs. This is precisely the statement of Theorem 3.2.2 applied to $G \wr \mathcal{P}_n$.

Corollary 8.2.3 (cf. Theorem 3.2.2)**.** *Let* $\{A_e \leq G^e \mid e \in \mathcal{P}_n\}$ *be a* $\mathcal{P}_n$*-closed set of subgroups, and let* $T = \mathcal{C}(\mathcal{P}_n, A_e, \phi_{e,f}|_{A_e}) \subseteq G \wr \mathcal{P}_n$ *be the corresponding full inverse subsemigroup. Let* τ *be a congruence on* $\mathcal{P}_n$*. Then* (T, τ) *is an inverse congruence pair for* $G \wr \mathcal{P}_n$ *if and only if for* $f \leq e$

$$e \;\tau\; f \implies A_e = \{g \in G^e \mid g_f \in A_f\}.$$

8.3 Inverse congruence pairs for $G \wr \mathcal{I}_n$

We now combine the descriptions of left congruences on the two semigroups $\mathcal{I}_n$ and $G \wr \mathcal{P}_n$ to determine left congruences on $G \wr \mathcal{I}_n$. First we try to justify why we have looked at these subsemigroups. We notice that the semilattices of idempotents for $G \wr \mathcal{P}_n$ and $\mathcal{I}_n$ are isomorphic, both being isomorphic to $\mathcal{P}_n$. Furthermore, recalling that $G \wr \mathcal{P}_n \cong (G_0)^n$, elements of $G \wr \mathcal{I}_n$ are described by a pair consisting of an element in $G \wr \mathcal{P}_n$ and an element in $\mathcal{I}_n$. We make the observation that

$$(g; a) = (g; aa^{-1})(1_{aa^{-1}}; a).$$

We think of $G \wr \mathcal{I}_n$ as a "gluing" of $G \wr \mathcal{P}_n$ and $\mathcal{I}_n$ along the semilattices of idempotents.

We proceed to classify inverse congruence pairs on $G \wr \mathcal{I}_n$. Initially we give a pair of elementary lemmata.

Lemma 8.3.1. *Let τ be a congruence understood to be on both $E(G \wr \mathcal{I}_n)$ and $E(\mathcal{I}_n)$. Then*

$$N_{G\wr\mathcal{I}_n}(\tau) = \{(g, a) \mid a \in N_{\mathcal{I}_n}(\tau)\} = G \wr N_{\mathcal{I}_n}(\tau).$$

Proof. This is immediate noting that $(g; a)(1_e; e)(g; a)^{-1} = (1_{aea^{-1}}; aea^{-1})$. $\square$

Recall that $\Psi\colon G \wr \mathcal{I}_n \to \mathcal{I}_n$ is the function that ignores the group component and for a left congruence ρ on $G \wr \mathcal{I}_n$ we write $\rho\Psi$ for the left congruence

$$\{(a, b) \in \mathcal{I}_n \times \mathcal{I}_n \mid \exists g, h \in (G_0)^n \text{ such that } (g; a)\ \rho\ (h; b)\}.$$

Using Ψ, a consequence of Lemma 8.3.1 is that

$$N_{\mathcal{I}_n}(\tau) = N_{G\wr\mathcal{I}_n}(\tau)\Psi.$$

In the following lemma we recall that $E\zeta$, the centraliser of the idempotents in $G \wr \mathcal{I}_n$, is $G \wr \mathcal{P}_n$.

Lemma 8.3.2. *Let τ be a congruence on $\mathcal{P}_n$ and let $T \subseteq G \wr \mathcal{I}_n$ be a full inverse subsemigroup such that (τ, T) is an inverse congruence pair for $G\wr\mathcal{I}_n$. Then $(\tau, T\Psi)$ is an inverse congruence pair for $\mathcal{I}_n$, and $(\tau, T \cap E\zeta)$ is an inverse congruence pair for $G \wr \mathcal{P}_n$. Moreover, if $\rho = \rho_{G\wr\mathcal{I}_n}(\tau, T)$, then*

$$\rho\Psi = \rho_{\mathcal{I}_n}(\tau, T\Psi),$$
$$\rho|_{E\zeta} = \rho \cap (E\zeta \times E\zeta) = \rho_{G\wr\mathcal{P}_n}(\tau, T \cap E\zeta).$$

Proof. The lemma makes two statements, the first, concerning the projection of a left congruence on $G \wr \mathcal{I}_n$ onto $\mathcal{I}_n$, is a restatement of Lemma 8.1.1. The second is about the restriction of a left congruence on $G \wr \mathcal{I}_n$ to $E\zeta$. This is an application of the general result Theorem 2.6.3, which says that if υ is a left congruence on an inverse semigroup S with corresponding inverse congruence pair (τ, T), and $V \subseteq S$ is an inverse subsemigroup then $\upsilon|_V$ is a left congruence on V with inverse congruence pair $(\tau|_{E(V)}, T \cap V)$. Applying this with $S = G \wr \mathcal{I}_n$, $\rho(\tau, T) = \upsilon$ and $V = E\zeta$ implies that $\rho|_{E\zeta} = \rho_{G\wr\mathcal{P}_n}(\tau, T \cap E\zeta)$. $\square$

We now can give a classification of left congruences on $G \wr \mathcal{I}_n$ in terms of inverse congruence pairs.

Theorem 8.3.3. *Let τ be a congruence on $\mathcal{P}_n$ and let T be a full inverse subsemigroup of $G \wr \mathcal{I}_n$. Then (τ, T) is an inverse congruence pair for $G \wr \mathcal{I}_n$ if and only if $(\tau, T\Psi)$ is an inverse congruence pair for $\mathcal{I}_n$ and $(\tau, T \cap E\zeta)$ is an inverse congruence pair for $G \wr \mathcal{P}_n$.*

Proof. We have seen that if (τ, T) is an inverse congruence pair for $G \wr \mathcal{I}_n$ then the other two pairs are also inverse congruence pairs, hence it remains to show that this is sufficient.

To this end we suppose that $T \in \mathfrak{V}(G \wr \mathcal{I}_n)$ and $\tau \in \mathfrak{C}(\mathcal{P}_n)$ are such that $(\tau, T\Psi)$ and $(\tau, T \cap E\zeta)$ are inverse congruence pairs for $\mathcal{I}_n$ and $G \wr \mathcal{P}_n$ respectively. We will show that (τ, T) is an inverse congruence pair for $G \wr \mathcal{I}_n$. As $(\tau, T\Psi)$ is an inverse congruence pair for $\mathcal{I}_n$ we have that $T\Psi \subseteq N_{\mathcal{I}_n}(\tau)$. From the description of $N_{G\wr\mathcal{I}_n}(\tau)$ from Lemma 8.3.1 we see that this implies that $T \subseteq N_{G\wr\mathcal{I}_n}(\tau)$, so (ICP1) is satisfied.

To prove (ICP2) we suppose $(g; a) \in G \wr \mathcal{I}_n$ and $(1_e; e), (1_f; f) \in E(G \wr \mathcal{I}_n)$ are such that $a^{-1}a \;\tau\; e$, $aa^{-1} \;\tau\; f$, and $(g; a)(1_e; e)$, $(1_f; f)(g; a) \in T$. By replacing e, f with $ea^{-1}a, faa^{-1}$ if necessary, without loss of generality we may assume that $e \leq a^{-1}a$ and $f \leq aa^{-1}$.

We notice that $ae, fa \in T\psi$ and as $(\tau, T\Psi)$ is an inverse congruence pair for $\mathcal{I}_n$ it follows that $a \in T\Psi$. Therefore there is some $h \in (G_0)^n$ such that $(h; a) \in T$. Then we have that

$$(1_f; f)(h; a) = (h_f; fa) \in T \quad \text{and} \quad (1_f; f)(g; a) = (g_f; fa) \in T.$$

Therefore $(g_f; fa)^{-1} = (g^{-1}_{a^{-1}f}; a^{-1}f) \in T$. Further,

$$(h_f; fa)(g^{-1}_{a^{-1}f}; a^{-1}f) = (h_f (g^{-1}_{a^{-1}f})_{fa}; faa^{-1}) = (h_f g^{-1}_{faa^{-1}}; f) = ((hg^{-1})_f; f),$$

where for the second equality we use Lemma 5.2.11, the fact that the function $\mathcal{I}_n \to \mathrm{End}((G_0)^n)$ defined by $a \mapsto [g \mapsto g_a]$ is an antihomomorphism, so $(g^{-1}_{a^{-1}f})_{fa} = g^{-1}_{faa^{-1}}$. For the final equality in the previous displayed equation we use the facts that $faa^{-1} = f$ and that $(hg^{-1})_f = h_f g^{-1}_f$. Thus we have that $((hg^{-1})_f; f) \in T$.

Our next step is to use the description of inverse congruence pairs for $G \wr \mathcal{P}_n$ from Corollary 8.2.3. For $e \in \mathcal{P}_n$, let $A_e = \{g \in G^e \mid (g; e) \in T\}$, so that

$$\bigcup_{e \in \mathcal{P}_n} \{(g; e) \mid g \in A_e\} = T \cap E\zeta.$$

As $(\tau, T \cap E\zeta)$ is an inverse congruence pair for $G \wr \mathcal{P}_n$ we may apply Corollary 8.2.3. We have $f \; \tau \; aa^{-1}$ and $f \leq aa^{-1}$, so

$$A_{aa^{-1}} = \{g \in G^{aa^{-1}} \mid g_f \in A_f\}.$$

As $((hg^{-1})_f; f) \in T$ we have $(hg^{-1})_f \in A_f$. By definition $h, g \in G^{aa^{-1}}$ (as $(g; a), (h; a) \in G \wr \mathcal{I}_n$) so $hg^{-1} \in G^{aa^{-1}}$, and, as $(hg^{-1})_f \in A_f$, it follows that $hg^{-1} \in A_{aa^{-1}}$. Therefore $(hg^{-1}; aa^{-1}) \in T$. As $(h; a) \in T$ (by assumption) and

$$(g; a) = (hg^{-1}; aa^{-1})^{-1}(h; a),$$

we have that $(g; a) \in T$. Thus (ICP2) is satisfied, so (τ, T) is an inverse congruence pair for $G \wr \mathcal{I}_n$. □

8.4 FULL INVERSE SUBSEMIGROUPS OF $G \wr \mathcal{I}_n$

We have seen that to be an inverse congruence pair for $G \wr \mathcal{I}_n$ it suffices for a pair $(\tau, T) \in \mathfrak{C}(\mathcal{P}_n) \times \mathfrak{V}(G \wr \mathcal{I}_n)$ to give inverse congruence pairs via the obvious maps to $\mathfrak{C}(\mathcal{P}_n) \times \mathfrak{V}(\mathcal{I}_n)$ and $\mathfrak{C}(\mathcal{P}_n) \times \mathfrak{V}(G \wr \mathcal{P}_n)$. The remaining questions are to do with what full inverse subsemigroups of $G \wr \mathcal{I}_n$ look like, and how they relate to subsemigroups of $\mathcal{I}_n$ and $G \wr \mathcal{P}_n$.

We recall that full inverse subsemigroups of $G \wr \mathcal{P}_n$ are described by $\mathcal{P}_n$-closed sets. Let $T \subseteq G \wr \mathcal{P}_n$ be a full inverse subsemigroup and let $\{A_e \leq G^e \mid e \in \mathcal{P}_n\}$ be the corresponding $\mathcal{P}_n$-closed set, so that $A_e = \{g \in G^e \mid (g; e) \in T\}$. We extend this notation to a subsemigroup $T \subseteq G \wr \mathcal{I}_n$: for $a \in \mathcal{I}_n$, we write A_a for the set

$$\{g \in G^{aa^{-1}} \mid (g; a) \in T\}$$

and we write $\mathcal{A}(T) = \{A_a \mid a \in T\Psi\}$. In particular, when $e \in E(\mathcal{I}_n)$ the set A_e is a group.

Continue to let $T \subseteq G \wr \mathcal{P}_n$ be a full inverse subsemigroups and suppose that $(g; a), (h; a) \in T$ (so $(h; a)^{-1} = (h^{-1}_{a^{-1}}; a^{-1}) \in T$). Then also

$$(g; a)(h^{-1}_{a^{-1}}; a^{-1}) = (gh^{-1}_{aa^{-1}}; aa^{-1}) = (gh^{-1}; aa^{-1}) \in T,$$

so $gh^{-1} \in A_{aa^{-1}}$. We have shown that if $g, h \in A_a$ then $gh^{-1} \in A_{aa^{-1}}$, which precisely says that the set A_a is a right coset of the subgroup $A_{aa^{-1}}$. Furthermore, we note that $g \in A_a$ precisely when $g^{-1}_{a^{-1}} \in A_{a^{-1}}$, so

$$A_{a^{-1}} = \{g^{-1}_{a^{-1}} \mid g \in A_a\}.$$

Since $A_{a^{-1}}$ is a right coset of $A_{a^{-1}a}$ this implies that $\{g^{-1}_{a^{-1}} \mid g \in A_a\}$ is a right coset of $A_{a^{-1}a}$, or equivalently, $\{g_{a^{-1}} \mid h \in A_a\}$ is a left coset of $A_{a^{-1}a}$.

Corollary 8.4.1. *Let $T \subseteq G \wr \mathcal{I}_n$ be a full inverse subsemigroup, and $a \in \mathcal{I}_n$. Then A_a is a right coset of $A_{aa^{-1}}$.*

This is somewhat reminiscent of the description of full inverse subsemigroups of $\mathcal{I}_n$ in Chapter 3, where we described the subsemigroups via an equivalence relation on $\mathcal{P}_n$ and a mapping from $\mathcal{P}_n$ into the set of cosets of symmetric groups. The approach used to describe full inverse subsemigroups of $\mathcal{I}_n$ may be mirrored for $G \wr \mathcal{I}_n$ using an analogous notion of a Brandt decomposition for $G \wr \mathcal{I}_n$. Proceeding in this manner one obtains full inverse subsemigroups of $G \wr \mathcal{I}_n$ via a partition of $\mathcal{P}_n$ and a mapping from $\mathcal{P}_n$ into the set of cosets of subgroups of $G \wr \mathcal{S}_k$ for k between 1 and n. However, our main result (Theorem 8.3.3) relates inverse congruence pairs for $G \wr \mathcal{I}_n$ to inverse congruence pairs for $\mathcal{I}_n$ and $G \wr \mathcal{P}_n$, and the aforementioned approach to describing full inverse subsemigroups does not directly tell us anything about how a subsemigroup of $G \wr \mathcal{I}_n$ relates to subsemigroups of $\mathcal{I}_n$ and $G \wr \mathcal{P}_n$.

Instead, our approach shall draw inspiration from the description of subgroups of semidirect products of groups due to Usenko [77], which is introduced in Theorem 5.4.6. We recall the main definition and result. Let $H \rtimes_\phi P$ be a semidirect product and recall that for $p \in P$ and $h \in H$ we write $p\phi = \phi_p$ and $h\phi_p = h^p$. Let $Q \leq P$ and $J \leq H$ be subgroups. We say that a function $\psi\colon Q \to H$ is a *normal crossed* $\mathfrak{R}_\phi^J$ *homomorphism* and the triple (J, Q, ψ) a *normal crossed* $\mathfrak{R}_\phi^J$ *(NCR) triple* if

(i) for all $r, q \in Q$ there is $j \in J$ such that $(rq)\psi = j(r\psi)(q\psi)^r$;

(ii) for all $q \in Q$ and $j \in J$ we have $(q\psi)j^q(q\psi)^{-1} \in J$.

For an NCR triple (J, Q, ψ) define the set

$$\mathbf{L}(J, Q, \psi) = \{(j(q\psi), q) \mid j \in J,\ q \in Q\}.$$

Theorem 5.4.6 states that the sets $\mathbf{L}(J, Q, \psi)$ are all the subgroups of $H \rtimes_\phi P$.

Definition 8.4.2. Let $S \subseteq \mathcal{I}_n$ be a full inverse subsemigroup, and let $T \subseteq G \wr \mathcal{P}_n$ be a full inverse subsemigroup. For each $e \in \mathcal{P}_n$ define $A_e \leq G^e$ as $A_e = \{g \in G^e \mid (g; e) \in T\}$ (so $T = \bigcup_{e \in \mathcal{P}_n} \{(g; e) \mid g \in A_e\}$). We say that a function $\phi\colon S \to (G_0)^n$ is a T *crossed homomorphism* and that (T, S, ϕ) is a *crossed triple* if

(i) for each $b \in S$, $b\phi \in G^{bb^{-1}}$;

(ii) for all $b, c \in S$ there is $h \in A_{bcc^{-1}b^{-1}}$ such that $(bc)\phi = h(b\phi)(c\phi)_b$;

(iii) for all $b \in S$, $e \in \mathcal{P}_n$ and $h \in A_e$ we have $(b\phi)h_b(b\phi)^{-1} \in A_{beb^{-1}}$.

For a crossed triple (T, S, ϕ) we define

$$\mathbf{V}(T, S, \phi) = \{(h(a\phi); a) \mid a \in S, h \in A_{aa^{-1}}\}.$$

We make an initial observation.

Lemma 8.4.3. *Let (T, S, ϕ) be a crossed triple and for each $e \in \mathcal{P}_n$ let $A_e = \{g \in G^e \mid (g; e) \in T\}$. Let $V = \mathbf{V}(T, S, \phi)$. Then for each $e \in \mathcal{P}_n$*

$$\{g \in G^e \mid (g; e) \in V\} = A_e.$$

In particular, for $e \in \mathcal{P}_n$, $e\phi \in A_e$.

Proof. Let $e \in \mathcal{P}_n$. By Definition 8.4.2(ii), there is $h \in A_e$ such that

$$e\phi = (ee)\phi = h(e\phi)(e\phi)_e.$$

By (i), $e\phi \in G^e$ so we have $(e\phi)_e = e\phi$ so $e\phi = h(e\phi)^2$. Since $h \in A_e \subseteq G^e$ and G^e is a group this implies that $1_e = h(e\phi)$, so $e\phi = h^{-1}$. In particular, we have that $e\phi \in A_e$.

Suppose that $(g; e) \in V$, then there is $k \in A_e$ such that $g = k(e\phi)$. Then $g = k(e\phi) = kh^{-1} \in A_e$, so we have shown that

$$\{g \in G^e \mid (g; e) \in V\} \subseteq A_e.$$

For the reverse inclusion we suppose that $k \in A_e$. We know that $e\phi \in A_e$ so we have that $k(e\phi)^{-1} \in A_e$ so

$$(k; e) = (k(e\phi)^{-1}(e\phi); e) \in V.$$

Therefore we have that

$$A_e \subseteq \{g \in G^e \mid (g; e) \in V\}$$

so the two are equal. □

Lemma 8.4.4. *Let $V \subseteq G \wr \mathcal{I}_n$ be a full inverse subsemigroup and define $\phi\colon V\Psi \to (G_0)^n$ as follows. For each $a \in V\Psi$ choose $g \in G^{aa^{-1}}$ such that $(g; a) \in V$, and let $a\phi = g$. Then $(V \cap E\zeta, V\Psi, \phi)$ is a crossed triple.*

Furthermore, $V = \mathbf{V}(V \cap E\zeta, V\Psi, \phi)$.

Proof. We certainly have that $V\Psi \subseteq \mathcal{I}_n$ is a full inverse subsemigroup and, noting $E\zeta = G \wr \mathcal{P}_n$, we see that $V \cap E\zeta \subseteq G \wr \mathcal{P}_n$ is also a full inverse subsemigroup. For $e \in \mathcal{P}_n$ let $A_e = \{g \in G^e \mid (g; e) \in V \cap E\zeta\}$ which we note is equal to $\{g \in G^e \mid (g; e) \in V\}$. We notice that (i) from Definition 8.4.2 is true by construction, so we must prove (ii) and (iii).

We start with (ii). Take $a, b \in V\Psi$ so by construction $(a\phi; a), (b\phi; b) \in V$. Then we note that

$$(a\phi; a)(b\phi; b) = ((a\phi)(b\phi)_a; ab) \in V.$$

On the other hand $V\Psi \subseteq \mathcal{I}_n$ is a subsemigroup, so $ab \in V\Psi$ which means that $((ab)\phi; ab) \in V$. Then

$$\begin{aligned}((a\phi)(b\phi)_a; ab)((ab)\phi; ab)^{-1} &= ((a\phi)(b\phi)_a; ab)((((ab)\phi)^{-1}_{b^{-1}a^{-1}}; b^{-1}a^{-1}) \\ &= ((a\phi)(b\phi)_a((ab)\phi)^{-1}_{abb^{-1}a^{-1}}; abb^{-1}a^{-1}) \in V.\end{aligned}$$

From the definition of ϕ we have that $(ab)\phi \in G^{abb^{-1}a^{-1}}$, which implies that $((ab)\phi)^{-1}_{abb^{-1}a^{-1}} = ((ab)\phi)^{-1}$. Also, by definition of $A_{abb^{-1}a^{-1}}$, we have that

$$\{g \in G^{abb^{-1}a^{-1}} \mid (g; abb^{-1}a^{-1}) \in V\} = A_{abb^{-1}a^{-1}},$$

thus we have that

$$(a\phi)(b\phi)_a((ab)\phi)^{-1} \in A_{abb^{-1}a^{-1}}.$$

We let $h = (a\phi)(b\phi)_a((ab)\phi)^{-1}$. As, for instance, $((a\phi)(b\phi)_a; ab) \in G \wr \mathcal{I}_n$ we observe that $(a\phi)(b\phi)_a \in G^{abb^{-1}a^{-1}}$. Thus we have that $(ab)\phi = h^{-1}(a\phi)(b\phi)_a$ so we have that (ii) holds.

For (iii) we take $a \in V\Psi$ and $h \in A_e$, so that $(a\phi; a)$ and $(h; e)$ are elements of V. Then

$$(a\phi; a)(h; e)(a\phi; a)^{-1} = ((a\phi)h_a(a\phi)^{-1}_{aea^{-1}}; aea^{-1}) \in V.$$

Therefore

$$(a\phi)h_a(a\phi)^{-1}_{aea^{-1}} \in A_{aea^{-1}}.$$

We observe that, as $h \in A_e \subseteq G^e$ we have that $h_e = h$, so

$$h_a = (h_e)_a = h_{ae} = (h_{ae})_{aea^{-1}} = h_{ae}1_{aea^{-1}} = h_a 1_{aea^{-1}}.$$

It follows that

$$(a\phi)h_a(a\phi)^{-1} = (a\phi)h_a 1_{aea^{-1}}(a\phi)^{-1} = (a\phi)h_a(a\phi)^{-1}1_{aea^{-1}} = (a\phi)h_a(a\phi)^{-1}_{aea^{-1}}.$$

Thus we have that $(a\phi)h_a(a\phi)^{-1} \in A_{aea^{-1}}$, so condition (iii) holds. This completes the proof of the first statement.

We now prove that $V = \mathbf{V}(E\zeta \cap V, V\Psi, \phi)$. We first suppose that $(h(a\phi); a) \in \mathbf{V}(E\zeta \cap V, V\Psi, \phi)$ so $a \in V\Psi$ and $h \in A_{aa^{-1}}$. Then $(h; aa^{-1}) \in V$, and also, by the definition of ϕ, $(a\phi; a) \in V$. Therefore

$$(h; aa^{-1})(a\phi; a) = (h(a\phi); a) \in V,$$

so we have that $\mathbf{V}(T, S, \phi) \subseteq V$.

For the reverse inclusion we suppose that $(g; a) \in V$. We certainly have that $a \in V\Psi$ so we have to show that there is $h \in A_{aa^{-1}}$ such that $g = h(a\phi)$. We know, by the definition of ϕ, that $(a\phi; a) \in V$, so

$$(g; a)(a\phi; a)^{-1} = (g; a)((a\phi)^{-1}_{a^{-1}}; a^{-1}) = (g(a\phi)^{-1}; aa^{-1}) \in V.$$

This implies that $g(a\phi)^{-1} \in A_{aa^{-1}}$, so we may take $h = g(a\phi)^{-1}$. This completes the proof. $\square$

Theorem 8.4.5. *Let (T, S, ϕ) be a crossed triple. Then $\mathbf{V}(T, S, \phi)$ is a full inverse subsemigroup of $G \wr \mathcal{I}_n$.*

Consequently, the subsemigroups $\mathbf{V}(T, S, \phi) \subseteq G \wr \mathcal{I}_n$ for crossed triples (T, S, ϕ) are all the full inverse subsemigroups of $G \wr \mathcal{I}_n$.

Proof. Let $V = \mathbf{V}(T, S, \phi)$ and let $A_e = \{g \in G^e \mid (g; e) \in T\}$. We show that V is a full inverse subsemigroup of $G \wr \mathcal{I}_n$. Suppose $(g(a\phi); a), (h(b\phi); b) \in V$, so $a, b \in S$, $g \in A_{aa^{-1}}$ and $h \in A_{bb^{-1}}$. Then we observe that

$$(g(a\phi); a)(h(b\phi); b) = (g(a\phi)h_a(b\phi)_a; ab).$$

By (ii) from Definition 8.4.2 we know that there is $k \in A_{abb^{-1}a^{-1}}$ such that

$$(ab)\phi = k(a\phi)(b\phi)_a.$$

By Definition 8.4.2(i), $b\phi \in G^{bb^{-1}}$ and (with reference to Lemma 5.2.12) it follows that $(b\phi)_a \in G^{abb^{-1}a^{-1}}$. As $k \in A_{abb^{-1}a^{-1}}$ and $a\phi \in G^{aa^{-1}}$ we have $k(a\phi) \in G^{abb^{-1}a^{-1}}$. Thus we have that $(ab)\phi$, $k(a\phi)$ and $(b\phi)_a$ are elements in $G^{abb^{-1}a^{-1}}$ and so

$$(a\phi)^{-1}k^{-1}(ab)\phi = (b\phi)_a.$$

Then

$$g(a\phi)h_a(b\phi)_a = g(a\phi)h_a(a\phi)^{-1}k^{-1}(ab)\phi.$$

By Definition 8.4.2(iii), as $h \in A_{bb^{-1}}$, we have $(a\phi)h_a(a\phi)^{-1} \in A_{abb^{-1}a^{-1}}$. We know $\{A_e \mid e \in \mathcal{P}_n\}$ is $\mathcal{P}_n$-closed (since T is a full inverse subsemigroup of $G \wr \mathcal{P}_n$), so, as $g \in A_{aa^{-1}}$ and $(a\phi)h_a(a\phi)^{-1}, k \in A_{abb^{-1}a^{-1}}$, we observe that

$$l = g(a\phi)h_a(a\phi)^{-1}k^{-1} \in A_{abb^{-1}a^{-1}}.$$

Then

$$(g(a\phi)h_a(b\phi)_a; ab) = (g(a\phi)h_a(a\phi)^{-1}k^{-1}(ab)\phi; ab) = (l(ab)\phi; ab) \in V,$$

so we have that V is a subsemigroup of $G \wr \mathcal{I}_n$.

That V is full follows from Lemma 8.4.3, as $1_e \in A_e$ for each $e \in \mathcal{P}_n$ so $(1_e; e) \in V$. We now show that V is inverse. Suppose that $(g(a\phi); a) \in V$, where $g \in A_{aa^{-1}}$. In $G \wr \mathcal{I}_n$,

$$(g(a\phi); a)^{-1} = ((a\phi)^{-1}_{a^{-1}} g^{-1}_{a^{-1}}; a^{-1}),$$

so we need to show that there is $h \in A_{a^{-1}a}$ such that

$$(a\phi)^{-1}_{a^{-1}} g^{-1}_{a^{-1}} = h(a^{-1}\phi).$$

By Definition 8.4.2(ii), noting that $a, a^{-1} \in S$, there is $k \in A_{a^{-1}a}$ such that

$$(a^{-1}a)\phi = k(a^{-1}\phi)(a\phi)_{a^{-1}}.$$

By Lemma 8.4.3, $(a^{-1}a)\phi \in A_{a^{-1}a}$, and, from Definition 8.4.2(i), $a\phi \in G^{aa^{-1}}$ and $a^{-1}\phi \in G^{a^{-1}a}$. Then (by Lemma 5.2.12) $(a\phi)_{a^{-1}} \in G^{a^{-1}a}$, so, from $(a^{-1}a)\phi = k(a^{-1}\phi)(a\phi)_{a^{-1}}$, we obtain that

$$(a\phi)^{-1}_{a^{-1}} = ((a^{-1}a)\phi)^{-1}k(a^{-1}\phi) = l(a^{-1}\phi)$$

where $l = ((a^{-1}a)\phi)^{-1}k \in A_{a^{-1}a}$. Furthermore, $g \in A_{aa^{-1}}$ so by Definition 8.4.2(iii), we have that $q = (a^{-1}\phi)g_{a^{-1}}(a^{-1}\phi)^{-1} \in A_{a^{-1}a}$. Then

$$(a\phi)^{-1}_{a^{-1}} g^{-1}_{a^{-1}} = l(a^{-1}\phi)g^{-1}_{a^{-1}}(a^{-1}\phi)^{-1}(a^{-1}\phi) = lq^{-1}(a^{-1}\phi).$$

Taking $h = lq^{-1}$ completes the proof that V is inverse. Thus we have that V is a full inverse subsemigroup. That every full inverse subsemigroup is of this form follows from Lemma 8.4.4, so we have completed the proof. $\square$

We remark that just as in the description of subgroups of semidirect products using the analogous crossed triples (Theorem 5.4.6) we do not have that each full inverse subsemigroup of $G \wr \mathcal{I}_n$ is described by a unique crossed triple. Given $V = \mathbf{V}(T, S, \phi) \subseteq G \wr \mathcal{I}_n$, the subsemigroups $T \subseteq G \wr \mathcal{P}_n$ and $S \subseteq \mathcal{I}_n$ are uniquely determined, but there is freedom with the choice of ϕ.

8.5 Subsemigroups of $(G_0)^n$

Full inverse subsemigroups of $(G_0)^n$ play an important role in our description of left congruences on $G \wr \mathcal{I}_n$. It behoves us to pay more attention to such subsemigroups; our current description is that they exactly correspond to $\mathcal{P}_n$-closed sets of subgroups. With this as motivation, we devote our final section to expanding upon the description of subgroups of direct products from [2], given in Theorem 5.4.3. While the description reached is technical it should - I hope - at least indicate the complexity of this problem.

We recall that each for each $e \in \mathcal{P}_n$ the group G^e is isomorphic to G^m for some m; it makes sense therefore to appeal to Theorem 5.4.3 and describe subgroups of G^e via the Goursat's decomposition. We recall that given a subgroup $H \leq G_1 \times \cdots \times G_m$ the Goursat's decomposition is a $(3m-2)$-tuple:

$$(A_1; A_2, B_2, \theta_1; \ldots; A_m, B_m, \theta_{m-1})$$

where for $1 \leq i \leq m$ we have $A_i \leq G_i$, for $2 \leq i \leq m$ we have $B_i \trianglelefteq A_i$ and for $1 \leq i \leq m-1$ we have that $\theta_i \colon \Lambda_i \twoheadrightarrow A_{i+1}/B_{i+1}$ is a surjective homomorphism with $\Lambda_i \leq A_1 \times \cdots \times A_i$ defined recursively: $\Lambda_1 = A_1$ and $\Lambda_{i+1} = \mathbf{\Gamma}(\Lambda_i, A_{i+1}, B_{i+1}, \theta_i)$ where

$$\mathbf{\Gamma}(A, B, C, \theta) = \{(a, b) \in A \times B \mid a\theta = bC\}.$$

The subgroup H is recovered from the Goursat's decomposition as $H = \Lambda_m$ and is equal to

$$\{(a_1, \ldots, a_m) \in A_1 \times \cdots \times A_m \mid \forall\, 1 \leq i \leq m-1,\ (a_1, \ldots, a_i)\theta_i = a_{i+1}B_{i+1}\}.$$

Further, for each $1 \leq i \leq m$, Λ_i is equal to the projection of H onto the first i coordinates.

To avoid having to differentiate between the first coordinate and other coordinates we remark that we can "symmetrise" the Goursat's decomposition. Formally we do this by appending a trivial group to the front of the direct product. Given $H \leq G_1 \times \cdots \times G_m$ a subgroup with Goursat's decomposition $(A_1; A_2, B_2, \theta_1; \ldots; A_m, B_m, \theta_{m-1})$ we note that

$$H \cong \{1\} \times H \subseteq \{1\} \times G_1 \times \cdots \times G_m.$$

Further it is straightforward to see that $\{1\} \times H$ has Goursat's decomposition

$$(A_0; A_1, A_1, \theta_0; A_2, B_2, \theta_1'; \ldots; A_m, B_m, \theta_{m-1}')$$

where θ_0 is the only function $\{1\} \to A_1/A_1$ and $\theta_i' \colon \{1\} \times \Lambda_i \to A_{i+1}/B_{i+1}$ is defined by $(1, l)\theta_i' = l\theta_i$. The A_0 is always a subgroup of $\{1\}$ so is redundant so we "remove" it. From now by Goursat's decomposition we refer to this symmetric version, so our Goursat's decompositions have the form

$$(A_1, B_1, \theta_0; A_2, B_2, \theta_1; \ldots; A_m, B_m, \theta_{m-1}).$$

The case we are interested in is when $G_i = G$ for each $1 \leq i \leq n$. Since full inverse subsemigroups of $G \wr \mathcal{P}_n$ correspond to $\mathcal{P}_n$-closed sets of subgroups it will be useful to amend our notation in order to give subgroups of the groups G^e. We use 0 to denote the zero in G_0 and write $\mathbf{0}$ for the zero function

$$\mathbf{0} \colon \bigcup_{1 \leq i \leq n} (G_0)^i \to 0.$$

Using this we may define Goursat's-type decompositions for subgroups of G^e in a (hopefully) obvious way. We recall the function $\Omega \colon (G_0)^n \to \bigcup_{0 \leq i \leq n} G_i$, which ignores zero entries.

Definition 8.5.1. Let $e \in E(\mathcal{I}_n)$ have $\operatorname{rank}(e) = m$ and let $H \leq G^e$ be a subgroup. Then the *special decomposition* for H is the $3n$-tuple

$$(A_1, B_1, \theta_0; A_2, B_2, \theta_1; \ldots; A_n, B_n, \theta_{n-1}),$$

where A_i and B_i are 0 and $\theta_{i-1} = \mathbf{0}$ whenever $i \notin \operatorname{Dom}(e)$, and the $3m$-tuple

$$(A_{i_1}, B_{i_1}, \theta_{i_0}; A_{i_2}, B_{i_2}, \theta_{i_1}; \ldots; A_{i_m}, B_{i_m}, \theta_{i_{m-1}})$$

obtained by ignoring the 0 and **0** entries is the Goursat's decomposition for $H\Omega$.

It follows immediately from Theorem 5.4.3, the result that states that subgroups of a direct product correspond to Goursat's decompositions, that subgroups of G^e correspond to special decompositions. The relationship between the special decomposition and the subgroup is also very similar to the groups case. Given a subgroup $H \leq G^e$, the associated special decomposition is

$$(A_1, B_1, \theta_0; A_2, B_2, \theta_1; \ldots; A_n, B_n, \theta_{n-1}),$$

where A_i is the projection of H onto the i^{th} coordinate and

$$B_i = \{a \in A_i \mid \exists x_j \in A_j \text{ for } j > i \text{ with } (\epsilon_1, \ldots, \epsilon_{i-1}, a, x_{i+1}, \ldots, x_n) \in H\},$$

where for $1 \leq j \leq i-1$, $\epsilon_j = 0$ if $A_j = \{0\}$ and $\epsilon_j = 1$ if $A_j \neq \{0\}$. Finally, we recover θ_i in the following way. Let Λ_i be the projection of H onto the first i coordinates. Then θ_i is defined by

$$\theta_i\colon \Lambda_i \to A_{i+1}/B_{i+1}; \quad (a_1, \ldots, a_i) \mapsto hB_{i+1} \text{ where } (a_1, \ldots, a_i, h) \in \Lambda_{i+1}.$$

Conversely, starting with $(A_1, B_1, \theta_0; A_2, B_2, \theta_1; \ldots; A_n, B_n, \theta_{n-1})$ a special decomposition, we construct Λ_i inductively as $\Lambda_1 = A_1$ and Λ_i is

$$\{(a_1, \ldots, a_i) \in A_1 \times \cdots \times A_i \mid \forall\, 1 \leq j \leq i-1,\ (a_1, \ldots, a_j)\theta_j = a_{j+1}B_{j+1}\}.$$

Then the subgroup H is equal to Λ_m and Λ_i is the projection of H onto the first i coordinates.

Our next result is very similar to, and follows directly from, Theorem 5.4.4, which describes the ordering of special decompositions of subgroups of G^e induced by the inclusion ordering on the subgroups.

Proposition 8.5.2. *Let H, K be subgroups of G^e with associated special decompositions*

$$\begin{gathered}(A_1^H, B_1^H, \theta_0^H; A_2^H, B_2^H, \theta_1^H; \ldots; A_n^H, B_n^H, \theta_{n-1}^H)\\ \textit{and} \quad (A_1^K, B_1^K, \theta_0^K; A_2^K, B_2^K, \theta_1^K; \ldots; A_n^K, B_n^K, \theta_{n-1}^K).\end{gathered}$$

Then $H \leq K$ if and only if for each $i \in \mathrm{Dom}(e)$ we have $A_i^H \leq A_i^K$, $B_i^H \leq B_i^K$, and for $x \in \Lambda_{i-1}$ with $x\theta_{i-1}^H = yB_i^H$ we have $x\theta_{i-1}^K = yB_i^K$ (in other words θ_{i-1}^K is an extension of the composition of θ_{i-1}^H with the quotient map $A_i^H/B_i^H \to A_i^H/(A_i^H \cap B_i^K)$).

Our attention now turns to considering what it means to be $\mathcal{P}_n$-closed. Let $H \leq G^e$ be a subgroup and let

$$(A_1, B_1, \theta_0; A_2, B_2, \theta_1; \ldots; A_n, B_n, \theta_{n-1})$$

be the special decomposition for H. Let $f \in E(\mathcal{I}_n)$ be such that $f \leq e$. Let Λ_i be the projection of H onto the first i coordinates. We define the subgroups of G and the functions which will appear in the special decomposition for $H1_f$. Relevant functions and groups used in this construction are shown in Fig. 8.1.

$$\begin{tikzcd}
\Lambda_i \arrow[r, "\theta_i"] & A_{i+1}/B_{i+1} \arrow[r, "\upsilon_{i+1}"] & A_{i+1}/B_{i+1}^f \\
\Lambda_i^f \arrow[u, "\sigma_i"] & A_{i+1} \arrow[u, "\pi_{i+1}"'] &
\end{tikzcd}$$

Figure 8.1: Functions for special decomposition projection

- If $i \notin \mathrm{Dom}(f)$ then $A_i^f = B_i^f = 0$ and $\theta_{i-1}^f = \mathbf{0}$.
- If $i \in \mathrm{Dom}(f)$ then $A_i^f = A_i$.
- If $1 \in \mathrm{Dom}(f)$ then let $B_1^f = B_1$ and let $\theta_0^f = \theta_0$.
- Let $1 \leq i \leq n-1$ be such that $i+1 \in \mathrm{Dom}(f)$. Then

  $$B_{i+1}^f = \{c \in A_{i+1} \mid \exists (g_1, \ldots, g_i, c) \in \Lambda_{i+1},\ \text{if } i \in \mathrm{Dom}(f) \text{ then } g_i = 1\}.$$

  We comment further on this expression for B_i^f. Let

  $$D = \{(g_1, \ldots, g_i) \in \Lambda_i \mid g_i = 1 \text{ when } i \in \mathrm{Dom}(f)\}$$

then let $C = D\theta_i$. Note that D is normal in Λ_i, so it follows that C is normal in A_{i+1}/B_{i+1}. Let π_{i+1} be the standard quotient map $A_{i+1} \to A_{i+1}/B_{i+1}$ Then, by the definitions of the relevant subgroups and functions, we see that

$$B^f_{i+1} = C\pi^{-1}_{i+1}.$$

As C is normal in A_{i+1}/B_{i+1} we have that B^f_{i+1} is normal in A_{i+1}

- For each $1 \leq i \leq n-1$ with $i+1 \in \mathrm{Dom}(f)$ let Λ^f_i be the projection of $H1_f$ onto the first i coordinates, so

$$\Lambda^f_i = \{(g_1, \ldots, g_i)1^i_f \mid (g_1, \ldots, g_i) \in \Lambda_i\}$$

where 1^i_f is the projection of 1_f onto the first i coordinates. Then let $\sigma_i \colon \Lambda^f_i \to \Lambda_i$ be any function such that $(g\sigma_i)1'_f = g$ and let $\upsilon_{i+1} \colon A_{i+1}/B_{i+1} \to A_{i+1}/B^f_{i+1}$ be the standard quotient homomorphism. Then define $\theta^f_i : \Lambda^f_i \to A_{i+1}/B^f_{i+1}$ by

$$\theta^f_i = \sigma_i\theta_i\upsilon_{i+1}.$$

Proposition 8.5.3. *Let $H \leq G^e$ be a subgroup with special decomposition $(A_1, B_1, \theta_0; A_2, B_2, \theta_1; \ldots; A_n, B_n, \theta_{n-1})$. Let $f \in E(\mathcal{I}_n)$ be such that $f \leq e$. Let A^f_i, B^f_i and θ^f_i be as defined above. Then the special decomposition for $H1_f$ is*

$$(A^f_1, B^f_1, \theta^f_0; A^f_2, B^f_2, \theta^f_1; \ldots; A^f_n, B^f_n, \theta^f_{n-1})$$

Proof. The proof is an exercise in definition chasing, and follows from the construction of the special decomposition. Say $H1_f$ has special decomposition

$$(C_1, D_1, \phi_0; C_2, D_2, \phi_1; \ldots; C_n, D_n, \phi_{n-1}).$$

Then as explained previously (following Definition 8.5.1) we obtain C_i, D_i and ϕ_i from $H1_f$ in the following way.

(i) For each i, C_i is the projection of $H1_f$ onto the i^{th} coordinate.

(ii) For each i

$$D_i = \{c \in C_i \mid \exists(\epsilon_1, \ldots, \epsilon_{i-1}, c, x_{i+1}, \ldots, x_n) \in H1_f\}$$

where for $1 \leq j \leq i-1$ in $\epsilon_i = 0$ if $C_j = \{0\}$ and $\epsilon_j = 1$ if $C_j \neq \{0\}$, and $x_j \in C_j$ for $i+1 \leq j \leq n$.

(iii) For each i, writing Λ_i^f for the projection of $H1_f$ onto the first i coordinates,

$$\phi_i \colon \Lambda_i^f \to C_{i+1}/D_{i+1}; \quad (g_1, \ldots, g_i) \mapsto h$$

where $(g_1, \ldots, g_i, h) \in \Lambda_{i+1}^f$.

It is immediate that $C_i = A_i$ when $i \in \mathrm{Dom}(f)$ and $C_i = 0$ when $i \notin \mathrm{Dom}(f)$, thus $C_i = A_i^f$ as claimed. It is also clear, for $0 \leq i \leq n-1$, that when $i \notin \mathrm{Dom}(f)$ both C_i and D_i are 0 and that ϕ_{i-1} is the unique map $\Lambda_{i-1}^f \to 0$, for which we are writing $\mathbf{0}$. Thus we have, when $i \notin \mathrm{Dom}(f)$, that $D_i = B_i^f = 0$ and $\phi_{i-1} = \theta_{i-1}^f = \mathbf{0}$ as claimed. If $1 \in \mathrm{Dom}(f)$ then, as D_1 and ϕ_0 are “artificial” elements, it is easily seen that $D_1 = B_1 = B_1^f$ and $\phi_0 = \theta_0 = \theta_0^f$.

Suppose that $i \in \mathrm{Dom}(f)$ and $i \geq 2$. We observe that $c \in D_i$ if and only if $c \in C_i$ with $(\epsilon_1, \ldots, \epsilon_{i-1}, c, x_{i+1}, \ldots, x_n) \in H1_f$ for some $x_j \in C_j$ for $i+1 \leq j \leq n$ and ϵ_j as defined above for $1 \leq j \leq i-1$. This precisely says that $(\epsilon_1, \ldots, \epsilon_{i-1}, c) \in \Lambda_i^f$, which in turn is equivalent to there being $(g_1, \ldots, g_i, c) \in \Lambda_i$ such that if $i \in \mathrm{Dom}(f)$ then $g_i = 1$. This is exactly the formulation for B_i^f, so we have shown that $D_i = B_i^f$.

We now assume that $i+1 \in \mathrm{Dom}(f)$ and we take $g = (g_1, \ldots, g_i) \in \Lambda_i^f$. Let $h = g\sigma_i$, so $h = (h_1, \ldots, h_i) \in \Lambda_i$ and $h1_f^i = g$ (where we recall 1_f^i is the projection of 1_f onto the first i coordinates), and say that $h\theta_i = cB_{i+1}$, so $(h_1, \ldots, h_i, c) \in \Lambda_{i+1}$. Since $i+1 \in \mathrm{Dom}(f)$, $(h_1, \ldots, h_i, c) \in \Lambda_i$ implies that $(g_1, \ldots, g_i, c) \in \Lambda_i^f$. This says that $h\phi_i = cD_{i+1} = cB_{i+1}^f$. As $g\sigma_i = h$ we have that

$$g(\sigma_i\theta_i\upsilon_{i+1}) = h\theta_i\upsilon_{i+1} = (cB_{i+1})\upsilon_{i+1} = cB_{i+1}^f = g\phi_i.$$

Therefore $\phi_i = \theta_i^f$. This completes the proof. $\square$

To conclude our discussion of full inverse subsemigroups of $(G_0)^n$ via the generalised Goursat's approach we combine Proposition 8.5.2 and Proposition 8.5.3 to describe the sets of special decompositions that arise from full inverse subsemigroups. We regard "0" as being included in the ordering of "subgroups" of G as the empty set.

Corollary 8.5.4. *For each $e \in \mathcal{P}_n$ let $H_e \leq G^e$ be a subgroup with special decomposition*

$$(A_1^e, A_1^e, \theta_0^e; A_2^e, B_2^e, \theta_1^e; \dots; A_n^e, B_n^e, \theta_{n-1}^e).$$

For $f \leq e$ define $(A_i^e)^f$, $(B_i^e)^f$ and $(\theta_i^e)^f$ as before Proposition 8.5.3. The union $\bigcup_{e \in \mathcal{P}_n} H_e$ is a full inverse subsemigroup of $(G_0)^n$ if and only if when $f \leq e$ and $i \in \mathrm{Dom}(f)$:

(i) $(A_i^e)^f \leq A_i^f$*;*

(ii) $(B_i^e)^f \leq B_i^f$*;*

(iii) if $x(\theta_{i-1}^e)^f = y(B_i^e)^f$ then $x\theta_{i-1}^f = yB_i^f$ (or equivalently $x(\theta_{i-1}^e)^f \subseteq x\theta_{i-1}^f$).

Proof. This follows from the definition of $\mathcal{P}_n$-closed and Proposition 8.5.2. Since $\{H_e \mid e \in \mathcal{P}_n\}$ is $\mathcal{P}_n$-closed we have for all $f \leq e$ that $H_e 1_f \subseteq H_f$. Applying Proposition 8.5.3 we obtain a special decomposition for $H_e 1_f$, which we write as

$$((A_1^e)^f, (B_1^e)^f, (\theta_0^e)^f; (A_2^e)^f, (B_2^e)^f, (\theta_1^e)^f; \dots; (A_n^e)^f, (B_n^e)^f, (\theta_{n-1}^e)^f).$$

We then apply Proposition 8.5.2 to obtain that $H_e 1_f \subseteq H_f$ if and only if $(A_i^e)^f \leq A_i^f$, $(B_i^e)^f \leq B_i^f$ and, for $x \in (\Lambda_{i-1}^e)^f$, if $x(\theta_{i-1}^e)^f = y(B_i^e)^f$ then $x\theta_{i-1}^f = yB_i^f$, which is exactly the claim in the result. $\square$

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